

CANONICAL FORMS FOR BOUNDARY CONDITIONS OF SELF-ADJOINT DIFFERENTIAL OPERATORS

Yorick Hardy^{1,†} and Bertin Zinsou¹

Abstract Canonical forms of boundary conditions are important in the study of the eigenvalues of boundary conditions and their numerical computations. The known canonical forms for self-adjoint differential operators, with eigenvalue parameter dependent boundary conditions, are limited to 4-th order differential operators. We derive canonical forms for self-adjoint $2n$ -th order differential operators with eigenvalue parameter dependent boundary conditions. We compare the 4-th order canonical forms to the canonical forms derived in this article.

Keywords Canonical forms, boundary conditions, self-adjoint operators, CS-decomposition, Hermitian.

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1. Introduction

Canonical forms of boundary conditions are important in the study of the eigenvalues of boundary conditions [17, Section 1.3], their numerical computations [1], characterizations of self-adjoint extensions of systems [16] and the study of transmission conditions [19]. In [4], Hao, Sun and Zettl investigate canonical forms of self-adjoint boundary conditions for fourth order differential operators. They derive three mutually exclusive types of boundary conditions, which are separated, coupled and mixed boundary conditions. In [2] Bao, Hao, Sun and Zettl provide new canonical forms of self-adjoint boundary conditions for regular differential operators of order two and four, while [12] considers canonical forms for regular differential operators of order three. These results are of importance in the study of dependence of eigenvalues with respect to the parameters of the differential equation [7–9, 18]. In particular, this dependence arises in n -th order boundary value transmission problems [7, 9].

In this paper, we extend the study conducted in [4] to $2n$ -th order differential operators. We start our investigation with sixth order differential operators with self-adjoint boundary conditions that we extend to $2n$ -th differential operators with self-adjoint boundary conditions and we show equivalence between the separated and coupled forms presented in [4] and those obtained during our investigation.

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In Section 2, we introduce the self-adjoint sixth order differential operators with eigenvalue dependent boundary conditions under consideration. In Section 3, we present the types of boundary conditions for the self-adjoint sixth order differential operators. Next, using the CS-decomposition, we provide a classification of the different types of canonical forms for self-adjoint sixth order differential operators in Section 4 that we extend in Section 5 to canonical forms for self-adjoint $2n$ -th order differential operators. Finally, in Section 6 we show equivalences between the separated and coupled forms provided in [4] with those obtained in this paper.

2. Self-adjoint sixth order boundary value problems

We consider on the interval $J = (a, b)$, $-\infty \leq a < b \leq \infty$, the sixth order differential equation with formally self-adjoint differential expression (with smooth coefficients) [11, Remark 3.2]

$$My = -(p_3 y''')''' + (p_2 y'')'' + (p_1 y')' + p_0 y = \lambda w y, \quad (2.1)$$

where $\frac{1}{p_3}$ exists on J , $p_j \in C^j(J)$ are sufficiently smooth real-valued functions on J and $w \in L(J, \mathbb{R})$ is a real-valued Lebesgue integrable function on J , $w > 0$ a.e. on J . If the coefficients are not smooth, we introduce the quasi-derivatives

$$\begin{aligned} y^{[0]} &= y, \quad y^{[1]} = y', \quad y^{[2]} = y'', \quad y^{[3]} = -p_3 y''', \\ y^{[4]} &= (-p_3 y''')' + p_2 y'', \quad y^{[5]} = (-p_3 y''')'' + (p_2 y'')' + p_1 y', \\ y^{[6]} &= -(p_3 y''')''' + (p_2 y'')'' + (p_1 y')' + p_0 y, \end{aligned}$$

and (2.1) is replaced by the equation $y^{[6]} = \lambda w y$ where $1/p_3, p_2, p_1, p_0, w \in L(J, \mathbb{R})$, $p_3 > 0$, $w > 0$ a.e. on J [11, p. 3]. In either case, the boundary conditions have the same form. Let $Y = (y, y', y'', y^{[3]}, y^{[4]}, y^{[5]})^\top$. We now consider the sixth order boundary value problem defined by (2.1) and the boundary conditions

$$AY(a) + BY(b) = 0, \quad A, B \in M_6(\mathbb{C}). \quad (2.2)$$

For the boundary conditions (2.2) with the assumptions made so far, [10, Theorem 2.4] leads to

Proposition 2.1. *Let C_6 be the symplectic matrix of order 6 defined by*

$$C_6 = ((-1)^r \delta_{r,7-s})_{r,s=1}^6, \quad (2.3)$$

where δ is the Kronecker delta. Then problems (2.1)–(2.2) are self-adjoint if and only if

$$\text{rank}(A : B) = 6 \quad \text{and} \quad AC_6 A^* = BC_6 B^*. \quad (2.4)$$

3. Types of boundary conditions of sixth order differential operators

The following theorem gives conditions satisfied by the matrices A, B for the problems (2.1)–(2.2) to be self-adjoint.

