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On the invariant analysis and integrability of the time-fractional potential KdV equation

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Abstract. We perform a Lie point symmetry analysis of a time-fractional potential Korteweg–de Vries (FPKdV) equation with the Riemann–Liouville derivative. By transforming the dependent variable, we map the time-fractional FP-KdV equation to a nonlinear ordinary differential equation (ODE) of fractional order using underlying symmetry generators. We obtain the derivative in the Erdélyi–Kober operator. We then construct the solution of the reduced fractional ODE by applying the power series method. The conservation laws (CLs) for the time-fractional FP-KdV equation are determined via Ibragimov's non-local conservation method to time-fractional partial differential equations (FPDEs). Solutions for FPDEs via CLs have yet to be explored. Additionally, we present graphical representations of the results obtained using the power series solution method.

Keywords. Potential Korteweg–de Vries; Lie symmetries; symmetry reduction; Riemann–Liouville fractional derivative; explicit power series solutions; conservation laws.

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1. Introduction

The well-known Korteweg–de Vries (KdV) equation [1] generates an infinite class of conservation laws (CLs) and their integrability; consequently, it is well known and documented. A large class of solutions is of the soliton or travelling wave type. Whether such solitons are obtained in what follows is yet to be seen. Recently, it has been shown that many complex nonlinear phenomena [2–4] and dynamics in engineering, viscoelasticity, electrochemistry, physics, electromagnetics, *inter alia*, have been successfully modelled using fractional calculus [5]. In particular, physical phenomena often depend on their current and historical states, which can be effectively modelled using the theory of fractional-order derivatives and integrals. Consequently, several analytical methods [6–12] are employed to derive exact, explicit and numerical solutions for nonlinear time-fractional partial differential equations (FPDEs) [13,14]. A few studies have been conducted on the symmetry analysis of FPDEs and their group properties are not well understood [15–19].

CLs play an essential role in the analysis of nonlinear partial differential equations (PDEs) in a physical sense. If the examined system generates CLs, it is feasible to discuss its integrability. Noether or variational symmetries can be obtained in variational systems, in which one studies the invariance of the action integral. According to Noether's theorem, Noether symmetries admit CLs [20]. In addition, many approaches do not require a Lagrangian to determine the CLs of nonlinear PDEs [21,22].

We can replace the time derivative of the nonlinear PDEs with the fractional derivative to obtain time-fractional nonlinear PDEs. In this work, we perform a Lie symmetry analysis to obtain reductions and then construct explicit solutions using the power series method and Ibragimov's non-local CLs [23] for the time-fractional FPKdV equation given in [24,25].

In the model

$$\mathcal{D}_t^\sigma \psi + a(\psi_x)^2 + b\psi_{xxx} = 0, \quad 0 < \sigma \leq 1, \tag{1}$$

where a and b are real-valued constants that serve as nonlinear models with significant physical rele-

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vance, particularly in water wave studies. In eq. (1), the first term signifies the evolution, the second term encapsulates nonlinearity and the last term denotes the third-order dispersion. Here, x and t are independent variables, representing spatial and temporal domains, respectively, while $\psi(x,t)$ denotes the dependent variable representing the wave profile. In eq. (1), $\mathcal{D}_t^{\sigma}$ denotes the RL partial derivative of order $\sigma (0 < \sigma \le 1)$ of the function $\psi(x,t)$ which can be defined as [26]

$$\mathcal{D}_t^{\sigma}\psi(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\sigma)}\dfrac{\partial^m}{\partial t^m}\displaystyle\int_0^t (t-\varrho)^{m-\sigma-1} \\ \quad \psi(x,\varrho)\mathrm{d}\varrho, & m-1<\sigma<m, \\ \dfrac{\partial^m\psi}{\partial t^m}, & \sigma = m \in \mathbb{N}. \end{cases} \tag{2}$$

Lie symmetry analysis of (1) for $\sigma = 1$ was performed in [27]. The results are interesting and open new literature on the invariant theory in this model. Additionally, this study adds new CLs and soliton solutions. To further explore this direction, we considered the fractional form of the desired model, which explains the phenomenon more closely.

The remainder of this paper is organised as follows. We present some basic definitions in §2. The fractional form of the desired model that explains the phenomena more closely. In §3, we obtain the symmetries and simple form of (1). The possible reductions by the optimal system are described in §4. We obtain explicit power series solutions to reduce fractional ODEs in §5. In addition, we discuss the solutions physically using graphs and convergence analysis. In §6, we derive the conservation laws of (1) using the formal Lagrangian method.

2. Preliminaries

We consider some basic definitions that are required in our work.

Definition 1. (*Fractional derivative*) The Riemann–Liouville (RL) derivative can be described as [26]

$$\mathcal{D}^{\sigma}h(t) = \begin{cases} \dfrac{\mathrm{d}^m}{\mathrm{d}t^m}I^{m-\sigma}h(t), & 0 \le m-1 < \sigma < m, \\ \dfrac{\mathrm{d}^m h}{\mathrm{d}t^m}, & \sigma = m \in \mathbb{N}, \end{cases} \tag{3}$$

where m is a natural number and $I^{\varphi}h(t)$ is the RL fractional integral of order φ given by

$$I^{\varphi}h(t) = \frac{1}{\Gamma(\varphi)}\int_0^t (t-\upsilon)^{\varphi-1}h(\upsilon)\mathrm{d}\upsilon, \qquad \varphi > 0,$$
$$I^{\varphi}h(t) = h(t), \qquad \varphi = 0,$$

where $\Gamma(\varphi)$ represents the Gamma function.

Definition 2. (*Lie group of transformations*) Given a one-parameter Lie group of transformation, we extend

$$\hat{\psi} = G(\psi; s), \tag{4}$$

into its Taylor series in parameter s, in the neighbourhood $s = 0$. Then the infinitesimal transformation of the Lie group of transformation is

$$\hat{\psi} = \psi + s\theta(\psi) + O(s^2), \tag{5}$$

where

$$\theta(\psi) = \left.\frac{\partial\hat{\psi}}{\partial s}\right|_{s=0}. \tag{6}$$

The components of the vector $\theta(\psi) = (\theta_1(\psi), \theta_2(\psi), \ldots, \theta_n(\psi))$ are called the infinitesimals of (4).

Definition 3. (*Infinitesimal generator*)

The operator

$$\mathbf{V} = \sum_{l=1}^{n}\theta_l(\psi)\frac{\partial}{\partial\psi_l}, \tag{7}$$

is known as the infinitesimal generator of the one-parameter Lie group of transformations (4), where $\psi = (\psi_1, \psi_2, \ldots, \psi_n) \in \mathbf{R}^n$ and $\theta(\psi) = (\theta_1(\psi), \theta_2(\psi), \ldots, \theta_n(\psi))$ are the infinitesimals of (4).

Definition 4. (*Lie bracket*) Let $\mathfrak{L}$ be the one-parameter Lie group of transformations (4) with the symmetry generators $\mathbf{V}_d, d = 1, 2, \ldots, \Theta$ given by (7). The commutator (Lie bracket) $[.,.]$ of the two symmetry generators $\mathbf{V}_e, \mathbf{V}_o$ is the first-order operator generated by

$$[\mathbf{V}_e, \mathbf{V}_o] = \mathbf{V}_e\mathbf{V}_o - \mathbf{V}_o\mathbf{V}_e.$$

Definition 5. (*Lie algebra*) A Lie algebra $\mathfrak{L}$ is a vector space over a field F with a bi-linear commutation law (the commutator) satisfying the following properties:

- Closure: For $L, M \in \mathfrak{L}$ it implies that $[L, M] \in \mathfrak{L}$.
- Bilinearity: $[L, yM + zN] = y[L, M] + z[L, N]$, $y, z \in F$, and $L, M, N \in \mathfrak{L}$.
- Skew symmetry: $[L, M] = -[M, L]$
- Jacobi identity: $[L, [M, N]] + [M, [N, L]] + [N, [L, M]] = 0$.

