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AN EXPLORATION OF FIRST YEAR UNIVERSITY STUDENTS' CONSTRUCTIONS OF PROOFS IN ELEMENTARY SET THEORY

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A thesis submitted to the Wits School of Education, Faculty of Humanities, University of the Witwatersrand in fulfilment of the requirements for the degree of Doctor of Philosophy.

Johannesburg, 2015.

ABSTRACT

In this thesis I explored the development of first year university students' proof construction abilities in the context of consultative group sessions. In order to do this I investigated students' difficulties in proof construction in the area of elementary set theory and the forms of guidance offered as they participated in consultative group sessions. Vygotsky's (1987) socio-cultural theory is the theoretical framework for the study. His premise that all higher mental functions which include the activity of mathematical proof construction, develop as a result of mediated activity in the context of more knowing others, motivated my exploration. Ten students purposefully chosen from a first year mathematics major class at the University of Limpopo (a historically disadvantaged university) participated in weekly consultative sessions. Students were encouraged to share their thoughts and ideas and critique other students as they attempted proof construction exercises. The lecturer (myself) was present to offer guidance whenever necessary. By establishing the sociomathematical norms pertinent to successful proof construction, my aim was to support students in becoming intellectually autonomous and to empower those with the potential to become more knowing peers to develop their capabilities. With this in mind I investigated the nature of the interactions of the students and lecturer in the consultative sessions. I also traced the journeys of two case study students as they progressed in the first two sessions.

Two complementary analytical frameworks incorporating social and cognitive aspects of students' development enabled me to obtain a holistic picture of the development and scaffolding of proof construction abilities in consultative group sessions.

Students' difficulties were found to be similar to those reported in the literature and included difficulties within meanings of mathematical terms, symbols, signs and definitions, logical reasoning and proof methods and deductive reasoning processes and justification. The most persistent of these difficulties seemed to be the challenge of knowing how to use the knowledge of the definitions of relevant mathematical objects, proof methods, deductive reasoning processes and justification. This is also referred to as strategic knowledge (Weber, 2001).

The two case study students showed great improvement in all aspects of their proof construction abilities as they progressed from the first to the second session. This highlighted the effectiveness of the consultative sessions in facilitating access to the observed students' zones of proximal development and in allowing students to make functional use of the various mathematical objects and processes needed in successful proof construction. This functional use together with the scaffolding received from their peers and the lecturer enabled students to develop and internalise proof construction skills and abilities.

Investigation of the nature of the interactions in the consultative sessions examined the lecturer's use of requests for clarification, reflection on proof construction strategy, critique and justification, while eliciting elaboration of contributions which could drive the proof construction process forward. The importance of the correct interpretation of definitions and their role in providing the logical structure and the justification of each step of the proof construction was emphasized. As the sessions progressed more knowing peers emerged from the group who took over the role and responsibilities of

the lecturer and provided most of the scaffolding to their peers. I often called upon these more knowing peers to explain and elaborate on completed proof constructions. Their presentations were observed to be effective learning opportunities for other students.

DECLARATION

I declare that this thesis is my own unaided work. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other university.

_____

(Signature of candidate)

_____ day of _____, _____

Dedicated to my parents Nasser and Mahnaz Khayyam whose unconditional love and encouragement have always supported my efforts in the acquisition of knowledge.

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Chapter 1: Introduction

1.1 The Research Problem

My philosophical beliefs on the importance of education have been shaped by the writings of Bahá'u'lláh who wrote: "Knowledge is as wings to man's life, and a ladder for his ascent. Its acquisition is incumbent upon everyone." (Bahá'u'lláh, 1988, pp.51-52). This belief has motivated me to explore one of the problematic areas in mathematics, particularly for first year undergraduate students; that of proof construction.

My primary concern in this thesis is exploring how mathematics lecturers at first year university level at the University of Limpopo can help to advance students' understanding of mathematical proof. This should ultimately help to make higher level mathematics courses more accessible to university students.

Students who major in mathematics related areas at university are expected to have a certain level of competence in proof comprehension and construction. Much research on investigating proof construction and the reasoning abilities of students across all grades including college and university levels indicates that the use of empirical arguments is prevalent at all levels (Coe & Ruthven, 1994; Stylianou, Blanton & Knuth, 2011; Healy & Hoyles, 2000; Harel & Sowder, 2007; Kuchemann & Hoyles, 2011). Empirical proofs are those that rely on inductive or perceptual examples, for example students' attempts to prove that a set is a subset of another set using Venn diagrams. Empirical reasoning using diagrams or perceptual and inductive examples is not acceptable as proof at university level however. At university level students are expected to read and produce mathematical proofs that obey well defined conventions in line with the acceptable practices of the mathematical community (Weber & Alcock, 2011). According to Tall (1989) for an argument to be considered a mathematical proof it must be based on accepted axioms and definitions. Furthermore the proof needs to proceed using deductive reasoning and employ the appropriate mathematical notation and proof techniques. These stringent requirements coupled with the newly met

mathematical objects contained within the mathematical area in which the proof is to be constructed can often be overwhelming for first year university students.

Research carried out at college and university level on students' challenges in proof construction has found three major areas of difficulty. The first is the mathematical language, symbols and signs that students are introduced to in the specific area of proof construction. Students' lack of understanding of this mathematical language and notation is one of the factors that inhibits their ability to understand definitions, and definitions play a pivotal role in the proof construction process (Moore, 1994; Dreyfus, 1999; Stylianou, Blanton & Knuth, 2011). Secondly logical reasoning processes and proof methods involved in the proof construction process which may be likened to road maps, essential on the journey through proof construction can pose serious challenges (Solow, 1981; Moore, 1994; Stylianou, Blanton & Knuth, 2011). Thirdly students' lack of deductive reasoning abilities and an appreciation for the need for justification of each deduction in the proof construction process can be a hindrance (Dreyfus, 1999; Moore, 1994; Stylianou, Blanton & Knuth, 2011). These challenges are exacerbated for first year students at the University of Limpopo as the majority of these students do not have English as their first language. Mathematicians such as Thurston (1994, p.164) have observed that one's proficiency in language does not just affect one's communication skills but also has a direct influence on one's thinking ability. The schools these students attended are situated in rural areas and have been historically disadvantaged. These schools still experience many challenges, particularly a shortage of well qualified mathematics, science and English teachers. Students entering the university are often under-prepared and this makes their transition to tertiary education even more daunting. The dilemma of under-prepared students enrolling for mathematics courses although exaggerated among historically disadvantaged students in South Africa, is not unique. It is a global problem. Tall (1995, p.13) reports that "...there is a general consensus among university mathematicians in England that students arrive at university to study mathematics with less understanding of proof, less proficiency in handling arithmetic... and less facility with algebraic manipulation." Hillel (as cited in Mamona-Downs and Downs, 2002, p.166) notes "The problem of the mathematical preparation of incoming students, their different social-cultural background, age, and expectations is evidently a

worldwide phenomenon. The traditional image of a mathematics student as well-prepared, selected, and highly motivated simply doesn't fit present-day realities."

Researchers such as Stylianou, Blanton and Knuth (2011) have argued that research is necessary so that mathematics educators might gain some understanding of the forms of thought processes which should be cultivated to promote the understanding and development of proof construction abilities as well as curricular and pedagogical interventions that might enrich students' conceptions of proof, and facilitate their progress to higher level mathematics courses. Scholars in mathematics education such as Alibert and Thomas (1991); Schoenfeld (1999); Tall (1991) and Blanton, Stylianou and David (2011) have identified the necessity for research on the development of students' proof construction abilities while emphasizing the social nature of proof construction. Alibert and Thomas (1991) argue that the formulation of conjectures and the development of proofs have two facets: the personal, where the mathematician sets out to convince himself, and the collective, where the mathematician sets out to convince others of the truth of his argument. They propose that mathematics courses offered at the undergraduate level often present the mathematical argument as a finished theory, thus omitting these two facets of the developmental path of the argument (ibid., p.215). These scholars advocate that research be carried out on developing students' proof construction abilities in environments which encourage students' active participation and engagement with the task of proof construction.

Social constructivist theories are a foundation for most studies researching students' cognitive development together with the nature of their social interaction with peers and more knowing others. My research is framed by Vygotsky's socio-cultural theory. While using this social perspective I study cognitive aspects of students' development of their proof construction abilities. In this way the social and cognitive aspects are brought together to obtain a holistic picture of students' development.

Vygotsky's premise that all higher mental functions arise as a result of mediated processes and through co-operative activity (Vygotsky, 1987, p.126) motivated my investigation of students' proof construction abilities in the context of *consultative group sessions*. A small group of purposefully chosen students was brought together in these sessions under the guidance of the lecturer in an environment where the active

participation of all students was encouraged as they engaged with proof construction exercises. Vygotsky proposed that when children receive scaffolding from more knowing others while participating in purposeful social interaction, they might be enabled to access their zones of proximal development where maturing functions are developed and internalized. Chaiklin (2003) argues that Vygotsky considered it a well-known fact that a child would be better able to solve more difficult tasks with some form of collaboration and help than he/she would be able to do independently. More important in Vygotsky's view is why and how this happens (Miller, 2003; Chaiklin, 2003). The question is how can mathematics educators create environments which encourage the forms of discourse that would lead to students' engagement with the construction of their own knowledge, and the transformation of their interpretive and analytical skills. This critical question is at the heart of my study.

My research starts off by attempting to identify the difficulties and challenges (under-prepared) students experience, and the forms of guidance offered when they are engaged with proof construction tasks in the context of consultative group sessions. This provides an important starting point because our understanding of how to support students' learning begins with an understanding of where their difficulties lie. I then explore how students' proof construction abilities are developed and evolve as they participate in the consultative group sessions. By studying the nature of the interactions in the consultative sessions, I try to identify the manner in which the lecturer encouraged the establishment of the necessary socio-mathematical norms to move the agenda of mathematical proof construction forward, and supported students in becoming intellectually independent. In addition I attempt to identify the characteristics and modes of reasoning observed in students who go on to become more knowing others, taking over the role and responsibilities of the lecturer and becoming active agents in the development of mathematical reasoning both for themselves and others. I also use case study methodology to trace the developmental paths of two of the participants of my study.

My research questions are:

Research Question 1

Investigating students' difficulties in proof construction, and the forms of guidance offered in the context of consultative group sessions:

a) What are the challenges and difficulties students face as they engage with proof construction in the area of elementary set theory?

b) What forms of guidance do the lecturer and students offer?

Research Question 2

Investigating the development of students' proof construction abilities as they participate in consultative group sessions through the use of two case studies:

How do the proof construction abilities of two case studies, Frank and Maria evolve and develop as they progress through the sessions?

Research Question 3

Investigating the nature of the interactions in the consultative group to explore how students' construction of proof might be facilitated:

a) How can lecturers encourage and support students who are engaging with proof construction while participating in consultative group discussions, to become intellectually autonomous?

b) What are the characteristics and modes of reasoning prevalent in students who seem to have the potential to become more knowing peers?

1.2 Rationale

Proof is considered to be a fundamental notion, a central idea of modern mathematics (Tall, 2002, p.3). The ability to construct proofs is therefore a crucial skill and a primary goal of a pure mathematics course. The accepted definition of proof may vary across different age groups and levels of education; arguments of varying degrees of formality may be acceptable in different contexts and communities. As my study is

concerned with proof comprehension and construction at a first year university level, the following definitions of mathematical proof are applicable:

- "a logical argument that one makes to justify a claim in mathematics and to convince oneself and others" (Stylianou, Blanton & Knuth, 2011, p.12).
- "an unbroken sequence of steps that establish a necessary conclusion, in which every step is an application of truth-preserving rules of logic" (Hanna & de Villiers, 2012, p.3).
- "a sequence of assertions, the last of which is the theorem that is proved and each of which is either an axiom or the result of applying a rule of inference to previous formulas in the sequence" (Tall, Yevdokimov, Koichu, Whiteley, Kondratieva & Cheng, 2012, p.15).

Mathematics lecturers at tertiary level have difficulty engendering an appreciation for and the necessity of the process of reasoning and proof. Research has shown that the task of proof construction poses great difficulty for students at all levels (Weber, 2001, p.101). Studies done all over the world, have shown that even high attaining students have difficulty with the task of proof construction (Healy & Hoyles, 2000; Weber, 2001; Hart, 1994; Harel, 2007; Selden & Selden, 2008; Recio & Godino, 2001).

Research on the proof construction abilities and processes used in proof construction on students attending college and pre-college has been carried out in other countries such as the U.S.A. and U.K., for example Healy and Hoyles (2000); Hart (1994); Harel and Sowder (1998, 2007); Blanton and Stylianou (2011); Recio and Godino (2001); Coe and Ruthven (1994); Dreyfus (1999); Selden and Selden (2008); Weber (2001) and Tall (2007). With the exception of a few studies such as those by de Villiers (2004) on prospective secondary school teachers and their understanding of geometrical proofs, there has been very little research on this subject in the South African context, particularly among under prepared students. The plight of South African students' challenges in proof construction in Algebra, especially those at a previously disadvantaged institution remains virtually unexplored. This study, set in a South African context, examines students' difficulties and challenges with proof construction in the area of elementary set theory.

Researchers such as Selden (2012) and Tall (1991) have identified many reasons which could account for the struggle university students have when introduced to proof construction. According to Selden (2012, p. 392) proof construction at the tertiary level requires the correct interpretation and use of definitions and established theorems as well as two essential ingredients: creativity and insight. In addition there is a general lack of explicit instruction in formal proof for students making the transition to formal proof construction. Furthermore, proofs at tertiary level tend to be far more complex than those encountered at lower levels of schooling. Selden (2012, p.393) argues that when one compares the typical proofs in geometry that students have come across in high school with proofs at tertiary level, "one sees that the objects in geometry are idealisations of real things (points, lines, planes), whereas objects in real analysis, linear algebra, abstract algebra or topology (functions, vector spaces, groups, topological spaces) are abstract reifications". Proofs at tertiary level also require students to have a deeper knowledge of the mathematical objects in the particular area of proof construction. Clark and Loveric (2008) explore the many challenges students face as they make the transition to proof construction in university level mathematics. They propose that this transition requires students to change the kinds of reasoning used, to shift from informal to formal language, to reason from mathematical definitions, to understand and apply theorems and make connections between mathematical objects (ibid., pp.28-29).

Although most mathematics educators acknowledge the difficulty of proof, there is also widespread agreement that the understanding and reasoning skills developed in the process of generating a proof are a highly beneficial and an irreplaceable foundation for any student wishing to advance their mathematical studies. Hanna (2007, p.15) proposes that although teaching students to recognize and produce valid mathematical arguments is a challenge, we need to find ways through research and classroom experience to help students master the skills and gain the understanding they need. Failure to do this, would deny students access to a crucial element of mathematics.

Several researchers have advocated investigation of students' notions and understanding of proof. Alibert and Thomas (1991, p.215) encourage mathematics educators to investigate students' views of the necessity for mathematical proof and their preference

of one type of proof over another. They emphasize the importance of such study in a first year university course where students are exposed to the rigour of formal proof for the first time. Harel and Sowder (2007, p.4) argue that there needs to be research on investigating students' difficulties in proof construction, and the type of instructional interventions which would be beneficial to the development of students' conceptions of proof.

Furthermore there has been little research on instructional scaffolding in tertiary mathematics contexts and the factors that affect the scaffolding process (cf. Blanton, Stylianou & David, 2004, p.119). Section 2.4 presents a discussion of studies on the ways in which classes at high school level, and collaborative groups at college and university level could incorporate social and sociomathemathical norms such as those encouraging critique, explanation and justification. These studies emphasize the pivotal role communication and social interaction play in mathematics learning (cf. Goos, 2004). Many of these studies also focus on how students can be supported to develop and become intellectually autonomous (Yackel & Cobb, 1996). Through the pedagogical choices made by teachers and lecturers such as incorporating activities such as explanation and justification, these environments strive to encourage the active participation of all the students in the development of their own cognitive abilities. Blanton et al. (2004, 2011) studied how undergraduate students' appropriation of advanced mathematical reasoning skills could be supported by instructional scaffolding. They found that an environment which encourages discussions that include metacognitive acts such as questioning, critiquing and providing justification of their own and their peers' arguments, as well as instructional scaffolding allows students to make gains in their proof construction abilities.

According to my theoretical framework environments such as those discussed above, facilitate students' access to their zones of proximal development. I have referred to these as environments which enable students' access to their zones of proximal development (or EZPD, discussed in Section 3.3.5). One of the incentives behind my study was to investigate the manner in which an EZPD leading to students' efficient development of proof construction abilities could be created in the form of consultative group sessions.

According to my theoretical framework, communication of ideas and thoughts using the psychological tools of speech and language is the primary means by which individual students might develop their understanding of (mathematical) objects and processes. Vygotsky's theory of concept formation describes how concepts are formed and undergo (non-linear) development through three basic phases: heap, complex and concept. In line with Vygotsky's theoretical premise, a main motivation behind my study is to show that an essential requisite of concept formation (in proof construction) is the functional use of mathematical terms, symbols, logical and deductive reasoning processes, proof methods and practices of justification. Functional use, in the context of my thesis refers to the use and application of (mathematical) signs, processes and practices while engaging in problem solving tasks such as proof construction, before one has complete understanding of these signs, processes and practices. I hope to show that students can be empowered to reach a more complete understanding of the underlying mathematical objects and processes while working on proof construction tasks, using and applying newly met terms, symbols, logical and deductive reasoning processes, proof methods and justification, when interacting with each other in discussion and consultation.

Research on instructional interventions such as the consultative group sessions investigated in my study which is situated in a historically disadvantaged South African university is critical. In such universities I believe that instructional scaffolding is even more essential. This study will explore how students progress with the task of proof construction when working together in a collaborative group under the guidance and help of the lecturer in consultative group sessions.

1.3 Context of the study

I will be exploring first year university students' difficulties and challenges in proof construction and the forms of guidance they receive in the area of elementary set theory in the context of consultative sessions. Furthermore I will focus on how these students' proof construction abilities developed as they took part in consultative group sessions.

In this section I will briefly describe the school background of these students, the way in which the first year mathematics course in which the study is situated is taught and the

manner in which the consultative sessions were set up. This is aimed at informing the reader of all the factors that could influence the mathematical thinking of the students who participated in my study.

This study focusses on students enrolled for a first year mathematics major course at the University of Limpopo, a historically disadvantaged university in South Africa. Such universities were established as separate institutions for black students under the apartheid regime and were distinct from institutions for white students. These universities were generally under-resourced and located in poor rural areas and the majority of student enrolments comprised of under-prepared black students. Although it has been a major goal of the new government (which came into power in 1994) to redress the inherited inequalities through social and educational reforms, schools in rural areas still suffer from a lack of resources and well qualified and competent mathematics and science teachers. The shortage of competent mathematics teachers has been the result of the singular lack of African students with higher level mathematics who could enrol in higher education and teacher education programmes for mathematics and science at universities, technical institutes and colleges of education (Howie, Marsh, Allummoottil, Glencross, Deliwe & Hughes, 2000, pp. 63-64). The severity of this problem has led to a shortage of qualified teachers teaching in schools with predominantly African students (ibid, pp. 63-64). In 2001 only 14% of schools reported that all their mathematics and science educators had what the government considered the minimum level of qualifications (CDE, 2004, p.11). In addition many schools in Limpopo do not have adequate facilities such as libraries, laboratories, telephones, water and electricity and even lavatories (Department of Education, 2008). According to the report; "From Laggard to World Class" (CDE, 2004), the entry of newly qualified mathematics and science educators in South Africa is not keeping pace with retirements, retrenchments and losses to other sectors. In 2000 there were 56% fewer students at teacher training colleges than in 1994 (ibid., p.10). As a result of the decline in enrolments in teacher training colleges (now amalgamated with universities), most analysts have been led to believe that learners will not achieve better results in mathematics and science in the near future (ibid., p.10). The students studying at the University of Limpopo are chiefly from these under-resourced schools.

The educationally disadvantaged backgrounds of the incoming students, is a major contributor to the difficult transition between secondary school and first year university. The challenges of this transition are augmented, among others, by the large numbers of students enrolled in first year mathematics courses at the University of Limpopo. Students entering first year have to adapt to large impersonal classes; typically over 300 students per class. This makes it almost impossible for lecturers to reach students effectively. Students also have to cope with covering large amounts of new material in a short time and continuous assessments in the form of tests or assignments on a weekly basis. This can be highly stressful. The academic staff, generally overworked and heavily involved in teaching and research, might seem to be unapproachable. Students having difficulties probably feel they have nowhere to turn. Lecturers expect students to be mature enough to shoulder the daunting responsibility of being at university and to do much of the work on their own. Most students fall short of these high expectations. It is also the first time that many students are away from their families and their responsibilities for those daily chores necessary in the rural setting. This sense of freedom together with the bombardment of differing values and opinions from peers, could be a factor that derails the 'weaker' students and takes them even further away from really applying themselves to the important task of learning. English, the language of teaching and learning at the University of Limpopo is not the first language of the majority of learners enrolled. This is another significant factor. The Third International Mathematics and Science Study (TIMSS) which took place in 1994 highlighted the importance of English language proficiency as a foundation for the development of mathematical fluency and skill (Howie et al., 2000, p.64).

All of these factors lead to an extremely high failure rate especially in the first year of study at most South African universities; this is even more accentuated in historically disadvantaged universities. Naledi Pandor, Minister of Education until 2009 noted that more than 50% of first year students drop out of higher education institutions in South Africa (SABC News, October 12, 2006). She suggested that alcohol abuse was one of the possible causes of the high dropout rate and proposed that "many young people are not prepared in academic terms for study at higher education institutions." Other reasons cited for the crippling first year university dropout rate were poor career guidance at school, poverty, a sense of alienation expressed by some pupils and a failure

by universities to tailor their output to under-prepared students (Cape Argus Sept. 21, 2008).

It is in the context of this challenging transition from secondary school to university that students meet proofs in the first year major mathematics course. These proofs are very different to the mainly algorithmic mathematics they had encountered at school. Proofs in the South African school curriculum are confined to the proof of a few theorems in Euclidean geometry and the proof of trigonometric identities. Proofs at tertiary level are more abstract, requiring students to understand definitions of concepts and link these definitions to the steps required in the theorem. These proofs also have a rigid axiomatic structure. My experience, which includes over 17 years of teaching at the University of Limpopo, is that, most university students studying mathematics, particularly first year university students find these proofs very challenging. If first year students do not receive the necessary guidance to enable them to overcome these challenges, these difficulties become aggravated as they progress to higher level mathematics courses. Instructional interventions such as consultative group sessions which could be offered in addition to the traditional methods of instruction could be very useful in helping to advance first year students' conceptions and abilities in proof construction.

Consultative group sessions were set up for this study, with a group of ten purposefully chosen participants from the first year mathematics major group. The aim was to create a warm and tolerant environment where every contribution would be welcome. The students were selected from different strata of mathematical ability (according to their first semester results) in order to investigate the effectiveness of the sessions for students who had varied mathematical abilities. The students were encouraged to take ownership of the proof construction process from the very beginning of the sessions. A volunteer from the participants would come up to the board to attempt the proof of a proposition or theorem while other students (and lecturer whenever necessary) made contributions. These contributions questioned points of confusion, provided guidance towards proof construction strategy and clarification of mathematical terms, definitions and proof methods. The lecturer and the student's peers would offer advice on the way in which the proof construction should proceed by using logical and deductive

reasoning and justification. The lecturer encouraged students to critique and question proof construction steps which did not make sense, and provided guidance whenever necessary, such as when incorrect ideas and proof methods persisted. In this way the social and sociomathematical norms pertinent to successful proof construction were set up and it was hoped that the participants would adopt these norms and would gradually take more leading roles in providing the necessary scaffolding for their peers.

1.4 Outline of thesis

The Literature Review Chapter (Chapter 2) begins by providing a discussion of various studies on the challenges and difficulties students experience with proof construction all over the world. This chapter also provides a discussion of various frameworks which categorize and analyse students' proof construction attempts. From amongst these frameworks I adopted a framework to use and build on for my analysis of students' proof construction actions and contributions. Studies on pedagogical interventions aimed at leading to an improvement of students' proof comprehension and construction abilities are also included. My research draws from such studies to set up a collaborative inquiry-based intervention in the form of consultative group sessions. Frameworks for the analysis of the discourse in classrooms situated in studies that take the social aspect of proof into account are also discussed. A framework was adopted for the analysis of the utterances of the lecturer and students as they interacted in the consultative sessions. Overall the Literature Review Chapter provided the background for my study, exposing gaps in the literature and pointing to possible interventions which could prove useful in research on the development of proof construction abilities. It also enabled me to draw on other mathematics educators' research for my analytical frameworks. These adopted frameworks were adapted and extended according to the requirements of my analysis.

Chapter 3 discusses the theoretical framework on which my thesis is based. The task of proof comprehension and construction is considered to implement higher mental functions which according to Vygotsky (1987) develop as a result of the mediated processes of speech and language and through cooperative activity. There is a discussion on Vygotsky's theory of concept formation and its adaptation to the mathematical realm. The central role of the functional use of the sign as being a

necessary vehicle for mathematical conceptual development is discussed and extended to include mathematical terms, symbols, signs, logical reasoning processes, proof methods and the practice of justification. I discuss Vygotsky's notion of the zone of proximal development (ZPD) as being the space where an individual's maturing functions are developed with the help of more knowing others. The ultimate aim of the consultative sessions is put forward as providing an environment where students' access to their ZPDs is facilitated and encouraged.

Chapter 4 presents the methodology and methods I used to address my research questions. My ontological and epistemological assumptions are discussed in this chapter. As my primary concern was to understand and explain how individual students interpreted the mathematical activity of proof construction as well as to investigate the nature of the interactions in the consultative sessions, my study was based in an interpretive paradigm. A detailed presentation of the consultative group method is included in this chapter. I also acknowledge that my interpretation of my observations and my analysis of the data collected is shaped by my theoretical framework. At the same time I recognize the possible effects that I might have inadvertently brought to the research.

Chapter 5 sets out in detail the two analytical frameworks used for the analysis of data collected in the form of transcripts. These frameworks were modified and extended as further categories and their corresponding indicators emerged as I worked with the data. An example of coded and analysed transcript is included in this chapter while the full transcripts with detailed coding of Episodes 1, 2, 3, 4 and 5 are in Appendix 1.

Chapter 6 contains the detailed analysis and discussion of the transcripts with a focus on the emergent themes of the difficulties students have with proof construction in the area of elementary set theory, and the forms of guidance offered in the consultative sessions. The analysis is used primarily to address my first research question in this chapter. It is also used towards addressing research questions 2 and 3 in chapters 7 and 8.

Chapter 7 presents the discussion addressing my second research question concerned with investigating how students working together on proof solving exercises in a consultative group develop their abilities of proof comprehension and construction. I

use case studies to focus on two of the participants of my study and track their progress as they attempt proof construction exercises in the first two consultative sessions.

Chapter 8 presents the discussion aimed at addressing my third research question concerned with investigating the nature and pattern of student and lecturer interactions. I try to trace the patterns of the lecturer's utterances as I attempt to establish sociomathematical norms pertinent to the successful development of proof construction abilities. I also attempt to identify the characteristics and modes of reasoning of those students who show potential in becoming more knowing peers.

In Chapter 9 I discuss the issues surrounding the trustworthiness of my research. I discuss the descriptive, interpretive, theoretical validity and internal generalizability of my research (Maxwell, 1992) as well as issues related to reliability.

Finally in Chapter 10 I present conclusions drawn from my research regarding my three research questions. I summarize the numerous difficulties students encountered and the useful forms of guidance they received. I point to the effectiveness of the consultative sessions in advancing students' development of proof construction abilities by facilitating access to their ZPDs and by promoting the functional use of newly met terms, signs, symbols, definitions, logical reasoning processes and proof methods and the practice of justification. This confirms Vygotsky's theory that the development of (mathematical) objects towards concept level understanding is accelerated by the individual's functional use of these objects while participating in consultation and collaboration with peers and more knowing others. The efficacy of the consultative sessions is further emphasized as a means of promoting the sociomathematical norms necessary for successful proof construction, supporting students to become intellectually independent and empowering those showing potential in becoming more knowing peers to develop their capabilities. Finally I point to contributions to mathematical education scholarship and mathematical pedagogy that my research might have made and elaborate on possible areas for future research.

Chapter 2 Literature review

2.1 Introduction

As a result of the increased attention to the role and nature of proof in mathematics education, many researchers such as Stylianou, Blanton and Knuth (2011) have observed a growing call for students to engage in proof at all grade levels at school and university. Stylianou et al. (2011) surmise that this increased attention towards the centrality of proof in mathematics education has led to greater research in three major areas. The first is a focus on the types of mathematical thinking processes needed for proof comprehension. The second area is concerned with the improvement of students' proof construction abilities and the third on investigating the curricular and pedagogical interventions which could possibly lead to developing and improving students' understanding in proof comprehension and construction. My study will involve aspects of all three areas because I will be looking at the difficulties which challenge first year university students as they engage in the task of formal proof construction in the area of elementary set theory. I will also be investigating a pedagogical intervention in the form of consultative group sessions and examining how students' difficulties are addressed and how their proof construction abilities are developed as they interacted in these sessions with one another and the lecturer.

In my literature review and theoretical framework chapters I have not focussed specifically on the course content of elementary set theory but have instead chosen my focus to be proof comprehension and construction. This is because set theory and logic are the foundations of mathematics. According to Hale (2003) the proofs dealt with in set theory provide a foundation on which all other branches of mathematics and their related proof constructions can operate. For example, most mathematical theorems are of the general form: "If P then Q". This is the implication proof discussed in depth in my study ($P \Longrightarrow Q$). The first statement P is the hypothesis, that is a statement that is assumed to be true. The last statement Q is the conclusion. In order to reach the conclusion, one starts with the hypothesis and proceeds with steps arrived at through deductive reasoning. These steps are justified by one or more of the following: rules of logic, previous steps in the current proof, previous theorems proved, axioms and

previous definitions. This form of proof is the cornerstone of all (direct) mathematical proofs.

In this chapter I will try to provide a brief yet comprehensive overview of the studies carried out by various researchers in the following areas: students' difficulties with proof construction, students' abilities and their proof schemes, the various analytical frameworks used to analyse students' proof construction actions and contributions and their discourse as they interact in collaborative sessions, and pedagogical interventions which could possibly lead to the development and improvement of students' proof comprehension and construction abilities. This will provide the background for my research.

2.1.1 Proof and curriculum

Proof has historically been included in geometry instruction in high schools all over the world because it was believed that deductive reasoning could be most effectively taught in the context of formal geometry (Stylianou et al. 2011, p.2). Many researchers view the fact that there is an absence of proof outside of high school geometry as one of the blatant deficiencies in mathematics education (Wu, 1996). Stylianou et al. (2011, p.3) observe that new curricula and trends advocating instruction that is more student-orientated have often meant a decrease in even this small presence of proof in mathematics courses in high school.

Moore (1994) describes the abrupt transition to proofs experienced by students studying mathematics at university in the United States. He states that many students in the United States enter university mathematics courses having only been exposed to proof in high school geometry.

The situation in South Africa is very similar. Although the first item in the scope for grades 10 to 12 in the NCS (National Curriculum Statement) curriculum in South Africa (Department of Education, 2003, p.10) is for students to work towards being able to "competently use mathematical process skills such as making conjectures, proving assertions and modelling situations", the actual learning outcomes refer specifically to proof only in the areas of mathematical number patterns and Euclidean geometry. In the NCS curriculum the content area involving Euclidean geometry was only included

in Mathematics Paper 3 which was intended for those students showing exceptional mathematical ability, while the majority of students were only taught the content material covered in papers 1 and 2. Students studying the mathematical content of paper 3 were expected to develop the ability to generalize, justify and prove mathematical number patterns and justify and prove conjectures plus a few theorems relating to 2 and 3-dimensional figures in Euclidean geometry (ibid., pp.12-13 and p.55). The grade 10 to 12 CAPS curriculum (Curriculum Assessment Policy Statement) published in 2011, intended to help "teachers unlock the power of NCS" will have its first matriculants in 2014. This curriculum has been extended to include the proof of theorems and their converses and riders in the areas of the Euclidean Geometry of parallelograms, circles and triangles (which were previously only covered in Mathematics Paper 3) (Department of Education, 2011). Proof of some trigonometric identities is also included.

In the English curriculum of the 1950's and '60s, most high school students met proof in the context of classic Euclidean geometry (Kuchemann & Hoyles, 2011). In the 1970's and '80's however, proof disappeared from the curriculum and is only making a comeback in this century in a less formal way. Proof is now taught in English high schools mainly in the context of algebra rather than geometry as is the case in most other countries.

Stylianou et al. (2011, pp.3-4) note that recent reform movements have called for changes that encourage increased engagement of students and teachers with proof. They argue that sound research that would provide guidance on understanding the teaching and learning of proof would be a catalyst for these changes. It is hoped that my study based on first year university students' experiences with proof construction, will in a small way contribute to the understanding of teaching and learning of proof.

2.2 Students' abilities, difficulties and notions of proof

2.2.1 Students' difficulties in the area of proof comprehension and construction

There has been considerable research in students' difficulties in the area of proof comprehension and construction in the context of undergraduate mathematics and high

school. What follows is a discussion of research in students' difficulties conducted by Solow (1981); Moore (1994); Dreyfus (1999); Weber (2001); Heinze and Reiss (2011) and Stylianou et al. (2011).

One of the difficulties students have when introduced to proofs at first year university is the particular mathematical language or discourse used. Thurston (1994) holds the view that linguistic ability does not only play a role in communication skills but is an important tool in one's thinking processes, and that our knowledge of mathematical terms and symbols is closely connected with our language facility. He cites an example: when students are introduced to calculus, the only "mathematical symbolese" available to them is the equals sign, which they use in place of a verb when writing expressions such as "$x^3 = 3x$" (Thurston, 1994, p.164). The challenge of linguistic ability is exacerbated at the University of Limpopo by the fact that the first language of the majority of students is not English. Most students prefer to use their mother tongue when conversing with one another and only speak, see and hear English in lectures and tutorials or when reading their text books. This makes the challenge of mathematical discourse even more difficult as students now have to surmount two hurdles; one being the 'ordinary' English language often taken for granted and the other the specific mathematical language and notation involved in proof construction. Wenger (2007) has discussed the challenge of learning to align one's discourse with that of the larger mathematical community in order to become a member of that community. In addition to the unfamiliarity of the mathematical discourse, several studies have shown that students' understanding of proof and its function are inadequate, and the need for formalism and rigour is not appreciated (Dreyfus 1999, Solomon, 2006). Studies have shown that students' understanding of the mathematical objects and definitions involved in the particular proof construction exercise are also a huge challenge. Experienced mathematicians who have developed the capacity to understand newly met abstract mathematical objects, have achieved powerful cognitive growth by developing their ability to compress abstract ideas into more accessible objects that can be connected together in increasingly flexible ways (Tall, 2007). This is a quite a tall order for the student encountering proof writing at first year university level. A summary of the difficulties that are discussed further below is provided in tabular form at the end of this section.

Solow (1981, p. vii) acknowledges the difficulty most students have in coming to grips with the proving process and attributes some of this to the fact that the knowledge with which students should be equipped is often "partially concealed" and not readily available. He likens students' struggles with the proving process to being asked to play a game where one has no knowledge of the rules. Solow (1981, p.1) describes mathematicians as those whose aim is to discover and communicate certain truths. These are communicated (using the language of mathematics) in the form of proofs to others who also speak the same language. He argues that students should be introduced to the basic grammar of the language of mathematics to ease the transition to proof construction. He recommends that students should be given a thorough and detailed explanation of some of the methods that they can employ to unravel the strategies behind the various proving techniques. Having mastered these techniques, the students are then enabled to apply them creatively to formulate their own proofs and to understand and appreciate the proofs they read in mathematical text books and literature.

Moore's (1994) research takes place in an undergraduate mathematics course that attempts to bridge the gap by teaching students how to communicate effectively using mathematical language and how to write proofs similar to those they would encounter in upper level university courses. Moore (1994) notes that much empirical research on high school students' difficulties with proof, uncovered five potential areas of difficulties most students encounter. These are perceptions of the nature of proof, logic and methods of proof, problem solving skills, the mathematical language used in proofs and mathematical object understanding. He observes that relatively few studies have focussed on university students and how the difficulties mentioned are related to one another. Moore (1994) collected data by acting as a non- participant observer of the students and the lecturer in class and tutorial sessions. His intention was to develop a theory of students' difficulties with proof, emerging from his analysis of data gathered. The course covered topics in mathematical logic and methods of proof, the principle of mathematical induction, elementary set theory, relations and functions and the real number system. This is very close to the course content of the Algebra semester course in which my study takes place.

Moore's analysis of the results of his study showed that there were seven major sources of students' difficulties in the areas mentioned above. Five difficulties were identified in the area of mathematical object understanding. These were:

- D1: Students were not able to state definitions. Proofs depend largely on definitions and Moore notes that not knowing the appropriate definitions often accounted for students' failures to produce a proof. Moore (ibid., p.257-258) emphasizes that a good knowledge and understanding of definitions is essential to students as this would provide the specific language and notation used in proof construction and also play a role in providing justification for each step or deduction. Moreover it is from definitions that one can extract the overall structure of the proof.
- D2: There was a lack of intuitive understanding of mathematical objects. Students found it difficult to learn the written form of the definition, as they did not have an informal understanding of the mathematical objects involved and therefore could not find or create mental pictures of the mathematical objects.
- D3: Students' mental images of the relevant mathematical objects were inadequate for doing proofs. Often because of the mathematical language and symbols used in the definition, students found these difficult to understand and so form an adequate image of the mathematical objects.
- D4: Students failed to generate and use their own examples. Moore (ibid. p.257) observed that students really appreciated the value of examples in helping them understand mathematical objects and their definitions and enabling them to use these objects in proof construction. He noted that, although the lecturer encouraged the students to generate and use examples as an aid in understanding the mathematical objects, definitions, theorems, problems and notation used in proof construction, the students often lacked this ability and this was a hindrance to their progress (ibid. p.260). Moore proposes that one reason might be that students have a "limited repertoire of domain-specific knowledge from which to pull examples" (ibid. p.260).
- D5: Students did not know how to structure a proof from a definition. Moore proposed that students' inability to use definitions to provide the overall structure, logic and proof method suitable for a particular proof (the skeleton of

a proof) was another great impediment. The definitions pertinent to a particular proof together with the rules of logic and knowledge of the necessary theorems and axioms generally provide a strategy with which to link the beginning to the end the proof (ibid. p.261). Moore observed that in some cases students knew a definition and could explain it but were unable to use the definition to write a proof. Students did not seem to "know how to use their mathematical knowledge to produce a proof" (ibid. p.262).

Moore (ibid. p.257) emphasizes that although examples, images of the mathematical objects and informal approaches were helpful, these supporting ideas did not guarantee that a student could construct the proof correctly. As mentioned previously the correct interpretation of definitions is extremely important as this plays a prominent role in providing the language and revealing the logical structure of the proof while giving students an intimation of the sequence of steps required and providing the justification for each step. Moore argued that students' beliefs about proof in mathematics could explain why they neglected to learn and understand the definitions. Students often felt that their images of mathematical objects were sufficient and that the added burden of knowing the notation of the definition was not necessary (ibid. p.257).

The sixth major source of difficulty was:

- D6: Students were unable to understand and use mathematical language and notation and this in turn led to further difficulties in the area of mathematical object understanding.

Difficulties in the area of mathematical object understanding were closely connected to difficulties in the area of proof methods and logic which all led to the seventh major source of difficulty:

- D7: Students did not know how to begin the proof.

Moore (1994) mentions the cognitive overload students undergo as they grapple with domain-specific knowledge such as terms, language and notation of the area in which they are doing the proof construction as well as extracting images of the newly met

mathematical objects from the definitions and using them appropriately while trying at the same time to learn what a proof is and how to write one.

Moore's analysis also showed that one of the consequences of students' inability to understand the mathematical language and notation was that they found it challenging to understand, remember and use definitions in their proof construction tasks (ibid. p.263). Furthermore Moore found that students' inability to start a proof was a direct result of their lack of understanding of all aspects of the (mathematical) objects and processes involved in proof construction, such as a lack of logical reasoning and awareness of the correct proof methods, and difficulties with the particular language and notation involved in proof construction (ibid. p.263). He also found that the level of rigour which students thought sufficient in proof construction was influenced by their perceptions of mathematics and proofs (ibid. p.263).

Dreyfus (1999) attributes the difficulty that students have with proof primarily to their lack of exposure to forms of knowledge on which proof depends. He draws on studies regarding forms of knowledge in mathematics which show that a large part of students' mathematical knowledge is tacit, so that although it is likely to be used correctly in applications, it cannot be used explicitly in reasoning. In addition students' explicit mathematical knowledge is largely not deductive, but inductive, abductive or generalized from experience. He further identifies some of the reasons for students' limited understanding of proof. He proposes that giving an explanation of a mathematical argument is very difficult even for reasonably proficient students as they lack the cognitive ability to interpret and use the relevant mathematical objects in a mathematical argument and more generally, that students have had little opportunity to learn the characteristics of a mathematical explanation (ibid. p.91). Furthermore linguistic ability plays a crucial role as proof writing requires good language skills in order to provide clear and concise explanations. According to his findings, which resonate strongly with my experiences, some typical practices students engage in when constructing proofs are vagueness (which points to a lack of conceptual clarity or linguistic ability or a combination) and proofs that are not substantial enough (either giving no explanation at all and including only computations or just repeating the claim rather than giving an explanation). Unfortunately mathematics text books often add to

the problem because they generally don't distinguish between formal arguments, visual or intuitive justification, generic examples or naive induction (ibid. p.97).

Weber (2001, pp.101-102) acknowledges that among the difficulties students have with proof construction is firstly that they do not have an accurate conception of a mathematical proof, that is, students are unsure about the validity and generality of proof, and often have a pre-conceived notion of its form. Secondly students lack real understanding of definitions and theorems and are therefore unable to apply them correctly. They lack the necessary syntactic knowledge. Syntactic knowledge refers to the particular mathematical content knowledge necessary for proof construction. He points out, however that there are instances where students know what a proof is, can reason logically and are aware of the pertinent definitions, mathematical objects and theorems relevant to the proof, but are still unable to construct the proof. This finding agrees closely with difficulty 'D5' identified by Moore (1994) as discussed earlier. Thus Weber (2001) observed that even though students seemed to have the necessary knowledge for proof construction, they often failed because they reached an impasse. Weber refers to this as a failure to invoke the syntactic knowledge the student has at his/her disposal. Weber proposes that there is a need for 'strategic knowledge' which is "knowledge of how to choose which facts and theorems to apply" (ibid., p. 101).

Stylianou, Blanton and Knuth (2011) identify (mostly at high school level) several areas of difficulty that challenge students when attempting to read or construct proofs. The first area of difficulty is the understanding of what can be classified as a proof and the appreciation that a proof is a generalized argument which covers all possible cases (Stylianou et al. 2011, p.4). Another problematic area (in agreement with Schoenfeld, 1985) is the logic and reasoning abilities involved in problem-solving or argument construction, together with the various methods of proof required in proof reading and construction. Third the grasp of the mathematical language, signs and symbols impedes students' understanding of definitions which play a pivotal role in proof construction (Stylianou et al. 2011, p.5). Last they mention that students' understanding of the mathematical objects involved in proof construction exercises becomes an inhibiting factor in their proof construction capabilities (ibid. p.5).

Heinze and Reiss (2011) argue that apart from the cognitive aspects and challenges students face when introduced to the task of proof construction, there is the additional challenge of the motivational state of mind of the student and his/ her "willingness and social readiness to adequately perform…" (ibid., p.192). They point out that various prominent mathematicians such as B. L. van der Waerden (1903 - 1996) and Henri Poincare (1854 - 1912) who described their struggle with mathematical proof "implicitly gave the reader a good idea of their positive attitude towards mathematics" (ibid., p.192). They suggest that such a positive attitude would also be very important for students studying mathematics. They propose that mathematics achievement is linked to students' interest in the subject and their motivation and emotional state towards it.

A perusal of the studies discussed earlier and the summary provided in Table 2.1 allows us to recognize three major areas of difficulty that emerge and are common in all the findings: mathematical language and notation, understanding of mathematical objects involved in the proof construction exercise including understanding of definitions, ability to generate own examples and the ability to apply these definitions correctly and finally deductive reasoning abilities and knowledge of proof methods, techniques and strategies. Armed with this knowledge I am not only better prepared as a researcher and lecturer having more insight into the types of problems which would most probably challenge my own students but also better equipped to develop an analytical framework which will enable me to track proof comprehension and construction abilities as students engaged with proof construction over the semester. Research regarding analytical frameworks will be discussed in Section 2.3.

Table 2.1: Students' difficulties and challenges in the area of proof comprehension and construction

Researchers	**Students' difficulties and challenges in the area of proof comprehension and construction**
Solow (1981)	• Knowledge that students need is partially concealed and not readily available. • Inadequate knowledge of proof methods, techniques and strategies.
Moore (1994)	• Inadequate knowledge and understanding of definitions • Inadequate intuitive understanding of mathematical

	objects. • Inadequate mental images of mathematical objects. • Failure to generate and use students' own examples. • Inability to extract the structure of a proof from a definition. • Inability to understand and use mathematical language and notation. • Inability to start a proof. • Lack of logic reasoning abilities and knowledge of proof methods. • Cognitive overload.
Dreyfus (1999)	• Lack of exposure to forms of knowledge on which proof depends. • Lack of deductive reasoning abilities. • Lack of conceptual clarity and knowing how to use relevant mathematical objects in mathematical arguments. • Lack of an understanding of the characteristics of a mathematical explanation. • Inadequate linguistic ability.
Weber (2001)	• Lack of an accurate conception of what a mathematical proof is. • Lack of a real understanding of definitions and theorems and the ability to apply them correctly, that is, a lack of syntactic knowledge. • Lack of strategic knowledge, that is, the knowledge of how to choose and apply the assumptions, definitions and theorems at one's disposal.
Blanton, Stylianou and Knuth (2011)	• Lack of an understanding of what constitutes a proof. • Inadequate logic and reasoning abilities. • Inadequate understanding of mathematical language and notation which hinders students' understanding of definitions. • Inadequate understanding of the mathematical objects contained in the proof construction task.
Heinze and Reiss (2011)	• Cognitive challenges related to mathematical reasoning and proof construction. • Affective challenges such as a lack of motivation and unwillingness to perform and engage with proof construction tasks.

2.2.2 Students' abilities, notions and beliefs in proof construction and their proof schemes

Research carried out on students' proof construction abilities has shown that even students at college level have difficulties constructing deductive proofs based on logical reasoning and justification. Students favoured empirical proofs based on examples, ritual proofs based on the perceived form of the proof and authoritarian proofs based on an authority such as a teacher or a text book. Below I outline and discuss studies carried out by Coe and Ruthven (1994); Healy and Hoyles (2000); Kuchemann and Hoyles (2011) and Harel and Sowder (2007).

Coe and Ruthven (1994) examined the proof practices and constructs of advanced mathematics students in their final year at college. The students were encouraged to justify the solutions they gave using rigour and a convincing argument. They found however that the majority provided at best empirical proof, that is, appealing to examples as a source of proof, with very few giving any further justification for the truth of the conjecture.

In their comprehensive review of literature, Healy and Hoyles (2000) found that empirical research has tended to focus on describing and analysing students' responses to proof construction tasks. These studies provide evidence that most students have difficulties in following or constructing formally presented deductive arguments. They found that little attention had been paid to documenting students' views of the meaning of mathematical proof, and that the relationship between these views and students' approaches to proof construction had not been empirically investigated (ibid., p.397). They then set about investigating these aspects of proof solving. They studied proof solving characteristics of high achieving 14-15 year olds in the United Kingdom. They investigated the way students constructed proofs for themselves as well as how the students judged given proofs. They found that the majority of the students were unable to construct algebraic proofs. Students valued general and explanatory arguments and predominantly used empirical arguments for their own proofs although the majority seemed to be aware that empirical arguments were not general. Students also seemed to prefer arguments presented in words for their own approaches to proof and had the most

success in constructing proofs when they used this narrative form, possibly including examples and diagrams. On the other hand students found that arguments containing algebra were hard to follow, but believed that the use of complicated algebraic expressions would get the best marks from their teachers. These results are indicative of the ritual and authoritarian proof schemes put forward by Harel and Sowder (1998, 2007) as described in the next section, since students seemed to share the belief that proofs having a certain form would be more appreciated by their teacher.

Kuchemann and Hoyles (2011, p. 171) state that a major challenge in mathematics education is for students to develop their ability in structural reasoning, which is to "reason mathematically,… make inferences and deductions from a basis of mathematical structures" rather than using empirical reasoning where examples are given to argue the validity of an argument. Results from a longitudinal study they conducted on high school students' use of structural reasoning in the field of number/algebra indicates that although there was a modest increase in structural reasoning over the three years of their study, empirical reasoning remained widespread. Interviews with students showed that they lacked confidence and had a poor understanding of structural reasoning (ibid., p.188). They argue that a reason for students' widespread tendency to use empirical methods might be that students could be using the empirical evidence to check whether their structural arguments were in fact valid (ibid., p.189). They observed that even students who seemed to have some understanding of a structural proof still used calculations to check their proof as if acknowledging that their structural argument might have some flaws.

Harel and Sowder (2007) have done a comprehensive literature review of studies on college and pre-college students' proof construction abilities and conceptions of proof. In all the studies they considered on both groups of students, they found that the external conviction proof scheme class (relying on an external authority such as the teacher or a textbook) and the empirical proof scheme class (relying on inductive and perceptual examples) dominated the students' approach to proof construction. These classes are described in detail in the Section 2.3.1.

De Villiers (1999) has suggested that the fundamental reason behind students' problems with proof could be attributed to their lack of appreciation of the various functions of

proof. He has identified a number of functions of proof as a useful model for research. These are verification, explanation, systematization, discovery, communication and intellectual challenge. In his research on South African mathematics teachers, de Villiers (2004) found that most of the teachers in his study saw the function of proof only in terms of verification, justification or conviction. Mathematicians, however, often see conviction as a prerequisite for finding a proof of a conjecture. He designed activities using Geometer's Sketchpad to encourage an appreciation of proof and to develop the understanding of the different functions of proof and structured these activities according to the Van Hiele levels for learning geometry. The Geometer's Sketchpad is a computer program used as a dynamic tool for exploring geometry and algebraic graphs. The Van Hiele levels for learning geometry developed by Dina and Pierre Van Hiele (ibid., p.706) are five different levels of thought by which students' understanding of geometry could be classified.

2.3 Frameworks for the categorization and analysis of students' proof constructions

There has been a large amount of research on the development of frameworks to aid in the analysis and categorization of student' proof construction attempts. This section has been divided into two main subsections: the first will include frameworks which help to categorize students' proof construction attempts (found in Section 2.3.1) and the other includes frameworks which might be used to explain and analyse students' proof constructions (found in Section 2.3.2). A distinction between these frameworks is that those described in Section 2.3.2 do not attempt to categorize the type of proof that students have constructed but rather give more detailed explanations on the path taken by the student.

2.3.1 Frameworks for categorizing students' proof constructions

Researchers such as Tall (2002), Harel and Sowder (1998), and Harel (2007) have developed schemes to help to categorize students' proof attempts. The proof schemes developed by Harel and Sowder (1998, 2007) appear to be the most comprehensive although they do have much in common with the ideas posited by Tall.

Three worlds of mathematics

Tall (2007) sees human learning begin with competencies which are genetic and develop by successively building on knowledge. According to Tall (2007, p.1), individuals who have developed an increased sophistication in reasoning ability are able to compress knowledge of abstract mathematical objects into simpler to use and hence more powerful mathematical objects. Tall (2002) proposes that mathematical thinking evolves through "three linked mental worlds of mathematics". These three worlds can be summed up as:

- An object based conceptual-embodied world. Here the individual reflects on observations made by physical senses to describe, define and deduce properties. These are then developed from thought experiments to Euclidean proof.

- An action based proceptual-symbolic world that compresses action schemas into thinkable notions operating dually as process and mathematical object (procept).

- A property based formal-axiomatic world of formal definitions and set-theoretic proof building axiomatic theories.

High school students are expected to operate cognitively by reflecting on the properties of processes and objects they encounter and by building inferences. Observations such as `if two numbers are odd, then their sum is even' or `if a triangle has two equal angles then it will have two equal sides' lead students to Euclidean proof. Similarly students observe regularities in the symbolic world such as `*5+2=2+5*', which leads them to the more general `$x + y = y + x$' and the principle of commutativity. In the formal axiomatic world which students usually first encounter at first year university level, the starting point is the definitions and axioms of the particular mathematical structure and proofs are then constructed by means of deduction from these definitions. Thus the transition to the formal world requires a considerable change of approach. Students trying to make sense of this new culture of mathematics must build on their experience of embodiment and symbolism. Tall advocates that one of the major factors affecting students' performance is their embodiments and the underlying knowledge they bring with them.

Students' proof schemes

Harel and Sowder (1998, 2007) have classified proofs according to different proof schemes where each category represents a cognitive stage in the students' development. They define the aim of instruction as progressively developing the proof schemes currently held by students towards proof schemes that are practised by contemporary mathematicians. Three categories (which are not mutually exclusive) have been classified each of which has several sub categories. These are the external conviction proof scheme, the empirical proof scheme and the deductive proof scheme.

The external conviction proof scheme has the sub categories: ritual, authoritarian and non-referential symbolic. The authoritarian proof scheme depends on an authority such as a teacher or a book. The ritual proof scheme is based on the strict appearance of the argument and the non-referential symbolic proof scheme depends on symbolic manipulation having no real coherent meaning for the student.

The empirical proof scheme has two sub categories: inductive and perceptual. The inductive proof scheme relies on evidence from examples, direct measurements of quantities, substitutions of specific numbers in algebraic expressions and so on, while the perceptual proof scheme relies on perceptions.

The deductive proof scheme has two sub categories each consisting of various proof schemes: the transformational proof scheme category, and the axiomatic proof scheme category. The three essential characteristics of transformational proof schemes are generality, operational thought and logical inference. The student satisfies the generality characteristic when he/she understands the need to establish the argument 'for all' allowing no exceptions. Operational thought is manifested when an individual forms goals and sub goals on the path to prove. The logical inference characteristic is manifested when he/she realizes that proving or justifying in mathematics is based on the rules of logical inference. The transformational proof scheme is further sub divided into contextual (when there is a restriction of the context of the argument), generic (when there is a restriction of the generality of the arguments' justification) and casual (when there is a restriction on the mode of the justification). The axiomatic proof schemes share the three features of generality, operational thought and logical inference

and include the premise that any proving process must have a set of accepted principles or axioms as a starting point.

2.3.2 Frameworks for the analysis of students' proof constructions

Several analytical frameworks put forward by various researchers with the aim of analysing students' proof construction abilities and their reasoning processes are discussed below. Among these is a comprehensive assessment model for proof comprehension developed by Meija-Ramos, Fuller, Weber, Rhoads and Samkoff (2012) which I have found offers the best basis from which to develop my analytical framework for the analysis of students' proof comprehension and construction attempts.

A framework that takes into account the formal-rhetorical and problem oriented parts of proof

Selden, Selden and Mckee (2008a, p.305) have developed a framework for distinguishing between different parts of a proof. They separate proofs into a formal-rhetorical part and a problem-oriented part. The formal-rhetorical part is the part that can be written using the formal aspects of the definitions and theorems without much attention to their deeper meanings or to problem solving. The remaining problem-oriented part depends on problem solving and a deeper understanding of the mathematical objects. Students seem to progress in constructing these two parts of proof independently.

A framework that takes into account the mathematical, psychological and pedagogical components

Stylianides and Silver (2011) use an analytic framework developed by Stylianides and Stylianides (2008) having three components: mathematical, psychological and pedagogical. This framework was developed on the basis of a conceptualization of reasoning and proving that proposes that mathematicians spend the majority of their time on activities which involve exploring and conjecturing (Stylianides & Silver, 2011, 236). They give examples of experienced mathematicians who have observed the thinking processes involved in proof construction. Lampert (1991) observes that

mathematicians produce new knowledge by testing assertions in a reasoned argument (ibid. p.125). Polya (1954b, p.vi) describes the steps mathematicians may use to arrive at a proof: guess a mathematical theorem, guess the idea of the proof before going into detail, combine observations and follow analogies, try and try again. Similarly Schoenfeld (1983) presented the various stages mathematicians might traverse before the end result of proof is attained. He proposed that there needs to be identification of a pattern, using these patterns to formulate conjectures and testing these against new empirical evidence and finally working to understand why the conjecture ought to be true.

Stylianides and Silver (2011) adopt the steps of 'identifying patterns' and 'making conjectures' grouped under the category of 'making mathematical generalizations' and the activities of 'providing proof' and 'providing non-proof arguments' grouped under the category of 'providing support to mathematical claims' in their analytic framework. These activities form the mathematical component of the framework. The psychological component of this framework is concerned with students' perceptions of the mathematical nature of these activities. The pedagogical component of the framework is concerned with how the students' perception of the mathematical nature of the activities discussed can be aligned with those of the mathematical community in general.

As this analytic framework focusses on activities such as identifying patterns and making conjectures, it is not very useful in the analysis of proof of theorems and propositions in the context of my study which focusses on the area of elementary set theory. However the rationale behind the framework, that proof is not arrived at in its finished form but that it is worked on instead, passing through several stages, and at times proceeding by trial and error until the correct proof is attained, is a very useful underpinning. Students who are being introduced to formal proof should be aware of this rationale.

Syntactic and semantic understanding of proof and the representation system of mathematical proof

Weber and Alcock (2004, 2011) argue that learning to prove at college level "requires students to work within a new representation system" (in the sense of Goldin, 1998).

Such a representation system as described by Goldin (1988) has as its basis primitive characters, configurations and structures. *Characters* are the elementary entities, the building blocks of the representation system such as elements of a well- defined set, for example the symbols and signs used in the algebra of set theory. Certain rules are used to combine characters into *permitted configurations,* such as sentences (formed from words) or mathematical equations (formed from variables, numbers and operation signs). In general, representation systems have some imposed *structure* and rules governing the movement from one set of configurations to another. Some examples of well- known representation systems are mathematical logic, group theory, set theory and derivational calculus. Weber and Alcock (2011, p.326) describe some of the defining characteristics of their representation system of mathematical proof:

- Characters: These include mathematical symbols, logical symbols and the mathematical language related to the context of the proof construction.
- Permissible configurations: These consist of mathematically correct sentences that might combine English words and logical symbols.
- Valid proofs: These follow acceptable proof frameworks that specify the assumptions at the beginning of the proof and the desired conclusion of the proof construction (ibid., p.326).
- Reasoning: This should be based on the definitions of mathematical objects or using established theorems.
- Assertions: These can either be assumptions clearly set out at the beginning of the proof or statements that have been deduced from previous steps or deductions in the proof.

