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Invariance Analysis and Dynamics of Difference Equations

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Abstract

In this paper, we use the invariance method to obtain symmetries and exact solutions for a class of difference equations with variable coefficients. We look at the stability of the equilibrium points admitted by this class. Behavior and periodicity of their solutions are investigated. Consequently, results in existing literature [Elsayed, E.M., Alofi, B. S. and Khan,A. Q. (2022), Qualitative behavior of solutions of tenth-order recursive sequence equation, *Mathematical Problems in Engineering*, ID 5242325] are generalized.

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1 Introduction

The theory of the sets of transformations of the solutions of differential equations served as the foundation for Lie group theory. Over time, it has developed into a potent tool for classifying and solving differential equations. It has evolved into a powerful tool for categorizing and solving differential equations over the years. More recently, Lie group theory have been applied to differential equations and difference equations [1–6]. Since their introduction in the middle of the seventeenth century, the area of differential equations have been a significant area of pure and applied mathematics. Despite the fact that their history has been extensively studied, it continues to be an important area of research because of the emergence of new connections with other branches of mathematics, the fertile interaction with applied fields, the interesting reformulation of fundamental issues and theories over time, the opening of new vistas in the twentieth century, and so on. The ideas that have been used for differential equations like symmetry based methods have been extended to difference equations in recent times. For more, see [2]. The approach is to apply the group of point transformations on the difference equations to leave them invariant. A differential or difference equation's symmetry is a transformation that preserves its family of solutions. Although finding the symmetries can be computationally time consuming in comparison to other solution methods, symmetry analysis can be utilized to solve various ordinary and partial differential/difference equations. The analysis of Lie symmetry is a really crucial method

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for differential equations that can also be extended to difference equations. There are important applications of difference equations in areas like numerical methods to solve partial differential equation, Fourier series, Algebra and Analysis.

Oftentimes, difference equations arise as the discrete counterpart of differential equations. In [7], the authors discretized a differential equation while maintaining all of its Lie point symmetries using well-proven techniques. They begin with Lie's categorization of second-order ordinary differential equations into categories based on the point symmetries in which they occur. The authors' aim is to provide a similar classification of second-order difference schemes. It turns out that they both have the same symmetry group and symmetry algebra. The authors present a general theory and outline the classification method by looking at the construction of invariant difference schemes. For related work on symmetry analysis and analytical solutions of difference and differential equations, see [8–14].

1.1 Preliminaries

The concepts discussed in this paper are periodicity, Lie group, group invariant and symmetries. To comprehend these concepts, we provide some background information.

Definition 1. A group invariant is a function $u(x,y)$ whose value at an image point is the same as its value at the source point

$$u(x',y') = u(x,y). \tag{1}$$

Definition 2. A parametrized set of point transformations,

$$\Gamma_\varepsilon : \mathbf{u} \mapsto \hat{\mathbf{u}}(\mathbf{u};\varepsilon), \quad \varepsilon \in (\varepsilon_0, \varepsilon_1),$$

where $\varepsilon_0 < 0 < \varepsilon_1$, is a one-parameter local Lie group if the following conditions are satisfied:

1. Γ_0 is the identity map, so that $\hat{\mathbf{u}} = \mathbf{u}$ when $\varepsilon = 0$.
2. $\Gamma_\delta \Gamma_\varepsilon = \Gamma_{\delta+\varepsilon}$ for every δ, ε sufficiently close to zero.
3. Each $\hat{u}_\alpha$ can be represented as Taylor series in ε (in a neighborhood of $\varepsilon = 0$ that is determined by $\mathbf{u}$), and therefore

$$\hat{u}_\alpha(\mathbf{u};\varepsilon) = u_\alpha + \varepsilon\ \xi_\alpha(\mathbf{u}) + O(\varepsilon^2),\ \alpha = 1,\ldots,N. \tag{2}$$

Definition 3. The infinitesimal operator is defined by

$$U = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{3}$$

Theorem 1. *The function $f(u)$ is invariant under the one-parameter Lie group of transformations $\boldsymbol{F}$ if and only if*

$$\tilde{U} f(u) = 0,$$

where $\tilde{U}$ is the prolongation of U.

For some smooth function ω, let the difference equation have a general form of

$$u_{n+5k} = \omega(n, u_n, u_{n+k}, u_{n+2k}, u_{n+3k}, u_{n+4k}). \tag{4}$$

As long as (4) holds, we can use the linearized symmetry condition

$$Q(n+5k, u_{n+5k}) - \tilde{U}\omega = 0, \tag{5}$$

to solve for the characteristic $Q = Q(n, u_n)$. Here, the prolonged generator is given by

$$\begin{aligned}\tilde{U} =& Q(n,u_n)\frac{\partial}{\partial u_n} + Q(n+k,u_{n+k})\frac{\partial}{\partial u_{n+k}} + Q(n+2k,u_{n+2k})\frac{\partial}{\partial u_{n+2k}} \\ &+ Q(n+3k,u_{n+3k})\frac{\partial}{\partial u_{n+3k}} + Q(n+4k,u_{n+4k})\frac{\partial}{\partial u_{n+4k}}. \end{aligned} \tag{6}$$

If an ordinary difference equation admits a lie group of transformations, one can construct special cases of solutions or invariant solutions.

Definition 4. An equilibrium point $\overline{u}$ of (4) is said to be locally stable if for every $\varepsilon > 0$ such that if $\{u_n\}_{n=0}^{\infty}$ is a solution of (4) with

$$|u_0 - \overline{u}| + |u_1 - \overline{u}| + \cdots + |u_{5k-1} - \overline{u}| < \delta,$$

then

$$|u_n - \overline{u}| < \varepsilon \text{ for all } n \geq 0. \tag{7}$$

Definition 5. An equilibrium point $\overline{u}$ of (4) is said to be a global attractor if for every solution $\{u_n\}_{n=0}^{\infty}$ of (4), we have

$$\lim_{n\to\infty} u_n = \overline{u}. \tag{8}$$

Definition 6. An equilibrium point $\overline{u}$ of (4) is globally asymptotically stable if $\overline{u}$ is locally stable and is a global attractor of (4).

Now, we let

$$p_i = \frac{\partial \omega}{\partial u_{n+i}}(\overline{u}, \ldots, \overline{u}), \quad i = 0, k, 2k, 3k, 4k. \tag{9}$$

It follows that

$$\lambda^{5k} - p_{4k}\lambda^{4k} - p_{3k}\lambda^{3k} - p_{2k}\lambda^{2k} - p_k\lambda^k - p_0 = 0 \tag{10}$$

is the characteristic equation near the equilibrium point $\overline{u}$.

Definition 7. An equilibrium point $\overline{u}$ of (4) is unstable if $\overline{u}$ is not locally stable.

Theorem 2. *Assume that ω is a continuously differentiable function defined over a wide area around $\overline{u}$. We have:*

(i) The equilibrium point $\overline{u}$ of (4) is locally asymptotically stable if all the roots of (10) have absolute values less than one.

(ii) The equilibrium point $\overline{u}$ of (4) is unstable if at least one root of (10) has an absolute value greater than one.

Definition 8. The equilibrium point $\overline{u}$ is referred to as non-hyperbolic if there is a root of (10) with absolute value equal to one.

2 Symmetries, reductions and exact solutions of the difference equation in concern

One of the goals of this paper is to generalize the work done in the article [15]. We achieve this by focusing on the ordinary difference equation

$$u_{n+5k} = \frac{u_n u_{n+2k} u_{n+4k}}{u_{n+k} u_{n+3k}(A_n + B_n u_n u_{n+2k} u_{n+4k})}, \tag{11}$$

where A_n and B_n are arbitrary sequences. We first start by obtaining the symmetries, then go on to getting the reductions and exact solutions where we will be looking into the cases for which A_n and B_n are 1-, 6k-periodic sequences. For an in-depth understanding, we will provide analysis of the cases where $k = 1$ and $k = 2$.