2.1 *Computation of the Lie point symmetries*

In this section, we discuss the general technique to find out the Lie symmetries of $(1+1)$-dimensional FPDE [26]

$$\partial_t^{\sigma}\psi(x,t) = E(t, x, \psi, \psi_x, \psi_{xx}, \ldots), \quad (0<\sigma\le 1), \tag{8}$$

with a one-parameter symmetry group of transformations as follows:

$$\begin{aligned}
t^\star &= t + s\theta^1(x,t,\psi) + O(s^2),\\
x^\star &= x + s\theta^2(x,t,\psi) + O(s^2),\\
\psi^\star &= \psi + s\Upsilon(x,t,\psi) + O(s^2),\\
\frac{\partial^\sigma \psi^\star}{\partial t^{\star\sigma}} &= \frac{\partial^\sigma \psi}{\partial t^\sigma} + s\Upsilon^0_\sigma(x,t,\psi) + O(s^2),\\
\frac{\partial \psi^\star}{\partial x^\star} &= \frac{\partial \psi}{\partial x} + s\Upsilon^x(x,t,\psi) + O(s^2),\\
\frac{\partial^2 \psi^\star}{\partial x^{\star 2}} &= \frac{\partial^2 \psi}{\partial x^2} + s\Upsilon^{xx}(x,t,\psi) + O(s^2),\\
\frac{\partial^3 \psi^\star}{\partial x^{\star 3}} &= \frac{\partial^3 \psi}{\partial x^3} + s\Upsilon^{xxx}(x,t,\psi) + O(s^2),\\
&\vdots,
\end{aligned} \tag{9}$$

where s is the group parameter and

$$\begin{aligned}
\Upsilon^x &= \mathbf{D}_x(\Upsilon) - \psi_t \mathbf{D}_x(\theta^1) - \psi_x \mathbf{D}_x(\theta^2),\\
\Upsilon^{xx} &= \mathbf{D}_x(\Upsilon^x) - \psi_{tx} \mathbf{D}_x(\theta^1) - \psi_{xx} D_x(\theta^2),\\
\Upsilon^{xxx} &= \mathbf{D}_x(\Upsilon^{xx}) - \psi_{txx} \mathbf{D}_x(\theta^1) - \psi_{xxx} \mathbf{D}_x(\theta^2).
\end{aligned} \tag{10}$$

Here, the total differential operator is denoted by the symbol $\mathbf{D}_j$ which can be defined as

$$\mathbf{D}_j = \frac{\partial}{\partial x^j} + \psi_j \frac{\partial}{\partial \psi} + \psi_{jk}\frac{\partial}{\partial \psi_k} + \cdots, \quad j,k = 1,2,3. \tag{11}$$

The vector fields related to the preceding transformation groups can be defined as

$$\mathbf{V} = \theta^1 \frac{\partial}{\partial t} + \theta^2 \frac{\partial}{\partial x} + \Upsilon \frac{\partial}{\partial \psi}\cdot \tag{12}$$

The operator given in eq. (12) is a point symmetry of eq. (8) iff,

$$\mathbf{V}^{[\sigma,3]}(\nabla)|_{\nabla=0} = 0, \tag{13}$$

where $0 < \sigma \le 1$ and

$$\nabla := \partial_t^\sigma \psi(x,t) - E(x,t,\psi,\psi_x,\psi_{xx},\ldots).$$

The invariance condition exhibits

$$\theta^1(x,t,\psi)|_{t=0} = 0 \tag{14}$$

and the σth extended infinitesimal related to RL fractional time derivative with eq. (14) is given by [26]

$$\begin{aligned}
\Upsilon^0_\sigma &= \frac{\partial^\sigma \Upsilon}{\partial t^\sigma} + \left(\Upsilon_\psi - \sigma \mathbf{D}_t(\theta^1)\right)\frac{\partial^\sigma \psi}{\partial t^\sigma} - \psi \frac{\partial^\sigma \psi_\psi}{\partial t^\sigma}\\
&\quad + \sum_{m=1}^{\infty}\left[\binom{\sigma}{m}\frac{\partial^m \Upsilon_\psi}{\partial t^m} - \binom{\sigma}{m+1}\mathbf{D}_t^{m+1}(\theta^1)\right]\\
&\quad \times \mathcal{D}_t^{\sigma-m}(\psi) - \sum_{m=1}^{\infty}\binom{\sigma}{m}\mathbf{D}_t^m(\theta^2)\mathcal{D}_t^{\sigma-m}(\psi_x) + \vartheta,
\end{aligned} \tag{15}$$

where

$$\binom{\sigma}{m} = \frac{(-1)^{m-1}\sigma\Gamma(m-\sigma)}{\Gamma(1-\sigma)\Gamma(m+1)}$$

and

$$\begin{aligned}
\vartheta &= \sum_{m=2}^{\infty}\sum_{l=2}^{m}\sum_{\phi=2}^{l}\sum_{r=0}^{\phi-1}\binom{\sigma}{m}\binom{m}{l}\binom{\phi}{q}\frac{1}{\phi!}\\
&\quad \times \frac{t^{m-\sigma}}{\Gamma(m+1-\sigma)}(-1)^q \psi^q \frac{\partial^l}{\partial t^l}[\psi^{\phi-q}]\frac{\partial^{m-l+\phi}}{\partial t^{m-l}\partial \psi^\phi}\cdot
\end{aligned} \tag{16}$$

Definition 6. $\psi = \Phi(x,t)$ is known as an invariant solution of eq. (8) related with the symmetry generator (12) if

- $\psi = \Phi(x,t)$ is the invariant surface of eq. (9), mathematically,

$$\mathbf{V}\Phi = 0 \Leftrightarrow \left(\theta^1 \frac{\partial \Phi}{\partial t} + \theta^2 \frac{\partial \Phi}{\partial x} + \Upsilon \frac{\partial \Phi}{\partial \psi}\right) = 0,$$

- $\psi = \Phi(x,t)$ satisfies eq. (8).

3. Symmetries of the FPKdV eq. (1)

Suppose that eq. (1) remains invariant under the transformation equation (9). We have

$$\mathbf{D}_t^\sigma \bar{\psi} + a(\bar{\psi}_x)^2 + b\bar{\psi}_{xxx} = 0, \quad 0 < \sigma \le 1, \tag{17}$$

such that $\psi = \psi(x,t)$ satisfies eq. (1). Then, by inserting eq. (9) into eq. (17), we can get the subsequent invariant equation

$$\Upsilon^0_\sigma + 2a\psi_x \Upsilon^x + b\Upsilon^{xxx} = 0. \tag{18}$$

By substituting the values of Υ^x, Υ^{xxx} and Υ^0_σ into eq. (18) and putting all the values of the monomials of derivatives of ψ equal to zero, we get the following system:

$$
\begin{aligned}
&\partial_t^\sigma \Upsilon + b\Upsilon_{xxx} - \psi \partial_t^\sigma \Upsilon_\psi = 0,\\
&\binom{\sigma}{m}\partial_t^m \Upsilon_\psi - \binom{\sigma}{m+1} D_t^{m+1}(\theta^1) = 0,\ m=1,2,3,\ldots\\
&\Upsilon_{\psi\psi} = \theta_\psi^1 = \theta_\psi^2 = \theta_t^2 = \theta_x^1 = 0,\\
&-a\Upsilon_\psi + a\sigma\theta_t^1 + 2a\Upsilon_\psi - 2a\theta_x^2 + 3b\Upsilon_{\psi\psi x} = 0,\\
&b\sigma\theta_t^1 - 3b\theta_x^2 = 0,\\
&2a\Upsilon_x + b(3\Upsilon_{\psi xx} - \theta_{xxx}^2) = 0,\\
&-2a\theta_\psi^2 + b(\Upsilon_{\psi\psi\psi} - 3\theta_{\psi\psi x}^2) = 0.
\end{aligned}
\tag{19}
$$

