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Mediating primary mathematics: theory, concepts, and a framework for studying practice

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Abstract In this paper, we present and discuss a framework for considering the quality of primary teachers' mediating of primary mathematics within instruction. The "mediating primary mathematics" framework is located in a sociocultural view of instruction as mediational, with mathematical goals focused on structure and generality. It focuses on tasks and example spaces, artifacts, inscriptions, and talk as the key mediators of instruction. Across these mediating strands, we note trajectories from error and a lack of coherence, via coherence localized in particular examples or example spaces, towards building a more generalized coherence beyond the specific example space being worked with. Considering primary mathematics teaching in this way foregrounds the nature of the mathematics that is made available to learn, and we explore the affordances of attending to both coherence and structure/generality. Differences in ways of using the framework when either considering the quality of instruction or working to develop the quality of instruction are taken up in our discussion.

Keywords Primary mathematics teaching · Sociocultural theory · Mediation · South Africa · Artifacts · Inscriptions · Instructional quality

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1 Introduction

In this paper, we discuss the theoretical antecedents, literature bases, and empirical motivations underpinning a framework we have developed for considering the quality of teachers' mediation of primary mathematics. Our focus is on how mathematics can be taught in ways that pay attention to, and help develop, understanding of the underlying structures of mathematics, and number in particular, at the primary school level, ways that lead learners to consider generality and to understand number as a scientific concept in the Vygotskian sense. Such attention to generality and structure is in contrast to the continued emphasis in many primary classrooms on teaching calculation procedures as the main purpose or goal, a stance that is particularly prevalent in early primary schooling in South Africa, but not unique to there.

We approach this problem from a sociocultural perspective and view the teacher as the main agent mediating between the objects of learning of mathematics and the particular learners in the classroom. We see mediation as central to instruction and involving teachers working with and coordinating various mediational means that are historically grounded in sociocultural materials and practices. Based on theory and empirical evidence, we take key mediational means to be tasks and example spaces, artifacts, inscriptions, and talk, which we draw together to form the "mediating primary mathematics" (MPM[1]) framework. In this paper, we present and discuss the MPM in relation to two key "grounds": firstly, our grounding of mathematics teaching in sociocultural theory leads to a focus on goals and means of mediation in instruction; second, there is a grounding of the MPM framework in empirical realities of primary mathematics teaching in South Africa and internationally. When considered together, these two grounds allow for a specification of goals of mathematics instruction. These grounds also direct our gaze to a consideration of key mediating means or "strands," and of levels of instructional quality within, and across, the mediational strands both within lessons and also over time.

A brief outlining of the background that motivates our attention to instructional quality leads into our thinking about key mediating strands and the kinds of instructional phenomena of interest within them. In the body of the paper, we present and discuss these strands and levels of address within them to mathematical structure/generality. We incorporate attention in our analysis to how other frameworks in the field have conceptualized empirical phenomena related to primary mathematics teaching, as this helps us to understand aspects that the MPM framework foregrounds, and thus, what affordances it offers as extension to and adaptation of other models.

In this paper, a range of theoretical writing on mediation is drawn together to identify goals of mathematical instruction and key mediational means in the first instance. Interpretations of the nature of instruction in empirical episodes, set within the mediating strands of the MPM framework, are then used both to illustrate the framework and its orientations to disaggregating instructional quality, and also to discuss its overlaps and contrasts with Hill, Blunk, Phelps, Sleep, and Ball's (2008) MQI model and Rowland, Turner, Thwaites, and Huckstep's

[1] In previous writing developing this work, we have used the term "Mathematical Discourse in Instruction—Primary," (MDI-P). The history of this work is a co-development of MDI frameworks between Hamsa Venkat and Jill Adler, which shared roots in sociocultural theory but differed in specific formulations across work in secondary and primary mathematics. In order to avoid confusions between the secondary and primary level models, we have changed our titling of the framework to MPM. Writing with Adler and her team is underway, detailing the histories and trajectories of development of both MPM and MDI.

(2009) Knowledge Quartet (KQ). While there are similarities between the MPM and other such frameworks, a key distinguishing feature of the MPM lies in its foregrounding of a mathematical emphasis on structure and generality. The model is designed to provide a nuanced reading of the ways in which key mediational means are assembled in instruction in ways that allow for structure and generality to be brought into focus.

This analysis and exemplification highlights the MPM's focus on the nature of the mathematics that is made available to learn and allows for an exploration of the quality of primary mathematics instruction at a fine grain level. We conclude with a discussion of what is afforded through the MPM framework in terms of thinking about the quality of primary mathematics instruction in ways that magnify attention to structure/generality, and also on coherence as an important staging post towards this goal. We also note options for using the framework as a developmental tool alongside its more evaluative possibilities.

1.1 Motivations for the MPM framework

The motivation for devising a framework for considering the quality of mathematics instruction began in longitudinal development work in primary mathematics teaching in South Africa in the Wits Maths Connect-Primary (WMC-P) project. The South African context has particularities linked to the legacy of apartheid on the preparation and development of teachers, but shares, with many contexts in the developing and developed world features that include: evidence of low performance in mathematics at all levels; problems with equitable access to resources; gaps in primary teachers' mathematical knowledge; widespread evidence of teacher-centered forms of instruction. Our evaluations of learner performance in the early primary years, after 5 years of professional development work in ten government primary schools, pointed to gains in early number learning. In this context, we set out to explore whether there was evidence of differences in teaching practices that could have contributed to these learning gains.

Differences in teaching are based in empirical distinctions observed in our videotaped lesson data. Changes in the direction of improvement are interpretations of difference drawing on theorization of what counts as "better" mathematics teaching. In order to study the quality of mathematics teaching, we therefore needed to identify aspects of practice understood as important for mathematical learning in the kinds of contexts we have outlined above and to think about how "improvement" in the orchestration of these aspects could be conceived.