Weber and Alcock (2011) argue that constructing a mathematical proof using this representation system increases the reliability of proofs and makes them more acceptable to the mathematics community. Interestingly this representation system is in concord with the proof comprehension model developed by Mejia-Ramos, Fuller, Weber, Rhoads and Samkoff (2012) outlined below which incorporates this representation system in the first three levels of the local comprehension of a proof.

According to Goldin (1998) one can operate within and reason about characters and configurations within a particular representation system in two ways. **Syntactic** understanding refers to working within the representation system of the proof itself,

using one's understanding of the characters and configurations to manipulate and construct permissible configurations. Alternatively **semantic** understanding involves the knowledge of representation systems other than the one in which the problem is met and using relevant configurations in those systems to link and develop understanding of the configuration in the original system.

Weber and Alcock (2004, p.210) have defined a syntactic proof production as one which logically manipulates relevant definitions and other facts while a semantic proof production is one in which individuals use their internally meaningful thinking about the mathematical objects involved in the proof to guide the deductions required for each step in the proof construction.

A comprehensive assessment model for proof comprehension

Mejia-Ramos, Fuller, Weber, Rhoads and Samkoff (2012) seek to fill the existing gap in current literature for assessing proof comprehension in advanced mathematics at the undergraduate level. They have introduced a comprehensive assessment model which can benefit lecturers and educational researchers by revealing the points of difficulty blocking students' understanding of a particular proof. The model can also be used to explore the effectiveness of instruction and do further research on how novel means of presentations and instruction on proof construction can affect and improve proof comprehension.

The first three levels of the model are concerned with the *local comprehension* of a proof; that is, at the level of specific terms and statements in the proof addressing "what they mean, what their logical status is and how they connect to preceding and succeeding statements" (ibid., p.7). These levels comprise the following:

The first level of **meaning of terms and statements** measures students' understanding of key terms and statements in the proof. This may be assessed by asking students to identify definitions of key terms, identify examples illustrating particular terms or statements or identify trivial implications of a given statement.

The second level of the **logical status of statements and proof framework** is concerned with students' understanding of the different assertions in the proof. This is assessed by asking students to explain their understanding of the purpose of making a

particular assumption and then asking them to identify the type of proof framework for example direct proof, proof by contradiction, proof by contraposition and so on.

The third level of **justification of claims** measures students' understanding of how each assertion in the proof follows on from previous statements in the proof and other proven or assumed statements. Students' comprehension of this aspect can be tested by questioning them on: what justifies claims made in the proof, the identification of the specific data supporting a claim and asking them to determine the specific claims that are supported by a given statement, that is, to identify given information in the proof which is used for the justification of new claims.

The next four levels of the model are concerned with the students' *holistic comprehension* of a proof; that is whether the student grasps the central ideas and methods of the proof and is able to apply this understanding to other proofs and other contexts. The model here is concerned with students' understanding of the proof as a whole.

The first level of the holistic comprehension of the proof is **summarizing using high level ideas.** One of the assessment methods includes asking the student to either provide a summary or identify the best summary from several summaries given. This reveals the students' understanding of the bigger picture or over-arching idea rather than the specific logical details. The idea comes from Leron's (1983) notion of structured proofs where an overview of the proof is first given and the main ideas are communicated to give students a better understanding.

The next level is concerned with **identifying the modular structure** of the proof or breaking down the proof into more manageable components. Questions that could be used to assess students' understanding of this aspect include asking them to identify the purpose of a module of the proof, and to describe the logical relation between two or more components of the proof.

The next level in determining students' holistic grasp of the proof is whether the student is able to **transfer general ideas or methods to other contexts.** All the mathematicians interviewed in the study listed identifying procedures used in the proofs they read in order to see if they could apply them to solve other proving problems as a

primary reason for studying the proofs of others. Students could be assessed by asking them to transfer the method or to apply the same method successfully in solving a different proof task, or to identify how the proof method could be applied in other proving tasks. They could also be asked whether they recognized the assumptions that need to be in place to allow the particular method to be used, that is whether they appreciated the scope of the method.

The last level in assessment of students' holistic comprehension is **illustrating with examples.** This is concerned with assessing whether students understand how a proof can be illustrated by using specific examples. Many mathematicians mentioned that they used examples to understand proofs that they read in order to make sense of the proof. Some would relate the proof to a diagram to develop understanding. Students could be assessed by asking them to use a specific example to illustrate a sequence of inferences or to use a diagram to interpret a statement or its proof. This involves being able to relate the statement or proof to an appropriately chosen diagram.

I have elaborated on this model to build an analytic framework which will allow me to gauge students' proof construction abilities as they engage in proof construction exercises. This will form part of my analytical framework and will be further described in Section 5.2.2.

2.4 Pedagogical interventions leading to the development and improvement of students' proof comprehension and construction abilities

There has been considerable research on how the concept of proof can be taught in such a way as to bring about an improvement in students' understanding in proof construction. Much research in this area highlights sociocultural aspects which promote better learning in the classroom.

There has also been research aimed at making proof writing more accessible to students where the perspective of the researcher is not aligned to the socio-cultural. The common thread connecting these studies is that they all strive to raise students' awareness, enabling them to reflect on and develop their thinking and reasoning

processes on proof and proof writing. I only mention some such studies very briefly here due to space constraints. Melis and Leron (1999) advocate structuring a proof in such a way as to enable students to get a global overview as well as insight into the sequential view, while Kuntze (2008) found that when students write texts on different aspects of proof their proof related meta-knowledge is stimulated hence resulting in improvement in proof construction abilities. Stylianides and Stylianides (2008) propose that 'pivotal counter examples' promote 'cognitive conflict' which encourages students to reflect and modify their understandings enabling them to develop and progress in terms of proof construction abilities. Soto-Johnson, Dalton and Yestness (2007) discuss three types of assessments positively affecting students' proof writing abilities. These are presentations of proofs by peers, practice in writing proofs and receiving prompt and meticulous feedback.

Sections 2.4.1 and 2.4.2 present discussion of studies where the sociocultural perspective is incorporated. The sociocultural perspective puts forward the idea that social interaction and collaboration with peers and more knowing others is a necessary prerequisite for effective learning. Thus the quality of the interactions and the scaffolding received by students is central to our understanding of how students learn. Section 2.4.2 outlines studies in which frameworks for the analysis of the discourse taking place in these classrooms have been developed. I have adopted the framework developed by Blanton, Stylianou and David (2011) as the best way I could study the utterances of students and the lecturer as they interacted in the consultative group sessions.

2.4.1 Studies incorporating the socio-cultural perspective

Sociocultural factors play an important role as the task of proving is a social one where interaction with others is essential (Blanton, Stylianou & David, 2004; 2011). Sociocultural factors related to students' transition to mathematical proof particularly in undergraduate settings, have been virtually unexplored and have only recently been brought to the fore. In this literature review I have also included studies in classrooms (conducted at university, high school and primary school level) which are not specifically involved in the teaching of proof as the particular classroom practices

described incorporate the sociocultural perspective and make them pertinent to my study.

Sociomathematical norms

Yackel and Cobb (1996) report on research they conducted at elementary school level. Although this research is not concerned with proof in the undergraduate setting in particular, their investigation of classroom environments and the norms established in these, leading to improved and more effective learning, makes it very pertinent to my research.

Yackel and Cobb (1996) put forward the notion of *sociomathematical* norms with the aim of describing how students develop and become "intellectually autonomous in mathematics" (ibid. p.458). Sociomathematical norms are "normative aspects of mathematical discussions that are specific to mathematical activity" (ibid., p.458). These are distinguished from general classroom social norms as they relate in particular to the mathematical facets of students' activity (ibid. p.458).

They begin by using the theoretical framework of constructivism, but they broaden this by including a sociological perspective on mathematical activity. They draw on constructs derived from symbolic interactionism whose primary contribution is the interactive constitution of meaning, and ethnomethodology with the main contribution being the notion of reflexivity. This sociological perspective proved to be central to the development of the notion of sociomathematical norms (ibid., p.459).

Yackel and Cobb (1996) propose that promoting a sense of social autonomy in children is one of the outcomes of establishing social norms in an inquiry based approach to mathematics instruction (ibid. p.473). In addition they propose that teachers foster the development of intellectual autonomy by establishing sociomathematical norms in an inquiry based tradition of mathematics instruction. They view students' development in achieving intellectual mathematical autonomy as synonymous with their being able to make decisions and judgements in mathematical problem solving activities by using their own reasoning processes rather than relying on an authority for assistance or confirmation (ibid., p.473).

In conclusion Yackel and Cobb (1996) emphasize the importance of the notion of sociomathematical norms as a way of examining the mathematical aspects of teacher's and students' activity in the mathematics classroom (ibid., p.474). These sociomathematical norms are constituted by the interaction of the students and the teacher in the classroom. In the process of interactively establishing these norms, students are empowered to become increasingly intellectually autonomous in mathematics. The role of teachers in establishing environments where positive sociomathematical norms are generated and students' intellectual autonomy is encouraged is paramount. This is in conflict with the view that students can reach an understanding compatible with the practices of the mathematical community on their own (ibid., p.474).

In a later study Yackel, Rasmussen and King (2000) extend the analysis of social interaction patterns that had been successful in primary and high school classrooms to the context of undergraduate mathematics. Their study was conducted on an undergraduate class in which differential equations was being taught. They recorded the social and sociomathematical norms detected in students' explanations when involved in problem solving and discuss how these norms were constituted. They focussed on how social and sociomathematical norms encouraged the practices of meaning-making and sophisticated mathematical reasoning in their analysis (ibid., p.286). They highlight the importance of social aspects of the classroom and encourage university lecturers to examine and reflect carefully on discussions which encourage interactions where explanation and justification are prominent. They encourage lecturers to be proactive in promoting such interactions in their classrooms (ibid., p.286).

Collaborative zones of proximal development

Goos, Galbraith and Renshaw (2002) carried out a three year study on patterns of student-student social interaction in secondary school mathematics classrooms. They investigated students working on problem solving exercises in small groups to determine how a 'collaborative zone of proximal development' could be fostered between students with similar levels of competence. They found that there was a successful problem solving outcome when students openly offered their thoughts to

each other to be accepted or discarded and acted as critics of one another's thinking. Unsuccessful problem solving was characterized by a lack of critical engagement with their peers' thinking processes. The researchers note that although they focussed on student-student interaction, the teacher's role is crucial in bringing about fruitful interaction (Goos et al., 2002, p. 220). They reiterate that the teacher offered much scaffolding in the lessons to help students select strategies, identify errors and evaluate answers. I suggest that the lecturer's role in the difficult task of proof construction is even more crucial and that he/she should be in the foreground to help students understand definitions, identify misconceptions, recall previous related results if needed and develop the correct strategy to prove statements.

Investigating the teacher's pedagogical choices

Martin, McCrone, Bower and Dindyal (2005) investigated the interplay of teacher and student actions in a high school geometry class and identified factors pertinent to the development of the students' understanding of proof. Their approach was based on Vygotsky's theory that gains in knowledge and understanding are often made with the assistance of other peers or lecturers who are more knowing. By analysing the actions of teachers and students (predominantly their discourse) they sought to understand what leads to gains in proof construction abilities, and what hinders the development of these abilities. They found that the teacher, through the pedagogical choices he/she makes, and the set-up of the classroom environment was able to engage the students in verbal reasoning, whole class argumentation and proof construction. The environment created encouraged the students and teacher to participate and contribute actively to the development of the students' ability to construct formal proofs. The teacher's important role of "analyzing, coaching and revoicing questions back to students" (ibid., p. 121) effectively monitored and influenced the students' reasoning and proof construction abilities. I used aspects of such an environment in my consultative group sessions and monitored the students' participation through their discourse and their development in proof construction as the course progresses over the second semester.

The Modified Moore Method

A mode of teaching that is based on the belief that students do not learn about mathematical objects and processes in proof construction by passively writing down the

proofs that the lecturer writes on the blackboard but rather by trying to construct the proofs themselves, is the modified Moore method (Weber, 2003, p.5). The lecturer presents the students with the definitions of mathematical objects and perhaps a few motivating examples regarding those objects and it is then left to the students to prove or disprove a set of related propositions. Students are asked to present their solutions to the class followed by critique and discussion by all participants, with the lecturer remaining in the background providing little or preferably no help. Advocates of the modified Moore method (MMM) claim that the personal engagement of the students achieved in this way, promotes ownership and might result in a deeper understanding of mathematical ideas and processes. Studies done by Smith (2006) and Selden, Selden and McKee (2008b) show that students could develop proof conceptions which are more meaningful to them in an MMM course. The MMM course was later referred to as IBL or inquiry based learning by Smith, Nichols, Yoo, and Oehler (2011). Although I believe that in the context of under-prepared students, courses taught in the style of MMM offer too little scaffolding by the lecturer or tutors, the studies offer some evidence that when students engage actively in the task of proof construction and validation, these tasks become more meaningful and real to them.

Scientific Debate

A similar method aimed at improving students' proof construction abilities is that of scientific debate introduced by Alibert and Thomas (1991, p.230). After the introduction of a mathematical conjecture, an environment is created where students are encouraged to put forward arguments and convince their classmates of the truth or falsehood of such arguments. Students begin to realize the need for precise definitions, clear arguments and rigorous proofs as a means of deciding on the correctness of conjectures. The organization of such a debate involves precise techniques and rules if it is to succeed but it is a powerful tool as the students are actively involved in proof construction. One very useful 'side-effect' is the observation that students no longer regard erroneous ideas as faults but as normal scientific events. The theoretical framework on which scientific debate was based includes these pertinent points:

1) Constructivism: "students construct their own knowledge through interactions and conflicts and re-equilibriation involving mathematical knowledge, other students and

problems" (ibid., pp.224-225). The teacher manages the interactions and guides the process by setting up the problems and the teaching environment, and is an active part of the discussions when necessary.

2) Learning is enhanced when students actively participate and apply themselves in knowledge construction.

3) Contradictions help to clarify and elucidate proof construction steps and thus facilitate the process of knowledge construction.

4) Working in a group is important and helps students gain personal meaning.

5) Meta-mathematical factors such as systems of representations are significant in the promotion of learning.

6) Mathematical objects are given meaning as students engage with one another and the problem set.

All the above points were incorporated when facilitating the consultative group sessions.

2.4.2 Studies in which frameworks for analysing discourse in collaborative classrooms have been developed

While investigating classrooms where the sociocultural perspective was taken into account, and where student collaboration and cooperation were encouraged with regard to problem solving or proof construction tasks, some researchers have developed frameworks that might be used for the analysis of the discourse taking place during the discussions taking into account students and teacher interaction. The analysis of the discourse during such interactions and the scaffolding received by students from their peers and lecturer is central to our understanding of how effective learning takes place. All the studies discussed in this section while advocating a very active and participatory role for the students also promote a very present and active role for the teacher who facilitates the discussion while providing a well-organized and encouraging environment. As mentioned previously I have adopted the framework developed by Blanton, Stylianou and David (2011) as this was the most comprehensive and incorporated most of the categories put forward by other researchers. The framework also concurred with my own investigation of categories which emerged as I went through transcripts of the consultative group sessions. A summary of the broad

categories that these researchers have identified will be given in Table 2.2 at the end of the section.

Establishing a culture of inquiry in secondary school classrooms: analysis of teacher's practices

Goos (2004) investigates the types of actions that teachers might take in establishing a culture of inquiry in secondary school classrooms. Sociocultural theories of learning are used to provide a framework for analysing teaching and learning practices over a two year period (Goos, 2004, p.258). Goos emphasizes the pivotal role played by communication and social interaction in mathematics learning. The notion of the ZPD which Vygotsky defined as the distance between the problem solving capabilities of children when working alone and with the assistance of more knowing others is also used to describe students' learning as they increasingly participate in class discussions. Goos (2004) characterizes classrooms having an inquiry based approach to mathematics as those where students learn to communicate mathematically while participating in discussions where new or unfamiliar problems are discussed and solved. Goos (2004) investigated patterns of discourse arising when students in an Australian secondary school worked together collaboratively on challenging problems, and reports on the practices of the teacher as he strives to establish a classroom culture of inquiry.

The teacher would start by challenging students with problems involving a new mathematical object, initiating discussions where he would withhold his own ideas and elicit students' thinking. Goos discovered three ways in which ZPDs were set up: through scaffolding, peer collaboration and interweaving of spontaneous and theoretical concepts (ibid. p.282). Initially the teacher scaffolded students' thinking processes by enacting his expectations as regards to making sense of their own and other's explanations and seeking justification for statements. As time passed the teacher's support was gradually withdrawn and students completed tasks on their own with the help of more capable peers who took over the scaffolding by asking questions which allowed them to recognize errors and reflect on their plan of action. By interweaving spontaneous and theoretical concepts, the teacher encouraged connections between every-day and scientific concepts. For example the teacher would paraphrase students' every-day language used for the ideas they expressed by introducing the appropriate

mathematical terms for those ideas. Goos (2004) emphasizes that the sociocultural approach has great potential to inform our understanding of how we can propel students towards becoming mature members of the wider mathematical community.

Establishing collaborative classrooms: analysis of teacher's contributions

Staples (2007) has done an in-depth case study of a collaborative high school mathematics classroom which attempted to answer questions on what collaborative practices are required of teachers and students for fruitful results, and how a community's capacity to engage in these collaborative practices develops over time. The study focussed on a highly accomplished teacher whose task was to teach a ninth grade pre- algebra class of lower-attaining group of students. The theoretical perspective of the study is based on sociocultural and situative perspectives which consider participation as fundamental to the social process of learning (ibid., p.163).

Analysis of the data revealed that the teacher's role in organizing collaborative participation in class was found to fall into three categories: supporting students in making contributions, establishing and monitoring a common ground, and guiding the mathematics (ibid., p.172). This study promotes a "very active and present role of the teacher throughout providing a well-defined structure within which students conduct their mathematical work" (ibid., p.213). I believe that such an active role is needed during the mathematics tutorials at the University of Limpopo to direct and help under-prepared students develop their proof construction abilities.

Investigating student collaboration: analysis of students' actions

Mueller, Maher and Yankelewitz (2009) base their research on the importance of communication in developing mathematical students' understanding and the increasingly accepted view that students should be encouraged to participate in mathematical discussions sharing their views and analysing and evaluating each other's ideas (p.276). They draw on the ideas of other researchers to construct a framework for analysis of student collaboration highlighting three modes: **co-construction** of ideas**, integration** and **modification.**

In the first form of collaboration: co- construction, students exchange ideas back and forth, building an argument together from the ground up. In the second form of

collaboration: integration, ideas from the student's peers strengthen the (originator) student's argument. In co-construction, the participating students are creators of the argument, but in the process of integration, the argument put forward by the originator is enhanced by the other participants' contributions (ibid. p.277). In the third form of collaboration: modification, after a student has put forward an argument that had not been expressed clearly or correctly, his peers help to make sense of it, correcting the error/s and creating a sound argument as a result.

The researchers encourage further study and analysis of how students collaborate on proof construction exercises in order to understand the necessary factors for the promotion of effective mathematical reasoning and argumentation abilities of all students (ibid., p.282). This study served to strengthen my belief that collaboration between students is a most effective way of promoting students' development of proof construction abilities. This framework did not provide sufficient detail for my purposes.

Inquiry-based learning : analysis of teacher's actions

Smith, Nichols, Yoo, and Oehler (2011, p.307) present an exploration of how the actions of the instructor changed during the semester in terms of taking up and handing over control in class discussions in an inquiry-based learning course (IBL previously referred to as the modified Moore method). The instructor who taught the IBL course believed that IBL courses enabled students to become independent thinkers (Smith et al., 2011, p. 311). They observed that the instructor consciously decided at times to forfeit his role and then reclaim his position as an authority during discussions (ibid., p.311). In order to analyse patterns in his instruction, they categorized his actions in the classroom under the following categories: motivating participation, facilitating whole group discussions and discussing and questioning students' strategies for proof construction. These emerged from the analysis of the data and resonated with the studies of Goos (2004) and Yackel, Rasmussen and King (2000). Smith et al. (2011) expected to find the instructor asserting his role of leadership at the beginning of the course, tapering off towards the end as students advanced in their understanding and gained more confidence. They found instead that the instructor only seemed to take on a leadership role during the middle of the semester and that this was relinquished at the beginning and end of the semester. At the outset he acted more as a facilitator

encouraging students to critique the proofs presented and offer their opinions. Towards the middle of the semester his comments were more frequent, and were centred on the mathematics being presented, encouraging students to examine and reflect on their proof construction strategies (ibid., p.321). As the students' confidence and participation increased towards the end of the semester, he again relinquished his leadership role. By the end of the course students showed increased understanding that mathematics is a social activity requiring active participation. Smith et al. (2011) conclude that although classroom environments such as those described here are rare in universities, they are effective in steering students towards more mature ways of mathematical thinking and participation (ibid, p. 322).

Active participation in mathematical proof construction: analysing student and teacher utterances

Blanton, Stylianou and David (2004, 2011) put forward the view that "proof is ultimately a socially constructed object whose purpose is to communicate the validity of a statement to a community based on established criteria by that community" (Blanton et al. 2011, p.290). They also argue that proof is not arrived at as a finished product but rather as an argument that evolves dynamically. When teaching proofs however, lecturers usually demonstrate completed proofs to students without any intimation of how the proof was developed. They agree with other researchers such as Schoenfeld (1986) that this could be one reason that the more traditional methods of instruction in proof are not effective. They report on an alternative mode of instruction where students actively participate in proof construction in line with the view that proof is a social activity. They investigated the nature of scaffolding in undergraduates' transition to mathematical proof in these classrooms.

Their results indicate that students engaging in discussions in which they are conscious of their thinking processes and encouraged to question, critique and provide justification of their own and their peers' arguments, make gains in their proof construction abilities. They also found that through the teacher's prompts and facilitative utterances, the students' capacity to engage in these types of discussions could become a habit of mind.

The study is positioned within Vygotsky's theoretical perspective that development cannot be separated from the social context in which it occurs. They use the notion of

the Zone of Proximal Development (ZPD) defined as the area of potential cognitive development through the help of more knowing others (Blanton et al. 2011, p.292). They argue that since the ZPD intends to measure abilities which are in the process of development through students' interactions with more knowing others, the quality of these interactions and the scaffolding received by the student is central to the understanding of how they learn. The study thus assesses students' development by investigating how students interact with more knowing others and how the instructor provides scaffolding or guidance and support to the student in proportion to his/her needs.

Blanton Stylianou and David (2011) developed a coding scheme which lends itself well to highlighting evidence of student development within the ZPD. Instructional scaffolding which refers to the guidance and support given to the student to develop understanding that he/she might potentially possess, is inferred from the coded utterances to show where and how development takes place within the ZPD. The framework was initially developed and is based on the work of Kruger (1993) and Goos, Galbraith and Renshaw (2002) who focussed on scaffolding taking place between peers. Blanton et al. (2011) extended this by developing categories and codes for the teacher's/lecturer's utterances to take into account instructional scaffolding arising particularly from the teacher's discourse in the classroom.

Teacher's utterances are categorized by the following:

- Transactive prompts: defined as requests for critique, explanations, justifications, clarifications, elaborations and strategies where the teacher's intention is to prompt students' transactive reasoning.
- Facilitative utterances: where the teacher re-voices or confirms student ideas or attempts to structure classroom discussion.
- Didactive utterances: utterances on the nature of (mathematical) knowledge, axioms and principles or historically developed ideas that students are not expected to re-invent.
- Directive utterances: providing students with either immediate or corrective feedback or information towards solving a problem.

Student utterances are categorized as follows:

- Proposal of a new idea: where students bring new and potentially useful information to the discussion. This could be a new mathematical object or a new representation which could potentially reveal a different aspect of existing information, an extension of a new idea or an elaboration of an existing idea towards a new direction. (New ideas can be correct or incorrect)
- Proposal of a new plan or strategy: where students suggest a course of action for developing a proof, or some aspect of the proof.
- Contribution to or development of an idea: where students add to existing ideas. These are often made by students other than those who made the initial suggestions, indicating that the suggested ideas are embraced by others.
- Transactive questions: where students ask for clarification, elaboration, critique, justification or explanation of peer utterances.
- Transactive responses: where students directly or indirectly respond to explicit or implicit transactive questions- these serve to clarify, elaborate, critique, justify or explain one's thinking.

Table 2.2 below summarizes categories for analysis of student and teacher discourse put forward by the researchers that have been discussed above.

Table 2.2: Categories for analysis of discourse in classrooms where student participation and collaboration is encouraged

Researcher/s	**Categories for analysis of student/ teacher discourse**	
	Students' discourse	**Teacher's discourse**
Goos (2004)		1) Scaffolding through enacting expectations with regard to making sense of own and others' explanations and seeking justification for statements made, 2) Encouraging peer collaboration, 3) Interweaving of spontaneous and theoretical objects.
Staples (2007)		1) Supporting students in making contributions, 2) Establishing and monitoring a common ground, 3) Guiding the mathematics.
Mueller, Maher and Yankelewitz (2009)	1) Co-construction of ideas 2) Integration of ideas 3) Modification of ideas	

Smith, Nichols, Yoo, and Oehler (2011)		1) Motivating participation in class, 2) Facilitating whole group discussions, 3) Discussing and questioning students' strategies for proof construction.
Blanton Stylianou and David (2004, 2011)	1) Proposal of a new idea, 2) Proposal of a new plan or strategy, 3) Contribution to or development of an idea, 4) Transactive questions requesting clarification, elaboration, critique, justification and so on, 5) Transactive responses serving to clarify, elaborate or justify one's ideas.	1) Transactive prompts requesting critique, explanation, justification, clarifications, strategies and so on, 2) Facilitative utterances re-voicing or confirming ideas or structuring class discussions, 3) Didactive utterances on the nature of mathematical knowledge such as axioms and developed ideas, 4) Directive utterances providing immediate or corrective feedback.

2.5 Summary

This literature review provides a background for the research questions addressed in my study. Section 2.2 highlights the difficulties and challenges undergraduate students all over the world experience when engaging with formal proof construction as well as students' abilities and conceptions of proof and the proof schemes prevalent in their proof construction attempts. This provides the background for my first research question concerning first year students' challenges with proof construction in the area of elementary set theory at the University of Limpopo and the forms of guidance observed.

Section 2.3 discusses frameworks developed by researchers to categorize and analyse students' proof constructions. The framework I used as a basis for the analysis of students' proof comprehension and construction abilities is based on an assessment model for assessing students' proof comprehension developed by Meija-Ramos, Fuller, Weber, Rhoads and Samkoff (2012) and is one of the frameworks discussed in this section.

Section 2.4 discusses pedagogical interventions conducive to the improvement and development of students' proof comprehension and construction abilities. In this section I focus on those studies where researchers incorporated a socio-cultural

perspective in these pedagogical interventions. My framework for the analysis of student and teacher discourse is based on the framework developed by Blanton, Stylianou and David (2011) and is one of the frameworks discussed here. My methodology for an intervention to be put in place in the form of consultative group sessions was also drawn in some ways from the ideas put forward by researchers discussed in this section.

This literature review has thus provided the necessary background for my research in providing relevant literature on students' difficulties, abilities and notions in proof construction and the proof schemes prevalent in their proof construction attempts. Research on pedagogical interventions that have been found to be effective in advancing students' conceptions and abilities in proof construction has been reported on. The possible frameworks for the analysis of proof construction actions and contributions, as well as frameworks for the analysis of the discourse occurring in collaborative sessions, have been discussed.

Chapter 3: Theoretical Framework

3.1 Introduction

As a result of numerous studies, examples of some of which were described in Section 2.4 which highlight the social aspect of learning in general and of proof itself, I have decided to use Vygotsky's sociocultural perspective as the theoretical framework of my study on the teaching and learning of proof. This is largely because Vygotsky's pivotal idea that the context in which learning occurs is vital to our understanding of *how* it occurs underpins my study.

3.1.1 Higher mental functions

One of Vygotsky's central notions in his study on the processes of development is his argument that all higher mental functions arise as a result of mediated processes and through co-operative activity (Vygotsky, 1987, p.126). With the passage of time these functions are transformed and become integrated into the child's own mental activity (Vygotsky, 1987, p. 259). Vygotsky identified language as being the most crucial of all mediated processes; he emphasized that participation in social interactions mediated by speech is a pre-requisite for higher voluntary forms of human behaviour (Minick, 2005, p.36). Vygotsky argued that language and speech are the most crucial mediational means that promote the development of higher mental functions in social and collaborative settings (Daniels, 2008, p.48). Kozulin (1994) commenting on the special role played by language and speech in Vygotsky's psychological system, notes that they play a double role; the first being a psychological tool forming other mental functions, and the second arising as a result that they themselves are among these mental functions, and hence are also undergoing development. Vygotsky refers to the central aspect of the sign as a vehicle for guiding and developing mental processes and he points to the use of tools including psychological tools in the development of higher mental functions (Daniels, 2008, p.26). In this context tools might refer to the actions of individuals as they effect change in their environment while sign systems include language, writing and number systems (Vygotsky, 1978, p.7). Vygotsky proposes that in the process of conceptual development in children, "the most significant moment in the course of intellectual development which gives birth to the purely human forms of

practical and abstract intelligence occurs when speech and practical activity, two previously completely independent lines of development converge" (Vygotsky, 1978, p.24). In the context of my study practical activity includes the mathematical activity of proof construction using mathematical terms, symbols, signs, logical reasoning processes, proof methods and justification. This is coupled together with speech as students interact with each other and the lecturer in consultative group sessions studied in this thesis.

Proof comprehension and construction

As discussed previously when speech and the use of signs are combined in any activity, such activity is transformed and made more productive. According to the definition of higher mental functions, mathematical proof comprehension and construction can be viewed as activities involving higher mental functions. When undergraduate students who have recently been introduced to the mathematical objects and processes involved in proof construction, are brought together in small groups and participate in solving proof construction tasks, an environment is created in which social and collaborative activity is encouraged. I therefore propose that, in line with Vygotsky's theory, in the course of their participation in consultative group sessions, through their speech and communication (written and spoken) together with their use of the newly met terms, symbols, signs, logical and deductive reasoning processes and the newly met proof techniques, their learning will be greatly supported.

3.2 Vygotsky's theory of concept formation

Vygotsky's study of the process of concept formation continued from the research done by various other researchers interested in the process of concept formation including Ach (1921). The study was based on the assumption that "a concept is not an isolated, ossified and changeless formation, but an active part of the intellectual process, constantly engaged in serving communication, understanding and problem solving" (Vygotsky, 1986, p.98). Vygotsky surmised from Ach's experiments that concept formation is not a mechanical process, but a creative one taking place in the course of problem solving. Another researcher; Uznadze, (1966) depicts children beginning to speak as using words which gradually develop meaning allowing children to set them

apart from other words over time (Uznadze, 1966, p.77). He goes on to conjecture that in a similar fashion the development of concepts begins with "forms of thinking which are not conceptual but which provide a functional equivalent of concepts" (Uznadze, 1966, p.77). These functional equivalents are similar to the mature concepts held by adolescents and adults in functional use, but differ in structure and quality (ibid, p.101). Vygotsky held the view that the central question in concept formation was how this process was accomplished (ibid, p.102). This was the question which then urged him to do further experiments to study the process of concept formation and as a result saw the emergence of the developmental phases I will be describing below. It is also this question which forms part of my research which takes place in the context of proofs in elementary set theory for first year undergraduates, as I attempt to investigate how students' proof construction abilities develop over the semester course in Algebra.

Proof construction and comprehension encompasses a range of skills and abilities (as described in Section 2.2). According to Meija-Ramos et al. (2012), these include three major aspects: the first is the ability to use often newly met mathematical language, symbols and signs. The second is the ability to recognize the correct proof framework or method of the proof and follow the logical structure of proofs, that is, the ability to proceed logically from the beginning of the proof to the conclusion. The third is the notion that all claims must be justified and that deductions must follow previous statements based on reason and logic. These abilities form part of the students' local comprehension of proof (as described in my analytical framework described in Section 5.2.2). Students' holistic comprehension of proof includes being able to describe and explain the over-arching approach used in the proof, being able to transfer the ideas and methods used in previous proof solving activities in other similar proof construction activities and being able to illustrate statements and inferences with examples or interpret statements with the help of diagrams. In this thesis I would like to extend Vygotsky's theory of concept formation and the developmental phases he proposed to the development of skills students need as they attempt proof constructions in the area of elementary set theory. This includes students' use of newly met terms, symbols, signs, logical and deductive reasoning processes, proof methods and the practice of justification.

Vygotsky's study on the process of concept formation revealed that although processes leading to concept formation begin in early childhood, the intellectual functions needed to form a psychological basis for these only develop and mature at puberty. The study also revealed that "it is a functional use of the word, or any other sign as means of focusing one's attention, selecting distinctive features and analysing and synthesizing them that plays a central role in concept formation" (Vygotsky, 1986, p.106). Vygotsky further elaborated that words and verbal thinking are the main processes which lead to concept formation and its generative cause, is a specific use of words as functional tools. In terms of this theory it is important that while students are struggling to develop the various abilities needed for proof construction, the various mathematical objects that are in the process of formation should be discussed with peers and more knowing others, so that through this discussion and communication these objects are gradually brought to life. I hope to show that this is precisely what was taking place in the consultative discussions within the small group which met weekly to work on proof construction exercises.

Vygotsky's experimental analysis led him to believe that there are three basic phases involved in the individual's development of concepts, with each of these phases having several stages. The first is the **heap** phase where objects are linked together without having any inherent connections. Objects are grouped together through vague connections that happen by chance in unorganized heaps, a trait known as "syncretism" (ibid. p.110).

The second phase comprises several variations of thinking in **complexes**. These are the functional equivalents of real concepts as the objects in a complex are related by actual connections, a big step from the syncretic thinking of the heap phase. Primarily what sets a complex and a concept apart is that the links between objects in a complex are any number of diverse or various concrete and actual existing connections while links between objects in a concept are based on a single attribute (ibid. p.113). Bonds between objects in complexes are factual and concrete, whereas in a concept they are logical and abstract. Five different types of complexes were identified by Vygotsky: associative, collection, chain, diffuse and pseudoconcept. The last type of complex, the **pseudoconcept,** was termed by him the bridge between "complexes and the final

highest stage in the development of concept formation" (Vygotsky, 1986, p.119). The pseudoconcept is a generalization formed in the child's mind. This has the outward appearance of a concept but inwardly is still a complex, in that the processes used to guide its realization are still the concrete bonds of a complex. An illustration of pseudoconceptual thinking in Vygotsky's experiments is the child grouping objects together with the sample object as though according to an abstract concept. On further detailed study it becomes apparent that the child has only done this grouping because of the existence of some concrete bonds between the objects, thus reflecting thinking of the complex form. Vygotsky noted that the pseudoconcept plays a dominant role in the child's thinking and is a vital link in the journey to true concept formation.

Vygotsky noted that were it not for the functional equivalence of concepts and pseudoconcepts which ensures a successful dialogue between the child and the adult, mutual understanding would be impossible (cf. Vygotsky, 1986, p.123). In the specific case of mastering language and words, Vygotsky echoed Uznadze's sentiments that it is this functional understanding which enables words to acquire meaning and concepts to come into being. He quotes Uznadze (1966, p.177): "Obviously even before it reaches the state of a mature concept, a word is able to substitute functionally for the concept, serving as a tool of mutual understanding between people". Thus the pseudoconcept comprising the essential functional characteristics of the concept, when used in verbal communication with adults is a "powerful factor in the development of the child's concepts" (Vygotsky, 1986, p.123). Similarly I propose that while students are engaged in proof construction exercises and involved in written (on the board) and verbal communication such as: talking together, consulting, reasoning, explaining and clarifying, albeit with limited understanding, they are often operating with the functionally equivalent pseudoconcepts. This usage is indispensable in the formation of mature concepts. I hope to show that the consultative group discussions are very powerful factors in the development of mathematical objects and processes necessary for proof reading, comprehension and construction.

The third major phase is that of thinking in true **concepts**.

Briefly; throughout Vygotsky's theory of conceptual development, the use of the word is emphasised as an essential part of both developing processes, playing a guiding role

in the formation of genuine concepts. In the domain of mathematical education, the term 'word' is interpreted less broadly as 'mathematical terms, symbols and signs'. This is further elaborated in Section 3.2.1.

3.2.1 Adaptation of Vygotsky's theory of concept formation to the mathematical education realm

Vygotsky's theory of concept formation has been adapted to the realm of mathematical education by Berger (2004a, 2004b, 2004c, 2006) while considering the question of how a university student makes sense of what is a new mathematical sign (Berger, 2004a, p.81). Drawing an analogy with the focus of Vygotsky's experiments, that of the child learning a new word, she argues similarly that the student uses a newly met mathematical sign as a means of communication as well as an "object on which to focus and to organize her or his mathematical ideas even before she or he fully comprehends the meaning of this sign" (Berger, 2004a, p.81). Berger (2004a, 2004b, 2006) further discusses the problem of how an individual learner who initially only has access to the newly met mathematical object through its definition, comes to know or understand that mathematical object. She argues and demonstrates through examples that the 'functional use' of a mathematical sign is both necessary for and productive of mathematical meaning-making for a university mathematics student (Berger, 2004a, p.82). Activities such as imitation, association, template matching and manipulation are all incorporated in functional usage of a mathematical sign (Berger, 2004a, p.83). These are the tools which enable students to make the transition from their own personal meaning of the newly met object to an understanding more in line with the object's use in the mathematical community.

In developing her theory on the functional use of a mathematical sign, she draws on Vygotsky's (1986, 1994) notion of the functional use of the word. In her analysis of mathematical interviews with undergraduate first year university students, Berger demonstrates how usage of a newly met mathematical sign evolves primarily through activities such as template-matching, association, manipulation and imitation using resources such as the definition which was provided in the task, and examples in the text book. I extend the notion of the functional use of a mathematical sign in my study to include functional use of newly met mathematical terms, definitions, logical

reasoning processes, proof methods and justification. I argue that students' functional use of the skills necessary for successful proof construction plays an important role in the formation of true concepts, bringing students' proof construction capacities closer to those expected of members of the mathematical community.

Berger adapts and extends Vygotsky's theory of concept formation making it more suitable for the mathematical domain, naming this "appropriation theory" (Berger, 2004b, p.4). The main reason for this extension of Vygotsky's classification (which came about from experiments with concrete objects) is that it does not consider what happens when students meet abstract objects with concrete representations. She has also "distinguished between the signifier-orientated aspects of object appropriation (where the student's primary focus is on the symbol) and signified-orientated aspects of object appropriation (where student's primary focus is on the idea conjured up by the symbol)" (Berger, 2004b, p.4). A brief description of appropriation theory follows.

The **heap** stage which according to Vygotsky was characterized by the grouping together of unrelated objects which are linked by chance in the child's perception is adapted to the mathematical context as a stage where learners "associate one sign with another because of physical context or circumstance" rather than based on a mathematical property of the signs (ibid. p.5). Thus an indicator for the heap stage is the use of non-mathematical criteria when engaging in reasoning and mathematical activities.

Berger identifies six non-linear stages of **complex** thinking where objects are grouped together by actual bonds which exist between them (Vygotsky, 1986, p.112). Complex thinking is the essential pre-curser to conceptual thinking. In this phase the student associates newly met signs with more familiar ones by abstracting or isolating the particular properties of these signs. The importance of complex thinking is that it enables students to communicate with their peers and more knowledgeable others using words and symbols. In this way their understanding of these newly met objects moves towards an understanding in common with the wider mathematical community. Whereas Vygotsky identified five stages of complex thinking; the associative complex, the chain complex, the collection complex, the diffuse complex and the pseudoconcept, Berger (2004b, 2004c) posits that these categories are not sufficiently adequate to

characterize the type of sign usage in the mathematical domain, and the need to distinguish between signifier-oriented and signified-oriented usage. Berger (2004b, 2004c), while discarding the diffuse complex category, adds the representation complex and the template complex. Berger also adds three sub-categories to the associative complex: surface association, example-centred association and artificial association.

I will be categorizing students' reasoning in my analysis as falling into heap, complex, pseudoconcept or concept thinking categories. As I will not be differentiating between the different types of complex thinking, because of space constraints, I will not go into detail about the various types of complexes.

The **pseudoconcept**, the final type in the complex stage, which has the appearance of a concept to the observer while still a complex because of incomplete or contradictory knowledge about the object, forms a bridge between complex thinking and conceptual thinking. Berger (2004c) proposes that students using pseudoconceptual thinking in the mathematical realm "are able to use and communicate about a mathematical notion as if they fully understand that notion, even though their knowledge of that notion may be riddled with contradictions and connections that are not based in logic" (ibid. p.14). Berger (2004c) argues that although all complex thinking allows students to communicate with others and develop their knowledge about the mathematical objects they are grappling with, pseudoconceptual thinking in particular allows students to engage and develop their knowledge in a way that is both personally and culturally meaningful. Detecting students' use of pseudoconcepts is difficult as the pseudoconcept has the outer appearance of a concept. The existence of a pseudoconcept can be empirically detected by investigating students' understanding of the mathematical object either before or after the student has used what seems to be a concept in an appropriate way, that is, before or after completion of a task which could be the construction of a proof or a portion thereof.

A mathematical **concept** is formed when the internal links between the different properties and attributes of the object as well as the external links between that object and other objects are consistent and logical (ibid. p.16).

3.3 The Zone of Proximal Development

In addition to the processes related to the development of higher mental functions and the theories of concept formation developed by Vygotsky, his theory on learning and development and the central notion of the Zone of Proximal Development (ZPD) are pertinent to my study. I will be using these important ideas as I discuss the learning that which place when students engage in proof construction exercises in the consultative group sessions.

Vygotsky argues that the relation between learning and development is complex. He proposes that in trying to match learning with the developmental level of the child, there are two developmental levels that need to be determined: the actual developmental level and the potential developmental level. The first is indicative of the child's mental functions and abilities when working entirely on his/her own and the second his/her mental capability when working under the guidance of a teacher or more knowing other. Vygotsky argued that children's true mental capability is better determined by observing what they can do with the assistance of others, rather than what they can do alone (1978, p.85). He referred to the difference between the two developmental levels as the zone of proximal development (ZPD) and defined it as "those functions that have not yet matured but are in the process of maturation", functions that are "currently in an embryonic state" (ibid., p. 86). Vygotsky argued that a child's mental development can only be determined when one has established both the actual developmental level (when the child is working alone) and the ZPD. The ZPD can therefore play a central role in research on the development of learning processes (ibid., p. 87). He proposed that education should operate on a few levels above children's current developmental levels and that "the only good learning is that which is [slightly] in advance of development" (ibid., p. 89). Wertch and Stone (1985, p.165) interpret this to mean that good teaching "awakens and raises to life those functions which are in a stage of maturing, which lie in the zone of proximal development". Vygotsky argued that an investigation of the ZPD is more helpful in revealing how intellectual progress occurs than just a measure of the mental age of a child (Vygotsky, 1986, p. 187). Del Rio and Alvarez propose that Vygotsky articulated the notion of Zone of Proximal Development "in order to deal methodologically with the need to anticipate the course of development" (Del Rio &

Alvarez, 2007, p. 280). Different educational settings give rise to different ZPD attainments and so researchers can aspire to find better or more ideal conditions for ZPD creation. I hope to show in my thesis that one such setting is obtained when a group of students with different abilities together with a more knowing other such as a lecturer participate in proof construction exercises actively, and engage each other with questions and prompts requesting clarification, reflection, explanation and justification.

A study concerned with the creation of zones of proximal development

In her research Miller (2003) endeavours to provide evidence for the view originated by Vygotsky that social interactions that take place in discussions with purpose, allow students to internalize and develop their cognitive abilities (ibid., p.290). Using a series of ethnographic classroom studies she posits the view that discussions in literature are important in developing students' reflective thought processes. In the classes studied, she reports on key issues about how teachers successfully mediated discussions to create a zone of proximal development in which students' capacities were developed. She found that classes which were successful in encouraging fruitful discussion had the following characteristics: the teacher made it clear and emphasized that the group would be working together, the teacher asked authentic questions about what was puzzling her and listened carefully to students, providing support when needed after waiting to see whether other students might provide a next step or move. Teachers in these classrooms showed great respect for students, nurturing their potential abilities and allowing them to take growing responsibility in critical enquiry (ibid., p.296). Teachers who were unsuccessful in creating zones of proximal development often answered students' questions themselves, headed off student interaction and discouraged students' initiatives and questions that did not agree with their own reasoning processes.

3.3.1 Consultative group sessions and the notion of EZPD

In the consultative group sessions in my study, my intention was to allow students to have a very active role and a prominent voice. Similar to the teachers described in Miller's (2003) research who were successful in mediating discussions to create zones of proximal development where students' capacities were developed, I tried to create a warm encouraging atmosphere where the views and questions of all participants were

invited and appreciated. In this way I tried to create an environment where active engagement, discussion and consultation were encouraged as students interacted with one another while doing proof construction tasks. I hoped that students would be enabled to access their zones of proximal development during these sessions. For ease of reference I designate an environment in which access to students' zones of proximal development is encouraged and promoted as the **EZPD**, that is, an **environment** in which students are enabled to access their **zones of proximal development**.

3.3.2 The role of imitation

The ZPD also highlights the importance of imitation in learning. Previously it had been thought that children's independent activity, not their imitative activity was indicative of their mental development, but Vygotsky argued that psychologists had shown that a "person can imitate only that which is within her developmental level" (Vygotsky, 1978, p. 88). He gives an example of a child who has difficulty solving a problem in arithmetic but grasps it as soon as the teacher has solved it on the blackboard. The same child would not be able to grasp a problem in higher mathematics solved on the board. He suggests that, when working with peers or adults in collaborative activities, children are able to accomplish a lot more by using imitation. According to Confrey (1995, p.40), the central role that imitation plays in cognitive development, leading to true concept formation, contributed towards Vygotsky's creation of the notion of the zone of proximal development. Chaiklin (2003, p.52) postulates that the assumption underlying the possibility of imitation is the existence of maturing psychological functions that are not yet able to operate independently but have developed to an adequate level enabling a person to make use of the scaffolding received.

3.3.3 The notion of internalization

Another key aspect in Vygotsky's theory of cognitive development is that of internalization. Vygotsky argued that all higher functions originate as social processes and are gradually internalized as children interact with more knowing others and master these functions for themselves (Confrey 1995, p.40). Vygotsky emphasized that the creation of the ZPD while the child is interacting with peers and more knowing others in his/her environment sets into motion a number of internal processes and once these are

internalized they become part of the child's independent area of operations. The guidance and assistance received during this interaction is referred to as scaffolding.

3.4 Summary

I have briefly discussed Vygotsky's socio-cultural theory and his theory of concept formation and its adaptation to the mathematical education realm. I have also discussed his emphasis on the central role of the functional use of the word in the process of concept formation and extended this to include functional use of newly met mathematical terms, symbols, signs, logical reasoning processes, proof methods and justification. It is this theory that underpins the entire study.

Chapter 4: Methodology and Methods

4.1 Introduction

In this chapter I describe how I investigated my research questions and why I used particular methods and methodology. Cohen and Manion (1994, pp.38-39) describe methods as the various means by which data is to be gathered by the researcher while methodology is summed up as the processes or techniques researchers use in their investigation. Sikes (2005, p.16) writes that methodology refers to the theory of how the researcher intends to gather knowledge in his/her research or investigation. Methodology is thus focussed on the description and analysis of research methods and aimed at understanding the process of scientific enquiry (Cohen & Manion, 1994, p.39).

My first research question focussed on the challenges and difficulties first year undergraduate students have with proof construction in the area of elementary set theory and the forms of effective guidance offered to them as scaffolding. The second and third research questions focussed on the development of students' proof construction abilities and the nature of the interactions of students and the lecturer in the context of consultative group sessions. With this in mind I investigated how students could be more effectively enabled to make progress and become intellectually autonomous, and how those showing potential in becoming more knowing peers could be empowered and supported.

These questions all relied on students' experiences with proof construction in a group context. I investigated students' views, thought processes and actions using methods based in an interpretive paradigm as my intention was to make sense of the subjective experiences and meanings my participants had with proof construction in the consultative group sessions (Creswell 2007, p.21, Cohen, Manion & Morrison 2011). My primary concern was to understand and explain how individual students interacted with and interpreted the mathematical activity of proof construction as they took part in consultative group sessions. My methods of analysis were thus qualitative. My analysis of the data collected in my study is interpretive: based on inferences I made as a result of my observations of students' proof construction actions and utterances using the constructs described in my analytical frameworks (see Chapter 5).

I have used the case study method to investigate the characteristics of my individual students (as related to my research questions) with the hope that this research will help inform other researchers' and teachers' views on the difficulties and challenges undergraduate students might experience and will offer insight into proof construction in collaborative group processes particularly in settings similar to the consultative group sessions.

My ontological and epistemological assumptions are discussed in Section 4.2. The methodology of the case study approach will be discussed in Section 4.3.1. Section 4.4 gives some background on the setting of study, while Section 4.5 describes the methods used. Section 4.6 discusses the methods used in the consultative group sessions.

4.2 Ontological and epistemological assumptions

My ontological assumptions are based in a social constructivist paradigm. As a researcher I do not see the world as having a universal absolute reality, but a reality which is dependent on individual perspectives and developed in each of us constructively (Hatch 2002, p.15). My epistemological assumptions arose from my ontological assumptions: they required me to get as close as possible to the students participating in my study so that I could collect their subjective accounts, experiences and actions (Sikes 2004, p.20, Creswell 2007, p.20). It was necessary to carry out my research by collecting data which focussed on how the participants of my study experienced and developed proof construction and proof comprehension abilities as they participated in consultative group sessions. I was a participant observer. I then attempted to make sense of my observations and interpret them in order to generate meaning from the data collected (Creswell 2009, p.9). My ontological assumptions imply that my interpretation of my observations is not purely objective but subjective and has been shaped by my own experiences and background, coloured by my own particular perspective (Creswell 2009, p.8). This perspective was largely shaped by my theoretical framework; Vygotsky's socio-cultural framework. Vygotsky's socio-cultural theory also underpinned the analytical frameworks used for analysis of students' and lecturer discourse and students' proof construction actions and contributions in the consultative sessions. The analytical framework for analysing

student and lecturer utterances is based on the understanding that the teaching and learning of proof is a social process. It assumes Vygotsky's thesis that all higher forms of cognitive learning have their origins in social interaction and are mediated by speech (Blanton, Stylianou & David, 2011). My second analytical framework used for analysis of students' proof construction actions and contributions incorporated the Vygotskian notion of the functional use of the sign to interpret students' use and application of newly met terms, symbols, logical reasoning processes, proof methods and the practice of justification. While analysing and interpreting my data, in addition to the effects of my particular perspective, I also considered the well-established argument that the very act of a researcher acting as an observer in a particular practice affects that practice (Brown & Dowling, 2001, p.47). Hence there was the possibility that the participants of my study would have acted differently if they had not been aware that the consultative sessions were being video recorded. I need to thus acknowledge the possible effects that I the researcher introduced to what was being researched.

On the basis of my ontological and epistemological assumptions, I am well aware that my study cannot convey the whole or absolute truth. The interpretations, discussions and conclusions I have offered arose from my attempt to discover and describe emergent ideas from my research as viewed from my own particular outlook. However I have tried to deliver an honest, trustworthy and coherent account while taking into consideration the implications of my ontological and epistemological assumptions.

4.3 Methodology

As outlined above, I engaged in qualitative research in this study, using a naturalistic or interpretive paradigm. This paradigm rejects the belief that human behaviour is governed by general universal laws. Rather the position is that individuals' points of view and interpretations of events must arise from the individuals themselves (Cohen, Manion & Morrison 2011). I thus explored students' difficulties and challenges in proof construction, and investigated students' progress in terms of the changes in their proof construction activities in the area of elementary set theory while acting as a participant observer in the intervention. I explored an intervention comprising a consultative method of group work and its effect on students' development regarding proof

construction abilities. I focussed on the processes of interaction in these consultative group sessions to gain a holistic picture of how lecturers could support students to become independent thinkers and empower those students who showed promise in becoming more knowing peers to develop their capabilities. My methodological approach in this study consisted of case studies. Section 4.3.1 discusses the methodology of case studies.

4.3.1 Methodology of case studies

The case study approach allows the researcher an opportunity to fully investigate an aspect with which one is concerned within a limited time scale (Bell, 2001, p.10). The aim of a case study approach is "to illuminate the general by looking at the particular" (Denscombe 2007, p.36). The primary case study was the consultative group sessions attended by students who were drawn from the first year mathematics major class at the University of Limpopo in 2010. There were also two smaller case studies in which the proof construction activities of two specially chosen students (Frank and Maria) were investigated. See Chapter 7 for the latter two case studies.

It is often problematic to generalize qualitative studies as the particular contexts and characteristics of individual participants of different cases are different (Creswell 2007, p.74). The deep level of investigation and intensive analysis involved in case study research however, hopefully enables this study to add to the growing literature on undergraduates' experiences with proof construction in a collaborative group context. The participants of my study were purposefully selected to be representative of their class in terms of mathematical ability (according to their first semester results) and gender. I chose students who were high attaining: 75% - 90% (category A), middle attaining: 60% -75% (category B) and low attaining: 45% - 60% (category C) to give me the opportunity of assessing proof comprehension and construction capabilities and how these abilities progressed in the course of the semester for students of varying levels of competence. They were also representative in terms of gender as half of the students were women although I did not focus on gender as an issue of interest.

I acted as a participant observer and developed less formal relations with the participants. Although I actively participated as a facilitator during the consultative

sessions, on viewing the video recorded sessions and studying the transcripts of these sessions, I as researcher also acted as a data gathering instrument. I lectured and tutored the students during the second semester in contact periods other than the consultative group sessions and therefore had sufficient time to engage with them in other settings. Studying social phenomena in a qualitative study necessarily influences the behaviour of those being studied (Hatch 2002, p.10). While interpreting and making sense of the data I had to be reflective, keeping track of my influence on the setting. I attempted to ensure that my own bias and emotional responses did not affect the research, that they did not affect the students adversely and that these biases were taken into account in my interpretation of events.

4.4 Setting of the study

The study took place at the University of Limpopo in a first year mathematics major course which is taught in two parts: a Calculus portion in the first semester and an Algebra portion in the second semester of the year. The study took place in the second semester where the material on elementary set theory and relations was taught under the umbrella of topics in Algebra. As mentioned previously in Chapter 1, the University of Limpopo is a previously disadvantaged university situated in a rural setting about 30 minutes' drive from Polokwane, the capital city of Limpopo. Students at the university mainly come from previously disadvantaged schools in the Limpopo and Mpumalanga provinces. A shortage of well-qualified mathematics and science teachers in South Africa means that mathematics teachers at these schools are often not as well qualified as those teaching at schools in urban areas.

Students registered for this course were divided into two groups and taught concurrently in two lecture venues by two lecturers: a colleague and myself. In 2010, the year that my data collection took place, there were 985 students registered for this course. The lectures are taught in a standard lecture format where the students sit quietly, listening to the lecturer and taking notes as he/she talks and writes on the blackboard. More recently tablets and data projectors have been introduced. Students were also assigned to tutorial groups and attended one three hour tutorial session per week. In the tutorial sessions, students worked on the exercises relevant to each section of the course with

the help of a lecturer and over 20 tutorial assistants. There was roughly a ratio of over 20 students per assistant. In these sessions students were expected to sit quietly attempting all the questions in the exercise set given that week and make use of the assistance of the lecturer and tutorial assistants by raising their hands and asking questions. The tutorial assistants were chosen from a pool of students who had passed the course well and were interested in tutoring. The lecturers involved met with the tutorial assistants once a week to revise and go over all the pertinent material before the tutorial thus ensuring that the assistants would be of help to the students. Continuous assessment took the form of weekly tests (covering material done in the previous week) written at the end of the tutorial sessions. The students also wrote two comprehensive tests during the course of the semester and an exam at the end of the semester. A study guide containing all the relevant notes and exercises was made available to the students at the beginning of the semester. In 2010 42% of the students passed the course at the end of the semester. This pass rate was quite normal; pass rates for this course (taught in the format that it was taught in 2010) ranged from 23% to 40% in the three previous years before the study took place.

4.5 Methods

The study took place in the year 2010, and was piloted in year 2009 (see Section 4.5.1 for discussion of the pilot study). Permission to conduct the research at the University of Limpopo was obtained from the Head of the Department of Mathematics and Applied Mathematics in the School of Computational and Mathematical Sciences. Ethical clearance was also obtained from the Ethics Office of the University of the Witwatersrand under whose auspices this research was carried out.

Twelve first year students were identified and selected based on their performance or marks in the first semester of that year. Four students were selected from each of the categories A (75% - 90%), B (60% - 75%) and C (45% - 60%).

Once the twelve students had been selected, they were invited to an information session where they were presented with all the information about the study. I explained the aims and purpose of the study, the methodology I would be using, their part in the study and how confidentiality and anonymity would be ensured. It was also explained that

participation was purely voluntary and that non-participation would not affect their marks adversely. Potential participants were assured that they could withdraw from the study at any time without any consequences whatsoever. Each student received a participant information sheet with all the information on the study. The students were asked to consider carefully whether they wanted to participate. Those who were willing to participate were asked to sign two consent forms: one for general consent for participation in the study and another giving their permission for me to video record the consultative group sessions.

Although all the students who had been selected signed the consent forms, only 10 of them actually came to the sessions. The students were asked to participate in a group which met once a week (in addition to regular tutorials) while the material on the section on elementary set theory and relations was being taught in class. Each session lasted about three hours. The students came together in a room equipped with a white board and markers to work together on proof solving tasks with my guidance and help. Here I implemented the consultative method as described in Section 4.6. Proof solving tasks included proof solving exercises which were included in tutorial exercises, as well as proofs of theorems and propositions discussed in class and included in the lecture notes. As discussed earlier, the aim of these sessions was to gain more understanding of the processes used by students and the challenges and difficulties they encountered when doing proof construction. Hence I needed to observe the students' interactions and discourse in these scaffolded sessions and to identify potential benefits and constraints. I attempted to create an environment which encouraged students' active participation and closely monitored the processes and factors which seemed to enable students to make progress in their proof construction abilities. There were four consultative group sessions in which the proofs of propositions and theorems relating to the Chapter on Elementary Set Theory were covered in detail.

Ten students attended the first session: two from category A, four from category B and four from category C. The second session was attended by six students: two from each of the categories A, B and C. The third session was attended by five students: two from category A and three from category C. The final session was attended by eight students: two from category A, two from category B and four from category C. The

pseudonyms of the students, their respective categories and the sessions they attended are shown in Table 4.1.

Table 4.1: Pseudonyms of participants, the category of their respective first semester results and the sessions they attended

Students who participated	Category of first semester exam results	Sessions that students attended
Frank	A	Sessions 1,2,3 and 4
Joseph	A	Sessions 1,2,3 and 4
Christine	B	Sessions 1,2 and 3
Maria	B	Sessions 1, 2 and 4
Bonnie	B	Sessions 1 and 4
Laura	B	Session 1
Gary	C	Sessions 1,2,3 and 4
Edgar	C	Sessions 1,2,3 and 4
Helen	C	Session 1
Kenny	C	Session 1,3 and 4

The students were invited to a session at the end of the semester, where they gave their feedback on the effectiveness and the value of the consultative sessions.