Solution of the above system yields

$$\theta^1 = c_1 + 3tc_2,$$

$$\theta^2 = c_3 + \sigma x c_2,$$

$$\Upsilon = -\sigma\psi c_2 + d(t),$$

where $0 < \sigma \leq 1$, c_1, c_2 and c_3 are constants. Using the invariant condition (14), we get $c_1 = 0$ and thus we have infinite-dimensional symmetry algebra for the FPKdV eq. (1) given by

$$
\begin{aligned}
\mathbf{V_1} &= \frac{\partial}{\partial x},\\
\mathbf{V_2} &= 3t\frac{\partial}{\partial t} - \sigma\psi\frac{\partial}{\partial \psi} + \sigma x\frac{\partial}{\partial x},\\
\mathbf{V}_\infty &= d(t)\frac{\partial}{\partial \psi}\cdot
\end{aligned}
\tag{20}
$$

To make this finite-dimensional, we take $d(t) = t$ and the obtained symmetry algebra is

$$
\begin{aligned}
\mathbf{V}_1 &= \frac{\partial}{\partial x},\\
\mathbf{V}_2 &= 3t\frac{\partial}{\partial t} - \sigma\psi\frac{\partial}{\partial \psi} + \sigma x\frac{\partial}{\partial x},\\
\mathbf{V}_3 &= t\frac{\partial}{\partial \psi}\cdot
\end{aligned}
\tag{21}
$$

Theorem 1. *The vector fields* $\mathbf{V}_i$ $(i = 1, 2, 3)$ *form a three-dimensional Lie algebra.*

Proof. Table 1 shows antisymmetry and zero diagonal elements. One can easily derive the structure constants from the commutator table and verifying the Jacobi identity is a straightforward process. □

4. One-dimensional system of subalgebras and symmetry reductions

In this section, using the symmetry generators obtained in the previous section, we determine the optimal algebra and reduced forms for eq. (1). As our original PDE has two independent variables, after transformation or reduction, we obtain it as an ODE.

Table 1. Commutator table.

$[\cdot,\cdot]$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$
$\mathbf{V}_1$	0	$\sigma\mathbf{V}_1$	0
$\mathbf{V}_2$	$-\sigma\mathbf{V}_1$	0	$(\sigma+3)\mathbf{V}_3$
$\mathbf{V}_3$	0	$-(\sigma+3)\mathbf{V}_3$	0

Table 2. Adjoint table.

$Ad(e^s)$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$
$\mathbf{V}_1$	$\mathbf{V}_1$	$-\sigma s\mathbf{V}_1 + \mathbf{V}_2$	$\mathbf{V}_3$
$\mathbf{V}_2$	$e^{\sigma s}\mathbf{V}_1$	$\mathbf{V}_2$	$e^{-(\sigma-3)s}\mathbf{V}_3$
$\mathbf{V}_3$	$\mathbf{V}_1$	$\mathbf{V}_2 + (\sigma+3)s\mathbf{V_3}$	$\mathbf{V}_3$

The adjoint representation of a Lie group or algebra can be determined using the Lie series, which is expressed as

$$\mathrm{Ad}\,(\exp{(s\mathbf{V}_e)}\,\mathbf{V}_o) = \mathbf{V}_o - s\,[\mathbf{V}_e, \mathbf{V}_o] + \frac{s^2}{2}\,[\mathbf{V}_e, [\mathbf{V}_e, \mathbf{V}_o]] - \cdots. \tag{22}$$

Table 1 lists the commutator table for the Lie algebra (21).

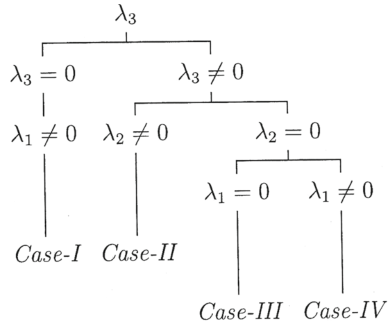
Theorem 2. *A system of one-dimensional subalgebras for the FP-KdV eq.* (1) *is given by* $\langle\mathbf{V}_1\rangle$, $\langle\mathbf{V}_2\rangle$, $\langle\mathbf{V}_3\rangle$ *and* $\langle\mathbf{V}_1 \pm \mathbf{V}_3\rangle$.

Proof. We deal with the symmetry algebra $\mathfrak{L}^3$ of FP-KdV eq. (1), for which the adjoint representation was connected in table 2 and non-zero commutators are

$$[\mathbf{V}_1, \mathbf{V}_3] = 3\mathbf{V}_1,\ [\mathbf{V}_2, \mathbf{V}_3] = \mathbf{V}_2.$$

We start with an arbitrary element $\mathbf{V}$ of the symmetry algebra $\mathfrak{L}^3$ which is defined by

$$\mathbf{V} = \lambda_1\mathbf{V}_1 + \lambda_2\mathbf{V}_2 + \lambda_3\mathbf{V}_3. \tag{23}$$



Case I. $\lambda_3 = 0$ and $\lambda_1 \neq 0$. Then we get

$$\mathbf{V} = \lambda_1\mathbf{V}_1. \tag{24}$$

Then by adjoint actions we get a subalgebra

$\mathbf{P}_1 = \langle \mathbf{V}_1 \rangle$.

Case II. $\lambda_1 = 0,\ \lambda_2 \neq 0, \lambda_3 \neq 0$. Then we get

$$\mathbf{V} = \lambda_2 \mathbf{V}_2 + \lambda_3 \mathbf{V}_3. \tag{25}$$

We take the action of $\mathbf{V}_3$ from the possible actions $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$. This results in

$$Ad(e^s \mathbf{V}_3)\mathbf{V} = \lambda_2 \mathbf{V}_2 + [s\lambda_2(\sigma+3) + \lambda_3]\mathbf{V}_3. \tag{26}$$

Taking

$$s = -\frac{\lambda_3}{\lambda_2(\sigma+3)},$$

we get

$\mathbf{P}_2 = \langle \mathbf{V}_2 \rangle$.

Case III. $\lambda_1 = \lambda_2 = 0, \lambda_3 \neq 0$. Then we get

$$\mathbf{V} = \lambda_3 \mathbf{V}_3. \tag{27}$$

Then by adjoint actions we get a subalgebra

$\mathbf{P}_3 = \langle \mathbf{V}_3 \rangle$.

Case IV. $\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 \neq 0$. Then we get

$$\mathbf{V} = \lambda_1 \mathbf{V}_1 + \lambda_3 \mathbf{V}_3. \tag{28}$$

Since $\mathbf{V}_1$ and $\mathbf{V}_3$ are not effecting, we take the action of $\mathbf{V}_2$

$$Ad(e^s \mathbf{V}_2)\mathbf{V} = \lambda_1 \mathbf{V}_1 + \lambda_3 e^{s(-2\sigma-3)}\mathbf{V}_3. \tag{29}$$

This results in a new subalgebra

$\mathbf{P}_4 = \langle \mathbf{V}_1 \pm \mathbf{V}_3 \rangle$.

