Advances in Applied Mathematics 164 (2025) 102835

Contents lists available at ScienceDirect

Advances in Applied Mathematics

journal homepage: www.elsevier.com/locate/yaama

Cells of fixed height in Catalan words and restricted growth functions

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A R T I C L E I N F O

Article history:
Received 14 October 2024
Received in revised form 13 December 2024
Accepted 16 December 2024
Available online xxxx

MSC:
05A15
05A18

Keywords:
Catalan words
Restricted growth functions
Polyominoes

A B S T R A C T

A word $w = w_1w_2 \cdots w_n$ of length n over the set of positive integers is called a Catalan word whenever $w_1 = 1$ and $1 \leq w_k \leq w_{k-1} + 1$ for $k = 2, 3, \ldots, n$. A restricted growth function is defined as a word $w = w_1w_2 \cdots w_n$ of length n over the set of positive integers where $w_1 = 1$ and for $k \geq 2$ we have $1 \leq w_k \leq \max\{w_1, w_2, \cdots, w_{k-1}\} + 1$. We also define cells and heights of cells and we represent such words as bargraphs (otherwise known as polyominoes) where the ith column contains w_i cells for $1 \leq i \leq n$ and where all columns have their bottom cell on the x-axis. In the case of Catalan words, we prove a relationship between the number of cells at different heights and first terms of the expanded polynomial $(1+x)^{2n}$. In the case of restricted growth functions we find polynomials $F_n(x)$ where the coefficient of x^j counts the number of cells of height j across all rgfs with n parts. In this case we also find bivariate generating functions for rgfs with k blocks, where the generating functions tracks the number of cells at a given height as well as the number of parts.

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https://doi.org/10.1016/j.aam.2024.102835
0196-8858/

1. Introduction

In this paper we study two types of words. Namely Catalan words and restricted growth functions and their representation as bargraphs or polyominoes. More specifically, we study the number of cells at particular heights in these polyominoes. For any part in these words, we call each square in the grid contained inside the bargraph of the Catalan word or restricted growth functions, a cell. We define the height h of any particular cell with top boundary at $y = b$, by $h = b$. This is illustrated in Fig. 2. In Sections 2-4 we study Catalan polyominoes and in Sections 5-7 we study restricted growth function polyominoes. These polyominoes are defined respectively in Section 2 and Section 5.

For Catalan polyominoes, the number of cells at height i coincides with the first $n+1$ coefficients of the binomial expansion of $(1+x)^{2n}$. This is proved in the theorems and Corollaries of Section 4. For example Corollary 5, asserts that the number of cells at height $n-i$, where $0 \le i \le n$, in all Catalan polyominoes of length n is given by the binomial coefficient $\binom{2n}{i}$. Our later analysis requires the notion of a Dyck path. A Dyck path is a path in the first quadrant which begins at the origin, ends at $(2n, 0)$, and consists of steps $(1, 1)$ (North-East), called rises, and $(1, -1)$ (South-East), called falls. We also need the associated notion of a Dyck walk which is defined just before Corollary 5. The method of proving Corollary 5 is to show in Theorem 6 that the generating function for Dyck walks ending at height $i \ge 0$ is the same as the generating function for the number of cells at height $i \le n$ in Catalan words with n letters. The former is counted by $\binom{2n}{i}$.

The analysis for restricted growth functions follows the format of that for Catalan polyominoes and there are analogues in this case for the number of cells at height i. This is presented in Theorem 9. Thereafter, we derive further results which includes a simpler form of Theorem 9, namely Corollary 19.

2. Catalan polyominoes

A word $w = w_1 w_2 \cdots w_n$ of length n over the set of positive integers is called a Catalan word whenever $w_1 = 1$ and $1 \le w_k \le w_{k-1} + 1$ for $k = 2, 3 \ldots, n$. We represent such words as bargraphs (otherwise known as polyominoes) where the ith column contains w_i cells for $1 \le i \le n$ and where all columns have their bottom cell on the x-axis. In Fig. 1 we illustrate polyominoes associated with Catalan words of length 4. For $1 \le i \le n$, the generating functions $H_i(x, q)$ which track the number of cells of height i with tracker q and number of parts tracked by x are defined in Theorem 3. This generating function yields the following series as per Equation (9):

$$\begin{aligned} H_2(x,q) &= x + (1+q)x^2 + \left(1 + q^2 + q(2+q)\right) x^3 \\ &\quad + \left(\mathbf{1} + \mathbf{3q} + \mathbf{5q^2} + \mathbf{5q^3}\right) \mathbf{x^4} + O\left(x^5\right). \end{aligned} \tag{1}$$

The bold coefficients count all cases of Catalan words with exactly 4 parts and cell height of 2 (see definition in Section 3) as illustrated in Fig. 1.

