Stabilisation and Lower Dimensional Matrix String Models.

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Declaration

I declare that this dissertation is my own unaided work. It is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg, South Africa. It has not been submitted before for a degree or examination at any other University.

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CONTENTS

1. Introduction

1.1. The matrix model approach to the sum over random surfaces

The introduction of extended one dimensional objects (popularly called "strings") to replace point particles as fundamental physical objects has been met with mixed success. The rationale behind such a replacement has no direct experimental evidence,[1] (although the necessity of the string approach was first implied by a relatively successful attempt to explain resonances, due to Veneziano) and has its roots in an attempt to combat the pathological behaviour of a quantum field theoretic formulation of gravity. This undesirable behaviour stems from the ultraviolet divergences which occur when computing quantum corrections about some leading classical configuration. This behaviour is a feature of many quantum field theories; however, in most cases renormalization schemes permit the removal of unwanted infinities, a notable exception being a quantum field theoretic formulation of gravity which is non-renormalizable.

The rudiments of the string picture are as follows. One introduces a one dimensional object, which sweeps out a two dimensional worldsheet in a physical spacetime. Excitations of the string manifest themselves as the appearance of particles such as photons or electrons. This is to be contrasted with the approach of traditional quantum field theory, where the electron is described by a quantum field distinct from that of the photon, and the Lagrangian corresponding to the dynamics of the interacting particles would contain both quantum fields.

Thus one sees that some form of unification has been attempted, whereby distinct excitations of one underlying object give rise to the spectrum of observed particles. Apart from this aesthetically pleasing aspect of the string approach, one finds that, by and large, the pathology encountered in traditional QFT's[2] no longer arises. One may attempt to explain this heuristically by noting that

[1] It does, however, have convincing theoretical evidence; calculations within the string framework have reproduced Hawking radiation. The original calculation was an uneasy marriage of general relativity and quantum mechanics.

[2] QFT is taken to denote "quantum field theory"

string interactions are represented by topology changes on the string worldsheet manifold. As opposed to the interaction vertices of QFT's the interaction does not take place at a single spacetime point, and thus is not Lorentz invariant. One may regard the interaction as 'smeared out'. Ultraviolet divergences are found to arise in QFT's when two vertices are found to approach each other with arbitrary closeness. The prescription to regularize a QFT simply corresponds to 'smearing out' the interaction vertices, such that the interactions no longer take place at a single spacetime point, a feature which is naturally incorporated in the string picture.

In addition, one finds that the spectrum of excitations associated with strings necessarily contains a spin 2 vector boson, the virtual particle which is thought to mediate the gravitational interaction. This is greatly encouraging in view of the fact that one is able to define a perturbative expansion of string theory about a classical background which is finite and anomaly free at 1-loop [1], thus avoiding the staggering technical difficulties associated with point particle theories of quantum gravity.

However, the string approach is not without its subtleties. In particular, one runs into difficulties when quantizing strings, since conformal symmetry is not respected in an arbitrary number of spacetime dimensions. One requires conformal symmetry to ensure the diffeomorphism invariance of the string worldsheet. Dimensions in which the conformal anomaly does not arise are denoted "critical" dimensions. The critical dimension depends on the number of local supersymmetries defined on the worldsheet. One may construct conformally invariant string actions for $N = 0$, 1, 2 and 4 local supersymmetries (on the worldsheet); the critical dimension for $N = 0$ (the bosonic string) is $d = 26$. $N = 1$ supersymmetry requires 10 dimensions, $N = 2$ requires 4, while $N = 4$ does not seem to make sense, requiring a negative spacetime dimension. Thus, if one wishes to eliminate the conformal anomaly, the level of local supersymmetry uniquely fixes the dimensionality of spacetime. In order to make contact with the four physical spacetime dimensions, a reasonable mechanism of compactification must be postulated.

In the non-critical theories, one does not have the luxury of such cancellations, and the conformal anomaly is manifest. The significance of this is that the world sheet metric and matter fields no longer decouple. One is now no longer able to integrate out the world sheet metric completely, and consequently one is required to quantize these additional degrees of freedom. Quantizing the worldsheet metric is equivalent to a theory of two dimensional euclidean quantum gravity. Moreover, the string co-ordinates may be reinterpreted as conformal matter fields coupled to the world sheet gravity.

The motivation for studying theories of non-critical strings arose primarily in an attempt to obtain non-perturbative insights into string models; formerly, there were five consistent string theories, defined only in terms of perturbative expansions. In the case of non-critical string "toy models", the double scaling limit allows one to obtain contributions from all topologies, and thus must be regarded as a non-perturbative technique. Such non perturbative insights into critical strings have been realised only recently, with the advent of M theory, which appears to be the unique theory underlying the five "different" string theories. In particular, a supersymmetric matrix model has been proposed, which purports to describe M theory in the light cone gauge [37].

In this dissertation, our emphasis will focus mainly on the $d = 0$ model, which may be interpreted as pure two dimensional quantum gravity, or equivalently, as the propagation of a string in a zero dimensional spacetime.

1.1.1. The Correspondence between the $d = 0$ Hermitian Matrix Model and $2d$ Quantum Gravity

The following discussion is a synthesis of the approaches in refs [35], [2].

Let us consider the continuum approach. One starts with the Einstein-Hilbert action

$$S = \Lambda \int d^2x\sqrt{g} - \frac{1}{4\pi G}\int d^2x\sqrt{g}R \tag{1.1}$$

where g is the determinant of the metric tensor corresponding to the two dimensional euclidean space, and R is the Ricci curvature scalar. In two dimensions, the Gauss-Bonnet theorem requires that

$$\int d^2x\sqrt{g}R = 4\pi(2 - 2h) \tag{1.2}$$

where h is the genus of the surface. The first term straightforwardly yields the area of the surface. The partition function is written

$$Z_{qg} = \int Dg_{\mu\nu}\, e^{-S} \tag{1.3}$$

At this stage a caveat should be entered regarding the definition of the integration measure. It is implicit in the above writing of Z_{qg} that the measure be defined modulo diffeomorphisms. Formally, one is required to factor out the volume of the group of diffeomorphisms, and thus one has

$$Z_{qg} = \int \frac{Dg_{\mu\nu}}{V_d} e^{-S} = \sum_{g=0} \int_0^\infty dA\; e^{2(1-g)/G - \Lambda A} \int_{S_{g,A}} \frac{Dg_{\mu\nu}}{V_d} \tag{1.4}$$

where the functional integration over the metric is now confined to a surface of fixed area and genus. In other words, one would like to eliminate equivalent reparametrisations to consider only the volume of the moduli space. This is rather non–trivial in the continuum, and consequently one might rather consider the partition function defined on a lattice.

Let us consider constructing surfaces of any given genus via a "random triangulation", [3] where the triangles may be taken to be equilateral with a unit area. Consequently, the area of a surface comprising α triangles is simply α. The prescription implied by the measure $\int_{S_{g,A}} \frac{Dg_{\mu\nu}}{V_d}$ now becomes a sum over inequivalent triangulations corresponding to a surface of fixed genus, constructed from α triangles. Therefore, the equivalent lattice form of (1.4) is written

$$Z_{qg} = \sum_{g=0}^{\infty} \sum_{\alpha=0}^{\infty} e^{2(1-g)/G-\Lambda\alpha}\ G_{g,\alpha} \tag{1.5}$$

where $G_{g,\alpha}$ corresponds to the discretized measure. Thus, we have restated the problem in the hope that the computation of $G_{g,\alpha}$ will prove to be more tractable than the sum over metrics in the continuum.

Let us consider triangulating a compact, 2 sided genus g surface without a boundary. The topological properties of the triangulated surface are precisely those of the underlying surface, since the two are homeomorphic. One is easily able to verify the genus of any such surface by means of the Euler–Poincare′ characteristic. Specifically, one has the relation

$$V - E + F = 2 - 2h \tag{1.6}$$

where V corresponds to the number of vertices, E the number of edges, and F the number of triangles comprising the surface. Further, in the specific case of a triangulation, one has the identity $3F = 2E$, (which is easily seen to hold if one considers the simplest case of an unbounded triangulated surface, the tetrahedron) permitting one to rewrite the above as

$$V - \frac{1}{2}F = 2 - 2h \tag{1.7}$$

To proceed further, let us consider the $d = 0$ hermitian matrix model, with partition function

$$Z = \int dM\ e^{-S} \tag{1.8}$$

where M is a $N \times N$ hermitian matrix, with the action

$$S = \frac{1}{2}M^2 - \frac{g}{3\sqrt{N}}M^3 \tag{1.9}$$

One may now consider traditional perturbation theory in the coupling g, in which case one has

$$Z = \sum_\alpha \int dM \; e^{-\frac{1}{2}M^2} \left(g^\alpha N^{-\frac{\alpha}{2}} \frac{Tr(M^{3\alpha})}{(3!)^\alpha \alpha!} \right) \tag{1.10}$$

If one wished to evaluate the α'th term, one would lay down α three point vertices and connect them via propagators in the usual way, bearing in mind that the diagrammatic structure of the propagator and vertex has been altered since the matrices carry two indices. The diagrams now require double lines to specify the flow of the matrix indices. One notes further that any such diagram yields a factor $N^{-\frac{\alpha}{2}}$ corresponding to the vertices, a factor N^0 for p propagators, and N^l for l closed loops, as the factor N^0 for each propagator arises since the propagator is formally the inverse of the quadratic term in the matrix action, while the perturbative expansion (1.10) demands that each vertex contribute a factor of $N^{-1/2}$ to the diagram. Furthermore, each loop requires a sum over contracted matrix indices, yielding $\delta_{ii} = N$. Now, let us consider the lattice dual to the Feyman diagram. It is not difficult to see that the interaction vertices correspond to the faces of triangles, the propagators are dual to the edges, and loops dual to the vertices. Thus, the powers of N corresponding to the Feynman diagram may be rewritten in terms of the geometric components of the dual lattice as follows,

$$-\frac{\alpha}{2} + l = -\frac{F}{2} + V \tag{1.11}$$

which is nothing but the Euler–Poincare′ characteristic of the dual lattice, as discussed above. If we were to rescale $M \to \sqrt{N}M$, the action scales as $S \to NS$, and thus we have made the N dependence of the action explicit. It is then clear that, considering the Feynman diagram, one has a factor of N^{-p} for p propagators, N^α for α interaction vertices, and a factor N^l for l closed loops. Thus, the overall power of N is $\alpha - p + l$, which may be written in terms of the geometrical components of the dual lattice as follows

$$F - E + V = 2 - 2h \tag{1.12}$$

One sees that our rescaling has not affected the topological expansion. Thus, if one considers a surface of genus h or higher, any diagram carrying the factor $N^{2(1-h)}$ may be inscribed upon it without self intersection.

The matrix structure of the associated diagrams are crucial in the identification of such diagrammatic expansions with surfaces. The scalar integral

$$Z = \int d\varphi \; e^{-\frac{1}{2}\varphi^2 + \frac{g}{3!}\varphi^3} \tag{1.13}$$

admits the same diagrammatic expansion without the double line "flow" and is thus unable to specify the orientation of the surface.

A diagrammatic expansion to represent surfaces must necessarily include only connected diagrams. Thus it is expedient to consider the free energy, which is defined in the usual way by

$$Z = e^{N^2 F} \tag{1.14}$$

One may now consider a perturbative expansion in the free energy, which generates *only connected diagrams*. As before, we have

$$F_\alpha = \frac{N^\alpha g^\alpha}{(3!)^\alpha} \langle M^{3\alpha} \rangle_c \tag{1.15}$$

where the subscript c denotes the connected diagrams.

In summary, in order to evaluate F_α, and consequently the free energy, one is required to draw all distinct connected diagrams with α vertices, and adjoin them by means of propagators. In view of the fact that one wishes to consider a sum over surfaces, one may choose to arrange contributions corresponding to specific genuses as an expansion in the powers of N associated with each diagram,

$$N^2 F = \sum_{h=0}^{\infty} \sum_{\alpha=0}^{\infty} N^{2(1-h)} g^\alpha F_{h,\alpha} \tag{1.16}$$

where F_h represents the sum of all distinct diagrams which may be inscribed on a surface of genus h without self-crossing. One sees immediately that the large N limit yields the sum of all distinct connected diagrams which may be inscribed on a planar surface.[3] Furthermore, if one considers the dual lattice corresponding to the the three point vertices, one sees that the diagrams inscribed on a surface of fixed genus may be taken to represent a random triangulation in the spirit of discretized quantum gravity.

One now realises that the expansion (1.16) corresponds exactly to the lattice description of $2d$ quantum gravity, provided that one makes the identifications

$$e^{1/G} = N \tag{1.17}$$

[3]The realisation that one may construct a perturbation expansion in N was first explored by t'Hooft, in the context of QCD [4].

$$e^{-\Lambda} = g \tag{1.18}$$

noting that the object $F_{h,\alpha}$ corresponds precisely to $G_{h,\alpha}$. Consequently, the partition function of discretized $2d$ quantum gravity is described by the free energy of the $d=0$ hermitian matrix model. In the above, we have considered random triangulations of the surface. The above argument is perfectly general, and extends to the discretization of a surface by any general n–gon. The couplings in the matrix potential may always be chosen to yield the correct powers of N under the rescaling $M \to \sqrt{N}M$, thus ensuring the existence of a topological expansion. Indeed, for an arbitrary power n, the correct scaling is ensured by the choice $\frac{g_n}{N^{n/2-1}}Tr(M^n)$. Thus one can reconsider the above discussion with the general action $S = \sum_n \frac{g_n}{N^{n/2-1}}Tr(M^n)$ (and thus a discretization involving different polygons) without threatening the generality of the above arguments.

In the next section, we will discuss a mechanism of obtaining continuum results from the hermitian matrix model. In the continuum limit, the ability to generate diagrams with an admixture of vertices can be shown to represent the coupling of conformal matter fields to gravity. The cubic action used above yields a continuum limit which describes pure gravity in the absence of matter fields.

1.1.2. The Continuum Limit of the $d=0$ model

In order to make contact with continuum descriptions of quantum gravity (specifically, the Liouville model) one would like to define a continuum limit. The outcome of any such limit must be insensitive to the specific nature of the discretization employed.

Heuristically, such a limit involves tuning the couplings in the matrix action to a critical point, such that the perturbation series defined by (1.15) diverges. Naively, one then expects the area of the discretized surface to diverge, since the number of vertices has become infinite. To define a continuum surface of finite area, one would consider scaling the area of the elementary n–gons to zero.

It has been demonstrated that, for some coupling g in the vicinity of the critical coupling g_c, the genus h contribution to the free energy, $F_h[g]$, scales as

$$F_h[g] \sim (g - g_c)^{(2-\gamma_o)(1-h)} \tag{1.19}$$

where the critical exponent γ_o labels the universality class. In the case of random triangulations generated by the cubic potential (1.9), $\gamma_o = -1/2$. The average area of a surface is clearly proportional to the number of vertices α in the diagram, and therefore one has

$$\delta < \alpha >= \delta \left[g \frac{\partial}{\partial g} \ln F_h[g] \right] \sim \frac{\delta}{g - g_c} \tag{1.20}$$

where we have chosen to exhibit the unit area of the triangles explicitly as δ. Clearly, as $g \to g_c$ the area diverges for fixed δ. If one wishes to regain a continuum surface of finite area, (1.20) suggests that one take $\delta \to 0$ in a correlated manner, thus keeping the product $\delta < \alpha >$ finite. In summary, one tunes the coupling constants to their critical values, while simultaneously rescaling the area of the "elementary" polygons to zero, in order to obtain a continuum surface.

There are strong indications that the matrix model does indeed describe continuum gravity in the continuum, the first of which is that the scaling behaviours of both models are found to coincide. Additionally, continuum correlation functions computed in the matrix model scheme are found to correspond with those obtained via the Liouville picture. (See the review [35] for a discussion.)

Unfortunately, the above considerations are relevant only in the limit $N \to \infty$, and consequently involves only contributions corresponding to spherical topologies. Naturally, one would prefer to capture contributions from all genuses. To do so, let us reconsider the scaling behaviour of (1.19), which implies that

$$F_h[g] = f_h(g - g_c)^{(2-\gamma_0)(1-h)} \tag{1.21}$$

Further, one knows that

$$N^2 F = \sum N^{2(1-h)} F_h[g] = \sum N^{2(1-h)} f_h(g - g_c)^{(2-\gamma_0)(1-h)} \tag{1.22}$$

and therefore, one sees that if one defines the string coupling constant,,

$$\kappa = N^{-2}(g - g_c)^{\gamma_0 - 2} \tag{1.23}$$

(1.22) may be rewritten as a genus expansion in this coupling as follows

$$N^2 F = \sum_{h=0} \kappa^{h-1} f_h \tag{1.24}$$

One sees that if one were to take $N \to \infty$, and $g \to g_c$ in such a way as to keep κ constant, one would obtain contributions to the free energy from all genus surfaces. This technique of retaining contributions from higher genus surfaces in the limit of large N is known as the double scaling limit, first implemented in refs [32], [33] and [34]. One sees that such a technique is a way of performing the sum over all topologies, and thus may be regarded as yielding a non perturbative insight into non-critical strings.

As discussed earlier, the above techniques are applicable to general potentials, and not merely the cubic potential. However, the introduction of more general potentials introduces differing scaling properties, such that the scaling exponent γ_o takes on different values. To be more specific, one has the general relation $\gamma_o = -\frac{1}{m}$, where m is the m'th multicritical point. Such theories describe the coupling of conformal matter to gravity, with the central charge $c = 1 - \frac{3(3-2m)^2}{2m-1}$.

In general, one has the so called *minimal conformal field theories* (the 'minimal' requires a finite number of primary fields) which are labelled by two relatively prime integers, (p, q). Unitary discrete theories make up a subset of this general class, labelled by $(m+1, m)$. The hermitian one matrix model describes a class of minimal conformal field theories labelled by $(2m-1, 2)$ at its double scaling multicritical points. Thus, the theories defined by the matrix model are generally not unitary, although one sees that the $m = 2$ multicriticality of the hermitian one matrix model describes a unitary theory.

In this dissertation, we will mainly confine ourselves to the discussion of pure gravity, with central charge $c = 0$, and thus $m = 2$, unless stated otherwise.

2. The Continuum limit of the d =0 Hermitian Matrix model

The $d = 0$ model describes a string worldsheet embedded in zero spacetime dimensions. The model is defined by the partition function

$$Z = \int dM\ e^{-TrV(M)} \tag{2.1}$$

where M is an $N \times N$ hermitian matrix, and the potential is taken to be of the form

$$V(M) = \frac{\omega}{2}M^2 - \left(\frac{g}{N}\right)^{k/2-1}\frac{M^k}{k} \tag{2.2}$$

As discussed earlier, the rescaling $M \to \sqrt{N}M$ will permit us to classify diagrams corresponding to a specific genus according to their powers of N.

The measure dM has the explicit form

$$dM = \prod_{i<j} dRe(M_{ij})dIm(M_{ij})\prod_i dM_{ii} \tag{2.3}$$

One sees that in zero spacetime dimensions the path integral is merely a weighted integral over the space of hermitian matrices M, since M does not depend on any spacetime parameters.

Note that the action $TrV(M)$ displays a manifest invariance under the group $U(N)$, due to the cyclic nature of the trace. The invariance under this symmetry suggests the change of variables from the matrices M to the eigenvalues λ_i, in which case one only integrates over N degrees of freedom. Performing such a change of variables, one arrives at the well known result

$$Z = \int \prod_{i=1}^{N} d\lambda_i \prod_{i<j}(\lambda_i - \lambda_j)^2\ e^{-\sum_m V(\lambda_m)} \tag{2.4}$$

where we have discarded a numerical factor corresponding to the volume of the group $U(N)$. One recognises the factor $\prod_{i<j}(\lambda_i - \lambda_j)^2$ as the Jacobian of the transformation.

One may perform a further straightforward rewriting of (2.4), to yield

$$Z = \int \prod_{i=1}^{N} d\lambda_i \, e^{\sum_{i\neq j} \ln|\lambda_i - \lambda_j| - \sum_m V(\lambda_m)} \tag{2.5}$$

Let us consider recasting (2.5) in such a way as to exploit this symmetry, by transforming to the collective variable

$$\phi(x) = \sum_{i}^{N} \delta(x - \lambda_i) \tag{2.6}$$

This object obeys, by definition,

$$\int \phi(x)dx = N \tag{2.7}$$

Writing (2.5) in terms of the density variable, one has

$$Z = \int d\phi e^{\int dx\phi(x) \int \phi(y)dy \ln|x-y| - \int dx\phi(x)V(x)} \tag{2.8}$$

Now, in order to identify the dominant large N configuration, *one must exhibit* N explicitly. One knows that the potential depends on N implicitly, and further N dependence is found in the contraint. Thus, one must rescale such that both the potential and the constraint no longer depend on N. To do so, one performs the independent rescalings

$$x = \sqrt{\beta}\bar{x} \tag{2.9}$$

and

$$\phi(x) = \phi\left(\sqrt{\beta}\bar{x}\right) = \sqrt{\beta}\bar{\phi}(\bar{x}) \tag{2.10}$$

where we have defined $\beta \equiv N/g$. The choice of couplings in the relation (2.2) ensures that under such a rescaling,

$$V(x) = V\left(\sqrt{\beta}\bar{x}\right) = \beta\bar{V}(\bar{x}) \tag{2.11}$$

and

$$\int \phi(\bar{x})d\bar{x} = g \tag{2.12}$$

The barred fields have no implicit dependence on β. Further, (2.8) becomes

$$Z = \int d\phi e^{\beta^2\left(\int d\bar{x}\phi(\bar{x}) \int \phi(\bar{y})d\bar{y} \ln|\bar{x}-\bar{y}| - \int d\bar{x}\phi(\bar{x})V(\bar{x})\right)} \tag{2.13}$$

In order to compute the leading configuration associated with this "effective action", one considers the variation with respect to a single eigenvalue λ_k :

$$\frac{d}{d\lambda_k}\left[\sum_{i,j\ (i\neq j)} \ln|\lambda_i - \lambda_j| - \sum_i V(\lambda_i)\right] = 0 \tag{2.14}$$

One must therefore solve the *equation*

$$\sum_{i\neq j} \frac{1}{\lambda_i - \lambda_j} = \frac{1}{2} V'(\lambda_i) \tag{2.15}$$

One sees that (2.15) may be re-written as an integral equation for the rescaled density variable[1]

$$⨍ dy \frac{\phi(y)}{x-y} = \frac{1}{2} V'(x) \tag{2.16}$$

This is known as the saddlepoint equation, which was first solved in [6]. (The bars have been dropped for convenience; however, use of the constrained fields is implied.) The function $\phi(y)$ describes the density of eigenvalues of a hermitian matrix. In the limit of large N, one requires that the eigenvalues of a hermitian matrix be distributed along a finite interval on the real axis.[2] Thus, $\phi(y)$ must have compact support on the real axis. In order to solve (2.16) let us introduce the function

$$G(z) = \int_a^b dy \frac{\phi(y)}{z-y} \tag{2.17}$$

One sees that the analytic structure of $G(z)$ is such that there is a cut along the real axis from a to b.

Let us consider evaluating (2.17) at the point $z = x + i\epsilon$:

$$G(x+i\epsilon) = \int dy \frac{\phi(y)}{x-(y-i\epsilon)} \tag{2.18}$$

Written in this fashion, one sees that we have displaced the pole above the real axis. Consequently, this integral may be partitioned in the following way,

$$G(x+i\epsilon) = ⨍ dy \frac{\phi(y)}{x-y} + i\phi(x)\int_\pi^0 d\theta \tag{2.19}$$

[1]The dash superimposed on the integral sign implies the principal value prescription.
[2]The eigenvalues of hermitian matrices are real.

where we have parametrized the contour by taking $x = y + \epsilon e^{i\theta}$. One has the result

$$G(x + i\epsilon) = \fint dy \frac{\phi(y)}{x - y} - i\pi\phi(x) \tag{2.20}$$

A similar analyis allows us to conclude

$$G(x \pm i\epsilon) = \fint dy \frac{\phi(y)}{x - y} \mp i\pi\phi(x) \tag{2.21}$$

Hence, one sees that the imaginary part of the operator $G(z)$ yields the density of eigenvalues. One is now able to suggest a form for $G(z)$,

$$G(z) = \frac{1}{2}V'(z) + \sqrt{\Delta(z)} \tag{2.22}$$

with $\pi^2\phi^2(x) = -\Delta(x)$, for x inside the cut.

Depending on the form of the potential, one may introduce an ansatz for the function $\Delta(z)$. We require a one cut solution[3] which posesses the symmetry of the potential. Such requirements will permit us to determine the structure of $\Delta(z)$, up to some unknown constants. However, one requires further information regarding the behaviour of $G(z)$ in order to fix the values of such unknown constants in terms of the continuum values of the "physical" couplings. Such information is supplied by the asymptotic behaviour of $G(z)$:

$$G(z)_{\lim\ z\to\infty} = \frac{1}{z}\int_a^b dy \frac{\phi(y)}{1 - y/z} \approx \frac{1}{z}\int_a^b dy\phi(y) = \frac{g}{z} \tag{2.23}$$

where we have employed the constraint (2.12). One expands $\sqrt{\Delta(z)}$ asymptotically in powers of z, requiring that (2.22) correspond with (2.23). In such a way one is able to express the coupling constants in terms of the interval of compact support of $\phi(x)$. One is then able to determine the critical value of the couplings, following which it is possible to realise the continuum limit.

Let us specialise to the quartic potential, to give a clear indication of how the continuum limit is achieved.

[3]The one cut structure is equivalent to requiring the exponential falloff of loop amplitudes in the limit of infinite loop size. [15]

2.1. The $d = 0$ Continuum Limit corresponding to the Quartic Potential

We seek to solve the equation[4]

$$\fint dy \frac{\phi(y)}{x-y} = \frac{1}{2}\left(z - z^3\right) \tag{2.24}$$

for the density, $\phi(x)$. In the spirit of the preceding discussion, we take

$$G(z) = V'(z) + \sqrt{\Delta(z)} \tag{2.25}$$

where $\Delta(z)$ is related to the density in the simple fashion described above. The form of $\Delta(z)$ corresponding to a quartic potential is, for a symmetric ansatz,

$$\Delta(z) = \frac{1}{4}(z^2 - \alpha^2)^2(z^2 - \gamma^2) \tag{2.26}$$

where the constants α, γ remain to be fixed in terms of g.[5] Plotting $\Delta(z)$ yields a graph symmetric about the vertical axis, with four roots. The roots labelled by $\pm\gamma$ bracket the cut interval, with the roots $\pm\alpha$ exterior to it. To achieve the $k = 2$ multicritical point, one wishes incur the accumulation of an extra root at the cut endpoint.

In order to do so, let us expand (2.25) about the point $z = \infty$, to yield

$$G(z)_{\lim_{z\to\infty}} = \frac{z}{2}\left(1 - \frac{1}{2}\gamma^2 - \alpha^2\right) + \frac{\gamma^2}{4z}\left(\alpha^2 - \frac{\gamma^2}{4}\right) + \frac{\gamma^4}{16z^3}\left(1 - \frac{1}{2}\gamma^2\right) \tag{2.27}$$

Requiring agreement with the asymptotic expansion (2.23) gives rise to the following equations:

$$1 = \frac{\gamma^2}{2} + \alpha^2 \tag{2.28}$$

$$\frac{\gamma^2}{4}\left(\alpha^2 - \frac{\gamma^2}{4}\right) = g \tag{2.29}$$

and

$$W_2 = \frac{\gamma^4}{16}\left(1 - \frac{1}{2}\gamma^2\right) \tag{2.30}$$

[4] We have fixed ω=1 for convenience.

[5] It is also conventional to denote $V(z) = -\Delta(z)$, as will be shown later in the context of the $d = 1$ model.

Thus, we have expressed the interval of support of $\phi(x)$ in terms of the physical couplings. One may further eliminate α from (2.29), and solve for γ in terms of g,

$$\gamma^2 = \frac{2}{3} \pm \frac{2}{3}\sqrt{1-12g} \tag{2.31}$$

It has been demonstrated that the critical value of the coupling is reached when γ^2 becomes complex, since the energy is no longer analytic there. Thus, g_c for the quartic potential is found to be

$$g_c = \frac{1}{12} \tag{2.32}$$

The value g_c corresponds to the value of the coupling which ensures the divergence of the planar diagrams, as explained above. This value of the coupling requires that an extra zero of the density accumulate at one of the endpoints of its support. Quite generally, the m'th multicritical point requires an accumulation of $m-1$ zeros at the endpoints of the support of the density. For convenience, (2.31) may be re-written as

$$\gamma^2 = \frac{2}{3} \pm 2f(g) \tag{2.33}$$

with a similar expression for α:

$$\alpha^2 = \frac{2}{3} \mp f(g) \tag{2.34}$$

The cut is located in the interval $(-\gamma, \gamma)$. A plot of V_{eff} One would like to tune $\alpha \to \gamma$, and consequently we note that as one varies g from zero to g_c, the function $f(g)$ varies smoothly through the interval $\frac{1}{3} < f(g) < 0$. Therefore the choice of signs

$$\gamma^2 = \frac{2}{3} - 2f(g) \tag{2.35}$$

$$\alpha^2 = \frac{2}{3} + f(g) \tag{2.36}$$

ensures consistency with figure (1.1). With the benefit of hindsight, one sees that tuning g to its critical value g_c is equivalent to requiring $\alpha \to \gamma$, as $\alpha_c^2 = \frac{2}{3} = \gamma_c^2$. Alternatively, one might first have determined the critical value of γ^2,and hence g_c, by forcing the local maximum ξ to zero, which is equivalent to requiring $\alpha \to \gamma$.

The scaling limit is defined by

$$g = g_c - \frac{3}{16}\Lambda a^2 \tag{2.37}$$

and

$$x = \gamma_c - \frac{a}{2\gamma_c}\zeta \tag{2.38}$$

where a a plays the role of a short distance cut-off with dimensions of length, which is taken to zero in the continuum limit. From (2.37) one sees that, in the vicinity of the critical point, equations (2.35) and (2.36) become

$$\gamma^2 = \gamma_c^2 - \sqrt{\Lambda}a \tag{2.39}$$

$$\alpha^2 = \alpha_c^2 + \frac{1}{2}\sqrt{\Lambda}a$$

Thus, inserting these expressions into the ansatz for the density,

$$\pi\phi(x) = \frac{1}{2}(\alpha^2 - x^2)\sqrt{\gamma^2 - x^2} \tag{2.40}$$

one obtains

$$\pi\phi(\zeta) = \frac{a^{3/2}}{2}\left(\zeta + \frac{\sqrt{\Lambda}}{2}\right)\sqrt{\zeta - \sqrt{\Lambda}} + O(a^{5/2}) \tag{2.41}$$

for $\zeta > \sqrt{\Lambda}$.

Thus, the universal form for the density corresponding to the $m = 2$ multicritical point is, up to a rescaling of a,

$$\phi(\zeta) = \sqrt{\zeta^3 - \frac{3}{4}\Lambda\zeta - \frac{\Lambda^{3/2}}{4}} \tag{2.42}$$

as given in [15].

2.2. The $d = 0$ Continuum Limit corresponding to the Cubic Potential

Let us now consider extracting the continuum form of the density by means of the cubic potential, in order to explicitly demonstrate the equivalence between the cubic and quartic continuum limits.