4.5.1 The pilot study

The pilot study took place in the second semester of 2009. Fifteen first year students were purposefully selected on the basis of their first semester marks and invited to an information session. They were told about the study's purposes and their roles in it and how it would affect them. They were assured of confidentiality and anonymity and that participation in the study was entirely voluntary, that they could withdraw at any time without any adverse consequences. Each participant received an information sheet detailing all the information just discussed and they were asked to carefully consider whether they would like to take part in the study. All the students expressed their willingness and were given two consent forms to read and sign; one was a general consent form and the other asked for permission to allow video recordings of the consultative group sessions. We discussed and decided on the most convenient time for everyone to attend these sessions, and the students generally expressed their excitement and enthusiasm at being involved in a novel intervention.

Three consultative group sessions took place in which 8 participants attended the first session, 12 attended the second session and 9 attended the third session. Of the 8

participants in the first session three were from category A, two from category B and 3 from category C. Of the 12 attending the second session, there were 4 students from each of the categories A, B and C. Of the 9 attending the third session, five students were from category A, two from category B and two from category C. The sessions were therefore quite representative in terms of students' mathematical ability.

The pilot study was undertaken with several aims in mind. The first was to investigate the feasibility and the do-ability of the consultative sessions. It was particularly important to explore whether students would be willing to participate and attend these sessions regularly. This was paramount if the consultative sessions were to yield reliable information. I also needed to pilot the novel intervention method and familiarize myself with this method, attempting to learn from my experiences the best ways in which to conduct the sessions to create an environment that would be conducive to students' active participation and engagement. I also needed to identify actions and habits (on my part and the students) which might hamper the progress of the sessions rendering them unproductive in terms of student engagement. These piloted sessions also helped to inform my decisions on the optimal duration of each session and whether my recording instruments were adequate and would provide accurate and detailed depictions of what actually occurred. I also wanted to ensure that my research would benefit the participants contributing to their welfare and not causing them any harm.

Regarding the study's viability, the students who attended the sessions were always very keen, punctual and eager and often formed study groups of their own when they went back to their places of residence, helping other students who had not had the opportunity of participating. I learned a great deal from the piloted consultative sessions. I had thought that the students would be shy and uneasy about the video recorder and that it would be difficult to get them to participate and discuss the proof construction tasks openly. Although they were a little awkward at first, as soon as the first student came up and attempted the first proof construction exercise (10-15 minutes into the first session), they were at ease. They offered their contributions without hesitation and seemed to forget about the video camera completely. The general mood was buoyant and happy and the students often told jokes and laughed. The feedback

was very positive, not confined to those attending these sessions but also from other students whom the participants had helped. Most of the students expressed their gratitude for having been involved in this form of instructional intervention. News of these sessions must have spread because many students approached me at the beginning of 2010 and asked whether these sessions would be continued that year.

I made brief field notes and memos during and after the pilot group consultative sessions. I also kept a reflective journal in which I recorded pertinent points regarding my methods and data collection. These are described in greater detail in Section 4.5.4. The piloted consultative sessions which were transcribed by a professional transcriber, allowed me to start thinking about which analytical frameworks I would use and how I would code and analyse the transcripts.

4.5.2 Video records and transcriptions of consultative group sessions

As with the pilot sessions, the consultative group sessions in the actual study were video recorded. As I was taking an active role in organizing and participating in the group sessions I was neither able to observe the students systematically nor make detailed field notes during the course of the sessions. Video recordings were required so that I could do detailed observation after the sessions. Videos offer a powerful medium for recording and analysing evolving situations and interactions (Cohen, Manion & Morrison 2011, p.530, Derry 2007, p.1). Their use allows the researcher an observational record which is more unfiltered than human observation, and has the advantage of being able to be viewed many times (Cohen, Manion & Morrison 2011, p. 470). For accurate analysis of the transcripts, the researcher can review the recorded sessions several times scrutinizing them with due care when attempting to code the transcripts in terms of the categories and their indicators.

Videos also allow the researcher to capture non-verbal data. These include the tone of voice, inflections and emphases of the speaker, pauses and silences, interruptions and mood of the speaker (whether they are excited, angry or happy), speed of talk and how many people are talking at the same time (Cohen, Manion & Morrison 2011, p.427). This obviously makes transcribing the sessions and analysis of the transcripts more

time-consuming. However even when this non-verbal data is not captured in the transcripts of the sessions, the researcher is reminded every time he/she reviews the tapes and he/she is therefore able to gain more insight. The data is subsequently richer and has more depth.

I used a small video camera equipped with a rechargeable long-life battery and a tripod. The sound quality was checked with earphones at the beginning of the session, and periodically during recording. The sessions were recorded continuously with the camera in a fixed position on the tripod on the periphery of the 'circle' of the group of participants as they faced the board. The camera was operated by a young student (the same age as the participants), and was only moved and zoomed very occasionally and carefully when focussing on someone who was offering a contribution, or on the board when one of the participants was writing out his/her proof construction attempt. The recording instrument was made as unobtrusive as possible. The room was not sound proof and at times, the sounds of birds chirping at the window, or chairs and doors creaking could be heard in the recorded sessions making transcription at these points challenging. Students' speech was also sometimes inaudible as some spoke very softly. Fortunately these occurrences did not happen often and I could see and hear the students clearly in most of the recordings.

As my video recordings would be my major source of information, and the means by which my data could be stored and retrieved, I had to be systematic in selecting all the detail necessary to support my analysis and interpretation of students' proof construction activities (Goldman, Erickson, Lemke & Derry, 2007, p.15). Each three hour consultative session was recorded completely and continuously.

When attempting to transcribe the sessions, I soon realized the enormity of my task in terms of time and effort. I therefore enlisted the help of a professional to whom I gave a detailed and comprehensive list of the nomenclature used in the sessions (including the mathematical terms, symbols and signs). Once I had received the completed transcripts, I viewed the video recorded sessions together with the transcripts several times, checking for errors and making corrections to ensure the accuracy of the transcripts. It was of the utmost importance to ensure the accuracy of the transcripts as I relied on these transcripts to inform my investigation in several ways. First: when investigating

students' difficulties with proof construction, difficulties related to the language and terms used when referring to newly met mathematical terms, symbols and signs and their development in this regard were important. Second: I was interested in how students offered and received guidance from their peers and so the language they used and the forms of guidance they gave were also very important. The transcripts went through an iterative process of revisions and corrections until I was satisfied that they were indeed accurate depictions of each session.

4.5.3 Selection of video recorded events and analytical frameworks

Once I was satisfied that the transcriptions were highly accurate and contained no incongruences, I went through them mindful of my research questions and the possible frameworks of analysis as proposed by other researchers. These are discussed in my Literature Review Chapter (Sections 2.3 and 2.4). I did some preliminary coding of the transcripts according to analytical frameworks which seemed to be congruent with the purposes of my study and the data I had collected.

I realized that I needed to use two complementary analytical frameworks to address my research questions fully. The first (based on a framework developed by Stylianou, Blanton and David (2011)) would allow me to analyse the students' and lecturer's discourse in order to categorize the nature of their utterances with the aim of tracing patterns of scaffolding between the lecturer and the students and between the students themselves. This would help me to gain an understanding of how students could be enabled to access their zones of proximal development. I hoped this analysis would also reveal how the norms pertaining to the consultative sessions were established, and how students in general were supported to become intellectually independent while those showing the potential of becoming more knowing others were empowered to develop their capabilities (research question 3). The second analytical framework (based on an assessment model developed by Meija-Ramos, Fuller, Weber, Rhoads and Samkoff (2012)) would be used to focus on analysing students' proof construction actions and contributions to trace the development of students' proof construction abilities as they progressed through the sessions (research question 2). Furthermore this analysis could help inform me about the difficulties and challenges that hindered

students' proof construction attempts and the various forms of scaffolding which could benefit them (research question 1). A more detailed elaboration of how the analytical frameworks were selected is given in Section 5.2.

Once these analytical frameworks had been identified and preliminary coding of all transcripts done, I needed to select the particular events that would form the basis for my study so that I could carry out more detailed coding and analysis. With this in mind, I searched the transcripts for the events which best illustrated the challenges and difficulties, the forms of guidance and scaffolding offered by the lecturer and the students' peers, and which showed how the norms pertaining to the consultative sessions were established. The selection of these events was not only based on the transcripts and video records but also on my own experiences in the consultative sessions and my field notes and memos written during and after each session. The quality of the video clips in terms of their clarity of picture and sound was also an important consideration, but this was secondary and did not bias the selection of the events. The events selected were Episodes 1 and 2 from session 1 and Episode 3 from session 2. I then looked through the transcripts for events that showed the first signs of visible and obvious improvements in proof construction ability of the two students observed in Episodes 1 and 2. These took place in Episodes 4 and 5 in session 2. I also wanted to focus on the ways in which the norms established by the lecturer were taken up by students. These included encouraging students to clarify, explain and justify deductions and conclusions while questioning and critiquing their peers, and proceeding from one step to the next using sound logical reasoning. I searched for those events where more knowing students began to assume the role and responsibility of the lecturer, becoming active agents offering the required scaffolding to their peers. This was clearly evident in Episodes 3, 4 and 5 of session 2. Hence my complete selection of transcribed video material comprised Episodes 1, 2, 3, 4 and 5 from sessions 1 and 2. To present events holistically, each of these episodes contained a completed proof construction attempted by students from the beginning to the conclusion. The episodes were consecutive.

As I engaged with the detailed coding and analysis of the events, using a grounded approach I allowed further categories and indicators to emerge from the data I was

working with (the transcripts). Both analytical frameworks were extended and adapted in order to better capture all aspects of the discourse and proof construction actions and contributions observed in the consultative sessions. A brief discussion of grounded theory as put forward by Corbin and Strauss (1990) is included below.

Corbin and Strauss (1990, p.5) explain that the procedures of grounded theory (first introduced in 1967), are geared towards providing a "thorough theoretical explanation of social phenomena under study". They put forward eleven procedures or canons to be followed by those carrying out grounded theory studies. These are briefly discussed:

- In grounded theory research data collection and analysis are interrelated and the analysis should begin with the very first data collected. This is necessary as this analysis is used to 'fine tune' questions to be asked and observations to be made with the next set of data to be collected. Corbin and Strauss stress that the interrelation of data collection and analysis is one of the most important factors ensuring the effectiveness of the grounded theory approach.
- The notions that form the basic units of analysis are brought to the surface gradually by comparing incidents and calling the same phenomena by the same terms as the analysis progresses.
- Categories are generated by grouping notions pertaining to the same phenomenon together. These categories are related to one another over time to form a theory.
- In grounded theory, it is not the groups of individuals, units of time and so on that determine how sampling proceeds, but the notions and phenomena that surface from the study as data collection and analysis go ahead together.
- As data collection continues and various incidents are noted, these are constantly compared to other incidents for similarities and differences resulting in notions that are more precise and consistent.
- The data should be examined for patterns of regularity of incidents occurring and the researcher should account for when there are variations in the original pattern.
- The researcher needs to be alert to actions and interactions that change when the prevailing conditions change.

- The researcher should keep track of all the notions and categories as well as the analytical process by using memos throughout the study. These memos are essential when reporting on the research and its implications.
- The hypotheses which are developed as analysis is ongoing are constantly revised by being taken back into the field until they are verified and shown to hold true for all evidence collected.
- As far as possible grounded theorists should share the outcome of the (ongoing) analysis with colleagues who are experienced in the same area of research. This will guard against bias and allow for new insights making for a richer and more collaborative analysis.
- Broader structural conditions must be analysed. The researcher has the responsibility to show in their analysis the specific links between conditions, actions and consequences (Corbin & Strauss (1990), p.11).

Corbin and Strauss (1990) put forward three types of coding that may be used as the fundamental analytical process, by the researcher; open, axial and selective. I have not gone into much detail about the coding process as my analytical approach was fundamentally typological and theory-driven. The procedures and canons outlined above were followed as much as possible but my analysis was primarily guided by the categories and indicators described in my analytical frameworks (as described above and in Section 5.2). As the analysis was ongoing categories and indicators that did not appear in these frameworks and which emerged according to the canons described above were noted, compared and developed into additional categories and indicators. These additional categories and their indicators are further discussed in Section 5.2.

Table 4.2 shows all the proof construction exercises in sessions 1 and 2, in the order in which they were attempted and the pseudonyms of the students who attempted the exercises. It should be noted that discussions between the proof construction attempts and the brief introduction given at the beginning of the sessions were not included and that the duration of each of each of these sessions was over three hours. As can be seen the proof methods of implication, double implication, equality and showing that one set is a subset of another set were the main proof methods contained in all the proof construction exercises. In line with the material covered in class, as each session

progressed new terms, symbols and concepts were gradually introduced: the Cartesian product and the power set in the second session, equivalence classes in the third session and some simple number theory proofs in the fourth session.

Table 4.2: Proofs attempted in the first and second sessions and their duration

Session	**Episode chosen for detailed coding and analysis**	**Participant attempting the proof construction**	**Proofs attempted in each session**	**Duration of each episode**
Session 1	Episode 1	Frank	$A \subseteq B$ and $B \subseteq C \Longrightarrow A \subseteq C$	23 minutes
		Frank and Joseph	$A \cup (B \cup C) = (A \cup B) \cup C$	28 minutes
	Episode 2	Maria attempted to prove a) $\Leftrightarrow$ b)	If A, B, C are sets, the following are equivalent: a) $A \subseteq B$ b) $A \cap B = A$ c) $A \cup B = B$	63 minutes (for: a) $\Leftrightarrow$b)) Remainder of proof took about 30 minutes
Session 2	Episode 3	Edgar	$(A \cup B) \times C = (A \times C) \cup (B \times C)$	22 minutes
	Episode 4	Maria	$(A \cap B) \times C = (A \times C) \cap (B \times C)$	13 minutes
		Gary	$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$	16 minutes
		Joseph	$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$	5 minutes
	Episode 5	Frank	$A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$	19 minutes
		Joseph	$P(A) \cup P(B) \subseteq P(A \cup B)$	17 minutes
		Christine	$P(A) \cap P(B) = P(A \cap B)$	11 minutes
		Frank	$Z \subseteq X$ and $Z \subseteq Y \Longrightarrow Z \subseteq X \cap Y$	4 minutes
		Gary and	$S \subseteq T \Longrightarrow S \cup A \subseteq T \cup A$ where A	5 minutes

		Joseph	is any set.	
		Joseph	If S ⊆ T and B is any set, then S ⋂ B⊆ T ⋂ B	4 minutes

4.5.4 Field notes, memos and reflective journals

Although my time was restricted by my role as a participant observer during the consultative sessions, I kept a notebook at all times in which I recorded brief field notes. This was done periodically and informally with a great deal of flexibility, to capture interesting and significant phenomena which occurred during the sessions (Brown & Dowling 2001, Cohen, Manion & Morrison 2011, p. 466-467). For each session I recorded the following:

- The pseudonyms of all the participants who attended,
- A chronicle of the proof construction tasks attempted in each session, a summary of the activities of the student who attempted the proof construction as well as the major contributors,
- Significant interactions between me and the students and between the students themselves,
- Major difficulties and challenges that students experienced as they engaged with proof construction tasks,
- My observations on how more knowing peers grew in their understanding that every proof construction step needed to be justified and explained and how they took over the scaffolding of students,
- Significant improvements in particular students' proof construction abilities,
- The general mood and feeling of the students at various stages, particularly when they seemed to be tired, happy, excited or enthusiastic.

I wrote memos at the end of each session on the pertinent points of observation and discussion during the session. These memos gave more detail to the field notes made during the sessions. I also kept a reflective journal which contained:

- Records of the participants and the selection process as well as my reflections and thoughts on the methodology, data collection and analysis of my data.

- Reflections on the methods being used, the data I was collecting and my intended analysis.
- Points which needed to be clarified. I also made a note if any ethical issues about the study as a whole occurred to me at any stage.

The field notes, memos and reflective journal were very helpful as I referred to them, particularly in the early stages of analysis to guide me in the coding and analysis.

4.6 Consultative group method

The methodology for the consultative group is based on the belief that learners learn best when they interact with one another and experts; consulting on the problems and questioning one another, while making functional use of newly met terms, symbols, signs, logical reasoning processes, proof methods and justification. According to Vygotsky (1978, p.90), an essential feature of learning occurs when one is interacting with adults and peers, within a zone of proximal development. This sets in motion a variety of "internal developmental processes" that become part of one's independent development once they are internalized.

Methodologies similar to the one that I have used in the consultative group sessions are found in the studies mentioned in Section 2.4. The studies surveyed were those in which a socio-cultural perspective was incorporated in pedagogical interventions leading to the development and improvement of students' proof comprehension and construction abilities. In line with my theoretical framework (Vygotsky's socio-cultural theory), these studies all advocate establishing classrooms where collaboration and consultation among students and more knowing others is encouraged. Students are gradually accustomed to explain and justify their reasoning and reflect on and critique their own thinking and the explanations given by their peers. In this way students are enabled to access their zones of proximal development leading to a more optimal development of their abilities in proof construction which is in fact a very social activity. A common thread running through all these studies is that they strive to establish environments in which students can more easily access their zones of proximal development (EZPD) and so bring about acceleration in their abilities to reason mathematically and develop proof construction abilities. All of these studies advocate

that the teacher/lecturer plays a very active role in guiding and managing an environment which encourages active student participation.

I argue that in the consultative sessions focussing on proof construction, the presence of an expert such as the lecturer, tutorial assistant or more competent peer is essential. Such an expert could provide scaffolding and help when required, but even more importantly help establish the norms that would enable and support students to become independent thinkers and empower those with the potential to become more knowing peers to develop that capacity. Proof construction is a difficult task posing numerous challenges to students all over the world (see Section 2.2.1) and students generally need much practice before they can accomplish this on their own. As the consultative sessions progress more knowing peers can be encouraged to take the lead by the lecturer or tutor who would now remain mainly in the background to guide and steer the process and intervene when necessary; that is, when the students need help.

In these sessions I tried to establish such an environment where fruitful discussions could take place, and introduced a process of consultation helpful to students when constructing proofs. The concept of consultation I used is one gleaned from the Baha'i writings. Consultation is understood to be a method of discussion where all the members in the group are encouraged to offer their views and listen to each other's views in a friendly and tolerant atmosphere while investigating the truth (Baha'i International Community 1989, p.1). In this manner every member of the group was encouraged to express his/her views and understanding freely as a contribution to the search for the right result. I had a critical role to play throughout this process; helping the students whenever necessary to reach the desired output and encouraging everyone to participate and give their views without fear of criticism or ridicule. I tried to put students at ease and constantly asked for their input, ideas, thoughts, comments and questions. I thus transferred the responsibility of finding the correct solution to the students themselves. I questioned and critiqued their thinking processes when they were using incorrect strategies or making deductions and conclusions without the necessary justification. I encouraged and praised students who made valid contributions while persistently eliciting their thinking, making it clear that what they had to say was valued and that they were expected to explain their thinking fully. Thus students were

encouraged to consult and learn from one another on the strategies of how to approach and construct proofs, what to do when stuck, how to correctly interpret and apply definitions and other pertinent results and many other strategic skills. These skills are invaluable and cannot really be explicitly taught in a formal classroom setting. I will refer to the method just described as the consultative method from now on.

4.7 Concluding Summary

In this chapter I have described the methodology and methods used in the study. My ontological assumptions based in a social constructivist paradigm gave rise to my epistemological assumption which required me to collect subjective accounts, experiences, actions and utterances of students and the lecturer as they interacted in the consultative sessions while engaging in proof construction exercises. Accordingly the analysis of my data is interpretive; based on the inferences I made as a result of my observations of students' actions, utterances and contributions together with the constructs from my analytical frameworks.

My ontological assumptions implied that my interpretations could not be purely objective but subjective, shaped by my own experiences, background and theoretical perspective (Vygotsky's socio-cultural theory). Vygotsky's thesis that all higher forms of cognitive learning have their origins in social interaction and that language and speech are the main psychological tools which mediate this learning, is at the heart of my study. Vygotsky's notion of the functional use of the sign was extended to refer to students' use and application of newly met (mathematical) terms, symbols, logical reasoning processes, proof methods and justification before they have a complete understanding of these objects and processes. Case study methodology was used to gain a holistic picture of how lecturers could support students in becoming independent thinkers and empower those showing the potential to becoming more knowing others to gradually take on the role and the responsibilities of the lecturer in providing scaffolding to their peers.

This was done in the context of consultative group sessions, an intervention based on creating an environment where active consultation and collaboration of students with their peers and the lecturer was encouraged. Students' difficulties and challenges with

proof construction in the area of elementary set theory and the various forms of guidance offered are also investigated. The sessions were video recorded and transcribed so that detailed observation could be carried out after the sessions. Additional data was in the form of brief field notes made during the sessions, memos written at the completion of each session, and a reflective journal kept throughout the study. These recorded the pseudonyms of the students in each session, a chronicle of proof construction exercises attempted in each session, summaries of proof construction attempts and major contributions from peers, significant interactions, major difficulties and challenges encountered, the evolving understanding of students in general and more knowing peers in particular, the general mood of students as they expressed feelings such as happiness, enthusiasm or frustration and reflections and thoughts on the methodology, data collection, ethical issues and analysis of data.

The transcripts of the video recordings underwent an intensive process of correction and revision so that any incongruence of spoken and written actions and utterances could be pinpointed and the accuracy of the transcripts ensured. Once I was satisfied that the transcriptions were accurate I then did some preliminary coding according to analytical frameworks which were congruent with my theoretical framework, my research questions and the data I had collected. To address my research questions fully, I realized I needed to use two complementary analytical frameworks; one for the analysis of the utterances of the lecturer and students and the other for analysis of students' proof comprehension actions and contributions. The first framework allowed me to trace the patterns of scaffolding offered to the students by the lecturer as well as that of the students to their peers and so reveal the effective ways in which the students were supported to access their zones of proximal development. The second framework allowed me to pinpoint students' difficulties and challenges in proof construction in the area of elementary set theory and the scaffolding beneficial to them and traced the development of students' proof construction abilities.

I then went through a systematic selection process on the basis of my research questions to select the events that would be coded and analysed in detail. The transcripts were thoroughly examined for those events that best illustrated the challenges and difficulties students experienced with proof construction, forms of guidance from the lecturer and

peers and the ways in which the norms pertaining to the consultative sessions were established. I also searched for those events which showed students beginning to assume the role and responsibilities of the lecturer, becoming active agents for the development of their own and their peers' proof construction abilities by offering scaffolding to their peers when necessary. Five consecutive episodes were thus selected for detailed coding and analysis and these episodes form the basis of the findings of the study.

A discussion of the trustworthiness of the study including concerns about the validity, reliability and generalizability of the methodology, methods and analysis of the study is found in Chapter 9.

Chapter 5: Analytical frameworks and coding of video recorded transcripts

5.1 Introduction

As discussed in Chapter 1 the skills of being able to read and write proofs are among the outcomes expected of students majoring in pure mathematics courses. My study first focussed on students' difficulties with proof construction and comprehension in the area of elementary set theory and the forms of guidance they received from their peers and the lecturer as they engaged in proof construction exercises in the consultative sessions (Research question 1). The study was also concerned with how students' proof construction abilities developed and evolved in the context of consultative group sessions (Research question 2). Research question 3 is concerned with studying the nature of the interactions in the consultative groups which might contribute to the establishment of sociomathematical norms. I also attempted to identify characteristics and modes of reasoning observed in students who showed the potential to become more knowing others.

This chapter discusses the analytical frameworks used to code and analyse the transcripts of video recorded consultative sessions held with my small group of participants. A sample of coded transcript is included in this chapter while the complete record of the coded transcripts of five episodes which occurred in the first two consultative sessions is in Appendix 1. There were four three hour consultative group sessions altogether. These occurred at one week intervals at the same time that the section on elementary set theory was being taught in formal lectures in the pure mathematics course in the second semester of 2010. All four sessions were transcribed and coded in brief. After a thorough perusal of these transcripts, I chose to focus on the first two sessions. The reason was that apart from time and space constraints, the first two sessions best illustrated students' challenges and difficulties and how establishing an environment encouraging students' active participation and engagement was conducive to rapid progress in their proof construction abilities. This is discussed in detail in Section 4.5.3.

5.2 Analytical Frameworks

In Section 4.5.2.1 I briefly described the process which led to my choice of analytical frameworks. I will now elaborate on this process. Originally I started off by using just one analytical framework, the framework for the analysis of students' and lecturer's discourse as proposed by Blanton, Stylianou and David (2004, 2011). As I worked on the detailed coding and analysis of the selected transcripts with my research questions in mind, further categories and their corresponding indicators emerged from the data. These were primarily concerned with the analysis of students' proof construction actions and contributions such as making correct/ incorrect deductions, correction of mistakes or errors, references to definitions or explanation of definitions, giving narrative or pictorial examples, correct/ incorrect use of mathematical language and symbols, explanations of mathematical objects and the structure of the proof and providing justification for deductions. Working back and forth several times from analysis of my data to attempting to address my research questions, I decided that I needed to extend the original framework by adding the categories mentioned above but I soon realized that this would be cumbersome and not very elegant. I was reading up on emerging literature on proof construction in undergraduate mathematics at that time. I came across an assessment model developed by Meija-Ramos, Fuller, Weber, Rhoads and Samkoff (2012) which incorporated almost all the categories lacking in the first analytical framework. The fact that I had identified these categories before I came across this assessment model pointed to my personal alignment and agreement with the framework developed by Mejia-Ramos et al. (2012). I therefore adopted this model and adapted it for use as my analytical framework for proof comprehension and construction. I now had two complementary analytical frameworks; one concerned with the social aspect (analysing students' and lecturer's utterances and discourse during the consultative group discussions) and the other with the cognitive aspect of students' proof construction actions and contributions. The first framework allows the researcher a window into how scaffolding takes place in the zones of proximal development created while students are engaged in group discussions. I hoped this would enable me to gain an understanding of how students could be supported in accessing their zones of proximal development. It would help me to see how the norms relating to the consultative sessions were established, point to the ways in which students were

supported to become intellectually independent and how more knowing peers were empowered to develop their capabilities. The second framework analyses students' proof comprehension and construction actions and contributions and allows us to track how proof construction abilities of students evolved as they participated in proof construction exercises in the consultative sessions. This framework also enabled me to identify the characteristics of those students who went on to become more knowing peers.

5.2.1 Analysis of student and lecturer discourse

Blanton et al.'s analytic framework is consistent with Vygotsky's framework of socially constructed knowledge where speech is a psychological tool of development and the unit of analysis is the utterances of students and lecturer (Blanton et al, 2011, p.291). The zone of proximal development (ZPD) was defined by Vygotsky as the space where the possibility of learning beyond the learner's own abilities takes place with the assistance of more knowledgeable others: in this case their peers and the lecturer. Looking through the lens of the ZPD we would like to encourage the development of cognitive abilities which have not yet fully matured in the course of the interaction of the learner with more knowing others (Kozulin, 1998). Diaz, Neal and Ameya-Williams (1999) argue that as learning occurs through social interaction, the quality and suitability of this interaction is an important factor in the development of students' abilities. The analysis of the utterances of students and the lecturer in the consultative sessions using the categories developed and their indicators should help me address how students' proof construction and reasoning abilities are scaffolded by teacher and peer utterances.

Analytical framework for the analysis of teacher and student utterances

The framework was initially developed and is based on the work of Kruger (1993) and Goos, Galbraith and Renshaw (2003) who focussed on scaffolding taking place between peers. Kruger (1993) defined the transactive nature of reasoning observed in the dialogue used by the teacher and students as reasoning which was characterized by "clarification, elaboration, and justification of one's own or one's partner's reasoning" (Goos, Galbraith & Renshaw, 2003, p.199). Blanton et al. (2011) extended this by

developing categories and codes for the teacher's/ lecturer's utterances to take into account instructional scaffolding arising particularly from the teacher's discourse in the classroom. A detailed description of the framework and the categories and codes for analysis of these utterances can be found in Section 2.4.2.

A few other categories and indicators or actions emerged as I worked on the analysis (mostly on the part of the students) which I realized were important and pertinent to the study. These were: **Taking on the role of the more knowing other** and **the moment of realization.** In the former, instances where students are seen taking responsibility for their own and other's learning and generally taking on the transactive prompts and facilitative utterances of the teacher were tracked as these are indicative of students taking a lead in scaffolding other students' learning. The moments of realization tracked instances where students are seen to be grasping ideas with which they had previously been battling. My aim will be to try to pinpoint what exactly leads to this realization. These last two categories were added to the framework developed by Blanton et al. In addition the category of **transactive arguments** which includes utterances said in **the process of writing and doing proof construction and not necessarily in response to questions,** and the act of **reflecting on one's own thinking process and actions** (included under the transactive response category) and requests for **reflection** and **strategy** (included in the transactive questions category) also emerged from the analysis. The following shows the extended framework for discourse analysis with my own additions in bold.

Table 5.1: Categories and indicators of students' utterances

Category of students' utterances	**Indicators: Actions/utterances encompassed in the category**
Proposal of a new idea	Students bring to the discussion • a new idea or representation, • extend a new idea or elaborate on an existing idea towards a new direction.
Proposal of a new plan or strategy	Students suggest a course of action aimed at developing the proof or some aspect of the proof.
Contribution to or development of an idea	Students contribute towards furthering or adding to an existing idea.
Transactive question	Students request **reflection on proof construction actions**, clarification, elaboration, critique, justification, **strategy** or explanation of peer's utterances

Transactive response	Students directly or indirectly • clarify, elaborate, critique, justify, explain or **reflect on** their thinking, • **give an answer, agree**.
Transactive argument (usually uttered in the process of writing and doing proof construction and not in response to other's questions)	**Students** • **say what is being written,** • **explain their reasoning, explain mathematical objects, give justification,** • **describe the structure of the proof for example start, continuation/ conclusion of plan of proof.**
Taking on the role of more knowing other	**Students** • **take responsibility for their own and other's learning,** • **involve other students, questioning, pointing out errors and requesting justification, clarification and so on, confirm other students' ideas,** • **take on the transactive prompts and facilitative utterances of the teacher.**
Moment of realization	**Students make gains in their understanding in terms of use and interpretation of mathematical objects, definitions, proof methods and so on. These could take place either through their own or someone else's contributions. [Aha moment]**

With respect to the teacher's utterances, examination of the transcripts resulted in the following further categorizations: **requests for reflection and examples** (falling into the category of transactive prompts), **attempting to structure proof writing, highlight learning and misconceptions and provide encouragement** (falling into the facilitative utterances category) and **making reference to definitions and explanation of definitions and illustrating and clarifying mathematical objects using examples** (falling under the didactive utterances category). Table 5.2 shows the extended framework with my additions in bold.

Table 5.2: Categories and indicators of teacher's utterances

Category of teacher's utterances	**Indicators: Actions/utterances encompassed in the category**
Transactive prompts	The teacher requests **reflection**, critique, justification, clarification, elaboration, strategy, **examples.**
Facilitative	The teacher • re-voices or confirms student ideas, • attempts to structure the discussion and **proof**

	writing, • **highlights learning and misconceptions,** • **provides encouragement.**
Didactive	The teacher • offers ideas on the nature of mathematics, axioms and principles or historically developed ideas, • **makes references to definitions and explains definitions,** • **illustrates and clarifies mathematical objects using examples.**
Directive	The teacher provides immediate corrective feedback or information towards solving a problem

Once all utterances had been coded I searched for evidence of patterns of scaffolding by the lecturer and peers. In their study Blanton et al. (2011) did not find any specific patterns occurring in the discourse and they point out that this is not surprising as it is difficult if not impossible to make connections between new ideas or plans and previous specific utterances in complex discussions in the classroom. Since my study involved a small group of students (an average of 7 students taking part in each session), I hope that I have been better placed to observe the origins of thoughts and patterns of students' utterances as they developed solutions to proof exercises (ibid., p.303). Blanton et al. (ibid.) did find however that a characteristic of successful collaborative sessions on proof construction was a high incidence of transactive reasoning together with new ideas and elaborations. They proposed that the teacher's prompts encouraging discussion and drawing students to present new ideas or to provide clarification, justification, elaboration and so on, were the most crucial in developing students' proof construction abilities.

5.2.2 Analysis of students' proof construction and comprehension abilities

As discussed in Chapters 1 and 2, students introduced to formal proof construction in advanced mathematics courses need to produce arguments based on accepted axioms and definitions, to proceed using clear deductive logical reasoning and use standard mathematical notation and proof methods (Weber & Alcock, 2011, p.323). Researchers have compared learning to construct mathematical proofs to learning a new language (Mamona-Downs & Downs, 2002) or mastering a different genre of speech or writing.

Arriving at an understanding of a sound mathematical proof, being able to read proofs, and construct similar proofs by oneself is a huge challenge for most students meeting formal mathematical proof for the first time. It is however, a necessary part of undergraduate mathematics curricula in most parts of the world because it may be argued that proof is a crucial element and the most characteristic feature of mathematics (Solow, 1981).

As discussed in Section 2.2.1 when striving to come to grips with the proving process, students are challenged by a host of obstacles. Since the analytical framework I used for analysis of students' proof construction abilities had to encompass all of the requirements for proof construction, I urge the reader to refer to Section 2.2.1 and also to Table 2.1 which summarizes the difficulties students experience when faced with these requirements.

Analytical framework for analysis of proof construction attempts

In striving to develop an analytical framework that would take cognizance of all these factors and enable the researcher to achieve an understanding of individual student's proof construction and comprehension skills and to see how these developed during the weekly consultative group discussions, I adapted a model aimed at the assessment of students' proof comprehension skills. Mejia-Ramos et al (2012) have developed a comprehensive assessment model for assessing proof comprehension in advanced mathematics at an undergraduate level (cf. Section 2.3.2). Since the model's aim is assessment, I have adapted it to enable its use in the analysis of students' attempts at proof construction. I have used the model in combination with a grounded approach as well, allowing sub-categories to emerge as I worked with the data. I have also expanded the model by using the Vygotskian notion of the functional use of the sign and the theory of concept formation (Vygotsky,1986, 1994; Berger, 2004a, 2004b, 2004c, 2006) to interpret the students' evolving understanding of the meaning of terms, signs, symbols logical reasoning processes, proof methods and justification. The extended and expanded analytical framework used to analyse students' proof construction attempts will now be discussed.

The model considers two aspects of students' understanding of proof in advanced mathematics. The first aspect focusses on students' understanding of the local characteristics of the proof such as the meaning of specific terms and statements, the logical reasoning employed in connecting statements, and whether each statement and conclusion has been made with the necessary justification. These first three categories of the model are concerned with the *local comprehension/ construction* of a proof (Mejia-Ramos et al., 2012, p.7). The second aspect; *holistic comprehension/ construction* focusses on students' holistic understanding of the proof. This relates to notions such as the main ideas or methods behind the proof, or parts of the proof and the ability to transfer these ideas or methods to other proofs which are similar or presented in different contexts.

As discussed earlier, Vygotsky's theory of concept formation and the central role of the functional use of the word was extended and included in the local aspect of the framework. When investigating how students understand newly met mathematical objects and processes related to proof construction, I have used the Vygotskian notion of the functional use of an object or process (Vygotsky,1986, 1994; Berger, 2004a, 2006) which refers directly to my theoretical framework. This category tracks the student's progress by examining how the student uses an object or process prior to their complete understanding. We track this non-linear progression of the students' use of the object or process between heaps, complexes, pseudoconcepts and concepts. A summary of these stages as elaborated to the mathematical domain and their indicators follows. A more detailed description is found in Sections 3.2 and 3.2.1.

In the heap stage, the student associates the new object with a previously encountered object where there is a chance or circumstantial connection between the two. Thinking in the heap stage is characterised by associating the newly met mathematical object or term with previously encountered terms/objects based on non-mathematical criteria.

In complex thinking the links between objects are based on the actual attributes of these objects. Complex thinking is further categorised into various sub-categories. The most refined form of complex is the pseudoconcept, a special complex which enables students to make the transition from complexes to concepts (Berger, 2004c, p.14). The unique feature of the pseudoconcept is its dual nature: it appears to be a concept to the

observer whereas it is actually a complex because of the student's incomplete and contradictory knowledge of the object. The pseudoconcept enables students to engage fruitfully and use the object/process in their discussions and deliberations. This engagement hopefully guides them on the path of developing their use and application of the object/process to gradually correspond to the way that the object/process is used by the mathematical community (Berger, 2004c, p.15).

The (mathematical) concept is formed when its internal and external links are consistent and logical. Internal links refer to the links between the "different properties and attributes of the concept" while external links refer to links between the concept and other concepts (Berger 2004c, p.16).

The local and holistic aspects of my analytical framework for analysis of proof construction abilities and contributions will now be described in detail below.

The first category of the students' *local comprehension/ construction* of a proof is concerned with the **meaning of terms, symbols and signs** (L1) and measures students' understanding of key terms, symbols and definitions in the proof. Here we are concerned with students' use of new and unfamiliar terminology, signs and symbols and also students' knowledge of definitions, that is, students' ability to explain definitions in their own words and in more formal language. An example of a definition in set theory is the definition of subset, where a set A is defined to be a subset of a set B ($A \subseteq B$) if and only if for each element x in A, x is in B.

The next category in the assessment model is concerned with the **logical status of statements and proof framework** (L2)**.** I propose that this criterion is aimed at assessing the student's mastery of the methodologies and techniques described by Solow (1981) in Section 2.2.1. An example of a type of proof frequently occurring in set theory is 'P implies Q' or '$P \Rightarrow Q$' in which one assumes that the statement to the left of the word 'implies' (namely P) is true while the goal is to conclude that the statement to the right (namely Q) is true. That is, we show that Q is true as a logical result of P being true (Solow, 1981, p.5).

The last category in the local aspect of proof comprehension is: **justification of claims** (L3). This category explores whether the student is able to justify and provide reasons

for new assertions based on previous assumptions or statements. I have developed sub-categories and indicators of students' actions in each of these sub-categories and these are described in Table 5.3 below.

Table 5.3: Categories and indicators for analysis of the local comprehension/ construction aspect of proof

Category	Sub-category	Indicators
L1: Meaning of terms, symbols and signs Purpose: In this category we consider whether the student can identify the definition of key terms in the proof or specify what is meant by signs, symbols or terms that are met in the proof.	**L1a: Using newly met terms, symbols and signs**	Students correctly use newly met terms and symbols in the proof construction process (written or spoken). This is interpreted using the functional use of the object. Indicators are given in Table 5.5 below.
	L1b: Mathematical Definitions	Students • describe or explain the meaning of terms or symbols, • provide definitions of symbols or terms used in the proof using formal language or in their own words, • make reference to or call to mind definitions appropriate to the proof construction, • question the meaning of terms, symbols and signs.
	L1c: Illustrating mathematical objects and definitions with examples	Students illustrate a mathematical object or definition with examples.
L2: Logical status of statements and proof framework Purpose: In this category we consider whether the student is able to follow the logical reasoning behind the proof and is able to identify the logical relationship between the statement that is to be proved, the	**L2a: Selecting correct and appropriate statements and phrases**	Students identify and select correct/ appropriate statements or phrases which make sense and add to the logic of the proof construction process.
	L2b: Selecting useful and appropriate aspects of definitions, selecting appropriate assumptions	Students select useful or appropriate aspects of definitions, select appropriate assumptions (also known as strategic knowledge (cf. Weber (2001)),
	L2c: Proof methods	Students

assumptions made and the conclusions of the proof. This category is indicative of students' grasp or lack of the strategic knowledge that they need to successfully complete the proof construction process.		• clarify or identify the type of proof framework or method of proof, • follow the reasoning process and methodology of the proof or component of the proof, • seek clarification on the reasoning process and methodology of the proof.
L3: Justification of claims Purpose: In this category we consider whether students can provide justifications for making new assertions or deductions following from previous steps in the proof construction process.	**L3a: Making correct deductions from previous statements providing the necessary justification**	Students make correct assertions/deductions from previous statements and definitions, recognizing and providing the necessary justification, providing correct explanations when asked.
	L3b: Questioning deductions made without justification	Students question and clarify when assertions or deductions have been made without any basis.
	L3c: Identifying basis for conclusions	Students identify the basis for a claim, or identify the reasons why a conclusion can be made.

As discussed above, when investigating how students interpret and apply the newly met mathematical terms, symbols, signs, logical reasoning processes, proof methods and justification I would like to use the Vygotskian notion of the functional use of the object or process. The indicators for the various categories are outlined in Table 5.4 below. Note that in the table below the term 'object' includes mathematical terms, definitions, symbols and signs while the term 'process' includes logical and deductive reasoning processes, proof methods and the practice of justification.

Table 5.4: Categories and indicators for the functional use of objects and processes

Categories for functional use of object or process	Indicators
Heap level thinking	Students associate the newly met object/process with a more familiar object/process having a vague or chance connection and based on non-mathematical criteria.
Complex level thinking	Students associate the newly met object/process with:

	• an object/process which shares a similar attribute, • an object/process previously met in an example, • a more familiar object/process which reminds the student of the newly met object/process in some way, • a more familiar object having a similarity of templates. This last complex is signifier oriented as it specifically refers to the template of the newly met object.
Pseudoconcept level thinking	Students might be able to use or apply the newly met object/process correctly (thus giving the appearance of concept level knowledge) but reveal their incomplete or contradictory knowledge (revealing complex level knowledge) in earlier or later activities.
Concept level thinking	Students are able to: • correctly and logically explain or describe the properties and attributes of the newly met object/process, • correctly identify and appreciate differences in properties of the newly met object/process as distinguished from other newly met or more familiar objects/processes, • correctly and logically use or apply the object/process.

Table 5.5 contains the sub-categories and indicators I developed for analysis of the holistic aspect of proof construction based on the Meija-Ramos et al. proof comprehension assessment model.

Table 5.5: Categories and indicators for the holistic aspect of proof comprehension

Category	Sub-category	Indicators
H1: Main ideas behind the proof and the modular structure of the proof Purpose: In this category we would like to see whether the student grasps the main ideas and methods of the proof.	**H1a: Main ideas of the proof**	The student is able to describe or explain the main ideas or the over-arching approach used in the proof.
	H1b: Proof components	The student is able to break down the proof into components (where applicable).
	H1c: Purpose of each	The student is able to

	proof component	identify the purpose or the role of a module or particular part of the proof.
	H1d: logical relationship between proof components	The student is able to recognize the logical relation between two or more modules of the proof.
H2: Transferring general ideas or methods to another context Purpose: In this category we would like to see whether the student can apply his/her understanding of the proof as a whole to other proofs and other contexts.	**H2a: Transferring ideas and methods to other proofs and contexts**	The student is able to use ideas and methods that he/she grappled with in previous proof construction exercises in subsequent exercises.
	H2b: Appreciating scope of methods	The student appreciates the scope of methods encountered by recognizing the assumptions which need to be in place to allow the method to be used.
H3: Illustrating with examples Purpose: In this category we would like to see whether the student uses examples to improve his understanding of the proof and statements within the proof.	**H3a: Illustrating proof construction steps with examples**	The student illustrates sequences of inferences with examples, and uses examples to better understand the statements and inferences made.
	H3b: Illustrating with diagrams	The student interprets statements in the proof or the proof itself with the help of diagrams.

In all the categories encompassed in both the local and holistic aspects of proof comprehension and construction, the student's ability to do so will be indicated by the code while the students' **inability** to perform these actions will be indicated by an **x** attached to the code. For example L1ax will indicate a student's incorrect use of newly met terms and symbols in the proof construction process (written or spoken). L2ax will indicate a student's incorrect or inappropriate selection of statements or phrases which do not make sense and therefore do not add to the logic of the proof construction process. L2bx will indicate a student's selection of non-useful/ inappropriate or trivial aspects of definitions. L3ax will indicate when a student makes incorrect deductions from previous statements or definitions, and so on. In this way students' incorrect actions and contributions can also be coded.

5.3 Coding of the video recorded transcripts

The proofs attempted in the four sessions were predominantly proofs involving implication, double implication (or equivalence), equality and showing that one set is a subset of another set. These proofs were done in the context of elementary set theory and relations and the areas covered included sets, Cartesian products of sets, power sets and equivalence classes. The proofs that I chose to focus on and analyse involved all the proof methodologies of implication, equivalence, equality and subsets, first in the area of sets covering the newly met concepts of intersection, union and subsets (in session 1), then going on to the Cartesian product of sets and power sets (in session 2). As discussed (in Sections 5.2 and 4.5.3) I chose to focus on the first two sessions of the four weekly consultative sessions. The reason was that my purpose was to identify students' challenges in this particular area of proof construction and show how establishing an environment encouraging students' active participation and engagement led to students' accelerated progress in their proof construction abilities. This was best illustrated in the first two sessions. The third and fourth sessions saw a continuation of the habits established during the first two sessions.

The weekly sessions were held in a small room in the mathematics department equipped with a white board. Although every precaution was taken to minimise noise and disturbances during video recording, there were times when students' voices were inaudible. There were also times when one could not see exactly which student was making the contribution, for example when the video camera was focussed on the person working on the board. In these instances the contribution is attributed to 'S' to stand for any of the participating students. Pseudonyms were used throughout the transcripts to ensure confidentiality and anonymity.

The videos were transcribed by a professional transcriber. I then went through the transcriptions viewing the videos several times to check these transcriptions and ensure their accuracy. A more detailed discussion of the process of transcription is presented in Section 4.5.2 while a description of how the transcripts of video recorded events were selected is included in Section 4.5.3. A discussion of the consultative group method appears in Section 4.6.

During these consultative sessions, after a very brief revision of definitions that were covered in the class, a proposition or theorem would be put on the board and students would volunteer to come up and attempt the proof construction while receiving help and guidance from all the participants including the lecturer.

The proof of each theorem constitutes a separate episode. The proof transcripts have been divided into sub-episodes according to the following criteria:

- A sub-episode contains a complete proof or component of proof (where applicable) attempted by the student at the board.
- Digressions which were omitted from the proof construction were instances where students' attention was diverted to something non-mathematical and completely irrelevant for example asking the person who is doing the proof construction at the board to move or write more clearly. Discussions which did not concern the actual proof construction being attempted but were still concerned with the general mathematical agenda have been allocated to separate sub-episodes which were still part of the overall analysis. This was because I wanted to be able to focus on the actual proof construction as a separate entity. Once these discussions ended and proof construction resumed, new sub-episodes were begun.
- When the discussion focussed on different themes such as a particular misconception or a more in-depth look at a different mathematical object, these different notions were also isolated, so that each notion or misconception could be discussed in its own sub-episode before going on to the next mathematical object and the next sub-episode.

An example of the coded transcript of sub-episode 2.1 has been included below. The full record of coded transcripts of the five episodes is in Appendix 1.

5.3.1 An example of coded transcript

An example of coded transcript of sub-episode 2.1 which took place in the first consultative group session follows. Maria makes a first attempt at the proof of the proposition '$A \subseteq B \Leftrightarrow A \cap B = A$'.

Session 1: sub-episode 2.1

In episode 2 Maria attempted the proof construction of the following theorem: $A \subseteq B \Leftrightarrow A \cap B = A$. This proof construction encompasses the method of proof of an implication, the method of proof of equality of sets and the method of proof of showing that one set is a subset of another. A successful proof construction also requires knowledge of the precise definitions of set equality, subset and intersection and the ability to use these definitions in the logical reasoning and justification of each step in the proof.

Sub-episode 2.1: Maria's first attempt at proof of *a) $\Rightarrow$ b)* or $A \subseteq B \Rightarrow A \cap B = A$

	Speech and actions	**Student and teacher utterances**	**Proof comprehension**	**Interpretation according to Theoretical Framework (T.F.) and general comments on the proof construction process**
	This theorem is put up on the board to be proved: *Theorem If A, B and C are sets, the following are equivalent.* *a) $A \subseteq B$* *b) $A \cap B = A$* c) $A \cup B = B$			
1	**Maria**: *[goes to the front and attempts the proof starting with (a) $\Rightarrow$ (b)]* I think you have to show that (a) is equal, implies (b) and (b) implies (c). And this would mean that (a) implies (c) *[writes: (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c) (a) $\Rightarrow$ (c)]*	*Proposal of new idea* *Transactive argument-reasoning and explaining while writing*	***H1ax**: the main approach to be used is explained with some flaws* ***H1bx**: breaking down the proof into components: implication is used instead of double implication* ***L1a**: correctly uses mathematical terms/ symbols/ signs*	*Recognizes that (a) $\Rightarrow$ (b) translates to $A \subseteq B \Rightarrow A \cap B = A$* *She interchanges the term "equal" with the term "implies" showing that the two are associated together. This*

				might indicate complex thinking.
	So you start by saying to show that, Ok, *A* is a subset of *B* implies that *A* is a… I forgot that name, what is it? *[writes:* $A \subseteq B \Rightarrow A \cap B = A$ *]*		***L2a****: selects correct opening statement for starting the proof showing apparent knowledge of what needs to be done* ***L1a****: correct use of mathematical symbols/signs (written)*	
2	**Student**: intersection	*Contribution to an idea*		
3	**Maria**: *A* intersection *B* which is equal to *A*. So from this if *A* is a subset of *B*	*Transactive argument-reasoning and explaining while writing*	***L2ax****: selecting incorrect statement to start the proof showing* ***L2cx:*** *lack of logical approach in the method of the proof*	*Adopting a method of proof which involves showing that the two sides of the implication are equivalent. This might indicate complex thinking of the proof method for proving an implication since Maria associates this method with the more familiar method of proving an identity.* *The need for justification of each statement is not well grasped.*
	this means that, mmm, *x* is an element of *A*, which implies that *x* is also an element of *B*. And… *[writes: If* $A \subseteq B \quad x \in A \Rightarrow x \in B$ *]*		***L2bx****: selects non-useful or trivial deductions from previous statements (spoken)* ***L1a****: the statement and non-useful deduction correctly written using mathematical symbols/signs*	
	Then we come to this side. That if *A* is an intersection of *B* which is equals to *A*		***L2ax****:selects incorrect statement to continue the proof showing* ***L2cx:*** *lack of logical approach in the method of the proof;*	
	it will mean that *A* is a		***L3ax****: makes an*	

	subset of B. And this would mean that x is an element of A.		*incorrect deduction from previous statement (spoken)*	
	If and if x is an element of A it implies that it is also an element of B. *[writes on the other side of the board so it looks like this:* *If* $A \subseteq B$ *if* $A \cap B = A$ $x \in A$ $A \subseteq B$ $\Rightarrow x \in B$ $\Rightarrow x \in A$ $\Rightarrow x \in B$ *]*		***L2bx**: selects non-useful or trivial deductions from previous statements (spoken)* ***L1a**: the statement and non-useful deduction correctly written using mathematical symbols/signs*	
	Then I've proven this one and I come to the (b). Again let A and the intersection of B which equals to A and would imply that A is a union of B which is equals to B *[writes: let* $A \cap B = A \Rightarrow A \cup B = B$ *]* Then again if…		***L3cx**: Proof is concluded without any basis* ***L2cx**: Incorrect proof method and logic process*	

5.4 Concluding Summary

In this chapter I have provided the motivation and described the analytical frameworks I have used to address the research questions contained in my study. I have described how engagement with the analysis of the transcripts while considering how to address my research questions, led me to realize that I needed two different types of analysis to address my research questions fully; one that would consider the social aspect and the other considering the cognitive aspect. My first research question addressed the difficulties that students have with proof construction in the area of elementary set theory and the forms of guidance they received. My second research question focused on the developing proof construction abilities of students while the third research question studied the interactions of the lecturer and students to explore in what ways

students were empowered to become intellectually autonomous and how students showing potential to become more knowing others were encouraged to develop their capacity.

To identify students' difficulties and the types of scaffolding found to be effective, I needed a framework which would allow analysis of students' proof construction attempts in terms of the following categories: students' use of correct/incorrect mathematical terms, symbols and signs, students' ability/ inability to use logical reasoning and proof methods and students' ability/ inability to provide justification for deductions and conclusions. The framework I have developed is based on a comprehensive assessment model for proof construction at the undergraduate level developed by Meija-Ramos et al. (2012). I have adapted this model for use in the analysis of students' attempts at proof construction. This framework allowed me to track students' proof construction and comprehension abilities as they progressed through the consultative sessions. It is hoped that this analysis will shed some light on whether the consultative sessions are effective in promoting improvement of students' proof construction abilities.

The framework used to study interactions of the lecturer and students is based on research done by Blanton et al. (2004, 2011) and provided the analytical tool that I needed to analyse students' and lecturer's discourse during consultative group sessions. Using this framework I searched for emerging patterns of scaffolding by the lecturer and students as well as patterns showing how the sociomathematical norms relevant to successful proof construction such as critique, justification and verification of their own and their peers' reasoning processes were established and were gradually adopted by the participants. I also attempted to find the primary factors which might lead students to become intellectually autonomous and empower those showing the potential in becoming more knowing peers, thus enabling them to take responsibility for their own and others' learning.

I expected that the use of these two analytic instruments would provide me a holistic view of students' difficulties and challenges and how students could be enabled to develop their proof construction abilities more effectively.

It must be noted that in developing both of these frameworks based on research done by the researchers mentioned, I have used a grounded approach, allowing additional categories, sub-categories and indicators to emerge from the data.

Chapter 6: Analysis and discussion of students' difficulties and forms of guidance offered

6.1 Introduction

In this chapter I attempt to analyse and present a discussion on themes related to the difficulties students have with proof construction in the area of elementary set theory and the forms of guidance offered in the consultative group sessions. These themes emerged from the coding and preliminary analysis of the transcripts of the video recorded sessions in Appendix 1. This chapter attempts to provide a comprehensive discussion of findings related to my first research question which is set out below for ease of reference.

Research Question 1

Investigating students' difficulties in proof construction and the forms of guidance offered in the context of consultative group sessions:

a) What are the challenges and difficulties students experience as they engage with proof construction in the area of elementary set theory?

b) What forms of guidance do lecturer and students offer?

The analysis in this chapter together with the coded transcripts in Appendix 1 will also be used to address research questions 2 and 3 in Chapters 7 and 8.

As discussed in Section 4.5.3 after close and repeated scrutiny of the complete transcripts, I decided to focus on the first two consultative sessions as these were the most fruitful in terms of significant occurrences related to the research questions. The majority of the challenges and difficulties were exposed and discussed in the first session. This session was also the primary arena where the norms pertaining to the consultative sessions were set up. The second session witnessed a great improvement in general of students' proof construction abilities. This session also revealed how several more knowing peers assumed the role and responsibility of guiding and offering scaffolding to their peers while adopting the norms established in the first session. The

five proof construction exercises (each is regarded as a different Episode) which were analysed in detail are shown in Table 6.1 below.

Table 6.1: Proof construction exercises analysed in the first two consultative sessions

Proof construction exercise	**Student who attempted the majority of the proof**	**Total time taken from start to completion of proof**
Session 1		
Episode 1: ***If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$***	Frank	23mins
Episode 2: ***If A, B and C are sets, the following are equivalent:*** ***d) $A \subseteq B$*** e) $A \cap B = A$	Maria	1hour 3mins
Session 2		
Episode 3: $(A \cup B) \times C = (A \times C) \cup (B \times C)$	Edgar	22mins
Episode 4: $(A \cap B) \times C = (A \times C) \cap (B \times C)$	Maria	13mins
Episode 5: $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$**.**	Frank	19mins

It must be noted that throughout this chapter the language used by participants of the study (whose first language was not English) has not been altered in any way and is an exact reflection of these students' speech. Pseudonyms were used to refer to the participants of the study to ensure confidentiality and anonymity. The lecturer is referred to by the letter 'T' in the transcripts, and whenever it is not clear which of the participants is making a contribution, he/she is referred to by the letter 'S' standing for student.

6.2 Challenges and difficulties students face and the forms of guidance offered

Difficulties that researchers such as Solow (1981), Moore (1994), Dreyfus (1999) and Weber (2001) have identified in the area of proof construction were discussed in Section 2.2.1. These were primarily the specific mathematical language used in proof construction, lack of logical reasoning abilities and lack of the knowledge of proof methods and insufficient appreciation of the need for justification of each deduction or conclusion made during the course of proof construction. I have reported on illustrative examples of students' difficulties and challenges under the various categories of my

analytical framework discussed in Section 5.2.2. Exemplars of difficulties in each of the categories are presented in boxes. Following each example and whenever possible, significant contributions by the lecturer and students directed at guiding and developing students' understanding are cited. I was not able to do this in some instances where guidance was not explicit, even where it was evident that these students had made gains in the development of their proof construction abilities. I argue that gains in understanding (throughout participation in the consultative sessions) have been made as a result of the students' functional use of mathematical terms, definitions, symbols, signs, logical reasoning processes, proof methods and justification while interacting with their peers and the lecturer. I argue that it is this functional use together with their interaction with the lecturer and their peers which enabled the student to make the transition to usage of the mathematical objects (including terms, definitions, symbols and signs) and processes (including logical reasoning processes, proof methods and justification) more aligned with their usage by the mathematical community.

It will be noted that there are instances where the same example appears and is discussed under several categories. The reason for this is that often while engaging in a particular discussion (while attempting to solve proof construction exercises), students experienced a whole range of difficulties. In order to make the analysis more systematic, instead of reporting on the difficulties related to a particular discussion all at once, I have reported on each of the difficulties under different categories separately. I have attempted to present all examples which emerged in the various relevant categories while analysing the transcripts.

6.2.1 L1: Meaning of mathematical terms, symbols and signs

The category L1 is concerned with the meaning of mathematical terms, symbols and signs. This category focusses on the students' use of new and unfamiliar terminology, symbols and signs (L1a) and also students' knowledge of definitions (L1b). This category also includes examining how students illustrated mathematical objects such as mathematical terms, symbols and definitions with examples (L1c). Under the category L1, the Vygotskian notion (discussed in Chapter 3) of the functional use of mathematical terms, definitions, symbols and signs, is incorporated. Students' thinking processes, inferred from their usage of signs, words and symbols, are broadly tracked

using Vygotsky's stages of concept formation: heap, complex, pseudoconcept and concept.

The difficulties students experience in understanding the correct meaning of mathematical terms, definitions, symbols and signs were most notably evident in the first three episodes. Most of the difficulties were centred on the use and interpretation of the symbols and definitions of the implication sign, the double implication sign, subsets, equality of sets, the union and Cartesian product.

6.2.1.1: L1a: Using newly met terms, symbols and signs

When students are introduced for the first time to formal proof at first year university, one of the difficulties they experience concerns the usage of the particular mathematical language or discourse as well as newly met mathematical symbols and signs. As stated earlier, students at the University of Limpopo have to overcome the additional hurdle of English being the language of teaching and learning. This is not the first language of the majority of the students.

