The Distinguished Guests of Giants

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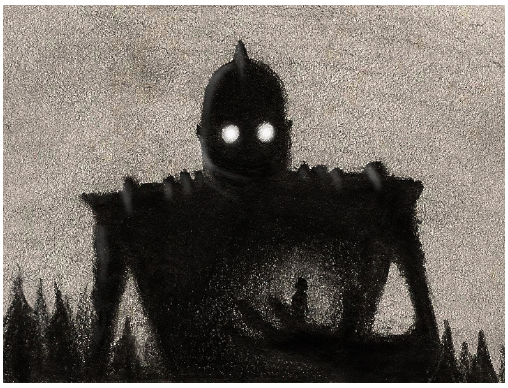

A Dissertation submitted to the Faculty of Science,
University of the Witwatersrand,
in fulfillment of the requirements for
the degree of Doctor of Philosophy.

Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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June 3, 2016

Abstract

The convenient pictorial descriptions of the half-BPS and near-BPS sectors of the AdS/CFT equivalent theories of $\mathcal{N} = 4$, $D = 4$ super Yang-Mills and $D = 10$ Type IIB superstring theory on $AdS_5 \times S^5$ are exploited in this thesis by using Schur polynomials labelled by Young diagrams as a basis for the gauge invariant operators in the field theory.

We use a "Fourier transform" on these operators to construct asymptotic eigenstates of the dilatation operator, the spectrum of which agrees precisely with the first two leading order terms in the small-coupling expansion of the exact result determined by symmetry. Motivated by the geometric description of the systems of open strings with magnon excitations to which the operators are dual, we propose a simple and minimal all-loop expression that interpolates between anomalous dimensions computed in the gauge theory and energies computed in the string theory. The connection to the string theory result provides the insight necessary to understand the interpretation of our Gauss graphs in the magnon language. Symmetry determines the two-body scattering matrix for the magnons up to a phase, and it is demonstrated that integrability is spoiled by the boundary conditions on the open strings.

The Schur polynomial construction is then applied to the study of closed strings on a class of half-BPS excitations of the $AdS_5 \times S^5$ background. The string theory predictions for the magnon energies are again reproduced by calculating the anomalous dimensions of particular linear combinations of our operators. Group theoretic quantities which can be read off the Young diagram labels provide the correct modification of terms in the dilatation action to account for the energies of magnons at different radii on the LLM plane. The representation theory implies a natural splitting of the full symmetry group - the distinction between what is the background and what is the excitation is accomplished in the choice of the subgroup and representations used to construct the operator.

Connecting the descriptions utilised in obtaining these results is expected to allow the construction of operators dual to general open string configurations on the class of backgrounds considered.

Acknowledgements

An insurmountable debt of gratitude to my supervisor, Prof. Robert de Mello Koch - a distinguished physicist and a committed Professor - for your continued guidance, patience, encouragement and inspiration.

To my significant other, the magical Ms. Senatore; the most valued and welcome guest in my universe, who illuminates elements more beautiful than even math could describe - I love you.

To the giants whose shoulders I truly and literally stood on, my parents Marlene & Ken - the life you built has afforded more opportunity than one could wish for, and I am grateful beyond expression.

To my sister Adele; sibling rivalry is a powerful motivator - thank you for giving me so much to aspire to.

Contents

Chapter 1

Introduction

In nature (or, perhaps, just our study of it), the existence of symmetry plays an undeniably vital role in understanding the processes by which phenomena occur. In addition to the proven predictive power of techniques involving its analysis, the notion of every system in existence being adequately described as a group of fundamental objects which are related to each other by some fundamental transformation is conceptually very attractive, and permits infinite adaptation to any set of constituents and transformations we may wish to consider. In a theory where the essence of the involved processes is captured by specifying a symmetry group, the exact nature of the constituents as physical objects is all but irrelevant - the physics is determined entirely by the dynamics encoded in the symmetry transformations. The language of representation theory allows this abstracted notion to be formulated concretely as an algebraic description, where the constituents can be assigned physical meaning under a particular representation. In the pursuit of a grand unified theory, which should allow predictions for the behaviour of any system regardless of its composition, scale, or the details of how it interacts, this language may prove an indispensable tool. If there is a universal symmetry group, every conceivable object or system of objects at any scale or combination of scales should be described as a particular representation thereof.

Of course, the standard model is currently the most recognized and tested example of these principles at work. The fundamental particles of which it is supposed that atoms consist are represented in a multiplet transforming under a fundamental gauge symmetry group. We have given names to the elements of these multiplets, and associated them with particular sub-atomic particles interacting with each other by means of the forces to which the symmetry groups correspond. The near-perfect correspondence with experimental observation has by now been quite rigorously verified. The development of string theory, hampered by the troublesome tachyon of the bosonic realization, underwent a revolution thanks to the advent of the aptly named supersymmetry; it is now widely studied as the most likely candidate for the next vital unification. As one plunges into the rabbit hole that is high energy physics, the degree of symmetry necessary for consistency of the theories developed at the smallest scales seems to increase at a rate which is difficult to quantify (how much more symmetric is a supersymmetric theory?).

This dissertation represents further development of a theory where symmetry groups have been utilised in a very concrete and noteworthy manner for the purposes of verifying the AdS/CFT correspondence [1]. There is significant motivation to infer that the gauge invariant operators on the field theoretic side of the correspondence which are dual to known objects in the string theory are constructed in terms of traces of the constituent fields. In the current construction, a basis obtained by assembling linear combinations of all possible multi-trace structures is a vital component, as it overcomes many limitations of formulating dual descriptions in terms of single trace operators. These combinations are conveniently generated by the use of the symmetric group; representation theory provides a mathematical and visual description of the organisation of the fields in these operators.

The field theory in question for the current study is $\mathcal{N} = 4$ super Yang-Mills theory, a 4-dimensional supersymmetric conformal field theory (CFT) with $U(N)$ gauge group. One of the most striking features of the commonly studied realisation of the AdS/CFT duality between this theory and Type IIB string theory is that the field theory does not include gravity; understanding how the gravitational description of the dual theory is reconstructed in the CFT is an open problem, which the theory of this dissertation ultimately aims to help resolve. In our construction, the proposed dual operators to string theory configurations are (restricted) Schur polynomials, assembled from complex combinations of the 6 scalar Higgs fields. These fields transform under the adjoint of $U(N)$.

There exists a fascinating relationship between the unitary and symmetric groups, known as Schur-Weyl duality, which has proven invaluable in the formulation and understanding of the interplay between the transformations of the matrix-valued CFT fields and the symmetric group transformations which permute the indices of the fields in the operator to produce different trace structures (see e.g. [2]). The Young diagrams, which specify a representation of both groups, consistently provide an excellent visual description of the underlying physics, and further can be directly compared to visual descriptions developed in the dual string theory [3, 4, 5]. The Schur polynomial operators we construct are labelled by Young diagrams, and the representation theory encoded therein is found to be a natural language in which to phrase the discussion and realization of the duality.

The primary task undertaken in the early stages of this study was to consider the various possible extensions of the results of [2]. In this article, the energy spectrum of arbitrary systems of giants with fundamental strings attached in the $AdS_5 \times S^5$ geometry was calculated in the dual field theory using the AdS/CFT correspondence. The Schur polynomial dual to a ground state of a giant graviton system is constructed from a single species of field, conventionally Z, and is exactly half-BPS. The excitations are represented by the insertion of impurities, each being a single complex field of the SYM theory of another species, into the operator dual to the ground state. This represents a "near-BPS" configuration; the expectation that results obtained in this regime can be extrapolated to strong coupling without modification still holds at large N. A novel application of Schur-Weyl duality allowed the calculation and diagonalization of the dilatation action on these operators for any system that can be specified by the representation labels, generalizing the earlier results [6, 7]. The results of the diagonalization suggest a natural, physical structure to the linear combinations of operators forming energy eigenstates, relating to the restriction imposed by Gauss' law on the string configurations. Operators which directly diagonalize the dilatation operator were constructed by the use of a double coset ansatz in [8], which applies this restriction using group theory.

When this physical structure was first discovered in [2], it was noted that the equations resulting from the diagonalization of the dilatation action on the operators written in the $S_n \times S_m$ basis used therein could be put into correspondence with simple diagrams representing the physical configurations consistent with the Gauss law. A point is drawn for each giant graviton in the system; joining these points using as many oriented connections as there are bit-strings in the system, in as many ways as possible under the constraint that each giant must have an equal number of incoming and outgoing lines, the full set of diagonalized equations can be read off the diagrams. These are referred to as *Gauss graphs*. The intuition for the group theoretic diagonalization being obtained in terms of a double coset construction is apparent after drawing these diagrams with the lines cut in half and each half numbered; viewing the system this way, it is clear that the diagrams are in correspondence with the elements of a particular double coset of the S_m group, up to conjugacy.

This is a remarkable result - the linear combination of restricted Schur polynomials, labelled by representations, which diagonalizes D can be obtained by specifying a permutation. The existence of a duality between sums of operators labelled by representations and those specified by any permutation in a particular conjugacy class, and the inverse relation between sums of permutations and a definite representation (as in the definition of a Schur polynomial) is a recurring and essential element of the current theory. The relation is between representations and conjugacy classes because permutations in the same conjugacy class produce the same trace structure and thus label the same operator. In this dissertation, it will be demonstrated how this relationship can be gainfully applied to the study of more general systems of excited giant gravitons.

Some logical extensions of the results of [2] have already been obtained. A sector of the theory that was not initially fully considered arises when all 3 bosonic scalar fields of the SYM are included in the model. Calculations have been performed on systems of this type [9], where the giants are considered to consist of Z fields, and both X and Y impurities are inserted as string bits corresponding to excitations. In these calculations, terms corresponding to the interaction between the X and Y impurity fields were neglected by a simple argument in terms of the number of these fields present in relation to the number of Z fields. An article [10] was published in which the exact contribution of these terms is computed in the framework of the double coset construction, the conclusion being that integrability is indeed preserved when including these terms in calculation. The additional inclusion of the 2 fermionic fields of the gauge theory is more complicated due to the different statistics they obey, but the existence of integrability and the relevance of the double coset were shown to persist for this sector in [11].

The problem of computing higher loop corrections to the Dilatation operator in the $SU(2)$ sector was tackled in [12], by appealing to the symmetry algebra of the theory. The generators $\vec{J}$ of the $SU(2)$ subgroup of the $SU(4)$ $\mathcal{R}$-symmetry group of the theory are known to be uncorrected, since their eigenvalues are fixed by the $su(2)$ algebra. Since the (corrected) dilatation operator commutes with these generators, higher loop corrections can be studied by considering the recursion relations that are the result of obtaining explicit expressions for the commutator. The result of this article was a proof that the piece of the dilatation operator that acts on the impurity fields is given by the one loop expression at any loop order.

Having calculated some of the subleading and higher loop corrections, the sector of the theory where single-species bit strings are attached to the giants has now been quite thoroughly investigated. The extension which we seek to understand in this dissertation is the generalization of this theory to construct and obtain the anomalous dimensions of operators dual to systems where the excitations are extended strings, corresponding to the insertion of single trace operators built from multiple field species. The case where the strings consist of a single species of field was considered first, during which an intuitive construction for operators of this type was achieved by a straightforward modification of the labelling of the states. In doing this, the physical configuration to which the operator corresponds actually becomes clearer than in the case of indistinguishable strings, due to the fact that a symmetry of the theory (that of swapping the strings with each other) is no longer necessary. The key observation is that it is possible to identify each of the states appearing on either side of the trace, which are labelled by the $S_{n+m} \to S_n \times (S_1)^m$ multiplicity indices of the restricted Schur polynomial, with one of the endpoints of the strings.

The actions of the $SU(2)$ generators $\vec{J}$ in the description where the excitations are each a string of Y fields, tied together by a permutation into a single trace structure (this represents a particular restriction of the operators studied in [13, 14, 15]), have been calculated, and a few interesting properties of their action were deduced. Thought was then given to the organisation of the results into physically meaningful pieces. It should be noted that describing the system in this way corresponds also to a restriction of the operators with bit-strings excitations, since the restricted Schurs sum over all possible trace structures - a class of the symmetric group elements summed in the bit string operators will include a factor that connects all the Y fields. An attempt to determine a transformation between the operators with single field impurities and those with single-trace impurities was undertaken. In doing so, it became apparent that some additional organisation of the fields composing the string word may be necessary to realise the expected emergent string field theory.

In calculating the action of the dilatation operator using the recent methods on systems of this type, it was found that we could identify terms in the solution as relating either to the kinetics of the system, or to the splitting of strings. Estimations of the typical size of these terms indicated that the string splitting terms are subleading, and can hence be neglected in final calculations. Collecting the leading (kinetic) terms, two types of string-brane interactions were evident: The first are the kissing interactions, where a string comes into contact with a brane, but does not exchange momentum with it. The second are the hopping interactions, where it appears that a unit of angular momentum is transferred from the brane to the string. These are exactly the interactions that arose under the dilatation action when considering more general, multi-species string words under the earlier formalism.

A natural conjecture for the transformation relating the bit-string and single-trace sectors and the form of the operators in terms of which it should be phrased was conceived - the expectation being that the additional organisation required to relate operators with single-trace and single-field insertions can be realised by modifying the Young diagram labels of the restricted Schur operators with trace insertions such that the fields associated to the trace are assigned boxes as well. It was believed that the extra boxes should take the form of "fluff", appearing as additional rows being generically $O(\sqrt{N})$ in length. The joined-string operators should then be expressed as a linear combination of bit-string operators, in which the fluff representation is modified in each term such that the resulting sum provides the correct trace structure to describe the particular string configuration. However, it was not at all obvious how this could be conveniently included into the current description, and it is even more non-trivial to imagine how the additional organisation could emerge when restricting to the sector where bit-strings are joined. Methods by which this description can be realised, and utilised in conjunction with earlier developments in the study of the symmetry algebra and dual string configuration to calculate anomalous dimensions of even more general systems than were initially considered, are presented in this dissertation.

The gauge theory result of [16] for the exact energies of the individual impurities on a spin chain corresponding to a single trace operator dual to a closed string, obtained using only a cunning modification of the symmetry algebra under which the operator constituents transform, provides an exceptionally useful intuition for the conceptual logic required to build energy eigenstates for any system involving $SU(2|3)$ excitations of string-dual operators. The key observation is that each impurity transforms under its own centrally extended representation of a reduced symmetry group, which arises after one makes a concrete distinction between background and excitation transformations. The powerful symmetry arguments allow also the construction of the exact two-particle scattering matrix which relates states having asymptotic excitations with permuted momenta, up to an overall phase. Integrability of the system is proven by noting that this S-matrix satisfies the Yang-Baxter equation. An application of Bethe ansatz techniques then provides a means by which to construct exact eigenstates.

The appropriate limit of this result which the AdS/CFT correspondence conjectures should be available in the string theory has been reproduced in [4]; the solution having the desired properties for identification with the "magnons" of the spin chain description corresponds to a string in $AdS_5 \times S^5$ with ends on antipodal points of one of the 3-spheres of the geometry, which possesses infinite charge under one of the symmetry generators and executes motion in the angular coordinate shifted by this generator. In the coordinates of [3], which are specifically designed to describe half- and near-BPS excitations of the ground state supergravity solution, the equations of motion gain a particularly simple and elegant geometric description. The magnons are represented on the LLM plane as a line joining two points on the boundary of a droplet; the central charges of the representation under which they transform are encoded in the components of the vector specifying this line. The semi-classical S-matrix as determined in the string theory provides a visualization of the scattering process in terms of consistent transformations of these diagrams; the interactions were studied also for open strings on maximal giants in [17], where the existence of "boundary magnons" which connect the boundary to the origin of the plane are included. The super-Poincare algebra of this $2+1$-dimensional model permits an exact identification with the field transformation algebra of the gauge theory.

Using the Schur polynomial construction, operators dual to a generalization of the systems studied in [16] which include open strings can be constructed, and the anomalous dimensions thereof compared to the weak-coupling limit of the exact result. The strings are modeled as a word containing two matrix types, Y and Z, in the same way as in the early papers on this particular application of gauge-gravity duality. The approach which has been most successful in describing the string interactions in this case entails associating the string word to a Cuntz oscillator chain, where the Y fields set up a lattice that is populated with Z's. The bulk Hamiltonian for the string system in terms of these oscillators is known, and the boundary interactions occurring under the action of the dilatation operator have also been studied. The state-operator correspondence can be used to write the results obtained in this way in terms of normalized Schur polynomial operators if desired. It is important to note that, since the giant is composed of Z fields, the endpoints of the string word must be occupied by a Y-field - were a Z to appear here, it would be better described as being part of the giant itself. A modification of this interpretation is also possible in the displaced corners limit, where Zs at the endpoints are thought of as still being part of the string lattice, and the process involved in the hopping interaction need not be viewed directly as a momentum exchange.

The first of two distinct results presented in this dissertation, which introduces this new interpretation, provides the understanding necessary to diagonalize the dilatation operator directly in terms of the string word description; that is, in a basis where representation and trace structures appear simultaneously. This is achieved by "Fourier-transforming" the restricted Schurs with single trace impurities. The operators thus constructed correspond to systems of open strings connecting giant gravitons in the $AdS_5 \times S^5$ geometry, and the agreement with the exact energies and predictions from the dual string theory is shown to be correct. The scattering matrix for magnons on general open strings is derived by the same procedure as in [16] using a suitable parameterization for the coefficients specifying the magnon representations. In addition, the connection to the string theoretic result presents a natural advancement and refinement of the Gauss graphs obtained previously.

The second result generalizes the construction for closed strings (without giants) to produce the field theory dual operators to closed strings on a more general class of half-BPS backgrounds with rotational symmetry, corresponding to LLM boundary conditions specified by a series of thick, well-separated concentric annuli. It is in performing this computation, for which the arguments are developed entirely in the representation basis, that a deeper understanding of the group theoretic structure was developed, concretely illustrating the means by which "fluff" which organises the fields of the excitation appears under the theory. The existence of identities relating trace operators specified by a permutation, and the Schur polynomials which are labelled by representations, plays a central role.

This dissertation is organised as follows: In Chapter 2 the background necessary to understand the concepts and constructions of the body is thoroughly reviewed. The following two Chapters form the body, and present the details of the computations and conceptualizations which provide new results. The findings of these chapters represent a natural extension of the article [2], upon which the author's MSc dissertation was based. General conclusions which can be understood by analysis of the two novel results are presented in Chapter 5. The Appendices collect some more elementary background material, as well as some calculations supporting the background section.

The first primary novel result of this dissertation, being the construction of asymptotic eigenstates of the dilatation operator in the Schur polynomial basis which are dual to systems of infinite open strings in $AdS_5 \times S^5$ with momentum-carrying magnon excitations, appears in Section 3.4. The general two-body scattering matrix for the magnons of these systems, which theoretically allows a completion to exact eigenstates, has been computed using symmetry; the result is quoted in Section 3.7. Integrability is not a feature of these systems, as demonstrated in Section 3.E. A proposal for operators in our basis which are dual to closed strings in more general LLM geometries is stated in Section 4.4, and subsequently confirmed in Section 4.5. An identity which plays an important role in establishing the validity of the proposed operators is proved in Section 4.B. These results have been submitted to the pre-print archive [18, 19], and are currently under consideration for publication in JHEP.

Chapter 2

Background

2.1 Supergravity Basics

The concept of gauge symmetry; that is, symmetry for which the transformation parameter depends on the spacetime point at which the transformation is applied, is well-known to be responsible for numerous crucial developments in the field of theoretical physics. Around the same time that the first gauge theories were formulated, various groups began studying a new type of global symmetry, the transformations of which allowed particles of different spin to be unified in representations of the underlying algebra. This was dubbed supersymmetry, and the remarkable progress that was made in the study of gauge theory, culminating in the standard model, inspired the pursuit of a gauged (local) form of the transformations. This theory would necessarily contain a gauge field for spacetime translations; this implies a gauged form of general relativity, where a fermionic "superpartner" of the graviton mediates supergravity interactions, in the same way that the graviton mediates non-supersymmetric gravitation. This section provides a brief overview of the simplest case of a global supersymmetric theory, and the effects of promoting to a local supersymmetry in this regime.

2.1.1 $\mathcal{N} = 1$, $D = 4$ Global Supersymmetry

The inclusion of global supersymmetry (SUSY) in a 4 dimensional quantum field theory requires the extension of the allowed symmetries from the group of Poincare and internal symmetry transformations to include additional, spinor-valued (See Appendix B for an introduction to spinor theory) supercharges, usually denoted Q^i_α. $\alpha = 1..2^{D/2}$ is a spacetime spinor index, while $i = 1..\mathcal{N}$ labels distinct supercharges that can exist within the theory. The study of theories possessing $\mathcal{N} > 1$ supercharges is referred to as extended supersymmetry; for the purposes of this introduction we will not be concerned with the ramifications of such a construction.

Including the single supercharge Q_α (a four component Majorana spinor) as an additional generator of symmetries of the theory, one finds that the charges associated with the Poincare and supersymmetry transformations are combined to form a new defining algebra, referred to as the superalgebra. The structure relations contain commutators and anticommutators of bosonic (B) and fermionic (F) charges which take the forms $[B, B] = B$, $[B, F] = F$ and $\{F, F\} = B$, where the purely bosonic commutators are simply the unmodified Lie algebra of the Poincare group. The new structure relations involving the fermionic supercharges are:

$$\{Q_\alpha, \bar{Q}^\beta\} = -\frac{1}{2}(\gamma_\mu)_\alpha^\beta P^\mu$$

$$[M_{\mu\nu}, Q_\alpha] = -\frac{1}{2}(\gamma_{\mu\nu})_\alpha^\beta Q_\beta$$

$$[P_\mu, Q_\alpha] = 0$$

These relations provide the most general superalgebra possible that respect the constraints implied by the Haag-Lopuszanski-Sohnius theorem[20], which states limitations on the charges and algebras that may appear in an interacting relativistic quantum field theory with supersymmetry in 4 dimensions.

In the SUSY formalism, bosons and fermions are united as basis states of a particle representation of the superalgebra - in the same way that the various particles of the standard model exist within multiplets of their respective symmetry groups, both bosons and fermions appear together in supermultiplets of the superalgebra. Consider a particle state having a definite momentum $\vec{p}$ (and hence a definite energy); under SUSY the state can be specified by its momentum and its helicity (which specifies also its nature as a boson or fermion) as $|\vec{p}, B\rangle$ or $|\vec{p}, F\rangle$. The spinor supercharge Q_α has spin 1/2, and its action on particle states is thus to transform between bosons and fermions. The commutation of the supercharge with the momentum operator indicates that the momentum and energy of the resulting state does not change. This makes manifest the realisation of boson-fermion symmetry under the application of supersymmetry.

As always, different representations of the algebra lead to different particle multiplets for the corresponding supersymmetric field theory. In all theories the requirement of supersymmetry is imposed in the Lagrangian formalism, as for any other symmetry, by requiring invariance of the action under SUSY transformations. We will not be concerned with the study of any particular multiplet, and this subsection is concluded with a particular result that follows directly from the definition of the algebra. We will first need the expression for the infinitesimal variation of a generic field under a supersymmetry transformation; the parameter for the transformation is a constant, anticommuting Majorana spinor, commonly denoted ϵ_α. For a generic field $\phi(x)$:

$$\delta(\epsilon)\phi(x) = -i[\bar{\epsilon}^\alpha Q_\alpha, \phi(x)]_{\text{qu}}.$$

There is an interesting structure to the action of the commutator of two successive SUSY variations on fields of supersymmetric theories, which generalizes naturally from this simple, globally supersymmetric algebra:

$$\begin{aligned}
[\delta_1, \delta_2]\phi(x) &= [\bar{\epsilon}_1 Q, [\bar{Q}\epsilon_2, \phi(x)]] - (\epsilon_1 \leftrightarrow \epsilon_2) \\
&= \bar{\epsilon}_1^\alpha [\{Q_\alpha, \bar{Q}^\beta\}, \phi(x)]\epsilon_{2\beta} \\
&= -\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu\phi(x).
\end{aligned} \tag{2.1}$$

It is clear from this result that the commutator of two supersymmetry variations produces an infinitesimal spacetime translation, with parameter $-\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2$.

2.1.2 $\mathcal{N} = 1,\ D = 4$ Supergravity

As mentioned, a theory of supergravity is obtained when one promotes the global supersymmetry presented in the previous section to a symmetry which holds only locally in the associated field theory. The action of the theory must therefore be invariant under supersymmetry transformations for which the spinor parameters $\epsilon(x)$ are arbitrary functions of the spacetime coordinates. This is the origin of the appearance of a description of gravity in any gauged supersymmetric theory, as is now explicitly demonstrated.

The gauge multiplet of any supergravity theory must contain the frame field $e^a_\mu(x)$, which describes the graviton, plus $\mathcal{N}$ vector-spinor fields $\psi^i_\mu(x)$, the quanta of which are gravitinos, the superpartner of the graviton. In the present case of $\mathcal{N} = 1, D = 4$ SUGRA, we will need the transformation rules for the two fields comprising the gauge multiplet:

$$\delta(\epsilon)e^a_\mu(x) = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu$$

$$\delta(\epsilon)\psi_\mu = D_\mu\epsilon(x) = \partial_\mu\epsilon + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\epsilon.$$

These rules follow by recalling that the gravitino is the gauge field of local supersymmetry, while the frame field variation is the simplest possible form consistent with the tensor structure and the requirement that $B \to F$ under a SUSY transformation.

We can now compute the commutator action of two supersymmetry variations on the fields involved; the frame field computation is simple and provides an illustration of the generalization of the relation (2.1):

$$\begin{aligned}
[\delta_1, \delta_2] e_\mu^a &= \frac{1}{2}\bar{\epsilon}_2 \gamma^a D_\mu \epsilon_1 - (\epsilon_1 \leftrightarrow \epsilon_2) \\
&= \frac{1}{2}(\bar{\epsilon}_2 \gamma^a D_\mu \epsilon_1 + D_\mu \bar{\epsilon}_2 \gamma^a \epsilon_1 \\
&= D_\mu \xi^a \\
\xi^a &= \frac{1}{2}\bar{\epsilon}_2 \gamma^a \epsilon_1 = -\frac{1}{2}\bar{\epsilon}_1 \gamma^a \epsilon_2.
\end{aligned} \tag{2.2}$$

Identities for spinor bilinears have been used to obtain this result. In order to locate the commonality in the structure of this expression and that of the previous subsection, we should now compare this to the expression for a general coordinate transformation of the frame field:

$$\delta(\xi) e_\mu^a = \xi^\rho \partial_\rho e_\mu^a + \partial \xi^\rho e_\rho^a$$

A direct comparison to (2.2) is possible after "covariantising" the derivatives in this expression by adding and subtracting the connection terms[21]:

$$\delta(\xi) e_\mu^a = \Delta_\mu \xi^\rho e_\rho^a - \xi^\rho \omega_\rho^a{}_b e_\mu^b + \xi^\rho T_{\rho\mu}^a$$

The first term matches the supergravity result after using the frame field to convert to frame indices, $e_\rho^a \Delta_\mu \xi^\rho = D_\mu \xi^a$. We see that the commutator of two infinitesimal supersymmetry transformations yields a space-time dependent vector field $(\bar{\epsilon}_2 \gamma^a \epsilon_1)(x)$, which is an element of the infinitesimal version of the group of local diffeomorphisms on space-time. A theory of supergravity will hence necessarily be diffeomorphism invariant; we are thus required to treat the space-time metric as a dynamical object - the appearance of gravity is a result of the spacetime dependence of ϵ. The second term is easily identified as a local Lorentz transformation with field-dependent parameter $\hat{\lambda}_{ab} = \xi^\rho \omega_{\rho ab}$. The explicit form of the torsion tensor can be used to rewrite the final term as $\frac{1}{2}(\xi^\rho \bar{\psi}_\rho)\gamma^a \psi_\mu$, which is a local supersymmetry transformation with parameter $\hat{\epsilon} = \xi^\rho \psi_\rho$. Thus, we arrive at the result

$$[\delta_1, \delta_2] e_\mu^a = (\delta(\xi) - \delta(\hat{\lambda}) - \delta(\hat{\epsilon})) e_\mu^a. \tag{2.3}$$

The commutator of two supersymmetries produces a sum of the gauge symmetries of the theory; this is a general feature of the commutator variation of any field in any supergravity theory, *so long as that field satisfies its equations of motion.* This caveat is particularly important.

One can illustrate the principle of an "on-shell" multiplet by performing the computation of the commutator action of two supersymmetries on the gravitino. The result includes the analogous terms as for the frame field, plus a number of additional terms which do not correspond to any of the gauge symmetry transformations:

$$[\delta_1, \delta_2]\psi_\mu = (\delta(\xi) - \delta(\hat{\lambda}) - \delta(\hat{\epsilon}))\psi_\mu + \cdots .$$

The additional terms take a form such that each involves a factor equal to a quantity R_μ; the field equation for ψ_μ obtained by requiring invariance of the action is $R_\mu = 0$, so that these terms vanish if ψ_μ satisfies its equation of motion. Closure of the algebra obviously requires the vanishing of these terms, and we say that the algebra of $\mathcal{N} = 1$, $D = 4$ supergravity closes only on-shell.

2.2 Type IIB Supergravity

This section presents a review and relevant details of the article [22], in which the field equations of Type IIB supergravity were first derived. This serves as an example of the amazing power of symmetry in physics, and captures some important features of the theory that will be useful when comparing our results to those obtained on the gravity side of the AdS/CFT correspondence. Of importance to the primary results of this dissertation are the expressions for the commutator parameter of the $NS - B_{\mu\nu}$ gauge transformation, which is derived in detail, and the expression for the supersymmetry variation of the gravitino.

2.2.1 Definition

$D = 11$ supergravity produces, by dimensional reduction, a ten dimensional theory with two supersymmetries, each of which transforms as a Weyl spinor having opposite handedness. The study of superstring theories predicts an additional supergravity theory of the same dimensionality having also 2 supersymmetries, in this case transforming with the same handedness. This theory should be regarded as a classical approximation to Type IIB superstring theory, since a one-loop analysis of the supergravity theory produces non-renormalizable quadratic divergences in its physical S-matrix elements, in contrast to the string theory which is finite in this regime.

The spectrum of fields of the theory can be read from the superfield expression. The charges of the components of the superfield are measured by an operator $U = 2 - \frac{1}{2}\theta\frac{\partial}{\partial\theta}$; this tells us how each of the component fields transforms under a $U(1)$ rotation of the fermionic superspace coordinates - the derivative appearing counts the number of superspace fermionic coordinates which multiply the component in the superfield. We are considering transformations of the form $\theta \to \theta e^{i\phi}$; components of the superfield will transform as $\Phi_i \to e^{iq\phi}\Phi_i$ under this action. The charge is determined by the value of q, which is the power of θ which multiplies Φ_i in the superfield expansion.

The theory contains a single complex scalar field A having $U = 2$; this can be equivalently described as a pair of real scalar fields ϕ and χ transforming as a doublet representation of $SU(1,1)$. A complex antisymmetric rank-two tensor $A_{\mu\nu}$ with $U = 1$ (or two real rank-2 tensors $B^{(a)}_{\mu\nu}$, again transforming as a doublet of the global symmetry group) and a real rank-4 antisymmetric tensor $A_{\mu\nu\rho\lambda}$ with $U = 0$ exist as the gauge fields of the string and $D3$ brane respectively. The $10D$ metric $g_{\mu\nu}$ of the spacetime on which the theory is defined is the last bosonic field included in the theory - as is common in describing these theories, the metric is not used directly; a real-valued "zehnbein" e^{τ}_{μ} with $U = 0$, which provides the transformation from the curved $g_{\mu\nu}$ to flat Minkowski space and simplifies the form of the relativistic action by removing the square root, is instead introduced as the dynamical field of the theory. It is related to the metric by $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, where η_{ab} is the flat metric - written in the basis of zehnbeins, the metric is locally flat.

The theory additionally possesses two fermionic fields, a complex Weyl spinor λ with $U = \frac{3}{2}$ and a complex Weyl gravitino ψ_μ with $U = \frac{1}{2}$.

Initial attempts at directly constructing this supergravity theory in the Lagrangian formalism failed due to the requirement of self-duality of the five-form field strength associated to the $D3$-brane gauge symmetry; it was shown that it is not possible to obtain this constraint in the context of a manifestly Lorentz-covariant action[23]. A technique by which the complete set of field equations could be derived, which does not rely on the introduction of a Lagrangian for the theory, was presented in [22]. The method involves only the analysis of the supersymmetry transformations of the fields; from the requirement that the field equations must transform into one another under SUSY variations, the complete set of covariant field equations for the theory were obtained. The method uses the fact that there are fields for which the commutator of two supersymmetry variations does not close the algebra to determine the equations of motion.

2.2.2 Global $SU(1,1)$ and Super-coordinate $U(1)$ Rotation Field Variations

Central to the ability of the author of the aforementioned article to obtain the set of symmetry transformations of the fields was the existence of a global $SU(1,1)$ invariance of the chiral supergravity theory considered. It was known that in lower dimensional theories ($D < 10$), the scalar fields could be associated with a coset group, taken as the quotient of the theories' non-compact global symmetry group with their maximal compact subgroup - the denominator group was additionally found to be the same group occurring as a linear symmetry of the free-theory spectrum. Owing to the inclusion of a time direction in the global symmetry group $SU(1,1)$ of the present theory, which removes the compactness present in the "dual" $SU(2)$, and the known existence of a linearly realised local $U(1)$ symmetry, it was proposed that the complex scalar A could be put into correspondence with the coset group $SU(1,1)/U(1)$.

This correspondence allows for the scalar fields to be written as a matrix representing the group action of $SU(1,1)$; it is simpler to realise this symmetry linearly by introducing an auxiliary real scalar field φ. This auxiliary field is then compensated for by the implementation of the $U(1)$ symmetry. The auxiliary scalar can be thought of as the transformation parameter for a rotation of the form $a = e^{i\varphi}$, which is equivalently represented by the 2×2 matrix in the natural $U(1)$ subgroup of $SU(1,1)$, $\Gamma_{SU(1,1)} = \begin{bmatrix} a & 0 \\ 0 & a^* \end{bmatrix}$, via the isomorphism that maps the group $U(1)$ of unit complex numbers to the maximal compact subgroup of the global symmetry. The group matrix is constructed analogously to the standard exponential map for $SU(2)$:

$$\begin{bmatrix} V_-^1 & V_+^1 \\ V_-^2 & V_+^2 \end{bmatrix} = \exp\left(\kappa \begin{bmatrix} i\varphi & A \\ A^* & -i\varphi \end{bmatrix}\right)$$

The analogy can be seen by recalling that Lorentz boosts may be described as "rotations by a complex angle". The exponentiation is of the form $e^{i\vec{\theta}.\vec{\sigma}}$, where if the σ_i are taken to be the standard $SU(2)$ generators, then setting $\vec{\theta} = [-i\alpha_1, i\alpha_2, \gamma]^T$ results in the first order expansion of the exponential satisfying the equation for the infinitesimal generators of $SU(1,1)$. κ is an overall normalization, related to the coupling. Since the exponent must be dimensionless and the fields A and φ are bosons in a ten dimensional spacetime (thus having dimension L^{-4}), we can deduce that κ has dimension L^4.

The super- and sub-script indices appearing on the matrix $V_\pm^\alpha$ can be understood to each correspond to states in representations of the two groups appearing in the coset; the superscript α indexes states in a doublet representation of $SU(1,1)$ while the subscript labels a pair of local $U(1)$ representations. In other words, the columns of the matrix are multiplets of the global symmetry, each of which is associated to a different representation of the local $U(1)$ symmetry. This suggests that the group elements of $SU(1,1)$ transform covariantly under its $U(1)$ subgroup - in order to understand this, we must consider transformations of the matrix $V_\pm^\alpha \in SU(1,1)$ under the action of another $SU(1,1)$ matrix.

This seems puzzling; how do the elements of a group transform under the action of the group? A natural candidate for this transformation is obtained by considering the expression for the matrix element in terms of the basis vectors:

$$\langle \vec{v} | V_\pm^\alpha | \vec{v} \rangle$$

We can (and usually do) choose the basis where $(v_i)_j = \delta_{ij}$ - this is not the only basis one can use. Since a composition of rotations is simply another rotation, the basis vectors can be any $\vec{v}' = R|\vec{v}\rangle$ for $R \in SU(1,1)$. We might therefore consider the transformation of the group elements to take the form

$$\langle \vec{v} | V_\pm^\alpha | \vec{v} \rangle \to \langle \vec{v} | R^T V_\pm^\alpha R | \vec{v} \rangle$$

i.e. $V_\pm^\alpha \to R^T V_\pm^\alpha R$. If we perform this transformation with $R = \begin{bmatrix} a & 0 \\ 0 & a^* \end{bmatrix}$, however, the result does not follow the expectation implied by the fact that the columns of R are associated with different $U(1)$ representations. Another transformation we may consider, which is compatible with the construction, is available due to the knowledge that the matrix is an element of the coset $SU(1,1)/U(1)$ - the matrix must maintain its transformation properties under the right action of the $U(1)$ subgroup:

$$\begin{aligned} V^\alpha_\pm R' &= \begin{bmatrix} V^1_- & V^1_+ \\ V^2_- & V^2_+ \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^* \end{bmatrix} \\ &= \begin{bmatrix} aV^1_- & a^*V^1_+ \\ aV^2_- & a^*V^2_+ \end{bmatrix}. \end{aligned}$$

This does indeed satisfy the requirement, and provides an illustration of the effect of the $U(1)$ subgroup, justifying the association of the columns with representations of opposite sign.

We can now consider infinitesimal transformations under $SU(1,1)$ by introducing two constant parameters γ and α which are real and complex respectively. A matrix representing the transformation can be constructed as:

$$m^\alpha_\beta = \begin{bmatrix} i\gamma & \alpha \\ \alpha^* & -i\gamma \end{bmatrix}$$

This may be compared with the standard general form of the infinitesimal generators of $SU(2)$:

$$n^\alpha_\beta = \begin{bmatrix} i\gamma & -\alpha^* \\ \alpha & -i\gamma \end{bmatrix}$$

The difference can be seen to arise by recalling that $SU(2)$ is defined as the set of 2×2 unitary matrices with unit determinant, i.e. elements of the group satisfy $U^\dagger U = I$, while $SU(1,1)$ is defined with the identity replaced by a $(1+1)$-dimensional Minkowski metric - they satisfy $\tilde{U}^\dagger J \tilde{U} = J$ with $J = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Equivalently, the group can be defined as the set of 2×2 complex matrices with entries of the form $\left[\begin{smallmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{smallmatrix}\right]$, having $|\alpha|^2 - |\beta|^2 = 1$ (for $SU(2)$, the condition is $|\alpha|^2 + |\beta|^2 = 1$).

This allows us to determine the $SU(1,1)$ variation of the scalar field matrix $V^\alpha_\pm$ as

$$\delta(m^\alpha_\beta)V^\alpha_\pm = m^\alpha_\beta V^\beta_\pm$$

i.e. under an $SU(1,1)$ variation, a linear combination of the fields in the $SU(1,1)$ multiplet weighted by the two parameters of the transformation and corresponding to the same representation of $U(1)$ is produced. We can also define a parameter Σ for the local $U(1)$ variation, under which V transforms as

$$\delta(\Sigma)V^\alpha_\pm = \pm i\Sigma V^\alpha_\pm$$

so that this action generates only an infinitesimal rotation of the original field in the complex plane.

One may wonder how the spinor indices of the $SU(1,1)$ matrix elements are raised and lowered; what this means, is that we seek to find a definition for the dual vectors that satisfies the requirement of providing a map to invariant scalars when its indices are contracted with those of the vector. This is achieved by the use of the antisymmetric symbol $\epsilon_{\alpha\beta}$, which is expected due to the isomorphism between $SU(1,1)$ and $SU(2)$. One can verify that this is an invariant of the transformation:

$$\begin{aligned} V^\alpha_\pm V^\beta_\pm \epsilon_{\alpha\beta} &= V^1_\pm V^2_\pm - V^2_\pm V^1_\pm \\ &= \epsilon_{+-}, \end{aligned}$$

where the last line follows from the commutation of the matrix elements and the condition of unit determinant:

$$\epsilon_{\alpha\beta} V^\alpha_- V^\beta_+ = \mathrm{Det}(V) = 1.$$

The invariance can be seen even more clearly by considering the infinitesimal $SU(1,1)$ transformation of the dual vectors. Since this definition implies $V^1_\pm = V_{2\pm}$ and $V^2_\pm = -V_{1\pm}$, we have for the transformation:

$$\delta V_{1\pm} = -i\gamma V_{1\pm} - \alpha^* V_{2\pm} \qquad , \qquad \delta V_{2\pm} = -\alpha V_{1\pm} + i\gamma V_{2\pm}$$

It is then completely straightforward to verify that $\delta(U^\alpha_\pm V_{\alpha\pm}) = 0$. We thus have $\epsilon_{\alpha\beta}V^\alpha_\pm = V_{\beta\pm}$ - it is conventional to raise indices by contraction with the first index of the non-singular form. When raising indices, another standard convention is to define the raising tensor as the negative of the inverse of its dual, so that $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta^\alpha_\gamma$. To compensate for the minus sign appearing in this identity, we contract the second index of the antisymmetric tensor when raising indices, so that this operation is defined by $\epsilon^{\alpha\beta}V_{\beta\pm} = -\epsilon^{\beta\alpha}V_{\beta\pm} = V^\alpha_\pm$.

It will become useful when supersymmetry variations are studied to define combinations of the scalar fields which have clean transformation properties under the action of the two quotient groups of the coset, since these symmetries must be respected when determining the allowed field combinations that appear under SUSY transformations. They must transform covariantly under the action of the global $SU(1,1)$ symmetry group and the supersymmetry coordinate rotation implemented by the local $U(1)$. One can easily check that

$$Q_\mu = -i\epsilon_{\alpha\beta}V^\alpha_-\partial_\mu V^\beta_+$$

is invariant under $SU(1,1)$ variations, $\delta(m^\alpha_\beta)Q_\mu = 0$, and additionally acts as a gauge field for the $U(1)$ transformations:

$$\delta(\Sigma)Q_\mu = \partial_\mu\Sigma$$

This field therefore accounts for the gauge freedom arising due to the required invariance of the theory under rotation of the fermionic superspace coordinates into each other - the two supersymmetries have the same handedness, and are thus indistinguishable. This implies that there is no $U(1)$ symmetry present in Type IIA supergravity, which is expected since the theory has only even-dimensional stable branes - there are thus no brane potentials which transform in a doublet representation with the dilaton and $F1$ string fields. The second $SU(1,1)$ invariant is

$$P_\mu = -\epsilon_{\alpha\beta}V^\alpha_+\partial_\mu V^\beta_+ \qquad , \qquad \delta(m^\alpha_\beta)P_\mu = 0$$

which transforms covariantly with charge $U = 2$ under the action of the $U(1)$:

$$\delta(\Sigma)P_\mu = 2i\Sigma P_\mu$$

It is clear that these are not the only $SU(1,1)$ and $U(1)$ covariant combinations that can be defined; one could have used

$$J_\mu = -i\epsilon_{\alpha\beta}V^\alpha_+\partial_\mu V^\beta_- \qquad , \qquad \delta(m^\alpha_\beta)J_\mu = 0$$

which is also $SU(1,1)$ invariant and acts as a gauge field under $U(1)$ transformations. The combination

$$K_\mu = \epsilon_{\alpha\beta}V^\alpha_-\partial_\mu V^\beta_- \qquad , \qquad \delta(m^\alpha_\beta)K_\mu = 0$$

is again $SU(1,1)$ invariant, and transforms covariantly with charge $U = -2$ under the action of $U(1)$. Comparing this result and the form of the expressions, it seems that the particular combinations favoured by Schwartz are essentially just a choice of convention, viz. to work with fields which transform covariantly under $U(1)$ with positive charges. J_μ and Q_μ are equivalent by the $U(1)$ symmetry of the theory, and either one could be used as the covariant gauge field in calculation.

Having understood the variations of the scalar fields, it is also necessary to present the $SU(1,1)\times U(1)$ transformation formulas for the remaining fields of the theory. Being that it is a complex field, the antisymmetric tensor $A_{\mu\nu}$ is described as an $SU(1,1)$ doublet - an additional index labelling the states in this representation is added to the set of tensor indices, and we decompose the field as $A_{\mu\nu} = B^1_{\mu\nu} + iB^2_{\mu\nu}$. The $SU(1,1)\times U(1)$ variation of this field is therefore:

$$\delta(m^\alpha_\beta)B^\alpha_{\mu\nu} = m^\alpha_\beta B^\beta_{\mu\nu} \qquad , \qquad \delta(\Sigma)B^\alpha_{\mu\nu} = 0$$

Descriptions via the coset-scalar association always have the property that fermionic fields are inert under the action of the global $SU(1,1)$, and transform covariantly under the $U(1)$ with charge as determined by the appearance of superspace fermion coordinates multiplying the component in the superfield:

$$\delta(\Sigma)\psi_\mu = \frac{i}{2}\Sigma\psi_\mu \qquad , \qquad \delta(m^\alpha_\beta)\psi_\mu = 0$$

$$\delta(\Sigma)\lambda = \frac{3i}{2}\Sigma\lambda \qquad , \qquad \delta(m^\alpha_\beta)\lambda = 0$$

This completes the determination of the transformation formulas for all the fields of the theory under the action of the $SU(1,1)$ and $U(1)$ symmetries. The rest of the bosonic fields are real-valued, and therefore do not transform under either of the unitary group actions.

2.2.3 Gauge Field Variations

The next symmetry of the theory that will be studied is the gauge invariance associated to the 2 antisymmetric tensor fields $B^\alpha_{\mu\nu}$ and $A_{\mu\nu\rho\lambda}$; these correspond to the gauge degrees of freedom associated to the string and $D3$-brane respectively. This can be understood in analogy with classical electromagnetism: point particles are 0-dimensional, have one-dimensional world-lines and carry electric charge if they couple to a one-index massless gauge field. In this case, the parameter for the gauge transformation is a constant, i.e. effectively a 0-index tensor. Strings have two-dimensional world-sheets and carry electric charge if they couple to the Kalb-Ramond (K-R) gauge field $B^\alpha_{\mu\nu}$ (a massless, two-index antisymmetric tensor field), and the gauge parameter for the transformation will be a 1-index tensor. A $D3$-brane has a 4-dimensional world-volume and is said to be electrically charged if it couples to a massless antisymmetric tensor field with 4 indices; the gauge parameter for these transformations will be a 3-index tensor.

The existence of field strengths in the theory having a higher number of tensor indices implies that the conserved charges of the theory are not just numbers as in 4-dimensional theories. The charge associated to a two-index field is obtained from the Noether current determined by setting the variation of the action to zero, which implies $\partial_\mu F^{\mu\nu} = j^\nu \rightarrow \partial_\nu j^\nu = 0 \Rightarrow Q = \int d^3x j^0$. Now consider the five-form field strength associated to the Ramond-Ramond (R-R) field; we have $\partial_\mu F^{\mu\nu\rho\sigma\tau} = j^{\nu\rho\sigma\tau} \rightarrow \partial_\nu j^{\nu\rho\sigma\tau} = 0 \Rightarrow Q^{\rho\sigma\tau} = \int d^3x j^{0\rho\sigma\tau}$ - the charge associated to this field is a 3-index tensor quantity.

In the democratic formulation of Type II string theory (and therefore in its massless limit, Type II supergravity), the R-R gauge transformations of the potentials which leave the theory invariant take the form

$$A_{(p)} \rightarrow A_{(p)} + d\Lambda_{(p-1)} + H \wedge \Lambda_{(p-3)}$$

H is the Neveu-Schwartz 3-form field strength, and the notation $A_{(p)}$ indicates that A is a p-index tensor. The following results for the variations of the gauge fields under the K-R gauge transformation are thus not unexpected[1]:

$$\delta(\Lambda_{(1)})B^\alpha_{\mu\nu} = 2\partial_{[\mu}\Lambda^\alpha_{\nu]} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu$$

$$\delta(\Lambda_{(1)})A_{\mu\nu\rho\lambda} = \frac{-i\kappa}{4}\epsilon_{\alpha\beta}\Lambda^\alpha_{[\mu}F^\beta_{\nu\rho\lambda]} = \frac{-i\kappa}{16}\epsilon_{\alpha\beta}(\Lambda^\alpha_\mu F^\beta_{\nu\rho\lambda} - \Lambda^\alpha_\nu F^\beta_{\rho\lambda\mu} + \Lambda^\alpha_\rho F^\beta_{\lambda\mu\nu} - \Lambda^\alpha_\lambda F^\beta_{\mu\nu\rho}) \qquad (2.4)$$

Note that the variation is non-zero for both gauge fields; this is related to the appearance of the $(p-1)$-form parameter, which corresponds to a valid number of indices for the parameters of both transformations. The term corresponding to variation of the 4-form potential is deduced from supersymmetry considerations, details of which are given in Section 2.2.5. For the R-R variation:

$$\delta(\Lambda_{(3)})B^\alpha_{\mu\nu} = 0$$

$$\delta(\Lambda_{(3)})A_{\mu\nu\rho\lambda} = 4\partial_{[\mu}\Lambda_{\nu\rho\lambda]} = \partial_\mu\Lambda_{\nu\rho\lambda} - \partial_\nu\Lambda_{\rho\lambda\mu} + \partial_\rho\Lambda_{\lambda\mu\nu} - \partial_\lambda\Lambda_{\mu\nu\rho}$$

The variation for the K-R field vanishes - this can be traced back to the fact that when $p = 2$, $\Lambda_{(p-3)} = \Lambda_{(-1)}$ does not have an index structure corresponding to one of the gauge parameters. One may have expected this to correspond to a gauge parameter for an instanton field, since such a field is effectively a zero-index tensor, but there appears to be no consistent way to define such a parameter.

[1]The index antisymmetrization is defined by

$$A_{[i_1 i_2 i_3 \cdots i_n]} = \frac{1}{n!}(A_{[i_1 i_2 i_3 \ldots i_n]} - A_{[i_2 i_1 i_3 \ldots i_n]} + A_{[i_2 i_3 i_1 \ldots i_n]} - \cdots) = \frac{1}{n!}\sum_{\sigma \in S_n} \chi(\sigma) A_{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(n)}}.$$

It is necessary in order to obtain the explicit expressions for the variation of the gauge fields to recall that the parameter $\Lambda_{(3)}$ and the field strength $F_{\mu\nu\rho}$ are assumed to be antisymmetric themselves.

It is again convenient for later purposes to define combinations which are invariant under the transformations considered. In the case of the gauge fields, the useful definitions are given as field strengths which are invariant under both R-R and K-R transformations; it can be checked that

$$F^{\alpha}_{\mu\nu\rho} = 3\partial_{[\mu}A^{\alpha}_{\nu\rho]} = \partial_{\mu}A^{\alpha}_{\nu\rho} - \partial_{\nu}A^{\alpha}_{\mu\rho} + \partial_{\rho}A^{\alpha}_{\mu\nu}$$

and

$$\begin{aligned} F_{\mu\nu\rho\lambda\sigma} &= 5\partial_{[\mu}A_{\nu\rho\lambda\sigma]} + \frac{5i\kappa}{8}\epsilon_{\alpha\beta}A^{\alpha}_{[\mu\nu}F^{\beta}_{\rho\lambda\sigma]} \\ &= \partial_{\mu}A_{\nu\rho\lambda\sigma} - \partial_{\nu}A_{\mu\rho\lambda\sigma} + \partial_{\rho}A_{\mu\nu\lambda\sigma} - \partial_{\lambda}A_{\rho\mu\nu\sigma} + \partial_{\sigma}A_{\lambda\rho\mu\nu} \\ &+ \frac{i\kappa}{16}(A^{\alpha}_{\mu\nu}F^{\beta}_{\rho\lambda\sigma} - A^{\alpha}_{\mu\rho}F^{\beta}_{\nu\lambda\sigma} + A^{\alpha}_{\mu\lambda}F^{\beta}_{\nu\rho\sigma} - A^{\alpha}_{\mu\sigma}F^{\beta}_{\nu\rho\lambda} + A^{\alpha}_{\nu\rho}F^{\beta}_{\mu\lambda\sigma} \\ &- A^{\alpha}_{\nu\lambda}F^{\beta}_{\mu\rho\sigma} + A^{\alpha}_{\nu\sigma}F^{\beta}_{\mu\rho\lambda} - A^{\alpha}_{\lambda\rho}F^{\beta}_{\mu\nu\sigma} + A^{\alpha}_{\sigma\rho}F^{\beta}_{\mu\nu\lambda} - A^{\alpha}_{\sigma\lambda}F^{\beta}_{\mu\nu\rho}) \end{aligned}$$

satisfy this condition. Additionally, the modified field strength

$$G_{\mu\nu\rho} = -\epsilon_{\alpha\beta}V^{\alpha}_{+}F^{\beta}_{\mu\nu\rho}$$

satisfies the requirement of $SU(1,1)$ invariance and is $U(1)$ covariant with $U = 1$.

2.2.4 Supersymmetry Variation

Whenever one defines local supersymmetry transformations, the transformation of the supersymmetry partner of the graviton is determined by consistency to be:

$$\delta(\epsilon)\psi_{\mu} = \frac{1}{\kappa}D_{\mu}\epsilon + \cdots .$$

This requirement follows from the fact that, since the spin of the gravitino is greater than 1, the time component of the field ($\mu = 0$) produces modes having negative norm. This is unphysical, and must be compensated by a gauge symmetry - this symmetry is expressed by the above equation. Given that the gravitino is a complex spinor with chirality defined by $\gamma_{11}\psi_{\mu} = -\psi_{\mu}$[2], it follows that the infinitesimal supersymmetry transformation parameter ϵ has these same properties - it is a complex Weyl spinor of the same handedness as the gravitino field.

The derivative appearing in the above transformation formula must be covariant under local Lorentz transformations as well as local $U(1)$ transformations. The Lorentz covariance is implemented in the usual way via the introduction of the spin connection, which acts as a gauge field generated by the Lorentz transformations and serves to couple the spinors to gravity. We have already defined two fields which transform covariantly under the action of $U(1)$; we additionally include the one which transforms as a gauge field in order to ensure gauge covariance under this group. The expression for the derivative is thus:

$$D_{\mu}\epsilon = (\partial_{\mu} + \frac{1}{4}\omega^{rs}_{\mu}\gamma_{rs} - \frac{i}{2}Q_{\mu})\epsilon$$

Some comments on the term involving the spin connection regarding the interpretation of its presence are useful to improve the picture of the theory. One may recall from general relativity that the covariant derivative in the case where we study transformations in the Poincare group includes a term of a similar form, where the spin connection is replaced by the Christoffel symbol. In that case, the term is necessary in order to preserve the coordinate-free description which is the reason for the use of tensors in describing physical fields. The Christoffel symbol provides the necessary structure to preserve invariance under vector coordinate transformations, i.e. it ensures that the theory remains invariant under transformations of the basis vectors by providing the connection on the tangent bundle at each point in the coordinate system - the vector undergoes parallel transport when a change of coordinates is implemented. The spin connection provides the same invariance in the context of the spinor bundle.

[2] γ_{11} is a notation for a product of 10D gamma matrices; these generalize the Dirac gamma matrices to higher dimensions. They are the generators of the Clifford algebra $Cl_{1,9}(\mathbb{R})$ satisfying the generalised anticommutation relation $\{\gamma_a, \gamma_b\} = 2\eta_{ab}I_N$ where $a, b = 0..9$ and $N = 2^{\frac{d}{2}}$ (the dimension of the gamma matrices)

The spin connection is determined by metric compatibility to be given in the form $\omega_{\mu\nu\rho} = \Omega_{\nu\rho\mu} + \Omega_{\nu\mu\rho} + \Omega_{\mu\rho\nu}$, where the Ωs are defined in terms of the zehnbein as $\Omega_{\mu\nu\rho} = e^{\tau}_{\rho}\partial_{[\mu}e_{\nu]\tau}$. Due to the inclusion of supersymmetry in the theory, it is convenient to modify the standard definition of these tensors to include supercovariant dependence on the gravitino field. Knowing that the supersymmetry variation of the zehnbein is given by[3]

$$\delta(\epsilon)e^{r}_{\mu} = -2\kappa \mathrm{Im}(\bar{\epsilon}\gamma^{r}\psi_{\mu})$$

it is clear that terms of the form $\partial\epsilon$ in the variation of

$$\Omega_{\mu\nu\rho} = e^{r}_{\rho}\partial_{[\mu}e_{\nu]r} + \kappa^{2}\mathrm{Im}(\bar{\psi_{\mu}}\gamma_{\rho}\psi_{\nu})$$

will cancel - this is the meaning of supercovariance, that gauge transformations in the variation of Ω (and thus in the covariant derivative), arising due to the presence of the gravitino field in the zehnbein variation, are eliminated.

Continuing with the analysis of the supersymmetry in the theory, it is now necessary to determine the transformation formulas for the variations of the rest of the fields. Bearing in mind that the global $SU(1,1)$ symmetry as well as the $U(1)$ superspace fermion rotation symmetry must be preserved throughout, one can determine that there are only 3 possible field combinations valid for appearance in the SUSY variation of V^{α}_{+}. Two of these are excluded as candidates by considering a second SUSY variation of the field, since this would produce terms proportional to $\partial\epsilon$ - the scalars are not gauge fields of the theory, and should not produce terms of this kind. The SUSY variation of the scalar field matrix elements is thus given by

$$\delta(\epsilon)V^{\alpha}_{+} = \kappa V^{\alpha}_{-}\bar{\epsilon}^{*}\lambda \qquad , \qquad \delta(\epsilon)V^{\alpha}_{-} = \kappa V^{\alpha}_{+}\bar{\epsilon}\lambda^{*}.$$

Of highest relevance to our purposes is the determination of the supersymmetry variation of the two-form gauge field, $B^{\alpha}_{\mu\nu}$. In the dual SYM theory, the supersymmetry algebra under which the fields of an excitation transform is extended by two central charges, which will be shown to correspond to the non-vanishing components of the parameter of a $B_{\mu\nu}$ gauge transformation in geometries with maximal supersymmetry. By determining this parameter in terms of ϵ using the analog of (2.2) for Type IIB supergravity, the requirement of supersymmetry invariance can be directly enforced by substituting the Killing spinor for the geometries under study.

Owing to the fact that the K-R field is itself a gauge field, we expect that terms of the form $\partial\epsilon$ can occur in its variation, since it must still transform as a gauge field under the action of the supersymmetry group. In order to determine the form of the transformation, we must arrange that the appearance of the supersymmetry gauge freedom in its variation is consistent with gauge transformations under the other groups. This is achieved by enforcing the requirement that the commutator of two local supersymmetries gives rise to a combination of all the local symmetry transformations of the theory, resulting in an expression for this relation of the supersymmetry algebra as:

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta(\xi) + \delta(l) + \delta(\epsilon) + \delta(\Lambda_{(1)}) + \delta(\Lambda_{(3)}) + \delta(\Sigma). \tag{2.5}$$

To understand this, recall that a supersymmetry changes bosons to fermions and vice versa - when the action of two supersymmetries is composed the result of the action on any of the bosonic fields is thus to produce another boson. The supersymmetry action is complicated, however, and will not simply transform $B \to F \to B$. Under the composed action, the field may undergo any of the other gauge transformations allowed within the theory - here we are requiring that the composition of supersymmetry transformations returns the same physical state, i.e. the same field up to local gauge transformations.