It is worthwhile noting that by letting $l = 5k, A_n = \frac{1}{\mu_n}, B_n = \frac{\mu_n - 1}{\mu_n k_n}$ and $r_n = u_n u_{n+2k} u_{n+4k}$ in the higher order Berverton-Holt difference equation [16]

$$r_{n+l} = \frac{\mu_n K_n r_n}{K_n + (\mu_n - 1) r_n}, \tag{12}$$

where μ_n (resp. K_n) represent the growth rate (carrying capacity) and $r_0, r_1, \ldots, r_{l-1}$ are the initial conditions, one can obtain Equations (11).

2.1 Symmetry generators

With $\omega = \omega(u_n, u_{n+k}, u_{n+2k}, u_{n+3k}, u_{n+4k})$ being the right hand side of (11), the symmetry condition (5) leads to

$$\begin{aligned} &Q(n+5k,\omega) - Q(n,u_n)\frac{\partial \omega}{\partial u_n} - Q(n+k,u_{n+k})\frac{\partial \omega}{\partial u_{n+k}} - Q(n+2k,u_{n+2k})\frac{\partial \omega}{\partial u_{n+2k}} \\ &- Q(n+3k,u_{n+3k})\frac{\partial \omega}{\partial u_{n+3k}} - Q(n+4k,u_{n+4k})\frac{\partial \omega}{\partial u_{n+4k}} = 0. \end{aligned} \tag{13}$$

Replacing the partial derivatives of ω with there expressions and further simplifying the above, we get:

$$\begin{aligned} &- \frac{Q(n,u_n)A_n u_{n+k}u_{n+2k}u_{n+4k} - Q(n+k,u_{n+k})(A_n u_n u_{n+2k}u_{n+4k} + B_n u_n^2 u_{n+2k}^2 u_{n+4k}^2)}{u_{n+3k}(A_n u_{n+k} + B_n u_n u_{n+k} u_{n+2k} u_{n+4k})^2} \\ &+ \frac{Q(n+2k,u_{n+2k})A_n u_n u_{n+3k}u_{n+4k} - Q(n+3k,u_{n+3k})(A_n u_n u_{n+2k}u_{n+4k} + B_n u_n^2 u_{n+2k}^2 u_{n+4k}^2)}{u_{n+k}(A_n u_{n+3k} + B_n u_n u_{n+2k} u_{n+3k} u_{n+4k})^2} \\ &+ \frac{Q(n+4k,u_{n+4k})A_n u_n u_{n+k}u_{n+2k}u_{n+3k}}{(A_n u_{n+k}u_{n+3k} + B_n u_n u_{n+k} u_{n+2k} u_{n+3k} u_{n+4k})^2} + Q(n+5k,u_{n+5k}) = 0. \end{aligned} \tag{14}$$

In (14) above, we need to eliminate $Q(n+5k, u_{n+5k})$. By considering u_{n+k} as a function of u_n, u_{n+2k}, u_{n+3k}, u_{n+4k} and u_{n+5k}, we differentiate the functional Eq. (14) with respect to u_n implicitly. Thus, we force the operator $L = \frac{\partial}{\partial u_n} - (\frac{\partial \omega}{\partial u_n} / \frac{\partial \omega}{\partial u_{n+k}}) \frac{\partial}{\partial u_{n+k}}$ on (14) to get (after multiplying the resulting equation by $u_n(A_n + B_n u_n u_{n+2k} u_{n+4k})^3 / (u_{n+k} u_{n+3k}^2 u_{n+2k} u_{n+4k})$) the following:

$$\begin{aligned} &- A_n u_n u_{n+k}(A_n + B_n u_n u_{n+2k} u_{n+4k}) \frac{\mathrm{d}Q(n,u_n)}{\mathrm{d}u_n} + A_n u_{n+k}(A_n + 2B_n u_n u_{n+2k} u_{n+4k}) Q(n,u_n) \\ &+ A_n B_n u_n^2 u_{n+k} u_{n+4k} Q(n+2k,u_{n+2k}) + A_n B_n u_n^2 u_{n+k} u_{n+2k} Q(n+4k,u_{n+4k}) + A_n u_n u_{n+k}(A_n \\ &+ B_n u_n u_{n+2k} u_{n+4k}) \frac{\mathrm{d}Q(n+k,u_{n+k})}{\mathrm{d}u_{n+k}} - A_n u_n (A_n + B_n u_n u_{n+2k} u_{n+4k}) Q(n+k,u_{n+k}) = 0. \end{aligned} \tag{15}$$

We now differentiate (15) thrice with respect to u_n and then set the coefficients of the products of shifts of u_n to 0 get the following system:

$$u_{n+k} : u_n Q^{(4)}(n,u_n) + 2Q^{(3)}(n,u_n) = 0 \tag{16}$$

$$u_{n+k}u_{n+2k}u_{n+4k} : u_n Q^{(4)}(n,u_n) + 4Q^{(3)}(n,u_n) = 0. \tag{17}$$

We find the general solution of (16) and (17) to be

$$Q(n,u_n) = \alpha_n + \beta_n u_n + \gamma_n u_n^2 \tag{18}$$

for some functions α_n, β_n and γ_n of n. Substitute (18) into (14) and further simplify to get the following:

$$\begin{aligned}
&\alpha_{n+5k}B_n^2u_n^2u_{n+k}^2u_{n+2k}^2u_{n+3k}^2u_{n+4k}^2 + 2A_nB_n\alpha_{n+5k}u_nu_{n+k}^2u_{n+2k}u_{n+3k}^2u_{n+4k} + A_n^2\alpha_{n+5k}u_{n+k}^2u_{n+3k}^2\\
&+A_nu_nu_{n+k}u_{n+2k}u_{n+3k}u_{n+4k}(\beta_{n+5k}-\beta_n+\beta_{n+k}-\beta_{n+2k}+\beta_{n+3k}-\beta_{n+4k})\\
&+B_nu_n^2u_{n+k}u_{n+2k}^2u_{n+3k}u_{n+4k}^2(\beta_{n+5k}+\beta_{n+k}+\beta_{n+3k}) - A_n\alpha_nu_{n+k}u_{n+2k}u_{n+3k}u_{n+4k}\\
&+A_n\alpha_{n+k}u_nu_{n+2k}u_{n+3k}u_{n+4k} + B_n\alpha_{n+k}u_n^2u_{n+2k}^2u_{n+3k}u_{n+4k}^2 - A_n\alpha_{n+2k}u_nu_{n+k}u_{n+3k}u_{n+4k}\\
&+A_n\alpha_{n+3k}u_nu_{n+k}u_{n+2k}u_{n+4k} + B_n\alpha_{n+3k}u_n^2u_{n+k}u_{n+2k}^2u_{n+4k}^2 - A_n\alpha_{n+4k}u_nu_{n+k}u_{n+2k}u_{n+3k}\\
&+\gamma_{n+k}B_nu_n^2u_{n+k}^2u_{n+2k}^2u_{n+3k}u_{n+4k}^2 + \gamma_{n+3k}B_nu_n^2u_{n+k}u_{n+2k}^2u_{n+3k}^2u_{n+4k}\\
&+A_n(\gamma_{n+k}u_nu_{n+k}^2u_{n+2k}u_{n+3k}u_{n+4k} - \gamma_nu_n^2u_{n+k}u_{n+2k}u_{n+3k}u_{n+4k} - \gamma_{n+2k}u_nu_{n+k}u_{n+2k}^2u_{n+3k}u_{n+4k}\\
&-\gamma_{n+4k}u_nu_{n+k}u_{n+2k}u_{n+3k}u_{n+4k}^2) + \gamma_{n+5k}u_n^2u_{n+2k}^2u_{n+4k}^2 = 0. \qquad (19)
\end{aligned}$$