Theorem 3.1. Assume that the matrices $A, B \in M_6(\mathbb{C})$ satisfy (2.4). Then

- (i) $3 \leq \text{rank } A \leq 6, 3 \leq \text{rank } B \leq 6$;
- (ii) let $0 \leq r \leq 3$; if $\text{rank } A = 3 + r$, then $\text{rank } B = 3 + r$.

Proof. See [15, Theorem 3]. □

Note that the boundary conditions (2.2) are invariant under left multiplication by a non singular matrix $G \in M_6(\mathbb{C})$ and if $AC_6A^* = BC_6B^*$, then

$$(GA)C_6(GA)^* = (GB)C_6(GB)^*.$$

Therefore, the boundary condition form (2.4) is invariant under elementary matrix row transformations of $(A : B)$.

Next, we define the different types of boundary conditions based on Theorem 3.1.

Definition 3.1. Let the hypotheses and notation of Theorem 3.1 hold. Then the boundary conditions (2.2), (2.4) are

- (1) separated if $r = 0$,
- (2) mixed if $r = 1, 2$,
- (3) coupled if $r = 3$.

Remark 3.1. Note that the boundary conditions (2.2) are separated if each of the six boundary conditions involves only one endpoint, coupled if each of the six boundary conditions involves both endpoints, while they are mixed if there is at least one separated and one coupled boundary conditions.

4. Canonical forms for sixth order differential operators

Here we follow the approach of [6], the matrices concerned are neither centro-unitary nor pseudo-unitary but they do share some algebraic characteristics of these matrices. Equation (2.4) can be written in the form

$$\text{rank}(A : B) = 6, \quad (A : B) \begin{pmatrix} C_6 & 0 \\ 0 & -C_6 \end{pmatrix} (A : B)^* = 0, \quad (4.1)$$

where $\begin{pmatrix} C_6 & 0 \\ 0 & -C_6 \end{pmatrix}$ is a skew-Hermitian matrix with eigenvalues i and $-i$. Thus, each column vector $\mathbf{x}_j^*$ of $(A : B)^*$ may be written in the form

$$\mathbf{x}_j^* = \mathbf{x}_{j,i}^* + \mathbf{x}_{j,-i}^* \quad (4.2)$$

where $\mathbf{x}_{j,\pm i}^*$ belongs to the eigenspace corresponding to the eigenvalue $\pm i$. Condition (4.1) may now be written

$$\mathbf{x}_{j,i} \mathbf{x}_{k,i}^* = \mathbf{x}_{j,-i} \mathbf{x}_{k,-i}^*. \quad (4.3)$$

Taking $\mathbf{x}_{j,i}$ as the rows of X_i and similarly for X_{-i} , (4.2) may be summarized as $(A : B) = X_i + X_{-i}$ and (4.3) as

$$X_i X_i^* = X_{-i} X_{-i}^*. \quad (4.4)$$

Now decompose

$$\begin{pmatrix} C_6 & 0 \\ 0 & -C_6 \end{pmatrix} = V \begin{pmatrix} iI_6 & 0 \\ 0 & -iI_6 \end{pmatrix} V^* \quad (4.5)$$

where V is an arbitrary unitary matrix providing the diagonalization. From the ordering of eigenvectors (columns of V) in (4.5) and the solution (4.2) in terms of eigenvectors, the matrix V may be chosen so that $(A : B)$ has the form

$$(A : B) = (C : D)V^*. \quad (4.6)$$

Writing

$$V = \begin{pmatrix} \mathbf{y}_{1,i}^* & \cdots & \mathbf{y}_{6,i}^* & \mathbf{y}_{1,-i}^* & \cdots & \mathbf{y}_{6,-i}^* \end{pmatrix}, \quad V^* = \begin{pmatrix} \mathbf{y}_{1,i} \\ \vdots \\ \mathbf{y}_{6,i} \\ \mathbf{y}_{1,-i} \\ \vdots \\ \mathbf{y}_{6,-i} \end{pmatrix},$$