Thus, we have a system of one-dimensional subalgebra for the symmetry algebra (21) given by

$$\begin{aligned} &\mathbf{P}_1 = \langle \mathbf{V}_1 \rangle,\\ &\mathbf{P}_2 = \langle \mathbf{V}_2 \rangle,\\ &\mathbf{P}_3 = \langle \mathbf{V}_3 \rangle,\\ &\mathbf{P}_4 = \langle \mathbf{V}_1 \pm \mathbf{V}_3 \rangle. \end{aligned} \tag{30}$$

4.1 *Symmetry reductions*

Reduction by $\mathbf{P}_1 = V_1$:

The characteristics formula for the infinitesimal generator $\mathbf{V_1}$ can be expressed symbolically as

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}\psi}{0}. \tag{31}$$

To theorise this clearly, solving eq. (31) yields $t = z$ and $\psi = g_1(z)$ with $g_1(z)$ satisfying completely $\mathbf{D}_t^\sigma g_1(z) = 0$, where $0 < \sigma \le 1$. Therefore, the group-invariant solution congruent with $\mathbf{V_1}$ can be expressed as $\psi_1(x,t) = k_1 t^{\sigma-1}$, where k_1 is an arbitrary parameter.

Reduction by $\mathbf{P}_2 = V_2$: Specifically, for the symmetry $\mathbf{V_2}$, we can write it as

$$\frac{\mathrm{d}t}{3t} = \frac{\mathrm{d}x}{\sigma x} = \frac{\mathrm{d}\psi}{-\psi\sigma}.$$

From this, we can get the similarity transformation and similarity variables which are given by

$$\xi = x t^{-\frac{\sigma}{3}}, \quad \psi(x,t) = t^{-\frac{\sigma}{3}}\mathbf{F}(\xi), \tag{32}$$

where $\mathbf{F}$ can be regarded as a function of independent variable ξ. By substituting eq. (32) into eq. (11), we obtain a new nonlinear fractional ODE.

Presently, we can obtain the essential theorem defined in the following theorem.

Theorem 3. *Equation* (32) *transforms eq.* (1) *to a nonlinear fractional ODE which can be written as*

$$\left(\mathcal{P}_{\frac{3}{\sigma}}^{1-\frac{4\sigma}{3},\sigma}\mathbf{F}\right)(\xi) + a(\mathbf{F}_\xi)^2 + b\mathbf{F}_{\xi\xi\xi} = 0, \quad 0 < \sigma \le 1 \tag{33}$$

with the Erdélyi–Kober (EK) fractional operator which can be defined by

$$\left(\mathcal{P}_\beta^{\theta^1,\sigma}\mathbf{F}\right)(\xi) = \prod_{j=0}^{m-1}\left(\theta^1 + j - \frac{1}{\beta}\frac{d}{d\xi}\right) \times \left(\mathcal{K}_\beta^{\theta^1+\sigma,m-\sigma}\mathbf{F}\right)(\xi), \tag{34}$$

$$m = \begin{cases} [\sigma]+1, & \sigma \notin \mathbf{N},\\ \sigma, & \sigma \in \mathbf{N}, \end{cases} \tag{35}$$

such that

$$\left(\mathcal{K}_\beta^{\theta^1,\sigma}\mathbf{F}\right)(\xi) = \begin{cases} \frac{1}{\Gamma(\sigma)}\int_1^\infty (\psi-1)^{\sigma-1}\psi^{-(\theta^1+\sigma)} \mathbf{F}(\xi\psi^{\frac{1}{\beta}})\mathrm{d}\psi, & \sigma > 0\\ \mathbf{F}(\xi), & \sigma = 0, \end{cases} \tag{36}$$

is the EK fractional integral operator.

Proof. Assume that $m-1 < \sigma < 1, m = 1$ and order (m)=order (h) where $h \in \mathbb{R}$. Using the RL definition in view of eq. (32), one can obtain at a stage

$$\frac{\partial^\sigma \psi}{\partial t^\sigma} = \frac{\partial^m}{\partial t^m}\left[\frac{1}{\Gamma(m-\sigma)}\int_0^t (t-\nu)^{m-\sigma-1} \times \nu^{\frac{-\sigma}{3}}\mathbf{F}\left(x\nu^{\frac{-\sigma}{3}}\right)\mathrm{d}\nu\right]. \tag{37}$$

Substituting $\zeta = t/\mu \Rightarrow \mathrm{d}\mu = -(t/\zeta^2)\mathrm{d}\zeta$, eq. (37) becomes

$$\frac{\partial^\sigma \psi}{\partial t^\sigma} = \frac{\partial^m}{\partial t^m}\left[t^{m-\frac{4\sigma}{3}}\frac{1}{\Gamma(m-\sigma)}\int_1^\infty (\zeta-1)^{m-\sigma-1}\right.$$

$$\times\zeta^{-(1-\frac{\sigma}{3}+m-\sigma)}\mathbf{F}\left(\xi\zeta^{\frac{\sigma}{3}}\right)\mathrm{d}\zeta\Bigg]. \quad (38)$$

Applying the EK fractional integration eq. (36) to eq. (38), one can have

$$\frac{\partial^\sigma\psi}{\partial t^\sigma}=\frac{\partial^m}{\partial t^m}\left[t^{m-\frac{4\sigma}{3}}\left(\mathcal{K}_{\frac{3}{\sigma}}^{1-\frac{\sigma}{3},m-\sigma}\mathbf{F}\right)(\xi)\right],\quad 0<\sigma\le 1. \quad (39)$$

Presently, we sort the right-hand side of eq. (39). Now, using the relation $\xi=xt^{-\frac{\sigma}{3}}$, $\Omega\in(0,\infty)$, one can have

$$\begin{aligned}t\frac{\partial}{\partial t}\Omega(\xi)&=tx\left(-\frac{\sigma}{3}\right)t^{-\frac{\sigma}{3}-1}\Omega'(\xi)\\&=-\frac{\sigma}{3}\xi\frac{\partial}{\partial\xi}\Omega(\xi).\end{aligned} \quad (40)$$

Therefore, one can get

$$\begin{aligned}\frac{\partial^\sigma\psi}{\partial t^\sigma}&=\frac{\partial^{m-1}}{\partial t^{m-1}}\left[\frac{\partial}{\partial t}\left(t^{m-\frac{4\sigma}{3}}\left(\mathcal{K}_{\frac{3}{\sigma}}^{1-\frac{\sigma}{3},m-\sigma}\mathbf{F}\right)(\xi)\right)\right],\\&\quad 0<\sigma\le 1,\\&=\frac{\partial^{m-1}}{\partial t^{m-1}}\left[t^{m-1-\frac{4\sigma}{3}}\left(m-\frac{4\sigma}{3}-\frac{\sigma}{3}\xi\frac{\partial}{\partial\xi}\right)\right.\\&\quad\left.\times\left(\mathcal{K}_{\frac{3}{\sigma}}^{1-\frac{\sigma}{3},m-\sigma}\mathbf{F}\right)(\xi)\right].\end{aligned} \quad (41)$$

Repeat order $m-1$ times yields

$$\begin{aligned}&\frac{\partial^m}{\partial t^m}\left[t^{m-\frac{4\sigma}{3}}\left(\mathcal{K}_{\frac{3}{\sigma}}^{1-\frac{\sigma}{3},m-\sigma}\mathbf{F}\right)(\xi)\right]\\&=\frac{\partial^{m-1}}{\partial t^{m-1}}\left[\frac{\partial}{\partial t}\left(t^{m-\frac{4\sigma}{3}}\left(\mathcal{K}_{\frac{3}{\sigma}}^{1-\frac{\sigma}{3},m-\sigma}\mathbf{F}\right)(\xi)\right)\right],\\&=\frac{\partial^{m-1}}{\partial t^{m-1}}\left[t^{m-1-\frac{4\sigma}{3}}\left(m-\frac{4\sigma}{3}-\frac{\sigma}{3}\xi\frac{d}{d\xi}\right)\right.\\&\quad\left.\times\left(\mathcal{K}_{\frac{3}{\sigma}}^{1-\frac{\sigma}{3},m-\sigma}\mathbf{F}\right)(\xi)\right],\\&\quad\vdots\\&=t^{-\frac{4\sigma}{3}}\prod_{j=0}^{m-1}\left[\left(1-\frac{4\sigma}{3}+j-\frac{\sigma}{3}\xi\frac{\partial}{\partial\xi}\right)\right.\\&\quad\left.\times\left(\mathcal{K}_{\frac{3}{\sigma}}^{1-\frac{\sigma}{3},m-\sigma}\mathbf{F}\right)(\xi)\right].\end{aligned}$$