Similarly, setting the coefficients of the products of shifts of u_n to 0 and solving the resulting system of equations, it follows that

$$\alpha_n = \gamma_n = 0 \qquad (20)$$

and

$$\beta_n + \beta_{n+2k} + \beta_{n+4k} = 0. \qquad (21)$$

We have that

$$\beta_n = e^{\frac{i\pi n}{k}\left(\pm\frac{1}{3}+p\right)}, p = \overline{0,\ldots,2k-1}. \qquad (22)$$

So, using (18), (20) and (22), we have that

$$Q(n,u_n) = e^{\frac{i\pi n}{k}\left(\pm\frac{1}{3}+p\right)}u_n, \qquad (23)$$

where $p = 0,\ldots,2k-1$ and $k \neq 0$. Therefore the $4k$ symmetry generators are:

$$U_{1p} = e^{\frac{i\pi n}{k}\left(-\frac{1}{3}+p\right)}u_n\frac{\partial}{\partial u_n}, \quad U_{2p} = e^{\frac{i\pi n}{k}\left(\frac{1}{3}+p\right)}u_n\frac{\partial}{\partial u_n}, \qquad (24)$$

where $p = 0,\ldots,2k-1$ and $k \neq 0$. Defining the canonical coordinate as

$$s_n = \int\frac{du_n}{Q(n,u_n)} = \int\frac{du_n}{\beta_nu_n} = \frac{1}{\beta_n}\int\frac{du_n}{u_n} = \frac{1}{\beta_n}\ln|u_n|,$$

we have

$$\beta_ns_n = \ln|u_n|. \qquad (25)$$

Letting

$$\widetilde{v}_n = \beta_ns_n + \beta_{n+2k}s_{n+2k} + \beta_{n+4k}s_{n+4k}, \qquad (26)$$

we have that $\tilde{U}\tilde{v}_n = 0$. This means that $\tilde{v}_n$ is invariant under the Lie group of transformations whose action can be obtained from the generator U. For aesthetic purposes, we introduce the variable

$$\begin{aligned}
|v_n| &= \exp\{-\widetilde{v}_n\} &(27)\\
&= \exp\{-\ln|u_nu_{n+2k}u_{n+4k}|\} &(28)\\
&= \left|\frac{1}{u_nu_{n+2k}u_{n+4k}}\right| &(29)
\end{aligned}$$

and, without loss of generality, we have

$$v_n = \frac{1}{u_nu_{n+2k}u_{n+4k}}. \qquad (30)$$

By Theorem 1 and Definition 1, then (30) is invariant. Using (11) and (30), we have that

$$v_{n+k} = A_n v_n + B_n \tag{31}$$

and

$$u_{n+6k} = \frac{v_n}{v_{n+2k}} u_n. \tag{32}$$

The solution for (31) is as follows:

$$v_{kn+j} = v_j(\prod_{k_1=0}^{n-1} A_{kk_1+j}) + \sum_{l=0}^{n-1}(B_{kl+j} \prod_{k_2=l+1}^{n-1} A_{kk_2+j}), \quad j = 0,\ldots,k-1. \tag{33}$$

Carrying out the iterations on (32), leads to the following:

$$u_{6kn+j} = u_j(\prod_{k_1=0}^{n-1} \frac{v_{6kk_1+j}}{v_{6kk_1+2k+j}}), \quad j = 0,\ldots,6k-1. \tag{34}$$

To eliminate some possible confusion, we rewrite (34) in the following forms:

$$\begin{aligned}
u_{6kn+j} &= u_j(\prod_{k_1=0}^{n-1} \frac{v_{k(6k_1)+j}}{v_{k(6k_1+2)+j}}), \quad j = 0,\ldots,k-1,\\
u_{6kn+j} &= u_j(\prod_{k_1=0}^{n-1} \frac{v_{k(6k_1+1)+j-k}}{v_{k(6k_1+3)+j-k}}), \quad j = k,\ldots,2k-1,\\
u_{6kn+j} &= u_j(\prod_{k_1=0}^{n-1} \frac{v_{k(6k_1+2)+j-2k}}{v_{k(6k_1+4)+j-2k}}), \quad j = 2k,\ldots,3k-1,\\
u_{6kn+j} &= u_j(\prod_{k_1=0}^{n-1} \frac{v_{k(6k_1+3)+j-3k}}{v_{k(6k_1+5)+j-3k}}), \quad j = 3k,\ldots,4k-1,\\
u_{6kn+j} &= u_j(\prod_{k_1=0}^{n-1} \frac{v_{k(6k_1+4)+j-4k}}{v_{k(6k_1+6)+j-4k}}), \quad j = 4k,\ldots,5k-1,\\
u_{6kn+j} &= u_j(\prod_{k_1=0}^{n-1} \frac{v_{k(6k_1+5)+j-5k}}{v_{k(6k_1+7)+j-5k}}), \quad j = 5k,\ldots,6k-1.
\end{aligned} \tag{35}$$

We now use (33) in the equations in (35) to obtain the following:

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{v_j(\prod_{s=0}^{6k_1-1} A_{ks+j}) + \sum_{l=0}^{6k_1-1}(B_{kl+j} \prod_{k_2=l+1}^{6k_1-1} A_{kk_2+j})}{v_j(\prod_{s=0}^{6k_1+1} A_{ks+j}) + \sum_{l=0}^{6k_1+1}(B_{kl+j} \prod_{k_2=l+1}^{6k_1+1} A_{kk_2+j})}, j = 0,\ldots,k-1$$

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{v_{j-k}(\prod_{s=0}^{6k_1} A_{ks+j-k}) + \sum_{l=0}^{6k_1}(B_{kl+j-k} \prod_{k_2=l+1}^{6k_1} A_{kk_2+j-k})}{v_{j-k}(\prod_{s=0}^{6k_1+2} A_{ks+j-k}) + \sum_{l=0}^{6k_1+2}(B_{kl+j-k} \prod_{k_2=l+1}^{6k_1+2} A_{kk_2+j-k})}, j = k,\ldots,2k-1$$

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{v_{j-2k}(\prod_{s=0}^{6k_1+1} A_{ks+j-2k}) + \sum_{l=0}^{6k_1+1}(B_{kl+j-2k} \prod_{k_2=l+1}^{6k_1+1} A_{kk_2+j-2k})}{v_{j-2k}(\prod_{s=0}^{6k_1+3} A_{ks+j-2k}) + \sum_{l=0}^{6k_1+3}(B_{kl+j-2k} \prod_{k_2=l+1}^{6k_1+3} A_{kk_2+j-2k})}, j = 2k,\ldots,3k-1$$

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{v_{j-3k}\left(\prod_{s=0}^{6k_1+2} A_{ks+j-3k}\right) + \sum_{l=0}^{6k_1+2} \left(B_{kl+j-3k} \prod_{k_2=l+1}^{6k_1+2} A_{kk_2+j-3k}\right)}{v_{j-3k}\left(\prod_{s=0}^{6k_1+4} A_{ks+j-3k}\right) + \sum_{l=0}^{6k_1+4} \left(B_{kl+j-3k} \prod_{k_2=l+1}^{6k_1+4} A_{kk_2+j-3k}\right)}, j = 3k, \ldots, 4k-1$$

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{v_{j-4k}\left(\prod_{s=0}^{6k_1+3} A_{ks+j-4k}\right) + \sum_{l=0}^{6k_1+3} \left(B_{kl+j-4k} \prod_{k_2=l+1}^{6k_1+3} A_{kk_2+j-4k}\right)}{v_{j-4k}\left(\prod_{s=0}^{6k_1+5} A_{ks+j-4k}\right) + \sum_{l=0}^{6k_1+5} \left(B_{kl+j-4k} \prod_{k_2=l+1}^{6k_1+5} A_{kk_2+j-4k}\right)}, j = 4k, \ldots, 5k-1$$

