COMBINATORICS OF GEOMETRICALLY DISTRIBUTED RANDOM VARIABLES

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DECLARATION

I declare that this Dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

(Signature of candidate)

day of 2011 in Johannesburg.

ABSTRACT

The aim of this thesis is to study various combinatorial problems relating to geometrically distributed random variables. In particular, we study sequences of geometric random variables with respect to the left-to-right maxima of the elements of the sequence. Traditionally, left-to-right maxima or records have been studied for permutations rather than sequences. For each of the parameters explained, we compute the mean and the variance both explicitly and asymptotically.

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Chapter 1

Introduction

As an introduction to the thesis on the *Combinatorics of Geometrically Distributed Random Variables*, let us firstly define a geometric random variable. Suppose $0 \leq p \leq 1$ is fixed. A random variable X is *geometric* if it is supported by the positive integers such that $\mathbb{P}(X = i) = pq^{i-1}$ for all $i \geq 1$, where $q = 1 - p$. A word $w = w_1 w_2 \ldots$ over the alphabet of positive integers is *geometrically distributed* if the positions of w are independent and identically distributed geometric random variables.

The study of geometrically distributed words has been a recent topic of research in enumerative combinatorics. In fact, the combinatorics of n geometrically distributed independent variables $X_1 \ldots X_n$ is becoming of greater and greater significance [14], especially due to its applications in 'probabilistic counting' [3] and the 'skip list' [23] in the area of computer science. In probabilistic counting [3], we let $p = q = \frac{1}{2}$. We are then interested in the smallest natural number that does not appear as an output of any of the X_i's. The skip list [23] is a data structure for searching. To each of the n elements that are stored, there will be some pointer fields, and the number of those is chosen according to a geometric random variable. The 'horizontal search costs' of a particular element are just the number of left-to-right maxima of the truncated and reversed sequence.

We begin with Chapter 2: an explanation of how Euler's partition identities can be combinatorially interpreted; that is, Euler's partition identities can be translated in

terms of geometric random variables. Chapter 3 reviews an asymptotic technique called Rice's method. To illustrate this technique, we consider the number of winners in a discrete geometrically distributed sample - how often a maximum occurs in the sample of n geometric random variables. Chapter 4 then launches into the central topic of left-to-right maxima or records. In this chapter, we evaluate, according to various different cases, how often we meet a number that is larger than all the elements to the left. Continuing in this manner, Chapter 5 studies the sum of positions of the records, using a new method of substitution. We then investigate, in Chapter 6, the average value and the average position for the rth record. Chapter 7 considers d-records. A d-record is the number of times the variable maintaining the dth largest value changes as we go through the sample from left to right. Thereafter, we discuss left-to-right maxima, not of geometric random variables, but of compositions. Although compositions are different from geometric random variables, a composition behaves a lot like a geometric random variable with $p = \frac{1}{2}$. The final chapters analyze skip list variants and consecutive records. Chapter 9 studies the number of weak consecutive records, while Chapter 10 considers a further new parameter - the largest *consecutive* record in a random word of length n. Much of these techniques are based on the symbolic methods in the Flajolet style [4].

Chapter 2

A Combinatorial Interpretation of Euler's Partition Identities

To commence our discussion of geometric random variables, we look at a relatively simple section within Helmut Prodinger's paper *Combinatorial Problems Related to Geometrically Distributed Random Variables and Applications in Computer Science* [19]. This chapter deals with the following identities with $q < 1$:

$$\prod_{n\geq 0}(1+tq^n) = \sum_{n\geq 0}\frac{t^n q^{\binom{n}{2}}}{Q_n(q)} \tag{2.0.1}$$

and

$$\prod_{n\geq 0}\frac{1}{1-tq^n} = \sum_{n\geq 0}\frac{t^n}{Q_n(q)} \tag{2.0.2}$$

where

$$Q_n(q) = (1-q)(1-q^2)\dots(1-q^n);$$

that is, Euler's partition identities can be interpreted in terms of geometric random variables.

Now consider the probability that the geometric random variables $X_1, \dots, X_n$ are strictly increasing (it must be said that this is quite unlikely for large n).
Consider:

$$\mathbb{P}\{X_1 < \dots < X_n\} \tag{2.0.3}$$

and consider its generating function

$$M_<(z) = \sum_{n\geq 0} P\{X_1 < \cdots < X_n\} z^n \tag{2.0.4}$$

and respectively the analogous quantities

$$\mathbb{P}\{X_1 \leq \cdots \leq X_n\} \tag{2.0.5}$$

and its generating function

$$M_\leq(z) = \sum_{n\geq 0} P\{X_1 \leq \cdots \leq X_n\} z^n. \tag{2.0.6}$$

Note that the case of the probability of geometric random variables weakly increasing is quite unlikely also.

NOTE A collection of n geometric random variables is sometimes called a *sample* of n of these variables; or a *word* where the letters of the alphabet are numbers rather than letters.

Now, we can set up appropriate languages. First, we set up a symbolic equation for a geometric sample that is strictly increasing:

$$\mathcal{M}_< = (\epsilon + \mathbf{1})(\epsilon + \mathbf{2})(\epsilon + \mathbf{3}) \ldots \tag{2.0.7}$$

where ϵ represents an 'empty letter'.

So this means that the smallest letter could be no letter (the 'empty letter') or a 1; the second letter could be a 2 or just epsilon (ϵ) (the 'empty letter') again.

So for our first generating function, ϵ will correspond to a 1 in the generating function. The probability of getting a 1 is p, and we have one geometric random variable so put a z; that is, pz is the first term in the product. The probability of getting a 2 is pq and we have one geometric random variable so mark it with a z; i.e., $(1+pqz)$. Hence, we can write the generating function for strictly increasing words as

$$M_<(z) = \prod_{k\geq 1} (1 + pq^{k-1} z). \tag{2.0.8}$$

Second, the symbolic decomposition of the less-than-or-equal-to case is

$$\mathcal{M}_{\leq} = \mathbf{1}^* \cdot \mathbf{2}^* \cdot \mathbf{3}^* \dots \tag{2.0.9}$$

where the star represents a sequence of zero or more items in the set.

So the sequence of 1's is represented by $1 + pz + (pz)^2 + \dots$, where $(pz)^2$ has two 1's. Accordingly, the generating function for the sequence of 1's would be $\frac{1}{1-pz}$. The generating function for the sequence of 2's would be $\frac{1}{1-pqz}$. Hence, we have the generating function for the less-than-or-equal-to case:

$$M_{\leq}(z) = \prod_{k\geq 1} \frac{1}{1 - pq^{k-1}z}. \tag{2.0.10}$$

Equations (2.0.8) and (2.0.10) are exactly Euler's partition identities. Therefore, using identities (2.0.1) and (2.0.2) and a shift (replacing $k-1$ by a new k), we can further write

$$M_{<}(z) = \prod_{k\geq 0}(1 + pq^k z) = \sum_{n\geq 0} \frac{p^n z^n q^{\binom{n}{2}}}{Q_n(q)} \tag{2.0.11}$$

and

$$M_{\leq}(z) = \prod_{k\geq 0} \frac{1}{1 - pq^k z} = \sum_{n\geq 0} \frac{p^n z^n}{Q_n(q)}. \tag{2.0.12}$$

If now we extract the coefficients of z^n, then we get the answers we are looking for (from Euler's identities); that is, we have the explicit formulae

$$\mathbb{P}\{X_1 < \cdots < X_n\} = \frac{p^n q^{\binom{n}{2}}}{Q_n(q)}$$

and

$$\mathbb{P}\{X_1 \leq \cdots \leq X_n\} = \frac{p^n}{Q_n(q)}.$$

Considering the fact that p and q are percentages, these probabilities are extremely small. In fact, it is clear from the above equations that the probability that the geometric random variables do increase is exponentially small.

Chapter 3

The Number of Winners in a Discrete Geometrically Distributed Sample

In this chapter, we want to introduce the asymptotic technique commonly used for the analysis of alternating sums. This is called Rice's method. We will use Rice's method in many of the problems studied in this thesis. As an example of the application of Rice's method, we consider the following problem: how often does the maximum occur in a sample of n geometric random variables.

Let the random variable G be geometrically distributed; that is $\mathbb{P}\{G = k\} = pq^{k-1}$, with $q = 1-p$. Also, assume that n independent copies are given. Then, let X count the number of winners; that is, the number of random variables with highest value. We shall consider the expectation of X and the asymptotic behaviour of $\mathbb{P}\{X = 1\}$ (the probability of a single winner) for $n \to \infty$. Surprisingly, this probability does not converge as $n \to \infty$, but rather has an oscillating behaviour. These oscillations are usually small. In fact, this fluctuating behaviour in asymptotic expansions is not at all uncommon and can be found in digital sums [2], for example.

In this paper, Kirschenhofer and Prodinger [8] outline an asymptotic technique that successfully yields the Fourier expansions of the fluctuating functions. This technique is called Rice's method. To show the advantages of this technique, we will

rederive some results on the earlier mentioned problem, and thereafter obtain some results concerning higher moments of distribution.

Let $Q := \frac{1}{q}$ (so if $q < 1$, then $Q > 1$) and $L := \log Q$ (since the log of a number greater than 1 must be positive and since $\log q$ would be negative, the latter abbreviation makes sense).

Further, let p_m denote the probability $p_m = \mathbb{P}\{X = m\}$; that is, the probability to have m winners (or m largest values) amongst n players. Then we can derive a formula for p_m:

$$p_m = p^m \binom{n}{m} \sum_{j \geq 1} q^{(j-1)m} (1 - q^{j-1})^{n-m}.$$

Explanation: We say the winning value is j, n people altogether, and m are winners. We choose the m winners out of the n people in $\binom{n}{m}$ ways and we observe that m out of n people have a winning value j and the probability that they got this value is given by $p^m q^{(j-1)m}$. The remaining people, of which there are $n - m$, get a strictly smaller value. These $n - m$ people either get a 1 with probability p, a 2 with probability pq, etc; that is,

$$\begin{aligned} \sum_{i=0}^{j-2} pq^i = p + pq + \cdots + pq^{j-2} &= \frac{p(1 - q^{j-1})}{1 - q} \\ &= 1 - q^{j-1} \quad \text{since } 1 - q = p. \end{aligned}$$

Now set $N := n - m$, expand the binomial and sum over j.
i.e., $N = n - m$
$p^m \binom{n}{m}$ has not changed so consider

$$\begin{aligned} &\sum_{j \geq 1} q^{(j-1)m} (1 - q^{j-1})^{n-m} \\ &= \sum_{j \geq 1} q^{(j-1)m} (1 - q^{j-1})^{N} \\ &= \sum_{j \geq 1} q^{(j-1)m} \sum_{k=0}^{N} \binom{N}{k} (-1)^k q^{(j-1)k} \quad \text{by the binomial theorem} \\ &= \sum_{k=0}^{N} \binom{N}{k} (-1)^k \sum_{j \geq 1} \left(q^{k+m}\right)^{j-1} \quad \text{by interchanging the sums.} \end{aligned}$$

Since $\sum_{j\geq 1}\left(q^{k+m}\right)^{j-1}$ is just a geometric series, we have

$$\sum_{j\geq 1} q^{(j-1)m}(1-q^{j-1})^{n-m}$$
$$=\sum_{k=0}^{N}\binom{N}{k}(-1)^k\frac{1}{1-q^{k+m}}.$$

Now by using $Q=\frac{1}{q}$, we can rewrite the previous probability to obtain

$$p_m = p^m\binom{n}{m}\sum_{k=0}^{N}\binom{N}{k}(-1)^k\frac{1}{1-Q^{-k-m}}. \tag{3.0.1}$$

We shall now analyze this alternating sum asymptotically using the following lemma.

Lemma 3.0.1 *Let $f(z)$ be a function that is analytic on $[n_0,+\infty]$. Assume that $f(z)$ is meromorphic (function whose singularities are zero's in the denominator) in the whole of $\mathbb{C}$ and analytic on $\Omega=\bigcup_{j=1}^{\infty}\gamma_j$, where the γ_j are concentric circles whose radius tends to infinity. Let $f(z)$ be of polynomial growth on Ω. Then, for n sufficiently large [5], we have the exact equation*

$$\sum_{k=n_0}^{n}\binom{n}{k}(-1)^k f(k)=\sum_{z} Res[n;z]f(z), \tag{3.0.2}$$

where

$$[n;z]=\frac{(-1)^{n-1}n!}{z(z-1)\dots(z-n)}=\frac{\Gamma(n+1)\Gamma(-z)}{\Gamma(n+1-z)}$$

and the sum is extended to all poles not on $[n_0,+\infty]$.

Moving the contour of integration to the left, it turns out that an asymptotic equivalent of (3.0.2) can be given by the residues of $[n;z]f(z)$ at the additional poles. We can always apply this lemma to $f(z)$ because the suitable growth conditions on $f(z)$ will be fulfilled in all our examples.

By equation (3.0.1), we need to consider only the residues of $[N;z]f(z)$; that is, by equation (3.0.1), replace a k by a z

$$[N;z]\frac{1}{1-Q^{-z-m}}.$$

We must compute residues at points that are not inside the sum. In (3.0.1), the sum went from $k = 0$ up till N, therefore we must consider the residues to the left of the line $\Re z = 0$; that is, we must look for the residue when the denominator of $f(z)$ is 0.
We now solve the equation:

$$Q^{-m-z} = 1 = e^{2j\pi i}.$$
$$z = -m + \frac{2j\pi i}{L}.$$

Therefore, it is now clear that the residues are at $z = -m + \chi_j$ with $\chi_j = \frac{2j\pi i}{L}$.

Now compute the residues. Plug the value of z into $[N; z]f(z)$.

$$\begin{aligned}\underset{z=-m+\chi_j}{Res}[N; z]f(z) &= \frac{\Gamma(N+1)\Gamma(m-\chi_j)}{\Gamma(N+1+m-\chi_j)} \underset{z=-m+\chi_j}{Res} \frac{1}{1-Q^{-z-m}} \\ &= \frac{\Gamma(N+1)\Gamma(m-\chi_j)}{\Gamma(N+1+m-\chi_j)} \frac{1}{\log Q} \\ &= \frac{\Gamma(N+1)\Gamma(m-\chi_j)}{\Gamma(N+1+m-\chi_j)} \frac{1}{L}\end{aligned}$$

after computing the residue by Mathematica or by hand.

Then, after multiplying by the factor $p^m\binom{n}{m}$ and returning to n instead of N, we obtain

$$\begin{aligned}&\frac{1}{L}p^m \frac{n!}{m!(n-m)!} \frac{\Gamma(n-m+1)\Gamma(m-\chi_j)}{\Gamma(n-m+1+m-\chi_j)} \\ &= \frac{1}{L}p^m \frac{n!}{m!(n-m)!} \frac{(n-m)!\Gamma(m-\chi_j)}{\Gamma(n+1-\chi_j)}\end{aligned}$$

where $\Gamma(n+1) = n!$, hence $\Gamma(n-m+1) = (n-m)!$

Now sum over all the j's.

$$p_m = \frac{1}{L}\frac{p^m}{m!} \sum_{j\in\mathbb{Z}} \frac{\Gamma(n+1)}{\Gamma(n+1-\chi_j)} \Gamma(m-\chi_j) + \text{smaller order terms.} \tag{3.0.3}$$

Equation (3.0.3) is the asymptotic formulae for m fixed and $n \to \infty$. In a later equation, we will see that there will be no such smaller order terms.

Often, it is convenient to extract the term with index $j = 0$ (the mean), then the remaining periodic function (of mean zero) can be named $\delta_m(x)$.

Since we know that a ratio of gamma functions (as in (3.0.3)) behaves like $n^{1-1+\chi_j} = n^{\chi_j}$ as $n \to \infty$, and by

$$e^{2j\pi i \frac{\log n}{\log Q}} = n^{2j\pi i \frac{1}{\log Q}} = n^{\chi_j}.$$

Equation (3.0.3) becomes

$$\begin{aligned} &\frac{1}{L}\frac{p^m}{m} + \frac{p^m}{L}\frac{1}{m!}\sum_{j\neq 0}\Gamma(m-\chi_j)n^{\chi_j} \\ &= \frac{1}{L}\frac{p^m}{m} + \frac{p^m}{L}\frac{1}{m!}\sum_{j\neq 0}\Gamma(m-\chi_j)e^{2j\pi i\frac{\log n}{\log Q}} \end{aligned}$$

Let

$$\delta_m(x) = \frac{1}{m!}\sum_{j\neq 0}\Gamma(m-\chi_j)e^{2j\pi ix} \quad \text{where } x = \frac{\log n}{\log Q}.$$

This $\delta_m(x)$ function is a continuous, periodic function of period 1, mean zero and small amplitude (for m not too large) and hence, we can write

$$p_m \sim \frac{1}{L}\frac{p^m}{m} + \frac{p^m}{L}\delta_m(\log_Q n) + O\left(\frac{1}{n}\right) \quad m \text{ fixed}, n \to \infty. \tag{3.0.4}$$

The first term is obtained when $j = 0$ because when $j = 0$, $\chi_j = 0$ so $\Gamma(m - \chi_j)$ is just $\Gamma(m) = (m-1)!$ i.e., $\frac{1}{L}\frac{p^m}{m!}(m-1)! = \frac{1}{L}\frac{p^m}{m}$. The second term in (3.0.4) corresponds to when $j \neq 0$, so use (3.0.3) and $\delta_m(x)$ i.e., $\frac{p^m}{L}\delta_m(\log_Q n)$. Plotting a few values of m in Mathematica, with $Q = 2$, we can note that when $m = 1$, the amplitude is 0.00001; when $m = 2$, the amplitude equals 0.0000425; and when $m = 3$, the amplitude equals 0.00014.

Considering the error term, there would be more poles if we had chosen a point more to the left. We only looked at the poles at $-m+\chi_j$. If we consider poles when $z = -1$ (because the gamma function has poles at all the negative values), we would have an extra term in (3.0.4), which would be something multiplied by $\frac{1}{n}$.

So we have used Rice's method to obtain (3.0.4), which is the answer to the question: what is the probability that we have m winners amongst n players.

Interpreting the answer in (3.0.4), since the fluctuations (the δ function) are tiny,

$$p_m \approx \frac{1}{L}\frac{p^m}{m}.$$

So, in particular, the probability that you have exactly one maximum would be $p_1 \approx \frac{p}{L}$. This is a fairly large value. Letting $p = \frac{1}{2}$, we have $p_1 \approx \frac{1}{2\ln 2} \approx 0.72$. The largest number is fairly likely to only occur once when you toss a fair coin; or, approximately 72 percent of the time, only once do you get your largest number.

It is interesting to note that the alternating sum (3.0.1) can be rewritten using the partial fraction decomposition of the meromorphic function $\frac{1}{1-q^{z+m}}$.
Rewrite (3.0.1) as an infinite sum over j:

$$\frac{1}{1-q^{z+m}} = \frac{1}{2} + \frac{1}{L}\sum_{j\in\mathbb{Z}} \frac{1}{z+m+\chi_j}, \quad \text{with } \chi_j = \tfrac{2j\pi i}{L}. \tag{3.0.5}$$

Now substitute (3.0.5) into (3.0.1).

$$p_m = p^m\binom{n}{m}\sum_{k=0}^{N}\binom{N}{k}(-1)^k\frac{1}{2} + \frac{1}{L}p^m\binom{n}{m}\sum_{j\in\mathbb{Z}}\sum_{k=0}^{N}\binom{N}{k}(-1)^k\frac{1}{k+m+\chi_j}.$$

- Looking at the first term above, we know that $\binom{n}{m} = 1$ when $n = m$; and that
  $(1-1)^N = \begin{cases} 0 & \text{when } N \geq 0, \\ 1 & \text{when } N = 0; \end{cases}$
  so
  $$\sum_{k=0}^{N}\binom{N}{k}(-1)^k = \delta_{N,0}.$$
  where $\delta_{N,0}$ is 1 when $N = 0$, zero otherwise; similarly for $\delta_{N,M}$.

- Looking at the second term, by replacing x by $m+\chi_j$ for $x \in C\backslash\{-N,\ldots,0\}$, we can write
  $$\sum_{k=0}^{N}\binom{N}{k}(-1)^k\frac{1}{k+x}$$
  $$= \frac{\Gamma(N+1)\Gamma(x)}{\Gamma(N+1+x)} \quad \text{by equation 5.41 in [7]}$$

$$
\begin{aligned}
&= \frac{N!(x-1)!}{(N+x)!} \\
&= \frac{N!(x-1)(x-2)\dots 1}{(N+x)(N+x-1)\dots(x)(x-1)\dots 1} \\
&= \frac{N!}{(N+x)(N+x-1)\dots(x)} \\
&= \frac{N!}{(N+x)(N+x-1)\dots(N+x-N-1+1)} \\
&= \frac{N!}{(N+x)_{N+1}} \quad \text{where} (a)_n = (a)(a-1)\dots(a-n+1).
\end{aligned}
$$

So, after letting $N = n - m$, we can write

$$
\begin{aligned}
p_m &= \frac{p^m}{2}\delta_{n,m} + \frac{1}{L}p^m \frac{n!}{m!(n-m)!} \sum_{j\in\mathbb{Z}} \frac{(n-m)!}{(n-m+m+\chi_j)_{n-m+1}} \\
&= \frac{p^m}{2}\delta_{n,m} + \frac{1}{L}\frac{p^m}{m!} \sum_{j\in\mathbb{Z}} \frac{n!}{(n+\chi_j)(n+\chi_j-1)\dots(n+\chi_j-n+m-1+1)} \\
&= \frac{p^m}{2}\delta_{n,m} + \frac{1}{L}\frac{p^m}{m!} \sum_{j\in\mathbb{Z}} \frac{n!}{(m+\chi_j)\dots(n+\chi_j)} \\
&= \frac{p^m}{2}\delta_{n,m} + \frac{1}{L}\frac{p^m}{m!} \sum_{j\in\mathbb{Z}} \Gamma(n+1)\frac{\Gamma(m+\chi_j)}{\Gamma(n+1+\chi_j)}.
\end{aligned}
$$

This makes sense since

$$
\begin{aligned}
&\frac{\Gamma(m+\chi_j)}{\Gamma(n+1+\chi_j)} \\
&= \frac{(m+\chi_j-1)!}{(n+\chi_j)!} \\
&= \frac{(m+\chi_j-1)(m+\chi_j-2)\dots 1}{(n+\chi_j)(n+\chi_j-1)\dots 1} \\
&= \frac{1}{(m+\chi_j)\dots(n+\chi_j)}.
\end{aligned}
$$

After letting j become $-j$, we obtain

$$
p_m = \frac{p^m}{2}\delta_{n,m} + \frac{1}{L}\frac{p^m}{m!} \sum_{j\in\mathbb{Z}} \Gamma(m-\chi_j)\frac{\Gamma(n+1)}{\Gamma(n+1-\chi_j)}. \tag{3.0.6}
$$

If we compare (3.0.6) to Rice's method in (3.0.3), it is clear that the smaller order terms from (3.0.3) equal zero since (3.0.6) has no smaller terms.

Up till now, we have studied the probability of having m largest values. Now, we want to know what is the average number of largest values; that is, consider the expectation.

$$\begin{aligned}\mathbb{E}_n &= \sum_{m\geq 1} m p_m \quad \text{by definition, the average is the sum of probabilities multiplied by } m \\ &= \sum_{m\geq 1} m p^m \binom{n}{m} \sum_{j\geq 1} q^{(j-1)m}(1-q^{j-1})^{n-m}.\end{aligned}$$

Letting $j-1 \to j$ so that $j \geq 1$ changes to $j \geq 0$ and using $m\binom{n}{m} = n\binom{n-1}{m-1}$, by factorials, we have

$$\mathbb{E}_n = n \sum_{m\geq 1} \binom{n-1}{m-1} p^m \sum_{j\geq 0} q^{jm}(1-q^j)^{n-m}.$$

Replace $m-1$ by m and interchange sums to get

$$\begin{aligned}\mathbb{E}_n &= n \sum_{m\geq 0} \binom{n-1}{m} p^{m+1} \sum_{j\geq 0} q^{j(m+1)}(1-q^j)^{n-(m+1)} \\ &= \frac{np}{q} \sum_{j\geq 0} q^{j+1}(1-q^j)^{n-1} \sum_{m\geq 0} \binom{n-1}{m} \left(\frac{pq^j}{1-q^j}\right)^m.\end{aligned}$$

Now, since

$$\sum_{m\geq 0} \binom{n-1}{m} \left(\frac{pq^j}{1-q^j}\right)^m = \left(1+\frac{pq^j}{1-q^j}\right)^{n-1} = \frac{(1-q^{j+1})^{n-1}}{(1-q^j)^{n-1}},$$

we have

$$\begin{aligned}\mathbb{E}_n &= \frac{np}{q} \sum_{j\geq 0} q^{j+1}(1-q^j)^{n-1} \frac{(1-q^{j+1})^{n-1}}{(1-q^j)^{n-1}} \\ &= \frac{np}{q} \sum_{j\geq 1} q^j(1-q^j)^{n-1} \quad \text{after replacing } j+1 \text{ by } j\end{aligned}$$

$$= \frac{n(1-\frac{1}{Q})}{\frac{1}{Q}} \sum_{j\geq 1} Q^{-j} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k Q^{-jk} \quad \text{by the binomial theorem and } p = 1 - \tfrac{1}{Q}$$

$$= n(Q-1) \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \sum_{j\geq 1} Q^{-j-jk}$$

$$= n(Q-1) \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{Q^{-k-1}}{1-Q^{-(k+1)}}$$

$$= n(Q-1) \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{1}{Q^{(k+1)}-1} \quad \text{by multiplying numerator and denominator by } Q^{k+1}.$$

If we put $m = 1$, then by (3.0.1),

$$p_1 = p\binom{n}{1} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{1}{1-Q^{-k-1}} \times \frac{Q^{k+1}}{Q^{k+1}}$$

$$= pn \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{Q^{k+1}}{Q^{k+1}-1}$$

$$= \left(1 - \frac{1}{Q}\right) n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{(Q^{k+1}-1)+1}{Q^{k+1}-1}$$

$$= \frac{Q-1}{Q} n \left(\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k + \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{1}{Q^{k+1}-1} \right)$$

$$= \frac{Q-1}{Q} n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{1}{Q^{k+1}-1} \quad \text{since } \sum_{k=0}^{n-1} \tbinom{n-1}{k} (-1)^k = 0.$$

Therefore, we can rewrite $\mathbb{E}_n$ as

$$\mathbb{E}_n = Q(p_1 - p\delta_{n,1}).$$

Here $\delta_{n,1}$ equals 1 if $n = 1$ or 0 otherwise.
If $n \geq 1$, we obtain $\mathbb{E}_n = Qp_1$ ($\delta_{n,1} = 0$).
If $n = 1$, then $\mathbb{E}_n = Qp_1 - Qp\delta_{n,1}$; that is, we have an extra term i.e., we have exactly one geometric random variable and so exactly one maximum; so $Qp_1 - p = \mathbb{E}_1 = 1$ when $n = 1$.

By (3.0.4), when $m = 1$ we can then conclude the asymptotic formula

$$\mathbb{E}_n = Qp_1 \quad \text{for } n \geq 1 \tag{3.0.7}$$

$$= \frac{p}{q}\frac{1}{L}\left(1 + \delta_1(\log_q n)\right) + O\left(\frac{1}{n}\right). \tag{3.0.8}$$

Ignoring fluctuations, the average number of maxima is $\frac{p}{q}\frac{1}{L}$. For the standard case when $p, q = \frac{1}{2}$, $\mathbb{E}_n \approx \frac{1}{\ln 2} \approx 1.4$ so therefore maxima are not common. Most of the time, there is only one maximum or sometimes there are two to three maxima.

Next, compute the variance. To do so, compute the second factorial moment M_n for $n \geq 2$:

$$M_n = \sum_{m \geq 2} m(m-1)p_m.$$

The second moment is $m^2 p_m$ but because we want the second factorial moment, we have $m(m-1)p_m$.

Similar to the expectation, we substitute in (3.0.1) for p_m and use the fact that $m(m-1)\binom{n}{m} = n(n-1)\binom{n-2}{m-2}$ by factorials, to obtain

$$\begin{aligned}
M_n &= n(n-1)\sum_{m\geq 2}\binom{n-2}{m-2}p^m \sum_{j\geq 0} q^{jm}(1-q^j)^{n-m} \\
&= \frac{n(n-1)p^2}{q^2}\sum_{j\geq 1} q^{2j}(1-q^j)^{n-2} \quad \text{as before, using the binomial theorem} \\
&= \frac{n(n-1)p^2}{q^2}\sum_{k=0}^{n-2}\binom{n-2}{k}(-1)^k \frac{1}{Q^{k+2}-1} \quad \text{by the binomial theorem} \\
&= 2Q^2 p_2 - 2\frac{p^2}{q^2}\delta_{n,2} \\
&= 2Q^2(p_2 - p^2\delta_{n,2}).
\end{aligned}$$

To compute the variance, compute $\mathbb{V}_n = M_n + \mathbb{E}_n - {\mathbb{E}_n}^2$.
Hence, by using the asymptotic formula (3.0.4) to calculate p_1 and p_2 in $\mathbb{E}_n$ and M_n, we have

$$\mathbb{V}_n = \frac{p^2}{q^2}\frac{1}{L} + \frac{p}{q}\frac{1}{L} - \frac{p^2}{q^2}\frac{1}{L^2} + \tau(\log_Q n) + O\left(\frac{1}{n}\right) \tag{3.0.9}$$

where $\tau(x) = \frac{p^2}{q^2}\frac{2}{L}\delta_2(x) + \frac{p}{q}\frac{1}{L}\delta_1(x) - \frac{p^2}{q^2}\frac{1}{L^2}\delta_1^2(x)$.

We indicate another approach using generating functions and the Poisson Transform for finding the expected number of maxima.

Let the coefficient of z^k in $F_n(z)$ denote the probability that out of n players, there are k winners. Then consider the following recursion:

$$F_n(z) = \sum_{k=1}^{n} \binom{n}{k} p^{n-k} q^k F_k(z) + p^n z^n, \quad n \geq 1, \; F_0(z) = 1. \tag{3.0.10}$$

Explanation: The term $\binom{n}{k}$ is the number of ways of choosing k winners out of n players. There are q people that are winners, hence q^k. The remaining $n-k$ people are not winners with probability p each, hence p^{n-k}. The p^n in the second term refers to the case that when everybody wrote down a 1 as their geometric random variable, there are n winners.

When at a specific time the remaining players all fail, we label each by a z and leave the recursion.

The expectation E_n is found by $F_n'(1)$; that is,

$$E_n = \sum_{k=1}^{n} \binom{n}{k} p^{n-k} q^k E_k + np_n, \quad n \geq 1.$$

Define the exponential generating function to be

$$E(z) = \sum_{n \geq 0} E_n \frac{z^n}{n!}$$

and so we have

$$\begin{aligned} E(z) &= \sum_{n \geq 0} \left(\sum_{k=1}^{n} \binom{n}{k} p^{n-k} q^k E_k + np_n \right) \frac{z^n}{n!} \\ &= \sum_{n \geq 0} \sum_{k=1}^{n} \binom{n}{k} p^{n-k} q^k E_k + \sum_{n \geq 0} n \frac{(pz)^n}{n!} \\ &= e^{pz} E(qz) + pze^{pz}. \end{aligned}$$

NOTE By convolution, if $\hat{\mathrm{A}}(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$, $\hat{\mathrm{B}}(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!}$ and $\hat{\mathrm{C}}(z) = \hat{\mathrm{A}}(z) \times \hat{\mathrm{B}}(z)$, then $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$.