[3]The Latin index on the gamma matrix appearing here indicates that this is a purely numerical, field-independent matrix; the indices can be converted using the zehnbein as $\gamma^{a} = \gamma^{\mu}e^{a}_{\mu}$

The terms in (2.5) correspond to the following transformations (determined by considering the commutator action on the fields in brackets):

$$\begin{aligned}
\delta(\xi) &- \text{General local coordinate transformations (zehnbein)}\\
\delta(l) &- \text{Local Lorentz transformation (spin connection)}\\
\delta(\epsilon) &- \text{Local supersymmetry transformation (gravitino)}\\
\delta(\Lambda_{(1)}) &- \text{Local gauge transformations (string - 2-index tensor field)}\\
\delta(\Lambda_{(3)}) &- \text{Local gauge transformations (D3-brane - 4-index tensor field)}\\
\delta(\Sigma) &- \text{Local superspace fermionic coordinate rotations (scalar field)}
\end{aligned}$$

Note that the $SU(1,1)$ transformation does not appear; this is a global symmetry of the theory, and thus the transformation of a physical state under its action will not produce the same physical state.

The general coordinate transformation is completely determined by the formulae for the SUSY variation of the zehnbein and gravitino, the result of which being that the parameter for the transformation of the zehnbein under the action of the commutator is

$$\xi^\mu = 2\mathrm{Im}(\bar{\epsilon_1}\gamma^\mu\epsilon_2)$$

Obtaining this result requires the use of the identity for the bilinear form of two spinors: $\bar{\epsilon_1}\gamma^\tau\epsilon_2 = -\bar{\epsilon_2}\gamma^\tau\epsilon_1$. In implementing this identity, one must also recall that the covariant derivative of a spinor is itself a spinor. This implies that $[\delta(\epsilon_1), \delta(\epsilon_2)]e^a_\mu = D_\mu(\xi^\nu e^a_\nu)$, as is the case in all supergravity theories. This demonstrates the satisfaction of (2.5), since one can show by writing out the terms contributing to the covariant derivative and performing some manipulations that the terms obtained on the right hand side correspond to gauge transformations of the theory, with field-dependent parameters.

The variation of the gauge fields $A^\alpha_{\mu\nu}$ and $A_{\mu\nu\rho\sigma}$ are determined to be given by (See Section 2.2.5 for the details of this derivation for the two-form):

$$\delta(\epsilon)A^\alpha_{\mu\nu} = V^\alpha_+\bar{\epsilon}^*\gamma_{\mu\nu}\lambda^* + V^\alpha_-\bar{\epsilon}\gamma_{\mu\nu}\lambda + 4iV^\alpha_+\bar{\epsilon}\gamma_{[\mu}\psi^*_{\nu]} + 4iV^\alpha_-\bar{\epsilon}^*\gamma_{[\mu}\psi_{\nu]} \tag{2.6}$$

$$\delta(\epsilon)A_{\mu\nu\rho\lambda} = 2\mathrm{Re}(\bar{\epsilon}\gamma_{[\mu\nu\rho}\psi_{\lambda]}) - \frac{3i\kappa}{8}\epsilon_{\alpha\beta}A^\alpha_{[\mu\nu}\delta A^\beta_{\rho\lambda]}. \tag{2.7}$$

The parameters for the gauge transformations under the action of the supersymmetry commutator are thus:

$$\Lambda^\alpha_\mu = A^\alpha_{\mu\rho}\xi^\rho + \frac{2i}{\kappa}(V^\alpha_+\bar{\epsilon}_1\gamma_\mu\epsilon^*_2 + V^\alpha_-\bar{\epsilon}^*_1\gamma_\mu\epsilon_2) \tag{2.8}$$

$$\Lambda_{\mu\nu\rho} = A_{\mu\nu\rho\lambda}\xi^\lambda - \frac{1}{2\kappa}\mathrm{Re}(\bar{\epsilon}_1\gamma_{\mu\nu\rho}\epsilon_2) + \frac{3}{8}\epsilon_{\alpha\beta}A^\alpha_{[\mu\nu}(V^\beta_+\bar{\epsilon}_1\gamma_{\rho]}\epsilon^*_2 + V^\beta_-\bar{\epsilon}^*_1\gamma_{\rho]}\epsilon_2). \tag{2.9}$$

It can be checked that these transformations have the correct properties by considering the commutator of local supersymmetry variation with a Λ_μ gauge transformation, $[\delta(\epsilon), \delta(\Lambda_{(1)})]$; the result is a $\Lambda_{\mu\nu\rho}$ gauge transformation, and we can assert that the algebra closes for these two transformations.

The next variation to be determined is that of the fermionic field λ. This derivation illustrates an important principle; applying the requirements of $SU(1,1)$ invariance, $U(1)$ conservation and supercovariance, the general form of the variation must be given by

$$\delta(\epsilon)\lambda = a_1\gamma^\mu\epsilon^*\hat{P}_\mu - a_2\gamma^{\mu\nu\rho}\epsilon\hat{G}_{\mu\nu\rho}$$

where $\hat{P}_\mu$ and $\hat{G}_{\mu\nu\rho}$ are obtained from P_μ and $G_{\mu\nu\rho}$ by requiring supercovariance of those field combinations; they are given by

$$\hat{P}_\mu = P_\mu - \kappa^2\bar{\psi}^*_\mu\lambda$$

$$\hat{G}_{\mu\nu\rho} = G_{\mu\nu\rho} - 3\kappa\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\lambda - 6i\kappa\bar{\psi}^*_{[\mu}\gamma_\nu\psi_{\rho]}.$$

In order to determine the values of the coefficients, one can use the requirement of closure of the algebra on the bosonic fields - substituting this variation into the commutator of two transformations acting on the bosonic fields, the coefficients are uniquely determined by enforcing this constraint. The result is:

$$\delta(\epsilon)\lambda = \frac{i}{\kappa}\gamma^{\mu}\epsilon^{*}\hat{P}_{\mu} - \frac{i}{24}\gamma^{\mu\nu\rho}\epsilon\hat{G}_{\mu\nu\rho}. \tag{2.10}$$

Determining the transformations and parameters for the remaining fields proceeds in a similar fashion; the problem of determining each and every one was not pursued as part of this dissertation, since at this point there is only one remaining transformation which is of relevance. Specifically, the explicit form of the variation of the gravitino field is required to determine the Killing spinor equation that must be satisfied when classifying half-BPS geometries in Section 2.4. This variation is given by:

$$\begin{aligned}\delta(\epsilon)\psi_{\mu} = &\frac{1}{\kappa}D_{\mu}\epsilon + \frac{i}{480}\gamma^{\rho_1\ldots\rho_5}\gamma_{\mu}\epsilon\hat{F}_{\rho_1\ldots\rho_5} + \frac{1}{96}(\gamma_{\mu}^{\nu\rho\lambda}\hat{G}_{\nu\rho\lambda} - 9\gamma^{\rho\lambda}\hat{G}_{\mu\rho\lambda})\epsilon^{*} \\ &- \frac{7\kappa}{16}(\gamma_{\rho}\lambda\bar{\psi}_{\mu}\gamma^{\rho}\epsilon^{*} - \frac{1}{1680}\gamma_{\rho_1\ldots\rho_5}\lambda\bar{\psi}_{\mu}\gamma^{\rho_1\ldots\rho_5}\epsilon^{*}) \\ &+ \frac{i\kappa}{32}\Big[(\frac{9}{4}\gamma_{\mu}\gamma^{\rho} + 3\gamma^{\rho}\gamma_{\mu})\epsilon\bar{\lambda}\gamma_{\rho}\lambda - (\frac{1}{24}\gamma_{\mu}\gamma^{\rho_1\rho_2\rho_3} \\ &+ \frac{1}{6}\gamma^{\rho_1\rho_2\rho_3}\gamma_{\mu})\epsilon\bar{\lambda}\gamma_{\rho_1\rho_2\rho_3}\lambda + \frac{1}{960}\gamma_{\mu}\gamma^{\rho_1\ldots\rho_5}\epsilon\bar{\lambda}\gamma_{\rho_1\ldots\rho_5}\lambda\Big]\end{aligned} \tag{2.11}$$

Once the final form of the variation of each field is determined, the author of [22] uses the notion of an "on-shell" symmetry as a means by which to obtain the equations of motion. The commutator of two supersymmetries acting on each field is calculated in full, and any terms appearing which cannot be identified as gauge transformations with field-dependent parameters (i.e. terms that spoil the closure of the algebra) are required to vanish under the field equations. In this way, by requiring that the additional terms vanish, the field equations of Type IIB Supergravity were derived. These expressions are not of relevance to this dissertation; one need take away only a feeling of awe at the ingenious application of symmetry to solve a problem which had seemed impossible using the standard formalism. Without a concrete formulation of Type IIB supergravity, the most well-tested sector of the AdS/CFT correspondence would not have been available.

2.2.5 $NS{-}B_{\mu\nu}$ Supersymmetry Variation and SUSY Commutator Parameters

In this subsection the form of the supersymmetry variation of the gauge field $A^{\alpha}_{\mu\nu}$ and the associated commutator action parameter will be explicitly determined. The gauge parameter for $A_{\mu\nu\rho\sigma}$ is also calculated; this allows a demonstration of the closure of the supersymmetry algebra, and serves as an example of the methods employed in the article [22].

2.2.5.1 Supersymmetry Variation of the Gauge Field

The first consideration from which some structure for the variation of the 2 index gauge field can be obtained is the fact that we are implementing a supersymmetry transformation - the gauge field is bosonic, and therefore only fermionic fields may appear in the variation. We thus know that one of the factors in each term must be either λ, ψ_{μ} or one of their complex conjugates. We should also expect that a form of the supersymmetry parameter must appear; it may be ϵ, $\bar{\epsilon}$ or one of the complex conjugates thereof.

The index structure requires that one of the scalar field matrix columns should participate, since this is the only other field in the theory with a contravariant spinor index, and we additionally must insert gamma matrices such that bilinear forms with the correct covariant index structure appear.

Since a pair of bosons appear in the action (which is integrated over a 10 dimensional manifold) with two derivatives, the dimension of the gauge fields is $[A^{\alpha}_{\mu\nu}] = L^{\frac{-8}{2}}$. The fermions appear in pairs with a single derivative, and therefore have dimension $[\lambda] = [\psi_{\mu}] = L^{\frac{-9}{2}}$. The dimension of the SUSY parameter is known due to the form of the expression for the general coordinate SUSY variation parameter; this parameter must have dimension $[\xi^\mu] = L$, and the parameter ϵ appears twice in the expression - we thus have $[\epsilon] = L^{\frac{1}{2}}$. The scalar field matrix elements are dimensionless. From the preceding considerations, we can thus assert that combinations with single powers of the fermions and SUSY parameters satisfy the requirement of dimensional consistency.

Before considering the constraints imposed by the symmetry requirements of $SU(1,1)$ and $U(1)$ covariance, we are thus allowed the following field combinations in the variation of $A^\alpha_{\mu\nu}$:

$$\begin{array}{ccccccc}
V^\alpha_\pm \epsilon\gamma_{\mu\nu}\lambda & , & V^\alpha_\pm \epsilon\gamma_{\mu\nu}\lambda^* & , & V^\alpha_\pm \epsilon\gamma_\mu\psi_\nu & , & V^\alpha_\pm \epsilon\gamma_\mu\psi^*_\nu \\
V^\alpha_\pm \bar{\epsilon}\gamma_{\mu\nu}\lambda & , & V^\alpha_\pm \bar{\epsilon}\gamma_{\mu\nu}\lambda^* & , & V^\alpha_\pm \bar{\epsilon}\gamma_\mu\psi_\nu & , & V^\alpha_\pm \bar{\epsilon}\gamma_\mu\psi^*_\nu \\
V^\alpha_\pm \epsilon^*\gamma_{\mu\nu}\lambda & , & V^\alpha_\pm \epsilon^*\gamma_{\mu\nu}\lambda^* & , & V^\alpha_\pm \epsilon^*\gamma_\mu\psi_\nu & , & V^\alpha_\pm \epsilon^*\gamma_\mu\psi^*_\nu \\
V^\alpha_\pm \bar{\epsilon}^*\gamma_{\mu\nu}\lambda & , & V^\alpha_\pm \bar{\epsilon}^*\gamma_{\mu\nu}\lambda^* & , & V^\alpha_\pm \bar{\epsilon}^*\gamma_\mu\psi_\nu & , & V^\alpha_\pm \bar{\epsilon}^*\gamma_\mu\psi^*_\nu
\end{array}$$

In order for consistency under $U(1)$ transformations, we require that the $U(1)$ charges of the involved fields in each term must sum to zero, since $\delta(\Sigma)A^\alpha_{\mu\nu} = 0$. The charges of the fields are given by:

$$\begin{array}{ccccc}
\psi_\mu : U = \dfrac{1}{2} & , & \lambda : U = \dfrac{3}{2} & , & V^\alpha_+ : U = 1 \\
\psi^*_\mu : U = -\dfrac{1}{2} & , & \lambda^* : U = -\dfrac{3}{2} & , & V^\alpha_- : U = -1
\end{array}$$

The $U(1)$ charge of all the forms in which the SUSY parameter ϵ appears can be determined by considering the variation of the field combination P_μ,

$$\delta(\epsilon)P_\mu = \kappa\partial_\mu\bar{\epsilon}^*\lambda,$$

and the variation of the fermionic field λ:

$$\delta(\epsilon)\lambda = \frac{i}{\kappa}\gamma^\mu\epsilon^*\hat{P}_\mu - \cdots$$

Since P_μ has charge $U = 2$ and λ has charge $U = \frac{3}{2}$ it is easily deduced that:

$$\begin{array}{ccc}
\epsilon : U = \dfrac{1}{2} & , & \epsilon^* : U = -\dfrac{1}{2} \\
\bar{\epsilon} : U = -\dfrac{1}{2} & , & \bar{\epsilon}^* : U = \dfrac{1}{2}
\end{array}$$

After analysing the $U(1)$ transformation properties of each of the possible combinations for the variation, we find that the only terms with the required covariance are:

$$\begin{array}{ccccccc}
V^\alpha_+ \epsilon\gamma_{\mu\nu}\lambda^* & , & V^\alpha_+ \bar{\epsilon}^*\gamma_{\mu\nu}\lambda^* & , & V^\alpha_- \epsilon^*\gamma_{\mu\nu}\lambda & , & V^\alpha_- \bar{\epsilon}\gamma_{\mu\nu}\lambda \\
V^\alpha_+ \epsilon^*\gamma_\mu\psi^*_\nu & , & V^\alpha_+ \bar{\epsilon}\gamma_\mu\psi^*_\nu & , & V^\alpha_- \epsilon\gamma_\mu\psi_\nu & , & V^\alpha_- \bar{\epsilon}^*\gamma_\mu\psi_\nu
\end{array}$$

The reason for eliminating the remaining terms in the above are found by considering the requirement of Lorentz invariance. As discussed in Appendix B, only combinations including products of spinors and adjoint spinors will transform covariantly under the Lorentz group action - the adjoint spinors play the role of the dual vectors to the un-barred spinors. We have thus reduced the allowed terms in the variation to

$$\begin{array}{ccc}
V^\alpha_+ \bar{\epsilon}^*\gamma_{\mu\nu}\lambda^* & , & V^\alpha_- \bar{\epsilon}\gamma_{\mu\nu}\lambda \\
V^\alpha_+ \bar{\epsilon}\gamma_\mu\psi^*_\nu & , & V^\alpha_- \bar{\epsilon}^*\gamma_\mu\psi_\nu
\end{array}$$

We must also require that the terms in the variation have the same symmetricity of indices as the gauge field; this is already satisfied as they are presented above for the terms involving λ, since the gamma matrix product $\gamma_{\mu\nu}$ is antisymmetric. The indices involved in the terms multiplying ψ_μ must be antisymmetrized in order to respect this constraint.

The only remaining task necessary to obtain the final form of the variation is the fixing of the coefficients associated to each of the terms allowed by symmetry. The general form of the variation is

$$\begin{aligned}\delta(\epsilon)A^{\alpha}_{\mu\nu} &= a_1 V^{\alpha}_{+}\bar{\epsilon}^*\gamma_{\mu\nu}\lambda^* + a_2 V^{\alpha}_{-}\bar{\epsilon}\gamma_{\mu\nu}\lambda \\ &\quad + a_3 V^{\alpha}_{+}\bar{\epsilon}\gamma_{[\mu}\psi^*_{\nu]} + a_4 V^{\alpha}_{-}\bar{\epsilon}^*\gamma_{[\mu}\psi_{\nu]}\end{aligned}$$

Since we have that $(A^1_{\mu\nu})^* = A^2_{\mu\nu}$ in the $SU(1,1)$ basis we are using, we must require that this condition holds for the variation also (remembering that the gamma matrices are imaginary in a Majorana basis):

$$\begin{aligned}&a_1^* V^2_{-}\bar{\epsilon}\gamma_{\mu\nu}\lambda + a_2^* V^2_{+}\bar{\epsilon}^*\gamma_{\mu\nu}\lambda^* - a_3^* V^2_{-}\bar{\epsilon}^*\gamma_{[\mu}\psi_{\nu]} - a_4^* V^2_{+}\bar{\epsilon}\gamma_{\mu}\psi^*_{\nu} \\ &= a_1 V^2_{+}\bar{\epsilon}^*\gamma_{\mu\nu}\lambda^* + a_2 V^2_{-}\bar{\epsilon}\gamma_{\mu\nu}\lambda + a_3 V^2_{+}\bar{\epsilon}\gamma_{[\mu}\psi^*_{\nu]} + a_4 V^2_{-}\bar{\epsilon}^*\gamma_{[\mu}\psi_{\nu]}\end{aligned}$$

$$\Rightarrow a_1^* = a_2 \qquad , \qquad a_3^* = -a_4$$

The simplest way to satisfy these conditions is to take a_1 to be real and equal to a_2, with a_3 imaginary and equal to a_4. An appropriate choice is $a_1 = a_2 = 1$, $a_3 = a_4 = 4i$, thus (up to arbitrariness in field normalizations) leading to the expression given in (2.6).

2.2.5.2 Commutator Gauge Parameters

Having determined the form of the supersymmetry variation for all the fields of the theory, the calculation of the commutator parameter proceeds as expected; we begin by writing out the second variation of the gauge field (corresponding to the first term in the commutator; the second term will differ only by swapping the indices on the ϵ parameters):

$$\begin{aligned}\delta(\epsilon_1)(\delta(\epsilon_2)A^{\alpha}_{\mu\nu}) &= V^{\alpha}_{+}\bar{\epsilon_2}^*\gamma_{\mu\nu}(\delta(\epsilon_1)\lambda^*) + (\delta(\epsilon_1)V^{\alpha}_{+})\bar{\epsilon_2}^*\gamma_{\mu\nu}\lambda^* \\ &\quad + V^{\alpha}_{-}\bar{\epsilon_2}\gamma_{\mu\nu}(\delta(\epsilon_1)\lambda) + (\delta(\epsilon_1)V^{\alpha}_{-})\bar{\epsilon_2}\gamma_{\mu\nu}\lambda \\ &\quad + 4iV^{\alpha}_{+}\bar{\epsilon_2}\gamma_{[\mu}(\delta(\epsilon_1)\psi^*_{\nu]}) + 4i(\delta(\epsilon_1)V^{\alpha}_{+})\bar{\epsilon_2}\gamma_{[\mu}\psi^*_{\nu]} \\ &\quad + 4iV^{\alpha}_{-}\bar{\epsilon_2}^*\gamma_{[\mu}(\delta(\epsilon_1)\psi_{\nu]}) + 4i(\delta(\epsilon_1)V^{\alpha}_{-})\bar{\epsilon_2}^*\gamma_{[\mu}\psi_{\nu]}\end{aligned}$$

Consider the second and fourth terms; inserting the expression for the variation of the scalars we obtain

$$\begin{aligned}(\delta(\epsilon_1)V^{\alpha}_{+})\bar{\epsilon_2}^*\gamma_{\mu\nu}\lambda^* - (1\leftrightarrow 2) &= \kappa V^{\alpha}_{-}\bar{\epsilon_1}^*\lambda\bar{\epsilon_2}^*\gamma_{\mu\nu}\lambda^* - (1\leftrightarrow 2) = -\kappa V^{\alpha}_{-}\bar{\epsilon_1}^*\bar{\epsilon_2}^*\lambda\gamma_{\mu\nu}\lambda^* + (1\leftrightarrow 2) = 0 \\ (\delta(\epsilon_1)V^{\alpha}_{-})\bar{\epsilon_2}\gamma_{\mu\nu}\lambda - (1\leftrightarrow 2) &= \kappa V^{\alpha}_{+}\bar{\epsilon_1}\lambda^*\bar{\epsilon_2}\gamma_{\mu\nu}\lambda - (1\leftrightarrow 2) = -\kappa V^{\alpha}_{+}\bar{\epsilon_1}\bar{\epsilon_2}\lambda^*\gamma_{\mu\nu}\lambda + (1\leftrightarrow 2) = 0\end{aligned}$$

where we have used the fact that $\epsilon_1\epsilon_2 = \epsilon_1^{\alpha}\epsilon_{2\alpha} = -\epsilon_{1\alpha}\epsilon_2^{\alpha} = \epsilon_2^{\alpha}\epsilon_{1\alpha} = \epsilon_2\epsilon_1$. Precisely the same procedure leads to the cancellation of the sixth and eighth terms under the commutator variation:

$$\begin{aligned}4i(\delta(\epsilon_1)V^{\alpha}_{+})\bar{\epsilon_2}\gamma_{[\mu}\psi^*_{\nu]} - (1\leftrightarrow 2) &= 4i\kappa V^{\alpha}_{-}\bar{\epsilon_1}^*\lambda\bar{\epsilon_2}\gamma_{[\mu}\psi^*_{\nu]} - (1\leftrightarrow 2) = -4i\kappa V^{\alpha}_{-}\bar{\epsilon_1}^*\bar{\epsilon_2}\lambda\gamma_{[\mu}\psi^*_{\nu]} + (1\leftrightarrow 2) = 0 \\ 4i(\delta(\epsilon_1)V^{\alpha}_{-})\bar{\epsilon_2}^*\gamma_{[\mu}\psi_{\nu]} - (1\leftrightarrow 2) &= 4i\kappa V^{\alpha}_{+}\bar{\epsilon_1}\lambda\bar{\epsilon_2}^*\gamma_{[\mu}\psi_{\nu]} - (1\leftrightarrow 2) = -4i\kappa V^{\alpha}_{-}\bar{\epsilon_1}\bar{\epsilon_2}^*\lambda\gamma_{[\mu}\psi_{\nu]} + (1\leftrightarrow 2) = 0\end{aligned}$$

All the field combinations appearing in the commutator gauge parameter must therefore be determined by the commutator of the remaining terms. In fact, the only variations that can contribute terms of the correct form to be identified as gauge transformations (i.e. terms including derivatives of ϵ) are those which act on ψ_μ. This follows because these terms are the only ones that involve the variation of a gauge field.

Proceeding with the calculation for term 5:

$$\begin{aligned}4iV^{\alpha}_{+}\bar{\epsilon_2}\gamma_{[\mu}(\delta(\epsilon_1)\psi^*_{\nu]}) - (1\leftrightarrow 2) &= \frac{4i}{\kappa}V^{\alpha}_{+}\bar{\epsilon_2}\gamma_{[\mu}D_{\nu]}\epsilon_1^* - (1\leftrightarrow 2) \\ &= \frac{2i}{\kappa}V^{\alpha}_{+}[-(D_\nu\bar{\epsilon_1})\gamma_\mu\epsilon_2^* - \bar{\epsilon_1}\gamma_\mu D_\nu\epsilon_2^* + (D_\mu\bar{\epsilon_1})\gamma_\nu\epsilon_2^* + \bar{\epsilon_1}\gamma_\nu D_\mu\epsilon_2^*] \\ &= \frac{2i}{\kappa}V^{\alpha}_{+}[D_\mu(\bar{\epsilon_1}\gamma_\nu\epsilon_2^*) - D_\nu(\bar{\epsilon_1}\gamma_\mu\epsilon_2^*)].\end{aligned}$$

This is obtained using the spinor identities given in the Appendix of [22], specifically the Majorana flip rule gives $\bar{\epsilon}_2\gamma_\mu\epsilon_1^* = -\bar{\epsilon}_1\gamma_\mu\epsilon_2^*$.

Following the same procedure for the seventh term yields:

$$\begin{aligned}4iV_-^\alpha\bar{\epsilon_2}^*\gamma_{[\mu}(\delta(\epsilon_1)\psi_{\nu]})-(1\leftrightarrow 2)&=\frac{4i}{\kappa}V_-^\alpha\bar{\epsilon_2}^*\gamma_{[\mu}D_{\nu]}\epsilon_1-(1\leftrightarrow 2)\\&=\frac{2i}{\kappa}V_-^\alpha[-(D_\nu\bar{\epsilon_1}^*)\gamma_\mu\epsilon_2-\bar{\epsilon_1}^*\gamma_\mu D_\nu\epsilon_2+(D_\mu\bar{\epsilon_1}^*)\gamma_\nu\epsilon_2+\bar{\epsilon_1}^*\gamma_\nu D_\mu\epsilon_2]\\&=\frac{2i}{\kappa}V_-^\alpha[D_\mu(\bar{\epsilon_1}^*\gamma_\nu\epsilon_2)-D_\nu(\bar{\epsilon_1}^*\gamma_\mu\epsilon_2)].\end{aligned}$$

It is clear at this point that these two contributions will be responsible for the appearance of the second term of (2.8). Expanding out the covariant derivatives and pulling the scalar fields inside the resulting spacetime derivatives, we find the terms having the necessary form to be interpreted as field-dependent parameters of a $\Lambda_{(1)}$ gauge transformation:

$$[\delta(\epsilon_1),(\delta(\epsilon_2)]A_{\mu\nu}^\alpha=\partial_\mu[\frac{2i}{\kappa}(V_+^\alpha\bar{\epsilon_1}\gamma_\nu\epsilon_2^*+V_-^\alpha\bar{\epsilon_1}^*\gamma_\nu\epsilon_2)]-\partial_\nu[\frac{2i}{\kappa}(V_+^\alpha\bar{\epsilon_1}\gamma_\mu\epsilon_2^*+V_-^\alpha\bar{\epsilon_1}^*\gamma_\mu\epsilon_2)]+(\delta(\epsilon)\lambda\text{−terms})+\cdots$$

The first term appearing in the gauge parameter (and in all other parameters as determined in [22]) has a familiar form; it is the scalar product of the parameter for a general coordinate transformation (gct) with the gauge field for the transformation to which the parameter corresponds. This is exactly the parameter that appears in the definition of the *covariant* general coordinate transformation (cgct):

$$\delta_{\text{cgct}}(\xi)=\delta_{\text{gct}}(\xi)-\delta(\xi^\mu B_\mu).$$

The origin of the need for this definition when a theory possesses internal gauge symmetry arises by considering the transformation properties of the general coordinate transformation of the fields. For $A_{\mu\nu}^\alpha$, the general coordinate transformation is given as an infinitesimal variation by

$$\delta_{\text{gct}}(\xi)A_{\mu\nu}^\alpha=\xi^\rho\partial_\rho A_{\mu\nu}^\alpha+\partial_\mu\xi^\rho A_{\nu\rho}^\alpha-\partial_\nu\xi^\rho A_{\mu\rho}^\alpha.$$

This form of the expression for the gct in not covariant with respect to $\Lambda_{(1)}$ gauge transformations:

$$\delta(\Lambda_{(1)})\delta_{\text{gct}}(\xi)A_{\mu\nu}^\alpha=\xi^\rho\partial_\rho(\partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu)+\partial_\mu\xi^\rho(\partial_\nu\Lambda_\rho-\partial_\rho\Lambda_\nu)-\partial_\nu\xi^\rho(\partial_\mu\Lambda_\rho-\partial_\rho\Lambda_\mu).$$

This problem is corrected if we modify the definition of the gct in line with the expression for the covariant gct, i.e. we include an additional set of terms, in the form of a variation with parameter $\xi^\mu A_{\mu\nu}^\alpha$, which cancel with the expression given above when the variation with respect to the gauge transformation is applied. It is simple to check that the appropriate modification is:

$$\begin{aligned}\delta_{\text{cgct}}(\xi)A_{\mu\nu}^\alpha&=(\delta_{\text{gct}}(\xi)-\delta(\xi^\mu A_{\mu\nu}^\alpha))A_{\mu\nu}^\alpha\\&=\delta_{\text{gct}}(\xi)A_{\mu\nu}^\alpha-(\partial_\mu(\xi^\rho A_{\nu\rho})-\partial_\nu(\xi^\rho A_{\mu\rho})).\end{aligned}$$

Clearly, the added terms have the correct form to be interpreted as a $\Lambda_{(1)}$ gauge transformation with parameter $\Lambda_\mu=\xi^\rho A_{\mu\rho}$. This feature of the construction is common to its application on all the gauge fields of the theory; the modification of the gct acting on a gauge field is a variation with respect to the associated gauge transformation, having parameter equal to the scalar product of the gauge field with the gct parameter:

$$\delta_{\text{cgct}}(\xi)B_{\mu\nu\cdots\rho}^A=(\delta_{\text{gct}}(\xi)-\delta_B(\xi^\rho B_{\mu\nu\cdots\rho}))B_{\mu\nu\cdots\rho}$$

Bearing this in mind, we can now compare the two sides of (2.5) to obtain the form of the $\Lambda_{(1)}$ gauge parameter. We have for the RHS:

$$\begin{aligned}(\delta_{\text{cgct}}(\xi)+\delta(\Lambda_{(1)})+\delta(\Lambda_{(3)})+\delta(l)+\delta(\epsilon))+\delta(\Sigma))A_{\mu\nu}^\alpha&=\delta_{\text{gct}}(\xi)A_{\mu\nu}^\alpha-2\partial_{[\mu}(A_{\nu]\rho}\xi^\rho)+2\partial_{[\mu}\Lambda_{\nu]}+\cdots\\&=\partial_\mu(-A_{\nu\rho}\xi^\rho+\Lambda_\nu)-\partial_\nu(-A_{\mu\rho}\xi^\rho+\Lambda_\mu)+\cdots\end{aligned}$$

Comparing with the relevant terms in the expression from the commutator variation on the LHS, the parameter is easily determined:

$$\partial_\nu[\frac{2i}{\kappa}(V_+^\alpha\bar{\epsilon_1}\gamma_\mu\epsilon_2^*+V_-^\alpha\bar{\epsilon_1}^*\gamma_\mu\epsilon_2)]+\cdots=\partial_\nu(-A_{\mu\rho}\xi^\rho+\Lambda_\mu)+\cdots$$

$$\Lambda_\mu=A_{\mu\rho}\xi^\rho+\frac{2i}{\kappa}(V_+^\alpha\bar{\epsilon_1}\gamma_\mu\epsilon_2^*+V_-^\alpha\bar{\epsilon_1}^*\gamma_\mu\epsilon_2)$$

This completes the computation of the parameter with which Kalb-Ramond gauge transformations of a field occur under the action of the commutator of two supersymmetry transformations - equation (2.8) has been reproduced.

The 3-index parameter associated to Ramond-Ramond gauge transformations can be obtained by exactly the same procedure using (2.7), again remembering always to use the covariant general coordinate transformation when acting on fields which transform under an internal symmetry of the theory. The first term in (2.9) arises due to this consideration. The other terms which can contribute to the gauge parameter are

$$\frac{2}{\kappa}\mathrm{Re}(\bar{\epsilon}_2\gamma_{[\mu\nu\rho}D_{\lambda]}\epsilon_1) - (1 \leftrightarrow 2),$$

from the second variation of the first term in (2.7), and

$$-\frac{3i\kappa}{8}\epsilon_{\alpha\beta}A^{\alpha}_{[\mu\nu}\left(\frac{4i}{\kappa}V^{\beta}_{+}\bar{\epsilon}_2\gamma_{[\rho}D_{\lambda]]}\epsilon_1^* + \frac{4i}{\kappa}V^{\beta}_{-}\bar{\epsilon}_2^*\gamma_{[\rho}D_{\lambda]]}\epsilon_1\right) - (1 \leftrightarrow 2),$$

from the variation of the ψ dependent terms in $\delta(\epsilon)A^{\alpha}_{\rho\lambda}$, which appears in the 4-index gauge field's SUSY variation. To identify a parameter from the form given, it is easiest to consider a single term in the antisymmetric sum over tensor index permutations. For the first term, recalling that the 3-index gamma matrix product is antisymmetric, we have for the term involving D_λ:

$$\frac{1}{2\kappa}\mathrm{Re}(\bar{\epsilon}_2\gamma_{\mu\nu\rho}D_\lambda\epsilon_1 - \bar{\epsilon}_1\gamma_{\mu\nu\rho}D_\lambda\epsilon_2) = -\frac{1}{2\kappa}\mathrm{Re}(D_\lambda(\bar{\epsilon}_1\gamma_{\mu\nu\rho}\epsilon_2)).$$

The evaluation of the parameter contribution arising from the next term is similar; first expand the inner antisymmetrization bracket, then use the fact that $A^{\alpha}_{\mu\nu}$ is antisymmetric in its tensor indices to obtain

$$\begin{aligned}
&\frac{2\times 3}{8}\epsilon_{\alpha\beta}A^{\alpha}_{[\mu\nu}\left(V^{\beta}_{+}\bar{\epsilon}_2\gamma_\rho D_{\lambda]}\epsilon_1^* + V^{\beta}_{-}\bar{\epsilon}_2^*\gamma_\rho D_{\lambda]}\epsilon_1\right) - (1 \leftrightarrow 2)\\
&\qquad \to \frac{2\times 3}{8\times 12}\epsilon_{\alpha\beta}A^{\alpha}_{\mu\nu}\left(V^{\beta}_{+}\bar{\epsilon}_2\gamma_\rho D_\lambda\epsilon_1^* + V^{\beta}_{-}\bar{\epsilon}_2^*\gamma_\rho D_\lambda\epsilon_1\right) - (1 \leftrightarrow 2)\\
&\qquad \to \frac{2\times 3\times 6}{8\times 12}\epsilon_{\alpha\beta}A^{\alpha}_{[\mu\nu}\left(V^{\beta}_{+}\bar{\epsilon}_2\gamma_{\rho]}D_\lambda\epsilon_1^* + V^{\beta}_{-}\bar{\epsilon}_2^*\gamma_{\rho]}D_\lambda\epsilon_1\right) - (1 \leftrightarrow 2)\\
&\qquad = \frac{3}{8}\epsilon_{\alpha\beta}A^{\alpha}_{[\mu\nu}D_\lambda\left(V^{\beta}_{+}\bar{\epsilon}_1\gamma_{\rho]}\epsilon_2^* + V^{\beta}_{-}\bar{\epsilon}_1^*\gamma_{\rho]}D_\mu\epsilon_2\right).
\end{aligned}$$

Thus we have determined the parameter for the R-R gauge transformation in the commutator of two supersymmetries to be:

$$\Lambda_{\mu\nu\rho} = A_{\mu\nu\rho\lambda}\xi^\lambda - \frac{1}{2\kappa}\mathrm{Re}(\bar{\epsilon}_1\gamma_{\mu\nu\rho}\epsilon_2) + \frac{3}{8}\epsilon_{\alpha\beta}A^{\alpha}_{[\mu\nu}(V^{\beta}_{+}\bar{\epsilon}_1\gamma_{\rho]}\epsilon_2^* + V^{\beta}_{-}\bar{\epsilon}_1^*\gamma_{\rho]}\epsilon_2). \quad (2.12)$$

We must obviously require that the algebra of all the variations of the theory closes; one non-trivial verification that the variations defined are fully consistent is thus to confirm that the commutator of a local supersymmetry with a transformation under another symmetry of the theory produces a transformation which can be identified with a third symmetry of the theory. One such check is performed by considering the commutator

$$[\delta(\epsilon), \delta(\Lambda_{(1)})]A_{\mu\nu\rho\lambda}.$$

Recall that the form of the $\Lambda_{(1)}$ variation of the R-R gauge field, (2.4), was indicated to be determined by supersymmetry considerations; this is exactly the consideration that was referred to - the requirement of closure of the superalgebra under all the variations. Substituting (2.4) and (2.7) into the commutator expression,

$$\begin{aligned}
[\delta(\epsilon), \delta(\Lambda_{(1)})]A_{\mu\nu\rho\lambda} &= -\frac{3i\kappa}{4}(\epsilon_{\alpha\beta}\Lambda^{\alpha}_{[\mu}\partial_{[\nu}\delta(\epsilon)A^{\beta}_{\rho\lambda]]} + \epsilon_{\alpha\beta}\partial_{[[\mu}\Lambda^{\alpha}_{\nu]}\delta(\epsilon)A^{\beta}_{\rho\lambda]})\\
&= -\frac{3i\kappa}{4}\epsilon_{\alpha\beta}\partial_{[\mu}(\Lambda^{\alpha}_{\nu}\delta(\epsilon)A^{\beta}_{\rho\lambda]})\\
&\to -\frac{3i\kappa}{16}\epsilon_{\alpha\beta}\partial_{\mu}(\Lambda^{\alpha}_{[\nu}\delta(\epsilon)A^{\beta}_{\rho\lambda]}),
\end{aligned}$$

one finds that the resulting transformation is a $\Lambda_{(3)}$ gauge transformation with parameter

$$\Lambda_{\mu\nu\rho} = -\frac{3i\kappa}{16}\epsilon_{\alpha\beta}\Lambda^{\alpha}_{[\mu}\delta\epsilon A^{\beta}_{\nu\rho]}.$$

Closure of the algebra is thus evident using the transformations defined.

2.3 The AdS/CFT Correspondence

2.3.1 Statement

The AdS/CFT correspondence was originally submitted as a conjecture by J.M Maldacena in his paper [1]. It proposes an exact equivalence between gauge theories in large-N limits and string theories. The most well-known and tested of this class of equivalences is that between Type-IIB superstring theory on spacetimes that are asymptotically[4] $AdS_5 \times S^5$ and Conformally Invariant $\mathcal{N} = 4$ Supersymmetric Yang-Mills Field Theory (SYM) on 4-dimensional Minkowski spacetime.

2.3.2 Conformal Field Theory

A conformal transformation is a coordinate transformation $x^\alpha \to \tilde{x}^\alpha(x)$ which causes a rescaling of the metric:

$$g_{\alpha\beta}(x) \to \Omega(x)^2 g_{\alpha\beta}(x)$$

A conformal field theory is a field theory which is invariant under these transformations, and hence is scale invariant - physical predictions of the theory do not depend on lengths, only angles. The interpretation of this metric rescaling depends on the properties of the metric; if the metric is the solution to some equations of motion, i.e. is dynamical, the transformation is a diffeomorphism and the conformal symmetry is a gauge symmetry, if the metric is fixed, we have a global symmetry which has associated conserved currents. A key feature of conformal field theories is that the stress-energy tensor associated to the conserved current is traceless, i.e. $T^\mu_\mu = 0$ - this implies scale invariance. The particular CFT we are considering is $\mathcal{N} = 4$ Super Yang-Mills theory.

$\mathcal{N} = 4$ SYM is special in the sense that its β function vanishes so that conformal invariance is not just a feature of the classical limit, but extends into the quantum regime. The beta function is defined as a function of the coupling parameter, g, and the energy scale μ as $\beta(g) = \frac{\partial g}{\partial \log(\mu)}$; examining this formula, it is clear that in theories with a vanishing beta function the coupling does not depend on the scale. Since the coupling sets the strength of interactions in the theory, we say that the theory possesses scale invariance - measuring the effects of interactions produces the same result regardless of the scale at which the experiment is performed. In some cases, a scale-invariant classical field theory can lose this property after quantization; the beta function is defined in the quantum theory and will not vanish in this case.

$\mathcal{N} = 4$ (the theory enjoys 4 supersymmetries, that is, there are 4 supersymmetry operators Q^i_α) is the maximal supersymmetry possible for theories describing particles with spin ≤ 1. This is because each of the Q_α couples with spin $\frac{1}{2}$ to the particles of the theory - if all 4 supersymmetries are applied, it can raise the spin of a particle by at most 2. If there were more supersymmetries, the spin of any particle could be increased to that of a graviton, at which point the theory contains a description of gravity. The particle multiplet of this theory contains 1 vector boson, A_μ, 6 scalar bosons ϕ_i which transform under the fundamental representation of the group $SO(6)$ and 4 fermions λ_a which transform under the fundamental of $SU(4)$. There are 15 generators of the $SO(2,4)$ conformal algebra: the energy-momentum 4 vector P^μ, associated to translations, angular momentum $\vec{L}$ for rotations, $\vec{K}$ for Lorentz boosts, $\mathcal{K}_\mu$ which generate special conformal transformations and the dilatation operator D, which generates scale transformations.

This is not the full symmetry of the theory, as it does not include supersymmetry and hence only describes the bosonic sector. The full superconformal algebra includes a number of other generators, but we will not be concerned with these - states in the theory are completely specified by the energy E, total$(\vec{L} \cdot \vec{L})/z$-component (L_z) angular momentum, a set of $SO(6)$ quantum numbers and scaling dimension Δ, as the set of operators for which these quantities are eigenvalues is the maximal set of

[4] The conjecture as originally stated identifies the ground state in the SYM theory with $AdS_5 \times S^5$; excited states in the dual theory correspond to deformations of this geometry which still possess the properties of $AdS_5 \times S^5$ in the asymptotic limit

mutually commuting operators, even after the generators of the supersymmetry are included. For the purposes of this dissertation, only the dilatation operator is of interest.

2.3.3 String Theory

String theory, at its essence, attempts to describe all matter as consisting of tiny, fundamental vibrating strings, i.e. it asserts that electrons and quarks are not in fact point particles, but are extended in one dimension. The entire spectrum of known particles can be associated with the vibrational modes of the strings - each possible excitation of a string corresponds to a different particle species. Originally the strings were considered to be bosonic, such that only bosons were described by the theory. In this case the theory has critical dimension $D = 26$, and was widely considered as a failure as a candidate for the description of reality due to the inescapable presence of tachyons, which signal inherent instabilities which have not yet been reconciled.

The advent of supersymmetry, which allows for transformations between bosonic and fermionic states, led to the possibility of describing all types of particles within the string theoretical framework. The consequences of such a model are vast and often quite strange. The appearance of massless spin-2 mediating bosons, for which the only consistent mode of interaction is gravity, cannot be eliminated. These are the gravitons, and their natural emergence from the mathematics of string theory provides significant motivation to consider this theory as a candidate for integrating gravity with the strange world of quantum mechanics.

This unification has long been sought after in the physics community, but has been fraught with difficulty. This is a consequence of the mutual incompatibility between the theories of quantum mechanics (QM) and general relativity (GR); there exist limiting cases in which both theories should apply, but here their predictions diverge and cannot be reconciled. String theory is thought to be an underlying theory which can accurately describe these limiting cases, and from which both QM and GR can be extracted in the regimes at which each provides a full description. Five distinct consistent string theories have been found, named Type I, Type IIA, Type IIB, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic. Each one of these requires spacetime supersymmetry on $9 + 1$ dimensional backgrounds for their consistency. Duality transformations between these 5 theories were found to relate the 5 theories to each other, and to a particular 11-dimensional supergravity theory that arises as the strong coupling limit of Type IIA string theory. These dualities led to the postulation of *M-theory*, a $(10 + 1)$-dimensional theory for which each of the 5 theories above is a limiting case [24]. The significant duality for our purposes is that between open and closed strings - open string theories reduce to field theories without gravity at low energy, while closed strings are described by theories of gravity in this limit.

The inclusion of supersymmetry in string theory leads to the result that the spacetime on which the theory is defined must be $(9 + 1)$-dimensional. Various theories exist which attempt to explain the appearance of our universe as existing on a $(3 + 1)$-dimensional spacetime by considering that the additional dimensions predicted by string theory are compactified ("curled up") on a certain 6-dimensional manifold, and hence not observable. These theories claim that our universe, gravitational interactions included, can be entirely described as existing on the remaining 4-dimensional spacetime. However, the AdS/CFT correspondence together with the holographic principle may in fact provide a far more elegant and natural explanation, albeit with the emergence of a description of gravity in our universe that is less obvious - the CFT defined in our 4-dimensional spacetime does not account for gravity, but operators defined in this theory can be used to describe gravitational effects in the dual string theory.

2.3.4 5-dimensional anti-de Sitter Space (AdS_5)

A brief description of the spacetime we consider is now given. The full spacetime is a product space of 5-dimensional anti-de Sitter Space (AdS_5) with a 5-dimensional sphere (S^5). A 5-sphere of radius R is simply the set of points which are a distance R from a fixed point in 6-dimensional Euclidean space.

The AdS_5 is a maximally symmetric Lorentzian manifold with constant negative scalar curvature. In plain English, it is a general relativity-like spacetime where time and space are mathematically equivalent in all directions (there are obviously still distinctions between time and space, such as the sign in the metric, but the space possesses the most symmetries possible between them), which is curved such that the curvature is constant across the entire spacetime in the absence of energy (empty spacetime), and is negative. The space essentially describes “gravity in a box”; if one were to stand at the centre and throw an object in any direction, the object would always return to the centre. A negative curvature corresponds to an attractive force, and it can be thought that AdS_5 has an inherent negative energy associated to it, even when empty.

Mathematical Description There exists a broad class of homogeneous spaces that can be described as a quadric surface (a D-dimensional hypersurface in $D+1$-dimensional space for which all points on the surface are zeros of a quadratic polynomial) embedded in a flat vector space. AdS_5 is such a space, as it can be mathematically defined as the quadric

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 - (t^1)^2 - (t^2)^2 = R^2 \tag{2.13}$$

embedded in 6-dimensional flat Minkowski space with metric

$$dS^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2 - (dt^1)^2 - (dt^2)^2$$

For our purposes, it is useful to calculate the induced metric on the submanifold in what are known as global co-ordinates (so named because they cover the entire manifold). This entails using the parameterization:

$$\begin{aligned}
t^1 &= R\sinh\rho\cos\tau \\
t^2 &= R\sinh\rho\sin\tau \\
X^1 &= R\cosh\rho\sin\theta\sin\beta\sin\alpha \\
X^2 &= R\cosh\rho\sin\theta\sin\beta\cos\alpha \\
X^3 &= R\cosh\rho\sin\theta\cos\beta \\
X^4 &= R\cosh\rho\cos\theta
\end{aligned}$$

It is easily verified that this satisfies equation (2.13), and that the induced metric can be written:

$$dS^2 = R^2 d\rho^2 + R^2\sinh^2\rho d\tau^2 - R^2\cosh^2\rho d\theta^2 - R^2\cosh^2\rho\sin^2\theta d\beta^2 - R^2\cosh^2\rho\sin^2\theta\sin^2\beta d\alpha^2$$

One can check that this metric is non-degenerate and has Lorentzian signature [25]. Another important set of coordinates to consider is the Poincare coordinates. Though with this coordinate choice, the space is divided into two regions that cannot be described simultaneously, it is nonetheless very useful in our study of the AdS/CFT correspondence - the boundary of the AdS_5 has a different structure when described by Poincare coordinates, and the correlation functions of the super Yang-Mills theory can be calculated with the interpretation that the CFT lives on this boundary ([26],[27]). The metric of AdS_5 can be obtained in Poincare coordinates by first introducing the light cone coordinates ([28]):

$$\begin{aligned}
u &= \frac{t^1 - X^4}{R^2} \\
v &= \frac{t^1 + X^4}{R^2}
\end{aligned}$$

The coordinates not included in these expressions are defined as $x^i = \frac{X^i}{Ru}$ and $\tilde{t} = \frac{t^2}{Ru}$, which yields the following equation upon substitution into (2.13):

$$R^4 uv + R^2 u^2(\tilde{t}^2 - \bar{x}^2) = R^2$$

where $\bar{x}^2 = \sum_{i=1}^{3}(x^i)^2$. This allows us to write v in terms of u, $\tilde{t}$ and x^i. After making the substitution $z = \frac{1}{u}$, we obtain the coordinate transformation from which we can calculate the induced metric:

$$\begin{aligned} t^1 &= \frac{1}{2z}(z^2 + R^2 + \bar{x}^2 - \tilde{t}^2) \\ t^2 &= \frac{R\tilde{t}}{z} \\ X^i &= \frac{Rx^i}{z} \qquad i = 1..3 \\ X^4 &= \frac{1}{2z}(z^2 - R^2 + \bar{x}^2 - \tilde{t}^2) \end{aligned}$$

The metric of AdS_5 in Poincare coordinates is then:

$$ds^2 = \frac{R^2}{z^2}(-dz^2 - (d\bar{x})^2 + d\tilde{t}^2)$$

We see that the space is divided into two Poincare charts, the first being the region $z > 0$, the second $z < 0$. Usually the $z > 0$ region is used, and the Poincare AdS space is then the region of the full space corresponding to that chart.

AdS_5 in Poincare patch coordinates has 4D Minkowski space as its boundary (note this is the spacetime on which our SYM theory is defined). Performing a Wick rotation, this can be transformed to 4D Euclidean spacetime, having metric

$$ds^2 = d\tilde{t}^2 + d\bar{x}^2$$

which can be written in spherical coordinates as

$$ds^2 = dr^2 + r^2 d\Omega_3^2.$$

If we make the substitution $t = \ln r$, we obtain

$$ds^2 = e^{2t}(dt^2 + d\Omega_3^2)$$

on which we can apply a conformal transformation to find

$$ds^2 = dt^2 + d\Omega_3^2$$

the metric of $R \times S^3$. This is the boundary of the AdS_5 in global coordinates. The existence of this relationship is central to our cause, as it allows for the identification between operators in the SYM theory and states in the string theory (This is discussed further in Section 2.8.2). The central identification necessary to obtain the important results of this thesis is between the Hamiltonian in the string theory and the dilatation operator in the Yang-Mills theory. This follows from the identification of the global symmetries of the field theory with the isometries of the spacetime in the gravity theory - both the Hamiltonian in the string theory and the Dilatation operator in the field theory correspond to generators of transformations which do not affect the spatial components (owing to the scale invariance of the SYM theory). That we present the Schur Polynomials (Section 2.7) in the Yang-Mills theory as dual to giant graviton states in string theory, and claim that they can be used to describe the full dynamics of these objects, is only possible due to the AdS/CFT correspondence.

2.3.5 Heuristic Motivation for the Conjecture

The argument arises by the consideration of the low energy limit of a system of N parallel D3 branes existing in a 10-dimensional spacetime ([1], reviewed in [29]). Energies we consider must be smaller than the string energy scale, $\frac{1}{l_s}$, or alternatively

$$E \ll \frac{1}{\sqrt{\alpha'}}$$

Working in this limit, we need only consider massless excitations of the system of D3 branes. The two possible excitations that are relevant in this limit are closed string excitations which propagate throughout the bulk of the 10 dimensional spacetime, and open string excitations on the branes themselves. The open strings on the branes do not oscillate due to the low energies, and look like particles in this limit. This implies that the theory describing dynamics on the branes should be one of quantum fields which admits the symmetries relevant to a theory living on a D3 brane. It turns out that this is a maximally supersymmetric ($\mathcal{N} = 4$) Yang-Mills theory with $U(N)$ gauge group. The closed strings are massless states of a theory of Type IIB supergravity. The dynamics on the brane is decoupled from the dynamics in the bulk in the low energy limit.

We now consider the same system from a different point of view; one where the D3 branes are considered as solutions to the equations of type IIB supergravity - massive charged objects which deform spacetime and are sources of closed strings. The geometry of these solutions in the space of the dimensions transverse to the brane can be viewed as consisting of an infinite throat at the D3-brane surrounded by flat Minkowski spacetime. If we consider closed strings in this configuration, as perceived by an observer at infinity[5], two possible excitations are apparent; those far from the throat horizon, in the bulk Minkowski spacetime, and those emanating from near the horizon. The excitations in the bulk will be described by Type IIB supergravity. Those near the horizon are redshifted, such that finite energy excitations of the strings appear to our observer to fall in the low energy limit. These two types of excitations decouple. Studying the metric of the gravitational solution describing the D3 brane system reveals that the near-horizon geometry is in fact $AdS_5 \times S^5$.

So we see that from both viewpoints, the low energy limit results in the D3 brane system splitting into two decoupled subsystems. In both cases, the closed string excitations far from the branes are described by Type IIB supergravity on 10 dimensional Minkowski spacetime. The AdS/CFT conjecture proposes that the other subsystems should exhibit an equivalence too. This is how the conclusion that $\mathcal{N} = 4$ SYM with $U(N)$ gauge group and Type IIB String theory on an $AdS_5 \times S^5$ background should be equivalent was reached. The strong form of the conjecture states in addition that it should hold for all values of the gauge group rank N and 't Hooft coupling λ.

2.3.6 Discussion

The argument presented above is valid for the specific case of the duality that we consider; between Type-IIB String Theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM on 4-d Minkowski spacetime. A more general argument indicating a connection between string theories and large N gauge theories is now briefly explained. The observation that $SU(N)$ gauge theories simplify in the large N limit was first made by Gerard 't Hooft in [30], where it is shown that in this limit an expansion in $\frac{1}{N}$ is admitted. It was also shown that the perturbative expansion for any Feynman diagram of the gauge theory in this limit has precisely the same form as the perturbative expansion over surfaces of increasing genus obtained from oriented closed string theory with string coupling equal to $\frac{1}{N}$. This is compelling evidence that string theories and field theories are related, at least in the perturbative regime.

[5] We must make observations relative to an observer at infinity, for reasons due to mathematical concerns. Recall that gauge transformations do not represent actual symmetries of the theory - they are associated to redundancies in the physical description. In the case where a gauge transformation does not go to the identity at infinity, this can no longer be said of the transformation; its action then implements a real symmetry of the theory. In this case it transforms the fields non-trivially over an infinite spatial region and acquires properties more consistent with a global symmetry. When calculating the variation of the action in a theory of gravity, the only terms that survive are surface terms arising from integration by parts. Hence, invoking Noether's Theorem, any observable quantity must be associated to these terms, so that only an observer at the surface (i.e. at infinity) is able to measure it. Intuitively this can be understood as a result of the fact that spacetime is flat at spatial infinity, so that an observer there is not influenced by contributions associated to the curvature of the background.

Another interesting point to note is that the AdS/CFT correspondence is a weak/strong coupling correspondence [31]. The parameters of the two theories are related by:

$$g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4}$$

Perturbation theory in the SYM theory requires $g_{YM}^2 N \ll 1$ for its validity, while in string theory $\frac{R^4}{l_s^4} \gg 1$ is the prerequisite for weakly curved geometries. Due to this, and the given relationship between the parameters of the two theories, studying one of the theories at weak coupling gives insight into the other theory at strong coupling. It is thus possible to make statements about the inaccessible, non-perturbative sector of one theory by performing calculations in the well-understood perturbative sector of the other. This makes the correspondence potentially very powerful, however, it also introduces a difficulty in testing the validity of the correspondence; one must find objects in both theories with some properties which are protected from corrections when extrapolating the perturbative result to the non-perturbative regime. One class of objects having such a protected property are the *D-branes*.

2.3.7 D-branes

By studying the equations of motion of string theory, one finds that open string endpoints must satisfy one of two types of boundary conditions: Neumann, corresponding to the endpoints being free to move through spacetime, and Dirichlet, where the endpoints are constrained to a fixed submanifold. D-branes (introduced in [32]) are objects that have arisen in string theory due to the assignment of Dirichlet boundary conditions to the endpoints of open strings. A number indicating the number of spatial dimensions of the brane is often appended to the name, so that a Dp-brane has p spatial dimensions. We can imagine that we have open strings propagating in a $(p+q)$-dimensional spacetime, and that we require the open strings to satisfy Dirichlet boundary conditions in q of the coordinates. The strings then satisfy Neumann boundary conditions in the other p co-ordinates, implying that they are free to move on a p-dimensional hypersurface - this hypersurface provides a description of a Dp-brane.

It was also shown in [32] that the conformal dimension of these branes is in fact protected, because they are BPS states. This is a consequence of supersymmetry and the quantization of $\mathcal{R}$-charge. BPS states are states for which the condition $\Delta = J$ is satisfied - the conformal dimension is equal to the $\mathcal{R}$-charge. The anticommutator relation of the supersymmetry operators is proportional to $(\Delta - J)$, thus the supersymmetry multiplet of BPS states is shortened - the actions of some combinations of the supersymmetry operators are null; for instance, "$\frac{1}{2}$-BPS" refers to the case where half the states in the supersymmetry multiplet vanish. $\mathcal{R}$-charge is known to be quantized and hence cannot vary smoothly, and we must infer that the conformal dimension also has this property. Thus, small variations in the value of the coupling will not in fact change the value of Δ, hence we say that this quantity is protected for BPS states. The conformal dimension of these objects can thus be reliably extrapolated from weak to strong coupling, which makes them ideal candidates for probing the AdS/CFT correspondence.

We hence can see that BPS D-branes provide a satisfactory means by which the AdS/CFT conjecture can be tested. This provides some motivation for the study of operators dual to giant gravitons, which are $D3$-branes wrapping an S^3, for which the relevant protected charge is angular momentum. To summarize, in the Yang-Mills theory conformal dimension is protected when extrapolated from weak to strong coupling because it is equal to $\mathcal{R}$-charge for BPS states, and $\mathcal{R}$-charge is quantized. Thus, on the string theory side, angular momentum on the brane is equal to the energy, which is dual to the conformal dimension. Since renormalized quantities are those for which loop corrections must be applied, we have a case of non-renormalization. For our purposes, we can also apply this same logic to non-BPS states, since we study systems where N, and hence J, go to infinity - these states are "nearly BPS", since small corrections in this limit don't affect the BPS properties of the state. We are in fact concerned with the calculation of the small corrections to Δ in this dissertation, with the interpretation that they correspond to excitations of a BPS state.

2.4 LLM Geometries

In the article [3], the authors have developed a method by which to construct regular $\frac{1}{2}$-BPS solutions to Type IIB string theory; that is, they derive a system of equations which describe the general properties of geometries satisfying the requirements of the $\frac{1}{2}$-BPS sector by supergravity solution classification techniques, which can be solved with specific boundary conditions to obtain a description of any state of the geometry. Each state corresponds to a $\frac{1}{2}$-BPS excitation of the $AdS \times S$ configuration. The symmetry requirements provide strong constraints on the form of the metric for these solutions, which, together with the requirement of regularity, result in a particularly simple classification of the solutions. These solutions are dual to operators in the gauge theory constructed from a single complex scalar (conventionally Z), and an exact map between the space of states on both sides of the correspondence manifests in terms of a phase space of free fermions.

2.4.1 Half-BPS States in Gauge Theory

To begin, we must consider the states in the gauge theory to which the string theory solutions that will be constructed are dual. The half-BPS states in the gauge theory ($\mathcal{N} = 4$ SYM compactified on $R \times S^3$) are associated to chiral primary operators, built as products of traces of powers of a single scalar field Z. Our Schur polynomial operators provide a basis in which all possible multi-trace structures are summed over; as discussed in Section 2.7.5, this results in an association between the representations organising the fields and the corresponding string solutions. The most useful description for the present purpose is in terms of free fermions[33], the origin of which is now briefly explained.