where V is unitary and each $\mathbf{y}_{j,\pm i}$ is an eigenvector corresponding to the eigenvalue $\pm i$. Equation (4.4) yields

$$X_i = C \begin{pmatrix} \mathbf{y}_{1,i} \\ \vdots \\ \mathbf{y}_{6,i} \end{pmatrix}, \quad X_{-i} = D \begin{pmatrix} \mathbf{y}_{1,-i} \\ \vdots \\ \mathbf{y}_{6,-i} \end{pmatrix}$$

so that (4.4) becomes

$$CC^* = DD^*,$$

and, since positive definite square roots are unique, the singular value decompositions $C = U_C \Sigma_C V_C^*$ and $D = U_D \Sigma_D V_D^*$ show that

$$\Sigma_C = (U_C^* U_D) \Sigma_D (U_C^* U_D)^*.$$

Hence

$$\begin{aligned} (A : B) &= (U_C \Sigma_C : U_D \Sigma_D) \begin{pmatrix} V_C & 0 \\ 0 & V_D \end{pmatrix}^* V^* \\ &= U_C (\Sigma_C : U_C^* U_D \Sigma_D) \begin{pmatrix} V_C & 0 \\ 0 & V_D \end{pmatrix}^* V^* \\ &= U_C (\Sigma_C : \Sigma_C) \begin{pmatrix} V_C & 0 \\ 0 & V_D (U_C^* U_D)^* \end{pmatrix}^* V^* \end{aligned}$$

$$= U_C \Sigma_C (I_6 : I_6) \begin{pmatrix} V_C & 0 \\ 0 & V_D (U_C^* U_D)^* \end{pmatrix}^* V^*$$

yields the solution (4.2) and satisfies (4.3). Since $\text{rank}(A : B) = 6$, we have $\text{rank}(\Sigma_C) = 6$ and hence Σ_C is invertible. By invariance of the boundary conditions under elementary row operations, we obtain the general form

$$(A : B) = (I_6 : I_6) \begin{pmatrix} V_X & 0 \\ 0 & V_Y \end{pmatrix}^* V^* \quad (4.7)$$

where V_X and V_Y are arbitrary unitary matrices. Here, the first 6 columns of V are eigenvectors corresponding to the eigenvalue i of $C_6 \oplus (-C_6)$, and the remaining 6 columns correspond to the eigenvalue $-i$. We write V as the block matrix

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

so that (4.7) becomes

$$(A : B) = (V_X^* V_{11}^* + V_Y^* V_{12}^* : V_X^* V_{21}^* + V_Y^* V_{22}^*)$$

where

$$\begin{pmatrix} C_6 & 0 \\ 0 & -C_6 \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} = i \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix}, \quad \begin{pmatrix} C_6 & 0 \\ 0 & -C_6 \end{pmatrix} \begin{pmatrix} V_{12} \\ V_{22} \end{pmatrix} = -i \begin{pmatrix} V_{12} \\ V_{22} \end{pmatrix}.$$

Again, since the boundary conditions are invariant under row operations, we will assume

$$(A : B) = (V_{11}^* + W V_{12}^* : V_{21}^* + W V_{22}^*) \quad (4.8)$$

where $W = V_X V_Y^*$ is unitary. Choosing a particular V provides some additional insight. For the purpose of illustration, we also set $W = I_6$ in the following example. Let

$$V_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -i \\ 0 & i & 0 \\ -i & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_3 \\ i C_3 \end{pmatrix},$$

$$V_{-i} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & i \\ 0 & -i & 0 \\ i & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_3 \\ -iC_3 \end{pmatrix}$$

where

$$C_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Now consider V given by

$$V = \begin{pmatrix} V_i & 0 & V_{-i} & 0 \\ 0 & V_{-i} & 0 & V_i \end{pmatrix}. \quad (4.9)$$

Hence

$$(A : B) = \begin{pmatrix} V_i^* + V_{-i}^* & 0 \\ 0 & V_i^* + V_{-i}^* \end{pmatrix} = \begin{pmatrix} \sqrt{2}I_3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}I_3 & 0 \end{pmatrix}.$$

Here $\text{rank}(A) = \text{rank}(B) = 3$. Choosing V as above, leads to a canonical form for separated boundary conditions in Lemma 4.1.

Let

$$W = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}.$$

Then

$$(A : B) = \begin{pmatrix} V_i^* & V_{-i}^* \\ V_{-i}^* & V_i^* \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_3 & -iC_3 & I_3 & -iC_3 \\ I_3 & iC_3 & I_3 & iC_3 \end{pmatrix}.$$

Here we obtain coupled boundary conditions, leading to a canonical form in Lemma 4.1.