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{v_{j-5k}\left(\prod_{s=0}^{6k_1+4} A_{ks+j-5k}\right) + \sum_{l=0}^{6k_1+4} \left(B_{kl+j-5k} \prod_{k_2=l+1}^{6k_1+4} A_{kk_2+j-5k}\right)}{v_{j-5k}\left(\prod_{s=0}^{6k_1+6} A_{ks+j-5k}\right) + \sum_{l=0}^{6k_1+6} \left(B_{kl+j-5k} \prod_{k_2=l+1}^{6k_1+6} A_{kk_2+j-5k}\right)}, j = 5k, \ldots, 6k-1. \quad (36)$$

109 Substituting (30) into equations (36), we obtain the following:

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{\left(\prod_{s=0}^{6k_1-1} A_{ks+j}\right) + u_j u_{j+2k} u_{j+4k} \sum_{l=0}^{6k_1-1} \left(B_{kl+j} \prod_{k_2=l+1}^{6k_1-1} A_{kk_2+j}\right)}{\left(\prod_{s=0}^{6k_1+1} A_{ks+j}\right) + u_j u_{j+2k} u_{j+4k} \sum_{l=0}^{6k_1+1} \left(B_{kl+j} \prod_{k_2=l+1}^{6k_1+1} A_{kk_2+j}\right)}, j = 0, \ldots, k-1$$

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{\left(\prod_{s=0}^{6k_1} A_{ks+j-k}\right) + u_{j-k} u_{j+k} u_{j+3k} \sum_{l=0}^{6k_1} \left(B_{kl+j-k} \prod_{k_2=l+1}^{6k_1} A_{kk_2+j-k}\right)}{\left(\prod_{s=0}^{6k_1+2} A_{ks+j-k}\right) + u_{j-k} u_{j+k} u_{j+3k} \sum_{l=0}^{6k_1+2} \left(B_{kl+j-k} \prod_{k_2=l+1}^{6k_1+2} A_{kk_2+j-k}\right)}, j = k, \ldots, 2k-1$$

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{\left(\prod_{s=0}^{6k_1+1} A_{ks+j-2k}\right) + u_{j-2k} u_j u_{j+2k} \sum_{l=0}^{6k_1+1} \left(B_{kl+j-2k} \prod_{k_2=l+1}^{6k_1+1} A_{kk_2+j-2k}\right)}{\left(\prod_{s=0}^{6k_1+3} A_{ks+j-2k}\right) + u_{j-2k} u_j u_{j+2k} \sum_{l=0}^{6k_1+3} \left(B_{kl+j-2k} \prod_{k_2=l+1}^{6k_1+3} A_{kk_2+j-2k}\right)}, j = 2k, \ldots, 3k-1$$

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{\left(\prod_{s=0}^{6k_1+2} A_{ks+j-3k}\right) + u_{j-3k} u_{j-k} u_{j+k} \sum_{l=0}^{6k_1+2} \left(B_{kl+j-3k} \prod_{k_2=l+1}^{6k_1+2} A_{kk_2+j-3k}\right)}{\left(\prod_{s=0}^{6k_1+4} A_{ks+j-3k}\right) + u_{j-3k} u_{j-k} u_{j+k} \sum_{l=0}^{6k_1+4} \left(B_{kl+j-3k} \prod_{k_2=l+1}^{6k_1+4} A_{kk_2+j-3k}\right)}, j = 3k, \ldots, 4k-1$$

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{\left(\prod_{s=0}^{6k_1+3} A_{ks+j-4k}\right) + u_{j-4k} u_{j-2k} u_j \sum_{l=0}^{6k_1+3} \left(B_{kl+j-4k} \prod_{k_2=l+1}^{6k_1+3} A_{kk_2+j-4k}\right)}{\left(\prod_{s=0}^{6k_1+5} A_{ks+j-4k}\right) + u_{j-4k} u_{j-2k} u_j \sum_{l=0}^{6k_1+5} \left(B_{kl+j-4k} \prod_{k_2=l+1}^{6k_1+5} A_{kk_2+j-4k}\right)}, j = 4k, \ldots, 5k-1$$

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{\left(\prod_{s=0}^{6k_1+4} A_{ks+j-5k}\right) + u_{j-5k} u_{j-3k} u_{j-k} \sum_{l=0}^{6k_1+4} \left(B_{kl+j-5k} \prod_{k_2=l+1}^{6k_1+4} A_{kk_2+j-5k}\right)}{\left(\prod_{s=0}^{6k_1+6} A_{ks+j-5k}\right) + u_{j-5k} u_{j-3k} u_{j-k} \sum_{l=0}^{6k_1+6} \left(B_{kl+j-5k} \prod_{k_2=l+1}^{6k_1+6} A_{kk_2+j-5k}\right)}, j = 5k, \ldots, 6k-1. \quad (37)$$

3 The case when A_n and B_n are 1-periodic

Assume that $\{A_n\} = \{A, A, A, \dots\}$ and $\{B_n\} = \{B, B, B, \dots,\}$. Then the equations in (37) become:

$$
\begin{aligned}
u_{6kn+j} &= u_j \prod_{k_1=0}^{n-1} \frac{A^{6k_1} + Bu_j u_{j+2k} u_{j+4k} \sum_{l=0}^{6k_1-1} A^l}{A^{6k_1+2} + Bu_j u_{j+2k} u_{j+4k} \sum_{l=0}^{6k_1+1} A^l}, j = 0, \dots, k-1 \\
u_{6kn+j} &= u_j \prod_{k_1=0}^{n-1} \frac{A^{6k_1+1} + Bu_{j-k} u_{j+k} u_{j+3k} \sum_{l=0}^{6k_1} A^l}{A^{6k_1+3} + Bu_{j-k} u_{j+k} u_{j+3k} \sum_{l=0}^{6k_1+2} A^l}, j = k, \dots, 2k-1 \\
u_{6kn+j} &= u_j \prod_{k_1=0}^{n-1} \frac{A^{6k_1+2} + Bu_{j-2k} u_j u_{j+2k} \sum_{l=0}^{6k_1+1} A^l}{A^{6k_1+4} + Bu_{j-2k} u_j u_{j+2k} \sum_{l=0}^{6k_1+3} A^l}, j = 2k, \dots, 3k-1 \\
u_{6kn+j} &= u_j \prod_{k_1=0}^{n-1} \frac{A^{6k_1+3} + Bu_{j-3k} u_{j-k} u_{j+k} \sum_{l=0}^{6k_1+2} A^l}{A^{6k_1+5} + Bu_{j-3k} u_{j-k} u_{j+k} \sum_{l=0}^{6k_1+4} A^l}, j = 3k, \dots, 4k-1 \\
u_{6kn+j} &= u_j \prod_{k_1=0}^{n-1} \frac{A^{6k_1+4} + Bu_{j-4k} u_{j-2k} u_j \sum_{l=0}^{6k_1+3} A^l}{A^{6k_1+6} + Bu_{j-4k} u_{j-2k} u_j \sum_{l=0}^{6k_1+5} A^l}, \quad j = 4k, \dots, 5k-1 \\
u_{6kn+j} &= u_j \prod_{k_1=0}^{n-1} \frac{A^{6k_1+5} + Bu_{j-5k} u_{j-3k} u_{j-k} \sum_{l=0}^{6k_1+4} A^l}{A^{6k_1+7} + Bu_{j-5k} u_{j-3k} u_{j-k} \sum_{l=0}^{6k_1+6} A^l}, j = 5k, \dots, 6k-1.
\end{aligned} \tag{38}
$$