Now, use the Poisson Transform and multiply the function by e^{-z}; that is,

$$\begin{aligned}
\hat{\mathrm{E}}(z) &= e^{-z} E(z) \\
&= \sum_{n\geq 0} \hat{\mathrm{E}}_n \frac{z^n}{n!} \\
&= e^{-z} e^{pz} E(qz) + pze^{-z} e^{pz} \\
&= e^{-qz} E(qz) + pze^{-qz} \\
&= \hat{\mathrm{E}}(qz) + pze^{-qz} \quad \text{by definition.}
\end{aligned}$$

Extracting the coefficient of $\frac{z^n}{n!}$ (since we are dealing with **exponential** generating functions) and equating coefficients, we have

$$\begin{aligned}
\hat{\mathrm{E}}_n &= q^n \hat{\mathrm{E}}_n + p(-q)^{n-1} n \\
\hat{\mathrm{E}}_n(1-q^n) &= pn(-1)^{n-1} q^{n-1} \\
\text{so} \quad \hat{\mathrm{E}}_n &= \frac{np}{q}(-1)^{n-1} \frac{q^n}{1-q^n} \qquad \text{for} \quad n \geq 1.
\end{aligned}$$

NOTE The exponential generating function e^{-qz} has coefficient $(-q)^n$; but because of the extra z we want the coefficient of $(-q)^{n-1}$. So therefore the coefficient of pze^{-qz} is given by $p(-q)^{n-1}n$. The n variable comes from the fact that we will now have $(n-1)!$; so to obtain an $n!$, we compensate by putting a n in the numerator.

Recall that by definition $E(z) = e^z \hat{\mathrm{E}}(z)$, so then the mean is

$$E_n = \sum_{k=0}^{n} \binom{n}{k} \hat{\mathrm{E}}_k \quad \text{by convolution.}$$

Substitute in the $\hat{\mathrm{E}}_k$ formula and switch from q to Q where $q = \frac{1}{Q}$, to get

$$E_n = \sum_{k=0}^{n} \binom{n}{k} \frac{kp}{q}(-1)^{k-1} \frac{1}{Q^k - 1}.$$

Use the identity $k\binom{n}{k} = n\binom{n-1}{k-1}$ and replace $k-1$ by k to obtain

$$E_n = \frac{np}{q} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{1}{Q^{k+1}-1}.$$

This E_n is the same formula as found previously on the top of page 23!

Chapter 4

Left-to-Right Maxima

4.1 Introduction

Suppose we have the sequence 1 3 2 1 4 6 5 1 1. The maximum is 6 and there is only one maximum, as discussed in the previous chapter. Now, in this chapter, we consider the sequence of words from left-to-right; as follows: 1 is the largest when we start from the left, 3 is then larger, then 4 is larger, then 6 is larger and after there are numbers that are not larger. So 4 is the number of left-to-right maximum in this example. Following the paper by Prodinger [20], we study how many left-to-right maxima there are on average.

Let X denote a geometrically distributed random variable, i.e., $\mathbb{P}\{X = k\} = pq^{k-1}$ with $k \in \mathbb{N}$ and $q = 1 - p$. The combinatorics of n geometrically distributed independent variables $X_1, \ldots, X_n$ is becoming of greater and greater significance, especially due to its applications in 'probabilistic counting' in the area of computer science. In 'probabilistic counting', we let $p = q = \frac{1}{2}$. We are then interested in the smallest natural number that does not appear as an output of any of the X_i's.

Here is a particular result, the asymptotic formula of which we will use further in this chapter.

Theorem 4.1.1 *(Szpankowski and Rego) The expected value* $\mathbb{E}_n$ *of max* $\{X_1, \ldots, X_n\}$

is given by

$$\mathbb{E}_n = \sum_{k\geq 0}[1-(1-q^k)^n] = \log_Q n + \frac{\gamma}{L} + \frac{1}{2} - \delta(\log_Q n) + O\left(\frac{1}{n}\right) \tag{4.1.1}$$

where $Q = \frac{1}{q}$ *and* $L = \log Q$*;* γ *is Euler's constant and* $\delta(x)$ *is a periodic function of period 1 and mean 0 which is given by the Fourier series*

$$\delta(x) = \frac{1}{L}\sum_{k\neq 0}\Gamma(-\chi_k)e^{2k\pi ix}. \tag{4.1.2}$$

The complex numbers χ_k *are given by* $\chi_k = \frac{2k\pi ix}{L}$.

In this paper, Prodinger [20] focuses on the number of left-to-right maxima; that is, how often we meet a number that is larger than all the elements to the left. This is a particularly relevant parameter in the context of random permutations. The results for random permutations are as follows:

- The expectation equals the harmonic number H_n where

$$H_n = \sum_{j=1}^{n}\frac{1}{j} \approx \log n + \gamma.$$

  This is the number of left-to-right maxima on average.

- The variance equals the harmonic number minus the nth harmonic number of second order

$$H_n - H_n^{(2)}$$

  where

$$H_n^{(2)} = \sum_{j=1}^{n}\frac{1}{j^2} = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2}.$$

  So the variance is close in size to the mean; they both roughly equal $\log n$.

When defining left-to-right maxima for words, there are two cases:

- A number is a strict left-to-right maximum if it is strictly larger than the elements to the left. This is the 'strict' case or the case 'without repetitions.'

- A number is a weak left-to-right maximum if it is larger than or equal to the elements to the left. This is the 'weak' case or the case 'with repetitions.'

To demonstrate: consider the geometric random sequence 1 3 2 3 4 6 5 1 1 6.
For strict left-to-right maxima, ignore repeats. There are four records: $1, 3, 4, 6$.
For weak left-to-right maxima, there are six: $1, 3, 3, 4, 6, 6$.

Note that random permutations never have repeats, so this distinction is irrelevant for random permutations. For random permutations, left-to-right maxima are the same as right-to-left minima; but this is not the case for geometric random variables. When the variable q tends to 1, the geometric random sample behaves in the limit like a permutation. If we let q tend to 0, then p tends to 1 (every time we toss a coin, we immediately get a head), and we will get a string of all 1's in the sample. A string of all 1's has one strict left-to-right maximum (at the beginning) and n weak left-to-right maxima.

This chapter will be divided into sections. 4.2 and 4.3 deal with expectations while 4.4 and 4.5 deal with variances. In 4.6 and 4.7, we consider the uniform distribution and use the model in which the random variable X takes the values $1, \ldots, M$ each with probability $\frac{1}{M}$. Thereafter, we return to the original model of geometrically distributed variables and consider left-to-right maxima. The computations for maxima require Mellin transforms and Rice's method, amongst others, while the minima calculations are quite elementary.

4.2 Expectation in the Strict Case

Set up a certain 'language' $\mathcal{L}$ and then translate it to generating functions.
Denote the 'letters' by $\mathbf{1}, \mathbf{2}, \ldots$
Decompose all sequences $x_1, x_2, \ldots$ in a canonical way as follows: combine each left-to-right maximum $\mathbf{k}$ with the following (smaller or equal) elements. Such a part is described by

$$\mathcal{A}_k := \mathbf{k}\{\mathbf{1}, \ldots, \mathbf{k}\}^*; \tag{4.2.1}$$

that is, after k, we can have a sequence of numbers 1 up to k. Such a group may be present or not.
Then we have our desired language,

$$\mathcal{L} := (\mathcal{A}_1 + \epsilon) \cdot (\mathcal{A}_2 + \epsilon) \cdot (\mathcal{A}_3 + \epsilon) \cdot \dots \tag{4.2.2}$$

with ϵ denoting the 'empty word' (not to be confused with the letter $\mathbf{0}$).
In the concatenation in (4.2.2), the first bracket suggests a sequence that either starts with 1's (i.e., $\mathcal{A}_1$) or has no 1's (ϵ) at the beginning; similarly for the second and third brackets.

Now let each letter be marked by a z and each left-to-right maximum by a y, so that the coefficient of $z^n y^k$ is the probability that n random variables have k left-to-right maxima. Hence, we obtain the generating function

$$F(z, y) = \prod_{k \geq 1} \left(1 + \frac{yzpq^{k-1}}{1 - z(1 - q^k)} \right). \tag{4.2.3}$$

Explanation: The ϵ or 'empty word' in (4.2.2) corresponds to a 1+ every time, for every $\mathcal{A}_k$; while the $\mathcal{A}_k$ corresponds to the term $\frac{yzpq^{k-1}}{1-z(1-q^k)}$. The term pq^{k-1} is the probability for a letter $\mathbf{k}$. The set $\{\mathbf{1}, \dots, \mathbf{k}\}$ maps into $z(1 - q^k)$ while $\{\mathbf{1}, \dots, \mathbf{k}\}^*$ maps into $\frac{1}{1-z(1-q^k)}$.

If we let $y = 1$ in (4.2.3), we obtain a telescoping product i.e., $F(z, 1) = \frac{1}{1-z}$. Note that if $y = 1$, we get all the geometric random variables.

We now wish to consider the expectation:
Let $\mathbb{E}_n = [z^n] f(z)$, where $f(z) = \frac{\partial F(z,y)}{\partial y}|_{y=1}$. So differentiate (4.2.3) and let $y = 1$.
Note that

$$\partial \prod_{i=1}^{n} g_i = \left(\prod_{i=1}^{n} g_i \right) \left(\sum_{i=1}^{n} \frac{g_i'}{g_i} \right).$$

Apply this differentiation rule to (4.2.3).

$$\begin{aligned} \left. \frac{\partial F(z, y)}{\partial y} \right|_{y=1} &= \left(\frac{1}{1 - z} \right) \left(\sum_{k \geq 1} \frac{\frac{zpq^{k-1}}{1-z(1-q^k)}}{1 + \frac{zpq^{k-1}}{1-z(1-q^k)}} \right) \\ &= \left(\frac{1}{1 - z} \right) \left(\sum_{k \geq 1} \frac{\frac{zpq^{k-1}}{1-z(1-q^k)}}{\frac{1-z(1-q^k)+zpq^{k-1}}{1-z(1-q^k)}} \right) \end{aligned}$$

$$= \left(\frac{1}{1-z}\right)\left(\sum_{k\geq 1}\frac{zpq^{k-1}}{1-z+zq^k+zpq^{k-1}}\right)$$
$$= \frac{pz}{1-z}\sum_{k\geq 1}\frac{q^{k-1}}{1-z+zq^{k-1}} \quad \text{since } p+q=1;$$

that is,

$$f(z) = \frac{pz}{1-z}\sum_{k\geq 0}\frac{q^k}{1-z(1-q^k)} \quad \text{after replacing } k-1 \text{ by } k \tag{4.2.4}$$
$$= p\sum_{k\geq 0}\left[\frac{1}{1-z}-\frac{1}{1-z(1-q^k)}\right] \tag{4.2.5}$$

by partial fractions or by using Mathematica. Here $f(z)$ is the generating function for the expected value or average of left-to-right maxima. Now extract coefficients from $f(z)$. This is easy since there are only geometric series.

$$\mathbb{E}_n = [z^n]f(z) = p\sum_{k\geq 0}[1-(1-q^k)^n]. \tag{4.2.6}$$

Equation (4.2.6) is the exact formula for the average. It is almost the same as (4.1.1). The only difference is the variable p before the sum. Therefore, we can immediately state:

Theorem 4.2.1 *The average number $\mathbb{E}_n$ of left-to-right maxima (in the strict case) in the context of n independently distributed geometric random variables has the asymptotic expansion*

$$\mathbb{E}_n = p\left[\log_Q n + \frac{\gamma}{L} + \frac{1}{2} - \delta(\log_Q n) + O\left(\frac{1}{n}\right)\right] \tag{4.2.7}$$

with the periodic function $\delta(x)$ from (4.1.2).

Note that $\log_Q n = \frac{\log_e n}{\log_e Q}$. Also note that the factor of the leading term $\frac{p}{\log Q} = \frac{(q-1)}{\log Q}$ goes monotonically from 0 to 1 as q varies between 0 and 1. When $q \to 0$, $p \to 1$, and we get the case when the word is only 1's with only one left-to-right maximum. Thus the logarithmic term is 0 i.e., $\frac{p}{\log Q}$ must tend to 0 when $q \to 0$.

4.3 Expectation in the Weak Case

Recall in the weak case, every time we repeat the largest value, we count it again. Again, as in the strict case, for the weak case we set up an appropriate language $\mathcal{L}$ and then translate it to a generating function $F(z, y)$.

$$\mathcal{A}_k := \mathbf{k}\{\mathbf{1}, \ldots, \mathbf{k\text{-}1}\}^*,$$

then

$$\mathcal{L} := {\mathcal{A}_1}^* \cdot {\mathcal{A}_2}^* \cdot {\mathcal{A}_3}^* \ldots$$

This language implies that we start with some 1's ($\mathcal{A}_1$). The $\mathcal{A}_2$ implies we have a 2 followed by some 1's, but if we have a 2 again, we must start with a new sequence.

Now, convert to a generating function: we are considering a sequence of numbers therefore we have $\frac{1}{1-\square}$. For each number, there exists a z and in $\mathcal{A}_k$, the probability is pq^{k-1}. After the number k, we already have a sequence from 1 up till $k-1$. The k itself is in the numerator and $\{\mathbf{1}, \ldots, \mathbf{k\text{-}1}\}^*$ has the generating function $\frac{1}{1-z(1-q^{k-1})}$. Hence the generating function for $\mathcal{A}_k$ is $\frac{pq^{k-1}z}{1-z(1-q^{k-1})}$. Yet in this language, we have a sequence of these things so the generating function for ${\mathcal{A}_k}^*$ is $\frac{1}{1-\frac{pq^{k-1}z}{1-z(1-q^{k-1})}}$.

Therefore,

$$\begin{aligned} F(z,y) &= \prod_{k\geq 1} \frac{1}{1-\frac{yzpq^{k-1}}{1-z(1-q^{k-1})}} \\ &= \prod_{k\geq 0} \frac{1-z(1-q^k)}{1-z+zq^k(1-py)} \quad \text{after simplifying.} \end{aligned} \tag{4.3.1}$$

We want the average number. We therefore follow the method in section 4.2 and differentiate with respect to y and then let $y = 1$ with $F(z, 1) = \frac{1}{1-z}$ again; that is, apply the same differentiation rule as in the strict case:

$$\begin{aligned} f(z) = \left.\frac{\partial F(z,y)}{\partial y}\right|_{y=1} &= \frac{pz}{1-z} \sum_{k\geq 0} \frac{q^k}{1-z(1-q^{k+1})} \\ &= \frac{p}{q} \sum_{k\geq 1} \left[\frac{1}{1-z} - \frac{1}{1-z(1-q^k)}\right] \end{aligned} \tag{4.3.2}$$

by partial fractions.

Then read off the coefficients because the above is just geometric series.

$$\mathbb{E}_n = [z^n]f(z) = \frac{p}{q}\sum_{k\geq 1}[1-(1-q^k)^n]. \tag{4.3.3}$$

Equation (4.3.3) is the exact formula for the expectation. Apart from the $\frac{p}{q}$ term, (4.3.3) matches (4.1.1). Consequently, we can state the following:

Theorem 4.3.1 *The average number $\mathbb{E}_n$ of left-to-right maxima (in the weak case) in the context of n independently distributed geometric random variables has the asymptotic expansion*

$$\mathbb{E}_n = \frac{p}{q}\left[\log_Q n + \frac{\gamma}{L} - \frac{1}{2} - \delta(\log_Q n) + O\left(\frac{1}{n}\right)\right] \tag{4.3.4}$$

with $\delta(x)$ from (4.1.2).

The function $q \to \frac{p}{q}\log Q$ is monotone decreasing from infinity to 0 as q varies from 0 to 1. When $p = 1$, then $q = 0$ and we get just a row of 1's in the geometric sample. The number of weak left-to-right maxima here will just be n. Hence, since $\frac{n}{\log n} \to \infty$, the coefficient multiplied by $\log n$ must tend to ∞.

4.4 Variance in the Strict Case

We must use (4.2.3) in the following formula for the variance $\mathbb{V}_n$

$$\mathbb{V}_n = [z^n]\frac{\partial^2 F(z,y)}{\partial y^2}\bigg|_{y=1} + \mathbb{E}_n - \mathbb{E}_n^2. \tag{4.4.1}$$

Then, we must use Leibniz's formula to differentiate a product; that is,

$$\frac{\partial^2 F(z,y)}{\partial y^2}\bigg|_{y=1} = g(z) + h(z)$$

where $g(z)$ includes all terms where different factors are differentiated once, and $h(z)$ includes all terms where one factor is differentiated twice. Here $h(z) = 0$, since (4.2.3) is a linear function in terms of y so differentiated twice, we get zero. Hence, we just need to compute $g(z)$:

$$g(z) = \frac{2}{1-z} \sum_{1 \le i < j} \frac{zpq^{i-1}}{1 - z(1-q^{i-1})} \frac{zpq^{j-1}}{1 - z(1-q^{j-1})} \tag{4.4.2}$$

in which the first term is the result when you differentiate the ith term of the product once with respect to i; and the second term is the result when you differentiate the jth term with respect to j. The generating function $g(z)$ has a double sum since we are summing over i and summing over j.

Then we separate using partial fractions (in Mathematica) to obtain

$$g(z) = 2p^2 \sum_{0 \le i < j} \left[\frac{1}{1-z} + \frac{1}{q^{i-j} - 1} \frac{1}{1 - z(1-q^i)} + \frac{1}{q^{j-i} - 1} \frac{1}{1 - z(1-q^j)} \right]. \tag{4.4.3}$$

Since we know that $\frac{1}{1-z} = \sum z^n$ and $\frac{1}{1-z(1-q^i)} = \sum (1-q^i)^n z^n$ and $\frac{1}{q^{i-j}-1}$ is a constant; it does not depend on z; we have

$$g_n := [z^n] g(z) = 2p^2 \sum_{0 \le i < j} \left[1 + \frac{1}{q^{i-j} - 1} (1-q^i)^n + \frac{1}{q^{j-i} - 1} (1-q^j)^n \right]. \tag{4.4.4}$$

We must now derive asymptotic results for g_n. Decompose the double sums. This must be done with care to ensure no mistakes regarding convergence. Consider the second sum and let $h = j - i$ to eradicate the j variable. Since $j > 1$, then h starts from 1.

$$\sum_{0 \le i < j} \frac{1}{q^{i-j} - 1} (1-q^i)^n = \sum_{i \ge 0} (1-q^i)^n \sum_{h \ge 1} \frac{1}{q^{-h} - 1} = \alpha \sum_{i \ge 0} (1-q^i)^n \tag{4.4.5}$$

where

$$\alpha = \sum_{h \ge 1} \frac{1}{Q^h - 1}.$$

Now consider the first and third terms in the sum and to eradicate the i variable, let $j - i = h$. Since $0 < i < j - 1$ and $h = j - i$, h is at least one and could go up

to j.

$$\begin{aligned}
&\sum_{0\le i<j}\left[1+\frac{1}{q^{j-i}-1}(1-q^j)^n\right]\\
&=\sum_{j\ge 1}\left[j+\sum_{h=1}^{j}\frac{1}{q^h-1}(1-q^j)^n\right]\\
&=\sum_{j\ge 1}\left[j+\sum_{h=1}^{j}\frac{1}{Q^{-h}-1}(1-q^j)^n\right].
\end{aligned}$$

Consider

$$\begin{aligned}
&\sum_{h=1}^{j}\frac{1}{Q^{-h}-1}\\
&=\sum_{h=1}^{j}\frac{Q^h}{Q^h-1}\\
&=\sum_{h=1}^{j}\left(1+\frac{1}{Q^h-1}\right)\\
&=j+\sum_{h=1}^{j}\frac{1}{Q^h-1}.
\end{aligned}$$

Therefore, we now have

$$\begin{aligned}
&\sum_{j\ge 1}\left[j-\left(j+\sum_{h=1}^{j}\frac{1}{Q^h-1}\right)\left(1-q^j\right)^n\right]\\
&=\sum_{j\ge 1}j[1-(1-q^j)^n]-\sum_{j\ge 1}\sum_{h=1}^{j}\frac{1}{Q^h-1}(1-q^j)^n. \qquad (4.4.6)
\end{aligned}$$

Collect terms in (4.4.5) and (4.4.6), then

$$\frac{g_n}{2p^2}=\sum_{i\ge 0}\sum_{h\ge 1}\frac{1}{Q^h-1}(1-q^i)^n+\sum_{j\ge 1}j[1-(1-q^j)^n]-\sum_{j\ge 1}\sum_{h=1}^{j}\frac{1}{Q^h-1}(1-q^j)^n.$$

Let h be bigger than j.

$$\frac{g_n}{2p^2}=\sum_{j\ge 1}j[1-(1-q^j)^n]+\sum_{j\ge 1}\sum_{h>j}\frac{1}{Q^h-1}(1-q^j)^n. \qquad (4.4.7)$$

Now to apply Rice's method, we restate the following lemma:

Lemma 4.4.1 *Let $\mathcal{C}$ be a curve surrounding the points $1, 2, \ldots, n$ in the complex plane and let $f(z)$ be analytic inside $\mathcal{C}$. Then*

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k f(k) = -\frac{1}{2\pi i} \int_{\mathcal{C}} [n; z] f(z) dz, \tag{4.4.8}$$

where

$$[n; z] = \frac{(-1)^{n-1} n!}{z(z-1)\ldots(z-n)} = \frac{\Gamma(n+1)\Gamma(-z)}{\Gamma(n+1-z)}. \tag{4.4.9}$$

If we extend the contour of integration, we find that the asymptotic expansion of the alternating sum is

$$\sum Res\left([n; z] f(z)\right) + \text{smaller order terms}, \tag{4.4.10}$$

where the sum is over all poles z_0 different from $1, \ldots, n$.

Therefore, rewrite the terms in (4.4.7) as alternating sums:

- Firstly

  $$\sum_{j \geq 1} j[1 - (1 - q^j)^n]$$
  $$= \sum_{j \geq 1} j \left[- \sum_{k=0}^{n} \binom{n}{k} q^{jk} (-1)^k \right] \quad \text{by the binomial theorem}$$

  then, after switching sums and letting $k = 1$ since $k = 0$ term is killed, we have

  $$= - \sum_{k=1}^{n} \binom{n}{k} (-1)^k \sum_{j \geq 1} j q^{jk}.$$

  Now, since

  $$\sum_{j \geq 1} j q^{jk} = \frac{x}{(1-x)^2} \quad \text{where } x = q^k,$$

  then, after moving the negative inside,

  $$\sum_{j \geq 1} j[1 - (1 - q^j)^n] = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \varphi(z) \tag{4.4.11}$$

where

$$\varphi(z) = -\frac{Q^z}{(Q^z-1)^2}. \tag{4.4.12}$$

- Now for the other sum of (4.4.7):

$$\begin{aligned}
&\sum_{j\geq 1}\sum_{h>j}\frac{1}{Q^h-1}(1-q^j)^n \\
&= \sum_{h>j}\frac{1}{Q^h-1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k q^{jk} \quad \text{by the binomial theorem on } (1-q^j)^n \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^k \sum_{h>j\geq 1}\frac{1}{Q^h-1}Q^{-jk} \quad \text{after letting } \tfrac{1}{q}=Q \text{ and swapping sums} \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\psi(k). \qquad (4.4.13)
\end{aligned}$$

For $\psi(z)$, expand out $\sum \frac{1}{Q^h-1}$ as a geometric series and introduce a new sum which will be m i.e., Q^{-hm}, so

$$\begin{aligned}
\psi(z) &= \sum_{m\geq 1}\sum_{h>j\geq 1} Q^{-hm-jz} \quad \text{we replace the } k\text{'s by } z\text{'s by Rice's method} \\
&= \sum_{m,s,j\geq 1} Q^{-sm-jm-jz} \quad \text{letting } s=h-j.
\end{aligned}$$

Now sum Q^{sm} for the first term and sum $Q^{-j(m+z)}$ for the second term by geometric series to obtain

$$\psi(z) = \sum_{m\geq 1}\frac{1}{Q^m-1}\frac{1}{Q^{m+z}-1}. \tag{4.4.14}$$

Now compute the residues. From (4.4.11), we see that there is a triple pole at $z=0$ (one pole from the Gamma function $z\neq 0,-1,-2,\ldots$ and two poles from $\varphi(z)$ since the denominator is squared) and a double pole at $z=\frac{2k\pi i}{L}$, $k\in\mathbb{Z}$, $k\neq 0$. The computation of the residue for the first sum and then the second sum of (4.4.12) is as follows in the programme Mathematica:

p:=1 − q;

Q:=1/q;

$$\text{Residue}\left[\frac{\text{Gamma}[n+1]*\text{Gamma}[-z]}{\text{Gamma}[n+1-z]} * \frac{Q^z}{(Q^z-1)^2}, \{z, 0\}\right]$$

$$\frac{-6\text{EulerGamma}^2-\pi^2+\text{Log}\left[\frac{1}{q}\right]^2-12\text{EulerGamma}\text{PolyGamma}[0,1+n]-6\text{PolyGamma}[0,1+n]^2+6\text{PolyGamma}[1,1+n]}{12\text{Log}\left[\frac{1}{q}\right]^2}$$

$$\text{FullSimplify}[\text{PolyGamma}[0, 1 + n] - \text{HarmonicNumber}[n]]$$

$-\text{EulerGamma}$

So replace PolyGamma[0,1+n] by HarmonicNumber[n]+EulerGamma.

Now for PolyGamma[1,1+n], so

$$\text{FunctionExpand}[\text{HarmonicNumber}[n, 2] + \text{PolyGamma}[1, 1 + n]]$$

$\frac{\pi^2}{6}$

so then we have

$$\frac{1}{12L^2}\left(-6\text{EulerGamma}^2 - \pi^2 + L^2 - 12\text{EulerGamma}(\text{HarmonicNumber}[n] - \text{EulerGamma})\right.$$
$$-6\left(\text{HarmonicNumber}[n]^2 - 2\text{EulerGammaHarmonicNumber}[n] + \text{EulerGamma}^2\right)$$
$$\left.+6\left(-\text{HarmonicNumber}^{(2)}[n] + \frac{\pi^2}{6}\right)\right)$$

which simplifies to

$$\frac{1}{L^2}\left(\frac{-L^2}{12} + \frac{H_n{}^2}{2} + \frac{H_n{}^{(2)}}{2}\right).$$

Now for the residue for the second sum. Consider

$$\text{Residue}\left[\frac{\text{Gamma}[n+1]*\text{Gamma}[-z]}{\text{Gamma}[n+1-z]} * \frac{1}{Q-1} * \frac{1}{Q^{1+z}-1}, \{z, -1\}\right].$$

This residue is insignificant since

$$\frac{\text{Gamma}[n+1]*\text{Gamma}[-z]}{\text{Gamma}[n+1-z]} = n^z.$$

Therefore, the contribution of the second sum is only $O\left(\frac{1}{n}\right)$

The computations for the double poles $z = \chi_k$ are similar and are given by the periodic functions $\delta_1(x)$ and $\delta_2(x)$ below.

Replacing H_n by $\log n + \gamma$ and looking back at (4.4.4), we have

$$g_n \sim p^2 \left({\log_Q}^2 n + \frac{2\gamma}{L} \log_Q n + \frac{\gamma^2}{L^2} + \frac{\pi^2}{6L^2} - \frac{1}{6} + \log_Q n \cdot \delta_1(\log_Q n) + \delta_2(\log_Q n) \right) \tag{4.4.15}$$

with

$$\delta_1(x) = -2\delta(x) \quad \text{and} \quad \delta_2(x) = \frac{2}{L^2} \sum_{k \neq 0} \Gamma'(-\chi_k) e^{2k\pi i x}.$$

Using the formula for the variance given in (4.4.1); that is, from (4.4.15) and (4.2.7), we can now state the following:

Theorem 4.4.2 *The variance $\mathbb{V}_n$ of the number of left-to-right maxima (in the strict sense) in the context of n independently distributed geometric random variables has the asymptotic expansion for $n \to \infty$:*

$$\mathbb{V}_n = pq \log n + p^2 \left(\frac{-5}{12} + \frac{\pi^2}{6L^2} - \frac{\gamma}{L} - [\delta^2]_0 \right) + p \left(\frac{\gamma}{L} + \frac{1}{2} \right) + \delta_3(\log_Q n) + O\left(\frac{1}{n} \right). \tag{4.4.16}$$

Here $[\delta^2]_0$ is the mean of the square of $\delta^2(x)$, a very small quantity that can be neglected for numerical purposes. Furthermore, $\delta_3(x)$ is a periodic function with mean 0; its Fourier coefficients could be described if needed. It must be noted that delta is as defined in equation (4.1.2).

4.5 Variance in the Weak Case

We must use $F(z, y)$ from (4.3.1) in

$$\mathbb{V}_n = [z^n] \frac{\partial^2 F(z, y)}{\partial y^2} \bigg|_{y=1} + \mathbb{E}_n - \mathbb{E}_n^2, \tag{4.5.1}$$

then

$$\frac{\partial^2 F(z, y)}{\partial y^2} \bigg|_{y=1} = g(z) + h(z) \tag{4.5.2}$$

where $g(z)$ includes all terms where different factors are differentiated once and $h(z)$ includes all terms where one factor is differentiated twice.

The method here is very similar to other sections in this chapter, so we will use previous equations to briefly state the results.

$$g(z) = \frac{2p^2}{q^2}\frac{z^2}{1-z}\sum_{1\le i<j}\frac{q^{i+j}}{(1-z(1-q^i))(1-z(1-q^j))}. \tag{4.5.3}$$

Then we write $g(z) = g_1(z) - g_2(z)$ with

$$g_1(z) = \frac{2p^2}{q^2}\frac{z^2}{1-z}\sum_{0\le i<j}\frac{q^{i+j}}{(1-z(1-q^i))(1-z(1-q^j))} \tag{4.5.4}$$

and

$$g_2(z) = \frac{2p^2}{q^2}\frac{z^2}{1-z}\sum_{j\ge 1}\frac{q^j}{1-z(1-q^j)}. \tag{4.5.5}$$

The function $g_1(z)$ has more terms than (4.5.3) because (4.5.4) includes the extra case when $i = 0$. Then, $g_2(z)$ contains only the extra terms that are included in (4.5.4) so that $g(z) = g_1(z) - g_2(z)$.

Note that $g_1(z)$ is very similar to (4.4.3) so we may use those results; that is, use (4.4.15) and divide by q^2.