We are interested in states having $\Delta - J = 0$, where J is the $\mathcal{R}$-charge associated to the field Z and Δ is the scaling dimension. Explicitly we can define $Z = \phi_5 + i\phi_6$ in terms of the original scalars of the theory, and the charge J can be thought of as an $SO(2)$ generator in a plane with coordinates associated to ϕ_5,ϕ_6. The only state which respects the property $\Delta - J = 0$ is the lowest momentum mode in a partial wave decomposition of the field Z on the 3-sphere. Under the AdS/CFT correspondence, the Z field is dual to a graviton, and for this reason the authors of [3] refer to the modes of the wave expansion as Kaluza-Klein (KK) modes. KK-modes correspond to integer multiples of momentum in extra, compactified (and thus periodic) dimensions, which produce oscillation modes of a standing wave on these dimensions - the standing waves correspond to the charge, and thus charge quantization is implied by the theory. The lowest "KK"-mode of Z corresponds to the case where the standing wave is in its lowest oscillation mode, and thus does not carry the associated charge; it is the first term in the expansion of the field in terms of spherical harmonics on the 3-sphere, which transform in irreducible representations of the $SO(4)_c$ isometry group [34][6]. It is important to clarify the role of this group, since there are in fact two $SO(4)$ groups which are involved in this discussion.

The first is the isometry group of the 3-sphere forming the spatial components of the spacetime on which the SYM theory is defined, which has been denoted $SO(4)_c$; this corresponds in the dual theory to the symmetry of a three-sphere in the AdS_5 component of the geometry. The second will be denoted $SO(4)_{\tilde{J}}$, and corresponds to the $SO(4) \subset SO(6)_{\mathcal{R}}$ which rotates the X and Y fields into each other - this is associated with the symmetry of a 3-sphere in the S^5 component of the dual geometry. The lowest "KK" mode in the expansion of the field Z is not transformed under the $SO(4)_c$ action and is thus a constant mode on the associated S^3; since it is of course also a singlet under the action of the $\mathcal{R}$-symmetry in the transverse ϕ_i ($SO(4)_{\tilde{J}}$), the mode is invariant under both groups of $SO(4)$ transformations. Invoking the state-operator correspondence, this mode corresponds to a local field operator $Z^i_j(0)$, while the higher modes correspond to covariant derivatives of the field. The $SO(2)_J \leftrightarrow U(1)_J$ charge of the local field is of course 1 (it corresponds to the first oscillation mode on the $S^3 \subset S^5$), which is indeed equal to the engineering dimension. The covariant derivatives do not possess the correct charge under the $\mathcal{R}$-symmetry generator to satisfy the BPS condition.

[6] The authors of this article also show directly that the operators corresponding to the truncated Kaluza-Klein tower satisfy equations of motion consistent with a particular plane-wave matrix theory in supergravity.

That this mode satisfies the condition $\Delta - J = 0$ is perhaps more easily understood after using the AdS/CFT correspondence, by recalling that the symmetries of the gauge theory can be associated to isometries in the string geometry, while Δ and J now correspond to energy and angular momentum respectively. The Z field carries $\mathcal{R}$-charge = 1, and thus admits transformations under an $SO(2)$ in the gauge theory, which must be identified with an $SO(2)$ in the dual string theory. This $SO(2)$ generates rotations in a particular plane in the geometry, so that operators composed only of Zs are identified with states carrying angular momentum (= J) only in the coordinates of this plane. The lowest KK-mode thus satisfies $\Delta - J = 0$, since there is no angular momentum in the transverse planes (which define an S^3, associated with the $SO(4)$ which rotates X and Y fields) contributing to the energy.

As a result of the conformal coupling of the state to the 3-sphere, the lowest "KK" mode of Z possesses a harmonic oscillator potential[26]. The motivation for studying single trace operators follows by considering the Hamiltonian for a general single matrix model in a harmonic oscillator potential in terms of creation and annihilation operators. After making a particular gauge choice, each creation operator has one upper $U(N)$ index and one lower $U(N)$ index; acting on the vacuum with k such operators produces a state which transforms as a tensor with k upper $U(N)$ indices and k lower $U(N)$ indices. To make a gauge invariant state, we need to contract the tensor indices of these states with an appropriate invariant tensor of $U(N)$. These invariant tensors have to be formed by different possible orderings of δ^μ_ν, which contract all the upper indices with all the lower indices.

There is another gauge choice which can be applied to this single matrix model, under which the matrix is diagonal. In this description, the wave functions for the Schrödinger equation are defined as functions of the eigenvalues λ_i of the matrix. Gauge invariance in this case is applied by requiring only that the matrix is invariant under the action of the subgroup of $U(N)$ which permutes the eigenvalues, but does not contradict the gauge choice. This requires that the wavefunction is completely symmetric under such permutations, and the classical Lagrangian for the eigenvalues is thus that of a harmonic oscillator. Quantum mechanically, one must implement a change of measure when moving from the matrix basis to the eigenvalue basis; the inclusion of this measure induces a change in the quantum Hamiltonian as compared to the classical. After absorbing the measure, the wavefunction for the eigenvalues is completely antisymmetric, and we see that the gauge-invariant states of the $N \times N$ matrix reduces to a system of N free fermions in a harmonic oscillator potential.

If one chooses a particular time slicing in $\mathcal{N} = 4$ SYM theory, the form of the gauge invariant operators as being defined in terms of a single complex combination of the scalar fields requires the description to be phrased in terms of a single-matrix quantum mechanical model with harmonic oscillator potential, and the above arguments can be applied. The relevant set of local operators forms a decoupled, low energy sector of the theory, which is dual to a sector in the string theory which is easily identified with the half-BPS states of the $AdS_5 \times S^5$ geometry. The gauge-invariant states of Z are thus also equivalent to a system of N fermions in a harmonic oscillator potential. This huge number of fermions forms droplets in phase space, which can be represented as a colouring of the phase space of the eigenvalues indicating regions occupied by the fermions. It is this picture which is reproduced in the string theory in the article [3].

One must recall that we intend to match the gauge symmetries of the SYM theory to the isometries of the metric constructed in the string theory; the relevant symmetry group for a half-BPS state in the gauge theory is $SO(4) \times SO(4) \times R$, where the first $SO(4)$ corresponds to $\mathcal{R}$-charge, the second to a subgroup of the conformal transformations, and R to the Hamiltonian $H' = H - J$. Of course, the state must also preserve 16 non-trivial supersymmetries.

2.4.2 Derivation Highlights

The goal of the following calculation is a general construction of geometries in 10 dimensions which satisfy the conditions associated to being half-BPS. In order to match the isometries of the metric to the gauge symmetries, the geometry must contain two 3-spheres (for the $SO(4)$ groups) and a killing vector (for the time translation generator, H'). An ansatz for the metric is thus given as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + e^{H+G}d\Omega_3^2 + e^{H-G}d\tilde{\Omega}_3^2$$

with the five form field strength being the only one excited; this corresponds to the inclusion of only giant graviton excitations on the geometry, which are well-known to be BPS objects. Since the field sourced by the giants exists over all of spacetime and the metric ansatz includes spherical symmetry, the five-form field strength must also admit spherical symmetry such that it is invariant under the metric isometries. The ansatz for the field strength is:

$$F_{(5)} = F_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3.$$

One must be careful when interpreting this expression as some indices have been suppressed: $F_{\mu\nu\phi_1\phi_2\phi_3} = \sqrt{g_{S^3}}\epsilon_{\phi_1\phi_2\phi_3}F_{\mu\nu}$ - the contribution to the field strength associated with the 3-sphere components is related to the volume of the S^3; this quantity is of course invariant for rotations under $SO(4)$. $F_{(5)}$ must be self-dual; applying this constraint to the ansatz implies that $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ are dual to each other in the four dimensions not corresponding to the spheres:

$$F = e^{3G}_{*4}\tilde{F} \quad , \quad F = dB \quad , \quad \tilde{F} = d\tilde{B}.$$

This can be understood by recognizing that the Hodge dual of a differential five-form in 10 dimensions must be a five-form on the coordinates transverse to those of the original; for the 3-spheres this immediately implies $d\Omega_3 \to d\tilde{\Omega}_3$ under action of the Hodge star, while the sum over the remaining 4-dimensions which are contracted with the two-index F and $\tilde{F}$ produce the same sums contracted with $\star\tilde{F}$ and $\star F$ respectively.

This completes the part of the construction necessary to ensure the required bosonic symmetries are respected. The requirement of maximal supersymmetry is imposed by ensuring that the supersymmetry variation of all the fields in the theory vanishes. We should think about the spacetime in a theory of quantum gravity as corresponding to the graviton field; since the graviton has spin 2, and is thus bosonic, we will require that the fermionic fields vanish in the classical background. This constraint forces the variation of all the bosonic fields to vanish identically, so that the only non-trivial enforcement of this invariance is implemented by the vanishing of the variations of the fermions. The dilaton and axion are assumed constant, and the three-form field strengths (corresponding to string excitations) are set to zero, so that the variation of λ (2.10) is immediately zero. There is thus only one Killing equation for spinors on the geometry thusfar constructed, given by setting the variation (2.11) of ψ_μ equal to zero. The resulting Killing spinor equation (KSE) is

$$\nabla_M\eta + \frac{i}{480}\Gamma^{M_1M_2M_3M_4M_5}F^{(5)}_{M_1M_2M_3M_4M_5}\Gamma_M\eta = 0 \tag{2.14}$$

The solutions η to this equation are the Killing spinors on manifold M, which in our case is the manifold having geometry defined by the ansätze. Forms constructed using the Killing spinors are automatically SUSY invariant; this allows the construction of vectors and tensors transforming under the isometries of the solution which preserve maximal supersymmetry. The analysis of this equation in [3] follows closely the methods of [35, 36, 37, 38, 39], with the result that forms which can be related to the fields of the theory via (2.14) are constructed. This permits expressions for the fields which are automatically SUSY invariant to be obtained, and produces a system of equations which relate various quantities appearing in the metric ansatz amongst each other. The methods employed are briefly summarized in the remainder of this subsection.

Choose a suitable basis for the 10D gamma matrices (explicit constructions given in Appendix D):

$$\Gamma_\mu = \gamma_\mu \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \quad , \quad \Gamma_a = \gamma_5 \otimes \sigma_a \otimes \mathbb{1} \otimes \hat{\sigma}_1 \quad , \quad \Gamma_{\tilde{a}} = \gamma_5 \otimes \mathbb{1} \otimes \tilde{\sigma}_a \otimes \hat{\sigma}_2.$$

The σ's are ordinary Pauli matrices, γs are 4-dimensional gamma matrices, and the indices μ, a and $\tilde{a}$ refer the components of the 4D space and the two 3-spheres respectively. In this basis, they satisfy the relations:

$$\Gamma_{11} = \Gamma_0 \cdots \Gamma_4 \prod \Gamma_a \prod \Gamma_{\tilde{a}} = \gamma^5 \hat{\sigma}^3 \quad , \quad \gamma^5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3.$$

The spinor η must satisfy the chirality condition $\Gamma_{11}\eta = \eta$.

One can first consider the implications of the KSE on the spherical components. The high degree of spherical symmetry present in the geometry as specified by the ansätze allows the re-expression of the 10-dimensional, 32 -component Killing spinor as a product of simpler spinors. Suppose we have an arbitrary spinor χ which exists on a unit 3-sphere; the symmetry present in the underlying manifold allows one to easily identify an equation whose solutions transform in a spinor representation of the rotation group associated to the isometries. Solutions of

$$\nabla_c \chi = a\frac{i}{2}\gamma_c \chi \quad , \quad a = \pm 1$$

where a is correlated with the chirality, transform in the correct representation. ∇_c is the covariant derivative with spin connection relevant to the 3-sphere; due to the warp factors appearing in the full metric, the spin connection must be modified to obtain the derivative covariant with respect to transformations of the sphere in the geometry under study. The Killing spinor of (2.14) can now be decomposed into a product of 2 "sphere-spinors" with an additional spinor which lives on the remaining four dimensions:

$$\eta = \epsilon_{a,b} \otimes \chi_a \otimes \chi_b \tag{2.15}$$

This decomposition induces a reduction of the KSE to a system of equations involving only ϵ - the problem has been reduced from the full 10D problem to one for an effectively 4-dimensional system. This system involves only the 4 dimensional metric $g_{\mu\nu}$, one gauge field B and two scalar fields, H and G. The reduced KSE equations are:

$$\begin{aligned}
(iae^{-\frac{1}{2}(H+G)}\gamma_5\hat{\sigma}_1 + \frac{1}{2}\gamma^\mu\partial_\mu(H+G))\epsilon + 2M\epsilon &= 0 \\
(ibe^{-\frac{1}{2}(H-G)}\gamma_5\hat{\sigma}_2 + \frac{1}{2}\gamma^\mu\partial_\mu(H-G))\epsilon - 2M\epsilon &= 0 \\
\nabla_\mu\epsilon + M\gamma_\mu\epsilon &= 0
\end{aligned} \tag{2.16}$$

where $M = \frac{i}{480}\Gamma^{M_1M_2M_3M_4M_5}F^{(5)}_{M_1M_2M_3M_4M_5}$.

One now constructs spinor bilinears using the reduced spinor; those which turn out to permit useful relations are ($\bar{\epsilon} = \epsilon^\dagger\Gamma^0$):

$$K_\mu = -\bar{\epsilon}\gamma_\mu\epsilon \quad , \quad L_\mu = \bar{\epsilon}\gamma^5\gamma_\mu\epsilon \quad , \quad Y_{\mu\nu} = \bar{\epsilon}\gamma_{\mu\nu}\hat{\sigma}_1\epsilon$$

$$f_1 = i\bar{\epsilon}\hat{\sigma}_1\epsilon \quad , \quad f_2 = i\bar{\epsilon}\hat{\sigma}_2\epsilon \quad , \quad \omega_\mu = \epsilon^t\Gamma^2\gamma_\mu\epsilon.$$

Differentiating each of these bilinears and then using the reduced KSE (2.16) and the technique of Fierz rearrangement, one obtains a system of equations relating the above to the fields of the 4D theory and amongst themselves.

It is easily verified that K_μ satisfies the relevant equation to be identified as a Killing vector. Contracting L_μ with an infinitesimal 4D coordinate vector produces a locally exact form (the equation tells us it is a closed form; locally the domain is contractible), and can be used to define a coordinate y via $L_\mu dx^\mu = \gamma dy$. A metric for the 4 dimensional space can now be defined in terms of this coordinate, where the 3 remaining coordinates are chosen to be orthogonal to y, and the metric in these directions is independent of y:

$$ds^2 = h^2dy^2 + \hat{g}_{\alpha\beta}dx^\alpha dx^\beta \quad , \quad \alpha, \beta = 1..3$$

Fierz rearrangement allows identities between L and K to be proven, one of which implies their orthogonality $K \cdot L = 0$, and thus eliminates the y component of the Killing vector so that $K_\mu = K_\alpha$. The other gives $K^2 = -L^2$. Setting one of the components of the 3-dimensional space to be the time component, $x^\alpha \to x^t$, and recalling that we require only a single timelike Killing vector corresponding to the Hamiltonian, the Killing vector is given by

$$K^\alpha = \delta^\alpha_t$$

The existence of this time-like Killing vector implies certain restrictions on the form of the metric; there is a particular metric decomposition which applies. The result of applying this decomposition to the x^α components of the metric produces the "conforma-stationary" form:

$$ds^2 = h^2 dy^2 + f^2(dt + V_i dx^i)^2 - f^{-2} h_{ij} dx^i dx^j \quad , \quad i, j = 1, 2.$$

The function f, the one form $V_i dx^i$ and the two-dimensional metric h_{ij} must all be independent of t due to the existence of the Killing vector. Using $K^2 = -L^2$ to relate the g_{yy} and g_{tt} components, one can show that $f^2 = -h^{-2}$, so that the metric becomes

$$ds^2 = h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + h_{ij} dx^i dx^j) \quad , \quad i, j = 1, 2.$$

The KSE relations for the bilinears f_1 and f_2 can be used to derive explicit expressions for these forms in terms of the scalar fields of the 4D theory, as well as a relation between the scalars and the gauge potentials B and $\tilde{B}$. The reduced KSE (2.16) can then be used in conjunction with these expressions to determine the coordinate dependence of the scalar H as $y = e^H$. Note that y is in fact equal to the product of the radii of the two 3-spheres; this will be a particularly important observation.

The f_1, f_2 relations can also be applied to obtain projector conditions which determine the explicit form of the Killing spinor in terms of the gamma matrices, a parameter δ (related to the scalar G) and a second spinor ϵ_1:

$$\epsilon = e^{i\delta\gamma^5\Gamma^3\hat{\sigma}_1}\epsilon_1 \quad , \quad \Gamma^3\hat{\sigma}^1\epsilon_1 = a\epsilon_1 \quad , \quad \sinh(2\delta) = ae^{-G}.$$

Inserting this into the expression for the bilinear f_2 yields a constraint on the scale of ϵ_1 as

$$\epsilon_1 = e^{\frac{1}{4}(H+G)}\epsilon_0 \quad , \quad \epsilon_0^\dagger\epsilon_0 = 1.$$

The phase of ϵ_0 can be set to zero by a local Lorentz rotation in the 1-2 plane, so that ϵ_0 takes a constant value over all of spacetime. In this way, by exploiting supersymmetry, we have arrived at a form for the Killing spinor for which the only coordinate dependence comes from its dependence on the fields of the theory.

This result for the form of the Killing spinor can be inserted into the expressions for the closed one-form ω_μ, the resulting expression is only non-vanishing for the $i, j = 1, 2$ components of the metric:

$$\omega_{\hat{1}} = \epsilon^t\Gamma^2\Gamma^1\epsilon = -iah^{-1}\epsilon_0^t\epsilon_0 \tag{2.17}$$

$$\omega_{\hat{2}} = \epsilon^t\Gamma^2\Gamma^2\epsilon = h^{-1}\epsilon_0^t\epsilon_0 \tag{2.18}$$

$$\omega_\mu = \omega_{\hat{c}} e^{\hat{c}}_\mu dx^\mu = (\text{const})(\tilde{e}^{\hat{1}}_i + ia\tilde{e}^{\hat{2}}_i)dx^i \tag{2.19}$$

The fact that ω is closed, $d\omega = 0$, implies that the vielbeins are independent of y and the 2-dimensional metric h_{ij} is flat. This expression will be of utmost importance for the analyses and results of this dissertation; the closed one-forms effectively define a pair of "coordinates" in analogy with the definition of y arising from L_μ. The difference is that in the case of y, this was a true spacetime coordinate of the geometry that was determined purely by the use of the fact that the form is exact. In this case, the "coordinates" arise due to the form of the Killing spinor, and are related to the spacetime coordinates in a non-trivial way. We can think of a parameter space laid over the 1-2 plane of the spacetime coordinates, which reinterprets this plane as the complex plane. Quantities measured with the "metric" defined by this closed one form admit an interpretation where they do not correspond only to lengths in the geometry: as we will see in the following section, the projections along each component of the two dimensional plane will be associated with the central charges of a (2+1)-dimensional superalgebra.

Using self-duality of the five-form field strength, it is then possible to obtain expressions for the gauge fields B, $\tilde{B}$ in terms of the vectors V_i. Using these together with the expression for K_μ obtained from the reduced KSE (2.16), one relates the exterior derivative of the V_i, dV, to the coordinate y and the function z (defined below). An expression for the gauge fields as a function of these same variables is then derived from these relations. When considering the final form of the metric, and the full set of field relations, it becomes clear that the full solution is completely specified by a single function $z(x_1, x_2, y)$. The metric and those relations which will be useful for the rest of this study are:

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2 \,, \tag{2.20}$$

where

$$h^{-2} = 2y \cosh G, \qquad z = \frac{1}{2} \tanh G, \tag{2.21}$$

$$y\partial_y V_i = \epsilon_{ij} \partial_j z, \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij} \partial_y z. \tag{2.22}$$

The function z is determined by solving the differential equation

$$\partial_i \partial_i z + y \partial_y \frac{\partial_y z}{y} = 0. \tag{2.23}$$

which is obtained by requiring the consistency condition for the V_i, $d(dV) = 0$, to hold. This is a remarkable result: the entire 10 dimensional supergravity solution is completely determined by a single function of 3 variables, which obeys a simple linear partial differential equation - in fact, this is Laplace's equation.

2.4.3 Explicit Constructions

In Appendix D, a Mathematica notebook containing explicit constructions of a particular basis of gamma matrices allowed by the definitions given in LLM is attached to this dissertation. The expressions for the closed one-form (2.19) in terms of the scalars are re-derived using sensible constructions for the reduced spinors. This serves to demonstrate the methods of the article [3], and highlights some issues regarding convention arising due to the arbitrary correlations with the chirality of the spinor representations, and the freedom one has in choosing a basis for the 4-dimensional gamma matrices used in the construction. Importantly, it is confirmed that the ω_μ are non-vanishing only in the components corresponding to the $x_1 - x_2$ plane.

2.4.4 Regularity of Solutions

Having related all of the quantities appearing in the metric through a single function z, it is now theoretically possible to construct any half-BPS metric in the string theory by solving (2.23) and plugging the result into the expression for the metric. There is one more vital constraint on the allowed forms which must be considered: the solutions must correspond to *regular* geometries.

The prototypical example used to illustrate the concept of regularity is that of a two dimensional conic manifold. Spacetime points on the cone can be specified by a radius and an angle. Imagine releasing a free-falling probe on the cone; there is nothing in the description which determines how one should define its trajectory when it reaches the tip of the cone - this is one indication that the metric contains a singularity. A possible interpretation is that the geodesic of such a particle abruptly terminates if it passes through the tip of the cone, since at this point there is no smooth path for the probe to continue along; this concept is referred to as "geodesic incompleteness". One could "complete" the mathematical description such that there is a smooth definition of the metric through the point at the tip, but the metric as initially defined is incomplete. The existence of a discontinuity at this point is also illustrated by studying the curvature; consider cutting the cone along a meridian and laying flat the resulting surface

- it will have the form of flat space with a wedge cut out. It is clear that the metric for this surface can be given as

$$ds^2 = dr^2 + r^2 d\phi^2 \tag{2.24}$$

with the range of definition of the coordinates restricted as $(r, \phi) \in (0, \infty) \times [0, 2\pi - \alpha)$ (where α is the angle deficit due to the missing wedge), and is also a valid description of a conical manifold excluding the points along the "seam" - the metric for the interior of the sheet is unchanged by the cutting procedure since it does not involve stretching or topological changes, and thus we see that the cone metric is flat everywhere on the interior of the sheet. For the points along the seam except the tip, the rotational symmetry of the cone implies that one can rescale the angular coordinate as $\phi = k\alpha$ such that each of these points is identified with another point where the metric is known to be flat. The tip is preserved by these rotations, and thus cannot be identified with any other point. When considering parallel transport around the tip of the cone as a means to determine the curvature, note that the rescaling preserves continuity by effectively performing a rotation of the points along the seam by the angle deficit in flat space; this angle is the same for arbitrarily small paths around the tip, and we must deduce that all the curvature is concentrated exactly at the tip of the cone. One can also see the emergence of the discontinuity by considering the ratio of the circumference of a circle to its radius on this manifold; this ratio is constant for every value of r until it reaches 0, due to the angular deficit and the rotational invariance of the point at the tip.

The above considerations signify that the metric has a singular point at the tip of the cone. At this point the radius goes to zero; the component of the metric for the angle vanishes, and it appears as if the spacetime "loses" one of its dimensions - this on its own does not imply a singularity, since the same is true of flat space when the metric is written in spherical coordinates. The difference is seen after rewriting the cone metric (2.24) with $\phi = k\alpha$ in Cartesian coordinates: when $k = 1$, one recovers Euclidean space; when $k \neq 1$ the metric has a discontinuity at $(0, 0)$. The presence of this singularity signals that the metric is irregular, and several methods exist for the resolution of such singularities.

From the LLM metric it is clear that there are two spheres in the geometry and further, that they have radii squared equal to ye^G and ye^{-G} respectively. Thus the product of the radii squared $= y^2$ vanishes on the LLM $(1-2)$ plane. This would produce a singularity in the metric - at these points the spacetime looks like it is "losing" 6 dimensions. The way the singularity is avoided is that one of the vanishing spheres combines with the dy^2 term to produce flat space, while the second sphere's radius squared does not shrink to zero. This can only happen if $G \to \infty$ ($z \to \frac{1}{2}$) or $G \to -\infty$ ($z \to -\frac{1}{2}$). Thus, for small y we must have

$$z = \pm\frac{1}{2} + \beta(x^1, x^2) y^\alpha + ... \tag{2.25}$$

Plugging this into (2.12) we find

$$\partial_i \partial_i z + y \partial_y \left(\frac{\partial_y z}{y} \right) = 0 = y^\alpha \partial_i \partial_i \beta + \alpha \beta y \partial_y (y^{\alpha - 2}) \tag{2.26}$$

which is solved by choosing

$$\alpha = 2 \qquad \partial_i \partial_i \beta = 0 \tag{2.27}$$

$z = \frac{1}{2}$**:**

$$\frac{1}{2} + \beta(x^1, x^2) y^2 = \frac{1}{2} \frac{e^G - e^{-G}}{e^G + e^{-G}} = \frac{1}{2} \frac{1 - e^{-2G}}{1 + e^{-2G}} = \frac{1}{2}(1 - 2e^{-2G} + \cdots)$$
$$\Rightarrow -e^{-2G} = \beta y^2 \tag{2.28}$$

Since e^{-2G} must be positive we set $\beta = -b^2$ so that $e^{-G} = by$ and

$$h^{-2} = ye^G + ye^{-G} = \frac{y}{by} + y^2 b \to \frac{1}{b} \tag{2.29}$$

and the metric behaves as

$$
\begin{aligned}
ds^2 &= -\frac{1}{b}(dt + V_i dx^i)^2 + b(dy^2 + dx^i dx^i) + \frac{1}{b}d\Omega_3^2 + y^2 b d\tilde{\Omega}_3^2 \\
&= -\frac{1}{b}(dt + V_i dx^i)^2 + b(dy^2 + y^2 d\tilde{\Omega}_3^2) + b dx^i dx^i + \frac{1}{b}d\Omega_3^2 +
\end{aligned} \tag{2.30}
$$

The $\tilde{\Omega}_3$ sphere has combined with y to give flat space and the Ω_3 sphere has a radius squared $\frac{1}{b}$ which does not vanish at $y = 0$.

$z = -\frac{1}{2}$**:**

$$
\begin{aligned}
-\frac{1}{2} + \beta(x^1, x^2)y^2 = -\frac{1}{2}\frac{1 - e^{2G}}{1 + e^{2G}} &= -\frac{1}{2}(1 - 2e^{2G} + \cdots) \\
&\Rightarrow e^{2G} = \beta y^2
\end{aligned} \tag{2.31}
$$

Since e^{2G} must be positive we set $\beta = b^2$ and

$$
h^{-2} = y e^{G} + y e^{-G} = y^2 b + \frac{y}{by} \to \frac{1}{b} \tag{2.32}
$$

and the metric behaves as

$$
\begin{aligned}
ds^2 &= -\frac{1}{b}(dt + V_i dx^i)^2 + b(dy^2 + dx^i dx^i) + \frac{1}{b}d\tilde{\Omega}_3^2 + y^2 b d\Omega_3^2 \\
&= -\frac{1}{b}(dt + V_i dx^i)^2 + b(dy^2 + y^2 d\Omega_3^2) + b dx^i dx^i + \frac{1}{b}d\tilde{\Omega}_3^2 +
\end{aligned} \tag{2.33}
$$

The Ω_3 sphere has combined with y to give flat space and the $\tilde{\Omega}_3$ sphere has a radius squared $\frac{1}{b}$ which does not vanish at $y = 0$. We see that $dx^{\hat{1}} = \sqrt{b}dx^1$ and $dx^{\hat{2}} = \sqrt{b}dx^2$.

2.4.4.1 Formulae for Solutions

The conditions for regularity, that $z = \pm\frac{1}{2}$ on the $y = 0$ plane, supplies boundary conditions for the equation (2.23). One can define a variable $\phi = z/y^2$, in terms of which (2.23) becomes the Laplace equation for ϕ in six dimensions, four of which have spherical symmetry. These boundary z values are then charge sources for the equation in in 6D, and the general solution is given after specifying the boundary values on the $x_1 - x_2$ plane as

$$
z(x_1, x_2, y) = \frac{y^2}{\pi}\int_{\mathcal{D}} \frac{z(x_1', x_2', 0)dx_1' dx_2'}{|(\mathbf{x} - \mathbf{x}')^2 + y^2|^2} = -\frac{1}{2\pi}\int_{\partial\mathcal{D}} dl n_i' \frac{x_i - x_i'}{|(\mathbf{x} - \mathbf{x}')^2 + y^2|} + \sigma \tag{2.34}
$$

$$
V_i(x_1, x_2, y) = \frac{\epsilon_{ij}}{\pi}\int_{\mathcal{D}} \frac{z(x_1', x_2', 0)(x_j - x_j')dx_1' dx_2'}{|(\mathbf{x} - \mathbf{x}')^2 + y^2|^2} = \frac{\epsilon_{ij}}{2\pi}\int_{\partial\mathcal{D}} \frac{dx_j'}{(\mathbf{x} - \mathbf{x}')^2 + y^2}. \tag{2.35}
$$

2.4.4.2 Example: $AdS_5 \times S^5$

As demonstrated in [3], the metric of $AdS_5 \times S^5$, which should be viewed as the ground state of the half-BPS excitations of itself, can be obtained by imposing the boundary conditions where $z = -\frac{1}{2}$ on a disc of radius 1 centred at the origin of the $y = 0$ plane, surrounded by an infinite region where $z = \frac{1}{2}$. The solution for z in this case is ed at the origin of the $y = 0$ plane, surrounded by an infinite region where $z = \frac{1}{2}$. The solution for z in this case is

$$
z(r, y; r_0 = 1) = \frac{r^2 - 1 + y^2}{2\sqrt{(r^2 + 1 + y^2)^2 - 4r^2}}. \tag{2.36}
$$

As a quick aside, writing out the first two terms in a small y expansion

$$
z = \frac{1}{2} - y^2\frac{1}{(r^2 - 1)^2} + ... \tag{2.37}
$$

we see that this solution has the properties expected from the analysis of general z solutions under the same expansion (2.25); that is, (1) the first correction is indeed quadratic in y and (2) this correction term is indeed negative. We can read off $b^{-1} = r^2 - 1$.

One can show, as in [3], that after substituting (2.36) into the general ansatz, applying a change of coordinates and some manipulations leads to an expression for the metric which exactly reproduces the standard form for $AdS_5 \times S^5$. The usefulness of this solution does not end here; it is trivial to construct any element of the class of excitations of $AdS_5 \times S^5$ corresponding to alternating rings on the $y = 0$ plane where $z = \pm\frac{1}{2}$ from this solution by taking a linear superposition of the $AdS_5 \times S^5$ solution (2.36) as

$$z(r, y; \{r_0^{(i)}\}) = \sum_i (-1)^{i+1} z(r, y; r_0^{(i)}). \tag{2.38}$$

2.4.5 Diagrammatic Classification: Reproducing Free Fermions

Since the functions giving the solutions for z are determined by the boundary condition $z = \pm\frac{1}{2}$ at $y = 0$, where different functions defining the boundary conditions produce the metrics of different half-BPS excitations of the $AdS \times S$ configuration, one can give the complete description of any particular solution by specifying the regions on the $y = 0$ plane where $z = \pm\frac{1}{2}$. An obvious way to display this definition is with a representation of the plot of the function z over this special plane; colour regions where $z = -\frac{1}{2}$ in black and those with $z = +\frac{1}{2}$ in white. Interpreting such a diagram as a picture of the fermion occupation droplets arising in the analysis of the dual gauge theory operators, the agreement is immediately manifest. The ground state in the gauge theory fermion droplet picture is a unit radius black disk; this is exactly the diagram obtained by plotting (2.36) in this way - this extends to all possible half-BPS excitations on both sides of the correspondence. These plots can also be perfectly matched to half-BPS excitations in the gauge theory by comparing to the representation theoretic labels of the expected duals in our Schur polynomial basis (see Section 4.1 for details).

2.4.6 Discussion

Lin, Lunin and Maldacena's highly technical application of symmetry analysis techniques explicitly confirms the proposal of [33] for the dual string geometric description to the matrix model studied therein. This allows one to obtain an intuitive classification of the entire half-BPS sector of Type IIB string theory geometries, and is arguably one of the most important contributions to the advancement of our understanding of the AdS/CFT correspondence. The fact that any of the infinite number of possible 10-dimensional solutions can be completely given by simply specifying the boundary conditions of a function as a 2-dimensional diagram simplifies the analysis of these solutions, and most importantly for our purposes, small deformations thereof, immeasurably. The diagrams further have established interpretations in terms of the expected dual states in the gauge theory, so that intuitive visual associations for new insights on either side of the correspondence can be developed, and may be expected to guide the building of expectation for the reconstruction of results in the dual theory. Of course, the theories are fundamentally different both conceptually and mathematically; the pictorial description collects aspects of the theories which transcend these divisions. The results of this dissertation, by utilising a basis where it is the representation theory which provides the pictorial description on the field theory side, achieve an interesting application of these principles.

2.5 Giant Magnons

Under the duality between string theory on $AdS_5 \times S^5$ and $\mathcal{N}=4$ SYM, the problem of obtaining the spectrum of excitations on a closed string has been extensively studied. In a limit where one of the $SO(6)$ charges is taken to be very large, corresponding to states having a large angular momentum in the directions of a subset of the coordinates, the BPS state having $E-J=0$ corresponds to a long chain of Zs; that is, a single trace operator $\mathrm{Tr}(Z^J)$. One can introduce a finite number of other fields W which propagate along the chain with a definite momentum:

$$O_p \sim \sum_l e^{ilp}(\cdots ZZZWZZZ\cdots).$$

The planar Hamiltonian for this excited state can be diagonalized by realizing a spin chain description [40, 41, 42]; with this interpretation, the fields W are "magnons". In this section, recent progress in the field theory using symmetry which results in an exact expression for the dispersion relation of the magnons, and the subsequently developed string theory description of the same system is reviewed.

2.5.1 Exact Energies from Symmetry

2.5.1.1 Asymptotic states

Previous approaches to the study of the spin chain description of closed string states has involved the analysis of states transforming under the subalgebra $su(2|3)$ of the full superconformal algebra $psu(2,2|4)$ of $\mathcal{N}=4$ SYM. In this model, the spin at each site can take one of five orientations, three of which correspond to bosonic excitations and two to fermions. Single trace, gauge invariant local operators are constructed as a linear combination of basic spin chain states:

$$|\Psi\rangle = *|Z\phi^1 ZZ\psi^2 Z\cdots\phi^1\rangle + *|\psi^1\phi^1 ZZZ\psi^2\cdots Z\rangle + \cdots.$$

The new insight of [16] involves the introduction of "Asymptotic states", for which there is a clear distinction between the manner in which the background of Zs and the excitations W are handled at the level of the algebra. Begin by defining a vacuum for the spin chain as an infinitely long chain of Zs:

$$|0\rangle^I = |\cdots ZZ\cdots ZZZ\cdots ZZ\cdots\rangle.$$

The superscript I refers to the first level of "screening"; the Zs forming the background have been neglected in the state labels. The author of [43] has shown that it is sufficient to consider periodic states on an infinite spin chain to obtain the correct physical spectrum up to a certain accuracy. Asymptotic states are now defined as excitations of this background, taking the form

$$|\chi_1\cdots\chi''_K\rangle^I = \sum_{n_1\ll\cdots\ll n_K} e^{ip_1 n_1}\cdots e^{ip_K n_K}|\cdots ZZ\cdots\chi(n_1)\cdots\chi'(..)\cdots\chi''(n_K)\cdots ZZZ\cdots\rangle.$$

The subscripts in the state label indicate the momentum carried by each magnon; the n_i appearing in brackets on the RHS indicate the position of the excitation along the chain. An additional simplifying constraint is imposed to allow the study of interactions acting independently on only a single magnon; the condition $n_k \ll n_{k+1}$ assumes that the magnons are well-separated along the chain.

2.5.1.2 The Asymptotic Algebra

The number of excitations, K, appearing in an asymptotic state is not preserved by transformations under the full $SU(2|3)$ symmetry group, since this group can generate additional excitations from the background by acting on the Z fields, and can remove excitations by transforming the other fields into Zs. The symmetry algebra which will be studied is thus that of the subgroup $SU(2|2)$ which leaves the Zs inert. The action of the generators of this group is summarized in the relations

$$[R^a{}_b, T^c] = \delta^c_b T^a - \frac{1}{2}\delta^a_b T^c, \qquad [L^\alpha{}_\beta, T^\gamma] = \delta^\gamma_\beta T^\alpha - \frac{1}{2}\delta^\alpha_\beta T^\gamma \tag{2.39}$$

where T is any tensor transforming as advertised by its index. The algebra also includes two sets of super charges Q_a^α and S_β^b. These close the algebra

$$\{Q_a^\alpha, S_\beta^b\} = \delta_a^b L_\beta^\alpha + \delta_\beta^\alpha R_a^b + \delta_a^b \delta_\beta^\alpha C\,, \tag{2.40}$$

where C is a central charge, and the standard commutator relations include

$$\{Q_a^\alpha, Q_b^\beta\} = 0\,, \qquad \{S_\alpha^a, S_\beta^b\} = 0. \tag{2.41}$$

Vitally, it was found that this algebra can be enlarged by two extra central charges to $su(2|2) \ltimes \mathcal{R}^2$:

$$\{Q_a^\alpha, S_\beta^b\} = \delta_a^b L_\beta^\alpha + \delta_\beta^\alpha R_a^b + \delta_a^b \delta_\beta^\alpha C_i\,, \tag{2.42}$$

$$\{Q_a^\alpha, Q_b^\beta\} = \epsilon^{\alpha\beta}\epsilon_{ab}\frac{k_i}{2}\,, \qquad \{S_\alpha^a, S_\beta^b\} = \epsilon_{\alpha\beta}\epsilon^{ab}\frac{k_i^*}{2}\,. \tag{2.43}$$

It is this extension which allows the determination of the exact dispersion relation; each magnon transforms in a definite representation of the centrally extended algebra, while physical states correspond to tensor products of the representations of the individual magnons. The physical states are constrained by the requirement that the total momentum of the constituent excitations vanishes; we will soon see that this physical constraint can be mapped to the requirement of vanishing total central charge for the tensor product of magnon representations, while each magnon may have non-zero values for these charges. The central charge C is identified with the energy of the magnon; the requirement to recover the original $su(2|2)$ algebra for physical states, together with conservation of energy, implies the constraints:

$$C = \sum_i C_i\,, \qquad \sum_i k_i = 0 = \sum_i k_i^*\,. \tag{2.44}$$

In order to match the number of fermions and bosons of $\mathcal{N} = 4$ SYM, the residual algebra which preserves the excitation number is in fact $su(2|2)^2$; each charge in both copies are set equal following [17]. One may at this point wonder how the bosonic part of the $SU(2|2)^2$ symmetry acts in the gauge theory. Consider the bosonic fields; there are 6 hermitian adjoint scalars ϕ^i that transform as a vector of $SO(6)$. We have combined them into the complex fields as follows

$$\begin{array}{ll} X = \phi^1 + i\phi^2 & \bar{X} = \phi^1 - i\phi^2 \\ Y = \phi^3 + i\phi^4 & \bar{Y} = \phi^3 - i\phi^4 \\ Z = \phi^5 + i\phi^6 & \bar{Z} = \phi^5 - i\phi^6 \end{array} \tag{2.45}$$

The symmetry group that we study transforms $Y, X, \bar{Y}, \bar{X}$ but does not act on $Z, \bar{Z}$. Thus, the bosonic piece of the symmetry group is the $SO(4)$ that rotates $\phi^1, \phi^2, \phi^3, \phi^4$ as a vector. Now, recall that $SO(4)$ is equivalent to $SU(2) \times SU(2)$. We usually call these two $SU(2)$ left and $SU(2)$ right. It will be useful to explicitly describe the relation between the $SO(4)$ generators (M_{pq}) and the $SU(2)$ generators (J_a^L, J_a^R). Let us use normalizations of the $SU(2)$ generators as

$$[J_3, J_\pm] = \pm 2 J_\pm \qquad [J_+, J_-] = J_3 \tag{2.46}$$

In this normalization J_3 is always integral for finite dimensional irreps. The generators of the left and right $SU(2)$ are

$$\begin{aligned} J_3^L &= -i(M_{12} + M_{34}) \\ J_+^L &= \frac{-1}{2}\left((M_{13} - M_{24}) + i(M_{14} + M_{23})\right) \\ J_-^L &= \frac{1}{2}\left((M_{13} - M_{24}) - i(M_{14} + M_{23})\right) \\ J_3^R &= -i(M_{12} - M_{34}) \\ J_+^R &= \frac{1}{2}\left(-(M_{13} + M_{24}) + i(M_{14} - M_{23})\right) \\ J_-^R &= \frac{1}{2}\left((M_{13} + M_{24}) + i(M_{14} - M_{23})\right) \end{aligned} \tag{2.47}$$

One way to understand this formulae is to use $M_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$ and then change to complex variables

$$\begin{aligned} z_1 &= x_1 + ix_2 \\ z_2 &= x_3 + ix_4 \\ \bar{z}_2 &= x_3 - ix_4 \\ \bar{z}_1 &= x_1 - ix_2 \end{aligned} \tag{2.48}$$

In terms of these, the $SU(2) \times SU(2)$ generators are

$$\begin{aligned} J_3^L &= z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \\ J_3^R &= z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \\ J_+^L &= z_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial \bar{z}_2} \\ J_-^L &= \bar{z}_1 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial z_1} \\ J_+^R &= \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial z_2} \\ J_-^R &= \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} - z_2 \frac{\partial}{\partial z_1} \end{aligned} \tag{2.49}$$

The (J_3^L, J_3^R) charges are

$$z_1 \to (1,1) \quad z_2 \to (1,-1) \quad \bar{z}_2 \to (-1,1) \quad \bar{z}_1 \to (-1,-1) \tag{2.50}$$

These charges show that the vector of $SO(4)$ transforms as $(1/2, 1/2)$ of $SU(2) \times SU(2)$. In our normalization the usual 1/2 has become 1. Given these charges, the individual terms in $J_\pm^{L,R}$ are clear. Since the transformation of z_1 is the same as that of X and the transformation of z_2 is the same as that of Y, we have understood (3.13) of [17]. The states $|\phi^a\rangle$ and $|\psi^\alpha\rangle$ that we use to discuss the $SU(2|2)$ symmetry are not easily related to fields of the super Yang-Mills theory - it is products of pairs of these states that is related.

2.5.1.3 The Representation

To specify the representation that each magnon transforms in, following [16, 44] we specify parameters a_k, b_k, c_k, d_k for each magnon, where

$$Q_a^\alpha |\phi^b\rangle = a_k \delta_a^b |\psi^\alpha\rangle\,, \qquad Q_a^\alpha |\psi^\beta\rangle = b_k \epsilon^{\alpha\beta} \epsilon_{ab} |\phi^b\rangle\,, \tag{2.51}$$

$$S_\alpha^a |\phi^b\rangle = c_k \epsilon_{\alpha\beta} \epsilon^{ab} |\psi^\beta\rangle\,, \qquad S_\alpha^a |\psi^\beta\rangle = d_k \delta_\alpha^\beta |\phi^a\rangle\,, \tag{2.52}$$

for the kth magnon. We are using the non-local notation of [44]; an alternative known as the twisted notation can also be used, where markers $Z^\pm$ are inserted into the state labels. The two notations are related by recognising an important manipulation which can be performed on the asymptotic states, under which a convenient description of the action of the central charges is emergent. Consider an asymptotic state with a single excitation of definite momentum:

$$|\chi\rangle^I = \sum_n e^{ipn} |\cdots ZZZ \cdots \chi(n) \cdots ZZZ \cdots\rangle$$

When a Z is added or removed in front of an excitation in the twisted notation, it is specified by inserting an operation $Z^\pm$ to the left of the excitation in the state label. Using the fact that the excitation has a well-defined momentum, this can be rewritten as

$$|Z^\pm \chi\rangle^I = \sum_n e^{ipn} |\cdots ZZZ \cdots \chi(n \pm 1) \cdots ZZZ \cdots\rangle = \sum_n e^{ipn \mp ip} |\cdots ZZZ \cdots \chi(n) \cdots ZZZ \cdots\rangle.$$

Thus, the operation $Z^\pm$ can always be shifted to the far right of the asymptotic state by introducing products of phases in terms of the magnon momenta as e^{ip}:

$$|Z^\pm \chi\rangle^I = e^{\mp ip} |\chi Z^\pm\rangle \tag{2.53}$$

In this dissertation, the non-local notation is used. The expressions above can be compared to those given in [16], where they are given in twisted notation. The markers $Z^{\pm}$ missing in the above as compared to [16] are compensated by absorbing the phases into the coefficients - in later manipulation, we simply multiply phases on states having these markers using (2.53).

Continuing with the algebra relations:

$$Q_1^1 Q_2^2 |\phi^2\rangle = a_k Q_1^1 |\psi^2\rangle = b_k a_k \epsilon^{12}\epsilon_{12} |\phi^2\rangle\,, \qquad Q_2^2 Q_1^1 |\phi^2\rangle = 0\,, \tag{2.54}$$

so that $k_k = 2\, a_k\, b_k$. An identical argument using the S_α^a supercharges gives $k_k^* = 2\, c_k\, d_k$. Consider next a state with a total of K magnons. If we are to obtain a representation without central extension, we must require that the central charges vanish

$$\begin{aligned} \frac{k}{2} &= \sum_{k=1}^{K} \frac{k_k}{2} = \sum_{k=1}^{K} a_k b_k = 0\,, \\ \frac{k^*}{2} &= \sum_{k=1}^{K} \frac{k_k^*}{2} = \sum_{k=1}^{K} c_k d_k = 0\,. \end{aligned} \tag{2.55}$$

Acting with the central charge k on an asymptotic state with K excitations, i.e. a tensor product representation, one obtains

$$k|\chi_1 \cdots \chi_K\rangle^I = \kappa |\chi_1 \cdots \chi_K\rangle^I \quad , \quad \kappa = \sum_{k=1}^{K} a_k b_k \prod_{l=k+1}^{K} e^{-ip_l}$$

This should vanish when acting on physical states. The condition for this to happen is compatible with the physical zero-momentum condition - it in fact corresponds to the same constraint if one sets $a_k b_k = \alpha(e^{-ip_k} - 1)$, since then

$$\begin{aligned} \frac{k}{2} &= \sum_{k=1}^{K} a_k b_k \prod_{l=k+1}^{K} e^{-ip_l} \\ &= \alpha \sum_{k=1}^{K} (e^{-ip_k} - 1) \prod_{l=k+1}^{K} e^{-ip_l} \end{aligned} \tag{2.56}$$

To motivate the final step in proving that this matches the vanishing momentum condition, consider a specific example. For $K = 4$ we have

$$\begin{aligned} \alpha \sum_{k=1}^{4} (e^{-ip_k} - 1) \prod_{l=k+1}^{4} e^{-ipl} &= \alpha(e^{-ip_1} - 1) e^{-i(p_2+p_3+p_4)} \\ &+ \alpha(e^{-ip_2} - 1) e^{-i(p_3+p_4)} \\ &+ \alpha(e^{-ip_3} - 1) e^{-ip_4} \\ &+ \alpha(e^{-ip_4} - 1) \\ &= e^{-i(p_1+p_2+p_3+p_4)} - 1 \end{aligned} \tag{2.57}$$

Notice that the second and third, third and fourth, and the fifth and sixth terms cancel in the second last expression. Thus we have

$$\frac{k}{2} = \alpha(e^{-i\sum_k p_k} - 1) = 0 \Rightarrow \sum_k p_k = 0 \tag{2.58}$$

completing the proof. A completely parallel argument using the S_α^a supercharges shows that

$$c_k d_k = \beta(e^{ip_k} - 1) \tag{2.59}$$

is the correct solution to ensure that the vanishing of central charges coincides with the physical momentum constraint.

These results can be used to rewrite the action of the new central charges (in twisted notation for clarity) as

$$\begin{aligned} k|\chi\rangle^I &= \alpha|Z^+\chi\rangle^I - \alpha|\chi Z^+\rangle^I \\ k^*|\chi\rangle^I &= \beta|Z^-\chi\rangle^I - \beta|\chi Z^-\rangle^I . \end{aligned} \tag{2.60}$$

This makes it clear that the operators generate a gauge transformation of the asymptotic states, corresponding to the addition or removal of Zs on either side of the state - of course, since the chain is infinite and periodic, adding or removing an arbitrary number of Zs at the ends of the state does not change the physical configuration to which it corresponds.

To obtain a formula for the central charge C consider

$$Q^\alpha_a S^b_\beta|\phi^c\rangle = c_k Q^\alpha_a \epsilon^{bc}\epsilon_{\beta\gamma}|\psi^\gamma\rangle = c_k b_k \epsilon^{bc}\epsilon_{\beta\gamma}\epsilon^{\alpha\gamma}\epsilon_{ad}|\phi^d\rangle . \tag{2.61}$$

Now set $a = b$ and $\alpha = \beta$ and sum over both indices to obtain

$$Q^\alpha_a S^a_\alpha|\phi^c\rangle = 2b_k c_k|\phi^c\rangle . \tag{2.62}$$

Very similar manipulations show that

$$S^a_\alpha Q^\alpha_a|\phi^c\rangle = 2a_k d_k|\phi^c\rangle \tag{2.63}$$

so that we learn the value of the central charge C_k

$$\{Q^\alpha_a, S^a_\alpha\}|\phi^c\rangle = 4C|\phi^c\rangle = 2(a_k d_k + b_k c_k)|\phi^c\rangle \qquad \Rightarrow \qquad C_k = \frac{1}{2}(a_k d_k + b_k c_k) . \tag{2.64}$$

One should note that the solutions for this central charge under the unmodified $su(2|2)$ algebra, with $\{Q, Q\} = \{S, S\} = 0$ which fixes $ab = cd = 0$, are $C = \pm\frac{1}{2}$. These values are only valid for the description of the gauge theory at leading order in the coupling, where the excitation transforms in the fundamental representation. This highlights the importance of the novel central extensions; they lift the overly restrictive conditions of the fundamental representation to allow an analysis of a less limited model of the excitations.

Using

$$\{S^1_2, Q^1_1\} = L^1_2 \qquad L^1_2|\psi^2\rangle = |\psi^1\rangle \tag{2.65}$$

we easily find

$$\{S^1_2, Q^1_1\}|\psi^2\rangle = (a_k d_k - b_k c_k)|\psi^1\rangle \qquad \Rightarrow \qquad a_k d_k - b_k c_k = 1 . \tag{2.66}$$

This is also the condition to get an atypical representation of $su(2|2)$ [44].

2.5.1.4 Parameterization

Following [16], a useful parameterization for the parameters of the representation is given by

$$a_k = \sqrt{g}\eta_k , \qquad b_k = \frac{\sqrt{g}}{\eta_k} f_k \left(1 - \frac{x^+_k}{x^-_k}\right) , \tag{2.67}$$

$$c_k = \frac{\sqrt{g} i\eta_k}{f_k x^+_k} , \qquad d_k = \frac{\sqrt{g} x^+_k}{i\eta_k}\left(1 - \frac{x^-_k}{x^+_k}\right) . \tag{2.68}$$

The spectral parameters $x^\pm_k$ are set by the momentum p_k of the magnon

$$e^{ip_k} = \frac{x^+_k}{x^-_k} . \tag{2.69}$$

The parameter f_k is a pure phase, given by the product $\prod_j e^{ip_j}$, where j runs over all magnons to the left of the magnon considered. To ensure unitarity $|\eta_k|^2 = i(x^-_k - x^+_k)$. The condition $a_k d_k - b_k c_k = 1$ to get an atypical representation implies that

$$x^+_k + \frac{1}{x^+_k} - x^-_k - \frac{1}{x^-_k} = \frac{i}{g} . \tag{2.70}$$

Using this parameterization, the expression for the central charge C in terms of the representation parameters can be rewritten in terms of the momentum

$$\begin{aligned} a_k b_k c_k d_k &= g^2(e^{-ip_k}-1)(e^{ip_k}-1) = 4g^2 \sin^2 \frac{p_k}{2} \\ &= \frac{1}{4}\Big[(a_k d_k + b_k c_k)^2 - (a_k d_k - b_k c_k)^2\Big] = \frac{1}{4}\Big[(2C_k)^2 - 1\Big] \end{aligned} \tag{2.71}$$

so that

$$C_k = \pm\sqrt{\frac{1}{4} + 4g^2 \sin^2 \frac{p_k}{2}} \tag{2.72}$$

This is the promised exact result for the dispersion relation satisfied by an asymptotic magnon. Using nothing other than the symmetry algebra, a prediction for the energy of a fundamental excitation of an infinite string at all values of the coupling is obtained.

2.5.1.5 The S-matrix

The asymptotic states, which transform as a tensor product of $SU(2|2)$ representations, treat excitations as fully independent, non-interacting components of the full state. When the exact action of the algebra is to be determined, such that eigenstates with well-defined energy can be constructed, one cannot disregard states with nearby excitations. A generic state is formed by "sewing" together the asymptotic regions in a manner compatible with the algebra. As an example, consider

$$|\psi\rangle = a|\cdots\chi_k\chi'_l\cdots\rangle^I + b|\cdots(\chi\chi')_{kl}\cdots\rangle^I + c|\cdots\chi''_l\chi'''_k\cdots\rangle^I$$

In the above, the first and last terms correspond to states in the left and right asymptotic regions, while the central term corresponds to a non-asymptotic state. The coefficients a and c are transformed under the exact algebra according to the asymptotic rules. a is related to c through relations to b given by requiring consistency with the algebra - it is thus not necessary to directly consider non-asymptotic states, as the relation between asymptotic regions captures the fact that excitations cross the asymptotic boundaries when the transformation from L-asymptotic to R-asymptotic is performed.

The completion of asymptotic states is therefore performed by the S-matrix, which relates the two asymptotic regions:

$$|\psi\rangle = a|\cdots\chi_k\chi'_l\cdots\rangle^I + (\mathrm{non-asympt.}) + S^I_{kl}|\cdots\chi_k\chi'_l\cdots\rangle^I$$

The requirement for asymptotic consistency is that the S-matrix must commute with all the generators of the $SU(2|2)^2$ algebra as $[J_k + J_l, S^I_{kl}] = 0$. The results and some discussion of this calculation are given explicitly in Section 3.7. It is important to note that these arguments determine the scattering matrix up to an as yet undetermined phase; that is, schematically, $S = \hat{S}_{ij}S_0$, where S is the full scattering matrix, $\hat{S}_{ij}$ is the S-matrix determined by these symmetry arguments, and S_0 is the phase which remains to be determined.

A generic eigenstate is a linear combination of basic states; the residual S-matrix S^I acting on some residual state $|\psi\rangle^I$ generates the correct linear combination to produce an eigenstate. The author of [16] has verified that the S-matrix satisfies the Yang-Baxter equation:

$$S^I_{12}S^I_{13}S^I_{23} = S^I_{23}S^I_{13}S^I_{12}.$$

This property, together with the fact that the momenta are only permuted under the action of S^I, imply that it is valid to assume that the S-matrix factorises; that is, any multi-particle scattering interaction can be decomposed into a composition of 2-particle scattering processes. Asymptotic eigenstates of the infinite spin chain are thus determined by the S-matrix.

The full diagonalization of the S-matrix is achieved in [16] by means of a nested Bethe Ansatz technique. The results were found to agree with an earlier conjecture [45].

2.5.1.6 Discussion

In yet another example of an ingenious application of symmetry, the exact energy and asymptotic eigenstates of excitations propagating on an infinite string have been determined in the dual gauge theory. The novel central extension of the algebra allows an analysis in which individual magnons each transform under their own distinct representation, specified by the central charges. The requirement that these charges vanish for physical states (to recover the standard algebra) can be chosen to coincide with the zero-momentum constraint; the symmetry algebra can then be used to obtain the dispersion relation that a magnon obeys in terms of the central charges. The powerful symmetry arguments have resulted also in an exact expression for the 2-particle S-matrix, which allows the asymptotic states to be completed to exact eigenstates.

2.5.2 String Theory Description

Applying the strong/weak nature of the AdS/CFT duality, the large 't Hooft coupling limit of the magnon dispersion relation (2.72) (in the notation of [3])

$$E - J = \frac{\sqrt{\lambda}}{\pi} |\sin\frac{p}{2}| \qquad (2.73)$$

should be possible to reproduce on the string theory side. The authors of [4] have constructed solutions in the string theory which are argued to correspond to the magnon excitations in the dual gauge theory; the energy of the string configuration matches the dispersion relation once an interesting identification between momentum and a geometric quantity is imposed. This subsection provides a brief review of this construction.

2.5.2.1 Limits

The first limit which is taken by the authors is the ordinary 't Hooft limit, so that they are considering free strings in $AdS_5 \times S^5$ and planar diagrams in the gauge theory. One of the $SO(6)$ generators is taken to infinity such that they are considering strings of infinite extent (infinite Zs in the dual theory), and can consider each of the excitations independently. The 't Hooft coupling $\lambda = g^2 N$ is held fixed, and the momentum p of any excitation considered is also assumed to be fixed so that the magnons have a solitonic quality, as in the gauge theory.

2.5.2.2 String Excitations in Flat Space

In order to build the intuition necessary to identify the configuration corresponding to an elementary excitation propagating on an infinite string on an $AdS_5 \times S^5$ geometry, it is instructive to first consider the situation in flat space. Consider a string with two localized excitations having momentum p and $-p$ propagating on its worldsheet; in light cone gauge, the spacetime picture is initially that of two particles moving at the speed of light along two trajectories each having a different constant value in one of the coordinates. The two trajectories are connected by a string corresponding to the magnons - the precise shape of this string will depend on details of the set of transverse excitations.

When the string is finite, the trajectories cross each other at periodic intervals as momentum is transferred between the two particles, and the excitations exchange their relative positions along the worldsheet. In the infinite J limit, this momentum exchange can happen forever without the excitations exchanging position, so that no crossing will be observed when considering the spacetime trajectories in this regime - one sees simply two trajectories connected by a string, eternally separated by a constant measure in one of the coordinates.

2.5.2.3 String Excitations in $AdS_5 \times S^5$

When one now considers the situation in the geometry we are studying, these same intuitions should apply in the context of the new topology. Consider the metric of an S^5, it can be written:

$$ds^2 = \sin^2\theta d\varphi^2 + d\theta^2 + \cos^2\theta d\Omega_3^2.$$

In these coordinates, φ is the coordinate which is shifted by J; that is, $J \to \infty$ implies that the string has a large angular momentum in the $\hat{\varphi}$ direction. The string ground state with $E-J=0$ will be represented in a spacetime picture as a lightlike trajectory moving along φ, with $\varphi - t$ constant. The trajectory is fixed to sit at $\theta = \frac{\pi}{2}$. Note that the 3-sphere piece of the metric vanishes for this value of θ. Since we are at the origin of the AdS_5, it is clear that in the LLM coordinates this corresponds to the string orbiting on the $r = 1$ boundary. This can be understood as a result of the centrifugal force due to the string's angular momentum causing most of the string to be pushed to this boundary [17].

The approach employed by the authors of [4] entails finding a string solution having the expected properties to be identified as describing elementary excitations, and then verifying that it is the minimal energy configuration to ensure that this is the case. Picking a pair of antipodal points on the S^3, the space parameterized by these points together with the coordinates θ and φ form an S^2; including time, the motion of a string with endpoints on the antipodal points takes place in $R \times S^2$.