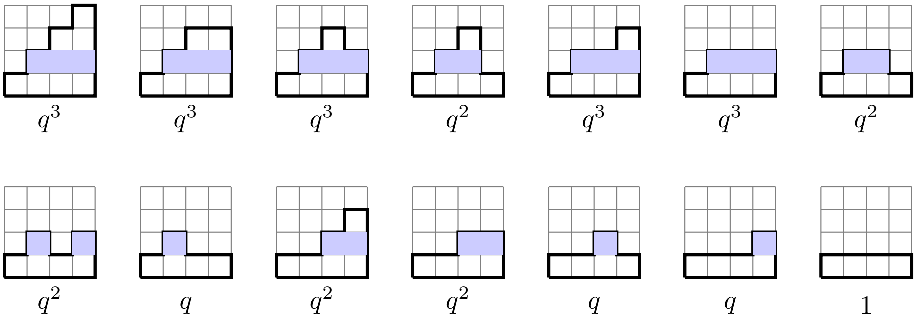

Fig. 1. All 14 Catalan words with 4 parts where those cells at height 2 are labeled with the variable which tracks the number of cells at this height.

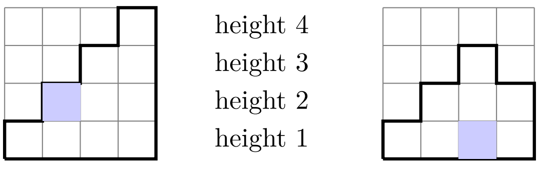

Fig. 2. Heights labelled for Catalan words or rgfs with 4 parts.

In Theorem 2.1 of [4], the current authors obtained a generating function tracking the number of letters of size i in Catalan words with n letters. The number of letters of size $\geq i$ is identical to the number of cells at height i. However, in Section 4 of the current paper, we use a different much simpler approach to count cells at height i and establish an interesting connection between cells at height i in Catalan words with n letters and central binomial coefficients.

3. Definition of cells and cell heights in both Catalan words and restricted growth functions

Consider any Catalan word or restricted growth function (rgf) with n parts. From the definition of both of these, the largest possible part across all such rfgs has size $k = n$. As examples set $k = n = 4$ and compare two Catalan words or rgfs, 1234 and 1232. These are illustrated in Fig. 2.

For any part in these words, we call each square in the grid contained inside the Catalan word or rgf, a cell. We define the height h of any particular cell with top boundary at $y = b$, by $h = b$. So the shaded cell in the left half of Fig. 2 is at height 2 and that one in the right half is at height 1.

4. Height of cells in Catalan words

In [2], the authors of that paper find a surprising connection between cells at height i in Motzkin words and central trinomial coefficients. That paper is the inspiration for this

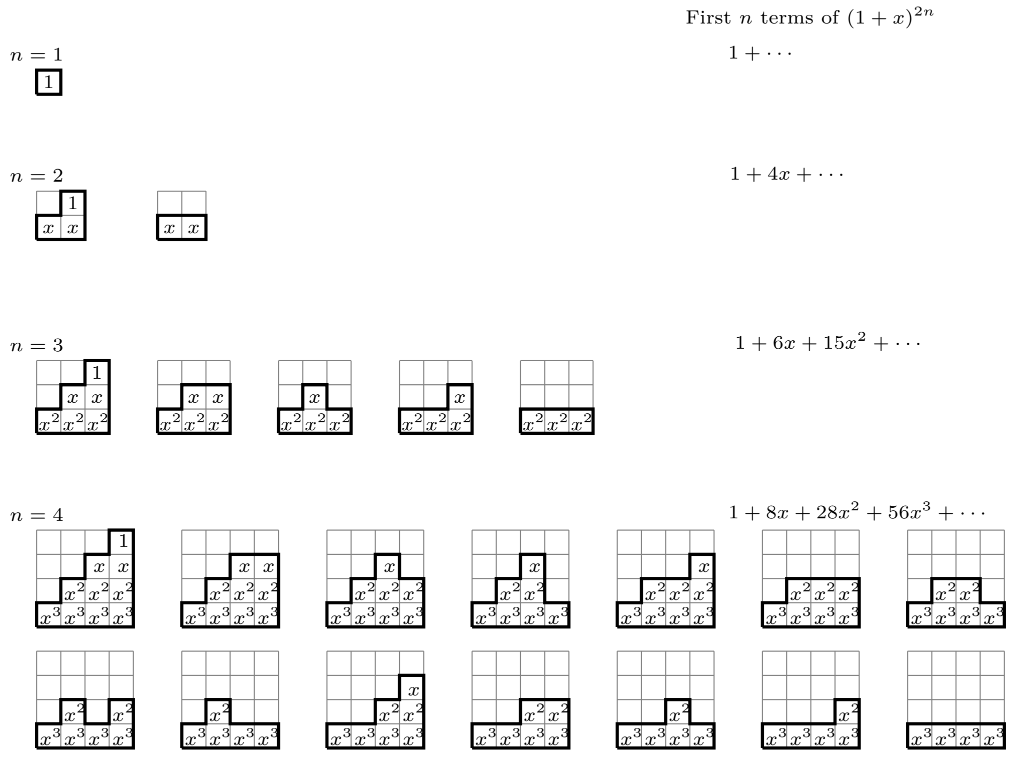

Fig. 3. Surprising correspondence between Catalan words of size n (where the label x^i marks a cell at height $n-i$) and the first n terms of $(1+x)^{2n}$ in increasing powers of x.

section which finds a connection between cells at height i in Catalan words and central binomial coefficients. In fact the open question is posed in [2] about whether binomial coefficients play a similar role in related situations. This section is an answer to that open question.