3.1 Case where the constant coefficient $A = 1$

For $A = 1$, the equations in (38) can be rewritten in the form

$$
\begin{aligned}
u_{6kn+j} &= u_j \prod_{k_1=0}^{n-1} \frac{1 + 6k_1 Bu_j u_{j+2k} u_{j+4k}}{1 + (6k_1+2) Bu_j u_{j+2k} u_{j+4k}}, \quad j = 0, \dots, k-1 \\
u_{6kn+j} &= u_j \prod_{k_1=0}^{n-1} \frac{1 + (6k_1+1) Bu_{j-k} u_{j+k} u_{j+3k}}{1 + (6k_1+3) Bu_{j-k} u_{j+k} u_{j+3k}}, j = k, \dots, 2k-1 \\
u_{6kn+j} &= u_j \prod_{k_1=0}^{n-1} \frac{1 + (6k_1+2) Bu_{j-2k} u_j u_{j+2k}}{1 + (6k_1+4) Bu_{j-2k} u_j u_{j+2k}}, j = 2k, \dots, 3k-1 \\
u_{6kn+j} &= u_j \prod_{k_1=0}^{n-1} \frac{1 + (6k_1+3) Bu_{j-3k} u_{j-k} u_{j+k}}{1 + (6k_1+5) Bu_{j-3k} u_{j-k} u_{j+k}}, j = 3k, \dots, 4k-1
\end{aligned}
$$

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{1+(6k_1+4)Bu_{j-4k}u_{j-2k}u_j}{1+(6k_1+6)Bu_{j-4k}u_{j-2k}u_j}, \quad j = 4k,\ldots,5k-1$$
$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{1+(6k_1+5)Bu_{j-5k}u_{j-3k}u_{j-k}}{1+(6k_1+7)Bu_{j-5k}u_{j-3k}u_{j-k}}, j = 5k,\ldots,6k-1. \tag{39}$$

3.2 Case where the constant coefficient $A \neq 1$

For $A \neq 1$, the equations in (38) take the form

$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{A^{6k_1}+Bu_ju_{j+2k}u_{j+4k}(\frac{1-A^{6k_1}}{1-A})}{A^{6k_1+2}+Bu_ju_{j+2k}u_{j+4k}(\frac{1-A^{6k_1+2}}{1-A})}, j = 0,\ldots,k-1$$
$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{A^{6k_1+1}+Bu_{j-k}u_{j+k}u_{j+3k}(\frac{1-A^{6k_1+1}}{1-A})}{A^{6k_1+3}+Bu_{j-k}u_{j+k}u_{j+3k}(\frac{1-A^{6k_1+3}}{1-A})}, j = k,\ldots,2k-1$$
$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{A^{6k_1+2}+Bu_{j-2k}u_ju_{j+2k}(\frac{1-A^{6k_1+2}}{1-A})}{A^{6k_1+4}+Bu_{j-2k}u_ju_{j+2k}(\frac{1-A^{6k_1+4}}{1-A})}, j = 2k,\ldots,3k-1$$
$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{A^{6k_1+3}+Bu_{j-3k}u_{j-k}u_{j+k}(\frac{1-A^{6k_1+3}}{1-A})}{A^{6k_1+5}+Bu_{j-3k}u_{j-k}u_{j+k}(\frac{1-A^{6k_1+5}}{1-A})}, j = 3k,\ldots,4k-1$$
$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{A^{6k_1+4}+Bu_{j-4k}u_{j-2k}u_j(\frac{1-A^{6k_1+4}}{1-A})}{A^{6k_1+6}+Bu_{j-4k}u_{j-2k}u_j(\frac{1-A^{6k_1+6}}{1-A})}, \quad j = 4k,\ldots,5k-1$$
$$u_{6kn+j} = u_j \prod_{k_1=0}^{n-1} \frac{A^{6k_1+5}+Bu_{j-5k}u_{j-3k}u_{j-k}(\frac{1-A^{6k_1+5}}{1-A})}{A^{6k_1+7}+Bu_{j-5k}u_{j-3k}u_{j-k}(\frac{1-A^{6k_1+7}}{1-A})}, j = 5k,\ldots,6k-1. \tag{40}$$

Theorem 3. *The non-constant solution of the ordinary difference equation*

$$u_{n+5k} = \frac{u_nu_{n+2k}u_{n+4k}}{u_{n+k}u_{n+3k}(A+Bu_nu_{n+2k}u_{n+4k})}, \tag{41}$$

where $|A| \neq 1$ and $B \neq 0$, is:

1. *$6k$-periodic if the initial conditions satisfy the conditions $x_i \neq x_{i+3k}$ and $u_ju_{j+2k}u_{j+4k} = (1-A)/B$.*
2. *$3k$-periodic if the initial conditions satisfy the conditions $x_i = x_{i+3k}$ and $u_ju_{j+k}u_{j+2k} = (1-A)/B$; and $k \not\equiv 0(mod\ 3)$.*

Proof. Assume that the initial conditions satisfy $u_ju_{j+2k}u_{j+4k} = (1-A)/B$. On this condition, using the first equation in (40), we have, for all n,

$$u_{6kn+j} = u_j, \quad j = 0,\ldots,6k-1. \tag{42}$$

This implies that the period of the solution divides $6k$. If the period is $1, k$ or $2k$, then $u_ju_{j+2k}u_{j+4k} = (u_j)^3 = \frac{(1-A)}{B}$. Therefore, all the terms (including the initial conditions) will be equal. This is a contradiction since all the terms are not equal. It follows that the period can only be $3k$ or $6k$. Thanks to (42), on condition that $u_ju_{j+2k}u_{j+4k} = \frac{1-A}{B}$, the solution is $6k$-periodic when the initial conditions satisfy $u_i \neq u_{i+3k}$; and it is $3k$-periodic when the initial conditions satisfy $u_i = u_{i+3k}$ with k not divisible by 3. ■

Theorem 4. *The non-constant solution of the ordinary difference equation*

$$u_{n+5k} = \frac{u_nu_{n+2k}u_{n+4k}}{u_{n+k}u_{n+3k}(-1+Bu_nu_{n+2k}u_{n+4k})}, \tag{43}$$

where $B \neq 0$, is $6k$-periodic provided that that the initial conditions satisfy $u_iu_{j+2k}u_{j+4k} \neq 2/B$.

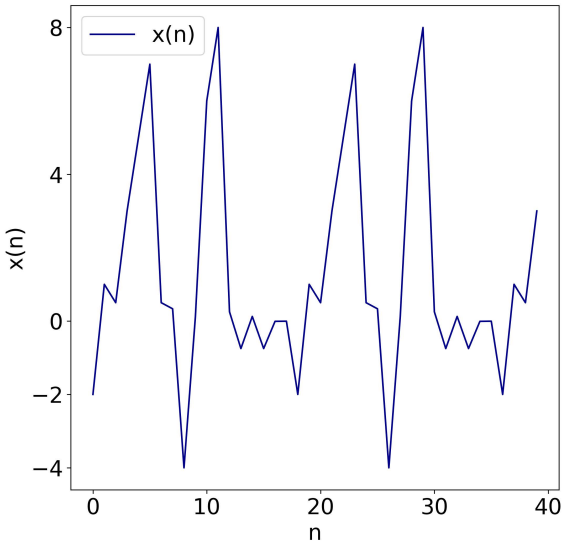


Fig. 1 Graph of $x_{n+15} = \frac{x_n x_{n+6} x_{n+12}}{x_{n+3} x_{n+9}(2+4x_n x_{n+6} x_{n+12})}$ with $x_0 = -2$, $x_1 = 1$, $x_2 = 1/2$, $x_3 = 3$, $x_4 = 5$, $x_5 = 7$, $x_6 = 1/2$, $x_7 = 1/3$, $x_8 = -4$, $x_9 = 1/9$, $x_{10} = 6$, $x_{11} = 8$, $x_{12} = 1/4$, $x_{13} = -3/4$, $x_{14} = 1/8$ satisfying (1) in Theorem 3.