Using newly met terms, symbols and signs: Incorrect language use

Use of the word approximate when referring to the implication sign

Frank's use of the word 'approximate' when referring to the double implication sign was indicative of incorrect language use. Below is his proof construction (line 1, sub-episode 1.1) and his explanation:

> **[1] Frank**: I can show you the proof, the steps we can take to solve this proposition. So the first step is to let x be an element of A. *[writes: let* $x \in A$*]* The first step that we must take it to let x be an element of A. So we approximate since x is an element of A, then x is an element of B. Then since here A is a subset of B. *[writes:* $\Leftrightarrow x \in B$ *(since* $A \subseteq B$*)].* Since x is an element of B then we can approximate that x is an element of C since B is a subset of C. *[writes:* $\Leftrightarrow x \in C$ *(since* $B \subseteq C$*)].*

In sub-episode 1.3 Gary (line 11) questioned Frank for clarification on the double implication sign. Frank's response in line 14 confirmed that he might be referring to the double implication symbol as 'approximate'. He could also be associating the word 'approximate' with the actions of deducing or implying.

[11] Gary: Can you explain about you're, you're saying the double implication signs?

[12] Frank: OK, this one?

[13] Gary: Ja

[14] Frank: This is for approximately. If you, we approximate that x an element of B *[points to* $\Leftrightarrow x \in B$ *(since* $A \subseteq B$*) on the board]* it can be implied, like you are implying that this x is an element of B since x is an element of A, this element can be in B *[points to the board]* since A is a subset of B, you see? Ja, you know what I'm saying? I suppose. You agree with me the way I did it?

I refrained from categorizing this as heap or complex level thinking as Frank (a second language English speaker) was probably referring to the double implication symbol or the actions of deducing or implying as approximation because he was not familiar with the correct terminology. Presumably as a result of his interaction with his peers while making functional use of the implication symbol, he gradually made the transition (see line 14) to refer to the implication symbol as 'implying'.

Using newly met terms, symbols and signs: understanding of mathematical terminology often taken for granted by lecturers

Use of the word 'suppose' and the implication symbol

Words used in the proof construction process often taken for granted by lecturers might not be fully understood by students. This could be exacerbated by the fact that English is not their first language. An example of this was found in sub-episode 1.2, line 4, when at Edgar's suggestion to start the proof construction attempt with a statement which would add to the logic of the proof construction process, Frank added a statement to the proof construction, containing the word 'suppose'. Frank explained as he wrote on the board:

[4] Frank: Okay you want me to write suppose A is a subset of B and B is a subset of C implies that A is a subset of C.

[writes as he is speaking directly above his proof attempt: Suppose $A \subseteq B$ *and* $B \subseteq C \Rightarrow A \subseteq C$*]*

Frank's use of the word 'suppose' and the implication symbol in mathematics discourse was inappropriate and he did not seem to be aware that there was a contradiction in the mathematical statement he had written on the board.

Guidance offered: More capable peers offer clarification and explanation

Frank was guided by his more capable peers who offered contributions from the categories: *L1a: clarifying the use of newly met terms, L2c: clarifying logical proof framework.* In line 5 (sub-episode 1.2) Gary clarified that the statement on the right of the implication sign was one that "we are supposed to prove" and hence should not be included in the supposition:

> **[5] Gary**: The first thing when you said 'Suppose that *A* is a subset of *B*, right? And *B*'s a subset of *C*, so we don't have to say 'it implies' that. OK, we're thinking that if we're saying *A* is a subset of *B* and *B* is a subset of *C* it implies that we are supposed to prove that *A* is a subset of *C* so we don't have to say we suppose that it implies that.

Helen in line 7 (sub-episode 1.2) also clarified that the statement on the right of the implication sign was one that "we need to show":

> **[7] Helen**: But also be like, no, for the fact that we're saying that we need to show that *A* is a subset of *C* we don't, you don't have to say 'it implies, implies…'

Using newly met terms, symbols and signs: use of mathematical terminology often taken for granted by lecturers

Use of the word 'assume'

Another example which clearly showed that students often do not understand and correctly use words peculiar to formal proof construction is that of Maria in sub-episode 2.4, line 51, when she answered the question posed by the lecturer: "What do you assume?" Below is the transcript including lines 48 to 51.

> **[48] T**: So if a) is true then b) is true. That's what you're trying to prove, right? If a) is true, then b) is true. So you start off with assuming something. What is what you start off with? What do you assume?
>
> **[49] Maria**: I assume that…
>
> **[50] T**: Don't rub everything out. Let's leave it. What do you assume?
>
> **[51] Maria**: I assume that (a) implies (b) and (b) implies (c) and I want to show that (a) implies (c).

Maria's erroneous response clearly showed that the use of the word 'assume' was not correctly interpreted as instead of stating the assumption, she described the plan of action that she had previously discussed (in sub-episode 2.1).

Using newly met terms, symbols and signs: Association of new or unfamiliar mathematical terms, symbols and signs with the more familiar (complex level thinking)

The double implication associated with implication

Many students did not differentiate between the implication and the double implication and seemed to regard them as the same entity. This seemed to indicate complex thinking, where newly met terms, symbols and signs are grouped together because they look similar. For example when Maria questioned the difference between the implication and double implication, Frank (in line 32 of sub-episode 1.3) replied that there was no difference. The transcript from lines 31 to 33 is included below:

> **[31] Maria**: Ja, what's the difference between?…
> *[referring to the implication and double implication]*
> **[32] Frank**: Oh there's no difference.
> **[33] Maria**: There's no difference?

Similarly when asked by the lecturer what P implies Q means, Edgar (in line 81 of sub-episode 1.3) replied:

> **[81] Edgar**: I think that in this case if we say that if P implies Q that means… after proving that it's true that P implies Q, we need to also prove the opposite side and the opposite way of Q being, implying to P.

Edgar was clearly describing the double implication in response to the question asked about the implication indicating that the two were regarded as identical.

The double implication associated with the notion of equality

Many students associated the double implication with the more familiar notion of equality, or to an equation (lines 50 and 66 of sub-episode 1.3). Maria in line 66 described her thinking of the implication sign:

> **[66] Maria**: It means that… like if you are proving something which is, like you've got an equal sign like this side is equal to this, so if you put that double implication it means that what you are proving on the left you are sure that is equal to what you are proving on the right. Ja.

Guidance offered: More capable peers exhibiting complex, pseudoconceptual and concept level knowledge offer clarification and explanation

As the discussion continued and the notions of the implication and double implication were further discussed, several students offered their contributions. Although these might have been at complex or pseudoconceptual level, they seemed to play an important role in deepening students' understanding. Some such as Gary also offered contributions that seemed to reveal concept level thinking. Gary offered an example to clarify and distinguish the difference between the implication and the double implication.

a) The double implication is associated with arrows

In lines 34 to 42 (sub-episode 1.3) Edgar and Helen explained their thinking on the difference between the implication and the double implication. They associated the double implication with a double arrow, and gave the method of proof as proving one side and then the other, going forward and back.

> **[34] Edgar**: OK, let me actually now try to explain *(pointing to the board)*. Actually you see this one which shows an arrow going to that forward one, that one, if you use that one you are going to make sure that you prove this side, you prove that one there.
> **[35] Helen**: Yes and then the other…
> **[36] Edgar**: And then you are going to prove again on the other side.
> **[37] Helen**: Yes
> **[38] Edgar**: So if you are using the double one
> **[39] Helen**: That means you have already shown…
> **[40] Edgar**: Yes, if you are using the double one with arrows you know, that one is like what applies on one side will also apply on the other side.
> **[41] Frank**: Ok
> **[42] Edgar**: So this one is a shortcut but as our lecturer has said, actually the best way is to use the longest method, because the other one you can explain more to, make you to understand.

The newly met terms, symbols and signs appeared to remind the students of more familiar symbols and signs and were thus associated with these. The implication sign was associated with a single arrow and the double implication sign with the double arrow. However the method of proof described seemed to be correct. In these instances students were interpreting the implication and double implication sign correctly even

though their description and explanation of these terms was flawed. This could indicate pseudoconceptual thinking. The double implication was also described as a shortcut when doing proof construction (line 42, sub-episode 1.3). Edgar and Helen's contributions were from the categories: *L1b: describing mathematical terms in own words.*

b) The double implication is associated with an equation

In line 70 of sub-episode 1.3 Gary associated the double implication with an equation having a left and right hand side. He interpreted the method of proof of the double implication as using the right hand side to go to the left hand side and vice versa. This seems to be indicative of pseudo-conceptual thinking on the proof method of an implication as in an implication proof, the statement that appears on the left of the implication sign is assumed and then one moves towards proving the statement on the right of the implication sign. In a double implication proof one would have to do both. Thus Gary's interpretation and description of the method of proof of an implication appeared to be correct. Gary's contributions were from the category: *L1b: describing mathematical terms in own words.*

> **[70] Gary**: Uh a double implication sign it simply means let's say if on the left hand side you have an equation, it means you can use the right hand side to go, to go back to the right hand side, to the left hand side and the other way round, you must leave the right hand side. That's how we do it.

c) Illustrating mathematical objects with examples

As seen above Gary associated the double implication with an equation. Following this, in lines 78 and 79, Gary corrected the proof attempt on the board and applied the double implication correctly. He ably explained why the double implication used by Frank should be replaced by an implication symbol and clarified the difference between the implication and double implication signs by giving an example. This seemed to be indicative of concept level thinking:

> **[78] Gary**: *[erases the* $\Leftrightarrow$*]* I'll start by removing the double implication sign because if let's say we say let x be an element in B *[writes: let* $x \in B$ *]* we are talking about if A is a subset of B *[points to* $A \subseteq B$ *]* and B is a subset of A, *[writes* $B \subseteq A$ *]* then we'll say if x is in B it means that we will have x in A, right?

S: Yes

[79] Gary: If we are given that B is a subset of A. But in this case we are not given that B is a subset of A *[points to: $B \subseteq A$]* so we cannot use the double implication sign here *[writes $\Leftrightarrow$ next to $x \in A$]* because in this case we'll say x is an element of B. And if we use this sign *[points to $\Leftrightarrow$]* it means that we will find that x is an element of A *[points to $x \in A$]* But we are not given this statement that B is a subset of A *[points to $B \subseteq A$]* That's why I undo that double implication sign *[points to $\Leftrightarrow$]* I simply use the single *[changes $\Leftrightarrow$ to $\Rightarrow$]*.

Gary's interpretation and application of the implication sign was correct and seems to indicate concept level understanding. Gary's contributions were from the category: *L1c: illustrating mathematical objects with examples.*

Summaries of difficulties and guidance in category L1a

Summary of difficulties experienced by students

Difficulties observed in this category included:

- **Incorrect language use**

  Incorrect language use was observed when for example the term 'approximate' was used to refer to the double implication sign and possibly to the actions of implying or deducing.
- **Inappropriate use of terms and symbols**

  There were many instances of inappropriate use of terms and symbols, the knowledge of which is often taken for granted by lecturers such as the words 'suppose' and 'assume' and the implication and double implication symbols.
- **Association of newly met terms, symbols and signs with more familiar terms, symbols and signs**

  Students associated newly met terms, symbols and signs with more familiar terms, symbols and signs. For example the notion of the double implication was associated with the notion of equality. This was probably based on the similar appearance of the two symbols which would indicate complex level reasoning.

Summary of guidance offered

Forms of guidance included:

- **Functional use of the terms, symbols and signs while interacting with peers and the lecturer in the consultative group sessions**

  Functional use of the terms, symbols and signs while interacting with peers and the lecturer in the consultative group sessions gradually brought students' use of these terms, symbols and signs closer to concept level.
- **More capable peers offer explanations of mathematical terms, symbols and signs using simple every-day language**

  An example of this was when Gary and Helen clarified that the statement on the right of the implication sign was to be proved or shown.
- **Complex, pseudoconcept or concept level contributions help to clarify heap or complex level use and interpretation of mathematical objects**

  Students offered complex, pseudoconcept or concept level contributions that helped to clarify heap or complex level use and interpretation of mathematical objects. An example was the association of the double implication symbol with a double headed arrow and the single implication symbol with a single headed arrow or the association of the double implication with an equation having a left and right hand side. The method of proof emerging from this was described as using one side to prove the other. Similarly the double implication symbol was associated with the notion of an equation and interpreted as using the right hand side to go to the left hand side and vice versa. Together with students' functional use of the notions of the implication and double implication as they interacted in the consultative sessions, these contributions seemed to play an important role in deepening students' understanding of these terms.
- **Illustrating mathematical objects with examples**

  In order to clarify the difference between the implication and the double implication, a more knowing peer offered an example.

6.2.1.2: L1b: Mathematical Definitions

In the course of the analysis of the five episodes, it became increasingly clear that definitions often hindered rather than supported students' proof construction attempts because of the incomplete and contradictory knowledge (indicating complex level or pseudoconceptual thinking) that students have of the mathematical terms, signs and symbols and the definitions of mathematical objects involved in the proof construction.

Knapp (2006) suggests that in order for students to be able to meaningfully use a definition to prove a statement, three skills are necessary. First they need to know the definition, that is, they should be able to give the definition in their own words and give examples and non-examples. Second they should be able to choose the appropriate definition and be able to identify the aspects in the definition which are useful in the proving process. Third they should know how to use the definition in the proof construction process.

Definitions also play a crucial role in providing the structural framework of a proof. Moore (1994) emphasised that the correct interpretation of a definition reveals the logical structure of a proof and gives students an intimation of the sequence of steps required in the proof (cf. Section 2.2.1). This was confirmed in my study as I observed students arriving at an incorrect method of proof for an implication as a result of their incorrect interpretation of the definition of the notion of the implication in Episode 2.

Analysis and discussion of illustrative examples in this category is given below.

Mathematical definitions: Instances where definitions become stumbling blocks

The union of two sets

In sub-episode 2.6 during the course of proof construction, a question about the difference between the notions of the union and intersection sparked an interesting discussion where it became evident that even though these apparently simple mathematical objects were covered at the beginning of the course, most students did not have a complete understanding of them and there were in fact quite a few misconceptions which could be attributed to the students' incomplete understanding or

misinterpretation of the definitions. In line 99, episode 2.6, Gary made the following statement:

> **[99] Gary**: OK, ah, can I say something about the union and intersection. At the union it's either let's say for example x, let's say *x* is in *A*, right? If we say union it's either in *A* or in *B*, it cannot be in both *A* and *B*. But then if you say intersection it means *A* and *B*, all of them, they contain *x*.

At the lecturer's request (*transactive request for examples*), Gary (in line 101) went to the board to illustrate his interpretation using a Venn diagram. The depiction of the union of two sets clearly showed what seemed to be a commonly held misconception: that the union does not contain elements in the intersection of the two sets. I suggest that this misunderstanding could have been brought about as a result of students' misinterpretation of the definition of the union of two sets: $A \cup B = \{x: x \in A \text{ or } x \in B\}$. Students might be getting confused and think that *x* may be in *A* or in *B* but not in both (exclusive 'or' versus inclusive 'or'). This is an example of how the definition of a mathematical object, instead of shedding light and clarity on the object introduces misunderstandings in the students' thinking.

Guidance: Using examples to illustrate the notion of the union to arrive at the correct interpretation of the definition

While trying to clarify and reach an understanding of the definitions of the union and intersection, there was a widespread use of examples. This was initiated by the lecturer in line 100 (*transactive request for an example*), and really helped to bring to light many of the students' misconceptions.

Edgar (lines 117 and 119) then made a positive contribution by doing another example which showed the intersection and union of two sets correctly. This example and his correct use of the notion of union was confirmed and highlighted by the lecturer (using a *facilitative utterance*). Edgar's contributions were from the category: *L1c: illustrating the notions of union and intersection using examples*.

Mathematical definitions: Associating mathematical objects with a word contained in their definitions

Association of the Cartesian product with the notion of the intersection

In sub-episode 3.1 Edgar exhibited complex level thinking when he associated the Cartesian product with the notion of intersection and I suggest that this was because both definitions contain the word 'and'. I have included Edgar's incorrect deduction made in line 3 (below) as he attempted to do the proof of: $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$.

> **[3] Edgar**: OK, that's what I need to show. So to prove that, um... Firstly we let these Cartesian points; x and y be an element of A union B brackets…*[writes: let* $(x, y) \in (A \cup B) \times C$ *]* This will imply that x and y are both elements of A union B and (x, y) an element of, and both of them are an element of C. *[writes:* $\Rightarrow (x, y) \in (A \cup B)$ *and* $(x, y) \in C$ *]* Yes?

Guidance: More knowledgeable peers pinpoint the cause of the misconception, encourage reflection through transactive prompts, make reference to the definition and explain it in their own words

In sub-episode 3.1 Gary (in lines 24, 26, 28, 30, 32 and 34) and Joseph (in lines 39, 41, 43, 45 and 47) acted as more knowing peers and pinpointed the cause of the misconception. They referred to the definition of the Cartesian product reminding the other students of the importance of the definition and helping them to have a better understanding of what it meant. In this episode which occurred in the second weekly session, a marked change was observed in the way that more knowing peers assumed the transactive prompts and utterances of the lecturer as their own and took over the role of scaffolding and guiding their peers through the proof construction exercises. Referring to the definitions of mathematical objects and clarifying and explaining these seemed to have become one of the habits established in these students. Since the relevant transcript and further discussion is included in Section 8.2.3 I will not repeat it here.

Mathematical definitions: Associating mathematical objects with a word contained in their definition

Association of the notion of the Cartesian product with the notion of intersection

In sub-episode 4.2 Christine asked whether instead of the Cartesian product symbol, one could use the intersection symbol. She seemed to be associating both of these notions with the word 'and' since the definitions of the intersection: $A \cap B = \{x: x \in A \text{ and } x \in B\}$ and the Cartesian product: $A \times B = \{(x, y): x \in A \text{ and } y \in B\}$ both contain the word 'and'. The discussion from lines 8 to 12 (sub-episode 4.2) is included here:

> **[8] Christine:** Can I ask something?
>
> **[9] Maria**: Ja
>
> **[10] Christine**: Because 'and' means intersection can we say, in the bracket say *A* intersection *C*. Can you say that?
>
> **[11] Maria:** Hmm?
>
> **[12] Christine:** That cross stands for an intersection, right? Can we put intersections in the bracket?

Guidance: More knowing peer identifies the cause of the misconception

Joseph offered an explanation in line 22 and this able explanation was confirmed by the lecturer who also referred to the definition of the Cartesian product and wrote it on the board again for easy reference. Joseph's contribution in line 22 (sub-episode 4.2) is included below.

> **[22] Joseph:** I think in terms of the intersection it is when you say like one variable, suppose *x* is in both sets *A* and *B*. Now when you have the crosses where you have two variables, *x* is in *A* and *y* is in *C*. So we've got there, we have *x, y – x* is the set of, I mean is an element of the set before the cross. And *y* is an element of the set after the cross. When you see a cross we actually speak of two variables.

Mathematical definitions: Associating and interchanging symbols or signs with a word contained in the definition of the symbol or sign

Interchanging the symbols of the union and Cartesian product with words contained in their definitions

Another error which revealed complex thinking occurred several times in episode 3. Symbols or signs were associated and interchanged with a word found in the symbol's definition. This type of thinking causes students to substitute certain words found in the definition of terms, signs or symbols with that symbol or sign and vice versa. An example of this was observed in line 78 of sub-episode 3.1 where Edgar wrote:

[78] Edgar: *[(x, y) ∈ (A×C) or (x, y) ∈ (B×C) ⇒ (x, y) ∈ (A×C) ∪ (x, y]*

Before he could finish writing he was stopped and corrected by his peers in line 79:

[79] S: Just write *B* cross *C*

It seems that Edgar thought that the word 'or' could be simply replaced by the symbol of union because of his association of the union with the word 'or' found in the definition of the union.

This type of reasoning occurred again further in the proof construction (sub-episode 3.2) when Edgar associated the Cartesian product (that is, the symbol '×') with the word 'and' in line 104. His repeated errors regarding the use and application of the Cartesian product demonstrated that reaching concept level understanding of newly met terms, symbols and signs is no easy task but one which takes time and practice. The transcript from lines 100 to 104 has been included below.

[100] Edgar: This can be that, ja. Thanks. That's a mistake I've been making on the right, yes. Let me *[writes: ⇒ x∈A and y∈C or x∈B and y∈C ⇒ x∈A or x∈B and y∈C ⇒ x∈ (A∪B) × C].*

[101] Gary: *y* must be an element of *C*.

[102] Edgar: Pardon?

[103] Gary: an element of *C*.

[104] Edgar: *[erases the C and puts y∈C. Thus the statement now reads: ⇒ x∈(A∪B)× y∈C ⇒ (x, y) ∈ (A∪B) × C]*

Guidance: More knowing peers illustrate mathematical objects using examples

Near the end of sub-episode 3.2 after Edgar had completed the proof, Joseph acting as more knowing other picked up the errors made in lines 78 and 104 and illustrated these with his own examples in lines 111 and 113. The transcript and further discussion can be found in Section 8.2.3 which contains a discussion of the characteristics of the interactions of the lecturer and peers in Episode 3.

Mathematical definitions: The link between definitions and proof methods

The implication

There was much discussion in episode 1 on the notion of the implication. By using transactive prompts for clarification and explanation, the lecturer tried to probe the students' ideas about this notion and gradually guide their understanding towards concept level. The discussion described below took place in sub-episode 1.3 and shows that even when students appeared to know the correct definition of a term, they had difficulty arriving at the correct proof method as a result of an important misinterpretation.

In sub-episode 1.3, line 95, Joseph gave his explanation of the statement $P \Longrightarrow Q$:

> **[95] Joseph**: If P is true then Q will be true but you can't say if Q is true then P is true.

The correct definition of P implies Q is: 'If P is true then Q is true'. Joseph's seemingly insignificant departure from this definition:" If P is true then Q **will** be true" seemed to cause him to arrive at the incorrect proof method for proving an implication (cf. sub-episode 2.4). This is further discussed in Section 6.2.2.3.

Guidance offered: Lecturer tries to show the connection between the definition of the implication and the method of proof

After Joseph's contribution in line 95 the lecturer (lines 96-98) tried to bring to light the correct proof method to be used.

> **[96] T**: Yes, I like that. If P is true
>
> **[97] Joseph**: Q is also
>
> **[98] T**: then Q is true *[writes If P is true, then Q is true]* That is a good definition. So that's all that this means. P implies Q means that if P, if P is right, if P is true, then Q is true. *[Points to*

$P \Rightarrow Q$] So when you want to prove this kind of thing that is why we start off with assuming that P is true. And then we move towards proving that Q is true.

In addition to highlighting the meaning of the definition of the implication symbol (using a *facilitative contribution*), the lecturer also tried to impart the understanding of the method one would use to prove an implication (using a *didactive contribution*). She advised the students that they should start off with assuming that P is true, and then move towards proving that Q is true.

Summaries of difficulties and guidance in category L1b

Difficulties experienced by students

- **Misinterpretation of the definition**

  The definition of the union of two sets: $A \cup B = \{x: x \in A \text{ or } x \in B\}$ seemed to introduce misunderstandings in students' thinking. Students seemed to hold the view that the union of two sets comprised all the elements of both sets except the elements in their intersection. This was surmised to be because students' interpretation of the definition might be that x may be in A or in B but not in both (exclusive 'or' versus inclusive 'or'). Understanding and interpreting mathematical definitions which include unfamiliar mathematical notation and terminology is puzzling and confusing to the average student who has not been exposed to mathematical definitions before.

- **Association of mathematical objects with a word contained in their definitions**

  Some students showed a tendency to associate mathematical objects with a word contained in the object's definition. For example the notion (and symbol) of the union (whose definition is: $A \cup B = \{x: x \in A \text{ or } x \in B\}$) was associated with the word 'or'. Similarly the notions (and symbols) of the intersection (whose definition is: $A \cap B = \{x: x \in A \text{ and } x \in B\}$) and the Cartesian product (whose definition is: $A \times B = \{(x, y): x \in A \text{ and } y \in B\}$) were both associated with the word 'and'. Students interchanged these symbols with the words associated with them (and vice versa). As a result of this association the notions of the Cartesian product and the intersection were associated with each other. Many

students showed a tendency to link the notion of the Cartesian product with the notion of the intersection and one student asked whether the symbol of the Cartesian product could be replaced by the symbol of the intersection. This notwithstanding the fact that the Cartesian product is a binary operation acting on two sets (for example A and B) to create a new set ($A \times B$), whose elements are ordered pairs (x , y), x being an element of A and y an element of B, whereas the intersection of two sets comprises single elements that are found in both sets. Students' association of these two mathematical objects with the word 'and', highlighted the great difficulty that students have in understanding and processing the full mathematical definition. They appeared to rather focus on one word that was common to both definitions (but in very different contexts) and based all their thinking on this limited understanding. The opportunity offered to students in the EZPD to interact with one another and develop their understanding of these notions through the functional use of the terms, signs, symbols and their definitions seemed to play a vital role in the development of their proof construction abilities.

- **Misinterpretation of the definition of the notion of implication giving rise to incorrect proof method**

  Students' description of their interpretation of the notion of the implication revealed almost imperceptible deviations from the correct definition. For example Joseph's explanation of '*P* implies *Q*' in line 95 of sub-episode 1.3 as: "If *P* is true then *Q* will be true". Similarly in sub-episode 2.4, line 46, Maria gave her explanation of *a)* implies *b)* as: "…if *a)* is true, then we know *b)* is true". These seemingly insignificant departures from the correct definition: 'If *P* is true then *Q* is true' could have led Joseph and Maria to believe (in sub-episode 2.4) that the method of proof of an implication '*P* implies *Q*' would be to first prove that *P* is true and that this will then mean that *Q* is also true.

Guidance offered to students

Forms of guidance included:

- **Using examples to illustrate mathematical objects**

  Examples were used to clarify the definition of mathematical objects such as the intersection and the union. Although the use of examples was initiated by the lecturer at the beginning of the discussion to clarify the notions of intersection and union, the students enthusiastically took over this activity and seemed to enjoy doing examples on the board. Judging by the whole-hearted participation and excitement observed, it was clear that students were able to discuss their perceptions and conceptions of confusing terms much more easily by using examples and were eager to get clarity on these notions. With the aid of examples, students' understanding of these mathematical objects hopefully progressed from complex thinking towards true concept level understanding.

- **More knowing peers encourage reflection on the definition of mathematical objects through transactive prompts and by referring to the definition**

  More capable peers (such as Gary and Joseph in Episodes 3 and 4) gradually assumed the role and responsibilities of the lecturer by adopting the transactive requests for clarification, reflection and justification provided guidance and scaffolding to their peers. They also referred to the definition of mathematical objects such as the Cartesian product (in episodes 3 and 4) and clarified and explained this definition in their own words, showing how it could be applied to the particular proof construction exercise with which the group was engaged. It seemed that the importance of definitions had been made apparent to them and this seemed to be quickly extended to the other participants through their interaction with their peers. This increased appreciation of definitions of mathematical objects could be the result of a growing understanding of the necessity for justification of each statement or deduction in the proof. The correct interpretation of definitions seemed to take on an increased significance and meaning as they may have now realized that definitions are valuable tools which allow them to map the way forward and justify deductions in the proof

construction process as opposed to meaningless bits of information that they had to memorize and deliver in a test or exam.

- **More knowing peers identify the cause of the misconception**

  More knowing peers (such as Joseph in episodes 3 and 4) exhibiting concept level understanding of the notions of intersection and Cartesian product were often able to identify the cause of their peers' difficulties. By pinpointing the root cause of students' confusion and association of the notion of the intersection with the notion of the Cartesian product as a result of the word 'and' common to both definitions of these mathematical objects, Joseph (in episodes 3 and 4) was able to help his peers to make progress in the proof construction process. Joseph's swiftness in detecting the root of the misconception and his patience and thoroughness of explanation illustrated the effectiveness of peer scaffolding in the EZPD.

- **More knowing peers illustrate mathematical objects using examples**

  More knowing peers gave examples to illustrate erroneous proof construction steps made as a result of students' association of a mathematical object with a word contained in its definition and their tendency to want to replace the symbol by this word or vice versa.

- **Lecturer highlights the definition of a mathematical object and tries to show the distinction between the definition and the method of proof**

  There was a great deal of discussion on the notions of the implication and double implication in the first session and the lecturer tried to elicit students' conceptions and thoughts of these notions through requests for clarification and explanation. Towards the end of the first proof construction exercise the lecturer drew the students' attention to the correct definition of the notion of the implication and highlighted its importance. She then continued to clarify the distinction between the definition of this mathematical object and the method of proof of an implication. She attempted to make the distinction (in general) between the definition of the object and the method of proof that one would use to prove the validity of a statement involving that object.

6.2.1.3: L1c: Illustrating mathematical objects and definitions with examples

As is evident from the transcript of the episodes analysed here, using examples to illustrate terms, symbols or definitions is a particularly useful tool which can help students gain clarity about the whole proof process and should be encouraged by lecturers. Students should be aided to realize that, when used in clarifying and exploring newly met mathematical objects and definitions, the use of examples is permissible and productive. One of the characteristics of potential more knowing peers was their tendency to turn to examples to illustrate mathematical objects and definitions in their explanations to the other students and also to clarify and reach improved understanding for themselves. This ability seemed to be further strengthened as a result of their participation in the consultative sessions. This will be more fully discussed in Chapter 8.

Students' difficulties in this category resulted mainly from their inability to generate useful examples due to their inexperience with newly met mathematical objects and their definitions. This finding is in accord with Moore (1994) who found that, although lecturers encouraged students to generate and use examples to aid to their understanding of the mathematical objects involved in proof construction, they were often hindered because of their inability to do so (cf. Section 2.2.1). He proposes that this was a result of students having a "limited repertoire of domain-specific knowledge from which to pull examples" (Moore, 1994, p. 260). Analysis and discussion of illustrative instances where students experienced challenges in this area is presented below.

Illustrating mathematical objects and definitions with examples: making maximal use of examples

Failure to make maximal use of examples when illustrating the notions of subset and equality

The first example occurring in the first session was that of Joseph in sub-episode 1.3, who offered an example clarifying the notions of subset and equality, after an emphatic contribution from Helen on this subject. Unfortunately this example got lost in the discussion and was not acknowledged by his peers. I have included the discussion taking place from lines 56 to 60 in sub-episode 1.3:

[56] Helen: Can I say something? Before you write can I, can I, can I, can I say something? We have been given that *A* is a subset of *B*. And then we can't say *A* is equal, is not equal, is equals to *B* because we are not given that *B* is a subset of *A*.

[57] Frank: OK

[58] Helen: Yes. We're only given that *A* is a subset of *B*, that is why we can't say that *A* is equal to *B* because we don't have *B* as a subset of *A*

[59] Frank: OK

[60] Joseph: Ja, it seems to say *A* is a subset of *B*, it doesn't necessarily mean that in every element that is in *A* are the same element that are in *B*. There may be…let's say *B* consists of elements of natural numbers and then *A* consists of elements that are even numbers.

The example given by Joseph was a very good one but as later discussion showed, seemed to slip past unnoticed. The reason could have been because the example was not done on the board. The lecturer (in line 63) did try to give Joseph another chance to mention the example hoping that Joseph's peers would take note of it but she was not explicit when asking Joseph to repeat what he had said. Joseph repeated his argument but did not mention the example. The discussion unfortunately continued without maximal use being obtained from this particular example.

Illustrating mathematical objects and definitions with examples: Students having a limited repertoire of examples

Illustrating the notion of the power set with an unhelpful example

Frank completed the first component of the proof ($A \subseteq B \Longrightarrow P(A) \subseteq P(B)$) correctly in sub-episode 5.1. His peers including Gary and Joseph tried to build up their understanding of the notion of the power set by reflecting on Frank's proof construction actions and asking for clarification and explanation. In sub-episode 5.2, line 26, Joseph tried to strengthen his understanding further by putting an example of a power set on the board. The example he chooses was not very helpful as he drew the Venn diagram of the power set of a set *A*, and tried to populate it with elements without first drawing the Venn diagram of the set *A* itself.

Guidance offered: A further example done by the lecturer

Although some clarity seemed to have been gained with Joseph's example, the lecturer sensing that there was a need for further clarification of the notion of the power set, did

another example on the board (making a *didactive contribution* in line 41). She drew the Venn diagrams of the set *A* containing two elements (*1* and *2*) and the power set of the set *A* containing all the subsets of *A*, better illustrating the relationship between elements of the set and the elements of the power set of that set.

Summary of difficulties and guidance in category L1c

Difficulties experienced by students

- **Failing to make maximal use of examples offered by students in an attempt to clarify mathematical objects**

  Joseph, a potential more knowing peer, offered a very good example in episode 1, clarifying the distinction between the notions of equality and subset. The example was not illustrated on the board. The participants unfortunately did not appear to take note of this example. I suggest that lecturers may need to explicitly highlight the importance of examples when these are offered, in order to obtain optimal benefit from these. Examples should be done on the board and not merely verbalised as it appears that students do not pay much attention to narrated examples.

- **Illustrating a mathematical object or definition with an unhelpful example**

  More knowing peers (such as Joseph in episode 5) used examples to illustrate newly met objects and definitions when striving to clarify these notions for themselves and other participants. Joseph offered an example of a power set by drawing the power set of a set *A* on the board and tried to populate it with elements. This proved to be difficult as he had not depicted the set *A* and its elements first. This difficulty probably arose because the notion of the power set was still very new and unfamiliar and Joseph's understanding of the notion was not very complete.

Guidance offered to students

Forms of guidance included:

- **Lecturer offers another more illuminating example**

  When the lecturer realized that the notion of the power set needed further clarification (in sub-episode 5.2), she did another example on the board. This time the set *A* is drawn followed by the power set of *A*, thus enabling connections to be made between the elements of the set *A* and the elements of its power set *P(A)*.

6.2.2 L2: Logical status of statements and proof framework

The difficulties that students had with regards to the logical reasoning and proof methodology required in the proof construction process were largely manifested in Episodes 1 and 2 where the proofs of: *If* $A \subseteq B$ *and* $B \subseteq C$*, then* $A \subseteq C$ and *a)* $A \subseteq B \Leftrightarrow$ *b)* $A \cap B = A$ were attempted. The category L2 focusses on the selection of correct and appropriate statements and phrases which add logic to the proof construction process (L2a), selecting useful and appropriate aspects of definitions and assumptions (L2b) and selection and application of the correct proof methods (L2c).

As in category L1 examples of the difficulties are indicated by the use of boxes for each one. Contributions made by students and the lecturer which seemed to lead to students' increased understanding are then identified and made explicit. It is assumed that in addition to the scaffolding received from their peers and the lecturer, students gained better understanding through their functional use of mathematical objects and processes which include logical reasoning processes and proof methods. I argue that the functional use of these processes is one of the important factors enabling the student to make the transition to a usage in line with their usage by the mathematical community.

6.2.2.1 L2a: Selecting correct and appropriate statements and phrases that make sense and add to the logic of the proof construction

Statements or phrases that add to the logic of the proof construction process include:

- statements usually written at the beginning of the proof attempt, where the assumptions and what is to be proved are clearly stated and set forth.
- statements giving the justification for deductions for example use of an assumption or a theorem previously proved.
- statements usually written at the conclusion of the proof attempt where reference is made to the components of the proof construction (where applicable) and justification is provided for the conclusion.

Although the inclusion of such statements is not strictly necessary for the proof construction to be correct, these statements make the proof much easier to read and understand, and are very helpful especially to those who are new to the process of proof construction. The use of such statements and phrases also helps to clarify whether the student who is attempting the proof construction actually knows the overall plan and is able to justify deductions and conclusions made. Lecturers often encourage students to use such statements in their proof construction attempts. Analysis and discussion of illustrative instances where students experienced difficulties and challenges in this area is given below.

Selecting correct and appropriate statements and phrases that make sense and add to the logic of the proof construction: clearly stating the assumptions and the statement to be proved

In sub-episode 1.1, line 1, Frank attempted the proof of the proposition: *If* $A \subseteq B$ *and* $B \subseteq C$, *then* $A \subseteq C$. His proof attempt on the board is set out below. As can be seen although the proof was done correctly with the exception of the double implication sign being used instead of the single implication sign, there was no statement clearly stating the assumptions and presenting what had to be proved at the beginning of the proof. This would add to the logic of the proof construction and would also give others, including the lecturer, the reassurance that the student does indeed have some idea of the overall approach or plan behind the proof.

[1] Frank: *let* $x \in A$

$\Leftrightarrow x \in B$ *(since* $A \subseteq B$*)*

$\Leftrightarrow x \in C$ *(since* $B \subseteq C$*)*

then $A \subseteq C$.

After this proof attempt the lecturer in sub-episode 1.2 asked for input from all the participants. Edgar in line 3 suggested the addition of a statement which would state the assumptions and what was needed to be shown. Edgar's contribution is included below.

> **[3] Edgar**: Ja, I just want to like, I don't know whether we need to start our proof… We said suppose that *A* is a subset of *B* and also *B* is a subset of *C* and then we specify what we need to do, what is it that we need to do in order to actually come up with something that completes the equation. I don't know, do we, don't we start by saying, 'Suppose is a subset of *A* and also that's a subset of *A*?

However when Frank in line 4 tried to add this statement to his proof construction, it became obvious that his understanding of the word 'suppose' and the implication symbol was incomplete, as he wrote a contradictory statement on the board. This incomplete understanding of words such as 'suppose' is discussed in Section 6.2.1.1.

> **[4] Frank:** Okay you want me to write suppose *A* is a subset of *B* and *B* is a subset of *C* implies that *A* is a subset of *C*. *[writes as he is speaking directly above his proof attempt: Suppose* $A \subseteq B$ *and* $B \subseteq C \Rightarrow A \subseteq C$*].*

<u>Guidance offered: Peers offer contributions that help to improve understanding of mathematical notation and statements that add to the logic of the proof construction while Frank makes functional use of these</u>

Gary and Helen in lines 5 and 7 of sub-episode 1.2 contributed towards guiding Frank's understanding. Gary in line 5 clarified that the statement on the right of the implication sign ($A \subseteq C$) was 'to be proved' and Helen in line 7 also confirmed that this statement needed 'to be shown'. Frank appeared to understand and made the correction. The transcript of lines 4, 5 and 7 is included in Section 6.2.1.1.

Selecting correct and appropriate statements and phrases that make sense and add to the logic of the proof construction: Following the steps in previous proof construction rather than showing evidence of following logical reasoning processes

In sub-episode 3.2 Edgar appeared to follow the steps used in the previous section of the proof construction and did not show evidence that he was thinking and reasoning about the mathematical objects just engaged with in the first part of the proof (sub-episode

3.1). To illustrate this: in sub-episode 3.1, after Edgar (line 3) chose an arbitrary element from the Cartesian product $(A \cup B) \times C$,

[3] Edgar: *let* $(x, y) \in (A \cup B) \times C$

Edgar was guided to make this deduction in line 48:

[48] Edgar: $\Rightarrow x \in (A \cup B)$ *and* $y \in C$

In sub-episode 3.2 in line 88, he carried out similar steps which were incorrect in this context:

[88] Edgar: *let* $(x, y) \in (A \times C) \cup (B \times C)$
$\Rightarrow x \in (A \times C)$ *or* $y \in (B \times C)$

This seems to show that instead of using his reasoning ability to apply the knowledge of the newly met terms symbols and signs to this particular situation, he followed the steps and procedure that he had used before.

Guidance offered: More knowing peer requests clarification, justification and reflection on actions taken

Seemingly through Gary's prompts and questions encouraging Edgar to reflect on his actions and the notion of the Cartesian product, Edgar realized his error. Gary's contributions were from the categories: *L3b: questioning and requesting clarification for incorrect deductions made without any basis,* and *L1b: prompts from peers encouraging reflection on the meaning of the notion of the Cartesian product*.

Interestingly in line 98 (sub-episode 3.2) Edgar's discourse showed that he did not appear to be aware of the cross, and was merely paying attention to the union symbol. The transcript from lines 91 to 100 is included below:

[91] Gary: Oh this statement after letting *x, y* be an element of *A* cross *C* union *B* cross *C* ah, can you clarify?
[92] Edgar: Which one?
[93] Gary: The first statement, *x*…. after that
[94] Edgar: After the left
[95] Gary: Ja. We have made
[96] Edgar: Oh this is a union and this is *x, y*
[97] Gary: We have *A* cross *C* meaning…
[98] Edgar: There's no cross here, it's a union, it's an "or". That means that this can be this, or
[99] Gary: Ja

[100] Edgar: This can be that, ja. Thanks. That's a mistake I've been making on the right, yes. Let me

[erases: $\Rightarrow x \in (A \times C)$ *or* $y \in (B \times C)$ *and* $\Rightarrow x \in A$ *and* $x \in C$ *or* $y \in B$ *and* $y \in C$

and writes: $\Rightarrow x \in A$ *and* $y \in C$ *or* $x \in B$ *and* $y \in C$

$\Rightarrow x \in A$ *or* $x \in B$ *and* $y \in C$

$\Rightarrow x \in (A \cup B) \times C$ *]*

Summary of difficulties and guidance in category L2a

Difficulties experienced by students

- **Failure to clearly state the assumptions and the statement to be proved at the beginning of the proof**

  Frank's attempted proof construction in the first episode did not contain an opening statement clearly stating the assumptions in the proof and the goal of the proof construction. At Edgar's suggestion to include such a statement Frank wrote a contradictory statement clearly showing an incomplete understanding of the word 'suppose' and the implication symbol. This strengthens my view that such statements are helpful to students (helping to clarify and add logic to the proof construction) as well as the lecturer in revealing the problematic areas in students' understanding.

- **Following the steps or procedure from previous components and not showing evidence of applying logical reasoning in the proof construction process**

  In episode 3 Edgar appeared to follow the steps just used in the previous component of the proof construction process rather than applying logical reasoning. Perhaps this is also an indication that students need to develop their sense of accuracy when writing mathematical statements and deductions. Students are often unaware that every written symbol and sign has a meaning and a consequence.

Guidance offered to students

- **Peers offer contributions clarifying newly met terms and symbols**

  Peers offered contributions clarifying incorrect and contradictory understanding of terms and symbols using simple every-day language. A vast improvement

was observed in students' abilities to select statements that added to the logic of the proof construction process in subsequent proof construction exercises. This was presumably as a result of the guidance that students received from their peers while they made functional use of newly met objects and processes.

- **More knowing peers request clarification, justification and reflection on actions taken in proof construction**

  More knowing peers prompted their counterparts to reflect on their proof construction actions and reasoning processes used. They also prompted their peers to reflect on the definition of mathematical objects involved in the proof construction exercise. In this way they helped their peers to recognize errors made in their logical reasoning processes.

6.2.2.2: L2b: Selecting useful and appropriate aspects of definitions and selecting appropriate assumptions (strategic knowledge)

Selecting correct and appropriate assumptions or aspects of definitions to use in the proof construction process is a huge challenge for students. It was evident in my analysis of the five episodes, that even when students had a good grasp of all the various categories of proof comprehension and construction, the proof could still remain challenging because of the lack of strategic knowledge, that is, knowing how to use the definitions and assumptions at their disposal to achieve the desired goal**.** The process of proof construction is not an algorithmic one where the appropriate knowledge and information at one's disposal guarantees success. Quite often one needs to think creatively and be able to reason in an 'out of the box' manner to find the way forward. I argue that this could be one of the key aspects of proof construction ability developed over time through practice and seems to be greatly aided by working with peers and experts. Analysis and discussion of examples of difficulties in this category is given below**.**

Selecting useful and appropriate aspects of assumptions and definitions: trying to use non-useful or trivial aspects of definitions

In sub-episode 2.3 the lecturer (lines 17 and 21) tried to clarify the correct proof method to be used by drawing Maria's attention to what she actually needed to prove and what

her assumption was (using *transactive requests for clarification, reflection and strategy*). However Edgar (in lines 22, 24 and 28) made a contribution that was not at all helpful and actually derailed the whole proof construction process. He suggested that Maria use the facts that A is a subset of A and that B is a subset of B in her proof construction attempt of $A \subseteq B \Rightarrow A \cap B = A$. The transcript from lines 17 to 28 is included below.

[17] T: Maybe if we just go back to the beginning. What are you trying to show, first of all?

[18] Maria: Here?

[19] T: Mmm

[20] Maria: I was trying to show that this [*underlines* $A \subseteq B$*]* implies this *[underlines* $A \cap B = A$ *in statement:* $A \subseteq B \Rightarrow A \cap B = A$*]*

[21] T: So that's the first thing you want to show that A subset of B implies A intersection B equals A. So what do we start off with?

[22] Edgar: Isn't it that we know that A will always be a subset of A.

[23] Maria: Hmm?

[24] Edgar: A will always be a subset of A. Always. In other words always start with A being a subset of A.

[25] Maria: Oh, here? Or there? *[points to the board]*

[26] Edgar: Ja, the first one.

[27] Maria: OK

[28] Edgar: And also B is a subset of B

Selecting useful and appropriate aspects of assumptions and definitions: Considering a statement that is supposed to be proved as an assumption

One of the most common errors made when students are starting out on their journeys in proof construction is using the statements that are to be proved as assumptions in the actual proving process.

An example of this was seen in sub-episode 2.1. In sub-episode 2.1 Maria made her first attempt at the proof construction: $A \subseteq B \Rightarrow A \cap B = A$. When attempting this proof construction she assumed the statement on the right of the implication symbol which is supposed to be proved, to be true and made assertions and deductions that were not justified. As she wrote on the board (in line 3), she explained her thought process:

[3] Maria: A intersection B which is equal to A. So from this if A is a subset of B. This means that, mmm, x is an element of A, which implies that x is also an element of B. And *[writes: If*

$A \subseteq B \quad x \in A \implies x \in B$ *].* Then we come to this side. That if *A* is an intersection of *B* which is equals to *A* it will mean that *A* is a subset of *B*. And this would mean that *x* is an element of *A*. If and if *x* is an element of *A* it implies that it is also an element of *B*. *[writes on the other side of the board so it looks like this*

If $A \subseteq B$ *if* $A \cap B = A$

$x \in A$ $A \subseteq B$

$\Rightarrow x \in B$ $\Rightarrow x \in A$ $\Rightarrow x \in B$ *]*

She re-iterated this thought process in line 5 of sub-episode 2.2.

Guidance offered: Peers critique and question proof construction actions

In this sub-episode Christine (line 6) referred to an implication as an equals sign, but later (in line 8) she described the method for the proof correctly, thus suggesting pseudoconceptual thinking. Christine questioned Maria on her logical reasoning and justification and was thus instrumental in helping to create the EZPD in which Maria's learning developed. As a result of Christine's questions and critique Maria began to realize that she might have made an inappropriate and incorrect deduction. In line 11 Maria began to doubt her thought processes. The transcript from lines 6 to 11 is included below.

[6] Christine: How? Isn't it that *A* intersection *B* is equal to *A* on the other side of the equals sign?

[7] Maria: Mmm?

[8] Christine: Aren't you supposed to say that *A* intersection *B* is a subset of *A* and the other way round?

[9] Maria: Ja, but we've got an equals sign here, meaning that *A* is a subset of *A* intersection *B*. At the same time *A* intersection *B* is a subset of *A*.

[10] Christine: Would you say *A* is a subset of *B*?

[11] Maria: Ja. Ok

[writes: $A \cap B = A$

a) $A \cap B \subseteq A$ *and*

b) $A \subseteq A \cap B$ *]*

A intersection *B* is equals *A* which means that *A* intersection *B* subset of *A*. Again *A* intersection, OK, again *A* is a subset of *A* intersection *B*. Ok, from this *[points to* $A \cap B \subseteq A$*]* would I be wrong if I say *A* is a subset of *B*? *[adds* $\Rightarrow A \subseteq B$ *]*

Selecting useful and appropriate aspects of assumptions and definitions: Considering a statement that is supposed to be proved as an assumption

Another example is found in sub-episode 2.8 when Maria was trying to prove: $A \cap B \subseteq A$. In line 177 after picking x as an arbitrary element of A intersection B, she went on to say: "So this would imply that x is an element of A intersection B which is a subset of A *[writes:* $\Rightarrow x \in A \cap B \subseteq A$*]*" where $A \cap B \subseteq A$ is what she is trying to prove. Maria's proof construction actions fall under the categories: *L2bx: selecting the statement to be proved as an assumption, L3ax: assertion made without any basis* and *L1ax: incorrect use of mathematical notation and symbols*.

Guidance offered: Lecturer and peers promptly remind the student that the statement that she has used in the proof construction still needs to be proved

One of the benefits to students working on proof construction exercises in the EZPD was that they received prompt and corrective feedback from the lecturer and their peers. After Maria's incorrect statement in line 177, the lecturer (in line 178) made a *transactive prompt requesting justification and critique* from Maria and her peers, as well as a *directive contribution* introducing the notion that students must be sure of the truth of statements and assertions that they write. Maria's peers also made positive contributions from the category: *L3b: clarifying and explaining that every statement needs to have a justification and logical reasoning behind it*. The transcript from lines 178 to 181 is included below.

> **[178] T**: Is that true? *[Maria looks at T]* Is it true? Every step of the way you must be sure that it is true. Is that true?
> **[179] S**: No
> **[180] Helen**: Not yet, because we're trying to prove that.
> **[181] S**: No, we're trying to prove.

Selecting useful and appropriate aspects of assumptions and definitions: Considering a statement that is supposed to be proved as an assumption

Similarly in sub-episode 2.9, line 261 when Maria was trying to prove that A is a subset of A intersection B, she once again used the statement she was trying to prove as an assumption. Maria's incorrect proof construction actions are from the categories: ***L2bx:*** *selecting the statement to be proved as an assumption,* ***L3ax****: makes an assumption of a*

statement that she is trying to prove, ***H2ax****: unable to transfer the ideas just used in the previous proof component to this component.*

[261] Maria: Ok. So I let x be an element of A. If x is an element of A and A is a subset of A intersection B *[writes: let* $x \in A$ *]* Hmm?

Guidance offered: The lecturer promptly clarifies and highlights the misconception and repeatedly draws attention to the assumption

After Maria's action of treating the statement which needed to be proved as an assumption, the lecturer promptly interjected making a *directive* contribution offering immediate feedback on the incorrect assumption. She also made a *facilitative* contribution highlighting the fact that one cannot assume what one needs to prove and asking her, using a *transactive prompt* to recall the correct assumption. This question was repeated in line 264 again as a *facilitative* contribution highlighting the fact that students need to always be aware of their assumptions in the proof construction process. The transcript from lines 262 to 265 is included below. Incidentally Maria's repetition (in line 265) of her peer's utterance in line 263 appeared to be a clear example of students learning through the process of imitation, one of the activities encompassed by the functional use of a mathematical object or process. This is also discussed in Section 7.2.2.1.

[262] T: It's not! That's what you're trying to show… that's what you're trying to show… So please don't get confused with what you are trying to show, you cannot assume that. But what have you assumed, what have you got?

[263] S: *(Some comments)* A is a subset of B.

[264] T: What have you assumed?

[265] Maria: A is a subset of B

Selecting useful and appropriate aspects of assumptions and definitions: Difficulty in using the definitions and assumptions at one's disposal to get to the desired goal

In sub-episode 2.11, Maria continued with the second component of the proof, that is, showing: $A \cap B = A \Rightarrow A \subseteq B$ and showed much more confidence. She appeared to have mastered the proof methods used in the last components of the proof (that is the implication proof method and the subset proof method) and was able to transfer these methods to the next component of the proof.

Maria started the proof correctly choosing an arbitrary element x in the set A in line 310. The deduction which followed this was written in line 324 when Maria made the deduction: $\Rightarrow x \in B$. This deduction did not follow simply from the previous statement: *Let* $x \in A$, and the assumption: $A \cap B = A$, but was the desired outcome or deduction for the proof to be complete. Maria appeared to have realized that she needed to use the assumption made at the outset of the proof to get to the desired goal, and she also recognized that the deduction $x \in B$ would allow her to reach the correct conclusion. She was unable however to proceed with this knowledge to reach the correct conclusion logically and sensibly.

[324] Maria: OK. And this, from this it would imply that x is an element of B because here it says that A intersection B is equal to A. *[writes:* $\Rightarrow x \in B$*]*

[325] T: Break it down into simple steps for us…

[326] Maria: *[completes the statement she was writing:* $\Rightarrow x \in B$ *(since* $A \cap B = A$*)]*

Guidance offered: The lecturer and other students try to guide Maria by reminding her of the assumption and its correct implication and urging her to use logical reasoning

The lecturer tried to get Maria to reach the correct deduction using *transactive prompts requesting clarification, explanation, reflection and logical reasoning* (lines 327, 331, 333, 337, 339, 343, 345 and 347) and also using *facilitative contributions structuring the proof writing and highlighting learning* (line 335). By prompting Maria to reflect on her actions, to ensure that every step made sense and to remember the assumption made while proceeding with logical reasoning, she tried to develop the strategic knowledge needed. Maria's peers also contributed in this regard. The transcript from lines 327 to 356 has been included.

[327] T: Are we clear? I think you missed a step.

[328] Frank: Since A is a subset of B

[329] Maria: Hmm?

[330] Frank: Since A is a subset of B. We want to show that. We want to show that.

[331] T: And is it clear for all of us, is it? Is it?

[332] Christine: No it's not

[333] T: OK, just go back and think about how to make that a bit more clear.

[334] Maria: Like?

[335] T: What do we know? What are our assumptions?

[336] Maria: You know that *A* intersection *B* is equal to *A*.

[337] T: OK, now you have let *x* to be?

[338] Maria: an element of *A*.

[339] T: What does that mean then?

[340] Maria: Like I've let *x* be, to be contained in *A*.

[341] T: Mmm

[342] Maria: And since I've already assumed that *A* intersection *B* is equal to *A* it means that *x* is also contained in *B*.

[343] T: Ok, that's not clear. You have written *x* is an element of *A*

[344] Student: Right

[345] T: And *A* is…?

[346] Maria: Is a subset of

[347] T: *A* is…?

[348] S: equals to

[348] Maria: is equal to *A* intersection *B*

[349] T: So what is *x* an element of?

[350] S: *x* is an element of *A* intersect *B*

[351] T: Right. Write that down.

[352] Edgar: *x* is an element of *A* intersect *B*.

[353] Maria: *[erases:* $\Rightarrow x \in B$ *(since* $A \cap B = A$*)]*

[354] Edgar: And… that since *x*, since *x* is an element in *A*

[355] Maria: *[writes:* $\Rightarrow x \in A \cap B$ *(since* $A = A \cap B$*)]*

Selecting useful and appropriate aspects of assumptions and definitions: Starting the proof incorrectly and reaching an impasse

Frank's attempt at the proof of $P(A) \subseteq P(B) \Rightarrow A \subseteq B$ in sub-episode 5.3:

After successfully completing the proof construction of the first component of the proof in sub-episode 5.1, Frank attempted the proof construction of the second component in sub-episode 5.3. When attempting to prove $P(A) \subseteq P(B) \Rightarrow A \subseteq B$, Frank struggled to start the proof correctly. Although Frank appeared to know the definition of the power set and was able to apply it in sub-episode 5.1, he seemed unable to use this definition to provide a strategy for doing the proof in sub-episode 5.3. He was unable to work out how to use the assumption $P(A) \subseteq P(B)$ to prove $A \subseteq B$. I have included lines 47 to 55 of sub-episode 5.3 below. Here Frank struggled to start the proof correctly and when

guided to make the correct first step, he faltered again and made a deduction without any justification (line 55).

[47] Frank: *[writes: let {x} ∈ A]*

[48] T: Now, think about that. What does everybody say about that? *[silence]* We want to show… What do we want to show?

[49] Frank: *A* is a subset of *B*

[50] T: So then you have to pick any element…

[51] Frank: Ja

[52] T: in where?

[53] Frank: *A*

[54] T: In *A*, right? And *A* is just a set

[55] Frank: Ok.

[erases the brackets so it now reads: let $x \in A \Rightarrow x \in B$ *(since* $P(A) \subseteq P(B)$*)]*

As seen above Frank (line 47) started the proof of showing $A \subseteq B$ by choosing the set $\{x\}$ as an element of A. Frank's mistake here was that $\{x\}$ is a set and cannot be an element of A. The correct course of action would have been to choose x to be an element of A, and then make the connection that $\{x\}$ is a subset of A and hence an element of the power set of A, that is an element of $P(A)$. When prompted by the lecturer and reminded that A was simply a set, Frank (line 55) correctly chose x to be an element of A, but then immediately made a deduction without the necessary justification which would lead him to the correct conclusion. I suggest that Frank did not know what the next appropriate step or deduction should be after the first step and he jumped to what he knew was the correct conclusion without any justification.

Guidance offered: lecturer asks peers to reflect on reasoning and strategy and make contributions towards proof construction and use examples to illustrate mathematical objects and processes

The lecturer used *transactive prompts* (lines 56, 57 and 59) asking students to reflect on their *reasoning and strategy*, and also *requesting them to use examples* (lines 64, 66, 68, 72, 74, 76 and 80) to clarify the notion of the power set and its application in the proof construction. The transcript referred to is included below in the discussion on further guidance.

Further guidance offered: peers offer contributions on strategy and reasoning, clarifying the proof construction process by using examples

Gary and Joseph in lines 61, 62 and 63 offered their contributions on finding the correct strategy to progress in the proof showing how much their understanding had progressed as they were now able to make positive contributions in the proof construction. Whereas in sub-episode 5.1 they were observed trying to build their understanding of the power set through engagement and reflection on the notion of the power set as they interacted with their peers, in sub-episode 5.3 they were able to use their knowledge of the newly met term to find a way forward in the proof construction process. Joseph then went up and completed the proof correctly in line 77 and when he realized that Frank was still not clear about the proof construction and the reasoning that he had used, he seemed to realize that the justification behind his proof construction steps would best be clarified by reflection on the example of a power set. In line 81 he altered the example the lecturer had asked Frank to do on the board by replacing the elements 1 and 2 by the general variables x and y. By doing this the relationship between the elements of a set and the elements of its power set were better demonstrated. This showed that Joseph's understanding of the power set had evolved to concept level as he was able to explain and apply the mathematical object correctly and generate well-thought of examples which clarified the object for his peers. This is further discussed in Section 7.2.1.2 and 8.2.5. The transcript from lines 56 to 81 is included below.

> **[56] T:** Do we agree with that?
>
> **[57] T:** So you wanted to show that A is a subset of B. You've taken an element in A and then you immediately go to say that element is in B. Since…
>
> **[58] Student:** Is x not in power set B?
>
> **[59] T:** Since what? Does it follow immediately?
>
> **[60] Student:** No, it does not follow immediately
>
> **[61] Gary:** I was thinking; x being an element of A, right? Ah, since x is an element of A, what it means that…
>
> **[62] Joseph:** $\{x\}$ is an element of the power set…
>
> **[63] Gary:** subset $\{x\}$ can be an element of power set of A, subset $\{x\}$ is an element of the power set of A.