Then for $g_2(z)$, by partial fractions on (4.5.5)

$$g_2(z) = \frac{2p^2}{q^2}\sum_{j\ge 1}\left[\frac{q^j}{1-q^j} + \frac{1}{1-z} - \frac{1}{1-q^j}\frac{1}{1-z(1-q^j)}\right]. \tag{4.5.6}$$

Then

$$\begin{aligned}
[z^n]g_2(z) &= \frac{2p^2}{q^2}\sum_{j\ge 1}\left[1 - \frac{1}{1-q^j}(1-q^j)^n\right] \\
&= \frac{2p^2}{q^2}\sum_{j\ge 1}\left[1-(1-q^j)^{n-1}\right] \\
&\quad \text{by substituting } (n-1) \text{ for } n \text{ in (4.3.4), we have} \\
&= \frac{2p^2}{q^2}\left[\log_Q(n-1) + \frac{\gamma}{L} - \frac{1}{2} - \delta(\log_Q(n-1))\right] + O\left(\frac{1}{n}\right) \\
&\quad \text{and since } \log_Q(n-1) = \log_Q n + O(\tfrac{1}{n}), \text{ we obtain} \\
&= \frac{2p^2}{q^2}\left[\log_Q n + \frac{\gamma}{L} - \frac{1}{2} - \delta(\log_Q n)\right] + O\left(\frac{1}{n}\right).
\end{aligned} \tag{4.5.7}$$

Thus, adding together (4.5.7) and our previous result in (4.4.15), we have

$$g_n = \frac{p^2}{q^2}\left(\log_Q{}^2 n + \frac{2\gamma}{L}\log_Q n + \frac{\gamma^2}{L^2} + \frac{\pi^2}{6l^2} - \frac{1}{6} + \log_Q n \cdot \delta_1(\log_Q n) + \delta_2(\log_Q n)\right)$$
$$- \frac{2p^2}{q^2}\left(\log_Q n + \frac{\gamma}{L} - \frac{1}{2} - \delta(\log_Q n)\right) + O\left(\frac{1}{n}\right). \tag{4.5.8}$$

The function $h(z)$ is not a linear function of y (y is in the denominator of (4.3.1)) so after differentiating one factor of $F(z,y)$ in (4.3.1) twice, we get

$$h(z) = \frac{2p^2}{q^2}\frac{z^2}{1-z}\sum_{k\geq 1}\frac{q^{2k}}{(1-z(1-q^k))^2}.$$

We get q^k for the first derivative, so then we get q^{2k} for the second derivative. We have a z^2 and the denominator is squared because we have differentiated twice. Further, the index has been shifted from $k \geq 0$ to $k \geq 1$.

Using partial fractions, we obtain

$$h(z) = \frac{2p^2}{q^2}\sum_{k\geq 1}\left[\frac{1}{1-z} - \frac{q^k}{1-q^k}\frac{1}{(1-z(1-q^k))^2} + \frac{2q^k-1}{1-q^k}\frac{1}{1-z(1-q^k)}\right]. \tag{4.5.9}$$

After extracting coefficients and using geometric series,

$$h_n = \frac{2p^2}{q^2}\sum_{k\geq 1}\left[1 - \frac{q^k}{1-q^k}(n+1)(1-q^k)^n + \frac{2q^k-1}{1-q^k}(1-q^k)^n\right]$$
$$= \frac{2p^2}{q^2}\sum_{k\geq 1}\left[1 - (1-q^k)^n - n\frac{q^k}{1-q^k}(1-q^k)^n\right] \tag{4.5.10}$$

after simplifying.

Consider the third term: $l_n := -n\sum_{j\geq 1}\frac{q^j}{1-q^j}(1-q^j)^n$

$$= -n\sum_{j\geq 1} q^j(1-q^j)^{n-1}$$

then use the binomial theorem and swap sums:

$$= -n\sum_{k\geq 0}^{n-1}\binom{n-1}{k}(-1)^k\sum_{j\geq 1} q^{j(k+1)}$$

$$= \sum_{k=0}^{n-1} \binom{n}{k+1} (k+1)(-1)^k \frac{1}{Q^{k+1}-1} \quad \text{letting } k+1 \text{ be } k$$

$$= \sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{k}{Q^k - 1}. \tag{4.5.11}$$

Now (4.5.11) is a typical alternating sum where Rice's method would be used. By Ricc's mcthod,

$$l_n = -\frac{1}{L} - \delta_4(\log_Q n) + O\left(\frac{1}{n}\right)$$

with

$$\delta_4(x) = \frac{1}{L} \sum_{k \neq 0} \Gamma(1 - \chi_k) e^{2k\pi i x}. \tag{4.5.12}$$

The first two terms of (4.5.10) are similar to that of g_2. Consequently we use (4.5.7) in h_n.

Collecting the expansions in (4.5.12) and (4.5.7), we get

$$h_n = \frac{2p^2}{q^2} \left[\log_Q n + \frac{\gamma}{L} - \frac{1}{2} - \delta(\log_Q n) - \frac{1}{L} - \delta_4(\log_Q n) \right] + O\left(\frac{1}{n}\right). \tag{4.5.13}$$

Finally, after summing $\mathbb{V}_n = g_n + h_n + \mathbb{E}_n - {\mathbb{E}_n}^2$, we have

Theorem 4.5.1 *The variance $\mathbb{V}_n$ of the number of left-to-right maxima (in the weak case) in the context of n independently distributed geometric random variables has the asymptotic expansion for $n \to \infty$:*

$$\mathbb{V}_n = \frac{p}{q^2} \log_Q n + \frac{p^2}{q^2} \left(\frac{-5}{12} + \frac{\pi^2}{6L^2} + \frac{\gamma}{L} - \frac{2}{L} - [\delta^2]_0 \right) + \frac{p}{q} \left(\frac{\gamma}{L} - \frac{1}{2} \right) + \delta_5(\log_Q n) + O\left(\frac{1}{n}\right). \tag{4.5.14}$$

Here $[\delta^2]_0$ is the mean of the square of $\delta^2(x)$, a very small quantity that can be neglected for numerical purposes. Furthermore, $\delta_5(x)$ is a periodic function with mean 0. Again, it must be noted that delta is as defined in equation (4.1.2).

4.6 Uniform Distribution: Left-to-Right Maxima in the Strict Case

We now consider uniformly distributed random variables. We have a fixed alphabet with letters $1, \ldots, M$ and each one can occur equally often, with the same probability of $\frac{1}{M}$, hence they are **uniformly distributed**.
Thus, let for $k = 1, \ldots, M$,

$$\mathcal{A}_k := \mathbf{k}\{\mathbf{1}, \ldots, \mathbf{k}\}^*. \tag{4.6.1}$$

Then, we use the same language decomposition as before:

$$\mathcal{L} := (\mathcal{A}_1 + \epsilon) \cdot (\mathcal{A}_2 + \epsilon) \cdot (\mathcal{A}_3 + \epsilon) \cdot \cdots \cdot (\mathcal{A}_M + \epsilon). \tag{4.6.2}$$

The generating function is then

$$F(z, y) = \prod_{k=1}^{M} \left(1 + \frac{\frac{yz}{M}}{1 - \frac{zk}{M}} \right). \tag{4.6.3}$$

Explanation: From (4.6.2), the ϵ translates to a $1+$ in (4.6.3). Equation (4.6.1) implies a sequence so we use the expression of $\frac{1}{1-\square}$. Then, the letter k itself comes with probability $\frac{1}{M}$. The k must be multiplied by a z since for each number, there is a z and then this product must be divided by M because of the probability of $\frac{1}{M}$. The y variable in the numerator counts the left-to-right maxima.

Now $f(z) = \left.\frac{\partial F(z,y)}{\partial y}\right|_{y=1}$ as before and since $f(z, 1)$ is a telescoping product and equals $\frac{z}{1-z}$, we find

$$f(z) = \frac{1}{M} \frac{z}{1-z} \sum_{k=0}^{M-1} \frac{1}{1 - \frac{kz}{M}} \tag{4.6.4}$$

and by partial fractions

$$= \sum_{k=0}^{M-1} \left[\frac{1}{M-k} \frac{1}{1-z} - \frac{1}{M-k} \frac{1}{1 - \frac{kz}{M}} \right]. \tag{4.6.5}$$

We want the coefficient of z^n. So expand the geometric series in (4.6.4). Therefore,

Theorem 4.6.1 *The expected number $\mathbb{E}_n$ of left-to-right maxima (in the strict case) where each element $1, \ldots, M$ can occur with probability $\frac{1}{M}$ is for $n \to \infty$ given by, where H_M is the Mth harmonic number,*

$$\begin{aligned}
\mathbb{E}_n = [z^n]f(z) &= \sum_{k=0}^{M-1} \frac{1}{M-k}\left[1 - \left(\frac{k}{M}\right)^n\right] \\
&= \sum_{k=1}^{M} \frac{1}{k}\left[1 - \left(1 - \frac{k}{M}\right)^n\right] \quad \textit{where } M-k \textit{ is replaced by } k \\
&= \sum_{k=1}^{M} \frac{1}{k} - \sum_{k=1}^{M} \frac{1}{k}\left(1 - \frac{k}{M}\right)^n \\
&= H_M + O\left(\left(\frac{M-1}{M}\right)^n\right) \quad \textit{where } H_n = \textstyle\sum_{j=1}^{n} \frac{1}{j}.
\end{aligned}$$

The terms in $\sum_{k=1}^{M} \frac{1}{k}\left(1 - \frac{k}{M}\right)^n$ are becoming smaller and smaller so we want the error of the largest term, which is when $k = 1$; so we get $\left(\frac{M-1}{M}\right)^n$.

From the above, it is clear that it is unnecessary to determine all the terms in the partial fraction decomposition explicitly; the precise expressions for the exponentially small terms are unnecessary. We will avoid computing unnecessary expressions in calculating the variance for uniform distribution.

Using Leibniz' formula (where $h(z) = 0$ again), we get

$$\begin{aligned}
g(z) &= \left.\frac{\partial^2 F(z,y)}{\partial y^2}\right|_{y=1} \\
&= \frac{2}{M^2}\frac{z^2}{1-z} \sum_{0 \le i < j < M} \frac{1}{1 - \frac{iz}{M}} \cdot \frac{1}{1 - \frac{jz}{M}}.
\end{aligned}$$

To find the coefficient of $\frac{1}{1-z}$, keep $\frac{1}{1-z}$ and find out how everything else behaves when z tends to 1. Let $z = 1$ and then eradicate the M's in the denominator.

$$g(z) = 2 \sum_{0 \le i < j < M} \left[\frac{1}{(M-i)(M-j)}\frac{1}{1-z} + \ldots\right] \quad \text{by partial fractions.} \qquad (4.6.6)$$

NOTE Further terms in (4.6.6) would be $\frac{1}{1-\frac{iz}{M}}$ and $\frac{1}{1-\frac{jz}{M}}$, but the coefficients of these terms would be so small, so therefore we choose to omit them.

Then, since (4.6.6) is a geometric series,

$$g_n = [z^n]g(z) = 2 \sum_{0 \le i < j < M} \left[\frac{1}{(M-i)(M-j)} + \dots \right]. \tag{4.6.7}$$

Now consider

$$2 \sum_{0 \le i < j < M} \frac{1}{(M-i)(M-j)}$$

letting $k = M - i$ and $l = M - j$

$$= 2 \sum_{1 \le l < k \le M} \frac{1}{kl}$$

$$= \left(\sum_{1 \le k \le M} \frac{1}{k} \right) \left(\sum_{1 \le l \le M} \frac{1}{l} \right) - \sum_{k=l} \frac{1}{kl} \quad \text{(the case where } l \ge k \text{ is taken care of by the 2)}$$

$$= (H_M)(H_M) - \sum \frac{1}{k^2}$$

$$= {H_M}^2 - H_M^{(2)} \qquad \text{since} \quad H_M^{(2)} = \sum_{j=1}^{M} \frac{1}{j^2}. \tag{4.6.8}$$

Using (4.4.1), we have the following theorem on the variance:

Theorem 4.6.2 *The variance $\mathbb{V}_n$ of the number of left-to-right maxima (in the strict case) where each element $1, \dots, M$ can occur with probability $\frac{1}{M}$ is for $n \to \infty$ given by*

$$\begin{aligned} \mathbb{V}_n &= g_n + \mathbb{E}_n - {\mathbb{E}_n}^2 \\ &= {H_M}^2 - H_M^{(2)} + H_M - {H_M}^2 \\ &= H_M - H_M^{(2)} + O\left(\left(\frac{M-1}{M} \right)^n \right). \end{aligned} \tag{4.6.9}$$

By comparison, if we select a permutation at random and the mean number of left-to-right maxima are uniformly distributed, the variance is, in this case, $\mathbb{V}_n = H_n - H_n^{(2)}$.

4.7 Uniform Distribution: Left-to-Right Maxima in the Weak Case

In the same way as before, let, for $k = 1, \ldots, M$,

$$\mathcal{A}_k := \mathbf{k}\{\mathbf{1}, \ldots, \mathbf{k\text{-}1}\}^*,$$

then

$$\mathcal{L} := \mathcal{A}_1^* \cdot \mathcal{A}_2^* \cdot \cdots \cdot \mathcal{A}_M^*$$

and

$$F(z, y) = \prod_{k=1}^{M} \frac{1}{1 - \frac{\frac{yz}{M}}{1 - \frac{z(k-1)}{M}}} = \prod_{k=0}^{M-1} \frac{1 - \frac{zk}{M}}{1 - \frac{zk}{M} - \frac{yz}{M}}. \tag{4.7.1}$$

Then

$$\begin{aligned} f(z) &= \left.\frac{\partial F(z, y)}{\partial y}\right|_{y=1} \\ &= \frac{1}{M} \frac{z}{1-z} \sum_{k=1}^{M} \frac{1}{1 - \frac{kz}{M}} \end{aligned}$$

and using the fact that when $k = m$, we just have $\frac{1}{1-z}$, by partial fractions

$$= \frac{1}{M} \frac{z}{(1-z)^2} + \sum_{k=1}^{M-1} \left[\frac{1}{M-k} \frac{1}{1-z} + \ldots \right]. \tag{4.7.2}$$

Extract coefficients to get

$$\begin{aligned} \mathbb{E}_n &= \frac{n}{M} + \sum_{k=1}^{M-1} \frac{1}{M-k} \quad \text{since the coefficient of } \frac{z}{(1-z)^2} \text{ is just } n \\ &= \frac{n}{M} + H_{M-1} \quad \text{where } H_{M-1} \text{ is the (M-1)st harmonic number.} \end{aligned} \tag{4.7.3}$$

Consequently,

Theorem 4.7.1 *The expected number $\mathbb{E}_n$ of left-to-right maxima (in the weak case) where each element $1, \ldots, M$ can occur with probability $\frac{1}{M}$ is for $n \to \infty$ given by*

$$\mathbb{E}_n = \frac{n}{M} + H_{M-1} + O\left(\left(\frac{M-1}{M}\right)^n\right). \tag{4.7.4}$$

The term $\frac{n}{M}$ is a bigger term than that in the strict case because here, in section 4.7, we are counting repeats and as expected, the largest element M is repeated $\frac{n}{M}$ times.

For the variance, we have by Leibniz

$$\left.\frac{\partial^2 F(z,y)}{\partial y^2}\right|_{y=1} = g(z) + h(z)$$

where

$$g(z) = \frac{2}{M^2}\frac{z^2}{1-z}\sum_{1\le i<j\le M}\frac{1}{1-\frac{iz}{M}}\frac{1}{1-\frac{jz}{M}}$$

and $h(z)$ is no longer a linear function of y (y is now in the denominator of (4.7.1)) so we do not get $h(z) \neq 0$, but

$$h(z) = \frac{2}{1-z}\sum_{k=0}^{M-1}\left(\frac{\frac{z}{M}}{1-\frac{z(k+1)}{M}}\right)^2. \tag{4.7.5}$$

- Now consider $g(z) = g_1(z) + g_2(z)$ where we split off the case when $j = M$, as follows:

$$g_1(z) = \sum_{1\le i<j<M}\left[\frac{2}{(M-j)(M-i)}\frac{1}{1-z} + \dots\right]. \tag{4.7.6}$$

  Then, using the same method as for (4.6.8), we obtain

$$\begin{aligned}[z^n]g_1(z) &= \sum_{1\le i<j<M}\left[\frac{2}{(M-j)(M-i)} + \dots\right] \\ &= H_{M-1}^2 - H_{M-1}^{(2)} + \dots\end{aligned} \tag{4.7.7}$$

- Then, for the case when $j = M$: $\frac{1}{1-jz}$ is just another $\frac{1}{1-z}$ so therefore we have a $\frac{1}{(1-z)^2}$ in $g_2(z)$; that is,

$$\begin{aligned}g_2(z) &= \frac{2}{M^2}\frac{z^2}{(1-z)^2}\sum_{1\le i<M}\frac{1}{1-\frac{iz}{M}} \\ &= \sum_{1\le i<M}\left[\frac{2}{M(M-i)}\frac{1}{(1-z)^2} - \frac{2(2M-i)}{M(M-i)^2}\frac{1}{1-z} + \dots\right]\end{aligned} \tag{4.7.8}$$

by partial fractions. Then using the fact that the coefficient of $\frac{1}{(1-z)^2}$ is $(n+1)$ and that $\frac{1}{1-z}$ is just the usual geometric series, we obtain

$$[z^n]g_2(z) = \sum_{1\le i<M} \left[\frac{2}{M(M-i)}(n+1) - \frac{2(2M-i)}{M(M-i)^2} + \dots\right].$$

Consider (for the first term)

$$\begin{aligned}
&\sum_{1\le i<M} \frac{1}{M(M-i)} \\
&= \sum_{1\le i<M} \left(\frac{1}{M} + \frac{1}{M-i}\right)\left(\frac{1}{M}\right) \\
&= \frac{M-1}{M^2} + \frac{1}{M}\sum_{1\le i<M} \frac{1}{M-i}.
\end{aligned}$$

Letting $M-i=k$, the second term becomes $\frac{H_{M-1}}{M}$.

For the second term, replace $M-i$ by a new i and swap limits, so that

$$\sum_{1\le i<M} \frac{(2M-i)}{(M-i)^2} = \sum_{i=1}^{M-1} \frac{M+i}{i^2}.$$

Hence,

$$[z^n]g_2(z) = \frac{2(n+1)H_{M-1}}{M} - \frac{2}{M}\sum_{i=1}^{M-1} \frac{M+i}{i^2}.$$

Since $\sum \frac{M}{i^2}$ gives $H^{(2)}_{M-1}M$ and $\sum \frac{i}{i^2} = \sum \frac{1}{i}$ gives H_{M-1},

$$\begin{aligned}
[z^n]g_2(z) &= \frac{2(n+1)H_{M-1}}{M} - \frac{2}{M}\left(MH^{(2)}_{M-1} + H_{M-1}\right) \\
&= \frac{2nH_{M-1}}{M} - 2H^{(2)}_{M-1} \quad \text{after multiplying out.} \qquad (4.7.9)
\end{aligned}$$

Now use $[z^n]g_1(z)$ in (4.7.7) and $[z^n]g_2(z)$ in (4.7.9) to easily obtain

$$\begin{aligned}
g_n = [z^n]g(z) &= H^2_{M-1} + H^{(2)}_{M-1} + \frac{2nH_{M-1}}{M} - 2H^{(2)}_{M-1} \\
&= H^2_{M-1} - 3H^{(2)}_{M-1} + \frac{2nH_{M-1}}{M}. \qquad (4.7.10)
\end{aligned}$$

Thereafter, we must look at $h(z)$ in (4.7.5). Splitting off the case when $k = 0$ and using partial fractions, we have

$$h(z) = \frac{2z^2}{M^2(1-z)^3} + 2\sum_{k=1}^{M-1}\left[\frac{1}{(M-k)^2}\frac{1}{1-z} + \dots\right].$$

Then, using the fact that the coefficient of $\frac{z^2}{(1-z)^3}$ is $n(n-1)$, we have

$$h_n = [z^n]h(z) = \frac{n(n-1)}{M^2} + 2H^{(2)}_{M-1}, \tag{4.7.11}$$

similar to the method in (4.7.9).

As usual, we now collect terms to get the variance and the theorem that follows.

$$\begin{aligned}\mathbb{V}_n &= g_n + h_n + E_n - E_n^2 \\ &= H^2_{M-1} - 3H^{(2)}_{M-1} + \frac{2nH_{M-1}}{M} + \frac{n(n-1)}{M^2} + 2H^{(2)}_{M-1} + \frac{n}{M} + H_{M-1} - \left(\frac{n}{M} + H_{M-1}\right)^2.\end{aligned} \tag{4.7.12}$$

Theorem 4.7.2 *The variance $\mathbb{V}_n$ of the number of left-to-right maxima (in the weak case) where each element $1, \dots, M$ can occur with probability $\frac{1}{M}$ is, for $n \to \infty$, given by*

$$\mathbb{V}_n = n\frac{M-1}{M^2} + H_{M-1} - H^{(2)}_{M-1} + O\left(\left(\frac{M-1}{M}\right)^n\right) \quad \textit{where } H_{M-1} \textit{ is the (M-1)st harmonic number.} \tag{4.7.13}$$

4.8 Left-to-Right Minima in the Strict Case

We return to geometric random variables and discuss left-to-right minima.
As in section 4.2, let

$$\mathcal{A}_k := \mathbf{k}\{\mathbf{k}, \mathbf{k{+}1}, \mathbf{k{+}2}, \dots\}^*,$$

then

$$\mathcal{L} := \dots (A_3 + \epsilon) \cdot (A_2 + \epsilon) \cdot (A_1 + \epsilon). \tag{4.8.1}$$

In $\mathcal{A}_k$, instead of a large number followed by smaller ones (as for maxima), we have here a small number followed by larger ones. We do not include k again because we are dealing with strict minima at the moment. For our language, as expected, the product goes in the reverse order to the language for maxima.
Then,

$$F(z,y) = \prod_{k\geq 1}\left(1 + \frac{yzpq^{k-1}}{1-zq^{k-1}}\right) = \prod_{k\geq 0}\frac{1-zq^k(1-py)}{1-zq^k}. \tag{4.8.2}$$

The denominator is what differs from the generating function for maxima in section 4.2.

Therefore,

$$\begin{aligned} f(z) = \left.\frac{\partial F(z,y)}{\partial y}\right|_{y=1} &= \frac{1}{1-z}\sum_{k\geq 0}\frac{zpq^k}{1-zq^{k+1}} \quad \text{since } f(z,1) = \tfrac{1}{1-z} \\ &= \frac{p}{q}\sum_{k\geq 1}\frac{q^k}{1-q^k}\left[\frac{1}{1-z} - \frac{1}{1-zq^k}\right] \end{aligned} \tag{4.8.3}$$

by partial fractions.

Consequently, the coefficients of the above geometric series are

$$\mathbb{E}_n = [z^n]f(z) = \frac{p}{q}\sum_{k\geq 1}\frac{1}{Q^k-1}[1-Q^{-kn}]. \tag{4.8.4}$$

The second part of the sum only produces an exponentially small contribution, so therefore, we are only looking at $\frac{p}{q}\sum_{k\geq 1}\frac{1}{Q^k-1}$ where $\sum_{k\geq 1}\frac{1}{Q^k-1}$ is called α, as it was earlier in (4.4.5). Hence the following theorem:

Theorem 4.8.1 *The average number $\mathbb{E}_n$ of left-to-right minima (in the strict case) is, for $n \to \infty$, given by*

$$\mathbb{E}_n = \frac{p}{q}\alpha + O\left(Q^{-n}\right) \tag{4.8.5}$$

where

$$\alpha = \alpha_Q = \sum_{k\geq 1}\frac{1}{Q^k-1}.$$

It is evident that calculating the average number of left-to-right minima is an easier problem than that of maxima. We did not even need to use Rice's method in the above.

Now, consider the variance. Consider

$$g(z) = \left.\frac{\partial^2 F(z,y)}{\partial y^2}\right|_{y=1} = \frac{2p^2}{q^2}\frac{z^2}{1-z}\sum_{1\le i<j}\frac{q^{i+j}}{(1-zq^i)(1-zq^j)} \tag{4.8.6}$$

like (4.5.3), by differentiating using Mathematica.

Then, we confine ourselves to the main term $\frac{1}{1-z}$ in the following partial fraction decomposition, since we are not interested in the explicit computation of exponentially small terms.

By substituting in $z = 1$ everywhere other than $\frac{1}{1-z}$ and using the variable Q ($\log q$ would be a negative number, therefore we must use capital Q), we obtain the following partial fraction decomposition

$$g(z) = \frac{2p^2}{q^2}\sum_{1\le i<j}\left[\frac{1}{(Q^i-1)(Q^j-1)}\frac{1}{1-z} + \dots\right]. \tag{4.8.7}$$

Thus,

$$g_n = [z^n]g(z) = \frac{2p^2}{q^2}\sum_{1\le i<j}\left[\frac{1}{(Q^i-1)(Q^j-1)} + \dots\right]. \tag{4.8.8}$$

Consider

$$\begin{aligned}
&\sum_{1\le i<j}\frac{1}{(Q^i-1)(Q^j-1)}\\
&= \frac{1}{2}\left[\left(\sum\frac{1}{Q^i-1}\right)\left(\sum\frac{1}{Q^j-1}\right) - \sum_{i=j}\frac{1}{(Q^i-1)(Q^j-1)}\right]\\
&= \frac{1}{2}\left[\alpha^2-\beta\right]
\end{aligned}$$

where

$$\beta = \beta_Q = \sum_{k\ge 1}\frac{1}{(Q^k-1)^2}.$$

In $\alpha^2 - \beta$, we have included the case where $i > j$, but we only need the case where $i < j$ so therefore half it; so we have $\frac{\alpha^2-\beta}{2}$.

$$g_n = \frac{2p^2}{q^2}\frac{\alpha^2 - \beta}{2}$$

and

$$\begin{aligned}\mathbb{V}_n &= g_n + \mathbb{E}_n - \mathbb{E}_n^2 \\ &= \left(\frac{p}{q}\right)^2 (\alpha^2 - \beta) + \frac{p}{q}\alpha - \left(\frac{p}{q}\alpha\right)^2 \\ &= \frac{p}{q}\alpha - \left(\frac{p}{q}\right)^2 \beta.\end{aligned}$$

Therefore, we have the following theorem:

Theorem 4.8.2 *The variance $\mathbb{V}_n$ of left-to-right minima (in the strict case) is, for $n \to \infty$,*

$$\mathbb{V}_n = \frac{p}{q}\alpha - \left(\frac{p}{q}\right)^2 \beta + O\left(Q^{-n}\right). \tag{4.8.9}$$

where

$$\alpha = \alpha_Q = \sum_{k\geq 1} \frac{1}{Q^k - 1}$$

and

$$\beta = \beta_Q = \sum_{k\geq 1} \frac{1}{(Q^k - 1)^2}.$$

4.9 Left-to-Right Minima in the Weak Case

Let

$$\mathcal{A}_k := \mathbf{k}\{\mathbf{k+1}, \mathbf{k+2}, \dots\}^*,$$

then

$$\mathcal{L} := \dots \mathcal{A}_3{}^* \cdot \mathcal{A}_2{}^* \cdot \mathcal{A}_1{}^*$$

and since the $\mathcal{A}_k$'s are now a sequence, we have an extra $\frac{1}{1-\square}$; that is,

$$F(z,y) = \prod_{k\geq 1} \frac{1}{1-\frac{yzpq^{k-1}}{1-zq^k}} = \prod_{k\geq 1} \frac{1-zq^k}{1-zq^{k-1}(py+q)}. \tag{4.9.1}$$

Then

$$\begin{aligned} f(z) = \left.\frac{\partial F(z,y)}{\partial y}\right|_{y=1} &= \frac{1}{1-z}\sum_{k\geq 1}\frac{zpq^{k-1}}{1-zq^{k-1}} \\ &= p\frac{z}{1-z}\sum_{k\geq 0}\frac{q^k}{1-zq^k} \quad \text{replacing } k-1 \text{ by } k \\ &= p\frac{z}{(1-z)^2} + p\sum_{k\geq 1}\left[\frac{1}{Q^k-1}\cdot\frac{1}{1-z}+\dots\right]. \end{aligned} \tag{4.9.2}$$

Therefore,

$$\mathbb{E}_n = [z^n]f(z) = pn + p\alpha + \dots \tag{4.9.3}$$

Theorem 4.9.1 *The average number $\mathbb{E}_n$ of left-to-right minima (in the weak case) is, for $n \to \infty$, given by*

$$\mathbb{E}_n = p(n+\alpha) + O\left(Q^{-n}\right) \tag{4.9.4}$$

where

$$\alpha = \alpha_Q = \sum_{k\geq 1}\frac{1}{Q^k-1}.$$

Now for the variance:

$$\left.\frac{\partial^2 F(z,y)}{\partial y^2}\right|_{y=1} = g(z) + h(z)$$

where

$$\begin{aligned} g(z) &= 2p^2\frac{z^2}{1-z}\sum_{0\leq i<j}\frac{q^{i+j}}{(1-zq^i)(1-zq^j)} \\ &= 2p^2\frac{z^2}{(1-z)^2}\sum_{j\geq 1}\frac{q^j}{1-zq^j} + 2p^2\frac{z^2}{1-z}\sum_{1\leq i<j}\frac{q^{i+j}}{(1-zq^i)(1-zq^j)} \end{aligned}$$

(where the first sum is the case when $i=0$ and the second when $i\geq 1$)

$$= 2p^2\sum_{j\geq 1}\left[\frac{1}{Q^j-1}\frac{1}{(1-z)^2} - \left(\frac{2}{Q^j-1}-\left(\frac{1}{Q^j-1}\right)^2\right)\frac{1}{1-z}+\dots\right]$$

$$+ 2p^2 \sum_{1\le i<j} \left[\frac{1}{(Q^i-1)(Q^j-1)} \frac{1}{1-z} + \dots \right] \tag{4.9.5}$$

by partial fractions again,

and

$$\begin{aligned} h(z) &= 2p^2 \frac{z^2}{1-z} \sum_{k\ge 0} \frac{q^{2k}}{(1-zq^k)^2} \qquad i=j=k \text{ in } h(z) \\ &= 2p^2 \frac{z^2}{(1-z)^3} + 2p^2 \sum_{k\ge 1} \left[\frac{1}{(Q^k-1)^2} \cdot \frac{1}{1-z} + \dots \right]. \end{aligned} \tag{4.9.6}$$

Therefore, after extracting coefficients,

$$\begin{aligned} g_n = [z^n]g(z) &= 2p^2 \sum_{j\ge 1} \frac{1}{Q^j-1}(n+1) - \frac{2}{Q^j-1} - \left(\frac{1}{Q^j-1}\right)^2 \\ &= +2p^2 \sum_{1\le i<j} \frac{1}{(Q^i-1)(Q^j-1)} + 2p^2[(n-1)\alpha - \beta] + 2p^2 \frac{\alpha^2-\beta}{2} + O\left(Q^{-n}\right) \end{aligned} \tag{4.9.7}$$

and

$$h_n = [z^n]g(z) = 2p^2 \binom{n}{2} + 2p^2\beta + O\left(Q^{-n}\right). \tag{4.9.8}$$

Hcncc,

Theorem 4.9.2 *The variance $\mathbb{V}_n$ of left-to-right minima (in the weak case) is, for $n \to \infty$, given by*

$$\begin{aligned} \mathbb{V}_n &= g_n + h_n + \mathbb{E}_n - \mathbb{E}_n^2 \\ &= p^2[2(n-1)\alpha - 2\beta + \alpha^2 - \beta + n(n-1) + 2\beta] + p[n+\alpha] - p^2[n+\alpha]^2 \end{aligned} \tag{4.9.9}$$

$$= pqn + p\alpha - p^2(2\alpha+\beta) + O\left(Q^{-n}\right) \tag{4.9.10}$$

after simplifying, where

$$\alpha = \alpha_Q = \sum_{k\ge 1} \frac{1}{Q^k-1}$$

and

$$\beta = \beta_Q = \sum_{k\ge 1} \frac{1}{(Q^k-1)^2}.$$

Chapter 5

Records in geometrically distributed words: sum of positions

A 'record' could be defined as the largest value so far in a sample of geometric random variables. Looking retrospectively then, from Chapter 3 the 'number of winners' could be considered the 'overall record.' In Chapter 4, we looked at left-to-right maxima or records. In this chapter, Chapter 5, we look again at Prodinger's study [22] of left-to-right maxima or records but we use a different parameter to that in Chapter 4: we are going to study here the sum of the positions of the maxima and use a method of substitution.