From (4.9), we have

$$\begin{aligned} V_{11} &= \frac{1}{\sqrt{2}} \begin{pmatrix} I_3 & 0 \\ iC_3 & 0 \end{pmatrix}, & V_{12} &= \frac{1}{\sqrt{2}} \begin{pmatrix} I_3 & 0 \\ -iC_3 & 0 \end{pmatrix}, \\ V_{21} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & I_3 \\ 0 & -iC_3 \end{pmatrix}, & V_{22} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & I_3 \\ 0 & iC_3 \end{pmatrix}. \end{aligned}$$

Choosing appropriate W provides the remaining canonical forms. Thus

$$A = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} I_3 & -iC_3 \\ 0 & 0 \end{pmatrix} + W \begin{pmatrix} I_3 & iC_3 \\ 0 & 0 \end{pmatrix} \right],$$

$$B = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 0 & 0 \\ I_3 & iC_3 \end{pmatrix} + W \begin{pmatrix} 0 & 0 \\ I_3 & -iC_3 \end{pmatrix} \right].$$

Let

$$W = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}.$$

It follows that

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} W_1 & I_3 \\ W_3 & 0 \end{pmatrix} \begin{pmatrix} I_3 & I_3 \\ I_3 & -I_3 \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ 0 & iC_3 \end{pmatrix},$$

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & W_2 \\ I_3 & W_4 \end{pmatrix} \begin{pmatrix} I_3 & I_3 \\ I_3 & -I_3 \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ 0 & iC_3 \end{pmatrix},$$

and hence

$$\begin{aligned} \text{rank}(A) &= \text{rank}(I_3) + \text{rank}(W_3), \\ \text{rank}(B) &= \text{rank}(I_3) + \text{rank}(W_2). \end{aligned}$$

Necessarily $\text{rank}(W_3) = \text{rank}(W_2)$. The CS-decomposition, described in detail in [13] and [5, Theorem 2.7,1], provides a useful way to speak about rank. In particular, we obtain the CS-decomposition of W using [3, Corollary 3,1]

$$W = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} C & S \\ -S & C \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$$

for some unitary matrices U_1 , U_2 , V_1 and V_2 , and positive semi-definite diagonal matrices C and S satisfying $C^2 + S^2 = I_3$. Hence, up to elementary row operations,

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} C & U_1^* \\ -S & 0 \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & I_3 \end{pmatrix} \begin{pmatrix} I_3 & I_3 \\ I_3 & -I_3 \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ 0 & iC_3 \end{pmatrix},$$

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & S \\ U_2^* & C \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ 0 & V_2 \end{pmatrix} \begin{pmatrix} I_3 & I_3 \\ I_3 & -I_3 \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ 0 & iC_3 \end{pmatrix}$$

with

$$\text{rank}(A) = \text{rank}(B) = \text{rank}(I_3) + \text{rank}(S).$$

When $\text{rank } S = 0$, then

$$W = \begin{pmatrix} W_1 & 0 \\ 0 & W_4 \end{pmatrix} \tag{4.10}$$

where $W_1 = U_1 V_1$ and $W_2 = U_2 V_2$ are unitary. If $\text{rank } S \neq 0$, then W does not simplify in an obvious way. Thus we have the following Lemma.

Lemma 4.1. *Let A and B be 6×6 matrices satisfying*

$$\text{rank}(A : B) = 6 \quad \text{and} \quad AC_6 A^* = BC_6 B^*.$$

Let Z be the matrix

$$Z = \frac{1}{\sqrt{2}} \begin{pmatrix} I_3 & I_3 & 0 & 0 \\ I_3 & -I_3 & 0 & 0 \\ 0 & 0 & I_3 & I_3 \\ 0 & 0 & I_3 & -I_3 \end{pmatrix} \begin{pmatrix} I_3 & 0 & 0 & 0 \\ 0 & iC_3 & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & iC_3 \end{pmatrix}.$$

There exist a 6×6 non singular matrix U , 3×3 unitary matrices U_1, U_2, V_1 and V_2 , and positive semi-definite diagonal matrices C and S with $C^2 + S^2 = I_3$, such that

$$(A : B) = U \begin{pmatrix} C & I_3 & 0 & S \\ -S & 0 & I_3 & C \end{pmatrix} \begin{pmatrix} V_1 & 0 & 0 & 0 \\ 0 & U_1^* & 0 & 0 \\ 0 & 0 & U_2^* & 0 \\ 0 & 0 & 0 & V_2 \end{pmatrix} Z$$

and, the boundary conditions are

1. *separated, if and only if $S = 0$.*
2. *mixed, if and only if $0 < \text{rank}(S) < 3$.*
3. *coupled, if and only if $\text{rank}(S) = 3$.*

Remark 4.1. It may be assumed, in this representation of $(A : B)$, that the diagonal entries of C are non-increasing and that the diagonal entries of S are non-decreasing.