As before, our starting point is the equation

$$⨍ dy \frac{\phi(y)}{x - y} = \frac{1}{2}\left(z - z^2\right) \tag{2.43}$$

One chooses the ansatz

$$\sqrt{\Delta(z)} = \frac{1}{2}\left(z - 1 + \frac{1}{2}(\alpha+\gamma)\right)\sqrt{(z-\alpha)(z-\gamma)} \tag{2.44}$$

noting that it is no longer symmetric. The factor in front of the square root is chosen to ensure that no powers of z and z^2 contribute asymptotically. As before, one expands $G(z) = \frac{1}{2}V'(z) + \sqrt{\Delta(z)}$ about the point $z = \infty$, requiring the correct asymptotic behaviour to yield the following set of equations:

$$\frac{3}{4}(\alpha+\gamma)^2 - (\alpha+\gamma) = \alpha\gamma \tag{2.45}$$

$$\frac{1}{16}(\alpha-\gamma)^2(1-(\alpha+\gamma)) = g \tag{2.46}$$

Introducing the variable $\sigma = \alpha + \gamma$, one is readily able to derive the equation

$$\sigma(\sigma-1)(\sigma-2) - 8g = 0 \tag{2.47}$$

with the help of relations (2.45) and (2.46). One can show that the energy picks up a singularity where equation (2.47) acquires a double root. Thus, one finds the critical value of the coupling is given by[6]

$$g_c^2 = \frac{1}{432} \tag{2.48}$$

The reader may easily verify that equation (2.47) acquires a double root for this value of the coupling.

In the vicinity of the critical point, one may write

$$g = g_c - \Lambda a^2 \tag{2.49}$$

In what follows, we choose to take $g_c = 1/12\sqrt{3}$. One then readily finds that the critical value of σ for this value of the coupling is $\sigma_c = 1 - 1/\sqrt{3}$. Thus, to obtain an expansion in powers of a, one notes that iterating (2.47) about the point $\sigma = \sigma_c - \sqrt{\Lambda}a$ yields

$$\sigma = \sigma_c - \sqrt{\Lambda}a + O(a^2) \tag{2.50}$$

One is interested only in powers of $O(a)$ since, due to the structure of the ansatz, higher powers cannot yield the correct scaling of $a^{3/2}$ at the critical point.

[6]Choosing the sign of g_c corresponds to choosing which zero will accumulate at the endpoint of support of ϕ.

Using the equations (2.45) and (2.46) one is able to claim series expansions for both α and γ, in terms of a, where we have implicitly taken $\alpha > \gamma$:

$$\alpha = \frac{\sqrt{3}+1}{2\sqrt{3}} - \sqrt{\Lambda}a + O(a^2) \tag{2.51}$$

$$\gamma = \frac{1-\sqrt{3}}{2} + O(a^2) \tag{2.52}$$

It is seen that γ does not scale, to order $O(a^2)$. In the limit $a \to 0$, one requires $\alpha \to \alpha_c$, and thus $\alpha_c = (\sqrt{3}+1)/2\sqrt{3}$.

To consider the continuum limit more closely, one writes

$$\sqrt{\Delta(z)} = \frac{1}{2}(z-\mu)\sqrt{(\alpha - z)(z-\gamma)} \tag{2.53}$$

with $\mu = 1 - \frac{1}{2}\sigma$, in which case one sees that, in the vicinity of the critical point, μ scales as

$$\mu = \alpha_c + \frac{1}{2}\sqrt{\Lambda}a \tag{2.54}$$

Thus, as one tunes $g \to g_c$,the density ansatz acquires a double root at the endpoint of its support, which is a signature of the continuum limit.

In the vicinity of the critical point, we take

$$z = \alpha_c - a\zeta \tag{2.55}$$

The density has the form

$$\pi\phi(z) = \frac{1}{2}(\mu - z)\sqrt{(\alpha - z)(z-\gamma)} \tag{2.56}$$

noting that it is the imaginary part of $G(z)$.

Substituting the relations (2.51), (2.52) and (2.55) in the above expression yields

$$\pi\phi(\zeta) = \frac{a}{2}\left(\zeta + \frac{\Lambda^{1/2}}{2} + O(a)\right)\sqrt{\frac{2}{\sqrt{3}}a\left(\zeta - \Lambda^{1/2}\right) + O(a^2)} \tag{2.57}$$

and thus one finally obtains

$$\pi\phi(\zeta) = a^{3/2}\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^{1/2}\left(\zeta + \frac{\Lambda^{1/2}}{2}\right)\sqrt{\zeta - \Lambda^{1/2}} + O(a^{5/2}) \tag{2.58}$$

with $\zeta > \Lambda^{1/2}$. This result is equivalent to the form of the density obtained by means of the quartic potential, up to an irrelevant numerical factor which may be absorbed into a. Thus one sees that the density has a universal form regardless of the underlying matrix action, where we consider only potentials associated with the $m = 2$ multicritical point.

3. Stabilisation and Stochastic Quantization

In our discussion of the previous section, the equivalence of the cubic and quartic continuum limits was demonstrated. There are, however, various difficulties associated with the cubic action, which defines a "bottomless" theor. Naturally, the "bottomless" nature of the action is invisible in perturbation theory, and a signature of bottomless theories are the appearance of instantons, which are intrinsically non perturbative effects. These correspond to the tunneling of eigenvalues into the infinite potential well. In general, critical matrix potentials of even multicriticality k are bottomless, and will admit such nonperturbative instabilities. [15], [14], [16], [18].

Various attempts to "stabilise" such bottomless theories have been proposed, in particular the method of Greensite *et al* [13].

Let us briefly review the method of stochastic quantization, which is employed in the study of euclidean field theory, and is of key importance in the understanding of stabilisation mechanisms.

Stochastic quantization is generally used in the context of a d dimensional euclidean field theory, defined by an action $S[\varphi]$. As usual in field theory, one would wish to compute correlators of some observable $G[\varphi]$,

$$\langle G[\varphi]\rangle = \int \mathcal{D}\varphi G[\varphi]e^{-S[\varphi]} \tag{3.1}$$

To do so, let us consider the Fokker–Planck formalism. In this case, correlators are computed in the following manner,

$$\langle G[\varphi]\rangle = \lim_{\tau\to\infty} \int \mathcal{D}\varphi G[\varphi]\Psi[\varphi,\tau] \tag{3.2}$$

where $\Psi[\varphi,\tau]$ is considered to be a probability distribution, and τ is to be regarded as a 'fictitious' time. The propagation of $\Psi[\varphi,\tau]$ in the 'fictitious' time is governed by the equation

$$\frac{\partial\Psi[\varphi,\tau]}{\partial\tau} = -H_{FP}^{\dagger}\Psi[\varphi,\tau] \tag{3.3}$$

where $H_{FP}^{\dagger}$ is a Hamiltonian of the Fokker–Planck type. It is of the form

$$H_{FP} = -\frac{1}{2}e^{S}\frac{\partial}{\partial\varphi}e^{-S}\frac{\partial}{\partial\varphi} \tag{3.4}$$

which is clearly not hermitian. In fact, the hermitian conjugate of (3.4) is

$$\begin{aligned} H_{FP}^{\dagger} &= \left(-\frac{1}{2}e^{S}\frac{\partial}{\partial\varphi}e^{-S}\frac{\partial}{\partial\varphi}\right)^{\dagger} \\ &= -\frac{1}{2}\left(e^{-S}\frac{\partial}{\partial\varphi}\right)^{\dagger}\left(e^{S}\frac{\partial}{\partial\varphi}\right)^{\dagger} \\ &= -\frac{1}{2}\frac{\partial}{\partial\varphi}e^{-S}\frac{\partial}{\partial\varphi}e^{S} \end{aligned} \tag{3.5}$$

Note that $\left(\frac{\partial}{\partial\varphi}\right)^{\dagger} = -\frac{\partial}{\partial\varphi}$, but in the above the overall *minus sign remains unchanged*, since we conjugate twice.

One readily sees from (3.3) that the propagator may formally be written

$$\Psi[\varphi,\tau] = e^{-H_{FP}^{\dagger}\tau}\Psi[\varphi,0] \tag{3.6}$$

It is important to note that $H_{FP}^{\dagger}$ is related by a similarity transformation to a hermitian Hamiltonian:

$$H = e^{S/2}H_{FP}^{\dagger}e^{-S/2} \tag{3.7}$$

The significance of this is that the eigenvalues of H_{FP} must be real. The non-hermeticity of H_{FP} implies that one should be cautious when considering the states upon which one allows the Hamiltonian to act. In particular, consider the following inner product, where the *hermitian* Hamiltonian acts on its eigenstates,

$$\langle n|\, H\, |n\rangle \tag{3.8}$$

which is readily rewritten with the aid of (3.7) as

$$\langle n_L|\, H_{FP}^{\dagger}\, |n_R\rangle \tag{3.9}$$

where we have scaled

$$\langle n_L| = \langle n|\, e^{S/2} \tag{3.10}$$

and

$$|n_R\rangle = e^{-S/2}\,|n\rangle \tag{3.11}$$

Thus, one readily sees that the eigenstates of H_{FP} and $H^{\dagger}_{FP}$ are $|n_L\rangle$, and $|n_R\rangle$, repectively. It is also clear that if

$$\langle n | m \rangle = \delta_{nm} \tag{3.12}$$

then so too one has

$$\langle n_L | m_R \rangle = \delta_{nm} \tag{3.13}$$

from the above definitions.

A key point regarding the stochastic formalism is simply that the hermitian form of the Fokker–Planck Hamiltonian may be written in a positive, semi-definite form

$$H = -\frac{1}{2} \hat{A}^{\dagger} \hat{A} \tag{3.14}$$

where the operator $\hat{A}$ takes the form

$$\hat{A} = e^{-S/2} \frac{\partial}{\partial \varphi} e^{S/2} \tag{3.15}$$

implying that the energy spectrum is not only real, but positive semi-definite, *ie* that $\varepsilon_n \geq 0$. If one were to assume the existence of a ground state wavefunctional with zero eigenvalue with respect to the hermitian Hamiltonian, one could write

$$\langle \varphi | H | 0 \rangle = 0 \tag{3.16}$$

Inserting the explicit form of H, one obtains an equation for the ground state wavefunctional,

$$\left(\frac{\partial}{\partial \varphi} - \frac{1}{2} \frac{\partial S}{\partial \varphi} \right) \left(\frac{\partial}{\partial \varphi} + \frac{1}{2} \frac{\partial S}{\partial \varphi} \right) \langle \varphi | 0 \rangle = 0 \tag{3.17}$$

The above equation has two possible solutions, namely $\langle \varphi | 0 \rangle = e^{-S/2}$ and $\langle \varphi | 0 \rangle = e^{S/2}$. For a well defined (bounded) action $S[\varphi]$ one would choose $\langle \varphi | 0 \rangle = e^{-S/2}$ $e^{-S/2}$ is crucial to the recovery of the correct correlators in the limit $\tau \rightarrow \infty$.

From the definitions (3.10) and (3.11) it becomes clear that $\langle \varphi | 0 \rangle = e^{-S/2}$ implies

$$\langle \varphi | 0_R \rangle = e^{-S} \tag{3.18}$$

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implies

$$\langle \varphi | 0_R \rangle = e^{-S} \tag{3.18}$$

and

$$\langle 0_L | \varphi \rangle = 1 \tag{3.19}$$

Let us consider the object

$$\lim_{\tau\to\infty}\langle G\,|\Psi(\tau)\rangle = \langle G|\,e^{-H^{\dagger}_{FP}\tau}\,|\Psi(0)\rangle \tag{3.20}$$

Inserting a complete set of states, the above becomes

$$\lim_{\tau\to\infty}\langle G\,|\Psi(\tau)\rangle = \lim_{\tau\to\infty}\sum_{n}\frac{\langle G|\,e^{-H^{\dagger}_{FP}\tau}\,|n_R\rangle\,\langle n_L\,|\Psi(0)\rangle}{\langle n_L\,|n_R\rangle} \tag{3.21}$$

Since the ket $|n_R\rangle$ is an eigenstate of $H^{\dagger}_{FP}$, one may further write

$$\lim_{\tau\to\infty}\langle G\,|\Psi(\tau)\rangle = \lim_{\tau\to\infty}\sum_{n}e^{-\varepsilon_n\tau}\frac{\langle G|\,n_R\rangle\langle n_L\,|\Psi(0)\rangle}{\langle n_L\,|n_R\rangle} \tag{3.22}$$

Taking the limit as $\tau\to\infty$, and assuming that there is an energy gap to the first excited state,[1] one has

$$\lim_{\tau\to\infty}\langle G\,|\Psi(\tau)\rangle = \frac{\langle G|\,0_R\rangle\langle 0_L\,|\Psi(0)\rangle}{\langle 0_L\,|0_R\rangle} \tag{3.23}$$

Inserting a complete set of states allows us to rewrite $\langle 0_L\,|0_R\rangle$ as

$$\begin{aligned}\langle 0_L\,|0_R\rangle &= \int d\varphi\,\langle 0_L|\,\varphi\rangle\langle\varphi\,|0_R\rangle\\ &= \int d\varphi e^{-S}\end{aligned} \tag{3.24}$$

where we have employed the relations (3.18) and (3.19), valid for $\varepsilon_0 = 0$. Similarly, one has

$$\begin{aligned}\langle F|\,0_R\rangle &= \int d\varphi\,\langle F|\,\varphi\rangle\langle\varphi\,|0_R\rangle\\ &= \int d\varphi F[\varphi]e^{-S}\end{aligned} \tag{3.25}$$

Furthermore, a consistent notion of probability requires that

$$\langle 0_L\,|\Psi(0)\rangle = \int d\varphi\,\langle 0_L|\,\varphi\rangle\langle\varphi\,|\Psi(0)\rangle = 1 \tag{3.26}$$

Thus, one sees that the object

$$\lim_{\tau\to\infty}\langle G\,|\Psi(\tau)\rangle = \langle G[\varphi]\rangle = \frac{\int d\varphi G[\varphi]e^{-S}}{\int d\varphi e^{-S}} \tag{3.27}$$

[1]That is, assume no degeneracy of the ground state.

as claimed earlier. Thus, one notes that stochastic quantization provides an alternative means of quantizing a theory, which is particularly appropriate in the case of the $d = 0$ model. This issue will be addressed later.

Additionally, stochastic quantization plays a key role in stabilising a theory. When considering the case of a bottomless action in the context of standard QFT, the theory is not well defined. Ideally, what one would wish to do is to incorporate some method of stabilisation, such that the effective action so obtained yields the same perturbative expansion as that of the original bottomless action. Furthermore, one might reasonably require consistent large N and classical limits.

The manner in which this is accomplished relies on the replacement of the euclidean correlator as the defining object of the theory by the semi positive definite Fokker–Planck Hamiltonian.

Let us first consider the case of a bounded action. One works from the perspective of a standard quantum mechanics. In other words, one inserts the action in the Fokker–Planck action, and solves for the corresponding wavefunctionals. As discussed above, the energy eigenvalue $\varepsilon_0 = 0$ has, quite generally, the corresponding wavefunctional $\langle\varphi|0\rangle = e^{-S[\varphi]/2}$. The central issue here, however, is the normalisability of the Boltzmann weight. In the case of a bounded action, the Boltzmann weight is normalisable, and thus the wavefunctional $\langle\varphi|0\rangle = e^{-S[\varphi]/2}$ corresponds to a physical state. The ground state energy is thus given by $\varepsilon_0 = 0$.

In the case of a bottomless action, one again finds the solution $\langle\varphi|0\rangle = e^{-S[\varphi]/2}$ corresponding to $\varepsilon_0 = 0$. However, this wavefunctional is not normalisable, and consequently cannot represent a physical state. Thus the "true" groundstate of the system must satisfy $\varepsilon_0 > 0$ due to the semi positive definite nature of the hamiltonian.[2] The wavefunctional which corresponds to the first non zero energy eigenstate may be regarded as the physical ground state. In the interests of obtaining a stabilised theory, one may write this wavefunctional in the suggestive form

$$\langle\varphi|1\rangle = e^{-S_{eff}/2} \tag{3.28}$$

The proposal for stabilisation is thus to replace the previous bottomless action with the effective action corresponding to the physical ground state. One may thus rather consider the stabilised euclidean correlators

$$\langle G[\varphi]\rangle = \int [d\varphi]\; G[\varphi] e^{-S_{eff}[\varphi]} \tag{3.29}$$

as being meaningful.

[2] We tacitly assume that the groundstate is not degenerate.

In the following sections we will recast the $d = 0$ model, employing the formalism of stochastic quantization. At this stage, it is not at all clear whether the large N behaviour of the original $d = 0$ theory is modified by this rewriting of the theory.[3] In what follows, we will demonstrate that they are, in fact, consistent. In general, we will not employ the full machinery of stabilisation in this dissertation–rather, we will use the term to imply the rewriting of the theory in terms of the Fokker–Planck Hamiltonian.

For the moment, let us discuss the $d = 1$ model, which has great relevance for what follows.

3.1. Collective Field Theory and the $d = 1$ model

The $d = 1$ model is defined by the quantum mechanics of a $N \times N$ hermitian matrix,

$$H = Tr\left(-\frac{1}{2}\frac{\partial^2}{\partial M^2} + V(M)\right) \tag{3.30}$$

In order to facilitate the study of this model, it is expedient to make a change of variables to invariant collective variables [28], since it is the singlet sector which has relevance for string theory. The collective field approach to $d = 1$ strings was first considered in [29]. Collective field theory is based on the identification of a symmetry of the theory, following which a change of variables is made to a set of collective fields which are manifestly invariant under this symmetry. For the moment, we will consider making a change of variables to generic collective fields ϕ_i, which are assumed to be invariant under $U(N)$ transformations.[4]

Let us first consider the kinetic term,[5]

$$H_o = -\frac{1}{2}\frac{\partial^2}{\partial M_{ij}\partial M_{ji}} \tag{3.31}$$

Invoking the chain rule permits us to write

$$H_o = -\frac{1}{2}\frac{\partial}{\partial M_{ij}}\left(\frac{\partial\phi_m}{\partial M_{ji}}\frac{\partial}{\partial\phi_m}\right) \tag{3.32}$$

[3] The correspondence on the level of the sphere was first demonstrated in [14] in the context of the Marinari Parisi model.

[4] One may think of this as reformulating the theory in terms of "gauge invariant" quantities.

[5] Repeated indices imply summation, unless expressly stated to the contrary.

and again

$$H_o = -\frac{1}{2}\left(\frac{\partial^2 \phi_m}{\partial M_{ij}\partial M_{ji}}\frac{\partial}{\partial \phi_m} + \frac{\partial \phi_n}{\partial M_{ij}}\frac{\partial \phi_m}{\partial M_{ji}}\frac{\partial}{\partial \phi_m}\frac{\partial}{\partial \phi_n}\right) \tag{3.33}$$

In what follows, it will prove convenient to define the following variables

$$\Omega_{nm} \equiv \frac{\partial \phi_n}{\partial M_{ij}}\frac{\partial \phi_m}{\partial M_{ji}} \tag{3.34}$$

and

$$\omega_m \equiv -\frac{\partial^2 \phi_m}{\partial M_{ij}\partial M_{ji}} \tag{3.35}$$

The Ω_{nm} may be considered to describe loops joining, while conversely the ω_m depict loop splitting processes, which will become clearer as one works with the specific forms of the invariant variables. The kinetic piece of the Hamiltonian thus takes the form

$$H_o = \frac{1}{2}\left(\omega_m \frac{\partial}{\partial \phi_m} - \Omega_{nm}\frac{\partial}{\partial \phi_m}\frac{\partial}{\partial \phi_n}\right) \tag{3.36}$$

In order to canonically quantize this Hamiltonian, one is required to impose the commutator

$$[\pi_n, \phi_m] = -i\delta_{nm} \tag{3.37}$$

in which case one readily sees that $\pi_n = -i\frac{\partial}{\partial \phi_n} \equiv -i\partial_n$. One is now able to rewrite (3.36) in terms of the conjugate momentum as follows

$$H_o = \frac{1}{2}\left(i\omega_m \pi_m + \Omega_{nm}\pi_n\pi_m\right) \tag{3.38}$$

One notes that the above Hamiltonian no longer retains the property of explicit hermiticity, following the change of variables to collective fields. In order to understand why this is so, let us consider the expectation value of an observable computed prior to the change of variables:

$$\langle \hat{O} \rangle = \int dM\, \Psi[M]\hat{O}[M]\Psi[M] \tag{3.39}$$

Clearly, if one wished to rewrite the above integral in terms of the ϕ_i's, one must include the Jacobian $J[\phi]$ of the transformation

$$\langle \hat{O} \rangle = \int d\phi J[\phi]\Psi[\phi]\hat{O}[\phi]\Psi[\phi] \tag{3.40}$$

One readily sees that the above may be rewritten

$$\langle\hat{O}\rangle = \int d\phi\Phi[\phi]\hat{O}'[\phi]\Phi[\phi] \tag{3.41}$$

where we have defined $\hat{O}'[\phi] \equiv J[\phi]^{1/2}\hat{O}[\phi]J[\phi]^{-1/2}$, and $\Phi[\phi] \equiv J[\phi]^{1/2}\Psi[\phi]$. Thus, to write the inner product in the space of the new invariant variables with respect to a trivial measure requires that one rescale the wavefunctionals and operators accordingly.

As a result, one sees that the Hamiltonian (3.38) must be transformed as

$$H_0' \equiv J^{1/2}H_0J^{-1/2} \tag{3.42}$$

in order to retain hermiticity. In principle, knowing the Jacobian would permit us to write down a hermitian Hamiltonian. In practice, one enforces the hermiticity requirement as a means of obtaining a differential equation to compute J.

Let us consider transforming the conjugate momentum

$$\pi_n' = J^{1/2}\pi_nJ^{-1/2} \tag{3.43}$$

which becomes

$$\pi_n' = \frac{i}{2}\partial_n \ln J + \pi_n \tag{3.44}$$

One is able to transform (3.38) by inserting the relation (3.44), and requiring all explicitly non–hermitian terms to vanish. Thus, one has

$$H_o' = \frac{1}{2}\left(i\omega_m\pi_m' + \Omega_{nm}\pi_m'\pi_n'\right) \tag{3.45}$$

which, after expanding the above, and using the identity $\pi_n\Omega_{nm}\pi_n = -i\partial_n\Omega_{nm}\pi_m + \Omega_{nm}\pi_n\pi_m$, yields the hermitian kinetic piece of the Hamiltonian

$$H_o' = \frac{1}{2}\pi_n\Omega_{nm}\pi_n + \frac{1}{4}\Omega_{nm}\partial_n\partial_m \ln J - \frac{1}{4}\omega_n\partial_n \ln J - \frac{1}{8}(\partial_n \ln J)\Omega_{nm}(\partial_m \ln J) \tag{3.46}$$

on condition that J satisfy

$$\begin{aligned} 0 &= \partial_m\left(\Omega_{mn}J\right) + \omega_nJ \\ &= \hat{O}_i^\dagger J \end{aligned} \tag{3.47}$$

where we have defined the operator $\hat{O}_i^\dagger \equiv \partial_m\Omega_{mn} + \omega_n$. One can now formally eliminate all mention of the Jacobian from the Hamiltonian, by means of the relation (3.47), to yield

$$H_o' = \frac{1}{2}\pi_n\Omega_{nm}\pi_n - \frac{1}{4}\partial_m\partial_n\Omega_{nm} - \frac{1}{4}\partial_n\omega_n + \frac{1}{8}(\omega_n + \partial_m\Omega_{mn})\Omega_{nm}^{-1}(\omega_m + \partial_n\Omega_{nm}) \tag{3.48}$$

This is the kinetic term of the hermitian $d = 1$ Hamiltonian, in terms of generic invariant variables. One may rewrite the above in the manifestly hermitian form

$$H_o' = \frac{1}{2}\tilde{O}^h_\alpha \Omega^{-1}_{\alpha\beta}\tilde{O}^{h\dagger}_\alpha \tag{3.49}$$

where we have defined the operator

$$\tilde{O}^h_\alpha = -\partial_k\Omega_{k\alpha} + \frac{1}{2}(\omega_\alpha + \partial_m\Omega_{m\alpha}) \tag{3.50}$$

The superscript h serves to emphasize the association of the above operator with the manifestly hermitian form of the Hamiltonian. One should note that the hermitian conjugate of $\tilde{O}^h_\alpha$ has the following action on the square root of the Jacobian,

$$\tilde{O}^{h\dagger}_\alpha J^{1/2} = \frac{1}{2J^{1/2}}\hat{O}^\dagger_\alpha J = 0 \tag{3.51}$$

We have constructed the Hamiltonian on the reduced invariant subspace, and in so doing obtained a differential equation specifying the Jacobian associated with the change to invariant collective fields. Thus far, we have employed generic invariants; however, we shall now specialise to the density description.

It proves convenient to compute ω_n, and Ω_{mn} in terms of the invariant variables $\phi_k = Tr(e^{ikM})$, and to then transform the result to x space, since

$$\phi(x) = \int \frac{dk}{2\pi} e^{-ikx}\phi_k \tag{3.52}$$

We consider computing ω_k, where

$$\omega_k \equiv -Tr\left(\frac{\partial^2\phi_k}{\partial M^2}\right) \tag{3.53}$$

One obtains

$$\omega_k = k\int_0^k dk'\phi_{k'}\phi_{k-k'} \tag{3.54}$$

where we have made use of the identity

$$\partial_{ij}\left(e^{aM}\right)_{ab} = a\int_0^1 dt\left(e^{taM}\right)_{ac}\partial_{ij}M_{cd}\left(e^{(1-t)aM}\right)_{db} \tag{3.55}$$

It is now a simple matter to compute the form of ω_k in x space, by simply fourier transforming (3.54). The result is

$$\omega(x) = 2\partial_x ⨍ dy\frac{\phi(x)\phi(y)}{x-y} \tag{3.56}$$

One now computes the form of $\Omega(x,x')$ in a similar fashion. In k space, one easily finds

$$\Omega(k,k') = -kk'\phi_{k+k'} \tag{3.57}$$

which takes the form

$$\Omega(x,x') = \partial_x\partial_{x'}\left(\delta(x-x')\phi(x')\right) \tag{3.58}$$

in x space.

Further, note that

$$\partial_m\Omega_{nm} \rightarrow \int dx' \frac{\partial\Omega(x,x')}{\partial\phi(x')}$$

which is readily shown to vanish, since

$$\int dx' \frac{\partial}{\partial\phi(x')}\left(\partial_x\partial_{x'}(\delta(x-x')\phi(x'))\right) = -\partial_x[\partial_x\delta(x)|_{x=0}]. \tag{3.59}$$

The Hamiltonian may now be written

$$\begin{aligned} H_o' &= \frac{1}{2}\int dxdx'\pi(x)\Omega(x,x')\pi(x') - \frac{1}{4}\int dx\frac{\partial}{\partial\phi(x)}\omega(x) \\ &\quad +\frac{1}{8}\int dxdx'\omega(x)\Omega^{-1}(x,x')\omega(x') \end{aligned} \tag{3.60}$$

Let us treat the term

$$\frac{1}{8}\int dxdx'\omega(x)\Omega^{-1}(x,x')\omega(x') \tag{3.61}$$

Inserting the form of $\omega(x)$, and integrating by parts twice, permits us to write the above as

$$\frac{1}{2}\int dxdx'\left(\fint dy\frac{\phi(x)\phi(y)}{x-y}\right)\partial_x\partial_{x'}\Omega^{-1}(x,x')\left(\fint dy\frac{\phi(x')\phi(y)}{x'-y}\right) \tag{3.62}$$

Inspecting the form of $\Omega(x,x')$, one observes that $\Omega^{-1}(x,x')$ must involve the inverse operations $\partial_x^{-1}\partial_{x'}^{-1}$, in which case one requires the derivatives acting on the inverse to nullify the effect of such operators. Thus, the above expression assumes the following form

$$\frac{1}{2}\int dxdx'\left(\fint dy\frac{\phi(x)\phi(y)}{x-y}\right)\frac{\delta(x-x')}{\phi(x)}\left(\fint dy\frac{\phi(x')\phi(y)}{x'-y}\right) \tag{3.63}$$

which, after integrating over the delta function, yields

$$\frac{1}{2}\int dx\left(\fint dy\frac{\phi(y)}{x-y}\right)^2 \tag{3.64}$$

The last term which requires consideration is

$$\begin{aligned}\int dx\frac{\partial\omega(x)}{\partial\phi(x)} &= 2\int dx\phi(x)\int dx'\frac{\delta(x-x')}{(x-x')^2}\\ &= -2\int dx\phi(x)\lim_{x\to x'}\partial_x\partial_{x'}\ln|x-x'| \end{aligned}\tag{3.65}$$

It is a simple matter to verify that the action $S=\sum_n g_n TrM^n$ assumes the form $\int V(x)\phi(x)dx$ in x space. The final form of the Hamiltonian is obtained by including a lagrange multiplier μ_F to enforce the constraint $\int dx\phi(x)=N$. The full $d=1$ Hamiltonian in x space thus reads

$$\begin{aligned}H' &= \frac{1}{2}\int dx(\partial_x\pi)\phi(x)(\partial_x\pi)+\frac{1}{2}\int dx\phi(x)\left(\fint dy\frac{\phi(y)}{x-y}\right)^2\\ &+\int dxV(x)\phi(x)-\mu_F\left(\int dx\phi(x)-N\right) \qquad (3.66)\\ &+\frac{1}{2}\int dx\phi(x)\lim_{x\to x'}\partial_x\partial_{x'}\ln|x-x'| \qquad (3.67)\end{aligned}$$

The well known identity

$$\frac{1}{2}\int dx\phi(x)\left(\fint dy\frac{\phi(y)}{x-y}\right)^2=\frac{\pi^2}{6}\int dx\phi^3(x). \tag{3.68}$$

permits us to write

$$\begin{aligned}H' &= \frac{1}{2}\int dx(\partial_x\pi)\phi(x)(\partial_x\pi)+\frac{\pi^2}{6}\int dx\phi^3(x)+\int dxV(x)\phi(x)\\ &-\mu_F\left(\int dx\phi(x)-N\right)+\frac{1}{2}\int dx\phi(x)\lim_{x\to x'}\partial_x\partial_{x'}\ln|x-x'| \end{aligned}\tag{3.69}$$

The $d=1$ Hamiltonian finally is

$$\begin{aligned}H' &= \frac{1}{2}\int dx(\partial_x\pi)\phi(x)(\partial_x\pi)+\frac{\pi^2}{6}\int dx\phi^3(x)+\int dx\left(V(x)-\mu_F\right)\phi(x)\\ &+\frac{1}{2}\int dx\phi(x)\lim_{x\to x'}\partial_x\partial_{x'}\ln|x-x'| \end{aligned}\tag{3.70}$$

In order to determine the leading configuration in the limit of large N, one rescales $x \to \sqrt{\beta}x, \phi \to \sqrt{\beta}\phi, \pi \to \sqrt{\beta}\pi$, and $\mu_F \to \beta\mu_F$, following which, the above becomes

$$H' = \frac{1}{2\beta^2}\int dx (\partial_x\pi)\phi(x)(\partial_x\pi) + \beta^2\left(\frac{\pi^2}{6}\int dx\phi^3(x) + \int dx\,(V(x) - \mu_F)\,\phi(x)\right) + 2\int dx\phi(x)\lim_{x\to x'}\partial_x\partial_{x'}\ln|x - x'| \tag{3.71}$$

where the potential scales as (2.11), and the constraint is again given by (2.12). One notes that the logarithmically divergent term is subleading in N in comparison to the bracketed term, and thus only contributes at the one loop level.