The Nambu action for this configuration can now be written down. Choosing the parameterization $t = \tau$, $\varphi - t = \varphi'$, $\theta =$const., one obtains

$$S = \frac{\sqrt{\lambda}}{2\pi} \int dt d\varphi' \sqrt{\cos^2\theta\theta'^2 + \sin^2\theta}.$$

After evaluating the integral, the equations of motion are found to be given by

$$\sin\theta = \frac{\sin\theta_0}{\cos\varphi'} \quad , \quad -(\frac{\pi}{2} - \theta_0) \le \varphi' \le \frac{\pi}{2} - \theta_0 \tag{2.74}$$

where $0 \le \theta_0 \le \frac{\pi}{2}$ is an integration constant. At a given time, the endpoints of the string are separated in the angle φ as $\Delta\varphi' = \Delta\varphi = 2(\frac{\pi}{2} - \theta_0)$; in terms of this quantity, the energy is

$$E - J = \frac{\sqrt{\lambda}}{\pi} \cos\theta_0 = \frac{\sqrt{\lambda}}{\pi} \sin\frac{\Delta\varphi}{2} \tag{2.75}$$

The critical observation, which allows a matching between the string and gauge descriptions, is the identification of the momentum of the string excitation with the angular separation of the endpoints; $p = \Delta\varphi$. The above energy formula then exactly matches (2.73). The periodicity in the momentum seen in the dispersion relation determined by symmetry in the field theory was a result of the discrete lattice structure; in formulating the dual string description, rather than requiring any kind of discreteness of the string worldsheet which would cause fundamental issues with the analysis, this periodicity is found to correspond to the periodicity of one of the coordinates in the geometry.

2.5.2.4 Geometric Description

An elegant geometric description of the string solution and the central charges associated to the dual state is obtained by rewriting the solution in the LLM coordinates. The metric on the $y = 0$ plane [7] is:

$$ds^2 = R^2 \left[-(1-r^2)\left(dt - \frac{r^2}{(1-r^2)} d\varphi'\right)^2 + \frac{dr^2 + r^2 d\varphi'^2}{(1-r^2)} + (1-r^2) d\Omega_3^2 + \cdots \right].$$

In this expression, $r^2 = \sin^2\theta = x_1^2 + x_2^2$ and the dots represent the components of the metric not relevant to this analysis, i.e. the y coordinate and the second 3-sphere (which vanishes for $r \le 1$).

In these coordinates, the previous solution corresponds simply to a line joining two points on the $r = 1$ boundary - this is easily seen by noting that (2.74) becomes $r\cos\varphi' = x_1 =$ constant. The energy is given by the length of this line, measured with the flat metric on the $x_1 - x_2$ plane. Of course, this energy still matches the central charge C in the strong coupling limit. One can check, by recalling that

[7] A study of the geodesics of a half BPS probe in this geometry reveals that states composed mostly from Zs exhibit motion preferentially on this plane.

a line subtending an angle θ at the boundary of a circle has length $2\sin\frac{\theta}{2}$, that the length of the line corresponding to the solution gives this result. Since the central charges k, k^* are the origin of this term in the dispersion relation, one can associate either one with the projection of the line onto each of the $\hat{1},\hat{2}$ axes in the LLM plane - that is, the complex central charge $k = k_{\hat{1}} + ik_{\hat{2}}$, while k^* is the complex conjugate of k. These expressions can be compared with those for the closed one form which can be constructed from the LLM Killing spinor (2.19); the correspondence is immediately clear for the projections onto the axes of the complex plane, and we see that either k or k^* determine the vector (under a reinterpretation of the LLM plane as the complex plane) which specifies the solution in LLM coordinates, depending on the $SO(4)$ chirality of the sphere spinors in the decomposition (2.15). This chirality thus relates to the orientation of the line segment, as pointed out in Appendix D. In the following subsection, this closed one form is explicitly shown to reproduce the central charges in the string theory.

2.5.2.5 Central Charges from Phases in LLM

We now seek to understand the appearance of the extra central charges of the gauge theory supersymmetry algebra from the point of view of the string theory. In order to make this concrete, a detailed explanation of their emergence under the gauge transformations of Type IIB supergravity is now presented. The task necessary to achieve this, is to compute how the string state transforms under the $NS - B_{\mu\nu}$ gauge transformation, which was argued to be the source of the emergence of the central charges that appear in the $SU(2|2)$ algebra on the string theory side in [4].

Warm-up : Point Particle Phase under a Gauge Transformation

Before considering the action of gauge transformations on the string state, it is helpful to first review the point particle. The gauge transformation acts as

$$\begin{aligned} A_\mu \to A'_\mu &= A_\mu + \partial_\mu \chi \\ |\psi(x)\rangle \to |\psi(x)\rangle' &= e^{-ie\chi(x)}|\psi(x)\rangle. \end{aligned} \tag{2.76}$$

We are using the notation $|\psi(x)\rangle$ to denote a particle state tightly concentrated around the point x. This is not, of course, a wave function in position space. Under a gauge transformation the transition amplitude transforms as

$$\langle\psi(x_2)|e^{-iHt}|\psi(x_1)\rangle \to e^{ie\chi(x_2)-ie\chi(x_1)}\langle\psi(x_2)|e^{-iHt}|\psi(x_1)\rangle. \tag{2.77}$$

We want to reproduce this from the path integral formalism, since the action is what we will study when considering the string transformation. In the path integral formalism

$$\begin{aligned} \langle\psi(x_2)|e^{-iHt}|\psi(x_1)\rangle &= \int_{x(0)=x_1, x(t)=x_2} [Dx] e^{iS} \\ S &= m\int ds + e\int dx^\mu A_\mu. \end{aligned} \tag{2.78}$$

Under a gauge transformation, it is the second term in the action which changes:

$$\delta S = e\int_{x_1}^{x_2} \partial_\mu \chi dx^\mu = e\chi(x)|_{x_1}^{x_2} = e\chi(x_2) - e\chi(x_1). \tag{2.79}$$

This depends only on the end points of the paths we integrate over; since these are fixed, we pick up an overall phase as we do the path integral. The phase can be pulled outside of the integral:

$$\begin{aligned} \int_{x(0)=x_1, x(t)=x_2} [Dx] e^{iS} &\to \int_{x(0)=x_1, x(t)=x_2} [Dx] e^{iS+i\delta S} \\ &= e^{ie\chi(x_2)-ie\chi(x_1)} \int_{x(0)=x_1, x(t)=x_2} [Dx] e^{iS}. \end{aligned} \tag{2.80}$$

This reproduces the transformation of the transition amplitude that we found above. Thus, the net effect of the gauge transformation is obtained by computing the change in the action.

String Phase under $NS - B_{\mu\nu}$

Proceeding with the same computation, but for the string, the action is

$$S = T\int_{\Sigma} d^2\sigma\sqrt{h}\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X) + \int_{\Sigma} B_{\mu\nu}dx^\mu \wedge dx^\nu \tag{2.81}$$

where Σ denotes the worldsheet of the string. Under a gauge transformation with parameter ω_μ we have

$$\delta B_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu = d\omega. \tag{2.82}$$

A simple application of Stokes' Theorem then gives

$$\delta S = \int_{\Sigma} d\omega = \int_{\partial\Sigma}\omega = \int_{\partial\Sigma}\omega_{\hat{a}}dx^{\hat{a}} = \int_{\partial\Sigma}(dx^2 - iadx^1) \tag{2.83}$$

where $\partial\Sigma$ is the boundary of the worldsheet of the string. It is this result that proves that the energy central charge is simply given by the length of the magnon, and indeed it is clear how the association of k, k^* with the components of a vector corresponding to the magnon on the complex plane is realized.

2.5.2.6 Algebra Equivalence

The authors of [4] have argued that the algebra relevant for the description of the $2+1$-dimensional string theory on the LLM plane is identical to the extended $SU(2|2)$ algebra of [16]. In the $2+1$-dimensional super-Poincare algebra, the string winding charges k^1, k^2 play the role of spatial momenta, and are the analog of the novel central charges in the gauge theory algebra. Thinking of the stretched string (the magnon) as specifying a vector of size

$$k^1 + ik^2 = i\frac{\sqrt{\lambda}}{\pi}e^{i\frac{p}{2}}\sin\frac{p}{2}$$

where these charges are supposed to specify momenta, one would expect that the exact result (2.72) is implied by the usual relativistic dispersion relation - however, Lorentz invariance is not a symmetry of the problem, but rather an outer automorphism of the algebra. The result (2.72) must thus be understood to arise by analysis of the supersymmetry algebra, exactly as in the gauge theory[16], and is thus a BPS formula. The 1 under the square root in (2.72) which is missing for the classical string energy formula (2.75) can also be recovered in the small momentum, strong coupling limit (enforced in the derivation) by employing a plane wave approximation; with this in place, quantization leads to the reappearance of the 1 in the string theory formula[46, 47].

2.5.2.7 Semi-classical S-matrix

The S-matrix was determined by symmetry arguments (up to a phase) for the dual gauge theory system in [16] (see Section 2.5.1.5); as has been pointed out, the symmetry algebra under which the system on the string theory side transforms is identical at the classical level to that under which this S-matrix was derived. It is natural to assume that sigma model quantization[8] will not modify the algebra, and therefore that the symmetry constraint leading to the expression for the S-matrix will be unchanged. It can thus be expected that the S-matrix for the string magnons will admit the same structure as in the gauge theory case, with only the phase S_0 being modified - this phase should be responsible for interpolating between weak and strong coupling. The scattering kinematics can be illustrated explicitly by showing the transition of the worldsheet projection (the magnons on the LLM plane) under a scattering; after the transition, the two magnons which were scattered exchange their momentum and thus their central charges, leading to a closed polygon with the same set of edges, but with two of the line segments having exchanged places. A particular magnon state is specified by giving its momentum p. Under a scattering, the phase associated with each involved magnon (since a magnon is identified by its momentum) will change, corresponding to its change in location relative to the rest of the configuration. It is therefore important to keep track of the orientations of the magnons under a series of scattering interactions.

[8]This corresponds to the inclusion of non-classical trajectories in the worldsheet path integral sum, arising due to the addition of corrections in the string tension.

The authors of [4] have calculated the leading contribution to the semi-classical S-matrix at strong coupling, which in fact comes from the phase S_0, by mapping to a similar problem in Sine-Gordon theory. The resulting picture thus obtained agrees with the expectation from the gauge theory analysis when expressed in convenient coordinates: the magnons undergo solitonic scattering, and thus simply exchange positions along the string worldsheet. The same picture of the scattering was presented from the gauge theory perspective in [44].

2.5.2.8 Discussion

By considering a suitably simple description of elementary excitations on a closed string in $AdS_5 \times S^5$, the authors of [4] postulate a string configuration which is a natural candidate for identification with the magnons of the dual spin chain. After deriving the equations of motion for the proposed dual state, an identification of an angle in the string geometry with the momentum of the string excitation exactly reproduces the strong coupling limit of the exact magnon energy formula (2.72). In suitable coordinates, the solution permits a simple geometric description. An analysis of the symmetries explains the appearance of the extra central charges from the string theory side, and identifies them with geometric quantities associated to the drawing of the solution on the LLM plane. Despite the huge differences between the technical and conceptual details of the dual theories, the exact matching of the supersymmetry algebras for both systems provides the explanation as to why they can be used to study the same systems. Of course, the situation in which this equivalence arises represents a very special, highly symmetric sector of both theories, and we cannot claim this as a proof of the fundamental principle involved. However, this still illustrates beautifully the motivation for considering symmetry and its application to group theory as the prime candidate for encoding the nature of the universe - while vastly many other details of equivalent descriptions of a system may diverge, the symmetry of the problem captures the underlying equivalence.

2.6 Giant Gravitons

The first description of giant gravitons appeared in [48]. They are spherical $D3$ branes which orbit in either of the two component factors of the $AdS_5 \times S^5$ background (although initially only the aptly named sphere giants were known, AdS counterparts were quick to follow in [49] and [50]), and are the result of the expansion of point gravitons due to the presence of the background Ramond-Ramond five form flux, by a process analogous to the Myers' effect for dielectric branes [51]. This section is intended as a brief summary of the fundamentals of the theory of giant gravitons.

2.6.1 The Stringy Exclusion Principle

2.6.1.1 Development

A study of a particular example of the AdS/CFT correspondence relating near-horizon microstates of black holes (obtained as orbifolds of a subset of AdS_3) to the states of a conformal field theory [52] led to the discovery that there exists an upper bound on the BPS particle number in the space, following from the unitarity of the superconformal algebra. The requirement of unitarity implies that states in the theory must have positive norm; if one observes the norm of states in our Super Yang-Mills theory as their energy is raised, it is found that the norm remains positive up to a certain point, after which it becomes negative - these states of negative norm are not valid for the theory, and hence must be excluded. The existence of this upper bound is commonly referred to as the Stringy Exclusion Principle. The physical origins of this principle had never been satisfactorily clarified, but they were long thought to be associated with physics at very small distance scales. Although the results were obtained in the context of an AdS_3 geometry, the Stringy Exclusion Principle is a property of any theory where operators are built out of matrices. This can be shown by considering the trace of powers of matrices - suppose we consider an $N \times N = 2 \times 2$ matrix X with eigenvalues λ_1 and λ_2, then:

$$\begin{aligned} \mathrm{Tr}(X) &= \lambda_1 + \lambda_2 \\ \mathrm{Tr}(X^2) &= \lambda_1^2 + \lambda_2^2. \end{aligned}$$

It is useful for us to consider traces, since these are the natural observables of the SYM theory - in order to be gauge invariant, observables in this theory must be invariant under the action of multiplication by unitary matrices, i.e. the gauge transformation is of the form $Z \to UZU^\dagger$. The fact that traces are cyclic implies this invariance for operators built out of traces. It is now a simple algebraic matter to determine that $\mathrm{Tr}(X^3)$ can be written in terms of the first two traces - the resulting equations are referred to as *trace relations*. Their existence implies that the state to which the trace of $X^3 = X^{(N+1)}$ corresponds is not in fact a new state, but rather a bound state of the other two systems. This can be logically extended to the traces of all higher powers of X, so that we see the natural emergence of an upper bound on the number of particle states arising due to the fact that we are working with matrices. Upper bounds on possible particle states were previously encountered in other string theoretical contexts, including the case of the duality between IIB strings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM.

2.6.1.2 Invasion of the giant gravitons

The result of [48] is the acquisition of a new perspective on this matter, one in which the principle emerges as a macroscopic effect, and the physical meaning of the bound is clear. It is considered that the massless single particle states to which the bound is applicable (i.e. the gravitons) in the S^5 component of the background expand as their angular momentum is increased. The 5-sphere has a fixed radius, so that the expansion must stop when the radius of the graviton matches that of the sphere. This cut-off in angular momentum was found to agree with the predictions of the stringy exclusion principle, thus validating the theory. Since the Kaluza-Klein (ordinary point-like) graviton is a BPS state, its transformation from point to membrane should not change its energy, and it would be expected that the energy calculated at a given momentum under this interpretation should match the energy of a KK graviton having that same momentum. The fact that these were found to match classically for the case of maximal angular momentum lends further credit to the theory.

In order to best understand how this process occurs, it is useful to review another case where particles undergo spatial extension proportional to their angular momentum - non-commutative field theories. These are field theories in which the operators corresponding to the spacetime coordinates do not commute with each other: $[x^\mu, x^\nu] = i\theta^{\mu\nu}$. The non-vanishing commutator between the coordinate function implies that the geometry has a "fuzzy" structure - in analogy with the Heisenberg uncertainty principle, which results from the non-commutativity of momentum and position in a quantized theory, one finds that positions can only be measured exactly along a single particular axis. Once this measurement is specified, there is some measure of uncertainty, encoded in the quantity on the RHS of the commutator, in the position measurement along any other axes. The basics of the theory are embodied in the case of a dipole moving through a magnetic field, which we can define on the surface of a 2-sphere to observe the emergence of an angular momentum bound corresponding to the separation of the ends of the dipole to the antipodes of the sphere. Below is a short summary of the physics of this situation:

Dipole Moving in a Magnetic Field

The setup we consider consists of a pair of unit charges of opposite sign that are moving on a plane in a constant magnetic field B. The Lagrangian is given as:

$$\mathcal{L} = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{B}{2}\epsilon_{ij}(\dot{x}_1^i x_1^j - \dot{x}_2^i x_2^j) - \frac{K}{2}(x_1 - x_2)^2$$

Coulomb and radiation contributions are assumed to be negligible. The terms correspond respectively to kinetic energy, interaction with the background magnetic field and the harmonic potential between the charges. We make the approximation that the identical particle masses are very small, so that the first term vanishes. Note that, since we are assuming negligible mass, we should in fact perform a relativistic analysis. It would also be more correct to perform the analysis, and then set the mass equal to zero. However, this happens not to affect the outcome, and so this section will be presented as it was read in [48]. It is useful to introduce centre of mass and relative coordinates, defined as:

$$X = \frac{(x_1 + x_2)}{2}$$
$$\Delta = \frac{(x_1 - x_2)}{2}.$$

Applying these approximations and the change of coordinates to the Lagrangian we obtain:

$$\mathcal{L} = B\epsilon_{ij}\dot{X}^i\Delta^j - 2K\Delta^2.$$

Using this equation to calculate the commutator $[P, X]$, together with the known fact that this must equal $-i$ (when $\hbar$ is set to one), it is possible to determine that the operators X and Δ are non-commuting, and satisfy the relation

$$[X^i, \Delta^j] = i\frac{\epsilon_{ij}}{B}.$$

The centre of mass momentum conjugate to X is given by

$$P_i = \frac{\partial\mathcal{L}}{\partial\dot{X}^i} = B\epsilon_{ij}\Delta^j.$$

Noting that the coordinate Δ gives the position of the particles relative to each other, we can rearrange this equation and take absolute values to obtain a formula for the distance between the particles:

$$|\Delta| = \frac{|P|}{B}. \tag{2.84}$$

The particles thus separate in the direction perpendicular to the momentum vector by an amount linearly proportional to the momentum of the dipole.

We now imagine that we place the dipole on the surface of a sphere of radius R, which has magnetic flux N. This is equivalent to the statement that we place a magnetic monopole of strength $2\pi N = \Omega_2 BR^2$ at the centre of the sphere. In this arrangement, by symmetry, the centre of mass of the dipole will remain on the equator as the components of the dipole separate. By simply glancing at (2.84), we know that the dipole should be as big as the sphere when its momentum is about $2BR$. This corresponds to an angular momentum $L = PR \sim BR^2$. Comparing with the expression for the flux N, we see that the angular momentum can be said to be $O(N)$. A more precise analysis of the situation yields the result we want - the maximum angular momentum of the dipole is in fact exactly equal to the flux:

$$|L_{max}| = N.$$

This was demonstrated in [48]. It should be noted that this maximum occurs when the dipole ends sit at opposite poles of the sphere, where they are in fact stationary - this may seem odd, but one must recall that the magnetic field carries angular momentum as a result of its coupling with the dipole charges - the angular momentum is in fact associated with the field itself. The fact that the angular momentum of a single field quantum moving in a spherical space in a non-commutative field theory is bounded by N is well-known[53].

Dielectric Branes

A brief review of the methods presented in [48] for the case of $AdS_5 \times S^5$ is included following this discussion. The treatment follows a tight analogy with the calculation performed above for the dipole. Before beginning, it is useful to understand how we are able to use such an analogy - in what sense does a $D3$ brane possess a dipole charge? Consider a point particle moving along a worldline described by x^μ with 1-form gauge potential A_μ; the particle couples to the potential via a term in the action of the form

$$e \int A_\mu dx^\mu$$

where e represents the charge or coupling constant. Note that if the parameterization is changed such that $dx^{\mu\prime} = -dx^\mu$, then the particle will appear to couple to the potential with a negative charge. Appropriate convention choice in this regard allows the charge on a particle to be well defined.

We consider a BPS particle moving on the spherical component of the space, where the background tensor field plays a similar role to the magnetic field considered previously. This background field is the Ramond-Ramond field, and since we know that D-branes carry RR charge [32], we must include a term in the action that accounts for the coupling of brane to background field - this can be referred to as the Chern-Simons term, since it resembles the proper Chern-Simons action defined for (2 + 1) dimensional space. A Dp brane is naturally charged under the $(p+1)$-form RR potential $A_{(p+1)}$, so that the Chern-Simons action for a $D3$ brane should have the form:

$$S_{CS} = \mu_3 \int_{D3} A_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \mu_3 \int_{D3} A_{(4)}$$

where μ_3 is the RR charge of the $D3$ brane. We can define a 5-form field strength which is given as an antisymmetrized sum of derivatives of the potential which is a natural generalisation of the expression for the 2-form field strength in electrodynamics. Denote this as:

$$F_{\mu\nu\rho\sigma\tau} = 5\partial_{[\mu} A_{\nu\rho\sigma\tau]}, \quad \text{i.e.} \quad F_{(5)} = dA_{(4)}.$$

The five-form field strength is the exterior derivative of the RR potential. There are a number of ways to choose A such that this relation is satisfied. The one that is of interest to us is that in which variables that effectively parameterize the $D3$ brane appear - the most convenient parameterization of the brane is in terms of t and Ω_3, and the potential can be written:

$$A_{(4)} = BR\frac{r^4}{4} d\phi d\Omega_3 = BR\dot{\phi}\frac{r^4}{4} dt d\Omega_3.$$

Now consider that we begin with a small $D3$ brane and pick a specific point on its surface, and suppose that $d\psi$ is one of the angles parameterizing $d\Omega_3$. If we traverse the surface of the brane in the direction of the infinitesimal vector $d\psi$, we find that by the time we reach the antipode of the original position,

$d\psi$ must have changed sign. Since this is a parameterization variable, this change in sign implies that a negative is picked up by the Chern-Simons term in the action, which can be interpreted as corresponding to a change in the sign of the charge. Thus, the RR charge on the surface of a D-brane appears as being of opposite sign at antipodal points on the brane, and antipodal points will therefore expand in opposite directions as the brane moves through the background. This explains why the dipole analogy is instructive, and why a graviton moving through a background RR potential expands with increasing angular momentum.

Angular Momentum Bound for BPS Particles on $AdS_5 \times S^5$

The supergravity equations of motion give the radius of the 5-sphere as

$$R = (4\pi g_s N)^{\frac{1}{4}} l_s$$

where g_s and l_s are the string coupling constant and string length scale respectively. An exact classical analysis of a $D3$ brane wrapping an S^3 and moving in the 5-sphere background is performed. The bosonic Lagrangian for the system is given as the sum of the Dirac-Born-Infeld (DBI) Lagrangian corresponding to kinetic energy of the $D3$ brane and the Chern-Simons Lagrangian which is associated with the coupling of the brane to the background field:

$$\mathcal{L} = \mathcal{L}_{DBI} + \mathcal{L}_{CS} = -T_{D3}\Omega_3 r^3 \sqrt{1-(R^2-r^2)\dot{\phi}^2} + \dot{\phi} N \frac{r^4}{R^4}.$$

The Dirac-Born-Infeld Action

The derivation of the DBI term in the brane Lagrangian is now presented. Ignoring the world volume gauge field on the $D3$ brane, the DBI Lagrangian is given by the well-known relativistic formula

$$S_{DBI} = -T_{D3} \int_{WV} \sqrt{\det(|g_{ind}|)}.$$

The integral is over the worldvolume of the brane, and g_{ind} is the metric induced on the $D3$ brane worldvolume. We begin by embedding the 5-sphere in a $6D$ Euclidean space parameterized by $X_1, ..., X_6$. The usual coordinate transformation to a set of 5 angles $\theta_1, .., \theta_5$ that satisfies the sphere equation $\sum_{i=1}^{6} X_i^2 = R^2$ is used. The first 4 angles range from 0 to π, while the azimuthal angle θ_5 ranges from 0 to 2π. Since we are interested in the action for a $D3$ brane on the S^5, we now embed a 3-sphere in the space. The surface of the spherical membrane can be parameterized by the angles $\theta_3, \theta_4, \theta_5$, so that the brane is free to move in the $X_1 - X_2$ plane. The radius of the brane depends on its position in this plane according to[9]:

$$r = R \sin\theta_1 \sin\theta_2.$$

It is useful to note that

$$X_1^2 + X_2^2 = R^2 - r^2$$

since this tells us that the brane is free to move in circles on the $X_1 - X_2$ plane without its radius changing, and allows us to write:

$$X_1 = \sqrt{R^2 - r^2}\cos\phi$$
$$X_2 = \sqrt{R^2 - r^2}\sin\phi.$$

The metric of the S^5 is then given by:

$$ds_{S_5}^2 = \frac{R^2}{(R^2 - r^2)} dr^2 + (R^2 - r^2) d\phi^2 + r^2 d\Omega_3^2.$$

The metric is now in the form where the coordinates can be easily interpreted as corresponding to a 3-sphere moving on a 2-disc with radial coordinate r and angular coordinate ϕ. All that is left is to

[9]Note that this limitation on the geometry of the brane is an ansatz introduced by Susskind et. al. in [48]. We are assuming that the giant gravitons are not subject to major deformations from a spherical shape. The equations of motion should, to be rigorous, be calculated first without this assumption, and then have the ansatz plugged in afterwards. However, as is typical of his work, Susskind's informal treatment happens to produce the correct result.

calculate g_{ind}; this is achieved by embedding a 4-dimensional worldvolume, being parameterized by t and the 3 angles specifying the brane, into the $AdS_5 \times S^5$. It is assumed that the only coordinate that varies with time is ϕ, and we obtain:

$$ds^2_{D3} = [(R^2 - r^2)\dot{\phi}^2 - 1]dt^2 + r^2 d\Omega_3^2.$$

The DBI Action is thus given by:

$$S_{DBI} = -T_{D3} \int_{WV} \sqrt{1 - (R^2 - r^2)\dot{\phi}^2 dt^2} \sqrt{r^6 d\Omega_3^2} = -T_{D3}\Omega_3 r^3 \int dt \sqrt{1 - (R^2 - r^2)\dot{\phi}^2}.$$

In this expression, $\dot{\phi}$ is the angular velocity, T_{D3} is the tension of the brane and $\Omega_3 r^3$ is the volume of a 3-sphere ($= 2\pi^2 r^3$). This means that the coefficient of the integral is in fact the mass of the $D3$ brane, as measured by an observer at a point in the AdS_5. The giant graviton in a 10-dimensional geometry is in fact massless, but an observer that is unaware of the S^5 component, as the AdS_5 observer is, would measure a mass. This "mass" is in fact the energy of the brane due to its momentum in the additional dimensions. Since the action is the time integral of the Lagrangian, removing this integral from the above expression gives us $\mathcal{L}_{DBI}$. Terms containing derivatives of r have been dropped since we are interested in the case where the radius is constant and close to maximal.

The Chern-Simons Action

The term of the Lagrangian referred to in [48] as the Chern-Simons term (it is more commonly associated with Wess and Zumino, although the term bears a resemblance to the action of Chern-Simons theory) is defined as:

$$S_{CS} = \int_{WV} \mathcal{P}[A_4].$$

It is the integral over the $D3$ brane world volume of the pullback of the 4-form potential onto the worldvolume. The pullback is defined as (the Y's are the coordinates on the world volume):

$$\mathcal{P}[A_4] = A_{\mu\nu\rho\tau} \frac{\partial X^\mu}{\partial Y^\alpha} \frac{\partial X^\nu}{\partial Y^\beta} \frac{\partial X^\rho}{\partial Y^\gamma} \frac{\partial X^\tau}{\partial Y^\delta} dY^\alpha \wedge dY^\beta \wedge dY^\gamma \wedge dY^\delta.$$

It is apparent that this is analogous to the definition of the induced metric - in fact, the pullback of a metric onto a certain manifold is exactly the induced metric on that manifold. The relation $F_5 = dA_4$ allows us to implement a form of Stokes theorem, since the action of the curl operator naturally arises from the action of d on a 4-form as a result of the antisymmetricity of wedge products. This antisymmetricity is important for the same reason that any form must be antisymmetric - it ensures that the Jacobian for a coordinate transformation comes out correctly, such that the integral of the form is coordinate independent. The manifold we integrate over must satisfy the condition that its boundary is the world volume of the $D3$:

$$S_{CS} = \int_{WV} \mathcal{P}[A_4] = \int_\Sigma F_5.$$

We choose Σ to be a 5 dimensional manifold whose boundary is the 4 dimensional surface swept out by the brane as it completes one orbit of the 2-disk, D_2. The D3 brane is moving at a constant radius from the centre of the D_2, this radius is r. If we consider the motion of the brane through one orbit (ϕ goes from 0 to 2π), it will trace out a 4-dimensional surface that bounds the portion of the D_2 with radius less than r. Σ is the manifold consisting of the 4 dimensional boundary and this region of the disk. We have a constant flux density, so that in analogy with the case of constant magnetic field in electrodynamics, we can define the five-form field strength as $F_5 = BdV$, where dV is the volume form on the 5-sphere. The S^5 is a Riemannian manifold, and the volume form is hence given by $dV = \sqrt{|g|} dr \wedge d\phi \wedge d\Omega_3 = Rr^3 dr \wedge d\phi \wedge d\Omega_3$. The time coordinate that has apparently disappeared is contained in ϕ. We are now in a position to calculate the action; we simply have to integrate the coordinates given in the volume form over Σ.

$$\begin{aligned}
S_{CS} &= \int_{D3} d\Omega_3 \int_0^{2\pi} d\phi \int_0^r dr' BR(r')^3 \\
&= BR\frac{2\pi}{4}\Omega_3 r^4 \\
&= BR\Omega_5 r^4 \\
&= 2\pi N \frac{r^4}{R^4}.
\end{aligned}$$

The recursion relation for the hyperarea coefficient of an n-sphere ($\Omega_n = \frac{2\pi}{n-1}\Omega_{n-2}$) was used in the second last step, and the last step used the flux quantization condition ($\Omega_5 R^5 B = 2\pi N$). The Chern-Simons Lagrangian can be obtained from the action by dividing by the period of a single orbit (since the time integral which must be performed to obtain the action from this expression contributes a T):

$$\mathcal{L}_{CS} = \frac{S_{CS}}{T} = \frac{\dot{\phi}}{2\pi} S_{CS} = \dot{\phi} N \frac{r^4}{R^4}.$$

Angular Momentum and Energy

The tension of the brane is given in 10 dimensional Planck units by

$$T_{D3} = \frac{1}{(2\pi)^3 l_s^4 g_s}.$$

Combining the equations for the brane tension and radius of the sphere we obtain the relation

$$T_{D3}\Omega_3 = \frac{N}{R^4}.$$

We can thus obtain the angular momentum from the Lagrangian:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{m\dot{\phi}(R^2 - r^2)}{\sqrt{1 - \dot{\phi}^2(R^2 - r^2)}} + N\frac{r^4}{R^4}.$$

We introduce the parameter $m = T_{D3}\Omega_3 r^3 = (\frac{N}{R^4})r^3$, which is the mass of the $D3$ brane. Since a radius r that is greater than R would produce a non-physical result, we see that the radius is bounded by $0 \leq r \leq R$. We must also include the constraint that the linear velocity cannot exceed the speed of light, that is $0 \leq \dot{\phi} r \leq 1$. Plugging the maximal radius into the angular momentum equation, we find the limit of the angular momentum:

$$L_{max} = N$$

This bound is applicable in the regime where $N \gg 1$. We can determine the energy of the brane to be given by

$$E = \sqrt{m^2 + \frac{(L - N\frac{r^4}{R^4})^2}{R^2 - r^2}}.$$

Analysing the variation of the energy with respect to r at a constant angular momentum, a stable minimum is found when

$$r^2 = \frac{L}{N}R^2 \tag{2.85}$$

indicating the expansion of the brane with increasing angular momentum. The existence of the minimum implies the existence of a stable brane configuration, at least in the classical regime. A quantum mechanical analysis may reveal the possibility of tunnelling, which would compromise the stability of the configuration. Inserting the expression for r at this minimum into the energy equation, and applying suitable approximations for the $N \gg 1$ limit, we obtain $E = \frac{L}{R}$; this is the energy of the brane for all values of angular momentum - indeed, it is the energy of a Kaluza-Klein graviton at angular momentum L.

The angular momentum bound $L_{max} = N$ is in perfect agreement with the bound enforced by the Stringy Exclusion Principle for a BPS particle. It is interesting to note that, since the angular momentum cannot become infinitely large, Heisenberg's uncertainty principle implies that we can never truly resolve positions to a single point - the geometry is "fuzzy". The implications of this may be interesting to consider, but do not affect the calculations presented in this dissertation. The trivial minimum at $r = 0$ corresponds to the ordinary point-like graviton solution, but this solution is subject to uncontrolled quantum corrections, since it describes a huge energy concentrated at a single point.

Thus we have obtained the angular momentum bound expected from the Stringy Exclusion Principle by envisioning the particle as expanding into a $D3$ brane at high angular momentum. The bound is then a result of the limit on the degree of expansion due to the finite size of the 5-sphere in which it is moving. The argument outlined above provides substantial evidence that for KK gravitons at large angular momenta, a description in terms of expanded $D3$ branes is preferable to one in terms of strings.

2.6.1.3 Discussion

The results obtained by McGreevy, Susskind and Toumbas, while they ingeniously recreate the physics expected by the Stringy Exclusion principle, were in no way rigorously tested nor did they admit much generality. The assumptions made were numerous and restricting - the situation considered takes into account only exactly spherical gravitons of constant radius. In general, the coordinates ϕ and r should both be functions of time and the world volume coordinates, and a concrete test of the proposal would be to determine if the giant graviton appears as an exact solution of the resulting equations of motion. However, perhaps by luck or possibly powerful intuition, the results obtained by their minimally technical derivation turn out to be correct. Several more detailed calculations have been performed which show the validity of their interpretation of the graviton as expanding into a brane at large angular momentum. It should also be noted that the Lagrangian derived contains only terms corresponding to bosonic interactions, however, a study of the supersymmetry of the giant graviton allows for the reappearance of the fermionic fields. An analysis of the supersymmetries admitted by giant gravitons in various $AdS \times S$ backgrounds was performed in [50], where it is confirmed that the giant gravitons in fact preserve all the same supersymmetries as the point-like graviton. This is expected, since from the point of view of an observer doing supergravity calculations in the AdS_5 the sphere giant graviton is a massive state with charge equal to its mass, and therefore one can determine that they satisfy the appropriate BPS bound (the same bound as the point gravitons).

2.6.2 Branes in AdS_5

Soon after Susskind and friends released their paper introducing sphere giants to the world, several authors (notably those of [49] and [50]) discovered that the gravitons could also expand in the AdS_5 component of the background. This is seen by following a treatment very similar to that used for the sphere giants; a spherical $D3$ brane wrapping the Ω_3 of the AdS_5 background is embedded into the spacetime. The same ansatz of constant radius and time-dependence being limited to the coordinate ϕ is substituted into the action, and we consider the same brane configuration of a giant orbiting along the equator of the S^5. The Lagrangian of this brane configuration is of the form:

$$\mathcal{L} = -T\Omega_3 R^4 \left(\tan^3\rho \sqrt{\sec^2\rho - \dot{\phi}^2} - \tan^4\rho \right).$$

The energy can then be calculated and is given by:

$$E = N \left(\sec\rho \sqrt{\frac{L^2}{N^2} + \tan^6\rho} - \tan^4\rho \right).$$

The energy has local minima at $\tan\rho = 0$ and $\tan\rho = \sqrt{\frac{L}{N}}$, for which the energy takes value $E = L$. The existence of these minima establishes that there exists a stable configuration of a spherical $D3$ brane embedded in the AdS_5. These states are labelled by exactly the same quantum numbers as the sphere giant states and the point graviton states. The AdS-branes are often referred to as "dual" to their S^5 counterparts, in the sense that the AdS-branes couple electrically to the RR field, and can be thought of as dielectric branes, while the S-branes couple magnetically and can be called dimagnetic. The authors of [50] also analysed the supersymmetry of the giants expanding in the AdS_5 component of the spacetime, and found them to preserve the exact same supersymmetries as the point-like and sphere giant gravitons.

We thus are in a situation where there are three distinct brane configurations (The AdS giant, S giant and point graviton) all of which share the same quantum numbers, and quantum mechanics leads us to expect these three states will mix. It therefore seems prudent to seek instanton solutions describing tunnelling between these states. Explicit expressions for the instantons evolving between the S-giant and point-like state, as well as between the AdS-giant and point-like graviton have been derived in [50], and were found to be $\frac{1}{4}$-BPS states, preserving 8 of the 32 supersymmetries. Instantons involved in direct transitions between the two types of brane gravitons have been sought, but numerical simulations have been performed ([54]) that demonstrate that the direct tunnelling solution does not exist - tunnelling between AdS- and S- giants is a two-instanton transition, with the point graviton acting as an intermediate state. This does not mean, however, that the transition is possible; as stated, the point graviton solution is unstable due to the massive gravitational field resulting from the concentration of energy at a single point. This transition may only be mathematically possible, and does not necessarily have a physical interpretation. We can visualize the transition by considering a plot of the potential as function of radius, which will be a triple potential well, symmetric about the vertical axis, with one of the minima at $r = 0$. The outer minima correspond to the stable giant graviton configurations, one being the sphere giant and the other the AdS, while the centre minimum is the point graviton.

We should note at this point that no bound on the expansion of AdS giants has been observed nor required in any of the analyses performed during the study of systems involving giant gravitons using the methods presented in this dissertation. One can construct operators in the field theory with any large value of $\mathcal{R}$-charge, corresponding to AdS giants with any momentum in the string theory, and perform the same computations and manipulations as in the case that the momentum respects the bound in the AdS directions. In light of results presented in this dissertation, which provide a clear pictorial description of the giants configuration in the geometry on both sides of the AdS/CFT correspondence, it seems reasonable that giants may expand indefinitely in the AdS directions. The stringy exclusion principle for AdS giants has a natural interpretation in our framework: it enforces a limit on the number of AdS giants that can exist. This is a result of the flux available in any given system; no more than N AdS giants may occur, since there will be no flux available to support their existence.

One may also care to note that the two types of branes collapse into the same point graviton state. This is motivated by the matrix description of M-theory in light-cone gauge, where different branes with different geometries can be represented using non-commutative geometry, but when any of the branes are shrunk to zero size, the same state emerges. This state is described by a set of commuting matrices, the entries of which are independent of which geometry it emerges from.

2.7 Schur Polynomials

2.7.1 Definition

These operators are constructed from complex combinations of the six scalar adjoint Higgs fields of the super Yang-Mills theory, and labelled by Young diagrams corresponding to irreducible representations of the symmetric group. The reason we must use complex combinations is to prevent mixing between the component scalar fields, thus causing the expectation value of the Z's and Y's with themselves to vanish. If these expectation values did not vanish, we would be required to Wick contract these fields within a single Schur polynomial, resulting in the occurrence of UV divergences that we would have to resolve. Schur Polynomials are built using a single complex combination of two of the scalar adjoint Higgs fields of the SYM theory([55]); for example, if we define $Z = \phi_1 + i\phi_2$, then the operator has the form

$$\chi_R(Z) = \frac{1}{n!}\sum_{\sigma \in S_n} \chi_R(\sigma)\mathrm{Tr}(\sigma Z^{\otimes n}) \tag{2.86}$$

where R is a Young diagram, having n boxes and at most N rows (N is the rank of the gauge group of the Yang-Mills theory), and $\chi_R(\sigma)$ is the character of σ in the representation R, being the trace over the matrix representation in the carrier space of R of the group element σ. For the purposes of this dissertation, the useful Schur polynomials are those which generate a sum over all possible multi-trace structures of $O(N^2)$, $O(N)$ or $O(\sqrt{N})$ fields, with particular limitations on the structure of the Young diagram labels being enforced. These operators correspond to the ground state of certain dual string theory systems - the particular system described depends on the number of fields and the structure of the Young diagram.

2.7.2 A Generalization: Restricted Schur Polynomials

2.7.2.1 Definition

A generalization of these operators, the Restricted Schur Polynomials, has been useful in describing excitations of the dual string theory systems. Originally this was achieved for systems of open strings suspended between giant gravitons by inserting open string words into the operator with $O(N)$ fields which describes the giant graviton system (corresponding to excitations arising by the attachment of strings to the surface of the ground-state giant)([13]). In this approach, the relevant restricted Schur polynomials are defined in terms of a restriction of the full symmetry group generating all possible trace structures for the involved fields, S_{n+m} (where n is the number of fields comprising the giants, and m is the number of string words attached), to the subgroup $S_n \times (S_1)^m$. This is the relevant subgroup in this case, since the string words are assumed to be distinguishable. The subduction multiplicity for this restriction is naturally enumerated by specifying a pair of partially labelled Young diagrams (See Appendix A); these labels additionally supply physical data when one recognises that each can be associated with a specification of the configuration of one of the endpoints of the open string word - this correspondence is demonstrated with a figure in Section 3.2. The restricted Schur polynomial relevant for this description is thus given by

$$\begin{aligned}&\chi_{R,R_1^k,R_2^k}(Z,\{\{n_i\}_1,\{n_i\}_2,\cdots,\{n_i\}_k\})\\ &=\frac{1}{n!}\sum_{\sigma\in S_{n+k}}\chi_{R,R_1^k,R_2^k}(\sigma)Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}(W_k)^{i_{n+1}}_{i_{\sigma(n+1)}}\cdots(W_2)^{i_{n+k-1}}_{i_{\sigma(n+k-1)}}(W_1)^{i_{n+k}}_{i_{\sigma(n+k)}}\end{aligned} \tag{2.87}$$

where $\chi_{R,R_1^k,R_2^k}(\sigma)$ is the restricted character (explained under the following heading), and the open string words are

$$(W_I)^i_j = (YZ^{n_1}YZ^{n_2-n_1}Y\cdots YZ^{n_{M_I}-n_{M_I-1}}Y)^i_j \tag{2.88}$$

where the n_i specific to word I are defined by the operator label $\{n_i\}_I$. It is these operators which are used in Chapter 3, where we revisit the attachment of open string words to systems of giant gravitons.

Much effort was then dedicated to understanding systems where the strings attached consisted of only a single field, rather than a string word; these systems correspond to the situation where it is the excitations themselves that connect between giants, rather than a more complicated string state. The study of these configurations required an advancement in the understanding of the underlying symmetries of the problem, which has led to numerous interesting and important identities and results being formulated. In this case, we define the restricted Schur polynomial in terms of some number of different complex scalar field combinations (for our purposes, we work with models in which only two of the field combinations participate; we choose $Z = \phi_1 + i\phi_2$ and $Y = \phi_3 + i\phi_4$) , yielding a multi-matrix model. The dual interpretation of these added "impurities" was not fully appreciated until the powerful symmetry arguments of Beisert published in [16], together with the elegant geometric picture of the dual string physics presented by Hofman and Maldacena in [4], concretely identified them as excitations propagating on the worldsheet of a closed string. One of the results of this dissertation is an extension of these results to open strings which connect a general system of giant gravitons; we can now state unequivocally that the impurities correspond to excitations which propagate on the worldvolume of the system of giant gravitons. The impurities are exactly the giant magnons of the previous section.

The restricted Schur polynomials relevant for this approach are labelled by a set of 3 Young diagrams:

$$\chi_{R,(r,s)jk}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s)jk}(\sigma) \mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}).$$

R labels an irreducible representation of S_{n+m}, r an irreducible representation of S_n and s an irreducible representation of S_m. The latter two Young diagrams together label an irreducible representation of $S_n \times S_m$. R may in general subduce the same irreducible representation of the $S_n \times S_m$ subgroup more than once - the indices jk are multiplicity labels which keep track of these copies, necessary when studying systems containing more than 2 giant gravitons. Restricted Schurs provide a complete basis for gauge invariant operators built from the Higgs fields due to the fact that any linear combination of multitrace operators can be written as a linear combination of the Schur Polynomials. This basis of operators is used in Chapter 4; in the conclusions, Chapter 5, a brief (but hopefully convincing) description of how we expect that the results of Chapter 3 can be re-derived using similar methods is presented. Analogous to the character appearing in the Schur polynomial, we define a *restricted character* $\chi_{R,(r,s)jk}(\sigma)$.

2.7.2.2 The Restricted Character

A brief overview of what it means to take a restricted trace in either of the descriptions[10] described above is now given. The restricted character is defined as the trace over the carrier space of a representation of the subgroup to which we restrict - the subspace is specified by the labels R_1^k, R_2^k in the first description, and by $(r,s)jk$ in the second. This corresponds to a trace over the subspace of the vector space associated with R. It is here that the multiplicity indices are important; the trace must be performed over the space associated with the correct copy of the subgroup representation. Consider this for the second description: suppose that upon restricting R to the $S_n \times S_m$ subgroup, we have irreducible representation (r,s) subduced once, and irreducible representation (t,u) is subduced twice. The matrix representation $\Gamma_R(\sigma_{nm})$ of an $S_n \times S_m$ group element σ_{nm} can then be written in a suitable basis in block diagonal form, with the diagonal blocks being the matrix representations of the subgroups:

$$\begin{bmatrix} \Gamma_{(r,s)}(\sigma_{nm})_{i_1 j_1} & 0 & 0 \\ 0 & \Gamma_{(t,u)}(\sigma_{nm})_{i_2 j_2} & 0 \\ 0 & 0 & \Gamma_{(t,u)}(\sigma_{nm})_{i_3 j_3} \end{bmatrix}, \qquad \sigma_{nm} \in S_n \times S_m.$$

In this case, the restricted character can be obtained simply by summing over the diagonal elements of the matrix on the diagonal corresponding to the subgroup representation we want. It would seem that only one multiplicity index is required - to specify which copy of (t,u) we want to trace over. However, not all S_{n+m} elements that are summed over will be members of the $S_n \times S_m$ subgroup, and it will not

[10]The $S_n \times (S_1)^m$ operators do not satisfy a completeness relation, and thus do not technically provide a basis of operators.

generically be possible to block diagonalize the matrix $\Gamma_R(\sigma)$:

$$\begin{bmatrix} A^{(1,1)}_{i_1j_1} & A^{(1,2)}_{i_1j_2} & A^{(1,3)}_{i_1j_3} \\ A^{(2,1)}_{i_2j_1} & A^{(2,2)}_{i_2j_2} & A^{(2,3)}_{i_2j_3} \\ A^{(3,1)}_{i_3j_1} & A^{(3,2)}_{i_3j_2} & A^{(3,3)}_{i_3j_3} \end{bmatrix}, \qquad \sigma \notin S_n \times S_m.$$

We now see the need for two multiplicity indices. The restricted trace over the subspace (t,u) can conceivably be performed over any of the 4 lower right entries of the above matrix. To compute the restricted character $\chi_{R,(r,s),jk}(\sigma)$, we trace the row index of $\Gamma_R(\sigma)$ only over the subspace associated to the j^{th} copy of (t,u) and the column index over the subspace associated to the k^{th} copy of (t,u). When performing the "trace" over the carrier space of (t,u) the row and column indices can come from different copies of (t,u) so that if $i \neq j$ we are not in fact summing diagonal elements of $\Gamma_R(\sigma)$. Operators constructed by summing these "off diagonal" elements are needed to obtain a complete basis of local operators [56].

In the case of the first description, we again give two labels, which each can be associated with a different copy of the $S_n \times (S_1)^m$ subgroup. Knowing the order in which the boxes of R are removed is all that is needed to uniquely specify a copy of the subduced representation. $\Gamma_R(\sigma)$ ($\sigma \in S_{n+m}$) will again not be generically block diagonalizable, and the trace may be performed over off-diagonal block matrices corresponding to subduced copies of the subgroup. The two labels fulfil the same role as the two indexes in the second description - they specify the row and column of the matrix representation $\Gamma_R(\sigma)$ over which the restricted trace must be applied.

The restricted character can be obtained from the normal character by the introduction of intertwining operators, $P_{R\to(R_1^k,R_2^k)}$ or $P_{R\to(r,s)jk}$, which take a matrix representation of the group element in the carrier space of R, and project it to the carrier space of the (r,s) subgroup. They obey

$$\begin{aligned} \Gamma_{(r,s)j}(\sigma)P_{R\to(r,s)jk} &= P_{R\to(r,s)jk}\Gamma_{(r,s)k}(\sigma) \qquad \sigma \in S_n \times S_m \\ \Gamma_{(r,s)l}(\sigma)P_{R\to(r,s)jk} = 0 &= P_{R\to(r,s)jk}\Gamma_{(r,s)q}(\sigma) \quad \sigma \in S_n \times S_m \quad l \neq j, \quad k \neq q. \end{aligned}$$

We can write the restricted characters in terms of these operators as

$$\chi_{R,R_1^k,R_2^k}(\sigma) = \text{Tr}\left(P_{R\to(R_1^k,R_2^k)}\Gamma_R(\sigma)\right)$$

or

$$\chi_{R,(r,s),ji}(\sigma) = \text{Tr}\left(P_{R\to(r,s)ji}\Gamma_R(\sigma)\right).$$

When there are no multiplicities, these are honest projection operators which project from the carrier space of R to the relevant subspace. When there are multiplicities they are actually intertwiners[57] - we are projecting onto one of the copies of a subspace, and the non-zero component of the matrix action of the operator will not necessarily be on the diagonal (see Section 2.8.3.1). However, it is constructed in essentially the same way as a projector and satisfies very similar identities. For these reasons we will sometimes be guilty of an abuse of language and refer to these simply as projectors even when there are multiplicities. These operators are not easy to construct explicitly, particularly in the second description, and this is the most significant obstacle when working with the restricted Schur polynomials. The new version of Schur-Weyl duality presented in [2] provides an efficient, transparent method by which the operators can be built in the $S_n \times S_m$ basis.

2.7.2.3 The Multiplicity Problem

The multiplicity indices jk appearing in the $S_n \times S_m$ description must be chosen such that they take the correct values to organise the multiplicities arising by the subduction of $S_n \times S_m$ representations for systems containing $p > 2$ giant gravitons. A proposal to resolve these multiplicities was given in [58], where they are labelled by the eigenvalues of the Cartan subalgebra of elements in the group algebra CS_{n+m} which are invariant under conjugation by $CS_n \times CS_m$. However, in the article [2], a much simpler method was shown to manifest itself by considering a novel application of Schur-Weyl duality.

The usual application of Schur-Weyl duality is in the construction of projectors onto good $U(p)$ irreducible representations using the Young symmetrizers i.e. by symmetrizing and antisymmetrizing indices on a tensor. The use of the duality as a means by which we can construct the intertwining operators appearing in the restricted Schur polynomials, while also resolving this multiplicity problem, turns this argument on its head: by using the irreducible representations of the unitary group, it is possible to build symmetric group projectors.

2.7.3 Schur Polynomials as Duals to Giant Graviton Systems

Many reasons to consider the Schur Polynomials defined by (2.86) as dual to systems of giant gravitons have emerged, and in many cases it occurs that the mathematics associated to the description is completely tractable, even simple in some cases. It was shown in [55] that the space of $\frac{1}{2}$-BPS representations in $\mathcal{N} = 4$ Super Yang-Mills is in one-to-one correspondence with the space of $U(N)$ Young diagrams, and hence with the single matrix Schur Polynomials being labelled by these diagrams. Insights gained by studying the dual quantum gravity have revealed that it is sensible to identify excitations of these $\frac{1}{2}$-BPS states with restricted Schur Polynomials [59]. Most importantly, it was shown that the Neumann and Dirichlet boundary conditions of the D-brane excitations (being that they can be described as strings propagating on the brane) must emerge dynamically in the Yang-Mills theory. This was done in the plane wave limit, using operators dual to giant gravitons that are defined as subdeterminants of one of the complex scalar fields, with the attached string world-volume built from another complex scalar and an impurity (corresponding to further oscillator excitations of the string) inserted into this string of scalars, which together represent the excitation ([60]):

$$\begin{aligned}\hat{O}_k^{Z,Y,X} &= \epsilon_{i_1 \ldots i_N}^{j_1 \ldots j_N} Z_{j_1}^{i_1} \cdots Z_{j_{N-1}}^{i_{N-1}} (Y^k X Y^{J-k})_{j_N}^{i_N} \\ \hat{O}_k^{Z,Y,X} &= \epsilon_{i_1 \ldots i_N}^{j_1 \ldots j_N} Z_{j_1}^{i_1} \cdots Z_{j_{N-1}}^{i_{N-1}} (Y^k Z Y^{J-k})_{j_N}^{i_N}.\end{aligned}$$

The first operator corresponds to a fluctuation parallel to the brane, and hence Neumann boundary conditions emerge, while the second operator corresponds to transverse fluctuations which produce Dirichlet boundary conditions. The matrix of normalized two point correlators of these operators was calculated, where the indices i, j of the matrix M_{ij} correspond to the position of the impurity within the string. Upon diagonalizing this matrix, one can obtain a basis of energy eigenstates, the form of which reflects the emergence of the desired boundary conditions. The authors of [59] go on to argue that the formalism developed using these operators is equivalent to one arising by the use of operators labelled by Young diagrams corresponding to irreducible representations of the symmetric group - these are basically the Schur Polynomial operators that are the focus of this dissertation.

It is simply shown that the Z's and Y's are each of dimension 1, and contribute one unit of angular momentum in the $1-2$ and $3-4$ plane respectively to the giant graviton system. With this association of the fields to angular momentum, and knowing that each box in R corresponds to a single field, we can infer an association of the Schur polynomial labels with giants expanded in either the AdS_5 or the S^5 component of the dual geometry. The 5-sphere has a certain radius (R_{S^5}), and hence gravitons expanding within it have a maximal size, bounded by N due to the Stringy Exclusion Principle. The relevant relation is

$$R = \sqrt{\frac{J}{N}} R_{S^5}$$

where R is the radius of the giant graviton, and J is angular momentum. It is clear then that it is sensible to label Schur polynomials describing sphere giant gravitons by a Young diagram having long ($O(N)$ boxes) columns, since this implements a natural bound on the angular momentum by the properties of the Young diagram. It also seems natural to associate those Schurs labelled by Young diagrams with long rows as AdS giants, since there is no bound on the radius of these giants and also no bound on the length of a row. This solves an important problem in the theory of giant gravitons as it provides a method by which the three different giant graviton states discussed previously can be distinguished from the point of view of the boundary field theory.

2.7.4 Restricted Schur Polynomials as duals to String-Giant Configurations

The two-point function of the single-matrix Schur polynomials was found to be diagonal in [55]. This result was extended to restricted Schurs - their two-point correlator was determined in the free field limit with all Feynman diagrams, non-planar included, summed over. The resulting expression is [61]:

$$\langle \chi_{R,(r,s)jk}(Z,Y)\chi_{T,(t,u)lm}(Z,Y)^\dagger\rangle = \delta_{R,(r,s)T,(t,u)}\delta_{kl}\delta_{jm}f_R\frac{\text{hooks}_R}{\text{hooks}_r\text{hooks}_s}.$$

f_R is the product of factors in Young diagram R, and hooks_R is the product of the hook lengths. Classically then, we see that there is no mixing between restricted Schurs being labelled by differing Young diagrams. The two point correlators of restricted Schur polynomials were calculated to one-loop level in [15], by associating a Cuntz oscillator chain state to each restricted Schur polynomial, and it was found that mixing of the operators at this level is highly constrained: only restricted Schurs with Young diagram labels that differ by the placement of a single box have non-vanishing two-point functions at this level. It has also been shown [56], by considering the restricted Schurs to have a partonic structure and generalizing the known product rule for single-matrix Schurs, that any higher-point correlator of these operators can be expressed as a sum of two-point correlators.

These are all important mathematical identities which provide evidence that the restricted Schur polynomials are a natural generalization of the operators which describe the ground state of a giant graviton system. They have been put to good use over the course of the development of our approach to the construction of operators dual to excited systems, and it has been apparent throughout the development of this theory that the correspondence can be phrased in terms of fascinating equivalences between the geometric string theory descriptions and the discrete pictorial structure given by the Young diagrams of the representation theory which underpins the Schur polynomial construction. Numerous examples demonstrating that the geometry of the dual string theory is encoded in the Young diagram label have been found, one of the more significant being linked to the emergence of locality in our model. The study of string splitting and joining interactions in the context of excited giant graviton systems produces evidence for the interpretation of the number of boxes traversed in moving from one corner of the Young diagram to another as being related to a radial separation distance in the string theory ([13]). This is clearly seen by recalling that the Myers' effect results in the expansion of a giant graviton with increased angular momentum, and that each box on the Young diagram corresponds to a unit of angular momentum. The remaining discussion of this section focuses on the additional physical structure first uncovered in [2], which provides a clear connection to one of the fundamental physical requirements that the theory must satisfy.

D-branes are surfaces on which strings, being oriented in terms of the charge they carry, can affix via their endpoints. The interactions between giant gravitons depend on the existence of charge-carrying open strings stretching between their surfaces. Since giant gravitons have a compact worldvolume, we require the total charge localised on their surface to vanish due to Gauss' Law. One of the results of [2] is the appearance of a direct and natural connection between the action of the dilatation operator on the restricted Schur polynomials and this constraint. It was conjectured in [59], and later proven in [8] by the introduction of a double coset ansatz that the restricted Schur Polynomials provide the correct number of states expected from this constraint. Via the "counting to construction" philosophy, using branching coefficients of the double coset to generate the correct linear combination of restricted Schur polynomials, the authors of [8] provide a basis of operators which directly diagonalize the dilatation operator - these are referred to as the *Gauss operators*. The fact that this completely physical constraint arises naturally out of the direct computation of the dilatation operator action on restricted Schurs, and is perfectly reproduced by sensible group theoretic considerations, is some of the strongest evidence available prior to the development of the articles on which this dissertation is based for the validity of our proposed correspondence.

An important observation which should be made when interpreting the Gauss operators is the fact that, in moving to this new basis, some of the symmetry which is imposed by the definition of the restricted

Schur polynomial with minimal strings attached is undone. In restricting to the $S_n \times S_m$ subgroup, we have implemented an expectation that the minimal strings are all indistinguishable, and thus can be freely permuted by the S_m component. This is not the physical situation - while the strings are indeed indistinguishable when considered on their own, once their attachment to the system of giants is considered, the relevant symmetry group is that of the double coset, which permits permutations of the strings such that only endpoints attached to the same giant graviton brane are exchanged. A more natural language to describe these operators is thus that in which the construction of restricted Schurs involves the subduction $S_{n+m} \to S_n \times (S_1)^m$, since in this case the labelling provides a clear link to the endpoint configurations. A rewriting of the Gauss operators in terms of this basis of restricted Schurs is presented in Appendix C to illustrate this point. This consideration also motivated the use of the $S_n \times (S_1)^m$ basis in Chapter 3

In this dissertation, we have further demonstrated the perfect correspondence between results obtained using operators constructed using the Schur polynomial technology, and those obtained in the dual string theory. The current results have not, to our knowledge, been possible to obtain (at least not to the same level of precision) by any other methods employing AdS/CFT.

2.7.5 Schur Polynomials as Duals to LLM Geometries

It is interesting in our theory, where AdS/CFT is a key premise, to ask how the additional dimensions and local physical interactions of the string theory emerge from the lower-dimensional field theory description. In [5], this question was considered for the particular case where giant gravitons in $AdS_5 \times S_5$ are described by Schur Polynomials in a 4-D super Yang-Mills theory. The dynamical content of any QFT is extracted via the calculation of correlation functions - in this case, we compute correlation functions of Schur Polynomials at weak coupling, with the knowledge that they are preserved even at strong coupling. The article [5] considers Schur Polynomials consisting of $O(N^2)$ fields (operators of this type correspond to new spacetime geometries in the dual string theory), and shows that the Young diagram label of these operators encodes useful details of the emergent geometry, and additionally that local physical interactions can be represented by the manipulation of these labels.