1.2 Goals of mathematics instruction, mediating means, and empirical realities

Our starting points lie in two key tenets of Vygotskian theory. Firstly, Vygotsky (1987) formulates educational goals in terms of appropriation of disciplinary ideas set within networks of scientific concepts. Kozulin (2003) notes that working towards disciplines viewed as networks of scientific concepts can be seen within instruction exhibiting a press towards generality and/or domain-specific ways of thinking, which for mathematics includes attention to generality, structure, and relationships, three highly intertwined constructs. As Watson and Mason (2006a) have argued, attention to structure provides the mechanism through which mathematical extensions can be brought into play. In the context of early number, for example, attention to structural properties related to commutativity or compensation—based on awareness of the relationships between quantities in expressions—builds flexibility and efficiency in calculation and supports extension of the number range that can be calculated with.

Conversely, concrete counting strategies (common in South Africa and elsewhere) ignore the efficiencies that can be leveraged through awareness of using structure and relationships, for example, using ten as a structuring benchmark in the place value system. Generality thus represents a key outcome of attention to structure and relationships.

A second tenet of Vygotskian theory is that learning comes about through mediated transactions (Wertsch, 1991). Kozulin (2003) describes this mediation as occurring through two key avenues—via cultural, artifact-based mediation and via human mediation—two aspects which we explore shortly.

Before turning to discuss forms of mediation, we need to make an important distinction between a focus on mediating means in particular and pedagogic forms more broadly. There is a significant body of evidence pointing to pedagogy as a culturally situated activity (Stigler & Hiebert, 1999). Culturally situated notions of pedagogy are important because there is ongoing and widespread evidence that largely "traditional" teacher-directed whole-class mathematics teaching is a predominant pedagogic form in South Africa (Adler & Pillay, 2007) and many other countries (e.g., see Alexander, 2000). Tabulawa (2013), arguing for sociocultural approaches to studying pedagogy, has described the lack of take up of reforms promoting learner-centered pedagogies in sub-Saharan Africa in terms of: "tissue rejection"—a metaphor for a philosophical rejection linked to "distinct and incompatible views of what constitutes legitimate knowledge, how that knowledge should be transmitted and how it is subsequently evaluated" (Tabulawa, 2013, p. 47).

Given this rejection, using frameworks for assessing the quality of teaching that privilege aspects of learner-centered pedagogic forms are likely to result, as Graven (2014) has noted, in unhelpful deficit readings based on absences. We say "unhelpful" because the international evidence suggests that more traditional teacher-led pedagogic forms are associated not only with high degrees of mathematical coherence and progression in many nations (Andrews, 2009; Stigler & Hiebert, 1999) but also with the poor coherence and low levels of progression seen in countries like South Africa. Such contradictory evidence of any direct association between pedagogic form/organizational format and what is substantially made available to learn mathematically, coupled with the prevalance of traditional pedagogic forms in contexts like South Africa, led to our central attention on teachers' ways of mediating mathematical ideas within instruction rather than to broader elements of their culturally situated pedagogic and classroom organization forms.

Turning to our elaboration of mediating means, our position aligns with Arzarello's (2006) interest in agents' dynamic activity with mediating forms as the means through which mathematical learning is made possible. Arzarello argues that this dynamic process involves a need to go beyond what he calls the "classical" semiotic focus on mathematical inscriptions to a broader range of mediating means that he refers to as a multimodal "semiotic bundle" that incorporates cultural artifacts, talk, and gestures. In line with this, a central part of our methodological approach to exploring the quality of primary mathematics instruction is via attention to teachers' ways of assembling, within lesson enactment, the range of mediating forms that make up semiotic bundles. Arzarello's theorization leads to our initial refinement of Kozulin's two avenues of mediation into three strands of instructional mediation—mediation with artifacts, inscriptions, and talk/gesture. While inscriptions, in being at least semi-permanent in lessons and not as evanescent as talk/gesture, could be considered part of artifactual mediation, for reasons set out below, we find it helpful to consider these separately from artifacts.

To these three strands, we add a fourth: mediation through tasks and examples. Often, the literature on mediation treats tasks/examples as the objects needing to be mediated but drawing on variation theory (Marton & Booth, 1997) and the work of theorists like Watson and Mason (2005), we argue that an important aspect of the quality of instruction relates to teachers' selections of tasks and associated ranges of examples which then act as a mediating means between teaching and learning mathematics. Tasks/examples provide the base upon which the other mediating strands come into play, with Mason and Pimm (1984) illustrating that talk relating to examples, as well as examples themselves, can range across the specific ("the even number 6"), the generic ("an even number like 6"), and the general ("any even number"). Hence, our framework addresses four overarching strands of mediation:

- Tasks/examples
- Artifacts
- Inscriptions
- Talk/gesture

With regard to instructional goals, we consider all four of these mediating strands as forms that can support moves to generality, with the tasks/examples range mediated via the other three strands. In methodological terms, we used Goldenberg and Mason's (2008) notions of "dimensions of possible variation" and "range of permissible change" to study the nature and extent to which, firstly, a coherent, and then second, a well-connected assembly of mediating forms were enacted within instruction in ways that encouraged mathematical expansions and a responsive mathematical instruction.

1.3 Mediation with tasks/examples

An extensive body of work carried out by Anne Watson and John Mason (some of this collated in their 2005 volume), highlights tasks and examples as the "raw material" upon which instruction is overlaid. Analysis of the potential of tasks has a long history in mathematics education (e.g., Stein, Smith, Henningsen, & Silver, 2000), and in research at the secondary level, Adler and Ronda (2015) analyze tasks for the extent to which they can leverage higher cognitive demand. Adler and Venkat's (2014) analysis also shows, however, as Watson and Mason (2006a) have noted, that the potential for focus on structure needs to be leveraged through instruction focused on looking for connections and relationships between instances.