The leading configuration is obtained by minimising the dominant term

$$\frac{\delta V_{eff}[\phi]}{\delta\phi} = \frac{\delta}{\delta\phi}\left(\frac{\pi^2}{6}\int dx\phi^3(x) + \int dx\,(V(x) - \mu_F)\,\phi(x)\right) = 0 \tag{3.72}$$

which yields the solution

$$\pi\phi_0(x) = \sqrt{2(\mu_F - V(x))} \tag{3.73}$$

It is well known that the $d = 1$ model may be regarded as the theory of non-interacting fermions in a potential well $V(x)$, where the eigenvalues of the hermitian matrix M represent the fermionic co-ordinates, and the collective field $\phi_0(x)$ plays the role of a (classical) fermionic density. The designation of the lagrange multiplier μ_F then becomes evident, since it plays the role of the fermi level. The continuum limit is approached in this instance by tuning μ_F to a quadratic maximum. The designation $V(x)$ in the discussion of the $d = 0$ model should now become clear, since one has

$$\pi\phi_0(x) = \sqrt{-2V(x))} \tag{3.74}$$

where we have set $\mu_F = 0$. The absence of the lagrange multiplier in this instance will be addressed in the following section, when discussing stabilised $d = 0$ models.

3.2. The Stabilised $d = 0$ model

Let us consider the semi positive definite, hermitian Fokker–Planck Hamiltonian

$$H_{fp} = \frac{1}{2}\hat{A}_i^\dagger\hat{A}_i \tag{3.75}$$

with $\hat{A}_i = e^{-S/2}\frac{\partial}{\partial\phi_i}e^{S/2}$. Permitting the derivatives to act, the above Hamiltonian may equivalently be written

$$H_{fp} = \frac{1}{2}Tr\left(-\frac{\partial^2}{\partial M^2} - \frac{1}{2}\frac{\partial^2 S}{\partial M^2} + \frac{1}{4}\left(\frac{\partial S}{\partial M}\right)^2\right) \tag{3.76}$$

where we work with $N \times N$ hermitian matrices M. The semi positive definite nature of this Hamiltonian is the key issue as regards stabilisation, since it has a bounded energy spectrum regardless of the nature of the action. Let us consider recasting the Hamiltonian (3.76) in terms of invariant collective variables ϕ_i. Consider first the term

$$H_{fp}^1 = \frac{1}{8}\left(\frac{\partial S}{\partial M}\right)^2 \tag{3.77}$$

Making use of the chain rule, one is easily able to demonstrate that the above becomes

$$H_{fp}^1 = \frac{1}{8}\Omega_{ij}\frac{\partial S}{\partial\phi_i}\frac{\partial S}{\partial\phi_j} \tag{3.78}$$

We now specialise to the density variables

$$\phi(x) = \sum_i \delta(x - \lambda_i) \tag{3.79}$$

It is easily seen that, in x space, the action becomes $S = \int dx\phi(x)V(x)$, in which case (3.78) is written

$$H_{fp}^1 = \frac{1}{8}\int dx \int dy\ \Omega(x,y)\frac{\delta}{\delta\phi(x)}\left(\int dx'\phi(x')V(x')\right)\frac{\delta}{\delta\phi(y)}\left(\int dy'\phi(y')V(y')\right) \tag{3.80}$$

Using the previous result

$$\Omega(x,y) = \partial_x\partial_y(\delta(x-y)\phi(x)) \tag{3.81}$$

and inserting the above into (3.80) and performing a few partial integrations, one obtains

$$H_{fp}^1 = \frac{1}{8}\int dx\phi(x)V'^2(x) \tag{3.82}$$

Let us now consider the term

$$H_{fp}^2 = -\frac{1}{4}\frac{\partial^2 S}{\partial M^2} \tag{3.83}$$

which is again easily transformed via the chain rule to the invariant collective variables

$$H_{fp}^2 = \frac{1}{4}\left(\omega_i \frac{\partial S}{\partial \phi_i} - \Omega_{ij}\frac{\partial^2 S}{\partial \phi_i \partial \phi_j}\right) \tag{3.84}$$

As before, in x space one has

$$\omega(x) = 2\partial_x \fint dy \frac{\phi(x)\phi(y)}{x-y} \tag{3.85}$$

Writing H_{fp}^2 in x space, one finds

$$H_{fp}^2 = \frac{1}{4}\left(\int dx \omega(x)\frac{\partial S}{\partial \phi(x)} - \int\int dx dy \Omega(x,y)\frac{\partial^2 S}{\partial \phi(x)\partial \phi(y)}\right) \tag{3.86}$$

For an action linear in ϕ, (which we will consider) it is clear that the second term above does not contribute, and one is left with

$$H_{fp}^2 = -\frac{1}{2}\int dx V'(x) \fint dy \frac{\phi(x)\phi(y)}{x-y} \tag{3.87}$$

All that remains to consider is the kinetic term. This term was analysed above and, following the restoration of hermeticity, was found to take the form

$$\begin{aligned} H' &= \frac{1}{2}\int dx (\partial_x \pi)\phi(x)(\partial_x \pi) + \frac{1}{2}\int dx \phi(x)\left(\fint dy \frac{\phi(y)}{x-y}\right)^2 \\ &\quad + 2\int dx \phi(x) \lim_{x\to x'} \partial_x \partial_{x'} \ln|x-x'| \end{aligned} \tag{3.88}$$

The complete stochastic Hamiltonian thus reads

$$\begin{aligned} H_{fp} &= \frac{1}{2}\int dx (\partial_x \pi)\phi(x)(\partial_x \pi) + \frac{1}{2}\int dx \phi(x)\left(\fint dy \frac{\phi(y)}{x-y}\right)^2 \\ &\quad -\frac{1}{2}\int dx V'(x)\fint dy \frac{\phi(x)\phi(y)}{x-y} \qquad (3.89) \\ &\quad +\frac{1}{8}\int dx \phi(x) V'^2(x) + 2\int dx \phi(x) \lim_{x\to x'} \partial_x \partial_{x'} \ln|x-x'| \qquad (3.90) \end{aligned}$$

which may equivalently be written

$$\begin{aligned} H_{fp} &= \frac{1}{2}\int dx (\partial_x \pi)\phi(x)(\partial_x \pi) + 2\int dx \phi(x) \lim_{x\to x'} \partial_x \partial_{x'} \ln|x-x'| \\ &\quad +\frac{1}{2}\int dx \phi(x)\left(\fint dy \frac{\phi(y)}{x-y} - \frac{V'(x)}{2}\right)^2 \end{aligned} \tag{3.91}$$

In this instance, one takes $\mu_F = 0$, otherwise the positive definite nature of the Fokker–Planck Hamiltonian would be lost. Consequently, one finds the following effective potential for the stabilised $d = 0$ model:

$$V_{eff} = 2\int dx\phi(x) \lim_{x\to x'} \partial_x\partial_{x'} \ln|x - x'| + \frac{1}{2}\int dx\phi(x)\left(\fint dy\frac{\phi(y)}{x-y} - \frac{V'(x)}{2}\right)^2 \tag{3.92}$$

We wish to demonstrate correspondence between the $d = 0$ and stabilised $d = 0$ results at large N. Thus, performing the rescalings $x \to \sqrt{\beta}x$, $\phi \to \sqrt{\beta}\phi$ to exhibit β explicitly yields

$$V_{eff} = \frac{\beta^2}{2}\int dx\phi(x)\left(\fint dy\frac{\phi(y)}{x-y} - \frac{V'(x)}{2}\right)^2 \tag{3.93}$$

with the logarithmically divergent term and the kinetic piece subleading in N. Thus, in the limit of large N, the effective potential determines the leading configuration.
Heuristically, the above expression implies that $\phi = 0$, or $\int dy\frac{\phi(y)}{x-y} = \frac{V'(x)}{2}$, since $\phi(x) \geq 0$. One should thus expect that $d = 0$ solutions at large N must be reproduced by this term.

Explicitly, one requires

$$\frac{\delta V_{eff}[\phi(x)]}{\delta\phi(x')} = 0 \tag{3.94}$$

for the leading configuration, with

$$\begin{aligned}\frac{1}{\beta^2}V_{eff} \quad &= \quad \frac{1}{2}\int dx\phi(x)\left(\fint dy\frac{\phi(y)}{x-y}\right)^2 - \frac{1}{4}\int dx\phi(x)\fint dy\phi(y)\frac{V'(x)-V'(y)}{x-y} \\ &\quad + \frac{1}{8}\int dx\phi(x)V'^2(x)\end{aligned} \tag{3.95}$$

where we have symmetrized the second term, and used the identity (3.68).

Let us now consider the case $V'(x) = \omega x - x^3$. Performing the functional differentiations yields

$$\pi^2\phi(x)^2 + \left(x^2 g + xW_1 + W_2 - \omega g\right) + \frac{1}{4}V'(x)^2 = 0 \tag{3.96}$$

where $W_i \equiv \int dx x^i\phi(x)$. Since one is considering the case of a symmetric potential, one may set $W_1 = 0$.

One finally obtains

$$4\pi^2\phi^2 + 4(x^2 g + W_2 - \omega g) + \omega^2 x^2 - 2\omega x^4 + x^6 = 0 \tag{3.97}$$

In which case, one may solve for ϕ. Note, however, that any such solution has a self consistency condition due to the appearance of ϕ in the definition of W_2. In order to confirm that this stabilised solution does reproduce $d = 0$, we consider mapping each of the above coefficients of x onto the corresponding coefficients of the $d = 0$ ansatz,

$$\pi\phi(z) = \frac{1}{2}(z^2 - \alpha^2)\sqrt{\gamma^2 - z^2} \tag{3.98}$$

to ensure that they reproduce the correct equations defining the position of the cut in terms of the couplings. Furthermore, by considering the definition of $G(z)$ for z large, $G(z) = \sum_n \frac{W_n}{z^{n+1}}$, one is able to read off the equation defining W_2 by considering the coefficients z^{-2} in the large z expansion of the $d = 0$ ansatz, and confirming whether the mapping in the two schemes corresponds.

Equating coefficients of z in the two schemes yields the equations

$$\omega = \frac{\gamma^2}{2} + \alpha^2 \tag{3.99}$$

and

$$\omega^2 + 4g = 2\alpha^2\gamma^2 + \alpha^4 \tag{3.100}$$

One sees that the first equation corresponds precisely with $d = 0$. Squaring (3.99), inserting the result in (3.100) and performing some simple algebra yields

$$g = \frac{\gamma^2}{4}\left(\alpha^2 - \frac{\gamma^2}{4}\right) \tag{3.101}$$

which is the correct $d = 0$ equation describing the position of the cut as a function of the coupling.

The expression for the W_2 in the stabilised model is

$$W_2 = \omega g - \frac{\alpha^4\gamma^2}{4} \tag{3.102}$$

which, after inserting equations (3.99) and (3.101) is readily shown to correspond with (2.30). Thus, in the limit $N \to \infty$, the cubic case of the $d = 0$ model is equivalently described by the stabilised model.

Let us now consider the case of the cubic potential, where one takes

$$V(x) = \frac{\omega}{2}x^2 - \frac{1}{3}x^3 \tag{3.103}$$

following the usual rescalings such that the collective field obeys

$$\int dx\phi(x) = g \tag{3.104}$$

Following the same procedure as above, one is able to obtain the $d = 1$ expression for the density, which is given by

$$\pi^2\phi^2 = \omega g - W_1 - gx - \frac{\omega^2 x^2}{4} + \frac{\omega x^3}{2} - \frac{x^4}{4} \tag{3.105}$$

As before, in order to make contact with the $d = 0$ solution, one maps the $d = 1$ expression for the density onto the ansatz

$$\pi^2\phi^2(z) = \frac{1}{4}(z-\mu)^2(\alpha - z)(z-\gamma) \tag{3.106}$$

Doing so, yields the equations

$$\omega = \frac{\sigma}{2} + \mu \tag{3.107}$$

$$\omega^2 = \alpha\gamma + 2\mu\sigma + \mu^2 \tag{3.108}$$

$$g = -\frac{\mu}{2}\left(\alpha\gamma + \frac{\mu}{2}\sigma\right) \tag{3.109}$$

and

$$W_1 = \omega g + \frac{1}{4}\mu^2\alpha\gamma \tag{3.110}$$

One sees that the equations (3.107) and (3.108) imply a consistency condition. Squaring (3.107) and equating the result with (3.108) yields

$$\alpha\gamma = \frac{\sigma^2}{4} - \sigma\mu \tag{3.111}$$

This equation may be further refined by inserting the equation for μ,

$$\alpha\gamma = \frac{3\sigma^2}{4} - \sigma\omega \tag{3.112}$$

This equation complies with the first of the $d = 0$ results.

Further, inserting the relations (3.111) and (3.107) in (3.109), and performing some manipulations yields

$$\sigma(\sigma - 2\omega)(\sigma - \omega) = 8g \tag{3.113}$$

in agreement with (2.47).

All that remains is to verify the equivalence of the moments. The expression obtained in the context of the $d = 0$ model is

$$W_1 = \frac{-5}{256}(\alpha^4 + \gamma^4) + \frac{\alpha^2\gamma^2}{128} + \frac{\alpha\gamma}{64}\left(\alpha^2 + \gamma^2\right) + \frac{\mu}{32}(\alpha - \gamma)(\alpha^2 - \gamma^2) \quad (3.114)$$

Now, using the identity $\alpha^4 + \gamma^4 = (\alpha - \gamma)^2\sigma^2 + 2\alpha^2\gamma^2$, and performing various manipulations, yields

$$W_1 = \frac{-5}{256}(\alpha - \gamma)^2\sigma^2 + \frac{\alpha\gamma}{64}(\alpha - \gamma)^2 + \frac{\mu}{32}(\alpha - \gamma)^2\sigma \quad (3.115)$$

Employing equations (3.112) and (3.107), one obtains

$$W_1 = \frac{\sigma^2}{64}(3\sigma - 2\omega)(\sigma - 2\omega) \quad (3.116)$$

The expression for W_1 in the $d = 1$ model is

$$W_1 = \omega g + \frac{1}{4}\mu^2\alpha\gamma \quad (3.117)$$

Employing the relations (3.112), (3.113) and (3.107) one regains the result (3.116).

Thus one sees that the stabilised model is in full agreement with the $d = 0$ solution.

3.3. The Schwinger Dyson (Loop) Equations

Consider the general correlation function

$$\langle F(\varphi)\rangle = \int d\varphi\ F(\varphi)e^{-S(\varphi)} \quad (3.118)$$

Let us make a variation in the field $\varphi \to \varphi + \epsilon$, where ϵ is understood to be an infinitesimal constant. Expanding to order ϵ yields

$$\int d\varphi F(\varphi + \epsilon)e^{-S(\varphi+\epsilon)} = \int d\varphi F[\varphi]e^{-S[\varphi]} + \epsilon \int d\varphi \left(\frac{dF[\varphi]}{d\varphi} - \frac{dS[\varphi]}{d\varphi}F[\varphi]\right) e^{-S} \quad (3.119)$$

If one requires the invariance of (3.119) under such a transformation, one sees that this requirement is expressed as[6]

$$0 = \int d\varphi\ \frac{d}{d\varphi}\left(F(\varphi)e^{-S(\varphi)}\right) \quad (3.120)$$

[6] More trivially, one might note that this is simply the integral of a total derivative, and thus vanishes.

This is known as the Schwinger–Dyson or loop equation, which the correlators must obey. In full,

$$\left\langle \frac{dF[\varphi]}{d\varphi} - \frac{dS[\varphi]}{d\varphi} F[\varphi] \right\rangle = 0 \tag{3.121}$$

If one were to consider $F[\varphi]$ as arising from the first variation of some quantity, say $G[\varphi]$, one could rewrite the above as follows

$$\left\langle \frac{d^2G[\varphi]}{d\varphi^2} - \frac{dS[\varphi]}{d\varphi}\frac{dG[\varphi]}{d\varphi} \right\rangle = 0 \tag{3.122}$$

One may link the above to stochastic quantization. Previously, we demonstrated that

$$\langle G[\varphi]\rangle = \lim_{\tau\to\infty} \langle G| e^{-H_{FP}^{\dagger}\tau} |\Psi(0)\rangle = \frac{\int d\varphi G[\varphi] e^{-S}}{\int d\varphi e^{-S}} \tag{3.123}$$

under the assumption $\varepsilon_0 = 0$, implying that the zero eigenstate has the form $\langle\varphi|0_R\rangle = e^{-S[\varphi]}$.

The correlator (3.123) cannot depend on the fictitious time τ in the limit $\tau \to \infty$. Thus, let us consider the object

$$\lim_{\tau\to\infty} \partial_\tau \langle G[\varphi]\rangle = 0 \tag{3.124}$$

which is seen to be

$$\lim_{\tau\to\infty} \partial_\tau \langle G[\varphi]\rangle = \lim_{\tau\to\infty} \sum_n e^{-\varepsilon_n\tau} \langle G| H_{FP}^{\dagger} |n_R\rangle \langle n_L|\Psi(0)\rangle = 0 \tag{3.125}$$

where we have inserted a complete set of states, and assumed that they are properly normalized. In the limit $\tau \to \infty$, one obtains

$$\lim_{\tau\to\infty} \partial_\tau \langle G[\varphi]\rangle = \lim_{\tau\to\infty} \langle G| H_{FP}^{\dagger} |0_R\rangle = 0 \tag{3.126}$$

where we have employed (3.26). Taking the complex conjugate of the above (where we assume real matrix elements) and inserting a complete set of states, yields

$$\begin{aligned} \lim_{\tau\to\infty} \partial_\tau \langle G[\varphi]\rangle &= \int d\varphi \langle 0_R|\varphi\rangle \langle\varphi| H_{FP} |G\rangle \\ &= \int d\varphi \left(\frac{\partial^2 G}{\partial\varphi^2} - \frac{\partial S}{\partial\varphi}\frac{\partial G}{\partial\varphi}\right) e^{-S} = 0 \end{aligned} \tag{3.127}$$

which are seen to be the Schwinger–Dyson equations of a euclidean quantum field theory, as derived above. It must be emphasized that this discussion of stochastic quantization is distinct from the issue of stabilisation, which is a modification of the theory. Here we are interested only in the link between the statement (3.124) and the Schwinger–Dyson equations of a field theory.

The stochastic quantization approach to loop equations in Yang–Mills theory was elucidated in [30]; [31] employed this approach in a numerical analysis of Schwinger–Dyson equations in the one matrix case.

The stochastic Fokker–Planck Hamiltonian is easily rewritten in terms of collective fields[7] as

$$H_{FP} = \omega_i \frac{\partial}{\partial\phi_i} - \Omega_{ij}\frac{\partial^2}{\partial\phi_i\partial\phi_j} + \Omega_{S,i}\frac{\partial}{\partial\phi_i} \tag{3.128}$$

where we have defined $\Omega_{S,i} \equiv Tr\left(\frac{\partial S}{\partial M}\frac{\partial\phi_i}{\partial M}\right)$. The Hamiltonian may be written more compactly as

$$H_{FP} = \hat{O}_i^S \partial_i \tag{3.129}$$

with $\hat{O}_i^S \equiv -\Omega_{ij}\partial_j + \tilde{\omega}_i$, and $\tilde{\omega}_i \equiv \omega_i + \Omega(S,i)$. This partitioning is not quite as arbitrary as it might first appear, since the $\hat{O}_i^S$'s are required to close an algebra to ensure the consistency of the collective field theory description.

Taking $\phi_n = TrM^n$ as our collective variable, the above Hamiltonian is written

$$H_{FP} = \sum_{n\geq 0}\left(\sum_{m\geq 0} m\phi_{n+m-2}\frac{\partial}{\partial\phi_m} + \sum_{r=0}^{n-2}\phi_r\phi_{n-r} - \Omega(S,\phi_n)\right) n\frac{\partial}{\partial\phi_n} \tag{3.130}$$

One can easily show that the pieces independent of the action close a Virasoro algebra, taking

$$\hat{O}_{n+2} = \sum_{m=0}^{\infty} m\phi_{n+m}\frac{\partial}{\partial\phi_m} + \sum_{r=0}^{n}\phi_r\phi_{n-r} \tag{3.131}$$

and defining $\hat{O}_{n+2} = L_n$, it is easy to verify that

$$[L_n, L_m] = (n-m)L_{n+m} \tag{3.132}$$

for positive modes $n \geq 0$, which is the form of the classical Virasoro algebra. One may make contact with a conformal field theoretic description, by defining the

[7]The collective field treatment of the stochastic Hamiltonian was first given in 5.16.

modes

$$\alpha_n = -\sqrt{2}\phi_n, \ \alpha_{-n} = -\frac{1}{\sqrt{2}}n\pi_n. \tag{3.133}$$

valid for $n > 0$. It is easy to verify that the commutation relation

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0} \tag{3.134}$$

holds. The above expressions are reminiscent of the mode expansions of a conformal field theory. In particular, (3.131) may be rewritten as[8]

$$: L_n := \sum_{m=0}^{\infty} \alpha_{-m}\alpha_{n+m} + \frac{1}{2}\sum_{r=0}^{n} \alpha_r \alpha_{n-r} \tag{3.135}$$

which are recognised to be the Virasoro generators associated with the energy momentum tensor of a free boson in a conformal field theory description,

$$T(z) = \sum_{n=-\infty}^{\infty} : L_n : z^{-n-2} = \frac{1}{2} : \partial_z\phi(z)\partial_z\phi(z) : \tag{3.136}$$

where the colons are taken to imply normal ordering.

Let us consider the $d = 0$ partition function following a change of variables to collective fields ϕ_α, in the presence of sources j_α,

$$Z(j) = \int d\phi J(\phi) e^{-S(\phi)} e^{-j_\alpha \phi_\alpha} \tag{3.137}$$

One may rewrite (3.137) rather suggestively, by defining the object $J_S(\phi) = J(\phi)e^{-S(\phi)}$, in which case

$$Z(j) = \int d\phi J_S e^{-j_\alpha \phi_\alpha} \tag{3.138}$$

Thus, it would appear that the object $Z(j)$ is nothing but the Laplace transform of the rescaled Jacobian, J_S. In the previous section, it was demonstrated that the Jacobian must satisfy the differential equation

$$\hat{O}_i^\dagger J = 0 \tag{3.139}$$

to exhibit the explicit hermiticity of the kinetic piece of the collective $d = 1$ Hamiltonian. In turn, the object J_S is required to ensure the hermiticity of

[8]One notes that we have rescaled the fields to obtain the required factor of $\frac{1}{2}$ associated with Virasoro; this factor arises naturally when considering the hermitian form of the Hamiltonian.

the non-hermitian Fokker–Planck Hamiltonian, following the change to invariant collective variables.

In order to determine the equation which J_S must satisfy, multiply (3.139) on the left by the factor e^{-S}, to yield

$$J_S \partial_i \Omega_{ij} + \Omega_{ij} e^{-S} \partial_i J + \omega_i J_S = 0 \tag{3.140}$$

Employing the identity

$$J_S \partial_i \Omega_{ij} + \Omega_{ij} e^{-S} \partial_i J = \partial_i (\Omega_{ij} J_S) + \Omega_{ij} J_S \partial_i S \tag{3.141}$$

one may readily rewrite (3.140) in the form

$$\partial_i (\Omega_{ij} J_S) + (\Omega_{ij} \partial_i S + \omega_i) J_S = 0 \tag{3.142}$$

which is further written as

$$\partial_i (\Omega_{ij} J_S) + \tilde{\omega}_i J_S = 0 \tag{3.143}$$

in terms of our previous definitions. This result may be expressed more compactly as

$$\hat{O}_i^{S\dagger} J_S = 0 \tag{3.144}$$

It is simple to show that, as one expects, commutation of the operator $\hat{O}_{n+2}^S$ gives rise to a Virasoro structure.

The above implies that one may consider

$$\begin{aligned} 0 &= \int d\phi \left(\hat{O}_i^{S\dagger} J_S \right) e^{-j\cdot\phi} \\ &= \int d\phi \left(\partial_{\phi_k} (\Omega_{ik}(\phi) J_S) + \tilde{\omega}(\phi) J_S \right) e^{-j\cdot\phi} \\ &= \int d\phi \left(J_S \Omega_{ik} j_k + \tilde{\omega}(\phi) J_S \right) e^{-j\cdot\phi} \end{aligned} \tag{3.145}$$

where, in the last line, we have integrated by parts and allowed the derivative to act on the exponential. The relevance of the sources now becomes clear, since making the correspondences

$$\partial_{\phi_k} \to -j_k, \phi_k \to -\partial_{j_k} \tag{3.146}$$

permits one to write

$$\begin{aligned} 0 &= \left(\Omega_{ik}(-\partial_{j_k}) j_k + \tilde{\omega}(-\partial_{j_k}) \right) \int d\phi J_S \, e^{-j\cdot\phi} \\ &= \hat{O}_i^S Z(j) \end{aligned} \tag{3.147}$$

Thus, one sees that the operators $\hat{O}_i^S$ are required to annihalate the partition function. Furthermore, it should now become clear that the operators $\hat{O}_i^S$ must close an algebra, since (3.147) requires

$$[\hat{O}_i^S, \hat{O}_j^S] Z(j) = 0 \tag{3.148}$$

which implies in turn that

$$[\hat{O}_i^S, \hat{O}_j^S] = f_{ijk} \hat{O}_k^S \tag{3.149}$$

Thus, if (3.144) is to have a solution, the operators $\hat{O}_i^S$ must close an algebra. As stated above, the existence of a Virasoro structure in the one matrix model ensures that such a solution may be realised. The object J_S is the Jacobian required to ensure the hermiticity of the non-hermitian stochastic Hamiltonian, following the change to invariant collective variables.

The result (3.145) may easily be rederived in the context of the stochastic quantization formalism. Recall that the Schwinger Dyson equation may be written in the context of a $d+1$ dimensional quantum field theory as

$$\lim_{\tau \to \infty} \partial_\tau \langle G[M] \rangle = \int dM \, \langle 0_R | \, M \rangle \, \langle M | \, H_{FP} \, | G \rangle = 0 \tag{3.150}$$

where H_{FP} denotes the non-hermitian Fokker-Planck Hamiltonian, and

$\lim_{\tau \to \infty} \langle 0_R \, | M \rangle = e^{-S}$ is the groundstate wavefunctional. Now, let us make the change of variables $M \to \phi_i$, incurring a Jacobian $J(\phi)$:

$$0 = \int d\phi_i J(\phi) e^{-S} \langle \phi | \, H_{FP} \, | G \rangle$$

Redefining $J_S(\phi) = J(\phi) e^{-S}$, and taking the hermitian conjugate yields

$$0 = \int d\phi_i G(\phi) H_{FP}^\dagger J_S(\phi). \tag{3.151}$$

Due to the above definitions, $H_{FP}^\dagger = \partial_\alpha \hat{O}_\alpha^{S\dagger}$, and thus the result (3.144) is reclaimed, with the sources $j_\alpha = 0$.

In summary, the requirement that the stochastic Fokker-Planck Hamiltonian exhibit explicit hermiticity yields an equation for the Jacobian $J_S(\phi)$, which may be rewritten as a set of constraints acting on the partition function. In order to ensure the consistency of the collective field description, it is required that these constraints close an algebra, which is ensured in the one matrix model by the existence of a Virasoro structure.

4. The Continuum Form of the Fokker–Planck Hamiltonian

4.1. The Macroscopic Loop Approach

The above discussion of the equivalence of the expression (3.127) with that of the Schwinger–Dyson equations of a quantum field theory suggests that one might construct a string field theory by considering a stochastic quantization approach to the $d = 0$ model. This was accomplished independently by [10], [12]. The following discussion is a synthesis of the approaches of [10], [11] and [12].

In order to construct a continuum Fokker–Planck Hamiltonian, it is traditional to choose the macroscopic loop as the collective variable, $\phi(l) = Tr(e^{lM})$. As usual, one proceeds by computing the quantities $\Omega(l, l')$ and $\omega(l)$.

Proceeding by definition, it is easy to show that

$$\Omega(l, l') = ll'\phi(l + l') \tag{4.1}$$

and

$$\omega(l) = -l \int_0^l dl'\phi(l')\phi(l - l') \tag{4.2}$$

In order to achieve the continuum limit, it will prove convenient to work in the z representation. Laplace transforming (4.1) yields

$$\Omega(z, z') = \partial_z \partial_{z'} \frac{G(z') - G(z)}{z - z'}$$

where $G(z) \equiv Tr\frac{1}{Z-M}$. Further, noting that (4.2) is a convolution, the laplace transform is readily found:

$$\omega(z) = \partial_z G^2(z) \tag{4.3}$$

In order to proceed further, one would like to exhibit N explicitly. To this end, one should seek the analogue of the constraint on the density variable. The corresponding constraint in this case is given by

$$\frac{1}{2\pi i} \oint G(z)dz = N \tag{4.4}$$

where the contour is chosen to enclose the eigenvalues of M. The contribution from the contour at infinity vanishes. Thus, one need only consider the consequences of the residue theorem. It is then clear that, in order to rid the constraint of N, one must perform the independent rescalings $z \to \sqrt{N}\bar{z}$, $G(z) \to \sqrt{N}G(\bar{z})$, following which (4.4) becomes

$$\frac{1}{2\pi i}\oint G(\bar{z})d\bar{z} = 1 \tag{4.5}$$

All that remains is to determine the scaling of the conjugate momentum, which is readily inferred from the canonical commutator

$$[\pi(z), G(z')] = \frac{1}{z - z'} \tag{4.6}$$

Performing the above [illegible]ings on the commutator (4.6) yields

$$[\gamma\pi(\bar{z}), \sqrt{N}G(\bar{z}')] = \frac{1}{\sqrt{N}}\frac{1}{\bar{z} - \bar{z}'} \tag{4.7}$$

It is thus clear that, in order to preserve the canonical commutator between the rescaled operators, one demands $\gamma = N^{-1}$. Hence, we require $\pi(z) \to N^{-1}\pi(\bar{z})$.