In Fig. 3 the initial coefficients of $(1+x)^{2n}$ are seen to be counting the number of cells of height i in Catalan words of length n. This surprising assertion will be proved later in Corollary 5.

Let w be a Catalan word. We denote by $h_i(w)$ the number of cells of height i in the Catalan polyomino associated with w. We introduce the following generating functions:

$$H_i(x,q) := 1 + \sum_{w\in C} x^{|w|} q^{h_i(w)} \tag{2}$$

and

$$B_i(x) := \frac{\partial}{\partial q} H_i(x,q)\bigg|_{q=1}. \tag{3}$$

We define $h(n,i) := [x^n]B_i(x)$, where $h(n,i)$ is the number of cells of height i in all Catalan words of length n.

Before we consider the next theorem, we remind the reader of the first return decomposition for the generating function D of an arbitrary Dyck path where only U steps are tracked with invariant x. This is given by

$$D = 1 + xD + x(D-1)D,$$

where 1 is the empty Dyck path, xD is where the path before the first return consists only of steps UD, and where $x(D-1)D$ is where the path before the first return has more than two steps.

Theorem 1. *For $i \geq 2$ the generating function $H_i(x,q)$ defined in Equation (2) satisfies the functional equation*

$$H_i(x,q) := \frac{1}{1 - xH_{i-1}(x,q)} \tag{4}$$

and

$$H_1(x,q) = \frac{1-\sqrt{1-4xq}}{2xq} = C(xq), \tag{5}$$

where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the Catalan number generating function.

Proof. Catalan words of length n are in bijection with Dyck paths of length $2n$. The bijection maps the ith entry of the word w to the y coordinate of the endpoint of the ith up-step of its corresponding Dyck path. By factorizing out the first and last steps of the first return of any Dyck path it follows that

$$H_1(x,q) = 1 + xqH_1(x,q) + xq(H_1(x,q)-1)H_1(x,q). \tag{6}$$

Above, the first term 1 on the right side is the empty case. The second term $xqH_1(x,q)$ is where the path before the first return consists only of the steps UD followed by an arbitrary path. The final term $xq(H_1(x,q)-1)H_1(x,q)$ counts the case where the path before the first return has more than two steps followed by an arbitrary path. From the above we deduce that

$$H_1(x,q) = \frac{1-\sqrt{1-4xq}}{2xq} = C(xq). \tag{7}$$

By analogy with Equation (6), factorizing out the first and last steps of the first return of any Dyck path, we obtain for $i \geq 2$ that

$$H_i(x,q) = 1 + (x + x(H_{i-1}(x,q) - 1))H_i(x,q). \tag{8}$$

In this case, factorizing out the first step replaces xq with x because steps at height 1 do not contain cells at height $i \geq 2$. $\square$

Example 2. Using Equations (7) and (8) the series $H_2(x,q)$ begins

$$H_2(x,q) = x + (1+q)x^2 + \left(1 + q^2 + q(2+q)\right)x^3 + \left(\mathbf{1 + 3q + 5q^2 + 5q^3}\right)x^4 + O\left(x^5\right). \tag{9}$$

The bold term is illustrated in Fig. 1.

From definition (2), we obtain that for all i, $H_i(x,1) = C(x)$. From definition (3) and by differentiating Equation (4) and setting $q = 1$ we also obtain that

$$B_i(x) = \frac{xB_{i-1}(x)}{(1 - xC(x))^2}. \tag{10}$$

In addition, from the definition in Equation (3) and also using Equation (5), we obtain

$$B_1(x) = xC'(x) = \frac{1}{\sqrt{1-4x}} - C(x). \tag{11}$$

Iterating Equation (10), we obtain

Theorem 3. *The number of cells at height $i \leq n$ in Catalan words with n letters tracked by x is given by $h(n,i) = [x^n]B_i(x)$ where $B_i(x)$ is given by*

$$B_i(x) = \frac{x^i C'(x)}{(1 - xC(x))^{2i-2}} = xC'(x)(C(x) - 1)^{i-1}. \tag{12}$$

The last equality in the above Theorem follows from the identity

$$\frac{x}{(1 - xC(x))^2} = C(x) - 1.$$

The main result of this section is the following theorem.