5. Canonical forms for $2n$ -th order differential operators

We consider on the interval $J = (a, b)$, $-\infty \leq a < b \leq \infty$, the $2n$ -th order differential equation with formally self-adjoint differential expression (with smooth coefficients) [11, Remark 3.2]

$$My = (-1)^n (p_n y^{(n)})^{(n)} + (p_{n-1} y^{(n-1)})^{(n-1)} + \cdots + (p_1 y')' + p_0 y = \lambda w y, \quad (5.1)$$

where $\frac{1}{p_n}$ exists on J , $p_j \in C^j(J)$ and $w \in L(J, \mathbb{R})$, $w > 0$ a.e. on J . If the coefficients are not smooth, we introduce the quasi-derivatives [11]

$$y^{[1]} = y', \quad y^{[2]} = y'', \quad \dots, \quad y^{[n-1]} = y^{(n-1)},$$

$$\begin{aligned}
y^{[n]} &= (-1)^n p_n y^{(n)}, \\
y^{[n+1]} &= (-1)^n (p_n y^{(n)})' + p_{n-1} y^{(n-1)}, \\
y^{[n+2]} &= (-1)^n (p_n y^{(n)})'' + (p_{n-1} y^{(n-1)})' + p_{n-2} y^{(n-2)}, \\
&\vdots \\
y^{[2n]} &= (-1)^n (p_n y^{(n)})^{(n)} + (p_{n-1} y^{(n-1)})^{(n-1)} + \cdots + p_0 y,
\end{aligned}$$

and (5.1) is replaced by the equation $y^{[2n]} = \lambda w y$ where $1/p_n, p_{n-1}, \dots, p_1, p_0, w \in L(J, \mathbb{R})$, $p_n > 0$, $w > 0$ a.e. on J [11, p. 3]. In either case, the boundary conditions have the same form. Let $Y = (y^{[0]}, \dots, y^{[2n-1]})^\top$. We now consider the $2n$ -th order boundary value problem defined by (5.1) and the boundary conditions

$$AY(a) + BY(b) = 0, \quad A, B \in M_{2n}(\mathbb{C}). \quad (5.2)$$

For the boundary conditions (5.2), [10, Theorem 2.4] leads to

Proposition 5.1. *Let C_{2n} be the symplectic matrix of order $2n$ defined by*

$$C_{2n} = ((-1)^r \delta_{r, 2n+1-s})_{r,s=1}^{2n}. \quad (5.3)$$

Then problems (5.1)–(5.2) are self-adjoint if and only if

$$\text{rank}(A : B) = 2n \quad \text{and} \quad AC_{2n}A^* = BC_{2n}B^*. \quad (5.4)$$

The method in Section 4 generalizes in a straightforward way. Thus we obtain the following theorem.

Theorem 5.1. *Let A and B be $2n \times 2n$ matrices satisfying*

$$\text{rank}(A : B) = 2n \quad \text{and} \quad AC_{2n}A^* = BC_{2n}B^*.$$

Let Z be the matrix

$$Z = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & I_n & 0 & 0 \\ I_n & -I_n & 0 & 0 \\ 0 & 0 & I_n & I_n \\ 0 & 0 & I_n & -I_n \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & (-1)^{n+1} i C_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & (-1)^{n+1} i C_n \end{pmatrix}.$$

Then there exists a $2n \times 2n$ non singular matrix U and $n \times n$ unitary matrices V_1, U_1^, U_2^* and V_2 , and positive semi-definite diagonal matrices C and S with $C^2 + S^2 = I_n$, such that*

$$(A : B) = U \begin{pmatrix} C & I_n & 0 & S \\ -S & 0 & I_n & C \end{pmatrix} \begin{pmatrix} V_1 & 0 & 0 & 0 \\ 0 & U_1^* & 0 & 0 \\ 0 & 0 & U_2^* & 0 \\ 0 & 0 & 0 & V_2 \end{pmatrix} Z,$$

and the boundary conditions are

1. *separated, if and only if $S = 0$.*
2. *mixed, if and only if $0 < \text{rank}(S) < n$.*
3. *coupled, if and only if $\text{rank}(S) = n$.*

6. Revisiting canonical forms for fourth order differential operators

Hao, Sun and Zettl derived canonical forms for self-adjoint boundary conditions for differential equations of order four [4]. In this section we will show some equivalences between the canonical forms in [4] and the forms presented in Lemma 4.1. The following canonical forms are given [4].