The Fokker–Planck Hamiltonian is

$$H_{fp} = -\int dz' \int dz \frac{G(z') - G(z)}{z - z'}\partial_z\pi(z)\partial_{z'}\pi(z') + \int dz\ \partial_z G^2(z)\pi(z) \tag{4.8}$$

$$+ \int dz\Omega(S, z)\pi(z) \tag{4.9}$$

Rescaling the kinetic term yields

$$H_{fp}^{kin} = -\frac{1}{N^2}\int d\bar{z}' \int d\bar{z}\frac{G(\bar{z}') - G(\bar{z})}{\bar{z} - \bar{z}'}\partial_{\bar{z}}\pi(\bar{z})\partial_{\bar{z}'}\pi(\bar{z}') \tag{4.10}$$

It is straightforward to determine that

$$\int dz\ \partial_z G^2(z)\pi(z) \to \int d\bar{z}\ \partial_{\bar{z}} G^2(\bar{z})\pi(\bar{z}) \tag{4.11}$$

In order to consider the scaling of the last term, let us recall the definition $\Omega(S, z) = Tr\frac{\partial S}{\partial M}\frac{\partial G(z)}{\partial M}$. This term may be rewritten as

$$\Omega(S, z') = \int dz\Omega(z, z')\frac{\partial S}{\partial G(z)} \tag{4.12}$$

The action in z space is such that the potential resembles a source for the operators $G(z)$,

$$S = \frac{1}{2\pi i}\oint_C dz V(z) G(z) \tag{4.13}$$

where the contour C encloses the poles of $G(z)$. It is then clear that

$$\Omega(S, z') = \int dz \Omega(z, z') V(z) \tag{4.14}$$

in which case the rescalings $z \to \sqrt{N}\bar{z}$ and $G(z) \to \sqrt{N} G(\bar{z})$ imply

$$\Omega(S, z) \to \sqrt{N}\Omega(S, \bar{z}) \tag{4.15}$$

provided that the couplings in the potential are chosen such that $V(z) \to NV(\bar{z})$. It is then quite clear that

$$\int dz \Omega(S, z)\pi(z) \to \int d\bar{z}\Omega(S, \bar{z})\pi(\bar{z}) \tag{4.16}$$

Hence, the Fokker–Planck Hamiltonian exhibiting N explicitly is written

$$H_{fp} = -\frac{1}{N^2}\int d\bar{z}' \int d\bar{z} \frac{G(\bar{z}') - G(\bar{z})}{\bar{z} - \bar{z}'} \partial_{\bar{z}}\pi(\bar{z})\partial_{\bar{z}'}\pi(\bar{z}') + \int d\bar{z}\ \partial_{\bar{z}} G^2(\bar{z})\pi(\bar{z}) \tag{4.17}$$

$$+ \int d\bar{z}\Omega(S, \bar{z})\pi(\bar{z}) \tag{4.18}$$

In order to proceed further, we take

$$S = \frac{\omega}{2}\bar{z}^2 - \frac{1}{3}\bar{z}^3 \tag{4.19}$$

To treat the term incorporating the action, one recalls that

$$\Omega(S, l) = \frac{\partial Tr V(M)}{\partial M_{ij}} \frac{\partial \phi(l)}{\partial M_{ji}} \tag{4.20}$$

which may be written

$$\Omega(S, l) = l V'\left(\frac{\partial}{\partial l}\right)\phi(l) \tag{4.21}$$

In order to move to the z representation, one thus considers

$$\Omega(S, \bar{z}) = -\partial_{\bar{z}} \int_0^\infty dl V'\left(\frac{\partial}{\partial l}\right)\phi(l)\ e^{-l\bar{z}} \tag{4.22}$$

One may readily demonstrate that, integrating by parts,

$$\int_0^\infty dl\ e^{-l\bar{z}} \partial_l \phi(l) = \bar{z}G(\bar{z}) - \phi(l)|_{l=0} \tag{4.23}$$

Furthermore,

$$\int_0^\infty dl\ e^{-l\bar{z}} \partial_l^2 \phi(l) = \bar{z}^2 G(\bar{z}) - \bar{z}\phi(l)|_{l=0} - \partial_l \phi(l)|_{l=0} \tag{4.24}$$

where we integrated by parts, and employed the result (4.23).

Thus, one has

$$\Omega(S, \bar{z}) = -\partial_{\bar{z}} \left(\omega \bar{z} G(\bar{z}) - \omega \phi(l)|_{l=0} - \bar{z}^2 G(\bar{z}) + \bar{z}\phi(l)|_{l=0} + \partial_l \phi(l)|_{l=0} \right) \tag{4.25}$$

Now, we consider the quantity $\phi(l)|_{l=0}$. The rescalings $z \rightarrow \sqrt{N}\bar{z}$, $G(z) \rightarrow \sqrt{N}G(\bar{z})$ imply that $G(\bar{z}) = \frac{1}{N} Tr \frac{1}{\bar{z}-M}$. Thus, since $G(\bar{z}) = \frac{1}{N} \int dl e^{-l\bar{z}} Tr(e^{lM})$, one sees that $\phi(l) \equiv \frac{1}{N} Tr(e^{lM})$, in which case $\phi(l)|_{l=0} = \frac{1}{N} Tr \mathbf{1} = 1$. The object $\partial_l \phi(l)|_{l=0} = TrM$ may be regarded as an initial condition which remains to be fixed.

The Fokker–Planck Hamiltonian is now written

$$\begin{aligned} H_{fp} \quad = \quad & -\frac{1}{N^2} \int d\bar{z}' \int d\bar{z} \frac{G(\bar{z}') - G(\bar{z})}{\bar{z} - \bar{z}'} \partial_{\bar{z}} \pi(\bar{z}) \partial_{\bar{z}'} \pi(\bar{z}') + \int d\bar{z}\ \partial_z G^2(\bar{z}) \pi(\bar{z}) \\ & + \int d\bar{z} \partial_{\bar{z}} \left((\bar{z}^2 - \bar{z}\omega) G(\bar{z}) + \omega - \bar{z} - \partial_l \phi(l)|_{l=0} \right) \end{aligned} \tag{4.26}$$

Our goal is to compute a continuum form for the above Hamiltonian, and thus we consider the scaling relations

$$\bar{z} = \bar{z}_c + a\xi \tag{4.27}$$

and

$$\omega = \omega_c + a^2 \Lambda \tag{4.28}$$

We further introduce the shifted field $G(\bar{z})$:

$$G(\bar{z}) = -\frac{1}{2}(\bar{z}^2 - \bar{z}\omega) + a^{3/2} G(\xi) \tag{4.29}$$

The required scaling of the conjugate momentum may be obtained by analogy with the procedure used to obtain the scaling in N, namely, by considering the canonical commutator. One obtains

$$\pi(\xi) = a^{5/2} \pi(\bar{z}) \tag{4.30}$$

Let us consider the potential term

$$G^2(\bar{z}) + (\bar{z}^2 - \bar{z}\omega)G(\bar{z}) + (\omega - \bar{z}) - \partial_l \phi(l)|_{l=0} \tag{4.31}$$

which yields, following the substitution of (4.29),

$$a^3 \left(G^2(\xi) - \frac{1}{a^3} \left\{ \frac{(\bar{z}\omega - \bar{z}^2)}{4} + \bar{z} - \omega + W_1 \right\} \right) \tag{4.32}$$

In the above we have replaced $\partial_l \phi(l)|_{l=0}$ by the familiar notation W_1. It should be noted that the shift was made in such a way as to cancel the term linear in $G(\bar{z})$, thus eliminating unphysical tadpole interactions. Comparison of the term in curly brackets with expression (3.105) indicates that it is none other than the continuum expression for the disk amplitude derived via the stabilised $d = 0$ model.

The potential term reads

$$-a^{1/2} \int d\xi (G^2(\xi) - G_0^2(\xi)) \partial_\xi \pi(\xi) \tag{4.33}$$

Now, let us turn to the kinetic term. Inserting the expression for the field, (4.29) one notes that the shift cannot contribute, and one has

$$-\frac{a^{1/2}}{N^2 a^5} \int d\xi \int d\xi' \frac{G(\xi') - G(\xi)}{\xi - \xi'} \partial_\xi \pi \partial_{\xi'} \pi \tag{4.34}$$

where the expression $N^{-2}a^{-5}$ has the correct scaling dimension for the square of the coupling constant, κ. One takes $N \to \infty$, $a \to 0$ in a correlated fashion, such that κ is held fixed. Thus, the final form of the continuum Hamiltonian reads

$$H^c_{fp} = -a^{1/2} \left(\kappa^2 \int d\xi \int d\xi' \frac{G(\xi') - G(\xi)}{\xi - \xi'} \partial_\xi \pi \partial_{\xi'} \pi + \int d\xi (G^2(\xi) - G_0^2(\xi)) \partial_\xi \pi(\xi) \right) \tag{4.35}$$

The factor $a^{1/2}$ may be absorbed into the definition of the stochastic time. One notes that in the above the specific form of the potential used to discretize the surfaces was absorbed into the universal form of the disk amplitude. Furthermore, the above hamiltonian is cubic in the fields, which one requires to reproduce the possible string interactions. One sees that, in the continuum,

$$\Omega^c(\xi, \xi) = \partial_{\xi'} \partial_\xi \frac{G(\xi') - G(\xi)}{\xi - \xi'} \tag{4.36}$$

may be regarded as representing the merging of continuum loops, and that

$$\omega^c(\xi) = \partial_\xi G^2(\xi) \tag{4.37}$$

represents continuum loop splitting. One could thus rewrite the above as

$$H^c_{fp} = -\kappa^2 \int d\xi \int d\xi' \Omega^c(\xi,\xi)\pi(\xi)\pi(\xi') - \int d\xi \omega^c(\xi)\pi(\xi) + \int d\xi G_0^2(\xi)\partial_\xi \pi(\xi) \tag{4.38}$$

The processes encoded in the above Hamiltonian are more easily interpreted in terms of loop fields,

$$\begin{aligned} H^c_{fp} &= -\int_0^\infty dl_1 \int_0^\infty dl_2 \kappa^2 l_1 l_2 \Psi^\dagger(l_1+l_2)\Psi(l_1)\Psi(l_2) \\ &\quad - \int_0^\infty dl_1 \int_0^\infty dl_2 (l_1+l_2)\Psi^\dagger(l_1)\Psi^\dagger(l_2)\Psi(l_1+l_2) \qquad (4.39) \\ &\quad - \int_0^\infty dl \rho_0(l)\Psi(l) \qquad (4.40) \end{aligned}$$

where one now has the field $\Psi^\dagger(l')$ which is the inverse laplace transform of $G(\xi)$, and $\Psi(l)$, the inverse laplace transform of the momentum conjugate to $G(\xi)$. Additionally, $\rho_0(l)$ is the inverse laplace transform of $\partial_\xi G_0^2(\xi)$. Furthermore, one has

$$[\Psi(l), \Psi^\dagger(l')] = \delta(l - l') \tag{4.41}$$

which follows directly from the inverse laplace transform of the canonical commutator between the field $G(\xi)$ and its conjugate momentum.

We therefore interpret the operator $\Psi(l)$ as destroying a loop of length l, and $\Psi^\dagger(l')$ as creating *a loop of length* l'. The first term in the above Hamiltonian is then seen to represent the process whereby two loops of length l_1 and l_2 merge to form a single loop of length $l_1 + l_2$. The second term corresponds to the splitting of a loop of length $l_1 + l_2$, into two loops of lengths l_1 and l_2. The last term represents a *string tadpole interaction*, whereby an incident string disappears.

Noting that the pre-continuum Hamiltonian has a Virasoro structure, it is natural to inquire whether such a structure survives the transition to the continuum. To this end, one would like to construct *the continuum Schwinger–Dyson equation*, which is defined by

$$Z(j) = \lim_{\tau\to\infty} \langle 0|\, H^c_{fp} e^{-H^c_{fp}\tau} |G\rangle = 0 \tag{4.42}$$

which is simply the conjugate of (3.127), where one takes the state

$$|G\rangle = e^{\int dl J(l)\Psi^\dagger(l)}|0\rangle \tag{4.43}$$

The use of sources permits us to rewrite (4.40) as

$$\begin{aligned}
0 &= \int_0^\infty dl_1 \int_0^\infty dl_2 \kappa^2(l_1+l_2)J(l_1+l_2)\frac{\delta}{\delta J(l_1)}\frac{\delta}{\delta J(l_2)}Z(J) \\
&+ \int_0^\infty dl_1 \int_0^\infty dl_2 l_1 l_2 J(l_1)J(l_2)\frac{\delta}{\delta J(l_1+l_2)}Z(J) \\
&+ \int_0^\infty dl \rho(l)J(l)Z(J)
\end{aligned} \tag{4.44}$$

One wishes to consider connected correlators, and thus one should rather work with $\ln Z(J)$. The above then becomes

$$\begin{aligned}
0 &= \int_0^\infty dl_1 \int_0^\infty dl_2 \kappa^2(l_1+l_2)J(l_1+l_2)\frac{\delta \ln Z(J)}{\delta J(l_1)}\frac{\delta \ln Z(J)}{\delta J(l_2)} \\
&+ \int_0^\infty dl_1 \int_0^\infty dl_2 \kappa^2(l_1+l_2)J(l_1+l_2)\frac{\delta^2 \ln Z(J)}{\delta J(l_1)\delta J(l_2)} \\
&+ \int_0^\infty dl \rho(l)J(l) + \int_0^\infty dl_1 \int_0^\infty dl_2 l_1 l_2 J(l_1)J(l_2)\frac{\delta \ln Z(J)}{\delta J(l_1+l_2)}
\end{aligned} \tag{4.45}$$

By changing variables of integration, one may further rewrite (4.45) as

$$\begin{aligned}
0 &= \int_0^\infty dl J(l) l \int_0^l dl' \left(\frac{\delta^2 \ln Z(J)}{\delta J(l')\delta J(l-l')} + \frac{\delta \ln Z(J)}{\delta J(l')}\frac{\delta \ln Z(J)}{\delta J(l-l')}\right) \\
&+\kappa^2 \int_0^\infty dl J(l) l \int_0^l dl' l \int_0^\infty dl' J(l') l' \frac{\delta \ln Z(J)}{\delta J(l+l')} \\
&+ \int_0^\infty dl J(l) l \int_0^l dl' \rho(l)
\end{aligned} \tag{4.46}$$

We took $\ln Z(J)$ to be the generating functional for loops; that is, we coupled the sources to continuum loop operators, such that multiloop correlators are obtained by repeated differentiations of $Z(J)$. One should note that if $Z(J)$ satisfies

$$\begin{aligned}
0 &= l \int_0^l dl' \left(\frac{\delta^2 \ln Z(J)}{\delta J(l')\delta J(l-l')} + \frac{\delta \ln Z(J)}{\delta J(l)}\frac{\delta \ln Z(J)}{\delta J(l-l')}\right) \\
&+\kappa^2 l \int_0^l dl' l \int_0^\infty dl' J(l') l' \frac{\delta \ln Z(J)}{\delta J(l+l')} + l \int_0^l dl' \rho(l)
\end{aligned} \tag{4.47}$$

then it is clearly a solution of the Schwinger Dyson equation (4.46). Furthermore, the resemblance of the above to the pre-continuum loop equations of the matrix model is striking. In order to identify more clearly the appearance of a Virasoro structure it will prove convenient to work with the ξ representation.

In the continuum, the loop operator $\Psi^\dagger(l)$ which creates a loop of length l may be expressed as a sum of local operators ϑ_n, which may be thought of as inserting "punctures" in the string worldsheet. Specifically, the continuum form of the loop operator for the $m = 2$ critical point is given by $\Psi^\dagger(l) = \kappa^2 \sum_{n\geq 0} \frac{l^{n+1/2}}{\Gamma(n+3/2)}\vartheta_n$. This implies that requiring the source to satisfy

$$\int_0^\infty dl J(l) l^{n+1/2} = \frac{1}{\kappa^2}\Gamma(n+3/2)\mu_n \tag{4.48}$$

ensures that one may rewrite the generating functional $\ln Z(J)$, and its associated correlators, in terms of such local operators ϑ_n, since the source term takes the form

$$\int dl J(l)\Psi^\dagger(l) = \sum_{n\geq 0} \mu_n \vartheta_n \tag{4.49}$$

We thus consider the Laplace transform of the Schwinger–Dyson equation

$$\begin{aligned} 0 &= l\int_0^l dl' \left(\frac{\delta^2 \ln Z(J)}{\delta J(l')\delta J(l-l')} + \frac{\delta \ln Z(J)}{\delta J(l')}\frac{\delta \ln Z(J)}{\delta J(l-l')}\right) \\ &\quad + \kappa^2 l \int_0^\infty dl' J(l') l' \frac{\delta \ln Z(J)}{\delta J(l+l')} + \rho(l) \end{aligned} \tag{4.50}$$

Let us factor the partition function into a product of universal and non univeral parts,

$$Z(J) = Z_u(J) Z_{non}(J) \tag{4.51}$$

where Z_u (Z_{non}) corresponds to the universal (non–universal) part of the partition function. Thus, one has

$$\frac{\delta \ln Z(J)}{\delta J(l')} = \frac{\delta \ln Z_{non}(J)}{\delta J(l')} + \frac{\delta \ln Z_u(J)}{\delta J(l')} \tag{4.52}$$

Now, $Z_{non}(J)$ takes the form

$$Z_{non}(J) = e^{\int dl Q(l) J(l) + \frac{1}{2}\int dl dl' J(l) J(l') C(l,l')} \tag{4.53}$$

where $Q(l)$ refers to the non–universal piece of the disk amplitude, and $C(l,l')$ the non–universal contribution to the cylinder amplitude. For the $m = 2$ critical point, $Q(l)$ takes the form

$$Q(l)=\frac{l^{-5/2}}{\Gamma(-\frac{3}{2})}-\frac{3\Lambda}{8}\frac{l^{-1/2}}{\Gamma(\frac{1}{2})} \tag{4.54}$$

while

$$C(l,l')=\frac{\kappa^2}{2\pi}\frac{\sqrt{ll'}}{l+l'} \tag{4.55}$$

is valid for all critical points.

One therefore sees that (4.52) becomes

$$\frac{\delta \ln Z(J)}{\delta J(l)}=\frac{l^{-5/2}}{\Gamma(-\frac{3}{2})}-\frac{3\Lambda}{8}\frac{l^{-1/2}}{\Gamma(\frac{1}{2})}+\frac{\kappa^2}{2\pi}\int_0^\infty dl' J(l')\frac{\sqrt{ll'}}{l+l'}+\frac{\delta \ln Z_u(J)}{\delta J(l)} \tag{4.56}$$

Our choice of source permits us to rewrite the above in terms of correlators of local operators as

$$\frac{\delta \ln Z(J)}{\delta J(l)}=\frac{l^{-5/2}}{\Gamma(-\frac{3}{2})}-\frac{3\Lambda}{8}\frac{l^{-1/2}}{\Gamma(\frac{1}{2})}+\frac{\kappa^2}{2\pi}\int_0^\infty dl' J(l')\frac{\sqrt{ll'}}{l+l'}+\kappa^2\sum_{n\geq 0}\frac{l^{n+1/2}}{\Gamma(n+\frac{3}{2})}\frac{\delta \ln Z_u(J)}{\delta \mu_n} \tag{4.57}$$

Let us consider the first term of (4.50):

$$l\int_0^l dl' \frac{\delta^2 \ln Z_u(J)}{\delta J(l')\delta J(l-l')} \tag{4.58}$$

which, after substituting (4.57), takes the form

$$\begin{aligned} & l\int_0^l dl' \frac{\delta}{\delta J(l-l')}\left(\frac{\kappa^2}{2\pi}\int_0^\infty dl_1 J(l_1)\frac{\sqrt{l_1 l'}}{l_1+l'}\right) \\ & +l\int_0^l dl' \frac{\delta}{\delta J(l-l')}\left(\kappa^2\sum_{n\geq 0}\frac{l^{n+1/2}}{\Gamma(n+\frac{3}{2})}\frac{\delta \ln Z_u(J)}{\delta \mu_n}\right) \end{aligned} \tag{4.59}$$

The other terms vanish due to the action of the functional derivative. One has

$$\begin{aligned} & l\int_0^l dl' \frac{\delta}{\delta J(l-l')}\left(\frac{\kappa^2}{2\pi}\int_0^\infty dl_1 \delta(l_1-(l-l'))\frac{\sqrt{l_1 l'}}{l_1+l'}\right) \\ & +l\int_0^l dl' \frac{\delta}{\delta J(l-l')}\left(\kappa^2\sum_{n\geq 0}\frac{l^{n+1/2}}{\Gamma(n+\frac{3}{2})}\frac{\delta}{\delta \mu_n}\left(\frac{\delta \ln Z_u(J)}{\delta J(l-l')}\right)\right) \end{aligned} \tag{4.60}$$

where we have permitted the functional derivative to act on the source term, and commuted the order of differentiation. This yields

$$\frac{\kappa^2}{2\pi}\int_0^l \sqrt{l'(l-l')} + \kappa^4 l \sum_{m,n\geq 0}\int_0^l dl' \frac{l'^{n+1/2}(l-l')^{m+1/2}}{\Gamma(n+3/2)\Gamma(m+3/2)}\frac{\delta^2 \ln Z_u(J)}{\delta\mu_n\delta\mu_m} \tag{4.61}$$

after integrating over the delta function and inserting the ansatz (4.57).

In order to evaluate the above, let us recall the identity

$$\int_0^\infty dl e^{-l\xi}\int_0^l dl' F(l')G(l-l') = \int_0^\infty dl_1 e^{-l_1\xi}F(l_1)\int_0^\infty dl_2 e^{-l_2\xi}G(l_2) \tag{4.62}$$

which is readily proven, following a change of integration variables. Further, recalling the integral representation of the Γ function

$$\Gamma(x) = \int_0^\infty dy e^{-y}y^{x-1} \tag{4.63}$$

allows us to conclude that

$$\int_0^\infty dl e^{-l\xi}l^n = \frac{1}{\xi^{n+1}}\Gamma(n+1) \tag{4.64}$$

Since (4.61) is a convolution, using (4.62) yields

$$-\partial_\xi\left(\kappa^4\sum_{m,n\geq 0}\xi^{-m-n-3}\frac{\delta^2\ln Z_u(J)}{\delta\mu_n\delta\mu_m} + \frac{\kappa^2}{16}\frac{1}{\xi^2}\right) \tag{4.65}$$

as the Laplace transform of (4.61).

We now consider the second term of (4.50),

$$\partial_\xi\int_0^\infty dl e^{-l\xi}\int_0^\infty dl' \frac{\delta\ln Z_u(J)}{\delta J(l')}\frac{\delta\ln Z_u(J)}{\delta J(l-l')} \tag{4.66}$$

Using the identity (4.62) is is readily seen that one must evaluate

$$\partial_\xi\left(\int_0^\infty dl e^{-l\xi}\frac{\delta\ln Z_u(J)}{\delta J(l)}\right)^2 \tag{4.67}$$

Hence, we consider

$$\int_0^\infty dl e^{-l\xi}\left(\frac{l^{-5/2}}{\Gamma(-\frac{3}{2})} - \frac{3\Lambda}{8}\frac{l^{-1/2}}{\Gamma(\frac{1}{2})} + \frac{\kappa^2}{2\pi}\int_0^\infty dl' J(l')\frac{\sqrt{ll'}}{l+l'}\right)$$
$$+\int_0^\infty dl e^{-l\xi}\left(\kappa^2\sum_{n\geq 0}\frac{l^{n+1/2}}{\Gamma(n+\frac{3}{2})}\frac{\delta}{\delta\mu_n}\left(\frac{\delta\ln Z_u(J)}{\delta J(l-l')}\right)\right) \tag{4.68}$$

Of the above, we first evaluate

$$\int_0^{\infty} dl e^{-l\xi} \left(\frac{\kappa^2}{2\pi} \int_0^{\infty} dl' J(l') \frac{\sqrt{ll'}}{l+l'} \right) \tag{4.69}$$

Realising that the Laplace transform inhibits the magnitude of l through exponential damping, we may consider a small l expansion

$$\frac{\sqrt{ll'}}{l+l'} = \sqrt{\frac{l}{l'}} \sum_{n=0}^{\infty} (-)^n \left(\frac{l}{l'}\right)^n \tag{4.70}$$

Recalling the choice of the source,

$$\int_0^{\infty} dl J(l) l^{n+1/2} = \kappa^{-2} \Gamma\left(n + \frac{3}{2}\right) \mu_n \tag{4.71}$$

By definition of the loop operator, only modes with positive n contribute, and hence the expansion (4.70) implies that the term (4.69) does not contribute to the Laplace transform, since all powers of n in the expansion of l' are negative.

The remaining terms are

$$\int_0^{\infty} dl e^{-l\xi} \left(\frac{l^{-5/2}}{\Gamma(-\frac{3}{2})} - \frac{3\Lambda}{8} \frac{l^{-1/2}}{\Gamma(\frac{1}{2})} + \kappa^2 \sum_{n \geq 0} \frac{l^{n+1/2}}{\Gamma(n+\frac{3}{2})} \frac{\delta \ln Z_u(J)}{\delta \mu_n} \right) \tag{4.72}$$

The Laplace transform may then be straightforwardly performed, to yield

$$\left(\xi^{3/2} - \frac{3\Lambda}{8} \xi^{-1/2} + \kappa^2 \sum_{n \geq 0} \xi^{-n-3/2} \frac{\delta \ln Z_u(J)}{\delta \mu_n} \right) \tag{4.73}$$

Thus, the Laplace transform of (4.66) is

$$-\partial_\xi \left(\xi^{3/2} - \frac{3\Lambda}{8} \xi^{-1/2} + \kappa^2 \sum_{n \geq 0} \xi^{-n-3/2} \frac{\delta \ln Z_u(J)}{\delta \mu_n} \right)^2 \tag{4.74}$$

The third term of (4.50) is

$$\kappa^2 l \int_0^{\infty} dl' J(l') l' \frac{\delta \ln Z_u(J)}{\delta J(l+l')} \tag{4.75}$$

which becomes, after inserting the ansatz

$$\kappa^2 l \int_0^\infty dl' J(l') l' \left(\frac{(l+l')^{-5/2}}{\Gamma(-\frac{3}{2})} - \frac{3\Lambda}{8} \frac{(l+l')^{-1/2}}{\Gamma(\frac{1}{2})} + \frac{\kappa^2}{2\pi} \int_0^\infty dl_1 J(l_1) \frac{\sqrt{l_1(l+l')}}{l_1+l+l'} \right)$$
$$+\kappa^4 l \int_0^\infty dl' J(l') l' \left(\sum_{n\geq 0} \frac{(l+l')^{n+1/2}}{\Gamma(n+3/2)} \frac{\delta \ln Z_u(J)}{\delta \mu_n} \right) \tag{4.76}$$

Let us consider the term

$$\frac{\kappa^2 l}{\Gamma(-\frac{3}{2})} \int_0^\infty dl' J(l') l' (l+l')^{-5/2} \tag{4.77}$$

Again, performing a small l expansion, one has

$$(l+l')^{-5/2} = l'^{-5/2} \sum_{n=0}^{\infty} \alpha_n \left(\frac{l}{l'}\right)^n \tag{4.78}$$

where α_n represents the n'th binomial coefficient of the expansion. One notes that there are no contributions due to this integral, since all powers of l' are negative. Similarly, consider

$$-\frac{3\Lambda\kappa^2 l}{8\Gamma(\frac{1}{2})} \int_0^\infty dl' J(l') l' (l+l')^{-1/2} \tag{4.79}$$

One has the expansion

$$(l+l')^{-1/2} = l'^{-5/2} \sum_{n=0}^{\infty} \alpha_n \left(\frac{l}{l'}\right)^n \tag{4.80}$$

Clearly, the only contribution arises from the $n = 0$ term, which yields

$$-\frac{3\Lambda\kappa^2 l}{8\Gamma(\frac{1}{2})} \int_0^\infty dl' J(l') l'^{1/2} = -\frac{3\Lambda}{16} l \mu_0 \tag{4.81}$$

by definition of the source, $J(l)$. The Laplace transform of this term is then

$$-\frac{3\Lambda}{16}\mu_0 \int_0^\infty dl e^{-l\xi} l = -\frac{3l}{16}\mu_0 \frac{1}{\xi^2} \tag{4.82}$$

Next, we consider

$$\frac{\kappa^4 l}{2\pi} \int_0^\infty dl' J(l') l' \int_0^\infty dl_1 J(l_1) \frac{\sqrt{l_1(l+l')}}{l_1+l+l'} \tag{4.83}$$

To this end, one expands

$$l'\frac{l_1^{1/2}(l+l')^{1/2}}{l_1+l+l'} = l_1^{1/2}l'^{1/2}\frac{l'}{l'+l_1}\left[\left(1+\frac{l}{l'}\right)^{1/2}\left(1+\frac{l}{l'+l_1}\right)^{-1}\right] \tag{4.84}$$

in which case, the only contributions are

$$\frac{\kappa^4}{2\pi}\int_0^\infty dl' J(l')l'^{1/2}\int_0^\infty dl_1 J(l_1)l_1^{1/2} \tag{4.85}$$

Laplace transforming,

$$\frac{\mu_0^2}{2\pi}\left(\frac{\pi}{4}\right)\int_0^\infty dl e^{-l\xi}l = \frac{\mu_0^2}{8}\frac{\Gamma(2)}{\xi^2} = -\frac{\mu_0^2}{8}\partial_\xi\frac{1}{\xi} \tag{4.86}$$

We now consider

$$\kappa^4 l\int_0^\infty dl' J(l')l'\sum_{n\geq 0}\frac{(l+l')^{n+1/2}}{\Gamma(n+\frac{3}{2})}\frac{\delta \ln Z_u(J)}{\delta\mu_n} \tag{4.87}$$

Writing the binomial coefficients in terms of the gamma function yields

$$\left(1+\frac{l}{l'}\right)^{n+1/2} = \sum_{k=0}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+\frac{3}{2}-k)\Gamma(k+1)}\left(\frac{l}{l'}\right)^k \tag{4.88}$$

in which case,

$$\kappa^4 l\sum_{n\geq 0}\int_0^\infty dl' J(l')l'^{n-k+3/2}\sum_{k=0}^{\infty}\frac{l^k}{\Gamma(n+\frac{3}{2}-k)\Gamma(k+1)}\frac{\delta \ln Z_u(J)}{\delta\mu_n} \tag{4.89}$$

Recalling our definition of the source, one has the requirement

$$k \leq n+1 \tag{4.90}$$

and so, altering the sum over k:

$$\kappa^4 l\sum_{n\geq 0}\int_0^\infty dl' J(l')l'^{n-k+3/2}\sum_{k=0}^{n+1}\frac{l^k}{\Gamma(n+\frac{3}{2}-k)\Gamma(k+1)}\frac{\delta \ln Z_u(J)}{\delta\mu_n} \tag{4.91}$$

Employing the relation (4.71) permits us to write

$$\kappa^2 l\sum_{n\geq 0}\sum_{k=0}^{n+1}\mu_{n-k+1}\frac{\Gamma(n+\frac{5}{2}-k)l^k}{\Gamma(n+\frac{3}{2}-k)\Gamma(k+1)}\frac{\delta \ln Z_u(J)}{\delta\mu_n} \tag{4.92}$$

Making use of the recursion property $\Gamma(x+1) = x\Gamma(x)$,

$$\kappa^2 l \sum_{n\geq 0} \sum_{k=0}^{n+1} \mu_{n-k+1} \left(n - k + \frac{3}{2}\right) \frac{l^k}{\Gamma(k+1)} \frac{\delta \ln Z_u(J)}{\delta \mu_n} \tag{4.93}$$

The Laplace transform of this object is just

$$-\kappa^2 \partial_\xi \sum_{n\geq 0} \sum_{k=0}^{n+1} \mu_{n-k+1} \left(n - k + \frac{3}{2}\right) \frac{1}{\xi^{k+1}} \frac{\delta \ln Z_u(J)}{\delta \mu_n} \tag{4.94}$$

Choosing to set $m = n - k + 1$, and noting that the limits of the sum remain unaltered, yields

$$-\kappa^2 \partial_\xi \sum_{n\geq 0} \sum_{m=0}^{n+1} \mu_m \left(m + \frac{1}{2}\right) \frac{1}{\xi^{n-m+2}} \frac{\delta \ln Z_u(J)}{\delta \mu_n} \tag{4.95}$$

Lastly, $\partial_\xi G_0^2(\xi)$ is the laplace transform of $\rho_0(l)$, and is thus already known to be $3\xi^2 - \frac{3}{4}\Lambda$ from our previous matrix model calculations.