Theorem 4. *For $0 < i \leq n$, the coefficient $[x^n]B_i(x) = \binom{2n}{n-i}$. In other words*

$$B_i(x) = \sum_{n \geq 0} \binom{2n}{n-i} x^n.$$

This implies

Corollary 5. *The number of cells of height $n-i$ where $0 \leq i < n$ in all Catalan polyominoes of length n is given by the binomial coefficient $\binom{2n}{i}$.*

The surprising assertion made in Fig. 3 follows from Corollary 5. The proofs of Theorem 4 and Corollary 5 follow from Theorems 6 and 7 below.

In the introduction, we defined a Dyck path. Next, we define a Dyck walk of length n: this is a Dyck path prefix with n steps but without the condition that it never goes below the x-axis (see for instance [3]). A Dyck walk ending on the x-axis is also known as a Grand Dyck path.

Theorem 6. *For all $i \geq 0$, the generating functions $G_i(x)$ counting Dyck walks ending at height $2i$ is given by*

$$G_i(x) = \frac{(xC(x^2))^{2i}}{1-2x^2C(x^2)}$$

where x tracks up and down steps. Moreover,

$$[x^{2n}]G_i(x) = \binom{2n}{n-i}.$$

Proof. Let U and D be the notation for single up and down steps respectively and let P be a Dyck walk ending at height $2i$. It can be decomposed as $GUC_1UC_2U\cdots UC_{2i}$, where G is a maximal Dyck walk ending at height zero and C_j is a Dyck path for all $1 \leq j \leq 2i$. Therefore, we have the generating function $G_i(x) = x^{2i}G_0(x)C(x^2)^{2i}$, where $C(x^2)$ is the generating function for Dyck paths counted according to occurrences of U and D steps. Notice that $G_0(x) = 1 + 2x^2C(x^2)G_0(x)$, because a nonempty Dyck walk ending at height zero is either $UCDG_0'$, or $D\bar{C}UG_0'$ where G_0' is a Dyck walk ending at height zero, C is a Dyck path (shifted upwards by one unit) and $\bar{C}$ is an upside-down Dyck path (shifted downwards by one unit). So, we have $G_0(x) = \frac{1}{1-2x^2C(x^2)}$. Therefore,

$$G_i(x) = (xC(x^2))^{2i}\frac{1}{1-2x^2C(x^2)}.$$

Moreover, we can count all Dyck walks that end at coordinates $(2n, 2i)$ by Dyck walks which have $n+i$ up steps and $n-i$ down steps. The total number of such paths is given by $\binom{2n}{n-i}$. $\square$

In order to prove Theorem 4 we now show that

$$G_i(\sqrt{x}) = \frac{x^i(C(x))^{2i}}{1-2xC(x)} = xC'(x)(C(x)-1)^{i-1} = B_i(x) \tag{13}$$

where the right hand equality above comes from Equation (12).

Theorem 7. *We have*

$$\frac{x^i C(x)^{2i}}{1-2xC(x)} = xC'(x)(C(x)-1)^{i-1}.$$

Proof. Our proof is by induction on $i \geq 1$. Firstly, for $i=1$, by the formula for $C(x)$ that

$$\frac{xC(x)^2}{1-2xC(x)} = xC'(x).$$

Now, we assume that

$$\frac{x^{i-1} C(x)^{2(i-1)}}{1-2xC(x)} = xC'(x)(C(x)-1)^{i-2}.$$

From this induction step the ith case easily follows because $xC(x)^2 = C(x)-1$. □

The proof of Theorem 4 follows immediately from Equation (13).

We use this theorem to give an alternative proof of following corollary which appears in [5].

Corollary 8. *The total area $u(n)$ of Catalan polyominoes of length $n \geq 1$ is given by*

$$u(n) = \frac{1}{2}\left(2^{2n} - \binom{2n}{n}\right). \tag{14}$$

Proof. There is a simple relationship between cells at height i and the area of Catalan polyominoes of length n, namely

$$u(n) = \sum_{i=1}^{n} h(n,i). \tag{15}$$

Therefore, by Theorem 5,

$$u(n) = \sum_{i=1}^{n} \binom{2n}{n-i}.$$

So

$$u(n) = \sum_{i=0}^{n-1} \frac{1}{2}\left(\binom{2n}{i} + \binom{2n}{2n-i}\right).$$

By the symmetry of binomial coefficients $\binom{2n}{i} = \binom{2n}{2n-i}$; so we obtain

$$u(n) = \frac{1}{2}\left(2^{2n} - \binom{2n}{n}\right). \quad \square$$

5. Introduction to restricted growth functions

To explain the notion of a restricted growth function, we need the following concepts.