In relation to tasks/example spaces, the literature provides two methodological options—coding for the quality of task/example space per se as Stein et al. (2000) do in their coding of the cognitive demand of tasks, or coding on the basis of how the task/example space is taken up and worked with in instruction (which Stein et al. also do), including attention to how particularizing and generalizing are enacted in the context of the example spaces within sections of lessons. There is evidence that learner attention to patterns of similarity and difference requires instruction that draws attention to these patterns (Marton, 2014), and given that our empirical focus is on early primary mathematics teaching, it is likely that young learners have not yet been inducted extensively into looking for the connections and relationships that promote attention to structure and generality. Given this, we follow Watson and Mason's approach of seeing task/example spaces as "raw material" that we note and describe as a mediating strand but do not code example spaces per se for the extent to which structure and generality are inherent. Instead, our focus is on the ways in which the potential of the task/example space is leveraged within and across the other MPM strands.

1.4 Mediation with artifacts

Given our empirical focus on early grades mathematics instruction where an arsenal of mathematical artifacts are frequently newly encountered by learners, of particular interest is Cole's (1996) distinguishing "artifacts" from "tools," drawing on Bakhurst's (1991) distinction between the "materiality" and "ideality" of artifacts. Bakhurst notes that:

> as an embodiment of purpose and incorporated into life activity in a certain way—being manufactured for a reason and put into use—the natural object acquires a significance. This significance is the 'ideal form' of the object, a form that includes not a single atom of the tangible physical substance that possesses it. (Bakhurst 1991, p. 182)

The physical substance constitutes the "materiality" of an artifact and, for Cole, an artifact is transformed into a tool when the nature of its use corresponds with the purposes recognized by the culture—that is, when its use embodies ideality rather than materiality. Thus, one strand of early mathematical learning is an appropriation of the use of artifacts, over time, in their ideal forms.

In common with early years' curricula internationally, the Foundations for Learning policy in South Africa (DoE, 2008) supported the introduction of a range of "structured" number artifacts into foundation phase (grades 1–3) classrooms. By "structured" artifacts, we refer to artifacts inhering in their design elements of mathematical relationships and properties—for example, the decimal number system of number relationships that underpins the structure of 100 squares and abaci. Wertsch (1998) has written extensively about the role of artifacts in extending the range of human unaided performance, illustrating this with examples such as the long multiplication algorithm, the use of which expands the number range of multiplication calculations that can be carried out. Of relevance to younger learners would be the use of the abacus to extend the range of numbers that can be operated on without unit counting.

However, Gravemeijer (1997) reminds us that "concrete embodiments do not convey mathematical concepts" (p. 316): this "conveying" usually requires instruction. Moyer's (2001) study exemplifies research noting that teachers' understanding of the rationales for the use of resources are critical for sensitive and responsive incorporation of resources within early mathematical learning. We have argued in earlier writing (Venkat & Askew, 2012) that the "newness" of access to structured resources in South Africa means that the take up of these artifacts in their ideal form is likely to be unfamiliar for teachers as well as learners. Given also that a number of studies indicate gaps in South African primary teachers' mathematical content knowledge (Taylor, 2011; Venkat & Spaull, 2015), the push in our research is towards studying how artifacts are *used* in instruction rather than simply focusing on their availability. There are therefore, international literature-based and contextual imperatives pointing to the importance of focus on how recruited artifacts are utilized in early primary mathematics teaching.

1.5 Mediation with inscriptions

We separate teacher inscriptions from physical artifacts for a number of reasons. While both phenomena provide material representational mediations of mathematical objects, the inscriptions we focus on are predominantly teacher generated, rather than externally produced.

Further, they are more often generated in the "flow" of teaching, and thus, afford greater openings to be configured responsively. While in our operationalization, we interpret, for example, diagrams drawn on the board ahead of the lesson as inscriptions, the teacher-generated feature remains.

Inscriptions, as Arzarello (2006) points out, are in many ways the "bread and butter" of mathematics teaching and learning. An extensive literature base focuses on the centrality both of representations per se and of moves between representations within mathematical working (Duval, 2006). In the early number instruction literature, the importance of working flexibly between actions on manipulatives and written replays of these actions through inscribed diagrams, written language and symbolic forms (and in oral language—which we deal with below) has been widely acknowledged as critical to supporting students' appropriation of mathematical ideas, as Anghileri (1995) notes, "The facility to 'move round' [between representations], adjusting interpretations for each element until there is a 'comfortable fit', models the development of mathematical understanding of the operations of arithmetic" (Anghileri, 1995, p. 10).

The usefulness of inscriptions in reifying processes into mathematical objects has been widely acknowledged (Sfard, 2008), with Hughes (1986) pointing to the importance of written inscriptions in early counting activities with very young children for recording and thereby facilitating moves beyond the immediate presence of physical objects.

South African primary teaching is marked by widespread use of oral modes of communication in teaching, with more limited use of inscriptions, as seen in limited written work in literacy and in mathematics, with small numbers and sporadic sets of completed written examples in learners' workbooks (Hoadley, 2012). Further, Mathews (2014) points to problems within the dynamic assembly of inscriptions in instruction related to division, with mediating talk (as we describe below) compounding gaps in the coherence of the problem-solving processes that are communicated.

1.6 Mediation with talk/gesture

As Cole (1996, p.117) has pointed out, consideration of artifacts can "apply with equal force whether one is considering language or the more usually noted forms," that is, the forms that have a more tangible materiality such as counters or abaci. We chose to separate talk/gestures from artifacts and inscriptions as, in classrooms, talk and gesture are usually the principal mediating means for responsively developing shared meanings "in-the-moment" (Edwards & Mercer, 1987). In addition, talk is unique in terms of its self-referential nature: verbal explanations are used to build up, clarify, and develop other verbal explanations through "chains of signification" (Walkerdine, 1988) in ways that the fixity of material artifacts cannot usually do.