Hence, the Laplace transform of (4.50) is

$$\begin{aligned} 0 \;=\; & -\partial_\xi \left(\kappa^4 \sum_{m,n\geq 0} \xi^{-m-n-3} \frac{\delta^2 \ln Z_u(J)}{\delta \mu_n \delta \mu_m} + \frac{g}{16} \frac{1}{\xi^2} \right) \\ & -\partial_\xi \left(\xi^{3/2} - \frac{3\Lambda}{8} \xi^{-1/2} + \kappa^2 \sum_{n\geq 0} \xi^{-n-3/2} \frac{\delta \ln Z_u(J)}{\delta \mu_n} \right)^2 \\ & -\kappa^2 \partial_\xi \sum_{n\geq 0} \sum_{m=0}^{n+1} \mu_m \left(m + \frac{1}{2}\right) \frac{1}{\xi^{n-m+2}} \frac{\delta \ln Z_u(J)}{\delta \mu_n} \\ & -\partial_\xi \left(\left(\frac{\mu_0^2}{8} - \frac{3\Lambda}{16}\mu_0 \right) \frac{1}{\xi} \right) - \partial_\xi \left(-\xi^3 + \frac{3}{4}\Lambda\xi \right) \end{aligned} \tag{4.96}$$

Expanding the squared term, one notes that terms arising from the shift cancel the tadpole interaction, yielding

$$\begin{aligned} 0 \;=\; & -2\kappa^4 \partial_\xi \sum_{n\geq 0} \sum_{m=0}^{n} \xi^{-m-3} \frac{\delta \ln Z_u(J)}{\delta \mu_{m-n}} \frac{\delta \ln Z_u(J)}{\delta \mu_n} \\ & -\kappa^2 \partial_\xi \sum_{n\geq 0} \sum_{m=0}^{n+1} \mu_m \left(m + \frac{1}{2}\right) \frac{1}{\xi^{n-m+2}} \frac{\delta \ln Z_u(J)}{\delta \mu_n} \end{aligned}$$

$$-\partial_\xi\left(\left(\left(\frac{\mu_0}{2}-\frac{3\Lambda}{8}\right)^2+\frac{9\Lambda^2}{64}\right)\frac{1}{2\xi}+\frac{\kappa^2}{16}\frac{1}{\xi^2}\right)$$
$$-2\kappa^2\partial_\xi\sum_{n\geq 0}\left(\xi^{-n}-\frac{3\Lambda}{8}\xi^{-n-2}\right)\frac{\delta\ln Z_u(J)}{\delta\mu_n} \tag{4.97}$$

In the above, terms in ξ which are not summed over constitute the remnants of the shift. Requiring each coefficient corresponding to a particular power of ξ to vanish as a consequence of linear independence, one obtains an infinite series of constraints which the universal partition function must obey. Setting the coefficient of ξ^{-1} to zero readily yields

$$\left(2\frac{\partial}{\partial\mu_1}+\frac{1}{2\kappa^2}\left(\left(\frac{3t}{8}-\frac{\mu_0}{2}\right)^2+\frac{9}{64}\Lambda^2\right)+\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)\mu_n\frac{\partial}{\partial\mu_{n-1}}\right)Z_u=0 \tag{4.98}$$

which corresponds to the L_{-1} mode of the Virasoro algebra. Similarly, one obtains for ξ^{-2},

$$\left(2\left(\frac{\partial}{\partial\mu_2}-\frac{3t}{8}\frac{\partial}{\partial\mu_0}\right)+\frac{1}{16}+\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\mu_n\frac{\partial}{\partial\mu_n}\right)Z_u=0 \tag{4.99}$$

and lastly, for ξ^{-m} with $m\geq 3$,

$$\left(2\left(\frac{\partial}{\partial\mu_{p+3}}-\frac{3\Lambda}{8}\frac{\partial}{\partial\mu_{p+1}}\right)+\kappa^2\sum_{n=0}^{p}\frac{\partial^2}{\partial\mu_n\partial\mu_{p-n}}-\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\mu_n\frac{\partial}{\partial\mu_{n+p+1}}\right)Z_u=0 \tag{4.100}$$

where $p\geq 0$.

One may verify that the above constraints close a Virasoro algebra [10],[11]

$$[L_n,L_m]=(n-m)L_{n+m} \tag{4.101}$$

demonstrating that the Virasoro structure has indeed survived the transition to the continuum, and implying that the constraints may be solved consistently.

4.2. The Density Approach

In this section, we will consider the density approach to a continuum Fokker–Planck Hamiltonian, which is more desirable than the traditional z representation for a number of reasons. Firstly, the machinery of the Fourier transform is more easily dealt with, as opposed to the subtleties associated with the convergence of inverse Laplace transforms. Futhermore, in the previous section, in order to obtain the correct continuum limit it was necessary to subtract a non–universal piece from the one point disk amplitude. In what follows, it will be shown that this shift is somewhat troublesome. Additionally, in the z representation, the continuum Hamiltonian was found to be non–local. All of these difficulties are resolved satisfactorily when considering the density formalism, which has enjoyed great success in the study of the $c = 1$ string.

Central to our considerations is the existence of an algebraic structure, which arises in a natural way when considering the collective field theoretic description of matrix models. In particular, the existence of an algebraic structure obeyed by the operators $\hat{O}_i$ under commutation ensures that the equation

$$\hat{O}_i^\dagger J = 0 \tag{4.102}$$

has a solution, at least locally. If this were not so, one could not restore the explicit hermiticity of the Hamiltonian following the change to invariant collective fields. Previously, we showed that such an algebraic structure is realised in the $c = 0$ matrix model in the form of a Virasoro algebra; however, the choice of collective fields

$$\begin{aligned} \phi_n &= TrM^n \,,\; n > 0 \\ \phi(l) &= Tre^{lM},\; l > 0 \end{aligned} \tag{4.103}$$

permitted us to capture a single positive frequency Virasoro algebra.

In the density formulation, one recasts the theory in terms of the collective field

$$\phi(k) = Tr(e^{ikM}) \tag{4.104}$$

where k is able to assume both positive and negative values. Our first step in studying the density description will be to verify the existence of a Virasoro structure in k space.

To this end, recall that

$$\hat{O}_i = -\Omega_{ij}\partial_j + \omega_i \tag{4.105}$$

In k space, this operator is written as[1]

$$\hat{O}(k) = -\int_{-\infty}^{\infty} dk'\Omega(k,k';\phi)\pi(k') + k\int_0^k dk'\phi(k-k')\phi(k') \tag{4.106}$$

One would like to verify that the commutation of this operator with itself yields the required algebraic structure; we consider

$$\begin{aligned}
&\left[\bar{k}\int_{-\infty}^{\infty} dk''k''\phi(\bar{k}+k'')\pi(k''), k\int_{-\infty}^{\infty} dk'k'\phi(k+k')\pi(k')\right] \\
&= \bar{k}k\int_{-\infty}^{\infty} dk''k''\phi(\bar{k}+k'')\int_{-\infty}^{\infty} dk'k'\delta(k''-\bar{k}-k)\pi(k') \\
&\quad -\bar{k}k\int_{-\infty}^{\infty} dk'k'\phi(k+k')\int_{-\infty}^{\infty} dk''k''\delta(k'-\bar{k}-k'')\pi(k'')
\end{aligned} \tag{4.107}$$

Integrating over the delta functions yields

$$\begin{aligned}
&\bar{k}k\int_{-\infty}^{\infty} dk'k'(k+k')\phi(\bar{k}+k+k')\pi(k') \\
&-\int_{-\infty}^{\infty} dk'k'(k'-\bar{k})\phi(k+k')\pi(k'-\bar{k})
\end{aligned} \tag{4.108}$$

Changing variables in the second integral $k' \to k'-\bar{k}$, permits one to write

$$-k\bar{k}(\bar{k}-k)\int_{-\infty}^{\infty} dk'k'\phi(k'+k+\bar{k})\pi(k') \tag{4.109}$$

The next commutator we consider is

$$\begin{aligned}
&\left[-\bar{k}\int_{-\infty}^{\infty} dk''k''\phi(\bar{k}+k'')\pi(k''), \int_0^k dk'\phi(k-k')\phi(k')\right] \\
&= -\int_0^k dk'(k-k')\phi(\bar{k}+k-k')\phi(k') - \int_0^k dk'k'\phi(\bar{k}+k')\phi(k-k')
\end{aligned} \tag{4.110}$$

where we have integrated over the delta functions arising from the action of the conjugate momentum. The last commutator we need evaluate is

$$\begin{aligned}
&\left[\int_0^{\bar{k}} dk''\phi(\bar{k}-k'')\phi(k''), -\int_{-\infty}^{\infty} dk'k'\phi(k+k')\pi(k')\right] \\
&= +\int_0^{\bar{k}} dk''(\bar{k}-k'')\phi(\bar{k}+k-k'')\phi(k'') + \int_0^{\bar{k}} dk''k''\phi(k+k'')\phi(\bar{k}-k'')
\end{aligned} \tag{4.111}$$

[1]Note that we have set $i\pi(k) \to \pi(k)$. This is conventional, we shall introduce it later when required.

Making the change of variables $k'' = k' + \bar{k}$ in the second integral of (4.110), and combining the result with the first integral of (4.111) yields

$$\int_0^{\bar{k}+k} dk''(\bar{k} - k'')\phi(k + \bar{k} - k'')\phi(k'') \tag{4.112}$$

Similarly, setting $k + k'' = k'$ in the second integral of (4.111), and combining the result with the first integral of (4.110), yields

$$\int_0^{k+\bar{k}} dk'(k' - k)\phi(k')\phi(\bar{k} + k - k') \tag{4.113}$$

Thus, combining (4.111) and (4.113) one has

$$(\bar{k} - k)\int_0^{k+\bar{k}} dk'\phi(k')\phi(\bar{k} + k - k') \tag{4.114}$$

Assembling the commutators, one obtains

$$(\bar{k} - k)\left(-\int_{-\infty}^{\infty} dk'k'\phi(k' + k + \bar{k})\pi(k') + \int_0^{k+\bar{k}} dk'\phi(k')\phi(\bar{k} + k - k')\right) \tag{4.115}$$

It is thus clear that, under commutation,

$$[\hat{O}(\bar{k}), \hat{O}(k)] = (\bar{k} - k)\hat{O}(k + \bar{k}) \tag{4.116}$$

and thus the operator $\hat{O}(k)$ does indeed close a Virasoro algebra, without a central extension.

In what follows, it will prove useful to rewrite the operator $\hat{O}(k)$ by means of the following identity for $\omega(k)$:

$$\omega(k) = \int_{-\infty}^{\infty} dk'\Omega(k, k')\frac{1}{|k'|}\phi(-k') \tag{4.117}$$

This identity is seen to yield the usual expression for $\omega(k)$ by splitting the region of integration as follows: [2]

$$\begin{aligned} \omega(k) &= -k\int_{-\infty}^{-k} \frac{k'}{|k'|}\phi(k + k')\phi(-k')dk' - k\int_{-k}^{0} \frac{k'}{|k'|}\phi(k + k')\phi(-k')dk' \\ &\quad -k\int_0^{\infty} \frac{k'}{|k'|}\phi(k + k')\phi(-k')dk' \end{aligned} \tag{4.118}$$

[2] Without loss of generality, we consider $k > 0$.

A change of variables $k' \to -k - k'$ in the first integral yields

$$
\begin{aligned}
\omega(k) &= k\int_0^\infty \frac{k'}{|k'|}\phi(k+k')\phi(-k')dk' - k\int_{-k}^0 \frac{k'}{|k'|}\phi(k+k')\phi(-k')dk' \\
&\quad -k\int_0^\infty \frac{k'}{|k'|}\phi(k+k')\phi(-k')dk' \qquad (4.119)
\end{aligned}
$$

giving

$$\omega(k) = k\int_0^k \phi(k-k')\phi(k')dk' \qquad (4.120)$$

as required.

Hence, we may now rewrite the operator $\hat{O}(k)$ as follows,

$$\hat{O}(k) = -\int_{-\infty}^{\infty} dk'\Omega(k,k';\phi)\left(i\pi(k') - \frac{1}{|k'|}\phi(-k')\right) \qquad (4.121)$$

where we have reintroduced the factor of i mentioned earlier.

Recall that the Jacobian must satisfy

$$\hat{O}^\dagger(k)J = 0 \qquad (4.122)$$

The bracketed term of (4.121) looks reminiscent of the annilation operator corresponding to a massless boson. In fact, one can show that it is the operator $\tilde{O}_h^\dagger(k)$ associated with the hermitian form of the Hamiltonian which may be interpreted in this way. Recall that this operator satisfies

$$\tilde{O}_h^\dagger(k)J^{1/2} = 0 \qquad (4.123)$$

In order to see this more clearly, let us fluctuate about a trivial background

$$\phi(x) = \partial_x\eta \qquad (4.124)$$

and

$$p(x) = -\partial_x\pi \qquad (4.125)$$

with the corresponding mode expansions

$$\eta(x,t) = -i\int_{-\infty}^{\infty}\frac{dk}{2\pi k}\left(\alpha_k e^{-ik(t-x)} + \tilde{\alpha}_k e^{-ik(t+x)}\right) \qquad (4.126)$$

and

$$p(x,t) = -\frac{1}{2}\int_{-\infty}^{\infty} dk\left(\alpha_k e^{-ik(t-x)} + \tilde{\alpha}_k e^{-ik(t+x)}\right) \qquad (4.127)$$

The oscillators α_k, $\tilde{\alpha}_k$ satisfy the commutation relations

$$\begin{aligned} [\alpha_k, \alpha_{k'}] &= k\delta(k+k') \\ [\tilde{\alpha}_k, \tilde{\alpha}_{k'}] &= k\delta(k+k') \end{aligned} \tag{4.128}$$

and

$$[\alpha_k, \tilde{\alpha}_{k'}] = 0 \tag{4.129}$$

One can then verify that the fields satisfy the canonical commutation relation,

$$[p(x), \eta(x')] = -i\delta(x-x') \tag{4.130}$$

Let us now consider taking the spatial derivative of (4.126), yielding

$$\phi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(\alpha_k(t)e^{ikx} - \tilde{\alpha}_k(t)e^{-ikx}\right) \tag{4.131}$$

Defining

$$\alpha_+(t-x) = \int_{-\infty}^{\infty} dk \alpha_k e^{ik(t-x)} \tag{4.132}$$

and

$$\alpha_-(t+x) = \int_{-\infty}^{\infty} dk \tilde{\alpha}_k e^{ik(t+x)} \tag{4.133}$$

one sees that (4.131) may be written more succinctly as

$$\phi(x) = \frac{1}{2\pi}\left(\alpha_+ - \alpha_-\right) \tag{4.134}$$

The definitions (4.132), (4.133) correspond to right and left moving degrees of freedom, respectively.

Requiring the hermeticity of the field $\phi(x)$ forces the oscillators to obey

$$\alpha_k = \alpha^{\dagger}_{-k} \tag{4.135}$$
$$\tilde{\alpha}_k = \tilde{\alpha}^{\dagger}_{-k} \tag{4.136}$$

One may write $p(x,t)$ in terms of the left and right moving degrees of freedom as follows,

$$p(x,t) = -\frac{1}{2}\left(\alpha_+ + \alpha_-\right) \tag{4.137}$$

Thus, one sees that the right and left movers may be expressed in terms of the fields as

$$\alpha_{\pm} = p(x) \pm \pi\phi(x) \tag{4.138}$$

in x space.

In what follows, it will prove convenient to consider the Fokker–Planck Hamiltonian with a trivial background,

$$H = \frac{1}{2}\int dx (\partial_x \pi)^2 \phi + \frac{\beta^2 \pi^2}{6}\int dx \phi^3(x) \tag{4.139}$$

One can easily verify that this may be rewritten in terms of right and left movers by means of (4.138) as follows

$$H = \frac{1}{12\pi}\left(\alpha_+^3 - \alpha_-^3\right) \tag{4.140}$$

which will prove useful when expressing the continuum Hamiltonian in terms of right and left movers.

Recalling the definition of $\phi(k)$, (4.104), one sees that

$$\phi(k) = \int dx \phi(x) e^{ikx} \tag{4.141}$$

and thus one naturally defines the x space representation as

$$\phi(x) = \int \frac{dk}{2\pi}\phi(k) e^{-ikx} \tag{4.142}$$

One is then forced to choose the following transforms for the conjugate momentum,

$$\pi(x) = \int dk e^{ikx}\pi(k) \tag{4.143}$$

and

$$\pi(k) = \int \frac{dx}{2\pi} e^{-ikx}\pi(x) \tag{4.144}$$

since we wish to preserve the canonical commutator between the two fields in both x and k space.

It is then straightforward from (4.131) to show that in k space, one has

$$\phi(k) = \alpha_{-k} - \tilde{\alpha}_k \tag{4.145}$$

where we have suppressed the time dependence of the oscillators, and that

$$\pi(k) = \frac{1}{2ik}(\alpha_k + \tilde{\alpha}_{-k}) \tag{4.146}$$

recalling that $p(x) \equiv -\partial_x \pi(x)$.

In order to make contact with the interpretation of (4.123), let us consider the operator

$$\vartheta(x) = p(x) - i \fint dy \frac{\phi(y)}{x-y} \tag{4.147}$$

which is readily rewritten as

$$\vartheta(x) = -\int dk ik\pi(k)e^{ikx} - i\int \frac{dk}{2\pi}\phi(k) \fint dy \frac{e^{-iky}}{x-y} \tag{4.148}$$

By means of contour integration arguments, one can easily show that

$$\fint dy \frac{e^{-iky}}{x-y} = -i\pi\epsilon(-k)e^{-ikx} \tag{4.149}$$

where we have taken half of the residues at the poles in accord with the principal value prescription, and defined

$$\epsilon(k) \equiv \left\{ \begin{array}{c} 1 \; k < 0 \\ -1 \; k < 0 \end{array} \right\} \tag{4.150}$$

Hence, setting $k \to -k$ in the second integral of (4.148) one obtains

$$\vartheta(x) = -\int dk \left(ik\pi(k) + \frac{1}{2}\epsilon(k)\phi(-k) \right) e^{ikx} \tag{4.151}$$

Inserting (4.145) and (4.146) in the above, one obtains

$$-\frac{1}{2}\int dk \left(\tilde{\alpha}_{-k}(1-\epsilon(k)) + \alpha_k(1+\epsilon(k))\right) e^{ikx} \tag{4.152}$$

Thus, one sees that one has the correspondence

$$\frac{1}{2}\left(\alpha_k(1+\epsilon(k)) + \tilde{\alpha}_{-k}(1-\epsilon(k))\right) = ik\pi(k) + \frac{1}{2}\epsilon(k)\phi(-k) \tag{4.153}$$

Hence, for $k > 0$, one has

$$\alpha_k = ik\pi(k) + \frac{1}{2}\phi(-k) \tag{4.154}$$

and for $k < 0$,

$$\tilde{\alpha}_k = -ik\pi(-k) - \frac{1}{2}\phi(k) \tag{4.155}$$

Note that, from (3.50) one has

$$\tilde{O}_h^\dagger(k) = \int_{-\infty}^{\infty} dk'\Omega(k,k')\frac{\partial}{\partial\phi(k)} + \frac{1}{2}\omega(k) \tag{4.156}$$

which is readily rewritten as

$$\tilde{O}_h^\dagger(k) = -k\int_{-\infty}^{\infty} dk'\phi(k+k')\left(ik'\pi(k') + \frac{1}{2}\epsilon(k')\phi(-k')\right) \tag{4.157}$$

The relation (4.123) requires that

$$k\int_{-\infty}^{\infty} dk'\phi(k+k')\left(ik'\pi(k') + \frac{1}{2}\epsilon(k')\phi(-k')\right) J^{1/2} = 0 \tag{4.158}$$

which can be restated as

$$\left(ik\pi(k) + \frac{1}{2}\epsilon(k)\phi(-k)\right) |\rangle = 0 \tag{4.159}$$

for all k.

In terms of the relations (4.154) and (4.155) this may be rewritten as

$$\begin{aligned} \alpha_k |\rangle &= 0,\ k > 0 \\ \tilde{\alpha}_k |\rangle &= 0,\ k < 0 \end{aligned} \tag{4.160}$$

which is nothing but the defining statement of the Fock vacuum. The linearization of the constraint points to a possible $SL(2,R)$ representation.

4.2.1. The Continuum Form of the Fokker–Planck Hamiltonian in the Density Formalism and Yoneya's Multicritical Continuum Hamiltonian

In the previous section a continuum form of the Fokker–Planck Hamiltonian in the z representation was written down. To accomplish this, a shift was made to cancel the non universal contribution to the one point disk amplitude arising from the potential term.

Specifically, we shifted

$$\begin{aligned} G(z) &\to \frac{1}{2}v'(z) + G(a\xi) \\ &= \frac{1}{2}v'(z) + a^{3/2}G(\xi) + O(a^{5/2}) \end{aligned} \tag{4.161}$$

Recall that $G(z)$ consists entirely of negative powers of z, whilst the potential is generically a polynomial in z. The trouble arises when considering the powers of z in $G(z)$ to constitute a basis; one sees that such a basis is incapable of constructing a shift comprising positive powers of z. To overcome this difficulty we will consider constructing a continuum Hamiltonian in the density formalism, where such complications do not arise. Moreover, we will argue that such a Hamiltonian assumes the same local form for all criticalities.

The Fokker–Plank Hamiltonian is written in the density formalism as

$$H_{fp} = \frac{1}{2\beta^2}\int dx\phi(x)\,(\partial_x\pi)^2 + \frac{\beta^2}{2}\int dx\phi(x)\left(⨍ dy\frac{\phi(y)}{x-y} - \frac{v'(x)}{2}\right)^2 \tag{4.162}$$

Expanding the squared term, and using the identity

$$\int dx\phi(x)\left(⨍ dy\frac{\phi(y)}{x-y}\right)^2 = \frac{\pi^2}{3}\int dx\phi^3(x) \tag{4.163}$$

one may rewrite (4.162) as

$$\begin{aligned} H_{fp} &= \frac{1}{2\beta^2}\int dx\phi(x)\,(\partial_x\pi)^2 + \frac{\beta^2\pi^2}{6}\int dx\phi^3(x) \\ &-\frac{\beta^2}{2}\int dx\phi(x)\left(⨍ dy\frac{\phi(y)}{x-y}v'(x) - \frac{1}{4}v'^2(x)\right) \end{aligned} \tag{4.164}$$

As discussed previously, the leading configuration is given by

$$\frac{\delta V_{eff}}{\delta\phi} = 0 \tag{4.165}$$

which yields the following solution for ϕ_0,

$$\frac{\pi^2}{2}\phi_0^2(x) = \frac{1}{2}⨍ dy\frac{\phi_0(y)}{x-y}\left(v'(x) - v'(y)\right) + \frac{1}{8}v'(x)^2 \tag{4.166}$$

Let us perturb (4.164) about a classical background $\phi_0(x)$ by setting

$$\phi(x) = \phi_0(x) + \frac{1}{\beta}\partial_x\eta(x) \tag{4.167}$$

Notice that the factor of β^{-1} is consistent with the naive $N \to \infty$ limit, in that $\phi(x) \to \phi_0(x)$ in this limit. Further, one writes

$$\partial_x \pi(x) = -\beta p(x) \tag{4.168}$$

Before proceeding, let us briefly consider the time of flight formalism, which plays a large role in the continuum limit of the $d = 1$ model. Consider the classical motion of a particle of unit mass in a potential $v(x)$, which has a total energy

$$\mu_F = \frac{1}{2}\dot{x}^2 + v(x) \tag{4.169}$$

where μ_F plays the role of the Fermi level. This naturally implies that the classical velocity of the particle is given by

$$\frac{dx}{d\tau} = \sqrt{2(\mu_F - v(x)} \tag{4.170}$$

Recall that, in the $d = 1$ picture, the rooted term is none other than the classical background $\phi_0(x)$, which was found by minimising the effective potential. Thus, one has

$$d\tau = \frac{dx}{\pi\phi_0(x)} \tag{4.171}$$

In the continuum, approached in the case of $d = 1$ by adjusting μ to the quadratic maximum of the potential, it was demonstrated that the time of flight became infinite, thus generating an extra dimension, and elevating the $d = 1$ model to the status of a two dimensional theory. In the case of $d = 0$, we set

$$\tau = \int \frac{dx}{\pi\phi_0(x)} \tag{4.172}$$

Considering the standard symmetric ansatz, and evaluating the time of flight for one cycle, one has

$$\tau = 2\int_0^a \frac{dx}{(\alpha^2 - x^2)\sqrt{\gamma^2 - x^2}} = \frac{\pi}{\alpha\sqrt{\alpha^2 - \gamma^2}} \tag{4.173}$$

Recall that, when approaching the continuum limit for $m = 2$, one requires the accumulation of an extra root at the cut endpoint, implying $b \to a$. Naively, it would seem that one has the same mechanism as found in $d = 1$, whereby an extra dimension is generated; however, this is not the case. Let us scale $x \to a\bar{y}$,

$$
\begin{aligned}
d\tau &= \frac{dx}{\pi\phi_0(x)} = \frac{ad\bar{y}}{a^{3/2}\left(\bar{y}+\frac{\Lambda^{1/2}}{2}\right)\sqrt{\bar{y}-\Lambda^{1/2}}} \\
&= a^{-1/2}\frac{d\bar{y}}{\phi(\bar{y})} \qquad (4.174)
\end{aligned}
$$

Thus, if the stochastic time is to survive the continuum limit, one must rescale $d\tau'$ $= a^{1/2}d\tau$ when considering the $m = 2$ multicriticality. An interesting consequence of this observation is that, whilst τ is not finite when approaching the continuum, the rescaled stochastic time τ' is. To see this, recall that

$$
\tau = \frac{\pi}{\alpha\sqrt{\alpha^2-\gamma^2}} \qquad (4.175)
$$

Inserting the scaling relations for α and γ, one finds that

$$
\tau = \frac{\pi}{\sqrt{a}}\left(1-\frac{a}{4\gamma_c^2}+O(a)\right) \qquad (4.176)
$$

and hence, when considering $\tau' = \sqrt{a}\tau$, one notes that the rescaled time of flight remains finite as $\alpha \to 0$, and no extra dimension is generated.

Expanding the kinetic term of (4.164) about ϕ_0,

$$
\frac{1}{2\beta^2}\int dx\phi(x)(\partial_x\pi)^2 = \frac{1}{2}\int dx\phi_0(x)p^2 + \frac{1}{2\beta}\int dx(\partial\eta)p^2 \qquad (4.177)
$$

Now we consider the terms

$$
-\frac{1}{2}\beta^2\int dx\phi(x)\left(⨍ dy\frac{\phi(y)}{x-y}v'(x)-\frac{1}{4}v'(x)^2\right) \qquad (4.178)
$$

which may be rewritten as

$$
\begin{aligned}
&-\frac{\pi^2\beta^2}{2}\int\phi(x)\phi_0^2(x)-\frac{\beta^2}{2}\int dx\phi_0(x)⨍ dy\frac{\phi_0(y)}{x-y}v'(y) \\
&-\frac{1}{4}\int dx(\partial_x\eta)⨍ dy(\partial_y\eta)\frac{v'(x)-v'(y)}{x-y} \qquad (4.179)
\end{aligned}
$$

with the aid of the large N configuration (4.166). Note that the terms linear in $\partial\eta$ cancel, signifying that $\phi_0(x)$ is indeed a minimum of the effective potential.

The Hamiltonian may now be written as

$$\begin{aligned}
H_{fp} &= \frac{1}{2\beta^2}\int dx\phi(x)(\partial_x\pi)^2 + \frac{\beta^2\pi^2}{6}\int dx\phi^3(x) \\
&\quad -\frac{\beta^2\pi^2}{2}\int \phi(x)\phi_0^2(x) - \frac{\beta^2}{2}\int dx\phi_0(x)⨍ dy\frac{\phi_0(y)}{x-y}v'(y) \\
&\quad -\frac{1}{4}\int dx(\partial_x\eta)⨍ dy(\partial_y\eta)\frac{v'(x)-v'(y)}{x-y} \qquad (4.180)
\end{aligned}$$

Of particular interest are the terms quadratic in $\partial\eta$, since they permit us to read off the spectrum of the theory. Consider the last term of (4.180)

$$\frac{1}{4}\int dx\partial_x\eta(x)\left(⨍ dy\frac{\partial_y\eta}{x-y}\right)\frac{v'(x)-v'(y)}{x-y} \qquad (4.181)$$

This non-local term quadratic in $\partial\eta$ modifies the spectrum of the theory in a non-trivial way. Unless it can be argued to vanish, the interpretation of the theory as comprising a massless scalar is in jeopardy.

Explicitly inserting a cubic potential, one can easily demonstrate that this term vanishes, where we invoke $\int dx(\partial_x\eta) = 0$. However, the insertion of a quartic potential gives rise to terms of the form $\int dxx(\partial_x\eta)$, which cannot be argued to vanish. The only possibility which remains is that the term may not survive the continuum limit.

In order to consider the continuum limit corresponding to the $m = 2$ multicritical point, let us reconsider the shift (4.167), where we rescale $x \to a\bar{y}$ and ignore non universal contributions to the background,

$$\begin{aligned}
\bar{\phi}(a\bar{y}) &= a^{3/2}\bar{\phi}_0(\bar{y}) + \frac{1}{\beta a}\partial_{\bar{y}}\eta(\bar{y}) \\
&= a^{3/2}\bar{\phi}(\bar{y}) = a^{3/2}\left(\bar{\phi}_0(\bar{y}) + \frac{1}{\beta a^{5/2}}\partial_{\bar{y}}\eta(\bar{y})\right) \qquad (4.182)
\end{aligned}$$

Recall that in the previous section regarding the continuum form of the Fokker–Planck Hamiltonian, the product $\beta a^{5/2}$ was held fixed, while the limits $\beta \to \infty$, $a \to 0$ were taken. This product is the string coupling constant, κ. Thus, we define

$$\bar{\phi}(\bar{y}) = \bar{\phi}_0(\bar{y}) + \frac{1}{\kappa}\partial_{\bar{y}}\eta(\bar{y})$$

Let us consider the non-local term in the continuum limit. Following the discussion regarding the rescaling of the stochastic time, one sees that in order to contribute in the continuum it must scale as a half integer power, $\sqrt{a}$. Naively inspecting this term under the rescaling $x \to a\bar{x}$, one notes that powers of a in the measure cancel those arising from the derivatives; thus the only contributions obtained arise from the potential.

To infer the scaling of the potential, recall that

$$\fint dy \frac{\phi_0(y)}{x-y} = \frac{1}{2}v'(x) \tag{4.183}$$

A naive rescaling of the left hand side $y \to ay$ yields a power of $a^{3/2}$, whereas the right hand side, a polynomial potential, should give rise to integer powers of a. This apparent paradox is resolved by realising that, after rescaling $y \to ay$ on the left hand side, powers of a must enter into the limits of integration.

Whilst it is difficult to do this computation explicitly, one might rather consider

$$\int dx \phi(x) = g \tag{4.184}$$

which is a simpler integral to perform, but nevertheless in the same spirit. When considering the $m=2$ multicritical point, a naive rescaling of the left hand side gives an overall power of $a^{5/2}$, yet g scales as

$$g = g_c - a^2 \tag{4.185}$$

signifying that further contributions of a must reside in the limits of integration.

Let us employ the quartic ansatz

$$\phi(x) = \frac{1}{2}(\alpha^2 - x^2)\sqrt{\gamma^2 - x^2} \tag{4.186}$$

with the scaling relations in the vicinity of the critical point given by

$$\begin{aligned} \alpha^2 &= \gamma_c^2 + \frac{1}{2}a \\ \gamma^2 &= \gamma_c^2 - a \\ x^2 &= \gamma_c^2 - 2\delta x + \frac{\delta x^2}{\gamma_c^2} \end{aligned} \tag{4.187}$$

We wish to evaluate

$$\frac{1}{2}\int_{-\gamma}^{\gamma}(\alpha^2 - x^2)\sqrt{\gamma^2 - x^2}dx \tag{4.188}$$

which, after inserting the relations (4.187), becomes

$$\frac{1}{2}\int d(\delta x)\left(2\delta x+\frac{1}{2}a-\frac{\delta x^2}{\gamma_c^2}\right)\sqrt{2\delta x-a-\frac{\delta x^2}{\gamma_c^2}} \tag{4.189}$$

The dependence of the limits of integration on α may be determined by setting the argument beneath the rooted term to zero, and solving for the roots, which yield the limits of integration relevant to δx,

$$\begin{aligned}\delta x &= \gamma_c^2 \pm \gamma_c^2\sqrt{1-\frac{a}{\gamma_c^2}}\\ &= \gamma_c^2 \pm \gamma_c^2\left(1-\frac{a}{2\gamma_c^2}+O(a^2)\right)\end{aligned} \tag{4.190}$$

where we have used the fact that, in the continuum limit, $a \to 0$. Further, we may neglect terms of order $(\delta x)^2$ as yielding non–universal contributions for $m = 2$. To leading order, we evaluate

$$\frac{1}{2}\int_{a/2}^{\Gamma} d(\delta x)\left(2\delta x+\frac{1}{2}a\right)\sqrt{2\delta x-a} \tag{4.191}$$

where Γ is an arbitrary cut off, since we are only interested in extracting the powers of a.