Using the EK fractional derivative eq. (34), one can have

$$\begin{aligned}&\frac{\partial^m}{\partial t^m}\left[t^{m-\frac{4\sigma}{3}}\left(\mathcal{K}_{\frac{3}{\sigma}}^{1-\frac{\sigma}{3},m-\sigma}\mathbf{F}\right)(\xi)\right]\\&=t^{-\frac{4\sigma}{3}}\left(\mathcal{P}_{\frac{3}{\sigma}}^{1-\frac{4\sigma}{3},\sigma}\mathbf{F}\right)(\xi).\end{aligned} \quad (42)$$

Substituting eq. (42) into eq. (39), one can have

$$\frac{\partial^\sigma\psi}{\partial t^\sigma}=t^{-\frac{4\sigma}{3}}\left(\mathcal{P}_{\frac{3}{\sigma}}^{1-\frac{4\sigma}{3},\sigma}\mathbf{F}\right)(\xi),\quad 0<\sigma\le 1. \quad (43)$$

Hence, eq. (11) is transformed into the succeeding ODE of fractional order

$$\left(\mathcal{P}_{\frac{3}{\sigma}}^{1-\frac{4\sigma}{3},\sigma}\mathbf{F}\right)(\xi)+a(\mathbf{F}_\xi)^2+b\mathbf{F}_{\xi\xi\xi}=0,\quad 0<\sigma\le 1. \quad (44)$$

The proof of the theorem is completed.

5. Explicit power series solutions for the FPKdV eq. (1)

To reduce the FPDE into nonlinear FPDEs in the ending assessment, we construct an exact explicit solution of the nonlinear fractional differential equations using the power series technique. In this section, we consider the power series solutions of the original FDE when obtaining the series solution of the FPDE. Suppose

$$\mathbf{F}(\xi)=\sum_{\kappa=0}^{\infty}s_\kappa\xi^\kappa, \quad (45)$$

so that

$$\begin{aligned}(\mathbf{F}_\xi)^2&=\sum_{\kappa=0}^{\infty}\sum_{n=0}^{\kappa}(\kappa+1-n)(n+1)s_{n+1}s_{\kappa+1-n}\xi^\kappa\\\mathbf{F}_{\xi\xi\xi}&=\sum_{\kappa=0}^{\infty}(\kappa+1)(\kappa+2)(\kappa+3)s_{\kappa+3}\xi^\kappa.\end{aligned} \quad (46)$$

Putting eqs (45) and (46) into eq. (44), one obtains

$$\begin{aligned}&\sum_{\kappa=0}^{\infty}\frac{\Gamma(1-\frac{4\sigma}{3}+1-\frac{\kappa\sigma}{3})}{\Gamma(1-\frac{4\sigma}{3}+1+\sigma-\frac{\kappa\sigma}{3})}s_\kappa\xi^\kappa\\&\quad+a\sum_{\kappa=0}^{\infty}\sum_{n=0}^{\kappa}(\kappa+1-n)(n+1)s_{n+1}s_{\kappa+1-n}\xi^\kappa\\&\quad+b\sum_{\kappa=0}^{\infty}(\kappa+1)(\kappa+2)(\kappa+3)s_{\kappa+3}\xi^\kappa=0.\end{aligned} \quad (47)$$

Comparing coefficients in eq. (47), when $n=0$, we get

$$s_3=\frac{-1}{6b}\left(as_1^2+\frac{\Gamma(2-\frac{4\sigma}{3})}{\Gamma(2-\frac{\sigma}{3})}s_0\right). \quad (48)$$

When $\kappa>1$, we get

$$s_{\kappa+3}=\frac{-1}{b(\kappa+1)(\kappa+2)(\kappa+3)}$$

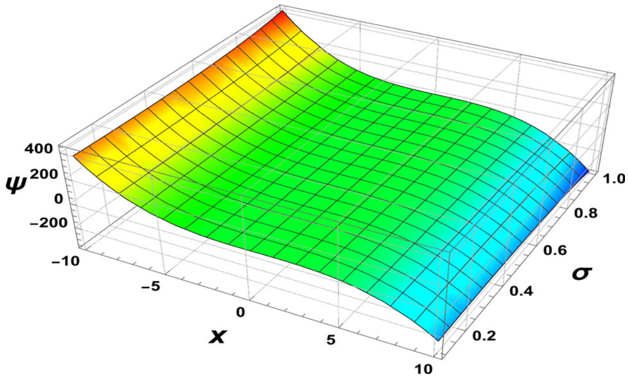


Figure 1. The impact of $0.1 < \sigma < 1$ on the solitary wave profile (51) of eq. (1) at a time $t = 1$, with constants $s_0 = 0.2$, $s_1 = 0.3$, $s_2 = 0.4$, $b = 1$ and $a = 1$.

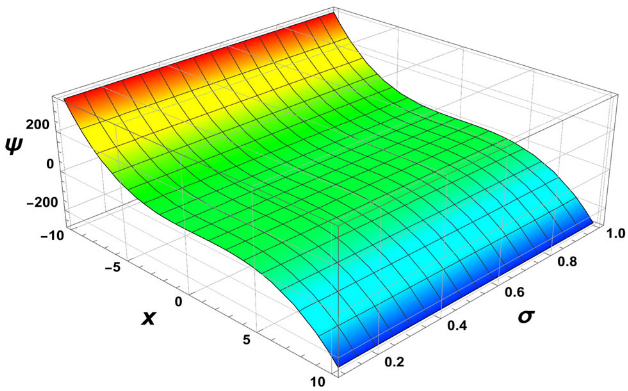


Figure 2. The impact of solution (51) of eq. (1) when $\sigma = 0.3$, $m = 4$ and the remaining have the same variable values.

$$\times\left(a\sum_{p=0}^{\kappa}(\kappa+1-p)(p+1)s_{p+1}s_{\kappa+1-p}\right.$$
$$\left.+\frac{\Gamma(2-\frac{4\sigma}{3}-\frac{n\sigma}{3})}{\Gamma(2-\frac{\sigma}{3}-\frac{n\sigma}{3})}s_\kappa\right). \tag{49}$$

After that, the series solution of eq. (44) can be written as

$$\mathbf{F}(\xi) = s_0 + s_1\xi + s_2\xi^2 + \left(\frac{-1}{6b}as_1^2 + \frac{\Gamma(2-\frac{4\sigma}{3})}{\Gamma(2-\frac{\sigma}{3})}s_0\right)$$
$$\times\xi^3 + \sum_{\kappa=1}^{\infty}\left[\frac{-1}{b(\kappa+1)(\kappa+2)(\kappa+3)}\right.$$
$$\times\left(a\sum_{p=0}^{\kappa}(\kappa+1-p)(p+1)s_{p+1}s_{\kappa+1-p}\right.$$
$$\left.\left.+\frac{\Gamma(2-\frac{4\sigma}{3}-\frac{n\sigma}{3})}{\Gamma(2-\frac{\sigma}{3}-\frac{n\sigma}{3})}s_\kappa\right)\right]\xi^{\kappa+3}. \tag{50}$$

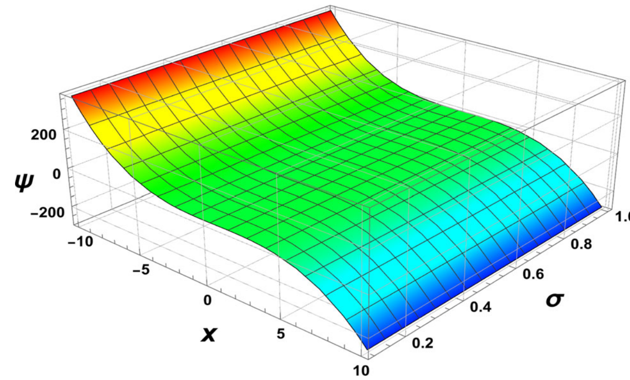


Figure 3. The impact of solution (51) of eq. (1) when $\sigma = 0.6$, $m = 4$ and same variable values.