[64] T: OK, do you want to write that down? Maybe… Draw the Venn diagram of that set *A* that I put up, that example. Ja, and see what… Do you remember how we got the power set? We had the elements 1 and 2

[65] Frank: *[draws a Venn diagram and writes Ø inside]*

[66] T: No, first draw *A*, the set of *A*

[67] Frank: Oh, *A*, here? *[labels the diagram A]*

[68] T: What did *A* have in it? 1 and 2.

[69] Frank: 1 and 2

[70] T: Right

[71] Frank: *[erases Ø, and writes 1, 2 in Venn diagram labelled A]* 1 and 2

[72] T: Uh huh. Now the power set is…

[73] Frank: *[writes: P(A) and draws a Venn diagram with Ø, {1}, {2}, {1, 2}]*

[74] T: Right. Does that give you a clue?

[75] Frank: *x* is in *A*.

[76] T: *x* is in *A*. So *x* is, it can be the 1 or the 2 in this case. Do you want to go up and show us?

[77] Joseph: *[goes to the board and says as he writes]* Subset $\{x\}$ is an element of power set *A*… is an element of the power set *B* since, and this is an element of *B* since this

[writes: $\Rightarrow \{x\} \in P(A)$

$\Rightarrow \{x\} \in P(B)$ *(since* $P(A) \subseteq P(B)$*)*

$\Rightarrow x \in B$

Thus $A \subseteq B$ *]*

[78] Frank: But at the beginning I was trying to show that the set was…

[79] Student: No you can't say that a set is an element of a set.

[80] T: Look at *A*…

[81] Joseph: The set, it's like saying this is an *x [in the circle labeled A, he erases 1and 2, and replaces this with x and y]* so that we say *x,* we say *y*. And then this *x* that's in here it can be considered as a subset so we say *x [draws the Venn diagram P(A) and writes {x}, {y}, {x, y} and Ø and erases the Ø, {1}, {2}, {1,2}]* and the subset will be *{x}, {y} and{x, y}* which is the set itself . And this one is just the same like we had a subset.

Summary of difficulties and guidance in category L2b

Difficulties experienced by students

- **Selecting non-useful or trivial aspects of definitions**

  One of the challenges students faced when first introduced to proof construction was the challenge of knowing which aspects of definitions would be useful and how to use them. In their quest to proceed in the proof construction process,

students sometimes presented all sorts of irrelevant and non-useful information. For example in sub-episode 2.3 Edgar's contributions included: *A* is a subset of *A* and *B* is a subset of *B*. These trivial contributions were not at all relevant to the proof construction.

- **Treating a statement that is supposed to be proved as an assumption**

  A common error by students in their initial attempts at proof construction was treating the statement that was supposed to be proved as a given and trying to use this statement in their proof construction attempt.

- **Difficulty in using the assumptions and definitions at one's disposal to make progress in proof construction**

  There were several instances where students demonstrated sound knowledge of the relevant proof methods and exhibited good understanding of the mathematical objects, definitions and assumptions relevant to the proof construction, and yet failed to drive the proof construction process forward. Even though all the other proof comprehension criteria appeared to be satisfied, the proof still remained challenging because of the lack of strategic knowledge, that is: knowing how to use the definitions and assumptions at their disposal to achieve the desired goal. Perhaps this is one of the key aspects of proof construction ability which is only developed with practice over time and may be expedited when working with peers and more knowing others in the EZPD (for example Joseph and Gary's contributions in sub-episode 5.3).

- **Starting the proof incorrectly and reaching an impasse**

  One occasion occurred when although the student's knowledge of the relevant definitions and proof method pertinent to the proof construction appeared to be sound, the student struggled to start the proof correctly (Frank in sub-episode 5.3). The difficulty of starting a proof correctly was one of the major sources of difficulty identified by Moore (1994). Even when Frank was assisted to begin the proof correctly, he struggled to continue, reaching an impasse. Weber (2001) discusses how quite often students who are aware of what a proof is, can reason logically,are aware of the pertinent definitions and have a good grasp of the mathematical objects relevant to the proof (students' syntactic knowledge) often fail as they reach an impasse. He refers to this failure to invoke their

syntactic knowledge as strategic knowledge. It is this strategic knowledge which seems to be lacking in this instance.

Guidance offered to students

- **Peers critique and question proof construction actions**

  Peers critiqued and questioned the proof construction actions of students doing the proof construction exercise and were instrumental in helping them to realize that each step should be accompanied by sound logical reasoning.

- **Lecturer provides prompt corrective feedback and highlights the importance of making sure of the truth of each statement made**

  When Maria repeatedly used the statement which was supposed to be proved as an assumption in the proof construction process even after the method of proof had been clarified, the lecturer and peers offered quick corrective feedback. The lecturer used transactive prompts requesting Maria to justify her actions and prompted her peers to critique incorrect actions. The lecturer also made a directive contribution emphatically reminding all the participants that they had to be sure of the truth of every statement. The lecturer repeated the question: "Is that true?". This might be significant as students often do not realize the importance of ensuring that each step taken is based on sound logical reasoning and the repeated question was aimed at emphasizing this message.

- **Lecturer offers prompt feedback and repeatedly draws attention to the assumption**

  In response to Maria's attempt to use the statement that was supposed to be proved as an assumption once again, the lecturer promptly interjected with a directive contribution offering immediate feedback on the incorrect action. The lecturer then made a facilitative contribution highlighting the fact that statements which are to be proved cannot be assumed, and a transactive prompt asking Maria to recall the correct assumption. This question was repeated. This could be significant as the lecturer was trying to emphasize that students had always to be aware of assumptions made in the proof construction process.

- **Lecturer and peers offer guidance by reminding the student of the assumption and its correct implication urging her to use logical reasoning**

When Maria had difficulty proceeding in the proof construction because of a lack in strategic knowledge in sub-episode 2.11, the lecturer used transactive requests for clarification, explanation, reflection and logical reasoning and facilitative contributions structuring proof writing and highlighting learning to steer Maria towards the correct deduction. Maria was prompted to reflect on her actions, to ensure that every step made sense and was logically sound and to remember the assumption made at the beginning of the proof. In this manner she was guided to make functional use of logical reasoning processes as she continued her proof construction attempt. Maria's peers made contributions recalling the assumption and together with the lecturer guided Maria towards the correct deduction.

- **Lecturer asks the student and peers to reflect on reasoning and strategy and use examples to illustrate mathematical objects**

  The lecturer used transactive prompts requesting students to reflect on their reasoning and find a strategy for the way forward in the proof construction. She also asked them to repeat an example of the newly met notion of the power set on the board hoping that this would clarify the interpretation and application of the mathematical object in the proof construction process.

- **More knowing peers offer contributions towards proof construction and clarify proof construction steps by using examples**

  More knowing peers offered contributions on strategy and reasoning and made improvements on the given example to better illustrate proof construction steps.

6.2.2.3: L2c: Proof methods

Students' knowledge and familiarity with proof methods such as the proof of an implication, proof of equality of sets and the proof of showing that one set is a subset of another, were found to be key to successful proof construction. It was evident however that these proof methods were a major challenge when students initially engaged with formal proof construction exercises. One reason could be that these proof methods were usually encountered in the course of covering chapters on different topics, in this case, set theory. Lecturers generally do not focus on the proof methods and discuss them as

these methods might seem rather obvious to them because most of them emerge from the definitions of the mathematical objects met in the area of study. Whatever the reasons, my study highlights the fact that lecturers do need to focus more on these proof methods and ensure that students understand and are comfortable with them.

As previously discussed there is also a link between having the correct understanding and interpretation of definitions and the ability to find the correct methods of proof. A good understanding of the definitions of mathematical objects involved in the actual proof construction enables one to make the connection to finding the correct proof methods. Moore (1994) emphasized that definitions not only provide the mathematical language and notation necessary for proof construction but also reveal the logical structure of the proof providing the justification for each step. Analysis and discussion of illustrative examples of the difficulties students had in this area is given below.

Proof methods: The proof methods of an implication and double implication

Associating the method of proof of an implication or double implication with that of an equation or identity

Many students associated the notions of the implication or double implication with the more familiar notion of an equality, or an equation in episode 1 (cf. lines 50 and 66 of sub-episode 1.3). The complex thinking observed in this episode leads to their use of an incorrect method of proof of an implication as seen later in episode 2. To illustrate, in sub-episode 1.3, Maria (line 66) described her thinking of the implication:

> **[66] Maria**: It means that… like if you are proving something which is, like you've got an equal sign like this side is equal to this, so if you put that double implication it means that what you are proving on the left you are sure that is equal to what you are proving on the right. Ja.

In episode 2 we observed Maria's interpretation of this mathematical object extended to her method of proving an implication. Her method of proof was similar to that of proving an equality or an identity as shown in line 3, sub-episode 2.1 where she attempted the proof of: $A \subseteq B \Rightarrow A \cap B = A$:

If $A \subseteq B$	*if* $A \cap B = A$
$x \in A$	$A \subseteq B$
$\Rightarrow x \in B$	$\Rightarrow x \in A$
	$\Rightarrow x \in B$

In sub-episode 2.2 when asked to explain her reasoning, Maria in line 5 said:

> **[5] Maria:** If, ok, here it says *A* is a subset of *B* and on this side it says *A* is an intersection of *B* which is equal to *A*. And if *A* is an intersection of *B* which is equal to *A* it means that *A* is a subset of *B*.

Maria's description and application of the proof method of the double implication clearly indicate that she regarded the implication as an equality or identity having two sides that she had to prove were 'equal' to each other. Subsequently she took the left hand side and the right hand side of the implication independently and by using incorrect deductions and trivial implications she attempted to show that each "side" resulted in the same statement: $x \in B$. This seems to indicate complex level thinking about her proof method of an implication.

Guidance offered: Peers question the logical reasoning in the proof method and critique deductions and assertions made without justification

After Maria's explanation of her reasoning in line 5 of sub-episode 2.2 shown above, Christine who also referred to the implication symbol as an equal sign, questioned the logical reasoning behind the proof method and critiqued deductions made without justification. Thus although Christine also had an incomplete understanding of the implication symbol, she appeared to have a better grasp of the method of proof and realized that the statement to the right of the implication symbol could not be taken as given but needed to be proved. Christine's contributions are from the category: *L2c: questions reasoning used and the methodology of the proof* and *L3b: questioning how an assertion is made from the previous statement without any basis*. The transcript from lines 6 to 13 is included in Section 6.2.2.1 and will thus not be repeated here.

Christine was instrumental in creating the EZPD where through her transactive questions for reflection, clarity and justification, Maria began to realize that she had made the assertion "A is a subset of B" without any justification (line 11). This seemed to be a clear indication of cognitive growth taking place in the EZPD as a result of interaction with Maria's peers (Christine in this case) and functional use of the proof method of an implication. Maria questioned the assertion she had made, asking in line 11: "…would I be wrong if I say A is a subset of B?"

Proof methods: The proof methods of an implication and double implication

Incorrect interpretation of the definition of implication to arrive at incorrect proof method

Although the correct proof method had been explained by the lecturer and Gary in sub-episode 1.3 and mentioned again by the lecturer in sub-episode 2.3, it appeared that Joseph and Maria had not grasped these explanations. In sub-episode 2.4 they revealed their persistent incomplete understanding of the proof method. The discussion from lines 42 to 54 is included here. This took place after the lecturer had once again asked Maria to clarify her attempted proof of *a)* $\Rightarrow$ *b)* (by making a *transactive request for clarification*).

[42] Maria: And if a) implies b) and b) implies c) we know that a) implies c). So I came here, I wanted to prove that a) is true.

[43] T: You want to prove that a) is true?

[44] Maria: Ja

[45] T: But write down for me, you remember we discussed what implication means? What does the a) implies b) mean?

[46] Maria: It means that if a) is true, then we know b) is true.

[47] Edgar: But if b) is true it doesn't mean that a) can be true.

[48] T: So if a) is true then b) is true. That's what you're trying to prove, right? If a) is true, then b) is true. So you start off with assuming something. What is what you start off with? What do you assume?

[49] Maria: I assume that…

[50] T: Don't rub everything out. Leave it. What do you assume?

[51] Maria: I assume that (a) implies (b) and (b) implies (c) and I want to show that (a) implies (c).

[52] T: Ok, somebody help her. What do you assume?

[53] Frank: Assume A is a subset of B – you'll assume that. Then you'll be fine.

[54] Joseph: You are saying if (a)'s true then (b) will be true. Now let's prove a) and why it's true, né? Then let x to be an element of A and see if it leads us to say x will be an element of B. Then if that is true it means that b) is true.

This excerpt clearly revealed that Maria was under the impression that she should first prove that the statement to the left of the implication sign was true. When asked to clarify what the statement a) implies b) means, she answered in line 46: "It means that if

a) is true, then we know b) is true". A seemingly insignificant departure from the wording in the definition of a) implies b): 'If a) is true, then b) is true' (by including the phrase '**then we know**') seems to be at the root of Maria's misunderstanding. Maria appeared to think that the proof method which follows from this definition is that she had to first prove a) to be true and from this it would automatically follow that b) was true. Joseph echoed this misunderstanding in line 54 (sub-episode 2.4). He agreed with Maria and repeated his previous reasoning from sub-episode 1.3: "You are saying if (a)'s true then (b) will be true. Now let's prove a) and why it's true, né? Then let x to be an element of A and see if it leads us to say x will be an element of B. Then if that is true it means that b) is true." Again the seemingly insignificant departure: '**will be true**' from the definition seems to mislead Joseph and cause him to use a totally incorrect method of proof.

Guidance offered: More knowing peer clarifies and elaborates the implication proof method

Gary assisted in lines 61, 62 and 67 of sub-episode 2.4 by clarifying and elaborating the implication proof method, showing true concept level thinking and creating the EZPD where Maria's proof construction abilities could develop. In fact Gary played a major role as a more knowing other throughout the proof construction process. Gary's contributions fall in the following categories: *H1a: explains the main idea behind the proof correctly, L2a: clarifies the reasons behind making a particular assumption and what needs to be proved, L1a: mathematical terms, symbols and signs correctly written and explained, L2c: correctly explains the method of proof of an implication*. Gary's contributions in lines 61, 62 and 67 of sub-episode 2.4 are included below.

> **[61] Gary**: First of all we are trying to show if A is a subset of B it will mean that it might take, it might lead us to A being an intersection B being equal to A. So what we must do now is that our assumption will be that A is a subset of B. After that we use our assumption to prove that A is an intersection of B which will be equal to A.
>
> **[62] Gary**: *[writes: Assume* $A \subseteq B$ *We show that* $A \cap B = A$ *]*
>
> *[above Assume* $A \subseteq B$ *writes:* $A \subseteq B \Rightarrow A \cap B = A$
>
> *Thus written on the board is (a)* $\Rightarrow$ *(b)*
>
> $$A \subseteq B \Rightarrow A \cap B = A$$
>
> *Assume* $A \subseteq B$. *We show that* $A \cap B = A$*]*

A is a subset of B.

[67] Gary: OK. First of all we say we must show that (a) implies (b) *[points to* (a) $\Rightarrow$(b)*]* meaning that A is a subset of B implies that the A intersection of B will give us A. Right? So first of all we must show that, we must assume that a) is true. That's why we say Assume that A is a subset of B. From this assumption we must show that it will lead us to A being a subset of B which will give us A…

Proof methods: The proof methods of an implication and double implication

Striving to grasp the correct proof method of an implication

In line 77 of sub-episode 2.5, Maria showed that she had not yet grasped the method of proof of an implication as she asked:

[77] Maria: So what I don't get here is that am I supposed to prove this or this *[points to: Assume $A \subseteq B$. We show that $A \cap B = A$]*

Guidance offered: more knowing peer gives a short simple rule using every-day language

Maria was guided by Helen (line 78, sub-episode 2.5) who told her that she should do "*the second intersection b) part*". Helen's contribution is from the category *L2a: correctly identifying what needs to be proved in the proof of an implication*. Maria seemed to identify with and appreciate this short simple rule using every-day language perhaps even more than all the explanations given before.

Proof methods: The proof methods of equality of sets and showing one set is a subset of another

Complex/pseudoconcept level reasoning of the proof method for showing equality of sets (incorporating method of proof of subset)

Having been guided by Helen to prove $A \cap B = A$, Maria encountered the method of proof of showing equality of two sets. She (in lines 83 and 85 of sub-episode 2.5) revealed complex or pseudoconceept level thinking about the proof methodology:

[83] Maria: I want to prove that like if like A and A intersection B, these things have something in common *(pointing to $A \cap B$ and A)*

[84] T: Mmm

[85] Maria: And that thing is x. So I want to show that if x is contained in A it will also be contained in B where they intersect.

She described the method of proof of equality of the two sets as showing that A and A intersection B have something in common, the element x. This might indicate complex/pseudoconcept level thinking of the methodology of the proof of equality of two sets as it seemed to give rise to the correct methodology of the proof of equality of sets but was not quite correct conceptually.

Proof methods: The proof methods of equality of sets and showing one set is a subset of another

Further complex/pseudoconcept level reasoning of the method of proof of equality of sets

In sub-episode 2.7 the proof method of showing equality of two sets was discussed further. Here Joseph exhibited complex level thinking about this method. In line 125 he talked about proving the equality of sets as proving the equality of the left and right of an equation or an identity.

[125] Joseph: Prove the left then prove the right.

He went on to elaborate in line 127:

[127] Joseph: It means if the two are equal, you find that if the left is true then the right must be true.

The actual proof method that he proposes to use, however, seems to be correct as in line 129 he elaborated further:

[129] Joseph: Say B intersection A is equals to A, then you let x to be in A, then you should show that x is also in the set of A intersection B.

Joseph described the proof method to prove that set A is a subset of A intersection B and it is presumed that he meant to do the converse also. Thus his reasoning might be pseudoconceptual as he appeared to describe the correct method but used an incorrect explanation.

Guidance offered: Lecturer makes reference to the definition of set equality and prompts students to arrive at the method of proof of showing equality of sets

In sub-episode 2.7, the lecturer brought to light the method of proof for showing equality of sets through an explicit reference to the definition of set equality (using a *transactive request for strategy and making reference to a definition* in line 139 shown below). She drew everyone's attention to how the definition of the notion of equality of two sets could be used to obtain the method of proof. I have included the conversation which took place in lines 139 to 155 to show how the correct method of proof is eventually arrived at by the use of *transactive prompts requesting clarification and explanation* (lines 141 and 145), *strategy and making reference to the definition* (lines 139, 143 and 154) as well as *facilitative contributions* highlighting learning while referring to the definition (lines 149 and 151).

[139] T: In other words, what are you trying to do? Go back to the definition guys, when are two sets equal? …When are two sets equal? … What does the definition of equality say?
[140] Edgar: When every element in the other one is also contained in the other one.
[141] T: Ok. Which means?
[142] Edgar: Which means that…
[143] T: How do we show that two sets are equal?
[144] Edgar: We need to prove that it is, this is true… When, when… Let's take a set A and a set B. We need to prove that every element in A is contained in B. And also every element in B is also contained in A.
[145] T: And what do we call that?
[146] Edgar: Um…
[147] T: Yes?
[148] Helen: Oh, I think we try to, to prove that if we have a, a subset A and a subset B we try to show that A is a subset of B and B is a subset of A.
[149] T: Good. Write that definition down for us please. That A equals B. It's very, very important and everybody is missing it here, you know. It's a fundamental definition. A equals B… You can write it right at the top there. Ja, at the top, even at the top - you're nice and tall so you can reach *[all laugh]*
[150] Helen: *[comes to the board and writes: $A=B$ when $A \subseteq B$ & $B \subseteq A$]*
[151] T: Beautiful, very nice. That's what I want. Does everybody remember that definition?
[152] S (chorus): Yes
[153] T: OK. Good. Now we are trying to show that A intersection B equals A. What do you think we're trying to show?
[154] Gary: That A is a subset of A intersection B and A intersection B is a subset of A.

Proof methods: The proof methods of equality of sets and showing one set is a subset of another

Losing sight of the correct proof method and the goal of the proof construction

Having been guided to realize that to prove equality of sets $A \cap B$ and A, one had to prove that $A \cap B \subseteq A$ and $A \subseteq A \cap B$, Maria started the proof of $A \cap B \subseteq A$ in sub-episode 2.8. Although she identified the correct plan of action and began the proof correctly by choosing x to be an element of $A \cap B$, she faltered and had to be guided to make the deduction that x will be an element of A and B. In lines 186 and 187 (included below) she again lost sight of what she had to prove and brought in the assumption $A \subseteq B$ and continued to try to arrive at what she had previously considered to be her desired goal, $x \in B$ (in sub-episode 2.1).

> **[186] Maria**: This will imply that x is an element of A and x is an element of B. *[writes:* $\Rightarrow x \in A$ *and* $x \in B$ *]*
>
> **[187] Maria:** And if x is an element of A and an element of B it will mean that x is a, A is a subset of B. That's what I think, because I say that if A is a subset of B *[writes:* $A \subseteq B$*]* it means that x is an element of A *[writes:* $x \in A$ *]* We should also imply that x is an element of B *[writes:* $\Rightarrow x \in B$*]*
>
> *[Now on board: to show* $A \cap B \subseteq A$
>
> $$\begin{aligned} &\text{let } x \in A \cap B \\ &\Rightarrow x \in A \text{ and } x \in B \\ &A \subseteq B \\ &x \in A \\ &\Rightarrow x \in B \ ] \end{aligned}$$

Thus she appeared to revert to complex level thinking again wanting to prove equality of both sides of the implication. This was once again indicative of Maria's lack of strategic knowledge and lack of a clear idea of the proof methodology.

<u>Guidance offered: Lecturer offers quick, direct and continuous assistance and repeatedly draws attention to the goal of the proof construction and peers offer contributions</u>

On seeing Maria reverting to her previous reasoning mode, the lecturer through *transactive prompts for strategy and elaboration* (line 190 and 192) and *facilitative utterances* attempting to restructure proof writing and highlight learning (lines 188, 194, 196 and 198) promptly asked Maria to reflect on her strategy. Joseph as more knowing peer also interjected with some contributions. The transcript from lines 188 to 201 is included below.

> **[188] T**: Go back to what you are trying to show
> **[189] Maria**: Ja
> **[190] T**: What are you trying to show?
> **[191] Joseph**: It will imply that x is an element of A…
> **[192] T**: Why?
> **[193] Joseph**: Because it's an element of A and B and we want to show that A intersection B is a subset of A.
> **[194] T**: You got it! What are we trying to show?
> **[195] S**: That A intersect B is a subset of A
> **[196] T**: So so far we've had that x is an element of A
> **[197] Maria**: And B
> **[198] T**: And B. So is it an element of A?
> **[199] S**: Yes
> **[200] T**:*[nods]* For sure. It's both. It's both an element of A and an element of B. So we can make the conclusion that x is an element of A, as you were saying, that x is an element of A. Is that right?
> **[201] Maria**: *[under $\Rightarrow x\in A$ and $x\in B$ writes: $\Rightarrow x\in A$]*

Further guidance offered: Detailed explanation and elaboration of proof presentation by a peer who has reached concept level understanding

At the completion of the proof of $A\cap B \subseteq A$ in sub-episode 2.8 the lecturer asked Christine to summarise the proof construction to clarify and elaborate for those who were still unsure. She did this in line 229.

> **[229] Christine**: *[goes up to the board and points to the statements]* Ok, we tried to show that A intersection B is a subset of A and then we let x be an element of A intersection B. This means that x is in both A and B and this *[points to $A\cap B$]* is an intersection of A. So x is an element of A and x is an element of B *[points to $x\in A$ and $x\in B$]* This means that since x is in both A and B, then x is also going to be in A, which you are trying to prove. Then we conclude by saying that

> *A* intersection *B* is a subset of *A*. *[points to Thus $A \cap B \subseteq A$]* So she was supposed to write this part *[draws a line under $A \cap B \subseteq A$]*

Christine comfortably interpreted and explained the newly met terms used in the proof construction such as 'implies' and 'intersection'. She used these terms with ease and was able to show in a very clear manner the connection between what needed to be proved and the statements made in the proof construction indicating concept level thinking for method of proof of subset. She identified the basis for each deduction and demonstrated as she went through the proof construction process that deductions had to be made with the necessary justification.

Proof methods: The proof methods of equality of sets and showing one set is a subset of another

Striving to grasp the correct proof method of showing one set is a subset of another

In sub-episode 2.9 Maria correctly identified that she now needs to prove $A \subseteq A \cap B$. However in line 249 she asked:

> **Maria:** So there *[points to: to show $A \subseteq A \cap B$]* I'm coming to show that *A* is a subset of *A* intersection *B*. Do I have to start with this side *[points to the first A]* or this side *[points to the $A \cap B$]*.

Thus Maria had not yet reached a full understanding of the method of proof of showing that one set is a subset of another and had not been able to transfer the proof method she had just used in the previous component of the proof (where she proved $A \cap B \subseteq A$) to this component.

Guidance offered: More knowing peer gives a short simple rule and lecturer and peers repeatedly draw attention to what needs to be proved and what is assumed

In answer to Maria's question (in line 249) of which 'side' to start with, Helen again offered a short simple rule using every-day language of 'starting with the left' in line 250. Helen's contributions are from the categories: *L2c (clarifies how to start proof in the proof framework), L2c (clarifying what needs to be shown in the proof framework).*

Proof methods: The proof methods of equality of sets and showing one set is a subset of another

Pseudoconcept level reasoning of the proof method of showing one set is a subset of another

In line 362 of sub-episode 2.11 we observed Maria's pseudoconcept level thinking of the proof method of showing that one set is a subset of another. After following the correct method of proof, which entailed picking an arbitrary element of set A and showing that this element was also contained in set B, Maria made the correct conclusion that $A \subset B$. As she was writing the conclusion on the board she explained:

> **[362] Maria**: if x is contained in A and in B it means that A is a subset of B.

She appeared to be under the impression that as x is contained in A and in B, that A would be a subset of B, which is not the correct description of the thinking behind the proof method.

Summary of difficulties and guidance in category L2c

Difficulties experienced by students

- **Association of the method of proof of a double implication and implication with the proof method of an equation or identity**

  Some students associated the notion and symbol of the double implication with the notion of equality. This association resulted in students following an incorrect method of proof for proving an implication. An example of this is that of Maria in sub-episode 2.1 who worked on the left hand side and the right hand side of the implication and tried to show that each side resulted in the same statement.

- **Incorrect interpretation of the notion of the implication to arrive at the incorrect proof method of an implication**

  In sub-episode 2.4 Maria revealed her evolving reasoning about the method of proof of an implication. In line 46 (sub-episode 2.4) she described her understanding of the definition of the implication a) implies b) to be: "If a) is true then we know b) is true". The seemingly insignificant departure from the wording in the definition of a) implies b): 'If a) is true, then b) is true' (by including the phrase '**then we know**') seems to be at the root of Maria's

misunderstanding. She appeared to think that the proof method following from this definition was that she first had to prove a) to be true and from this it would automatically follow that b) was true. This misunderstanding was echoed by Joseph in line 54 (sub-episode 2.4) who agreed with Maria and repeated his reasoning from sub-episode 1.3 when he said: "You are saying if (a)'s true then (b) will be true. Now let's prove a) and why it's true, né? …Then if that is true it means that b) is true." Again the seemingly insignificant departure from the words used in his interpretation of the definition: '**will be true**' appeared to mislead Joseph and causes him to use an incorrect method of proof.

- **Complex/pseudoconcept level reasoning of the proof methods of showing equality of sets and showing that one set is a subset of another**

  When encountering the proof of showing equality of sets, students revealed complex/pseudoconceptual reasoning. Although the method the students proposed seemed to be correct, their explanation and description of the reasoning behind the method appeared incomplete and incorrect. For example in sub-episode 2.5 (line 83), Maria conceptualized this proof method as showing that the sets had something in common.

- **Losing sight of the correct proof method and the goal of the proof construction**

  While striving to follow through the method of proof for showing that one set was a subset of another, Maria reverted to complex level reasoning and lost sight of her goal in the proof construction in sub-episode 2.8. This showed that these proof methods were still problematic for her and that her understanding of these methods as indicated by her application might still be at complex level.

Guidance offered to students

- **Peers question logical reasoning in the proof method and critique deductions and assertions made without justification**

  Although peers might have also had incomplete understanding of the notion of the implication, the questions they raised on the logical reasoning behind the proof method and their critique of deductions or assertions made without justification began to alert their struggling counterpart of the several incorrect

proof construction actions. For example Christine's questions about the logical reasoning behind the proof method and her critique of deductions and assertions made without justification prompted Maria to start questioning her thought processes. As students engaged with the proof method of an implication, making functional use of this method allowed them to address their incorrect misconceptions and move towards more correct conceptions.

- **More knowing peer clarifies and elaborates the implication proof method using his own words**

  A more knowing peer who seemed to exhibit concept level thinking on the implication proof method assisted by clarifying and elaborating on this method. These more knowing peers contributed towards the creation of an optimal environment (EZPD) where the proof construction abilities of all participants could develop. By explaining the proof method in their own words, they made it easier for their peers to follow and grasp the proof method through activities such as imitation.

- **More knowing peer gives a short simple rule using every-day language**

  When a more knowing peer offered a short simple rule in every-day language on the proof method to be followed, this seemed to be much appreciated by their struggling counterpart. The use of simple every-day language helped the student to gain understanding of previous explanations which may not have been completely understood.

- **Lecturer makes reference to the definition of set equality and prompts students to arrive at the method of proof of showing equality of sets**

  The lecturer guided the students towards discovering the method of proof for showing equality of two sets by referring to the definition of set equality and using transactive prompts requesting clarification, explanation and strategy (in sub-episode 2.7). Students participated by giving an informal definition which was gradually refined until the general method of proof was brought to light. Further transactive prompts for clarification and strategy then prompted students to apply this general method to the particular proof with which they were engaged. The practice of guiding students to develop and arrive at the correct method of proof, starting with the definition of a mathematical object

was an effective means of alerting all the participants to the importance of the definition of a mathematical object, and how that definition could be used to arrive at the method of proof. This is a valuable practice and it would be beneficial if students could be taken through such an exercise at least once. By carrying out such an exercise, in terms of the theoretical framework of my study, students' use and application of the proof method of equality of sets was brought closer to concept level understanding through the functional use of the method and reflection on the definition of set equality.

- **Lecturer offers quick, direct and continuous assistance and repeatedly draws attention to the goal of the proof construction and peers offer contributions**

  When Maria reverted to complex level reasoning on the methods of proof of an implication and showing that one set is a subset of another, the lecturer (in sub-episode 2.8) promptly asked her to reflect on her strategy and reasoning and to identify the goal of the proof construction. Maria's peers who were continuously offering their contributions answered this question correctly. The lecturer then repeated the question drawing the participants' attention to the importance of always keeping the goal of the proof construction in mind.

- **A detailed explanation and elaboration of the proof construction done by a more knowing peer**

  One of the most effective means of helping students develop and strengthen their proof construction abilities seemed to be having more knowing peers do detailed presentations on the proof construction exercise in which the students were engaged. An example of this was evident in sub-episode 2.8 when the lecturer asked Christine to do a detailed proof presentation of the component of the proof which had just been completed. The beauty of having these more knowing peers doing the proof presentation was that they used simple every-day language to explain the mathematical objects involved in the proof construction. At the same time they clarified the reasoning process and clearly showed their appreciation for the need of justification of all deductions and

conclusions. The other students were encouraged and strengthened as they realized the possibility that someone like themselves could reach concept level understanding. One of the crucial functions of lecturers could be the identification of more knowing students such as Christine. These students (with whom other students can identify) could be the key to helping other students reach similar levels of understanding through their very able presentations using every-day language. These presentations could also promote functional use of practices such as logical reasoning processes and proof methods as students learn from the more knowing peers through activities such as imitation.

- **More knowing peer gives a short simple rule and lecturer and peers repeatedly draw attention to what needs to be proved and what is assumed**

  When Maria needed more assistance with the proof of showing that one set is a subset of another, Helen (in sub-episode 2.9) again offered a short simple rule in every-day language. As Maria progressed further in the proof construction she needed further assistance and the lecturer repeatedly reminded her of the goal of the proof construction and that she had to be aware of the assumptions made, and to use these in the proof construction process. More knowing peers offered their assistance and guidance continuously until she completed the component of the proof construction in which she was engaged. The continuous and patient scaffolding from the lecturer and her peers seemed to be instrumental in the vast improvement of her proof construction ability as evidenced in sub-episode 2.11 and episode 4.

6.2.3 L3: Justification of claims

The categories: L3a (making correct deductions from previous statements and definitions while providing the necessary justification), L3b (questioning and clarifying assertions and deductions made without any justification) and L3c (identifying the basis for a claim or the reasons why conclusions can be made), are all very closely related. I will therefore include analysis of illustrative examples of the challenges and difficulties students met in these areas as a whole.

As in categories L1 and L2, students' use of deductive reasoning processes and their appreciation for the need for justification of deductions was strengthened through their

functional use of these processes as they engaged in proof construction in the consultative sessions. This functional use, in combination with the students' interaction and the guidance received from their peers and the lecturer enabled the students to make the transition to a usage of these processes in line with that of the mathematical community.

Justification of claims: Lack of appreciation for justification of assertions and deductions

Throughout episode 2 we observed Maria's lack of an appreciation for justification of assertions and deductions made in the proof construction. She also did not appear to be aware that she might be questioned and asked for justification of her deductions. In her first proof attempt at $A \subseteq B \Rightarrow A \cap B = A$ in line 3 of sub-episode 2.1, her proof construction attempt was as follows.

If $A \subseteq B$	*if* $A \cap B = A$
$x \in A$	$A \subseteq B$
$\Rightarrow x \in B$	$\Rightarrow x \in A$
	$\Rightarrow x \in B$

As she was writing on the board, she explained what she had written on the right: "Then we come to this side. That if A is an intersection of B which is equals to A it will mean that A is a subset of B. And this would mean that x is an element of A…". She seemed to think that $A \subseteq B$ followed from $A \cap B = A$. She repeated this reasoning in line 5.

Guidance offered: Peers question incorrect reasoning process and ask for justification

In sub-episode 2.2, lines 6, 8 and 10 Christine questioned Maria's proof method as well as her reasoning behind making the assertion $A \subseteq B$ following from $A \cap B = A$. This has been discussed in Section 6.2.2.2.

Justification of claims: Lack of appreciation for justification of assertions and deductions made

In sub-episode 2.8 Maria was still struggling to grasp the proof method of the implication as well as the logical reasoning and an appreciation of the need for

justification of each assertion and deduction made. Although she had now been guided to realize that she should be trying to prove $A \cap B \subseteq A$, she made an erroneous deduction without justification of the very statement she needed to prove (in line 177). The transcript from lines 172 to 177 is included below.

[172] Maria: *[Writes: to show before $A \cap B \subseteq A$]*

[173] T: Because that implies that that is true. So we don't know that that is true yet – we are trying to show that. Ok, go ahead.

[174] Maria: So let x be an element of A *[writes: let $x \in A$]* This would imply that… Ok *[underneath writes: $\Rightarrow$]* Let x be an element of A intersection B *[next to let $x \in A$ writes $\cap B$, thus we have let $x \in A \cap B$]*

[175] T: That's good. Everybody happy?

[176] S: Yes

[177] Maria: So this would imply that x is an element of A intersection B which is a subset of A *[writes: $\Rightarrow x \in A \cap B \subseteq A$]*

<u>Guidance offered: Lecturer promptly and repeatedly asks for justification and peers make suggestions of the correct deduction to make</u>

The lecturer (in line 178 of sub-episode 2.8) promptly interjected asking Maria to reflect on the truth of each statement and ensure the truth of each one. The lecturer's contributions in line 178 are in the form of *transactive prompt requesting critique and justification, and a directive contribution highlighting the fact that students need to be sure of the truth of every statement*. Joseph (in line 182) made a contribution and identified the correct deduction to be made and the reason behind this deduction. The transcript from lines 178 to 182 is included below.

[178] T: Is that true? *[Maria looks at T]* Is it true? Every step of the way you must be sure that it is true. Is that true?

[179] S: No

[180] Helen: Not yet, because we're trying to prove that.

[181] S: No, we're trying to prove

[182] Joseph: I would say x is an element of A and x is an element of B.

Justification of claims: Lack of appreciation for justification of assertions and deductions made

In sub episode 2.8 after having brought the proof of $A \cap B \subseteq A$ to conclusion, at Frank's suggestion, Maria (in line 219 as shown below in 'guidance offered') made the conclusion $A \subseteq A \cap B$ without any basis or justification. It was possible that Frank's suggestion could have been made because he still associated the implication with the double implication, and thus thought that the conclusion $A \subseteq A \cap B$ could also be made. Although Maria did try to clarify that this had not been proved yet in line 213, she did not appear to have the necessary conviction and later went on to write the erroneous conclusion on the board.

Guidance offered: Lecturer asks more knowing peer to clarify and elaborate while going through the proof construction

At the lecturer's transactive request for critique and justification, Christine (lines 223 and 227 of sub episode 2.8) firmly asserted that this conclusion could not be made and gave reasons for this. At the request of the lecturer Christine (line 229) went to the board and proceeded through the whole proof construction in detail, explaining at each step the reasons why deductions and the final conclusion could be made and clarifying the meaning of mathematical objects used in the proof construction as well as the proof method. As she proceeded through the proof construction she clearly identified the basis for each deduction and demonstrated that deductions had to be be made with the necessary justification. I argue that while Christine was proceeding with her clear elaboration of the proof method, the deductive reasoning processes used and the justification needed for each step in the proof construction, her peers were likely to be developing their proof construction skills by listening and watching attentively and developing their own ability to imitate these practices in their proof construction attempts. The transcript of line 229 was included in Section 6.2.2.3 and thus will not be repeated here.

Justification of claims: Lack of appreciation for justification of assertions and deductions

In sub-episode 5.3 Frank attempted the second component of the proof construction, that is the proof of: $P(A) \subseteq P(B) \Rightarrow A \subseteq B$. He began the proof incorrectly by choosing $\{x\}$ to be an element of A. He was guided to correct this and he started the proof correctly in line 55 by choosing $x \in A$. He then made a deduction which would lead him to the correct conclusion, but without the appropriate justification. He did provide the assumption $P(A) \subseteq P(B)$ as his justification, but the deduction did not simply follow from this.

> **[55] Frank:** Ok. *[erases the brackets so it now reads: let $x \in A$, writes: $\Rightarrow x \in B$ (since $P(A) \subseteq P(B)$)]*

Guidance offered: Lecturer requests students to reflect on proof construction actions and justify them, provides immediate corrective feedback, requests for an example to be done to clarify the proof construction

Through *transactive prompts requesting reflection and justification* (lines 56 and 59), and a *directive utterance* giving immediate corrective feedback on the proof construction (line 57), the lecturer tried to prompt Frank and the other participants to reflect on their proof construction actions and proceed logically. The transcript from lines 56 to 60 is included below.

> **[56] T:** Do we agree with that?
> **[57] T:** So you wanted to show that A is a subset of B. You've taken an element in A and then you immediately go to say that element is in B. Since…
> **[58] Student:** Is x not in power set B?
> **[59] T:** Since what? Does it follow immediately?
> **[60] S:** No, it does not follow immediately.

Further guidance offered: More knowing peers make contributions offering the correct strategy for completing the proof and build on the example given by the lecturer

More knowing peers now contributed significantly towards the correct proof construction strategy and Joseph completed the proof using logical reasoning and justifying each deduction. He also changed the example on the board slightly to make

the connection to the proof construction steps clearer. Since a detailed discussion of this was given in the section covering L2b (Section 6.2.2.2), I will not repeat the transcript or discussion here.

Summary of difficulties and guidance in category L3

Difficulties experienced by students

- **Lack of appreciation for the need of justification of assertions and deductions**

  The lack of an appreciation for the need of justification for each deduction was most obviously evident in Maria's proof attempt in Episode 2. From the very outset it was clear that Maria did not have any idea of how and why there should be any need for justification of a statement following from another. In certain instances, she appeared surprised to be questioned and asked for justification.

  A lack of an appreciation for justification of deductions was also manifested when students who had apparently developed this appreciation, experienced difficulty in making progress and became stuck due to a lack of strategic knowledge. Students in this predicament might be tempted to make a deduction they know would lead to the desired goal (while skipping a few crucial steps) citing as their justification the assumption at their disposal without really understanding how the deduction was arrived at.

Guidance offered to students

- **Peers question incorrect reasoning process and ask for justification of assertions and deductions**

  Students who themselves might have lacked clarity about some of the notions involved in the proof construction exercise but who did have a good understanding of the logical reasoning required and the need for justification of

each assertion and deduction, could be instrumental in encouraging their counterparts to reflect carefully on and be critical of proof construction steps taken.

- **Lecturer promptly and repeatedly asks for justification of deductions and peers make suggestions of the correct deduction to be made**

  When students persisted in making deductions without justification, the lecturer promptly made a directive contribution highlighting the fact that they needed to be sure of the truth of every statement they wrote in the proof construction. She also made transactive requests for critique and justification and repeatedly asked the question: "Is that true?" attempting to highlight the importance of the need for justification of deductions. It was hoped that this would prompt students to make functional use of the practice of justification for each step in the proof construction process.

- **Lecturer asks more knowing peer to clarify and elaborate while going through the proof construction**

  In order to clarify the logical reasoning used in the proof as well as the justification necessary for each step, the lecturer asked Christine to go through the whole proof construction of the component of the proof which had just been completed. As she proceeded through the proof construction she clearly identified the basis for each deduction and demonstrated that deductions had to be made with the necessary justification.

- **Lecturer provides immediate corrective feedback and requests students to reflect on proof construction actions and to provide justification for each step. She further requests an example to be done to clarify steps made in the proof construction**

  When students made deductions without the appropriate justification because of a lack of strategic knowledge, as in sub-episode 5.3, the lecturer made a directive contribution providing immediate corrective feedback. She then prompted Frank and the other students using transactive requests for reflection and justification to reflect on their proof construction actions and proceed logically. Once the proof had been successfully completed by a more knowing peer, sensing that there was still uncertainty about the proof construction steps

taken, the lecturer asked that an example be done to clarify the notion of the power set and the logical reasoning in the proof.

- **More knowing peers make contributions offering the correct strategy for completing the proof and build on an example given previously by the lecturer**

  When Frank (in sub-episode 5.3) had difficulty proceeding with the proof construction because of a lack of strategic knowledge, more knowing peers (Joseph and Gary in this case) contributed significantly towards proof construction strategy and completed the proof successfully. Sensing some uncertainty and confusion in some participants (most obviously Frank), the lecturer referred to the example illustrating the notion of the power set and asked Frank to reflect on this in order to clarify the proof construction steps taken. Joseph then altered this example slightly by replacing the elements contained in the set *A* by variables so that the connection between the proof construction steps taken and the example was better illustrated.

6.2.4: Students' holistic aspects of proof comprehension

The holistic comprehension of proof encompasses the categories H1, H2 and H3. H1 involves the main ideas behind the proof and the modular structure of the proof; H2 involves the capability of students to transfer ideas and methods of proof construction to other contexts; and H3 involves the use of examples to illustrate and improve one's understanding of the proof and statements within the proof. The category H1 which encompasses the extent to which students grasped the main ideas and methods of the proof has a close connection to category L2c which encompassed proof methods. Students having difficulty with the proof methods in a particular proof construction would also have difficulties with the main ideas of the proof, breaking the proof down into components and identifying the purpose of each component and the relation between the various components of the proof. Thus students with challenges and difficulties in the category L2c would implicitly have difficulties in category H1. Similarly the category H3 which involves the use of examples to improve students' understanding of the proof is very closely related to L1c which measures their abilities

to illustrate mathematical objects or definitions with examples. Thus difficulties in the category L1c would be common to difficulties in category H3.

In this section I will therefore focus on the difficulties and challenges students had with being able to transfer and apply ideas and methods used in previous proof construction exercises to other proofs and other contexts (category H2).

6.2.4.1: H2: Transferring general ideas or methods to subsequent proofs

Difficulties students experienced with transferring methods of proof and general ideas involved in proof construction to subsequent proof construction exercises were evident in episodes 2 and 3. In episode 2 it was observed that Maria apparently had difficulty in transferring the method of proof of showing that one set was a subset of another from the component of proof where she had to prove $A \cap B \subseteq A$ to the component of proof where proof of $A \subseteq A \cap B$ was attempted. We also observed Maria's inability to transfer ideas related to the need for justification of assertions and deductions made in the proof construction process. In episode 3 Edgar had continuing difficulty with the notion of the Cartesian product and was unable to transfer the correct interpretation and usage of this notion from the first component of the proof to the second component.

I venture that the difficulties and challenges students have with their ability to transfer general ideas or proof methods to other proofs might be related to the cognitive overload they experience while engaged with the process of proof construction. In trying to process all the requirements of formal proof construction, including the challenges of many newly met mathematical objects, proof methods, logical reasoning and the need for justification, it is understandable that they would have some difficulty in being able to transfer these ideas immediately after they have been introduced to them. It appears that students need time to internalize new notions such as newly met terms, symbols and proof methods, so that these form part of the students' own reasoning processes. In terms of Vygotsky's process of concept formation, I interpret this to mean that until full concept level thinking has been attained, pseudoconceptual thinking might easily revert to complex thinking. It is only when true concept level thinking has been reached by students, that they are able to fully internalize and transfer the knowledge they have gained successfully.

Transferring general ideas or methods to subsequent proofs: Method of proof of showing that one set is a subset of another set

In sub-episode 2.8 Maria started the proof of $A \cap B \subseteq A$. She received continuous guidance from her peers as she made many errors in both her logical reasoning and her lack of ability to provide justification for each deduction. Maria then started the proof of $A \subseteq A \cap B$ in sub-episode 2.9. She asked in line 247 and 249 which side she should start with, indicating that she had not been able to transfer the method of proof of showing one set is a subset of another set from the proof attempt of $A \cap B \subseteq A$ to the proof of $A \subseteq A \cap B$. Helen (line 250) made a contribution of a short simple rule using every-day language: "Start with *A*. Left" and Maria continued, still requiring a lot of help with all the proof construction steps. The relevant transcript is included in Section 6.2.2.3 and thus will not be repeated here.

Transferring general ideas or methods to subsequent proofs: Transferring ideas regarding the need for justification of assertions and deductions in the proof construction process

Directly following Maria's difficulty of transferring the method of proof for showing that one set is a subset of another as described above, we saw that she had also been unable to transfer ideas on the need for justification of assertions and deductions. In sub-episode 2.8 when Maria was attempting to prove $A \cap B \subseteq A$, in line 177 after choosing x to be an element of $A \cap B$ she made the deduction:

> **[177] Maria**: So this would imply that x is an element of A intersection B which is a subset of A *[writes:* $\Rightarrow x \in A \cap B \subseteq A$*]*

This action was addressed by the lecturer and her peers who reminded Maria that she needed to ensure the truth of every statement and justify each deduction. This has also been discussed in Section 6.2.3.

Now in sub-episode 2.9, as Maria was attempting to prove $A \subseteq A \cap B$, we saw that she made a very similar mistake in line 261 after choosing x to be an element of A. The lecturer immediately interjected (line 262) with a directive utterance providing feedback

on this error and drew Maria's attention to the assumption and urged her to use this in the proof construction.

[261] Maria: Ok. So I let x be an element of A. If x is an element of A and A is a subset of A intersection B *[writes: let $x \in A$]* Hmm?

[262] T: It's not! That's what you're trying to show… that's what you're trying to show… So please don't get confused with what you are trying to show, you cannot assume that. But what have you assumed, what have you got?

Transferring general ideas or methods to subsequent proofs: Transferring the correct interpretation and usage of the Cartesian product and its elements the ordered pairs

In sub-episode 3.1 while Edgar was attempting the proof of the first component of the proof construction: $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$, he was guided to develop his usage and interpretation of the Cartesian product by his more knowing peers: Gary and Joseph. Edgar was reminded several times in sub-episode 3.1, of the correct usage and interpretation of the notion of the Cartesian product and was finally able to complete the first component of the proof.

However in sub-episode 3.2 when he attempted the proof of the second component of the proof: $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$, we saw that he had been unable to transfer the correct interpretation and use of the Cartesian product to this component of the proof. He made the following incorrect deductions in line 88:

[88] Edgar: So the other one says
[writes: to show $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$
let $(x, y) \in (A \times C) \cup (B \times C)$
$\Rightarrow x \in (A \times C)$ *or* $y \in (B \times C)$
$\Rightarrow x \in A$ *and* $x \in C$ *or* $y \in B$ *and* $y \in C$ *]* So it's fine?

Summary of difficulties and guidance in the category H2

Difficulties experienced by students

- **<u>Transferring the method of proof of showing that one set is a subset of another</u>**

On completing the proof of $A \cap B \subseteq A$ with much guidance from the lecturer and her peers in sub-episode 2.8, Maria began the proof of $A \subseteq A \cap B$ in sub-episode 2.9. However it was evident that she had not been able to transfer this proof method as she asked for guidance on how to start the proof.

- **Transferring ideas regarding the need for justification of assertions and deductions**

  Throughout her proof construction attempt in episode 2 Maria was reminded and advised that any deduction or assertion made in the proof construction process needed to be justified. In particular in sub-episode 2.8 when Maria made an unjustified deduction, the lecturer wishing to stress the importance of the need for justification, repeatedly asked the question: "Is that true?" and made a directive contribution clearly stating that every step of the proof construction process had to be justified. However in sub-episode 2.9 Maria made the deduction which was the goal of the proof construction, once again without any justification.

- **Transferring the correct interpretation and usage of the notion of the Cartesian product**

  In sub-episode 3.1 Edgar was assisted to develop his understanding of the notion of the Cartesian product with the assistance of more knowing peers (Gary and Joseph) who referred to the definition and clarified its application in simple every-day language. They also identified the cause of Edgar's errors to be his association of the notion of the Cartesian product with the notion of the intersection. After the completion of the first component of the proof in sub-episode 3.1, Edgar started the proof of the second component in sub-episode 3.2. He continued to make incorrect deductions indicating his incomplete understanding of the notion of the Cartesian product and his inability to transfer the guidance he had received on the notion of the Cartesian product in the first component to the second component of the proof.

Guidance offered to students

- **More knowing peer offers a simple rule in ever-day language**

When Maria needed assistance with the proof method of showing that one set is a subset of another, Helen offered a sort simple rule in every-day language by telling her to start with the left hand side.

- **Lecturer provides prompt corrective feedback and draws attention to assumptions made**

  When Maria (in sub-episode 2.9) made a deduction which was actually the goal of the proof construction without any justification, the lecturer provided prompt feedback using a directive utterance stating that this statement still needed to be proved and drew attention to the assumption at Maria's disposal.

- **More knowing peer requests clarification and prompts struggling counterpart to reflect on the mathematical objects involved**

  When Edgar in sub-episode 3.2 faltered in his use and interpretation of the notion of the Cartesian product, Gary helped him realize his errors by asking him to clarify his thought processes and reflect on the definition of the mathematical object.

6.2.5: Difficulties not directly covered under the local or holistic proof comprehension and construction categories

There were just three note-worthy instances of difficulties which were not related to the categories used to analyse students' proof comprehension and construction attempts. One difficulty concerned students' confidence and belief in their own capabilities. Another difficulty concerned the negative aspects of the consultative group sessions. Although the creation of an environment where students came together and were encouraged to share their ideas on proof construction consulting freely and respectfully was of great benefit generally, there were instances where students might have been misled by incorrect ideas offered by their peers. Finally I consider the challenges that lecturers could have in striving to make optimal use of the consultative group sessions.

6.2.5.1: Students' confidence and belief in their own abilities

A common problem observed in students' attempts at proof construction was that they did not have the necessary confidence and belief in their own capabilities.

There was an example of this in sub-episode 2.8 when after Maria had concluded that $A \cap B \subseteq A$, Frank (lines 207 and 209) urged her to write the conclusion $A = A \cap B$. Helen (line 215) agreed with him and suggested that $A \subseteq A \cap B$ was also true. Although Maria did voice her opinion that $A \subseteq A \cap B$ had not been proved yet, she lacked conviction and acted on their suggestions. In line 219, she made the conclusion: *Thus* $A \subseteq A \cap B$.

Guidance offered: More knowing peer elaborates on proof construction

When the lecturer asked for critique and justification, Christine (line 223 and 227) explained that this conclusion could not be made because $A \subseteq A \cap B$ had not yet been proved. At the lecturer's request, Christine (line 229) elaborated and explained in detail the proof construction steps that had been taken. Christine did this very ably demonstrating the logical reasoning behind the proof construction and providing justification for each deduction and conclusion made. When a similar situation occurred in sub-episode 2.11 Maria was much more confident and appeared to realize that she should not just follow her peers' suggestions blindly if these suggestions did not make sense and were not justified.

6.2.5.2: Negative aspects of the consultative sessions

Students can be misled and confused by their peers in the EZPD even though they learned a great deal from them. One of the drawbacks of establishing an environment where all the students felt comfortable and welcome to contribute towards proof construction, sharing their thoughts on the task at hand, was that incorrect ideas could also be presented. The presence of more knowing others such as lecturers and tutors is thus very necessary to prevent these incorrect ideas from taking root in other students and becoming misconceptions. Analysis of some examples where this happened is given below.

Negative aspects of the consultative sessions: Incorrect ideas offered by students are taken up by peers

The double implication associated with the notion of subsets

In episode 1 (the first proof attempt), there was a great deal of discussion about the notions of implication and double implication. Initially in sub-episode 1.3 Edgar showed complex or pseudoconceptual reasoning when he described his thinking on the notion of the double implication as arrows going forward and back in lines 34, 36 and 40. Later in the same sub-episode where the lecturer asked for further clarification Edgar heard Helen's contribution in line 85 (included below) referring to what $P \Rightarrow Q$ means.

> **[85] Helen**: I think it actually means that P is not Q but it may be Q. I think.

It is likely that he might have been misled and he may have thought that Helen regarded P and Q to be sets. He then appeared to revert to complex level thinking associating the notion of the implication with the notion of subset. We note that the students had just recently been introduced to the new mathematical terminology of set theory where capital letters denote sets. Edgar's contribution in line 91 (sub-episode 1.3) is included below.

> **[91] Edgar**: I think that, I think if there are certain elements in P that means all of them they can be found in Q. But not all elements that are in Q can be found in P.

Negative aspects of the consultative sessions: Incorrect ideas offered by students are taken up by peers

Misinterpretation of the definition of union

When discussing the notions of intersection and union in sub-episode 2.6, Gary put forward the idea and then did an example showing that the union of two sets did not contain the elements from the intersection of the sets. Christine (who went to do another example on the board) confirmed this mode of thinking. Maria unfortunately seemed to be misled by Gary's and Christine's incorrect conceptions of the union and asked in line 113:

> **Maria**: So for union they don't have anything in common?

6.2.5.3: The challenges lecturers might face in making optimal use of consultative sessions

At the beginning of the proof construction attempt in episode 2, Maria did not have a clear idea of how to prove the equivalence of the statements: *a)* $A \subseteq B$, *b)* $A \cap B = A$ and *c)* $A \cup B = B$. Her plan of action was: *(a)* $\Rightarrow$ *(b), (b)* $\Rightarrow$ *(c), (a)* $\Rightarrow$ *(c)* as she described in line 1 of sub-episode 2.1. The lecturer allowed her to proceed, thinking that perhaps this would be clarified later or she had perhaps not noticed the discrepancy. In retrospect the students would have benefitted more if the notion of the equivalence of the statements had been elaborated on, and explained in detail at the beginning of the proof construction in order to help them have a clearer idea of what had to be done. There were also times when I felt that I fell short in the thoroughness of my explanations and in providing the appropriate feedback while the students were busy with the task of proof construction. This is an aspect that lecturers need to be aware of as they strive to make optimal use of consultative sessions.

Summary of difficulties and guidance falling outside the categories of my framework

Difficulties experienced by students

- **<u>Students lack confidence and belief in their own capabilities</u>**

  It was evident particularly in the first session that even though students (for example Maria) seemed to have the correct idea, they were easily misled by their peers as they lacked the conviction and belief in themselves and their own capabilities. It became apparent that through their engagement and interaction in the consultative group sessions, they gradually built up this confidence and belief in themselves. By the second session they were able to stand their ground when questioned, and were able to explain their reasoning in defence of their actions. The proof presentations done by more knowing peers who went through the proof construction with conviction and clarified the logical reasoning involved and the need for justification of each deduction, plus the encouragement received from the lecturer and peers throughout the proof construction process were also factors in building up students' confidence.

- **Incorrect ideas and conceptions may be propagated**

  One possible drawback of students working together in the consultative group sessions where participation and sharing of ideas was encouraged was that incorrect ideas could be offered by some of the participants and these could be adopted by the rest of the students. This was particularly evident in the first session where students were still very new to formal proof construction and all the notions involved in the process. The presence of a more knowing other such as a lecturer or tutor is vital at this stage to guide the students towards correct ideas and conceptions.

- **Challenges that lecturers might face in striving to make optimal use of consultative sessions**

  When I went through the transcripts of the video recorded sessions there were several instances I wished I could go back and handle the discussions differently. These were times where I felt that I fell short in the thoroughness of my explanations or where I felt I had not provided the appropriate feedback that one in hindsight realizes should have been provided.

  One of the possible reasons was that the methodology of doing proof solving in the context of consultative group discussions was new. I had used this methodology for the first time in the pilot study, a year earlier. Also the idea was to give as much room as necessary so that the students could try to figure out as much as possible for themselves. Consultative group sessions are very different to the traditional mode of lecturing where the class silently listens to everything the lecturer says, or to tutoring where there is usually one-on-one interaction between a student and the tutor. In consultative group sessions it is the students themselves who are the desired participants, as the whole idea is to encourage them to actively learn from each other and the lecturer (if necessary) as they work on proof construction exercises. The lecturer's task is to encourage students' participation and sharing of ideas while establishing the norms that would pertain to successful proof construction. These norms include encouraging certain modes of thinking and discussion such as using logical deductive reasoning and justification in the proof construction process. The lecturer also provides guidance on definitions and proof methods, when

necessary, and assists when incorrect ideas or proof construction actions are presented, or when incorrect strategies and proof methods are being used. This rather novel mode of discussion which has proved very effective and exciting, (as shown in this study) might also present challenges. Lecturers not only need a thorough understanding of the material to be covered, but they also need to be aware that the group brings with it a certain energy which requires the lecturer to be dynamic in his/her guidance.

In order for a lecturer to be effective in engaging students and driving discussions of optimal benefit to the students, I propose that not only thorough knowledge of the subject matter is needed, but also the lecturer needs to be able to think and make decisions on his/her feet on when and where to interject, and where it is best to leave students to come to an understanding by themselves.

I also suggest that it is important for teachers and lecturers to have a thoughtful attitude and awareness that they themselves are also always learning when teaching. A lecturer can and does make mistakes and omissions all the time, and if he/she adopts a reflective attitude of learning at all times, then he/she will be more inclined and empowered to improve.