Let us consider the rescaling $\delta x \to a\xi$, in which case obtains

$$\frac{a^{5/2}}{2}\int_{1/2}^{\frac{\Gamma}{a}} d\xi\left(2\xi+\frac{1}{2}\right)\sqrt{2\xi-1} \tag{4.192}$$

To perform this integral one may partition it in the following way,

$$\frac{a^{5/2}}{2}\int_{1/2}^{\frac{\Gamma}{a}} d\xi\left(2\xi-1+\frac{3}{2}\right)\sqrt{2\xi-1} \tag{4.193}$$

from which one obtains

$$\frac{a^{5/2}}{2}\left[\frac{1}{2}(2\xi-1)^{3/2}+\frac{1}{5}(2\xi-1)^{5/2}\right]_{1/2}^{\frac{\Gamma}{a}} \tag{4.194}$$

Inserting the limits of integration, one has the following

$$\frac{a^{5/2}}{2}\left[\frac{1}{2}\left(\frac{2\Gamma}{a}\right)^{3/2}\left(1-\frac{a}{2\Gamma}\right)^{3/2}+\frac{1}{5}\left(\frac{2\Gamma}{a}\right)^{5/2}\left(1-\frac{a}{2\Gamma}\right)^{5/2}\right] \tag{4.195}$$

Performing a small a expansion of the above, one obtains

$$\begin{aligned}&\frac{a}{4}(2\Gamma)^{3/2}\left(1-\frac{3}{4\Gamma}a+O(a^2)\right)+\frac{(2\Gamma)^{5/2}}{10}\left(1-\frac{5}{4\Gamma}a+O(a^2)\right)\\ =\ &\frac{(2\Gamma)^{5/2}}{10}-\frac{3}{16\Gamma}a^2+O(a^3)\end{aligned} \tag{4.196}$$

Thus, one notes that the terms linear in a cancel, and the correct scaling for g is obtained. Our motivation for studying the above integral is to emphasize that the above analysis suggests that the potential scales with integer powers of a. Examining the non local term quadratic in $\partial\eta$, one sees that this implies an overall integer scaling of

As a result, one sees that the non-local term quadratic in $\partial\eta$ cannot survive the continuum limit, since one requires $\sqrt{a}$ for $m = 2$.

In fact, one can argue that this term must vanish for all criticalities, since the stochastic time will always require rescaling by non integer powers of a, as is easily seen by adapting the time of flight argument given earlier, noting that the universal form of the density for the $m = 3$ multicritical point scales as $a^{5/2}$, implying that the stochastic time must be rescaled by $a^{3/2}$. In general, for the m'th multicritical point, one requires that we rescale $\tau' = a^{k-3/2}\tau$. Thus, since the non-local term will always scale with integer powers of a for all criticalities, it cannot contribute in the continuum limit.

One notes that, when considering integrated quantities in the continuum limit, the integrands must be multiplied by appropriate continuum powers of a, and/or κ, before driving the upper limit to infinity as $a \to 0$.

Similarly, the non-local term quadratic in the background cannot contribute in the continuum since, rescaling the stochastic time by a factor $\sqrt{a}$, one sees that one cannot obtain the integer powers of a required to construct κ^2.

In order to consider the continuum limit of (4.180), one scales $x \to a\bar{y}$, and sets $\phi(x) \to a^{3/2}\bar{\phi}(\bar{y})$, to obtain

$$H = \frac{1}{2}\int d\bar{y}\bar{\phi}(\bar{y})p^2 + \frac{\kappa^2\pi^2}{6}\int d\bar{y}\bar{\phi}^3(\bar{y}) - \frac{\kappa^2\pi^2}{2}\int d\bar{y}\bar{\phi}(\bar{y})\bar{\phi}_0^2(\bar{y}) \tag{4.197}$$

where we have absorbed the factor $\sqrt{a}$ into the redefinition of the stochastic time, and set $\kappa \equiv \beta a^{5/2}$.

The above expression coincides with Yoneya's suggestion for a multicritical continuum Hamiltonian consistent with the string equation [36]. One notes that, evaluating V_{eff} at $\bar{\phi}_0$, the effective potential is no longer positive definite, implying that one may no longer consider the above to be a Fokker–Planck type Hamiltonian.

However, one still has the condition

$$\frac{\delta V_{eff}}{\delta \bar{\phi}} = 0, \quad \Rightarrow \quad \bar{\phi} = \bar{\phi}_0 \tag{4.198}$$

and thus one sees that the continuum background minimises the effective potential, as it must.

Further, it is important to realise that the above Hamiltonian is valid for all criticalities. For a general multicritical point $m = l$, one scales $x \to a\bar{y}$, and $\phi(x) \to a^{l-\frac{1}{2}}\bar{\phi}(\bar{y})$, with the string coupling constant defined by $\kappa = \beta a^{l+\frac{1}{2}}$. As noted earlier, the stochastic time requires a rescaling $\tau \to a^{l-3/2}\tau$. One can easily check that inserting the above scalings in (4.180), one regains (4.197), recalling that the non–local terms vanish for all criticalities.

In order to consider the connection to the string equation, it proves convenient to reformulate the above in terms of right and left movers,

$$H = \frac{1}{4\pi}\int dy \int_{\alpha_-}^{\alpha_+} (p^2 - p_0^2(y))dp \tag{4.199}$$

where we have identified $p_0(y) = \bar{\phi}_0(y)$, and recalled the relation (4.138).

As stated above, Yoneya's motivation for introducing such a Hamiltonian rests on the fact that one must recover the string equation in the continuum.

To this end, we consider the stationary equation,

$$p^2(y) - p_0^2(y) = 0 \tag{4.200}$$

One can compare this with the string equation,

$$[P, Q] = 1 \tag{4.201}$$

which arises most naturally in the context of orthogonal polynomials.

One can show that, in the double scaling limit, Q is defined as a hermitian differential operator of degree 2,

$$Q = d^2 - u^2 \tag{4.202}$$

where d is a differential operator, and u is the specific heat.

A P which satisfies the Heisenberg algebra is

$$P = Q_{+}^{\frac{2m-1}{2}} \tag{4.203}$$

where the subscript indicates that, in the expansion of $Q^{\frac{2m-1}{2}}$ with respect to d, only the positive powers of d are retained.

For the $m = 2$ multicritical point one expands $Q^{3/2}$, bearing in mind that u and d do not commute, and discards negative powers of d, to obtain

$$P = d^3 - \frac{3}{4}\{u, d\} \tag{4.204}$$

Squaring P, and neglecting derivatives of the specific heat at tree level, one obtains

$$P_0^2 = d^6 - 3ud^4 + \frac{9}{4}u^2d^2 \tag{4.205}$$

The tree approximation is classical since it excludes one loop effects, and is thus equivalent to large N. Rewriting P_0^2 in terms of Q, one obtains

$$P_0^2 = Q^3 - \frac{3}{4}uQ + \frac{1}{4}u^3 \tag{4.206}$$

One notes that, making the identifications

$$P_0 \to \pi\bar{\phi}_0 \tag{4.207}$$

and

$$Q \to \xi \tag{4.208}$$

with $u = \sqrt{\Lambda}$, that we have directly obtained the continuum form of the density. For $m = 3$, one sets

$$P = Q_+^{5/2} \tag{4.209}$$

and expands, retaining only positive powers of d, as before. One can show, by the same method of calculation, that one obtains the result

$$P_0^2 = Q^5 + \frac{5}{8}u^3Q^2 - \frac{15}{64}u^4Q + \frac{9}{64}u^5 \tag{4.210}$$

where one considers contributions from tree level only.

Thus (4.206) implies, for the $m = 2$ multicritical density,

$$\pi\phi(\xi) = \left(\xi - \frac{1}{2}\sqrt{\Lambda}\right)\sqrt{\xi + \sqrt{\Lambda}} \tag{4.211}$$

and similarly (4.210) suggests that, for $m = 3$,

$$\pi\phi(\xi) = \left(\left(\xi^2 - \frac{1}{2}\xi\sqrt{\Lambda} + \frac{3}{8}\Lambda\right)\sqrt{\xi + \sqrt{\Lambda}}\right) \tag{4.212}$$

In what follows, this result will be compared with that obtained by more traditional methods.

One sees that the stationary equation of (4.199) implies the string equation, where one makes the above correspondences.

In summary, we have taken the continuum limit of the Fokker–Planck Hamiltonian in the density formulation, to yield a local continuum Hamiltonian valid for all criticalities, which is consistent with the string equation. This Hamiltonian is no longer of the Fokker–Planck type, since the effective potential evaluated at the classical configuration is non zero.

5. THE MOMENTS FORMALISM AND THE $m = 3$ MULTICRITICAL POINT

In this section, we will construct the form of the $m = 3$ multicritical density, by means of the moments formalism of Ambjørn *et al* [19]. The reasons for doing so are twofold; firstly, when considering higher criticalities, it provides a much more expedient method of extracting the critical values of the couplings than the traditional saddlepoint analysis. Secondly, the following formalism proves to generalise naturally in the case of a particular supersymmetric extension of the bosonic matrix model which will be examined later; hence it will be advantageous to develop the formalism in the context of the bosonic matrix model.

We consider the matrix model partition function written in terms of the eigenvalues,

$$Z(g) = \int \prod_i d\lambda_i \Delta^2(\lambda_i)\, e^{-S} \tag{5.1}$$

where the notation $Z(g)$ suggests that we consider the couplings themselves as sources for the field $\phi_n = \sum_i \lambda_i^n$, with $S = \sum_n g_n \phi_n$, and $\Delta^2(\lambda_i)$ the usual Vandermonde determinant. We showed earlier that the partition function obeys

$$O_n Z(g) = 0 \tag{5.2}$$

where the O_n's were found to close a Virasoro algebra under commutation, required for the consistency of the collective field approach. Specifically, in the previous section, we demonstrated that O_n could be written as

$$O_n = \sum_m \alpha_{-m}\alpha_{n+m} + \frac{1}{2}\sum_{r=0}^{n} \alpha_r \alpha_{n-r} \tag{5.3}$$

where the modes may be written in terms of the couplings as follows

$$\alpha_n = -\sqrt{2}\frac{\partial}{\partial g_n},\ n > 0 \tag{5.4}$$

and

$$\alpha_{-n} = -\frac{1}{\sqrt{2}} n g_n, \ n > 0 \tag{5.5}$$

Thus,

$$O_n Z(g) = \left(\sum_m m g_m \frac{\partial}{\partial g_{m+n}} + \sum_{r=0}^{n} \frac{\partial^2}{\partial g_r \partial g_{n-r}} \right) Z(g) = 0 \tag{5.6}$$

Allowing the derivatives to act, and performing some trivial rewritings, one has the correlator

$$\left\langle \sum_i v'(\lambda_i) \lambda_i^{n+1} - \sum_{i \neq j} \frac{\lambda_i^{n+1} - \lambda_j^{n+1}}{\lambda_i - \lambda_j} \right\rangle = 0 \tag{5.7}$$

Multiplying the above correlator by z^{-n-2}, and summing over n yields

$$\sum_i \frac{v'(\lambda_i)}{z - \lambda_i} - \sum_{i \neq j} \frac{1}{z - \lambda_i} \frac{1}{z - \lambda_j} = 0 \tag{5.8}$$

One can rewrite the above as

$$\oint_C \frac{v'(\omega)}{z - \omega} G(\omega) - G^2(z) = 0 \tag{5.9}$$

where we have chosen the contour C such that it encircles the cut,[1] yet excludes the point $z = \omega$. Further, rewriting the above in terms of connected correlators, one obtains

$$\oint_C \frac{v'(\omega)}{z - \omega} G(\omega) - G^2(z) + \frac{1}{N^2} G(z, z) = 0 \tag{5.10}$$

We would like to consider the planar limit of the above equation. Taking the limit $N \to \infty$, one sees that (5.10) becomes

$$\oint \frac{d\omega}{2\pi i} \frac{v'(\omega)}{z - \omega} G_0(\omega) = G_0(z)^2 \tag{5.11}$$

One can readily verify by direct substitution in (5.11) that the solution is

$$G_0(z) = \frac{1}{2} \oint_C \frac{d\omega}{2\pi i} \frac{v'(\omega)}{z - \omega} \left[\frac{(z - x)(z - x)}{(\omega - x)(\omega - y)} \right]^{\frac{1}{2}} \tag{5.12}$$

[1] We assume that the loop operator has a one cut structure.

where, as before, the contour C encircles the cut bounded by $[x, y]$, but the point z is excluded. This equation is accompanied by auxilliary conditions to ensure the correct asymptotic behaviour, $G_0(z) \sim z^{-1}$, as $z \to \infty$,

$$0 = \oint_C \frac{d\omega}{2\pi i} \frac{v'(\omega)}{\sqrt{(\omega - x)(\omega - y)}} \tag{5.13}$$

$$2 = \oint_C \frac{d\omega}{2\pi i} \frac{\omega v'(\omega)}{\sqrt{(\omega - x)(\omega - y)}} \tag{5.14}$$

These conditions are obtained by expanding (5.12) about $z = \infty$, and enforcing the required asymptotic z^{-1} behaviour. At first sight, this solution bears little resemblance to the ansatz proposed when solving the saddlepoint equation; however, we will demonstrate that the two are equivalent.

Let us reconsider (5.12), where one deforms the contour C to infinity, picking up the pole at $\omega = z$ as one does so. Thus we consider

$$\begin{aligned} & \frac{1}{2} \oint_C \frac{d\omega}{2\pi i} \frac{v'(\omega)}{z - \omega} \left[\frac{(z - x)(z - x)}{(\omega - x)(\omega - y)} \right]^{\frac{1}{2}} \\ = & -\frac{1}{2} \oint_{C''} \frac{d\omega}{2\pi i} \frac{v'(\omega)}{z - \omega} \left[\frac{(z - x)(z - y)}{(\omega - x)(\omega - y)} \right]^{\frac{1}{2}} \\ & + \frac{1}{2} \oint_{C'} \frac{d\omega}{2\pi i} \frac{v'(\omega)}{z - \omega} \left[\frac{(z - x)(z - y)}{(\omega - x)(\omega - y)} \right]^{\frac{1}{2}} \end{aligned} \tag{5.15}$$

where the contour C'' encircles the point $\omega = z$, and C' is taken to infinity. Performing the integral over C'' simply picks up the residue at $\omega = z$, and yields $\frac{1}{2} v'(z)$.

In order to consider the integral over C', note that, for $|\omega| > |z|$, where $[x, y] < |z|$, one has the expansion

$$\frac{1}{z - \omega} = -\frac{1}{2} \sum_{n=0}^{\infty} \left\{ \frac{(z - x)^n}{(\omega - x)^{n+1}} + \frac{(z - y)^n}{(\omega - y)^{n+1}} \right\} \tag{5.16}$$

Inserting the above in the integral over C', one is able to write

$$\begin{aligned} & \frac{1}{2} \oint_{C'} \frac{d\omega}{2\pi i} \frac{v'(\omega)}{z - \omega} \left[\frac{(z - x)(z - y)}{(\omega - x)(\omega - y)} \right]^{\frac{1}{2}} \\ = & -\frac{1}{4} \sqrt{(z - x)(z - y)} \sum_{n=1}^{\infty} \left\{ (p - x)^{n-1} M_n + (p - y)^{n-1} J_n \right\} \end{aligned} \tag{5.17}$$

where we have defined the moments,

$$M_k = \oint_C \frac{d\omega}{2\pi i} \frac{v'(\omega)}{(\omega - x)^{k+1/2}(\omega - y)^{1/2}} \tag{5.18}$$

$$J_k = \oint_C \frac{d\omega}{2\pi i} \frac{v'(\omega)}{(\omega - x)^{1/2}(\omega - y)^{k+1/2}} \tag{5.19}$$

Hence, the planar solution of the hermitian matrix model may be written as

$$G_0(z) = \frac{1}{2}v'(z) - \frac{1}{4}\Pi(z)\sqrt{(z-x)(z-y)} \tag{5.20}$$

where we have defined

$$\Pi(z) \equiv \sum_{n=1}^{\infty} \Big((z-x)^{n-1} M_n + (z-y)^{n-1} J_n \Big) \tag{5.21}$$

which is the form of $G_0(z)$ encountered in the context of the saddlepoint analysis. The above formalism, whilst appearing cumbersome, is useful in that it allows one to readily extract the critical values of the couplings and cut interval. The above form of $\Pi(z)$ may be seen to correspond to two series expansions, centred about each end of the cut. As will be shown, such a rewriting enables us to control the accumulation of roots at the endpoints with great ease.

In order to fully appreciate this, let us Taylor expand $\Pi(z)$ about the endpoint of the cut $z = x$,

$$\Pi(z) = \Pi(x) + \sum_{n=1}^{\infty} \frac{(z-x)^n}{n!} \left| \frac{d^n \Pi(z)}{dz^n} \right|_{z=x} \tag{5.22}$$

and about the endpoint $z = y$,

$$\Pi(z) = \Pi(y) + \sum_{n=1}^{\infty} \frac{(z-y)^n}{n!} \left| \frac{d^n \Pi(z)}{dz^n} \right|_{z=y} \tag{5.23}$$

One can combine the above expansions to yield

$$\Pi(z) = \frac{1}{2}(\Pi(x) + \Pi(y)) + \sum_{n=1}^{\infty} \frac{(z-x)^n}{2n!} \left| \frac{d^n \Pi(z)}{dz^n} \right|_{z=x} + \sum_{n=1}^{\infty} \frac{(z-y)^n}{2n!} \left| \frac{d^n \Pi(z)}{dz^n} \right|_{z=y} \tag{5.24}$$

Comparing the above with (5.21), one makes the identifications

$$M_1 + J_1 = \frac{1}{2}(\Pi(x) + \Pi(y)) \tag{5.25}$$

$$M_{k+1} = \frac{1}{2k!}\left|\frac{d^k\Pi(z)}{dz^k}\right|_{z=x} \tag{5.26}$$

$$J_{k+1} = \frac{1}{2k!}\left|\frac{d^k\Pi(z)}{dz^k}\right|_{z=y} \tag{5.27}$$

In order to specify the critical values of the couplings, recall that the m'th multicritical point calls for the accumulation of $m-1$ zeroes at one of the endpoints of support of the density, where the density is of the form

$$\pi\phi(z) = \Pi(z)\sqrt{(x-z)(z-y)} \tag{5.28}$$

Consider a polynomial of the form

$$\Pi(z) = (z-x)^{m-1}(z-\alpha)...(z-\beta) \tag{5.29}$$

where we have enforced the accumulation of zeroes at x. For such a function, one must have

$$\left|\frac{d^n\Pi(z)}{dz^n}\right|_{z=x} = 0,\ n = m-1 \tag{5.30}$$

$$\left|\frac{d^k\Pi(z)}{dz^k}\right|_{z=x} \neq 0,\ k \geq m \tag{5.31}$$

Thus, inspecting the identifications (5.25), (5.26) and (5.27), one sees that, for the m'th multicritical point, we require that

$$\begin{aligned} M_1(g_i^c, x_c, y_c) &= M_2(g_i^c, x_c, y_c) = ... = M_{m-1}(g_i^c, x_c, y_c) = 0 && (5.32)\\ M_k(g_i^c, x_c, y_c) &\neq 0,\ k \geq m && (5.33)\end{aligned}$$

with the J_k's generally non–zero. Using the conditions (5.32) in concert with (5.13) and (5.14) will permit us to solve for the g_i^c's and the continuum values of the cut interval, (x_c, y_c). For the $m = 2$ multicritical point, one simply requires that M_1 vanish. An analogous analysis exists for the accumulation of zeros at the opposite endpoint of the cut, whereby one replaces the M's with the J's. We will now extract the continuum form of the $m = 3$ multicritical density, by means of this formalism.

We employ a symmetric potential which will support the generation of a $k = 3$ multicriticality,

$$v(z) = \frac{1}{2}g_2 z^2 - g_4\frac{1}{4}z^4 + g_6\frac{1}{6}z^6 \tag{5.34}$$

where we have taken $g_i > 0$. The advantage of such a choice is that (5.13) is automatically satisfied. We thus consider the second condition, which now becomes

$$2 = \oint_C \frac{d\omega}{2\pi i} \frac{\omega v'(\omega)}{\sqrt{\omega^2 - \gamma^2}} \tag{5.35}$$

where we have set $y = -x$, and identified $x = \pm\gamma$ as the cut endpoints. The contour C is taken about infinity, and thus we may expand the denominator of the integrand (5.35) about the point $\omega = \infty$, to yield

$$\oint_C \frac{d\omega}{2\pi i} \left(1 + \frac{1}{2}\left(\frac{\gamma}{\omega}\right)^2 + \frac{3}{8}\left(\frac{\gamma}{\omega}\right)^4 + \frac{5}{16}\left(\frac{\gamma}{\omega}\right)^6 + O\left(\frac{\gamma}{\omega}\right)^7\right) v'(\omega) = 2 \tag{5.36}$$

Inserting the explicit form of the potential, we may pick up the poles to yield

$$\frac{1}{2}g_2\gamma^2 - \frac{3}{8}g_4\gamma^4 + \frac{5}{16}g_6\gamma^6 = 2 \tag{5.37}$$

as the condition on the couplings as a function of the cut endpoints. Recalling that our $G(z)$ is symmetric, one may write

$$G(z) = \frac{1}{2}v'(z) - \frac{1}{4}\sqrt{z^2 - \gamma^2}\sum_{n=1}^{\infty}\left((z-\gamma)^{n-1}M_n + (z+\gamma)^{n-1}(-1)^{n+1}M_n\right) \tag{5.38}$$

where we have noted that, for a symmetric potential, the moments satisfy $M_k = (-1)^{k+1}J_k$. Our next task is to construct the moments explicitly. This is readily done from (5.18), where we expand the integrand about $\omega = \infty$, and pick up the poles appropriately to yield

$$\begin{aligned} M_k &= g_{k+1} + k\gamma g_{k+2} + \gamma^2\left(\frac{1}{2} + \frac{k}{2}(k+1)\right)g_{k+3} \\ &\quad + \gamma^3\left(\frac{k}{2} + \frac{k(k+1)(k+2)}{6}\right)g_{k+4} && (5.39) \\ &\quad + \gamma^4\left(\frac{3}{8} + \frac{1}{4}k(k+1) + \frac{k(k+1)(k+2)(k+3)}{24}\right)g_{k+5} + \ldots && (5.40) \end{aligned}$$

Since we have only three couplings, the non zero moments are

$$\begin{aligned} M_1 &= g_2 - \frac{3}{2}\gamma^2 g_4 + \frac{15}{8}\gamma^4 g_6 \\ M_2 &= -2\gamma g_4 + 5\gamma^3 g_6 \end{aligned}$$

$$
\begin{aligned}
M_3 &= -g_4 + \gamma^2 \frac{13}{2} g_6 \\
M_4 &= 4\gamma g_6 \\
M_5 &= g_6
\end{aligned}
\tag{5.41}
$$

We may now explicitly construct the factor $\Pi(z)$,

$$
\Pi(z) = \sum_{n=1}^{5} \left((z-\gamma)^{n-1} M_n + (z+\gamma)^{n-1} (-1)^{n+1} M_n \right)
$$

$$
= 2g_2 - 2g_4 \left(z^2 + \frac{\gamma^2}{2} \right) + g_6 \left(2z^4 + z^2\gamma^2 + \frac{3}{4}\gamma^4 \right) \tag{5.42}
$$

Now, for the $k = 3$ multicritical point, one obtains the critical value of the couplings by setting $M_1 = M_2 = 0$, $M_i \neq 0$, for $i > 2$.

Thus, *at the critical point*, we have the conditions

$$
0 = g_2^c - \frac{3}{2}\gamma_c^2 g_4^c + \frac{15}{8}\gamma_c^4 g_6^c \tag{5.43}
$$

$$
0 = -2g_4^c + 5\gamma_c^2 g_6^c \tag{5.44}
$$

The critical value of the cut interval is given by (5.44),

$$
\frac{2}{5} \left(\frac{g_4^c}{g_6^c} \right) = \gamma_c^2 \tag{5.45}
$$

Inserting the above in (5.43) gives

$$
g_4^c = \sqrt{\frac{10}{3} g_2^c g_6^c} \tag{5.46}
$$

Now, (5.37) is valid for *all* couplings, including the critical point, and thus we may write

$$
\frac{1}{2}\gamma_c^2 g_2^c - \frac{3}{8} g_4^c \gamma_c^4 + \frac{5}{16} g_6^c \gamma_c^6 - 2 = 0 \tag{5.47}
$$

Inserting (5.45) and (5.46) in (5.47) allows us to conclude that

$$
g_6^c = \frac{(g_2^c)^3}{270} \tag{5.48}
$$

from which follows

$$
g_4^c = \frac{(g_2^c)^2}{9} \tag{5.49}
$$

For the sake of convenience, we fix $g_2 = 1$, which will simplify what follows.[2] One now finds, from (5.45), that the critical point is

$$\gamma_c^2 = 12 \tag{5.50}$$

Thus, we have determined the critical values of the couplings. One can readily verify that, taking $g_4^c = 1/9$, and $g_6^c = 1/270$, one may factor (5.47) as

$$\frac{1}{864}\left(\gamma_c^2 - 12\right)^3 = 0 \tag{5.51}$$

which demonstrates that we have tuned the couplings such that, on either side of the symmetry axis, we have three coincident roots, which is indicative of the fact that we have reached the $k = 3$ multicritical point. *The next step is to obtain* the continuum form of the density.

The $k = 3$ scaling limit is defined by choosing the couplings g_4, g_6 to scale as

$$g_i = g_i^c - \Lambda^{2/3} a^3 \tag{5.52}$$

and

$$z = \gamma_c - \frac{a}{2\gamma_c}\zeta \tag{5.53}$$

The approach to the continuum is controlled by setting

$$\gamma^2 = \gamma_c^2 - \Lambda^{1/3} a \tag{5.54}$$

Recalling that

$$G(z) = \frac{1}{2}v'(z) - i\pi\phi(z) \tag{5.55}$$

the density ansatz is written

$$\pi\phi(z) = \frac{1}{4}\Pi(z)\sqrt{\gamma^2 - z^2} \tag{5.56}$$

In order to obtain the continuum form of the $k = 3$ multicritical density, all that remains is to insert the above scaling relations into (5.42), and expand the ansatz in powers of a.

Lengthy computation verifies the cancelling of terms linear in a in $\Pi(z)$, and thus the leading order contribution is given by the coefficients of $a^{5/2}$,

[2] This is permitted, since only the relative strengths of the couplings are of significance, and are dictated by equation (5.37).

$$\begin{aligned}
\pi\phi(a\xi) &= \frac{\sqrt{a}}{4}\sqrt{\xi-\Lambda^{1/3}}\left[a^2\left(\frac{\xi^2}{135}+\frac{1}{270}\Lambda^{1/3}\xi+\frac{1}{360}\Lambda^{2/3}\right)+O(a^3)\right] \\
&= \frac{a^{5/2}}{540}\sqrt{\xi-\Lambda^{1/3}}\left(\xi^2+\frac{1}{2}\Lambda^{1/3}\xi+\frac{3}{8}\Lambda^{2/3}\right)+O(a^{7/2}) \qquad (5.57)
\end{aligned}$$

and hence we identify

$$\pi\phi(\xi)=\sqrt{\xi-\Lambda^{1/3}}\left(\xi^2+\frac{1}{2}\Lambda^{1/3}\xi+\frac{3}{8}\Lambda^{2/3}\right) \qquad (5.58)$$

as the universal form of the density corresponding to the $k=3$ multicritical point, which is seen to correspond with that derived earlier by means of the string equation.

6. Stabilisation and Supersymmetric Matrix Models

In the previous sections, we approached the issue of stabilisation via stochastic quantization. An alternative approach might be to consider supersymmetrizing the matrix model. The motivation for doing so may be realised by noting that, in the formalism of supersymmetric quantum mechanics, one can show that the operator describing time translations of the theory is written as

$$H_S = \frac{1}{2}\{Q, \bar{Q}\} \tag{6.1}$$

where Q and $\bar{Q}$ are the supercharges of the theory. Requiring that the vacuum be invariant under the action of the supercharges yields the condition

$$Q\,|0\rangle = 0 \tag{6.2}$$

One notes that, if the vacuum state is to leave supersymmetry unbroken, the action of the Hamiltonian on the vacuum is prescribed due to the above requirement,

$$H_S\,|0\rangle = 0 \tag{6.3}$$

and thus one has a zero ground state energy. If the vacuum does not preserve supersymmetry, the ground state energy will be positive, due to the positive definite nature of the Hamiltonian. This discussion has direct parallels with that of stabilisation by means of stochastic quantization.

In what follows, we will explore several approaches to this subject.

6.1. The Marinari Parisi Model

The following discussion is taken in part from [5], although related results were obtained independently in [9].