The number of records was first studied by Rényi [24]. Recently, Myers and Wilf extended the study of records to multiset permutations and words. Oliver and Prodinger [18] discussed set partition words: a word where all the numbers from one up to the maximum is a record (see also [15]); while Knopfmacher, Mansour, and Wagner [10] explored records in set partitions. In the context of random permutations, this problem was introduced by Kortchemski [12]. Prodinger then extended this idea to geometric words. We summarize Kortchemski's results for random permutations [12]:
An element x_i in $x_1, \ldots, x_n$ is a record if it is larger than $x_1, \ldots, x_{i-1}$. The position

of the record is then i. For example, 5 8 6 2 7 9 3 4 has the records 5, 8, and 9 in positions 1, 2, 6 and so the new parameter is $1 + 2 + 6 = 9$.

The probability generating function for random permutations is then

$$\prod_{k=1}^{n} \frac{z^k + k - 1}{k}$$

and the expectation is then

$$\sum_{k=1}^{n} \frac{d}{dz} \frac{z^k + k - 1}{k}\bigg|_{z=1} = \sum_{k=1}^{n} \frac{k}{k} = n \tag{5.0.1}$$

where the z variable indicates the sum of position of the records.

The study of records can be applied to observations of extreme weather problems, tests of randomness, determination of minimal failure, and stresses of electric components [6].

We will consider this parameter for words over the alphabet $\{1, 2, 3, \dots\}$, with geometric probabilities p, pq, $pq^2, \dots$ with $p + q = 1$. We will compute both explicitly and asymptotically, only the expectation, since higher moments require huge computations.

Again, as in Chapter 4, we distinguish between strong records (must be strictly larger than elements to the left) and weak records (must be only larger than or equal to elements to the left). This model may be considered to be a q-version; the model of permutations is approached for $q \to 1$. This holds true, because the positions and their sums are independent of the value.

We will find a functional relation for the bivariate generating function $F(z, u)$ where the coefficient of $z^n u^k$ is the probability that a random word of length n has the sum of the positions of the records (denoted **srec**) equal to k.

We abbreviate

$$[\![k]\!] := 1 - z(1 - q^k). \tag{5.0.2}$$

5.1 Strong Records

As stated above, the coefficient of $z^n u^l$ is the probability that a random word of length n has the srec parameter equal to l. This means that this word has a record k in its last letter. The recursion is then given by

$$f_k(z,u) = \sum_{1\le j<k} f_j(zu,u)\frac{zupq^{k-1}}{1-(1-q^j)zu} + zupq^{k-1}. \tag{5.1.1}$$

Explanation: In the first term of the recursion, j must be between 1 and k. The previous record was j, and we have a sequence of letters $\le j$, followed by the last letter k. The numerator takes care of the k. Since we had an arbitrary number of letters $\le j$, we have a sequence of letters so therefore use $\frac{1}{1-\square}$. The factor $(1-q^j)$ occurs because the letter was either $1, 2$, up to letter j. The variable z occurs because each letter in the sequence must be denoted by a z. The variable u occurs because each one of the intermediate letters adds one to the sum of the positions. Furthermore, all z's must be replaced by zu in $f_j(zu,u)$ to keep track of the sum of the positions of the records. The last term is the case in which there were no previous records. The last letter has record k, but there were no previous records, so we have a one letter word. Because the sum of positions equals 1, we have a single u and the probability is pq^{k-1}.

The generating function is then

$$F(z,u) = \sum_{k\ge 1} f_k(z,u)\frac{1}{[\![k]\!]}. \tag{5.1.2}$$

This follows since any k can be the last record, and thereafter, only letters $\le k$ can occur.

Note that

$$f_k(z,1) = \frac{zpq^{k-1}}{[\![k-1]\!]}. \tag{5.1.3}$$

In (5.1.3), we replaced u by 1, hence we are counting words that have k as their last letter.

Also, let

$$a_k(z) = \frac{\partial}{\partial u} f_k(z,u)\Big|_{u=1}. \tag{5.1.4}$$

Divide through the recursion by the numerator to obtain

$$\frac{f_k(z,u)}{zupq^{k-1}} = \sum_{1\le j<k} \frac{f_j(zu,u)}{1-(1-q^j)zu} + 1. \tag{5.1.5}$$

Replace k by $k-1$ and subtract the $k-1$ recursion from the k recursion. After subtracting, the only case left is when $j = k-1$.

$$\frac{f_k(z,u)}{zpq^{k-1}} - \frac{f_{k-1}(z,u)}{zpq^{k-2}} = \frac{f_{k-1}(zu,u)u}{1-(1-q^{k-1})zu}. \tag{5.1.6}$$

Using the product rule, differentiate (5.1.6) with respect to u, let $u = 1$ and use (5.1.4). When we differentiate u on its own, we obtain on the RHS

$$\frac{f_{k-1}(z,1)}{[\![k-1]\!]}.$$

When we differentiate the denominator, we obtain

$$\frac{f_{k-1}(z,1)(1-q^{k-1})z}{[\![k-1]\!]^2}.$$

Now, to differentiate $f_{k-1}(zu,u)$:
letting $u = 1$ gives $[\![k-1]\!]$ in the denominator, then use the chain rule:

$$\frac{\partial}{\partial u} f_{k-1}(zu,u) = \frac{\partial(zu)}{\partial u} \underbrace{\frac{\partial}{\partial(zu)} f_{k-1}(zu,u)}_{} \frac{\partial}{\partial u} f_{k-1}(zu,u)$$

$u = 1$ so we are really differentiating with respect to z to get

$$= z\frac{d}{dz} f_{k-1}(z,1) a_{k-1}(z).$$

Collecting terms after differentiating, we have

$$\frac{a_k(z)}{zpq^{k-1}} - \frac{a_{k-1}(z)}{zpq^{k-2}} = \frac{f_{k-1}(z,1)}{[\![k-1]\!]} + \frac{a_{k-1}(z)}{[\![k-1]\!]} + \frac{z}{[\![k-1]\!]}\frac{d}{dz} f_{k-1}(z,1) + \frac{f_{k-1}(z,1)(1-q^{k-1})z}{[\![k-1]\!]^2}. \tag{5.1.7}$$

We must now simplify and substitute in for $f_k(z,1)$ from (5.1.3).

$$\frac{a_k(z)}{zpq^{k-1}} - \frac{a_{k-1}(z)}{zpq^{k-2}}$$

$$= \frac{zpq^{k-2}}{[\![k-1]\!][\![k-2]\!]} + \frac{a_{k-1}(z)}{[\![k-1]\!]} + \frac{z}{[\![k-1]\!]} \underbrace{\left(\frac{pq^{k-2}}{[\![k-2]\!]} + \frac{(1-q^{k-2})(zpq^{k-2})}{[\![k-2]\!]^2} \right)}_{\text{by the chain rule}} + \frac{zpq^{k-2}(1-q^{k-1})z}{[\![k-2]\!][\![k-1]\!]^2}.$$

Now, multiply LHS by $[\![k-1]\!]$, so

$$\begin{aligned} \text{LHS} &= \frac{a_k(z)[\![k-1]\!]}{zpq^{k-1}} - \frac{a_{k-1}(z)}{zpq^{k-2}} \left([\![k-1]\!] + zpq^{k-2} \right) \\ &= \frac{a_k(z)[\![k-1]\!]}{zpq^{k-1}} - \frac{a_{k-1}(z)[\![k-2]\!]}{zpq^{k-2}} \end{aligned}$$

because

$$\begin{aligned} 1 - z(1-q^{k-1}) + zpq^{k-2} &= 1 - z + zq^{k-1} + zpq^{k-2} \\ &= 1 - z + zq^{k-2}(q+p) \\ &= 1 - z + zq^{k-2} \quad \text{since } p+q=1 \\ &= 1 - z(1-q^{k-2}) \\ &= [\![k-2]\!]. \end{aligned}$$

For the RHS:

$$\text{RHS} = \frac{zpq^{k-2}}{[\![k-1]\!][\![k-2]\!]} + \frac{zpq^{k-2}}{[\![k-2]\!]^2} \left[\frac{(1-q^{k-2})z}{[\![k-1]\!]} + \frac{[\![k-2]\!]}{[\![k-1]\!]} + \frac{(1-q^{k-1})z[\![k-2]\!]}{[\![k-1]\!]^2} \right].$$

Multiply through by $[\![k-1]\!]$.

$$\text{RHS} = zpq^{k-2} \left[\underbrace{\frac{1}{[\![k-2]\!]} + \frac{1}{[\![k-2]\!]^2} \left((1-q^{k-2})z \right) + \frac{1}{[\![k-2]\!]}}_{} + \underbrace{\frac{(1-q^{k-1})z}{[\![k-2]\!][\![k-1]\!]}} \right]$$

then put the first three terms together and split up the last term into two pieces:

$$\begin{aligned} &= zpq^{k-2} \left[\frac{2 - 2z(1-q^{k-2}) + z(1-q^{k-2})}{[\![k-2]\!]^2} - \frac{1}{[\![k-2]\!]} + \frac{1}{[\![k-1]\!][\![k-2]\!]} \right] \\ &= zpq^{k-2} \left[\frac{1}{[\![k-1]\!][\![k-2]\!]} + \frac{1}{[\![k-2]\!]^2} \right]. \end{aligned}$$

$$\text{i.e.,} \quad \frac{a_k(z)[\![k-1]\!]}{zpq^{k-1}} - \frac{a_{k-1}(z)[\![k-2]\!]}{zpq^{k-2}} = \frac{zpq^{k-2}}{[\![k-1]\!][\![k-2]\!]} + \frac{zpq^{k-2}}{[\![k-2]\!]^2}. \tag{5.1.8}$$

Sum these relations in (5.1.8) from 2 up to k. The RHS is then straightforward. The LHS telescopes and equals

$$\frac{a_k(z)[\![k-1]\!]}{zpq^{k-1}} - 1.$$

When $k = 2$, a_{k-1} becomes a_1. Then we must compute

$$a_1(z) = \frac{\partial}{\partial u} f_1(z,u)\Big|_{u=1}.$$

$f_1(z,u)$ is the generating function where the record has value 1 so the whole word must be ones (1 can only be a record if there is only one 1). Each one is counted with probability p. We must have a z for each letter in the word. The word is of random length n and n must be ≥ 1 because we don't want an empty word. Therefore,

$$\begin{aligned} f_1(z,u) &= \sum_{n\geq 1} (pz)^n \\ &= upz. \end{aligned}$$

Then

$$\frac{\partial}{\partial u} f_1(z,u) = pz = a_1(z).$$

Then multiplying $\frac{a_1(z)[\![k-2]\!]}{zpq^{k-2}}$ when $k = 2$ gives

$$\frac{a_1(z)[\![0]\!]}{zpq^0} = pz(1)\frac{1}{zp} = 1.$$

$$\text{i.e.,} \quad \frac{a_k(z)[\![k-1]\!]}{zpq^{k-1}} - 1 = \sum_{j=2}^{k} \frac{zpq^{j-2}}{[\![j-1]\!][\![j-2]\!]} + \sum_{j=2}^{k} \frac{zpq^{j-2}}{[\![j-2]\!]^2}. \tag{5.1.9}$$

Multiplying through by $\frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]}$ and taking the second term on the LHS across the equals sign to the RHS gives

$$\frac{a_k(z)}{[\![k]\!]} = \frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]} + \frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]} \sum_{j=1}^{k-1} \frac{zpq^{j-1}}{[\![j]\!][\![j-1]\!]} + \frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]} \sum_{j=2}^{k} \frac{zpq^{j-2}}{[\![j-2]\!]^2}. \tag{5.1.10}$$

Note that we have also used a shift from $j = 2$ to $j = 1$ on the third term.

Then, using (5.0.2), we can check that

$$\frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]} = \frac{1}{[\![k]\!]} - \frac{1}{[\![k-1]\!]}. \tag{5.1.11}$$

So then,

$$\sum_{j=1}^{k-1} \frac{zpq^{j-1}}{[\![j]\!][\![j-1]\!]} = \sum_{j=1}^{k-1} \left(\frac{1}{[\![j]\!]} - \frac{1}{[\![j-1]\!]} \right) \quad \text{by (5.1.11).}$$

Once again, this is a telescoping sum i.e., it telescopes to

$$\frac{1}{[\![k-1]\!]} - 1.$$

So, using (5.1.11) and the above telescoping sum, we obtain

$$\frac{a_k(z)}{[\![k]\!]} = \frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]} + \frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]} \left[\frac{1}{[\![k-1]\!]} - 1 \right] + \frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]} \sum_{j=2}^{k} \frac{zpq^{j-2}}{[\![j-2]\!]^2}. \tag{5.1.12}$$

Now, we wish to compute the generating functions of the expectations, that is compute $G(z)$. To find $G(z)$, we must differentiate $F(z,u)$ with respect to u, then substitute in $u = 1$.

$$\begin{aligned} \text{i.e.,} \quad G(z) &= \sum_{k\geq 1} \frac{a_k(z)}{[\![k]\!]} \quad \text{by definition of } F(z,u) \\ &= \sum_{k\geq 1} \frac{zpq^{k-1}}{[\![k-1]\!]^2[\![k]\!]} + \sum_{1\leq j<k} \left[\frac{1}{[\![k]\!]} - \frac{1}{[\![k-1]\!]} \right] \frac{zpq^{j-1}}{[\![j-1]\!]^2} \end{aligned}$$

by substituting in (5.1.12) and then combining the first two terms to get the first term and then applying (5.1.11) to the second term of (5.1.12) to get the second term above.

If we work out the double sum, we get that

$$\sum_{j+1}^{\infty} \left[\frac{1}{[\![k]\!]} - \frac{1}{[\![k-1]\!]} \right] = \frac{1}{1-z} - \frac{1}{[\![j]\!]}.$$

The term $j+1$ is the minimum term of telescoping and when k goes to ∞, we get $\frac{1}{1-z}$; that is,

$$G(z) = \sum_{k\geq 1} \frac{zpq^{k-1}}{[\![k-1]\!]^2[\![k]\!]} + \sum_{1\leq j} \left[\frac{1}{1-z} - \frac{1}{[\![j]\!]} \right] \frac{zpq^{j-1}}{[\![j-1]\!]^2}.$$

Then after separating the second sum into two sums, $G(z)$ is

$$G(z) = \sum_{k\geq 1} \frac{zpq^{k-1}}{[\![k-1]\!]^2[\![k]\!]} + \frac{1}{1-z}\sum_{1\leq j} \frac{zpq^{j-1}}{[\![j-1]\!]^2} - \sum_{1\leq j} \frac{zpq^{j-1}}{[\![j-1]\!]^2[\![j]\!]}$$

and since the first and third sum above cancel, we obtain

$$= \frac{1}{1-z}\sum_{k\geq 1} \frac{zpq^{k-1}}{[\![k-1]\!]^2}. \tag{5.1.13}$$

Substitute in for $z = \frac{w}{w-1}$, then

$$\begin{aligned} G(z) &= \frac{1}{1-\frac{w}{w-1}} \sum_{k\geq 1} \frac{\frac{w}{w-1}pq^{k-1}}{[1-\frac{w}{w-1}(1-q^k)]^2} \\ &= \frac{1}{\frac{-1}{w-1}} \sum_{k\geq 1} \frac{\frac{wpq^{k-1}}{w-1}}{[\frac{w-1-w(1-q^{k-1})}{w-1}]^2} \\ &= -(w-1)\frac{1}{w-1}\sum_{k\geq 1} \frac{wpq^{k-1}}{\frac{(w-1-w+wq^{k-1})^2}{(w-1)^2}} \\ &= -p\sum_{k\geq 0} \frac{wq^k}{(-1+wq^k)^2}. \end{aligned} \tag{5.1.14}$$

Now, use the following formula for coefficients, derived from Cauchy's Integral Formula.

$$[z^n]G(z) = (-1)^n[w^n](1-w)^{n-1}G(z(w)). \tag{5.1.15}$$

So, returning to $G(z)$ in (5.1.14) and replacing $G(z)$ by the RHS of (5.1.14), we have

$$\begin{aligned} [z^n]G(z) &= p(-1)^{n+1}[w^n](1-w)^{n+1}\sum_{k\geq 0}\frac{wq^k}{(1-wq^k)^2} \\ &= p(-1)^{n+1}[w^n]\sum_{l=0}^{n+1}\binom{n+1}{l}(-w)^{n+1-l}\sum_{k\geq 0}\frac{wq^k}{(1-wq^k)^2} \quad \text{by the binomial theorem.} \end{aligned}$$

Now rearrange the sum and remove the powers of w, to get

$$\begin{aligned} [z^n]G(z) &= p\sum_{l=1}^{n+1}\binom{n+1}{l}(-1)^l[w^{l-1}]\sum_{k\geq 0}\frac{wq^k}{(1-wq^k)^2} \\ &= p\sum_{l=2}^{n+1}\binom{n+1}{l}(-1)^l\sum_{k\geq 0}(q^k)^{l-1}[w^{l-1}]\frac{w}{(1-w)^2} \quad \text{every } w \text{ is with a } q^k \text{ so we have } (q^k)^{l-1}. \end{aligned}$$

Then use the binomial theorem on $\frac{w}{(1-w)^2}$, take the coefficient and sum the geometric series over k, to get

$$[z^n]G(z) = p\sum_{l=2}^{n+1}\binom{n+1}{l}(-1)^l\frac{l-1}{1-q^{l-1}}.$$

In the above, we have learnt the new method of substitution that Prodinger presents in his paper [22]. Consequently, we have the following theorem:

Theorem 5.1.1 *The expected value of the sum of the positions of records, in random words of length n, is given by*

$$\mathbb{E}_n = p\sum_{k=2}^{n+1}\binom{n+1}{k}(-1)^k\frac{k-1}{1-q^{k-1}}. \tag{5.1.16}$$

Looking back at (4.4.8) and (4.4.9) of Chapter 4, we can use those formulae in this instance; replacing k by z in $f(z)$ in (4.4.8) and replacing n by $n+1$ in (4.4.9). This gives (5.1.16)'s representation as a contour integral:

$$\mathbb{E}_n = \frac{p}{2\pi i}\oint_{\mathcal{C}}\frac{(-1)^{n+1}(n+1)!}{z(z-1)\dots(z-n-1)}\frac{z-1}{1-q^{z-1}}dz \tag{5.1.17}$$

where the curve $\mathcal{C}$ encircles the poles $2, 3, \dots, n+1$ and no others.

For an asymptotic expansion, we change the contour and take the extra residues into account (with a negative sign). The poles are at

- $z = 1 + \frac{2\pi ik}{\log Q} =: 1 + \chi_k$ for $k \in \mathbb{Z}$, $Q := \frac{1}{q}$
- and at $z = 0$ (which we ignore).

So the residue is

$$\frac{pn}{\log Q} + \frac{pn}{\log Q}\sum_{k\neq 0}\chi_k\Gamma(-1-\chi_k)e^{2\pi ik\cdot\log_Q n} + O(1).$$

Hence, we can state the following theorem:

Theorem 5.1.2 *The expected value of the sum of the positions of records, in random words of length n, has the asymptotic expansion*

$$\mathbb{E}_n = \frac{pn}{\log Q}\left(1 + \delta(\log_Q(n))\right) + O(1) \tag{5.1.18}$$

where the periodic function (of small amplitude) is given by

$$\delta(x) = \sum_{k\neq 0} \chi_k \Gamma(-1-\chi_k) e^{2\pi i k x}.$$

Now, letting $q \to 1$ in the explicit formula for $\mathbb{E}_n$ in (5.1.16), we obtain:
$\frac{1-q}{1-q^{k-1}} = \frac{1}{1+q+q^2+\cdots+q^{k-2}}$
which tends to $\frac{1}{k-1}$ when $q = 1$.
So, we basically have the binomial theorem

$$\sum_{k=2}^{n+1} \binom{n+1}{k} (-1)^k = n. \tag{5.1.19}$$

When $q \to 0$, **srec**$\to 1$. But when $q \to 1$, we get the case of permutations as predicted by equation (5.0.1) in the beginning of this chapter.

5.2 Weak Records

We use the same method as for strong records, thus we will briefly review the computations.

The recursion for weak records becomes

$$f_k(z,u) = \sum_{j\leq k} f_j(zu,u) \frac{zupq^{k-1}}{1-zu(1-q^{j-1})} + zupq^{k-1}. \tag{5.2.1}$$

Explanation: We will explain the two differences from (5.1.1). The previous record was j, but now we are counting weak records so the previous record could be equal to k; hence the index $j \leq k$. Then in the denominator, we have q^{j-1}: After the last record, we have a sequence of values after the last j that must be $\leq j-1$. Therefore, q^j from before, becomes q^{j-1}.

Therefore, we have the generating function

$$F(z,u) = \sum_{k\geq 1} f_k(z,u)\frac{1}{[\![k-1]\!]}. \tag{5.2.2}$$

After our last record, we are only able to have values $\leq k-1$.

Then, we write the recursion as

$$\frac{f_k(z,u)}{zpq^{k-1}} - \frac{f_{k-1}(z,u)}{zpq^{k-2}} = \frac{f_k(zu,u)u}{1-zu(1-q^{k-1})}$$

and apply the same method as for strong records:

$$\frac{a_k(z)}{zpq^{k-1}} - \frac{a_{k-1}(z)}{zpq^{k-2}} = \frac{f_k(z,1)}{[\![k-1]\!]} + \frac{a_k(z)}{[\![k-1]\!]} + \frac{z\frac{d}{dz}f_k(z,1)}{[\![k-1]\!]} + \frac{f_k(z,1)z(1-q^{k-1})}{[\![k-1]\!]^2}.$$

Then,

$$\frac{a_k(z)[\![k]\!]}{zpq^{k-1}} - \frac{a_1(z)[\![1]\!]}{zp} = \sum_{j=2}^{k}\frac{zpq^{j-1}}{[\![j]\!][\![j-1]\!]} + \sum_{j=2}^{k}\frac{zpq^{j-1}}{[\![j]\!]^2}.$$

Here,

$$a_1(z) = \sum_{n\geq 1}\binom{n+1}{2}(pz)^n = \frac{pz}{[\![1]\!]^3}.$$

So then we have

$$\frac{a_k(z)}{[\![k-1]\!]} = \frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]^2} + \frac{(pz)^2q^{k-1}}{(1-pz)^2[\![k]\!][\![k-1]\!]} + \frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]}\sum_{j=2}^{k}\frac{zpq^{j-2}}{[\![j]\!]^2}.$$

Summing over k, we obtain

$$\begin{aligned}
G(z) &= \sum_{k\geq 1}\frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]^2} + \frac{pz^2}{(1-pz)^2(1-z)} + \sum_{j\geq 2}\frac{zpq^{j-1}}{[\![j]\!]^2}\sum_{k\geq j}\frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]} \\
&= \sum_{k\geq 1}\frac{zpq^{k-1}}{[\![k-1]\!][\![k]\!]^2} + \frac{pz^2}{(1-pz)^2(1-z)} + \sum_{j\geq 2}\frac{zpq^{j-1}}{[\![j]\!]^2}\left[\frac{1}{1-z} - \frac{1}{[\![j-1]\!]}\right] \\
&= \frac{zp}{(1-pz)^2} + \frac{pz^2}{(1-pz)^2(1-z)} + \frac{1}{1-z}\sum_{k\geq 2}\frac{zpq^{k-1}}{[\![k]\!]^2} \\
&= \frac{pz}{(1-pz)^2(1-z)} + \frac{1}{1-z}\sum_{k\geq 2}\frac{zpq^{k-1}}{[\![k]\!]^2}.
\end{aligned} \tag{5.2.3}$$

Compare (5.2.3) to our previous result of (5.1.13), where strong refers to the case for strong records.

$$G^{\text{strong}}(z) = \frac{1}{1-z} \sum_{k \geq 0} \frac{zpq^k}{[\![k]\!]^2}.$$

Then

$$G(z) = \frac{1}{q} G^{\text{strong}}(z) - \frac{zp}{q(1-z)}. \tag{5.2.4}$$

We can then state the new expectations in terms of the old

$$\mathbb{E}_n = \frac{1}{q} \mathbb{E}_n^{\text{strong}} - \frac{p}{q}. \tag{5.2.5}$$

To obtain the asymptotic formula, divide $\mathbb{E}_n^{\text{strong}}$ in (5.1.18) by q; that is,

$$\mathbb{E}_n \sim \frac{pn}{q \log Q} \left(1 + \delta(\log_Q n)\right). \tag{5.2.6}$$

Chapter 6

Value and Position of the rth Left-to-Right Maximum

For words of length n, generated by independent geometric random variables, Knopfmacher and Prodinger [11] consider the average value and the average position of the rth left-to-right maximum, for fixed r and $n \to \infty$.

H. Wilf, in his paper *On the outstanding elements of permutations* [25], proved the formula $(1 - 2^{-r})n$ for the average value of the rth left-to-right maximum, and the asymptotic formula $\frac{(\log n)^{r-1}}{(r-1)!}$ for the average position. Motivated by Wilf's study [25], Knopfmacher and Prodinger [11] studied the two parameters **value** and **position** of the rth left-to-right maximum for geometric random variables.

NOTE Note that not all words of length n have r left-to-right maxima.

6.1 The Value

The generating function is given by

$$\text{Value}(z,u) := \frac{1}{1-z}\prod_{i=1}^{h-1}\left\{1+\frac{pq^{i-1}zu}{1-(1-q^i)z}\right\}zpq^{h-1}. \tag{6.1.1}$$

Explanation: This generating function originates from the unique decomposition of a string as $a_1w_1 \dots a_{r-1}w_{r-1}a_rw_r$, where $a_1, \dots, a_r$ are the left-to-right maxima, the w_i are the strings between them, and w can be anything. If $a_k = l$, then we have the term $pq^{l-1}zu$. Each letter in w_k comes with probability $1-q^k$ and so w_k corresponds to the denominator $\sum_{j\geq 0}(1-q^k)^j z^j = \frac{1}{(1-(1-q^k)z)}$. Since a particular value i does not necessarily occur as a left-to-right maxima, we have a 1+ in the product. However, when we look for the coefficient of u^{r-1}, we have $r-1$ left-to-right maxima and the rth has value h. In the generating function, u keeps track of the number of left-to-right maxima, but we want to study the value of the rth one.

Use the abbreviation

$$[\![i]\!] := 1-(1-q^i)z.$$

From (6.1.1), we wish to compute $\pi_n^{(r)}$, the probability that a string of length n has r left-to-right maxima.

The coefficients of $z^n u^{r-1}$ in Value(z,u) in (6.1.1) above, call them $\pi_{n,h}^{(r)}$, are not probabilities, because, as stated in the **NOTE**, there exist words of length n with fewer than r left-to-right maxima.

Compute $\pi_n^{(r)}$ from Value(z,u) in (6.1.1) by summing over all values h:

$$\pi_n^{(r)} := [z^n u^{r-1}]\frac{1}{1-z}\sum_{h\geq 1}\prod_{i=1}^{h-1}\left\{1+\frac{pq^{i-1}zu}{[\![i]\!]}\right\}zpq^{h-1} = \sum_{h\geq 1}\pi_{n,h}^{(r)}. \tag{6.1.2}$$

6.2 The Position

Introduce the variable v to keep track of the position. The generating function is given by

$$\sigma_{n,j}^{(r)} := [z^n u^{r-1} v^j]\frac{1}{1-z}\sum_{h\geq 1}\prod_{k=1}^{h-1}\left\{1+\frac{pq^{k-1}zvu}{1-(1-q^k)zv}\right\}zvpq^{h-1}, \tag{6.2.1}$$

then $\frac{\sigma_{n,j}^{(r)}}{\pi_n^{(r)}}$ is the probability that a random string of length n has the rth left-to-right maximum in position j.

Explanation: This is the same decomposition as before in section 6.1. However, here we are not interested in the value h, so we sum over it. Since the variable v keeps track of position, we must ensure that every z that does not appear in the factor $\frac{1}{1-z}$ must be multiplied by a v; that is, every letter up to the rth left-to-right maximum will be marked by a v, so that we have as many v's as the position.

Computationally, it is easier to work with the parameter 'position - r.' For this parameter, we do not label the letters that are left-to-right maxima by a v, so that we have as many v's as 'position - r' indicates.
The generating function is then

$$\text{Position}(z,u,v) := \frac{1}{1-z}\sum_{h\geq 1}\prod_{k=1}^{h-1}\left\{1+\frac{pq^{k-1}zu}{1-(1-q^k)zv}\right\}zpq^{h-1}. \tag{6.2.2}$$

This generating function is preferred since the variable v appears in fewer places, and hence when we differentiate with respect to v to calculate the expected value, we will have fewer terms with which to work.

Note that sometimes we will assume $r \geq 2$. This is no problem because the position of the first left-to-right maximum is obviously 1.

6.3 Three Technical Lemmas

To read off coefficients, we will need the following formula:

$$[w^n]\sum_i a_i f(b_i w) = \sum_i a_i b_i^n \cdot [w^n]f(w). \tag{6.3.1}$$

$\sum_i a_i b_i^n$ can always be summed in closed form.

We state two lemmas to facilitate our future reading off of coefficients in iterated sums.