Proof. When $A = -1$ in (40), we have, for all n, $u_{6kn+j} = u_j$, $j = 0, \ldots, 6k-1$. The $6k$-periodic solutions are given by:

$$\{u_0, u_1, \ldots, u_{5k-1}, u_{5k} = \frac{u_0 u_{2k} u_{4k}}{u_k u_{3k}(-1 + B u_0 u_{2k} u_{4k})}, u_{5k+1} = \frac{u_1 u_{1+2k} u_{1+4k}}{u_{1+k} u_{1+3k}(-1 + B u_1 u_{1+2k} u_{1+4k})}, \ldots,$$
$$u_{6k-1} = \frac{u_{k-1} u_{3k-1} u_{5k-1}}{u_{2k-1} u_{4k-1}(-1 + B u_{k-1} u_{3k-1} u_{5k-1})}, u_0, u_1, \ldots\}.$$

■

Fig.1 illustrates the graph of (41) with the initial conditions satisfying the first condition in Theorem 3. As expected, the solution is 18-periodic.

Fig.2 illustrates the graph of (41) with the initial conditions satisfying the second condition in Theorem 3. As expected, the solution is 6-periodic.

Fig.3 (resp. Fig.4) illustrates the graph of (43) with the initial conditions satisfying the condition in Theorem 4. As expected, the solution is 18-periodic (resp. 12-periodic).

Theorem 5. *The equilibrium point of the equation*

$$u_{n+5k} = \frac{u_n u_{n+2k} u_{n+4k}}{u_{n+k} u_{n+3k}(1 + B u_n u_{n+2k} u_{n+4k})}, \tag{44}$$

where $B \neq 0$ is a constant, is non-hyperbolic.

Proof. The equilibrium point of (44) is $\bar{u} = 0$ and the characteristic equation of (44) near $\bar{u} = 0$ is given by (see (10)):

$$\lambda^{5k} - \lambda^{4k} + \lambda^{3k} - \lambda^{2k} + \lambda^k - 1 = 0. \tag{45}$$

So $\lambda = 1$ is one of the roots for (45), therefore by Definition 8, the equilibrium point $\bar{u} = 0$ of (44) is non-hyperbolic. ■

Back-shifting (11) $5k-1$ times yields the equivalent difference equation

$$u_{n+1} = \frac{u_{n-5k+1} u_{n-3k+1} u_{n-k+1}}{u_{n-4k+1} u_{n-2k+1}(a + b u_{n-5k+1} u_{n-3k+1} u_{n-k+1})} \tag{46}$$

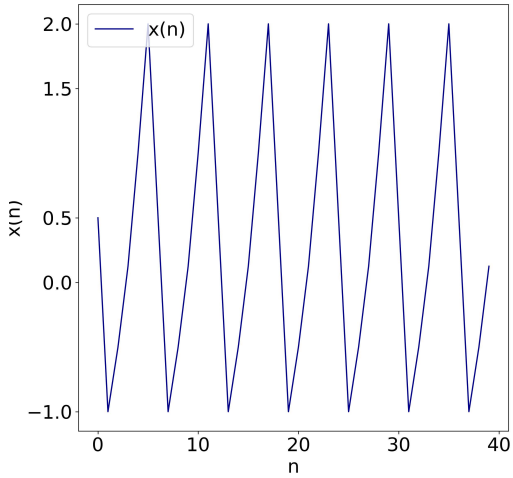


Fig. 2 Graph of $x_{n+10} = \frac{x_n x_{n+4} x_{n+8}}{x_{n+2} x_{n+6}(2+4x_n x_{n+4} x_{n+8})}$ with $x_0 = 1/2$, $x_1 = -1$, $x_2 = -1/2$, $x_3 = 1/8$, $x_4 = 1$, $x_5 = 2$, $x_6 = 1/2$, $x_7 = -1$, $x_8 = -1/2$, $x_9 = 1/8$ satisfying (2) in Theorem 3.

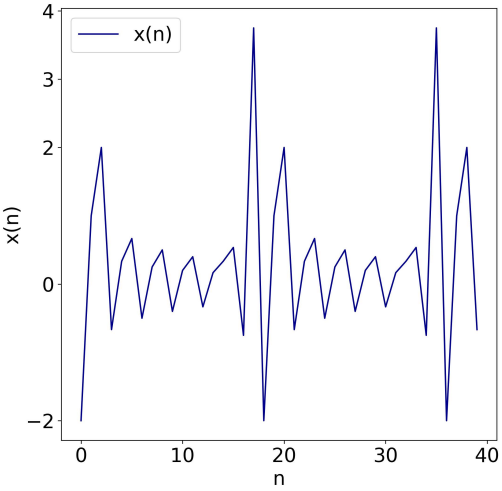


Fig. 3 Graph of $x_{n+15} = \frac{x_n x_{n+6} x_{n+12}}{x_{n+3} x_{n+9}(-1+4x_n x_{n+6} x_{n+12})}$ with $x_0 = -2$, $x_1 = 1$, $x_2 = 2$, $x_3 = -2/3$, $x_4 = 1/3$, $x_5 = 2/3$, $x_6 = -2/4$, $x_7 = 1/4$, $x_8 = 2/4$, $x_9 = -2/5$, $x_{10} = 1/5$, $x_{11} = 2/5$, $x_{12} = -2/6$, $x_{13} = 1/6$, $x_{14} = 2/6$.

whose solution is obtained by back-shifting that of (11) (see Eq. (4)) $5k-1$ times. As a consequence of this, the solution of (46) is given by:

$$u_{6kn+j-5k+1} = u_{j-5k+1} \prod_{k_1=0}^{n-1} \frac{a^{6k_1} + b u_{j-5k+1} u_{j-3k+1} u_{j-k+1} \sum_{l=0}^{6k_1-1} a^l}{a^{6_1+2} + b u_{j-5k+1} u_{j-3k+1} u_{j-k+1} \sum_{l=0}^{6k_1+1} a^l}, j = 0, \ldots, k-1 \tag{47}$$

$$u_{6kn+j-5k+1} = u_{j-5k+1} \prod_{k_1=0}^{n-1} \frac{a^{6k_1+1} + b u_{j-6k+1} u_{j-4k+1} u_{j-2k+1} \sum_{l=0}^{6k_1} a^l}{a^{6k_1+3} + b u_{j-6k+1} u_{j-4k+1} u_{j-2k+1} \sum_{l=0}^{6k_1+2} a^l}, j = k, \ldots, 2k-1 \tag{48}$$

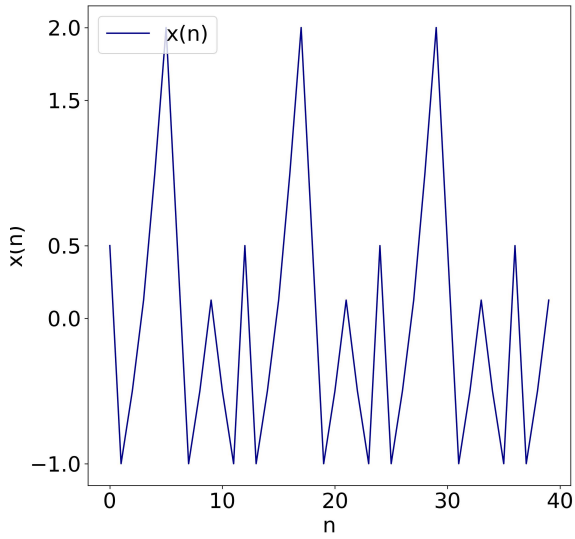


Fig. 4 Graph of $x_{n+10} = \frac{x_n x_{n+4} x_{n+8}}{x_{n+2} x_{n+6}(-1+4x_n x_{n+4} x_{n+8})}$ with $x_0 = 1/2, x_1 = -1, x_2 = -1/2, x_3 = 1/8, x_4 = 1, x_5 = 2, x_6 = 1/2, x_7 = -1, x_8 = -1/2, x_9 = 1/8$.