By defining coherent gauge theory states that are dual to KK graviton states in the string theory, the dual geometry can be explored, since gravitons follow null geodesics and hence their dynamics is influenced only by the geometry of the spacetime in which they move. The first step taken in defining these coherent states was to determine creation and annihilation operators on the gravitons (a and $a^\dagger$). The action of an operator of the form $\frac{\text{Tr}(Z)}{\sqrt{N}}$ provides the single graviton creation operator - the result of calculating the correlator between powers of this operator suggest that its action on the field theory vacuum matches with the action of the dual graviton creation operator on $AdS_5 \times S_5$ in quantum gravity. The annihilation operator has the same form, but with a derivative of Z. Defining the coherent state as a normalized exponentiation of the creation operator, the Lagrangian describing the low energy excitations of this state on $R \times S^3$ (to which we transform from the Euclidean spacetime by conformal mapping) was found to produce equations of motion agreeing with known results for gravitons. Writing the trace as a Schur polynomial labelled by a Young diagram with a single box, the same results were obtained. It was also shown that the creation operator involving a trace over a product of p Zs creates gravitons carrying p units of angular momentum.

The authors of [5] then seek to investigate the dynamics of gravitons propagating on new backgrounds - by considering the action on the SYM vacuum, the dynamics on $AdS_5 \times S^5$ and its half-BPS excitations were studied. The vacuum can be taken as the normalized state dual to a Schur polynomial $\chi_B(Z)$, which is labelled by a Young diagram B having $O(N^2)$ boxes. A convenient example to begin the study is given by considering the case where B is an $M \times N$ block. Using the product rule for Schur polynomials, an operator dual to a state of n gravitons in this background can be given in terms of a Schur polynomial labelled by a Young diagram which looks like B with n boxes attached. Since this Young diagram has only a single corner to which the boxes can attach, localization of the gravitons in the dual geometry is expected to be guaranteed. The two point function provides the normalization, and a definition for the action of the graviton creation operator in this background. Initially, the annihilation operator presented a discrepancy with the previous results, since one now obtains a non-zero result when acting with the Z-derivative on the background - the action removes boxes from B. A resolution was realized by the postulation of *countergravitons*, which orbit in the opposite direction to the gravitons. The correct results were obtained by arguing that $\text{Tr}(Z)$ creates gravitons and destroys countergravitons, while Z-derivatives have the opposite effect. The annihilation and creation operators for countergravitons (b and $b^\dagger$) were introduced and, making the obvious identification with $\text{Tr}(Z)$ and the Z derivative, the correct commutator relations were seen to emerge. The action of these operators correspond graphically to adding or removing boxes from the lower (upper) right corner of B for the countergraviton (graviton).

The implication of this identification is that the trace operators no longer have a local action on B - splitting each into two parts provided a clean way to diagrammatically represent their action. A coherent graviton state was again assembled so that the dynamics could be checked against known results. The comparison of the result with the dual quantum gravity required the translation of the background B to an LLM geometry. The Schur polynomials can be translated into a free fermion state, and the free fermion state into boundary conditions for an LLM geometry (See Fig. 4.4 for an illustration). For the example where B is an $M \times N$ block, the boundary conditions can be represented as a black annulus on a white background - this pattern is graphically related to the Young diagram B, so that locality on the Young diagram is shown to correspond to locality in the dual geometry. In this picture, the creation/annihilation operators produce small "ripples" on the inner and outer edges of the annulus, in agreement with the expectation for the diagrammatic representation of excitations in the fermion occupation picture of [33].

To see the agreement between the energy resulting from the Hamiltonian calculated using our methods and that from the LLM picture, a rescaled time coordinate is introduced. This transforms us from a regime where energy is measured with respect to the background, to one where it is measured with respect to the true vacuum of the theory. In order to handle the case where the gravitons we create carry p units of angular momentum, each box in the Schur polynomial representation of the trace is given one of two identities (a or b) - when using the Littlewood-Richardson rule in this case, the boxes with a particular identity can only be added to either the upper or lower right hand corners of the Young diagram. For example, $\text{Tr}(Z_a^p)$ may create a graviton carrying p units of momentum, localized on the outer ring of the annulus in the LLM picture.

The introduction of giant graviton creation operators requires the addition of operators which add or subtract units of energy from the giant - these are each associated with a Cuntz oscillator. The resulting Lagrangian was found to coincide with the known result for a giant graviton propagating in an LLM geometry. For strings in this background, which are not half-BPS, a Cuntz oscillator for the string lattice is defined, and the results of the previous articles are used to define coherent states for the Hamiltonian derived therein. The sigma model action governing the low energy dynamics of the lattice model thus defined was found to agree with the known result for strings in an LLM geometry. A generalization to the case where the Young diagram B has multiple corners, and thus corresponds to a series of concentric rings in a picture of the LLM boundary conditions, is obtained by decomposing $\text{Tr}(Z)$ into a sum of graviton and countergraviton creation and annihilation operators which each act on a particular corner (ring edge) of B (the LLM rings).

In the standard picture of the large N limit of a matrix model, a particular background can be associated to a particular distribution of eigenvalues. Due to the dual description of a Schur polynomial model as a system of N non-relativistic fermions, we can argue that the length of a particular row in the Young diagram label is proportional to the magnitude of the associated eigenvalue. If the Young diagram describing the background has q corners, the eigenvalues will be clumped around q values, and Z can be split into q submatrices that each have similar eigenvalues. It was proposed that this splitting corresponds to the splitting of Z into local operators, and some evidence for this proposal was given. This evidence suggests that locality on the Young diagram also corresponds to locality on the eigenvalues, when these eigenvalue "clumps" are well separated. A straightforward generalization to the case where the backgrounds are built from more than one matrix was given, and an example using trivial backgrounds was briefly investigated, where the results suggest that it may be possible to construct $\frac{1}{4}$-BPS geometries using clever combinations of the $\frac{1}{2}$-BPS LLM geometries.

In the article [62], the result of probing these LLM backgrounds using strings was investigated further. First, an example using a string containing only a single Z-impurity was considered in order to study the contractions between the string impurity and the Zs of the Schur polynomial operator. Splitting the operator into a string part and a background part, the calculation of the piece of the correlator corresponding to these contractions reduces to the expectation value of a derivative operator acting on Schur polynomials. A graphical notation was introduced for the purpose of determining the structure of the derivative operator when the string word has a more general form. An arbitrary derivative operator can be decomposed into a product of some combination of 8 basic types of derivatives, each of which has a known action on our Schur polynomials corresponding to the background. Applying this knowledge to the calculation of correlators where the string impurities can contract with fields in the Schur polynomial operator, it was found that these contributions only become important when working with operators composed of $O(N^2)$ fields. It is natural then to interpret these contractions as a means by which to account for backreaction of the strings on the geometry when working in the field theory.

By considering the case where the background geometry corresponds to the annulus LLM boundary conditions (rectangular Young diagram), the authors showed that the contribution to the correlator arising from any of the possible contractions between the string impurities and the fields on the giant are all the same size. This result allows one to write

$$\langle \chi^{(1)}_{R,R_1} \chi^{(1)}_{R,R_1}{}^{\dagger} \rangle = (M+N) N^n (1+\frac{M}{N})^n f_R.$$

The Young diagram R is an $M \times N$ block, labelling a representation of S_{NM}. R_1 has a single box labelled (it can only be the box in row M and column N), for which the associated index is that of the open string word $W^i_j = (Y(Z)^n Y)^i_j$. Recall that the Ys in the string can be associated with a lattice, populated by the Z impurities, each of which then corresponds to a Cuntz oscillator. Adding an extra impurity thus corresponds to applying another Cuntz oscillator to the state, and it is clear that the background is accounted for by rescaling the Cuntz oscillators corresponding to the string impurities. Ribbon diagrams can be employed to calculate this correlator in general, with the modification that each ribbon carries an extra factor of $\frac{c}{N}$, where c is the weight of the boxes added/removed by Z or $\frac{d}{dZ}$ (these are local operators in the annulus background).

The authors of [62] then considered the end point interactions of open strings in a regular LLM background by writing the Hamiltonian for string interactions derived in [14] in terms of the rescaled Cuntz oscillators. In terms of these operators, one can clearly see that there is nothing distinguishing end-point interactions from interactions in the bulk of the string. One can identify boxes that "hop off" the string as rather simply moving into the $(L+1)$th site of the string - this is possible because (1)whether a box is added to the string or to the background, it contributes the same factor to the correlator, and (2) because correlators between operators labelled by different Young diagrams vanish. This coincides well with the interpretation of the Schur polynomial operators as dual to a new background, since open string excitations are not expected in this case. It also means that the calculations for closed string interactions are identical to those already performed.

Correlation functions of background operators for LLM geometries corresponding to q thick, well-separated concentric rings were also considered. As previously discussed, Z can be replaced in this case by a block diagonal matrix with q block elements, each corresponding to a particular clump of eigenvalues. In the limit we consider, the off-diagonal modes connecting these subsectors decouple, so that this can be viewed as a set of q single matrix models, each corresponding to an annulus LLM geometry. The correlation functions can thus be calculated for each of these single matrix, annulus-geometry models using the methods of the article under review[62]. A factor of $\frac{N_i}{N}$ must be included for each trace in the local operator, where N_i is the number of eigenvalues in the clump associated to the ith ring.

A major obstacle in working with objects composed of $O(N^2)$ fields is the fact that the usual $\frac{1}{N^2}$ suppression of non-planar diagrams is overpowered by combinatorial factors, and an expansion in $\frac{1}{N}$ is no longer possible. In [63], the possibility of reorganizing the expansion in terms of some other small factor was investigated in the half-BPS and near-BPS limit for operators with $\mathcal{R}$-charge N^2. The half-BPS sector was studied first - using the product rule for Schur polynomials, the correlators of fields in the trivial (no boxes) and annulus ($N \times M$ rectangular Young diagram) backgrounds were calculated. Comparing the results of these two computations, it was demonstrated that a new expansion parameter for correlators in the annulus background is given by $\frac{1}{N+M}$, and further that the shift $N \to N + M$ is the only effect of this background. This is however not a property of the full theory; a general relation between amplitudes in the trivial and annulus backgrounds was not successfully determined. Comparing the LLM boundary conditions to which each of the backgrounds corresponds, one can make the conclusion that probes built out of Zs are only able to explore the outer edge of the annulus - the same computation using $\frac{d}{dZ}$ was expected to provide probes which explore the inner edge.

Considering that the relation implies an equivalence between the correlators in a field theory with gauge group $U(N)$ on the annulus background (or background of M giant gravitons), and correlators on the trivial background with gauge group $U(N + M)$, the authors were led to consider the infrared duality of [64]. The relation mentioned above shows that half-BPS correlators in $U(N)$ gauge theory with a background on M giant gravitons are exactly equal to the same correlators calculated in $U(M)$ gauge theory with a background of N giants. The correlators are extremal, and hence the calculations performed give correct values in the deep infrared limit. Some further arguments providing a link to this duality were given, but will not be included in this discussion.

The authors of [63] then proceed with the use of holography in the LLM background as a means to reproduce the results already obtained. Since there is a non-renormalization theorem protecting certain graviton three point functions, an attempt to show the agreement between these when calculated in the free field theory and in the strong coupling limit of supergravity was given. The direct computation of these functions was not obtained, but by relating them to one-point functions it was shown that the supergravity and gauge theory results do indeed agree - the supergravity calculations were performed under a free fermion description by rewriting the normalized supergravity state in terms of Schur polynomials.

Multi-ring LLM backgrounds were considered next; due to the fact that the Z matrices can be decomposed into pieces local to the edges of each ring, the generalization of the re-organization of the expansion is clear: the gravitons at the edge of each ring have their own expansion parameter equal to $\frac{1}{N+M_i}$, where M_i is the number of columns in the Young diagram at the corner corresponding to ring i. When another field Y is introduced into the operator, in the case where the Z and Y Young diagrams are both rectangular, the operator factorizes into a Y piece and a Z piece. These two types of observables admit a different reorganization of the perturbation theory in that the small parameters of the two expansions are not the same.

The study of the BMN-like limit proposed by the authors of [65] using the technology developed in [63] was then pursued. Using the same two-charge background as before, with the number of columns in the Z (M_1) and Y (M_2) Young diagrams both $O(\sqrt(N))$, it is immediately apparent that the strong and weak coupling results match, and that the anomalous dimension should arise from the dynamics of strings stretching between the Z giants and the Y giants ($\Delta\ M_1 M_2$). The two-point function of these operators where $M_1 = M_2 = M$ can be factorized into a perturbative piece and a non-perturbative piece, where the former admits an expansion in $\frac{1}{M}$, suggesting that the new effective genus counting parameter should be taken as $\frac{1}{M}$. This new parameter is sensible for the description of the large-M limit of a $U(M)$ gauge theory - the expected low energy worldvolume theory of a system of M giant gravitons.

The results obtained suggest that this theory is weakly coupled, even when the original SYM theory is strongly coupled.

In order to determine the sector of the full theory in which the reorganization of the expansion is valid, the two-point correlator of near-BPS BMN-like loops (string words) is calculated. The Y fields appearing as impurities on the Z-lattice of these operators are contracted planarly, leaving the correlator of a product of operators of the form $\mathrm{Tr}(Z^{n_i} Z^{\dagger n_i})$. Using the factorization of correlators of these operators, the methods of [63] were applied to write the correlator in terms of restricted Schur polynomials. To leading order in $N+M$, it was shown that the only effect of the background on the correlators is to rescale the matrix Z. The exact analysis again proved that the correlator admits an expansion in $\frac{1}{M+N}$, with an additional M-dependence which does not affect the previous conclusion. Using the Schwinger-Dyson equation, it was further possible to show that this reorganization of the perturbation theory should be valid in even more general settings. The result that the rescaling of Z is the only effect of the background is the same conclusion drawn previously in the context of rescaled Cuntz oscillators.

In Chapter 4 of this dissertation, the investigation into the Schur polynomial description of the dual states to LLM geometries with concentric ring boundary conditions is revisited. Drawing on the results described in this section, as well as the insights presented in Section 2.5 and Chapter 3, the full spectrum of one loop anomalous dimensions for magnons attached to the outer and inner edges of the concentric rings are calculated. Agreement with the string theory predictions is manifest, and a clear map between the pictorial descriptions of the string theory (LLM diagram with magnons) and of the representation theory (Young diagrams with local excitations) is apparent.

2.8 The Dilatation Operator

2.8.1 Definition

Consider the two-point function of a set of conformal fields O_α; it has the form

$$\langle O_\alpha(x) O_\beta(y) \rangle = \frac{\delta_{\alpha\beta}}{|x-y|^{2\Delta_\alpha}}$$

The quantity Δ_α is known as the scaling dimension of the field. There are numerous difficulties associated with the calculation of this quantity by the use of the two point function [66]: calculating the corrections to the two point function perturbatively introduces infinities, where the corrections themselves must be finite, leading to the necessity of complicated renormalization. However, instead of renormalizing two-point functions at each order of the perturbative calculation, one can rather introduce the dilatation operator: its action on a conformal field operator is given by

$$DO_\alpha = \Delta_\alpha O_\alpha.$$

The eigenvalue of the dilatation operator when acting on a conformal field gives the scaling dimension of the field. When considering an interacting theory, the scaling dimension is found to depend on the coupling constant, and by extension so does the dilatation generator. The dilatation operator can be expanded under perturbation theory in terms of powers of the coupling constant:

$$D = \sum_{k=0}^{\infty} (\frac{g_{YM}^2}{16\pi^2})^k D_{2k}$$

The operator D_{2k} is referred to as the k-loop dilatation operator. The 0-loop operator gives the classical scaling dimension, which for fields composed only of scalars (as our Schur operators are) is simply equal to the number of scalars comprising the field, since the engineering dimension of scalars is one. The action of the one-loop dilatation operator was determined in [66] to be

$$D_2 = -g_{YM}^2 \mathrm{Tr}[Y, Z][\partial_Y, \partial_Z]. \tag{2.89}$$

2.8.2 Why D?

The focus of this project is on the generalization of the study of the action of the dilatation operator on the Schur polynomial operators. The importance of the results obtained can be seen in the context of the AdS/CFT correspondence[11]: by performing a Wick rotation and conformal transformation on the metric of the SYM theory, we obtain an implementation of the gauge theory on $R \times S^3$. States of the theory on this space are in one-to-one correspondence with operators on the 4-D Euclidean space reached by the Wick rotation, by the state-operator correspondence. The boundary of the AdS_5 of the string theory can be shown to be $R \times S^3$, thus it is natural to postulate an identification between time translations and scaling. This explains the correspondence of gauge theory operators to string theory states, and allows for a direct comparison between operators of the two theories, where we see that the operator implementing scale transformations in the field theory on Minkowski space (the dilatation operator) is dual to the Hamiltonian of the string theory on $R \times S^3$.

Hence, if we can determine the eigenvalues of the dilatation operator acting on the restricted Schur polynomials in general, we can associate this with the energy of the giant graviton system in the string theory. We would thus have the energy eigenstates (constructed from restricted Schurs) and the associated eigenvalues for a giant graviton system as described by string theory. This is all the information necessary to determine the time evolution, and hence the dynamics, of any giant graviton system - a problem that had seemed intractable for many years due to the infinities which arise when attempting to sum all possible paths for an oscillating membrane. It was argued in [67] and [68] that the spectrum of anomalous dimensions (obtained by calculating the loop corrections to the dilatation operator) of the

[11] See 2.3.4 for further discussion of the points presented in this section

operators dual to giant graviton states reproduces the spectrum of vibrational modes of giant gravitons calculated in [69]. Numerical studies of the one loop dilatation operator acting on restricted Schurs dual to a two giant graviton system with 3 or 4 impurities [6] yields a surprising and powerful result - the dilatation operator was found to be equivalent to a set of decoupled harmonic oscillators, the frequencies of which are determined by the representation organising the Y-fields. The article [2] found the same result, arising in a continuum limit. In [70], the appearance of a set of decoupled harmonic oscillators describing the dynamics of the branes was obtained for the case of a general system of giants and strings, without invoking a continuum limit.

2.8.3 Action on Restricted Schur Polynomials

2.8.3.1 Exact Action in $S_n \times S_m$ Basis

The action of the dilatation operator on restricted Schur polynomials has been studied in [7], [6], [71], [2], amongst others. The discussion following uses restricted Schur polynomials having $O(1)$ long rows, however, the discussion for $O(1)$ long columns is very similar. Applying the operator defined by (2.89) to the restricted Schur Polynomial, we obtain:

$$D\chi_{R,(r,s)jk} = \frac{g_{\rm YM}^2}{(n-1)!(m-1)!}\sum_{\psi\in S_{n+m}} {\rm Tr}_{(r,s)jk}\left(\Gamma_R((1,m+1)\psi-\psi(1,m+1))\right)\times$$
$$\times\delta^{i_1}_{i_{\psi(1)}}Y^{i_2}_{i_{\psi(2)}}\cdots Y^{i_m}_{i_{\psi(m)}}(YZ-ZY)^{i_{m+1}}_{i_{\psi(m+1)}}Z^{i_{m+2}}_{i_{\psi(m+2)}}\cdots Z^{i_{n+m}}_{i_{\psi(n+m)}}. \qquad (2.90)$$

As a consequence of the $\delta^{i_1}_{i_{\psi(1)}}$ appearing in the summand, the sum over ψ runs only over permutations for which $\psi(1)=1$. To perform the sum over ψ, write the sum over S_{n+m} as a sum over cosets of the S_{n+m-1} subgroup obtained by keeping those permutations that satisfy $\psi(1)=1$. The result is derived in the same way as the reduction rule for Schur polynomials (see appendix C of [13]). A sum of the form $\sum_{\psi\in S_{n+m}}\chi_R(\psi)$ can be rewritten as $\sum_{\psi\in S_{n+m-1}}\sum_{j=1}^{n+m}\chi_R(\psi\cdot(1,j))$. Using this, together with the knowledge that the operator $\sum_{j=1,j\neq k}^{n+m}(k,j)$ acting on a particular Young-Yamonouchi state has as its eigenvalue the weight of the box labelled k in the diagram, one obtains:

$$D\,\chi_{R,(r,s)jk} = \frac{g_{\rm YM}^2}{(n-1)!(m-1)!}\sum_{\psi\in S_{n+m-1}}\sum_{R'}c_{RR'}\,{\rm Tr}_{(r,s)jk}\Big(\Gamma_R\left((1,m+1)\right)\Gamma_{R'}(\psi)$$
$$-\Gamma_{R'}(\psi)\Gamma_R\left((1,m+1)\right)\Big)Y^{i_2}_{i_{\psi(2)}}\cdots Y^{i_m}_{i_{\psi(m)}}(YZ-ZY)^{i_{m+1}}_{i_{\psi(m+1)}}Z^{i_{m+2}}_{i_{\psi(m+2)}}\cdots Z^{i_{n+m}}_{i_{\psi(n+m)}}.$$

The sum over R' runs over all Young diagrams that can be obtained from R by dropping a single box; $c_{RR'}$ is the factor associated to the box that must be removed from R to obtain R'. Since the dilatation operator has derivatives with respect to Z and Y in the same trace, it naturally mixes Zs and Ys. The appearance of $\Gamma_R\left((1,m+1)\right)$ is thus expected, as the group element of which this is a matrix representation is not an element of the $S_n\times S_m$ subgroup - it mixes indices belonging to Zs and indices belonging to Ys. To clarify, one can imagine we have an S_{n+m} representation R, which is q dimensional, i.e. there are q Young-Yamonouchi labels corresponding to vectors in the basis for its carrier space. We denote these basis vectors by $|YY_i\rangle$, where $i\in[1,q]$ and the YY_i are the Young Tableaux labels. Another set of basis vectors is obtained by numbering only m of the boxes in the Young diagram for R. Each possible way of numbering these m boxes provides a partially labelled Young diagram, and the labelled boxes can be removed and assembled in different ways to describe S_m representations. We know that in this way we can obtain $S_n\times S_m$ representations that are subduced by R, and that a state in the carrier space of an $S_n\times S_m$ representation can be decomposed as a linear combination of states in the carrier space of R:

$$|R,(r,s)\rangle = \sum_{i=1}^{q}C^{(r,s)}_{YY_i}|YY_i\rangle.$$

The carrier space of R, instead of being described as a q-dimensional vector space with basis vectors carrying Young-Yamonouchi labels, is split up into a number of lower dimensional carrier spaces, each being the carrier space of a representation of some $S_n\times S_m$ subgroup. Acting with $\Gamma_R\left((1,m+1)\right)$ on one of these states will produce a linear combination of states in the carrier space of R which may correspond to a linear combination of the states labelled by a different set of $S_n\times S_m$ labels, possibly even including Young Yamonouchi states that do not combine to form states of the subgroup - $\Gamma_R\left((1,m+1)\right)|R,(r,s)\rangle = \sum_{s'} C_{s'}|R,(r,s')\rangle + \sum_i C_i|YY_i\rangle$. The action of this element of the S_{n+m} group thus mixes states from different representations of the $S_n\times S_m$ subgroup.

We will make use of the following notation

$$\mathrm{Tr}(\sigma Z^{\otimes n}Y^{\otimes m}) = Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}Y^{i_{n+1}}_{i_{\sigma(n+1)}}\cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}}\,.$$

Now, use the identities (bear in mind that $\psi(1)=1$)

$$Y^{i_2}_{i_{\psi(2)}}\cdots Y^{i_m}_{i_{\psi(m)}}(YZ-ZY)^{i_{m+1}}_{i_{\psi(m+1)}}Z^{i_{m+2}}_{i_{\psi(m+2)}}\cdots Z^{i_{n+m}}_{i_{\psi(n+m)}} = \mathrm{Tr}\left(\Big((1,m+1)\,\psi-\psi\,(1,m+1)\Big)Z^{\otimes n}Y^{\otimes m}\right)$$

and (this identity is proved in [56]; the sum over T runs over all possible irreducible representations of S_{n+m})

$$\mathrm{Tr}(\sigma Z^{\otimes n}Y^{\otimes m}) = \sum_{T,(t,u)lq}\frac{d_T n!m!}{d_t d_u (n+m)!}\mathrm{Tr}_{(t,u)lq}(\Gamma_T(\sigma^{-1}))\chi_{T,(t,u)ql}(Z,Y)$$

to obtain

$$D\chi_{R,(r,s)jk}(Z,Y) = \sum_{T,(t,u)lq} M_{R,(r,s)jk;T,(t,u)lq}\chi_{T,(t,u)ql}(Z,Y)$$

where

$$M_{R,(r,s)jk;T,(t,u)lq} = g^2_{YM}\sum_{\psi\in S_{n+m-1}}\sum_{R'}\frac{c_{RR'}d_T nm}{d_t d_u(n+m)!}\mathrm{Tr}_{(r,s)jk}\Big(\Gamma_R((1,m+1))\Gamma_{R'}(\psi)-\Gamma_{R'}(\psi)\Gamma_R((1,m+1))\Big)\times$$
$$\times\,\mathrm{Tr}_{(t,u)lq}\left(\Gamma_{T'}(\psi^{-1})\Gamma_T((1,m+1))-\Gamma_T((1,m+1))\Gamma_{T'}(\psi^{-1})\right)\,.$$

The sum over ψ can be evaluated using the fundamental orthogonality relation

$$M_{R,(r,s)jk;T,(t,u)lq} = -g^2_{YM}\sum_{R'}\frac{c_{RR'}d_T nm}{d_{R'}d_t d_u(n+m)}\mathrm{Tr}\Big(\Big[\Gamma_R((1,m+1)),P_{R\to(r,s)jk}\Big]I_{R'\,T'}\times \tag{2.91}$$
$$\times\Big[\Gamma_T((1,m+1)),P_{T\to(t,u)ql}\Big]I_{T'\,R'}\Big)\,. \tag{2.92}$$

Sums of this type are discussed in the next subsection and the intertwiners $I_{R'\,T'}$ which arise are explained in detail. This expression for the one loop dilatation operator is exact in N. To obtain the spectrum of anomalous dimensions, we need to consider the action of the dilatation operator on normalized operators. It is only by using the normalized operators that the suppression of the operators labelled by Young diagrams with $n\neq p$ long columns or rows that can arise under the action of the dilatation operator is manifest. The two point function for the restricted Schur polynomials, as we have seen, is not unity. Normalized operators which do have unit two point function can be obtained from

$$\chi_{R,(r,s)jk}(Z,Y) = \sqrt{\frac{f_R\,\mathrm{hooks}_R}{\mathrm{hooks}_r\,\mathrm{hooks}_s}}O_{R,(r,s)jk}(Z,Y)\,.$$

In terms of these normalized operators

$$DO_{R,(r,s)jk}(Z,Y) = \sum_{T,(t,u)lq} N_{R,(r,s)jk;T,(t,u)ql}O_{T,(t,u)ql}(Z,Y)$$

$$N_{R,(r,s)jk;T,(t,u)ql} = -g^2_{YM}\sum_{R'}\frac{c_{RR'}d_T nm}{d_{R'}d_t d_u(n+m)}\sqrt{\frac{f_T\,\mathrm{hooks}_T\,\mathrm{hooks}_r\,\mathrm{hooks}_s}{f_R\,\mathrm{hooks}_R\,\mathrm{hooks}_t\,\mathrm{hooks}_u}}\times$$
$$\times\mathrm{Tr}\Big(\Big[\Gamma_R((1,m+1)),P_{R\to(r,s)jk}\Big]I_{R'\,T'}\Big[\Gamma_T((1,m+1)),P_{T\to(t,u)lq}\Big]I_{T'\,R'}\Big)\,.$$

It is this last expression that will be used when deriving the various character identities which will be useful in the body of this dissertation. The bulk of the work entails evaluating the trace. There are three objects which appear: the symmetric group projection operators $P_{R\to(r,s)jk}$, the intertwiners $I_{T'\,R'}$ and the symmetric group element $\Gamma_R((1,m+1))$. We have already discussed the operators $P_{R\to(r,s)jk}$. $\Gamma_R((1,m+1))$ corresponds to the matrix action of a permutation which mixes Y and Z fields. The next subsection provides a detailed discussion of the intertwiners $I_{T'\,R'}$; for our current purposes, the explicit form and action of these objects will not be used extensively, but an understanding of their function is important when interpreting and manipulating later results.

Intertwiners

In this section we will consider the sum over S_{n+m-1} which was performed to obtain (2.91). This will give a very explicit understanding of the intertwiners appearing in the expression for the dilatation operator. When S^n acts on $V^{\otimes n}$, $n>1$ it furnishes a reducible representation. Imagine that two of the irreducible representations subduced by this representation are R and S. Representing the action of σ as a matrix $\Gamma(\sigma)$, in a suitable basis we can write

$$\Gamma(\sigma)=\begin{bmatrix}\Gamma_R(\sigma) & 0 & \cdots\\ 0 & \Gamma_S(\sigma) & \cdots\\ \cdots & \cdots & \cdots\end{bmatrix}.$$

In our construction, we consider the action of $\Gamma(\sigma)$ to be that of a group element of S_{n+m}, in a reducible representation of the group. The carrier space of this representation will be a direct sum of the carrier spaces of irreducible representations subduced - the reducible representation acts in the space $V^{\otimes n+m}$, while the subduced representations will act in spaces spanned by various linear combinations of the vectors in $V^{\otimes n+m}$. The linear combinations are constructed such that they span the carrier space of a particular subgroup representation, in such a way that one is not transformed out of that carrier space by the action of the group elements. The representations subduced here are S_{n+m} representations formed by assembling $n+m$ boxes in different ways to form Young diagrams for all the possible irreducible representations of S_{n+m}. Continuing with the general discussion above, if we restrict ourselves to an S_{n-1} subgroup of S_n, then in general, both R and S will subduce a number of representations. Assume for the sake of this discussion that R subduces R'_1 and R'_2 and that S subduces S'_1 and S'_2. This is precisely the situation that arises in the sum performed to obtain (2.91), with the matrix $\Gamma(\sigma)$ being composed of the matrix representations corresponding to the particular R and T for which we are calculating $N_{R,(r,s)jk;T,(t,u)ql}$. Then, for $\sigma\in S_{n-1}$ we have

$$\Gamma(\sigma)=\begin{bmatrix}\Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \cdots\\ 0 & \Gamma_{R'_2}(\sigma) & 0 & 0 & \cdots\\ 0 & 0 & \Gamma_{S'_1}(\sigma) & 0 & \cdots\\ 0 & 0 & 0 & \Gamma_{S'_2}(\sigma) & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\end{bmatrix}.$$

Imagine that as Young diagrams $S'_1=R'_1$, that is, one of the irreducible representations subduced by R is isomorphic to one of the representations subduced by S. If this situation does not occur for the R and S considered, the term being calculated vanishes by the fundamental orthogonality relation. A simple application of this relation for non-zero terms gives

$$\sum_{\sigma\in S_{n-1}}\begin{bmatrix}\Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\end{bmatrix}_{ij}\begin{bmatrix}0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & \Gamma_{S'_1}(\sigma) & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\end{bmatrix}_{ab}$$

$$=\frac{(n-1)!}{d_{R'_1}}\delta_{R'_1S'_1}\begin{bmatrix}0 & 0 & \mathbf{1} & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\end{bmatrix}_{ib}\begin{bmatrix}0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ \mathbf{1} & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\end{bmatrix}_{aj}$$

$$\equiv \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 S'_1} (I_{R'_1 S'_1})_{ib} (I_{S'_1 R'_1})_{aj}$$

where the form of the intertwiners has been spelled out. The **1**'s in the first and second matrices are identity matrices of the same dimension as $\Gamma_{R'_1}(\sigma)$ and $\Gamma_{S'_1}(\sigma)$ respectively, and the 0's are zero matrices of appropriate dimension for the intertwiners to act on $\Gamma(\sigma)$. The intertwiners are analogous to the delta functions appearing under usual application of the fundamental orthogonality relation, where the matrices being multiplied do not appear as block diagonal elements of the carrier space of some reducible representation. They appear because we are comparing matrices for which the indices labelling element positions begin at different values. This is the situation that occurs when comparing two matrices corresponding to the same subduced representation, but which reside in different subspaces. The intertwiners can be understood as follows: suppose we use indices (a, b) to reference elements in the matrix $\Gamma(\sigma)$ corresponding to $\Gamma_{R'_1}(\sigma)$, and indices (i, j) for those corresponding to $\Gamma_{S'_1}(\sigma)$. Then a and b will run from $1 \cdots d_{R'_1}$, while i and j will run from $d_R + 1 \cdots d_R + d_{S'_1}$. It is necessary to remain in the vector space that $\Gamma(\sigma)$ lives in, rather than considering the subspaces independently, due to the appearance of both R and T in the same trace in (2.91). The fundamental orthogonality relation in this case is realised as:

$$\sum_{\sigma \in S_{n-1}} [\Gamma(\sigma)]_{ab} [\Gamma(\sigma^{-1})]_{ij} \sim \delta_{a+d_R,j} \delta_{b+d_R,i}$$

Obviously, these are not proper delta functions and cannot be represented as identity matrices - instead we find that they have only one non-zero block as indicated above, and we name them intertwiners. Intertwiners are maps between two isomorphic spaces. For $\sigma \in S_{n-1}$

$$I_{R'T'} \Gamma_{T'}(\sigma) = \Gamma_{R'}(\sigma) I_{R'T'}.$$

The box removed to obtain R' and T' can be removed from any corner of the Young diagram.

2.8.3.2 Planar Action on $S_n \times (S_1)^m$ Operators

We will now consider the action of the one loop dilatation operator, $D = -\frac{g_{YM}^2}{8\pi^2} \mathrm{Tr}[Y, Z][\partial_Y, \partial_Z]$, on restricted Schurs labelled by $S_n \times (S_1)^m$ representations. We must compute

$$\begin{aligned} & D\chi_{R,R_1^k,R_2^k}(Z, \{\{n_i\}_1, \{n_i\}_2, \cdots, \{n_i\}_k\}) \\ &= \frac{1}{n!} D(\sum_{\sigma \in S_{n+k}} \chi_{R,R_1^k,R_2^k}(\sigma) Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} (W_k)^{i_{n+1}}_{i_{\sigma(n+1)}} \cdots (W_2)^{i_{n+k-1}}_{i_{\sigma(n+k-1)}} (W_1)^{i_{n+k}}_{i_{\sigma(n+k)}}). \end{aligned} \quad (2.93)$$

The notation of this equation was explained in Section 2.7.2.1. The Y derivative appearing in each term of the dilatation operator will act on a Y belonging to a specific open string word, and it will thus be sufficient in the planar limit we consider to demonstrate the action on one of each of the two distinct types of impurities that appear; those in the bulk of the string word and those that sit at ends of the open string chain, when we have an operator with a single string word with m magnons:

$$D\chi_{R,R_1^1,R_2^1}(Z, \{n_1, n_2, \cdots, n_m\})$$

Bulk Magnons

Consider the action of the first term arising from the commutator expression for the action; the index structure is explicitly

$$YZ \frac{d}{dY} \frac{d}{dZ} = Y^i_j Z^j_k \frac{d}{dY^l_k} \frac{d}{dZ^i_l}.$$

The full action of this term on the operator (3.7) will consist of a sum of terms corresponding to the Z derivative acting on every Z in all the string words, followed by the Y derivative acting on every Y in each of these terms. Focus on a particular Y in a particular string word, say with upper index i_q, and act with the Z derivative on the field directly left adjacent to this magnon in the chain, then with the Y derivative on this impurity:

$$\cdots \frac{d}{dZ^i_l} (Z^{i_q-1}_{i_q}) \frac{d}{dY^l_k} (Y^{i_q}_{i_{q+1}}) \cdots = \cdots \delta^{i_q-1}_i \delta^l_{i_q} \delta^{i_q}_l \delta^k_{i_{q+1}} \cdots = \cdots \delta^l_l \delta^{i_q-1}_i \delta^k_{i_{q+1}} \cdots .$$

Of course, since the fields are $SU(N)$ matrices, the delta function $\delta_l^l = N$. This is just one of a huge number of terms that will arise under the full action of the dilatation operator, but this enhancement by a factor of N means that it is the one that dominates in the planar limit. By considering the product of delta matrices that appears for any other placement of the Z and Y which the operator acts on, it is clear that this is the only term that picks up this factor.

Completing the computation of this term by multiplying the YZ,

$$\cdots Y_j^i Z_k^j \delta_l^l \delta_i^{i_q-1} \delta_{i_{q+1}}^k \cdots = N(\cdots Y_j^{i_q-1} Z_{i_{q+1}}^j \cdots) = N(\cdots Y_{i_q}^{i_q-1} Z_{i_{q+1}}^{i_q} \cdots)$$

so that we see that the fields are replaced in opposite order, and we have:

$$\begin{aligned} YZ\frac{d}{dY}\frac{d}{dZ}\chi_{R,R_1^k,R_2^k}(Z,\{n_1,n_2,\cdots,n_m\}) = N(&\chi_{R,R_1^k,R_2^k}(Z,\{n_1-1,n_2+1,\cdots,n_m\}) \\ &+\chi_{R,R_1^k,R_2^k}(Z,\{n_1,n_2-1,n_3+1,\cdots,n_m\})+\cdots \\ &+\chi_{R,R_1^k,R_2^k}(Z,\{n_1+1,n_2,\cdots,n_m-1\})). \end{aligned} \tag{2.94}$$

The analysis for the other 3 terms in the commutator proceeds in the same way. For $ZY\frac{d}{dZ}\frac{d}{dY}$ one finds that the dominant contribution in the planar limit comes from acting with the Z derivative on the field directly right-adjacent to the Y acted on by the Y derivative, and the fields are again replaced in the opposite order, leading to

$$\begin{aligned} ZY\frac{d}{dZ}\frac{d}{dY}\chi_{R,R_1^k,R_2^k}(Z,\{n_1,n_2,\cdots,n_m\}) = N(&\chi_{R,R_1^k,R_2^k}(Z,\{n_1+1,n_2-1,\cdots,n_m\}) \\ &+\chi_{R,R_1^k,R_2^k}(Z,\{n_1,n_2+1,n_3-1,\cdots,n_m\})+\cdots \\ &+\chi_{R,R_1^k,R_2^k}(Z,\{n_1-1,n_2,\cdots,n_{m-1}+1,n_m-1\})). \end{aligned} \tag{2.95}$$

The terms which dominate the planar limit for $-YZ\frac{d}{dZ}\frac{d}{dY}$ come from acting on the right-adjacent Z, and in this case the fields are replaced in the same order:

$$-YZ\frac{d}{dZ}\frac{d}{dY}\chi_{R,R_1^k,R_2^k}(Z,\{n_1,n_2,\cdots,n_m\}) = -N(m-2)(\chi_{R,R_1^k,R_2^k}(Z,\{n_1,n_2,\cdots,n_m\}).$$

The fields are again replaced in the same order by $-ZY\frac{d}{dY}\frac{d}{dZ}$; the dominant contribution comes from acting on the left-adjacent Z:

$$-ZY\frac{d}{dY}\frac{d}{dZ}\chi_{R,R_1^k,R_2^k}(Z,\{n_1,n_2,\cdots,n_m\}) = -N(m-2)(\chi_{R,R_1^k,R_2^k}(Z,\{n_1,n_2,\cdots,n_m\}).$$

In the last two expressions, the factor of $(m-2)$ appears because there will be one term arising for the action of the Y derivative on each bulk magnon. We see explicitly that the action of the dilatation operator factorizes into an action on each magnon; this is in fact obvious from the outset when working in the planar limit, simply due to the way that derivatives act.

This gives the final result for the planar action of the bulk one loop dilatation operator on the restricted Schurs in this description; it is quoted below for the case where the open string has only 3 magnons (and hence a single bulk magnon) in order to simplify the expression:

$$\begin{aligned} D_{\text{bulk}}\chi_{R,R_1^1,R_2^1}(Z,\{n_1,n_2,\cdots,n_m\}) = \frac{g_{YM}^2 N}{8\pi^2}\Bigg[&2\chi_{R,R_1^1,R_2^1}(Z,\{(n_1),(n_2)\}) \\ -\chi_{R,R_1^1,R_2^1}(Z,\{(n_1-1),(n_2+1)\}) - \chi_{R,R_1^1,R_2^1}&(Z,\{(n_1+1),(n_2-1)\})\Bigg]. \end{aligned} \tag{2.96}$$

For a more general system with k bulk magnons, one obtains a sum of k expressions each taking the form of the above.

Boundary Magnons

The action of the dilatation action on the boundary magnons was determined in general in ([14], [15]). The method employed focuses on obtaining the Hamiltonian for a single string attached to an arbitrary bound state of giants, to leading order in g_{YM}^2. The string words are each associated to a Cuntz oscillator chain, which can be understood as a lattice of Y fields which can be populated with Z fields. The state operator correspondence then implies that the problem of determining the anomalous dimensions of operators in the Yang-Mills theory can be solved by determining the dynamics of the Cuntz oscillators. In this model, the adding and removal of fields implemented by the dilatation operator commutator terms is recast in terms of creation and annihilation operators satisfying the relations

$$\hat{a}_i \hat{a}_i^\dagger = I \quad , \quad \hat{a}_i^\dagger \hat{a}_i = I - |0\rangle\langle 0|.$$

The bulk interaction Hamiltonian, derived above by acting directly on the operators, was determined in this construction in [13]. There are three types of boundary interactions which were considered: hop-on and hop-off, interpreted as interactions where momentum is exchanged between the string and giants, and the kissing interaction, which can be understood as a composition of a hop-on with a hop-off resulting in zero net angular momentum exchange. Towards this end, formulas were developed to enable objects like $\chi_{R',R_1',R_2'}(Z,ZW)$ (note a Z has "hopped off" the giant) in terms of $\chi_{R,R_1,R_2}(Z,W)$. We can again consider a system with only a single string with 3 magnons and a single giant to simplify the demonstration:

$$\chi_{(1^{b_0+1}),(1_1^{b_0}),(1_2^{b_0})}(Z,W(\{n_1,n_2\})) = -\chi_{(1^{b_0}),(1_1^{b_0-1}),(1_2^{b_0-1})}(Z,ZW(\{n_1,n_2\}))+\chi_{(1^{b_0})}(Z)\mathrm{Tr}(W(\{n_1,n_2\})). \tag{2.97}$$

The term with ZW corresponds to a Z hopping off the giant and onto the string; the term with $\mathrm{Tr}(W)$ corresponds to closed string emission.

Using the two-point functions for the restricted Schurs, a relation between the operators and normalized states of the Cuntz oscillator chain can be shown to be given by

$$\chi_{(1^{b_0+1}),(1_1^{b_0}),(1_2^{b_0})}(Z,W(\{n_1,n_2\})) \leftrightarrow \sqrt{\left(\frac{4\pi\lambda}{N}\right)^{b_0+h} N^{h-1}(b_0+1)\frac{N!}{(N-b_0-1)!}}|b_0+1;\{n_1,n_2\};1\rangle \tag{2.98}$$

where h is the number of fields in W.

Now, the action of the term in the dilatation operator which generates a hop-off interaction acts on a Cuntz state as $H_{off}|b_0,W\rangle = |b_0,ZW^-\rangle$, where W^- is the string word with one Z field removed. The Z can be removed from between either the first and second or last and second-last magnons in the chain; we will obtain a term in the full action for each. We can relate this to a state where the Z removed from the word is placed on the giant by plugging (2.98) into (2.97) with W replaced by W^-, which gives (after using large N approximations to simplify the coefficients)

$$|b_0,ZW^-\rangle = -\sqrt{\frac{1}{N}(N-b_0)}|b_0+1,W^-\rangle + \sqrt{\frac{h}{b_0}}|\text{closedstring}\rangle.$$

Noting that $h \sim O(\sqrt{N})$ and $b_0 \sim O(N)$ so that the closed string contribution can be dropped at large N, we have the result for the hop-off interaction:

$$H_{off}|b_0,W\rangle = -\sqrt{1-\frac{b_0}{N}}|b_0+1,W^-\rangle.$$

The result for the hop-on interaction can be obtained by recognizing that anomalous dimensions are a real quantity, implying that the Hamiltonian must be Hermitian. The hop-on term is thus obtained by taking the Hermitian conjugate of the hop-off term. After writing the hop-off Hamiltonian in terms of creation and annihilation operators which add or remove Zs from the giant and string word, it is clear that this term is given by

$$H_{on}|b_0, W\rangle = -\sqrt{1-\frac{b_0}{N}}|b_0-1, W^+\rangle$$

where W^+ has an extra Z between the boundary magnon and the next magnon in the chain.

The final boundary interaction was referred to as a kissing interaction in [14]; it corresponds to a composition of a hop-on and hop-off interaction, and thus its action is given by

$$H_{kiss}|b_0, W\rangle = (1-\frac{b_0}{N})|b_0, W\rangle.$$

There is also one interaction for which the expression is the same as in the bulk, arising when the Z involved in a kissing interaction belongs to the string. Of course, this term is

$$H_{bb}|b_0, W\rangle = |b_0, W\rangle.$$

Consider now the application of this action on the first magnon in the chain. There is a clear association between the above four terms and the terms arising from the commutator when acting directly on the Schur polynomials operators. For the first magnon, the hop-off interaction is associated the negative term in the dilatation operator commutator for which the dominant term in the planar limit comes from acting on the left-adjacent Z, $-YZ\frac{d}{dY}\frac{d}{dZ}$. The hop on term is associated with the term $-ZY\frac{d}{dZ}\frac{d}{dY}$, and the kissing term with $ZY\frac{d}{dY}\frac{d}{dZ}$. The bulk-like term arises from the action of $YZ\frac{d}{dZ}\frac{d}{dY}$.

The final results for the boundary terms of the dilatation action on the first and last magnons, written in terms of normalized operators, are thus given by:

$$\begin{aligned} D_{\text{first magnon}} O_{1^{n+1},1^n,1^n}(Z, W(\{(n_1),(n_2)\}) &= \frac{g_{YM}^2 N}{8\pi^2}\Big[\left(1+1-\frac{n}{N}\right) O_{1^{n+1},1^n,1^n}(Z, W(\{(n_1),(n_2)\}) \\ -\sqrt{1-\frac{n}{N}}&\left(O_{1^{n+2},1^{n+1},1^{n+1}}(Z, W(\{(n_1-1),(n_2)\}) + O_{1^n,1^{n-1},1^{n-1}}(Z, W(\{(n_1+1),(n_2)\})\right)\Big] \end{aligned} \tag{2.99}$$

$$\begin{aligned} D_{\text{last magnon}} O_{1^{n+1},1^n,1^n}(Z, W(\{(n_1),(n_2)\}) &= \frac{g_{YM}^2 N}{8\pi^2}\Big[\left(1+1-\frac{n}{N}\right) O_{1^{n+1},1^n,1^n}(Z, W(\{(n_1),(n_2)\}) \\ -\sqrt{1-\frac{n}{N}}&\left(O_{1^{n+2},1^{n+1},1^{n+1}}(Z, W(\{(n_1),(n_2-1)\}) + O_{1^n,1^{n-1},1^{n-1}}(Z, W(\{(n_1),(n_2+1)\})\right)\Big] \end{aligned} \tag{2.100}$$

In general, the coefficient of the hopping term is related to the factor of the box on the Young diagram R to which the endpoint of the string corresponding to the magnon considered is associated. These results may be reproduced by using the relation (2.97), rewritten with normalized restricted Schurs $O_{R,R_1,R_2}(Z, W)$; performing the calculations in the state language in [14] is what led to the understanding of the physical importance and details of the normalization in our theory. The reason that the normalization becomes physically important for our operators is due to the fact that there is a simple relation between properly normalized operators and their adjoint. The results of this dissertation suggest a new understanding, under which it may be possible to perform the calculation of this section in a way that illustrates explicitly how the factors enter into the coefficients of the hopping terms.

Chapter 3

Anomalous Dimensions of Heavy Operators from Magnon Energies

3.1 Introduction

In this chapter we will connect two distinct results that have been achieved in the context of gauge/gravity duality. The first result, which is motivated by the Penrose limit in the $AdS_5 \times S^5$ geometry[47], is the natural language for the computation of anomalous dimensions of single trace operators in the planar limit provided by integrable spin chains (see [72] for a thorough review). For the spin chain models we study, using only the symmetries of the system, one can determine the exact large N anomalous dimensions and the two magnon scattering matrix. Using integrability one can go further and determine the complete scattering matrix of spin chain magnons[16, 44]. The second results which we will use are the powerful methods exploiting group representation theory, which allow one to study correlators of operators whose classical dimension is of order N. In this case, the large N limit is not captured by summing the planar diagrams. Our results allow a rather complete understanding of the anomalous dimensions of gauge theory operators that are dual to giant graviton branes with open strings suspended between them. These results generalize the analysis of [17] to systems that include non-maximal giant gravitons and dual giant gravitons. The boundary magnons of an open string attached to a maximal giant graviton are fixed in place - they cannot hop between sites of the open string. In the case of non maximal giant gravitons and dual giant gravitons there are non-trivial interactions between the open string and the brane, allowing the boundary magnons to move away from the string endpoints.

The operators we focus on are built mainly out of one complex $U(N)$ adjoint scalar Z, and a much smaller number M of impurities given by a second complex scalar field Y, which are the "magnons" that hop on the lattice of the Zs. The dilatation operator action on these operators matches the Hamiltonian of a spin chain model comprising of a set of defects that scatter from each other. The spin chain models enjoy an $SU(2|2)^2$ symmetry. The symmetries of the system determine the energies of impurities, as well as the two impurity scattering matrix[16, 44] (reviewed in Section 2.5.1). The $SU(2|2)$ algebra includes two sets of bosonic generators (${R^a}_b$ and ${L^\alpha}_\beta$) that each generate an $SU(2)$ group. The action of the generators is summarized in the relations

$$[{R^a}_b, T^c] = \delta^c_b T^a - \frac{1}{2}\delta^a_b T^c\,, \qquad [{L^\alpha}_\beta, T^\gamma] = \delta^\gamma_\beta T^\alpha - \frac{1}{2}\delta^\alpha_\beta T^\gamma \tag{3.1}$$

where T is any tensor transforming as advertised by its index. The algebra also includes two sets of super charges Q^α_a and S^b_β. These close the algebra

$$\{Q^\alpha_a, S^b_\beta\} = \delta^b_a L^\alpha_\beta + \delta^\alpha_\beta R^b_a + \delta^b_a \delta^\alpha_\beta C\,, \tag{3.2}$$

where C is a central charge, and

$$\{Q^\alpha_a, Q^\beta_b\} = 0\,, \qquad \{S^a_\alpha, S^b_\beta\} = 0. \tag{3.3}$$

We will realize this algebra on states that include magnons. When the magnons are well separated, each magnon transforms in a definite representation of $su(2|2)$ and the full state transforms in the tensor product of these individual representations. Acting on the ith magnon we can have a centrally extended representation[16, 44]

$$\{Q^{\alpha}_{a}, S^{b}_{\beta}\} = \delta^{b}_{a} L^{\alpha}_{\beta} + \delta^{\alpha}_{\beta} R^{b}_{a} + \delta^{b}_{a} \delta^{\alpha}_{\beta} C_i \,, \tag{3.4}$$

$$\{Q^{\alpha}_{a}, Q^{\beta}_{b}\} = \epsilon^{\alpha\beta}\epsilon_{ab}\frac{k_i}{2}\,, \qquad \{S^{a}_{\alpha}, S^{b}_{\beta}\} = \epsilon_{\alpha\beta}\epsilon^{ab}\frac{k_i^*}{2}\,. \tag{3.5}$$

The total multimagnon state must be in a representation for which the central charges k_i, k_i^* vanish. Thus the multi magnon state transforms under the representation with

$$C = \sum_i C_i\,, \qquad \sum_i k_i = 0 = \sum_i k_i^*\,. \tag{3.6}$$

A key ingredient to make use of the $su(2|2)$ symmetry entails determining the central charges k_i, k_i^* and hence the representations of the individual magnons. There is a natural geometric description of the system, first obtained by an inspired argument in[73] and later put on a firm footing in [4](see Section 2.5.2 for a review), which gives an elegant and simple description of these central charges. The two dimensional spin chain model that is relevant for planar anomalous dimensions is dual to the worldsheet theory of the string moving in the dual $\mathrm{AdS}_5\times\mathrm{S}^5$ geometry. This string is a small deformation of a $\frac{1}{2}$-BPS state. A convenient description of the $\frac{1}{2}$-BPS sector (first anticipated in [33]) is in terms of the LLM coordinates introduced in [3] (Section 2.4), which are specifically constructed to describe $\frac{1}{2}$-BPS states built mainly out of Zs. In the LLM coordinates, there is a preferred LLM plane on which states that are built mainly from Zs orbit with a radius $r = 1$ (in convenient units). Consider a closed string state dual to a single trace gauge theory operator built mainly from Zs, but also containing a few magnons M. The closed string solution looks like a polygon with vertices on the unit circle. The sides of the polygon are the magnons. The specific advantage of these coordinates is that they make the analysis of the symmetries particularly simple and allow a perfect match to the $SU(2|2)^2$ superalgebra of the gauge theory described above. Matching the gauge theory and gravity descriptions in this way implies a transparent geometrical understanding of the k_i and k_i^*, as we now explain. The commutator of two supersymmetries in the dual gravity theory contains NS-B_2 gauge field transformations. As a consequence of this gauge transformation, strings stretched in the LLM plane acquire a phase which is the origin of the central charges k_i and k_i^*. It follows that we can immediately read off the central charges for any particular magnon from the sketch of the closed string worldsheet on the LLM plane: the straight line segment corresponds to a complex number which is the central charge[4].

The gauge theory operators that correspond to closed strings have a bare dimension that grows, at most, as $\sqrt{N}$. We are interested in operators whose bare dimension grows as N when the large N limit is taken. These operators include systems of giant graviton branes. The key difference as far as the sketch of the state on the LLM plane is concerned, is that the giant gravitons can orbit on circles of radius $r < 1$ while dual giant gravitons orbit on circles of radius $r > 1$. The magnons populating open strings which are attached to the giant gravitons can be divided into boundary magnons (which sit closest to the ends of the open string) and bulk magnons. The boundary magnons will stretch from a giant graviton located at $r \neq 1$ to the unit circle, while bulk magnons stretch between points on the unit circle. We will also consider the case below that the entire open string is given by a single magnon, in which case it will stretch between two points with $r \neq 1$.

The computation of correlators of the corresponding operators in the field theory is highly non-trivial. Indeed, as a consequence of the fact that we now have order N fields in our operators, the number of ribbon graphs that can be drawn is huge. These enormous combinatoric factors easily overpower the usual $\frac{1}{N^2}$ suppression of non-planar diagrams so that both planar and non-planar diagrams must be summed to capture even the leading large N limit of the correlator[74]. This problem can be overcome by employing group representation theory techniques. The article [55] showed that it is possible to compute the correlation functions of operators built from any number of Zs exactly, by using the Schur polynomials as a basis for the local operators of the theory. In [75] these results were elegantly explained by pointing out that the organization of operators in terms of Schur polynomials is an organization in terms of projection operators. Completeness and orthogonality of the basis follows from the completeness and orthogonality of the underlying projectors. With these insights[55, 75], many new directions opened up. A basis for the local operators which organizes the theory using the quantum numbers of the global symmetries was given in [76, 77]. Another basis, employing projectors related to the Brauer algebra was put forward in [78] and developed in a number of interesting works[58, 79, 80, 81, 82, 83, 84]. For the systems we are interested in, the most convenient basis to use is provided by the restricted Schur polynomials. Inspired by the Gauss Law which will arise in the world volume description of the giant graviton branes, the authors of [59] suggested operators in the gauge theory that are dual to excited giant graviton brane states. This inspired idea was pursued both in the case that the open strings are described by an open string word[13, 14, 15] and in the case of minimal open strings, with each open string represented by a single magnon[61, 56]. The operators introduced in [13, 61] are the restricted Schur polynomials. Further, significant progress was made in understanding the spectrum of anomalous dimensions of these operators in the studies[14, 15, 7, 6, 71, 2, 8, 70]. Extensions which consider orthogonal and symplectic gauge groups and other new ideas, have also been achieved[85, 86, 87, 88, 89, 90].

In this chapter we will connect the string theory description and the gauge theory description of the operators corresponding to systems of excited giant graviton branes. Our study gives a concrete description of the central charges k_i and some of the consequences of the $su(2|2)$ symmetry. We will see that the restricted Schur polynomials provide a natural description of the quantum brane states. For the open strings we find a description in terms of open spin chains with boundaries and we explain precisely what the boundary interactions are. The double coset ansatz of the gauge theory, which solves the problem of minimal open strings consisting entirely of a single magnon, also has an immediate and natural interpretation in the same framework.

There are closely related results which employ a different approach to the questions considered in this chapter. A collective coordinate approach to study giant gravitons with their excitations has been pursued in [91, 92, 93, 94, 95]. This technique employs a complex collective coordinate for the giant graviton state, which has a geometric interpretation in terms of the fermion droplet (LLM) description of half BPS states[33, 3]. The motivation for this collective coordinate starts from the observation that within semiclassical gravity, we think of the D-branes as being localized in the dual spacetime geometry. It might seem however, that since in the field theory the operators we write down have a precise $\mathcal{R}$-charge and a fixed energy, they are dual to a delocalized state. Indeed, since gauge/gravity duality is a quantum equivalence it is subject to the uncertainty principle of quantum mechanics. The $\mathcal{R}$-charge of an operator is the angular momentum of the dual states in the gravity theory, so that by the uncertainty principle, the dual giant graviton-branes must be fully delocalized in the conjugate angle in the geometry. The collective coordinate parameterizes coherent states, which do not have a definite $\mathcal{R}$-charge and so may permit a geometric interpretation of the position of the D-brane as the value of the collective coordinate. With the correct choice for the coherent states, mixing between different states of a definite $\mathcal{R}$-charge would be taken into account and so when diagonalizing the dilatation operator (for example) the mixing between states with different choices of the values of the collective coordinate might be suppressed. This computation would be, potentially, much simpler than a direct computation utilizing operators with a definite $\mathcal{R}$-charge. Of course, by diagonalizing the dilatation operator for operators dual to giant graviton brane plus open string states, one would expect to recover the collective coordinates, but this may only be possible after a complicated mixing problem in degenerate perturbation theory is solved. Some of the details that have emerged from our study do not support this semiclassical reasoning. Specifically, we find that the brane states are given by restricted Schur polynomials and these do not receive any corrections when the perturbation theory problem is solved, so that there does not seem to be any need to solve a mixing problem which constructs localized states from delocalized ones. Our large N eigenstates do have a definite $\mathcal{R}$-charge. The nontrivial perturbation theory problem involves mixing between operators corresponding to the same giant graviton branes, but with different open string words attached. Thus, it is an open string state mixing problem, solved with a discrete Fourier transform, as it was for the closed string. However, there is general agreement between the approaches: the Fourier transform solves a collective coordinate problem which diagonalizes momentum, rather than position.

For an interesting recent study of anomalous dimensions, at finite N, using a very different approach, see [96].

This chapter is organized as follows: In Section 3.2 we recall the relevant facts about the restricted Schur polynomials. The action of the dilatation operator on these restricted Schur polynomials is studied in Section 3.3 and the eigenstates of the dilatation operator are constructed in Section 3.4. Section 3.5 provides the dual string theory interpretation of these eigenstates and perfect agreement between the energies of the string theory states and the corresponding eigenvalues of the dilatation operator is demonstrated. In Section 3.6 and Section 3.7 we consider the problem of magnon scattering, both in the bulk and off the boundary magnons. We have checked that the magnon scattering matrix we compute is consistent with scattering results obtained in the weak coupling limit of the theory. One important conclusion is that the spin chain is not integrable. In Section 3.8 we review the double coset ansatz and describe the dual string theory interpretation of these results. Our conclusions and some discussion is given in Section 3.9. The Appendices collect some technical details.