Lemma 6.3.1 *Assume that we have the power series:*

$$A^{(j)}(w) = \sum_{n\geq 1} a_n^{(j)} w^n, \quad j = 1,\ldots,s,$$

then

$$[w^n] \sum_{1\le i_1<i_2<\cdots<i_s} A^{(1)}(wq^{i_1})\dots A^{(s)}(wq^{i_s})$$
$$= \sum_{0=l_0<l_1<\cdots<l_{s-1}<l_s=n} \frac{a^{(s)}_{l_1-l_0}\dots a^{(1)}_{l_s-l_{s-1}}}{(Q^{l_1}-1)\dots(Q^{l_s}-1)}. \tag{6.3.2}$$

Proof 6.3.2 Without loss of generality, we prove the lemma for the case $s = 3$.

$$[w^n] \sum_{1\le i<j<h} A(wq^i)B(wq^j)C(wq^h)$$
$$= \sum_l \sum_{1\le i<j<h} [w^{n-l}]A(wq^i)B(wq^j)[w^l]C(wq^h) \quad \text{by convolution}$$
$$= \sum_l \sum_{1\le i<j<h} [w^{n-l}]A(wq^i)B(wq^j)q^{hl}[w^l]C(w) \quad \text{by formula (6.3.1)}$$
$$= \sum_l \frac{q^{jl+l}}{1-q^l} \sum_{1\le i<j} [w^{n-l}]A(wq^i)B(wq^j)[w^l]C(w)$$
$$= \sum_l \frac{1}{Q^l-1} \sum_{1\le i<j} [w^{n-l}]A(wq^i)B(wq^j)[w^l]C(w)q^{jl}$$
$$= \sum_l \frac{c_l}{Q^l-1} \sum_{1\le i<j} [w^n]A(wq^i)B(wq^j)(wq^j)^l \qquad \text{since } [w^l]C(w) = c_l$$
$$= \sum_{l,m} \frac{c_l}{Q^l-1} \sum_{1\le i<j} [w^{n-m}]A(wq^i)[w^m]B(wq^j)(wq^j)^l \quad \text{by convolution again}$$
$$= \sum_{l,m} \frac{c_l}{Q^l-1} \sum_{1\le i<j} [w^{n-m}]A(wq^i)q^{jm}[w^m]B(w)w^l \quad \text{by formula (6.3.1)}$$
$$= \sum_{l,m} \frac{q^{im+m}}{1-q^m} \frac{c_l}{Q^l-1} \sum_{1\le i} [w^{n-m}]A(wq^i)[w^m]B(w)w^l$$
$$= \sum_{l,m} \frac{c_l}{(Q^l-1)(Q^m-1)} \sum_{1\le i} [w^{n-m}]A(wq^i)q^{im}[w^m]B(w)w^l$$
$$= \sum_{l,m} \frac{c_l b_{m-l}}{(Q^l-1)(Q^m-1)} \sum_{1\le i} [w^n]A(wq^i)(wq^i)^m \quad \text{by formula (6.3.1)}$$
$$= \sum_{l,m} \frac{c_l b_{m-l} a_{n-m}}{(Q^l-1)(Q^m-1)(Q^n-1)}.$$

The second lemma to enable the reading off of coefficients in iterated sums is stated as follows:

Lemma 6.3.2

$$[w^n] \sum_{1\le i_1<i_2<\cdots<i_s} A^{(1)}(wq^{i_1})\dots A^{(s)}(wq^{i_s})i_s$$
$$= [t] \sum_{0=l_0<l_1<\cdots<l_{s-1}<l_s=n} a^{(s)}_{l_1-l_0}\dots a^{(1)}_{l_s-l_{s-1}} \prod_{i=1}^{s}\left(\frac{1}{Q^{l_i}-1}+t\frac{Q^{l_i}}{(Q^{l_i}-1)^2}\right).$$

Proof 6.3.4 Without loss of generality, we will show the proof in the case $s=2$.

$$[w^n]\sum_{1\le i<j} A(wq^i)B(wq^j)j$$
$$=\sum_l \sum_{1\le i<j} [w^{n-l}]A(wq^i)\cdot[w^l]B(wq^j)j$$
$$=\sum_l\sum_{1\le i}[w^{n-l}]A(wq^i)\cdot\sum_{j>1}[w^l]B(wq^iq^j)(i+j) \quad \text{after replacing } j \text{ by } j+i$$
$$=\sum_l\sum_{1\le i}[w^{n-l}]A(wq^i)\cdot\sum_{j>1}[w^l]B(wq^iq^j)i$$
$$+\sum_l\sum_{1\le i}[w^{n-l}]A(wq^i)\cdot\sum_{j>1}[w^l]B(wq^iq^j)j \quad \text{by multiplying out}$$
$$=\sum_l \frac{b_l}{Q^l-1}\sum_{1\le i}[w^{n-l}]A(wq^i)i\cdot q^{il}$$
$$+\sum_l \frac{b_lQ^l}{(Q^l-1)^2}\sum_{1\le i}[w^{n-l}]A(wq^i)\cdot q^{il} \quad \text{by Lemma 6.3.1}$$
$$=\sum_l \frac{b_l}{Q^l-1}\sum_{1\le i} a_{n-l}q^{i(n-l)}i\cdot q^{il}$$
$$+\sum_l \frac{b_lQ^l}{(Q^l-1)^2}\sum_{1\le i} a_{n-l}q^{i(n-l)}\cdot q^{il} \quad \text{since } [w^{n-l}]A(wq^i)=a_{n-l}q^{i(n-l)}$$
$$=\sum_l \frac{a_{n-l}b_l}{Q^l-1}\sum_{1\le i} q^{in}i+\sum_l\frac{a_{n-l}b_lQ^l}{(Q^l-1)^2}\sum_{1\le i}q^{in}$$
$$=\sum_l \frac{a_{n-l}b_l}{Q^l-1}\frac{Q^n}{(Q^n-1)^2}+\sum_l\frac{a_{n-l}b_lQ^l}{(Q^l-1)^2}\frac{1}{Q^n-1}.$$

The product in question is

$$\left(\frac{1}{Q^l-1}+t\frac{Q^l}{(Q^l-1)^2}\right)\left(\frac{1}{Q^n-1}+t\frac{Q^n}{(Q^n-1)^2}\right)$$

and the linear term in it is

$$\frac{1}{Q^l-1}\frac{Q^n}{(Q^n-1)^2}+\frac{Q^l}{(Q^l-1)^2}\frac{1}{Q^n-1}.$$

6.4 The Probability That There Are r Maxima

Recall the following formula:

$$[z^n]f(z)=(-1)^n[w^n](1-w)^{n-1}f\left(\frac{w}{w-1}\right) \tag{6.4.1}$$

and use it to read off the nth coefficients of equation (6.1.2).

Write (6.1.2) as an iterated sum:

$$\begin{aligned}\pi_n^{(r)} &= [z^n]\frac{1}{1-z}\sum_{1\le i_1<\cdots<i_{r-1}<h}\frac{pq^{i_1-1}z}{[\![i_1]\!]}\cdots\frac{pq^{i_{r-1}-1}z}{[\![i_r-1]\!]}zpq^{h-1}\\ &= \left(\frac{p}{q}\right)^r[z^n]\frac{1}{1-z}\sum_{1\le i_1<\cdots<i_{r-1}<h}\frac{q^{i_1}z}{[\![i_1]\!]}\cdots\frac{q^{i_{r-1}}z}{[\![i_r-1]\!]}zq^h\end{aligned}$$

substitute in $z=\frac{w}{w-1}$ and apply (6.4.1) to get

$$=(-1)^r\left(\frac{p}{q}\right)^r(-1)^n[w^n](1-w)^{n-1}\sum_{1\le i_1<\cdots<i_{r-1}<h}\frac{q^{i_1}w\ldots q^{i_{r-1}}w}{(1-q^{i_1}w)\ldots(1-q^{i_{r-1}}w)}wq^h$$

expanding using the binomial theorem gives

$$=(-1)^r\left(\frac{p}{q}\right)^r\sum_{k=r}^{n}\binom{n-1}{k-1}(-1)^k[w^k]\sum_{1\le i_1<\cdots<i_{r-1}<h}\frac{q^{i_1}w\ldots q^{i_{r-1}}w}{(1-q^{i_1}w)\ldots(1-q^{i_{r-1}}w)}wq^h. \tag{6.4.2}$$

Now, evaluate the inner sum using Lemma 6.3.1 with $k=n$, $r=s$, $A^{(1)}(w)=\cdots=A^{(r-1)}(w)=\frac{w}{1-w}$, $A^{(r)}(w)=w$.

Therefore,

$$
\begin{aligned}
&[w^k] \sum_{1\le i_1<\cdots<i_{r-1}<h} \frac{q^{i_1}w\ldots q^{i_{r-1}}w}{(1-q^{i_1}w)\ldots(1-q^{i_{r-1}}w)} wq^h \\
&= \sum_{0=l_0<1=l_1<l_2<\cdots<l_r=k} \frac{1}{(Q-1)(Q^{l_2}-1)\ldots(Q^{l_r}-1)} \\
&= \frac{q}{p} \sum_{2\le l_2<\cdots<l_r=k} \frac{1}{(Q^{l_2}-1)\ldots(Q^{l_r}-1)} \qquad \text{since } \tfrac{1}{Q-1}=\tfrac{q}{p}.
\end{aligned}
$$

Then, after substituting back and replacing $k-1$ by k in (6.4.2), we have

$$
\pi_n^{(r)} = (-1)^{r-1}\left(\frac{p}{q}\right)^{r-1} \sum_{k=r-1}^{n-1} \binom{n-1}{k} (-1)^k f(k) \tag{6.4.3}
$$

with

$$
f(k) = \sum_{2\le l_2<\cdots<l_r=k+1} \frac{1}{(Q^{l_2}-1)\ldots(Q^{l_r}-1)}. \tag{6.4.4}
$$

Next, in order to apply Rice's method here, we need the continuation of $f(k)$ to the complex plane. We can use symmetric functions, where we can always represent such iterated summations by powersums:

$$
\Theta(k) := \sum_{l=2}^{k} \frac{1}{(Q^l-1)^d}.
$$

To find an extension of this quantity $\Theta(k)$ valid in the complex plane, we have

$$
\Theta(z) := \sum_{l\ge 2} \frac{1}{(Q^l-1)^d} - \sum_{l\ge 1} \frac{1}{(Q^{l+z}-1)^d}.
$$

However, we only need the values $f(0),\ldots,f(r-2)$.

NOTE A definite sum $\sum_{a\ge k>b} g(k)$ is defined by $G(b)-G(a)$, with an indefinite sum function. In particular, for $a=b$ we get 0, and $\sum_{1\le k<-2} g(k) = -g(0)-g(1)-g(2)$. This idea can be iterated.

We have here a sum of $r-1$ indices, so therefore $f(1)=\cdots=f(r-2)=0$.

For $f(0)$, consider the following example.

NOTE

$$A(n) = \sum_{j=1}^{n-1} \left(\sum_{k=j+1}^{n} a_k a_j \right)$$

can be rewritten as

$$A(n) = \frac{1}{2} \left(\sum_{i=1}^{n} a_i \right)^2 - \frac{1}{2} \sum_{i=1}^{n} a_i^2,$$

and if we substitute in for $n = -2, -1, 0, 1, 2, 3$, we obtain the output

$$\frac{1}{2}(-a_{-1}-a_0)^2+\frac{1}{2}a_{-1}^2+\frac{1}{2}a_0^2, a_0^2, 0, 0, \frac{1}{2}(a_1+a_2)^2-\frac{1}{2}a_1^2-\frac{1}{2}a_2^2, \frac{1}{2}(a_1+a_2+a_3)^2-\frac{1}{2}a_1^2-\frac{1}{2}a_2^2-\frac{1}{2}a_3^2.$$

Therefore, we can see that

$$f(0) = (-1)^r \left(\frac{1}{Q-1} \right)^{r-1}. \tag{6.4.5}$$

Rice's method and the pole at $z = 0$ gives us

$$\pi_n^{(r)} = 1 + O(\frac{1}{n}). \tag{6.4.6}$$

Asymptotically, every geometric sample has r left-to-right maxima for a fixed r.

6.5 The Average Value of the rth Maximum

We compute the average value $\mathbb{E}_n^{(r)}$.

$$\mathbb{E}_n^{(r)} = \sum_{h \geq 1} h[z^n u^{r-1}]\text{Value}(z, u)$$

expand out to get an iterated sum

$$= [z^n] \frac{1}{1-z} \sum_{1 \leq i_1 < \cdots < i_{r-1} < h} \frac{pq^{i_1-1}z}{[\![i_1]\!]} \cdots \frac{pq^{i_{r-1}-1}z}{[\![i_r - 1]\!]} zpq^{h-1}h$$

$$= \left(\frac{p}{q} \right)^r [z^n] \frac{1}{1-z} \sum_{1 \leq i_1 < \cdots < i_{r-1} < h} \frac{q^{i_1}z}{[\![i_1]\!]} \cdots \frac{pq^{i_{r-1}}z}{[\![i_r - 1]\!]} zq^h h$$

again by (6.4.1) and substituting we get

$$= (-1)^r \left(\frac{p}{q}\right)^r (-1)^n [w^n](1-w)^{n-1} \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \frac{q^{i_1} w \ldots q^{i_{r-1}} w}{(1-q^{i_1}w)\ldots(1-q^{i_{r-1}})} w q^h h$$

and by the binomial theorem

$$= (-1)^r \left(\frac{p}{q}\right)^r \sum_{k=r}^{n} \binom{n-1}{k-1} (-1)^k [w^k] \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \frac{q^{i_1} w \ldots q^{i_{r-1}} w}{(1-q^{i_1}w)\ldots(1-q^{i_{r-1}})} w q^h h. \tag{6.5.1}$$

Now, evaluate the inner sum using Lemma 6.3.2:

$$[w^k] \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \frac{q^{i_1} w \ldots q^{i_{r-1}} w}{(1-q^{i_1}w)\ldots(1-q^{i_{r-1}})} w q^h h$$
$$= [t] \sum_{1 = l_1 < l_2 < \cdots < l_r = k} \prod_{i=1}^{r} \left(\frac{1}{Q^{l_i - 1}} + t \frac{Q^{l_i}}{(Q^{l_i}-1)^2} \right).$$

Therefore, after substituting in and replacing $k-1$ by k, we get

$$\mathbb{E}_n^{(r)} = (-1)^{r-1} \left(\frac{p}{q}\right)^r \sum_{k=r-1}^{n-1} \binom{n-1}{k} (-1)^k f(k) \tag{6.5.2}$$

with

$$f(k) = [t] \sum_{1 = l_1 < l_2 < \cdots < l_r = k+1} \prod_{i=1}^{r} \left(\frac{1}{Q^{l_i - 1}} + t \frac{Q^{l_i}}{(Q^{l_i}-1)^2} \right). \tag{6.5.3}$$

Like before, $f(1) = \cdots = f(r-2) = 0$ and since in general

$$\sum_{1 = l_1 < l_2 < \cdots < l_r = 1} a_{l_1} a_{l_2} \ldots a_{l_r} = (-1)^r a_1 a_1 \ldots a_1,$$

we find that

$$f(0) = [t] \sum_{1 = l_1 < l_2 < \cdots < l_r = 1} \prod_{i=1}^{r} \left(\frac{1}{Q^{l_i - 1}} + t \frac{Q^{l_i}}{(Q^{l_i}-1)^2} \right)$$
$$= (-1)^r [t] \left(\frac{1}{Q-1} + t \frac{Q}{(Q-1)^2} \right)^r$$

$$= (-1)^r r \frac{Q}{(Q-1)^{r+1}} \tag{6.5.4}$$

$$\text{i.e.,} \quad f(0) = (-1)^r r \frac{\frac{1}{q}}{(\frac{1}{q}-1)^{r+1}} = (-1)^r r \frac{q^r}{(1-q)^{r+1}} = (-1)^r r \frac{q^r}{p^{r+1}}.$$

Substituting in (6.5.2), we have

$$\mathbb{E}_n^{(r)} = (-1)^r \left(\frac{p}{q}\right)^r (-1)^r r \left(\frac{q}{p}\right)^r \frac{1}{p} = \frac{r}{p}(-1)^{-1}.$$

Since we are taking negative residues, we have

$$\mathbb{E}_n^{(r)} = \frac{r}{p}.$$

Consequently, we have the following theorem:

Theorem 6.5.1 *The average value $\mathbb{E}_n^{(r)}$ of the rth left-to-right maximum in a random sequence of n elements, generated by geometric random variables according to the geometric distribution is given by*

$$\mathbb{E}_n^{(r)} = \frac{r}{p} + O\left(\frac{1}{n}\right) \quad \textit{for fixed } r \textit{ and } n \to \infty. \tag{6.5.5}$$

Taking the limit $q \to 1$ would yield the corresponding result for permutations. Equation (6.5.5) would then yield infinity.

6.6 The Average Position of the rth Maximum

Differentiate the generating function Position(z, u, v) from equation (6.2.2) with respect to v and let $v = 1$.
Therefore,

$$[z^n u^{r-1}] \frac{\partial}{\partial v} \text{Position}(z, u, v)\Big|_{v=1}$$

differentiate with respect to v first then expand out the product to get

$$= [z^n]\left(\frac{p}{q}\right)^r \frac{1}{1-z}[t] \sum_{1\le i_1<\cdots<i_{r-1}<h} \prod_{j=1}^{r-1}\left(\frac{q^{i_j}z}{[\![i_j]\!]} + t\frac{q^{i_j}z(1-q^{i_j})z}{[\![i_j]\!]^2}\right) zq^h$$

then using (6.4.1) and substituting, we get

$$= (-1)^n[w^n](1-w)^{n-1}\left(\frac{p}{q}\right)^r (-1)^r[t] \sum_{1\le i_1<\cdots<i_{r-1}<h} \prod_{j=1}^{r-1}\left(\frac{q^{i_j}w}{1-q^{i_j}w} - t\frac{q^{i_j}w(1-q^{i_j})w}{(1-q^{i_j}w)^2}\right) wq^h$$

use the binomial theorem on $(1-w)^{n-1}$ and replace $k-1$ by k to obtain

$$= \left(\frac{p}{q}\right)^r (-1)^{r-1} \sum_{k=r-1}^{n-1} \binom{n-1}{k} (-1)^k f(k), \tag{6.6.1}$$

where

$$f(k) = [w^{k+1}][t] \sum_{1\le i_1<\cdots<i_{r-1}<h} \prod_{j=1}^{r-1}\left(\frac{q^{i_j}w}{1-q^{i_j}w} - t\frac{q^{i_j}w(1-q^{i_j})w}{(1-q^{i_j}w)^2}\right) \tag{6.6.2}$$

$$wq^h = f_1(k) + f_2(k)$$

with

$$f_1(k) = [t][w^k] \sum_{1\le i_1<\cdots<i_{r-1}<h} \prod_{j=1}^{r-1}\left(\frac{q^{i_j}w}{1-q^{i_j}w} - t\frac{q^{i_j}w}{(1-q^{i_j}w)^2}\right) wq^h \tag{6.6.3}$$

and

$$f_2(k) = [t][w^{k+1}] \sum_{1\le i_1<\cdots<i_{r-1}<h} \prod_{j=1}^{r-1}\left(\frac{q^{i_j}w}{1-q^{i_j}w} + t\frac{(q^{i_j}w)^2}{(1-q^{i_j}w)^2}\right) wq^h. \tag{6.6.4}$$

The numerator terms in the product; that is, $q^{i_j}w$ and $(q^{i_j}w)^2$, come about by multiplying out $q^{i_j}w(1-q^{i_j})w$ in (6.6.2). In $f_1(k)$, we take the coefficient of w^k, not w^{k+1}, because we have absorbed the second w from the term $q^{i_j}w$ (that is, after multiplying out (6.6.2), we should have had the terms $q^{i_j}w^2$ and $(q^{i_j}w)^2$). We can now apply Lemma 6.3.1 to $f_1(k)$ in (6.6.3) and $f_2(k)$ in (6.6.4).

For $f_1(k)$, $A_1(w) = \cdots = A_{r-1}(w) = \frac{w}{w-1} - \frac{tw}{(1-w)^2}$ and $A_r(w) = w$.
Taking coefficients, we have 1 from $\frac{w}{w-1}$ and tn from $\frac{tw}{(1-w)^2}$, where according to the lemma, n must be replaced by $l_2 - l_1$. Hence,

$$f_1(k) = [t] \sum_{1=l_1<l_2<\cdots<l_r=k} \frac{(1-t(l_2-l_1))\ldots(1-t(l_r-l_{r-1}))}{(Q^{l_1}-1)\ldots(Q^{l_r}-1)}$$

then since the numerator is a telescoping product, we end up with $l_r - l_1$ where r is k

$$= -(k-1) \sum_{1=l_1<l_2<\cdots<l_r=k} \frac{1}{(Q^{l_1}-1)\ldots(Q^{l_r}-1)}. \tag{6.6.5}$$

For $f_2(k)$, $A_1(w) = \cdots = A_{r-1}(w) = \frac{w}{w-1} + \frac{(tw)^2}{(1-w)^2}$ and $A_r(w) = w$.
Taking coefficients, we have 1 from $\frac{w}{w-1}$ and $t(n-1)$ from $\frac{(tw)^2}{(1-w)^2}$, where according to the lemma, n must be replaced by $l_2 - l_1$. Hence,

$$f_2(k) = [t] \sum_{1=l_1<l_2<\cdots<l_r=k+1} \frac{(1+t(l_2-l_1-1))\ldots(1+t(l_r-l_{r-1}-1))}{(Q^{l_1}-1)\ldots(Q^{l_r}-1)}$$

the numerator telescopes again to give $(k-r)$

$$= (k-r) \sum_{1=l_1<l_2<\cdots<l_r=k+1} \frac{1}{(Q^{l_1}-1)\ldots(Q^{l_r}-1)}. \tag{6.6.6}$$

We require the behaviour of $f(0)$. The function $f_1(0)$ is dominant here.
Using the general rule that

$$\sum_{2\le l_2<\cdots<l_{r-1}<0} a_{l_2}\ldots a_{l_{r-1}} = (-1)^r \frac{a_0^{r-1} - a_1^{r-1}}{a_0 - a_1},$$

we find that as $z \to 0$

$$f(z) \sim \frac{(-1)^r}{(Q-1)(Q^z-1)^{r-1}}. \tag{6.6.7}$$

Next, we must look at the contribution of an rth order pole at $z = 0$. In their paper, *Mellin transforms and asymptotics: finite differences and Rice's integrals*, P. Flajolet and R. Sedgewick [5] provide the following formula:

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^m} = \frac{\log^m n}{m!} + O(\log^{m-1} n). \tag{6.6.8}$$

Therefore, substituting (6.6.7) into (6.6.1) and using (6.6.8) (where m is $r-1$), we have the following theorem:

Theorem 6.6.1 *The average position of the rth left-to-right maximum in a random sequence of n elements, generated by geometric random variables according to the*

geometric distribution is given by

$$\frac{1}{(r-1)!}\left(\frac{p}{q}\log_Q n\right)^{r-1} + O(\log^{r-2} n) \quad \textit{for fixed } r \geq 2 \textit{ and } n \to \infty.$$

Letting $q \to 1$, we have, for the case of permutations,

$$\frac{(\log n)^{r-1}}{(r-1)!}.$$

Chapter 7

d-Records in Geometrically Distributed Random Variables

Again, we consider left-to-right maxima or records; although in this chapter we generalize the idea. Here, Prodinger [21] studies the left-to-right maxima of the d-largest record in sequences generated by independent geometric random variables. The explicit and asymptotic formulae for the expectations for both the strict and weak models will be calculated.

A d-record occurs when we compute the d-largest values, and the variable maintaining it changes its value while the sequence is scanned from left to right. A d-record is the number of times the variable maintaining the dth largest value changes as one goes through the sample from left to right. The case $d = 1$ is the case as studied in Chapter 4.

For example,
if C_i is the current value of the ith largest value and if we have the sample 1 3 4 3 3 7 5, then
C_1 has values $1, 3, 4, 7$ so there are 4 1-records; C_1 counts all the left-to-right maxima.
C_2 has values $1, 3, 4, 5$ so there are 4 2-records; C_2 counts the second largest values from left to right.
C_3 has values $1, 3, 4$, so there are 3 3-records; C_3 the third largest values from left to right.

The last value of C_d is the d-largest value.
Therefore, we are counting the number of records of the d-type.

Note that the last value of the variable C_d for the computation of d-records is the d-largest value. This is clear in the example above.

We use the following usual abbreviations: $Q = \frac{1}{q}$, $L = \log Q$.

7.1 Expectation for the Strict Model

Consider the random variable X, number of d-records. Write

$$X = \chi_1 + \chi_2 + \dots,$$

where χ_k is one if the variable C_d will eventually change to the value k; zero otherwise. The expectation $\mathbb{E}(\chi_k)$ is just the probability that this happens. We must compute $\mathbb{E}(\chi_k)$. Note that C_d, at some stage, takes on the value k. This cannot happen unless there are d values $\geq k$. Thus the d values must be $\geq k$, but not all of them $> k$. Let $a_1, \dots, a_d$ be the actual values that are greater than k_1 and $w_1, \dots, w_d$ be the values smaller than k. These could be single letters or a whole sequence of letters. Hence, we want to count the weight (probability) of all the words of length n of the form $w_1 a_1 \dots w_d a_d y$ with letters $a_i \in \{k, k+1, \dots\}$ (not all of them larger than k) and words $w_i \in \{1, \dots, k-1\}^*$, $y \in \{1, 2, \dots\}^*$. The generating function for the expectation $\mathbb{E}(X)$ for strict d-records is given by

$$\mathbb{E}(X) = [z^n] \sum_{k \geq 1} \left[(q^{k-1}z)^d - (q^k z)^d \right] \frac{1}{[1 - z(1 - q^{k-1})]^d} \frac{1}{1 - z}. \tag{7.1.1}$$

Explanation: The generating function for any word y is just $\frac{1}{1-z}$. For the word $a_1, \dots, a_d$, each has the probabilities $pq^{k-1} + pq^k + \dots$ which sums to q^{k-1}. This happens d times hence we raise to the power d. Each a_i is a letter, which is counted by the variable z. Hence the term $(q^{k-1}z)^d$. We must then subtract off the case that all letters $a_1, \dots, a_d$ are strictly greater than k i.e., $pq^k + pq^{k+1} + \dots$ which sums to q^k. Again there are d of them, each counted by a z, hence $(q^k z)^d$. Next for

$w_1, \ldots, w_d$. These are sequences, so the generating function is $\frac{1}{1-\square}$. The values of the w's are either $1, 2, \ldots, k-1$ so therefore the probabilities are $p+pq+\cdots+pq^{k-2}$, which sums to $(1-q^{k-1})$. Again there are d of them, each counted by a z, hence $\frac{1}{[1-z(1-q^{k-1})]^d}$.

Now, replace $k-1$ by k and take out $(q^k z)^d$ from the first bracket to obtain

$$\mathbb{E}(X) = [z^n] \sum_{k \geq 0} (q^k z)^d (1-q^d) \frac{1}{[1-z(1-q^k)]^d} \frac{1}{1-z}. \tag{7.1.2}$$

Like in Chapter 5, use the substitution

$$z = \frac{w}{w-1}$$

and the formula (as in (5.1.15))

$$[z^n] f(z) = (-1)^n [w^n] (1-w)^{n-1} f\left(\frac{w}{w-1}\right),$$

to get

$$\mathbb{E}(X) = [w^n](-1)^{n+d}(1-q^d)(1-w)^n \sum_{k \geq 0} (q^k w)^d \frac{1}{(1-wq^k)^d}.$$

Then, after taking w^{n-j} out of $(1-w)^n$, we obtain

$$\mathbb{E}(X) = (-1)^{n+d}(1-q^d) \sum_{j=0}^{n} [w^{n-j}](1-w)^n \cdot [w^j] \sum_{k \geq 0} (q^k w)^d \frac{1}{(1-wq^k)^d}.$$

The index $j = 0$ up to n are the possible powers of w^{n-j} that you can get out of $(1-w)^n$.

Now use the binomial theorem on $(1-w)^n$. By the binomial theorem, we obtain $(-1)^{n-j}$, which eradicates the terms $(-1)^n$; that is,

$$\mathbb{E}(X) = (-1)^d (1-q^d) \sum_{j=d}^{n} \binom{n}{j} (-1)^j [w^j] \sum_{k \geq 0} (q^k w)^d \frac{1}{(1-wq^k)^d}.$$

We have the index $j-d$ because inside the sum, all powers are at least w^d.

Consider the k-sum:

$$[w^j] \sum_{k \geq 0} (q^k w)^d \frac{1}{(1-wq^k)^d}$$

$$= [w^j]w^d \left[\frac{1}{(1-w)^d} + q^d \frac{1}{(1-wq)^d} + \frac{q^{2d}}{(1-wq^2)^d} + \left(\frac{q^3}{1-wq^3} \right)^d \right]$$
$$= [w^j]w^d \frac{1}{(1-w)^d}(1 + q^j + q^{2j} + \dots)$$
$$= [w^j]\frac{1}{1-q^j}\frac{w^d}{(1-w)^d}.$$

Therefore, we have

$$\mathbb{E}(X) = (-1)^d(1-q^d)\sum_{j=d}^{n}\binom{n}{j}(-1)^j[w^j]\frac{1}{1-q^j}\frac{w^d}{(1-w)^d}.$$

$$\text{Now} \quad [w^j]\frac{w^d}{(1-w)^d} = \binom{j-1}{d-1}.$$

$$\text{i.e.,} \quad \mathbb{E}(X) = (-1)^d(1-q^d)\sum_{j=d}^{n}\binom{n}{j}(-1)^j\frac{1}{1-q^j}\binom{j-1}{d-1} \tag{7.1.3}$$

and by Rice's method, we obtain

$$= (-1)^d(1-q^d)\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(-1)^n n!}{z(z-1)\dots(z-n)}\frac{(z-1)\dots(z-d+1)}{(d-1)!}\frac{1}{1-q^z}dz.$$

For the asymptotic evaluation, we must look at the residues outside of this curve, taking them with a negative sign. The most significant residue is when $z = 0$. To compute this by hand, first cancel the powers of $(z-1)$ from both the numerator and denominator. So in the denominator, we have $(z-d)(z-(d+1))\dots(z-n)$ and in the numerator, we have $n(n-1)\dots d$ as a product. Rewrite $\frac{n}{z-n}$ as $\left(\frac{1}{1-\frac{z}{n}}\right)(-1)$ and similarly for the rest of the terms. Hence, we will have d copies of (-1), and the $(-1)^d$ cancels with the $(-1)^d$ in the formula for $\mathbb{E}(X)$ above.
We have

$$(1-q^d)[z^{-1}]\frac{1}{z}\frac{1}{(1-\frac{z}{d})\dots(1-\frac{z}{n})}\frac{1}{1-q^z}.$$

Expand each of the terms above by geometric series.

$$\frac{1}{1-q^z} \sim \frac{1}{1-e^{z\log q}}$$
$$\sim \frac{1}{1-(1+z\log q+\dots)}$$

$$\sim \frac{1}{-z\log q - \frac{z^2}{2}(\log q)^2 + \dots}$$
$$\sim \frac{1}{Lz(1-\frac{Lz}{2})}$$
$$\sim \frac{1}{Lz}\left(1+\frac{Lz}{2}\right)$$

and the denominator becomes

$$\left(1-\frac{z}{d}\right)\dots\left(1-\frac{z}{n}\right) \sim \left(1+\frac{z}{d}\right)+\dots+\left(1+\frac{z}{n}\right)$$
$$\sim 1+z\left(\frac{1}{d}+\frac{1}{(d+1)}+\dots+\frac{1}{n}\right) \quad \text{after multiplying out.}$$

This can be rewritten as

$$1+z(H_n-H_{d-1}).$$

Therefore,

$$(1-q^d)[z^{-1}]\frac{1}{z}\frac{1}{(1-\frac{z}{d})\dots(1-\frac{z}{n})}\frac{1}{1-q^z}$$
$$\sim (1-q^d)[z^{-1}]\left(1+z(H_n-H_{d-1})\right)\frac{1}{Lz}\left(1+\frac{Lz}{2}\right)\frac{1}{z}$$
$$\sim (1-q^d)\frac{1}{L}\left(H_n-H_{d-1}+\frac{L}{2}\right)$$

now divide by L and note that $\frac{H_n}{L}$ behaves like $\log_Q n+\frac{\gamma}{L}$, to get

$$\sim (1-q^d)\left(\log_Q n+\frac{\gamma}{L}-\frac{H_{d-1}}{L}+\frac{1}{2}\right). \tag{7.1.4}$$

Letting $d=1$ in (7.1.3), we obtain the expectation for counting ordinary records

$$p\left(\log_Q n+\frac{\gamma}{L}+\frac{1}{2}\right). \tag{7.1.5}$$

Reassuringly, this is the exact formula we obtained in (4.2.7), where the extra terms in (4.2.7) are so small they are insignificant.