$$u_{6kn+j-5k+1} = u_{j-5k+1} \prod_{k_1=0}^{n-1} \frac{a^{6k_1+2} + bu_{j-7k+1}u_{j-5k+1}u_{j-3k+1} \sum_{l=0}^{6k_1+1} a^l}{a^{6k_1+4} + bu_{j-7k+1}u_{j-5k+1}u_{j-3k+1} \sum_{l=0}^{6k_1+3} a^l}, j = 2k, \ldots, 3k-1 \tag{49}$$

$$u_{6kn+j-5k+1} = u_{j-5k+1} \prod_{k_1=0}^{n-1} \frac{a^{6k_1+3} + bu_{j-8k+1}u_{j-6k+1}u_{j-4k+1} \sum_{l=0}^{6k_1+2} a^l}{a^{6k_1+5} + bu_{j-8k+1}u_{j-6k+1}u_{j-4k+1} \sum_{l=0}^{6k_1+4} a^l}, j = 3k, \ldots, 4k-1 \tag{50}$$

$$u_{6kn+j-5k+1} = u_{j-5k+1} \prod_{k_1=0}^{n-1} \frac{a^{6k_1+4} + bu_{j-9k+1}u_{j-7k+1}u_{j-5k+1} \sum_{l=0}^{6k_1+3} a^l}{a^{6k_1+6} + bu_{j-9k+1}u_{j-7k+1}u_{j-5k+1} \sum_{l=0}^{6k_1+5} a^l}, j = 4k, \ldots, 5k-1 \tag{51}$$

$$u_{6kn+j-5k+1} = u_{j-5k+1} \prod_{k_1=0}^{n-1} \frac{a^{6k_1+5} + bu_{j-10k+1}u_{j-8k+1}u_{j-6k+1} \sum_{l=0}^{6k_1+4} a^l}{a^{6k_1+7} + bu_{j-10k+1}u_{j-8k+1}u_{j-6k+1} \sum_{l=0}^{6k_1+6} a^l}, j = 5k, \ldots, 6k-1. \tag{52}$$

3.3 An In-depth Analysis of the Case where $k = 2$

We take a look at the case $k = 2$. The equations (47) to (52) simplify to the following when $k = 2$:

$$u_{12n+j-9} = u_{j-9} \prod_{k_1=0}^{n-1} \frac{a^{6k_1} + bu_{j-9}u_{j-5}u_{j-1} \sum_{l=0}^{6k_1-1} a^l}{a^{6_1+2} + bu_{j-9}u_{j-5}u_{j-1} \sum_{l=0}^{6k_1+1} a^l}, j = 0, 1,$$

$$u_{12n+j-9} = u_{j-9}\prod_{k_1=0}^{n-1}\frac{a^{6k_1+1}+bu_{j-11}u_{j-7}u_{j-3}\sum_{l=0}^{6k_1}a^l}{a^{6k_1+3}+bu_{j-11}u_{j-7}u_{j-3}\sum_{l=0}^{6k_1+2}a^l}, j=2,3,$$

$$u_{12n+j-9} = u_{j-9}\prod_{k_1=0}^{n-1}\frac{a^{6k_1+2}+bu_{j-13}u_{j-9}u_{j-5}\sum_{l=0}^{6k_1+1}a^l}{a^{6k_1+4}+bu_{j-13}u_{j-9}u_{j-5}\sum_{l=0}^{6k_1+3}a^l}, j=4,5,$$

$$u_{12n+j-9} = u_{j-9}\prod_{k_1=0}^{n-1}\frac{a^{6k_1+3}+bu_{j-15}u_{j-11}u_{j-7}\sum_{l=0}^{6k_1+2}a^l}{a^{6k_1+5}+bu_{j-15}u_{j-11}u_{j-7}\sum_{l=0}^{6k_1+4}a^l}, j=6,7,$$

$$u_{12n+j-9} = u_{j-9}\prod_{k_1=0}^{n-1}\frac{a^{6k_1+4}+bu_{j-17}u_{j-13}u_{j-9}\sum_{l=0}^{6k_1+3}a^l}{a^{6k_1+6}+bu_{j-17}u_{j-13}u_{j-9}\sum_{l=0}^{6k_1+5}a^l}, j=8,9.$$

$$u_{12n+j-9} = u_{j-9}\prod_{k_1=0}^{n-1}\frac{a^{6k_1+5}+bu_{j-19}u_{j-15}u_{j-11}\sum_{l=0}^{6k_1+4}a^l}{a^{6k_1+7}+bu_{j-19}u_{j-15}u_{j-11}\sum_{l=0}^{6k_1+6}a^l}, j=10,11. \quad (53)$$

3.3.1 The case when $a=1$

In the instance case, using (53), we have that:

$$u_{12n-9} = \frac{u_{-9}\prod_{k_1=0}^{n-1}(1+6k_1bu_{-9}u_{-5}u_{-1})}{\prod_{k_1=0}^{n-1}(1+(6k_1+1)bu_{-9}u_{-5}u_{-1})}, \qquad u_{12n-8} = \frac{u_{-8}\prod_{k_1=0}^{n-1}(1+6k_1bu_{-8}u_{-4}u_0)}{\prod_{k_1=0}^{n-1}(1+(6k_1+2)bu_{-8}u_{-4}u_0)},$$

$$u_{12n-7} = \frac{u_{-7}\prod_{k_1=0}^{n-1}(1+(6k_1+1)bu_{-9}u_{-5}u_{-1})}{\prod_{k_1=0}^{n-1}(1+(6k_1+3)bu_{-9}u_{-5}u_{-1})}, \qquad u_{12n-6} = \frac{u_{-6}\prod_{k_1=0}^{n-1}(1+(6k_1+1)bu_{-8}u_{-4}u_0)}{\prod_{k_1=0}^{n-1}(1+(6k_1+3)bu_{-8}u_{-4}u_0)},$$

$$u_{12n-5} = \frac{u_{-5}\prod_{k_1=0}^{n-1}(1+(6k_1+2)bu_{-9}u_{-5}u_{-1})}{\prod_{k_1=0}^{n-1}(1+(6k_1+4)bu_{-9}u_{-5}u_{-1})}, \qquad u_{12n-4} = \frac{u_{-4}\prod_{k_1=0}^{n-1}(1+(6k_1+2)bu_{-8}u_{-4}u_0)}{\prod_{k_1=0}^{n-1}(1+(6k_1+4)bu_{-8}u_{-4}u_0)}$$

$$u_{12n-3} = \frac{u_{-3}\prod_{k_1=0}^{n-1}(1+(6k_1+3)bu_{-9}u_{-5}u_{-1})}{\prod_{k_1=0}^{n-1}(1+(6k_1+5)bu_{-9}u_{-5}u_{-1})}, \qquad u_{12n-2} = \frac{u_{-2}\prod_{k_1=0}^{n-1}(1+(6k_1+3)bu_{-8}u_{-4}u_0)}{\prod_{k_1=0}^{n-1}(1+(6k_1+5)bu_{-8}u_{-4}u_0)},$$

$$u_{12n-1}=\frac{u_{-1}\prod_{k_1=0}^{n-1}(1+(6k_1+4)bu_{-9}u_{-5}u_{-1})}{\prod_{k_1=0}^{n-1}(1+(6k_1+6)bu_{-9}u_{-5}u_{-1})},\qquad u_{12n}=\frac{u_0\prod_{k_1=0}^{n-1}(1+(6k_1+4)bu_{-8}u_{-4}u_0)}{\prod_{k_1=0}^{n-1}(1+(6k_1+6)bu_{-8}u_{-4}u_0)},$$

$$u_{12n+1}=\frac{u_{-9}u_{-5}u_{-1}\prod_{k_1=0}^{n-1}(1+(6k_1+5)bu_{-9}u_{-5}u_{-1})}{u_{-7}u_{-3}(1+bu_{-9}u_{-5}u_{-1})\prod_{k_1=0}^{n-1}(1+(6k_1+7)bu_{-9}u_{-5}u_{-1})},$$

$$u_{12n+2}=\frac{u_{-8}u_{-4}u_0\prod_{k_1=0}^{n-1}(1+(6k_1+5)bu_{-8}u_{-4}u_0)}{u_{-6}u_{-2}(1+bu_{-8}u_{-4}u_0)\prod_{k_1=0}^{n-1}(1+(6k_1+7)bu_{-8}u_{-4}u_0)}. \tag{54}$$