Theorem 6.1 ([4, Theorems 3, 4 and 5]). *Let A and B be 4×4 matrices satisfying*

$$\text{rank}(A : B) = 4 \quad \text{and} \quad AC_4A^* = BC_4B^*.$$

Then, the boundary conditions are

1. *separated, if there exists 4×4 and 8×8 non singular matrices R and R' , respectively, such that*

$$(A : B) = R \begin{pmatrix} r_1 & \overline{a_{21}} & 0 & -1 & 0 & 0 & 0 & 0 \\ a_{21} & r_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_3 & \overline{b_{41}} & 0 & -1 \\ 0 & 0 & 0 & 0 & b_{41} & r_4 & 1 & 0 \end{pmatrix} R',$$

for some $r_1, r_2, r_3, r_4 \in \mathbb{R}$ and $a_{21}, b_{41} \in \mathbb{C}$;

2. *mixed, if there exist 4×4 and 8×8 non singular matrix R and R' , respectively, such that*

$$(A : B) = R \begin{pmatrix} r_1 & \overline{a_{21}} & 0 & -1 & -\overline{a_{31}} & -\overline{za_{31}} & 0 & 0 \\ a_{21} & r_2 & 1 & 0 & -\overline{a_{32}} & -\overline{za_{32}} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & r_3 & \overline{b_{41}} & 0 & -1 \\ za_{31} & za_{32} & 0 & 0 & b_{41} & r_4 & 1 & 0 \end{pmatrix} R',$$

for some $r_1, r_2, r_3, r_4 \in \mathbb{R}$ and $a_{21}, a_{31}, a_{32}, a_{41}, a_{42}, b_{41} \in \mathbb{C}$;

3. *coupled, if there exist 4×4 and 8×8 non singular matrix R and R' , respectively, such that*

$$(A : B) = R \begin{pmatrix} r_1 & \overline{a_{21}} & 0 & -1 & -\overline{a_{31}} & -\overline{a_{41}} & 0 & 0 \\ a_{21} & r_2 & 1 & 0 & -\overline{a_{32}} & -\overline{a_{42}} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & r_3 & \overline{b_{41}} & 0 & -1 \\ a_{41} & a_{42} & 0 & 0 & b_{41} & r_4 & 1 & 0 \end{pmatrix} R',$$

for some $r_1, r_2, r_3, r_4 \in \mathbb{R}$ and $a_{21}, a_{31}, a_{32}, a_{41}, a_{42}, b_{41} \in \mathbb{C}$.

We will consider these forms in the context of Theorem 5.1. In the case of differential equations of order four, Theorem 5.1 becomes

Corollary 6.1. *Let A and B be 4×4 matrices satisfying*

$$\text{rank}(A : B) = 4 \quad \text{and} \quad AC_4A^* = BC_4B^*.$$

Let Z be the matrix

$$Z = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & I_2 & 0 & 0 \\ I_2 & -I_2 & 0 & 0 \\ 0 & 0 & I_2 & I_2 \\ 0 & 0 & I_2 & -I_2 \end{pmatrix} \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & -iC_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & -iC_2 \end{pmatrix}.$$

Then there exists a 4×4 non singular matrix U and 2×2 unitary matrices V_1 , U_1^ , U_2^* and V_2 , and positive semi-definite diagonal matrices C and S with $C^2 + S^2 = I_2$, such that*

$$(A : B) = U \begin{pmatrix} C & I_2 & 0 & S \\ -S & 0 & I_2 & C \end{pmatrix} \begin{pmatrix} V_1 & 0 & 0 & 0 \\ 0 & U_1^* & 0 & 0 \\ 0 & 0 & U_2^* & 0 \\ 0 & 0 & 0 & V_2 \end{pmatrix} Z,$$

and the boundary conditions are

1. *separated, if and only if $S = 0$.*
2. *mixed, if and only if $\text{rank}(S) = 1$.*
3. *coupled, if and only if $\text{rank}(S) = 2$.*

There are 36 canonical forms according to [4, Theorem 2], which yield (by elementary operations) the forms listed in Theorem 6.1. To show that Theorem 6.1 and Corollary 6.1 are equivalent, we need to show that U (and C , S , U_1 , U_2 , V_1 and V_2) and R and R' exist for each canonical form (i.e. for each type of boundary conditions) which gives equality of the forms. The forms given in both the theorem and the corollary ensure that $AC_4A^* = BC_4B^*$. An exhaustive comparison of all 36 forms is too lengthy and cumbersome to pursue here. We will show equivalence for two of the forms, which show clear connections between the two representations of boundary conditions.