Theorem 7.1.1 *The parameter 'number of changes of variable C_d' (= number of d-records), for random strings of length n, produced by independent geometric random*

variables, has the following asymptotic equivalents for $n \to \infty$ for expectation

$$\mathbb{E}(X) \sim (1-q^d)\left(\log_Q n + \frac{\gamma}{L} - \frac{H_{d-1}}{L} + \frac{1}{2}\right) + \delta_E(\log_Q n). \tag{7.1.6}$$

The (small) periodic function $\delta_E(x)$ could be determined in principle in terms of its Fourier coefficients.

7.2 Expectation for the Weak Model

When we count weak records, we are, in fact, counting with multiplicity. If the one that is to be compared with the variable for the d-record is equal, we **also** count that as a change.

For the generating function for the expectation, we must note that $m \geq d$ elements must have been $\geq k$, but at most $d-1$ of them $> k$. By a similar method as in section 7.1 (using m's instead of d's), we obtain

$$\mathbb{E}(X) = [z^n] \sum_{k\geq 1} \sum_{m\geq d} \sum_{\lambda=0}^{d-1} \binom{m}{\lambda} (zq^k)^\lambda (zpq^{k-1})^{m-\lambda} \frac{1}{[1-z(1-q^{k-1})]^m} \frac{1}{1-z}. \tag{7.2.1}$$

Explanation: The term $\binom{m}{\lambda}$ implies that we are choosing λ out of the m values that are $> k$. For $(zq^k)^\lambda$: q^k is the probability if you sum the numbers that are $> k$, multiply by a z and raise to the power of λ, where λ counts values $> k$. Therefore, there are $m-\lambda$ values equal to k; hence $(zpq^{k-1})^{m-\lambda}$.

Again, like in Chapter 5, use the substitution

$$z = \frac{w}{w-1}$$

and the formula

$$[z^n]f(z) = (-1)^n [w^n](1-w)^{n-1} f\left(\frac{w}{w-1}\right),$$

to obtain

$$\mathbb{E}(X) = (-1)^n[w^n](1-w)^{n-1}\sum_{k\geq 1}\sum_{m\geq d}\sum_{\lambda=0}^{d-1}\binom{m}{\lambda}(wq^k)^\lambda(wpq^{k-1})^{m-\lambda}\frac{(-1)^m}{(1-wq^{k-1})^m}(1-w)$$

then by the binomial theorem on $(1-w)^n$, we obtain

$$= \sum_{j=d}^{n}\binom{n}{j}(-1)^j[w^j]\sum_{m\geq d}(-1)^m\sum_{\lambda=0}^{d-1}\binom{m}{\lambda}\sum_{k\geq 0}(wq^{k+1})^\lambda(wpq^k)^{m-\lambda}\frac{1}{(1-wq^k)^m}$$

then since w^j always goes together with a $(q^k)^j$, we obtain

$$= \sum_{j=d}^{n}\binom{n}{j}(-1)^j[w^j]\sum_{m\geq d}(-1)^m\sum_{\lambda=0}^{d-1}\binom{m}{\lambda}(wq)^k(wp)^{m-\lambda}\frac{1}{(1-w)^m}\sum_{k\geq 0}(q^j)^k$$

then after summing the geometric series,

$$= \sum_{j=d}^{n}\binom{n}{j}(-1)^j[w^j]\sum_{m\geq d}(-1)^m\sum_{\lambda=0}^{d-1}\binom{m}{\lambda}(wq)^k(wp)^{m-\lambda}\frac{1}{(1-w)^m}\frac{1}{1-q^j}$$

and after rearranging, we have

$$= \sum_{j=d}^{n}\binom{n}{j}(-1)^j\frac{1}{1-q^j}\underbrace{[w^j]\sum_{m\geq d}\sum_{\lambda=0}^{d-1}\binom{m}{\lambda}\left(\frac{q}{p}\right)^\lambda\left(\frac{pw}{1-w}\right)^m}_{\Psi(j)}. \tag{7.2.2}$$

Let us now evaluate $\Psi(j)$. For a general a and b, consider

$$\Xi = \sum_{0\leq\lambda<d}\sum_{m\geq d}\binom{m}{\lambda}a^\lambda b^m.$$

Consider the sum over m:

$$\sum_{m\geq\lambda}\binom{m}{\lambda}b^m = \frac{b^\lambda}{(1-b)^{\lambda+1}} \quad \text{by a combinatorial identity.}$$

We do not want the index $m \geq \lambda$, but rather $m \geq d$. So subtract off the sum when $m < d$.

$$\Xi = \sum_{0\leq\lambda<d}a^\lambda\left[\frac{b^\lambda}{(1-b)^{\lambda+1}} - \sum_{m<d}\binom{m}{\lambda}b^m\right]$$

$$= \frac{1}{1-b}\frac{1-(\frac{ab}{1-b})^d}{1-\frac{ab}{1-b}} - \underbrace{\sum_{0\leq\lambda\leq m<d}\binom{m}{\lambda}a^\lambda b^m}_{T_d}. \tag{7.2.3}$$

In (7.2.3), the first term is a geometric series, where the common ratio is $\left(\frac{ab}{1-b}\right)$. The second term is just left as a double sum and denoted T_d.
We now try to show that there is a recurrence for T_d:

$$T_0 = 0 \quad \text{and} \quad T_{d+1} = T_d + \sum_{0\le\lambda\le d} \binom{d}{\lambda} a^\lambda b^d \quad \text{where the sum is the case when } m = d$$
$$= T_d + (1+a)^d b^d \quad \text{by the binomial theorem.}$$

We have a simple recursion for T_{d+1} in terms of T_d. By iterating, we can solve for a recursion for T_d:

$$T_d = \sum_{0\le h<d} ((1+a)b)^h$$
$$= \frac{1-((1+a)b)^d}{1-(1+a)b} \quad \text{by geometric series.} \tag{7.2.4}$$

Therefore, substitute for T_d (7.2.4) into (7.2.3), to get

$$\Xi = \frac{1}{1-b}\frac{1-(\frac{ab}{1-b})^d}{1-\frac{ab}{1-b}} - \frac{1-((1+a)b)^d}{1-(1+a)b}$$
$$= \frac{((1+a)b)^d - \left(\frac{ab}{1-b}\right)^d}{1-(1+a)^b} \quad \text{after simplifying.} \tag{7.2.5}$$

Back to $\Psi(j)$ in (7.2.2). Let $a = \frac{p}{q}$, $b = \frac{pw}{w-1}$.
Then,

$$1 + a = \frac{1}{p},$$
$$(1+a)b = \frac{w}{w-1},$$
$$1 - b = \frac{w-1-pw}{w-1} = \frac{1-qw}{1-w},$$
$$\text{and} \quad \frac{ab}{1-b} = \frac{qw}{qw-1}.$$

Substitute into (7.2.5) and simplify to evaluate $\Psi(j)$.

$$\Psi(j) = [w^j]\frac{\left(\frac{w}{w-1}\right)^d - \left(\frac{qw}{qw-1}\right)^d}{1-\frac{w}{w-1}}$$

$$= (-1)^d [w^j](1-w)\left[\left(\frac{w}{w-1}\right)^d - \left(\frac{qw}{1-qw}\right)^d\right] \quad \text{after simplifying the denominator}$$

$$= (-1)^d \left[\binom{j-1}{d-1} - \binom{j-2}{d-1} - q^j\binom{j-1}{d-1} + q^{j-1}\binom{j-2}{d-1}\right]$$

then take $[w^j]$ in both terms to obtain

$$= (-1)^d \left[\binom{j-1}{d-1}(1-q^j) - \binom{j-2}{d-1}(1-q^{j-1})\right]. \tag{7.2.6}$$

Substitute (7.2.6) into (7.2.2).

$$\mathbb{E}(X) = \sum_{j=d}^{n}\binom{n}{j}(-1)^j\frac{1}{1-q^j}(-1)^d\left[\binom{j-1}{d-1}(1-q^j) - \binom{j-2}{d-1}(1-q^{j-1})\right] \tag{7.2.7}$$

$$= (-1)^d\sum_{j=d}^{n}\binom{n}{j}(-1)^j\binom{j-1}{d-1} - (-1)^d\sum_{j=d}^{n}\binom{n}{j}(-1)^j\frac{1}{1-q^j}\binom{j-2}{d-1}(1-q^{j-1})$$

$$= 1 - (-1)^d\sum_{j=d}^{n}\binom{n}{j}(-1)^j\frac{1-q^{j-1}}{1-q^j}\binom{j-2}{d-1}$$

(for the explanation of $(-1)^d\sum_{j=d}^{n}\binom{n}{j}(-1)^j\binom{j-1}{d-1} = 1$,

see the calculations leading to (7.3.3) in section 7.3)

$$= 1 - (-1)^d\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{(-1)^n n!}{z(z-1)\dots(z-n)}\frac{1-q^{z-1}}{1-q^z}\binom{z-2}{d-1}dz \quad \text{by Rice's method}$$

$$\sim 1 - (-1)^d[z^{-1}]\frac{n!\Gamma(-z)}{\Gamma(n+1-z)}\frac{1-q^{z-1}}{1-q^z}\binom{z-2}{d-1}$$

and by Mathematica, we have

$$\sim 1 + d\frac{p}{q}\log_Q n - \frac{1}{L}\left((H_{d-1}-\gamma)d\frac{p}{q} + \frac{p}{q} - d\frac{p}{q}\right) - \frac{d}{2}\left(1+\frac{1}{q}\right). \tag{7.2.8}$$

Theorem 7.2.1 *The parameter 'number of changes of variable C_d' (=number of d-records, weak model) for random strings of length n, produced by independent geometric random variables, has the following asymptotic equivalent for $n \to \infty$ for*

its expectation

$$\mathbb{E}(X) \sim d\frac{p}{q}\log_Q n - \frac{1}{L}\left((H_{d-1}-\gamma)d\frac{p}{q} + \frac{p}{q} - d\frac{p}{q}\right) - \frac{d}{2}\left(1+\frac{1}{q}\right) + 1 + \delta_{EW}(\log_Q n). \tag{7.2.9}$$

7.3 The Permutation Model

We now consider the case when the limit q tends to 1 (q the parameter of the geometric distribution). When $q \to 1$, the geometric random variable behaves like a permutation; that is, when $q \to 1$, then all the n numbers will be different. Hence, we are considering the model of random permutations.

Take the limit $q \to 1$ in (7.1.3) in the expectation for the strict model.
i.e., take

$$\lim_{q\to 1}\frac{1-q^d}{1-q^j} = \frac{d}{j} \quad \text{by L'hopital's rule}$$

to obtain

$$\mathbb{E}(X) = (-1)^d d\sum_{j=d}^{n}\binom{n}{j}(-1)^j\frac{1}{j}\binom{j-1}{d-1}. \tag{7.3.1}$$

Next, take the limit $q \to 1$ in (7.2.7) in the expectation for the weak model, to obtain

$$\mathbb{E}(X) = 1 - (-1)^d\sum_{j=d}^{n}\binom{n}{j}(-1)^j\frac{j-1}{j}\binom{j-2}{d-1}. \tag{7.3.2}$$

When $q \to 1$, there are no repeats in the random permutation; so the expectation for the strong and weak models should have the same answer. So (7.3.1) and (7.3.2) should coincide. To show this, we take (7.3.1) and subtract off (7.3.2) to get

$$(-1)^d d\sum_{j=d}^{n}\binom{n}{j}(-1)^j\frac{1}{j}\binom{j-1}{d-1} - 1 + (-1)^d\sum_{j=d}^{n}\binom{n}{j}(-1)^j\frac{j-1}{j}\binom{j-2}{d-1}$$
$$= \left\{(-1)^d\sum_{j=d}^{n}\binom{n}{j}(-1)^j\binom{j-1}{d-1}\left[\frac{d}{j}\binom{j-1}{d-1} + \frac{j-1}{j}\binom{j-2}{d-1}\right]\right\} - 1$$

then by simplifying with factorials, we get

$$
\begin{aligned}
&= \left\{ (-1)^d \sum_{j=d}^{n} \binom{n}{j} (-1)^j \binom{j-1}{d-1} \left[\frac{d}{j} \binom{j-1}{d-1} + \frac{j-d}{j} \binom{j-1}{d-1} \right] \right\} - 1 \\
&= \left\{ (-1)^d \sum_{j=d}^{n} \binom{n}{j} (-1)^j \binom{j-1}{d-1} \left[\frac{d}{j} + 1 - \frac{d}{j} \right] \right\} - 1 \\
&= \left\{ (-1)^d \sum_{j=d}^{n} \binom{n}{j} (-1)^j \binom{j-1}{d-1} [1] \right\} - 1 \\
&= (-1)^d \sum_{j=d}^{n} \binom{n}{j} (-1)^j \binom{j-1}{d-1} - 1 \\
&= (-1)^d n \binom{n-1}{d-1} \sum_{j=d}^{n} \binom{n-d}{j-d} (-1)^j \frac{1}{j} - 1 \quad \text{by rewriting binomial coefficients} \\
&= n \binom{n-1}{d-1} \sum_{j=d}^{n-j} \binom{n-d}{j} (-1)^j \frac{1}{j+d} - 1 \quad \text{by replacing } j-d \text{ by } j \\
&= n \binom{n-1}{d-1} \frac{1}{d\binom{n}{d}} - 1 \quad \text{by a standard identity} \\
&= 1 - 1 \qquad (7.3.3) \\
&= 0.
\end{aligned}
$$

Therefore, the two equations commute thus the strong and weak models have the same expectation when $q \to 1$.

We derive the probability generating function. Notice that the coefficient of x^k is the probability that the number of d-records in a permutation of n elements is equal to k.
The probability generating function is then

$$\prod_{k=d}^{n} \left(\frac{k-d}{k} + \frac{xd}{k} \right). \qquad (7.3.4)$$

Explanation: Consider a permutation $p_1, \ldots, p_n$. For the first $d-1$ elements, nothing happens. Now, if k is d, then p_d introduces a change $(0+x)$. If k is $d+1$, then p_{d+1} introduces no change with probability $\frac{1}{(d+1)}$ and a change with probability

$\frac{d}{(d+1)}$. Further, $p_{(d+2)}$ introduces no change with probability $\frac{2}{(d+2)}$ and a change with probability $\frac{d}{(d+2)}$; and so on. The index is $k = d$ up to n since n is always going to be the first record so it cannot be the dth record.

Then, after differentiating the probability generating function and substituting in $x = 1$, the expectation is

$$d(H_d - H_{d-1}). \tag{7.3.5}$$

Chapter 8

Record Statistics in a Random Composition

Once again, this chapter discusses left-to-right maxima, but this time, instead of geometric random variables, we consider compositions. It is well known that compositions of n behave asymptotically like geometric words of length $\frac{n}{2}$ with $p = \frac{1}{2}$. However, the intention here is to derive the results directly.

A composition $\sigma = a_1 a_2 \dots a_m$ of n is an ordered collection of positive integers whose sum is n. Therefore, a composition σ of n with parts in A is a restricted word over the alphabet A. An element a_i in σ is a strong (weak) record if $a_i > a_j$ $(a_i \geq a_j)$ for all $j = 1, 2, \dots, i-1$. Furthermore, the position of this record is i.

In this paper by Knopfmacher and Mansour [9], we will denote the set of all compositions of n with m parts in A by $C_A(n, m)$. It is well known that the number of compositions of $n \geq 1$ with m parts in $\mathbb{N}$ is given by $\binom{n-1}{m-1}$, and that the total number of compositions of n is 2^{n-1}. In this chapter, we will find generating functions for these parameters, the number of strong records, the number of weak records, the sum of positions of strong records, and the sum of positions of weak records in a random composition of n with parts in $A = [d] := \{1, 2, \dots, d\}$ or $A = \mathbb{N}$. This chapter also studies the average values as $n \to \infty$ in the case $A = \mathbb{N}$. The final section in this chapter presents a combinatorial explanation for the fact that the total number of weak records in compositions of n plus 2^{n-1} equal the total number

of strong records in compositions of $n+1$, for $n \geq 1$.

8.1 Number of Strong Records

Let $NSR_A(z, y, q)$ denote the generating function for the number of compositions of n with m parts in A:

$$NSR_A(z, y, q) = \sum_{n,m \geq 0} \sum_{\sigma \in C_A(n,m)} z^n y^m q^{nsr(\sigma)} \tag{8.1.1}$$

where $nsr(\sigma)$ is the number of strong records in composition σ.

We will find an explicit formula for this generating function; that is, we will find a generating function for the three variables: z, y, and q.

For the generating function $NSR_{[d]}(z, y, q)$, we are counting how many times the number d occurs. Thus, d stands for the largest part; we cannot have parts larger than d. The first time we encounter a d, we have a new left-to-right maximum. Denote the number of occurrences of the part d in the composition $\sigma \in C_d(n, m)$ by $l(\sigma)$. Therefore $l(\sigma)$ counts how many times part d occurs in the composition. The contribution of the case $l(\sigma) = 0$ is given by $NSR_{[d-1]}(z, y, q)$ because $l(\sigma) = 0$ only if there are no parts equal to d. On the contrary, assume $l(\sigma) > 0$. Then, σ can be decomposed as $\sigma' d \sigma''$, where σ' is a composition with parts in $[d-1]$, hence we must use the subscript $[d-1]$, and σ'' is a composition with parts in $[d]$. This means if $l(\sigma) > 0$, if there is at least one d in the composition, then σ' is the part before the d, thereafter we will have a d of course, and then after that we have the rest of the composition. d contributes d to the z variable, 1 to the number of parts and is a left-to-right maxima, represented by a q, i.e., we must have the term $z^d yq$. Hence, the contribution of the case $l(\sigma) > 0$ is

$$z^d yq NSR_{[d-1]}(z, y, q) NSR_{[d]}(z, y, 1)$$

where $z^d yq NSR_{[d-1]}(z, y, q)$ replaces σ' and $NSR_{[d]}(z, y, 1)$ replaces σ''.
For σ'', q has been replaced by a 1, which implies that we are not counting any left-to-right maxima. This makes sense because after d, there are no more left-to-right maxima.

Putting both cases $l(\sigma) = 0$ and $l(\sigma) > 0$ together, we have

$$NSR_{[d]}(z, y, q) = NSR_{[d-1]}(z, y, q) + z^d yq NSR_{[d-1]}(z, y, q) NSR_{[d]}(z, y, 1). \quad (8.1.2)$$

Now let $q = 1$, so we are now only counting compositions according to size and number of parts. Then, by definition, a composition is a sequence so again we have $\frac{1}{1-\square}$. Each part has a y and the parts of the composition are some number j between 1 and d. Thus, by induction we have

$$NSR_{[d]}(z, y, 1) = \frac{1}{1 - y\sum_{j=1}^{d} z^j}. \quad (8.1.3)$$

Now we can solve for $NSR_{[d]}$ in terms of $NSR_{[d-1]}$ using (8.1.2) and then

$$NSR_{[d]}(z, y, q) = \prod_{j=1}^{d} \left(1 + \frac{z^j yq}{1 - y\sum_{i=1}^{j} z^i}\right). \quad (8.1.4)$$

Hence, we have the following theorem:

Theorem 8.1.1 *The generating function $NSR_{[d]}(z, y, q)$ is given by*

$$NSR_{[d]}(z, y, q) = \prod_{j=1}^{d} \left(1 + \frac{z^j yq}{1 - y\sum_{i=1}^{j} z^i}\right).$$

Substituting $q = 1$ in (8.1.4) in the above theorem, we find that the generating function for the number of compositions of n with m parts in $[d]$ is given by (8.1.3); that is,

$$NSR_{[d]}(z, y, 1) = \frac{1}{1 - y\sum_{j=1}^{d} z^j}.$$

For the average number of strict left-to-right maxima, differentiate (8.1.4) using Leibniz formula.

$$\frac{\partial}{\partial q} NSR_{[d]}(z, y, 1) = \prod_{j=1}^{d} \left(1 + \frac{z^j y}{1 - y\sum_{i=1}^{j} z^i}\right) \left(\sum_{j=1}^{d} \frac{z^j y}{1 - y\sum_{i=1}^{j-1} z^i}\right)$$

put the first bracket over a common denominator to obtain

$$= \prod_{j=1}^{d} \left(\frac{1 - y\sum_{i=1}^{j-1} z^i}{1 - y\sum_{i=1}^{j} z^i} \right) \left(\sum_{j=1}^{d} \frac{z^j y}{1 - y\sum_{i=1}^{j-1} z^i} \right)$$

then since the first product is just a telescoping product, we get

$$= \frac{1}{1 - y\sum_{i=1}^{d} z^i} \left(\sum_{j=1}^{d} \frac{z^j y}{1 - y\sum_{i=1}^{j-1} z^i} \right). \tag{8.1.5}$$

Let $f_n = \sum_{m=0}^{n} \sum_{\sigma \in C_{\mathcal{N}(n,m)}} nsr(\sigma)$ denote the sum over all compositions σ of n of the number of strong records in σ.

Let $f(z) = \sum_{n \geq 0} f_n z^n$ be the generating function of the sequence f_n. Then after letting $y = 1$ in the derivative and replacing d by infinity, the generating function $f(z)$ for the average value of maxima is

$$\begin{aligned} f(z) :&= \frac{1}{1 - \sum_{i \geq 1} z^i} \sum_{j \geq 1} \frac{z^j}{1 - \sum_{i=1}^{j-1} z^i} \\ &= \frac{1 - z}{1 - 2z} \sum_{j \geq 1} \frac{z^j}{1 - \sum_{i=1}^{j-1} z^i} \end{aligned} \tag{8.1.6}$$

by geometric series.

Now to compute the average number asymptotically, first sum the finite geometric series in (8.1.6) and use partial fractions to obtain

$$\begin{aligned} f(z) &= \frac{z - z^2}{1 - 2z} + \frac{(1 - z)^2}{1 - 2z} \sum_{k \geq 2} \frac{z^k}{1 - 2z + z^k} \\ &= \frac{z - z^2}{1 - 2z} + (1 - z)^2 \sum_{k \geq 2} \left[\frac{1}{1 - 2z} - \frac{1}{1 - 2z + z^k} \right]. \end{aligned} \tag{8.1.7}$$

Then

$$\begin{aligned} \mathbb{E}_n^s &= \frac{1}{2^{n-1}} [z^n] f(z) \\ &= \frac{1}{2} + \frac{1}{2^{n-1}} [z^n] (1 - z)^2 \sum_{k=2}^{n} \left[\frac{1}{1 - 2z} - \frac{1}{1 - 2z + z^k} \right]. \end{aligned} \tag{8.1.8}$$

We will not show this asymptotic proof because we can simply use the asymptotic evaluation for geometric random variables given in (4.2.7) with $q = \frac{1}{2}$ and $\frac{n}{2}$ replacing n.

Theorem 8.1.2 *The average number $\mathbb{E}_n^s$ of strong left-to-right maxima in the context of compositions of n has the asymptotic expansion*

$$\mathbb{E}_n^s = \frac{1}{2}\left[\log_2 n - \frac{1}{2} + \frac{\gamma}{L} - \delta(\log_2 n)\right] + O(1). \tag{8.1.9}$$

Here, $L = \log 2$; γ is Euler's constant; and $\delta(x)$ is a periodic function of period 1 and mean 0 and small amplitude, which is given by the Fourier series

$$\delta(x) = \frac{1}{L}\sum_{k\neq 0}\Gamma(-\chi_k)e^{2k\pi ix},$$

where the complex numbers χ_k are given by $\chi_k = \frac{2k\pi i}{L}$.

8.2 Number of Weak Records

Let $NWR_A(z, y, q)$ denote the generating function for the number of compositions of n with m parts in A:

$$NWR_A(z, y, q) = \sum_{n,m\geq 0}\sum_{\sigma\in C_A(n,m)} z^n y^m q^{nwr(\sigma)} \tag{8.2.1}$$

where $nwr(\sigma)$ is the number of weak records in composition σ. We will find an explicit formula for this generating function.

Denote the number of occurrences of the part d in the composition $\sigma \in C_{[d]}(n, m)$ by $l(\sigma)$. The contribution of the case $l(\sigma) = 0$ is given by $NWR_{[d-1]}(z, y, q)$. If, on the contrary, $l(\sigma) = l > 0$, σ can be decomposed as $\sigma^{(1)}d\sigma^{(2)}d\ldots d\sigma^{(l+1)}$, where $\sigma^{(j)}$ is a composition with parts in $[d-1]$. Everytime a d occurs, it is a new weak left-to-right maximum so the above decomposition is just a labeling of parts: $\sigma^{(1)}$ is the first composition then a d occurs, $\sigma^{(2)}$ is the second composition then a d occurs etc, where the parts of all the compositions are strictly less than d (because all the d's are written out explicitly). Therefore, for the case where there are exactly l occurrences of d; that is, the contribution of the case $l(\sigma) = l$, equals

$$(z^d yq)^l NWR_{[d-1]}(z, y, q)(NWR_{[d-1]}(z, y, 1))^l.$$

The bracket is raised to the power of l because d occurs l times in the list.

Then, the generating function for $\sigma^{(1)}$ up to $\sigma^{(l+1)}$ is given by

$$NWR_{[d]}(z,y,q) = \frac{NWR_{[d-1]}(z,y,q)}{1 - z^d yq NWR_{[d-1]}(z,y,1)}. \tag{8.2.2}$$

$\sigma^{(2)}$ up to $\sigma^{(l+1)}$ all have the same generating function. For $\sigma^{(1)}$ however, we need to count all left-to-right maxima that came before the first part equal to d. After, $(\sigma^{(2)}, \ldots, \sigma^{(l+1)})$, we do not count left-to-right maxima because those numbers will not be bigger than d. Then sum over l to get (8.2.2), which is the sum of geometric series.

When $q = 1$ and by induction

$$NWR_{[d]}(z,y,1) = \frac{1}{1 - y\sum_{j=1}^{d} z^j}. \tag{8.2.3}$$

Thus,

$$NWR_{[d]}(z,y,q) = \prod_{j=1}^{d} \frac{1}{1 - \frac{z^j yq}{1 - y\sum_{i=1}^{j-1} z^i}}. \tag{8.2.4}$$

Hence, we have the following theorem:

Theorem 8.2.1 *The generating function $NWR_{[d]}(z,y,q)$ is given by*

$$NWR_{[d]}(z,y,q) = \prod_{j=1}^{d} \frac{1}{1 - \frac{z^j yq}{1 - y\sum_{i=1}^{j-1} z^i}}.$$

For an asymptotic expansion, we must differentiate with respect to q and substitute in $q = 1$. So then the generating function for the total number of weak records in compositions over $\mathbb{N}$ is then

$$\begin{aligned} g(z) :&= \left.\frac{\partial NWR_{[\mathbb{N}]}(z,1,q)}{\partial q}\right|_{q=1} \\ &= \frac{1-z}{1-2z} \sum_{k\geq 1} \frac{z^k}{1 - \sum_{i=1}^{k} z^i} \end{aligned}$$

$$= \frac{(1-z)^2}{1-2z} \sum_{k\geq 1} \frac{z^k}{1-2z+z^{k+1}} \quad \text{after working out the geometric series } \sum_{i=1}^{k} z^i$$

$$= \frac{(1-z)^2}{z} \sum_{k\geq 2} \left[\frac{1}{1-2z} - \frac{1}{1-2z+z^k} \right] \quad \text{by partial fractions.} \tag{8.2.5}$$

This new generating function $g(z)$ is almost the same as $f(z)$ in (8.1.7). Therefore, since we know the asymptotic expansion for $f(z)$ in (8.1.9), we can say

$$\begin{aligned} \mathbb{E}_n^w &= \frac{1}{2^{n-1}}[z^n]g(z) \\ &= \frac{1}{2^{n-1}}[z^n](1-z)^2 \sum_{k=2}^{n} \left[\frac{1}{1-2z} - \frac{1}{1-2z+z^k} \right]. \end{aligned} \tag{8.2.6}$$

Then,

$$\begin{aligned} \mathbb{E}_n^w &= \sum_{k=2}^{n+1} \left(1 - \frac{q_{n+1,k}}{2^{n-1}} \right) \\ &= 2\mathbb{E}_{n+1}^s - 1 \end{aligned} \tag{8.2.7}$$

where

$$q_{n,k} := [z^n] \frac{(1-z)^2}{1-2z+z^k}.$$

We replace $q_{n,k}$ by $q_{n+1,k}$ because we are taking the coefficient of z^{n+1}. So, using theorem 8.1.2, we can then obtain

Theorem 8.2.2 *The average number $\mathbb{E}_n^w$ of weak left-to-right maxima in the context of compositions of n has the asymptotic expansion*

$$\mathbb{E}_n^w = \log_2 n - \frac{3}{2} + \frac{\gamma}{L} - \delta(\log_2 n) + O(1). \tag{8.2.8}$$

8.3 The Sum of Positions of Strong Records (*ssrec*)

Let $PSR_A(z, y, q)$ denote the generating function for the number of compositions of n with m parts in A according to the statistic ssrec:

$$PSR_A(z, y, q) = \sum_{n,m\geq 0} \sum_{\sigma \in C_A(n,m)} z^n y^m q^{ssrec(\sigma)}. \tag{8.3.1}$$

We will find an explicit formula for this generating function.

recall that the number of occurrences of the part d in the composition $\sigma \in C_{[d]}(n, m)$ by $l(\sigma)$. So, as before, d is the largest part. The contribution of the case $l(\sigma) = 0$ is given by $PSR_{[d-1]}(z, y, q)$. We will assume $l(\sigma) > 0$, then σ can be decomposed as $\sigma' d \sigma''$, where σ' is a composition with parts in $[d-1]$ and σ'' is a composition with parts in $[d]$. Therefore, the contribution of the case $l(\sigma) > 0$ is given by

$$z^d yq PSR_{[d-1]}(z, qy, q) PSR_{[d]}(z, y, 1).$$

This is almost the same as that for strong records, except for the term qy. The variable y gives the number of parts, so for the position of d, we must have one q for each of the y's that came before, hence qy.
Therefore,

$$PSR_{[d]}(z, y, q) = PSR_{[d-1]}(z, y, q) + z^d yq PSR_{[d-1]}(z, qy, q) PSR_{[d]}(z, y, 1). \tag{8.3.2}$$

For $q = 1$ and by induction, we always have the same generating function for compositions into parts, where y counts the number of parts and z counts the size of the composition,

$$PSR_{[d]}(z, y, 1) = \frac{1}{1 - y\sum_{j=1}^{d} z^j}. \tag{8.3.3}$$

Hence, after substituting (8.3.3) in (8.3.2),

$$PSR_{[d]}(z, y, q) = PSR_{[d-1]}(z, y, q) + \frac{z^d yq}{1 - y\sum_{i=1}^{d} z^i} PSR_{[d-1]}(z, qy, q)$$

iterate the first term to get

$$= PSR_{[d-2]}(z,y,q) + \frac{z^{d-1}yq}{1-y\sum_{i=1}^{d-1} z^i} PSR_{[d-2]}(z,qy,q)$$
$$+ \frac{z^{d}yq}{1-y\sum_{i=1}^{d} z^i} PSR_{[d-1]}(z,qy,q)$$
$$= PSR_{[d-3]}(z,y,q) + \sum_{j=d-2}^{d} \frac{z^{j}yq}{1-y\sum_{i=1}^{j} z^i} PSR_{[j-1]}(z,qy,q)$$
$$= 1 + \sum_{j=1}^{d} \frac{z^{j}yq}{1-y\sum_{i=1}^{j} z^i} PSR_{[j-1]}(z,qy,q).$$

We must now iterate d times.

$$PSR_{[d]}(z,y,q) = 1 + \sum_{j_1=1}^{d} \frac{z^{j_1}yq}{1-y\sum_{i=1}^{j_1} z^i} \left[1 + \sum_{j_2=1}^{j-1} \frac{z^{j_2}yq^2}{1-yq\sum_{i=1}^{j_2} z^i} PSR_{[j_2-1]}(z,q^2y,q)\right]$$
$$= 1 + q\sum_{j_1=1}^{d} \frac{z^{j_1}y}{1-y\sum_{i=1}^{j_1} z^i} + q^2 \sum_{j_1=1}^{d} \frac{z^{j_1}y}{1-y\sum_{i=1}^{j_1} z^i} \sum_{j_2=1}^{j_1-1} \frac{z^{j_2}yq}{1-y\sum_{i=1}^{j_2} z^i}$$
$$\times \left[1 + \sum_{j_3=1}^{j_2-1} \frac{z^{j_3}yq^3}{1-yq^2\sum_{i=1}^{j_3} z^i} PSR_{[j_3-1]}(z,q^3y,q)\right]$$
$$= 1 + q\sum_{j_1=1}^{d} \frac{z^{j_1}y}{1-y\sum_{i=1}^{j_1} z^i} + q^2 \sum_{j_1=1}^{d} \frac{z^{j_1}y}{1-y\sum_{i=1}^{j_1} z^i} \sum_{j_2=1}^{j_1-1} \frac{z^{j_2}yq}{1-yq\sum_{i=1}^{j_2} z^i}$$
$$+ q^3 \sum_{d\geq j_1>j_2>j_3\geq 1} \prod_{i=1}^{3} \frac{z^{j_i}yq^{i-1}}{1-yq^{i-1}\sum_{l=1}^{j_i} z^l} PSR_{[j_3-1]}(z,q^3y,q).$$

The pattern is clear and we can state the following theorem:

Theorem 8.3.1 *The generating function $PSR_{[d]}(z,y,q)$ is given by*

$$1 + \sum_{k=1}^{d} q^k \left(\sum_{d\geq j_1>j_2>\cdots>j_k\geq 1} \prod_{i=1}^{k} \frac{z^{j_i}yq^{i-1}}{1-yq^{i-1}\sum_{l=1}^{j_i} z^l} \right). \tag{8.3.4}$$

This generating function is analogous to equation (5.1.2) in Chapter 5, but in Chapter 5 we never gave an explicit formula for $F(z,u)$; we used recurrences instead.