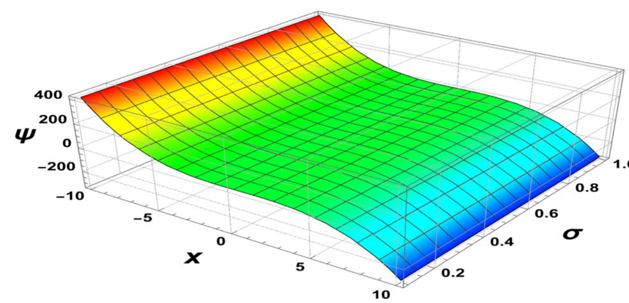


Figure 4. The impact of solution (51) of eq. (1) when $\sigma = 0.9$, $m = 4$ and same variable values.

As a result, we get the power series solution which can be written as

$$\psi(x,t) = s_0t^{\frac{-\sigma}{3}} + s_1xt^{\frac{-2\sigma}{3}} + s_2x^2t^{-\sigma}$$
$$+\left(\frac{-1}{6b}(as_1^2 + \frac{\Gamma(2-\frac{4\sigma}{3})}{\Gamma(2-\frac{\sigma}{3})}s_0\right)x^3t^{\frac{-4\sigma}{3}}$$
$$+\sum_{\kappa=1}^{\infty}\left[\frac{-1}{b(\kappa+1)(\kappa+2)(\kappa+3)}\right.$$
$$\times\left(a\sum_{p=0}^{\kappa}(\kappa+1-p)(p+1)s_{p+1}s_{\kappa+1-p}\right.$$
$$\left.\left.+\frac{\Gamma(2-\frac{4\sigma}{3}-\frac{n\sigma}{3})}{\Gamma(2-\frac{\sigma}{3}-\frac{n\sigma}{3})}s_\kappa\right)\right]x^{\kappa+3}t^{\frac{-\sigma(\kappa+4)}{3}}. \tag{51}$$

In eq. (51) the parameters s_0, s_1 and s_2 are arbitrary. Hence all the coefficients s_m with $\kappa = 2, 3, 4, \ldots$ can be obtained by applying eqs (48) and (49).

5.1 *Graphical representation of the explicit solutions*

To facilitate the evaluation of the characteristics of the power series solutions, we generate various measurement plots of solution (51), as depicted in figures 1, 2,

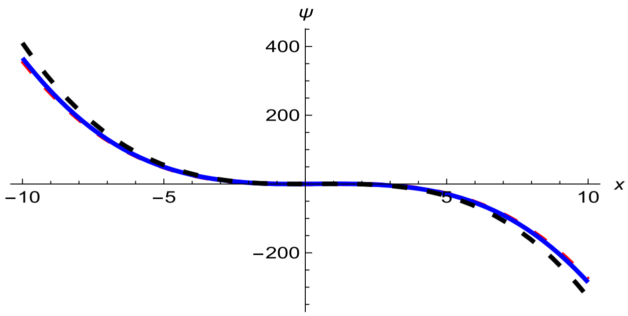


Figure 5. The 2D plot depicted in figure 1 at $t = 0.5$ indicating that as the value of σ increases, the amplitude of the wave diminishes. This observation suggests that solitary wave depiction exhibits characteristics akin to a dynamical system.

3, 4, 5. This involves selecting suitable parameter values to visualise the behaviour of the solution.

5.2 *Convergence analysis*

In this section, we explore the convergence properties of the explicit solution (51) using the coefficients defined by eqs (49) and (48). Equation (49) takes the following form:

$$|s_{\kappa+3}| \le \left|\frac{-1}{b}\right|\left(|a|\sum_{p=0}^{\kappa}|s_{p+1}||s_{\kappa+1-p}| + \frac{|\Gamma(2-\frac{4\sigma}{3}-\frac{n\sigma}{3})|}{|\Gamma(2-\frac{\sigma}{3}-\frac{n\sigma}{3})|}|s_\kappa|\right). \tag{52}$$

However,

$$\frac{\Gamma(p)}{\Gamma(q)} < 1$$

for any p and q values. Thereafter, eq. (52) becomes

$$|s_{\kappa+3}| \le \Lambda\left(\sum_{p=0}^{\kappa}|s_{p+1}||s_{\kappa+1-p}| + |s_\kappa|\right) \tag{53}$$

in which $\Lambda = \max\{2, a\}$. To proceed, we use another power series $N(\xi)$ as

$$N(\xi) = \sum_{\kappa=0}^{\infty} n_\kappa \xi^\kappa \tag{54}$$

with $n_i = |s_i|$, $i = 0, 1, 2 \ldots$. So, we have

$$n_{\kappa+2} \le \Lambda\left[|n_\kappa| + \sum_{p=0}^{m}|n_p||n_{\kappa-p}|\right], \tag{55}$$

where $\kappa = 0, 1, 2 \ldots$ and it is clear that

$$|n_{\kappa+2}| \le s_{\kappa+2} \Rightarrow |n_\kappa| \le s_\kappa.$$

The series given by eq. (54) is a major series of eq. (45). Presently, we have to demonstrate the series $N = N(\xi)$ whose radius of convergence is positive. Then, eq. (54) will be given in the following manner:

$$\begin{aligned} N(\xi) &= n_0 + n_1\xi + \sum_{\kappa=2}^{\infty} n_\kappa \xi^\kappa, \\ &= n_0 + n_1\xi + n_2\xi^2 \\ &\quad + \Lambda\sum_{\kappa=0}^{\infty} n_\kappa \xi^{\kappa+2} + \Lambda\sum_{\kappa=0}^{\infty}\sum_{l=0}^{\kappa} n_l n_{\kappa-l}\xi^\kappa. \end{aligned} \tag{56}$$

Presently, looking at an implicit functional system with respect to ξ, we have

$$\begin{aligned} \mathbf{R}(\xi, N) &= N(\xi) - n_0 - n_1\xi - n_2\xi^2 \\ &\quad -\Lambda(N - n_0)\xi^2 - \Lambda(N^2 - n_0^2)\xi^2 \end{aligned} \tag{57}$$

since $\mathbf{R}$ is an analytic function in the neighbourhood (nhbd) of $(0, n_0)$ where

$$\mathbf{R}(0, n_0) = 0$$

and

$$\frac{\partial \mathbf{R}}{\partial N}(0, n_0) \ne 0.$$

Then, using the implicit function theorem, we reach a convergence.