6.3 Concluding Summary

In conclusion I highlight key challenges evident as students attempted proof construction in the consultative sessions and I discuss the significant forms of scaffolding observed as effective in contributing to overcoming these challenges.

Difficulties with the meanings of newly met terms, symbols and signs included incorrect language use, inappropriate use of terms and symbols and association of newly met terms, symbols and signs with more familiar terms, symbols and signs. The functional use of terms, symbols and signs while interacting with peers and more knowing others helped bring students' use and interpretation of these mathematical objects closer to concept level. Clarification of newly met terms and symbols using simpler every-day language and pseudoconcept/ concept level explanations which likened the terms or symbols to more familiar terms and symbols while conveying the correct application and proof method were also very effective.

Difficulties with mathematical definitions included misinterpretation of definitions and association of mathematical objects and symbols with a word contained in their definitions. Another serious difficulty was students' misinterpretation of the definition of a mathematical object (relevant to a specific proof) resulting in an incorrect proof method. Forms of scaffolding included using examples to clarify definitions of mathematical objects. These were initiated by the lecturer but the students enthusiastically took over and participated in this activity whole heartedly. Using examples to clarify and discuss definitions of mathematical objects captured their interest and attention. The lecturer in the first episode also referred to the definitions of mathematical objects several times, alerting students' attention to their importance. When students had difficulty in discovering the method of proof of set equality, the lecturer prompted students to examine the definition carefully and reflect on the appropriate strategy that would result in the correct method of proof. More knowing peers assumed the role and responsibilities of the lecturer in the second session, adopting the norms established in the first session. They explained the meaning of definitions in their own words and seemed to pass on to others, their own growing appreciation of the importance of the correct understanding and interpretation of definitions of mathematical objects, and the implications of this correct interpretation for the justification of steps in the proof construction process. They also used examples to illustrate definitions and clarify misconceptions. The effectiveness of the learning environment in the consultative sessions where students made functional use of mathematical objects and definitions while interacting with one another, was clearly evident. Students appreciated the usefulness of definitions more fully and strove to clarify definitions of mathematical objects for themselves and their peers. An example of this occurred in Episode 5 where Gary and Joseph built their understanding of the notion of the power set through interaction with their peers and by reflecting on the definition of the notion.

Difficulties that students experienced with selecting examples to illustrate terms and symbols and proof construction steps were mainly a result of their struggle to generate appropriate examples. This could have been caused by their inexperience with the mathematical objects and the subject area. While encouraging the use of examples and alerting students to the value of examples in illustrating newly met mathematical

objects, lecturers have to be aware that students might not be able to generate helpful examples for themselves. Lecturers might need to provide this form of scaffolding for them. Oral examples from peers were generally ignored. Lecturers should draw attention to these examples and ask that they be done on the board.

The difficulties that students exhibited in the selection of statements and phrases which would add to the logic of the proof construction process were exposed by their struggles to clearly state the assumptions and the statement to be proved at the beginning of the proof construction. Such statements are especially important in the initial stages of proof construction as they help students to clarify for themselves and others, what needs to be proved and raise awareness of the assumptions at their disposal. Peers offered scaffolding in the form of contributions using simple every-day language to clarify unfamiliar terms and symbols and encouraged students to make functional use of statements which added to the logic of the proof construction process. Students made quick improvement in this aspect.

Difficulties that students had with the selection of useful or appropriate aspects of definitions and assumptions occurred mainly at the initial stages of their introduction to formal proof construction when the various proof methods had not yet been fully grasped. These included their use of statements that needed to be proved as statements that were given or assumed. As discussed earlier, the inclusion of statements that would add to the logic of the proof construction process, like statements at the beginning of the proof construction setting out what needs to be proved and the assumptions at the students' disposal would also help in this regard. Students also selected non-useful or trivial aspects of definitions and assumptions. They also had difficulty in some instances in starting the proof correctly and in driving the proof construction forward, even though they appeared to have a good understanding of the assumptions, definitions, newly met terms and symbols and proof methods relevant to the proof construction, thus revealing a lack of strategic knowledge (cf. Weber, 2001). Forms of guidance included critique from peers on logical reasoning and justification, prompt corrective feedback from the lecturer and peers, repeated reminders to students that every statement had to be true and justified, repeated reminders to be aware of assumptions made and the statement which needed to be proved and appeals to students

to be aware of their reasoning processes and use logical reasoning and reflect on strategy. The lecturer encouraged the use of examples to clarify the mathematical objects involved in the proof construction in order to shed light on proof construction steps. More knowing peers who assumed the role and responsibilities of the lecturer provided scaffolding to their peers by making contributions on strategy, and clarifying proof construction steps and reasoning by using examples.

Proof methods posed a major challenge especially in the initial stages of proof construction. The proof methods of the implication and double implication in particular posed great difficulties. As students interacted in the consultative group sessions, their functional use of proof methods enabled them to address their misconceptions and gain better understanding in terms of the use and application of these methods. For example, the notion of the implication and the proof method to be used in an implication proof was associated with equality and showing each 'side' of the implication gave rise to an identical statement. Further on in the proof construction process it became apparent that students' understanding of the method of proof of $P \Longrightarrow Q$ was to prove that P was true from which it would follow that Q was also true. This misunderstanding seemed to arise from students' departure from the correct wording of the definition of the notion of implication. The methods of proof for showing equality of sets and showing that one set was a subset of another also posed some difficulty. Once again students' functional use of these proof methods, as they engaged with the proof construction while interacting with their peers and the lecturer, allowed them to address misconceptions and arrive at the correct use and application of these methods. Students showed complex/pseudoconcept level reasoning on these proof methods describing the method of proof of set equality as showing that the two sets had an element in common. Although their description of the reasoning used was inappropriate, the method of proof that these students tended to pursue seemed to be correct.

Significant forms of guidance included peers questioning the logical reasoning used in the proof method, clarifying and elaborating the proof method and offering a short simple rule using every-day language. The lecturer helped students arrive at the correct method of proof of set equality by drawing their attention to the definition of set equality and prompting them to reflect on the strategy for proof construction and to

apply logical reasoning. The lecturer also asked more knowing peers who seemed to have reached concept level understanding in all aspects of proof construction to go through proof components which had just been completed and to elaborate on proof methods and proof construction steps. The identification of such peers is paramount as they are very useful for communicating and conveying to their classmates their own understanding and appreciation of all aspects of proof construction including the important aspect of proof methods. These students have the potential in helping their fellow students' hidden talents to emerge.

Difficulties students exhibited with regard to justification of claims included making deductions and conclusions without appropriate justification. Their appreciation of the need for justification of statements was abandoned when experiencing difficulty because of a lack of strategic knowledge. Those who were stuck and could not proceed often made unjustified deductions and conclusions. They were encouraged to make functional use of deductive reasoning processes and the practice of justification as they received scaffolding from their peers and the lecturer. Forms of guidance included peers questioning and critique of reasoning processes. When students persistently made assertions and deductions without justification, the lecturer interjected and asked students if they were certain of the truth of such statements reminding students that each statement had to be justified. More knowing peers doing proof presentations of proof components which had just been completed, demonstrated to their classmates that each step in the proof had to be accompanied by logical reasoning and the appropriate justification. At times the lecturer asked these peers to do examples on the board to clarify the mathematical objects related to the proof construction and thus clarify proof construction steps.

Difficulties students experienced with transferring methods and ideas to subsequent proof exercises included their inability to transfer methods of proof from one proof component to another as well as their inability to transfer knowledge and usage of mathematical objects involved in the proof construction as they proceeded through the proof. For example in the initial stages of proof construction the notion that each deduction and conclusion made had to be justified and based on logical reasoning, had to be repeated several times. I suggest that one reason for these difficulties may be the

cognitive overload that students experience as they engage with the process of proof construction. In the struggle to master all the requirements of formal proof construction, it is understandable that students would have some difficulty in transferring these ideas and methods so soon after they have been introduced to them. Time is needed for new mathematical objects such as newly met terms, symbols and proof methods, to become internalized and attain concept level realization. Before this happens, pseudoconceptual thinking might easily revert to complex thinking.

Finally difficulties or challenges outside the categories of the framework for analysis of proof construction and comprehension included students' lack of confidence and belief in their own abilities, the challenge of incorrect ideas that might be propagated in an environment where contributions from all students were welcome, and the challenges that lecturers have to keep in mind as they strive to make the very best use of consultative sessions. With respect to students' confidence and belief in their own capabilities, one of the most empowering learning opportunities seemed to be proof presentations done by more knowing peers which were delivered with conviction showing the others the possibility that students like themselves had been able to master proof construction abilities, and reason using sound logical processes. I also argue that the encouragement offered by the lecturer throughout the sessions (in the form of facilitative utterances) played a role in bolstering students' confidence and belief in their own capabilities. Encouragement is a powerful motivator and students should be encouraged as much as possible especially in the initial stages when obstacles often seem insurmountable. On the challenge of erroneous ideas and notions put forward which could be adopted by other students in the consultative sessions, it is suggested that the presence of lecturers or more knowing others is necessary to guide the students away from these misconceptions and re-direct them towards more correct ideas and methods.

Competencies that lecturers need to develop as they strive to make optimal use of consultative sessions include being able to make quick decisions while taking part in the consultative sessions, being able to provide the necessary guidance while at the same time allowing students to be active participants and empowering those showing the

potential to become more knowing peers to take over the responsibility of providing the necessary scaffolding.

The analysis in this chapter (together with the coded and briefly analysed transcripts found in Appendix 1) is also used to trace the paths of development in proof construction abilities of two case studies in particular. The analysis is used to point to the areas where there was evidence of transformation in the students' abilities. This was used to address my second research question in Chapter 7. The analysis in this chapter, together with the coded and briefly analysed transcripts found in Appendix 1 was also used to address my third research question (in Chapter 8) by investigating the nature of interactions between the lecturer and students, and between students themselves as they engaged in proof construction in the consultative sessions. This enabled me to trace the nature and patterns of scaffolding offered by the lecturer as she tried to create an environment where students might be best supported to access their zones of proximal development (EZPD), thereby making progress in their proof construction abilities. The analysis also enabled the researcher to identify the characteristics of and modes of reasoning used by those who seemed to show potential in becoming more knowing others.

Chapter 7: Investigating how students' proof construction abilities evolve and develop in the consultative group

7.1 Introduction

In this chapter I will be further analysing and discussing themes which emerged from the coding and analysis of video recorded consultative sessions (found in Chapter 6 and Appendix 1) to answer Research Question 2. My second research question is repeated below for ease of reference.

Research Question 2

Investigating the development of students' proof construction abilities as they participate in consultative group sessions through the use of two case studies:

How do the proof construction abilities of two case studies, Frank and Maria evolve and develop as they progress through the sessions?

In line with my theoretical framework (Vygotsky, 1986, 1994) the consultative sessions were intended to avail to students an environment which encouraged and facilitated access to their zones of proximal development and allowed for functional usage of newly met terms, symbols, logical reasoning processes, proof methods and the practice of justification. I will argue that these sessions seemed to be highly beneficial leading to more effective development of the students' higher mental functions which encompassed their proof construction abilities. I will be closely examining the journeys of two case study students, Frank and Maria, both of whom participated in proof construction exercises in the first and second sessions. Throughout the analysis and discussion of students' proof construction attempts I have made inferences about the categories to which students' usage, interpretation and application of newly met terms, symbols, signs, proof methods, logical reasoning and justification processes, belong, according to the indicators of my analytical framework which has been described in detail in Section 5.2.2.

7.2 Students' progression in the first two sessions

I have used the journeys of two students, Frank and Maria to illustrate how the consultative group sessions might be an efficient means of helping students gain understanding and confidence in proof construction. Students at all levels of mathematical proficiency appeared to benefit by their participation in the sessions and the interaction with their peers. Even students who were really struggling with all the aspects of proof construction like Maria, made large gains in a relatively short period of time.

Both Frank and Maria were first year students. In the first module of the Pure Mathematics course in the first semester Maria had achieved 67% in the final exam and was therefore in the B category (as described in Chapter 4) while Frank had achieved 80% in the final exam and was in the A category.

We follow Frank and Maria as they attempt proof construction exercises in the first two consultative sessions. The two sessions were just one week apart. Frank attempted the proof of the proposition: *If* $A \subseteq B$ *and* $B \subseteq C$, *then* $A \subseteq C$ in the first session in Episode 1 and the proof of the proposition: $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$ in the second session in Episode 5. Maria attempted the proof of the equivalence of the statements: *a)* $A \subseteq B$ and *b)* $A \cap B = A$ in the first session in Episode 2 and the proof of the proposition: $(A \cap B) \times C = (A \times C) \cap (B \times C)$ in the second session in Episode 4.

7.2.1 Frank's journey: evolution of proof construction abilities

Frank began with the proof of the following proposition: *If* $A \subseteq B$ *and* $B \subseteq C$, *then* $A \subseteq C$ in Episode 1 in the first session. A successful proof construction of this proposition requires knowledge of the methodologies of an implication proof and of showing that one set is a subset of another set. It also requires knowledge of the precise definition of subset plus the ability to use this definition in the logical reasoning and justification of each step in the proof. Frank returned to the board in the second weekly session to attempt the proof of the proposition: $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$ in episode 5. This proof requires knowledge of the proof methods of the double implication, implication and of

showing that one set is a subset of another set, as well as knowledge of the precise definitions of subset and power set. Such knowledge is necessary for the student to take appropriate actions that add to the logic of the proof construction and to make correct deductions based on the necessary justification. Below is a discussion of Frank's proof construction attempts in these episodes.

7.2.1.1: Episode 1: Frank's attempt at the proof: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Frank was the first participant to attempt a proof construction task in the first session on the board to attempt the proof of: *If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Sub-episode 1.1

In sub-episode 1.1, Frank's initial attempt at proof construction seemed to show some familiarity with the framework of an implication proof. The main written flaw in this proof construction was that instead of using the implication symbol, Frank used the double implication symbol. As he wrote the proof on the board, he repeatedly referred to the double implication symbol as approximation. The consistent use of the incorrect word when referring to the double implication sign was probably due to the fact that he was not familiar with the correct word.

Sub-episode 1.2

In sub-episode 1.2 it became clear that Frank's grasp of terms and symbols associated with the proof construction process (which lecturers often take for granted) such as 'suppose' and 'imply' and the implication and double implication symbols was incomplete. His incorrect use and incomplete explanation of the notions of the implication and double implication confirmed that his interpretation and use of these notions was at complex level. For example, when Edgar suggested the addition of a statement (containing the word 'suppose') at the beginning of the proof which would add to the logic of the proof construction process, by making clear the assumptions in this proof, and what had to be shown, Frank (line 4) wrote a contradictory statement showing that the word 'suppose' and the notion of the implication were incorrectly used and interpreted. This is shown below.

> **[4] Frank:** Okay you want me to write suppose A is a subset of B and B is a subset of C implies that A is a subset of C. *[writes as he is speaking directly above his proof attempt: Suppose* $A \subseteq B$ *and* $B \subseteq C \Rightarrow A \subseteq C$*].*

This could also indicate that Frank's understanding of the proof framework of an implication was at complex level. Frank received some guidance from his peers. Gary and Helen offered their more appropriate ideas about the proof framework for an implication. In line 5 Gary clarified that the statement on the right of the implication was the one "we are supposed to prove" and in line 7, Helen clarified that the statement on the right of the implication was the one "we need to show".

Sub-episode 1.3

When Gary in line 11 asked what Frank meant by the double implication sign, he answered in line 14 that it stood for approximation. Through the functional use of the newly met terms, symbols and signs as well as statements which added to the logic of the proof construction, together with the scaffolding received from his peers, he began to align his word usage to that of the mathematical community. In line 14 probably as a result of Frank's interactions in sub-episode 1.2 with Gary and Helen whose understanding of the notion of implication seemed to be at concept level, Frank made the transition from 'approximate' to 'imply' when referring to the implication.

In line 32 it became evident from Frank's discourse that he did not make any distinction between the implication and the double implication, which explained why he had used the double implication symbol instead of the implication throughout the proof. He appeared to be associating the two notions together and this confirmed that his understanding of the notions of the implication and double implication was incomplete and at complex level.

In this sub-episode the lecturer continued to ask questions (see Appendix for transcripts) probing students' understanding of the notions of the implication and double implication and at the conclusion of proof construction, she also tried to make the distinction between the definition of these mathematical objects and the actual proof method of an implication proof. The lecturer's transactive prompts revealed the students' various (complex and pseudoconceptual) interpretations of the notions of the implication and the double implication. These included associations with the notions of

equality, equations, subsets and arrows going forward and back. These contributions probably played a large role in developing Frank's understanding of the notions of the implication and double implication. As students engaged with the notions of the implication and double implication, they made functional use of these mathematical objects and the proof method of the implication which appeared to enable them to make progress in their use and application of these objects and processes.

In line 70 Gary contributed guiding Frank and the other participants towards a better understanding of the notions of the implication and double implication and their respective proof methods. Although Gary associated the implication proof with an equation having a left hand side and a right hand side, he described the method of proof as using the left hand to get to the right and vice versa. This complex/pseudoconcept level thinking about the proof framework of an implication was nevertheless helpful and a step towards a more correct interpretation and application of the implication proof framework. In line 72 Gary explained why the double implication sign should be replaced with the implication sign and in lines 78 and 79 he illustrated the notions of the implication and double implication by using an example, showing why the single implication had to be used instead of the double implication in Frank's proof construction attempt. He then corrected Frank's proof attempt by substituting implication signs for the double implication signs.

Summary of episode 1

As shown in Table 7.1 Frank made 24 proof construction actions, 13 of which were correct. In Table 7.2 we see that most of Frank's incorrect actions are from category L1 (meaning of terms, symbols and signs) while categories L2 (logical status of statements and proof framework) and L3 (justification of claims) are almost free of error (1 incorrect action in each category). Frank appeared to have a basic understanding of the logical reasoning and need for justification of each deduction (encompassed in categories L2 and L3). His use and interpretation of newly met terms, symbols and signs and definitions encompassed in category L1 appeared to be at complex level. Thus we observe that the categories L2 (logical status of statements and proof framework) and L3 (justification of claims) played an important role, allowing Frank to

make progress in the proof construction even when his understanding of the usage and interpretation of newly met terms, symbols and signs is incomplete.

As Frank advanced through the proof construction, he made functional use of the terms, signs and symbols such as the implication and double implication symbols, as well as statements which added to the logic of the proof construction. Through engagement and interaction with other participants together with his functional use of newly met terms, symbols, signs, logical reasoning processes, proof methods related to the proof construction and justification practices, these mathematical objects and processes were refined and could hopefully mature into genuine and true concepts, in line with their usage by the mathematical community. It was interesting to see how Frank's use and application of proof methods and statements which add to the logic of the proof construction as well as his use of the terms associated with the implication proof improved in the second session in Episode 5.

7.2.1.2: Episode 5: Frank's attempt at the proof of the proposition: $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$

The proof of this proposition involved the proof of the two implications $A \subseteq B \Rightarrow P(A) \subseteq P(B)$ and $P(A) \subseteq P(B) \Rightarrow A \subseteq B$ and required knowledge of the proof methods of the double implication, of implication and of how to show that one set was a subset of another set, as well as the knowledge of the precise definitions of subset and the power set, a new notion only covered in class very recently.

Sub-episode 5.1

Frank started the proof construction by breaking down the proof into components and beginning with the first component: $A \subseteq B \Rightarrow P(A) \subseteq P(B)$ showing that he had developed pseudoconcept/ concept level understanding of the double implication proof method.

There was a vast improvement in his use and interpretation of terms and symbols related to formal proof construction (category L1a) and his ability to select statements and phrases which added to the logic of the proof construction process (category L2a). This was presumably a result of the functional use of terms, symbols, signs, logical and deductive reasoning processes and proof methods during the proof construction

attempts in the first two sessions as well as all the interaction and scaffolding he received as he and the other students participated in the EZPD. Before starting the first component of the proof, Frank (in line 1) clearly indicated that he intended to prove the first component of the proof. He wrote out his assumption and what was needed to be proved. His explanation in line 3 confirmed his concept level understanding of being able to select appropriate statements which added to the logic of the proof construction process (category L2a). Frank now showed a much better use of notions, terms and symbols related to the proof construction process such as 'suppose', 'assume' and the implication symbol (category L1).

Frank then went on to start the first component of the proof correctly, and showed pseudoconcept/ concept level understanding of the proof method of an implication and the proof method of showing that one set is a subset of another. Frank's elaboration of his proof construction in line 3 together with his correct proof construction actions confirmed concept level understanding of all three proof methods. In contrast to his proof construction actions and elaborations in Episode 1, he had developed concept level use of all the proof methods presumably as a result of his functional use of the proof methods and his interactions in the consultative group sessions. He was now very comfortable and able to use all the correct terminology connected with the proof methods such as 'assume'.

He also exhibited correct interpretation and application of the definition of the newly met notion of the power set, translating elements of the power set of a certain set to be subsets of that set and vice versa. This appeared to indicate that his understanding of the newly met terms was at pseudoconcept or concept level. However in lines 5 and 22, his incomplete explanations of the notion and his uncertainty about his correct deductions in the proof construction seemes to indicate that his understanding of these mathematical objects was at pseudoconcept level. He finished the proof of the first component with 17 proof construction actions without any errors (see Table 7.1).

In this sub-episode Frank's peers took the opportunity to reflect on, and ask questions about his completed proof construction of the first component of the proof and strengthened their understanding of the newly met notion of the power set.

Sub-episode 5.2

In sub-episode 5.2 Joseph used his own initiative to do an example (on the board) to illustrate the notion of the power set both for himself and others. In the discussion that followed, as students such as Joseph, Gary, Frank and Maria made functional use of the notion of the power set, they appeared to have made gains in their understanding of this mathematical object. As the notion of the power set was very new, students did not have a large repertoire of useful examples to use and the example given by Joseph was not very helpful because he drew the Venn diagram of a power set of a set A trying to populate it with elements, without first drawing the Venn diagram of the set A with its elements. The lecturer did another example in order to illustrate the notion of the power set more clearly, in line 41. In this very simple example she first drew a Venn diagram of the set A, having two elements and then drew the corresponding power set of the set A, in the hope that this would help students to see the connection between the elements of a set and the elements of its power set and helping to clarify the mathematical object further.

Sub-episode 5.3

In sub-episode 5.3 Frank continued with the next component of the proof, the proof of $P(A) \subseteq P(B) \Rightarrow A \subseteq B$. He started the proof correctly showing concept level understanding in his application of the methodology of the implication proof. His ability to select statements and phrases which add to the logic of the proof construction process was again evident as he clearly stated the plan of action, the assumptions and what was needed to be proved. In order to prove $A \subseteq B$ he began by choosing $\{x\}$ to be an element of the set A. However $\{x\}$ is a set and cannot be an element of A. The correct course of action would have been to choose x to be an element of A, and then make the connection that $\{x\}$ was a subset of A and hence an element of the power set of A, that is, an element of $P(A)$. When prompted by the lecturer and reminded that A was a set and not a power set, Frank (line 55) correctly chose x to be an element of A, but then immediately made a deduction ($\Rightarrow x \in B$ *(since* $P(A) \subseteq P(B)$*)*) which would lead him to the correct conclusion while omitting several crucial steps and without having the necessary justification. It was interesting to note that although an appreciation for the need of justification while making deductions and conclusions had always been one

of his strengths, this was abandoned when he had difficulty making progress in attaining the desired goal.

Gary and Joseph who were trying to build their understanding of the power set through peer interaction as well as earnest reflection on the definition of the power set and Frank's proof construction of the first component of the proof (in sub-episodes 5.1 and 5.2), now came to Frank's aid. They contributed positively by suggesting the correct strategy and deductions for correct proof construction. Joseph went up and completed the proof in line 77 and when he realized that Frank was still unclear about the proof construction and the reasoning he had used, he altered the example (of a power set) which the lecturer had written on the board in sub-episode 5.2 by replacing the elements 1 and 2 with the general variables x and y. By doing this the relationship between the elements of a set and the elements of its power set was better demonstrated.

Summary of Episode 5

In this episode Frank showed concept level understanding of the proof methods of the double implication, implication and showing that one set is a subset of another (category L3). The functional use of these proof methods as he engaged with proof construction exercises (in session 1) while interacting with his peers had possibly enabled Frank to make a great deal of progress in this regard. He also showed sound reasoning abilities and was able to select appropriate statements and phrases which added to the logic of the proof construction (category L2) presumably as a result of guidance received from his peers and his functional use of such statements. His usage of terms, symbols and signs related to the proof construction process (category L1) was correct and appropriate. The functional use of the terms, symbols and signs involved in the proof construction process seemed to have enabled Frank to reach concept level use and interpretation. He also seemed to have pseudoconcept level understanding of the interpretation and application of the notion of the power set on which the proof construction depended. His application and use of the notion was correct as was evident in his correct proof construction of the first component of the proof in sub-episode 5.1. However in sub-episode 5.3 his difficulty in starting the second component correctly and making progress in the proof construction was probably due to an incomplete grasp

of the notion of the power set (thus leading me to conjecture that his understanding of this notion was pseudoconceptual) as well as a lack of strategic knowledge.

As seen in Tables 7.1 and 7.2 out of 45 proof construction actions in Episode 5, Frank did 41 correctly. His incorrect actions stemmed from his inability to use the assumption at his disposal (involving the notion of the power set) to proceed logically in the proof construction. He received most of his guidance from his peers who made 24 correct contributions. There were 15 transactive prompts from the lecturer asking for explanation, reflection, strategy, critique, justification and examples to clarify mathematical objects and proof construction steps. The lecturer also made 4 directive and didactive utterances (towards the end of the proof construction process) providing immediate feedback on incorrect actions and reminding students of the methods of proof.

7.2.1.3: Overall discussion of Frank's journey

Frank appeared to have made large gains in terms of his understanding of proof methods relevant to these sessions. Whereas in Episode 1 his understanding of the proof method of the implication and the terms and symbols used in the proof construction were at complex level, in Episode 5, we saw what appeared to be concept level understanding of all proof methodologies (including the implication, double implication, subset and equality). He also showed great improvement in his use and application of terms, symbols and signs involved in the proof construction process. He appeared to have reached concept level understanding through functional use of the terms, symbols, proof methods and logical and deductive reasoning processes related to the proof construction process. His understanding of the logical reasoning involved in proof construction and the need for justification of each statement of which he seemed to have some basic understanding in the first episode had been strengthened in Episode 5.

In sub-episode 5.3, when attempting to prove $P(A) \subseteq P(B) \Rightarrow A \subseteq B$, Frank struggled to start the proof correctly. Although he seemed to know the definition of the power set and was able to apply it in sub-episode 5.1, bringing the proof of the first component to completion with no errors, he was unable to work out how to use the assumption $P(A) \subseteq$

P(B) to prove $A \subseteq B$. It was apparent that although Frank was able to reason logically and was aware of the pertinent definitions and mathematical objects relevant to the proof (students' syntactic knowledge), he failed to make progress in the second component of the proof construction as he seemed to have reached an impasse. This was probably due to his pseudoconceptual grasp of the notion of the power set and a lack of strategic knowledge. When the lecturer helped him to realize the correct first step, he had difficulty in proceeding to the next, and instead made a deduction which would have led to the desired conclusion, but without the necessary justification as well as omitting several crucial steps. It was noted that when the students were stuck and proceeding with proof construction became challenging, the logical reasoning and justification of each step which seemed to be well established habits seem to be abandoned. This was aggravated by the introduction of new and unfamiliar mathematical objects which caused uncertainty and confusion. This could indicate that these deductive reasoning processes and the appreciation of the need for justification were not at concept level.

Table 7.1 below summarizes Frank's correct and incorrect proof construction actions and contributions in Episodes 1 and 5 showing his great improvement in proof construction abilities from the first session to the second. This is an indication that Frank had made large gains regarding all aspects of proof construction and was well on his way to becoming a member of the wider mathematical community. Contributions from Frank's peers and the lecturer are also shown to have decreased from the first to the fifth episode.

Table 7.1: A summary of Frank's journey in terms of proof construction actions and lecturer and peer's actions and utterances

Episodes	Proof construction actions and contributions				Lecturer Utterances		
	Frank's actions		Other participants		Facilitative	Transactive prompt	Directive/ Didactive
	Correct	Incorrect	Correct	Incorrect			
1	13	11	32	8	5	18	6
5	41	4	24	5	0	15	4

Table 7.2 below shows Frank's proof construction actions and contributions in the various proof comprehension and construction categories. This table does not include the categories where there were no proof construction actions on the part of the student. As can be seen there was a general increase in all correct actions and contributions and a general decrease in incorrect contributions. There is an exception in the category L2bx which focuses on students' inability to select useful or appropriate aspects of definitions and appropriate assumptions. As discussed above, it is evident that this aspect of proof construction remained a challenging one to students even when gains had been made in all other aspects.

As can be seen from Table 7.2 there was great improvement in the category L1 (meaning of terms, symbols and signs). Frank's use and interpretation of terms, symbols and signs had improved (categories L1a and L1b) and he was encouraged to use examples to illustrate mathematical objects (category L1c). In category L2 (logical status of statements and proof frameworks), he had improved in most aspects. He made great improvement in episode 5 in selecting appropriate statements which would add to the logic of the proof construction (L2a). His knowledge of proof methods (L2c) had also been strengthened and he showed concept level understanding of these in episode 5. Although he did select appropriate assumptions and aspects of definitions (L2b) in sub-episode 5.1, we can see that he experienced some difficulty with this category in sub-episode 5.3 where he needed some assistance. As mentioned previously, this aspect of proof construction remained challenging and I believe a lot more practice and time spent on proof construction is necessary for this aspect to be strengthened. In category L3 (justification of claims), there was an increase in Frank's ability to make correct deductions and conclusions based on the necessary justification (L3a and L3c). His only incorrect deduction was as a result of his difficulty with category L2b as discussed above.

Table 7.2: A summary of Frank's proof constructions actions and contributions according to the various categories

Frank's proof construction contributions and actions	Episode 1: Number of contributions	Episode 5: Number of contributions
Category L1		
L1a	4	9
L1b	1	5
L1c	0	4
L1ax	7	2
L1bx	2	0
Category L2		
L2a	2	6
L2b	0	1
L2c	3	8
L2ax	1	0
L2bx	0	1
Category L3		
L3a	2	5
L3c	1	2
L3ax	1	1
Category H1		
H1b	0	1

7.2.2 Maria's journey: evolution of proof construction abilities

Maria's first proof construction attempt in the first session was the proof of the proposition: *a)* $A \subseteq B \Leftrightarrow$ *b)* $A \cap B = A$. This proof construction encompassed the methods of proof of a double implication, an implication, equality of sets and the method of proof of showing that one set is a subset of another. A successful proof construction also required knowledge of the precise definitions of set equality, subset and intersection and the ability to use these definitions in the logical reasoning and justification of each step in the proof. In the second session Maria returned to the board to attempt the proof construction of $(A \cap B) \times C = (A \times C) \cap (B \times C)$ which involved the proof methodologies of set equality and showing that one set is a subset of another. A successful proof construction also required knowledge and application of the definitions of the notions of the Cartesian product, intersection, subset and set equality.

7.2.2.1: Episode 2: Proof of a theorem showing equivalence of two statements

The first theorem Maria volunteered to do on the board in the first session was the proof of *a)* $A \subseteq B$ $\Leftrightarrow$ *b)* $A \cap B = A$. The proof of the double implication *a)* $\Leftrightarrow$ *b),* entails proving both implications: *a)* $\Rightarrow$ *b)* and *b)* $\Rightarrow$ *a).* Maria's attempt to prove *a)* $\Leftrightarrow$ *b)* in the second episode was the main arena where most of her misconceptions and incorrect ideas on proof methods were revealed and worked on. This proof construction took place in the first consultative session where students were still very unfamiliar with the mathematical objects and methods involved in proof construction and was the most fruitful in terms of the amount of learning which seemed to take place. It was also the most lengthy of all five episodes in terms of the duration of the proof. The first session was also the arena where the lecturer established the norms relating to students' expected modes of interaction and participation. These were communicated to students through the lecturer's use of transactive prompts for reflection, clarification, explanation, strategy, justification and critique and facilitative utterances offering encouragement and highlighting misconceptions, as well as confirming and re-voicing correct ideas and conceptions.

Maria could be regarded as an average student, grappling and often failing in the task of proof construction. Throughout this proof there was much scaffolding and guidance by the lecturer and Maria's peers. At some points the lecturer wondered whether Maria would ever be capable of understanding the mathematical objects and the various methods encompassed within the proof. However surprisingly after the proof of a) $\Rightarrow$ b), Maria stayed on to do the proof of b) $\Rightarrow$ a) and here she showed a remarkable change in both her confidence and ability. Presumably as a result of her functional use of the newly met terms, symbols, proof methods, deductive reasoning processes and justification in activities including imitation (see sub-episode 2.9) while receiving scaffolding from the lecturer and her peers, we can see great development in her usage, interpretation and application of these mathematical objects and processes.

Sub-episode 2.1

Maria's association of the proof method of the implication with the proof method of an equality was already evident in episode 1, when in sub-episode 1.3, line 66, in response

to the lecturer's requests for students' explanations of the notions of the implication and double implication, Maria showed that she associated the implication with the more familiar notion of equality as she described the method to be used as proving that the two sides of an equation were equal to each other. In sub-episode 2.1 when Maria was attempting the proof of $A \subseteq B \Rightarrow A \cap B = A$, her proof construction attempt confirmed her complex level thinking and her association of the method of proof of an implication with the proof method of an equality or identity. She subsequently took the left hand side and the right hand side of the implication independently and by making incorrect deductions and using trivial implications of assumptions showed that each "side" resulted in the same statement: $x \in B$.

Out of 16 actions, Maria did 5 correctly (refer to Table 7.3). These were mostly from the L1 category (using mathematical terms, symbols and signs correctly), while correct actions in the L2 (the use of logical reasoning in the proof construction) and L3 (the need for justification of deductions) categories were seriously deficient.

Sub-episode 2.2

In sub-episode 2.2 Maria (in line 5) indicated that she believed that the equality of the sets $A \cap B$ and A which was supposed to be proved, was a given. This confirmed her incomplete understanding of the proof method of an implication. Her initial proof construction attempts and elaboration of her reasoning process, made it clear that the need for justification of each statement following from previous statements and assumptions in a logical and sensible way was not appreciated.

In this sub-episode although Christine also referred to the implication symbol as an equals sign, she questioned Maria on her logical reasoning and justification and was thus instrumental in helping to create the EZPD in which Maria's learning was developed. Probably as a result of Christine's questions and critique and the functional use of the method of proof of an implication, Maria (in line 11) began to realize that she might have made an inappropriate and incorrect deduction, and that she might not have the correct idea about the proof method.

Sub-episode 2.3

Not all the input from Maria's peers was helpful. In sub-episode 2.3 Edgar suggested that Maria uses a trivial and non-useful implication of a definition in the proof construction. Maria's attempt to use Edgar's suggestion leads her in a totally wrong direction. This showed that the strategic knowledge of knowing which assumptions and which aspects of definitions are useful as well as the ability to select the appropriate implication or deduction from previous statements need to be carefully scaffolded in the initial stages of the learning process in proof construction.

Sub-episode 2.4

In sub-episode 2.4 Maria revealed her evolving thinking process of the proof method of an implication. Whereas in the previous sub-episodes she appeared to view the proof method of an implication as similar to the method of proving that the two 'sides' of an equation or identity are equal, she now displayed her evolving understanding of the proof process: that is to first prove that the statement to the left of the implication sign was true and from there, she would know that the statement to the right of the implication sign would be true.

This incorrect method for proof of an implication seemed to be quite a common misunderstanding (as confirmed by Joseph in this sub-episode) and appeared to be a result of students' incorrect interpretation of the definition of an implication. Gary assumed the role of more knowing peer, clarifying and elaborating on the proof method. He showed true concept level thinking and contributed to creating the environment in which Maria was enabled to access her zone of proximal development and develop her proof construction abilities.

Sub-episode 2.5

However after all the scaffolding offered by Gary and the lecturer, Maria showed that she still had not yet grasped the method of proof of an implication as she asked whether she should prove $A \subseteq B$ or $A \cap B = A$.

She was guided by Helen who told her that she should do "the second intersection b) part". Maria seemed to identify with and appreciate this short simple rule perhaps even more than all the earlier explanations, reinforcing the impression that some students

appear to be looking for a set method, a standard formula or rule they can follow, and which will always work. The suggestion that she should prove 'the second part' appeared to have satisfied this search. Also this showed that we (lecturers) may be misled into assuming that students understand precisely what we have said. For example the word 'assume' might not have been part of Maria's vocabulary so she might not have been able to make sense of all the earlier explanations.

The next proof method that Maria encountered in this sub-episode was the method of proof of equality of two sets (which incorporates the method of proof of showing that one set is a subset of another). Maria revealed complex level thinking in her description of the proof methodology describing the method of proof of equality of the two sets (*A* and *A* intersection *B*) as showing that *A* and *A* intersection *B* had something in common and that was the element *x*.

The method she described gave rise to the correct methodology of the proof of equality of sets but her description and elaboration were not quite correct. When proving equality of two sets one has to prove that one set is a subset of the other and vice versa. In order to prove that set *A* is a subset of another set *B* one must show that any arbitrary element of set *A* can be found in set *B* and Maria has conceptualized this as showing that the sets have something in common.

Sub-episode 2.6

In sub-episode 2.6 a question on the difference between the intersection and union of sets led to a very interesting discussion and revealed the surprising fact that although these mathematical objects had been covered as the basic foundation of the coursework on elementary set theory and it was generally assumed that thestudents have a firm understanding of them, they were not at all well understood. While trying to clarify and reach an understanding of these notions, there was a widespread use of examples initiated by the lecturer, and these really helped to bring to light several of the students' misconceptions.

Students described and illustrated their understandings of the notions of union and intersection revealing both complex and pseudoconceptual thinking. Gary (lines 99 and 101) depicted the union of two sets by a Venn diagram to include all the elements in the

two sets except the elements in the intersection. Christine (line 110) also did an example giving the impression that when considering the union of two sets there could not be any elements in the intersection of the two sets. Maria (line 113) seemed to be misled by their incorrect conceptions of the union, and asked for confirmation that the union of two sets did not contain any elements that the sets might have in common. Edgar (lines 117 and 119) made a positive contribution by doing another example which correctly showed the intersection and union of two sets and this was confirmed and highlighted by the lecturer.

Sub-episode 2.7

Students continued discussing the proof method of showing equality of sets in sub-episode 2.7. The lecturer reminded the students to refer to the definition of equality and they were guided to use and refer to the definition in order to arrive at the method of proof. Maria (line 162) wrote the correct plan of action on the board; it was hoped that through observing the development or extraction of the proof method of equality of two sets from the definition, she and all the other students would now not only have developed a good concept level understanding of the proof method of equality of two sets, but also an understanding of how (in general) to develop the proof method from the definition of a mathematical object.

Sub-episode 2.8

In sub-episode 2.8 Maria started the proof of $A \subseteq B \Longrightarrow A \cap B = A$ once again, this time armed with the knowledge gained from the discussions in the previous sub-episodes, of the proof methodologies of an implication and method of proof of set equality. She still seemed to be battling with these proof methods and the need for justification of each statement and deduction, and made apparently non-useful deductions and assumptions. Maria had still not realized that every step of the proof construction had to be justified and that deductions had to be accompanied with logical reasoning (L2 and L3 categories). This also indicated Maria's lack of strategic knowledge as she was unable to take the proof further after the first correct step and needed continuous guidance from her peers, mostly Joseph at this stage. Although she started the proof correctly and gave the impression that she had now reached concept level understanding of the proof method of an implication, she appeared to revert back to complex level thinking after

the first few steps, again wanting to prove equality of both sides of the implication. In line 186 she again seemed to lose sight of what she was supposed to prove and brought in the assumption $A \subseteq B$ and continued to try to arrive at what she had previously considered to be her desired goal, $x \in B$ (in sub-episode 2.1). This was once more indicative of Maria's lack of strategic knowledge and lack of a clear idea of the proof framework or methodology.

Through continuous assistance from the lecturer and other more knowing peers Maria concludes the proof of $A \cap B \subseteq A$. Sensing that there might still be some confusion about the proof construction done so far, the lecturer asked Christine as more knowing peer to go over the proof and explain what had been done. Christine (in line 229) did this proof presentation with clarity and conviction, using newly met terms with ease and clearly showing the connection between what needed to be shown and the statements made in the proof construction (indicating her concept level thinking for methods and ideas relevant to the proof). She identified the basis for each deduction and demonstrated as she went through the proof construction process that deductions had to be made with the necessary justification. This form of explanation from students' peers presented an excellent learning opportunity for them. In this sub-episode Joseph, Gary and Christine acted as more knowing others, displaying concept level understanding of the proof method as well as good strategic knowledge, and guided Maria's efforts in the proof construction.

The proof of $A \cap B \subseteq A$ ended in line 206. Looking at all of Maria's actions in this sub-episode (highlighted in bold in the coded transcript found in the Appendix) pertinent to this proof construction, out of 23 actions, 14 were correct (see Table 7.3). The incorrect actions were mostly from the L2 and L3 categories, showing that Maria was still battling with the proof methodology and the logical reasoning of the proof process, as well as the ability to provide justification of assertions and deductions following from previous statements.

Sub-episode 2.9

In sub-episode 2.9 Maria started the proof of $A \subseteq A \cap B$ to complete the proof of $A = A \cap B$ and it became clear (in line 249) that she had not been able to transfer the proof method of proving that one set is a subset of another from the previous sub-episode as she could

not identify the statement she needed to prove. She was helped to start the proof by Helen and Christine and in line 261 she brought in as an assumption, the statement she was actually trying to prove. This was one of the few times that the lecturer (in line 262) made a directive statement providing immediate feedback on the error of this action. In line 262 the lecturer, after providing feedback, asked Maria what her assumption was. Another one of the participants answered this question, and when the lecturer again asked Maria what her assumption was, Maria was now able to answer and repeated the assumption just mentioned. This was a clear indication of imitation, an activity I presumed was happening throughout the sessions (but was often difficult to detect) as students learned from each other and the lecturer.

In line 267 Maria once more seemed to lose sight of the goal and what she needed to prove, and made a deduction leading to the correct conclusion without providing a basis for the deduction and conclusion. She still did not seem to realize that she could not make deductions and conclusions without the necessary justification. It seemed that it was very easy for students to revert to bad habits such as making assertions or deductions without justification and losing sight of their goal in the proof construction process. I suggest the reason was that Maria still had not formed a concept level understanding of the proof methodology, and seemed to have difficulty in making the transition from complex level to concept level in terms of proof methods and the logical reasoning and justification process. Her peers, Joseph, Helen and Christine, patiently pointed out the appropriate deduction she should be making from the previous statement (line 269) and thus they continued to create the learning environment which facilitated Maria's access to her zone of proximal development (EZPD).

Out of 29 actions taken by Maria in this component of proof construction 16 were correct (see Table 7.3), with most of the incorrect actions coming from the categories L2 (indicating Maria's difficulty in following the proof method, reasoning processes and logic of the proof), L3 (indicating Maria's difficulty in making correct deductions and conclusions from previous statements with the necessary justification) and H2 (not being able to transfer the methods and logical reasoning used in the previous component of the proof to this component). The scaffolding Maria received as she made functional use of newly met terms, symbols and proof methods was probably instrumental in the

fast and vast improvement in her proof construction ability which was evident later in sub-episode 2.11 and in the second session.

Sub-episode 2.11

In sub-episode 2.11 Maria began with the next component of the proof: $A \cap B = A \Longrightarrow A \subseteq B$. Now she seemed much more confident about the implication proof framework as well as the proof of subset framework and showed that she had transferred the methods met in the previous components of the proof to this component. She appeared to really believe in what she was doing and explained her proof construction actions with conviction as she proceeded. Her understanding and correct application of the proof methods of an implication and subset proof was confirmed as she correctly identified the assumption, what she needed to show or prove and the steps needed to reach the desired goal.

In lines 311, 313, 317 Frank tried to argue and persuade her that since the proof of *a)* $\Longrightarrow$ *b)* has been completed she should now do the proof of *b)* $\Longrightarrow$ *c).* In the past Maria might have acted on this suggestion as she had done in the previous proof component, but now (in lines 314, 316 and 319) she firmly explained that she was proving *b)* $\Longrightarrow$ *a)* and was not swayed by his insistent suggestions.

What Maria now lacked in her proof construction efforts was the logical reasoning ability and strategic knowledge of how to use the assumptions to proceed in the proof construction and how to make appropriate or correct deductions from previous statements. In line 324 Maria made a deduction which was not a direct logical deduction from the previous statement and assumption but was however the desired deduction which would enable the desired conclusion to be made. It was not clear whether Maria had made the deduction from logical reasoning or if she was just guessing as she knew what the conclusion should be. Although all the other proof comprehension criteria seemed to be satisfied, the proof still remained challenging because of the lack of strategic knowledge, that is: knowing how to use the definitions and assumptions at one's disposal to get to the desired goal (L2b). Perhaps this is one of the key aspects of proof construction ability which is only developed over time through practice and when working with others in the EZPD.

At the proof's conclusion Maria showed gains in the logical reasoning involved in the proof construction concluding the proof correctly and confirming that she had reached pseudoconcept/ concept level understanding of the proof methods of subset, implication and double implication. In lines 362 and 366 she gave the correct explanation behind the conclusions she was making for the double implication, confirming that her understanding for the proof of an implication had evolved to pseudoconcept/ concept level.

Maria had been developing her understanding of the application of logical reasoning processes and the proof methodologies of an implication, equality of sets and showing that one set was a subset of another during the course of the proof construction. In the same way mathematical terms, symbols and signs, functional use of these proof methodologies seemed to enable students' use and application of these methodologies to pass between the various stages of heap, complex and pseudoconcept eventually evolving into true concept level thinking. This will be seen in the proof done by Maria in the second session (Episode 4).

In sub-episode 2.11 out of 32 identified proof comprehension actions 28 were correct which demonstrated the vast improvement in Maria's proof construction ability (see Table 7.3). The only four incorrect actions occurred where deductions were made which did not follow simply from previous statements (category L3ax) and not being able to identify the correct assumption needed in the proof construction (category L2bx). Most other aspects such as: L1a (correctly using newly met terms and symbols (written and spoken) during the entire proof construction process), L2a (selecting correct or appropriate statements and phrases which make sense and add to the logic of the proof construction), L2b (selecting useful or appropriate deductions or aspects of definitions), L2c (selecting the correct proof framework and following the reasoning process and proof methodology), L3a (making correct deductions from previous statements and definitions), L3c (making correct conclusions with all the necessary justification), H1a,b,c,d (explaining the main ideas behind the proof, and identifying the role of different modules of the proof and how they relate to one another) and H2a,b (using ideas she had struggled with in the previous proof construction and recognizing the assumptions which needed to be in place for the method used), had been attained.

There were 13 contributions from other students, 8 of which were correct and appropriate, while the lecturer contributed 12 transactive prompts and 7 facilitative utterances (see Table 7.3).

Summary of episode 2

The proof construction Maria attempted in episode 2 encompassed and required knowledge of not just one, but four proof methodologies or frameworks: the proofs of an implication and double implication, the proof of showing equality of two sets and the proof of showing that one set is a subset of another. It was a rather complex proof especially for first year students who had only recently been introduced to the notion of proof. Along with the various proof methodologies, there were also the challenges of newly met terms, symbols and signs, the logical reasoning required in proof construction, the use of assumptions and definitions, and the appreciation of the need for justifying each deduction made from previous statements.

Maria like many other students battled with most of the aspects of proof construction, especially the proof methodologies and logical processes involved as well as the need for justification of deductions. The proof construction turned out to be rather longwinded and tiresome. There were times when the methodology used in the previous component of the proof needed to be used again in the next component. Maria was often unable to transfer this knowledge and this was a disappointment for the lecturer and perhaps the other students as well. When students work on their own or with peers with similar capabilities as their own, a serious burden is placed on their thought processes. Students doing these proof construction exercises face the combined challenge of many newly met notions, terms, symbols and signs, unfamiliar proof methods and the challenge of logical reasoning and justification required in the proof construction process, all within one proof construction exercise. It is very difficult for students to overcome these many and varied challenges on their own.

Moore (1994) identified this challenge as the problem of 'cognitive overload' that students suffer as they grapple with domain-specific knowledge of terms and notions contained in the proof construction exercise, as well as the interpretation of definitions and their appropriate use.

Looking at the proof construction actions or steps taken by Maria in Episode 2, from the beginning of proof construction up to the point that it was successfully completed, I have identified 133 actions (highlighted in bold in the full transcript contained in the Appendix) 76 of which were done correctly (see Table 7.3). The participation from others in the group, including Joseph, Gary, Edgar, Christine, Helen, Frank and others, came to a total of 113, most of which (90) were helpful and appropriate. This shows the high level of participation by all students ensuring a very effective EZPD where all participants benefitted from the interactions. The lecturer made a total of 65 transactive prompts asking for clarification, reflection, justification, critique, strategy, examples and use of reasoning ability. There were also 38 facilitative utterances highlighting learning, giving encouragement and confirming students' ideas and 5 directive and didactive utterances referring to definitions and elaborating on definitions of mathematical objects.

7.2.2.2: Episode 4: Proof of $(A \cap B) \times C = (A \times C) \cap (B \times C)$

In the second session which occurred one week after the first, Maria volunteered to do the proof of the proposition: $(A \cap B) \times C = (A \times C) \cap (B \times C)$. A successful proof construction of this proposition required knowledge of the proof method of set equality and the proof methodology of showing that one set is a subset of another as well as the precise definitions of subset, intersection and the Cartesian product and the ability to use these definitions in the logical reasoning and justification of each step in the proof.

Sub-episode 4.1

The proof of $(A \cap B) \times C = (A \times C) \cap (B \times C)$, a proof of showing equality of sets requires that one proves that $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$ and $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$. Hence the proof methodology of showing equality of sets also encompasses the proof methodology of showing that one set is a subset of another. In sub-episode 4.1, presumably as a result of her functional use of logical reasoning processes and proof methodologies in her previous attempt at proof construction in Episode 2, we observe that Maria had successfully mastered both these processes and methodologies and seemed to be very comfortable using them and explaining her reasoning to others. Her thorough explanations together with her correct use and application seemed to suggest

that her grasp of the proof methodologies encompassed in this proof was now at concept level.

She correctly described the approach that she was going to use (line 7) and broke down the proof into two components. She then systematically started the first component of the proof and followed the proof method for showing that one set is a subset of another correctly. Each deduction she made was accompanied by all the correct reasoning and justification, and her written use of newly met terms, symbols and signs was excellent. She appeared to be fully aware of the logical relationship between statements and deductions she had made in the proof and the conclusion she would like to make. This was in stark contrast to the proof she attempted in the first session (episode 2). Every deduction was accompanied by a detailed justification where she explained all her reasoning, showing that the justification process had really become a well- established habit for her now. This was presumably as a result of the functional use she had made of deductive reasoning processes and the practice of justification of each step of the proof construction, while receiving scaffolding from her peers and the lecturer.

Her written use and application of the newly met terms, symbols and signs was sound, but she continually referred to the Cartesian product as 'times' or 'multiply'. As she was able to use and apply the notion of the Cartesian product correctly and sensibly, this might indicate that she could work with and apply the mathematical object but was reluctant to use the symbol's longer name.

Sub-episode 4.2: Discussion of the association of the Cartesian product with the intersection

In sub-episode 4.2 it was evident how interaction and participation in the EZPD by Christine, Maria and more knowing others helped them develop their understanding of the definition of the Cartesian product. It became clear in sub-episode 4.2 that Maria's use and application of the notions of the Cartesian product and the intersection was incomplete.

Christine questioned whether she could substitute the intersection symbol for the symbol of the Cartesian product (in lines 10 and 12), showing that her understanding of the Cartesian product and its definition was at complex level. I believe her confusion arose because the definitions of both the intersection ($A \cap B = \{x: x \in A \text{ and } x \in B\}$) and

the Cartesian product ($A \times B = \{(x, y): x \in A \text{ and } y \in B\}$) contained the word 'and' and Christine seemed to be associating these mathematical objects with the word 'and' and thus with each other. This might be indicative of complex thinking (in particular associative complex thinking). Maria's response to Christine's question did not clarify this misunderstanding. Her explanation in line 18 (sub-episode 4.2) is given below.

> **[18] Maria:** I think that here because we're speaking of a multiplication…. *[points to: Proposition: $(A \cap B) \times C = (A \times C) \cap (B \times C)$]* here we started with a multiplication sign and we want to prove that you see this side here *[points to: $(A \cap B) \times C$]* we've got an intersection and here we've got a multiplication sign. And here we've got *[points to: $(A \times C) \cap (B \times C)$]* two multiplication signs. So if we prove this *[points to the lower part of the board]* we must prove this also looking at this side that what this side contains *[points to: $(A \times C) \cap (B \times C)$]*

As seen in her explanation she seemed to be looking at the expressions on each side of the equality, and when seeing that they both had 'multiplication signs', she felt that she had to keep those signs so that the two 'sides' will contain the same signs. Although she was able to use and apply the notion of the Cartesian product in the first component of the proof construction very well, she still had not grasped the correct use and application of this newly met mathematical object adequately. Thus she exhibited pseudoconcept or complex thinking about the newly met term, the Cartesian product.

Joseph taking on the role of more knowing other, now contributed in lines 22 and 24. Joseph's able explanation was confirmed by the lecturer who also referred to the definition of the Cartesian product, and wrote it on the board again for easy reference.

Mathematical definitions often pose a huge challenge to students who find them difficult to 'unpack' and correctly interpret. In this sub-episode we saw that engagement with proof construction tasks while interacting with peers and the lecturer in the EZPD, allowed students to make functional use of mathematical objects and definitions and greatly helped them with this challenge.

Sub-episode 4.3: Conclusion of the first component of the proof and attempt at proof construction of $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$

Joseph's able guidance and explanation was a great help to Maria and in sub-episode 4.3 it was clear that she had made progress in her understanding of the notions of the Cartesian product and the intersection. In line 28 of sub-episode 4.3 Maria started and

completed the proof of the second component of the proof correctly giving detailed justification of each deduction and conclusion made. She continued to show concept level thinking of the proof methodologies of equality and showing that one set was a subset of another. She explained her thinking process carefully and emphasized where and why she was using the intersection symbol and the Cartesian product symbol. She also referred to the definition of the Cartesian product and logically explained her deduction in terms of this definition. It appeared that her thinking on these newly met terms had evolved and had now reached concept level as she now very ably explained exactly what these mathematical objects meant and was able to use them correctly and with ease (category L1). This is a very good example that shows how scaffolding (by peers and the lecturer) in the EZPD allowed Maria (and hopefully all the other participants) to make functional use of the notion of the Cartesian product thereby enabling the evolution from complex or pseudoconcept level thinking to concept level thinking. Her logical reasoning ability (category L2) as well as her ability to provide sound justification for each step in the proof construction process (category L3) had also been strengthened through her increased understanding of the definition of the notion of the Cartesian product. She completed the proof successfully without any further interruptions.

Summary of episode 4

Maria's proof construction attempt contained 64 steps or actions (highlighted in bold) which were **all** done correctly except for one response in sub-episode 4.2 where she was not able to correctly explain the use of the Cartesian product and its difference with the notion of intersection (see Table 7.3). I have not taken into account simple writing errors or her repeated spoken misuse of the Cartesian product as 'multiply' or 'times' as I felt that this had not in any way hampered the written proof construction process and was a rather 'normal' misuse.

In this episode Maria seemed to have concept level understanding in terms of her use and application of all the proof methods relevant to the proof construction (category L2c). She also selected useful or appropriate deductions from definitions and assumptions, was able to explain her logical reasoning process as she proceeded with the proof and chose correct and appropriate statements which added logic to the proof

construction process (categories L2a and L2b). She identified the basis for all deductions and conclusions made from previous steps carefully explaining and providing justification (category L3). Her use and application of newly met terms, symbols and signs (category L1) also seemed to be at concept level, but was later revealed to be at complex or pseudoconcept level. In sub-episode 4.3 however she correctly described her thinking process behind the use of the intersection and the Cartesian product emphasizing their meanings and the reasons why she was using each of these symbols. This leads me to believe that, as a result of the scaffolding received from the lecturer and peers and her functional use of the notion of the Cartesian product, her understanding of this newly met term had evolved from complex or pseudoconcept level in sub-episodes 4.1 and 4.2 to concept level in sub-episode 4.3. The participation from Maria's peers came to a total of 10, 4 of which were correct and appropriate (see Table 7.3). On the whole, Maria's proof construction in episode 4 showed her great progress and development in all the categories of proof construction.

7.2.2.3: Overall discussion of Maria's journey

Maria's proof construction attempt in Episode 4 was a giant leap from her previous proof construction attempt in the first session (Episode 2). In the first session Maria's grasp of the proof methodologies of the implication, equality and subset proofs were all at complex or even heap level. She needed continuous guidance and assistance on how to proceed with the various proof components. She was also challenged by the need for justification of deductions, bringing in as assumptions statements that she needed to prove, and making deductions and conclusions without any basis.

Presumably her functional use (including the activity of imitation as seen in sub-episode 2.9) of unfamiliar terms, symbols signs, definitions, proof methods as well as deductive reasoning processes and the practice of justification during the consultative group sessions, plus her interactions with peers and more knowing others and the scaffolding received, had enabled her to make rapid progress in her proof construction abilities.

In the proof attempted in the second session (Episode 4), her grasp of the proof methodologies seemed to be at concept level. She ably and thoroughly explained her reasoning and logic as she proceeded with each deduction and each conclusion made in

the course of proof construction. She also appeared to have an excellent grasp of selecting correct and appropriate statements which added to the logic of the proof construction process, for example at the beginning and conclusion of each proof component as well as at the conclusion of the whole proof. Her use of newly met terms, symbols and signs (category L1) seemed to be at concept level, except her interpretation of the newly met notion of the Cartesian product which was revealed to be at complex or pseudoconcept level in sub-episode 4.2. In sub-episodes 4.2 and 4.3 we saw how Maria's complex or pseudoconceptual application of the notion of the Cartesian Product was developed to concept level usage and application through functional use of the terms and symbols, her interaction with the other participants and the scaffolding received from more knowing peers.

Maria's proof construction attempt in Episode 4 demonstrated her great progress and improvement in terms of all proof construction abilities. This was very striking and encouraging as it showed that even average students like Maria who had great difficulty with all aspects of proof construction could become capable of mastering these abilities in a very short time given the opportunity to attempt proof construction in an environment which encouraged interaction with peers and more knowing others.

Table 7.3 summarizes Maria's proof construction actions and the contributions from her peers and the lecturer. Her progression in terms of her proof construction abilities from Episode 2 (in the first of the weekly sessions) to Episode 4 (in the second weekly session) was quite striking even though the two sessions were only one week apart. In fact the change had already begun in sub-episode 2.11. When we examine the participation from Maria's peers in the second episode, which I surmise was largely responsible for this striking change, one sees a huge number of positive or helpful contributions in Episode 2 numbering 90. There were also 38 facilitative utterances, 65 transactive prompts and 5 directive/ didactive prompts from the lecturer in Episode 2.