Note that in u_{12n+1} (resp. u_{12n+2}), we replace u_1 (resp. u_2) with $u_{-9}u_{-5}u_{-1}/(u_{-7}u_{-3}(1+bu_{-9}u_{-5}u_{-1}))$ (resp. $\frac{u_{-8}u_{-4}u_0}{u_{-6}u_{-2}(1+bu_{-8}u_{-4}u_0)}$).

3.3.2 The case when $a \neq 1$

In the instance case, using (53), we have that:

$$u_{12n-9}=\frac{u_{-9}\prod_{k_1=0}^{n-1}(a^{6k_1}+bu_{-9}u_{-5}u_{-1}(\frac{1-a^{6k_1}}{1-a}))}{\prod_{k_1=0}^{n-1}(a^{6k_1+1}+bu_{-9}u_{-5}u_{-1}(\frac{1-a^{6k_1+1}}{1-a}))},\qquad u_{12n-8}=\frac{u_{-8}\prod_{k_1=0}^{n-1}(a^{6k_1}+bu_{-8}u_{-4}u_0(\frac{1-a^{6k_1}}{1-a}))}{\prod_{k_1=0}^{n-1}(a^{6k_1+2}+bu_{-8}u_{-4}u_0(\frac{1-a^{6k_1+2}}{1-a}))},$$

$$u_{12n-7}=\frac{u_{-7}\prod_{k_1=0}^{n-1}(a^{6k_1+1}+bu_{-9}u_{-5}u_{-1}(\frac{1-a^{6k_1+1}}{1-a}))}{\prod_{k_1=0}^{n-1}(a^{6k_1+3}+bu_{-9}u_{-5}u_{-1}(\frac{1-a^{6k_1+3}}{1-a}))},\qquad u_{12n-6}=\frac{u_{-6}\prod_{k_1=0}^{n-1}(a^{6k_1+1}+bu_{-8}u_{-4}u_0(\frac{1-a^{6k_1+1}}{1-a}))}{\prod_{k_1=0}^{n-1}(a^{6k_1+3}+bu_{-8}u_{-4}u_0(\frac{1-a^{6k_1+3}}{1-a}))},$$

$$u_{12n-5}=\frac{u_{-5}\prod_{k_1=0}^{n-1}(a^{6k_1+2}+bu_{-9}u_{-5}u_{-1}(\frac{1-a^{6k_1+2}}{1-a}))}{\prod_{k_1=0}^{n-1}(a^{6k_1+4}+bu_{-9}u_{-5}u_{-1}(\frac{1-a^{6k_1+4}}{1-a}))},\qquad u_{12n-4}=\frac{u_{-4}\prod_{k_1=0}^{n-1}(a^{6k_1+2}+bu_{-8}u_{-4}u_0(\frac{1-a^{6k_1+2}}{1-a}))}{\prod_{k_1=0}^{n-1}(a^{6k_1+4}+bu_{-8}u_{-4}u_0(\frac{1-a^{6k_1+4}}{1-a}))}$$

$$u_{12n-3}=\frac{u_{-3}\prod_{k_1=0}^{n-1}(a^{6k_1+3}+bu_{-9}u_{-5}u_{-1}(\frac{1-a^{6k_1+3}}{1-a}))}{\prod_{k_1=0}^{n-1}(a^{6k_1+5}+bu_{-9}u_{-5}u_{-1}(\frac{1-a^{6k_1+5}}{1-a}))},\qquad u_{12n-2}=\frac{u_{-2}\prod_{k_1=0}^{n-1}(a^{6k_1+3}+bu_{-8}u_{-4}u_0(\frac{1-a^{6k_1+3}}{1-a}))}{\prod_{k_1=0}^{n-1}(a^{6k_1+5}+bu_{-8}u_{-4}u_0(\frac{1-a^{6k_1+5}}{1-a}))},$$

$$u_{12n-1}=\frac{u_{-1}\prod_{k_1=0}^{n-1}(a^{6k_1+4}+bu_{-9}u_{-5}u_{-1}(\frac{1-a^{6k_1+4}}{1-a}))}{\prod_{k_1=0}^{n-1}(a^{6k_1+6}+bu_{-9}u_{-5}u_{-1}(\frac{1-a^{6k_1+6}}{1-a}))},\qquad u_{12n}=\frac{u_0\prod_{k_1=0}^{n-1}(a^{6k_1+4}+bu_{-8}u_{-4}u_0(\frac{1-a^{6k_1+4}}{1-a}))}{\prod_{k_1=0}^{n-1}(a^{6k_1+6}+bu_{-8}u_{-4}u_0(\frac{1-a^{6k_1+6}}{1-a}))},$$

$$u_{12n+1}=\frac{u_{-9}u_{-5}u_{-1}\prod_{k_1=0}^{n-1}(a^{6k_1+5}+bu_{-9}u_{-5}u_{-1}(\frac{1-a^{6k_1+5}}{1-a}))}{u_{-7}u_{-3}(a+bu_{-9}u_{-5}u_{-1})\prod_{k_1=0}^{n-1}(a^{6k_1+7}+bu_{-9}u_{-5}u_{-1}(\frac{1-a^{6k_1+7}}{1-a}))},$$

$$u_{12n+2} = \frac{u_{-8}u_{-4}u_0 \prod_{k_1=0}^{n-1} (a^{6k_1+5} + bu_{-8}u_{-4}u_0(\frac{1-a^{6k_1+5}}{1-a}))}{u_{-6}u_{-2}(a+bu_{-8}u_{-4}u_0) \prod_{k_1=0}^{n-1} (a^{6k_1+7} + bu_{-8}u_{-4}u_0(\frac{1-a^{6k_1+7}}{1-a}))}. \quad (55)$$

Remark 1.

- Setting $a = 1$ and $b = 1$, equations (54) reduce to Theorem 1 in [15].
- Setting $a = 1$ and $b = 1$ in Theorem 5 reduces to Theorem 2 in [15].
- Setting $a = 1$ and $b = -1$, the equations in (54) reduce to Theorem 3 in [15].
- Setting $a = 1$ and $b = -1$ in Theorem 5 reduces to Theorem 4 in [15].
- Setting $a = -1$ and $b = -1$ in the equations in (55) and Theorem 4 reduce to Theorems 5 and 6 in [15].
- Setting $a = -1$ and $b = 1$ in the equations in (55) and Theorem 4 reduces to Theorems 7 and 8 in [15].

4 Conclusion

We have reviewed the steps to determine the symmetry generators of the ordinary difference Eq. (11) and we have found its exact solutions. We have studied the periodicity of the solution and have proved the existence of k, $2k$, $3k$, $6k$-periodic solutions. Furthermore, we have looked at the stability of the equilibrium point. For a more in-depth understanding, we have analyzed the case where $k = 2$. We have provided different combinations of the values of $a = \pm 1$ and $b = \pm 1$ that reduce to all the eight theorems in [15], thus generalizing the findings in the article [15].

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