Theorem 4. *[28] Suppose that* $\mathbf{h}$ *is a* γ*-mapping of an open set* $X \subset R^{\alpha+\zeta}$ *into* R^ζ*, such that* $\mathbf{f}(\omega, \upsilon) = 0$ *for some point* $(\omega, \upsilon) \in \mathbf{g}$*. Put* $S = \mathbf{f}'(\omega, \upsilon)$ *and assume that* $S_\mathbf{v}$ *is invertible. Then there exists an open set* $A \subset R^{\alpha+\zeta}$*,* $Y \subset R^\alpha$ *for* $(\omega, \upsilon) \in A$*,* $\upsilon \in Y$ *which may have the subsequential characteristics*

- *To each* $\mathbf{u} \in Y$*, keep in contact with a unique* $\mathbf{v}$ *such that* $(\mathbf{v}, \mathbf{u}) \in A$*,* $\mathbf{f}(\mathbf{v}, \mathbf{u}) = 0$*,*
- *If this* $\mathbf{v}$ *is described as* $\mathbf{g}(\mathbf{u})$*, then*

  $$\mathbf{h}(\upsilon) = \omega,$$
  $$\mathbf{h}(\mathbf{h}(\mathbf{u}), \mathbf{u}) = 0 \quad (\mathbf{u} \in Y),$$
  $$\mathbf{h}'(\upsilon) = -(S_\mathbf{v})^{-1}S_\mathbf{u},$$

  where $\mathbf{h}$ *is the* γ*-mapping of* Y *into* R^ζ*.*
- *The function* $\mathbf{h}$ *is completely given by the above property.*

We can easily see that $N = N(\xi)$ is analytic in the nhbd of point $(0, n_0)$ and confirm that it has a positive radius. The above discussion clarifies that the power series equation (48) converges at the nhbd of point $(0, n_0)$.

6. Conservation laws

In this section, we compute the CLs [29] of FPKdV eq. (1) which assures the following condition:

$$\left[\mathbf{D}_t(\mathcal{X}^t)+\mathbf{D}_x(\mathcal{X}^x)\right]_{\text{Eq.}(1)}=0, \tag{58}$$

where $\mathcal{X}^t$ and $\mathcal{X}^x$ are the conserved quantities. With regard to eq. (1), we define a 'formal Lagrangian' of the form

$$\mathbf{L}=v(x,t)[\mathcal{D}_t^{\sigma}\psi+a(\psi_x)^2+b\psi_{xxx}],\quad 0<\sigma\leq 1, \tag{59}$$

where $v(x,t)$ is the new dependent variable. Assuming eq. (59), we can define the action integral of the form

$$\int_0^t\int_{\Omega}\mathbf{L}(x,t,\psi,v,\mathcal{D}_t^{\sigma}\psi,\psi_x,\psi_{xxx})\mathrm{d}x\,\mathrm{d}t. \tag{60}$$

The respective Euler–Lagrange operator (EL operator) can be written as

$$\frac{\delta}{\delta\psi}=\frac{\partial}{\partial\psi}-(\mathcal{D}_t^{\sigma})^*\frac{\partial}{\partial\mathcal{D}_t^{\sigma}\psi}-\mathbf{D}_x\frac{\partial}{\partial\psi_x}-\mathbf{D}_x^3\frac{\partial}{\partial\psi_{xxx}},\quad 0<\sigma\leq 1, \tag{61}$$

where $(\mathcal{D}_t^{\sigma})^*$ is regarded as an adjoint operator of $(\mathcal{D}_t^{\sigma})$ and we can describe the adjoint equation for eq. (1) as

$$\frac{\delta\mathbf{L}}{\delta\psi}=0. \tag{62}$$

The adjoint operator $(\mathcal{D}_t^{\sigma})^*$ with RL derivative has the structure

$$(\mathcal{D}_t^{\sigma})^*=(-1)^c I_T^{c-\sigma}(\mathcal{D}_t^c)={}_t^C\mathcal{D}_T^{\sigma},\quad 0<\sigma\leq 1, \tag{63}$$

where $\mathbf{I}_T^{c-\sigma}$ has following form:

$$\mathbf{I}_T^{c-\sigma}\psi(x,t)=\frac{1}{\Gamma(c-\sigma)}\int_0^t\frac{\psi(x,\Xi)}{(\Xi-t)^{\sigma-c+1}}\mathrm{d}\Xi. \tag{64}$$

Assuming dependent variable ψ, independent variables t and x we have

$$\mathbf{V}+\mathbf{D}_t(\theta^1)\hat{I}+\mathbf{D}_x(\theta^2)\hat{I}=\mathbf{U}\frac{\delta}{\delta\psi}+\mathbf{D}_t(\mathcal{X}^t)+\mathbf{D}_x(\mathcal{X}^x), \tag{65}$$

where $\frac{\delta}{\delta\psi}$ is the EL operator and $\hat{I}$ is the identity operator. Hence, the vector field $\mathbf{V}$ has the following structure:

$$\mathbf{V}=\theta^1\frac{\partial}{\partial t}+\theta^2\frac{\partial}{\partial x}+\Upsilon\frac{\partial}{\partial\psi}+\Upsilon_{\sigma}^0\frac{\partial}{\partial\mathcal{D}_t^{\sigma}\psi}+\Upsilon^x\frac{\partial}{\partial\psi_x}+\Upsilon^{xxx}\frac{\partial}{\partial\psi_{xxx}},\quad 0<\sigma\leq 1 \tag{66}$$

and $\mathbf{U}$ denotes the characteristic function given by

$$\mathbf{U}=\Upsilon-\theta^1\psi_t-\theta^2\psi_x.$$

To discuss the case of $\mathbf{V}_i$ with x,t independent variable, each symmetry which can be written in terms of the characteristic is

$$\mathbf{U}_1=-u_x,\quad \mathbf{U}_2=-\sigma\psi-\sigma x\psi_x-3t\psi_t,\quad \mathbf{U}_3=t. \tag{67}$$

By considering RL-fractional derivative, the density component $\hat{\mathcal{X}}^t$ of the conserved flow is given by

$$\hat{\mathcal{X}}^t=\theta^1\mathbf{L}+\sum_{l=0}^{c-1}(-1)^l{}_0\mathcal{D}_t^{\sigma-1-l}(\mathbf{U}_{\gamma})\mathbf{D}_t^l\frac{\partial\mathbf{L}}{\partial_0\mathcal{D}_t^{\sigma}\psi}-(-1)^c\mathbf{J}\left(\mathbf{U}_{\gamma},\mathbf{D}_t^n\frac{\partial\mathbf{L}}{\partial_0\mathcal{D}_t^{\sigma}\psi}\right), \tag{68}$$

where $0<\sigma\leq 1$ and $\mathbf{J}(\cdot)$ is defined by

$$\mathbf{J}(\mathbf{f},\mathbf{g})=\frac{1}{\Gamma(c-\sigma)}\int_0^t\int_t^T\frac{\mathbf{f}(\theta^1,x)\mathbf{g}(\nu,x)}{(\nu-\theta^1)^{\sigma-c+1}}\mathrm{d}\nu\,\mathrm{d}t \tag{69}$$

and the flux component $\hat{\mathcal{X}}^x$ for independent variable x yields

$$\begin{aligned}\hat{\mathcal{X}}^x&=\theta^i\mathbf{L}+\mathbf{U}_{\gamma}\left[\frac{\partial\mathbf{L}}{\partial\psi_i^{\gamma}}-\mathbf{D}_j\left(\frac{\partial\mathbf{D}}{\partial\psi_{ij}^{\gamma}}\right)+\mathbf{D}_j\mathbf{D}_k\left(\frac{\partial\mathbf{L}}{\partial\psi_{ijk}^{\gamma}}\right)-\cdots\right]\\&\quad+\mathbf{D}_j(\mathbf{U}_{\gamma})\left[\frac{\partial\mathbf{L}}{\partial\psi_{ij}^{\gamma}}-\mathbf{D}_k\left(\frac{\partial\mathbf{L}}{\partial\psi_{ijk}^{\gamma}}\right)+\cdots\right]\\&\quad+\mathbf{D}_j\mathbf{D}_k(\mathbf{U}_{\gamma})\left[\frac{\partial\mathbf{L}}{\partial\psi_{ijk}^{\gamma}}-\cdots\right]+\cdots,\end{aligned} \tag{70}$$

where $\theta^i=\theta^2$, $i,j,k=1,2,3$ and $\gamma=1,2,3$.