Table 7.3: A summary of Maria's journey in terms of proof construction actions and lecturer and peer's actions and utterances

Sub-episode	Proof construction actions and suggestions				Lecturer Utterances		
	Maria's actions		Other participants		Facilitative	Transactive prompt	Directive / Didactive
	Correct	Incorrect	Correct	Incorrect			
2.1	5	11	0	0	0	0	0
2.2- 2.7	13	20	37	14	9	37	0
2.8	14	9	26	4	15	13	4
2.9	16	13	19	0	7	3	1
2.11	28	4	8	5	7	12	0
Total:	**76**	**57**	**90**	**23**	**38**	**65**	**5**
4.1-4.3	**63**	**1**	**4**	**6**	**2**	**3**	**1**

Table 7.4 below shows Maria's proof construction actions and contributions in each of Episodes 2 and 4 according to the various proof construction categories. Since the two proofs attempted by Maria in the two sessions were not exactly the same in terms of their length and scope of proof methods and newly met mathematical objects, they necessarily required different proof construction abilities (in terms of number of actions and categories of proof construction). I will not therefore compare the number of proof construction actions taken by Maria in the various categories across the two proof constructions, but rather compare categories which indicated a lack of ability in proof construction and comprehension.

Focussing on the categories indicating lack of ability in the various proof construction actions, that is, those categories having an 'x' attached to them, we observe a huge improvement in all the categories. In episode 4, difficulties in category L1 encompassing the meaning of terms, symbols and signs (L1ax and L1bx) had largely been overcome, with the exception of the newly met notion, the Cartesian product. Similarly difficulties in category L2 encompassing logical status of statements and proof frameworks (L2ax, L2bx and L2cx) had been overcome. We were able to see how Maria's confidence grew in selecting appropriate statements, assumptions and aspects of definitions which added logic to the proof construction process as well as the proof methods relevant to the proof construction. Maria showed similar improvement in the category L3 which encompasses justification of claims (L3ax, L3bx and L3cx). In Episode 4, she had a much greater appreciation of the need for justification of each

deduction and conclusion and provided the necessary reasoning at each step of the proof. The number of incorrect actions in this category fell to zero. Similarly in the category H1 difficulties regarding the identification of main ideas and correctly breaking down the proof into components (H1ax and H1bx) were overcome. In the category H2, difficulties with the ability to transfer ideas and methods met in previous proof construction exercises to subsequent exercises (H2ax) were also resolved.

Table 7.4: A summary of Maria's proof constructions actions and contributions according to the various categories

Maria's proof construction contributions and actions	**Episode 2: Number of contributions**	**Episode 4: Number of contributions**
Category L1		
L1a	14	19
L1b	4	2
L1ax	3	0
L1bx	0	1
Category L2		
L2a	14	6
L2b	2	8
L2c	13	6
L2ax	8	0
L2bx	11	0
L2cx	17	0
Category L3		
L3a	13	10
L3b	1	0
L3c	5	4
L3ax	10	0
L3bx	1	0
L3cx	3	0
Category H1		
H1a	7	2
H1b	0	2
H1c	0	1
H1d	0	1
H1ax	1	0
H1bx	1	0
Category H2		
H2a	1	2
H2b	2	0
H2ax	2	0

7.3 Concluding summary

In Section 7.2 I Frank and Maria's progress in all aspects of proof comprehension and construction was observed. Frank made progress from complex thinking in the category of meanings of terms, symbols and signs (category L1) in Episode 1 to concept level in Episode 5. He also made progress in the area of proof methods and logical reasoning and selection of statements and phrases which add to the logic of the proof construction (categories L2a and L2c) to concept level in Episode 5. This was presumably a result of his functional use of the newly met terms, symbols, logical reasoning processes and proof methods while receiving scaffolding and guidance during his interactions with his peers and the lecturer. His appreciation for the need of justification (category L3) of which he seemed to have some basic understanding in Episode 1 was strengthened in Episode 5. The only area that that remained problematic was that of knowing how to use the assumptions, proof method and logical reasoning to proceed when the proof construction became a little more complicated (category L2b, also termed strategic knowledge). At these times, the established habit of justification of statements and deductions also seemed to be abandoned as he desperately tried to find a way forward. The difficulty experienced with category L2b led him to make incorrect and inappropriate deductions. The challenge of strategic knowledge appeared to be the most challenging of all the aspects of proof construction. Much practice and time is needed to be spent on proof construction in order to strengthen and improve this aspect. Interestingly this aspect of proof construction was also the most challenging in Maria's proof construction when she was attempting the final component of the proof in sub-episode 2.11. At this point all the other proof construction aspects seemed to have been well understood. I suggest that this aspect of proof construction is optimally developed when students interact and engage with their peers and more knowing others in an environment which facilitates students' access to their zones of proximal development (EZPD). Students are empowered to make gains in this category because they are surrounded by peers and more knowing others from whose experience, creativity and knowledge they can benefit.

Examining Maria's journey, we observed her persistent difficulty with proof methods and logical reasoning (category L2) and the justification of deductions and conclusions

(category L3) in Episode 2. She was also challenged in her inability to transfer the methods and ideas from previous proof components to subsequent components (category H2). This might be due to the cognitive overload (cf. Moore, 1994) students experience as they face the combined challenge of new mathematical objects, unfamiliar proof methods and the challenge of logical reasoning and justification required in the proof construction process, all contained in one proof exercise. These challenges place a heavy burden on the average student's cognitive abilities and could hamper their ability to internalize all the myriad aspects of the learning taking place during the course of the proof construction efficiently. This might be one reason that students battle to transfer methods and ideas from one proof or proof component to subsequent proof constructions. Another possible reason for the frustration they experience in their struggles to improve their proof construction abilities could be that students often consult their peers, who have similar difficulties, and whose proof construction abilities in terms of the categories involved in the local and holistic aspects of proof construction are as undeveloped as their own. For students to really be able to make strides in their development of these abilities, there has to be interaction and consultation with more knowing peers, lecturers and tutors (at least initially) in an environment where access to their ZPD is encouraged and facilitated. This was shown to be possible in the small consultative groups. The speed of Maria's transformation was truly amazing, leading me to believe in the effectiveness of the process. Her proof construction attempt in Episode 2 gave us a glimpse into how students really do battle with proof construction, and why it is so vital that they form working groups with other students with a range of capabilities. Working with peers and more knowing others, students are able to make functional use of newly met terms, symbols and signs and proof techniques while being continuously prompted and questioned on clarification, reflection and justification. I suggest that this accelerates their progress resulting in far less frustration.

In Episode 4 we saw Maria's vast improvement in her use and application of proof methods and reasoning processes as well as her appreciation of the need for justification of all statements and deductions. Her use and application of terms and symbols in the proof construction process also appeared to be at concept level except for the newly met notion of the Cartesian product. Her use and interpretation of this notion was quickly developed to concept level through her functional use of the notion of the Cartesian

product while interacting with all the participants and more knowing peers. A vital factor promoting student' development of proof construction is the opportunity offered to interact with one another while receiving scaffolding from peers and more knowing others in the EZPD. In this interaction they develop their understanding of notions and definitions as well as logical reasoning processes and the ability to justify proof construction steps, through the functional use of terms, signs, symbols, definitions, proof methods and deductive reasoning processes and the practice of justification.

Maria's journey could be compared to children's struggle when learning to walk for the first time (even though walking is not a cognitive ability). The patient care and encouragement the child receives from parents and other adults can be likened to the support that students receive from peers and more knowing others in the consultative sessions. The environment in the consultative sessions encouraged students to become active participants in the development of their proof construction abilities by facilitating access to their zones of proximal development and enabling the internalization of all the learning that is taking place with greater efficiency and speed.

To conclude, in Chapter 7 we observed how the learning environment created in the consultative sessions enabled the two case study students' development of proof construction abilities. I propose that this was due to the facilitation of students' access to their zones of proximal development and that this access allowed them to make functional use of newly met mathematical terms, symbols, signs and proof methods as well as deductive reasoning and justification processes. It was this functional use which promoted their learning.

Chapter 8: Facilitating students' construction of proof

8.1 Introduction

In this chapter I will present analysis and discussion aimed at addressing my third research question which is repeated below for ease of reference.

Research Question 3

Investigating the nature of the interactions in the consultative group to explore how students' construction of proof may be facilitated:

a) How can lecturers encourage and support students who are engaging with proof construction while participating in consultative group discussions, to become intellectually autonomous?

b) What are the characteristics and modes of reasoning prevalent in students who seem to have the potential to become more knowing peers?

To address these questions I will present an analysis and discussion of the nature of interactions taking place in the five episodes of consultative group sessions, in Section 8.2. The characteristics of the contributions and interactions of the lecturer and all participants have been analysed according to the categories (and their corresponding indicators) found in my analytical framework for analysis of student and teacher discourse found in Section 5.2.1. This was done in order to allow the patterns which established the norms leading to the learning environment described in this study to be brought to light. I searched for patterns of action by the lecturer as she encouraged and elicited students' ideas and contributions and established the norms which supported students in developing their proof construction abilities and enabled them to become intellectually autonomous. I have also tried to identify the characteristics and modes of reasoning of students who showed potential to become more knowing peers, and how these students could be empowered to develop this potential, through both their own endeavours and the opportunities available to them through the interaction with their peers and the lecturer in the consultative sessions. I have also presented significant

examples (in episodes 1 to 5) of the actions and modes or patterns of reasoning of those students who have appeared to have developed the capacity to become intellectually autonomous and could be well be on their way to becoming more knowing peers. These findings are organized and summarized and presented in Section 8.3 as an overall discussion addressing my third research question.

In the first session of the consultative group sessions (episodes 1 and 2) the lecturer played a leading role in prompting and eliciting students' contributions and ideas towards the proof construction exercises attempted by the students on the board. In the second session (episodes 3, 4 and 5) which occurred just a week after the first session, there was a marked decrease in the lecturer's contributions, while those students who showed potential to become more knowing peers, assumed the roles of scaffolding and leading the mathematical discussions forward. Table 8.1 below depicts the number of the lecturer's transactive prompts, facilitative utterances and directive and didactive utterances in the 5 analysed episodes. The table also shows the number of correct and incorrect contributions made by other students in their efforts to guide their peer who was attempting the proof construction exercise.

As shown in Table 8.1 in episodes 1 and 2 which took place during the first of the consultative sessions, there were a large number of transactive prompts and facilitative utterances from the lecturer. In episode 3 which took place in the second session we observe that the lecturer's contributions dramatically dropped to zero. Several more knowing peers adopted the transactive prompts and facilitative utterances which were previously contributed by the lecturer. These students assumed the role and responsibilities of driving the proof construction sessions forward according to the norms and criteria established in the first session. This pattern was repeated in episodes 4 and 5 where there was a low incidence of lecturer contributions, other than the transactive prompts in Episode 5.

Table 8.1 The number of lecturer's utterances and peer contributions in episodes in the first session and episodes in the second session

Episode	Lecturer's utterances			Other students' contributions	
	Transactive prompts	Facilitative	Directive/ Didactive	Correct	Incorrect
First session					
1	18	5	6	32	8
2	65	38	5	90	23
Second session					
3	0	0	0	28	0
4	3	2	1	4	6
5	15	0	4	24	5

8.2 Characteristics of interactions of lecturer and peers

A brief analysis and description of the interactions of the lecturer and students in episodes 1 to 5 is given below.

8.2.1 Characteristics of the interactions of lecturer and students in Episode 1

Frank's first attempt at proof construction occurred in episode 1, sub-episode 1.1. In sub-episode 1.2 the lecturer made only one transactive request prompting reflection and critique. This started a discussion where Frank's incomplete interpretation and application of the notion of implication and the implication proof method as well as the newly met terms and symbols related to the proof construction was revealed. Frank's peers, Gary and Helen used simple every-day language to clarify the words 'suppose' and 'imply' and the implication symbol.

Transactive requests from the lecturer occurred mostly in sub-episode 1.3, in which there was a total of 17. Of these 8 were requests for clarification, 3 for strategy and 6 for elaboration. Requests for strategy prompted students' thoughts and reasoning processes on the way forward in the proof construction process (for example lines 73 and 75). The lecturer asked for clarification and explanation of the meanings of newly met terms and clarification of definitions and proof methods such as those of the

implication and double implication. These prompted students to offer their thoughts and ideas of these notions. Whenever their contributions were helpful, transactive requests for clarification and elaboration prompted them to give further explanation (for example lines 69, 71 and 77). In this manner, norms were established, encouraging students to reflect on strategy, and elaborate on those contributions which made sense and were logical. Also by asking those students exhibiting pseudoconcept and concept level thought processes to elaborate and explain their reasoning, the lecturer encouraged the development of more knowing peers while the other students were made aware of these potential more knowing peers. There were 5 facilitative utterances, in which the lecturer highlighted learning, encouraged helpful and appropriate contributions and confirmed and re-voiced correct contributions. There were 4 directive utterances occurring towards the end of the proof construction attempt, where the lecturer gave corrective feedback on the proof construction (for example lines 80 and 84) and 2 didactive utterances where the lecturer shared information on the definition of the implication and the proof method of the implication (for example lines 98 and 104). Whenever students made incorrect contributions which the lecturer felt would not lead to useful discussions, the lecturer used facilitative utterances to try to restructure the proof construction allowing and encouraging more correct ideas to emerge. Directive contributions providing corrective feedback were made when incorrect ideas persisted (for example lines 80 and 84) or ideas that were totally incongruent with the meaning of mathematical objects were presented (for example lines 92 and 94).

There were 32 correct contributions from students, most notable of which were from Helen, Joseph and Gary. Helen (in lines 56 and 58) gave a correct interpretation of the notions of subset and set equality. In lines 60, 62 and 64, Joseph confirmed this mode of thinking by giving a narrative example. In lines 70, 72, 78 and 79, Gary gave his complex level/ pseudoconceptual understanding of the notions of the implication and double implication and when asked to elaborate, he explained well using an example to illustrate the distinction between the two mathematical objects. It is interesting to observe that both Gary and Joseph who emerged as prominent more knowing peers in the group, used examples to illustrate mathematical objects and ideas such as the proof methods relevant to the proof construction.

Key contributions made by the lecturer in Episode 1

The pattern of the lecturer's contributions in Episode 1 is outlined below.

- Transactive requests for strategy and clarification prompted students' ideas and contributions for the way forward in the proof construction.
- Transactive requests for clarification and elaboration prompted students who made helpful contributions that were at pseudoconcept or concept level to further explain their reasoning processes. In this way the students were encouraged to elaborate on contributions which made sense and were logical. This also encouraged the development and raised awareness of students who showed the potential to become more knowing peers.
- Facilitative utterances highlighted learning and encouraged students' proof construction attempts and confirmed correct and appropriate contributions.
- Directive utterances providing corrective feedback were offered towards the end of the proof construction when incorrect ideas persisted or ideas which were incongruent with the correct meaning of mathematical objects were presented.
- Didactive contributions offering clarification on the notion and proof method of the implication were presented towards the end of the proof construction after all participants had shared their views and contributions about these mathematical objects.

Key contributions made by students in Episode 1

- Peers offered contributions to clarify proof methods and meanings of newly met terms and symbols using simple every-day language.
- Peers offered contributions on strategy for the way forward in the proof construction process.
- More knowing peers offered elaborations on the notions of subset, set equality, the implication and the double implication and made use of examples to illustrate these mathematical objects.

8.2.2 Characteristics of the interactions of lecturer and students in Episode 2

A brief description and summary of the interactions in the sub-episodes of episode 2 is given below. Sub-episodes 2.1 and 2.10 have been omitted. Maria made her first proof construction attempt on her own in sub-episode 2.1. In sub-episode 2.10 the lecturer clarified the notion of equivalence and explained in more detail that Maria now needed to prove the converse. Sub-episodes 2.2 to 2.6 have been grouped together because various attempts were made to clarify and arrive at an understanding of the correct method for proving a theorem involving an implication in these five sub-episodes. The lecturer's contributions followed the same pattern in these sub-episodes: prompting students to give their ideas and contributions and reflect on and justify proof construction actions and eliciting elaboration and explanation from students who made correct or appropriate contributions while encouraging and confirming these contributions. In sub-episode 2.7 Maria eventually arrived at the realization that she had to prove the equality of A and $A \cap B$ and here the method of proof of equality of sets was brought to light.

Sub-episodes 2.2 to 2.6

In the initial stages of episode 2, the lecturer prompted students to clarify and describe their reasoning processes on methods of proof and newly met mathematical objects by using transactive requests for clarification, reflection and justification (for example in lines 4, 12, 14, 34, 43, 45, 56, 58, 65 and 82). In this way Maria's peers were encouraged to offer their contributions. The lecturer also made tranactive requests for reflection and strategy (for example in lines 17, 21, 37, 48 and 60) to Maria, confirming any correct ideas that she may have had and asking her to reflect on the way forward. As a result of these prompts Maria revealed her incorrect reasoning about the method of proof of an implication. In sub-episodes 2.1 and 2.2 we observed that her method was similar to that of proving an identity or equality, and in sub-episode 2.4 her explanation of the proof method erroneously hinged on first proving the statement to the left of the implication was correct in order to claim that the statement to the right was true. It was interesting to see that Joseph had the same incorrect idea of the method of proof of an implication. In sub-episode 2.2 Christine questioned Maria prompting her to realize

that her reasoning and proof method might not be correct. In sub-episode 2.3 Edgar offered contributions which were non-useful and trivial, de-railing the progress of proof construction until the lecturer made a transactive prompt requesting reflection and strategy for the way forward at the beginning of sub-episode 2.4.

In sub-episode 2.4 Gary acted as a potential more knowing other giving the correct method of proof of an implication. Transactive prompts and facilitative utterances from the lecturer encouraged him to elaborate on his ideas which he did in lines 61, 62 and 67. Initially even students such as Gary, who might have correct ideas about mathematical objects, definitions and proof methods were not very willing to elaborate on these. The lecturer had to urge them to elaborate several times. These students gradually became more confident in their own abilities and offered and elaborated on their contributions much more willingly. In sub-episode 2.4 we observe the active participation of Maria, Edgar, Frank, Joseph, Gary (who made 9 correct contributions) and other students who were not identified. This showed a very high level of participation by Maria's peers who were all involved in giving their thoughts and reasoning processes of the implication proof method.

Even after Gary's thorough explanation which was highlighted and confirmed by the lecturer at the end of sub-episode 2.4, Maria, in sub-episode 2.5, was still unsure whether she should prove the statement on the left or on the right of the implication. Maria's poor grasp of words such as 'assume' and 'imply' could be responsible for her continued misunderstanding of the proof method. Helen (in line 78) told her in brief everyday language that she should prove "the second intersection b) part". The lecturer continued giving her transactive prompts asking for clarification, allowing the students to make their contributions and confirmed and encouraged correct ideas and contributions. In this sub-episode students actively taking part in the discussions were Maria, Helen (making 2 correct contributions) and Edgar (making 1 incorrect contribution).

In sub-episode 2.6 Laura raised a question on the notions of intersection and union and the majority of the students got involved in the ensuing discussion. When Gary offered his complex level description of the notion of union, the lecturer asked him to illustrate the notion using an example of Venn diagrams on the board. Other students such as

Christine and Edgar gave further examples on the clarification of these mathematical objects. Edgar's description and example showed concept level understanding of the notion of union. The lecturer confirmed this mode of reasoning and highlighted the error of the other examples. In sub-episode 2.6 the students actively involved in the discourse were: Laura, Edgar, Gary, Kenny, Helen, Christine, Maria, Joseph and Bonnie. This shows the high level of student participation and was indicative that the attention of the majority of the students had been captured through the use of examples illustrating mathematical objects.

Key contributions made by the lecturer in sub-episode 2.2 to 2.6

The pattern of contributions from the lecturer from sub-episode 2.2 through to sub-episode 2.6, is outlined below.

- Transactive requests for clarification, explanation, reflection and illustration with examples prompted students to reflect and offer a strategy on the way forward. This was repeated until one of the students made a contribution that could (when the idea was at concept level or at pseudoconcept level) lead the mathematical discussion forward.
- Transactive requests for elaboration prompted students to pursue correct ideas (either pseudoconcept level or concept level) whenever these were offered.
- Once the correct understanding of mathematical objects or definitions or proof methods (as the case might be) had been reached, the lecturer highlighted this understanding by using facilitative utterances confirming and re-voicing these contributions.
- A potential more knowing peer was asked to present what had transpired in the proof construction thus far. The presentation of a completed component of the proof or of the whole proof by one of the more knowing students seemed to be an effective means of ensuring that all students moved forward together in developing their proof construction abilities.

Key contributions made by students in sub-episode 2.2 to 2.6

- Peers questioned Maria's reasoning and proof method.

- More knowing peers described and elaborated on the proof method of an implication.
- More knowing peers used simple every-day language to clarify the proof method and which statement had to be proved.
- Peers offered clarification of the notions of union and intersection using examples.

Sub-episode 2.7

In sub-episode 2.7 the lecturer continued with transactive prompts for clarification, reflection and strategy in an attempt to bring to light the correct proof method of showing equality of sets (lines 124, 126, 128,130, 134, 136, 138, 141, 143 and 145). She also referred to the definition of set equality implicitly guiding students to realize that a closer scrutiny of the definition would allow them to arrive at the correct method of proof. After much prompting, the definition of set equality and the proof method were brought to light. Facilitative contributions from the lecturer confirmed the definition and its importance in finding the proof method and highlighted all the learning that had taken place (149 and 151). The students were then prompted (through the transactive prompts for clarification and strategy in line 156) to apply the proof method to the context of the problem on which they were working. In this sub-episode the students actively engaged in the discourse were Joseph, Maria, Edgar, Helen and Gary who all showed a very high level of participation. There were a total of 8 correct contributions and 3 incorrect contributions from these students.

Key contributions made by the lecturer in sub-episode 2.7

The lecturer makes the following contributions:

- Transactive requests for reflection, strategy and reflection prompted students to work towards bringing to light the correct proof method of showing equality of sets.
- References made to the definition of set equality and transactive prompts for reflection and strategy urged students to extract the correct proof method from the definition.

- Facilitative contributions confirmed the importance of definitions and highlighted all the learning that had taken place.
- A transactive request for clarification and strategy prompted students to apply the proof method that has been brought to light to the context of the problem on which they were working.

Key contributions made by students in sub-episode 2.7

- Students engaged with the definition of the notion of set equality to arrive at the correct proof method and applied this method to the particular context.

Sub-episode 2.8

In this sub-episode the lecturer made a facilitative contribution (in line 163) encouraging and praising students for having been able to reach the proof method of showing equality of sets through their correct interpretation and use of the definition. Maria now continued with the proof. After beginning the proof correctly, in line 177 she assumed the statement that she needed to prove (line 177). At this point the lecturer made a transactive prompt for critique and justification and repeated the question: "Is that true?" twice (line 178). She also made a directive contribution (in line 178) trying to emphasize to Maria and the other students the importance of justifying the deductions that one has made. When Maria again seemed to revert to complex level reasoning about the proof method she was using, the lecturer made facilitative contributions (lines 188 and 194) and a transactive prompt requesting strategy (line 190) repeatedly drawing Maria's attention to what she needed to prove. The lecturer and other students such as Joseph now helped Maria to make the correct deductions at every step bringing one component of the proof to conclusion (lines 196 to 206).

The lecturer then asked Christine as a potential more knowing other to go up to the board and go through the proof construction of the component that had just been completed clarifying each step and showing exactly how the conclusion had been attained. Christine did this proof presentation very well (line 229). She explained and used newly met mathematical objects with ease, identified what needed to be shown in the proof, established the connection between this and statements made in the proof and justified each deduction and conclusion. She appeared to have reached concept level

use and interpretation of all the terms and proof methods relevant to the proof construction and communicated this to her peers in a very able manner.

Contributions from the lecturer and other students reached their highest levels in this sub-episode, with 13 transactive prompts and 15 facilitative utterances from the lecturer and 26 correct and 4 incorrect contributions from students.

Key contributions made by the lecturer in sub-episode 2.8

The lecturer made the following significant contributions in sub-episode 2.8:

- Facilitative contributions encouraged students to extract the proof method of showing equality of sets from the definition of set equality.
- A transactive prompt for critique and justification repeating the question: "Is that true?" emphasized the importance of justifying the deductions that one made.
- Facilitative contributions and transactive requests for strategy repeatedly drew Maria's attention to what she needed to prove.
- Transactive requests prompted a more knowing peer to go through the proof construction which had just been done to clarify each step and demonstrate in detail exactly how the conclusion had been attained.

Key contributions made by students in sub-episode 2.8

- More knowing peers (particularly Joseph) responded to the lecturer's transactive requests for strategy, clarification and justification by offering contributions on the correct deductions for the way forward. These students showed a good grasp of the proof method and the necessary strategic knowledge of how to use the assumption and the correct proof method to proceed with the proof construction.
- Christine's proof presentation delivered with conviction highlighted the assumption and the statement to be proved, explained the logical reasoning behind each step and clearly provided the necessary justification for each step in the proof construction.

Sub-episode 2.9

At the beginning of sub-episode 2.9 Maria identified the correct plan of action for the first time (line 245). The lecturer made facilitative contributions encouraging and

confirming the plan (line 246). Maria now revealed that she had not been able to transfer the method of proof of showing that one set is a subset of another set to this component of the proof (line 249). Helen (in lines 250 and 252) quickly interjected giving Maria short simple advice in every-day language. Maria showed that she had been unable to transfer the ideas discussed in the previous component to this component when she used the statement that she needed to prove as an assumption once again (line 261). The lecturer then made a directive contribution providing immediate feedback on this incorrect notion (line 262). She prompted Maria to remember and use the assumption that had been made. Through requests for reflection and strategy the lecturer (lines 262, 264, 266 and 268) and other students (in lines 269, 273, 275 and 280) took Maria through to the completion of the next component of the proof.

In this sub-episode Helen, Christine, Joseph and Frank actively participated in the discussions with Maria again indicating a very high level of continuous interaction and participation. The lecturer contributed 3 transactive prompts, 7 facilitative utterances and 4 directive utterances and there were 19 correct contributions from Maria's peers.

Key contributions made by the lecturer in sub-episode 2.9

The lecturer made the following significant contributions in this sub-episode:

- Facilitative contributions encouraged students and confirmed the correct plan of action.
- A directive contribution provided immediate feedback on the incorrect action of using the statement that one needed to prove as an assumption.
- Transactive requests prompted Maria to use the assumption.
- Transactive requests prompted Maria and other students to reflect on strategy for taking the proof forward.

Key contributions made by students in sub-episode 2.9

- Peers offered contributions in simple every-day language on the proof method of showing that one set is a subset of another.

- Peers offered suggestions on the correct strategy for the way forward in the proof construction and made contributions of correct deductions which helped Maria to make progress in the proof construction.

Sub-episode 2.11

In sub-episode 2.11 Maria showed a marked change in her confidence and ability. She identified the correct plan of action for the final component of the proof (line 310) and showed a vastly improved ability to use the implication proof method. After identifying the first correct step, she made an (unjustified) deduction which would lead to the desired conclusion of the proof in lines 324 and 326. She mentioned an appropriate assumption (in line 326) but did not appear to know how to use this assumption to make the correct deductions. The lecturer asked her through transactive prompts requesting clarification and explanation (lines 325, 327, 331, 333, 337 and 339) to reflect on her actions and follow logical reasoning (line 339, 343, 345, 347 and 350). The lecturer also made facilitative contributions attempting to restructure proof writing and highlighting assumptions made at the beginning of the proof construction (line 335). Other students participated and came to Maria's help in identifying the correct deduction to be made. Once the correct deduction had been identified, the lecturer again requested the students to reflect on their reasoning (line 363) and made facilitative contributions confirming and highlighting what had been learned (lines 352, 357, 359, 361, 365 and 367). Maria then successfully brought the final component of the proof to conclusion. The lecturer made a total of 12 transactive prompts and 7 facilitative contributions. Contributions from other students totalled 8 correct contributions from Christine, Gary, Edgar and other unidentified students and 5 incorrect contributions from Frank. This once again indicated a very high level of participation.

Key contributions made by the lecturer in sub-episode 2.11

The lecturer made the following significant contributions in this sub-episode:

- Repeated transactive requests for clarification and explanation and reflection on her reasoning processes prompted Maria to reflect on her actions and follow logical reasoning.

- Facilitative contributions prompted students to restructure proof writing and highlight assumptions.
- Transactive requests prompted all participants to reflect on their reasoning.
- Facilitative contributions confirmed and highlighted what had been learned.

Key contributions made by students in sub-episode 2.11

- Peers helped identify the correct assumption to be used in the proof construction.
- Peers helped provide the correct deductions to be made when Maria had difficulty proceeding with proof construction steps.

Summary of lecturer's contributions in episode 2

As seen in table 8.1 there were a total of 65 transactive prompts and 38 facilitative utterances in episode 2. Patterns of the lecturer's contributions in episode 2 saw the lecturer initially drive discussions forward through transactive prompts to all the participants asking for clarification and explanation. There were also continuous requests to the student constructing the proof to reflect on proof construction actions and offer a strategy for the way forward. The lecturer did not at any stage provide the answers but continuously prompted all participants to offer their thoughts and reasoning. The lecturer also encouraged the use of examples to clarify and illustrate notions related to the proof construction. When pseudoconcept or concept level contributions which could take the proof construction forward were made, the lecturer pursued these with transactive prompts for elaboration and explanation. Once correct understanding of mathematical objects, ideas or proof methods had been attained the lecturer solidified this understanding with facilitative utterances confirming, re-voicing and highlighting the learning in process. Once a module or component of the proof construction was brought to a conclusion, the lecturer asked potential more knowing peers to go through the proof construction and elaborate on this in detail.

Further on in the process of proof construction, reference was made to the definition of a mathematical object, and transactive prompts for reflection and strategy which guided and prompted students to use the definition to arrive at the correct method of proof. The lecturer continued using transactive prompts for clarification and strategy to prompt

students to use and apply the definition and proof method discovered to the particular context of the problem.

Once the proof methods relevant to the proof construction had been identified and clarified, then errors on the justification of deductions, and the logical reasoning process were increasingly evident. The lecturer used transactive prompts requesting critique and justification. She repeatedly asked the student doing the proof construction to reflect on the truth or correctness of deductions and drew attention to the assumptions and the goal of the proof construction. High levels of peer participation guided the student doing the proof construction at every step, through to the conclusion of another component. At the completion of the next component of the proof, the lecturer asked a more knowing peer to present the proof and this was done in a very capable manner.

The error of assuming the statement that was supposed to be proved was repeated further in the proof construction process and the lecturer made directive contributions providing immediate feedback and reminding the students to make use of the assumption. Transactive requests for reflection and strategy from the lecturer and step by step contributions from peers helped bring the next component of the proof to completion.

Maria's proof construction abilities improved greatly in the final component of the proof construction but even though Maria was following the correct proof method she still had difficulty in making the correct deduction which would drive the proof construction forward. The lecturer urged Maria through transactive requests to reflect, clarify, explain and use logical reasoning and drew attention to the assumption made. Maria's peers participated in guiding Maria to make the correct deductions and complete the proof.

8.2.3 Characteristics of the interactions of lecturer and students in Episode 3

In episode 3 Edgar attempted the proof of the proposition: $(A \cup B) \times C = (A \times C) \cup (B \times C)$. A successful proof construction of this proposition required knowledge of the proof method of proving equality of sets as well as knowledge of the precise definitions of union, subset and the Cartesian product, and the ability to use these definitions in the

logical reasoning and justification of each step in the proof. The method of proof of the equality of sets appeared to be well understood after all the practice in the first session and it was only the newly met notion of the Cartesian product of sets that posed a challenge to Edgar and hampered the proof construction.

In this episode Gary and Joseph assumed the same transactive prompts and facilitative utterances which the lecturer had contributed to scaffold students' thinking processes in the first session (in episodes 1 and 2). They were instrumental in guiding Edgar to realize his incorrect ideas about the use of the notions of the Cartesian product and ordered pairs. By using transactive prompts requesting reflection, clarification, logical reasoning and justification which now appeared to have become well established habits they took over the scaffolding altogether. Gary (in lines 18 to 34 shown below) did this by repeatedly asking Edgar to reflect on and justify his actions, and by referring to the definition of the Cartesian product and elaborating on this to apply to the particular context. Joseph (lines 39 to 47 shown below) on the other hand uncovered the root of the misconception and elaborated on this to enable Edgar to make the transition from complex level thinking toward concept level thinking.

Illustrative pattern of Gary's guidance and scaffolding

Edgar started the proof correctly in line 3 of sub-episode 3.1 by taking this proof construction action: *let* $(x, y) \in (A \cup B) \times C$. However he then made the incorrect deduction: $\Rightarrow (x, y) \in (A \cup B)$ *and* $(x, y) \in C$. In lines 10 and 12 Gary suggested selecting a statement at the beginning of the proof which clarified which component of the proof construction Edgar would be attempting first which would add logic to the proof construction process. He then continued in lines 18 to 34 shown below to help Edgar to realize the error of his deduction by asking him to reflect on and justify his actions. Gary explicitly referred to the definition of the Cartesian product (lines 24, 26 and 28) explaining it in simpler terms and tried to guide Edgar to apply this definition to the particular context (lines 32 and 34).

[18] Gary: Another thing. You say let x and y be an element of that, right?

[19] Edgar: Yes

[20] Gary: Then after that you say it implies that x, y is an element of A union B?

[21] Edgar: Yes

[22] Gary: Why do you say that?

[23] Edgar: It's an element of…. Oh, agiri if you look at this is an element of the whole of this, isn't it? *[points to* $(A \cup B) \times C$*]* So now, um, this, it implies that this one *[points to (x, y)]* is an element of this one *[points to* $(A \cup B)$ *]* and again is an element of C – both of them x and y.

[24] Gary: Let's go back to our actual definition *[referring to*$(x , y) \in (A \times B)$*]*.

[25] Edgar: The definition, ja.

[26] Gary: It says x is, x comes from A

[27] Edgar: Mmm

[28] Gary: And y will come from B

[29] Edgar: Mmm

[30] Gary: while working with Cartesian products, right?

[31] Edgar: Mmm

[32] Gary: So right now we're working with Cartesian products you tell us that A union B, it means that x must come from A union B.

[33] Edgar: x must come from x union B?

[34] Gary: x should come from A union B. And then y comes from C.

Illustrative pattern of Joseph's guidance and scaffolding

After Gary's guidance, Edgar stubbornly held on to his erroneous reasoning as shown in line 35 below.

[35] Edgar: Before, before you do that I think, I think according to my understanding I don't know, according to my understanding I think I have to… This one *[points to: let* $(x, y) \in (A \cup B) \times C$ *]* If the… this one *[points to* (x, y) *]* is an element of both these *[underlines* $(A \cup B) \times C$ *]* isn't it?

Now Joseph (in lines 39 to 47) offered his insight (Edgar's association of the notion of Cartesian product with the notion of intersection) on the cause of Edgar's erroneous reasoning and elaborated on this, guiding Edgar to realize the correct deduction (in line 48) as seen below.

[39] Joseph: You can say it's an element of both $A \cup B$ and C if we are talking of an intersection.

[40] Edgar: If we are talking of an intersection?

[41] Joseph: Ja. And if that cross wasn't there A union B intersection C

[42] Edgar: Ok

[43] Joseph: So in this case we are talking of cross product.

[44] Edgar: Yes

[45] Joseph: It means the element x belongs to the set that is before the cross.

[46] Edgar: Yes

[47] Joseph: And then *y* belongs to the set that is after the cross. So in this case he is right in saying *x* is an element of *A* union *B*.

[48] Edgar: Ok. I was confused. Thanks a lot for that *[erases :* $\Rightarrow (x, y) \in (A \cup B)$ *and* $(x, y) \in C$ *]* Ok *[writes:* $\Rightarrow x \in (A \cup B)$ *and* $y \in C$ *]* Is that what you are saying? So this implies that *x* is an element of *A* or *x* is an element of *B* and *y* an element of *C*. Ok *[writes:* $\Rightarrow x \in A$ *or* $x \in B$ *and* $y \in C$ *]*

As a result of all the scaffolding Edgar seemed to have an increased awareness and appeared to now "know" the correct usage of these terms, symbols and signs. However as he continued the proof, he repeatedly made mistakes that showed his incorrect use and interpretation of the notion of the Cartesian product and its elements, the ordered pairs. When his peers asked him to reflect on and justify his proof construction steps, he quickly corrected these mistakes indicating that he was developing an understanding of these notions and hopefully progressing from complex thinking towards concept level thinking.

Once Edgar correctly concluded the proof construction in sub-episode 3.2, Joseph drew attention to other errors made (in proof construction steps leading to the conclusion) possibly as a result of Edgar's association of the notion of the Cartesian product with the word 'and' contained in its definition. Joseph did this by giving several other examples of similar errors made with other mathematical objects using the same mode of reasoning. Joseph's use of examples to illustrate mathematical objects and their correct usage in the course of proof construction was another of the characteristics of these valuable more knowing peers. This is outlined below.

<u>Joseph's use of examples to illustrate mathematical objects and their usage in proof construction</u>

After the deduction obtained in line 100: "$\Rightarrow x \in A$ *or* $x \in B$ *and* $y \in C$", Edgar made the following deduction in line 104: "$\Rightarrow x \in (A \cup B) \times y \in C$". It was possible that Edgar was associating the notion of the Cartesian product (and its symbol) with the word 'and' contained in its definition and surmised that the two were interchangeable. After Edgar had concluded the proof construction Joseph prompted Edgar through transactive requests for clarification and used examples to clarify misconceptions about the usage

of terms (lines 105 to 113). In this way he drew attention to the root cause of the error (in line 104).

[105] Joseph: I've got a question

[106] Edgar: Yes

[107] Joseph: Can you say suppose you have x is an element of A intersect B

[108] Edgar: For example?

[109] Joseph: For example, ja. From there can you say x is an element of A, write the intersection sign x is an element of B?

[110] Edgar: You come and…

[111] Joseph: [goes to the board] Suppose you have x is an element of A intersect B *[writes:* $x \in (A \cap B)$ *]* You say it says to us that x is an element of A intersect x is an element of B *[writes:* $\Rightarrow x \in A \cap x \in B$ *]* Because I think this intersection *[points to:* $x \in (A \cap B)$ *]* tells us that we are thinking of one set

[112] Edgar: Ok

[113] Joseph: And can we say that? Can we move from there to there? *[points to* $x \in (A \cap B) \Rightarrow x \in A \cap x \in B$ *]* Why I'm asking this, I see this here *[underlines:* $\Rightarrow x \in (A \cup B) \times y \in C$ *]* So I'm happy that we came across this because I'm also getting confused. Can we say this? *[points to* $x \in (A \cap B) \Rightarrow x \in A \cap x \in B$ *]* Or even can we say x being an element of A union B implies that x is an element of A or x is an element of B? *[writes:* $x \in (A \cup B) \Rightarrow x \in A \cup x \in B$ *]* Can we say?

The correction was then made: that is $\Rightarrow x \in (A \cup B) \times y \in C$ was changed to $\Rightarrow x \in (A \cup B)$ *and* $y \in C$.

Key contributions made by students in episode 3

In episode 3 which occurred in the second consultative group session, the transactive prompts and facilitative utterances which previously characterized the lecturer's contributions in the first session seemed to have now become well established habits especially in the more knowing students (possibly through the use of activities such as imitation). They showed their ability to guide Edgar and provided the much needed scaffolding, successfully bringing the proof to completion without a single word from the lecturer. It was interesting that Gary and Joseph assumed the responsibility of guiding Edgar and acted as more knowing others using two very different but complementary styles.

- Gary used transactive prompts that asked the student to think and reflect on his actions. He also used the definition of the Cartesian product and elaborated on it and tried to show how one could apply the knowledge of the definition to the particular situation. In this way he encouraged the student to arrive at the correct solution or make the required correction himself.
- Joseph on the other hand, dug deeper to find the cause of the student's difficulty or misconception and by making the reason for the student's error apparent, tried to guide and ensure that the student really understood and would not make the same mistake again. He also referred to the definition of the Cartesian product previously referred to by Gary and used every-day language to interpret this definition and show how it could be applied.

Gary tried to drive the student to self-realization while Joseph revealed the essence of the student's error, and guided the student's development in this way. Both methods have their merits but it appeared that Edgar responded more positively or was more affected by Joseph's insight. For example in line 48, Edgar seemed to realize his error at last, after Joseph's explanations. We cannot however really distinguish which method was better as all three students were learning from each other in the EZPD, so it was very possible that both Joseph's explanation and Edgar's realization were also the result of Gary's earlier prompts for reflection and elaboration and his references to the definition and his explanation of the it.

In the course of this proof construction process, Gary made 14 contributions in the form of transactive requests for reflection, clarification and justification as well as making reference to and explaining definitions. Joseph also made 14 contributions in total and these included explaining definitions, pinpointing the cause of misconceptions and using examples to illustrate mathematical objects. The lecturer's role had receded so much into the background at this juncture that she did not make a single contribution. In fact, the other students (mostly Gary and Joseph in this case) seemed to have made the transactive prompts for reflection, justification and strategy their own. This was a surprising and remarkably fast transition as this was only the second time that the students had been together and it was very encouraging to see them develop attitudes of questioning and making certain of the truth of statements and deductions, as well as

referring to the definition of the Cartesian Product and ensuring that it was used and applied correctly.

8.2.4 Characteristics of the interactions of lecturer and students in Episode 4

In episode 4 Maria showed vast improvement in the areas of proof construction with which she had significant difficulty in episode 2. She showed great improvement particularly in her use and interpretation of proof methods and the logical reasoning processes relevant to the proof and in her appreciation of the need for justification of all assertions and deductions. Initially (in sub-episode 4.1) she also applied the notion of the Cartesian product correctly and completed the first component of the proof with no errors. In sub-episode 4.1 the only contributions from Maria's peers were to point out writing errors. However in sub-episode 4.2 when Christine questioned whether the Cartesian product symbol could be replaced by the intersection symbol, Maria displayed an incomplete understanding of the notion of the Cartesian product and was unable to explain the distinction between the notions of intersection and the Cartesian product. Joseph then contributed by giving his concept level reasoning of the distinction between the notions of the Cartesian product and intersection. He explained the definition in simple every-day language. The lecturer used facilitative contributions confirming and re-voicing Joseph's contributions and transactive prompts requesting clarification and elaboration. Presumably as a result of her functional use of the Cartesian product while she engaged in the proof construction, and the scaffolding from her peers, Maria's interpretation of the mathematical object developed. She then went on to complete the second component of the proof in sub-episode 4.3 showing concept level usage and interpretation of the Cartesian product as she now ably explained its use and meaning and even referred to the definition of the mathematical object in her explanation.

The lecturer's contributions in this episode were again minimal and served primarily to confirm appropriate contributions and request elaboration on these contributions from more knowing peers.

Key contributions by the lecturer in Episode 4

- Transactive requests prompted clarification and elaboration from the student doing the proof construction and her peers.
- Facilitative contributions confirmed correct ideas and highlighted misconceptions and the learning taking place.
- Didactive contributions referred to the definition of the Cartesian product and confirmed Joseph's interpretation of this notion and highlighted the distinction between the notions of the Cartesian product and the intersection.

Key contributions by students in Episode 4

- Joseph's insightful contributions helped to clarify the interpretation of the definition of the Cartesian product and pinpointed distinctions between the notions of the Cartesian product and the intersection.

8.2.5 Characteristics of the interactions of lecturer and students in Episode 5

In the initial stages of Episode 5 while Frank was attempting the proof construction of the first component of the proof of the proposition: $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$, his more knowing peers (Gary and Joseph) who had previously (in Episodes 3 and 4) taken over the responsibility of scaffolding and leading the mathematical discussions forward, seemed to be trying to build their own understanding of the newly met notion of the power set. The main challenge the students had when working with power sets was that of realizing that the power set of A (for example) was a set containing all of the subsets of the set A which were themselves sets. Thus contrary to elements of a simple set being single elements such as x, each element of a power set is a set. Frank's proof construction attempt in which he used and applied the notion of the power set correctly was a good opportunity for the other participants who might have been uncertain about the application of this mathematical object to build up their understanding and deepen it.

Possibly through their earnest inquiry and reflection on the definition of the mathematical object as they made functional use of the power set and through their

interaction with their peers and the lecturer, they were able to make gains in their understanding and resume their roles as more knowing peers in sub-episode 5.3 when Frank struggled to complete the proof of the second component of the proof. This is discussed in more detail below.

Sub-episode 5.1

In sub-episode 5.1 Frank started the first component of the proof construction and completed this proof construction without any errors. He displayed concept level usage of the proof methods of the double implication, implication and showing that one set is a subset of another as well as the logical reasoning process involved in the proof construction and showed a good appreciation for the need of justification of each deduction. His proof construction also showed correct interpretation and use of the notion of the power set and its definition. There were just two transactive prompts from the lecturer in this sub-episode urging Frank (who was silent while writing on the board throughout the proof construction attempt) to elaborate, explain and clarify his reasoning in the proof construction process.

In this sub-episode both Gary and Joseph who acted as more knowing others in episodes 3 and 4 tried to come to grips with the newly met notion of the power set, building their understanding as they made functional use of the newly met term while interacting with their peers (requesting clarification and explanation) and reflecting (on Frank's proof construction attempt) in the EZPD. In lines 8, 10, 14, 16, 18, 21 and 23, we saw them questioning and reflecting on Frank's responses and implicitly on the definition of the power set as they tried to develop their understanding. Maria (line 24) was also participating in the EZPD and seemed to have a good understanding of the newly met mathematical object.

Students reflecting on mathematical objects and definitions during the proof presentation

In sub-episode 5.1, Frank's peers including Gary and Joseph tried to build up their understanding of the notion of the power set by reflecting on Frank's proof construction actions and requesting clarification and explanation. I have included part of their interaction (from lines 8 to 21) which showed them using the same transactive prompts for clarification and explanation as the lecturer had used in earlier episodes, on aspects

of the power set and its elements and then making contributions of their own. Their actions and contributions could also possibly be as a result of imitation of the modes of reasoning and questioning the lecturer had used previously.

[8] Gary: So in your case X is an element?

[9] Frank: Yes, it's an element… in, of the power set, but it's a subset of…

[10] Gary: You create then your capital letter X – where is it?

[11] Frank: Oh, my capital letter X?

[12] Gary: Ja

[13] Frank: This one?

[14] Gary: What does it represent?

[15] Frank: When you, when you… when you are talking in terms of power set we want to make a variable to be an element of a power set, we must make it in a capital letter. You understand? You must not make it a…

[16] Joseph: You are saying X is a subset of A?

[17] Frank: Yes

[18] Joseph: Meaning that when you talk of power sets we say it consists of sets, a power set of A which means it consists of all possible sets of

[19] Frank and Edgar: A

[20] Frank: yes

[21] Joseph: Now you can't say X is in itself is an element of… you must say X is a subset since a power set consists of sets.

[22] Frank: Oh, you want me to say that X is a subset of this? *[points to let* $X \in P(A) \Rightarrow X \subseteq A$ *]*
Oh, here we must change this element to a subset?

[23] Joseph: Of course! Power sets consist of subsets.

[24] Maria: I think since a power set consists of sets and then I can say that X is an element of the power set of A it means that X is contained in the power set of A, not X being a subset of the power set of A, I think.

Gary and Joseph's transactive questions seemed to have progressed to a higher level of sophistication and their yearning to reach better understanding was obvious. I suggest that they themselves were now acting as their own guides, rather than relying on the lecturer or their peers. They seemed to have assumed the responsibility for developing their own understanding by examining the definition of the power set and reflecting on it (for example in lines 8, 10, 14, 16, 18, 21 and 23). They earnestly tried to reach a higher level of understanding while questioning and clarifying the mathematical object not only for themselves, but for the other students as well. I believe that this was

another indication of students becoming 'intellectually autonomous' as referred to by Yackel and Cobb (1996).

Sub-episode 5.2

In sub-episode 5.2 Joseph's quest for further clarification of the notion of the power set and its elements led him to illustrate this notion using an example (line 26). Using examples to illustrate and clarify mathematical objects had become one of his well-established habits. As the notion of the power set was very new, Joseph did not have a large repertoire of useful examples to use or generate. He drew the Venn diagram of the power set of a set A and tried to populate it without first depicting the set A and its elements. The difficulty that Joseph had in generating a useful example which would help him and others to understand the notion of the power set more completely was one of the difficulties identified by Moore (1994). He noted that although students valued using examples to help them understand mathematical objects and build their images of these objects, this was sometimes hindered by their limited experience in the particular mathematical field (cf. Section 2.2.1).

Realising that the Joseph's example was not that helpful and that the notion could be illustrated more clearly, the lecturer did another example (in line 41). This very simple example depicted the set A (shown having two elements: 1 and 2) and its corresponding power set using Venn diagrams, in the hope that this would help the students to see the connection between the elements of a set and the elements of its power set, and further clarify the mathematical object. The only contribution from the lecturer in this sub-episode was this didactive one illustrating the notion of the power set with an example.

Sub-episode 5.3

In sub-episode 5.3 Frank began the proof of the second component of the proof construction: $P(A) \subseteq P(B) \Longrightarrow A \subseteq B$. Following the correct method of proof of an implication, he (in line 47) chose an element in the set A: "let $\{x\} \in A$". However this choice was incorrect as this would be a subset, and not an element of the set A. The lecturer tried to guide the proof construction through transcative prompts for reflection, strategy and justification (lines 48, 52, 56 and 59) and directive contributions (line 50, 54 and 57) reminding Frank that he was choosing an element in the set A which was simply a set.

Probably as a result of their functional use of the power set while reflecting on the correct proof construction of the first component of the proof attempted by Frank in sub-episode 5.1 and the interaction with their peers and the lecturer in the EZPD, Gary and Joseph's understanding of the notion of the power set had advanced very swiftly. They now made constructive contributions towards the appropriate strategy for proof construction and determined the correct deductions and plan of action (lines 61, 62 and 63). They appeared to have reached concept level interpretation and use of the notion of the power set and resumed their roles as more knowing peers. The lecturer asked Frank to re-do (on the board) the example which had previously been done in order to shed clarity on the suggestions made by Gary and Joseph (line 64). She asked Frank to reflect on the example for inspiration on the way forward (lines 74 and 76). When Frank still showed signs of uncertainty, Joseph went up and completed the second component of the proof construction on the board correctly (line 77). Presumably in an attempt to remove Frank's uncertainty and confusion, he also altered the example on the board slightly (line 81) improving it to clarify the proof construction steps of this component even further.

It was very interesting to observe that although Gary and Joseph were not at concept level usage and interpretation of the notion of the power set in sub-episode 5.1, they succeeded in improving their understanding in a very short space of time and could help their peers with the proof construction in sub-episode 5.3. They were able to do this presumably through their earnest engagement and reflection, making functional use of the mathematical object and its definition while engaging with their peers in the group discussion. I argue that this shows the effectiveness of the EZPD in propelling the students' understanding of the mathematical objects and processes involved in the proof construction forward and having a very positive impact on their abilities in proof construction.

Key interactions of the lecturer in Episode 5

- Transactive requests prompted students to reflect, offer proof construction strategy and provide justification for deductions.
- Directive contributions provided guidance on the proof construction process.

- A didactive contribution offered a more helpful example to illustrate the newly met notion of the power set.

Key interactions of students in Episode 5

- More knowing peers (Joseph and Gary) initially built up their knowledge of the newly met mathematical object by reflecting on the proof construction done by their peer, and earnestly engaging with their peers and the lecturer, while making functional use of the object and its definition by enquiring about and examining the definition of the object.
- Joseph tried to build up and strengthen understanding of the notion of the power set by drawing an example of a power set on the board.
- Joseph and Gary contributed on the correct strategy for the way forward and suggested correct deductions to be made, and Joseph successfully completed the proof of the second component.
- Joseph improved on the example of the power set previously given by the lecturer to further clarify proof construction steps.

8.3 Overall Discussion

In Section 8.2 the nature of lecturer and student interactions in each of the five episodes was explored and discussed. Throughout these discussions it has been assumed that as students interacted with their peers and the lecturer in the consultative group sessions, they made functional use of newly met terms, symbols, signs, proof methods as well as logical and deductive reasoning processes and justification of each step in the proof construction process. With the help of the guidance and scaffolding in these sessions, students were enabled to make progress in their proof construction abilities. I have not highlighted students' functional use of mathematical objects and processes in these discussions but have rather focussed on student and lecturer interactions which were significant and promoted such progress and development.

In conclusion I summarize the lecturer's significant actions in these episodes which in my understanding, contributed towards the development of students in their journey of becoming intellectually autonomous. At the same time I will attempt to identify characteristics and modes of reasoning of students who showed potential to become

more knowing peers and observe how these students were empowered in the consultative sessions. Thus I will be answering my third research question which is repeated below for ease of reference:

a) How can lecturers encourage and support students who are engaging with proof construction while participating in consultative group discussions, to become intellectually autonomous?

b) What are the characteristics and modes of reasoning prevalent in students who act as more knowing peers and how are they empowered as they participate in the consultative group sessions?

8.3.1 Significant actions of the lecturer that may contribute to the development of intellectual autonomy in students

A summary of the lecturer's significant actions towards the development of students' intellectual autonomy in each of the five episodes is given below.

Episode 1

In Episode 1 Frank's proof construction attempt contained a few flaws; namely the use of the double implication symbol instead of the single implication symbol and the absence of statements or phrases at the beginning of the proof which would add to the logic of the proof construction. However the lecturer did not offer corrections nor did she verify whether the attempt had been correct or incorrect. Instead her actions elicited students' contributions and ideas by repeatedly making transactive requests for reflection and clarification on proof construction strategy. From the very beginning of the consultative sessions the lecturer transferred the responsibility for finding the correct way forward onto the students, and made it clear that by working and consulting together they would be able to reach their ultimate goal of a correct proof construction. She thus encouraged each student to develop his/her own capacity and take an active role in learning rather than relying on an external source such as the lecturer. Contributions from students which were pseudoconceptual or conceptual were encouraged using facilitative contributions confirming these ideas and the lecturer asked these students to further elaborate and explain their reasoning. The lecturer's actions implicitly made students realize that their ideas and contributions were valued and

respected. During the course of the proof construction, the lecturer, through requests for clarification and explanation, prompted students to give their ideas about newly met terms and symbols which had been observed to be challenging. Students offered further contributions clarifying the proof method and the meaning of newly met terms, symbols and signs using simple every-day language. There were also contributions from peers towards proof construction strategy and clarification of the notions of the implication and double implication. It was evident that students (such as Gary and Joseph) who showed great potential to become more knowing others, took their own initiative to use examples to illustrate newly met mathematical objects and proof methods.

Towards the end of the proof construction in Episode 1, after all the students had had a chance to offer their ideas and contributions, the lecturer summed up all the learning that had been discussed. This was to ensure that all the students were made aware of all the mathematical objects and processes such as proof methods discussed and that incorrect ideas or ideas incongruent with the true meaning of mathematical objects were addressed. As discussed in Section 6.2.5 one of the drawbacks with learning environments encouraging active participation from all students is that wrong ideas and notions could easily be propagated. The aim of the summary was to highlight incorrect ideas and provide the correct interpretation and application of the notions relevant to the proof construction hence preventing propagation of misconceptions to other students.

Episode 2

In this episode there was a similar pattern of lecturer contributions as in Episode 1. Using transactive requests for clarification, reflection and justification students were prompted to describe and clarify their reasoning processes about methods of proof and meanings of newly met terms and symbols. When incorrect proof methods were used or incorrect ideas introduced such as the use of trivial and non-useful aspects of assumptions and definitions, transactive requests for reflection and strategy urged students to reflect and offer their contributions on the way forward. This allowed students' conceptions of proof methods and meanings of newly met terms and symbols to emerge so that misconceptions could be addressed and clarified. I also suggest that in sharing these conceptions students were better able to engage with the mathematical objects and processes while their misconceptions were corrected and clarified.

When potentially more knowing peers offered contributions on the correct method of proof the lecturer's transactive requests for elaboration encouraged further explanation. Initially these students would offer very brief and cryptic answers and had to be coaxed to give fuller and deeper explanations (for example Gary in sub-episode 2.4). In this way students were made aware that those who had more understanding of the mathematical objects and proof methods involved in the proof construction would be responsible for clarifying and explaining these objects and methods to the whole group. This also seemed to contribute to the confidence of those students who displayed potential in their proof construction abilities. Their contributions gradually became more forthcoming and their explanations were always given in depth (for example see Christine in sub-episode 2.8). The lecturer confirmed and sometimes re-voiced these contributions using facilitative utterances.

When students had questions about mathematical objects and processes related to the proof construction and the area of set theory in general the lecturer used transactive requests asking students to use examples to illustrate these notions to gain more clarity. Students' participation was further encouraged in this way as they showed their eagerness to come up and do examples on the board depicting their conceptions of newly met terms and symbols.

During the course of proof construction when the proof method for showing equality of sets was met, the lecturer asked students to refer to the definition of set equality several times and use this definition to arrive at the proof method. In this way students were shown how the definition of a mathematical object could be examined and interpreted in order to extract the overall structure of the proof giving rise to the correct method of proof. Once the general proof method had become apparent, the students were asked to apply the method to the particular proof construction with which they were engaged.

Throughout the proof construction attempt in Episode 2, we saw Maria's persistent difficulties with respect to incomplete understanding of proof methods involved in the proof construction. She also lacked the ability to use deductive logical reasoning and justify her deductions and conclusions. Interactions with peers and the lecturer repeatedly reminded her to ensure the truth of statements she made, and to be continually aware of assumptions and the statement to be proved. When the first

component of the proof had been brought to conclusion with the continuous scaffolding received from lecturer and peers, the lecturer asked a more knowing peer (Christine in sub-episode 2.8) to go through the proof construction in detail to clarify and explain proof construction steps and show exactly how the conclusion was attained.

It was clear as Maria struggled on to further components of proof construction, that she had not been able to transfer proof methods and ideas related to logical reasoning and justification from previous components to subsequent ones. The lecturer again reminded students to be aware of assumptions and the statement to be proved, and urged them to use logical reasoning processes and justification. Peers offered scaffolding using simple every-day language and high levels of participation and interaction drove the proof construction forward.

Maria's use and application of proof methods improved towards the end of the proof construction attempt but challenges with regards to logical reasoning processes and the practice of justification persisted. The lecturer through transactive requests for clarification, explanation and justification, prompted reflection on proof construction steps and urged students to ensure the correctness or truth of each deduction in the proof construction process. She also raised their awareness of implicit assumptions. High levels of participation from her peers enabled Maria to conclude the proof construction.