First, we consider separated boundary conditions, i.e. $S = 0$:

$$U \begin{pmatrix} V_1 & U_1^* & 0 & 0 \\ 0 & 0 & U_2^* & V_2 \end{pmatrix} Z = R \begin{pmatrix} A_{11} & C_2 & 0 & 0 \\ 0 & 0 & B_{21} & C_2 \end{pmatrix} R'$$

where the left hand side is obtained from the separated boundary conditions form of Corollary 6.1 and the right hand side from the separated boundary conditions form of Theorem 6.1. Thus for example,

$$A_{11} = \begin{pmatrix} r_1 & \overline{a_{21}} \\ a_{21} & r_2 \end{pmatrix}$$

is Hermitian, and similarly for B_{21} . Without loss of generality we may assume $R = I_4$, and we will also assume $R' = I_4$. We obtain

$$U \begin{pmatrix} V_1 & U_1^* & 0 & 0 \\ 0 & 0 & U_2^* & V_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} A_{11} + iI_2 & A_{11} - iI_2 & 0 & 0 \\ 0 & 0 & B_{21} + iI_2 & B_{21} - iI_2 \end{pmatrix}.$$

From

$$\begin{aligned} U \begin{pmatrix} V_1 \\ 0 \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} A_{11} + iI_2 \\ 0 \end{pmatrix}, & U \begin{pmatrix} U_1^* \\ 0 \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} A_{11} - iI_2 \\ 0 \end{pmatrix}, \\ U \begin{pmatrix} 0 \\ U_2^* \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ B_{21} + iI_2 \end{pmatrix}, & U \begin{pmatrix} 0 \\ V_2 \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ B_{21} - iI_2 \end{pmatrix}, \end{aligned}$$

we find

$$U^{-1} = \frac{i}{\sqrt{2}} \begin{pmatrix} U_1^* - V_1 & 0 \\ 0 & V_2 - U_2^* \end{pmatrix}.$$

Thus,

$$\begin{aligned} \frac{i}{2}(U_1^* - V_1)(A_{11} + iI_2) &= V_1, & \frac{i}{2}(U_1^* - V_1)(A_{11} - iI_2) &= U_1^*, \\ \frac{i}{2}(V_2 - U_2^*)(B_{21} + iI_2) &= U_2^*, & \frac{i}{2}(V_2 - U_2^*)(B_{21} - iI_2) &= V_2, \end{aligned}$$

so that the matrices $W_1 = U_1 V_1$ and $W_2 = U_2 V_2$ in (4.10) are unitary, where

$$\begin{aligned} V_1^* U_1^* &= I_2 - 2i(A_{11} + iI_2)^{-1} = (A_{11} - iI_2)(A_{11} + iI_2)^{-1}, \\ U_2 V_2 &= I_2 - 2i(B_{21} + iI_2)^{-1} = (B_{21} - iI_2)(B_{21} + iI_2)^{-1}, \end{aligned}$$

are unitary since A_{11} and B_{21} are Hermitian. Conversely, the matrices

$$\begin{aligned} A_{11} &= -i(V_1^* U_1^* - I_2)^{-1}(V_1^* U_1^* + I_2), \\ B_{21} &= -i(U_2 V_2 - I_2)^{-1}(U_2 V_2 + I_2) \end{aligned}$$

are Hermitian whenever V_1, U_1^*, U_2^* and V_2 are unitary.