Now, we wish to find the generating function for the number of compositions of n according to the total of the statistic ssrec. Differentiate (8.3.2) with respect to q and let $q = 1$. We must differentiate using the chain rule.

Let

$$v_d(z) = \frac{\partial}{\partial q} PSR_{[d]}(z, 1, q)\Big|_{q=1},$$

then

$$v_d(z) = v_{d-1}(z) + \frac{z^d}{(1 - \sum_{j=1}^{d} z^j)(1 - \sum_{j=1}^{d-1} z^j)} \tag{8.3.5}$$

$$+ \frac{z^d}{1 - \sum_{j=1}^{d} z^j} \left(v_{d-1}(z) + \frac{\partial}{\partial y} PSR_{[d-1]}(z, y, 1)\Big|_{y=1} \right). \tag{8.3.6}$$

The first term $v_{d-1}(z)$ comes from differentiating the first term of (8.3.2). The second term in (8.3.5) is found by differentiating only q in the second term of (8.3.2). The final term of (8.3.5) is found by differentiating the second term of (8.3.2) in terms of the variable qy (since $q = 1$, we are really differentiating by y hence $\frac{\partial}{\partial y}$).

Substituting (8.3.3) in (8.3.5); that is, differentiating

$$PSR_{[d]}(z, y, 1) = \frac{1}{1 - y \sum_{j=1}^{d} z^j}$$

with respect to y, we get that

$$v_d(z) = \frac{1 - \sum_{j=1}^{d-1} z^j}{1 - \sum_{j=1}^{d} z^j} v_{d-1}(z) + \frac{z^d}{(1 - \sum_{j=1}^{d} z^j)(1 - \sum_{j=1}^{d-1} z^j)^2}. \tag{8.3.7}$$

We must now iterate the recurrence relation in (8.3.7) d times. There is only one thing to iterate because v_{d-1} is on the RHS and nothing else. After iterating and using the initial condition $v_0(z) = 0$, we obtain the following corollary:

Corollary 8.3.2 *The generating function* $v_d(z) = \frac{\partial}{\partial q} PSR_{[d]}(z, 1, q)|_{q=1}$ *is given by*

$$v_d(z) = \frac{z}{1 - \sum_{j=1}^{d} z^j} \sum_{j=0}^{d-1} \frac{z^j}{(1 - \sum_{i=1}^{j} z^i)^2}. \tag{8.3.8}$$

From this corollary and letting $d \to \infty$ (we are doing all compositions so we can then sum the geometric series), we can say that the generating function for the number of compositions of n according to the total of the statistic ssrec is given by

$$v(z) := \frac{z(1-z)}{1-2z} \sum_{j \geq 0} \frac{z^j}{(1-\sum_{i=1}^{j} z^i)^2}. \tag{8.3.9}$$

For the asymptotic expansion, we can once again let $q = \frac{1}{2}$ and replace n by $\frac{n}{2}$ in (5.1.18) to obtain the following theorem:

Theorem 8.3.3 *The average sum of the positions of the strong records e_n^s in compositions of n has the asymptotic expansion*

$$e_n^s = \frac{n}{4 \log 2} (1 + \delta_2(\log_2 n)) + O(1), \tag{8.3.10}$$

where $\delta_2(x)$ is a periodic function of period 1, mean zero and small amplitude, which is given by the Fourier series

$$\delta_2(x) = \sum_{k \neq 0} \chi_k \Gamma(-1-\chi_k) e^{2k\pi i x}.$$

8.4 The Sum of Positions of Weak Records (*wsrec*)

Let $PWR_A(z, y, q)$ denote the generating function for the number of compositions of n with m parts in A according to the statistic wsrec:

$$PWR_A(z, y, q) = \sum_{n,m \geq 0} \sum_{\sigma \in C_A(n,m)} z^n y^m q^{wsrec(\sigma)}. \tag{8.4.1}$$

We will find an explicit formula for this generating function.

Again we denote the number of occurrences of the part d in the composition $\sigma \in C_{[d]}(n, m)$ by $l(\sigma)$. Then the contribution of the case $l(\sigma) = 0$ is given by $PWR_{[d-1]}(z, y, q)$.

If, on the contrary, $l(\sigma) > 0$ then σ can be decomposed into $\sigma' d \sigma''$, where σ' is a composition with parts in $[d]$ and σ'' is a composition with parts in $[d-1]$. Previously, when counting how many d's, we had to count every d. Now, all the other d's are irrelevant, except for the d in $\sigma' d \sigma''$, which is the last d. Therefore, we are decomposing according to the last d, not the first one, so therefore we do not need to consider what comes after the d. Thus, the contribution of the case $l(\sigma) > 0$ is

$$z^d yq PWR_{[d]}(z, qy, q) PWR_{[d-1]}(z, y, 1).$$

Therefore,

$$PWR_{[d]}(z, y, q) = PWR_{[d-1]}(z, y, q) + z^d yq PWR_{[d]}(z, qy, q) PWR_{[d-1]}(z, y, 1). \tag{8.4.2}$$

Comparing (8.4.2) to (8.3.2), the only difference is that the subscripts d and $d-1$ in the last term switch positions.

For $q = 1$ and by induction,

$$PWR_{[d]}(z, y, 1) = \frac{1}{1 - y \sum_{j=1}^{d} z^j}. \tag{8.4.3}$$

Thus, we have the following theorem. Note that the method is similar to the previous section, so we just briefly discuss and state the results.

Theorem 8.4.1 *The generating function $PWR_{[d]}(z, y, q)$ satisfies the following recurrence relation*

$$PWR_{[d]}(z, y, q) = PWR_{[d-1]}(z, y, q) + \frac{z^d yq}{1 - y \sum_{i=1}^{d-1} z^i} PWR_{[d]}(z, qy, q). \tag{8.4.4}$$

Now, for

$$w_d(z) = \frac{\partial}{\partial q} PWR_{[d]}(z, 1, q)|_{q=1},$$

we use the same method as for section 8.3: by applying theorem 8.4.1,

$$w_d(z) = w_{d-1}(z) + \frac{z^d}{(1 - \sum_{j=1}^{d-1} z^j)(1 - \sum_{j=1}^{d} z^j)} \tag{8.4.5}$$

$$+\frac{z^d}{1-\sum_{j=1}^{d-1} z^j}\left(w_d(z)+\frac{\partial}{\partial y}PWR_{[d]}(z,y,1)\Big|_{y=1}\right). \tag{8.4.6}$$

Then, using

$$PWR_{[d]}(z,y,1)=\frac{1}{1-y\sum_{j=1}^{d} z^j},$$

we have

$$\frac{1-\sum_{j=1}^{d} z^j}{1-\sum_{j=1}^{d-1} z^j}w_d(z)=w_{d-1}(z)+\frac{z^d}{(1-\sum_{j=1}^{d} z^j)(1-\sum_{j=1}^{d-1} z^j)}. \tag{8.4.7}$$

$$\text{Therefore,}\quad w_d(z)=\frac{1-\sum_{j=1}^{d-1} z^j}{1-\sum_{j=1}^{d} z^j}w_{d-1}(z)+\frac{z^d}{(1-\sum_{j=1}^{d} z^j)^3}. \tag{8.4.8}$$

Iterating the above recurrence d times and using the initial condition $w_0(z)=0$, we obtain the following theorem:

Corollary 8.4.2 *The generating function* $w_d(z)=\frac{\partial}{\partial q}PWR_{[d]}(z,1,q)|_{q=1}$ *is given by*

$$w_d(z)=\frac{1}{1-\sum_{j=1}^{d} z^j}\sum_{j\geq 1}\frac{z^j}{(1-\sum_{i=1}^{j} z^i)^2}. \tag{8.4.9}$$

From this corollary and letting $d\to\infty$, we can see that the generating function for the number of compositions of n according to the total of the statistic wsrec is given by

$$w(z):=\frac{1-z}{1-2z}\sum_{j\geq 1}\frac{z^j}{(1-\sum_{i=1}^{j} z^i)^2}. \tag{8.4.10}$$

For the asymptotic expansion, we again let $q=\frac{1}{2}$ and replace n by $\frac{n}{2}$ in (5.2.6) to get the following theorem:

Theorem 8.4.3 *The average sum of the positions of the weak records* e_n^w *in compositions of* n *has the asymptotic expansion*

$$e_n^w=\frac{n}{2\log 2}\left(1+\delta_2(\log_2 n)\right)+O(1),$$

where $\delta_2(x)$ is a periodic function of period 1, mean zero and small amplitude, which is given by the Fourier series

$$\delta_2(x) = \sum_{k \neq 0} \chi_k \Gamma(-1 - \chi_k) e^{2k\pi i x}.$$

8.5 Combinatorial Explanation

In this section, we give a combinatorial explanation for the fact that the number (sum) of the positions of weak records in all compositions of n plus 2^{n-1} equals the number (sum) of the positions of strong records in all compositions of $n+1$, for $n \geq 1$.

Let $sw_{n,r}$ (or $ss_{n,r}$) be the sum of rth power of the positions of weak (or strong) records in all the compositions of n, that is we are summing the rth power of i, where i is the position.

$$sw_{n,r} = \sum_{\sigma \in C_n} \sum_{\sigma_i \text{ is a weak record of } \sigma} i^r, \tag{8.5.1}$$

$$ss_{n,r} = \sum_{\sigma \in C_n} \sum_{\sigma_i \text{ is a strong record of } \sigma} i^r, \tag{8.5.2}$$

$$sw'_{n,r} = \sum_{\sigma \in C_n(A)} \sum_{\sigma_i \text{ is a weak record of } \sigma, i > 1} i^r, \tag{8.5.3}$$

$$ss'_{n,r} = \sum_{\sigma \in C_n(A)} \sum_{\sigma_i \text{ is a strong record of } \sigma, i > 1} i^r, \tag{8.5.4}$$

where $C_n = \bigcup_{m=1}^{n} C_{n,m}$ is the set of all compositions of n.

From the definitions, the first entry is always a record. Therefore,

$$sw_{n,r} = |C_n| + sw'_{n,r} \tag{8.5.5}$$

and

$$ss_{n,r} = |C_n| + ss'_{n,r} \tag{8.5.6}$$

where $|C_n| = 2^{n-1}$ is the number of compositions of n.

In (8.5.1)-(8.5.4), if we want to sum the positions of records, let $r = 1$. On the other hand, if we want to sum the number of records, let $r = 0$. We are really only interested in the above two cases: when $r = 0$ or $r = 1$ but the above definitions will work for higher numbers, such as $r = 2, 3$.

Now, we can see that $\sigma_1 \dots \sigma_m$ is a composition of n. σ_i, $i > 1$, is a weak record if and only if $\sigma_1 \dots \sigma_{i-1}(\sigma_i + 1)\sigma_{i+1} \dots \sigma_m$ is a composition of $n + 1$. $\sigma_i + 1$, $i > 1$, is then a strong record. Thus, the multiset of all positions i of the weak records in all compositions of n is the same multiset as all positions i of the strong records in all compositions of $n + 1$.

Using the definitions, this means that for all n and r,

$$ss'_{n+1,r} = sw'_{n,r}. \tag{8.5.7}$$

Then using (8.5.5) and (8.5.6), we obtain

$$\begin{aligned} ss_{n+1,r} &= 2^n + ss'_{n+1,r} && \text{by (8.5.6)} \\ &= 2^n + sw'_{n,r} && \text{by (8.5.7)} \\ &= 2^{n-1} + 2^{n-1} + sw'_{n,r} && 2^n \text{ can be split up as } 2^{n-1} + 2^{n-1} \\ &= 2^{n-1} + sw_{n,r} && \text{by (8.5.5).} \end{aligned}$$

Therefore, we can state the following theorem:

Theorem 8.5.1 *For all* $n \geq 1$

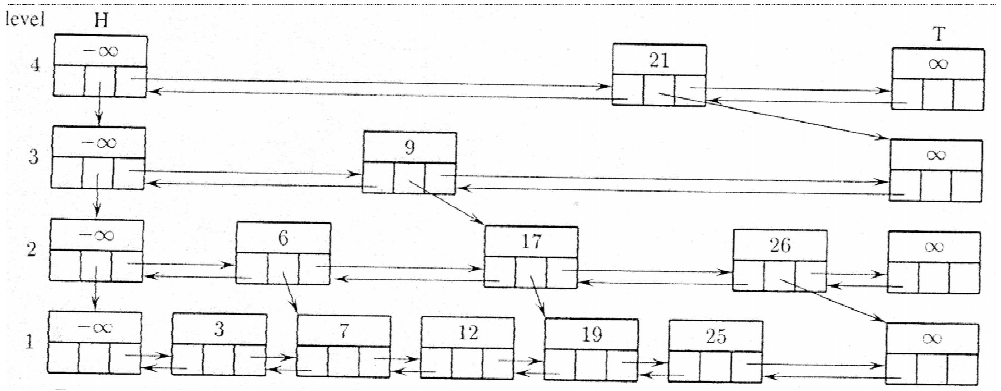
$$ss_{n+1,r} = sw_{n,r} + 2^{n-1}. \tag{8.5.8}$$

Chapter 9

Analysis of a New Skip List Variant

The skip list variant was introduced by Cho and Sahni [1]. In this paper [13], Louchard and Prodinger study the parameter $K(a_1 a_2 \dots a_n)$, which we can call the number of weak *consecutive* maxima. This K parameter is the analogue of the combined horizontal and vertical cost in the original skip list.

A skip list structure, as introduced by Pugh [23], can be understood by the following example. The data (3,6,7,9,12,17,19,21,25,26) have a certain level associated with them, that follows the geometric law $\mathbb{P}(\text{level is} = k) = pq^{k-1}$. They are linked as indicated from the following diagram taken from [13]:



We want to study the length of a path to reach a certain element. For example, to reach 25, follow the path 9−17−19−25. We record four steps then. The values of the data are irrelevant; we are only interested in their levels. Therefore, in the example above, the sequence is 1 2 1 3 1 2 1 4 1 2. We must start from the highest level that allows us to reach the desired element (here being 3) and remain there for as long as possible; otherwise, go down one level. We can more easily understand the skip list process if we consider the sequence in reverse, that is 2 1 4 1 2 1 3 1 2 1. The path then starts at the second element of the sequence, which has level 1, and we scan the sequence, counting elements on the same level, until we encounter an element on a higher level. Consequently, we visit the underlined elements: 2 $\underline{1}$ 4 $\underline{1}$ $\underline{2}$ 1 $\underline{3}$ 1 2 1. These underlined elements are the *consecutive* maxima. There are four underlined elements, the same number as the length of our search path.

To count the number of weak consecutive maxima, we count repetitions of the current maximum and only allow the maximum to change to the next value (=1 + the previous value).

Assume that the levels a_i are independently generated by the geometric law with parameter q with $p = 1 - q$. Consider two parameters: the level I that has already been reached; and the counter K that counts how often the current maximum I has been either repeated or replaced by $I + 1$.

If we start at level r and with $K = 1$ before we start to read the word, we denote this by $K^{\langle r\rangle}(n)$. For a skip list analysis, we must assume that the first symbol read defines the starting value, so we denote this by $\mathcal{K}(n)$. Thus, we can say that

$$\mathcal{K}(n) = \sum_{r\geq 1} pq^{r-1} K^{\langle r\rangle}(n-1).$$

Then

$$\mathbb{P}\{\mathcal{K}(n) = k\} = \sum_{r\geq 1} pq^{r-1} \mathbb{P}\{K^{\langle r\rangle}(n-1) = k\}.$$

We will use the following standard q-series notation:

$$(x)_n = (x;q)_n = \prod_{i=0}^{n-1}(1 - xq^i) \tag{9.0.1}$$

$$(x)_\infty = \prod_{i\geq 0}(1 - xq^i) \tag{9.0.2}$$

$$(x)_n = \frac{(x)_\infty}{(xq^n)_\infty}. \tag{9.0.3}$$

Furthermore, as previously

$$Q = \frac{1}{q} \quad \text{and} \quad L = \log Q.$$

Now, for the generating functions.
Consider the random variables $K^{\langle r \rangle}(n)$ and $I^{\langle r \rangle}(n)$.
Let

$$\pi(n;k,i) = \mathbb{P}\{K^{\langle r \rangle}(n) = k, I^{\langle r \rangle}(n) = i\}.$$

Start at level $r = 1$. Then

$$\pi(n;k,i) = \pi(n-1;k-1,i-1)pq^{i-1} + \pi(n-1;k-1,i)pq^{i-1} + \pi(n-1;k,i)(1-pq^{i-1}-pq^i) \tag{9.0.4}$$

where $\pi(0;1,1) = 1$.
Explanation: In the first term above, we attach a number i so as to increase all parameters by one. In the second term, we already have an i and we add another i to increase the number of consecutive maxima by one. The third term is the final case in which we do not have an i or an $i+1$, so we do not change any of the parameters.

Translate this into a trivariate generating function

$$F(z,u,v) = \sum_{n,k,i \geq 0} \pi(n;k,i) z^n u^k v^i. \tag{9.0.5}$$

Multiplying (9.0.1) by $z^n u^k v^i$, and after simplifying and using $F(0,u,v) = uv$, we obtain

$$F(z,u,v) = \frac{1}{1-z}\left[uv + pz\left(uv + \frac{1}{q}(u-q-1)\right)F(z,u,qv)\right]. \tag{9.0.6}$$

Iterating we get

$$F(z,u,v) = \sum_{j \geq 0} \frac{(pz)^j}{(1-z)^{j+1}} uvq^j \prod_{l=0}^{j-1}\left(uvq^l + \frac{u-q-1}{q}\right). \tag{9.0.7}$$

Ignore the parameter I (the level that has already been reached), which means set $v = 1$, to get

$$G(z,u) = F(z,u,1) = \sum_{j\geq 0} \frac{(pz)^j}{(1-z)^{j+1}} uq^j \prod_{l=0}^{j-1} \left(uq^l + \frac{u-q-1}{q} \right)$$

with $F(0,u,v) = uv^r$.
After iterating, we have the explicit form

$$F(z,u,v) = F^{\langle r \rangle}(z,u,v) = \sum_{j\geq 0} \frac{(pz)^j}{(1-z)^{j+1}} u(vq^j)^r \prod_{l=0}^{j-1} \left(uvq^l + \frac{u-q-1}{q} \right).$$

Therefore, the generating function relevant to the skip list is given by summing the above equation over r:

$$\begin{aligned} \mathcal{F}(z,u,v) &= z \sum_{r\geq 1} pq^{r-1} F^{\langle r \rangle}(z,u,v) \\ &= \frac{u}{q} \sum_{j\geq 1} \frac{(pz)^j}{(1-z)^j} \frac{q^j v}{1-q^j v} \prod_{l=0}^{j-2} \left(uvq^l + \frac{u-q-1}{q} \right). \end{aligned} \tag{9.0.8}$$

Since the I-parameter is not relevant, drop the v variable to obtain

$$\mathcal{G}(z,u) = \mathcal{F}(z,u,1) = \frac{u}{q} \sum_{j\geq 1} \frac{(pz)^j}{(1-z)^j} \frac{q^j}{1-q^j} \prod_{l=0}^{j-2} \left(uq^l + \frac{u-q-1}{q} \right). \tag{9.0.9}$$

Note that

$$[z^{n+1}] \frac{z^j}{(1-z)^j} = \binom{n}{j-1}$$

and then replacing $j-1$ by j we note that

$$[z^{n+1}]\mathcal{G}(z,u) = \frac{u}{q} \sum_{j=0}^{n} (pq)^{j+1} \binom{n}{j} \frac{1}{1-q^{j+1}} \prod_{l=0}^{j-1} \left(uq^l + \frac{u-q-1}{q} \right). \tag{9.0.10}$$

We can derive (9.0.9) combinatorially also, as follows:

There is a unique decomposition of words:

$$(r(\mathbb{N}\backslash\{r,(r+1)\})^*)^+ ((r+1)(\mathbb{N}\backslash\{(r+1),(r+2)\})^*)^+ \ldots (s(\mathbb{N}\backslash\{s,s+1\})^*)^+. \tag{9.0.11}$$

Explanation: The above decomposition indicates that the level starts at r. When the level is set to $r+1$, we stay at this level provided we do not hit a $r+1$ or a $r+2$. The power of the plus sign means that the bracket is a nonempty sequence. It is also clear that the level ends at s.

This symbolic form must then be summed over all possible choices of r and s, to get

$$\overline{\mathcal{G}}(z,u)=\sum_{1\le r\le s}\prod_{i=r}^{s}\frac{zupq^{i-1}}{1-z+zpq^{i-1}(1+q-u)}. \tag{9.0.12}$$

Explanation: The numerator above indicates that we had a part equal to i; while the denominator shows that we have numbers not equal to i or $i+1$, but if the number is a repeat of the number i we must mark it with a u.

It is necessary to show that the two forms of $\mathcal{G}(z,u)$ coincide; that is we will give a direct proof of the equality between (9.0.12) and (9.0.9).
Using formal residue calculus and by substituting $z=\frac{w}{w-1}$, we use the usual formula linking coefficients, as seen in (5.1.15) in Chapter 5.

$$\begin{aligned}[z^n]f(z)&=\frac{1}{2\pi i}\oint f(z)\frac{dz}{z^{n+1}}=-\frac{1}{2\pi i}\oint f(z(w))\frac{dw(w-1)^{n-1}}{w^{n+1}}\\&=(-1)^n[w^n]f(z(w))(1-w)^{n-1}=\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{n-k}[w^{n-k}]f(z(w)).\end{aligned}$$

Therefore, after replacing the z's by w's in (9.0.12),

$$\overline{\mathcal{G}}(z,u)=\sum_{1\le r\le s}(-1)^{s+1-r}\prod_{i=r}^{s}\frac{wupq^{i-1}}{1-wpq^{i-1}(1+q-u)}. \tag{9.0.13}$$

Now, using the formula linking coefficients just discussed, we have, after replacing n by $n+1$,

$$[z^{n+1}]\overline{\mathcal{G}}(z,u)=(-1)^{n+1}[w^{n+1}](1-w)^n\sum_{1\le r\le s}(-1)^{s+1-r}\prod_{i=r}^{s}\frac{wupq^{i-1}}{1-wpq^{i-1}(1+q-u)}$$

then expand out using the binomial theorem and replace $s-r+1$ by h

$$=(-1)^{n+1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k[w^{n+1-k}]\sum_{1\le r,h}(-1)^h\prod_{i=r}^{r+h-1}\frac{wupq^{i-1}}{1-wpq^{i-1}(1+q-u)}$$

now replace $n-k$ by k and i by $i+r-1$ to get

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} [w^{k+1}] \sum_{1 \le r,h} (-1)^h \prod_{i=1}^{h} \frac{wupq^{i+r-2}}{1 - wpq^{i+r-2}(1+q-u)}$$

and after evaluating the geometric series over r,

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} \frac{q^{k+1}}{1-q^{k+1}} [w^{k+1}] \sum_{h \ge 1} (-1)^h \prod_{i=1}^{h} \frac{wupq^{i-2}}{1 - wpq^{i-2}(1+q-u)}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} \frac{(pq)^{k+1}}{1-q^{k+1}} [w^{k+1}] \sum_{h \ge 1} (-1)^h \prod_{i=1}^{h} \frac{wuq^{i-2}}{1 - wq^{i-2}(1+q-u)}.$$

What remains is to prove that

$$\frac{u}{q} \prod_{l=0}^{k-1} \left(uq^l + \frac{u-q-1}{q} \right) = (-1)^{k-1} [w^{k+1}] \sum_{h \ge 1} (-1)^h \prod_{i=1}^{h} \frac{wuq^{i-2}}{1 - wq^{i-2}(1+q-u)}$$

and after replacing w by a $-w$, to prove that

$$= [w^{k+1}] \sum_{h \ge 1} \prod_{i=1}^{h} \frac{wuq^{i-2}}{1 - wq^{i-2}(1+q-u)}.$$

After multiplying through by the k and after shifting the l, h, and i variables, we have equivalently

$$u \prod_{l=1}^{k} \left(uq^l + u - q - 1 \right) = [w^{k+1}] \sum_{h \ge 0} \prod_{i=0}^{h} \frac{wuq^i}{1 + wq^i(1+q-u)}.$$

For this proof, with $v = 1 + q - u$, we set

$$H(w) := \sum_{h \ge 0} \prod_{i=0}^{h} \frac{wuq^i}{1 + wq^i v}.$$

Then,

$$H(w) = \frac{wu}{1+wv} + \frac{wu}{1+wv} H(wq). \tag{9.0.14}$$

To show equation (9.0.14) to be true, consider that

$$H(wq) = \sum_{h \ge 0} \prod_{i=0}^{h} \frac{wuq^{i+1}}{1 + wq^{i+1} v}.$$

So, we multiply $H(wq)$ by the q^0 term and add the missing $h = 0$ term, and hence, we obtain the above equation (9.0.14).

Simplifying (9.0.14), we have

$$(1 + wv)H(w) = wu + wuH(wq).$$

Now, set $a_k = [w^{k+1}]H(w)$. Therefore,

$$a_k + va_{k-1} = u[\![k = 0]\!] + uq^k a_{k-1}$$

with $[\![P]\!] = 1$ if P is true, zero otherwise.
Simplifying,

$$a_k = u[\![k = 0]\!] + (uq^k - v)a_{k-1}$$

and solved by iteration, we have the desired result:

$$a_k = u\prod_{l=1}^{k}(uq^l - v).$$

Consequently, it is clear from the above calculations that $\mathcal{G}(z, u) = \overline{\mathcal{G}}(z, u)$.

9.1 Expectation

Consider equation (9.0.10):

$$[z^{n+1}]\mathcal{G}(z, u) = \frac{u}{q}\sum_{j=0}^{n}(pq)^{j+1}\binom{n}{j}\frac{1}{1 - q^{j+1}}\prod_{l=0}^{j-1}\left(uq^l + \frac{u - q - 1}{q}\right).$$

Differentiate this and let $u = 1$ to obtain the average,

$$1 + Q(Q + 1)\sum_{j=1}^{n}(pq)^{j+1}\binom{n}{j}\frac{1}{1 - q^{j+1}}\prod_{l=1}^{j-1}(q^l - 1)$$

then by the definition in (9.0.1), we have

$$= 1 + p(Q + 1)\sum_{j=1}^{n}(pq)^{j}\binom{n}{j}\frac{1}{1 - q^{j+1}}(-1)^{j-1}(q)_{j-1}$$

$$= 1 + p(Q+1)\sum_{j=1}^{n}(pq)^j\binom{n}{j}\frac{1}{(1-q^{j+1})(1-q^j)}(-1)^{j-1}(q)_j \tag{9.1.1}$$

and by Rice's method

$$= 1 - p(Q+1)\frac{1}{2\pi i}\int \frac{n!(-1)^n}{z(z-1)\dots(z-n)}\frac{(pq)^z}{(1-q^{z+1})(1-q^z)}\frac{(q)_\infty}{(q^{z+1})_\infty}dz. \tag{9.1.2}$$

Then, compute the residues:

$$1 + p(Q+1)[z^{-1}]\frac{n!(-1)^n}{z(z-1)\dots(z-n)}\frac{(pq)^z}{(1-q^{z+1})(1-q^z)}\frac{(q)_\infty}{(q^{z+1})_\infty}$$
$$= 1 + \frac{Q+1}{L}H_n + \frac{Q+1}{L}\ln(p) - (Q+1)\alpha - \frac{(1+q)^2}{2pq} \quad \text{by Mathematica} \tag{9.1.3}$$

with

$$\alpha = \sum_{i\geq 1}\frac{q^i}{1-q^i}. \tag{9.1.4}$$

Hence, the average $\mathbb{E}\mathcal{K}(n)$ is asymptotic to

$$(Q+1)\log_Q n + \frac{(Q+1)\gamma}{L} + \frac{Q+1}{L}\ln(p) - (Q+1)\alpha - \frac{(1+q)^2}{2pq} + \delta(\log_Q n) + 1. \tag{9.1.5}$$

Consequently, we have the following theorem:

Theorem 9.1.1 *Expectation of the $\mathcal{K}(n)$-parameter is asymptotic to*

$$\mathbb{E}\mathcal{K}(n) \sim (Q+1)\log_Q n + \frac{(Q+1)\gamma}{L} + \frac{Q+1}{L}\ln(p) - (Q+1)\alpha - \frac{(1+q)^2}{2pq} + \delta(\log_Q n) + 1,$$

by where the constant α is given by

$$\alpha = \sum_{i\geq 1}\frac{q^i}{1-q^i}.$$

$\delta(x)$ is a small periodic function. Its Fourier coefficients could be given in principle.

Note that the computation is in fact for $\mathcal{K}(n+1)$, instead of $\mathcal{K}(n)$. However, that does not change our asymptotic result above.