Episode 3

In Episode 3 (taking place in the second session) the lecturer's transactive requests and facilitative utterances were adopted and completely taken over by more knowing peers (possibly using imitation). Requests for clarification, reflection and justification as well as reference to and elaboration of the definitions of the mathematical objects relevant to the proof construction had become well established habits in these more knowing peers who took over the responsibility of guiding their peers and bringing the proof to successful completion.

Episode 4

In Episode 4 the only contributions the lecturer made were facilitative utterances which confirmed and re-voiced concept level use and interpretation of the notion of the

Cartesian product (from more knowing peers) and transactive requests for clarification asking students to offer further elaboration on these mathematical objects.

Episode 5

Once Frank had completed the first component of the proof construction in Episode 5, the lecturer prompted him to clarify and explain the reasoning and logic behind his proof construction steps. Gary and Joseph (prominent more knowing peers) in this episode tried to develop their own understanding of the power set through their interaction with their peers as they reflected on Frank's proof construction steps of the first component of the proof. They also reflected on the definition of the notion of the power set and questioned its usage and interpretation as they strove to reach concept level understanding. They had clearly taken over the responsibility of developing their own understanding by earnestly engaging with the mathematical object through functional use while interacting and discussing the notion with their peers in the consultative sessions.

Joseph took his own initiative to use an example to illustrate the notion of the power set for himself and his peers. This was once more indicative that the use of examples to illustrate mathematical objects and ideas related to the proof construction has become a well-established habit for this more knowing peer.

When Frank started the second component of the proof construction which was a little trickier than the first component, Gary and Joseph's understanding of the notion of the power set had developed to such an extent that they were able to resume their roles as more knowing peers once again. They offered contributions on the correct strategy for the way forward. When Frank's perplexity persisted, Joseph took over the proof construction bringing the proof to completion. Realising Frank's uncertainty about the proof construction steps just completed, Joseph cleverly altered the example given by the lecturer on the board depicting a set A and its power set, by replacing the elements in set A with variables, thus making the connections between the proof construction steps and the example more obvious. Gary and Joseph had in a very short period of time been able to develop their own understanding to such a level that they were able to use and apply the mathematical object correctly in the proof construction of the second component. They had developed the skills within themselves and were able to take the

responsibility of developing their own understanding through earnest engagement and enquiry while making functional use of the mathematical object in their interaction with all participants in the consultative group sessions.

Summary of key actions and contributions

To conclude, key actions and contributions of the lecturer which supported students in their journeys to become intellectually autonomous are given briefly in point form below.

- The lecturer transferred the responsibility for finding correct strategies in proof construction to students themselves, by initially withholding giving direct corrective feedback on proof construction steps and instead eliciting students' contributions using transactive prompts for reflection, clarification and strategy.
- The lecturer encouraged and elicited in-depth explanation and elaboration from students who made positive contributions which indicated pseudoconcept/ concept level reasoning, nurturing these students to develop confidence in their own capabilities and at the same time making other students aware of these students' abilities.
- When incorrect methods or incorrect ideas were presented students were re-directed using facilitative and transactive utterances to reflect on strategy and find the correct way forward.
- The use of examples was greatly encouraged to illustrate newly met terms and symbols as well as proof construction steps, and students were made aware of the illuminating power of examples in this regard.
- Students were prompted using transactive requests for reflection and strategy to examine the definitions of mathematical objects closely in order to extract the overall structure and method of proof.
- There were continuous reminders throughout the proof construction process using transactive prompts to keep the students mindful of the assumptions and the statement to be proved.
- There were continuous reminders using transactive prompts to ensure the truth of each statement and deduction.

- Towards the end of components of proof construction the learning that had taken place was summed up and elaborated on by more knowing peers who were called on to do proof presentations. Initially when more knowing peers had not yet been identified, the lecturer discussed and summed up the ideas and proof methods which had been discussed during the course of proof construction in the hope that correct conceptions would be strengthened and incorrect or inappropriate notions would be addressed.

Some of the lecturer's key actions or contributions which appeared to empower more knowing peers to assume their roles are summarized in point form below. These are in addition to the actions and contributions of the lecturer listed above.

- The lecturer encouraged further elaboration from students showing the potential to become more knowing peers, nurturing them to gain confidence in their own capabilities and become responsible for clarifying the understanding of their peers. These students seemed to gain confidence and offered in-depth explanations much more readily as the sessions progressed.
- The lecturer encouraged the use of examples when more knowing others offered them to help clarify definitions of newly met terms and symbols and proof construction steps, and helped these students to select more illuminating examples when necessary.

8.3.2 Characteristics and modes of reasoning prevalent in potential more knowing students

Key findings regarding the characteristics and modes of reasoning prevalent in students who have the potential of acting as more knowing others are summarized in point form below.

- These students readily engaged with the consultative practices of the group sessions and critiqued students' actions, contributions and reasoning processes even though they themselves often had complex or pseudoconcept level knowledge of proof methods and reasoning processes (for example Christine in sub-episode 2.2). They were also eager to ask questions about mathematical objects and processes troubling them in the proof construction.

- These students were able to communicate very ably and effectively, delivering presentations of completed components of the proof with conviction. They could give a holistic picture of the completed proof (or component of proof) as they highlighted the assumptions and the statement to be proved while explaining in detail the logical reasoning behind each step of the proof and providing the necessary justification (for example Christine in sub-episode 2.8).
- These students often used their own initiative to offer examples to illustrate newly met terms, symbols and proof methods (for example Joseph and Gary in Episodes 1, 2, 3 and 5). They instinctively turned to the illuminating power of examples when they were having difficulty in understanding or communicating their understanding of these mathematical objects and processes. Their use of examples was strengthened by the encouragement and scaffolding received from the lecturer in the form of providing more helpful examples when needed.
- When having difficulty with newly met mathematical objects in the proof construction, more knowing peers assumed the responsibility for developing their own understanding of these objects, rather than relying on others such as the lecturer. Through earnest engagement with the mathematical objects related to the proof construction while interacting with their peers they were able to develop their own understanding as they made functional use of these objects (for example Gary and Joseph in sub-episode 5.2).
- These students showed an appreciation of the importance and usefulness of definitions in suggesting the sequence of steps to be followed in the proof construction and in providing the justification for each step (for example Gary and Joseph in Episode 3). When meeting newly met terms in the course of proof construction they strove to build their understanding of the mathematical object by examining and reflecting on the definition of the notion while interacting with their peers and the lecturer (for example Gary and Joseph in sub-episode 5.2).
- These students seemed to appreciate the importance of using statements that added to the logic of the proof construction such as statements at the beginning

of the proof construction stating all the assumptions and the statement which had to be proved. They never omitted such statements in their own proof constructions.

Chapter 9: Questions of Trustworthiness

9.1 Introduction

In this study I investigated difficulties and challenges that first year undergraduate students experienced as they engaged with proof construction tasks and explored how students' proof construction abilities developed as they interacted with peers and the lecturer in the context of consultative group sessions. In this chapter I address issues of trustworthiness in my methodology, methods and analysis to ensure the quality and credibility of my study.

As discussed in Chapter 4, my ontological assumptions imply that each individual constructively develops his/her own conceptions of reality. My interpretations of the world (and this study in particular) are shaped by my theoretical perspective which is Vygotsky's socio-cultural framework. In this chapter I will examine whether my use of analytical frameworks (which incorporate Vygotsky's theory of concept formation and his notion of the zone of proximal development), validly and reliably interpreted students' proof construction actions and their interactions within the consultative group sessions. I also examine the validity and reliability of my data (which primarily consists of transcripts of the video recorded sessions) and data collection methods which include video recording, transcribing and coding of transcripts.

9.2 Validity

Several researchers have argued for alternative terms for the important notion of validity in qualitative research. Lincoln and Guba (1985) have used the notion of authenticity. They put forward key criteria including credibility and transferability to be used to replace validity in qualitative research. They propose that to ensure the validity of a study, the researcher needs to show that he/she has depicted an accurate and true description and that his/her interpretation and reconstruction of events is accurate. The notion of credibility has, in particular, been suggested as a means of measuring the quality or goodness of case study research (Opie 2004, p.71). Credibility is defined by McMillan (1996, p.250) to be "the extent to which the data, data analysis and conclusions are believable and trustworthy". Maxwell (1992, p.281) in agreement with

Wolcott (1990) puts forward the notion of understanding as a more suitable conception than validity in qualitative research. Maxwell argues for five types of validity: descriptive validity, interpretive validity, theoretical validity, generalizability and evaluative validity. He further proposes that external generalizability and evaluative validity are not as central to qualitative research as the other categories of validity (Maxwell 1992, p. 295). I will be addressing descriptive validity, interpretive validity, theoretical validity and (internal) generalizability in this section.

9.2.1 Descriptive Validity

Descriptive validity is centred on the factual accuracy of the account and whether the researcher has reported on the events that ensued with complete honesty and integrity (Maxwell 1992, p. 285). Maxwell emphasizes the primary importance of this aspect of validity arguing that all other validity criteria are dependent upon it (ibid. p.286).

The notion of descriptive validity in my study primarily focusses on the accuracy of the transcripts of the video recorded sessions. The accuracy of these transcripts was indeed extremely important to me as I would be basing my analysis and outcome of the study on these. I do believe that the transcripts of the sessions reflect the actions, speech and writing of the participants of my study very accurately. The video recorded sessions were originally transcribed by a professional transcriber to whom I had supplied a detailed information sheet containing all the mathematical terminology which might have been unfamiliar to her. I then listened to the recordings and watched them in tandem with the transcriptions several times, correcting the transcripts whenever I detected incorrect mathematical terminology and language use (in students' spoken and written work). The transcripts went through several revisions until I could no longer detect any incongruence between what had transpired in the video recorded session and the transcript. At this stage I invited four of the students who had participated in the study and were still continuing with their studies at the University, to a session where they were able to view excerpts of the video recorded sessions together with the transcripts of these sessions to get their feedback and see whether they agreed with these transcriptions. These students viewed the video clips as they read through the transcripts. Two of these were Gary and Joseph who were major contributors

throughout the sessions, and the other two were Kenny and Bonnie who did not play as dominant a role in the sessions. These students agreed that the transcripts were accurate reflections of the video recorded sessions.

I do not claim that the transcripts are perfect reflections of all the actions and events which occurred as there were many features which could have been observed but were omitted for the most part. Non-verbal data was not recorded in the transcripts (as discussed in Section 4.5.2) such as the stress and pitch of students' voices, how much time each student took to answer or make a particular contribution and the feelings of students such as their excitement when they eventually discovered the correct deduction or next step (although I periodically noted the general mood of students in my brief field notes). Although these additional observations would have added to the overall quality of the transcripts providing a more holistic picture of students' actions and reactions, I do not think that they would have affected my interpretations or analysis of transcripts.

9.2.2 Interpretive Validity

Interpretive validity refers to the researcher's ability to interpret the meaning of the situations and events that participants are engaging in from the participants' perspective correctly (Maxwell 1992, p.288). In my study interpretive validity refers to how accurately my interpretations of what the participants are thinking reflected what was actually happening, based on my observations of the video recordings, transcripts and my own experiences while acting as a participant observer in the consultative sessions. Once the transcripts had gone through the rigorous iterative process of revision and correction (as described in Section 9.2.1) and were now data with which I could work, while coding and analysing students' actions and contributions, I returned (to the video recordings) several times as my interpretations evolved, trying to ensure that these interpretations were accurate and viable representations of what had really occurred (Barron & Engle 2007, p.24). I attempted to ensure that I had coded the transcripts strictly according to the indicators of the various categories of my analytical frameworks.

Although I was guided by my research questions I remained open to observing new phenomena as I began to code and analyse the transcripts (Barron & Engle 2007, p.25).

For example I observed that the emerging more knowing students seemed to share certain characteristics in the ways they attempted to resolve difficulties they met when engaging with proof construction for themselves. I therefore added this to my research questions. I kept an open mind regarding the categories of my two analytical frameworks, adapting and extending these to include additional phenomena that were important to the research questions addressed in my study.

While engaged with the analysis of my transcripts I tried to avoid statements such as: "Frank understood the notion of the power set and has reached concept level understanding", as his usage and application of the mathematical object might have been correct while he might still have been confused about the object. Instead I have made statements such as: "Frank used and applied the notion of the power set correctly and appeared to have concept level understanding" (in accordance to the indicators which relate to the categories in my analytical frameworks). I acknowledge that my inference was based on his actions, words and writing. I tried to be impartial while coding the transcripts of the video recorded sessions according to the indicators defined and discussed in Sections 5.2 and 5.3.

My interpretations have obviously influenced my decisions on placing the actions and spoken and written contributions of students in the particular categories I have chosen. In this regard I approached one of my colleagues (who has a Master's degree in mathematics) in the Mathematics department of the University of Limpopo, and has been teaching at this institution for over 37 years, to examine the coded transcripts together with a detailed description of my analytical frameworks including my categories and indicators. She read through the transcripts of all five episodes twice and although she acknowledged having some difficulty with the terminology of the coding, particularly with the terms 'complex' and 'concept', she was in agreement with the coding. As discussed in Section 9.2.1 four of the students who participated in the study were asked to view excerpts of the video recordings that I had selected for detailed coding and analysis, with the transcripts. In addition to being asked about their views on the accuracy of the transcriptions, they were also asked to provide brief interpretations of what was happening during those particular events. Their contributions confirmed my interpretations of the events in the consultative group

sessions. This, of course, does not mean that I can claim that my study is totally free from all threats to interpretive validity as other researchers might have completely different views. However I attempted to do my best in coding the transcripts as fairly as possible according to the indicators and categories of my analytical frameworks.

9.2.3 Theoretical Validity

Theoretical validity focusses on the theoretical constructions on which the researcher has based the study on and refers to the ability of the account to not only describe or interpret phenomena, but to offer an explanation for them (Maxwell 1992, p.291). In my study theoretical validity refers to the validity of the conceptions which I have imposed on my transcripts; that is the categories and indicators originating from my theoretical framework contained within my two analytical frameworks. The theoretical validity of my analysis therefore depends on how well the categories (with their indicators) reflect the events, activities, contributions and reasoning abilities of students as they engaged in proof construction exercises.

My analytical framework for the analysis of teacher and student utterances was based on the framework developed by Blanton, Stylianou and David (2011) who in turn based their framework on the work of Kruger (1993) and Goos, Galbraith and Renshaw (2002). The framework uses the term transactive reasoning to characterise clarification, elaboration, justification and critique of one's own or another's reasoning (Blanton, Stylianou & David, 2011, p.294). This framework was extended by additional categories and indicators as I worked on the transcripts of the video recorded consultative sessions. The framework was discussed in detail in Section 5.2.1. By categorizing the actions and contributions of the lecturer and students according to various indicators, such as students' requests for clarification and explanation of peers' utterances and actions, this coding scheme highlighted evidence of student development within the zone of proximal development. This has been done with the aim of addressing how students' proof construction and reasoning abilities are scaffolded by the lecturer and peers. The four broad categories for teacher's utterances are transactive prompts (requests for reflection, critique, justification, clarification, elaboration, strategy and examples), facilitative utterances (re-voicing and confirming students'

ideas, attempts to structure discussions and proof writing, highlighting learning and misconceptions and providing encouragement), didactive utterances (offering ideas on the nature of mathematics, axioms and historically developed ideas, making reference to and explaining definitions and illustrating mathematical objects using examples) and directive utterances (providing immediate corrective feedback or information). Transactive prompts transferred the responsibility of proof construction and verification to students themselves and built the practices of argumentation necessary for successful proof construction by prompting students' transactive reasoning (ibid., p.295). Facilitative utterances supported students' reasoning abilities by encouraging, repeating and rephrasing valid contributions and re-directing discussions to more correct avenues. Didactive utterances provided explanation of the notions that students needed to be aware of and were not expected to reinvent. Directive utterances provided students with immediate or corrective feedback or information towards solving a problem. The broad categories for students' utterances were proposals of new ideas, proposals of a new plan or strategy, contributions to or development of an idea, transactive questions (for clarification, explanation, justification and so on), transactive responses, transactive arguments, taking on the role of a more knowing other and moments of realization. I contend that the indicators for each of these categories are clear and unambiguous, and allowed the researcher to make correct judgements on the category to which the particular utterance belonged.

My analytical framework of students' proof construction and comprehension abilities was based on the assessment model developed by Meija-Ramos, Fuller, Weber, Rhoads and Samkoff (2012). The framework considers two main aspects of students' proof construction abilities: the local aspect (meaning of terms, symbols and signs, logical status of signs and proof framework and justification of claims) and a holistic aspect (main ideas or methods relevant to the proof, ability to transfer these to other proofs and illustrating mathematical objects and processes with examples). I adapted the framework to facilitate its use in the analysis of students' proof construction activities and contributions. The various categories contained in the local and holistic aspects of proof comprehension and construction were assigned clearly defined indicators as described in Section 5.2.2. The framework also expanded on the Vygotskian notion of the functional use of the sign to interpret students' evolving understanding of the

meaning of newly met mathematical objects and processes. My theoretical constructs attempted to categorize students' usage and interpretation of newly met mathematical terms, symbols, signs and the proof methods relevant to the various proof construction exercises according to Vygotsky's stages of concept formation adapted to the mathematical realm as discussed in Chapter 3. These categories and their indicators have been described in detail in Section 5.2.2. With regard to the theoretical validity of my account the link between the actions and contributions of students being accurately depicted as one of the phases of thinking (heap, complex, pseudoconcept or concept) is vital. For example was Maria's use and interpretation of the Cartesian product in sub-episode 4.1 truly pseudoconceptual? My justification for this categorization stemmed from the fact that in sub-episode 4.1 she seemed to use and apply the mathematical object correctly but when questioned about it in sub-episode 4.2, she offered an incorrect explanation and interpretation. My lengthy deliberations on students' actions and contributions and my elaborations justifying my categorizations of these actions and contributions have led me to believe that the categories and their indicators of my analytical frameworks are accurate reflections of the theoretical constructs underlying my frameworks.

9.2.4 Generalizability

Generalizability considers the extent to which one can relate the theory, findings and conclusions of the study to contexts other than the one directly studied (Maxwell 1992, p. 293). Maxwell distinguishes between two aspects of generalizability. The first is internal generalizability and refers to generalizing to other people, activities and settings within the community in which the study has taken place (first year mathematics classes at the University of Limpopo). The second is external generalizability and refers to generalizing to other communities or institutions (other universities and educational institutions in my case).

With regard to the findings and conclusions of the study, I believe my study will augment the existing literature on research in proof construction. As my study took place at a previously disadvantaged university, it is not possible to generalize these findings and conclusions externally. However I suggest (cf. Section 2.2.1 on students'

difficulties as reported in the literature in other parts of the world) that other students at other universities both in South Africa and elsewhere might experience similar difficulties and challenges as those discussed in Chapter 6. In addition the benefits experienced by the participants of my study while engaging with their peers and the lecturer in the consultative group sessions and the ways in which lecturers can support and empower students to become intellectually autonomous in the context of group consultative sessions as discussed in Chapters 7 and 8 would, I suggest, be similar to those experienced by students elsewhere (cf. Section 2.4.1 on studies where the socio-cultural aspect of proof is taken into account).

On the question of whether the theoretical framework supporting my study would be useful in making sense of students' engagement in similar activities or situations, I believe that Vygotsky's phases of concept development and their adaptations to the mathematical realm have widespread applicability to many other mathematical activities in which students engage. The notion of the EZPD, an environment which encourages and facilitates students' active participation, while promoting access to their zones of proximal development has widespread applicability to many other mathematical activities. This assertion is based on similar studies in the literature (cf. Section 2.4.1), my own experience as a student and teacher in mathematics and my observations of my students' development and progress during the course of my study. As discussed in Section 4.5 the students who participated in the study were purposefully chosen to be representative of mathematical ability (according to their first semester exam results) and gender. Students from all categories A, B and C were found to benefit from participation in the consultative sessions.

9.3 Reliability

According to Bell (2001, p.103) reliability is a measure of how well the procedures used yield similar results under the same conditions at all times. In the interpretive or naturalistic paradigm however, the world is seen as socially constructed and the accounts collected are the subjective experiences of the participants of the study. If a study were to be repeated with different participants, the conditions would no longer be the same. In naturalistic research, the researcher is concerned rather more with the

accuracy of the observations made, and whether the analysis of these observations is an accurate reflection of the actual events. Lincoln and Guba (1985) have introduced the criterion of dependability as an adaptation of the traditional concept of reliability. The concept of dependability emphasises the necessity for the researcher to be aware that the context of the study is subject to change and instability (Creswell, 2007, p.204).

Concerning the safe keeping and auditing of my enquiry process and to ensure dependability of data gathered for the study, all the raw data in the form of video recordings and the transcriptions of the video recordings was kept safely on my computer and on several backup hard drives. The brief field notes of my observations during the consultative sessions, memos and my reflective journal were kept in a safe place under lock to ensure that the enquiry process was well audited.

To establish the dependability of my study, my primary concern was whether the analysis of my data agreed well with what actually had occurred in the real life setting of my study (Cohen, Manion & Morrison 2011, p.202). When coding and analysing students' actions and contributions (written and spoken) with respect to proof construction, I categorized these according to the indicators and categories described in my analytical frameworks (cf. Sections 5.2.1 and 5.2.2). The reliability of my study will be determined by how well I used these indicators of my analytical frameworks to categorize students' proof construction actions and contributions and student and lecturer utterances. I tried my utmost to be consistent and rigorous when coding and categorizing the transcripts. On a practical level it was sometimes difficult to categorize actions and contributions when there was insufficient information, for example when the student gave no (oral) explanation on his/her (written) actions or contributions. I have done my best however to provide thorough justifications and explanations for my categorizations. In this way I hope I have greatly reduced the possibility of inconsistency in the analysis of the data.

9.4 Enhancing credibility

In this section I would like to discuss some of the ways in which the reliability and validity of my study could have been enhanced.

Firstly although I collected video records of all four consultative group sessions and these were transcribed, I carried out detailed coding and analysis of selected video clips of the first two sessions. The reasons for this selection were discussed in Section 4.5.3, and included factors relating to how I could best address the research questions given all the time and space constraints. I still believe that my selection was systematic and resulted in accurate representations of what had actually happened in the consultative sessions. However the credibility of my study would obviously have been be enhanced if I had carried out detailed coding and analysis of all the transcribed video sessions. As discussed in Section 4.5.2 the process of obtaining data from video records (the resources for developing data, and not data in themselves) has enormous time implications (Barron & Engle 2007, p.25). Hence I felt justified in making the selections I used for detailed coding and analysis.

Another way credibility could have been enhanced would have been by involving other mathematics educational researchers (other than my supervisor) to code and analyse selections of transcripts according to the indicators and categories contained in my analytical frameworks. Although as discussed in Section 9.2.2 I did ask an experienced lecturer who had taught at the University of Limpopo for many years to go through the transcripts once they had been coded, it could have been more beneficial if I had approached other researchers earlier and asked them to code the transcripts independently after providing them with a detailed description of my theoretical and analytical frameworks. One of the reasons that this did not happen was that the development of my analytical frameworks was a lengthy process involving several iterations of transcripts being coded and categories and indicators from the analytical frameworks being revised and extended. For each iteration, the coding (and preliminary analysis) of the transcripts took a great deal of time and effort and once I was eventually satisfied with the analytical frameworks and the coding of the transcripts, I was also very aware of time constraints. Fortunately I had opportunities to present my analytical frameworks, discuss my indicators and categories and apply these to an excerpt of transcript I had analysed, at PhD seminars at the University of the Witwatersrand.

9.5 Concluding summary

This chapter has addressed issues regarding the validity and reliability of my study. Improvements could have been made regarding the interpretive validity and reliability of the study. Regarding the issue of reliability, improvements could have been achieved by asking other mathematics education researchers to code excerpts of my transcripts using my analytic frameworks independently. I could have then compared these to my own coded transcripts and reached consensus on the coding process through discussion and consultation. Similarly with regard to interpretive validity, the study could have been improved by asking other mathematics education researchers to analyse the transcripts and develop codes (that is categories and indicators) within my analytic frameworks that they deemed appropriate and compared these to mine. It is possible that other researchers might disagree with some aspects of my analysis and interpretation. While acknowledging these limitations, my hope is that the study provides a coherent, believable and trustworthy account of an inquiry-based collaborative intervention in the context of proof construction in the area of elementary set theory.

Chapter 10: Conclusions

10.1 Introduction

In this study I have attempted to gain an understanding of first year mathematics students' difficulties and the forms of guidance beneficial to them at the University of Limpopo in the area of elementary set theory (Research question 1), and explored how participation in consultative group sessions supported students' development with regard to their proof construction abilities (Research question 2). The nature of student and lecturer utterances was examined to gain insights on how lecturers could support students' development and empower them to become intellectually autonomous (Research question 3). I have also highlighted some of the characteristics of students who showed potential in becoming more knowing peers and have identified some of the ways in which these students might be empowered to develop their capabilities (Research question 3).

In the consultative group sessions I hoped to create an environment which encouraged students' participation and their interaction with their peers and the lecturer, while they worked on proof construction exercises. In order to achieve a holistic picture of students' development of proof construction abilities in these sessions, I used two complementary analytical frameworks to incorporate both the social and the cognitive aspects related to students' development. The first (incorporating the social aspect) focussed on analysing the nature of students and lecturer utterances as they interacted. The second (incorporating the cognitive aspect) focussed on analysing the proof construction abilities of students implicit in their written and spoken actions and contributions. Vygotsky's theories of learning and development were integrated in both of these frameworks. In the first framework the analysis allowed the researcher a window into how the environment in the group sessions facilitated students' access to their zones of proximal development. The second framework incorporated Vygotsky's phases of concept development to track students' use and interpretation of newly met terms, symbols, signs, proof methods, deductive reasoning processes and the practice of justification as students made functional use of these during the sessions. Functional use refers to students' use of newly met (mathematical) objects and processes (in the

form of symbols and words) before they fully grasp the meaning of these objects and processes. It includes such activities as imitation. Imitation, as conceived by Vygotsky (1987, p.210) does not mean the mindless copying of actions, but pertains to an individual performing such activities mindfully, in cooperation with peers and more knowing others. According to Vygotsky (1987, p.209) imitation can only occur of such activities that are within the individual's range of potential intellectual attainment.

10.2 Reflections on research questions

In this chapter I reflect on the significant findings and discussion pertaining to my research questions. This will be done in the following summaries.

10.2.1 Students' difficulties and challenges

My first research question focussed on the challenges and difficulties that students experienced and the forms of guidance or scaffolding they received from their peers and the lecturer as they engaged with proof construction exercises. Investigating students' difficulties and challenges when introduced to formal proof construction and the forms of guidance helpful to them would, I hope enable mathematics teachers and lecturers to use this knowledge to address these challenges, and possibly adapt their modes of instruction. I include highlights of significant findings discussed in detail in Chapter 6. I will be reporting on both the difficulties that students experience and the forms of guidance offered to them together under the various categories of students' proof construction abilities, as described in my analytical framework (cf. Section 5.2.2). While highlighting these findings I have referred to similar or contrasting findings by other researchers. Most of these findings were discussed in my Literature Review Chapter (particularly Section 2.2.1).

The category L1 (meaning of terms, symbols and signs) encompasses using newly met terms, symbols and sign, use of mathematical definitions and using examples to illustrate mathematical terms, symbols and definitions. Difficulties students experienced in this category included problems with the mathematical terminology and discourse peculiar to mathematical proof construction. This was compounded by the fact that English was not the students' first language. Students used newly met terms,

symbols and signs incorrectly associating these with more familiar terms, symbols and signs. Students revealed complex level usage of newly met mathematical objects and processes for example referring to the implication sign as an equals sign and attempting to apply the proof of an equality or identity when proving an implication. These difficulties resonate closely with those reported by Moore (1994), Stylianou, Blanton and David (2011) and Dreyfus (1999). Moore (1994) found that one of the major sources of students' difficulty was that students were unable to use mathematical language and notation and this led to further difficulties in the area of mathematical object understanding.

Forms of guidance included peers offering explanations using simpler every-day language and offering their pseudoconcept or concept level interpretations which conveyed a more correct use and application of these mathematical objects and processes.

Students misinterpreted mathematical definitions. There were instances when this misinterpretation caused them to follow an incorrect method of proof. For example, the misinterpretation of the definition of the notion of implication led students to an incorrect proof method for proving an implication. This will be further discussed under proof methods. There were also times where students associated terms and symbols with a word contained in their definition, for example the notion of the union ($\cup$) was associated with the word 'or' and the notions of the intersection ($\cap$) and the Cartesian product ($\times$) with the word 'and'. Students' association of the notions of the intersection and the Cartesian product with the word 'and' and their subsequent tendency to want to interchange the symbol of the Cartesian product with the symbol of intersection, alerted me to the realisation of the great difficulty students have in understanding and interpreting the full mathematical definition. Students instead seemed to focus on one word common to both definitions and based all their thinking on this limited understanding. Difficulties that students have with interpretation and application of definitions have been reported extensively in the literature. Stylianou et al. (2011) have noted that students' difficulty in grasping mathematical language, signs and symbols hinders students' understanding of definitions. Similarly Weber (2001) found that students' lacked real understanding of definitions and were thus unable to apply them

correctly. He referred to the understanding of the necessary definitions and theorems related to the proof as syntactic knowledge. Moore (1994) found that students' inability to state and appropriate definitions was one of the causes of their failure to produce a proof. He emphasized that definitions often predict the sequence of the steps in the proof construction, and provide the justification for each step. Knapp (2006) suggests that in order for students to use definitions meaningfully to prove a statement, they need to know the definition (that is give examples and non-examples and define the mathematical object in their own words), determine which definition and which aspects of the definition would be useful and lastly know how to use the definition which is similar to the strategic knowledge referred to by Weber (2001).

As a form of guidance, the lecturer encouraged the use of examples to clarify mathematical objects and their definitions. Most students participated in this activity. In order to illuminate the link between the definition and the method of proof, the lecturer also prompted students to reflect on the definition (of set equality) and extract the overall structure of the proof framework. It was generally observed that students' use and interpretation of newly met terms, symbols and definitions were brought closer to concept level use through their functional use while interacting with their peers and the lecturer.

Students sometimes struggled to generate helpful examples to illustrate mathematical objects and definitions because of their incomplete knowledge in that particular area of mathematics. This was evident when Joseph tried to present an example of the newly met notion of the power set in Episode 5. Similarly Moore (1994) found that students failed to generate and use their own examples, even though they appreciated the value of examples in helping them understand mathematical objects. He proposed that a possible reason for this is that students have a 'limited repertoire' of knowledge in the required area of mathematics from which to draw such examples. I suggest that the use of examples should be encouraged whenever possible, and that this form of scaffolding be provided when students are inexperienced in the particular knowledge area.

The category L2 (Logical status of statements and proof framework) included the three aspects of selecting correct and appropriate statements which make sense and add to the

logic of the proof construction, selecting useful and appropriate aspects of definitions and assumptions and knowledge of correct proof methods.

Including statements that would add to the logic of the proof construction such as stating the assumptions and the statement that had to be proved at the beginning of the proof construction was very helpful to the students, as it clarified their goal in proof construction and the assumptions at their disposal, both initially and throughout proof construction as they referred back to these statements.

Peers offered guidance in this regard by critiquing other students' proof construction actions and raising awareness of the need for sound logical reasoning while keeping in mind assumptions and the statement to be proved.

One of the major challenges faced by students was selecting correct or appropriate aspects of definitions and appropriate assumptions in the process of proof construction. These difficulties included selection of non-useful or trivial aspects of definitions, treating statements which were supposed to be proved, as assumptions, and difficulties in using the relevant assumptions and definitions to start the proof or make progress in proof construction. Students also had difficulty in starting the proof construction correctly (one of the difficulties identified by Moore (1994)) and continuing with the proof construction process. This was the case even when they seemed to have grasped most proof construction requirements. According to Weber (2001), this is as a result of a lack of strategic knowledge which he describes as a failure to use the syntactic knowledge (knowledge of all the facts and theorems) that students have at their disposal. Difficulties in the area of logical reasoning in the proof construction process have also been reported by other researchers as discussed in Section 2.2.1. Stylianou, Blanton and Knuth (2011) reported that students lacked logic and reasoning abilities involved in problem-solving or argument construction. Similarly Kuchemann and Hoyles (2011) found that a major challenge for students was to develop mathematical reasoning and to make inferences and deductions on the basis of mathematical structures rather than empirical reasoning.

Forms of guidance included peers offering critique on proof construction actions. When difficulties persisted, the lecturer provided prompt corrective feedback. The lecturer

highlighted the need for ensuring the truth of each statement and that each step had to be accompanied by logical reasoning. The lecturer reminded students of assumptions, and urged them to reflect on proof construction actions.

Proof methods which are indispensable for students' smooth journey in proof construction posed a major challenge. Students struggled with the proof methods of the implication, double implication, equality of sets and the proof of showing one set was a subset of another. The proof methods of the implication and double implication in particular were very problematic. Students' complex level interpretation and association of the implication and double implication symbols with the symbol for equality led them to attempt to use the method for proving equality when attempting the proof of an implication. Students' misinterpretation of the definitions of the implication and double implication were also largely responsible for their challenges with these proof methods. An example of this was observed in Episode 2 when Maria attempted to prove *a)* $\Longrightarrow$ *b)*. Many students interpreted the definition of the implication as: 'if *a)* is true then we **know** *b)* is true' or 'if *a)* is true then *b)* **will** be true'. This simple misunderstanding led to an incorrect strategy for proof construction: that is to first prove the truth of *a)* from which the truth of *b)* would follow. There were also instances where students seemed to know the definition of a mathematical object, but were unable to extract the method of proof from this definition. Moore (1994) notes that students' inability to use definitions to provide the overall structure, logic and proof method suitable for a particular proof is another great hindrance to proof construction. Solow (1981) likens the students' inadequacy of the knowledge and skills required in proof construction to them being asked to play a game where they do not know the rules. He recommends that students be given a detailed explanation of methods they can use to unravel the strategies behind various proof techniques. My study has highlighted the need for lecturers to focus on proof methods, and ensure that students have a good understanding of these before proceeding to more advanced proof construction exercises.

Forms of guidance included peers offering critique of incorrect proof construction actions. When prompted for clarification, peers offered pseudoconcept or concept level interpretations of proof methods and reasoning processes. There were also

contributions which summarised the proof method into a short simple rule in every-day language. The lecturer also often drew attention to the goal of the proof construction and the assumptions that had initially been made. Another form of scaffolding was offered by the lecturer when she prompted students to arrive at the correct method of proof by examining and reflecting on the meaning of the definition, and extracting the method of proof in this way. I suggest that this particular strategy is one that lecturers can emphasize in class as a very beneficial practice. As students progressed through the proof construction exercises, they made functional use of the proof methods. Presumably it was this functional use including activities such as imitation which led to great improvement in their usage and application of these proof methods. The lecturer also identified several more knowing peers who periodically came to the board and gave a detailed presentation as proofs or components of proofs were completed. These students were able to clearly articulate and explain the link between the assumptions, the statement to be proved, the reasoning behind each step and the proof method, hence clarifying most aspects of the proof construction process and in particular, the proof methods.

Regarding the category L3 (Justification of claims) many instances were observed when students did not provide justification for each deduction and conclusion. This was not only confined to students' initial experiences with proof construction. Even students who had appeared to have gained an appreciation for justification but became stuck and were not able to continue with a particular proof construction, tried to make progress in the proof construction by making deductions without justification, thus abandoning this practise.

Forms of scaffolding included peers questioning and critiquing reasoning processes. When unjustified deductions were made persistently, the lecturer reminded students to use logical reasoning and ensure the truth of each statement, always bearing in mind assumptions which would help with the proof construction. Presentations by more knowing peers were also very useful in clarifying the deductive reasoning processes involved and the justification which had to be provided for each step.

With regards to the holistic categories of proof construction, I focussed on the category H2: transferring general ideas and methods to other contexts. Students had problems

with transferring the methods of proof of showing that one set is a subset of another and ideas regarding the need for justification of statements and using sound logical reasoning processes. I suggest that students' inability to transfer methods and ideas could be due to the cognitive overload (as referred to by Moore (1994)) students experience when introduced to formal proof construction. Further practice and more time are needed for the newly met objects, proof methods, logical reasoning processes and practices of justification to become internalized and reach concept level realization.

Challenges outside the categories of my analytical framework for proof construction fell into three broad categories. These were students' lack of confidence and belief in their own abilities, the challenge of incorrect ideas which might be propagated in an environment where contributions from all students were welcome, and the challenges lecturers need to keep in mind when striving to make optimal use of consultative sessions.

With regard to students' lack of confidence in their own abilities, proof presentations by more knowing peers delivered with confidence and conviction were beneficial. These portrayed to the others that students such as they themselves were comfortable about and could ably explain the reasoning processes and the justification involved in the proof construction process as well as the newly met terms, symbols and proof methods. These presentations not only helped to clarify the newly met objects and processes involved in proof construction in the every-day language which the students could relate to, but they were also a source of motivation to the others, encouraging them to try to reach that same level of understanding. Kajander and Lovric (2005) have similarly identified the beneficial practice of using tutors, a little older than first year students who could empathise with these students' experiences and assist them in a problem solving environment. I also suggest that the encouragement offered by the lecturer during the consultative sessions along with the continuous help and scaffolding from their peers were important factors in nurturing students' self-confidence.

In striving to make optimal use of consultative sessions, it is important to realize that these sessions are quite different from traditional modes of instruction and therefore require a very different set of skills and competencies which lecturers might need to develop within themselves, no matter how experienced they might be in other teaching

modes. The ultimate goal is to create an environment where students' access to their zones of proximal development would be facilitated. I suggest that this is best done by creating a warm and friendly atmosphere where all the students feel welcome and everyone's contributions are valued. Although lecturers should allow students to take responsibility for their own learning and development as much as possible by encouraging them to be the primary contributors in attempts of proof construction of the various exercises, they also need to clearly establish the norms pertaining to successful proof construction. Most important of these norms were: nurturing of students' abilities to examine each step in the proof construction process carefully and critically to ensure that each step is accompanied by logical reasoning and justification; raising students' appreciation of definitions and their importance in both revealing the structure of the proof and in the justification of deductions and raising awareness of the importance of keeping in mind the assumptions and the statement to be proved. Lecturers have to be alert and watchful in providing encouragement, confirmation and guidance whenever needed, thus driving mathematical discussions forward, and preventing the propagation of misconceptions. Students are thus supported to become independent thinkers and potential more knowing peers are empowered to develop that potential and gradually take on the roles and the responsibilities of the lecturer. In this regard, Mcclain (2011) proposed that teachers need to have a thorough understanding of the mathematics covered in discussion sessions in order to be able to raise students' abilities in mathematical argumentation to higher levels. She proposes that lecturers have to be able to make quick decisions regarding factors such as the speed, structure and the direction of the discussions, thus ensuring that the mathematical agenda moves ahead.

10.2.2 Students' evolving proof construction abilities

Chapter 7 addressed my second research question which investigated the development of students' proof construction abilities by following two of the participants of the consultative group sessions, Frank and Maria, who attempted proof construction tasks in the first and second sessions. I explored how these students' proof construction abilities evolved from one session to the next.

Frank made great progress in the categories of the meanings of terms, symbols and signs (L1 category) and logical status of statements and proof methods (L2 category). I argue that this was a result of Frank's functional use of newly met terms, symbols and proof methods as he engaged with proof construction exercises while receiving guidance and scaffolding from the lecturer and his peers. The difficulty which remained was that of knowing how to use all the mathematical information at the student's disposal (for example the assumptions, relevant definitions and proof methods) to proceed when the proof construction became challenging. This was termed strategic knowledge by Weber (2001) and has also been discussed in Section 10.2.1.

In the first session Maria had persistent difficulties with proof methods and logical reasoning (L2 category) and the need for justification of deductions (L3 category). She needed repeated guidance from the lecturer and her peers on proof methods, logical reasoning processes and the justification of each statement and deduction. It became clear that she had difficulty in transferring methods and ideas from one component of proof construction to the next. She made functional use of newly met terms, symbols, definitions, proof methods, logical reasoning processes and the practice of justification as she received continuous scaffolding from the lecturer and her peers throughout the first session. Maria used activities such as imitation as she interacted with her peers and the lecturer in the consultative session. She showed vast improvement in these areas in the second session and was able to explain the reasoning and justification process behind the proof very well. In the second session we also observed Maria's pseudoconceptual use and interpretation of the newly met term, the Cartesian product in the first component of her attempted proof construction. Possibly as a result of the scaffolding she received from a more knowing peer who explained and clarified the notions of the Cartesian product and the intersection, Maria appeared to reach concept level use and interpretation of the Cartesian product in the second component of the proof. This demonstrated the effectiveness of two aspects: first the consultative session as an environment which facilitated students' access to their zones of proximal development, and second, the effectiveness of the process of making functional use of the newly met term while interacting with the lecturer and peers. The opportunity offered to students as they participated in the EZPD allowed them to interact with one another while receiving scaffolding from their peers and more knowing others. Their

functional use of all of the various skills necessary for successful proof construction, such as the interpretation and application of mathematical objects and definitions, application of logical reasoning processes and the justification of proof construction steps enabled the accelerated development of these skills.

On examining the journeys of these two participants we observed how the environment created in the consultative sessions effectively enabled students' development of proof construction abilities. During Maria's struggle with the proof construction exercise in Episode 2, we perceived the serious burden students would have to shoulder when working on their own, or with other students having similar capabilities as their own. Students who are novices in formal proof construction face the combined challenge of many newly met terms, symbols and signs and unfamiliar proof methods plus the challenge of the logical reasoning and justification required in the proof construction process all within one proof construction exercise. The environment created in the consultative sessions encouraged students' active participation and interaction and facilitated access to their zones of proximal development and enabled their functional use (including activities such as imitation) of newly met terms, symbols, definitions, proof methods, deductive reasoning processes and practices of justification of deductions as they engaged with proof construction exercises.

10.2.3 Supporting students in becoming intellectually autonomous

In Chapter 8 which addressed my third research question, I examined the nature of the interactions in the consultative group sessions and addressed the question of how lecturers could support students in becoming intellectually autonomous. I also attempted to identify the characteristics and modes of reasoning of students who showed potential in becoming more knowing peers, and explored how these students could be empowered to develop that potential in the consultative group sessions.

By investigating the nature of student and lecturer utterances in each of the five episodes that were coded and analysed in detail, I tried to trace patterns in the discourse as the lecturer tried to establish the norms that were necessary in promoting successful proof construction. These included encouraging students' questioning and critiquing

the proof construction actions and contributions of one another and imparting the realization that students had to be responsible for ensuring that each step in the proof construction was accompanied by sound logical reasoning and justification.

General patterns that emerged during the sessions were that as proofs were attempted on the board the lecturer would not comment on the correctness or validity of the proof but rather encouraged students to ask for explanation, offer critique, make contributions or suggestions for improvement or alternative proof construction actions. From the very beginning of the sessions the lecturer transferred the responsibility for verifying the proof construction steps and finding the correct solution to the students themselves. The lecturer encouraged students' critique when they questioned actions which were not justified and asked for elaboration of contributions that offered pseudoconcept or concept level usage, interpretation and application of newly met terms, symbols, definitions and proof methods relevant to the proof construction. In this way the lecturer implicitly conveyed to students that their contributions were valued. On realising that mathematical objects and processes such as newly met terms, symbols and proof methods had not been completely understood, the lecturer asked other students to offer their ideas and explanations of these objects and processes. Students were encouraged to articulate their ideas by repeated transactive requests for reflection and strategy. The lecturer discussed and addressed misconceptions by repeatedly asking for alternative ideas until more correct ideas were offered in the form of pseudoconcept or concept level interpretation of mathematical objects. The students who made such contributions would then be prompted to elaborate and explain their ideas. If, at the conclusion of the proof construction attempt, incorrect ideas still persisted, then the lecturer would offer explanations of meanings of definitions and mathematical objects and processes. This only happened in Episode 1. From Episode 2 onwards, more knowing peers whom the lecturer had identified such as Christine, Joseph and Gary came up to do proof presentations in which all aspects of the proof construction were clearly explained and clarified in simple every-day language. They seemed to be highly effective in conveying the meanings of terms and symbols, the logical reasoning behind the proof framework, and explaining and clarifying why and how each step in the proof construction needed to be justified. The identification of students with the ability to explain and clarify proof construction is I suggest one of the lecturer's vital

responsibilities. If correctly identified these students are ideal in explaining the characteristics of successful proof construction in simple every-day language that the other students can relate to and understand. The lecturer would often confirm these contributions and encourage those who gave in-depth explanations. These students grew in confidence and the other students realised that fellow students, just like them, were able to successfully comprehend and construct proofs. This seemed to encourage them to press onwards in their efforts to better their understanding.

An activity initiated by the lecturer which students seemed to find very helpful and in which they engaged very enthusiastically was that of using examples to clarify newly met terms and symbols. Students' interest was heightened when their peers went to do examples on the board to clarify mathematical objects which had not been completely understood.

Another significant activity the lecturer used was that of prompting students to examine and reflect on the definition (of set equality) to extract the method of proof. This was effective in alerting students to the link between the definition of a mathematical object and the proof method to be used and allowed students to realize how the overall structure of the proof was in fact apparent in the definition. I believe that such an activity is important because proof methods are vital and can be likened to road maps in journeys in proof construction. When students have an idea how to set about extracting the method of proof from the definition, then they are better able to make progress in the proof construction process.

Some students had persistent difficulty with using logical reasoning, and showed a lack of appreciation that each step of proof construction had to be accompanied by the necessary justification. These students were repeatedly reminded in discussions with the lecturer and their peers that they had to make certain of the truth of each statement and had at all times to be aware of assumptions and the goal of the proof construction, that is, the statement or proposition to be proved. Presentations by more knowing peers were again very helpful in conveying the careful thinking, reasoning and justification behind each step in the proof construction. Such presentations could be considered as effective learning opportunities and a means of unlocking students' potential.

There were times that students still had difficulty in driving the proof construction forward even though they showed great improvement in most areas of proof construction, such as meanings of terms, symbols and signs, knowledge of the logical reasoning required in proof construction and proof methods and had gained an appreciation for the need for justification of deductions. This was possibly due to the fact that proof construction is not a linear algorithmic process where knowledge of the mathematical objects and processes involved in the proof construction will guarantee one's success. According to Selden (2012, p.392) creativity and insight are two essential ingredients allowing individuals to use the knowledge at their disposal to make progress in the proof. Weber (2001) refers to this ability as strategic knowledge. I suggest that one of the best ways that this strategic knowledge might be developed is by students' participation in collaborative inquiry-based discussions such as the consultative group sessions. Students can develop their strategic knowledge by working on several proof construction exercises which require the same proof methods, while being introduced to an increasing array of new terms, symbols and definitions, with each exercise gradually growing in difficulty. When students who had made gains in most aspects of proof construction skills, experienced difficulty in starting the proof or proceeding with the proof construction, they sometimes abandoned their practice of ensuring that each step in the proof construction was accompanied by logical reasoning and justification. These students made deductions which would lead to the desired conclusion while omitting certain crucial steps. Students received the necessary guidance and scaffolding from the lecturer and their peers in their interactions and discussion (where they were urged to reflect on their proof construction actions and strategy while highlighting the assumptions and the statement to be proved) enabling them to proceed and correct their actions. They were thus supported to make gains in their strategic knowledge which would add to their competence in proof construction.

In the second session several more knowing peers showed their readiness to assume the role and responsibilities of the lecturer by taking over the transactive prompts asking for clarification, explanation, reflection on proof construction actions, logical reasoning and justification, and providing scaffolding as needed. It is interesting to note that these students had been able to 'become' more knowing peers in a very short space of time, leading me to believe in the effectiveness of the consultative group sessions in allowing

students who exhibited potential to realise that potential and mobilising them to help their peers with the difficult task of proof construction. Characteristics of students who showed potential in becoming more knowing peers were identified. These students earnestly engaged with the consultative practices of the group sessions; critiqued other students' proof construction actions and reasoning processes and requested clarification of mathematical objects or processes which were a cause of confusion. They communicated a holistic understanding of the proving process very effectively by explaining the logical reasoning and the justification behind each step and made regular use of examples to illustrate newly met terms, symbols and proof methods. They showed an appreciation of the importance and usefulness of definitions in both suggesting the sequence of steps to be followed in proof construction and in providing the justification for each step and regularly used statements that added to the logic of the proof construction. They demonstrated their apparent realization that they themselves were responsible for developing their own understanding of newly met mathematical objects by deeper examination of definitions and the use of examples to illustrate these, and through the process of interaction and enquiry with their peers and the lecturer.

In the second session the lecturer receded into the background allowing these students to become active agents for the promotion of their own and their peers' learning and development. The few contributions made by the lecturer in these episodes included requests for clarification and elaboration of concept and pseudoconcept level contributions, facilitative utterances encouraging and confirming correct usage and interpretation of mathematical objects and proof construction processes and a few didactive contributions referring to definitions and clarifying mathematical objects by using examples.

The atmosphere in the consultative sessions was generally buoyant with a great deal of laughter and joking between the serious tasks of proof construction. Students were always welcome to offer their contributions as proof construction tasks were attempted on the board and all the participants were very tolerant of one other's ideas and contributions. Friendship and camaraderie developed very quickly between the participants of the group and this seemed to help students to tackle the sometimes frustrating and arduous task of proof construction. Kolstoe (1995, p.8) puts forward the

view that when two or more people consult under suitable conditions a new "intellectual power and emotional balance" come to the fore. Heinze and Reiss (2011) have argued that a positive attitude towards mathematical proof construction plays a significant role in the development of proof construction abilities.

10.3 Contributions to Mathematics Education Scholarship and Mathematical Pedagogy

By investigating challenges and difficulties of first year students at the University of Limpopo while engaged in proof construction exercises in the area of elementary set theory, the study contributes to the literature on undergraduate students' difficulties. It adds to this literature by considering the vantage point of students whose first language is not English and who have come from previously disadvantaged rural communities and schools. Helpful forms of scaffolding offered by the lecturer and peers were pinpointed and reported in the hope that this will contribute to pedagogical practices in similar situations. Interestingly many of the challenges and difficulties encountered by students at the University of Limpopo were very similar to the challenges and difficulties reported by researchers all over the world, as seen in Section 2.2.1. I have also identified possible solutions towards enhancing students' self confidence in their proof construction abilities. In addition I point to the challenges lecturers might have to keep in mind when attempting to facilitate students' proof construction abilities in collaborative environments. It is hoped that these submissions might be useful to lecturers who are contemplating setting up collaborative modes of instruction as they strive to improve the proof construction abilities of their students.

My study also contributes to the growing body of research on how proof construction abilities of students can be nurtured in collaborative inquiry-based classes. First year students who are introduced to formal proof construction often find the challenges posed by the mathematical language and definitions, newly met mathematical objects in the particular area of mathematics, proof methods and the logical reasoning and justification processes overwhelming. The consultative group sessions proved to be extremely effective in supporting students' development in general, and empowering those showing potential to become more knowing peers to develop their potential and

capabilities. This mode of instruction might be offered as an additional activity to the traditional modes of instruction currently in use. It was observed that most of the participants made huge gains in their proof construction abilities in just one session, for example Frank and Maria who made tremendous progress from session 1 to session 2. Thus even in institutions where resources are limited and tutors and lecturers already overloaded, students' participation in just one consultative group session could make a big difference. Also by identifying the characteristics of those students who might have the potential to become more knowing peers, lecturers could be enabled to become aware of these students and take action to nurture their capabilities, thus empowering them to become active agents in the development of their own and their peers' understanding and proof construction abilities.

The study identified the ways in which lecturers could create an environment conducive to students' development and empowerment by encouraging the establishment of norms pertaining to mathematical proof construction. These include encouraging students' engagement in the activities of consultation, justification, explanation and using sound logical reasoning. I argue that the effectiveness of the consultative group sessions is due to their success in facilitating students' access to their zones of proximal development (Vygotsky, 1978) where functions that have not yet matured can be developed in collaborative mathematical activity. The notion of the EZPD, an environment where students' access to their zones of proximal development is facilitated was introduced and elaborated on to elucidate the connection between the social (in the form of collaborative inquiry based modes of instruction) and the cognitive (in the form of promoting students' access to their zones of proximal development).

I contend that this study provides confirmation that higher mental functions such as proof construction abilities arise as a result of mediated processes and through co-operative activity and that language and speech are the means by which these functions are mediated (Vygotsky, 1987, p.126). The study seems to confirm Vygotsky's key principle that the development of practical and abstract intelligence takes place when speech and practical activity (in the context of the consultative group sessions) are brought together (Vygotsky, 1978, p.7). Vygotsky (1986) built on theories put forward

by Uznadze (1966) who conjectured that the development of objects and processes begins with a 'functional equivalent' of objects/processes which are similar to the mature concepts held by adults in functional use but differ in their structure and quality. Berger (2004a) argues that the functional use of a mathematical sign is necessary for and productive of meaning-making for a university mathematics student. In this study I have argued that the rapid development of the proof construction capabilities of students participating in the consultative group sessions was a result of their functional use of newly met terms, symbols, logical reasoning processes and proof methods and the practice of justification as they interacted with their peers and the lecturer when engaging in proof construction exercises.

I have shown that Vygotsky's phases of concept development and their adaptation to the mathematical realm can be applied to students' proof construction actions and contributions in the areas of meanings of newly met terms, symbols, signs and proof methods. It was demonstrated that students' use and interpretation of newly met terms, symbols, signs and proof methods could be described as evolving through complex, pseudoconcept and concept levels as students made progress in the sessions. Pseudoconcept level use and interpretation of newly met terms, symbols and proof methods presented interesting situations where students seemed to use and apply mathematical objects and processes correctly but subsequently revealed their incomplete understanding in later discussions. Students whose use and interpretation of newly met terms, symbols and proof methods were at pseudoconcept level were sometimes observed to revert to complex level use. It was argued that this was due to the fact that these ideas had not yet been sufficiently internalised and had not reached concept level usage and interpretation.

The type of analysis I have used in this study is rather novel as it attempts to draw together social and cognitive aspects of the development of students' reasoning and analytical abilities associated with proof comprehension and construction. I have built on existing analytic frameworks in the literature to develop two complementary analytical frameworks that impart a holistic analysis of students' evolving proof construction abilities and student and lecturer utterances as they participated in consultative group sessions. This could be useful to other researchers who would like to

study students' proof comprehension and construction abilities in the context of inquiry-based modes of collaborative classes.

10.4 Areas for future research

One possible area for future research is for an exploration of the difficulties students experience and the development of their proof construction abilities in mathematical fields other than elementary set theory such as in the areas of real analysis or abstract algebra, to be carried out in the context of consultative group sessions. Such research could enable us to establish whether the consultative group sessions which seemed to be very effective in the development of students' proof construction abilities in the area of elementary set theory, empowering students to become intellectually autonomous, could be effective in other areas of mathematical proof construction. This would also allow us to explore the usefulness and applicability of Vygotsky's theory of concept formation and the expanded notion of the functional use of mathematical objects and processes involved in formal proof construction, to other mathematical areas.

Moreover the investigation of undergraduate students being introduced to formal proof in the context of alternative topics to elementary set theory such as the topic of Boolean algebra which might be more meaningful to students as it has practical applications in real world contexts, could yield interesting research.

Another possible avenue of interesting research would be to track the progress of students who had the opportunity of participating in the consultative group sessions in their first year of university, as they advance to higher level mathematics courses. In this way one could explore whether the skills developed in the course of their participation were helpful as they progress further in their tertiary mathematics courses. Moreover the trialling and development of longer term interventions could demonstrate the optimum amount of time necessary for most students to become closer to being intellectually autonomous and hence make a significant impression on pass rates in higher level mathematics courses as well as make positive contributions to mathematics research.

10.5 Summarizing Conclusion

To conclude I suggest that this study has highlighted the numerous challenges many first year undergraduate students experience when introduced to formal proof construction in the context of elementary set theory. In particular it focussed on students whose first language was not English, and who commonly entered first year university with poor prior mathematics knowledge. Nonetheless many of its findings resonate with those of researchers in the developed world such as Moore (1994); Solow (1981); Dreyfus (1999); Weber (2001) and Blanton, Stylianou and David (2011).

To recap briefly, challenges were detected primarily with the meanings and interpretations of mathematical terms; symbols and definitions; the logical reasoning and proof methods relevant to the proof construction; the appreciation for the justification of deductions and conclusions and with the ability to transfer ideas and methods to subsequent proof constructions. These were overcome as students interacted with one another and the lecturer as they engaged with proof construction exercises while making functional use of mathematical terms, symbols, definitions, logical reasoning processes and proof methods and practices of justification.

I believe that the study has shown that the consultative group sessions provided an environment which was conducive to the development of participants' proof construction abilities in general, and to the empowerment of those showing potential in becoming more knowing peers. Those students who showed the potential in becoming more knowing peers emerged through discussions and consultation. They were distinguished by their ability to critique reasoning processes that were not sound and logical, ask questions about mathematical objects which were a source of confusion, and offer pseudoconcept or concept level interpretations of newly met terms, symbols, proof methods and deductive reasoning processes. They were able to communicate their understanding of terms, symbols and definitions with conviction, and explain the reasoning processes behind proof construction steps, and convey an appreciation of the need for justification of each step or deduction in the proof. These more knowing peers also showed an understanding of the illuminating power of examples to illustrate mathematical objects and proof methods. I suggest that the identification of such more knowing peers is one of the lecturer's vital responsibilities as these students might be

able to convey an appreciation for all aspects of proof construction effectively and motivate other students.

I suggest that the environment established in the consultative group sessions facilitated students' access to their zones of proximal development and that these consultative sessions could be highly beneficial if used as an additional mode of instruction to the traditional modes currently in use. My study has clearly shown that such an environment is conducive to interactions and discourse which could lead to students' engagement with the construction of their own knowledge (with regard to proof construction abilities) as well as the transformation of their interpretive and analytical skills. The responsibility for proof construction and verification was transferred to the participants by encouraging them to contribute towards the proof construction and to critique steps which did not make sense or did not follow logical reasoning. Students were enabled to actively engage with their difficulties and challenges effectively and improve their proof construction abilities quickly and with far less frustration. In making functional use of newly met terms, symbols, definitions, deductive reasoning processes and proof methods while receiving guidance and scaffolding from the lecturer and peers, students made large gains in all the aspects of proof construction ability, as well as making gains in their self-confidence. After a number of such sessions, I envisage that students could, of their own accord, use the practices introduced in the sessions to consult on newly met terms and symbols, definitions and strategies and go on to make further gains in their proof comprehension and construction abilities. This would naturally fuel their confidence and motivate them to pursue their mathematical studies with enthusiasm. I conjecture that the camaraderie and friendship developed while students struggle to find the way forward in the proof construction tasks by consulting together, could turn the drudgery of engaging with mathematical proof into a more enjoyable and creative experience. This sense of conviviality could help to unlock students' mathematical potential. It is hoped that this method would also encourage students to feel that they are part of the community of mathematicians and help them appreciate the beauty and elegance of mathematical proof.

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