Talk and gesture, at one level, are part of the semiotic bundle of mathematical sign systems. However, instructional talk as part of teacher's work goes beyond representations of mathematical ideas to include, among other aspects, the teaching of principles, with explanatory talk (Leinhardt, 1990) supporting both growing of awareness of critical and incidental features of problem situations (Marton & Booth, 1997), and also making connections (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997). Further, a substantial body of evidence points to the need for teacher talk to both connect with, remediate, or advance existing learner understandings in contingently responsive ways (Mason & Spence, 1999).

Given the complexities noted above, our reading of the literature, and our analysis of lessons observed in the WMC-P project, it became clear that considering talk/gesture as a blanket mediating means would be too broad and hence we sub-divided this strand of mediating means into three sub-strands:

Talk/gesture for generating solutions to problems
Talk/gesture for mathematical connection
Talk/gesture for advancing learning connections

1.6.1 Talk/gesture for generating solutions to problems

The first strand of talk/gesture is focused on the verbal/gestural aspects of teachers' generation of solutions to problems; it does not engage with elements of what Leinhardt (1990) has described as "instructional explanation," that is going beyond simply finding answers to problems to "convey, convince, and demonstrate." This focus is anchored in international evidence of primary teachers struggling simply to complete some mathematical tasks for themselves (e.g., Ma, 1999). Our early observations suggested that effective generating of solutions could not be taken for granted: there was a need to focus on the mathematical coherence present, or not, in generating solutions. In analysis of empirical excerpts of teaching, we noted that while correct answers were produced by teachers, these answers were sometimes produced in ways that lacked mathematical coherence. Mathews (2014) illustrates instances of talk that muddy the situations presented in division tasks. For example, he describes an episode where $10 \div 2$ was initially described in terms of sharing 10 between two people, but subsequent talk (and inscriptions) involving grouping actions produced a situation involving five people, and with the talk not distinguishing the move made between sharing and grouping actions. More broadly, givens and unknowns were not always made distinct in teacher talk (Venkat, 2013). Adler and Ronda (2015), looking at secondary mathematics teaching, have also noted error and ambiguity as features of teacher talk, with unclear referents for pronomials such as "it," "this," and "that" commonly seen in teachers' talk—a feature also noted in primary classrooms (Venkat & Naidoo, 2012). Given the evidence of primary school age children in many parts of the world making very limited progress across grades in terms of basic mathematical problem-solving performance (Pritchett & Beatty, 2015), and often in pedagogic cultures where oral instruction predominates—we consider mediating talk and gestures that establish problem-solving procedures to be an important part of our focus on instructional quality.

1.6.2 Talk/gesture for mathematical connection

Askew et al.'s (1997) research into effective primary mathematics teaching suggested that beliefs about mathematics teaching as "connectionist" were associated with higher learning gains. A key aspect of teacher connectionist beliefs related to connecting between mathematical ideas, with such connections reflected broadly as important within the development of conceptual understandings of mathematics. Watson and Mason have written extensively about the mechanisms through which such connections are fostered in instruction. One such mechanism, drawing from variation theory, is focused on connecting examples by looking for structural similarities and contrasts between the examples in "example spaces" as routes into generalization and abstraction. This kind of "connecting work" is viewed as necessary for moving beyond the "immediate doing" of individual tasks:

> We see generalization as sensing the possible variation in a relationship, and abstraction as shifting from seeing relationships as specific to the situation, to seeing them as potential properties of similar situations. Any task, particularly problem solving and modeling tasks, can focus learners' attention to the immediate 'doing' (calculations, representation, etc.) but unless special steps are taken to promote further engagement, there is seldom motivation for abstraction, rigor, or conceptualization beyond that necessary for the current problem. (Watson & Mason, 2006a, p. 94)

In parallel writing, Watson and Mason (2006b) provide a spatial metaphor for how these connections between examples can be made in instruction in terms of working with the "warp and weft" in examples spaces— that is, with examples literally listed vertically, drawing attention in instruction to the metaphorical horizontal structuring relationships within examples and the metaphorical vertical structuring relationships between examples. Ekdahl, Venkat, and Runesson (2016) note that horizontal connections within examples can be made through linking between representations or between various parts within the inscribed relation, with linking occurring through talk and gestural actions. This work highlights multi-directional connection possibilities.

In our earlier work, we have pointed to the presence of "extreme localization"—in which a series of examples are each treated from first principles through unit counting (Venkat & Naidoo, 2012). Similarly, Adler and Venkat (2014) illustrate and analyze highly disconnected ways of working across potentially "connectable" sets of examples. A key consequence of working in this way is that mathematics is presented as a series of discrete ideas with separate rules, rather than as a network of connected ideas in which operations become increasingly powerful in their generality.

1.6.3 Talk for advancing learning connections

Leinhardt's (1990) work on instructional explanations highlights the need to extend beyond mathematical connections in teacher talk into explanations that are attuned and responsive to students' current appropriations of the sub-skills needed to follow an offered process. This notion of appropriateness of explanation also features within Askew et al.'s (1997) "connectionist" orientations. Leinhardt (1990) emphasizes the need for rationales for action and instructional explanations that deal responsively with evidence of a lack of appropriation of an idea or a procedure, or to advance a current appropriation: "In instructional explanations what gets explained beyond what is presented is material not understood, or material which may not be understood or which may have future value not immediately relevant" (Leinhardt, 1990, p. 6).

In the South African context, this strand links to a ground in which there is some evidence of an absence of responsive evaluation in early years teaching in more disadvantaged school settings, that is, where student offers were simply accepted without discussion of whether they were appropriate at all (Hoadley, 2006). There is also evidence of teacher talk that "pulls back" towards more naïve strategies than those offered by students—for example, an insistence on concrete counting in the face of evidence of recalled arithmetic facts in early number instruction (Ensor et al., 2009). Responsive disconnection was also evident in Askew, Venkat, and Mathews' (2012) analysis of a teacher's use of artifacts, inscriptions, and talk to solve a missing addend task. Here, the teacher marshaled mediating forms in ways that were mathematically correct, but then stated rules for producing answers without accompanying rationales in either physical representations or oral argumentation.