Chapter 10

Consecutive Records in Geometrically Distributed Words

Clearly, the parameter $\mathcal{K}(a_1 a_2 \dots a_n)$, the number of weak *consecutive* records, is essential in the analysis of a skip list structure [17]. However, the parameter to be discussed in this chapter is the new parameter $\mathcal{M}$ - the largest *consecutive* record in a random word of length n. In this paper, Oliver and Prodinger [16] consider words $a_1 a_2 \dots a_n$ with independent letters a_k taken from the set of natural numbers, and a weight (probability) attached to it: the letter $i \in \mathbb{N}$ occurs with probability pq^{i-1} $(p + q = 1)$. Again, we derive exact and asymptotic formulae for the expectation and the variance, assuming random words of length n.

We will scan the word from left to right. Assume that the current record is value k. So any letter that is different from k, $k+1$ is ignored. However, if the letter scanned is one of these, we say that this letter is a *weak consecutive record* and set the value of the current maximum equal to this letter. Therefore, the value of the current record either remains k or advances to the next value $k + 1$. Essentially then, we only have a new record if there is an increase of one; otherwise the letter is not a record.

For example, consider the word 1 3 1 1 2 4 3 5 3 5 1 4 1 2 3 4 6 5 1 .
Underline each consecutive maximum $\underline{1}\,3\,\underline{1}\,\underline{1}\,\underline{2}\,4\,\underline{3}\,5\,\underline{3}\,5\,1\,\underline{4}\,1\,2\,3\,\underline{4}\,6\,\underline{5}\,1$.
The parameter $\mathcal{M}$ investigates the maximum of the underlined values. The maxi-

mum of the underlined values is the same as the number of strict consecutive maxima.

For the parameter $\mathcal{M}$, it is not necessary to underline repetitions, as done above. Rather, when our current maximum is $k-1$, we ignore all letters different from k, and when it occurs with probability pq^{k-1}, we set the current maximum equal to k.

Again, we will use the following notation:

$$(x)_n = (x;q)_n = \prod_{i=0}^{n-1}(1-xq^i) \tag{10.0.1}$$

$$(x)_\infty = \prod_{i\geq 0}(1-xq^i) \tag{10.0.2}$$

$$(x)_n = \frac{(x)_\infty}{(xq^n)_\infty}. \tag{10.0.3}$$

Furthermore, as previously

$$Q = \frac{1}{q} \quad \text{and} \quad L = \log Q.$$

10.1 Expectation

Before we can calculate the expectation, we must first analyze the $\mathcal{M}$-parameter.

Let $b_k(z)$ be the generating function. The coefficient of z^n is then the probability that a random word of length n has parameter $\mathcal{M}$ equal to k.

The recursion is

$$b_k(z) = b_{k-1}(z)\frac{zpq^{k-1}}{1-z[1-pq^k]} + \frac{zpq^{k-1}}{1-z[1-pq^k]}, \text{ for } k \geq 1,\ b_0 = 0. \tag{10.1.1}$$

Explanation: Consider firstly the first term. We previously had the record set to value $(k-1)$; hence $b_{k-1}(z)$. A new consecutive record k is attached to an existing word. So k occurs with probability pq^{k-1}. We multiply by z since again z counts

the letters. Therefore, we have the numerator zpq^{k-1}. In the word, the letter k is followed by letters different from $k+1$; thus all these letters are not consecutive records. So we can have any sequence of numbers, none of which is a $k+1$; hence $\frac{1}{1-z[1-pq^k]}$. The second term of the right-hand side corresponds to the situation that there was no previous record; that is, the word starts with letter k.

Simplify (10.1.1) to

$$b_k(z)[1-z(1-pq^k)] = b_{k-1}(z)zpq^{k-1} + zpq^{k-1}. \tag{10.1.2}$$

Substitute $z = \frac{w}{w-1}$ into (10.1.2).

$$b_k(1-wq^k) = -b_{k-1}wpq^{k-1} - wpq^{k-1} \quad \text{after simplifying} \tag{10.1.3}$$

where b_k stands for $b_k(z)$.

Multiply both sides of (10.1.3) by $\frac{(wpq;q)_{k-1}(-1)^k}{w^k p^k q^{\binom{k}{2}}}$ to obtain

$$\frac{b_k(wpq;q)_k(-1)^k}{w^k p^k q^{\binom{k}{2}}} = \frac{b_{k-1}(wpq;q)_{k-1}(-1)^{k-1}}{w^{k-1}p^{k-1}q^{\binom{k-1}{2}}} - \frac{(wpq;q)_{k-1}(-1)^k}{w^{k-1}p^{k-1}q^{\binom{k-1}{2}}}. \tag{10.1.4}$$

(On the LHS, we absorb in the $(1-wpq^k)$ term in (10.1.3) to get $(wpq;q)_k$.)
(On the RHS, the q^{k-1} in the numerator disappears in both terms.)

We need to solve the LHS of (10.1.4) for b_k. We will consider (10.1.3), divide through by $(1-wpq^k)$ and iterate:

$$\begin{aligned} b_k &= \frac{-wpq^{k-1}}{[1-wpq^k]} - \frac{b_{k-1}wpq^{k-1}}{[1-wpq^k]} \\ &= \frac{-wpq^{k-1}}{1-wpq^k} - \frac{wpq^{k-1}}{[1-wpq^k]}\left[\frac{-wpq^{k-2}}{1-wpq^{k-1}} - \frac{b_{k-2}wpq^{k-2}}{1-wpq^{k-1}}\right] \end{aligned}$$

multiplying out the brackets, we obtain

$$= \frac{-wpq^{k-1}}{1-wpq^k} + \frac{(wpq^{k-1})(wpq^{k-2})}{(1-wpq^k)(1-wpq^{k-1})} + \frac{b_{k-2}wpq^{k-1}wpq^{k-2}}{(1-wpq^k)(1-wpq^{k-1})}. \tag{10.1.5}$$

The pattern is clear and we have

$$b_k = \sum_{j=0}^{k-1} \frac{q^{\binom{k}{2}-\binom{j}{2}}(-pw)^{k-j}}{(wpq^{j+1};q)_{k-j}}. \tag{10.1.6}$$

To check that (10.1.5) is accurate, if we let $j = k - 1$ in (10.1.6) we obtain the first term in (10.1.5); and if we let $j = k - 2$ in (10.1.6) we obtain the second term in (10.1.5). The subsequent terms would come from doing further iterations in (10.1.5).

Notice that

$$\sum_{k\geq 1} \frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} = -pq^j w. \tag{10.1.7}$$

Therefore,

$$\begin{aligned}\sum_{k\geq 1} b_k &= \sum_{j\geq 0}\sum_{k\geq 1} \frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} \\ &= -\sum_{j\geq 0} pq^j w \quad \text{by (10.1.7)} \\ &= -w \\ &= \frac{z}{1-z}. \end{aligned} \tag{10.1.8}$$

Let us look at the combinatorial interpretation: Equation (10.1.8) tells us that the sum of b_k's is $\frac{z}{1-z}$, which is a generating function where all coefficients are 1. $\sum b_k$ is the sum of probabilities that the largest consecutive maximum parameter is k. Since every geometric word must have some value k for its consecutive maximum, this generating function therefore describes all possible words when summing over k. Our calculations thus far have verified a combinatorial explanation.

Now compute the average of the $\mathcal{M}$-parameter.

$$\begin{aligned}\text{average} &= \sum_{k\geq 1} k b_k \quad \text{by definition} \\ &= \sum_{k\geq 1} k \sum_{j=0}^{k-1} \frac{q^{\binom{k}{2}-\binom{j}{2}}(-pw)^{k-j}}{(wpq^{j+1};q)_{k-j}} \quad \text{by (10.1.6)} \\ &\quad \text{swap the two sums and replace } k \text{ by } k+j \\ &= \sum_{j\geq 0}\sum_{k\geq 1} (k+j) \frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} \\ &\quad \text{after multiplying out the bracket and using (10.1.7)}\end{aligned}$$

$$= \sum_{j\geq 0}\sum_{k\geq 1} k\frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} - \sum_{j\geq 0} jpq^j w$$

$$= \sum_{j\geq 0}\frac{d}{dt}\sum_{k\geq 0}\frac{q^{\binom{k}{2}}(-pq^j wt)^k}{(wpq^{j+1};q)_k}\bigg|_{t=1} - \frac{q}{p}w. \tag{10.1.9}$$

In the first term, we replaced the inner sum by a derivative while the second term is found by using a standard series.

Consider the inner term of (10.1.9). We will use **Heine's transform**:

$$\sum_{n\geq 0}\frac{(a)_n(b)_n}{(q)_n(c)_n}t^n = \frac{(b)_\infty(at)_\infty}{(c)_\infty(t)_\infty}\sum_{n\geq 0}\frac{(\frac{c}{b})_n(t)_n}{(q)_n(at)_n}b^n. \tag{10.1.10}$$

In the inner sum, we must replace $q^{\binom{k}{2}}(-1)^k$ by $\lim_{\epsilon\to 0}\epsilon^k(\frac{1}{\epsilon})_k$ since

$$\left(\frac{1}{\epsilon}\right)_k\epsilon^k = \prod_{i=0}^{k-1}\left(1-\frac{q^i}{\epsilon}\right)\epsilon = \prod_{i=0}^{k-1}(q^i-\epsilon)(-1) \quad \text{by (10.0.1)}$$

Taking the limit as $\epsilon\to 0$, we get

$$q\sum_{i=0}^{k-1} i(-1)^k = q^{\binom{k}{2}}(-1)^k.$$

So we have

$$\sum_{k\geq 0}\frac{q^{\binom{k}{2}}(-pq^j wt)^k}{(wpq^{j+1};q)_k}$$

$$= \lim_{\epsilon\to 0}\sum_{k\geq 0}\frac{(\frac{1}{\epsilon})_k(q)_k(pq^j wt\epsilon)^k}{(wpq^{j+1};q)_k(q)_k}$$

then using Heine's transform with $a=\frac{1}{\epsilon}$, $b=q$, $c=wpq^{j+1}$, and $t=(pq^j wt\epsilon)$

$$= \lim_{\epsilon\to 0}\frac{(q)_\infty(pq^j wt)_\infty}{(wpq^{j+1})_\infty(pq^j wt\epsilon)_\infty}\sum_{k\geq 0}\frac{(wpq^j)_k(pq^j wt\epsilon)_k q^k}{(q)_k(pq^j wt)_k}$$

$$= \frac{(q)_\infty(pq^j wt)_\infty}{(wpq^{j+1})_\infty}\sum_{k\geq 0}\frac{(0)_k(wpq^j)_k q^k}{(q)_k(pq^j wt)_k} \quad \text{after letting } \epsilon\to 0$$

then using Heine's transform again with $a=0$, $b=wpq^j$, $c=pq^j wt$, and $t=q$

$$= \frac{(q)_\infty (pq^j wt)_\infty}{(wpq^{j+1})_\infty} \frac{(wpq^j)_\infty}{(pq^j wt)_\infty (q)_\infty} \sum_{k\geq 0} \frac{(t)_k (q)_k (wpq^j)^k}{(q)_k}$$

$$= (1 - wpq^j) \sum_{k\geq 0} (t)_k (wpq^j)^k \quad \text{after cancelling.} \tag{10.1.11}$$

Next differentiate with respect to t and let $t = 1$.

$$\frac{d}{dt}(t)_k = \frac{d}{dt}\left[\prod_{i=0}^{k-1}(1 - tq^i)\right] \quad \text{by (10.0.1)}$$

$$= \underbrace{\prod_{i=0}^{k-1}(1 - tq^i)} \sum_{i=0}^{k-1} \frac{-q^i}{1 - tq^i}$$

$$= 0 \text{ for } t = 1$$

$$\text{So} \quad \frac{d}{dt}(t)_k \Big|_{t=1} = \prod_{i=1}^{k-1}(1 - q^i)(-q^0) = -(q)_{k-1}.$$

So, we have

$$-(1 - wpq^j) \sum_{k\geq 1} (q)_{k-1} (wpq^j)^k. \tag{10.1.12}$$

Substitute in (10.1.9). We now have

$$\sum_{k\geq 1} kb_k = -\sum_{j\geq 0}(1 - wpq^j) \sum_{k\geq 0} (q)_k (wpq^j)^{k+1} - \frac{q}{p}w \qquad k-1 \text{ was replaced by } k$$

$$= \sum_{j\geq 0}\sum_{k\geq 0} (q)_k (wpq^j)^{k+1} + \sum_{j\geq 0}\sum_{k\geq 0} (q)_k (wpq^j)^{k+2} - \frac{q}{p}w \quad \text{after multiplying in}$$

$$= -\sum_{k\geq 0} \frac{(q)_k (wp)^{k+1}}{1 - q^{k+1}} + \sum_{k\geq 1} \frac{(q)_{k-1}(wp)^{k+1}}{1 - q^{k+1}} - \frac{q}{p}w \quad \text{the } j \text{ sums are just geometric series}$$

we then separate out the $k = 0$ term from the first term and add it to the last term:

$$= -\sum_{k\geq 1} \frac{(q)_k (wp)^{k+1}}{1 - q^{k+1}} + \sum_{k\geq 1} \frac{(q)_{k-1}(wp)^{k+1}}{1 - q^{k+1}} - \frac{w}{p}.$$

Then combining the two sums and using the fact that $(q)_k = (q)_{k-1}(1 - q^k)$,

$$\sum_{k\geq 1} kb_k = \sum_{k\geq 1} \frac{(q)_{k-1} q^k (wp)^{k+1}}{1 - q^{k+1}} - \frac{w}{p}. \tag{10.1.13}$$

Extract the coefficients of the powers of w

$$[w^{j+1}]\sum_{k\geq 1} kb_k = \frac{(q)_{j-1}q^j p^{j+1}}{1-q^{j+1}} \quad \text{for } j \geq 1 \tag{10.1.14}$$

and

$$[w^1]\sum_{k\geq 1} kb_k = -\frac{1}{p}. \tag{10.1.15}$$

Now use the standard formula from Chapter 5.

$$\begin{aligned}
[z^{n+1}]\sum_{k\geq 1} kb_k &= (-1)^{n+1}[w^{n+1}](1-w)^n \sum_{k\geq 1} kb_k \\
&= (-1)^{n+1}\sum_{j=0}^{n}\binom{n}{n-j}(-1)^{n-j}[w^{j+1}]\sum_{k\geq 1} kb_k \quad \text{by the binomial theorem} \\
&= \sum_{j=0}^{n}\binom{n}{j}(-1)^{j+1}[w^{j+1}]\sum_{k\geq 1} kb_k \quad \text{by replacing } n-j \text{ by } j \\
&\text{then substitute in (10.1.13), to get} \\
&= \frac{1}{p} + \sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\frac{(q)_{j-1}q^j p^{j+1}}{1-q^{j+1}}.
\end{aligned} \tag{10.1.16}$$

By Rice's method, we can write

$$[z^{n+1}]\sum_{k\geq 1} kb_k = \frac{1}{p} + \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(-1)^{n+1}n!}{z(z-1)\dots(z-n)}\frac{(q)_{z-1}q^z p^{z+1}}{1-q^{z+1}}dz.$$

The curve $\mathcal{C}$ encircles the poles $1, 2, \dots, n$ and no others.
For the asymptotic evaluation, collect the (negative) residues at 0 and at $z = \chi_k = \frac{2\pi i k}{L}$ to obtain

$$\log_Q n - \alpha + \frac{\log(p)}{L} + \frac{\gamma}{L} + \frac{1}{2} + \delta(\log_Q n) \tag{10.1.17}$$

where

$$\alpha = \sum_{i\geq 1}\frac{q^i}{1-q^i}. \tag{10.1.18}$$

10.2 Variance

Compute the second factorial moment:

$$\begin{aligned}
&\sum_{k\geq 2} k(k-1)b_k \\
&= \sum_{k\geq 1} k(k-1) \sum_{j=0}^{k-1} \frac{q^{\binom{k}{2}-\binom{j}{2}}(-pw)^{k-j}}{(wpq^{j+1};q)_{k-j}} \quad \text{by (10.1.6)} \\
&= \sum_{j\geq 0}\sum_{k\geq 1} (k+j)(k+j-1)\frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} \quad \text{replace } k \text{ by } k+j \\
&= \sum_{j\geq 0}\sum_{k\geq 1} k(k-1)\frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} + 2\sum_{j\geq 0} j \sum_{k\geq 1} k\frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} - \frac{2q^2}{p^2}w. \qquad (10.2.1)
\end{aligned}$$

Consider the middle sum.

$$\begin{aligned}
&\sum_{j\geq 0} j \sum_{k\geq 1} k\frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} \\
&= -\sum_{j\geq 0} j(1-wpq^j)\sum_{k\geq 1}(q)_{k-1}(wpq^j)^k \quad \text{by (10.1.12)} \\
&= -\sum_{j\geq 0} j\sum_{k\geq 1}(q)_{k-1}(wpq^j)^k + \sum_{j\geq 0} j\sum_{k\geq 1}(q)_{k-1}(wpq^j)^{k+1} \quad \text{after multiplying out}
\end{aligned}$$

swap sums and evaluate the jth sum to obtain

$$= -\sum_{k\geq 1}\frac{(q)_{k-1}(wp)^k q^k}{(1-q^k)^2} + \sum_{k\geq 1}\frac{(q)_{k-1}(wp)^{k+1}q^{k+1}}{(1-q^{k+1})^2}$$

split off $k=1$ from the first sum to get $-\frac{wq}{p}$ and then combine the remaining sums:

$$= -\frac{wq}{p} + \sum_{k\geq 1}\frac{(q)_{k-1}(wp)^{k+1}q^{2k+1}}{(1-q^{k+1})^2}. \qquad (10.2.2)$$

Substituting (10.2.2) into (10.2.1), we obtain

$$\sum_{k\geq 2} k(k-1)b_k = \sum_{j\geq 0}\sum_{k\geq 1} k(k-1)\frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} + 2\sum_{k\geq 1}\frac{(q)_{k-1}(wp)^{k+1}q^{2k+1}}{(1-q^{k+1})^2} - \frac{2q}{p^2}w. \qquad (10.2.3)$$

Now we consider the first double sum above:

$$\sum_{j\geq 0}\sum_{k\geq 1} k(k-1)\frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} = \sum_{j\geq 0}\frac{d}{dt^2}(1-wpq^j)\sum_{k\geq 0}(t)_k(wpq^j)^k\Big|_{t=1} \quad \text{by (10.1.11).}$$

From here, we can see that we must differentiate $(t)_k$ twice with respect to t and then let $t=1$.

$$\frac{d}{dt}(t)_k = \prod_{i=0}^{k-1}(1-tq^i)\sum_{i=0}^{k-1}\frac{-q^i}{1-tq^i}.$$

By Leibniz rule,

$$\frac{d^2}{dt^2}(t)_k = 2\prod_{i=0}^{k-1}(1-tq^i)\sum_{0\leq i<j\leq k-1}\frac{(-q^i)}{1-tq^i}\frac{(-q^j)}{1-tq^j}.$$

For $t=1$, this is zero unless $i=0$ in which case we get

$$\frac{d^2}{dt^2}(t)_k = 2\prod_{i=0}^{k-1}(1-q^i)\sum_{j=1}^{k-1}\frac{q^j}{1-q^j} = 2(q)_{k-1}T(k-1).$$

Hence the first double sum is as follows:

$$\sum_{j\geq 0}\sum_{k\geq 1} k(k-1)\frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} = 2\sum_{j\geq 0}(1-wpq^j)\sum_{k\geq 2}(q)_{k-1}T(k-1)(wpq^j)^k$$

where

$$T(k) = \sum_{i=1}^{k}\frac{q^i}{1-q^i}.$$

Multiply out the bracket $(1-wpq^j)$ and replace k by $k+1$.

$$\sum_{j\geq 0}\sum_{k\geq 1} k(k-1)\frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} = 2\sum_{k\geq 1}\frac{(q)_k T(k)(wp)^{k+1}}{1-q^{k+1}} - 2\sum_{k\geq 1}\frac{(q)_{k-1}T(k-1)(wp)^{k+1}}{1-q^{k+1}}. \tag{10.2.4}$$

Since we know that $T(k) = \frac{q^k}{1-q^k} + T(k-1)$ and $(q)_k = (q)_{k-1}\times(1-q^k)$, substitute into (10.2.4) and take out a common factor of $(q)_{k-1}$, to obtain

$$\sum_{j\geq 0}\sum_{k\geq 1} k(k-1)\frac{q^{\binom{k}{2}}(-pq^j w)^k}{(wpq^{j+1};q)_k} = 2\sum_{k\geq 1}\frac{(q)_{k-1}\left[1-T(k-1)\right](wp)^{k+1}q^k}{1-q^{k+1}}. \tag{10.2.5}$$

Then substitute back into (10.2.3):

$$\sum_{k\geq 2} k(k-1)b_k = 2\sum_{k\geq 1} \frac{(q)_{k-1}\left[1-T(k-1)\right](wp)^{k+1}q^k}{1-q^{k+1}} + 2\sum_{k\geq 1} \frac{(q)_{k-1}(wp)^{k+1}q^{2k+1}}{(1-q^{k+1})^2} - \frac{2q}{p^2}w. \tag{10.2.6}$$

Taking the coefficient, we have for $j \geq 1$ and $\frac{-2q}{p^2}$ for $j = 0$

$$[w^{j+1}]\sum_{k\geq 2} k(k-1)b_k = 2\frac{(q)_{j-1}\left[1-T(j-1)\right]p^{j+1}q^j}{1-q^{j+1}} + 2\frac{(q)_{j-1}p^{j+1}q^{2j+1}}{(1-q^{j+1})^2}. \tag{10.2.7}$$

Therefore, we consider

$$\begin{aligned}
&[z^{n+1}]\sum_{k\geq 1} k(k-1)b_k \\
&= (-1)^{n+1}[w^{n+1}](1-w)^n \sum_{k\geq 1} k(k-1)b_k \\
&= (-1)^{n+1}\sum_{j=0}^{n}\binom{n}{n-j}(-1)^{n-j}[w^{j+1}]\sum_{k\geq 1} k(k-1)b_k \quad \text{by the binomial theorem} \\
&= \sum_{j=0}^{n}\binom{n}{j}(-1)^{j+1}[w^{j+1}]\sum_{k\geq 1} k(k-1)b_k \quad \text{by symmetry}
\end{aligned}$$

substitute in (10.2.7) to obtain

$$= \frac{2q}{p^2} + 2\sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}(q)_{j-1}p^{j+1}q^j\left[\frac{-T(j-1)}{1-q^{j+1}} + \frac{1}{(1-q^{j+1})^2}\right]. \tag{10.2.8}$$

Again, by Rice's method, there is an integral representation

$$\begin{aligned}
&[z^{n+1}]\sum_{k\geq 1} k(k-1)b_k \\
&= \frac{2q}{p^2} + \frac{2}{2\pi i}\int_{\mathcal{C}} \frac{(-1)^n n!}{z(z-1)\ldots(z-n)}(q)_{z-1}p^{z+1}q^z\left[\frac{T(z-1)}{1-q^{z+1}} - \frac{1}{(1-q^{z+1})^2}\right]dz.
\end{aligned} \tag{10.2.9}$$

Note that

$$T(k) = \sum_{j=1}^{k}\frac{q^j}{1-q^j} = \alpha - \sum_{j>k}\frac{q^j}{1-q^j} = \alpha - \sum_{j\geq 1}\frac{q^{j+k}}{1-q^{j+k}} \tag{10.2.10}$$

where

$$\alpha = \sum_{i \geq 1} \frac{q^i}{1-q^i}. \tag{10.2.11}$$

Then

$$T(z) = \alpha - \sum_{j \geq 1} \frac{q^{j+z}}{1-q^{j+z}}. \tag{10.2.12}$$

We expect there to be a triple pole at $z = 0$. Therefore, first expand $T(z-1)$.

$$\begin{aligned} T(z-1) &= T(z) - \frac{q^z}{1-q^z} \\ &= \alpha - \sum_{j \geq 1} \frac{q^{j+z}}{1-q^{j+z}} - \frac{q^z}{1-q^z} \quad \text{by (10.2.12).} \end{aligned}$$

Consider $\sum_{j \geq 1} \frac{q^{j+z}}{1-q^{j+z}}$ and do the Taylor expansion around $z = 0$
i.e., $f(z) = f(0) + f'(0)z + \ldots$.

Let

$$f(z) = \sum_{j \geq 1} \frac{q^{j+z}}{1-q^{j+z}}.$$

$$f(0) = \sum_{j \geq 1} \frac{q^j}{1-q^j} = \alpha.$$

$$f'(z) = \sum_{j \geq 1} \frac{(1-q^{j+z})q^{j+z} \log q + q^{(j+z)2} \log q}{(1-q^{j+z})^2}.$$

$$\text{Therefore} \quad f'(0) = \sum_{j \geq 1} \frac{q^j \log q}{(1-q^j)^2} = \beta L$$

$$\text{where} \quad \beta = \sum_{j \geq 1} \frac{q^j}{(1-q^j)^2}.$$

Hence,

$$T(z-1) \sim \beta L z - \frac{1}{Lz} + \frac{1}{2} - \frac{Lz}{12}. \tag{10.2.13}$$

Also note that

$$(q)_z = \frac{(q)_\infty}{(q^{z+1})_\infty} \quad \text{by definition.}$$

We do Taylor expansions. Let

$$f(z) = \frac{(q)_\infty}{(q^{z+1})_\infty}.$$

$$f(0) = 1.$$

$$f'(z) = \frac{(q)_\infty}{(q^{z+1})_\infty^2}(q^{z+1})_\infty \sum_{i\geq 0} \frac{-q^{z+1+i}\log q}{1-q^{z+1}q^i} = -L\frac{(q)_\infty}{(q^{z+1})_\infty}\sum_{i\geq 0}\frac{q^{z+1+i}}{1-q^{z+1+i}}.$$

$$f'(0) = \sum_{i\geq 0}\frac{q^{i+1}}{1-q^{i+1}}\log q = -\alpha L \quad \text{where } \alpha \text{ is given by (10.2.11).}$$

$$f''(z) = +L\frac{(q)_\infty}{(q^{z+1})_\infty}\left(\sum_{i\geq 0}\frac{q^{z+1+i}}{1-q^{z+1+i}}\right)^2 + L\frac{(q)_\infty}{(q^{z+1})_\infty}\sum_{i\geq 0}\frac{q^{z+1+i}(1-q^{z+1+i}) - q^{z+1+i}(-q^{z+1+i})}{(1-q^{z+1+i})^2}.$$

$$f''(0) = +L\alpha^2 + L\beta = (\alpha^2+\beta)L.$$

Hence,

$$(q)_z = \frac{(q)_\infty}{(q^{z+1})_\infty} \sim 1 - \alpha L z + \frac{\alpha^2+\beta}{2}L^2z^2. \tag{10.2.14}$$

Next, in Mathematica we expand everything else and compute the residue. Reading off the negative residue at $z=0$ gives

$$(\log_Q n)^2 + 2\left(\frac{\gamma+\log(p)}{L} - \alpha\right)\log_Q n + \frac{\log^2(p)+\gamma^2+\frac{\pi^2}{6}+\log(p)}{L^2} - \frac{2\alpha\,(\log(p)+\gamma)}{L} + \alpha^2 - \beta - \frac{1}{6}. \tag{10.2.15}$$

For the variance, add the expectation and subtract off the square of the expectation to obtain

$$\frac{\pi^2}{6L^2} + \frac{1}{12} - \beta. \tag{10.2.16}$$

Theorem 10.2.1 *The average and variance of the $\mathcal{M}$-parameter are asymptotically given by*

$$\mathbb{E}\mathcal{M}(n) \sim \log_Q n - \alpha + \frac{\log(p)}{L} + \frac{\gamma}{L} + \frac{1}{2} + \delta_E(\log_Q n) \tag{10.2.17}$$

$$\mathbb{V}\mathcal{M}(n) \sim \frac{\pi^2}{6L^2} + \frac{1}{12} - \beta + \delta_V(\log_Q n). \tag{10.2.18}$$

Here, $\delta(x)$ is an unspecified periodic function of period 1 and small amplitude. Its Fourier coefficients could be computed in principle. The poles come from the residues at $z = \chi_k = \frac{2\pi ik}{L}$.

Chapter 11

Appendix

We recalculate the mean for the strict case from Chapter 4, using the different method of substitution as introduced in Chapter 5.

Consider (4.2.4). Substitute in $\frac{w}{w-1}$ and simplify.

$$\begin{aligned} f(z) &= \frac{\frac{pw}{w-1}}{1-\frac{w}{w-1}} \sum_{k\geq 0} \frac{q^k}{1-\frac{w}{w-1}(1-q^k)} \\ &= -pw(1-w) \sum_{k\geq 0} \frac{q^k}{1-wq^k}. \end{aligned}$$

Then

$$\begin{aligned} [z^n]f(z) &= (-1)^{n+1}[w^n](1-w)^{n-1} f(z(w)) \\ &= (-1)^{n+1}[w^n](1-w)^n pw \sum_{k\geq 0} \frac{q^k}{1-wq^k} \\ &= p\sum_{l=0}^{n} \binom{n}{l} (-1)^{l+1}[w^l] \sum_{k\geq 0} \frac{wq^k}{1-wq^k} \\ &= p\sum_{l=0}^{n} \binom{n}{l} (-1)^{l+1} \sum_{k\geq 0} (q^k)^l [w^l] \frac{w}{w-1} \\ &= p\sum_{l=1}^{n} \binom{n}{l} (-1)^{l+1} \frac{1}{1-q^l} \end{aligned}$$

$$
\begin{aligned}
&= p\sum_{l=1}^{n}\binom{n}{l}(-1)^{l+1}\sum_{k\geq 0}(q^l)^k \\
&= p\sum_{k\geq 0}\sum_{l=1}^{n}\binom{n}{l}(-1)^{l+1}q^{kl} \\
&= p\sum_{k\geq 0}[1-(1-q^k)^n].
\end{aligned}
$$

We have therefore used the substitution method and rederived what is equivalent to (4.2.6).

For the expectation in the weak case, consider

$$f(z) = \frac{pz}{1-z}\sum_{k\geq 0}\frac{q^k}{1-z(1-q^{k+1})}.$$

Continuing as before,

$$
\begin{aligned}
f(z) &= \frac{\frac{pw}{w-1}}{1-\frac{w}{w-1}}\sum_{k\geq 0}\frac{q^k}{1-\frac{w}{w-1}(1-q^{k+1})} \\
&= -pw(1-w)\sum_{k\geq 0}\frac{q^k}{1-wq^{k+1}} \\
&= \frac{-pw}{q}(1-w)\sum_{k\geq 0}\frac{q^{k+1}}{1-wq^{k+1}} \\
&= \frac{-pw}{q}(1-w)\sum_{k\geq 0}\frac{q^{k+1}}{1-wq^{k+1}} \\
&= \frac{-pw}{q}(1-w)\sum_{k\geq 1}\frac{q^k}{1-wq^k} \\
&= \frac{-pw}{q}(1-w)\left[\sum_{k\geq 0}\frac{q^k}{1-wq^k}-\frac{1}{1-w}\right].
\end{aligned}
$$

The extra term above is just a constant, which does not contribute to the coefficient of z^n. So we really have the same sum as for the strict case; and hence, we will obtain the same answer as in (4.3.3).

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