Case 1. To deal with $\mathbf{V}_1$, we have $\mathbf{U}=-\psi_x$, $\theta^2=1$. Substituting all the values in eqs (68)–(70), we have the following result:

$$\hat{\mathcal{X}}^t=\mathbf{D}_t^{\sigma-1}(-\psi_x)v+\mathbf{J}(-\psi_x,v_t),\qquad 0<\sigma\leq 1,$$

$$\hat{\mathcal{X}}^x=v\mathbf{D}_t^{\sigma}(\psi)-av(\psi_x)^2+b(\psi_{xx}v_x-\psi_x v_{xx}).$$

Case 2. For $\mathbf{V}_2$, we have $\mathbf{U}=-\sigma\psi-\sigma x\psi_x-3t\psi_t$, $\theta^1=3t$, $\theta^2=\sigma x$ and $\Upsilon=-\sigma\psi$. Putting all the values in eqs (68)–(70), we have the following results:

$$\begin{aligned}\hat{\mathcal{X}}^t&=3t(v(\mathbf{D}_t^{\sigma}\psi+a(\psi_x)^2)+b\psi_{xxx})\\&\quad+\mathbf{D}_t^{\sigma-1}(-\sigma\psi-\sigma x\psi_x-3t\psi_t)v\\&\quad-\mathbf{J}(-\sigma\psi-\sigma x\psi_x-3t\psi_t,v_t),\end{aligned}$$

$$\begin{aligned}\hat{\mathcal{X}}^x&=\sigma xv\mathbf{D}_t^{\sigma}\psi-a\sigma xv(\psi_x)^2-2a\sigma\psi v\psi_x-\sigma b\psi v_{xx}\\&\quad-\sigma bx\psi_x v_{xx}-6atv\psi_x\psi_t-3bt\psi_t v_{xx}+\sigma bv_x\psi_x\\&\quad+\sigma bxv_x\psi_{xx}+3btv_x\psi_{tx}-\sigma bv\psi_{xx}-3btv_{txx}.\end{aligned}$$

Case 3. For $\mathbf{V}_3$, we have $\mathbf{U} = t$ and putting all the values in eqs (68)–(70), we have the following results:

$$\hat{\mathcal{X}}^t = \mathbf{D}_t^{\sigma-1}(tv) - \mathbf{J}(t, v_t),$$

$$\hat{\mathcal{X}}^x = t(2a\psi_x v_x + v\psi_{xx}).$$

Notes. The CLs are 'associated' with the symmetries through which they are constructed. For example, the conserved vector in Case 2 is inherently linked to the scale-invariant solution (51). Solutions for FPDEs via CLs are yet to be explored.

7. Conclusions

In this study, we investigated the time-fractional FP-KdV eq. (1) using the Lie symmetry analysis method and the RL derivative was applied. We reduced the concerned equation to a nonlinear ODE of fractional order of interest and is the scale-invariant exact reduction for which the fractional ordinary equation was solved by applying the power series method. We used Ibragimov's non-local conservation theorem to construct the CLs of the concerned equations. Graphical representations of the solutions are also presented. Finally, no soliton or tarvelling-wave solutions were obtained. Lie symmetry analysis of (1) for $\sigma = 1$ was performed in [27]. The results are interesting and open new literature on the invariant theory in this model. Additionally, this study adds new CLs and soliton solutions. To further explore this direction, we considered the fractional form of the desired model, which explains the phenomenon more closely.

Availability of data and materials All the data generated or analysed during in this study are included in this article.

References

[1] R M Miura, C S Gardner and M D Kruskal, *J. Math. Phys.* **9**, 1204 (1968)
[2] Z Yang *et al*, *Proc.* **11**, 1737 (2023)
[3] X Sui, Q Chen and G Gu, *Infrared Phys. Tech.* **60**, 155 (2013)
[4] T Hou, X Liu, Z Li, Y Wang, T Chen and B Huang, *Appl. Ocean Res.* **154**, 104361 (2025)
[5] I Podlubny, *Fractional differential equations* (Elsevier, 1998)
[6] J Liu, T Liu, C Su and S Zhou, *Case Stud. Therm. Eng.* **49**, 103210 (2023
[7] Y Yu, M Wan, J Qian, D Miao, Z Zhang and P Zhao, *IJAR* **169**, 109181 (2024)
[8] G Xie, B Fu, H Li, W Du, Y Zhong, L Wang, H Geng, J Zhang and L Si, *Eng. Anal. Bound. Elem.* **166**, 105802 (2024)
[9] Y Zhang, Z Gao, X Wang and Q Liu, *CMES* **134**, 821 (2023)
[10] S Meng, F Meng, F Zhang, Q Li, Y Zhang and A Zemouche, *Automatica* **162**, 111512 (2024)
[11] Q Chen *et al Adv. Eng. Inform.* **62**, 102741 (2024)
[12] Y Zhang, X Zhuang and R Lackner, *Int. J. Numer. Anal. Methods Geomech.* **42**, 1199 (2018)
[13] S Zhang and H-Q Zhang, *Phys. Lett. A* **375**, 1069 (2011)
[14] S Guo, L Mei, Y Li and Y Sun, *Phys. Lett. A* **376**, 407 (2012)
[15] M Inc, A Yusuf, A-I Aliyu and D Baleanu, *Phys. A: Stat. Mech. Appl.* **493**, 94 (2018)
[16] D Baleanu, M İnç, A Yusuf and A-I Aliyu, *Nonlinear Anal.: Model. Control.* **22**, 861 (2017)
[17] E Buckwar and Y Luchko, *J. Math. Anal. Appl.* **227**, 81 (1998)
[18] V D Djordjevic and T M Atanackovic, *J. Comput. Appl. Math.* **222**, 701 (2008)
[19] H Liu, *Stud. Appl. Math.* **131**, 317 (2013)
[20] E Noether, In *Gesammelte Abhandlungen-Collected Papers* (Springer, Berlin, 1983) pp. 231–239
[21] A H Kara and F M Mahomed *Nonlinear Dyn.* **45**, 367 (2006)
[22] S C Anco and G Bluman, *Eur. J. Appl. Math.* **13**, 545 (2002)
[23] N H Ibragimov, *J. Math. Anal. Appl.* **333**, 311 (2007)
[24] A Biswas, S Kumar, E-V Krishnan, B Ahmed, A Strong, S Johnson and A Yildirim, *Rom. Rep. Phys.* **65**, 1125 (2013)
[25] G W Wang, T-Z Xu, G Ebadi, S Johnson, A-J Strong and A Biswas, *Nonlinear Dyn.* **76**, 1059 (2014))
[26] M S Hashemi and D Baleanu, *Lie symmetry analysis of fractional differential equations* (Chapman and Hall/CRC, 2020)
[27] N Abbas, A Hussain, M-B Riaz, T-F Ibrahim, F-O Birkea and R-A Tahir. *Results Phys.* **56**, 107302 (2024)
[28] W Rudin, *Principles of mathematical analysis* (McGrawHill, New York, 1964) Vol. 3
[29] N Abbas, A Hussain, T-F Ibrahim, M-Y Juma and F-O Birkea. *Opt. Quantum Electron.* **56**, 809 (2024)