We will first consider the supersymmetric matrix model of Marinari and Parisi [7], with action

$$S[\Phi] = \int dt d\theta_1 d\theta_2 Tr\left(\frac{1}{2}D_1\Phi D_2\Phi + i\bar{W}(\Phi)\right) \tag{6.4}$$

where the covariant derivative D_j on superspace has the form

$$D_j = \partial_{\theta_j} + i\theta_j\partial_t \tag{6.5}$$

and the superpotential has the generic form

$$\bar{W}(\Phi) = \sum_n b_n Tr(\Phi^n) \tag{6.6}$$

Furthermore, the superfield Φ is known to have the expansion

$$\Phi = M - i\psi_1\theta_1 - i\psi_2\theta_2 + i\theta_1\theta_2 F \tag{6.7}$$

where M and F are bosonic matrices, the ψ_j's fermionic matrices, and the θ_j's real Grassman numbers. The Lagrangian is written

$$L = \int d\theta_1 d\theta_2 Tr\left(\frac{1}{2}D_1\Phi D_2\Phi + i\bar{W}(\Phi)\right) \tag{6.8}$$

It is possible to integrate out the Grassman co-ordinates. Expanding the kinetic term yields

$$D_1\Phi = i\theta_1\dot{M} + i\psi_1 - \theta_1\theta_2\dot{\psi}_2 + i\theta_2 F \tag{6.9}$$

and

$$D_2\Phi = i\theta_2\dot{M} + \dot{\psi}_1\theta_1\theta_2 + i\psi_2 - i\theta_1 F \tag{6.10}$$

Constructing the kinetic piece,

$$\begin{aligned} D_1\Phi D_2\Phi \quad = \quad & -\theta_1\theta_2\dot{M}^2 - \theta_1\psi_2\dot{M} + \theta_2\psi_1\dot{M} + i\psi_1\dot{\psi}_1\theta_1\theta_2 - \psi_1\psi_2 - \psi_1\psi_2 \\ & -\theta_1\psi_1 F - i\theta_1\theta_2\dot{\psi}_2\psi_2 - \theta_2\psi_2 F - \theta_1\theta_2 F^2 \end{aligned} \tag{6.11}$$

One may now integrate over the Grassman co-ordinates, recalling that

$$\int \theta_1\theta_2 d\theta_2 d\theta_1 = 1 \tag{6.12}$$

and

$$\int d\theta_j = 0 \tag{6.13}$$

One thus sees that the only terms that need be considered are those quadratic in θ_1 and θ_2, hence one readily obtains

$$\int D_1\Phi D_2\Phi d\theta_1 d\theta_2 = \dot{M}^2 - i\left(\psi_1\dot{\psi}_1 - \dot{\psi}_2\psi_2\right) + F^2 \tag{6.14}$$

The kinetic piece of the Lagrangian is thus written

$$L_{kin} = \frac{1}{2}\dot{M}_{ij}\dot{M}_{ji} - \frac{i}{2}\left(\psi_{1ij}\dot{\psi}_{1ji} - \dot{\psi}_{2ij}\psi_{2ji}\right) + \frac{1}{2}F_{ij}F_{ji} \tag{6.15}$$

In order to obtain an effective Lagrangian which does not depend on the Grassmann co-ordinates, one must still integrate out the dependence of the potential on the θ_i. In order to do so, it proves convenient to exploit the fact that Grassmann co-ordinates square to zero. Accordingly, let us consider the formal binomial expansion of $Tr\Phi^n$:

$$Tr(\Phi^n) = Tr(M+\alpha)^n = Tr\left(M^n\left(1 + \frac{n\alpha}{M} + \frac{n(n-1)}{2!}\left(\frac{\alpha}{M}\right)^2 + O\left(\frac{\alpha}{M}\right)^3\right)\right) \tag{6.16}$$

where we have defined $\alpha \equiv -i\psi_1\theta_1 - i\psi_2\theta_2 + i\theta_1\theta_2 F$. Evaluating the term quadratic in α, one obtains

$$\alpha^2 = 2\theta_1\theta_2\psi_1\psi_2 \tag{6.17}$$

It is thus clear that any terms of higher order in α must vanish, and thus (6.16) becomes

$$Tr(\Phi^n) = Tr\left(M^n + nM^{n-1}\alpha + n(n-1)M^{n-2}\theta_1\theta_2\psi_1\psi_2\right) \tag{6.18}$$

Substituting for α,

$$\begin{aligned} Tr(\Phi^n) &= Tr(M^n + inM^{n-1}\psi_1\theta_1 - inM^{n-1}\psi_2\theta_2 + inM^{n-1}\theta_1\theta_2 F \\ &\quad + n(n-1)M^{n-2}\theta_1\theta_2\psi_1\psi_2) \end{aligned} \tag{6.19}$$

The effective potential is now easily computed,

$$\begin{aligned} \int d\theta_1 \int d\theta_2 i\bar{W}(\Phi) &= -\int d\theta_1 \int d\theta_2 \theta_1\theta_2 n(M^{n-1})F \\ &\quad + in(n-1)\int d\theta_1 \int d\theta_2\theta_1\theta_2 nM^{n-2}\psi_1\psi_2 \end{aligned} \tag{6.20}$$

which, using (6.12) and (6.13) is

$$\int d\theta_1 \int d\theta_2 i\bar{W}(\Phi) = \sum_n nb_n\left(Tr(M^{n-1}F) - in(n-1)Tr(\psi_1 M^{n-2}\psi_2)\right) \tag{6.21}$$

In the interests of brevity, let us define

$$\bar{W}(M) = \sum_n b_n Tr(M^n) \tag{6.22}$$

in which case,

$$\frac{\partial \bar{W}(M)}{\partial M_{ij}} = \sum_n b_n n (M^{n-1})_{ji} \tag{6.23}$$

and

$$\frac{\partial^2 \bar{W}(M)}{\partial M_{\alpha\beta}\partial M_{ij}} = \sum_n b_n n(n-1)(M^{n-2})_{\beta i}\delta_{j\alpha} \tag{6.24}$$

Hence, the above may be rewritten as

$$\int d\theta_1 d\theta_2 i\bar{W}(\Phi) = \frac{\partial \bar{W}(M)}{\partial M_{ij}} F_{ij} - i\psi_{1ij}\frac{\partial^2 \bar{W}(M)}{\partial M_{ij}\partial M_{kl}}\psi_{2kl} \tag{6.25}$$

The full Lagrangian may now be written down,

$$\begin{aligned} L &= \frac{1}{2}\dot{M}_{ij}\dot{M}_{ji} - \frac{i}{2}\left(\psi_{1ij}\dot{\psi}_{1ji} - \dot{\psi}_{2ij}\psi_{2ji}\right) + \frac{1}{2}F_{ij}F_{ji} + \frac{\partial \bar{W}(M)}{\partial M_{ij}}F_{ij} \\ &\quad - i\psi_{1ij}\frac{\partial^2 \bar{W}(M)}{\partial M_{ij}\partial M_{kl}}\psi_{2kl} \end{aligned} \tag{6.26}$$

Notice that the field F has no time derivatives, and hence is not a dynamical variable. Thus, we may eliminate it via its equation of motion

$$\frac{\partial L}{\partial F_{kl}} = 0 \tag{6.27}$$

which yields

$$F_{ij} = -\frac{\partial \bar{W}(M)}{\partial M_{ji}} \tag{6.28}$$

Inserting (6.28) in (6.26) gives rise to the following Lagrangian,

$$L = \frac{1}{2}\dot{M}_{ij}\dot{M}_{ji} - \frac{i}{2}\left(\psi_{1ij}\dot{\psi}_{1ji} - \dot{\psi}_{2ij}\psi_{2ji}\right) - \frac{1}{2}\frac{\partial \bar{W}(M)}{\partial M_{ji}}\frac{\partial \bar{W}(M)}{\partial M_{ij}} - i\psi_{1ij}\frac{\partial^2 \bar{W}(M)}{\partial M_{ij}\partial M_{kl}}\psi_{2kl} \tag{6.29}$$

Let us compute the momenta conjugate to each field:

$$\Pi_{M_{ij}} = \frac{\partial L}{\partial \dot{M}_{ij}} = \dot{M}_{ji} \tag{6.30}$$

and

$$\Pi_{\psi_{1ij}} = \frac{\partial L}{\partial \dot{\psi}_{1ij}} = \frac{i}{2}\psi_{1ji} \tag{6.31}$$

Similarly,

$$\Pi_{\psi_{2ij}} = \frac{\partial L}{\partial \dot{\psi}_{2ij}} = \frac{i}{2}\psi_{2ji} \tag{6.32}$$

Before proceeding further, let us consider the equations of motion for the fermionic fields ψ_i :

$$\dot{\psi}_{1ij} = -\frac{\partial^2 \bar{W}(M)}{\partial M_{ji}\partial M_{kl}}\psi_{2kl} \tag{6.33}$$

and

$$\dot{\psi}_{2kl} = \frac{\partial^2 \bar{W}(M)}{\partial M_{lk}\partial M_{ij}}\psi_{1ij} \tag{6.34}$$

One may now eliminate the time derivatives of the fermionic fields, following which the Hamiltonian may be written down,

$$H = \frac{1}{2}\Pi_{M_{ij}}\Pi_{M_{ji}} + \frac{i}{2}[\psi_{1ij}, \psi_{2kl}]\frac{\partial^2 \bar{W}(M)}{\partial M_{kl}\partial M_{ij}} + \frac{1}{2}\frac{\partial \bar{W}(M)}{\partial M_{ji}}\frac{\partial \bar{W}(M)}{\partial M_{ij}} \tag{6.35}$$

where one imposes the canonical commutator

$$[\Pi_{M_{ij}}, M_{kl}] = -i\delta_{ik}\delta jl \tag{6.36}$$

It is convenient to introduce the fermionic operators

$$\Psi_{ij} = \frac{1}{\sqrt{2}}(\psi_{1ij} + i\psi_{2ij}) \tag{6.37}$$

and

$$\bar{\Psi}_{ij} = \frac{1}{\sqrt{2}}(\psi_{1ij} - i\psi_{2ij}) \tag{6.38}$$

following which, the fermionic anti-commutators may be concisely written as

$$\{\bar{\Psi}_{ij}, \Psi_{mn}\} = \delta_{im}\delta_{jn} \tag{6.39}$$

while the Hamiltonian takes the form

$$H = \frac{1}{2}\Pi_{M_{ij}}\Pi_{M_{ji}} + \frac{1}{2}[\bar{\Psi}_{ij}, \Psi_{kl}]\frac{\partial^2 \bar{W}(M)}{\partial M_{ij}\partial M_{kl}} + \frac{1}{2}\frac{\partial \bar{W}(M)}{\partial M_{ji}}\frac{\partial \bar{W}(M)}{\partial M_{ij}} \tag{6.40}$$

The Hamiltonian (6.40) may be written as

$$H = \frac{1}{2}\{Q, \bar{Q}\} \tag{6.41}$$

where the supercharges are[1]

$$Q^{\dagger} = \Psi_{ij}\left(\Pi_{M_{ji}} + i\frac{\partial W(M)}{\partial M_{ji}}\right) \tag{6.42}$$

and

$$Q = \bar{\Psi}_{ij}\left(\Pi^{*}_{M_{ji}} - i\frac{\partial \bar{W}(M)}{\partial M^{*}_{ji}}\right) \tag{6.43}$$

signifying that the Hamiltonian describes a supersymmetric quantum mechanics as discussed above.

The main difficulty associated with considering the complete set of states in this model is that the bosonic and fermionic components of the superfield Φ may not be simultaneously diagonalised. However, a consistent truncation of the theory exists [27], which permits one to consider some states of *non zero fermion* number whilst maintaining manifest supersymmetry.

Taking the matrix U which diagonalises M, let us now form the new matrix $U\Phi U^{\dagger}$. As stated above, since the fermionic matrices cannot share a common eigenbasis with M, this new matrix will not be diagonal. In what follows, we will restrict ourselves to considering only the diagonal entries of this matrix, that is

$$\left(U\Phi U^{\dagger}\right)_{ii} = \lambda_i + \bar{\theta}\left(U\Psi U^{\dagger}\right)_{ii} + \left(U\bar{\Psi}U^{\dagger}\right)_{ii}\theta + \bar{\theta}\theta\left(UFU^{\dagger}\right)_{ii} \tag{6.44}$$

where we have defined the complex Grassman numbers $\theta = \frac{1}{\sqrt{2}}\left(\theta_1 + i\theta_2\right)$, and the λ_i are the eigenvalues of M. Defining new fermionic operators $\psi_i = \left(U\Psi U^{\dagger}\right)_{ii}$, $\psi_i^{\dagger} = \left(U\bar{\Psi}U^{\dagger}\right)$, we consider only fermionic states which may be constructed by the application of these operators on the Hilbert space.

In order to consider the dynamics of the *singlet sector*, one must express the Hamiltonian in terms of the radial degrees of freedom. Decomposing M into its radial and angular parts, one can show that

$$\frac{\partial}{\partial M_{ij}} = \sum_{m} U^{\dagger}_{jm}U_{mi}\frac{\partial}{\partial\lambda_m} - \sum_{n\neq m,k}\frac{U^{\dagger}_{jm}U_{mk}U_{ni}}{\lambda_n - \lambda_m}\frac{\partial}{\partial U_{nk}} \tag{6.45}$$

[1] Note that for a hermitian matrix $A_{ij} = A^{*}_{ji}$.

Utilising the above, the supercharges may be written explicitly in terms of radial and angular variables,

$$Q = \psi_i^\dagger \left(-i\frac{\partial}{\partial\lambda_i} - i\frac{\partial \bar{W}(\lambda)}{\partial\lambda_i}\right) - i\sum_{k\neq n} \frac{\left(U\bar{\Psi}U^\dagger\right)^*_{kn}}{\lambda_k - \lambda_n}\left(U_{nm}\frac{\partial}{\partial U_{km}}\right)^* \tag{6.46}$$

$$Q^\dagger = \psi_i \left(-i\frac{\partial}{\partial\lambda_i} + i\frac{\partial \bar{W}(\lambda)}{\partial\lambda_i}\right) - i\sum_{k\neq n} \frac{\left(U\bar{\Psi}U^\dagger\right)_{kn}}{\lambda_k - \lambda_n}\left(U_{nm}\frac{\partial}{\partial U_{km}}\right) \tag{6.47}$$

The singlet sector is populated with wavefunctions depending only on $\left(\lambda, \psi^\dagger\right)$. Such a wavefunction is readily constructed,

$$\varphi(\lambda, \psi^\dagger) = \alpha(\lambda)\prod_i \psi_i^\dagger |0\rangle \tag{6.48}$$

It can be shown that the subspace of such wavefunctions is invariant under the action of (6.46) and (6.47) ensuring the consistency of the truncation, and furthermore that the form of the supercharges in the subspace is given by

$$Q = \psi_i^\dagger \left(-i\frac{\partial}{\partial\lambda_i} - i\frac{\partial \bar{W}(\lambda)}{\partial\lambda_i}\right) \tag{6.49}$$

$$Q^\dagger = \psi_i \left(-i\frac{\partial}{\partial\lambda_i} + i\frac{\partial \bar{W}(\lambda)}{\partial\lambda_i}\right) - i\sum_{k\neq n}\frac{1}{\lambda_k - \lambda_n} \tag{6.50}$$

One notes that conjugating (6.49) does not yield (6.50); however, this is as a result of the fact that the change of variables has incurred a non-trivial Jacobian in direct analogy with the $d = 1$ model. As before, in order to restore explicit hermiticity, one is required to scale the wavefunctions $\varphi \to J^{1/2}\varphi$ and the momenta $p_i \to J^{1/2}p_iJ^{-1/2}$. It is not necessary to scale the fermionic fields, since they may only enter linearly into the change of co-ordinates, and thus their contribution to the Jacobian is trivial.

Performing the rescalings, one obtains

$$Q \to Q' = \psi_i^\dagger \left(-i\frac{\partial}{\partial\lambda_i} - i\frac{\partial \bar{W}(\lambda)}{\partial\lambda_i} + \frac{i}{2}\frac{\partial \ln J}{\partial\lambda_i}\right) \tag{6.51}$$

$$Q^\dagger \to Q'^\dagger = \psi_i \left(-i\frac{\partial}{\partial\lambda_i} + i\frac{\partial \bar{W}(\lambda)}{\partial\lambda_i} + \frac{i}{2}\frac{\partial \ln J}{\partial\lambda_i} - i\sum_{k\neq n}\frac{1}{\lambda_k - \lambda_n}\right) \tag{6.52}$$

One may now require $Q'^\dagger = Q'$, and thus one finds that the Jacobian is given by

$$\frac{\partial \ln J}{\partial \lambda_i} = \sum_{i \neq n} \frac{1}{\lambda_i - \lambda_n} \tag{6.53}$$

Hence, in the singlet sector we have the supercharges

$$Q = \psi_i^\dagger \left(\pi_i - i \frac{\partial W_{eff}(\lambda)}{\partial \lambda_i} \right) \tag{6.54}$$

$$Q^\dagger = \psi_i \left(\pi_i + i \frac{\partial W_{eff}(\lambda)}{\partial \lambda_i} \right) \tag{6.55}$$

where

$$W_{eff}(\lambda) \equiv \bar{W}(\lambda) - \sum_{l<k} \ln |\lambda_k - \lambda_l| \tag{6.56}$$

The Hamiltonian describing the dynamics of the singlet sector is now readily constructed, following the prescription (6.41),

$$H = \frac{1}{2}\pi_\lambda \pi_\lambda + \frac{1}{2}[\psi_i^\dagger, \psi_j^\dagger] \frac{\partial^2 W_{eff}(\lambda)}{\partial \lambda_i \partial \lambda_j} + \frac{1}{2}\left(\frac{\partial W_{eff}(\lambda)}{\partial \lambda_i} \right)^2 \tag{6.57}$$

In order to consider the dominant behaviour at large N, it will prove convenient to recast the above in the collective field theory formalism. In order to do so, we introduce the fields

$$\begin{aligned} \phi(x) &\equiv \partial_x \varphi = \sum_i \delta(x - x_i) \\ \phi\sigma(x) &\equiv -\sum_i \delta(x - x_i)\pi_i \\ \psi(x) &= -\sum_i \delta(x - x_i)\psi_i \\ \psi^\dagger(x) &= -\sum_i \delta(x - x_i)\psi_i^\dagger \end{aligned} \tag{6.58}$$

The above fields have the following commutation relations,[2]

$$\begin{aligned} [\sigma(x), \sigma(y)] &= -i\delta(x - y), \\ [\sigma(x), \psi^\dagger(y)] &= i\frac{\psi^\dagger}{\phi}(x)\partial_x \delta(x - y), \\ [\sigma(x), \psi(y)] &= i\frac{\psi}{\phi}(x)\partial_x \delta(x - y), \\ \{\psi(x), \psi^\dagger(y)\} &= \phi(x)\delta(x - y). \end{aligned} \tag{6.59}$$

[2]These relations are readily shown with the aid of the identity, $\delta(x - x_i)f(x) = \delta(x - x_i)f(x_i)$.

One may now write the supercharges (6.54) and (6.55) in terms of the continuum fields by noting that, at the classical level, $\int dx \frac{\delta(x-x_i)\delta(x-x_j)}{\phi(x)} = \delta_{ij}$, and use of the chain rule allows us to write $\frac{\partial W}{\partial x_i} = \frac{\delta W}{\delta\varphi(x_i)}$, in which case one has

$$Q = \int dx \psi^\dagger(x) \left(\sigma(x) - i\frac{\delta W}{\delta\varphi(x)}\right) \tag{6.60}$$

and

$$Q^\dagger = \int dx \psi(x) \left(\sigma(x) + i\frac{\delta W}{\delta\varphi(x)}\right) \tag{6.61}$$

The Hamiltonian is now readily constructed from the supercharges, and is written

$$\begin{aligned} H &= \frac{1}{2}\int dx\phi(x)\sigma^2(x) + \frac{1}{2}\int dx\phi(x)\left(\frac{\delta W}{\delta\varphi(x)}\right)^2 \\ &-\frac{1}{2}\int \frac{dx}{\phi(x)}[\psi(x)^\dagger, \psi(x)]\partial_x\left(\frac{\delta W}{\delta\varphi(x)}\right) \\ &+\frac{1}{2}\int dx \int dy[\psi(x)^\dagger, \psi(y)]\left(\frac{\delta^2 W}{\delta\varphi(x)\delta\varphi(y)}\right) \end{aligned} \tag{6.62}$$

As usual, one has the contraint on the collective field $\int dx\phi(x) = N$.

6.2. Supersymmetrized Collective Field Theory

Another means by which one might construct a supersymmetrized matrix model, is to consider extending the bosonic collective field theory. This was first done in [17].

The key insight here is that the kinetic term

$$L = \frac{1}{2}\int \frac{dx}{\phi}\dot{\varphi}^2 - \frac{\pi^2}{6}\int dx\phi^3 - \int dx v(x)\phi \tag{6.63}$$

may be rewritten as

$$L = \frac{1}{2}\int dx \int dy \dot{\varphi}(x,t) g_{xy}\dot{\varphi}(y,t) \tag{6.64}$$

where, as before, $\phi = \partial_x\varphi$ and one has the metric

$$g_{xy} = \frac{1}{\phi(x)}\delta(x-y) \tag{6.65}$$

Such a theory has a standard supersymmetrization, given by the Lagrangian

$$\begin{aligned} L &= \frac{1}{2}\left(\dot{q}^a g_{ab}\dot{q}^b - g^{ab}\partial_a W \partial_b W\right) + i\psi^{\dagger a} g_{ab}\dot{\psi}^b + i\psi^{\dagger a}\Gamma_{bc,a}\dot{q}^c\psi^b \\ &+\psi^{\dagger c}\Gamma^a_{cd}\partial_a W\psi^d - \psi^{\dagger a}\partial_a\partial_b W\psi^b \end{aligned} \tag{6.66}$$

where $q^a(t)$ are the bosonic variables, $g_{ab}(q)$ is a metric, $W = W(q)$ the superpotential and Γ^a_{cd} are the Christoffel symbols. This can be obtained by a classical point transformation $x^i = x^i(q^a)$, $\psi^i = \left(\frac{\partial x^i}{\partial q^a}\right)\psi^a$ from the Lagrangian

$$L = \sum_{i=1}^{N}\left(\frac{1}{2}\dot{x}_i^2 - \frac{1}{2}(\partial_i W)^2\right) + i\sum_{i=1}^{N}\psi_i^\dagger\dot{\psi}_i - \sum_{ij}[\psi_i^\dagger, \psi_j]\partial_i\partial_j W \tag{6.67}$$

In this case, the bosonic variables $q^a \to \varphi(x)$, and $g_{ab} \to g_{xy} = \frac{1}{\phi(x)}\delta(x-y)$, in which case one has the Lagrangian

$$\begin{aligned} L &= \frac{1}{2}\int\frac{dx}{\phi}\dot{\varphi}^2 - \frac{1}{2}\int dx\phi\left(\frac{\delta W}{\delta\varphi(x)}\right)^2 + i\int\frac{dx}{\phi}\psi^\dagger\dot{\psi} + i\int\frac{dx}{\phi}\dot{\varphi}\partial_x\left(\frac{\psi^\dagger}{\phi}\right)\psi \\ &+\frac{1}{2}\int\frac{dx}{\phi}[\psi^\dagger, \psi]\partial_x\left(\frac{\delta W}{\delta\varphi(x)}\right) - \frac{1}{2}\int dx\int dy[\psi^\dagger(x), \psi(y)]\left(\frac{\delta W}{\delta\varphi(y)\delta\varphi(x)}\right) \end{aligned} \tag{6.68}$$

One is able to rewrite the above as

$$\begin{aligned} L &= \frac{1}{2}\int\frac{dx}{\phi}\dot{\varphi}^2 - \frac{1}{2}\int dx\phi\left(\frac{\delta W}{\delta\varphi(x)}\right)^2 + \frac{i}{2}\int\frac{dx}{\phi}\left(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi\right) \\ &+\frac{i}{2}\int\frac{dx}{\phi}\dot{\varphi}\left(\partial_x\left(\frac{\psi^\dagger}{\phi}\right) - \psi^\dagger\partial_x\left(\frac{\psi}{\phi}\right)\right) + \frac{1}{2}\int\frac{dx}{\phi}[\psi^\dagger, \psi]\partial_x\left(\frac{\delta W}{\delta\varphi(x)}\right) \\ &-\frac{1}{2}\int dx\int dy[\psi^\dagger(x), \psi(y)]\left(\frac{\delta W}{\delta\varphi(y)\delta\varphi(x)}\right) \end{aligned} \tag{6.69}$$

by means of a partial integration. The reason for doing so, is that the above forn ensures the manifest hermiticity of the momentum conjugate to φ. It can be demonstrated that the two forms of the Lagrangian are equivalent at the quantum level.

In order to canonically quantize the theory, one now computes the conjugate momenta in the usual way, to yield

$$\pi(x) = \frac{\dot{\varphi}}{\phi} + \frac{i}{2}\left[\partial_x\left(\frac{\psi^\dagger}{\phi}\right) - \psi^\dagger\partial_x\left(\frac{\psi}{\phi}\right)\right]$$

$$\begin{aligned}\Pi_\psi &= \frac{i}{2}\frac{\psi^\dagger}{\phi} \\ \Pi_{\psi^\dagger} &= -\frac{i}{2}\frac{\psi}{\phi}\end{aligned} \tag{6.70}$$

One must perform a constrained quantization, and thus in the above we identify the second class constraints

$$\begin{aligned}\chi &= \Pi_\psi - \frac{i}{2}\frac{\psi^\dagger}{\phi} \\ \bar{\chi} &= \Pi_{\psi^\dagger} + \frac{i}{2}\frac{\psi}{\phi}\end{aligned} \tag{6.71}$$

Employing Dirac brackets, one obtains

$$\begin{aligned}[\varphi(x),\varphi(y)] &= 0 \\ [\varphi(x),\pi(y)] &= i\delta(x-y) \\ \{\psi(x),\psi^\dagger(y)\} &= \phi(x)\delta(x-y) \\ [\varphi(x),\psi(y)] &= [\varphi(x),\psi^\dagger(y)] = 0 \\ [\pi(x),\psi(y)] &= \frac{i}{2}\frac{\psi(y)}{\varphi(y)}\partial_x\delta(x-y) \\ [\pi(x),\psi^\dagger(y)] &= \frac{i}{2}\frac{\psi^\dagger(y)}{\varphi(y)}\partial_x\delta(x-y)\end{aligned} \tag{6.72}$$

Taking

$$\sigma(x) = \pi(x) - \frac{i}{2}\left[\partial_x\left(\frac{\psi^\dagger}{\phi}\right) - \psi^\dagger\partial_x\left(\frac{\psi}{\phi}\right)\right] \tag{6.73}$$

one is able to write the Hamiltonian in the form

$$\begin{aligned}H &= \frac{1}{2}\int dx\sigma(x)\phi(x)\sigma(x) + \frac{1}{2}\int dx\phi(x)\left(\frac{\delta W}{\delta\varphi(x)}\right)^2 \\ &-\frac{1}{2}\int\frac{dx}{\phi(x)}[\psi(x)^\dagger,\psi(x)]\partial_x\left(\frac{\delta W}{\delta\varphi(x)}\right) \\ &+\frac{1}{2}\int dx\int dy[\psi(x)^\dagger,\psi(y)]\left(\frac{\delta^2 W}{\delta\varphi(x)\delta\varphi(y)}\right)\end{aligned} \tag{6.74}$$

with the commutation relations

$$\begin{aligned}[\sigma(x),\psi(y)] &= i\frac{\psi}{\phi}(x)\partial_x\delta(x-y) \\ [\sigma(x),\psi^\dagger(y)] &= i\frac{\psi^\dagger}{\phi}(x)\partial_x\delta(x-y)\end{aligned} \tag{6.75}$$

which follow directly from the commutators (6.72). One notes that the above Hamiltonian and commutation relations coincide precisely with the continuum description of the Marinari–Parisi model. Hence the supersymmetrization of the bosonic collective field theory is equivalent to the Marinari–Parisi model, as first shown in [5].

One should note that the above commutators indicate that the bosonic momentum does not commute with the fermionic fields, which is rather undesirable. Scaling $\psi \to \sqrt{\phi}\psi$, and $\psi^\dagger \to \sqrt{\phi}\psi^\dagger$, one sees that

$$[\pi(x), \sqrt{\phi}\psi(y)] = 0 \tag{6.76}$$

$$[\pi(x), \sqrt{\phi}\psi^\dagger] = 0 \tag{6.77}$$

as required.

Writing the Hamiltonian in terms of the rescaled fields,

$$\begin{aligned} H \;=\; & \frac{1}{2}\int \frac{dx}{\phi}\left(\phi p - \frac{i}{2}\left((\partial_x\psi^\dagger)\psi - \psi^\dagger(\partial_x\psi)\right)\right)^2 \\ & +\frac{1}{2}\int dx\phi(x)\left(⨍ dy\frac{\partial_y\varphi}{x-y} - \tilde{W}'(x)\right)^2 \\ & -\frac{1}{2}\int dx[\psi^\dagger,\psi]\frac{d}{dx}\left(⨍ dy\frac{\partial_y\varphi}{x-y} - \tilde{W}'(x)\right) \\ & +\frac{1}{2}\int dx\left[\psi^\dagger\sqrt{\phi(x)}, \frac{d}{dx}⨍ dy\frac{\psi(y)\sqrt{\phi(x)}}{x-y}\right] \end{aligned} \tag{6.78}$$

Let us consider the large N limit of the above Hamiltonian. One would like to exhibit N, in order to determine the planar contribution, and thus we must scale $x \to \sqrt{N}x$, $\phi \to \sqrt{N}\phi$, $\varphi \to N\varphi$, $\pi \to \pi N^{-3/2}$ and $\psi^\dagger \to \psi^\dagger N^{1/4}$, in which case one has the scaling behaviour

$$\begin{aligned} H \;=\; & \frac{1}{2N^2}\int \frac{dx}{\phi}\left(\phi p - \frac{i}{2}\left((\partial_x\psi^\dagger)\psi - \psi^\dagger(\partial_x\psi)\right)\right)^2 \\ & +\frac{N^2}{2}\int dx\phi(x)\left(⨍ dy\frac{\partial_y\varphi}{x-y} - \tilde{W}'(x)\right)^2 \\ & -\frac{1}{2}\int dx[\psi^\dagger,\psi]\frac{d}{dx}\left(⨍ dy\frac{\partial_y\varphi}{x-y} - \tilde{W}'(x)\right) \\ & +\frac{1}{2}\int dx\left[\psi^\dagger\sqrt{\phi(x)}, \frac{d}{dx}⨍ dy\frac{\psi(y)\sqrt{\phi(x)}}{x-y}\right] \end{aligned} \tag{6.79}$$

It is then clear that, in the limit of large N, there are no fermionic contributions to the Hamiltonian, and the dominant term

$$\lim_{N\to\infty} H = \frac{N^2}{2}\int dx\phi(x)\left(\fint dy\frac{\partial_y\varphi}{x-y} - \bar{W}'(x)\right)^2 \tag{6.80}$$

is the familiar large N expression for the Fokker–Planck Hamiltonian. In this instance, the existence of a zero energy eigenstate implies that the vacuum state is invariant under the action of the supercharges, by (6.41). A vacuum state with non vanishing energy therefore implies supersymmetry breaking. One sees from the above large N form of the Hamiltonian that unbroken supersymmetry at the classical level requires that the density satisfy

$$\int dy\frac{\phi_0(y)}{x-y} - \bar{W}'(x) = 0 \tag{6.81}$$

Hence, the Marinari–Parisi model in the limit of large N corresponds to the stabilisation of $d=0$ bosonic potentials. One sees that the classical backgrounds obtained via solutions of the stabilised $d=0$ models can be reinterpreted as large N solutions of the Marinari–Parisi model which preserve supersymmetry to leading order.

6.3. The Supereigenvalue Model

To conclude this dissertation, we will review some aspects of the other successful approach to constructing a supersymmetric analogue of the one matrix model, the supereigenvalue model, proposed by Alvarez–Gaume' *et al* [23],[24].

The term 'matrix model' is misleading, since the theory is cast in terms of bosonic and fermionic "eigenvalues", and a traditional formulation in terms of the original matrix variables (if, indeed, it exists) remains elusive. A somewhat unorthodox matrix formulation of the supereigenvalue model exists [20]; however, its form depends on N, and is consequently difficult to exploit.

The central idea behind this model is the generalisation of the algebraic structure of the bosonic matrix model. Specifically, in the bosonic case we demonstrated that the hermiticity requirement of the stochastic Hamiltonian in the collective field description led to the equation

$$\hat{O}_i^{S\dagger} J_S = 0 \tag{6.82}$$

Coupling the partition function to sources j, one was able to rewrite the above condition as a series of constraints acting on the partition function

$$\hat{O}_i^S Z(j) = 0 \tag{6.83}$$

For the consistency of the collective field theory description it is required that the operators $\hat{O}^S_i$ close an algebra, realised in the case of the one matrix model by the presence of a Virasoro structure. In the context of the supereigenvalue model, the approach is centred on generalising the Virasoro generators to the $N = 1$ supersymmetric case. Reasoning by analogy with the hermitian matrix model, one requires that these generators annihalate a suitably generalised partition function, following which the form of the Jacobian may be deduced. As discussed above, it is not known if such an object originates from a change of variables and hence we use the term provisionally.