2 Summary

The MPM strands that emerge from this discussion as key mediational means with backing in the literature and salience in our work for exploring the quality of primary mathematics instruction are thus:

Tasks/example spaces as the base upon which mediation occurs
Artifact mediation
Inscriptional mediation
Talk and gestural mediation, breaking down into: methods for generating/validating solutions, building mathematical connections, and building learning connections through explanations and evaluations.

2.1 Disaggregating the strands

Considering primary mathematics instruction in South Africa and internationally requires grappling with evidence of a lack of coherence and connections at the lower extreme while at the same time allowing for evidence of the promotion of the mathematical goals of structure and generality. It was therefore necessary to think about the quality of instruction in ways that went beyond simply identifying mediational means and mathematical goals related to structure and generality to ways of disaggregating the mediational means seen in different lesson enactments. This disaggregation needed to incorporate coherence and connection as interim staging posts towards the goals of structure and generality. Further, Mason's (2002) writing emphasizes the need to guard against viewing moves towards generality as uni-directional. Instead, he notes that mathematical instruction involves simultaneous awareness of the particular and the general—of "example" and "examplehood"—with responsive moves in both directions. This notion of responsive appropriateness was important to retain in our interpretations of instruction, with particularizing moves viewed as necessary in instances of evidence of incorrect answers to initially set tasks.

Nevertheless, expanding the scope of instruction to embrace increasingly powerful, general, and flexible mathematical problem-solving approaches remains a broadly accepted key goal, and one that underpinned our ways of thinking about the quality of primary mathematics instruction.

Thus the next step in developing the MPM framework involved looking at how the strands of mediation were enacted in different ways in lessons. We did this, initially, through close analysis of six videotaped lessons. Neither a "blanket" approach to interpreting the quality of mediation across a whole lesson, nor a time-coding approach for evaluating the mediation (for example, in 5 min segments) suited our purposes. Instead we "parsed" the lessons into episodes, with a break between episodes being marked by: a change in the example space being worked in; a change in the mathematical object; or a shift in the way of working, from, for example, whole class to individual work. We described and discussed differences between teachers in their ways of working with the strands of mediating means within episodes, and used the literature and theory detailed above to create a preliminary leveling of ways of working, and thus, a series of staging posts towards structure and generality.

We worked with our broader research team to apply the initial leveled framework to other lessons and through an iterative process of refining the levels and checking whether these applied to the data, a stable version of the MPM framework emerged that we present here. The framework in its entirety, and an example of MPM “maps” summarizing the coding of instructional episodes of two lessons taught by one teacher are presented in Appendices 1 and 2.

In the next section, we discuss and illustrate staging posts towards structure and generality within each of our leveled MPM strands (with tasks/example spaces, as noted above, forming the base for work within the other strands) using episode descriptions drawn from our empirical dataset.

2.2 Staging points within the MPM strands

2.2.1 Staging points: mediation with artifacts

We considered artifact-based mediation on the basis of the extent of structuring in the use of artifacts. Unstructured artifacts—such as counters and tally marks—used in unstructured ways (i.e., as artifacts for counting quantities) feature at the lower end of this strand. The evidence of structured artifacts (e.g., abaci, 100 squares) being used in unstructured ways in instruction provides an interim step, as the material nature of the artifact presents some possibilities for learners to attend to structure and relations even if the teacher does not make these explicit. Structured use of unstructured or structured artifacts (e.g., number lines in which jump strategies involving multiples of ten are used) features at the upper end, with this structuring fundamentally extending the range of the example spaces that can be worked with. Table 1 shows the four levels that we rated artifacts on, with descriptions of artifact use in episodes that illustrate each rating.

2.3 Staging points: mediation with inscriptions

Attention to structuring that brings possibilities for abstracting relationships and properties into play is also a focus in this strand. A range of teacher-mediating actions with inscriptions were seen that provided openings to distinguish the extent of attention to structure. For example, laying out examples systematically in ordered ways could be contrasted with more “random” inscriptions of examples offered or presented by learners. Systematic ordering supported attention to properties, to the behavior of operations, and to aspects such as completeness, contrasting with unstructured presentations where examples were largely treated as ends in themselves, with fewer possibilities for connecting between examples. In this strand, as in the artifact strand, we attended also to the use of unstructured number inscriptions focused on counting processes (e.g., tally marks), rather than on structured inscriptions. Further, given the concerns about teacher knowledge and coherence, and the limited extent of written registers in instruction in South Africa (Hoadley, 2012), we incorporated coding for an absence of inscriptions or problematic inscriptions (i.e., inscriptions that connected poorly in mathematical terms with the task/example) at the lowest end of the range, stepping up to inscriptions that simply recorded examples and/or response at the next level. In Table 2, these levels for inscriptions are presented and illustrated.

Table 1 Levels of artifact use, with indicators and illustrative excerpts

No artifacts or artifacts that are problematic/ inappropriate	Unstructured artifacts used in unstructured ways	Structured artifacts used in unstructured ways	Structured or unstructured artifacts used in structured ways
0	1 (bags of counters/tally marks)	2 (abaci, 100 squares, etc., used with unit counting, and without reference to structural properties)	3 (Abacus, 100 square/place value blocks/cards, number lines, etc.)
Lesson is conducted purely orally, with no artifacts or inscriptions	Pairs of numbers adding to nine are explored and counters used to check that a pair of numbers totals to 9.	Beads on the abacus used to add 4 and 8 by counting along 4 beads on the top row, 8 beads on the second row and counting all.	10s strips and unit squares used to support identification of value of underlined digit in several 2-digit numbers.