Firstly, a (set) partition Π of set $[n] = \{1, 2, \ldots, n\}$ of size j (i.e., having j blocks) is a collection $B_1, B_2, \ldots, B_j$ that satisfies: $\emptyset \neq B_i \subseteq [n]$, $B_{i_1} \cap B_{i_2} = \emptyset$ for $i_1 \neq i_2$ and $\cup_{i=1}^{j} B_i = [n]$. The elements B_i are called blocks and we list these blocks in increasing order of their minimal elements, that is, $\min B_1 < \min B_2 < \cdots < \min B_j$. We denote the set of all partitions of $[n]$ with exactly j blocks by $P_{n,j}$ and $|P_{n,j}| := S(n, j)$, which are known as the Stirling set numbers or Stirling numbers of the second kind. A set partition Π can be written as $\pi_1, \pi_2, \ldots, \pi_n$, where $\pi_j = i$ if $j \in B_i$ for $1 \leq j \leq n$. The latter form is called a restricted growth function, see [1] or [8]. The number of partitions of an n element set into j blocks is in bijection with restricted growth functions of length n using j different letters. For example, $\Pi = \{\{13\}, \{2\}, \{4\}\}$ is a partition of [4] and the restricted growth function form is $\Pi = 1213$. It may already be clear that a restricted growth function has a 1 in position 1 and any entry is a positive integer at most one larger than the maximum of the entries in the positions to its left. For the convenience of the reader, here is the formal definition: A restricted growth function is defined as a word $w = w_1 w_2 \cdots w_n$ of length n over the set of positive integers where $w_1 = 1$ and for $2 \leq k \leq n$ we have $1 \leq w_k \leq \max\{w_1, w_2, \cdots, w_{k-1}\} + 1$.

Henceforth we will abbreviate restricted growth function(s) as rgf(s).

The structure of the relevant sections is as follows: In Section 3, we already defined the notion of a cell and the height of a particular cell in restricted growth functions. In Theorem 9 of Section 6 we define polynomials $F_n(x)$ of degree n in which the number of cells of height j across all rgfs with n parts is given by $[x^j]F_n(x)$. In Section 7 we obtain bivariate generating functions which track the number of cells at height s using tracker q_s across all rgfs with number of parts n tracked by x.

Throughout these sections the letters i, j, k, ℓ represent integers in the range one or larger, if unspecified.

6. A generating function for the total number of cells at each height in rgfs

In order to study the height of cells in arbitrary rgfs with n parts, we need to track the total number of occurrences of parts of size $\geq j$, where $1 \leq j \leq n$ across all such rgfs. This is equal to the total number of cells at height j. Suppose the first occurrence of the letter j is in column i. Then we have $i \leq j \leq n$. The total number of rgfs to the left of column i is counted by the Stirling set partition number $S(j-1, i-1)$. This is therefore the number of possibilities of parts left of position i; then we have a j, and after the j are $n-i$ columns. From these we choose r columns with parts less than j. This may be done in $\binom{n-i}{r}(j-1)^r$ ways. For the parts to the right of column i that have not yet been assigned sizes, we let s be the number which are equal to j. The number of choices for the assignment of these s is chosen from the available $n-i-r$ positions with $\binom{n-i-r}{s}$ possibilities. The other parts in the remaining $n-i-r-s$ positions are all $\geq j+1$ and therefore can only increase in the mode required by the definition of restricted growth functions. These may be thought of as rgfs placed on top of a rectangle with height j and width $n-i-r-s$, and therefore in bijection with rgfs with $n-i-r-s$ parts which are counted by the Bell numbers $B_{n-i-r-s}$. Collecting together the generating function for each of the parts described above, we obtain $F_{j,n}$, i.e., the total number of parts greater than or equal to j in restricted growth functions with n parts as

$$F_{j,n} = \sum_{i=j}^{n}\sum_{r=0}^{n-i} S(j-1,i-1)\binom{n-i}{r}(j-1)^r \sum_{s=0}^{n-i-r}\binom{n-i-r}{s} B_{n-i-r-s}(n-i-r+1).$$

Next, we define the polynomial $F_n(x)$ of degree n as $F_n(x) := \sum_{j=1}^{n} F_{j,n}x^j$. From this definition it follows that $[x^j]F_n(x)$ is equal to the number of cells of height j across all rgfs with n parts. Equivalently $[x^j]F_n(x)$ is equal to the number of parts of size $\geq j$ across all rgfs with n parts.

Thus we have proved the following theorem.

Theorem 9. *For each integer $n > 0$ and for $1 \leq j \leq n$, define $F_{j,n}$ as*

$$F_{j,n} := \sum_{i=j}^{n}\sum_{r=0}^{n-i} S(j-1,i-1)\binom{n-i}{r}(j-1)^r \sum_{s=0}^{n-i-r}\binom{n-i-r}{s} B_{n-i-r-s}(n-i-r+1) \tag{16}$$

and define the polynomial $F_n(x)$ as $F_n(x) := \sum_{j=1}^{n} F_{j,n}x^j$. Then the number of cells of height j across all rgfs with n parts is given by $F_{j,n} = [x^j]F_n(x)$.