By direct analogy with the bosonic case, one starts by choosing an action of the form

$$V(\lambda,\theta) = \sum_k \sum_{i=1}^{N} \left(g_k \lambda_i^k + \zeta_{k+\frac{1}{2}} \theta_i \lambda_i^k \right) \tag{6.84}$$

where the $\zeta_{k+\frac{1}{2}}$ are the fermionic couplings, and θ_i, λ_i the fermionic and bosonic eigenvalues, respectively. One can then write the partition function as

$$Z = \int \prod_i d\lambda_i d\theta_i \Delta(\lambda,\theta) e^{-NV(\lambda,\theta)} \tag{6.85}$$

where we have introduced the measure $\Delta(\lambda,\theta)$, whose form remains unspecified. We now introduce the super–Virasoro generators, written in terms of the bosonic and fermionic couplings as follows,

$$G_m = \frac{1}{N^2} \sum_{i=1/2}^{m} \frac{\partial}{\partial\zeta_i \partial\zeta_{m-i}} + \sum_{i=1/2}^{\infty} \zeta_i \frac{\partial}{\partial g_{m+i}} + \sum_{k=1}^{\infty} k g_k \frac{\partial}{\partial \zeta_{k+m}} \tag{6.86}$$

$$\begin{aligned} L_n &= \frac{1}{2N^2} \sum_{k=0}^{n} \frac{\partial^2}{\partial g_k \partial g_{n-k}} + \sum_{k=1}^{\infty} k g_k \frac{\partial}{\partial g_{k+n}} + \frac{1}{2N^2} \sum_{i=1/2}^{n-1/2} \left(\frac{n}{2} - i\right) \frac{\partial}{\partial\zeta_i} \frac{\partial}{\partial\zeta_{n-i}} \\ &\quad + \sum_{i=1/2}^{\infty} \left(\frac{n}{2} + i\right) \zeta_i \frac{\partial}{\partial\zeta_{i+n}} \end{aligned} \tag{6.87}$$

The approach of Alvarez Gaumé *et al*, is to require that

$$G_{n-1/2} Z = 0 \tag{6.88}$$

for $n \geq 0$. It is sufficient to impose the relation (6.88), since $L_n Z = 0$ will then be satisfied automatically, since L_n may be expressed in terms of anticommutators of G_n. Hence, we require the consistency conditions to hold by construction, as means of determining the measure.

Permitting $G_{n-1/2}$ to act on the partition function, one readily obtains

$$\left\langle N\sum_{k=0}^{\infty}\sum_{i=1}^{N}\zeta_{k+1/2}\lambda_i^{k+n}+N\sum_{k=0}^{\infty}\sum_{i=1}^{N}kg_k\theta_i\lambda_i^{k+n-1}-\sum_{k=0}^{n-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\theta_i\lambda_i^k\lambda_j^{n-1-k}\right\rangle=0 \tag{6.89}$$

One can equivalently write the above as

$$\left\langle\sum_i\lambda_i^n\left(\frac{\partial}{\partial\theta_i}-\theta_i\frac{\partial}{\partial\lambda_i}\right)-\sum_{i\neq j}\theta_i\frac{\lambda_i^n-\lambda_j^n}{\lambda_i-\lambda_j}\right\rangle=0 \tag{6.90}$$

where we have employed the identity $\sum_{k=0}^{n-1}\lambda_i^k\lambda_j^{n-1-k}=\frac{\lambda_i^n-\lambda_j^n}{\lambda_i-\lambda_j}$, and it is understood that the derivatives operate on the action. Integrating the above by parts allows the derivatives to act on the measure, yielding

$$\sum_i\lambda_i^n\left(-\frac{\partial}{\partial\theta_i}+\theta_i\frac{\partial}{\partial\lambda_i}\right)\Delta(\lambda,\theta)=\Delta(\lambda,\theta)\sum_{i\neq j}\theta_i\frac{\lambda_i^n-\lambda_j^n}{\lambda_i-\lambda_j} \tag{6.91}$$

This equation has the unique solution

$$\Delta(\lambda,\theta)=\prod_{i<j}(\lambda_i-\lambda_j-\theta_i\theta_j) \tag{6.92}$$

Thus, we have determined the Jacobian of the model.

In view of the fact that the couplings in the bosonic potential are often regarded as sources for loop operators in the bosonic matrix model, in the supereigenvalue model it is then natural to define the connected 'hybrid' $(n|m)$'th superloop correlator as

$$W(z_1,\ldots,z_n|z_1',\ldots,z_m')=$$
$$=N^{n+m-2}\left\langle\sum_{i_1}\frac{\theta_i}{z_1-\lambda_{i_1}}\cdots\sum_{i_n}\frac{\theta_i}{z_n-\lambda_{i_n}}\sum_{k_1}\frac{1}{z_1'-\lambda_{k_1}}\cdots\sum_{k_m}\frac{1}{z_m'-\lambda_{k_m}}\right\rangle \tag{6.93}$$

Furthermore, it will prove convenient to define $V(\lambda_i)=\sum_{k=0}^{\infty}g_k\lambda_i^k$, and $\Psi(\lambda_i)=\sum_{k=0}^{\infty}\zeta_{k+1/2}\lambda_i^k$.

As in the bosonic case, the loop equations of the supereigenvalue model are encoded in the statements

$$G_{n+1/2}Z=0 \tag{6.94}$$

$$L_n Z = 0 \tag{6.95}$$

for $n \geq -1$. Thus we reconsider (6.89) shifting the index $n \to n+1$, to yield

$$\left\langle N \sum_{k=0}^{\infty} \sum_{i=1}^{N} \zeta_{k+1/2} \lambda_i^{k+n+1} + N \sum_{k=0}^{\infty} \sum_{i=1}^{N} k g_k \theta_i \lambda_i^{k+n} - \sum_{i \neq j} \theta_i \frac{\lambda_i^{n+1} - \lambda_j^{n+1}}{\lambda_i - \lambda_j} \right\rangle = 0 \tag{6.96}$$

Multiplying the above by $\sum_n z^{-n-2}$ and summing, yields

$$N \sum_{i=1}^{N} \Psi(\lambda_i) \frac{1}{z - \lambda_i} + N \sum_{i=1}^{N} V'(\lambda_i) \frac{\theta_i}{z - \lambda_i} - \sum_{i \neq j}^{N} \theta_i \frac{\lambda_i^{n+1} - \lambda_j^{n+1}}{\lambda_i - \lambda_j} \tag{6.97}$$

We now assume that the bosonic eigenvalues have compact support on the real axis. In what follows, our goal will be to consider the extension of Ambjørn's formalism to the supereigenvalue model, which was developed in refs [25],[26].

To this end, it proves fruitful to reformulate the above in terms of contour integrals, requiring that the contour C enclose all poles of the superloop operators[3], and that it incurs no contribution at infinity. One then obtains

$$\oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z - \omega} W(\omega|) + \oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{z - \omega} W(|\omega) = W(z|) W(|z) + \frac{1}{N^2} W(z|z) \tag{6.98}$$

where we have rewritten (6.97) in terms of connected correlators. Similarly, the condition (6.95) gives rise to the equation

$$\begin{aligned} &\oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega) W(|\omega) + \Psi'(\omega) W(\omega|)}{z - \omega} - \frac{1}{2} \frac{d}{dz} \oint \frac{d\omega}{2\pi i} \frac{\Psi(\omega) W(\omega|)}{z - \omega} \\ = \; & \frac{1}{2} \left(W(|z)^2 - W(z|) W'(z|) + \frac{1}{N^2} \left(W(|z, z) - \frac{d}{dz'} W(z, z')|_{z=z'} \right) \right) \end{aligned} \tag{6.99}$$

These 'superloop' equations are seen to be Grassmann odd and even, respectively.

The correlator (6.93) may be obtained by the application of superloop insertion operators, acting on the free energy F,

$$W(z_1, ..., z_n | z_1' ... z_m') = \frac{\delta}{\delta \Psi(z_1)} \cdots \frac{\delta}{\delta \Psi(z_n)} \frac{\delta}{\delta V(z_1')} \cdots \frac{\delta}{\delta V(z_m')} F \tag{6.100}$$

[3] As in the bosonic case, we assume a one cut structure.

where we have defined

$$\frac{\delta}{\delta\Psi(z)} = -\sum_{n=0} \frac{1}{z^{n+1}} \frac{\partial}{\partial\xi_{n+1/2}} \tag{6.101}$$

and

$$\frac{\delta}{\delta V(z)} = -\sum_{n=0} \frac{1}{z^{n+1}} \frac{\partial}{\partial g_n} \tag{6.102}$$

as the superloop insertion operators.

The key insight which allows for the construction of Plefka's iterative scheme is the knowledge that the free energy depends at most quadratically on the fermionic couplings, a result first obtained obtained by [21],[22]. One then sees that (6.100), along with equations (6.101) and (6.102) implies the following structure for the one superloop correlators

$$W(z|) = v(z) \tag{6.103}$$

$$W(z|) = u(z) + \hat{u}(z) \tag{6.104}$$

where the functions $v(p)$, $u(p)$ and $\hat{u}(p)$ are first, zeroth and second order in the fermionic couplings, respectively. In order to follow the notation of Plefka it will serve to rewrite equations (6.98) and (6.99) in terms of the above functions, following which, we rewrite them graduated in the order of fermionic couplings.

Doing so yields four equations, listed in ascending order in the couplings,

$$\oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} u(\omega) = \frac{1}{2}u(z)^2 + \frac{1}{2N^2}\frac{\delta}{\delta V(z)}u(z) - \frac{1}{2N^2}\frac{d}{dz'}\frac{\delta}{\delta\Psi(z)}v(z')|_{z=z'} \tag{6.105}$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} v(\omega) + \oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{z-\omega} u(\omega) = v(z)u(z) + \frac{1}{N^2}\frac{\delta}{\delta\Psi(z)}v(z) \tag{6.106}$$

$$\begin{aligned}
&\oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} \hat{u}(\omega) + \oint_C \frac{d\omega}{2\pi i} \frac{\Psi'(\omega)}{z-\omega} v(\omega) - \frac{1}{2}\frac{d}{dz}\oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{z-\omega} v(\omega) \\
= \;& u(z)\hat{u}(z) - \frac{1}{2}v(z)\frac{d}{dz}v(z) + \frac{1}{2N^2}\frac{\delta}{\delta V(z)}\hat{u}(z)
\end{aligned} \tag{6.107}$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{z-\omega} \hat{u}(\omega) = v(z)\hat{u}(z) \tag{6.108}$$

We would like to consider the planar[4] limit of the above equations. Taking the limit $N \to \infty$, one sees that (6.105) becomes

$$\oint \frac{d\omega}{2\pi i}\frac{V'(\omega)}{z-\omega}u_0(\omega) = \frac{1}{2}u_0(z)^2 \tag{6.109}$$

This equation, up to a factor of $\frac{1}{2}$, has the structure of the planar loop equation of the hermitian matrix model, which has been extensively studied. The solution is

$$u_0(z) = \oint_C \frac{d\omega}{2\pi i}\frac{V'(p)}{z-\omega}\left[\frac{(z-x)(z-x)}{(\omega-x)(\omega-y)}\right]^{\frac{1}{2}} \tag{6.110}$$

where, as in the bosonic case, the contour C' encircles the cut bounded by $[x, y]$, but the point z is excluded. This equation is accompanied by auxilliary conditions to ensure the correct asymptotic behaviour, $u_0(z) \sim z^{-1}$, as $z \to \infty$.

These are

$$1 = \oint_C \frac{d\omega}{2\pi i}\frac{\omega V'(\omega)}{\sqrt{(\omega-x)(\omega-y)}} \tag{6.111}$$

and

$$0 = \oint_C \frac{d\omega}{2\pi i}\frac{V'(\omega)}{\sqrt{(\omega-x)(\omega-y)}} \tag{6.112}$$

These conditions are easily obtained by expanding (6.110) about the point $z = \infty$, and tailoring the coefficients of z to obtain the required behaviour.

Let us consider the first order equation in the planar limit. Equation (6.106) becomes

$$\oint_C \frac{d\omega}{2\pi i}\frac{V'(\omega)}{z-\omega}v_0(\omega) + \oint \frac{d\omega}{2\pi i}\frac{\Psi(\omega)}{z-\omega}u_0(\omega) = v_0(z)u_0(z) \tag{6.113}$$

This equation has the solution

$$v_0(z) = \oint_C \frac{d\omega}{2\pi i}\frac{\Psi(\omega)}{z-\omega}\left[\frac{(\omega-x)(\omega-y)}{(z-x)(z-y)}\right]^{\frac{1}{2}} + \frac{\chi}{\sqrt{(z-x)(z-y)}} \tag{6.114}$$

where χ is an as yet undetermined constant, and C is the contour defined in the bosonic case. Notice that one does not have the analogue of the bosonic conditions to dictate the movement of the cut in response to tuning the couplings.

[4]The term 'planar' should be used with care in this instance; since a matrix formulation remains elusive, the connection between large N and a genus expansion remains obscure.

In order to verify the above solution, consider constructing the right hand side of (6.113) explicitly,

$$\begin{aligned} v_0(z)u_0(z) &= \oint_{C_1} \frac{d\omega}{2\pi i} \oint_{C_2} \frac{dz'}{2\pi i} \frac{V'(\omega)\Psi(z')}{(z-\omega)(z-z')} \left[\frac{(z'-x)(z'-y)}{(\omega-x)(\omega-y)}\right]^{\frac{1}{2}} \\ &+ \oint_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} \frac{\chi}{\sqrt{(\omega-x)(\omega-y)}} \end{aligned} \tag{6.115}$$

Constructing the left hand side yields

$$\begin{aligned} &\oint_{C_1} \frac{d\omega}{2\pi i} \oint_{C_2} \frac{dz'}{2\pi i} \frac{V'(\omega)\Psi(z')}{(z-\omega)(\omega-z')} \left[\frac{(z'-x)(z'-y)}{(\omega-x)(\omega-y)}\right]^{\frac{1}{2}} \\ &+ \oint_{C_1} \frac{d\omega}{2\pi i} \oint_{C_2} \frac{dz'}{2\pi i} \frac{V'(z')\Psi(\omega)}{(z-\omega)(\omega-z')} \left[\frac{(\omega-x)(\omega-y)}{(z'-x)(z'-y)}\right]^{\frac{1}{2}} \\ &+ \oint_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} \frac{\chi}{\sqrt{(\omega-x)(\omega-y)}} \end{aligned} \tag{6.116}$$

The equivalence of (6.115) and (6.116) is demonstrated by considering the second term of (6.116),

$$\oint_{C_1} \frac{d\omega}{2\pi i} \oint_{C_2} \frac{dz'}{2\pi i} \frac{V'(z')\Psi(\omega)}{(z-\omega)(\omega-z')} \left[\frac{(\omega-x)(\omega-y)}{(z'-x)(z'-y)}\right]^{\frac{1}{2}} \tag{6.117}$$

Stretching the contour C_2 "over" C_1, one obtains the contributions

$$\begin{aligned} &\oint_{C_1} \frac{d\omega}{2\pi i} \oint_{C_2'} \frac{dz'}{2\pi i} \frac{V'(z')\Psi(\omega)}{(z-\omega)(\omega-z')} \left[\frac{(\omega-x)(\omega-y)}{(z'-x)(z'-y)}\right]^{\frac{1}{2}} \\ &- \oint_{C_1} \frac{d\omega}{2\pi i} \oint_{C_2''} \frac{dz'}{2\pi i} \frac{V'(z')\Psi(\omega)}{(z-\omega)(\omega-z')} \left[\frac{(\omega-x)(\omega-y)}{(z'-x)(z'-y)}\right]^{\frac{1}{2}} \end{aligned} \tag{6.118}$$

Performing the integral over C_2'', one picks up the pole $z' = \omega$, yielding

$$\oint_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)\Psi(\omega)}{(z-\omega)} = 0 \tag{6.119}$$

since the point z is excluded from the contour C_1. In the remaining term, interchange z' and ω, to yield

$$\oint_{C_1} \frac{d\omega}{2\pi i} \oint_{C_2'} \frac{dz'}{2\pi i} \frac{V'(\omega)\Psi(z')}{(z-z')(z'-\omega)} \left[\frac{(z'-x)(z'-y)}{(\omega-x)(\omega-y)}\right]^{\frac{1}{2}} \tag{6.120}$$

One then sees that, combining the first and second terms of (6.116), yields (6.115). Hence the solution is verified.

The next step is to identify the fermionic analogue of the bosonic moments. Thus, we proceed in analogy with the bosonic case, considering $v_0(z)$, where the integral over the contour is taken to infinity. One has

$$v_0(z) = \oint_C \frac{d\omega}{2\pi i}\frac{\Psi(\omega)}{z-\omega}\left[\frac{(\omega-x)(\omega-y)}{(z-x)(z-y)}\right]^{\frac{1}{2}} + \frac{\chi}{\sqrt{(z-x)(z-y)}} \tag{6.121}$$

As before, one deforms the contour C to infinity, picking up the pole at $\omega = z$ as one does so. Thus, we consider

$$\begin{aligned} &\oint_C \frac{d\omega}{2\pi i}\frac{\Psi(\omega)}{z-\omega}\left[\frac{(\omega-x)(\omega-y)}{(z-x)(z-y)}\right]^{\frac{1}{2}} \\ = &-\oint_{C''} \frac{d\omega}{2\pi i}\frac{\Psi(\omega)}{z-\omega}\left[\frac{(\omega-x)(\omega-y)}{(z-x)(z-y)}\right]^{\frac{1}{2}} \\ &+\oint_{C'} \frac{d\omega}{2\pi i}\frac{\Psi(\omega)}{z-\omega}\left[\frac{(\omega-x)(\omega-y)}{(z-x)(z-y)}\right]^{\frac{1}{2}} \end{aligned} \tag{6.122}$$

Performing the integral over C'' simply picks up the residue at $\omega = z$, and yields $\Psi(z)$. In order to consider C', recall that, for $\omega > z$, where $[x, y] < |z|$, one has the expansion

$$\frac{1}{z-\omega} = -\frac{1}{2}\sum_{n=0}^{\infty}\left\{\frac{(z-x)^n}{(\omega-x)^{n+1}} + \frac{(z-y)^n}{(\omega-y)^{n+1}}\right\} \tag{6.123}$$

as in the bosonic case. Inserting the above in the integral over C', one is able to write

$$\oint_{C'} \frac{d\omega}{2\pi i}\frac{\Psi(\omega)}{z-\omega}\left[\frac{(\omega-x)(\omega-y)}{(z-x)(z-y)}\right]^{\frac{1}{2}} = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{\{(z-x)^{n-1}\Xi_n + (z-y)^{n-1}\Lambda_k\}}{\sqrt{(z-x)(z-y)}} \tag{6.124}$$

where we have defined the fermionic moments,

$$\Xi_n = \oint_{C'} \frac{d\omega}{2\pi i}\frac{\Psi(\omega)}{(\omega-x)^n}\sqrt{(\omega-x)(\omega-y)} \tag{6.125}$$

and

$$\Lambda_n = \oint_{C'} \frac{d\omega}{2\pi i}\frac{\Psi(\omega)}{(\omega-y)^n}\sqrt{(\omega-x)(\omega-y)} \tag{6.126}$$

for $n \geq 1$. Thus, one sees that $v_0(z)$ may be written as

$$v_0(z) = \Psi(z) - \frac{1}{2}\sum_{n=1}^{\infty} \frac{\{(z-x)^{n-1}\Xi_n + (z-y)^{n-1}\Lambda_n\}}{\sqrt{(z-x)(z-y)}} + \frac{\chi}{\sqrt{(z-x)(z-y)}} \quad (6.127)$$

To fully determine $v_0(z)$, all that remains is to uncover the explicit form of the unknown χ. Following Plefka, let us introduce the basis functions to facilitate this task. The full machinery of these basis functions is required for the scheme of iterative solution developed by Ambjørn in the bosonic case, and extended by Plefka in the case of the supereigenvalue model. We will not be concerned with developing the iterative scheme; however, the basis functions simplify the following computation somewhat.

$$\chi^{(n)}(z) = \frac{1}{M_1}\left(\phi_x^{(n)}(z) - \sum_{k=1}^{n-1}\chi^{(k)}(z)M_{n-k+1}\right) \quad (6.128)$$

$$\Psi^{(n)}(z) = \frac{1}{J_1}\left(\phi_y^{(n)}(z) - \sum_{k=1}^{n-1}\Psi^{(k)}(z)M_{n-k+1}\right) \quad (6.129)$$

where

$$\phi_x^{(n)}(z) = (z-x)^{-n}[(z-x)(z-y)]^{-1/2} \quad (6.130)$$

$$\phi_y^{(n)}(z) = (z-y)^{-n}[(z-x)(z-y)]^{-1/2} \quad (6.131)$$

Next, we define the linear operator

$$\hat{V}'f(z) = \oint \frac{d\omega}{2\pi i}\frac{V'(\omega)}{z-\omega}f(\omega) \quad (6.132)$$

which acts on the basis functions in the following way,

$$\begin{aligned} \left(\hat{V}' - u_0(z)\right)\chi^{(n)}(z) &= \frac{1}{(z-x)^n} \\ \left(\hat{V}' - u_0(z)\right)\Psi^{(n)}(z) &= \frac{1}{(z-y)^n} \end{aligned} \quad (6.133)$$

To consider the computation of χ, we must examine equations (6.107) and (6.108) in the planar limit. Thus, we consider

$$\begin{aligned} \left(\hat{V}' - u_0(z)\right)\hat{u}_0 &= \frac{1}{2}\frac{d}{dz}\oint_C \frac{d\omega}{2\pi i}\frac{\Psi(\omega)}{z-\omega}v_0(\omega) - \oint_C \frac{d\omega}{2\pi i}\frac{\Psi'(\omega)}{z-\omega}v_0(\omega) \\ &\quad -\frac{1}{2}v_0(z)\frac{d}{dz}v_0(z) \end{aligned} \quad (6.134)$$

After a good deal of computation, one can show that

$$\left(\hat{V}' - u_0(z)\right)\hat{u}_0 = \frac{1}{2}\frac{\Xi_2(\Xi_1-\chi)}{(x-y)}\chi^{(1)}(z) - \frac{1}{2}\frac{\Lambda_2(\Lambda_1-\chi)}{(x-y)}\Psi^{(1)}(z) \tag{6.135}$$

in which case, one may read off the solution for $\hat{u}_0$,

$$\hat{u}_0(z) = \frac{1}{2}\frac{\Xi_2(\Xi_1-\chi)}{(x-y)}\chi^{(1)}(z) - \frac{1}{2}\frac{\Lambda_2(\Lambda_1-\chi)}{x-y}\Psi^{(1)}(z) \tag{6.136}$$

Introducing the planar order 3 equation,

$$\oint_C \frac{d\omega}{2\pi i}\frac{\Psi(\omega)}{z-\omega}\hat{u}_0(\omega) = v_0(z)\hat{u}_0(z) \tag{6.137}$$

one can show, after inserting (6.136) and (6.127), that

$$0 = \frac{1}{2}\frac{\Xi_2(\Xi_1-\chi)(\Lambda_1-\chi)}{M_1(x-y)^3(p-y)} - \frac{1}{2}\frac{\Lambda_2(\Lambda_1-\chi)(\Xi_1-\chi)}{J_1(x-y)^3(p-x)} \tag{6.138}$$

The coefficient χ is thus found to be

$$\chi = \frac{1}{2}(\Xi_1+\Lambda_1) \tag{6.139}$$

Thus, (6.121) reads

$$v_0(z) = \oint_C \frac{d\omega}{2\pi i}\frac{\Psi(\omega)}{z-\omega}\left[\frac{(\omega-x)(\omega-y)}{(z-x)(z-y)}\right]^{\frac{1}{2}} + \frac{1}{2}\frac{\Xi_1+\Lambda_1}{\sqrt{(z-x)(z-y)}} \tag{6.140}$$

Re-examining (6.127), one sees that it may now be written as

$$\begin{aligned} v_0(z) &= \Psi(z) - \frac{1}{2}\sum_{n=2}^{\infty}\frac{\{(z-x)^{n-1}\Xi_n + (z-y)^{n-1}\Lambda_n\}}{\sqrt{(z-x)(z-y)}} \\ &= \Psi(z) - \frac{1}{2}\frac{\Pi(z)}{\sqrt{(z-x)(z-y)}} \end{aligned} \tag{6.141}$$

where we have defined $\Pi(z) = \sum_{n=2}\{(z-x)^{n-1}\Xi_n + (z-y)^{n-1}\Lambda_n\}$. Note that $v_0(z)$ has no explicit dependence on Ξ_1, Λ_1.

We have thus constructed the 'planar' one loop correlator linear in the fermionic couplings.

6.3.1. The Continuum Limit of the Supereigenvalue Model

The analysis of the scaling limit for the bosonic quantities of the model was first considered by Ambjørn in the context of the hermitian matrix model [19]. The scaling for the fermionic quantities was originally performed by Alvarez-Gaume *et al* [24], although the following presentation is based on the analysis of Plefka [25],[26]. We will discuss the continuum limit of the supereigenvalue model in a rather general way.

The earlier discussion of the role of the moments in defining the critical subspace of coupling constants in the hermitian matrix model is carried through unaltered to the bosonic sector of the supereigenvalue model, and thus, for the m'th multicritical point, where the extra zeroes are chosen to accumulate at x, we require that

$$\begin{aligned} M_1(g_i^c, x_c, y_c) &= M_2(g_i^c, x_c, y_c) = \ldots = M_{m-1}(g_i^c, x_c, y_c) = 0 \qquad (6.142) \\ M_k(g_i^c, x_c, y_c) &\neq 0, \; k \geq m \qquad (6.143) \end{aligned}$$

with $J_l^c \neq 0$, $l \geq 1$. One directly generalises this in the case of the fermionic couplings,

$$\begin{aligned} \Xi_2(\xi_i^c, x_c, y_c) &= \ldots = \Xi_{n-1}(\xi_i^c, x_c, y_c) = 0 \\ \Xi_k(\xi_i^c, x_c, y_c) &\neq 0, \; k \geq n \end{aligned}$$

with $\Lambda_l \neq 0$, $l \geq 2$. One notes that one does not set the moments Ξ_1, Λ_1 to zero, since the fermionic correlator $v_0(z)$ does not depend on them.

As usual, the approach to the continuum is controlled by setting

$$\begin{aligned} x &= x_c - a\Lambda^{1/m} \qquad (6.144) \\ z &= z_c + a\xi \qquad (6.145) \end{aligned}$$

One may now consider the scaling of y, the cut endpoint, by assuming that the critical coupling is reached as

$$g_k = f(a) g_k^c \qquad (6.146)$$

where the form of $f(a)$ has yet to be determined. Imposing the conditions (6.111) and (6.112), one can deduce that

$$y - y_c \sim a^m \qquad (6.147)$$

and

$$f - 1 \sim a^m \qquad (6.148)$$

It readily follows from the above that one has the following scaling behaviour for the bosonic moments,

$$M_k \sim a^{m-k} \tag{6.149}$$

while the higher M moments, and J moments do not scale.

In the case of the fermionic couplings, the situation is slightly different. In particular, one does not have an analogue of the relations (6.111), (6.112) enforcing the aymptotic behaviour of the bosonic one loop correlator, which constrain the movement of the cut in response to the adjustment of the coupling constants, g_k. In the above discussion, (6.111) and (6.112) were used to deduce the scaling of the bosonic couplings g_k in the approach to the continuum.

In the fermionic framework, one does not have any such relations, leaving the issue of the scaling of the fermionic coupling constants somewhat *ad hoc*.

One can therefore choose[5]

$$\xi_{k+1/2} = a^{1/2}\xi^c_{k+1/2} \tag{6.150}$$

At this stage, it is important to realise that the bosonic and fermionic critical points are labelled by the integers (m, n) respectively, which are independent of each other. One can then show that the following scaling behaviours are obtained,

$$\begin{aligned} \Xi_1 - \Lambda_1 &\sim a^{n-1/2} \\ \Xi_k &\sim a^{n-k+1/2},\ k \in [2, n-1] \end{aligned} \tag{6.151}$$

It is possible to relate the two scaling sectors by imposing the sensible requirement of Alvarez Gaume' *et al* [24]; that the one bosonic superloop correlator scale uniformly, *ie* that $u_0(z)$ and $\hat{u}_0(z)$ scale in the same way.

One is aware, from our previous calculations in the bosonic sector, that the purely bosonic correlator scales as

$$u_0(z) \sim a^{m-1/2} \tag{6.152}$$

where m is an integer labelling the bosonic criticality.

Requiring that the bosonic correlator quadratic in the fermionic couplings $\hat{u}_0(z)$ scale in the same way as $u_0(z)$ implies that the bosonic and fermionic scaling behaviours are no longer independent, and must satisfy

$$n = m + 1 \tag{6.153}$$

[5]In this instance, choosing the more conventional scaling as in the bosonic case leads to inconsistencies, since the integer n labelling the fermionic multicriticalities assumes half integer values.

One may then deduce the following scaling properties,

$$\begin{aligned} \Xi_k &\sim a^{m-k+3/2}, \ k \in [2, m] \\ \Xi_k &\sim a^{1/2}, \ k > m \\ \Lambda_l &\sim a^{1/2}, \ l > 1 \end{aligned} \tag{6.154}$$

Armed with these results, the authors of [8] were able to show that the central charge of the superconformal field theory obtained in the continuum is

$$\hat{c} = 1 - \frac{(2m-1)^2}{m} = 0, -\frac{7}{2}, \dots m \geq 1 \tag{6.155}$$

We are unable to analyse the continuum limit of the superloop equations which describe the $\hat{c} = 0$ theory, since the $m = 1$ criticality is not amenable to treatment by the techniques employed in this dissertation, as in the bosonic case. The $m = 2$ multicritical point corresponds to a central charge of $-\frac{7}{2}$, which describes a non-unitary superconformal field theory.

Conclusion

In this dissertation, we have reviewed many of the techniques applicable to the matrix model approach to non-critical strings. The saddlepoint equation of [6] was studied and solved for both cubic and quartic potentials, and the continuum limit corresponding to the $m = 2$ multicritical point was taken.

The collective field theory of the $d = 1$ model was developed, with an emphasis on the hermiticity requirement of the collective Hamiltonian, which leads naturally to the realisation of an algebraic structure in the one matrix model. The role of stochastic quantization in the mechanism of stabilisation mechanism was discussed. The $d = 0$ theory was then rewritten with the aid of the stochastic Hamiltonian, whereupon correspondence between the large N configurations of the '$d = 1$' approach and the saddlepoint equation was demonstrated explicitly for both cubic and quartic potentials.

The link between the stochastic Hamiltonian and the Schwinger–Dyson equations of a Euclidean quantum field theory was demonstrated. Employing the non hermitian stochastic Hamiltonian in the z representation, the continuum limit was taken for the $k = 2$ multicriticality, to yield a non–local continuum Hamiltonian. The continuum Schwinger–Dyson equations were then derived, with a view to extracting a continuum Virasoro structure.

In the following section, the possibility of constructing a density approach was explored. In particular, a local continuum Hamiltonian, valid for all criticalities and consistent with the string equation was found, in agreement with Yoneya [36].

The $m = 3$ multicritical density was then constructed, via the moments formalism of Ambjørn, as a prelude to the discussion of supersymmetry in the context of the matrix model; in particular, the supereigenvalue model.

The possibility of supersymmetrizing the matrix model as an alternative method of stabilisation was then explored in the following sections. The Marinari–Parisi model was studied, and a link was made to the supersymmetrized collective field theory. The large N configurations of the Marinari–Parisi model were shown to be equivalent to those obtained via the stochastic quantization approach.

Lastly, the supereigenvalue model of [23] was studied in the moments description, culminating in the realisation that one may not consider the continuum limit of the superloop equations for $\hat{c} = 0$, since it corresponds to the $m = 1$ multicriticality, which does not yield to traditional methods of analysis.

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