2.4 Staging points: mediating talk/gesture for generating and validating solutions

Here, we are interested in the ways in which teacher-mediating talk addressed the solutions to tasks/examples, either by generating solutions from scratch or by working with and validating solutions offered by learners. At the lower extreme, we included coding for episodes in which teachers just stated answers or accepted answers from learners without any discussion of how the answers were arrived at. This lower extreme also incorporated episodes in which the talk associated with solution methods was mathematically ambiguous or incorrect. In "stated" answers, the discourse presents mathematical results declaratively without associated explanations or rationales of the problem-solving processes involved; in the cases of ambiguity or error, unknowns were treated as givens within the problem-solving talk, or erroneous or ambiguous statements were offered.

We elected to place errors/ambiguity in instruction on problem-solving procedures within the generating/validating procedures strand where declarative statements of procedures are also located. In contexts of concerns about low mathematical performance, a coherent declarative presentation of a procedure for problem-solving, or some validation of a coherent procedure offers possibilities for some progress with individual problems or a class of problems, and thus can be viewed as a move forward. This contrasts with Hill et al.'s (2008) MQI framework in which declarative ways of dealing with mathematics in instruction are separated from errors/ambiguity in instruction. In the MQI model, erroneous solution procedures (captured in the "errors and imprecision" strand) are separated from the absence of a solution procedure (in the "richness of the mathematics" strand).

We then attended to the extent of generalizability of the method of solution that was offered (or validated) by the teacher. The task and example space came into play here in our consideration of whether a mathematically appropriate procedure had the potential to enable learners to generate an answer to another example in the immediate example space, whether it could be used to generate solutions beyond the immediate example space, and then beyond the immediate example space without being restricted to the immediate artifacts and inscriptions used in the episode. Thus, our attention in this strand is focused on the generality of applicability of the problem-solving procedure offered or validated. In Table 3, we summarize the ratings across this sub-aspect, and, as before, detail the indicators and exemplifications associated with them.

2.5 Staging points: mediating talk/gesture building mathematical connections

This strand is centrally concerned with the mathematical ways in which examples and example spaces are worked with, incorporating incoherent work with examples or oral recitation from learners with no further teacher talk at the lower extreme, and moving via coherent, but separate treatment of individual examples, towards uni-dimensional and then multi-dimensional connections across an example space. This strand leans on Watson and Mason's (2006b) notion of working with the horizontal and vertical "warp and weft" of connections in an example space that we noted earlier in order to support attention to abstraction and generalization, and Askew et al.'s (1997) identification of effective teaching involving a teacher knowledge base with multiple mathematical connections. Connections can be made in a range of ways: between the representations that feature in the semiotic bundle within examples, between examples in the task/example space, and between task/example spaces. Our approach here overlaps with the orientation described by Rowland (2013, p. 151) in his noting that it is "possible to comment on the examples actually deployed in the scenarios to be described, and how they compare with available alternatives." Although this quote suggests a focus on the examples per se, his empirical analyses include attention to the highlighting (or

Table 2 Levels of mediating inscriptions with indicators and illustrative excerpts

No inscriptions or problematic/incorrect inscriptions	Inscriptions only recording tasks or responses	Unstructured inscriptions	Structured inscriptions
0	1	2 (tally marks, circles)	3 (tables of ordered bonds; structured number lines, ENL; inscriptions underpinned by number relations)
Learners asked to give missing number in these sentences: $41 + 1 = \_ - 1 = 41$; $41 + 2 = \_ - 2 = 41$; $41 + 3 = \_ - 3 = 41$	With 117 and 170 written on board, teacher asks which is bigger. No further inscriptions are added.	In response to a task to "draw 16," a straggled line of circles from a learner is accepted.	Teacher reorders partitions of 7 offered by learners into ascending order and asks for missing pairs.

Table 3 Levels of generating/validating solutions with indicators and illustrative excerpts

No method or problematic generation/validation	Singular method/validation	Localized method/validation	Generalized method/validation
0	1	2	3
(mixing of knowns and unknowns; error/ambiguity)	(provides a method that generates/ validates the immediate answer; enables learner to produce the an swer in the immediate example space)	(provides a method that can generate/validate answers beyond the particular example space)	(provides/validates a strategy/method that can be generalized to both other example spaces and without restriction to a particular artifact/inscription)
Solution to 20 ÷ 4 begins with the teacher talking about the need for five groups to share across.	Teacher tells learners to use counters to find the answer to 4 + 5.	Teacher shows how adding 10 on a 1–100 square involves moving one row down.	Teacher works on adding 9 by adding 10 (as a quick fact) and then subtracting 1.

not) of connections and relationships present in teacher talk, alongside attention to the mathematical potential for connection within and related to, but beyond, the immediate example space. Table 4 shows the levels, indicators, and exemplars in this strand.

2.6 Staging points: mediating talk/gesture building learning connections

The "building mathematical connections" strand focuses on mathematical connections and coherence: the "building learning connections" strand complements this by attending to the ways in which teacher talk presents mathematical discourse with potential to: support learner progression, build on prior understandings, and work productively with learner offerings. Much of the coding in this strand was seen in the empirical space of teacher responses to learner offers. At the lower end, we incorporated South African evidence of teaching that "pulled back" towards more naïve strategies (Ensor et al., 2009) or failed to offer any evaluation of learner inputs (Hoadley, 2006). A step forward from this situation involved evaluations of offers as correct or incorrect. At the upper end, instructional talk worked to advance mathematical offers, with rationales provided or elicited from learners for their choices. Examples of "advancing" include talk and gestures that indicate responsive teaching moves beyond the use of counting artifacts, accompanied by moves to more structured artifacts and inscriptions, or possibilities for "fading" the use of external representations over time, and encouraging appropriations of these models as internalized tools. Responsive unpacking or particularizing moves in response to incorrect offers form part of the advancing learning talk repertoire. Table 5 summarizes the levels, indicators, and examples in this strand.