For different n values, here are some examples of the polynomials obtained from Theorem 9, in which $[x^j]$ is the total number of cells at height j across all rgfs with n parts.

Example 10. $n = 2$: $F_2(x) = 4x + x^2$
$n = 3$: $F_3(x) = 15x + 6x^2 + x^3$
$\mathbf{n = 4}$: $\mathbf{F_4(x) = 60x + 30x^2 + 9x^3 + x^4}$
$n = 5$: $F_5(x) = 260x + 148x^2 + 60x^3 + 13x^4 + x^5$
$n = 6$: $F_6(x) = 1218x + 755x^2 + 368x^3 + 115x^4 + 18x^5 + x^6$.

Example 11. In the case of $n = 4$ above, Theorem 9 yields that across rgfs with 4 parts, there are 60 cells at height 1; 30 cells at height 2; 9 cells at height 3 and 1 cell at height 4. This is displayed in Fig. 4 by simply counting the number of cells at each of these heights.

7. Bivariate generating function for number of cells at each height in rgfs

As may be found in [3] or [7], a decomposition for rgfs in which the max part size is k is

$$\mathbf{1}^{+}\,(\mathbf{2}(<2)^{*})^{+}\,(\mathbf{3}(<3)^{*})^{+}\cdots(\boldsymbol{k}(<k)^{*})^{+}. \tag{17}$$

Here we have used the standard symbolic decomposition notation, where $+$ or $*$ reflect respectively non-empty or possibly empty sequences as described in the preceding bracket pair. For example, $(<2)^{*}$ means a sequence of positive integers each <2 and it may be empty. $(2(<2)^{*})^{+}$ means a sequence of integers of the form $2(<2)^{*}$ where the $+$ indicates that there is at least one 2. This is emphasized using the boldface **2**. In accordance with the definition of an rgf, it can grow from left to right by at most 1. The bold integers in Decomposition (17) show the first occurrence of this integer.

Let's assume $s \le i \le k$ and focus on the decomposition subword

$$(\mathbf{i}(<i)^{*})^{+}. \tag{18}$$

We write a generating function tracking the number of parts $\geq s$ with tracker q and number of parts with tracker x. We decompose the parts smaller than i into a sequence of parts which are either smaller than s or parts $\geq s$.

Any part smaller than s does not contribute a q to the generating function but each part larger or equal to s contributes a single q.

Therefore the generating function for this sequence of parts is

$$qx\frac{1}{1-((s-1)+q(i-(s-1)))x}. \tag{19}$$

So the generating function for the full decomposition (17) is $B_{k,s}(q,x)$ given by

$$B_{k,s}(q,x) := \prod_{i=1}^{\min\{s-1,k\}} \frac{x}{1-ix}\prod_{i=s}^{k}\frac{qx}{1-((s-1)+q(i-(s-1)))x}. \tag{20}$$

Thus we have proved our next theorem.

Theorem 12. *For rgfs with k blocks, the generating function $B_{k,s}(q_s,x)$ where q_s tracks the number of cells of height s and x tracks the total number of parts is given by*

$$B_{k,s}(q_s,x) := \prod_{i=1}^{\min\{s-1,k\}} \frac{x}{1-ix}\prod_{i=s}^{k}\frac{q_s x}{1-((s-1)+q_s(i-(s-1)))x}. \tag{21}$$

Corollary 13. *For rgfs with at most j blocks, let $B_j(q_s,x)$ be the generating function tracking the number of cells at height s with tracker q_s. Then $B_j(q_s,x) := \sum_{k=1}^{j} B_{k,s}(q_s,x)$.*

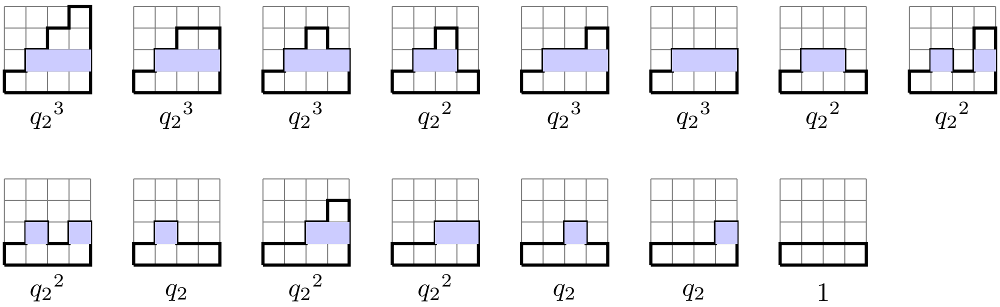

Fig. 4. All 15 rgfs with 4 parts where those cells at height 2 are labeled with the variable which tracks the number of cells at this height.