3.2 Giants with open strings attached

In this section we will review the gauge theory description of the operators dual to giant graviton branes with open string excitations. In this description, each open string is described by a word with order $\sqrt{N}$ letters. Most of the letters are the Z field. There are however $M \sim O(1)$ impurities which are the magnons of the spin chain. For simplicity we will usually take all of the impurities to be a second complex matrix Y. This idea was first applied in [67] to reproduce the spectrum of small fluctuations of giant gravitons [69]. The description was then further developed in [97, 68, 98, 99, 100]. The articles [98, 99, 100] in particular developed this description to the point where interesting dynamical questions[1] could be asked and answered. The open string words are then inserted into a sea of Zs which make up the giant graviton brane(s). Concretely, the operators we consider are

$$\begin{aligned}&O(R,R_1^k,R_2^k;\{n_i\}_1,\{n_i\}_2,\cdots,\{n_i\}_k)\\&=\frac{1}{n!}\sum_{\sigma\in S_{n+k}}\chi_{R,R_1^k,R_2^k}(\sigma)Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}(W_k)^{i_{n+1}}_{i_{\sigma(n+1)}}\cdots(W_2)^{i_{n+k-1}}_{i_{\sigma(n+k-1)}}(W_1)^{i_{n+k}}_{i_{\sigma(n+k)}}\end{aligned}\tag{3.7}$$

where the open string words are

$$(W_I)^i_j=(YZ^{n_1}YZ^{n_2-n_1}Y\cdots YZ^{n_{M_I}-n_{M_I-1}}Y)^i_j\,.\tag{3.8}$$

A more detailed introduction to our operators is given in Section 2.7. We have used the notation $\{n_i\}_I$ in (3.7) to describe the integers $\{n_1,n_2,\cdots,n_{M_I}\}$ which appear in the Ith open string word. This is a lattice notation, which lists the number of Zs appearing to the left of each of the Ys, starting from the second Y: the Zs form a lattice and the n_i give a position in this lattice. This notation is particularly convenient when we discuss the action of the dilatation operator. We will also find an occupation notation useful. The occupation notation lists the number of Zs between consecutive Ys, and is indicated by placing the n_i in brackets. Thus, for example $O(R,R_1^1,R_2^1,\{n_1,n_2,n_3\})=O(R,R_1^1,R_2^1,\{(n_1),(n_2-n_1),(n_3-n_2)\})$. R is a Young diagram with $n+k$ boxes. A bound state of p_s giant gravitons and p_a dual giant gravitons is described by a Young diagram R with p_a rows, each containing order N boxes and p_s columns, each containing order N boxes. $\chi_{R,R_1^k,R_2^k}(\sigma)$ is a restricted character [13] given by

$$\chi_{R,R_1^k,R_2^k}(\sigma)=\mathrm{Tr}_{R_1^k,R_2^k}\left(\Gamma_R(\sigma)\right)\tag{3.9}$$

[1]For example, one could consider the force exerted by the string on the giant.

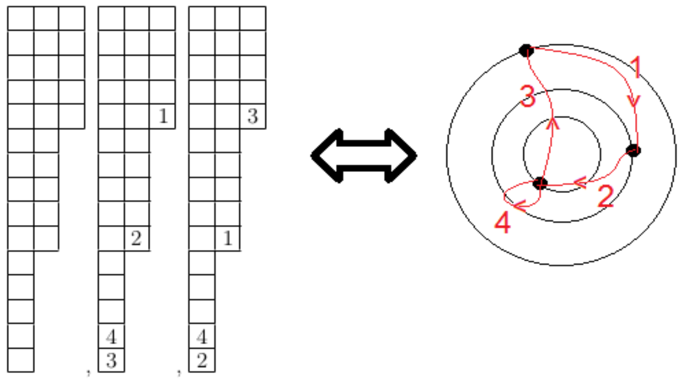


Figure 3.1: A cartoon illustrating the R, R_1^k, R_2^k labelling for an example with $k = 4$ open strings and 3 giant gravitons. The shape of the strings stretching between the giants is not realistic - only the locations of the end points of the open strings are accurate. The giant gravitons are orbiting on the circles shown; the radius shown for each orbit is accurate. They wrap an S^3 which is transverse to the plane on which they orbit. The smaller the radius of the giant's orbit, the larger the S^3 it wraps. The size of the S^3 that the giant wraps is given by its momentum, which is equal to the number of boxes in the column which corresponds to the giant. The numbers appearing in the boxes of R_1^4 tell us where the open strings start and the numbers appearing in the boxes of R_2^4 where they end.

R^k is a Young diagram with n boxes, that is, it is a representation of S_n. The irreducible representation R of S_{n+k} is reducible if we restrict to the S_n subgroup. R^k is one of the representations that arise upon restricting. In general, any such representation will be subduced more than once. Above we have used the subscripts 1 and 2 to indicate this. We have in mind a Gelfand-Tsetlin like labelling to provide a systematic way to describe the possible R^k we might consider. In this labelling, we use the transformation of the representation under the chain of subgroups $S_{n+k} \supset S_{n+k-1} \supset S_{n+k-2} \supset \cdots \supset S_n$. This is achieved by labelling boxes in R. Dropping the boxes with labels $\leq i$, we obtain the representation of S_{n+k-i} to which R^k belongs. We have to spell out how this chain of subgroups are embedded in S_{n+k}. Think of S_q as the group which permutes objects labelled $1, 2, 3, \cdots, q$. Here we have $q = n + k$ and the objects we have in mind are the Z fields or the open string words. We associate an integer to an object by looking at the upper indices in (3.7); as an example, the open string described by W_2 is object number $n+k-1$. To go from S_{n+k-i} to $S_{n+k-i-1}$, we keep only the permutations that fix $n+k-i$. We can put the states in R_1^k and R_2^k into a 1-to-1 correspondence. The trace $\mathrm{Tr}_{R_1^k, R_2^k}$ sums the column index over R_1^k and the row index over R_2^k. If we associate the row and column indices with the endpoints of the open string, we can associate the endpoints of the open string I with the box labelled I in R_1^k and R_2^k. The numbers appearing in the boxes of R_1^k literally tell us where the k open strings start and the numbers in R_2^k where the k open strings end. See Figure 3.1 for an example of this labelling. Each Y in an open string word is a magnon. We will take the number of magnons $M_I = O(1)\ \forall I$. The $Z^{i_j}_{i_{\sigma(j)}}$ with $1 \leq j \leq n$ belong to the system of giants and the Z's appearing in W_I belong to the Ith open string. It is clear that $n \sim O(N)$.

Each giant graviton is associated with a long column and each dual giant graviton with a long row in the Young diagrams labelling the restricted Schur polynomial. Our notation for the Young diagrams is to list row lengths. Thus a Young diagram that has two columns, one of length n_1 and the second of length n_2 with $n_2 < n_1$ is denoted $(2^{n_2}, 1^{n_1-n_2})$, while a Young diagram with two rows, one of length n_1 and one of length n_2 ($n_1 > n_2$) is denoted (n_1, n_2).

We want to use the results of [13, 14, 15] to study correlation functions of these operators. The correlators are obtained by summing all contractions between the Zs belonging to the giants, and by grouping the open string words in pairs and summing only the planar diagrams between the fields in each pair of the open string words. To justify the planar approximation for the open string words we take $n_i \geq 0$ and $\sum_{i=1}^{L} n_i \leq O(\sqrt{N})$. For a nice careful discussion of related issues, see [101].

We can put these operators into correspondence with normalized states

$$O(R, R_1^k, R_2^k; \{n_i\}_1, \{n_i\}_2, \cdots, \{n_i\}_k) \leftrightarrow |R, R_1^k, R_2^k; \{n_i\}_1, \{n_i\}_2, \cdots, \{n_i\}_k\rangle \tag{3.10}$$

by using the usual state-operator correspondence available for any conformal field theory. In what follows we will mainly use the state language.

3.3 Action of the Dilatation Operator

The one loop dilatation operator, in the $SU(2)$ sector, is[40, 66]

$$D = -\frac{g_{YM}^2}{8\pi^2}\mathrm{Tr}\left([Y, Z]\left[\frac{d}{dY}, \frac{d}{dZ}\right]\right) \tag{3.11}$$

Our goal in this section is to review the action of this dilatation operator on the restricted Schur polynomials, which was constructed in general in [14, 15] (reviewed in Section 2.8). When we act with D on $O(R, R_1^k, R_2^k; \{n_i\}_1, \{n_i\}_2, \cdots, \{n_i\}_k)$ the derivative with respect to Y will act on a Y belonging to a specific open string word. Thus, in the large N limit we can decompose the action of D into a sum of terms, with each individual term being the action on a specific open string. If we act on a magnon belonging to the bulk of the open string word, then the only contribution comes by acting with the derivative respect to Z on a field that is immediately adjacent to the magnon. We act only on the adjacent Z fields because to capture the large N limit we should use the planar approximation for the open string word contractions. To illustrate the action on a bulk magnon, consider the operator corresponding to a single giant graviton with a single open string attached. The giant has momentum n so that R is a single column with $n+1$ boxes: $R = 1^{n+1}$. Further, $R_1^1 = R_2^1 = 1^n$. The open string has three magnons and hence we can describe the corresponding state as $|1^{n+1}, 1^n, 1^n; \{n_1, n_2\}\rangle$. The action on the bulk magnon at large N is

$$D_{\text{bulk magnon}}|1^{n+1}, 1^n, 1^n; \{(n_1), (n_2)\}\rangle = \frac{g_{YM}^2 N}{8\pi^2}\Big[2|1^{n+1}, 1^n, 1^n; \{(n_1), (n_2)\}\rangle$$
$$-|1^{n+1}, 1^n, 1^n; \{(n_1-1), (n_2+1)\}\rangle - |1^{n+1}, 1^n, 1^n; \{(n_1+1), (n_2-1)\}\rangle\Big] \tag{3.12}$$

If we act on a magnon which occupies either the first or last position of the open string word, we realize one of the four possibilities listed below.

1. The derivative with respect to Z acts on the Z adjacent to the Y, belonging to the open string and the coefficient of the product of derivatives with respect to Y and Z replaces these fields in the same order. None of the labels of the state change. This term has a coefficient of 1[14, 15].

2. The derivative with respect to Z acts on the Z adjacent to the Y, belonging to the open string word and the coefficient of the product of derivatives with respect to Y and Z replaces these fields in the opposite order. In this case, a Z has moved out of the open string word and into its own slot in the restricted Schur polynomial - a hop off interaction in the terminology of [14]. In the process the Young diagram labelling the excited giant graviton grows by a single box. If the string is attached to a giant graviton, the column the endpoint of the relevant open string belongs to inherits the extra box. If the string is attached to a dual giant graviton, the row the endpoint of the relevant open string belongs to inherits the extra box. The coefficient of this term is given by minus one times the square root of the factor associated with the open string box divided by N[14, 15]. We remind the reader that a box in row i and column j is assigned the factor $N - i + j$.

3. The derivative with respect to Z acts on a Z belonging to the giant and the coefficient of the product of derivatives with respect to Y and Z replaces these fields in the opposite order. In this case, a Z has moved from its own slot in the restricted Schur polynomial and onto the open string word - a hop on interaction in the terminology of [14]. In the process the Young diagram labelling the giant graviton shrinks by a single box. The details of which column/row shrinks is exactly

parallel to the discussion in point 2 above. The coefficient of this term is given by minus one times the square root of the factor associated with the open string box divided by N[14, 15].

4. The derivative with respect to Z acts on a Z belonging to the giant and the coefficient of the product of derivatives with respect to Y and Z replaces these fields in the same order. This is a kissing interaction in the terminology of [14]. None of the labels of the state change. The coefficient of this term is given by the factor associated with the open string box divided by N[14, 15].

For the example we are considering the dilatation operator has the following large N action on the magnons closest to the string endpoints

$$D_{\text{first magnon}}|1^{n+1},1^n,1^n;\{(n_1),(n_2)\}\rangle = \frac{g_{YM}^2 N}{8\pi^2}\Big[\Big(1+1-\frac{n}{N}\Big)|1^{n+1},1^n,1^n;\{(n_1),(n_2)\}\rangle \\ -\sqrt{1-\frac{n}{N}}\left(|1^{n+2},1^{n+1},1^{n+1};\{(n_1-1),(n_2)\}\rangle + |1^n,1^{n-1},1^{n-1};\{(n_1+1),(n_2)\}\rangle\right)\Big] \tag{3.13}$$

and

$$D_{\text{last magnon}}|1^{n+1},1^n,1^n;\{(n_1),(n_2)\}\rangle = \frac{g_{YM}^2 N}{8\pi^2}\Big[\Big(1+1-\frac{n}{N}\Big)|1^{n+1},1^n,1^n;\{(n_1),(n_2)\}\rangle \\ -\sqrt{1-\frac{n}{N}}\left(|1^{n+2},1^{n+1},1^{n+1};\{(n_1),(n_2-1)\}\rangle + |1^n,1^{n-1},1^{n-1};\{(n_1),(n_2+1)\}\rangle\right)\Big] \tag{3.14}$$

There are a few points worth noting: The complete action of the dilatation operator can be read from the Young diagram labels of the operator. The factors of the boxes in the Young diagram for the endpoints of a given open string determine the action of the dilatation operator on that open string. When the labels $R_1^k \neq R_2^k$, the string end points are on different giant gravitons and the two endpoints are associated with different boxes in the Young diagram so that the action of the dilatation operator on the two boundary magnons is distinct. To determine these endpoint interactions we must go beyond the planar approximation. Notice that for a maximal giant graviton we have $n = N$. In this case, most of the boundary magnon terms in the Hamiltonian vanish and the boundary magnons are locked in place at the string endpoints. The giant graviton brane is simply supplying a Dirichlet boundary condition for the open string. For non-maximal giants, all of the boundary magnon terms are non-zero and, for example, Z fields that belong to the open string can wander into slots describing the giant. Alternatively, since the split between open string and brane is probably not very sharp, we might think that the magnons can wander from the string endpoints into the bulk of the open string. The coefficient of these hopping terms is modified by the presence of the giant graviton, so that the boundary magnons do not behave in the same way as the bulk magnons do.

As a final example, consider a dual giant graviton which carries momentum n. In this case, R is a single row of n boxes and we have

$$D_{\text{first magnon}}|n+1,n,n;\{(n_1),(n_2)\}\rangle = \frac{g_{YM}^2 N}{8\pi^2}\Big[\Big(1+1+\frac{n}{N}\Big)|n+1,n,n;\{(n_1),(n_2)\}\rangle \\ -\sqrt{1+\frac{n}{N}}\left(|n+2,n+1,n+1;\{(n_1-1),(n_2)\}\rangle + |n,n-1,n-1;\{(n_1+1),(n_2)\}\rangle\right)\Big] \tag{3.15}$$

In the appendix 3.A we discuss the action of the dilatation operator at two loops.

3.4 Large N Diagonalization: Asymptotic States

We are now ready to construct eigenstates of the dilatation operator. We will not construct exact large N eigenstates. Rather, we focus on states for which all magnons are well separated. From these states we can still obtain the anomalous dimensions. In section 3.6 we will describe how one might use these asymptotic states to construct exact eigenstates, following [16, 44]. In the absence of integrability however, this can not be carried to completion and our states are best thought of as very good approximate eigenstates.

The Zs in the open string word define a lattice on which the Ys hop. Our construction entails taking a Fourier transform on this lattice. The boundary interactions allow Zs to move onto and out of the lattice, so the lattice size is not fixed. It is not clear what the Fourier transform is, if the size of the lattice varies. The goal of this section is to deal with these complications. With each application of the one-loop dilatation operator, a single Z can enter or leave the open string word. At γ loops at most γ Zs can enter or leave. At any finite loop order (γ) the change in length $\Delta L = \gamma$ of the lattice is finite while the total length L of the lattice is $\sqrt{N}$. Thus, at large N the ratio $\frac{\Delta L}{L} \to 0$ and we can treat the lattice length as fixed. This observation is most easily used by first introducing "simple states" that have a definite number of Zs, in the lattice associated to each open string. This is accomplished by relaxing the identification of the open string word with the lattice. The dilatation operator's action now allows magnons to move off the open string, mixing simple states with states that are not simple. However, by modifying these simple states we can build states that are closed under the action of the dilatation operator. Our simple states are defined by taking a "Fourier transform" of the states (3.10). The simplest system to consider is that of a single giant, with a single string attached, excited by only two magnons (i.e. only boundary magnons - no bulk magnons). The string word is composed using J Z fields and the complete operator using $J+n$ Zs. Introduce the phases

$$q_a = e^{\frac{i2\pi k_a}{J}} \tag{3.16}$$

with $k_a = 0, 1, ..., J-1$. As a consequence of the fact that the lattice is a discrete structure, momenta are quantized with the momentum spacing set by the inverse of the total lattice size. This explains the choice of phases in (3.16). The simple states we consider are thus given by

$$\begin{aligned} |q_1, q_2\rangle &= \sum_{m_1=0}^{J-1} \sum_{m_2=0}^{m_1} q_1^{m_1} q_2^{m_2} |1^{n+m_1-m_2+1}, 1^{n+m_1-m_2}, 1^{n+m_1-m_2}; \{J - m_1 + m_2\}\rangle \\ &+ \sum_{m_2=0}^{J-1} \sum_{m_1=0}^{m_2} q_1^{m_1} q_2^{m_2} |1^{n+J+m_1-m_2+1}, 1^{n+J+m_1-m_2}, 1^{n+J+m_1-m_2}; \{m_2 - m_1\}\rangle \end{aligned} \tag{3.17}$$

This Fourier transform is a transform on the lattice describing the open string worldsheet. The two magnons sit at positions m_1 and m_2 on this lattice. If $m_2 > m_1$, there are $m_2 - m_1$ Zs between the magnons. If $m_1 > m_2$, there are $J + m_2 - m_1$ Zs between the magnons. The Zs before the first magnon of the string and after the last magnon of the string, are mixed up with the Zs of the giant - they do not sit on the open string word. All of the terms in (3.17) are states with different positions for the two magnons, but each is a giant that contains precisely n Zs with an open string attached, and the open string contains precisely J Zs. We can't distinguish where the string begins and where the giant ends: the open string and giant morph smoothly into each other. This is in contrast to the case of a maximal giant graviton, where the magnons mark the endpoints of the open string[2]. If this interpretation is consistent we must recover the expected inner product on the lattice and we do: Consider a giant with momentum n. An open string with a lattice of J sites is attached to the giant. The string is excited by M magnons, at positions $n_1, ..., n_{M-1}$ and n_M, with $n_{j+1} > n_j$. The corresponding normalized states, denoted by $|n; J; n_1, n_2, \cdots, n_k\rangle$ will obey[3]

$$\langle n; J; n_1, m_2, \cdots, m_M | n, J, n_1, n_2, \cdots, n_M\rangle = \delta_{m_2 n_2} \cdots \delta_{m_M n_M} \quad n_{k+1} > n_k, m_{k+1} > m_k\,. \tag{3.18}$$

This is the statement that, up to the ambiguity of where the open string starts, the magnons must occupy the same sites for a non-zero overlap. It is clear that ($G(x) \equiv 1^{x+1}, 1^x, 1^x$ and again, $n_{j+1} > n_j, m_{j+1} > m_j$)

$$\langle G(n + J + m_1 - m_2); \{m_2, \cdots, m_M\} | G(n + J + n_1 - n_2); \{n_2, \cdots, n_M\}\rangle = \delta_{m_2 n_2} \cdots \delta_{m_k n_k}$$

[2]For the maximal giant graviton, the boundary magnons are not able to hop and so sit forever at the end of the open string. For a non-maximal giant graviton the boundary magnons can hop. Even if they are initially placed at the string endpoint, they will soon explore the bulk of the string.

[3]As a consequence of the fact that it is not possible to distinguish where the open string begins and where the giant ends, there is no delta function setting the positions of the first magnons to be equal to each other - we have put this constraint in by hand in (3.18).

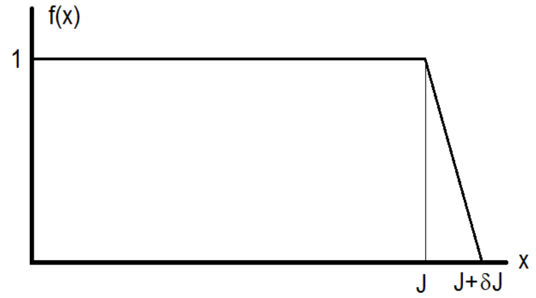


Figure 3.2: The cutoff function used in constructing large N eigenstates

reproducing the lattice inner product. The simple states are an orthogonal set of states. To check this, compute the coefficient c_a of the state $|1^{n+a+1}, 1^{n+a}, 1^{n+a}; \{J-a\}\rangle$. Looking at the two terms in (3.17) we find the following two contributions

$$\begin{aligned} c_a &= \sum_{m_1=a}^{J-1} q_1^{m_1} q_2^{m_1-a} + \sum_{m_1=0}^{a-1} q_1^{m_1} q_2^{m_1-a} \\ &= \begin{cases} J q_2^{-a} & \text{if} \quad k_1+k_2=0 \\ 0 & \text{if} \quad k_1+k_2 \neq 0 \end{cases} \end{aligned} \tag{3.19}$$

Thus, $q_1 = q_2^{-1}$ to get a non-zero result. We will see that this zero lattice momentum constraint maps into the constraint that the $su(2|2)$ central charges of the complete magnon state must vanish. Our simple states are then given by setting $q_2 = q_1^{-1}$ and are labelled by a single parameter q_1; denote the simple states using a subscript s as $|q_1\rangle_s$.

The asymptotic large N eigenstates are a small modification of these simple states. When we apply the dilatation operator to the simple states nothing prevents the boundary magnons from "hopping past the endpoints of the open string", so the simple states are not closed under the action of the dilatation operator. We need to relax the sharp cut off on the magnon movement, by allowing the sums that appear in (3.17) above to be unrestricted. We accomplish this by introducing a "cut off" function, shown in Figure 3.2. In terms of this cut off function $f(\cdot)$ our eigenstates are

$$\begin{aligned} |\psi(q_1)\rangle &= \sum_{m_2=0}^{n+J} \sum_{m_1=0}^{m_2} f(m_2) q_1^{m_1-m_2} |1^{n+J+m_1-m_2+1}, 1^{n+J+m_1-m_2}, 1^{n+J+m_1-m_2}; \{m_2-m_1\}\rangle \\ &+ \sum_{m_1=0}^{J+m_2} \sum_{m_2=0}^{n} f(m_1) f(J-m_1+m_2) q_1^{m_1-m_2} |1^{n+m_1-m_2+1}, 1^{n+m_1-m_2}, 1^{n+m_1-m_2}; \{J-m_1+m_2\}\rangle \end{aligned} \tag{3.20}$$

The dilatation operator can not arrange that the number of Zs between two magnons becomes negative. Thus, any bounds on sums in the definition of our simple states enforcing this are respected. On the other hand, the dilatation operator allows boundary magnons to hop arbitrarily far beyond the open string endpoint. Bounds in the sums for simple states enforcing this are not respected. Replace these bounds enforced as the upper limit of a sum, by bounds enforced by the cut off function. From Figure 3.2 we see that the cut off function is defined using a parameter δJ. We require that $\frac{\delta J}{J} \to 0$ as $N \to \infty$, so that at large N the difference between these eigenstates and the simple states $|q_1\rangle_s$ vanishes, as demonstrated in Appendix 3.B. We also want to ensure that

$$f(i) = f(i+1) + \epsilon \qquad \forall i \tag{3.21}$$

with $\epsilon \to 0$ as $N \to \infty$. (3.21) is needed to ensure that we do indeed obtain an eigenstate. It is straight forward to choose a function $f(x)$ with the required properties. We could for example choose δJ to be of order $N^{\frac{1}{4}}$. Our large N answers are not sensitive to the details of the cut off function $f(x)$. When $1/N$ corrections to the eigenstates are computed $f(x)$ may be more constrained and we may need to reconsider the precise form of the cut off function and how we implement the bounds.

It is now straight forward to verify that, at large N, we have

$$\begin{aligned} D|\psi(q_1)\rangle &= 2\times\frac{Ng_{YM}^2}{8\pi^2}\left(1+\left[1-\frac{n}{N}\right]-\sqrt{1-\frac{n}{N}}(q_1+q_1^{-1})\right)|\psi(q_1)\rangle \\ &= 2g^2\left(1+\left[1-\frac{n}{N}\right]-\sqrt{1-\frac{n}{N}}(q_1+q_1^{-1})\right)|\psi(q_1)\rangle \end{aligned} \tag{3.22}$$

The analysis for the dual giant graviton of momentum n leads to

$$\begin{aligned} D|\psi(q_1)\rangle &= 2\times\frac{Ng_{YM}^2}{8\pi^2}\left(1+\left[1+\frac{n}{N}\right]-\sqrt{1+\frac{n}{N}}(q_1+q_1^{-1})\right)|\psi(q_1)\rangle \\ &= 2g^2\left(1+\left[1+\frac{n}{N}\right]-\sqrt{1+\frac{n}{N}}(q_1+q_1^{-1})\right)|\psi(q_1)\rangle \end{aligned} \tag{3.23}$$

The generalization to include more magnons is straight forward. We will simply consider increasingly complicated examples and for each simply quote the final results. The discussion is most easily carried out using the occupation notation. For example, the simple states corresponding to three magnons are

$$\begin{aligned} |q_1,q_2,q_3\rangle &= \sum_{n_3=0}^{J-1}\sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_2} q_1^{n_1}q_2^{n_2}q_3^{n_3}|G(n+J+n_1-n_3);\{(n_2-n_1),(n_3-n_2)\}\rangle \\ &+ \sum_{n_1=0}^{J-1}\sum_{n_3=0}^{n_1}\sum_{n_2=0}^{n_3} q_1^{n_1}q_2^{n_2}q_3^{n_3}|G(n+n_1-n_3);\{(J+n_2-n_1),(n_3-n_2)\}\rangle \\ &+ \sum_{n_2=0}^{J-1}\sum_{n_1=0}^{n_2}\sum_{n_3=0}^{n_1} q_1^{n_1}q_2^{n_2}q_3^{n_3}|G(n+n_1-n_3);\{(n_2-n_1),(J+n_3-n_2)\}\rangle \end{aligned} \tag{3.24}$$

where we have again lumped together the Young diagram labels $G(x)=R,R_1^1,R_2^1=1^{x+1},1^x,1^x$. The coefficient of the ket $|G(n+J-a-b);\{(a),(b)\}\rangle$ is given by the sum

$$\sum_{n_1=0}^{J-1}(q_1q_2q_3)^{n_1}q_2^aq_3^{a+b} \tag{3.25}$$

which vanishes if $k_1+k_2+k_3\neq 0$. Consequently we can set $q_3=q_1^{-1}q_2^{-1}$. Including the cut off function, our energy eigenstates are given by

$$\begin{aligned} |\psi(q_1,q_2)\rangle = &\sum_{n_3=0}^{\infty}\sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_2} q_1^{n_1-n_3}q_2^{n_2-n_3}f(n_3)|G(n+J+n_1-n_3);\{(n_2-n_1),(n_3-n_2)\}\rangle \\ + &\sum_{n_1=0}^{J+n_2}\sum_{n_3=0}^{\infty}\sum_{n_2=0}^{n_3} q_1^{n_1-n_3}q_2^{n_2-n_3}f(n_1)f(J+n_3-n_1)|G(n+n_1-n_3);\{(J+n_2-n_1),(n_3-n_2)\}\rangle \\ + &\sum_{n_2=0}^{J+n_3}\sum_{n_1=0}^{n_2}\sum_{n_3=0}^{\infty} q_1^{n_1-n_3}q_2^{n_2-n_3}f(n_2)f(J+n_3-n_1)|G(n+n_1-n_3);\{(n_2-n_1),(J+n_3-n_2)\}\rangle \end{aligned}$$

It is a simple matter to see that

$$D|\psi(q_1,q_2)\rangle = (E_1+E_2+E_3)|\psi(q_1,q_2)\rangle \tag{3.26}$$

where

$$\begin{aligned} E_1 &= g^2\left(1+\left[1-\frac{n}{N}\right]-\sqrt{1-\frac{n}{N}}(q_1+q_1^{-1})\right) \\ E_2 &= g^2\left(2-q_2-q_2^{-1}\right) \\ E_3 &= g^2\left(1+\left[1-\frac{n}{N}\right]-\sqrt{1-\frac{n}{N}}(q_3+q_3^{-1})\right) \end{aligned} \tag{3.27}$$

Now consider the extension to states containing many magnons: For an M magnon state, consider all M cyclic orderings of the "magnon positions"

$$
\begin{array}{l}
n_1 \le n_2 \le n_3 \le \cdots \le n_{M-2} \le n_{M-1} \le n_M \le J-1 \\
n_M \le n_1 \le n_2 \le n_3 \le \cdots \le n_{M-2} \le n_{M-1} \le J-1 \\
n_{M-1} \le n_M \le n_1 \le n_2 \le n_3 \le \cdots \le n_{M-2} \le J-1 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
n_2 \le n_3 \le \cdots \le n_{M-2} \le n_{M-1} \le n_M \le n_1 \le J-1
\end{array} \tag{3.28}
$$

Construct the differences $\{n_2 - n_1, n_3 - n_2, n_4 - n_3, \cdots, n_M - n_{M-1}, n_1 - n_M\}$. Every difference except for one is positive. Add J to the difference that is negative, i.e. the resulting differences are $\{\Delta_2, \Delta_3, \Delta_4, \cdots, \Delta_M, \Delta_1\}$ with

$$
\Delta_i = \begin{cases} n_i - n_{i-1} & \text{if} \quad n_i \ge n_{i-1} \\ \\ J + n_i - n_{i-1} & \text{if} \quad n_i \le n_{i-1} \end{cases} \tag{3.29}
$$

For each ordering in (3.28) we have a term in the simple state. This term is obtained by summing over all values of $\{n_1, n_2, \cdots, n_L\}$ consistent with the ordering considered, of the following summand

$$
q_1^{n_1} q_2^{n_2} \cdots q_L^{n_M} |1^{n+\Delta_1+1}, 1^{n+\Delta_1}, 1^{n+\Delta_1}; \{(\Delta_2), (\Delta_3), \cdots, (\Delta_M)\}\rangle \tag{3.30}
$$

Repeating the argument we outlined above, this term vanishes unless $q_M^{-1} = q_1 q_2 \cdots q_{M-1}$ so that the summand can be replaced by

$$
q_1^{n_1-n_M} q_2^{n_2-n_M} \cdots q_{M-1}^{n_{M-1}-n_M} |1^{n+\Delta_1+1}, 1^{n+\Delta_1}, 1^{n+\Delta_1}; \{(\Delta_2), (\Delta_3), \cdots, (\Delta_M)\}\rangle \tag{3.31}
$$

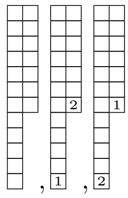
Finally, consider the extension to many string states and an arbitrary system of giant graviton branes. Each open string word is constructed as explained above. We add extra columns (one for each giant graviton) and rows (one for each dual giant graviton) to R. The labels R_1^k and R_2^k specify how the open strings are connected to the giant and dual giant gravitons. When describing twisted string states, the strings describe a closed loop, "punctuated by" the giant gravitons on which they end. As an example, consider a two giant graviton state, with a pair of strings stretching between the giant gravitons. The two strings carry a total momentum of J. Notice that we are using the two strings to define a single lattice of J sites. One might have thought that the two strings would each define an independent lattice. To understand why we use the two strings to define a single lattice, recall that we are identifying the zero lattice momentum constraint with the constraint that the $su(2|2)$ central charges of the complete magnon state must vanish. There is a single $su(2|2)$ constraint on the two string state, not one constraint for each string. We interpret this as implying there is a single zero lattice momentum constraint for the two strings, and hence there is a single lattice for the two strings. This provides a straight forward way to satisfy the $su(2|2)$ central charge constraints. The first giant graviton has a momentum of b_0 and the second a momentum of b_1. The first string is excited by M magnons with locations $\{n_1, n_2, \cdots, n_{M-1}, n_M\}$ and the second by $\tilde{M}$ magnons with locations $\{\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_{\tilde{M}-1}, \tilde{n}_{\tilde{M}}\}$ where we have switched to the lattice notation. We need to consider the $M + \tilde{M}$ orderings of the $\{n_i\}$ and $\{\tilde{n}_i\}$. Given a specific pair of orderings, we can again form the differences

$$
\begin{aligned}
\Delta_1 &= \begin{cases} n_1 - \tilde{n}_M & \text{if} \quad n_1 \ge \tilde{n}_M \\ J + n_1 - \tilde{n}_M & \text{if} \quad n_1 \le \tilde{n}_M \end{cases} \\
\Delta_i &= \begin{cases} n_i - n_{i-1} & \text{if} \quad n_i \ge n_{i-1} \\ \\ J + n_i - n_{i-1} & \text{if} \quad n_i \le n_{i-1} \end{cases} \qquad i = 2, 3, \cdots, M \\
\Delta_{M+1} &= \begin{cases} \tilde{n}_1 - n_M & \text{if} \quad n_M \le \tilde{n}_1 \\ J + \tilde{n}_1 - n_M & \text{if} \quad n_M \ge \tilde{n}_1 \end{cases} \\
\Delta_{M+i} &= \begin{cases} \tilde{n}_i - \tilde{n}_{i-1} & \text{if} \quad \tilde{n}_i \ge \tilde{n}_{i-1} \\ \\ J + \tilde{n}_i - \tilde{n}_{i-1} & \text{if} \quad \tilde{n}_i \le \tilde{n}_{i-1} \end{cases} \qquad i = 2, 3, \cdots, \tilde{M}
\end{aligned} \tag{3.32}
$$

For each ordering we again have a term in the simple state, obtained by summing over all values of $\{n_1, n_2, \cdots, n_M, \tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_{\tilde{M}}\}$ consistent with the ordering considered, of the following summand

$$q_1^{n_1} \cdots q_M^{n_M} \tilde{q}_1^{\tilde{n}_1} \cdots \tilde{q}_{\tilde{M}}^{\tilde{n}_{\tilde{M}}} |G(\Delta_1, \Delta_{M+1}); \{(\Delta_2), (\Delta_3), \cdots, (\Delta_M)\}, \{(\Delta_{M+2}), (\Delta_{M+3}), \cdots, (\Delta_{M+\tilde{M}})\}\rangle \tag{3.33}$$

where

$G(x,y) \equiv$  (3.34)

In the first Young diagram above there are $b_1 + y + 1$ rows with 2 boxes in each row and $b_0 + x - b_1 - y - 1$ rows with 1 box in each row. Repeating the argument we outlined above, this term vanishes unless $\tilde{q}_{\tilde{M}}^{-1} = q_1 \cdots q_M \tilde{q}_1 \cdots \tilde{q}_{\tilde{M}-1}$ so that the summand can be replaced by

$$q_1^{n_1 - \tilde{n}_{\tilde{M}}} q_2^{n_2 - \tilde{n}_{\tilde{M}}} \cdots \tilde{q}_{\tilde{M}-1}^{\tilde{n}_{\tilde{M}-1} - \tilde{n}_{\tilde{M}}} |G(\Delta_1, \Delta_{M+1}); \{(\Delta_2), (\Delta_3), \cdots, (\Delta_M)\}, \{(\Delta_{M+2}), (\Delta_{M+3}), \cdots, (\Delta_{M+\tilde{M}})\}\rangle \tag{3.35}$$

This completes our discussion of the large N asymptotic eigenstates. We will now consider the dual string theory description of these states.

3.5 String Theory Description

The string theory description of the gauge theory operators is most easily developed using the limit introduced by Maldacena and Hofman[4], in which the spectrum on both sides of the correspondence simplifies. The limit considers operators of large $\mathcal{R}$ charge J and scaling dimension Δ holding $\Delta - J$ and the 't Hooft coupling λ fixed. Both sides of the correspondence enjoy an $SU(2|2) \times SU(2|2)$ supersymmetry with novel central extensions as realized by Beisert in [16, 44]. Once the central charge of the spin-chain/worldsheet excitations have been determined, their spectrum and constraints on their two body scattering are determined. A powerful conclusion argued for in [4] using the physical picture developed in [73] is that there is a natural geometric interpretation for these central charges in the classical string theory. This geometric interpretation also proved useful in the analysis of maximal giant gravitons in [17]. In this section we will argue that it is also applicable to the case of non-maximal giant and dual giant gravitons.

Giant gravitons carry a dipole moment under the RR five form flux F_5. When they move through the spacetime, the Lorentz force like coupling to F_5 causes them to expand in directions transverse to the direction in which they move[51]. The giant graviton orbits on a circle inside the S^5 and wraps an S^3 transverse to this circle but also contained in the S^5. Using the complex coordinates $x = x^5 + ix^6$, $y = x^3 + ix^4$ and $z = x^1 + ix^2$ the S^5 is described by

$$|z|^2 + |x|^2 + |y|^2 = 1 \tag{3.36}$$

in units with the radius of the S^5 equal to 1. The giant is orbiting in the $1-2$ plane on the circle $|z| = r$. The size to which the giant expands is determined by cancelling the force causing them to expand, due to the coupling to the F_5 flux, against the D3 brane tension, which causes them to shrink. Since the coupling to the F_5 flux depends on their velocity, the size of the giant graviton is determined by its angular momentum n as [48, 49, 50]

$$|x|^2 + |y|^2 = \frac{n}{N} \tag{3.37}$$

Using (3.36) we see that the giant graviton orbits on a circle of radius[48]

$$r = \sqrt{1 - \frac{n}{N}} < 1 \tag{3.38}$$

Consider now the worldsheet geometry for an open string attached to a giant graviton. Following [4], we will describe this worldsheet solution using LLM coordinates[3]. The worldsheet for this solution, in these coordinates, is shown in Figure 3.3. The figure shows an open string with 6 magnons. Each magnon corresponds to a directed line segment in the figure. The first and last magnons connect to the giant which is orbiting on the smaller circle shown. Between the magnons we have a collection of $O(\sqrt{N})$ Zs. These are pushed by a centrifugal force to the circle $|z| = 1$ giving the string worldsheet the shape shown in the figure.

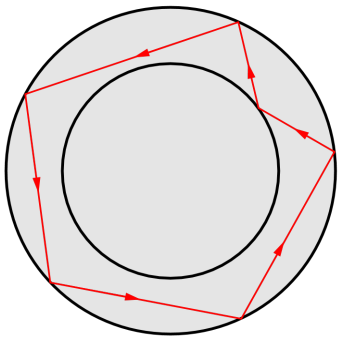


Figure 3.3: The giant is orbiting on the smaller circle shown. Each red segment is a magnon. The arrows in the figure simply indicate the orientation of the central charge k_i of the ith magnon.

In the limit that the magnons are well separated, each magnon transforms in a definite $SU(2|2)^2$ representation. The open string itself transforms as the tensor product of the individual magnon representations. The representation of each individual magnon is specified by giving the values of the central charges k_i, k_i^* appearing in (3.5). Regarding the plane shown in Figure 3.3 as the complex plane, k is given by the complex number determined by the vector describing the directed segment corresponding to the magnon. In particular, the magnitude of k is given by the length of the line corresponding to the magnon. The energy of the magnon, which transforms in a short representation, is determined by supersymmetry to be[16, 44]

$$E = \sqrt{1 + 2\lambda |k|^2} = 1 + \lambda |k|^2 - \frac{1}{2}\lambda^2 |k|^4 + ... \tag{3.39}$$

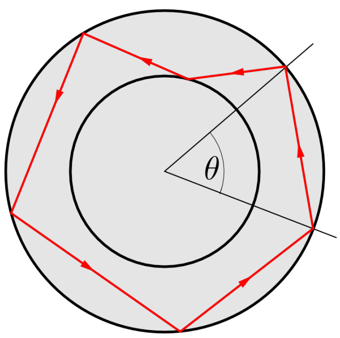


Figure 3.4: A bulk magnon subtending an angle θ has a length of $2\sin\frac{\theta}{2}$.

For a magnon which subtends an angle θ we find[4]

$$E = 1 + 4\lambda \sin^2\frac{\theta}{2} + O(\lambda^2) = 1 + \lambda(2 - e^{i\theta} - e^{-i\theta}) + O(\lambda^2) \tag{3.40}$$

This is in perfect agreement with the field theory answer (3.27) if we set $\lambda = g^2$ and

$$q = e^{i\frac{2\pi k}{J}} = e^{i\theta} \qquad \Rightarrow \qquad \theta = \frac{2\pi k}{J} \tag{3.41}$$

Thus the angle that is subtended by the magnon is equal to its momentum, which is the well-known result obtained in [4]. Consider now the boundary magnon, as shown in Figure 3.5. The circle on which the giant orbits has a radius given by

$$r = \sqrt{1 - \frac{n}{N}} \tag{3.42}$$

The large circle has a radius of 1 in the units we are using. Thus, the length of the boundary magnon is given by the length of the diagonal of the isosceles trapezium shown in Figure 3.5. Consequently

$$\begin{aligned} E &= 1 + \lambda((1-r)^2 + 4r\sin^2\frac{\theta}{2}) + O(\lambda^2) \\ &= 1 + \lambda\left(1 + r^2 - r(e^{i\theta} + e^{-i\theta})\right) + O(\lambda^2) \end{aligned} \tag{3.43}$$

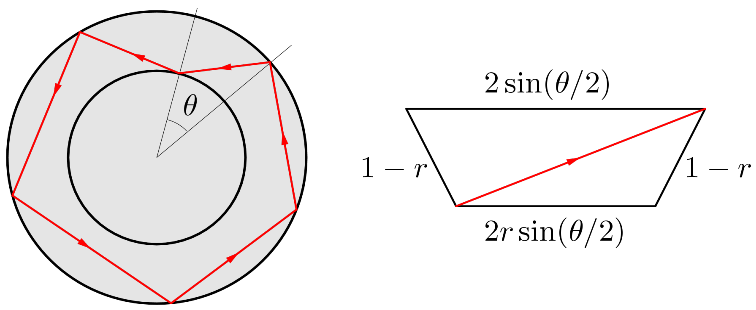


Figure 3.5: A boundary magnon subtending an angle θ has a length of $\sqrt{(1-r)^2 + 4r\sin^2\frac{\theta}{2}}$.

This is again in complete agreement with (3.27) after we set $\theta = \frac{2\pi k}{J}$ and recall that $r = \sqrt{1 - \frac{n}{N}}$. This is a convincing check of the boundary terms in the dilatation operator and of our large N asymptotic eigenstates. In the description of maximal giant gravitons, the boundary magnon always stretches from the centre of the disk to a point on the circumference of the circle $|z| = 1$. Consequently, for the maximal giant the boundary magnon subtends an angle of zero and it never has a non-zero momentum. For submaximal giants we see that the boundary magnons do in general carry non-zero momentum. This is completely expected: in the case of a maximal giant graviton, the boundary magnons are locked in the first and last position of the open string lattice. As we move away from the maximal giant graviton, the coefficients of the boundary terms which allow the boundary magnons to hop in the lattice, increase from zero, allowing the boundary magnons to move and hence, to carry a non-zero momentum. In the Appendix 3.A we have checked that the two loop answer in the field theory agrees with the $O(\lambda^2)$ term of (3.39).

Notice that the vector sum of the directed lines segments vanishes. This is nothing but the statement that our operator vanishes unless $q_M^{-1} = q_1 q_2 \cdots q_{M-1}$. This condition ensures that although each magnon transforms in a representation of $su(2|2)^2$ with non-zero central charges, the complete state enjoys an $su(2|2)^2$ symmetry that has no central extension. It is for this reason that the central charges must sum to zero and hence that the vector sum of the red segments must vanish. This is achieved in an interesting way for certain multi-string states: each open string can transform under an $su(2|2)^2$ that has a non-zero central charge and it is only for the full state of all open strings plus giants that the central charge vanishes. An example of this for a two string state is given in Figure 3.6.

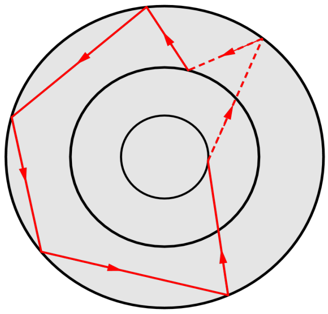


Figure 3.6: A two strings attached to two giant gravitons state. Both giants are submaximal and so are moving on circles with a radius $|z| < 1$. One of the strings has only two boundary magnons. The second string has two boundary magnons and three bulk magnons. Notice that each open string has a non-vanishing central charge. It is only for the full state that the central charge vanishes. See [95] for closely related observations.

To conclude this section, we will consider an example involving a dual giant graviton. In this case, the giant graviton orbits on a circle[49, 50]

$$r = \sqrt{1 + \frac{n}{N}} > 1 \tag{3.44}$$

The length of the line segment corresponding to the boundary magnon is again given by the length of the diagonal of an isosceles trapezium, as shown in Figure 3.7. Consequently

$$\begin{aligned} E &= 1 + \lambda((r-1)^2 + 4r\sin^2\frac{\theta}{2}) + O(\lambda^2) \\ &= 1 + \lambda\left(1 + r^2 - r(e^{i\theta} + e^{-i\theta})\right) + O(\lambda^2) \end{aligned} \tag{3.45}$$

which is in perfect agreement with (3.23) after we set $\theta = \frac{2\pi k}{J}$ and $r = \sqrt{1 + \frac{n}{N}}$.

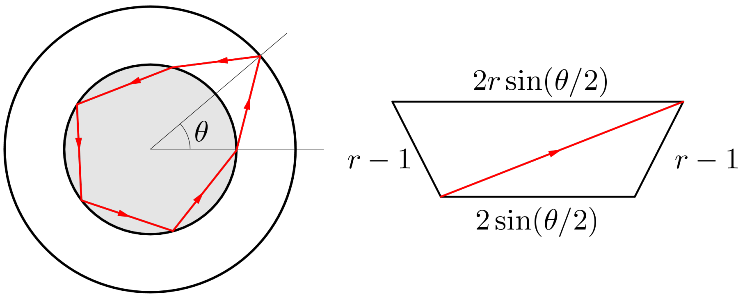


Figure 3.7: A boundary magnon subtending an angle θ has a length of $\sqrt{(r-1)^2 + 4r\sin^2\frac{\theta}{2}}$.

3.6 From asymptotic states to exact eigenstates

The states we have written down above are asymptotic states in the sense that we have implicitly assumed that all of the magnons are well separated. In this case the excitations can be treated individually and the symmetry algebra acts as a tensor product representation. However, the magnons can come close together and even swap positions. When they swap positions, we get different asymptotic states that must be combined to obtain the exact eigenstate. The asymptotic states must be combined in a way that is compatible with the algebra, as explained in [16]. This requirement ultimately implies a unique way to complete the asymptotic states to obtain the exact eigenstate.

When two bulk magnons swap positions, the corresponding asymptotic states are combined using the two particle S-matrix. The relevant two particle S-matrix has been determined in [16, 44]. It is also possible for a bulk magnon to reflect/scatter off a boundary magnon. For maximal giant gravitons[17], the reflection from the boundary preserves the fact that the boundary magnon has zero momentum and it reverses the sign of the momentum of the bulk magnon. In this section we would like to investigate the scattering of a bulk magnon off a boundary magnon for a non-maximal giant graviton.

We must require that the total central charge k of the state vanishes. Thus, after the scattering the directed line segments must still sum to zero. Further the central charge C of the state must remain unchanged. Taken together, these conditions uniquely fix the momentum of both bulk and boundary magnon after the scattering.

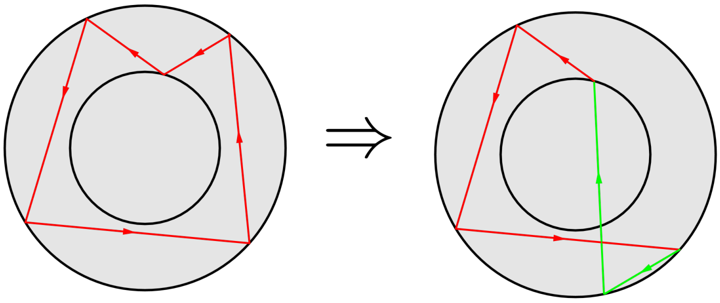


Figure 3.8: A bulk magnon scatters with a boundary magnon. In the process the direction of the momentum of the bulk magnon is reversed.

In Figure 3.8 the process of scattering a bulk magnon off the boundary magnon is shown. After the scattering the magnons have a different momentum, corresponding to line segments that have changed and these are shown in green. In this case the giant graviton is close enough to a maximal giant that the momentum of the boundary magnon is reversed, so this is a reflection-like scattering. Before and after the scattering the line segments line up to form a closed circuit, so that the central charge k of the state before and after scattering is zero. To analyze the constraint arising from fixing the central charge C, we parameterize the problem as shown in figure 3.9. There is a single parameter θ which is fixed by requiring

$$\begin{aligned}&\sqrt{1+8\lambda\sin^2\frac{\varphi_2}{2}}+\sqrt{1+8\lambda\left([1+r]^2+4r\sin^2\frac{\varphi_1}{2}\right)}\\&=\sqrt{1+8\lambda\sin^2\frac{\theta}{2}}+\sqrt{1+8\lambda\left([1+r]^2+4r\sin^2\left(\frac{\varphi_1+\varphi_2+\theta}{2}\right)\right)}\end{aligned} \tag{3.46}$$

which is the condition that the state has the correct central charge C. In the above formula we have

$$r=\sqrt{1-\frac{b_0}{N}}\,. \tag{3.47}$$

The equation (3.46) has two solutions, one of which is negative $\theta=-\varphi_2$ and describes the state before the scattering. We need to choose the solution for which $\theta\neq-\varphi_2$. Notice that for $b_0=N$ this condition implies that $\theta=\varphi_2$ which is indeed the correct answer[17]. In this case, the bulk magnon reflects off the boundary with a reverse in the direction of its momentum but no change in its magnitude. The momentum of the bulk magnon remains zero. When $b_0=0$ the momenta of the two magnons is exchanged which is again the correct answer [16, 44]. When $0<b_0<N$ we find the solution to (3.46) for the momentum of the bulk magnon interpolates between reflection like scattering (when the momentum of the magnon is reversed) and magnon like scattering (when the momenta of the two magnons are exchanged). In this case though, in general, the magnitude of the momenta of the bulk and the boundary magnons are not preserved by the scattering - the scattering is inelastic.

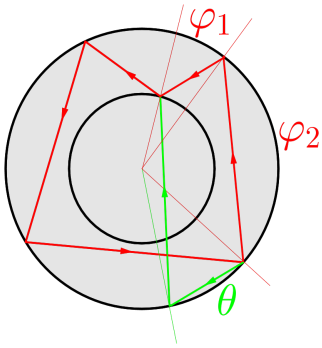


Figure 3.9: A bulk magnon scatters with a boundary magnon. In the process the direction of the momentum of the bulk magnon is reversed. Before the scattering the boundary magnon subtends an angle φ_1 and the bulk magnon subtends an angle φ_2. After the scattering the boundary magnon subtends an angle $\varphi_1 + \varphi_2 + \theta$ and the bulk magnon subtends an angle $-\theta$.

The fact that the scattering between boundary and bulk magnons is not elastic has far reaching consequences. First, the system will not be integrable. In the case of purely elastic scattering for all magnon scatterings, the number of asymptotic states that must be combined to construct the exact energy eigenstate is roughly $(M-1)!$ for M magnons. This is the number of ways of arranging the magnons (distinguished by their momentum) up to cyclicity. There are M magnon momenta appearing and these momenta are the same for all the asymptotic states. The exact eigenstates can then be constructed using a coordinate space Bethe ansatz. For the case of inelastic scattering, the momenta appearing depend on the specific asymptotic state one considers and there are many more than $(M-1)!$ asymptotic states that must be combined to construct the exact eigenstate. In this case constructing the exact eigenstates from the asymptotic states appears to be a formidable problem.

3.7 S-matrix and boundary reflection matrix

We have a good understanding of the symmetries of the theory and the representations under which the states transform. Following Beisert [16, 44], this is all that is needed to obtain the magnon scattering matrix. In this section we will carry out this analysis.

Each magnon transforms under a centrally extended representation of the $SU(2|2)$ algebra

$$\{Q^\alpha_a, Q^\beta_b\} = \epsilon^{\alpha\beta}\epsilon_{ab}\frac{k_i}{2}\,, \qquad \{S^a_\alpha, S^b_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta}\frac{k_i^*}{2}\,, \tag{3.48}$$

$$\{S^a_\alpha, Q^\beta_b\} = \delta^a_b L^\beta_\alpha + \delta^\beta_\alpha R^a_b + \delta^a_b\delta^\beta_\alpha C_i\,. \tag{3.49}$$

There are also the usual commutators for the bosonic $su(2)$ generators. There are three central charges k_i, k_i^*, C_i for each $SU(2|2)$ factor. Following [17] we set the central charges of the two copies to be equal. The action of the bosonic part of the $SU(2|2)^2$ symmetry in the gauge theory is reviewed in Section 2.5.1.2. To specify the representation that each magnon transforms in, following [16, 44] we specify parameters a_k, b_k, c_k, d_k for each magnon, where

$$Q^\alpha_a|\phi^b\rangle = a_k\delta^b_a|\psi^\alpha\rangle\,, \qquad Q^\alpha_a|\psi^\beta\rangle = b_k\epsilon^{\alpha\beta}\epsilon_{ab}|\phi^b\rangle\,, \tag{3.50}$$

$$S^a_\alpha|\phi^b\rangle = c_k\epsilon_{\alpha\beta}\epsilon^{ab}|\psi^\beta\rangle\,, \qquad S^a_\alpha|\psi^\beta\rangle = d_k\delta^\beta_\alpha|\phi^a\rangle\,, \tag{3.51}$$

for the kth magnon. We are using the non-local notation of [44]. In Section 2.5.1.3, it is shown how the algebra relations can be used to obtain expressions for the central charges in terms of these parameters as

$$k_k = 2\,a_k\,b_k$$

$$k_k^* = 2\, c_k\, d_k$$
$$C_k = \frac{1}{2}(a_k d_k + b_k c_k).$$

An equation which enforces the condition that we have an atypical representation of $su(2|2)$ is also found to be satisfied [44]:

$$a_k d_k - b_k c_k = 1.$$

Consider next a state with a total of K magnons; if we are to obtain a representation without central extension, we must require that the central charges vanish

$$\frac{k}{2} = \sum_{k=1}^{K} \frac{k_k}{2} = \sum_{k=1}^{K} a_k b_k = 0\,,$$
$$\frac{k^*}{2} = \sum_{k=1}^{K} \frac{k_k^*}{2} = \sum_{k=1}^{K} c_k d_k = 0\,. \tag{3.52}$$

Following [16], a useful parameterization for the parameters of the representation, when considering bulk magnons, is given by

$$a_k = \sqrt{g}\eta_k\,, \qquad b_k = \frac{\sqrt{g}}{\eta_k} f_k \left(1 - \frac{x_k^+}{x_k^-}\right)\,, \tag{3.53}$$

$$c_k = \frac{\sqrt{g} i \eta_k}{f_k x_k^+}\,, \qquad d_k = \frac{\sqrt{g} x_k^+}{i\eta_k} \left(1 - \frac{x_k^-}{x_k^+}\right)\,. \tag{3.54}$$

The parameters $x_k^\pm$ are set by the momentum p_k of the magnon

$$e^{i\frac{2\pi p_k}{J}} = \frac{x_k^+}{x_k^-}\,. \tag{3.55}$$

The parameter f_k is a pure phase, given by the product $\prod_j e^{ip_j}$, where j runs over all magnons to the left of the magnon considered. To ensure unitarity $|\eta_k|^2 = i(x_k^- - x_k^+)$. The condition $a_k d_k - b_k c_k = 1$ to get an atypical representation implies that

$$x_k^+ + \frac{1}{x_k^+} - x_k^- - \frac{1}{x_k^-} = \frac{i}{g}\,. \tag{3.56}$$

This equation will be very useful in verifying some of the S-matrix formulas given below. A useful parameterization for the parameters specifying the representation for a boundary magnon is given by

$$a_k = \sqrt{g}\eta_k\,, \qquad b_k = \frac{\sqrt{g}}{\eta_k} f_k \left(1 - r\frac{x_k^+}{x_k^-}\right)\,, \tag{3.57}$$

$$c_k = \frac{\sqrt{g} i \eta_k}{f_k x_k^+}\,, \qquad d_k = \frac{\sqrt{g} x_k^+}{i\eta_k} \left(1 - r\frac{x_k^-}{x_k^+}\right)\,, \tag{3.58}$$

where $r = \sqrt{1 - \frac{n}{N}}$ is the radius of the path on which the giant graviton of momentum n orbits[4] and the parameters $x_k^\pm$ are again set by the momentum carried by the boundary magnon according to (3.55). For the boundary magnon, f_k is again a phase as described above and now $|\eta_k|^2 = i(rx_k^- - x_k^+)$. For a maximal giant graviton $r = 0$ and the boundary magnon carries no momentum and $|\eta_k|^2 = -ix_k^+$. For the boundary magnon, the condition $a_k d_k - b_k c_k = 1$ to get an atypical representation implies that

$$x_k^+ + \frac{1}{x_k^+} - rx_k^- - \frac{r}{x_k^-} = \frac{i}{g} \tag{3.59}$$

[4]For an open string attached to a dual giant graviton, we would have $r = \sqrt{1 + \frac{n}{N}}$ where n is the momentum of the dual giant graviton.

This equation will again be useful below. Equation (3.59) interpolates between (3.56) for $r = 1$, which is the correct condition for a bulk magnon and the condition obtained for $r = 0$

$$x_k^+ + \frac{1}{x_k^+} = \frac{i}{g} \tag{3.60}$$

which was used in [17] for the boundary magnon attached to a maximal giant graviton.

Following [16, 44] one can check that the above parameterization obeys (3.52). Finally, the energy central charge for the boundary magnons is

$$\begin{aligned} a_k b_k c_k d_k &= g^2(re^{-ip_k} - 1)(re^{ip_k} - 1) = g^2((1-r)^2 + 4r\sin^2\frac{p_k}{2}) \\ &= \frac{1}{4}\Big[(a_k d_k + b_k c_k)^2 - (a_k d_k - b_k c_k)^2\Big] = \frac{1}{4}\Big[(2C_k)^2 - 1\Big] \end{aligned} \tag{3.61}$$

so that

$$C_k = \pm\sqrt{\frac{1}{4} + g^2((1-r)^2 + 4r\sin^2\frac{p_k}{2})}. \tag{3.62}$$

This expression again interpolates between the bulk ($r = 1$, (2.72)) and maximal boundary ($r = 0$) results[17].

The components of an energy eigenstate in different asymptotic regions are related by the bulk-bulk and boundary-bulk magnon scattering matrices S and R. S and R must commute with the $su(2|2)$ group. The labels of the representations of individual magnons can change under the scattering but they must do so in a way that preserves the central charges of the total state. In the picture of the energy eigenstates provided by the LLM plane, the central charges are given by the directed line segments (which are vectors and hence can also be viewed as complex numbers), one for each magnon. The fact that these line segments close into polygons is the statement that the central charges k and k^* of our total state vanish. The sum of the lengths squared of these line segments determines the central charge C. By scattering, these segments can rearrange themselves as long as the sums $\sum_i \sqrt{1 + 2\lambda l_i^2}$ with l_i the length of segment i is preserved and so long as they still form a closed polygon.