3 Discussion

The strands and illustrative excerpts within them presented in the previous section highlight the variety of mediating means within instruction that we are interested in and illustrate some of the empirical range that we saw within each strand. The MPM-based coding of lessons provides a lens that zooms in on the nature of the mathematics that is made available to learn in ways that place particular value on mathematics as being about attending to structure and generality, rather than to individual calculations. In our attention to the nature of mathematics seen within instruction, there is consonance with the foci of both Hill et al.'s (2008) MQI and Rowland et al.'s (2009) KQ. While the categories within each of these models have evolved over time, leading to different focal dimensions, both teams have emphasized fundamental attention to mathematics. Hill et al. (2008, p. 431) describe MQI as: "a composite of several dimensions that characterize the rigor and richness of the mathematics of the lesson." Rowland (2013, p. 18) similarly describes the KQ as an: "empirically based conceptual framework for lesson review discussions with a focus on the mathematics content of the lesson."

Within this broad overlap with the focus of MPM, there are, however, differences in our aims and emphases. The MPM framework is tilted towards mathematical emphasis on structure, relation, and generality, within a sociocultural view of mathematics as a network of scientific concepts. Evidence of limited attention in instruction to the efficiencies garnered by awareness of structure and relationships give this mathematical orientation strong purchase for studying changes in instruction over time. In contrast, both the MQI and KQ models are more general about the kinds of mathematics that they value, with Hill et al. (2008) emphasizing rigor and richness (underpinned by precision and connection) and Rowland et al. (2009) noting the need for connection and coherence across example sequences.

Table 4 Levels of mathematical connections with indicators and illustrative excerpts

Problematic use of examples	Separate examples	1-dimensional connections	2/multi-dimensional connections
0 (disconnected and/or incoherent treatment of examples or oral recitation with no additional teacher talk)	1 (every example treated from scratch)	2 (teacher talk connects between examples or artifacts/inscriptions or episodes)	3 (teacher talk makes vertical and horizontal (or multiple) connections between examples/artifacts/inscriptions/episodes)
Teacher asks what needs to be added to 4 to make 7 and tells learner to put out 4 fingers on one hand, and three on the other.	Learners are asked for pairs of numbers adding to 16. Each offer is checked by counting with cubes.	Teacher checks the number of 10s in 100 by counting out 10s strips.	Teacher working on place value records, say, 19 as 10 + 9, and accompanies this with base 10 blocks. She draws learners' attention to the similarities and differences in partitioning and representing other 2-digit numbers in similar fashion.

Table 5 Levels of learning connections with indicators and illustrative excerpts

Pull back	Accepts/evaluates offer	Advances or verifies offers	Advances **and** explains offers
0	1	2	3
Pull back to naïve methods or no evaluation of offers (correct or incorrect)	(accepts learner strategies or offers a strategy or notes or questions incorrect offer)	(builds on, acknowledges or offers a more sophisticated strategy or addresses errors/misconceptions through some elaboration, e.g., "Can it be—?" "Would—this be correct, or this?" Non-example offers)	(explains strategic choices for efficiency moves or provides rationales in response to learner offers related to common misconceptions or provides rationale in anticipation of a common misconception)
When 16 is offered immediately as the total of 9, 4, and 3, teacher demands a counting out of the quantities on an abacus.	With no answers offered for what needs to be added to 4 to make 7, teacher tells a child to show 7 fingers, close 4 of them 1 by 1, and then count the fingers left open.	Teacher responds to learner's count-on strategy offer with a suggestion to count on from larger number	After using learner inputs to represent 19 with 1 10-strip and 9 unit squares, teacher asks if it would be okay to add another unit square, following through on the class response that this "would be 10" with a collecting gesture of the 10 unit squares to the left to join the 10.

While less specific on mathematical goals, the MQI model is more prescriptive on learner participation forms, including explicit valuing of attention to what they describe as “common core-aligned student practices” which include coding for the extent of learner questions and explanations. This points to an aspect backgrounded in the MPM framework, in that there is less explicit attention to particular formats for classroom interaction and student practices. Overlaps with the KQ are in the transformation, connection, and contingency strands, with limited attention to foundation category knowledge emanating from our applications of the MPM framework.

A further important point to note relates to the purposes to which the various frameworks are geared. Hill et al. (2008, p. 436) explicitly note that their methodological approach is not geared towards studying individual teachers—the origins of their work on MQI were instead focused on exploring and quantifying the relationship between primary teachers’ mathematical knowledge and their mathematics instruction. The KQ, in contrast, has been used primarily as a tool for individual teacher-level development, in terms of knowledge and practice. Our approach, focused on coding at the level of task/example space demarcated episodes rather than in the time segment demarcations of lessons used in MQI, is closer to the KQ methodology with overlapping attention to example spaces across the KQ and MPM frameworks. Our aims in terms of applying the MPM thus straddle the two contrasting goals of the MQI and KQ studies.

As noted at the start of this paper, we had imperatives within the project to examine for evidence of changes in the quality of instruction that would allow us to explain changes in learner performance. In that examination, we are comparing earlier and later MPM maps for each teacher, with analysis of teachers’ “extent and breadth” profiles. For extent of mediation, our focus is on levels of mediation for structure/generality within strands; for breadth, we look at mediation transversally across strands. This allows us to gauge differences in the nature of use of specific mediating means as well as differences in the orchestration of mediating means in episodes. The analysis of the extent and breadth of mediation across lessons from the same teacher at two points in time reveal differences that are, in the main, in the direction of more extensive and broader mediation for structure and generality in the latter lesson. This suggests that the MPM framework is useful for documenting empirical changes in theoretically grounded ways.