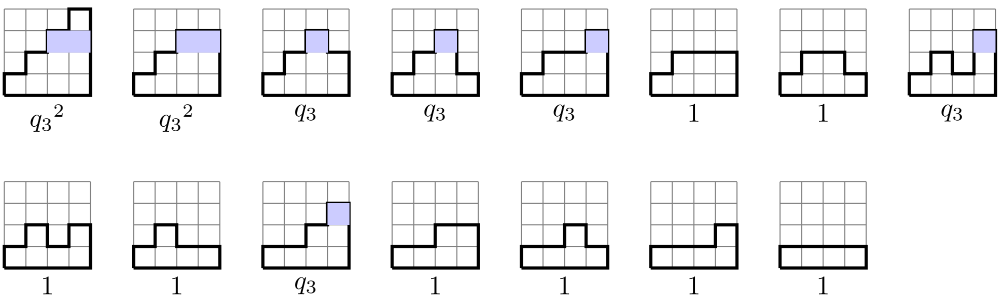

Fig. 5. All 15 rgfs with 4 parts where those cells at height 3 are labeled with the variable which tracks the number of cells at this height.

As an example we give

$$\begin{aligned} B_4(q_2, x) &:= \sum_{k=1}^{4} B_{k,s}(q_2, x) = x + (1+q_2)x^2 + \left(1 + q_2{}^2 + q_2(2+q_2)\right)x^3 \\ &+ \left(\mathbf{1} + \mathbf{3q_2} + \mathbf{6q_2{}^2} + \mathbf{5q_2{}^3}\right)\mathbf{x^4} + \left(1 + 4q_2 + 12q_2{}^2 + 20q_2{}^3 + 14q_2{}^4\right)x^5 \\ &+ O\left(x^6\right). \end{aligned} \tag{22}$$

The bold coefficients count all cases of rgfs with exactly 4 parts as illustrated in Fig. 4. These bold coefficients can be compared with the bold coefficients of Equation (1) for Catalan words. These coefficients are very similar since there is only one rgf of length 4 which is not a Catalan word. This can also be seen by comparing Fig. 4 with Fig. 1.

A second example is

$$\begin{aligned} B_4(q_3, x) &:= x + 2x^2 + (4+q_3)x^3 + \left(\mathbf{8} + \mathbf{5q_3} + \mathbf{2q_3^2}\right)\mathbf{x^4} + \left(16 + 17q_3 + 14q_3^2 + 4q_3^3\right)x^5 \\ &+ O\left(x^6\right), \end{aligned} \tag{23}$$

where the bold term is illustrated below in Fig. 5.

Next, we want to count the total number of cells at height s. So we need to differentiate the generating function $B_{k,s}(q_s,x)$ with respect to q_s and then set $q_s=1$:

$$\frac{\partial}{\partial q_s}B_{k,s}(q_s,x)=\prod_{i=1}^{\min\{s-1,k\}}\frac{x}{1-ix}\prod_{i=s}^{k}\frac{q_s x}{1-((s-1)+q_s(i-(s-1)))x}\times$$
$$\times\sum_{j=s}^{k}\frac{1+x-sx}{q_s(1-((s-1)+q_s(j-(s-1)))x)}. \tag{24}$$

Therefore we have

$$\frac{\partial}{\partial q_s}B_{k,s}(q_s,x)\big|_{q_s=1}=\prod_{i=1}^{k}\frac{x}{1-ix}\sum_{j=s}^{k}\frac{1+x-sx}{1-jx}. \tag{25}$$

It now follows that

$$\frac{\partial}{\partial q_s}B_{\ell}(q_s,x)\big|_{q_s=1}=\sum_{k=1}^{\ell}\prod_{i=1}^{k}\frac{x}{1-ix}\sum_{j=s}^{k}\frac{1+x-sx}{1-jx}. \tag{26}$$

Thus we have proved the following theorem:

Theorem 14. *For rgfs with at most $\ell\geq 1$ blocks, the generating function for the total number of cells of height s in rgfs of length n is given by*

$$\frac{\partial}{\partial q_s}B_{\ell}(q_s,x)\big|_{q_s=1}=\sum_{k=1}^{\ell}\prod_{i=1}^{k}\frac{x}{1-ix}\sum_{j=s}^{k}\frac{1+x-sx}{1-jx}. \tag{27}$$

As an example we have

Example 15.

$$\frac{\partial}{\partial q_2}B_4(q_2,x)\big|_{q_2=1}=\mathbf{x^2}+\mathbf{6x^3}+\mathbf{30x^4}+144x^5+680x^6+3168x^7+O\left(x^8\right), \tag{28}$$

where the bold coefficient of $30x^4$ counts the total number of cells at height 2 in Fig. 4. All these bold coefficients were also given in Example 10 when $n=2,3,4$ and $j=2$.