Consider now the scattering of two bulk magnons, magnon k and magnon $k+1$. The quantum numbers of the two incoming magnons and those of the outgoing magnons (denoted with a prime) are as follows

$$\begin{aligned} a_k &= \sqrt{g}\eta_k & a'_k &= a_k \\ b_k &= \frac{\sqrt{g}}{\eta_k} f_k \left(1 - \frac{x_k^+}{x_k^-}\right) & b'_k &= \frac{x_{k+1}^+}{x_{k+1}^-} b_k \\ c_k &= \frac{\sqrt{g} i \eta_k}{f_k x_k^+} & c'_k &= \frac{x_{k+1}^-}{x_{k+1}^+} c_k \\ d_k &= \frac{\sqrt{g} x_k^+}{i\eta_k}\left(1 - \frac{x_k^-}{x_k^+}\right) & d'_k &= d_k \end{aligned} \tag{3.63}$$

$$\begin{aligned} a_{k+1} &= \sqrt{g}\eta_{k+1} & a'_{k+1} &= a_{k+1} \\ b_{k+1} &= \frac{x_k^+}{x_k^-}\frac{\sqrt{g}}{\eta_{k+1}} f_k \left(1 - \frac{x_{k+1}^+}{x_{k+1}^-}\right) & b'_{k+1} &= \frac{\sqrt{g}}{\eta_{k+1}} f_k \left(1 - \frac{x_{k+1}^+}{x_{k+1}^-}\right) \\ c_{k+1} &= \frac{x_k^-}{x_k^+}\frac{\sqrt{g} i \eta_{k+1}}{f_k x_{k+1}^+} & c'_{k+1} &= \frac{\sqrt{g} i\eta_{k+1}}{f_k x_{k+1}^+} \\ d_{k+1} &= \frac{\sqrt{g} x_{k+1}^+}{i\eta_{k+1}}\left(1 - \frac{x_{k+1}^-}{x_{k+1}^+}\right) & d'_{k+1} &= d_{k+1}\,. \end{aligned} \tag{3.64}$$

We will also study the scattering of a bulk magnon with a boundary magnon. Denoting the quantum numbers of the boundary magnon with a subscript b and the quantum numbers of the bulk magnon without a subscript, the quantum numbers of the magnons before and after the reflection are as follows

$$a = \sqrt{g}\eta \qquad a' = \sqrt{g}\eta'$$

$$b = \frac{\sqrt{g}}{\eta'} f\left(1 - \frac{x^+}{x^-}\right) \qquad b' = \frac{\sqrt{g}}{\eta'} f\left(1 - \frac{x^{+\prime}}{x^{-\prime}}\right)$$
$$c = \frac{\sqrt{g} i\eta}{f x^+} \qquad c' = \frac{\sqrt{g} i\eta'}{f x^{+\prime}}$$
$$d = \frac{\sqrt{g} x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right) \qquad d' = \frac{\sqrt{g} x^{+\prime}}{i\eta'}\left(1 - \frac{x^{-\prime}}{x^{+\prime}}\right) \tag{3.65}$$

$$a_b = \sqrt{g}\eta_b \qquad a'_b = \sqrt{g}\eta'_b$$
$$b_b = \frac{x^+}{x^-}\frac{\sqrt{g}}{\eta_b} f\left(1 - r\frac{x_b^+}{x_b^-}\right) \qquad b'_b = \frac{\sqrt{g}}{\eta'_b} f \frac{x^{+\prime}}{x^{-\prime}}\left(1 - r\frac{x_b^{+\prime}}{x_b^{-\prime}}\right)$$
$$c_b = \frac{x^-}{x^+}\frac{\sqrt{g} i\eta_b}{f x_b^+} \qquad c'_b = \frac{x^{-\prime}}{x^{+\prime}}\frac{\sqrt{g} i\eta'_b}{f x_b^{+\prime}}$$
$$d_b = \frac{\sqrt{g} x_b^+}{i\eta_b}\left(1 - \frac{x_b^-}{x_b^+}\right) \qquad d'_b = \frac{\sqrt{g} x_b^{+\prime}}{i\eta'_b}\left(1 - \frac{x_b^{-\prime}}{x_b^{+\prime}}\right) \tag{3.66}$$

where $\frac{x^{+\prime}}{x^{-\prime}} = e^{-i\theta}$, $\frac{x_b^{+\prime}}{x_b^{-\prime}} = \frac{x^+ x_b^+ x^{-\prime}}{x^- x_b^- x^{+\prime}}$ and we solve (3.46) for θ.

Implementing the consequences of invariance under $SU(2|2)^2$ is exactly parallel to the analysis of [16, 44, 17]. For completeness we will review the S-matrix describing the scattering of two bulk magnons. Since the S-matrix has to commute with the bosonic $su(2)$ generators Schur's Lemma implies that it must be proportional to the identity in each given irreducible representation of $su(2)$. This immediately implies that

$$S_{12}|\phi_1^a \phi_2^b\rangle = A_{12}|\phi_{2'}^{\{a}\phi_{1'}^{b\}}\rangle + B_{12}|\phi_{2'}^{[a}\phi_{1'}^{b]}\rangle + \frac{1}{2}C_{12}\epsilon^{ab}\epsilon_{\alpha\beta}|\psi_{2'}^{\alpha}\psi_{1'}^{\beta}\rangle \tag{3.67}$$

$$S_{12}|\psi_1^\alpha \psi_2^\beta\rangle = D_{12}|\psi_{2'}^{\{\alpha}\psi_{1'}^{\beta\}}\rangle + E_{12}|\psi_{2'}^{[\alpha}\psi_{1'}^{\beta]}\rangle + \frac{1}{2}F_{12}\epsilon_{ab}\epsilon^{\alpha\beta}|\phi_{2'}^{a}\phi_{1'}^{b}\rangle \tag{3.68}$$

$$\begin{aligned} S_{12}|\phi_1^a \psi_2^\beta\rangle &= G_{12}|\psi_{2'}^{\beta}\phi_{1'}^{a}\rangle + H_{12}|\phi_{2'}^{a}\psi_{1'}^{\beta}\rangle \\ S_{12}|\psi_1^\alpha \phi_2^b\rangle &= K_{12}|\psi_{2'}^{\alpha}\phi_{1'}^{b}\rangle + L_{12}|\phi_{2'}^{b}\psi_{1'}^{\alpha}\rangle \end{aligned} \tag{3.69}$$

Next, demanding the S-matrix commutes with the supercharges implies[16, 44]

$$\begin{aligned}
A_{12} &= S_{12}^0 \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} \\
B_{12} &= S_{12}^0 \frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\left(1 - 2\frac{1 - \frac{1}{x_2^- x_1^+}}{1 - \frac{1}{x_2^- x_1^-}}\frac{x_2^+ - x_1^+}{x_2^+ - x_1^-}\right) \\
C_{12} &= S_{12}^0 \frac{2g^2\eta_1\eta_2}{f x_1^+ x_2^+}\frac{1}{1 - \frac{1}{x_1^+ x_2^+}}\frac{x_2^- - x_1^-}{x_2^- - x_1^+} \\
D_{12} &= -S_{12}^0 \\
E_{12} &= -S_{12}^0\left(1 - 2\frac{1 - \frac{1}{x_2^+ x_1^-}}{1 - \frac{1}{x_2^- x_1^-}}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\right) \\
F_{12} &= -S_{12}^0 \frac{2f(x_1^+ - x_1^-)(x_2^+ - x_2^-)}{\eta_1\eta_2 x_1^- x_2^-}\frac{1}{1 - \frac{1}{x_1^- x_2^-}}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+} \\
G_{12} &= S_{12}^0 \frac{x_2^+ - x_1^+}{x_2^- - x_1^+} \qquad H_{12} = S_{12}^0 \frac{\eta_1}{\eta_2}\frac{x_2^+ - x_2^-}{x_2^- - x_1^+} \\
K_{12} &= S_{12}^0 \frac{\eta_2}{\eta_1}\frac{x_1^+ - x_1^-}{x_2^- - x_1^+} \qquad L_{12} = S_{12}^0 \frac{x_2^- - x_1^-}{x_2^- - x_1^+}
\end{aligned} \tag{3.70}$$

Thus, the S-matrix is determined up to an overall phase. Here we have simply chosen $D_{12} = -S_{12}^0$ which specifies the overall phase. This overall phase is constrained by crossing symmetry[102].

When considering the equations for the reflection/scattering matrix describing the reflection/scattering of a bulk magnon from a boundary magnon, we need to pay attention to the fact that the central charges of the representation are no longer swapped between the two magnons. Rather, the central charges after the reflection are determined by solving (3.46). Denote the central charge of the boundary magnon before the reflection by p_B. Denote the central charge of the bulk magnon before the reflection by p_b. Denote the central charge of the boundary magnon after the reflection by k_B. Denote the central charge of the bulk magnon after the reflection by k_b. Denote the reflection/scattering matrix by $\mathcal{R}$. Invariance of the reflection/scattering matrix under the bosonic generators implies that

$$\mathcal{R}|\phi^a_{p_B}\phi^b_{p_b}\rangle = A^R_{12}|\phi^{\{a}_{k_B}\phi^{b\}}_{k_b}\rangle + B^R_{12}|\phi^{[a}_{k_B}\phi^{b]}_{k_b}\rangle + \frac{1}{2}C^R_{12}\epsilon^{ab}\epsilon_{\alpha\beta}|\psi^\alpha_{k_B}\psi^\beta_{k_b}\rangle \tag{3.71}$$

$$\mathcal{R}|\psi^\alpha_{p_B}\psi^\beta_{p_b}\rangle = D^R_{12}|\psi^{\{\alpha}_{k_B}\psi^{\beta\}}_{k_b}\rangle + E^R_{12}|\psi^{[\alpha}_{k_B}\psi^{\beta]}_{k_b}\rangle + \frac{1}{2}F^R_{12}\epsilon_{ab}\epsilon^{\alpha\beta}|\phi^a_{k_B}\phi^b_{k_b}\rangle \tag{3.72}$$

$$\begin{aligned}\mathcal{R}|\phi^a_{p_B}\psi^\beta_{p_b}\rangle &= G^R_{12}|\psi^\beta_{k_B}\phi^a_{k_b}\rangle + H^R_{12}|\phi^a_{k_B}\psi^\beta_{k_b}\rangle \\ \mathcal{R}|\psi^\alpha_{p_B}\phi^b_{p_b}\rangle &= K^R_{12}|\psi^\alpha_{k_B}\phi^b_{k_b}\rangle + L^R_{12}|\phi^b_{k_B}\psi^\alpha_{k_b}\rangle\end{aligned} \tag{3.73}$$

The analysis now proceeds as above. The result is

$$\begin{aligned}
A^R_{12} &= \frac{\eta_1\eta_2 x_1'^+ x_1^+(x_1^- - x_2^+)\left((x_2^+ - rx_2^-)(rx_2'^+ - x_2'^-)x_2^+ + (x_2^- - rx_2^+)(x_2'^+ - rx_2'^-)x_2'^+\right)}{\eta_1'\eta_2' x_2'^+ x_2^+(x_1^- - x_1^+)(x_1^+ - x_1'^+)(x_1^+(rx_2^+ - x_2^-) + x_2^-(rx_2^- - x_2^+))} \\
B^R_{12} &= A^R_{12}\left[1 + \frac{2x_2'^-(x_1'^- - x_1'^+)}{x_1'^+(x_1^- - x_2^+)(x_1'^- x_2'^- - rx_1'^+ x_2'^+)}\frac{B_1}{B_2}\right] \\
B_1 &= x_2^- x_1'^+\left[(x_1^- - x_1^+)(2x_1^- - x_1'^-)(x_2^+ x_1'^+ - x_1^+ x_2^+) - x_1'^+ x_1^-(x_2^+ - rx_2^-)(x_1^- - x_2^+)\right]\frac{rx_2'^+ - x_2'^-}{rx_2'^- - x_2'^+} \\
&+ \left[x_1^+ x_1'^+(x_1^- - x_2^+)(x_2^- - rx_2^+) + (x_1^- - x_1^+)x_2^- x_2^+(x_1'^+ - x_1^+)\right]x_1'^- x_2'^- \\
B_2 &= (rx_2^- - x_2^+)\left[x_1^+ x_2'^- x_1'^- \frac{rx_2^+ - x_2^-}{rx_2^- - x_2^+} - x_1'^+ x_1^- x_2^- \frac{rx_2'^+ - x_2'^-}{rx_2'^- - x_2'^+}\right] \\
C^R_{12} &= S^0_{12}\frac{2\eta_2\eta_1 C_1}{fx_2^+(x_1^+ - x_1'^+)(x_1^+(rx_2^+ - x_2^-) + x_2^-(rx_2^- - x_2^+))(x_1'^- x_2'^- - rx_1'^+ x_2'^+)} \\
C_1 &= x_1'^+\frac{x_1^- - x_2^+}{x_1^- - x_1^+}\left(x_1'^+ x_1^- x_2^-(x_2^+ - rx_2^-)(rx_2'^+ - x_2'^-) + x_1^+ x_1'^- x_2'^-(x_2^- - rx_2^+)(x_2'^+ - rx_2'^-)\right) \\
&+ x_2^- x_2^+(x_1^+ - x_1'^+)\Big(x_1^-(rx_1'^+ x_2'^+ + x_1'^- x_2'^- - 2x_1'^+ x_2'^-) + x_1'^- x_2'^-(rx_2'^- - x_1'^- + x_1'^+ - x_2'^+)\Big) \\
D^R_{12} &= -S^0_{12} \\
E^R_{12} &= -S^0_{12}\left[1 - 2x_1^+ x_2'^- \frac{\frac{x_1'^-}{x_1^-}(x_1'^- - x_1'^+ + x_2'^+ - rx_2'^-) - (x_1'^- - x_1'^+) - \frac{x_1'^+ x_2^-}{x_1^+ x_2'^-}\frac{x_2^+ - rx_2^-}{x_2^- - rx_2^+}(x_2'^- - rx_2'^+)}{\left[x_1^+ + x_2^-\frac{x_2^+ - rx_2^-}{x_2^- - rx_2^+}\right]\left[rx_1^{+\prime}x_2^{+\prime} - x_1^{-\prime}x_2^{-\prime}\right]}\right] \\
F^R_{12} &= S^0_{12}\frac{2x_1^+ x_1'^+ f(x_1^{-\prime} - x_1^{+\prime})(x_2'^- - rx_2'^+)(x_2^- - rx_2^+)}{\eta_1'\eta_2' x_1^- x_1^{-\prime}\left[x_1^+(x_2^- - rx_2^+) + x_2^-(x_2^+ - rx_2^-)\right]\left[x_1^{-\prime}x_2^{-\prime} - rx_1^{+\prime}x_2^{+\prime}\right]} \\
&\times\left[x_1^- - x_1'^- + \frac{rx_2^- - x_2^+}{x_2^- - rx_2^+}\frac{x_2^- x_1^-}{x_1^+} + \frac{x_2'^+ - rx_2'^-}{x_2'^- - rx_2'^+}\frac{x_1'^- x_2'^-}{x_1'^+}\right] \\
G^R_{12} &= S^0_{12}\frac{\eta_1 x_1^+\left[x_2^+(rx_2^- - x_2^+)(rx_2'^+ - x_2'^-) + x_2'^+(rx_2^+ - x_2^-)(x_2'^+ - rx_2'^-)\right]}{\eta_2' x_2'^+(x_1^- - x_1^+)\left[x_1^+(x_2^- - rx_2^+) + x_2^-(x_2^+ - rx_2^-)\right]} \\
H^R_{12} &= S^0_{12}\frac{\eta_1(x_1^{-\prime} - x_1^{+\prime})\left[x_1^- x_2^-(rx_2^- - x_2^+) + x_1^+ x_1^{-\prime}(rx_2^+ - x_2^-)\right]}{\eta_1' x_1^{-\prime}(x_1^- - x_1^+)\left[x_1^+(x_2^- - rx_2^+) + x_2^-(x_2^+ - rx_2^-)\right]} \\
K^R_{12} &= S^0_{12}\frac{\eta_2 x_2^-\left[x_1^- x_1^{+\prime}(rx_2^{+\prime} - x_2^{-\prime}) + x_1^{-\prime}x_2^{-\prime}(rx_2^{-\prime} - x_2^{+\prime})\right]}{\eta_2' x_1^{-\prime}x_2^{-\prime}\left[x_1^+(x_2^- - rx_2^+) + x_2^-(x_2^+ - rx_2^-)\right]}
\end{aligned}$$

$$L^R_{12} \;=\; S^0_{12}\frac{\eta_2 x_2^-(x_1^- - x_1^{-\prime})(x_1^{-\prime} - x_1^{+\prime})}{\eta_1' x_1^{-\prime}\Big[x_1^+(x_2^- - r x_2^+) + x_2^-(x_2^+ - r x_2^-)\Big]} \tag{3.74}$$

where

$$\frac{x_1^+}{x_1^-} = e^{ip_b} \qquad \frac{x_2^+}{x_2^-} = e^{ip_B}\;, \tag{3.75}$$

$$\frac{x_{1'}^+}{x_{1'}^-} = e^{ik_b} \qquad \frac{x_{2'}^+}{x_{2'}^-} = e^{ik_B}\;. \tag{3.76}$$

It is simple to verify that this R matrix is unitary for any value of r and any momenta, and further that it reproduces the bulk S matrix for $r = 1$ and the reflection matrix for scattering from a maximal giant graviton for $r = 0$. In performing this check we compared to the expressions in [103]. To provide a further check of these expressions, we have considered the case that the boundary and the bulk magnons have momenta that sum to π, as shown in Figure 3.10. In this situation it is very simple to compute the final momenta of the two magnons - the final momenta are minus the initial momenta. In Appendix 3.D

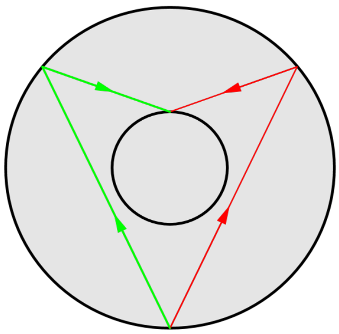


Figure 3.10: A bulk magnon scatters with a boundary magnon. The sum of the momenta of the two magnons is π. Here we only show two of the magnons; we indicate them in red before the scattering and in green after the scattering. In the process the direction of the momentum both magnons is reversed.

we have computed the value of $\frac{1}{2}\left(1+\frac{B^R_{12}}{A^R_{12}}\right)$ at one loop. We find this agrees perfectly with the answer obtained from (3.74). To perform this check, one needs to express $x^\pm$ in terms of p by solving $x^+ = x^- e^{ip}$ and (3.59) for the boundary magnon or (3.56) for the bulk magnon. Doing this we find

$$x^- = e^{-i\frac{p}{2}}\left(\frac{1}{2g\sin\frac{p}{2}} + 2g\sin\frac{p}{2}\right) + O(g^2), \tag{3.77}$$

for a bulk magnon and

$$x^- = -\frac{i}{g(r - e^{ip})} + ige^{-ip}(r - e^{ip})\frac{re^{ip} - 1}{r + e^{ip}} + O(g^2) \tag{3.78}$$

for a boundary magnon. Inserting these expansions into (3.74) and keeping only the leading order (which is g^0) at small g, we reproduce (3.117) for any allowed value of r.

It is a simple matter to verify that the boundary Yang-Baxter equation is not satisfied by this reflection matrix, indicating that the system is not integrable. This conclusion follows immediately upon verifying that changing the order in which the bulk magnons scatter with the boundary magnon leads to final states in which the magnons have different momenta. Consequently, the integrability is lost precisely because the scattering of the boundary and bulk magnons, for boundary magnons attached to a non-maximal giant graviton, is inelastic.

3.8 Links to the Double Coset Ansatz and Open Spring Theory

There is an interesting limiting case that we can consider, obtained by taking each open string word to simply be a single Y, i.e. each open string is a single magnon. In this case one must use the correlators computed in [61, 56] as opposed to the correlators computed in [13]. The case with distinguishable open strings is much simpler since when the correlators are computed, only contractions between corresponding open strings contribute; when the open strings are identical, it is possible to contract any two of them. In this case one must consider operators that treat these "open strings" symmetrically, leading to the operators constructed in [61]. In a specific limit, the action of the dilatation operator factors into an action on the Zs and an action on the Ys [71, 2]. The action on the Ys can be diagonalized by Fourier transforming to a double coset which describes how the magnons are attached to the giant gravitons[2, 8]. For an operator labelled by a Young diagram R with p long rows or columns, the action on the Zs then reduces to the motion of p particles along the real line with their coordinates given by the lengths of the Young diagram R, interacting through quadratic pair-wise interaction potentials [70]. For interesting related work see [104]. Our goal in this section is to explain the string theory interpretation of these results.

The conclusion of [2, 8] is that eigenstates of the dilatation operator given by operators corresponding to Young diagrams R that have p long rows or columns can be labelled by a graph with p vertices and directed edges. The number of directed edges matches the number of magnons Y used to construct the operator. These graphs have a natural interpretation in terms of the Gauss Law expected from the worldvolume theory of the giant graviton branes[59]. Since the giant graviton has a compact world volume, the Gauss Law implies the total charge on the giant's world volume vanishes. Each string end point is charged, so this is a constraint on the possible open string configurations: the number of strings emanating from the giant must equal the number of strings terminating on the giant. Thus, the graphs labelling the operators are simply enumerating the states consistent with the Gauss Law. To stress this connection we use the language "Gauss graphs" for the labels, we refer to the vertices of the graph as branes since each one is a giant graviton brane and we identify the directed edges as strings since each is a magnon. The action of the dilatation operator is nicely summarized by the Gauss graph labelling the operator. Count the number n_{ij} of strings (of either orientation) stretching between branes i and j in the Gauss graph. The action of the dilatation operator on the Gauss graph operator is then given by

$$DO_{R,r}(\sigma) = -\frac{g_{YM}^2}{8\pi^2}\sum_{i<j} n_{ij}(\sigma)\Delta_{ij}O_{R,r}(\sigma)\,. \tag{3.79}$$

The operator Δ_{ij} is defined in Appendix 3.C. For a proof of this, see [2, 8]. To obtain anomalous dimensions one needs to solve an eigenproblem on the R, r labels, which has been accomplished in [70] in complete generality.

For three open strings stretched between three giant gravitons we have to solve the following eigenvalue problem

$$\begin{aligned}
&\frac{g_{YM}^2}{8\pi^2}\Big[(2N-c_1-c_2+3)O(c_1,c_2,c_3)-\sqrt{(N-c_1+1)(N-c_2+1)}O(c_1+1,c_2-1,c_3)\\
&-\sqrt{(N-c_1)(N-c_2+2)}O(c_1-1,c_2+1,c_3)\Big]\\
&+\frac{g_{YM}^2}{8\pi^2}\Big[(2N-c_2-c_3+5)O(c_1,c_2,c_3)-\sqrt{(N-c_2+1)(N-c_3+3)}O(c_1,c_2-1,c_3+1)\\
&-\sqrt{(N-c_2+2)(N-c_3+2)}O(c_1,c_2+1,c_3-1)\Big]\\
&+\frac{g_{YM}^2}{8\pi^2}\Big[(2N-c_1-c_3+4)O(c_1,c_2,c_3)-\sqrt{(N-c_3+2)(N-c_1+1)}O(c_1+1,c_2,c_3-1)\\
&-\sqrt{(N-c_3+3)(N-c_1)}O(c_1-1,c_2,c_3+1)\Big]\\
&=\gamma O(c_1,c_2,c_3)
\end{aligned} \tag{3.80}$$

where c_1, c_2 and c_3 are the lengths of the columns = momenta of the three giant gravitons and γ is the anomalous dimension. At large N, approximating for example $O(c_1,c_2,c_3) = O(c_1+1,c_2,c_3-1)$ which amounts to ignoring back reaction on the giant gravitons, we have

$$\frac{g_{YM}^2 N}{8\pi^2}\Big[\sqrt{1-\frac{c_1}{N}}-\sqrt{1-\frac{c_2}{N}}\Big]^2 O(c_1,c_2,c_3)+\frac{g_{YM}^2 N}{8\pi^2}\Big[\sqrt{1-\frac{c_2}{N}}-\sqrt{1-\frac{c_3}{N}}\Big]^2 O(c_1,c_2,c_3)$$
$$+\frac{g_{YM}^2 N}{8\pi^2}\Big[\sqrt{1-\frac{c_3}{N}}-\sqrt{1-\frac{c_1}{N}}\Big]^2 O(c_1,c_2,c_3)=\gamma O(c_1,c_2,c_3)\,. \qquad (3.81)$$

The Gauss graph associated with this operator has a string stretching between the brane of momentum c_1 and the brane of momentum c_3, a string stretching between the brane of momentum c_1 and the brane of momentum c_2 and a string stretching between the brane of momentum c_2 and the brane of momentum c_3.

On the string theory side, since our magnons don't carry any momentum, we have three giants moving in the plane with magnons stretched radially between them. Identifying the central charges, we find they are radial vectors with length equal to the distance between the giants. With these central charges we can write down the energy

$$E=\sqrt{1+2\lambda(r_1-r_2)^2}+\sqrt{1+2\lambda(r_1-r_3)^2}+\sqrt{1+2\lambda(r_3-r_2)^2}\,. \qquad (3.82)$$

Using the usual translation between the momentum of the giant graviton and the radius of the circle it moves on

$$r_i=\sqrt{1-\frac{c_i}{N}} \qquad i=1,2,3 \qquad (3.83)$$

we find that the order λ term in the expansion of (3.82) precisely matches the gauge theory result (3.81).

If we don't ignore back reaction on the giant graviton, we find that (3.80) leads to a harmonic oscillator eigenvalue problem. In this case, we are keeping track of the Zs slipping past a magnon, from one giant onto the next. In this way, one of the giants will grow and one will shrink thereby changing the radius of their orbits and hence the length of the magnon stretched between them. In this process we would expect the energy to vary continuously, which is exactly what we see at large N. A specific harmonic oscillator state (see [70] for details) corresponds to two giant gravitons executing a periodic motion. In one period, the giants first come towards each other and then move away from each other again. Exciting these oscillators to any finite level, we find an energy that is of order the 't Hooft coupling divided by N. These very small energies translate into motions with a huge period.

There is an important point worth noting. The harmonic oscillator problem that arises from (3.80) is obtained by expanding (3.80) assuming that c_1-c_2 is order $\sqrt{N}$ and c_1,c_2 are of order N. The oscillator Hamiltonian then arises as a consequence of (and depends sensitively on) the order 1 shifts in the coefficients of the terms in (3.80). Thus to really trust the oscillator Hamiltonian we find we must be sure that (3.80) is accurate enough that we can expand it and the order 1 term we obtain is accurate. This is indeed the case, as we discuss in Appendix 3.C.

3.9 Conclusions

In this study we have used the descriptions of the action of the dilatation operator derived using an approach which relies heavily on group representation theory techniques, to study the anomalous dimensions of operators with a bare dimension that grows as N, as the large N limit is taken. For these operators, even just to capture the leading large N limit, we are forced to sum much more than just the planar diagrams and this is precisely what the representation theoretic approach manages to do. We have demonstrated an exact agreement with results coming from the dual gravity description, which is convincing evidence in support of this approach. It gives definite correct results in a systematic large N expansion, demonstrating that the representation theoretic methods provide a useful language and calculational framework with which to tackle the kinds of large N but non-planar limits we have studied in this chapter. Of course, we have mainly investigated the leading large N limit and the computation of $\frac{1}{N}$ corrections is an interesting problem that we hope to return to in the future.

The progress that was made in understanding the planar limit of $\mathcal{N} = 4$ super Yang-Mills theory is impressive (see [72] for a comprehensive review). Of course, much of the progress is thanks to integrability. There are however results that do not rely on integrability, only on the symmetries of the theory. In our study we clearly have a genuine extension of methods (giant magnons, the $SU(2|2)$ scattering matrix) that worked in the planar limit, into the large N but non-planar setting. Further, even though integrability does not persist, it is present when the radius r of the circle on which the graviton moves is $r = 0$ (maximal giant graviton) or $r = 1$ (point-like giant graviton). If we perturb about these two values of r, we are departing from integrability in a controlled way and hence we might still be able to exploit integrability. For more general values of r, we have managed to find asymptotic eigenstates in which the magnons are well separated and we expect these to be very good approximate eigenstates. Indeed, anomalous dimensions computed using these asymptotic eigenstates exactly agree with the dual string theory energies. Without the power of integrability it does not seem to be easy to patch together asymptotic states to obtain exact eigenstates.

We have a clearer understanding of the non-planar integrability discovered in [7, 6, 71, 2, 8, 70]. The magnons in these systems remain separated and hence free, so they are actually non-interacting. One of the giants would need to lose all of its momentum before any two magnons would scatter. It is satisfying that the gauge theory methods based on group representation theory are powerful enough to detect this integrability directly in the field theory. The results we have found here give the all loops prediction for the anomalous dimensions of these operators. In the limit when we consider a very large number of fields there would seem to be many more circumstances in which one could construct operators that are ultimately dual to free systems. This is an interesting avenue that deserves careful study, since these simple free systems may provide convenient starting points, to which interactions may be added systematically.

A possible instability associated to open strings attached to giants has been pointed out in [98]. In this case it seems that the spectrum of the spin chain becomes continuous, the ground state is no longer BPS and supersymmetry is broken. The transition that removes the BPS state is simply that the gap from the ground state to the continuum closes. Of course, the spectrum of energies is discrete but this is only evident at subleading orders in $1/N$ when one accounts for the back reaction of the giant graviton-branes. The question of whether these BPS states with given quantum numbers exist or not has been linked to a walls of stability type description [105] in [95]. It would be interesting to see if these issues can be understood using the methods of this chapter.

Appendices to Chapter 3

3.A Two Loop Computation of Boundary Magnon Energy

The dilatation operator, in the su(2) sector, can be expanded as[40, 66]

$$D = \sum_{k=0}^{\infty} \left(\frac{g_{YM}^2}{16\pi^2} \right)^k D_{2k} = \sum_{k=0}^{\infty} g^{2k} D_{2k} \,, \tag{3.84}$$

where the tree level, one loop and two loop contributions are

$$D_0 = \mathrm{Tr}\left(Z \frac{\partial}{\partial Z} \right) + \mathrm{Tr}\left(Y \frac{\partial}{\partial Y} \right) , \tag{3.85}$$

$$D_2 = -2 : \mathrm{Tr}\left([Z, Y] \left[\frac{\partial}{\partial Z}, \frac{\partial}{\partial Y} \right] \right) : , \tag{3.86}$$

$$D_4 = D_4^{(a)} + D_4^{(b)} + D_4^{(c)} \,, \tag{3.87}$$

$$\begin{aligned} D_4^{(a)} &= -2 : \mathrm{Tr}\left(\left[[Y, Z] \,, \frac{\partial}{\partial Z} \right] \left[\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z} \right] , Z \right] \right) : \\ D_4^{(b)} &= -2 : \mathrm{Tr}\left(\left[[Y, Z] \,, \frac{\partial}{\partial Y} \right] \left[\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z} \right] , Y \right] \right) : \\ D_4^{(c)} &= -2 : \mathrm{Tr}\left([[Y, Z] \,, T^a] \left[\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z} \right] , T^a \right] \right) : \,. \end{aligned} \tag{3.88}$$

The boundary magnon energy we computed above came from D_2. By computing the contribution from D_4 we can compare to the second term in the expansion of the string energies. Since we are using the planar approximation when contracting fields in the open string words, in the limit of well separated magnons, the action of D_4 can again be written as a sum of terms, one for each magnon. Thus, if we compute the action of D_4 on a state $|1^{n+1}, 1^n, 1^n; \{n_1, n_2\}\}\rangle$ with a single string and a single bulk magnon, it is a trivial step to obtain the action of D_4 on the most general state.

A convenient way to summarize the result is to quote the action of D_4 on a state for which the magnons have momenta q_1, q_2, q_3. Of course, we will have to choose the q_i so that the total central charge vanishes as explained in the article above. Thus we could replace $q_3 \to (q_1 q_2)^{-1}$ in the formulas below. We will write the answer for a general giant graviton system with strings attached. For the boundary terms, each boundary magnon corresponds to an end point of the string and each end point is associated with a specific box in the Young diagram. Denote the factor of the box corresponding to the first magnon by c_F and the factor of the box associated to the last magnon by c_L. A straight forward but somewhat lengthy computation, using the methods developed in [14, 15] gives

$$\begin{aligned} &(D_4)_{\text{first magnon}} |\psi(q_1, q_2, q_3)\rangle = \\ &\qquad -\frac{g^4}{2} \left[\left(1 + \frac{c_F}{N} \right)^2 - 2\left(1 + \frac{c_F}{N}\right) \sqrt{\frac{c_F}{N}} (q_1 + q_1^{-1}) + \frac{c_F}{N} (q_1^2 + 2 + q_1^{-2}) \right] |\psi(q_1, q_2, q_3)\rangle \\ &= -\frac{g^4}{2} \left[1 + \frac{c_F}{N} - \sqrt{\frac{c_F}{N}} (q_1 + q_1^{-1}) \right]^2 |\psi(q_1, q_2, q_3)\rangle \\ &= -\frac{1}{2} \left[g^2 \left(1 + \frac{c_F}{N} - \sqrt{\frac{c_F}{N}} (q_1 + q_1^{-1}) \right) \right]^2 |\psi(q_1, q_2, q_3)\rangle \end{aligned} \tag{3.89}$$

in perfect agreement with (3.39). The term $D_4^{(b)}$ does not make a contribution to the action on distant magnons, since we sum only the planar open string word contractions. The remaining terms $D_4^{(a)}, D_4^{(c)}$ both make a contribution to the action on distant magnons. For completeness note that

$$(D_4)_{\text{bulk magnon}} |\psi(q_1, q_2, q_3)\rangle = -\frac{1}{2} \left[2g^2 \left(2 - (q_2 + q_2^{-1}) \right) \right]^2 |\psi(q_1, q_2, q_3)\rangle \,. \tag{3.90}$$

3.B The difference between simple states and eigenstates vanishes at large N

In this section we want to quantify the claim made in section 3.4 that the difference between our simple states and our exact eigenstates vanishes in the large N limit. We will do this by computing the difference between the simple states and eigenstates and observing this difference has a norm that goes to zero in the large N limit.

For simplicity, we will consider a two magnon state. The generalization to many magnon states is straight forward. Our simple states have the form

$$\begin{aligned} |q\rangle \;&=\; \mathcal{N}\Big(\sum_{m_1=0}^{J-1}\sum_{m_2=0}^{m_1} q^{m_1-m_2}|1^{n+m_1-m_2+1},1^{n+m_1-m_2},1^{n+m_1-m_2};\{J-m_1+m_2\}\rangle \\ &+\; \sum_{m_2=0}^{J-1}\sum_{m_1=0}^{m_2} q^{m_1-m_2}|1^{n+J+m_1-m_2+1},1^{n+J+m_1-m_2},1^{n+J+m_1-m_2};\{m_2-m_1\}\rangle\Big)\,. \end{aligned} \tag{3.91}$$

Requiring that $\langle q|q\rangle = 1$ we find

$$\mathcal{N} = \frac{1}{J\sqrt{J+1}}\,. \tag{3.92}$$

With this normalization we find that the simple states are orthogonal

$$\langle q_a|q_b\rangle = \delta_{k_a k_b} + O\left(\frac{1}{J}\right) \qquad \text{where} \qquad q_a = e^{i\frac{2\pi k_a}{J}}, \quad q_b = e^{i\frac{2\pi k_b}{J}}\,. \tag{3.93}$$

This is perfectly consistent with the fact that in the planar limit the lattice states, given by

$$|1^{n+m_1-m_2+1},1^{n+m_1-m_2},1^{n+m_1-m_2};\{J-m_1+m_2\}\rangle$$

are orthogonal and our simple states are simply a Fourier transform of these.

Our eigenstates have the form (we will see in a few moments that the normalization in the next equation below is the same as the normalization in (3.92))

$$\begin{aligned} &|\psi(q)\rangle = \mathcal{N}\Big(\sum_{m_2=0}^{\infty}\sum_{m_1=0}^{m_2} f(m_2)q^{m_1-m_2}|1^{n+J+m_1-m_2+1},1^{n+J+m_1-m_2},1^{n+J+m_1-m_2};\{m_2-m_1\}\rangle \\ &+\sum_{m_1=0}^{J+m_2}\sum_{m_2=0}^{\infty} f(m_1)f(J-m_1+m_2)q^{m_1-m_2}|1^{n+m_1-m_2+1},1^{n+m_1-m_2},1^{n+m_1-m_2};\{J-m_1+m_2\}\rangle\Big) \\ &\equiv |q\rangle + |\delta q\rangle \end{aligned} \tag{3.94}$$

where

$$\begin{aligned} &|\delta q\rangle = \mathcal{N}\Big(\sum_{m_2=J}^{n+J+1}\sum_{m_1=0}^{m_2} f(m_2)q^{m_1-m_2}|1^{n+J+m_1-m_2+1},1^{n+J+m_1-m_2},1^{n+J+m_1-m_2};\{m_2-m_1\}\rangle \\ &+\sum_{m_1=J}^{J+m_2}\sum_{m_2=0}^{n+m_1} f(J-m_1+m_2)f(m_1)q^{m_1-m_2}|1^{n+m_1-m_2+1},1^{n+m_1-m_2},1^{n+m_1-m_2};\{J-m_1+m_2\}\rangle\Big) \\ &+\sum_{m_1=0}^{J-1}\sum_{m_2=m_1+1}^{n+m_1} f(J-m_1+m_2)q^{m_1-m_2}|1^{n+m_1-m_2+1},1^{n+m_1-m_2},1^{n+m_1-m_2};\{J-m_1+m_2\}\rangle\Big) \\ &= \mathcal{N}\Big(\sum_{m_2=J}^{J+\delta J}\sum_{m_1=0}^{m_2} f(m_2)q^{m_1-m_2}|1^{n+J+m_1-m_2+1},1^{n+J+m_1-m_2},1^{n+J+m_1-m_2};\{m_2-m_1\}\rangle \\ &+\sum_{m_1=J}^{l_-}\sum_{m_2=0}^{J+\delta J} f(J-m_1+m_2)f(m_1)q^{m_1-m_2}|1^{n+m_1-m_2+1},1^{n+m_1-m_2},1^{n+m_1-m_2};\{J-m_1+m_2\}\rangle\Big) \end{aligned}$$

$$+\sum_{m_1=0}^{J-1}\sum_{m_2=m_1+1}^{m_1+\delta J} f(J-m_1+m_2)q^{m_1-m_2}|1^{n+m_1-m_2+1},1^{n+m_1-m_2},1^{n+m_1-m_2};\{J-m_1+m_2\}\rangle\Big)$$

and l_- is the smallest of $J+m_2$ and $J+\delta J$. It is rather simple to see that $|\delta q\rangle$ is given by a sum of $O(J)$ terms and that each term has a coefficient of order δJ. Consequently, up to an overall constant factor $c_{\delta q}$ which is independent of J, we can bound the norm of $|\delta q\rangle$ as

$$\langle\delta q|\delta q\rangle \le c_{\delta q} J(\delta J)^2\mathcal{N}^2 = c_{\delta q}\frac{(\delta J)^2}{J(J+1)} \tag{3.95}$$

which goes to zero in the large J limit, proving our assertion that the difference between the simple states and the large N eigenstates vanishes in the large N limit.

3.C Review of Dilatation Operator Action

The studies [7, 6] have computed the dilatation operator action without invoking the distant corners approximation. The only approximation made in these studies is that correlators of operators with p long rows/columns with operators that have p long rows/columns and some short rows/columns, vanishes in the large N limit. These results are useful since they provide data against which the distant corners approximation could be compared. Further, we have demonstrated that the action of the dilatation operator reduces to a set of decoupled harmonic oscillators in [71, 2, 8, 70]. However, to obtain this result we needed to expand one of the factors in the dilatation operator to subleading order. The agreement of the resulting spectrum[5] is strong evidence that the distant corners approximation is valid. It is worth discussing these details and explaining why we do indeed obtain the correct large N limit. This point is not made explicitly in [71, 2, 8, 70].

In terms of operators belonging to the $SU(2)$ sector and normalized to have a unit two point function, the action of the one loop dilatation operator

$$DO_{R,(r,s)}(Z,Y)=\sum_{T,(t,u)} N_{R,(r,s);T,(t,u)}O_{T,(t,u)}(Z,Y)$$

is given by

$$N_{R,(r,s);T,(t,u)} = -g_{YM}^2\sum_{R'}\frac{c_{RR'}d_T nm}{d_{R'}d_t d_u(n+m)}\sqrt{\frac{f_T\,\text{hooks}_T\,\text{hooks}_r\,\text{hooks}_s}{f_R\,\text{hooks}_R\,\text{hooks}_t\,\text{hooks}_u}}\times$$
$$\times\text{Tr}\left(\left[\Gamma_R((n,n+1)),P_{R\to(r,s)}\right]I_{R'\,T'}\left[\Gamma_T((n,n+1)),P_{T\to(t,u)}\right]I_{T'\,R'}\right).$$

The above formula is exact. After using the distant corners approximation to simplify the trace and prefactor, this becomes

$$DO_{R,(r,s)\mu_1\mu_2} = -g_{YM}^2\sum_{u\nu_1\nu_2}\sum_{i<j}\delta_{\vec m,\vec n}M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2}\Delta_{ij}O_{R,(r,u)\nu_1\nu_2}\,. \tag{3.96}$$

Notice that we have a factorized action: the Δ_{ij} (explained below) acts only on the Young diagrams R, r and

$$M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2} = \frac{m}{\sqrt{d_s d_u}}\Big(\langle\vec m,s,\mu_2\,;\,a|E^{(1)}_{ii}|\vec m,u,\nu_2\,;\,b\rangle\langle\vec m,u,\nu_1\,;\,b|E^{(1)}_{jj}|\vec m,s,\mu_1\,;\,a\rangle$$
$$+\langle\vec m,s,\mu_2\,;\,a|E^{(1)}_{jj}|\vec m,u,\nu_2\,;\,b\rangle\langle\vec m,u,\nu_1\,;\,b|E^{(1)}_{ii}|\vec m,s,\mu_1\,;\,a\rangle\Big) \tag{3.97}$$

where a and b are summed, acts only on the s,μ_1,μ_2 labels of the restricted Schur polynomial. a labels states in the irreducible representation s and b labels states in the irreducible representation t. To spell out the action of operator Δ_{ij} it is useful to split it up into three terms

$$\Delta_{ij}=\Delta^+_{ij}+\Delta^0_{ij}+\Delta^-_{ij}\,. \tag{3.98}$$

[5] One can also compare the states that have a definite scaling dimension. The states obtained in the distant corners approximation are in perfect agreement with the states obtained in [7, 6] by a numerical diagonalization of the dilatation operator.

Denote the row lengths of r by r_i and the row lengths of R by R_i. Introduce the Young diagram r_{ij}^+ obtained from r by removing a box from row j and adding it to row i. Similarly r_{ij}^- is obtained by removing a box from row i and adding it to row j. In terms of these Young diagrams we have

$$\Delta_{ij}^0 O_{R,(r,s)\mu_1\mu_2} = -(2N + R_i + R_j - i - j) O_{R,(r,s)\mu_1\mu_2}\,, \tag{3.99}$$

$$\Delta_{ij}^+ O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N + R_i - i)(N + R_j - j + 1)}\, O_{R_{ij}^+,(r_{ij}^+,s)\mu_1\mu_2}\,, \tag{3.100}$$

$$\Delta_{ij}^- O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N + R_i - i + 1)(N + R_j - j)}\, O_{R_{ij}^-,(r_{ij}^-,s)\mu_1\mu_2}\,. \tag{3.101}$$

As a matrix Δ_{ij} has matrix elements

$$\begin{aligned} \Delta_{ij}^{R,r;T,t} &= \sqrt{(N + R_i - i)(N + R_j - j + 1)}\,\delta_{T,R_{ij}^+}\delta_{t,r_{ij}^+} \\ &+\sqrt{(N + R_i - i + 1)(N + R_j - j)}\,\delta_{T,R_{ij}^+}\delta_{t,r_{ij}^+} - (2N + R_i + R_j - i - j)\delta_{T,R}\delta_{t,r}\,. \end{aligned} \tag{3.102}$$

In terms of these matrix elements we can write (3.96) as

$$DO_{R,(r,s)\mu_1\mu_2} = -g_{YM}^2 \sum_{T,(t,u)\nu_1\nu_2} \sum_{i<j} \delta_{\vec{m},\vec{n}} M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2}\, \Delta_{ij}^{R,r;T,t} O_{T,(t,u)\nu_1\nu_2}\,. \tag{3.103}$$

Although the distant corners approximation has been used to extract the large N value of $M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2}$, the action of $\Delta_{ij}^{R,r;T,t}$ is computed exactly. In particular, the coefficients appearing in (3.102) are simply the factors associated with the boxes that are added or removed by $\Delta_{ij}^{R,r;T,t}$, and hence in developing a systematic large N expansion for $\Delta_{ij}^{R,r;T,t}$ we can trust the shifts of numbers of order N by numbers of order 1.

The limit in which the dilatation operator reduces to sets of decoupled oscillators corresponds to the limit in which the difference between the row (or column) lengths of Young diagram R are fixed to be $O(\sqrt{N})$ while the row lengths themselves are order N. The continuum variables are then

$$x_i = \frac{R_{i+1} - R_i}{\sqrt{R_1}} \qquad i = 1, 2, \cdots, p-1 \tag{3.104}$$

when R has p rows (or columns) and the shortest row (or column) is R_1. In this case, the leading and subleading (order N and order $\sqrt{N}$) contribution to $\Delta_{ij} O_{R,(r,s)\mu_1\mu_2}$ vanish, leaving a contribution of order 1. This contribution is sensitive to the exact form of the coefficients appearing in (3.102), and it is with these shifts that we reproduce the numerical results of [7, 6].

3.D One Loop Computation of Bulk/Boundary Magnon Scattering

In this appendix we will compute the scattering of a bulk and boundary magnon, to one loop, using the asymptotic Bethe ansatz. See [43] where studies of this type were first suggested and [106] for related systems. We can introduce a wave function $\psi(l_1, l_2, \cdots)$ as follows

$$O = \sum_{l_1, l_2, \cdots} \psi(l_1, l_2, \cdots) O(R, R_1^k, R_2^k; \{l_1, l_2, \cdots\})\,. \tag{3.105}$$

We assume that the boundary magnon (at l_1) and the next magnon along the open string (at l_2) are very well separated from the remaining magnons. These magnons are both assumed to be Y impurities.

To obtain the scattering we want, we only need to focus on these two magnons. The time independent Schrödinger equation following from our one loop dilatation operator is

$$E\psi(l_1,l_2) = \left(3+\frac{c}{N}\right)\psi(l_1,l_2) - \sqrt{\frac{c}{N}}\left(\psi(l_1-1,l_2)+\psi(l_1+1,l_2)\right) \\ -\left(\psi(l_1,l_2-1)+\psi(l_1,l_2+1)\right) \tag{3.106}$$

where c is the factor of the box that the endpoint associated to the magnon at l_1 belongs to. The equation (3.106) is valid whenever the two magnons are not adjacent in the open string word, i.e. when $l_2 > l_1+1$[6]. In the situation that the magnons are adjacent, we find

$$E\psi(l_1,l_1+1) = \left(1+\frac{c}{N}\right)\psi(l_1,l_1+1) - \sqrt{\frac{c}{N}}\psi(l_1-1,l_2) - \psi(l_1,l_1+2)\,. \tag{3.107}$$

We make the following Bethe ansatz for the wave function

$$\psi(l_1,l_2) = e^{ip_1l_1+ip_2l_2} + R_{12}\,e^{ip_1'l_1+p_2'l_2}\,. \tag{3.108}$$

It is straight forward to see that this ansatz obeys (3.106) as long as

$$E = 3+\frac{c}{N} - \sqrt{\frac{c}{N}}(e^{ip_1}+e^{-ip_1}) - (e^{ip_2}+e^{-ip_2}) \tag{3.109}$$

and

$$\sqrt{\frac{c}{N}}(e^{ip_1}+e^{-ip_1}) + e^{ip_2}+e^{-ip_2} = \sqrt{\frac{c}{N}}(e^{ip_1'}+e^{-ip_1'}) + e^{ip_2'}+e^{-ip_2'}\,. \tag{3.110}$$

Note that (3.109) is indeed the correct one loop anomalous dimension and (3.110) can be obtained by equating the $O(\lambda)$ terms on both sides of (3.46), as it should be. From (3.107) we can solve for the reflection coefficient R. The result is

$$R_{12} = -\frac{2e^{ip_2} - \sqrt{\frac{c}{N}}e^{ip_1+ip_2} - 1}{2e^{ip_2'} - \sqrt{\frac{c}{N}}e^{ip_1'+ip_2'} - 1} \tag{3.111}$$

Two simple checks of this result are

1. We see that $R_{12}R_{21} = 1$.
2. If we set $c = N$ we recover the S-matrix of [43].

We will now move beyond the $su(2)$ sector by considering a state with a single Y impurity and a single X impurity. The operator with a Y impurity at l_1 and an X impurity at l_2 is denoted $O(R,R_1^k,R_2^k;\{l_1,l_2,\cdots\})_{YX}$ and the operator with an X impurity at l_1 and a Y impurity at l_2 is denoted $O(R,R_1^k,R_2^k;\{l_1,l_2,\cdots\})_{XY}$. We now introduce a pair of wave functions as follows

$$O = \sum_{l_1,l_2,\cdots}\left[\psi_{YX}(l_1,l_2,\cdots)O(R,R_1^k,R_2^k;\{l_1,l_2,\cdots\})_{YX} \\ +\psi_{XY}(l_1,l_2,\cdots)O(R,R_1^k,R_2^k;\{l_1,l_2,\cdots\})_{XY}\right]\,. \tag{3.112}$$

From the one loop dilatation operator we find the time independent Schrödinger equation (3.106) for each wave function, when the impurities are not adjacent. When the impurities are adjacent, we find the following two time independent Schrödinger equations

$$E\psi_{YX}(l_1,l_1+1) = \left(2+\frac{c}{N}\right)\psi_{YX}(l_1,l_1+1) - \sqrt{\frac{c}{N}}\psi_{YX}(l_1-1,l_1+1) \\ -\psi_{XY}(l_1,l_1+1) - \psi_{YX}(l_1,l_1+2) \tag{3.113}$$

$$E\psi_{XY}(l_1,l_1+1) = \left(2+\frac{c}{N}\right)\psi_{XY}(l_1,l_1+1) - \sqrt{\frac{c}{N}}\psi_{XY}(l_1-1,l_1+1)$$

[6]Notice that we are associating a lattice site to every field in the spin chain and not just to the Zs.

$$-\psi_{YX}(l_1,l_1+1)-\psi_{XY}(l_1,l_1+2) \tag{3.114}$$

Making the following Bethe ansatz for the wave function

$$\begin{aligned}\psi_{YX}(l_1,l_2) &= e^{ip_1l_1+ip_2l_2}+Ae^{ip_1'l_1+ip_2'l_2}\\ \psi_{XY}(l_1,l_2) &= Be^{ip_1'l_1+ip_2'l_2}\end{aligned} \tag{3.115}$$

we find that the two equations of the form (3.106) imply that both $\psi_{XY}(l_1,l_2)$ and $\psi_{YX}(l_1,l_2)$ have the same energy, which is given in (3.109). The equations (3.113) and (3.114) imply that

$$\begin{aligned}A &= \frac{e^{ip_2'}+e^{ip_2}-1-\sqrt{\frac{c}{N}}e^{ip_1'+ip_2'}}{1+\sqrt{\frac{c}{N}}e^{ip_1'+ip_2'}-2e^{ip_2'}}\,,\\ B &= \frac{e^{ip_2}-e^{ip_2'}}{1+\sqrt{\frac{c}{N}}e^{ip_1'+ip_2'}-2e^{ip_2'}}\,.\end{aligned} \tag{3.116}$$

It is straight forward but a bit tedious to check that $|A|^2+|B|^2=1$ which is a consequence of unitarity. To perform this check it is necessary to use the conservation of momentum $p_1+p_2=p_1'+p_2'$, as well as the constraint (3.110). We now finally obtain

$$\frac{A}{R_{12}} = \frac{e^{ip_2'}+e^{ip_2}-1-\sqrt{\frac{c}{N}}e^{ip_1'+ip_2'}}{2e^{ip_2}-\sqrt{\frac{c}{N}}e^{ip_1+ip_2}-1}\,. \tag{3.117}$$

This should be equal to

$$\frac{1}{2}\left(1+\frac{B_{12}}{A_{12}}\right) \tag{3.118}$$

where A_{12} and B_{12} are the S-matrix elements computed in section 3.7, describing the scattering between a bulk and a boundary magnon. This allows us to perform a non-trivial check of the S-matrix elements we computed.

3.E No Integrability

The (boundary) Yang-Baxter equation makes use of the boundary magnon (B) and two bulk magnons (1 and 2). For our purposes, it is enough to track only scattering between bulk and boundary magnons. The Yang-Baxter equation requires equality between the scattering[7] which takes $B+1\to B'+1'$ and then $B'+2\to\tilde{B}'+\tilde{2}$ and the scattering which takes $B+2\to B'+2'$ and then $B'+1\to\tilde{B}'+\tilde{1}$. For the first scattering, given the initial momenta p_1,p_2,p_B, we need to solve

$$\begin{aligned}&\sqrt{1+8\lambda\sin^2\frac{p_1}{2}}+\sqrt{1+8\lambda((1+r)^2+4r\sin^2\frac{p_B}{2})}\\ &=\sqrt{1+8\lambda\sin^2\frac{k_1}{2}}+\sqrt{1+8\lambda((1+r)^2+4r\sin^2\frac{q}{2})}\end{aligned} \tag{3.119}$$

$$\begin{aligned}&\sqrt{1+8\lambda\sin^2\frac{p_2}{2}}+\sqrt{1+8\lambda((1+r)^2+4r\sin^2\frac{q}{2})}\\ &=\sqrt{1+8\lambda\sin^2\frac{k_2}{2}}+\sqrt{1+8\lambda((1+r)^2+4r\sin^2\frac{k_B}{2})}\end{aligned} \tag{3.120}$$

for the final momenta k_1,k_2,k_B. For the second scattering we need to solve

$$\begin{aligned}&\sqrt{1+8\lambda\sin^2\frac{p_2}{2}}+\sqrt{1+8\lambda((1+r)^2+4r\sin^2\frac{p_B}{2})}\\ &=\sqrt{1+8\lambda\sin^2\frac{l_2}{2}}+\sqrt{1+8\lambda((1+r)^2+4r\sin^2\frac{s}{2})}\end{aligned} \tag{3.121}$$

[7] There are some bulk magnon scatterings that we are ignoring as they don't affect our argument.

$$
\begin{aligned}
&\sqrt{1+8\lambda\sin^2\frac{p_1}{2}}+\sqrt{1+8\lambda((1+r)^2+4r\sin^2\frac{s}{2})}\\
&=\sqrt{1+8\lambda\sin^2\frac{l_1}{2}}+\sqrt{1+8\lambda((1+r)^2+4r\sin^2\frac{l_B}{2})}
\end{aligned}
\tag{3.122}
$$

for the final momenta l_1, l_2, l_B. It is simple to check that, in general, $k_1 \neq l_1$, $k_2 \neq l_2$ and $k_B \neq l_B$, so the two scatterings can't possibly be equal.

Chapter 4

LLM Magnons

4.1 Motivation

There is convincing support in favor of the duality between $\mathcal{N} = 4$ super Yang-Mills theory and IIB strings on the asymptotically $AdS_5 \times S^5$ spacetimes[1, 27]. The duality identifies a gauge theory operator for each state in the string theory, and the dimension of this operator with the energy of the string theory state. By comparing operator dimensions and string state energies, we find a wealth of non-trivial predictions that can be tested. Since the duality is a strong/weak coupling relation, the results of perturbative field theory and perturbative string theory are not related by the duality and checking the equality of dimensions and energies is in general highly nontrivial. In the case that we consider, gauge theory operators belonging to the $SU(2|3)$ sector of the theory, this check can be carried out in complete detail. This remarkable progress is possible by exploiting the symmetry of the problem in an interesting way[16, 44].

The gauge theory operator corresponding to a closed string state is a single trace operator[47] constructed using a very large number ($J \sim O(\sqrt{N})$) of scalar Z fields, with a few impurities (which may be the scalars X, Y, $X^\dagger$ or $Y^\dagger$ or of one of four possible fermions). The symmetry which plays a central role is an $SU(2|2)^2$ symmetry which acts on the impurities. When the impurities are well separated within the trace, each impurity transforms as a short multiplet of a centrally extended $SU(2|2)^2$ symmetry; the fact that the multiplet is short determines its anomalous dimension[16] (see Section 2.5.1 for a detailed review). The anomalous dimension of the single trace operator is then given by summing the dimensions of the impurities. The sum of the new magnon central charges[1] vanishes so that the closed string state is a representation of the original algebra. An inspired explanation for these dimensions was given in [73]. This physically motivated description matches the string description, which was worked out in [4](reviewed in Section 2.5.2). Exactly the same symmetry algebra plays a role and the central charges of the algebra acquire an elegant geometrical interpretation as is now explained in summary. In general, the anticommutator of two supersymmetries in 10 dimensional supergravity includes a gauge transformation of the NS-$B_{\mu\nu}$ field, which acts non-trivially on stretched strings. The parameter of the gauge transformation was computed in [22] in terms of the Killing spinors of the geometry. By inserting the explicit expression of the LLM Killing spinors[3] into the general formula for the gauge transformation produced by the anticommutator of two supercharges, one finds that the relevant NS gauge transformations on the LLM plane, are those with a constant gauge parameter. Thus any string stretched along the LLM plane will acquire a phase under these gauge transformations, as shown explicitly in Section 2.5.2.5. These are the central charges that we are after[4]. To develop the geometrical interpretation of these central charges, we need to develop the geometrical picture of the string worldsheet projected to the LLM plane. The Zs inside the trace carry the quantum numbers of gravitons, which map to a single point, orbiting in a circle of radius $r = 1$. The impurities inside the trace map into "giant magnons" - straight line segments that stretch between the points on the $r = 1$ circle where the Zs are located[4]. Thus, magnons

[1] i.e. the charges switched on for the representations of the magnon but not present in the original group.

are represented by directed line segments on the plane. Introduce a complex number k for each oriented segment, whose magnitude is the length of this segment and whose phase is the direction of this segment. Thanks to the fact that the gauge transformations giving rise to the central charges are constant, the central charge of the corresponding magnon is k[4]. The projected closed string is a polygon on the LLM plane with a consistent assignment of orientation to each side of the polygon, so that the central charges of the magnons sum to zero.

The powerful $SU(2|2)^2$ symmetry arguments developed above do not make use of the planar approximation. It is then natural to expect that the argument continues to work even for operators with such a large dimension that the planar approximation is no longer accurate[2]. A natural setting in which this expectation can be tested, is for open strings attached to giant gravitons. In this case, one is studying correlators of operators that have a bare dimension of order N. This has been carried out for the maximal giant in [17] and in complete generality (any number of giants and dual giants of any size) in [18](Chapter 3 of this dissertation). These studies confirm that the $SU(2|2)^2$ symmetry arguments continue to work. Another natural situation to consider is that of closed strings (and/or giant gravitons) exploring the LLM geometries. These correspond to operators with a bare dimension of order N^2. In this chapter we study this general class of problems.