This combination of theoretical derivation and empirical sensitivity is important in that it provides a hypothesis about what counts as improving quality of instruction (the *roots of* change) while empirically illuminating some *routes towards* change. Our broader work involves in-class coaching initiatives based on lesson observations using a modified version of the MPM framework. Transversal coding across the strands within specific episodes is being used in this work for thinking about enacted mediation of the task/example space in relation to alternatives that may extend learners’ possibilities for working within, and beyond, the example space presented. Hence, the framework is proving useful not only to retrospectively track changes but to prospectively support teaching development.

This developmental orientation informs our inclusion of errors and ambiguity within the mediating dimensions related to artifacts, inscriptions and talk. We build errors and imprecisions into a developmental trajectory that locates errors as the start of a journey into mathematical instruction. In this journey, coherent work with examples and connected work with, and then beyond, specific example spaces are key staging points that bring the attention to mathematical structure and generality that we value into the realms of possibility within instruction. Our reading of both South African and international primary mathematics instruction suggests that this trajectory provides a useful lens for exploring primary mathematics teaching in relation to these mathematical goals.

Appendix 1

Table 6 The MPM framework

Appendix 1: The MPM framework — MEDIATING TASKS/EXAMPLES

MEDIATING ARTIFACTS			
No artifacts used or artifacts that are problematic/ inappropriate	Unstructured artifacts used in unstructured ways	Structured artifacts used in unstructured ways	Structured artifacts used in structured ways/unstructured artifacts used in structured ways
0	1	2	3

MEDIATING INSCRIPTIONS			
No inscriptions or inscriptions that are problematic/ incorrect	Inscriptions that only record tasks or responses	Unstructured inscriptions	Structured inscriptions
0	1	2	3

MEDIATING TALK/GESTURES				
Method for generating/ validating solutions	No method or problematic generation/validation	Singular method/validation	Localized method/validation	Generalized method/validation
	0	1	2	3
Building mathematical connections	Disconnected and/or incoherent treatment of examples OR Oral recitation with no additional teacher talk	Every example treated from scratch	Talk connects between examples or artifacts/inscriptions or episodes	Talk makes vertical and horizontal (or multiple) connections between examples/ artifacts/inscriptions/episodes
	0	1	2	3
Building learning connections: explanations and evaluations - of errors/ for efficiency/ with rationales for choices	Pull back to naïve methods OR No evaluation of offers (correct or incorrect)	Accepts/evaluates offers Accepts learner strategies or offers a strategy OR Notes or questions incorrect offer	Advances or verifies offers Builds on, acknowledges or offers a more sophisticated strategy OR Addresses errors/misconceptions through some elaboration, e.g. 'Can it be ----?' 'Would – this be correct, or this?' Non-example offers	Advances **and** explains offers Explains strategic choices for efficiency moves OR Provides rationales in response to learner offers related to common misconceptions OR Provides rationale in anticipation of a common misconception
	0	1	2	3

Appendix 2

Table 7 MPM maps—Ms. M

Appendix 2: MPM maps - Ms M

2012 lesson

Math focus	Oral fwd count	Compare numbers	Inverse relations?	Oral fwd count	Identify number	Match words & symbols	Match words& symbols	Place value
Task	Choral forward count	Which is more (M)/ less (L)?	Fill in missing numbers	Choral forward count	Identify number on grid	Word to symbol match	Symbol to word match	Place value of digits
Example space	101-130 in 1s	L100/200; L213/231, M91/111, L95/51. (Six more e.g.s on worksheet not dealt with)	41+1=_-1=41 41+2=_-2=41 41+3=_-3=41	101-110 121-132 101-131	106, 166, 116, 161	162, 121, (189, 176, 110 also on sheet, not dealt with)	152, 121 (199, 187, 152 on sheet; not dealt with)	143 105
No of examples	1	4	3	3	4	2	2	2
OPPORTUNITIES FOR TEACHING								
Incorrect answers	N	N	Y	N	N	N	N	Y
Inefficient methods	N	N	Y	N	N	N	N	N
MEDIATING ARTIFACTS								
Physical artifacts	101-200 grid	No artifacts	No artifacts	101-200 grid	101-200 grid	101-200 grid	No artifacts	No artifacts
Artifact use	2	0	0	2	2	2	0	0
MEDIATING INSCRIPTIONS								
Inscriptions	None	None	Problematic inscription	None	None	Circle number on grids	Number words, symbols	HTU number format
Inscription type	0	0	0	0	0	1	1	3
MEDIATING TALK/GESTURE								
Generating solutions	0	0	0	0	1	0	0	2
Mathematical connections	0	1	0	0	1	2	2	2
Learner connections	0	0	0	0	1	1	1	2

2015 lesson

Math focus	Compare numbers	Add/take away 10	Closer to ..	Take away/ difference	Efficient addition	Indiv'l work
Task	Compare numbers orally	10 more or 10 less than …	Number is closer to ..	Subtract as take away/ difference	Efficient calculation methods	Efficient +/ - method
Example space	Bigger? 6/16; 49/19; 103/301 Smaller? 25/5; 83/80; 145/45	10 more than: 23, 45, 79, 109, 274? 10 less than: 15,32,43, 87,50,52?	21: closer to 19 or 25? 201 closer to 99 or 205?	22-3 & 22-19; 105-99 & 105-7	27+8; 6+25; 177+6; 9+163	7+94; 172+18; 29+235; 77-9; 102-97
No of examples	6	11	2	4	4	5
OPPORTUNITIES FOR TEACHING						
Incorrect answers	N	Y	Y	Y	Y	
Inefficient methods	N	N	N	Y	Y	
MEDIATING ARTIFACTS						
Physical artifacts	No artifacts	No artifacts	No artifacts	No artifacts	No artifacts	
Artifact use	0	0	0	0	0	
MEDIATING INSCRIPTIONS						
Inscriptions	None	None	Empty number line	Empty number line	Empty number line	
Inscription type	0	0	3	3	3	
MEDIATING TALK/GESTURE						
Generating solutions	0	2	3	3	3	
Mathematical connections	2	2	3	3	3	
Learner connections	2	2	3	3	3	

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