Our final example is

Example 16.

$$\frac{\partial}{\partial q_2}B_5(q_2,x)\big|_{q_2=1}=\mathbf{x^2}+\mathbf{6x^3}+\mathbf{30x^4}+\mathbf{148x^5}+750x^6+3918x^7+O\left(x^8\right), \tag{29}$$

where the bold coefficients were also counted in Example 10 when $n = 2, 3, 4, 5$ and $j = 2$.

Now we rewrite Equation (25) as

$$\frac{\partial}{\partial q_s} B_{k,s}(q_s, x)\big|_{q_s=1} = \left(\sum_{n\geq k} S(n,k)x^n \right) \sum_{j=s}^{k} \frac{1+x-sx}{1-jx}, \tag{30}$$

where $S(n,k)$ are the Stirling set numbers. Extracting the coefficient of $[x^n]$, we obtain

$$f_{k,n}(s) := S(n,k)(k-s+1) + \sum_{i=k}^{n-1} S(i,k) \sum_{j=s}^{k} j^{n-i-1}(j-s+1). \tag{31}$$

This counts the number of cells at height s across all rgfs with n parts and k blocks. So we have proved our next theorem.

Theorem 17. *The number of cells at height s across all rgfs with n parts and k blocks is given by*

$$f_{k,n}(s) = S(n,k)(k-s+1) + \sum_{i=k}^{n-1} S(i,k) \sum_{j=s}^{k} j^{n-i-1}(j-s+1). \tag{32}$$

Corollary 18. *For fixed k and for all rgfs with n parts and k blocks, the average number of cells at height s is asymptotic to*

$$\frac{k+1-s}{k} n$$

as $n \to \infty$.

Proof. Let k be fixed. By singularity analysis, (see [6]), applied to Equation (25), we obtain

$$\prod_{i=1}^{k} \frac{x}{1-ix} \sum_{j=s}^{k} \frac{1+x-sx}{1-jx} = \frac{k+1-s}{kk!(1-kx)^2} + O\left(\frac{1}{1-kx}\right). \tag{33}$$

Hence

$$[x^n] \frac{\partial}{\partial q_s} B_{k,s}(q_s, x)\big|_{q_s=1} = \frac{k+1-s}{kk!} nk^n + O\left(k^n\right). \tag{34}$$

Since $S(n,k)$ is asymptotic to $\frac{k^n}{k!}$ as $n \to \infty$, dividing the above equation by $S(n,k)$, we obtain that the average number of cells at height s as $n \to \infty$ is as stated in Corollary 18. $\square$

Similarly, rewriting (26) and extracting the coefficient of $[x^n]$, we obtain

$$\sum_{k=1}^{\ell} S(n,k)(k-s+1) + \sum_{k=1}^{\ell}\sum_{i=k}^{n-1} S(i,k)\sum_{j=s}^{k} j^{n-i-1}(j-s+1). \tag{35}$$

This counts the number of cells at height s across all rgfs with n parts and at most ℓ blocks. Taking $\ell = n$ we count the number of cells at height s across all rgfs with n parts. This is precisely $F_{s,n}$ as defined in Theorem 9. Also $F_{s,n} = \sum_{k=1}^{n} f_{k,n}(s)$.

So, we have proved the following simpler formula for $F_{s,n}$ defined in Theorem 9.

Corollary 19. *For each integer $n > 0$ and for $1 \le s \le n$, and we have*

$$F_{s,n} = \sum_{k=1}^{n} S(n,k)(k-s+1) + \sum_{k=1}^{n}\sum_{i=k}^{n-1} S(i,k)\sum_{j=s}^{k} j^{n-i-1}(j-s+1). \tag{36}$$

Returning once again to Equation (21) and summing over k we obtain

$$\sum_{k\ge 1} B_{k,s}(q_s,x) = \sum_{k\ge 1}\prod_{i=1}^{\min\{s-1,k\}} \frac{x}{1-ix}\prod_{i=s}^{k}\frac{q_s x}{1-((s-1)+q_s(i-(s-1)))x}, \tag{37}$$

which is the generating function obtained from Theorem 12, but now we allow any number of blocks. That is, Equation (37) is the bivariate generating function for rgfs with any number of blocks where q_s tracks the number of cells of height s and x tracks the number of parts.

Thus we have proved our final theorem:

Theorem 20. *The bivariate generating function for rgfs with any number of blocks where q_s tracks the number of cells of height s and x tracks the number of parts is given by*

$$\sum_{k\ge 1} B_{k,s}(q_s,x) = \sum_{k=1}^{s-1}\prod_{i=1}^{k}\frac{x}{1-ix} + \prod_{i=1}^{s-1}\frac{x}{1-ix}\sum_{k\ge s}\prod_{i=s}^{k}\frac{q_s x}{1-((s-1)+q_s(i-(s-1)))x}. \tag{38}$$

Remark 1. What the current researchers were unable to find for restricted growth functions was an analogue for Theorem 6 in the restricted growth function context. We pose this as an open question for the interested researcher as an answer may pave the way to an analogue of Corollary 5 in the restricted growth function case.

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