Next, we consider coupled boundary conditions, where $\text{rank}(S) = 2$:

$$U \begin{pmatrix} CV_1 & U_1^* & 0 & SV_2 \\ -SV_1 & 0 & U_2^* & CV_2 \end{pmatrix} Z = R \begin{pmatrix} A_{11} & C_2 & -A_{21}^* & 0 \\ A_{21} & 0 & B_{21} & C_2 \end{pmatrix} R'$$

where the left hand side is obtained from the coupled boundary conditions form of Corollary 6.1 and the right hand side from the coupled boundary conditions form of Theorem 6.1. Hence A_{11} and B_{21} are Hermitian and A_{21} is non singular. Without loss of generality we may assume $R = I_4$. We will also assume $R' = I_4$. Thus

$$U \begin{pmatrix} CV_1 & U_1^* & 0 & SV_2 \\ -SV_1 & 0 & U_2^* & CV_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} A_{11} + iI_2 & A_{11} - iI_2 & -A_{21}^* & -A_{21}^* \\ A_{21} & A_{21} & B_{21} + iI_2 & B_{21} - iI_2 \end{pmatrix}.$$

Since U is an arbitrary invertible matrix, we will begin with the equivalent form

$$\begin{aligned} U & \begin{pmatrix} V_2^* S^{-1} C V_1 & V_2^* S^{-1} U_1^* & 0 & I_2 \\ -I_2 & 0 & V_1^* S^{-1} U_2^* & V_1^* S^{-1} C V_2 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} A_{11} + iI_2 & A_{11} - iI_2 & -A_{21}^* & -A_{21}^* \\ A_{21} & A_{21} & B_{21} + iI_2 & B_{21} - iI_2 \end{pmatrix}. \end{aligned}$$

Let U' be invertible such that

$$U = \begin{pmatrix} I_2 & A_{21}^* (B_{21} + iI_2)^{-1} \\ -A_{21} (A_{11} - iI_2)^{-1} & I_2 \end{pmatrix}^{-1} U',$$

which is sensible since the matrix

$$\begin{aligned} & \begin{pmatrix} I_2 & A_{21}^* (B_{21} + iI_2)^{-1} \\ -A_{21} (A_{11} - iI_2)^{-1} & I_2 \end{pmatrix} \\ &= \left(\begin{pmatrix} A_{11} & 0 \\ 0 & B_{21} \end{pmatrix} + \begin{pmatrix} -iI_2 & A_{21}^* \\ -A_{21} & iI_2 \end{pmatrix} \right) \begin{pmatrix} (A_{11} - iI_2)^{-1} & 0 \\ 0 & (B_{21} + iI_2)^{-1} \end{pmatrix} \end{aligned}$$

is invertible since

$$\begin{pmatrix} -iI_2 & A_{21}^* \\ -A_{21} & iI_2 \end{pmatrix}$$

is skew-Hermitian and invertible (here we take the block determinant [14, Theorem 3] which yields a positive definite matrix $I_2 + A_{21}^* A_{21}$). Furthermore, we set

$$U' = -\sqrt{2}i \begin{pmatrix} (B_{21} + iI_2)(A_{21}^*)^{-1} & 0 \\ 0 & -(A_{11} - iI_2)A_{21}^{-1} \end{pmatrix}^{-1} U''.$$

It follows that,

$$U'' \begin{pmatrix} V_2^* S^{-1} C V_1 & V_2^* S^{-1} U_1^* & 0 & I_2 \\ -I_2 & 0 & V_1^* S^{-1} U_2^* & V_1^* S^{-1} C V_2 \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} & 0 & I_2 \\ -I_2 & 0 & K_{23} & K_{24} \end{pmatrix}$$

where

$$\begin{aligned} K_{11} &= \frac{i}{2} [(B_{21} + iI_2)(A_{21}^*)^{-1}(A_{11} + iI_2) + A_{21}], \\ K_{12} &= \frac{i}{2} [(B_{21} + iI_2)(A_{21}^*)^{-1}(A_{11} - iI_2) + A_{21}], \\ K_{23} &= -\frac{i}{2} [(A_{11} - iI_2)A_{21}^{-1}(B_{21} + iI_2) + A_{21}^*], \\ K_{24} &= -\frac{i}{2} [(A_{11} - iI_2)A_{21}^{-1}(B_{21} - iI_2) + A_{21}^*]. \end{aligned}$$

We must have $U'' = I_4$, and since $K_{24} = K_{11}^*$ we can directly find V_1 , V_2 and S and C using the singular value decomposition of K_{11} and the fact that $S^2 + C^2 = I_2$ where S and C are positive semi-definite diagonal matrices. It remains to show that unitary U_1 and U_2 exist and satisfy the equations, i.e. if and only if

$$K_{12}K_{12}^* - K_{11}K_{24} = I_2, \quad K_{23}K_{23}^* - K_{24}K_{11} = I_2.$$

Straight forward calculation establishes that these equalities hold. We note that $\text{rank}(S) = 2$ since $K_{12}K_{12}^* = I_2 + K_{11}K_{11}^*$ is positive definite and hence $S \leq I_2$.

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