The LLM geometries are regular $\frac{1}{2}$ BPS solutions of type IIB string theory that are asymptotically $AdS_5 \times S^5$. They are dual to operators constructed using a single complex field Z. These geometries enjoy an $R \times SO(4) \times SO(4)$ isometry group and have a metric which is given by $(i, j = 1, 2)$

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2\,, \tag{4.1}$$

where

$$h^{-2} = 2y \cosh G, \qquad z = \frac{1}{2} \tanh G, \tag{4.2}$$

$$y\partial_y V_i = \epsilon_{ij}\partial_j z, \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z. \tag{4.3}$$

Thus, the metric is completely determined by the function z, which is a function of y, x^1 and x^2. It is obtained by solving

$$\partial_i\partial_i z + y\partial_y \frac{\partial_y z}{y} = 0. \tag{4.4}$$

Regularity requires that $z = \pm\frac{1}{2}$ on the LLM $(y = 0)$ plane. As a consequence, the complete set of LLM solutions can be labelled by colouring the entire LLM plane into black (where $z = -\frac{1}{2}$) and white (where $z = +\frac{1}{2}$) regions[3](reviewed in Section 2.4).

Each LLM geometry corresponds to a specific $\frac{1}{2}$ BPS operator. We will restrict ourselves to geometries that correspond to colouring the plane with a collection of concentric annuli, surrounding a central disk which may be of either colour. Each such geometry corresponds to a Schur polynomial, labelled by a Young diagram B[55, 33]. Every plane colouring we consider can be translated into a Young diagram and hence into a definite gauge theory operator. An example of the translation is shown in Fig 4.4. The $AdS_5 \times S^5$ geometry corresponds to a black disk of radius 1. Gravitons dual to gauge theory operators built using only the Z field would follow circular orbits at the outer edge of any black region. We would also have closed string states, with worldsheet given by a polygon with every vertex on the outer edge of any black annulus. Each polygon edge is a magnon. Using the general form of the LLM Killing spinors, it is again true that the anticommutator of two supersymmetries includes a constant gauge transformation. Consequently the central charge of the magnon is still set by the corresponding line segment. The angle that the line segment corresponding to a magnon subtends with respect to the origin of the LLM plane,

[2]In this case there is in general no integrability and so the constraints implied by the $SU(2|2)^2$ analysis give results that would be difficult to obtain by any other method.

determines the momentum of the magnon. To turn this angle into a length (and hence an energy) we need to know the radius of the circle(s) that the line segment's end points are located on. Thus, the values of the central charges as well as the dispersion relation depend on the radii of the annuli in the LLM boundary condition.

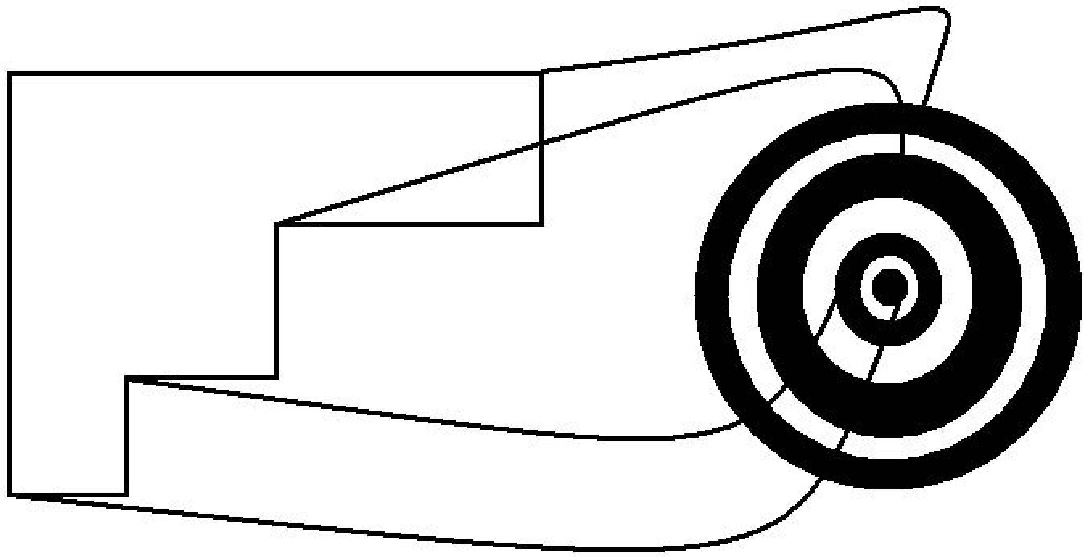


Figure 4.1: Inward pointing corners correspond to outer radii of black regions. Each black (white) region corresponds to a vertical (horizontal) line on the edge of the Young diagram. The area of each white (black) region divided by π is given by the number of columns (rows) in the corresponding line, divided by N. The area of the central black disk divided by π is given by N minus the total number of rows, divided by N. The total area of the black regions is π.

Our goal is to write down the operators dual to closed strings in the above LLM geometries. The first question we need to address is how to write down gauge theory operators that are localized at the edge of an annulus in the dual gravity description. Once these operators are constructed, we can consider the problem of determining their anomalous dimensions. By computing their one loop anomalous dimensions, we can determine the central charges of the representations in which the magnons transform and compare to the string theory predictions. We will find complete agreement. This both gives support that we have correctly constructed localized closed string states and that the $SU(2|2)^2$ symmetry analysis is still applicable in this case, as one would expect. For some of the closed string operators we consider, the problem of determining the anomalous dimensions is an integrable problem and the results obtained in the planar limit generalize immediately without any effort.

This chapter is organized as follows: In Section 4.2 we study the planar operator mixing problem at one loop. This discussion is usually phrased in terms of single trace operators, a language which is not useful outside of the planar limit. We translate this discussion into the language of restricted Schur polynomials. It is the restricted Schur language that will generalize. In Section 4.3 we point out, using simple examples, some of the issues related to constructing operators dual to excitations localized on the LLM plane. Using these insights we give our proposal for operators dual to localized closed string states in Section 4.4 and we compute their anomalous dimensions in Section 4.5. These all correspond to excitations localized at the outer edge of an LLM annulus. Although our computations are quite technical and make heavy use of group representation theory the result is striking in its simplicity: the problem in the nontrivial geometry is given by simply scaling the N dependence in the planar result. In Section 4.6 we consider an example that is simple enough that it can be treated without any use of group representation theory and we confirm our results in this simple setting. In Section 4.7 we explain how to describe excitations localized at the inner edge of an LLM annulus. Finally, in Section 4.8 we discuss our results and draw some conclusions.

4.2 Dilatation Operator in the Planar Limit using Restricted Schur Polynomials

To simplify our discussion we will restrict ourselves to the $su(2)$ sector of $\mathcal{N} = 4$ super Yang-Mills theory. Restricting to this sector does not interfere with our goal of computing the central charge appearing in the $SU(2|2)$ symmetry algebra, but it will significantly simplify our arguments. The local operators of interest to us are loops that correspond to closed string states[47]. In the planar limit, a basis for these operators is labelled by the ordered set of integers $\{n_k\}$

$$O(\{n_k\}) = \mathrm{Tr}(Z^{n_1} Y Z^{n_2} Y \cdots Z^{n_m} Y) \tag{4.5}$$

with $\sum_k n_k = n$. Not all sets correspond to distinct operators due to cyclicity of the trace. For string states we would hold $n + m \sim \sqrt{N}$ and for the states we are interested in, we take $m \ll n$. The planar approximation is valid as long as $\frac{(m+n)^2}{N} \ll 1$. We refer to these as "loop operators" and make use of the following notation

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = Y^{i_1}_{i_{\sigma(1)}} \cdots Y^{i_m}_{i_{\sigma(m)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}} \cdots Z^{i_{n+m}}_{i_{\sigma(n+m)}} \,. \tag{4.6}$$

Each of the operators (4.5) corresponds to a permutation that is a single $m + n$ cycle

$$O(\{n_k\}) = \mathrm{Tr}(\sigma_{\{n_k\}} Z^{\otimes n} Y^{\otimes m}) \tag{4.7}$$

The loop operators are a particularly convenient description of the planar limit. In particular, it was in terms of these variables that the link to the spin chain and the subsequent discovery of integrability was made[40, 72]. This is a consequence of the fact that there is a bijection between loop operators and spin chain states. In addition, these variables also provide an explicit and direct link to the string worldsheet[107].

In the planar limit the loop operators are orthogonal. As we increase $n + m$ beyond $O(\sqrt{N})$, the loop operators start to mix and they no longer provide a useful description[74]. The operators we are interested in correspond to closed string states in an LLM geometry[3], so that $n \sim O(N^2)$ and $\frac{m}{n} \ll 1$. The loop operators are useless in this limit. The basis provided by the restricted Schur polynomials is far more useful: these operators are exactly orthogonal in the free field theory and their mixing at loop level is tightly constrained[61, 2][3]. The definition of the restricted Schur polynomial is[61](see Section 2.7 for further details)

$$\chi_{R,(r,s)\alpha\beta}(Z, Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s),\alpha\beta}(\sigma) Y^{i_1}_{i_{\sigma(1)}} \cdots Y^{i_m}_{i_{\sigma(m)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}} \cdots Z^{i_{n+m}}_{i_{\sigma(n+m)}} \,. \tag{4.8}$$

In this definition R is a Young diagram with $n + m$ boxes and hence labels an irreducible representation (irrep) of S_{n+m}, r is a Young diagram with n boxes and labels an irrep of S_n and s is a Young diagram with m boxes and labels an irrep of S_m. The group S_{n+m} has an $S_n \times S_m$ subgroup. Taken together r and s label an irrep of this subgroup. A single irrep R will in general subduce many possible representations of the subgroup. A particular irrep of the subgroup may be subduced more than once in which case we must introduce a multiplicity label to keep track of the different copies subduced. The indices α and β appearing above are these multiplicity labels. The object $\chi_{R,(r,s)\alpha\beta}(\sigma)$ is called a restricted character[13]. At present there are no general powerful methods to compute these characters and this is one of the main obstacles that must be overcome when performing explicit computations. In this section there are a number of useful identities that we will prove, obtained by writing the known planar action of the dilatation operator on loop operators, in terms of the restricted Schur polynomial operators. In the Appendices we will derive these results, and more general exact identities, using the representation theory of the symmetric group. These identities are all we will need in this chapter, so that we entirely avoid the need to explicitly evaluate any restricted characters.

[3]There are a number of distinct bases that are exactly orthogonal in the free field limit[76, 77, 78, 79, 83, 108].

The restricted Schur polynomials provide a complete basis. As a consequence we can write the loop operators as a linear combination of restricted Schur polynomials. The basic identity we need to carry this out is[56]

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = \sum_{T,(t,u)\alpha\beta} \frac{d_T n! m!}{d_t d_u (n+m)!} \chi_{T,(t,u)\alpha\beta}(\sigma^{-1}) \chi_{T,(t,u)\beta\alpha}(Z,Y) \tag{4.9}$$

Applying this to a loop operator, we find

$$\begin{aligned} O(\{n_k\}) &= \sum_{T,(t,u)\alpha\beta} \frac{d_T n! m!}{d_t d_u (n+m)!} \chi_{T,(t,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}}) \chi_{T,(t,u)\beta\alpha}(Z,Y) \\ &= \sum_{T,(t,u)\alpha\beta} \sqrt{\frac{f_T \mathrm{hooks}_t \mathrm{hooks}_u}{\mathrm{hooks}_T}} \chi_{T,(t,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}}) O_{T,(t,u)\beta\alpha}(Z,Y) \end{aligned} \tag{4.10}$$

where $O_{T,(t,u)\beta\alpha}(Z,Y)$ is a restricted Schur polynomial normalized to have unit two point function.

The action of the dilatation operator on the loop operators in the planar limit is remarkably simple. The one loop dilatation operator, in the $SU(2)$ sector, is[66]

$$D = -\frac{g_{YM}^2}{8\pi^2} \mathrm{Tr}\left([Y,Z] \left[\frac{d}{dY}, \frac{d}{dZ} \right] \right) \tag{4.11}$$

Acting on the loop operators we have (Section 2.8 for details)

$$\begin{aligned} DO(\{n_k\}) = &\frac{g_{YM}^2 N}{8\pi^2} \left(2O(\{n_k\}) - O(\{n_1+1, n_2-1, \cdots, n_m\}) - O(\{n_1-1, n_2+1, \cdots, n_m\}) \right) \\ &+ \frac{g_{YM}^2 N}{8\pi^2} \left(2O(\{n_k\}) - O(\{n_1, n_2+1, n_3-1, \cdots, n_m\}) - O(\{n_1, n_2-1, n_3+1, \cdots, n_m\}) \right) \\ &+ \cdots\cdots\cdots \\ &+ \frac{g_{YM}^2 N}{8\pi^2} \left(2O(\{n_k\}) - O(\{n_1-1, n_2, \cdots, n_m+1\}) - O(\{n_1+1, n_2, \cdots, n_m-1\}) \right) \end{aligned} \tag{4.12}$$

This result is certainly not exact - only the planar contractions are retained. The subleading (non-planar) terms will induce a splitting of the above loop operator into a double trace operator.

When acting on restricted Schur polynomials the one loop dilatation operator takes the form[6]

$$DO_{R,(r,s)\mu_1\mu_2}(Z,Y) = \sum_{T,(t,u)\nu_1\nu_2} N_{R,(r,s)\mu_1\mu_2;T,(t,u)\nu_1\nu_2} O_{T,(t,u)\nu_1\nu_2}(Z,Y) \tag{4.13}$$

where

$$\begin{aligned} N_{R,(r,s)\mu_1\mu_2;T,(t,u)\nu_1\nu_2} &= -\frac{g_{YM}^2}{8\pi^2} \sum_{R'} \frac{c_{RR'} d_T nm}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T\, \mathrm{hooks}_T\, \mathrm{hooks}_r\, \mathrm{hooks}_s}{f_R\, \mathrm{hooks}_R\, \mathrm{hooks}_t\, \mathrm{hooks}_u}} \times \\ &\times \mathrm{Tr}\Big(\Big[\Gamma^{(R)}((1,m+1)), P_{R,(r,s)\mu_1\mu_2} \Big] I_{R'\,T'} \Big[\Gamma^{(T)}((1,m+1)), P_{T,(t,u)\nu_2\nu_1} \Big] I_{T'\,R'} \Big) \end{aligned} \tag{4.14}$$

The trace above is over irrep $R \vdash n+m$. $I_{R'\,T'}$ is a map from irrep R' to irrep T', where both are irreps of S_{n+m-1}. R' is one of the irreps subduced from R upon restricting to the S_{n+m-1} subgroup of S_{n+m} obtained by keeping only permutations that obey $\sigma(1) = 1$. T' is subduced by T in the same way. $I_{R'\,T'}$ is only non-zero if R' and T' have the same shape. Thus, to get a non-zero result R and T must differ at most by the placement of a single box. d_a denotes the dimension of symmetric group irrep a. f_S is the product of the factors in Young diagram S and hooks_S is the product of the hook lengths of Young diagram S. Finally, $c_{RR'}$ is the factor of the corner box that must be removed from R to obtain R'. Combining the above results, we now find

$$DO(\{n_k\}) = \sum_{R,(r,s)\mu_1\mu_2} \sqrt{\frac{f_R \mathrm{hooks}_r \mathrm{hooks}_s}{\mathrm{hooks}_R}} \chi_{R,(r,s)\mu_2\mu_1}(\sigma^{-1}_{\{n_k\}}) DO_{R,(r,s)\mu_1\mu_2}(Z,Y)$$

$$= \sum_{T,(t,u)\nu_1\nu_2} \sum_{R,(r,s)\mu_1\mu_2} \sqrt{\frac{f_R \text{hooks}_s}{\text{hooks}_{R/r}}} \chi_{R,(r,s)\mu_2\mu_1}(\sigma^{-1}_{\{n_k\}}) N_{R,(r,s)\mu_1\mu_2;T,(t,u)\nu_1\nu_2} O_{T,(t,u)\nu_1\nu_2}(Z,Y) \tag{4.15}$$

where

$$\text{hooks}_{R/r} = \frac{\text{hooks}_R}{\text{hooks}_r} \tag{4.16}$$

The equality of (4.12) and (4.15) now proves that[4]

$$\begin{aligned}
&\sum_{T,(t,u)\nu_1\nu_2} \sum_{R,(r,s)\mu_1\mu_2} \sqrt{\frac{f_T \text{hooks}_u}{\text{hooks}_{T/t}}} \chi_{T,(t,u)\nu_2\nu_1}(\sigma^{-1}_{\{n_k\}}) N_{T,(t,u)\nu_1\nu_2;R,(r,s)\mu_1\mu_2} O_{R,(r,s)\mu_1\mu_2}(Z,Y) \\
&= \frac{g_{YM}^2 N}{8\pi^2} \sum_{R,(r,s)\mu_1\mu_2} \sqrt{\frac{f_R \text{hooks}_s}{\text{hooks}_{R/r}}} O_{R,(r,s)\mu_1\mu_2}(Z,Y) \\
&\Big(\chi_{R,(r,s)\mu_2\mu_1}(2\sigma^{-1}_{\{n_k\}} - \sigma^{-1}_{\{n_1+1,n_2-1,\cdots,n_m\}} - \sigma^{-1}_{\{n_1-1,n_2+1,\cdots,n_m\}}) \\
&\quad + \chi_{R,(r,s)\mu_2\mu_1}(2\sigma^{-1}_{\{n_k\}} - \sigma^{-1}_{\{n_1,n_2+1,n_3-1,\cdots,n_m\}} - \sigma^{-1}_{\{n_1,n_2-1,n_3+1,\cdots,n_m\}}) + \cdots + \\
&\quad + \chi_{R,(r,s)\mu_2\mu_1}(2\sigma^{-1}_{\{n_k\}} - \sigma^{-1}_{\{n_1-1,n_2,\cdots,n_m+1\}} - \sigma^{-1}_{\{n_1+1,n_2,\cdots,n_m-1\}})\Big)
\end{aligned} \tag{4.17}$$

Finally, since the restricted Schur polynomials are independent, this implies

$$\begin{aligned}
&\sum_{T,(t,u)\nu_1\nu_2} \sqrt{\frac{f_T \text{hooks}_u}{\text{hooks}_{T/t}}} \chi_{T,(t,u)\nu_2\nu_1}(\sigma^{-1}_{\{n_k\}}) N_{T,(t,u)\nu_1\nu_2;R,(r,s)\mu_1\mu_2} \\
&= \frac{g_{YM}^2 N}{8\pi^2} \sqrt{\frac{f_R \text{hooks}_s}{\text{hooks}_{R/r}}} \Big(\chi_{R,(r,s)\mu_2\mu_1}(2\sigma^{-1}_{\{n_k\}} - \sigma^{-1}_{\{n_1+1,n_2-1,\cdots,n_m\}} - \sigma^{-1}_{\{n_1-1,n_2+1,\cdots,n_m\}}) \\
&\quad + \chi_{R,(r,s)\mu_2\mu_1}(2\sigma^{-1}_{\{n_k\}} - \sigma^{-1}_{\{n_1,n_2+1,n_3-1,\cdots,n_m\}} - \sigma^{-1}_{\{n_1,n_2-1,n_3+1,\cdots,n_m\}}) + \cdots + \\
&\quad + \chi_{R,(r,s)\mu_2\mu_1}(2\sigma^{-1}_{\{n_k\}} - \sigma^{-1}_{\{n_1-1,n_2,\cdots,n_m+1\}} - \sigma^{-1}_{\{n_1+1,n_2,\cdots,n_m-1\}})\Big)
\end{aligned} \tag{4.18}$$

which is the identity obeyed by restricted characters that we aimed to derive. We stress that this is not an exact result - we have used the simplifications of the planar limit to obtain it.

This rewriting of the planar dilatation operator is interesting. The above identity is written using Young diagrams and the language of restricted characters. However, the appearance of the permutations which label the loop operators keeps manifest the bijection to the spin chain. This is the restricted Schur polynomial way of mapping to the spin chain dynamics. In what follows, we will argue that the description of closed string states using a permutation is a useful description for operators dual to closed strings probing LLM backgrounds. The operators described using the permutation are certainly not single trace operators. Indeed, in the next section we will argue that single trace operators in the gauge theory are not dual to localized excitations in the dual gravity. This discussion will motivate the form of the operators that are dual to closed strings in the LLM geometries.

4.3 How not to Localize

The operators dual to the closed string states we study are composed of collections of Zs raised to some power, separated by Y fields. The worldsheet geometry of these strings has been studied in [4]. The pieces

[4]The use of notation expressing the linear combination of characters as the character of a sum of the permutations is justified by the linearity of the matrix representations of the permutations, which are used implicitly when writing a restricted character:

$$\chi_{R,(r,s)\mu_2\mu_1}(\sigma_1 + \sigma_2) = \text{Tr}(P_{R\to(r,s)\mu_2\mu_1}(\Gamma_R(\sigma_1) + \Gamma_R(\sigma_2))).$$

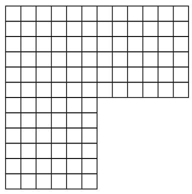
of the worldsheet constructed by the Zs are localized on a particular circle in the LLM plane, while the Y fields correspond to worldsheet magnons that stretch between these points. In this section we consider the problem of writing operators, composed entirely out of Zs, that are localized in the radial direction of the LLM plane. Using the intuition gathered from this toy problem we will construct operators dual to closed strings in the LLM geometries. This section is largely a review of relevant results from [5, 62, 63]. The papers [109, 110, 111, 112] include closely related ideas. These articles used an eigenvalue density description for the $\frac{1}{2}$ BPS operators in the gauge theory. The construction of localized giant graviton branes has been pursued in [91, 92, 93, 94, 95] using a collective coordinate description.

Consider the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, which is dual to the vacuum state of $\mathcal{N} = 4$ super Yang-Mills theory. Point like gravitons, with a momentum $p \sim O(1)$ are dual to operators

$$\mathrm{Tr}(Z^p) \tag{4.19}$$

They are localized at the radius $r = 1$ on the LLM plane. Thus, in this case, a single trace operator with $O(1)$ fields is dual to an object in the string theory, localized in the radial direction.

Single trace operators do not create localized states in general[5]. Consider a three ring geometry - two concentric rings with a central black disk. This corresponds to a Schur polynomial labelled by a Young diagram with the following shape

$R =$  (4.20)

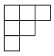
Applying $\mathrm{Tr}(Z) = \chi_{\square}(Z)$ to the above state, the product $\chi_R(Z)\chi_{\square}(Z)$ is easily computed using the Littlewood Richardson rule[55]. The product consists of three terms, one labelled by a Young diagram obtained by adding a box to row 1 of R, one labelled by a Young diagram obtained by adding a box to row 7 of R and one labelled by a Young diagram obtained by adding a box to row 13 of R. In the dual gravity this is a superposition of states. One state is the original geometry with a single graviton localized at the outer edge of the largest ring, one state is the original geometry with a single graviton localized at the outer edge of the smallest ring and the third state is the original geometry with a single graviton localized at the outer edge of the central disk. Thus, the single trace state can not be interpreted as an excitation of the original geometry, localized at some radius in the plane. This completely local and gauge invariant operator in the gauge theory maps to something non-local in the gravity dual.

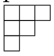
To localize the excitation in the geometry, we need to mix the indices of the operator creating the excitations with the indices of the fields making up the background in such a way that the boxes describing the excitation are only added at one location on the Young diagram describing the original geometry[5, 62, 63]. We need to mix the gauge group indices of the excitation and the background to produce a local excitation. The details of how these indices are mixed determines where the excitation is localized. We will not pursue the problem of explicitly constructing operators which create localized excitations further.

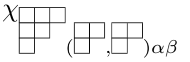
4.4 Localized Closed String States

Using the lessons of the previous section, we will now write down operators dual to closed string states localized on the LLM plane. We can write the loop operator as a linear combination of restricted Schur polynomials labelled by a triple of Young diagrams using the general result (4.10) which is true for any permutation σ. Our strategy is to write a local version of the restricted Schur polynomial, and then use (4.10) to obtain local loops. We will now motivate and explain our proposal for the localized version of a restricted Schur polynomial in the LLM backgrounds. Our arguments in what follows holds for general values of m and n_C. For simplicity we will however often consider an example with $m = 3$ and $n_C = 3$ before stating the general result.

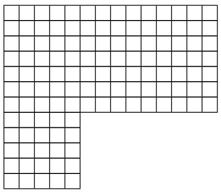
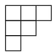
Consider a restricted Schur polynomial with labels

$R =$  $\qquad r =$  $\qquad s =$  (4.21)

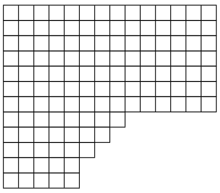
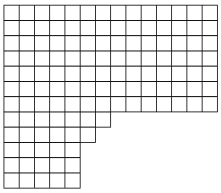
These would be summed (along with other possible restricted Schur polynomials) to produce a loop operator with three Ys and three Zs. To completely specify the possible operators we need a multiplicity label since two copies of ,  arise when we restrict  to the $S_3 \times S_3$ subgroup. We do the restriction by removing boxes from R to leave r. The removed boxes are then assembled to produce s. This way of embedding the representations that appear is particularly convenient, since it allows us to associate the multiplicity label with s matching the construction developed in [2]. Thus, the possible restricted Schur polynomials are

χ  $\qquad \alpha, \beta = 1, 2$ (4.22)

As a consequence of the fact that two copies of ,  arise, there are four possible operators we can define. We want to write down local versions of these restricted Schur polynomials, in the background corresponding to Young diagram B given below. B corresponds to two concentric black rings surrounding a black disk. We will write down local versions of the above four operators localized around the outer edge of the middle ring. These operators are again given by restricted Schur polynomials. R is replaced by the Young diagram obtained by adjoining R to B in the appropriate location and similarly for r. The new labels are denoted R_B and r_B. s is unchanged and since the problem of assembling the boxes removed from R_B to form s has exactly the same multiplicity as the problem of assembling the boxes removed from R to form s we assign it a multiplicity label that takes the same values as the original label did. For the example we are considering we have

$B =$  $\qquad R =$  $\qquad r =$  (4.23)

and

$R_B =$  $\qquad r_B =$  (4.24)

Although all Young diagrams shown have a finite number of boxes, all row and column lengths would scale as N as we take the large $N \to \infty$ limit. In terms of these labels, the original loop operator is

$$O(\{n_k\}) = \sum_{T,(t,u)\alpha\beta} \sqrt{\frac{f_T \text{hooks}_t \text{hooks}_u}{\text{hooks}_T}} \chi_{T,(t,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}}) O_{T,(t,u)\beta\alpha}(Z,Y) \tag{4.25}$$

Our proposal for the operator localized in the LLM background B is

$$O_B(\{n_k\}) = \sum_{T,(t,u)\alpha\beta} \sqrt{\frac{f_{T_B} \text{hooks}_{t_B} \text{hooks}_u}{\text{hooks}_{T_B}}} \chi_{T_B,(t_B,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}}) O_{T_B,(t_B,u)\beta\alpha}(Z,Y) \tag{4.26}$$

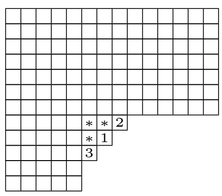
Some notation: $B \vdash n_B$, $T \vdash n_C + m$, $t \vdash n_C$ and $n = n_B + n_C$. n_B counts the number of Z fields in the background; n_C counts the number of Z fields in the excitation. Both sums above run over all Young diagrams $T \vdash m + n_C$, $t \vdash n_C$, $u \vdash m$ and the same multiplicity index. We are interested in the limit in which $n_B \sim N^2$ and $n_C \sim \sqrt{N}$. We want to compute the action of the dilatation operator on $O_B(\{n_k\})$. To accomplish this it is useful to simplify the above expression for $O_B(\{n_k\})$. The top most and left most box of R is added to column c and row r of B. In the above example $c = 6$ and $r = 8$; in the limit we consider both r and c are of order N. Let r_1 denote the length of the first row of B and let c_1 denote the length of the first column of B. The first simplification we use comes from noticing that, at large N we have

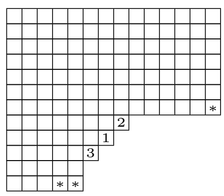
$$\begin{aligned}
\frac{d_{T_B} n! m!}{d_{t_B} d_u (n+m)!} &= \frac{\text{hooks}_{t_B} \text{hooks}_u}{\text{hooks}_{T_B}} \\
&= \left(\frac{c_1 - r}{c_1 - r + c} \frac{r_1 - c}{r_1 - c + r} \right)^m \frac{\text{hooks}_t \text{hooks}_u}{\text{hooks}_T} (1 + O(N^{-1})) \\
&= \kappa^m \frac{\text{hooks}_t \text{hooks}_u}{\text{hooks}_T} (1 + O(N^{-1})) \qquad (4.27)
\end{aligned}$$

The precise form of κ does depend on the details of B. However, this is the only source of B dependence and the structure

$$\frac{d_{T_B} n! m!}{d_{t_B} d_u (n+m)!} = \kappa^m \frac{\text{hooks}_t \text{hooks}_u}{\text{hooks}_T} (1 + O(N^{-1})) \qquad (4.28)$$

holds for any B[5]. The second simplification is in the value of the restricted character $\chi_{T_B,(t_B,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}})$. Here we will see that the fact that $\sigma^{-1}_{\{n_k\}}$ is an $n_C + m$ cycle plays a crucial role. In terms of the original loop operator in the original $\text{AdS}_5 \times \text{S}^5$ geometry, this is equivalent to the statement that the loop operator is a single trace. To compute the restricted character we will use a specific representation of the symmetric group called Young's orthogonal representation[113], which simplifies in the large N limit (see Appendix A). When we want to compute traces we are summing the diagonal elements of the representation of a permutation, so that we often need to keep the terms with the original pattern, which is suppressed at large N. This observation was used to obtain the large N limit of the one loop dilatation operator in [71, 2]. Using this representation, consider the computation of the restricted character $\chi_{T_B,(t_B,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}})$, given by taking a (restricted) trace. Recall that our goal is to write down the local versions of the restricted Schur polynomials given in (4.22). These are to be summed to produce a loop operator with three Ys and three Zs. The positions of the first three boxes to be removed (these are associated with the Ys; they are the boxes that are assembled to produce s) are fixed by the shapes of R and r. After removing these first three boxes, we need to label a further three boxes. This follows because $\sigma^{-1}_{\{n_k\}}$ describes a loop with three Ys and three Zs and is thus a six-cycle. All we need is the labels of the first six boxes in the Young-Yamonouchi pattern if we are to evaluate the action of $\sigma^{-1}_{\{n_k\}}$ on the state. We could remove them all from the vicinity of the first three boxes (see the diagram on the left below) or we could include some more distant boxes (see the diagram on the right below for an example).

$R_B =$ 

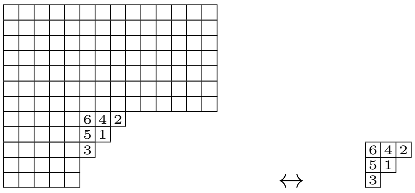
$R_B =$ 

(4.29)

The important observation is that only states of the form given in the diagram on the left above contribute. Concretely, after acting with $\sigma^{-1}_{\{n_k\}}$ on the states of the form given in the diagram on the right above, we always find (at the leading order in large N) that some of the first m labels (given in our example by 1,2,3) are transported into the distant boxes. This follows from the fact that (i) the difference in content between the distant boxes and the first m boxes is order N and hence (ii) in Young's orthogonal representation, transpositions between the first m and the remaining boxes always transport local boxes to the location of the distant boxes. These states will not contribute to the trace because the overlap of the state with the original box locations and the state with some local boxes swapped with distant boxes

[5]Each corner in the Young diagram would be associated with a different value for κ. For more than one excitation, at more than one corner, we'd have a product of terms, one for each corner. The term for a given corner is the value of κ for that corner raised to the power of the number of Ys appearing in the excitation at that corner.

vanishes. Thus, the trace only receives contributions from states of the form given in the diagram on the left above. The contribution from the states of the form given in the diagram on the right above are of order N^{-1}. In general, the $m+n_C$ cycle "ties" the n_C+m labelled boxes together. This forces the n_C Z boxes (which might have appeared in any distant corner of R_B) to sit adjacent to the Y boxes. At this stage it is useful to introduce a bijection between states in R and subspaces in R_B as follows



(4.30)

Each of these subspaces is an irrep of S_{n_B} equivalent to the irrep labelled by B. This map is equivariant[6] with respect to the action of the cycle $\sigma^{-1}_{\{n_k\}}$ so that we now find

$$\chi_{T_B,(t_B,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}}) = d_B \chi_{T,(t,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}}) \tag{4.31}$$

where d_B is the dimension of symmetric group representation B. This result holds only in the large N limit.

It is useful to introduce notation with which to describe the different vector spaces that enter into our analysis. The states labelled by patterns filling R_B are denoted $|R_B, a\rangle$ with $a = 1, ..., d_{R_B}$. The states of the type shown in (4.30) are written as $|\hat{R}_B, a\rangle$ with $a = 1, ..., d_R d_B$ and $|R, a\rangle$ with $a = 1, ..., d_R$ respectively. The result we have found above is written as

$$\chi_{T_B,(t_B,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}}) = \chi_{\hat{T}_B,(\hat{t}_B,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}}) = d_B \chi_{T,(t,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}}) \tag{4.32}$$

with the new notation. The intermediate step makes it explicit that only certain states contribute to the restricted trace. The first equality above is only true at large N; the second is exact.

The above discussion has shown that we can restrict the states participating in the trace; from now on we will restrict all traces in this way - it simplifies our discussion of the action of the dilatation operator dramatically. Using these simplifications, we can write our proposal for the operator localized in the LLM background B as

$$\begin{aligned} O_B(\{n_k\}) &= \sum_{T,(t,u)\alpha\beta} \sqrt{\frac{f_{T_B}\text{hooks}_{t_B}\text{hooks}_u}{\text{hooks}_{T_B}}} \chi_{\hat{T}_B,(\hat{t}_B,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}}) O_{\hat{T}_B,(\hat{t}_B,u)\beta\alpha}(Z,Y) \\ &= \kappa^{\frac{m}{2}} d_B \sum_{T,(t,u)\alpha\beta} \sqrt{\frac{f_{T_B}\text{hooks}_{t}\text{hooks}_u}{\text{hooks}_{T}}} \chi_{T,(t,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}}) O_{\hat{T}_B,(\hat{t}_B,u)\beta\alpha}(Z,Y) \end{aligned} \tag{4.33}$$

This looks remarkably similar to the original local loop operator

$$O(\{n_k\}) = \sum_{T,(t,u)\alpha\beta} \sqrt{\frac{f_{T}\text{hooks}_{t}\text{hooks}_u}{\text{hooks}_{T}}} \chi_{T,(t,u)\alpha\beta}(\sigma^{-1}_{\{n_k\}}) O_{T,(t,u)\beta\alpha}(Z,Y) \tag{4.34}$$

Naively one may have expected to treat all the Z fields in our operator, on the same footing. The arguments of this section motivate the fact that for the operators we propose, this is not the case. The reason why some of the Zs are treated differently is quite transparent in our analysis: the permutation $\sigma^{-1}_{\{n_k\}}$ has tied some of the Z fields with the Y fields. The Ys are localized on the Young diagram, so that the Zs tied to the Ys will be localized to this region too. This is rather natural as these fields are supposed to constitute a single object: the closed string. Recall that the number of Zs in the closed string is n_C and the number of Zs in the background is n_B. Thanks to the permutation $\sigma^{-1}_{\{n_k\}}$, the first $m+n_C$ labelled

[6]This follows from the fact that the action of a group element depends only on the differences of the content in the labelled boxes, and these differences are equal for the two states appearing in (4.30).

boxes are localized on R_B. r_B is filled with patterns that label states in an irreducible representation of $S_{n_C+n_B} = S_n$ while $\hat{r}_B$ is filled with patterns that label states in an irreducible representation of $S_{n_C} \times S_{n_B}$. These are two very different things. For further discussion of these localized restricted Schur polynomials, see Appendix 4.A.

4.5 LLM Magnons

In this section we would like to evaluate the action of the one loop dilatation operator on our proposed localized loops in the LLM background B. Our goal is to compute the one loop anomalous dimension of the localized loop, a quantity that can be compared to energies in the dual string theory, to either support or rule out our proposal. To begin we will compute $DO_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}(Z,Y)$ which is needed for the evaluation of $DO_B(\{n_k\})$ - the object of interest to us. Using the exact one loop result, we find[7]

$$DO_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}(Z,Y) = \sum_{T,(t,u)\nu_1\nu_2} N_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2;T,(t,u)\nu_1\nu_2} O_{T,(t,u)\nu_1\nu_2}(Z,Y) \tag{4.35}$$

where

$$N_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2;T,(t,u)\nu_1\nu_2} = -\frac{g_{YM}^2}{8\pi^2} \sum_{\hat{R}'_B} \frac{c_{R_B,R'_B} d_T nm}{d_{\hat{R}'_B} d_t d_u (n+m)} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_{r_B} \text{hooks}_s}{f_{R_B} \text{hooks}_{R_B} \text{hooks}_t \text{hooks}_u}}$$
$$\text{Tr}\left(\left[(1,m+1), P_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}\right] I_{R'_B T'} \left[(1,m+1), P_{T,(t,u)\nu_2\nu_1}\right] I_{T' R'_B}\right) \tag{4.36}$$

A few comments are in order. The one loop dilatation operator action was derived when acting on a restricted Schur polynomial, $O_{R_B,(r_B,s)\mu_1,\mu_2}(Z,Y)$. Above we are acting on $O_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}(Z,Y)$. By thinking of $O_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}(Z,Y)$ as a linear combination of restricted Schur polynomials, it is straightforward to repeat the derivation given in [6]. There are two possible swaps that contribute. The first swap is of the form $(1,m+1)$ which acts on a Y slot and a Z slot which belongs to the localized loop; this is the permutation that appears in (4.35). The second swap is of the form $(1,m+n_C+1)$ and it acts on a Y slot and a Z slot which belongs to the background; it gives a contribution that is suppressed in the large N limit and so we drop it. This follows from the fact that $m+n_C+1$ must appear in a corner, and after stripping off the first $m+n_C$ boxes, the only corners remaining are distant. In this way we learn that (4.35) is not exact: use of the large N limit has been made to discard a specific interaction between the background and the loop.

To make further progress, we need to study the matrix elements $N_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2;T,(t,u)\nu_1\nu_2}$. Our first task is to characterize the labels $T,(t,u)\nu_2\nu_1$ of operators that contribute in (4.35). The intertwiner $I_{T'R'_B}$ is only non-zero when R_B and T differ by the placement of at most one box - it is only in this case that a non-zero map can be defined. To proceed further, we need a little more notation. Specifically, we need to spell out which slots in the restricted Schur polynomial are associated to which representations. Writing the restricted Schur polynomial as

$$O_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}(Z,Y) = \frac{1}{(n+m)!} \sum_{\sigma\in S_{n+m}} \chi_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}(\sigma) Y^{i_1}_{i_{\sigma(1)}} \cdots Y^{i_m}_{i_{\sigma(m)}}$$
$$Z^{i_{m+1}}_{i_{\sigma(m+1)}} \cdots Z^{i_{m+n_C}}_{i_{\sigma(m+n_C)}} Z^{i_{m+n_C+1}}_{i_{\sigma(m+n_C+1)}} \cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}} \tag{4.37}$$

associates the first $m+n_C$ slots to the closed string excitation and the last n_B slots to the background. We will now make use of the Casimirs of the symmetric group, given by summing all the elements in a given conjugacy class. The symmetric groups and Casimirs which play a role are

1. $S_{n_B+n_C-1}$ which permutes the indices $m+2, m+3, \cdots, m+n$ has Casimirs $\mathcal{C}_i^{n_B+n_C-1}$. Recall that $n = n_B + n_C$.
2. S_{n_B} which permutes the indices $m+n_C+1, m+n_C+2, \cdots, m+n$ has Casimirs $\mathcal{C}_i^{n_B}$.

[7]The same result would be obtained by acting on $O_{R_B,(r_B,s)\mu_1\mu_2}(Z,Y)$ and using the simplifications of large N.

3. S_{n_C-1} which permutes the indices $m+2, m+3, \cdots, m+n_C$ has Casimirs $\mathcal{C}_i^{n_C-1}$.

4. S_{m-1} which permutes the indices $2, 3, \cdots, m$ has Casimirs $\mathcal{C}_i^{m-1}$.

These Casimirs distinguish a representation. Knowing the value of the Casimir associated to a given conjugacy class when acting on any state belonging to a particular representation is equivalent to knowing the value of the character in the representation for the relevant conjugacy class. Knowing the value of all the Casimirs is equivalent to knowing the complete set of characters, which specifies the representation completely. Denoting the state labelled by a in some representation R (not necessarily an irrep) we have

$$\mathcal{C}_i|R, a\rangle = \lambda_i^R |R, a\rangle \tag{4.38}$$

We get the same value of λ_i^R no matter what state (i.e. what value of a) we act on and knowing the complete set of λ_i^Rs allows us to determine R completely, up to equivalence. Consider the trace we wish to compute

$$\begin{aligned} &\mathrm{Tr}\left(\left[(1, m+1), P_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}\right] I_{R'_B T'} \left[(1, m+1), P_{T,(t,u)\nu_2\nu_1}\right] I_{T' R'_B}\right) \\ &= \mathrm{Tr}\left((1, m+1) P_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2} I_{R'_B T'} (1, m+1), P_{T,(t,u)\nu_2\nu_1} I_{T' R'_B}\right) + \ldots \end{aligned} \tag{4.39}$$

There are three more terms in the dots above. The projection operator $P_{\hat{R}_B \to (\hat{r}_B,s)\mu_1\mu_2}$ has the following form

$$P_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2} = \sum_i |i\rangle\langle i| \tag{4.40}$$

where the states $|i\rangle$ are linear combinations of states labelled by patterns with all states having exactly the same boxes filled with all integers from $m+1$ to $m+n$. It is thus possible to decompose this projector into a sum of terms (denoted $P^a_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}$) which each have their $m+1$ th box in a specific corner (with label a) of $\hat{r}_B$. Denote the representation obtained by removing this corner box from r_B by $r'_B(a)$. In this case we can write

$$\begin{aligned} P_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2} &= \sum_a P^a_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2} \\ \mathcal{C}_i^{n_B+n_C-1} P^a_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2} &= \lambda_i^{r_B(a)'} P^a_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2} \end{aligned} \tag{4.41}$$

A similar decomposition gives

$$\begin{aligned} P_{T,(t,u)\nu_2\nu_1} &= \sum_a P^a_{T,(t,u)\nu_2\nu_1} \\ \mathcal{C}_i^{n_B+n_C-1} P^a_{T,(t,u)\nu_2\nu_1} &= \lambda_i^{t(a)'} P^a_{T,(t,u)\nu_2\nu_1} \end{aligned} \tag{4.42}$$

Let us now show a concrete example of how these Casimirs can be used to derive restrictions on the labels $T, (t, u)$ given the labels $\hat{R}_B, (\hat{r}_B, s)$. For each of the four terms in (4.39) we can argue as follows

$$\begin{aligned} &\lambda_i^{r_B(a)'} \mathrm{Tr}\left((1, m+1) P^a_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2} I_{R'_B T'} (1, m+1) P^b_{T,(t,u)\nu_2\nu_1} I_{T' R'_B}\right) \\ &= \mathrm{Tr}\left((1, m+1) P^a_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2} \mathcal{C}_i^{n_B+n_C-1} I_{R'_B T'} (1, m+1) P^b_{T,(t,u)\nu_2\nu_1} I_{T' R'_B}\right) \\ &= \mathrm{Tr}\left((1, m+1) P^a_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2} I_{R'_B T'} (1, m+1) \mathcal{C}_i^{n_B+n_C-1} P^b_{T,(t,u)\nu_2\nu_1} I_{T' R'_B}\right) \\ &= \lambda_i^{t(b)'} \mathrm{Tr}\left((1, m+1) P^a_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2} I_{R'_B T'} (1, m+1) P^b_{T,(t,u)\nu_2\nu_1} I_{T' R'_B}\right) \end{aligned} \tag{4.43}$$

The first equality uses (4.41). The second equality is a consequence of the fact that

$$\left[\mathcal{C}_i^{n_B+n_C-1}, I_{\hat{R}'_B T'} (1, m+1)\right] = 0 \tag{4.44}$$

which follows because $(1, m+1)$ commutes with all elements of $S_{n_B+n_C-1}$ and the intertwiner maps between equivalent representations of $S_{n_B+n_C-1}$. The final equality uses (4.42). Thus, we have learned that if $\lambda_i^{r_B(a)'} \neq \lambda_i^{t(b)'}$, or equivalently if $r_B(a)' \neq t(b)'$ we have

$$\mathrm{Tr}\left((1, m+1) P^a_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2} I_{R'_B T'} (1, m+1) P^b_{T,(t,u)\nu_2\nu_1} I_{T' R'_B}\right) = 0 \tag{4.45}$$

Now, writing

$$\begin{aligned}&\mathrm{Tr}\Big((1,m+1)P_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}I_{R'_BT'}(1,m+1)P_{T,(t,u)\nu_2\nu_1}I_{T'R'_B}\Big)\\ &=\sum_a\sum_b \mathrm{Tr}\Big((1,m+1)P^a_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}I_{R'_BT'}(1,m+1)P^b_{T,(t,u)\nu_2\nu_1}I_{T'R'_B}\Big)\end{aligned} \tag{4.46}$$

we learn that the trace vanishes unless r_B and t differ by at most the placement of a single box. We can refine this analysis further. In constructing the operator $O_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}(Z,Y)$, we know that not all states in r_B participate: only the states in the irrep (r,B) of $S_{n_C}\times S_{n_B}$ participate. Our shorthand for "states in the irrep (r,B) of $S_{n_C}\times S_{n_B}$" is $\hat{r}_B$. We will demonstrate that t also belongs to an $S_{n_C}\times S_{n_B}$ representation. We know that

$$\mathcal{C}_i^{n_B}P_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}=\lambda_i^B P_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2} \tag{4.47}$$

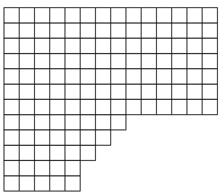
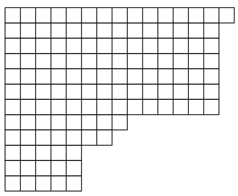
Now, decompose irreducible representation t in $P_{T,(t,u)\nu_2\nu_1}$ into a direct sum of irreducible representations of the S_{n_B} subgroup as follows

$$P_{T,(t^b,u)\nu_2\nu_1}=\sum_b P^b_{T,(t^b,u)\nu_2\nu_1} \tag{4.48}$$

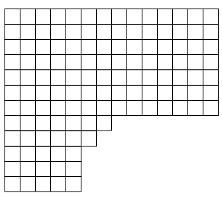
Arguing exactly as we did above we learn that the trace is only non-zero if $t^b=B$. This implies that we can replace T and t by $\hat{T}_B$ and $\hat{t}_b$ respectively. Finally, using the Casimirs $\mathcal{C}_i^{n_C-1}$ and $\mathcal{C}_i^{m-1}$ we can argue that r and t as well s and u differ by at most the placement of a single box. So, the mixing problem is tightly constrained: operators $O_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}(Z,Y)$ (using irrep (B,r,s) of the subgroup $S_{n_B}\times S_{n_C}\times S_m$) only mix with operators $O_{\hat{T}_B,(\hat{t}_B,u)\nu_1\nu_2}(Z,Y)$ (using irrep (B,t,u) of the subgroup) if the Young diagrams in the pairs s,u, as well as r,t and R,T differ by at most the placement of a single box.

A comment is in order. In the previous section we have proposed operators dual to local excitations on some background geometry. Our proposal puts the fields belonging to the background into one representation and the fields belonging to the excitation into a second representation. The result we have obtained above shows that the dilatation operator, which is to be identified with the Hamiltonian of the excitation, respects this structure: only operators that have this same make-up, with the same representation for the background, mix. This is support in favor of our construction.

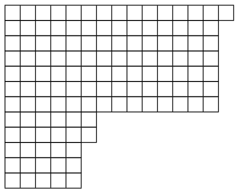
Since T_B and R_B can differ by at most one block we need to consider the following possibility

$R_B=$  $T_B=$  (4.49)

i.e. T_B has a distant box added. If this is a Y box, we will have

$t_B=$  (4.50)

i.e. t_B has no distant boxes. Since the box we must drop is a Y box, the swap $(1,m+1)$ appearing in (4.39) must leave the boxes inert, and this immediately implies that this dilatation operator matrix element is of order N^{-1}. This proves that under the action of the dilatation operator, none of the Y boxes are transported to distant locations on the Young diagram. Since the distant box in T_B is a Z box, we will have

$t_B =$  (4.51)

We have now learned enough about the matrix elements $N_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2;\hat{T}_B,(\hat{t}_B,u)\nu_1\nu_2}$ that we can consider the action of the dilatation operator on a localized loop

$$\begin{aligned} DO_B(\{n_k\}) &= \sum_{R,(r,s)\mu_1\mu_2} \sqrt{\frac{f_{R_B}\text{hooks}_{r_B}\text{hooks}_u}{\text{hooks}_{R_B}}} \chi_{\hat{R}_B,(\hat{r}_B,s)\mu_2\mu_1}(\sigma^{-1}_{\{n_k\}}) DO_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}(Z,Y) \\ &= \sum_{R,(r,s)\mu_1\mu_2} \sum_{T,(t,u)\nu_1\nu_2} \sqrt{\frac{f_{R_B}\text{hooks}_{r_B}\text{hooks}_u}{\text{hooks}_{R_B}}} \chi_{\hat{R}_B,(\hat{r}_B,s)\mu_2\mu_1}(\sigma^{-1}_{\{n_k\}}) \\ &\quad \times N_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2;\hat{T}_B,(\hat{t}_B,u)\nu_1\nu_2} O_{\hat{T}_B,(\hat{t}_B,u)\nu_1\nu_2}(Z,Y) \\ &= -\frac{g^2_{YM}}{8\pi^2} \sum_{p=1}^{m} \sum_{q=m+1}^{m+n} \sum_{T^+,t,u,\nu_1\nu_2} \sum_{T} d_B c_{T^+T} \frac{\text{hooks}_T}{\text{hooks}_{T^+}} \sqrt{\frac{f_{T_B}\text{hooks}_{t_B}\text{hooks}_u}{\text{hooks}_{T_B}}} \\ &\quad \times \chi_{T^+,(t,u,\square)\nu_1\nu_2}(\psi^{-1}) O_{\hat{T}_B,(\hat{t}_B,u)\nu_2\nu_1}(Z,Y) \end{aligned} \tag{4.52}$$

where to get the last line above we have made use of the identity proved in Appendix 4.B. Focus on the character $\chi_{T^+,(t,u,\square)\nu_1\nu_2}(\psi^{-1})$. There is an extra box in T^+ that must be dropped to obtain T. The permutation ψ depends on p and q. If the permutation ψ^{-1} consists of cycles that mix the indices of the local fields with the possible distant Z box, then we know by the arguments of section 4.4 that these contributions can be dropped at large N. In Appendix 4.B we argue that ψ^{-1} has two cycles, one of length k and one of length $n_C + m + 1 - k$. The distant box will not be suppressed as long as it appears in a cycle that does not tie it to any local boxes. There are only two such terms. At large N, there are $mn_C = O(N)$ terms appearing on the right hand side of (4.52), so that this is a subleading contribution. This proves that at large N, any terms with a distant Z box can be neglected. Our argument has demonstrated that only local loop operators, as we have defined them, contribute. This is enough to prove that

$$\begin{aligned} &\text{Tr}\left(\left[(1,m+1), P_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2}\right] I_{R'_B T'_B} \left[(1,m+1), P_{\hat{T}_B,(\hat{t}_B,u)\nu_2\nu_1}\right] I_{T'_B R'_B} \right) \\ &\quad = d_B \text{Tr}\left(\left[(1,m+1), P_{R,(r,s)\mu_1\mu_2}\right] I_{R'T'} \left[(1,m+1), P_{T,(t,u)\nu_2\nu_1}\right] I_{T'R'} \right) \end{aligned} \tag{4.53}$$

To prove the equality note that the map $I_{T'_B R'_B}$ has a trivial action: it simply maps between two subspaces. It is the permutation $(1, m+1)$ that has a nontrivial action. However, as we have discussed, there is a map between $\hat{R}_B$ and R that is equivariant with respect to the action of $(1, m+1)$. The only difference between the two sides of (4.53) is that the LHS has a contribution from each possible pattern of the background Young diagram B. This is the origin of d_B on the right hand side. Apart from the above trace, the matrix elements $N_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2;\hat{T}_B,(\hat{t}_B,u)\nu_1\nu_2}$ involve a few other factors. Looking back at (4.36), we will need the values of

$$\frac{d_{\hat{T}_B}}{d_{\hat{R}'_B} d_{\hat{t}_B} d_u} = \frac{d_T}{d_{R'} d_t d_u d_B} \tag{4.54}$$

$$\frac{n_C}{n_C + m} = 1 + O\left(\frac{m}{n_C}\right) = \frac{n}{n+m} \tag{4.55}$$

$$\frac{\text{hooks}_{T_B}\text{hooks}_{r_B}\text{hooks}_s}{\text{hooks}_{R_B}\text{hooks}_{t_B}\text{hooks}_u} = \kappa^m \frac{\text{hooks}_r\text{hooks}_s}{\text{hooks}_R} \kappa^{-m} \frac{\text{hooks}_T}{\text{hooks}_t\text{hooks}_u} = \frac{\text{hooks}_T\text{hooks}_r\text{hooks}_s}{\text{hooks}_R\text{hooks}_t\text{hooks}_u} \tag{4.56}$$

In the first formula above we used the fact that the first box dropped from $d_{\hat{R}'_B}$ is a Y box, i.e. it does not belong to B. In the second equation above we used the fact that we have a dilute magnon gas (i.e. $m \ll n_C$). In the last identity above we have used (4.27). Finally, the only factors which don't cancel in the ratio $\frac{f_{T_B}}{f_{R_B}}$ are the factors of boxes which are not common between T_B and R_B. Denote the factor of the box labelled $n_C + m$ in the Young-Yamonouchi pattern for the states in $\hat{R}_B$ by N_{eff}. The values of $\frac{f_{T_B}}{f_{R_B}}$ and $c_{R_B R'_B}$ are given by replacing N in $\frac{f_T}{f_R}$ and $c_{RR'}$ by N_{eff}. This now proves that

$$N_{\hat{R}_B,(\hat{r}_B,s)\mu_1\mu_2;\hat{T}_B,(\hat{t}_B,u)\nu_1\nu_2} = N_{R,(r,s)\mu_1\mu_2;T,(t,u)\nu_1\nu_2}\Big|_{N\to N_{\text{eff}}} \tag{4.57}$$

In the end, this is a remarkably simple result: the matrix elements of the dilatation operator acting on localized loops in the LLM geometry are given by replacing $N \to N_{\text{eff}}$ in the matrix elements of the dilatation operator in the trivial background! This provides a generalization of the $\frac{1}{2}$-BPS result proved in [63]. Performing this replacement we now find that

$$\begin{aligned}
&DO_B(\{n_k\})\\
=\;& \frac{g_{YM}^2 N_{\text{eff}}}{8\pi^2}\left(2O_B(\{n_k\}) - O_B(\{n_1+1, n_2-1, \cdots, n_m\}) - O_B(\{n_1-1, n_2+1, \cdots, n_m\})\right)\\
+\;& \frac{g_{YM}^2 N_{\text{eff}}}{8\pi^2}\left(2O_B(\{n_k\}) - O_B(\{n_1, n_2+1, n_3-1, \cdots, n_m\}) - O_B(\{n_1, n_2-1, n_3+1, \cdots, n_m\})\right)\\
&\vdots\\
+\;& \frac{g_{YM}^2 N_{\text{eff}}}{8\pi^2}\left(2O_B(\{n_k\}) - O_B(\{n_1-1, n_2, \cdots, n_m+1\}) - O_B(\{n_1+1, n_2, \cdots, n_m-1\})\right)
\end{aligned} \tag{4.58}$$

We can diagonalize this action exactly as we do it in the planar limit: by Fourier transforming to momentum space. For example, a state with two magnons, one of momentum p and one of momentum $-p$ is given by

$$O_B(p,-p) = \sum_{n_1=0}^{n_C} O_B(\{n_1, n_C - n_1\}) e^{in_1 p} \tag{4.59}$$

$$DO_B(p,-p) = (E(p) + E(-p))O_B(p,-p)$$

$$E = \frac{g_{YM}^2 N}{8\pi^2}\frac{N_{\text{eff}}}{N}\left(2\sin\frac{p}{2}\right)^2 \equiv \frac{N_{\text{eff}}}{N} g^2 \left(2\sin\frac{p}{2}\right)^2 \tag{4.60}$$

This diagonalization and its perfect agreement with a string worldsheet description has been discussed many times in the literature (see [114] for example).

Now that we have the gauge theory anomalous dimension associated to a magnon, we would like to compare it to the prediction from the string theory analysis. We will compare the gauge theory and the string theory for localized loops located in the three regions possible. For the gauge theory prediction we need the value of N_{eff} which is given by the factors of the boxes labelled in Fig 4.2. The factor of the box labelled i is denoted $N_{\text{eff}}^{(i)}$ with

$$N_{\text{eff}}^{(1)} = N + c_1 + c_2, \qquad N_{\text{eff}}^{(2)} = N + c_1 - r_1, \qquad N_{\text{eff}}^{(3)} = N - r_1 - r_2 \tag{4.61}$$

Noting that $N = r_1 + r_2 + r_3$ these can also be written as

$$N_{\text{eff}}^{(1)} = r_1 + r_2 + r_3 + c_1 + c_2, \qquad N_{\text{eff}}^{(2)} = r_2 + r_3 + c_1, \qquad N_{\text{eff}}^{(3)} = r_3 \tag{4.62}$$

These factors have a very natural geometrical interpretation in the dual gravity. The simplest way to map between the LLM boundary condition and the Young diagram describing the background is through the use of the free fermion language[3]. The LLM boundary condition is a picture of the phase space of N non-interacting fermions in an external harmonic oscillator potential. The central disk is a number of

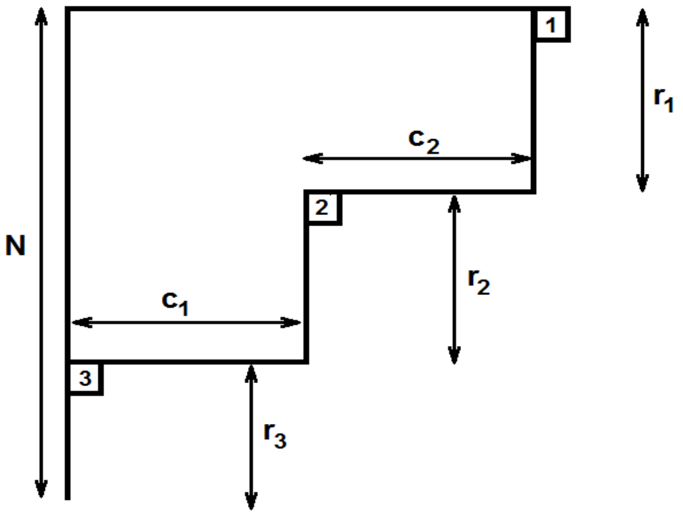


Figure 4.2: A Young diagram B which corresponds to the LLM background with boundary condition shown in Fig 4.3. The number of rows and columns defining B as well as N are shown.

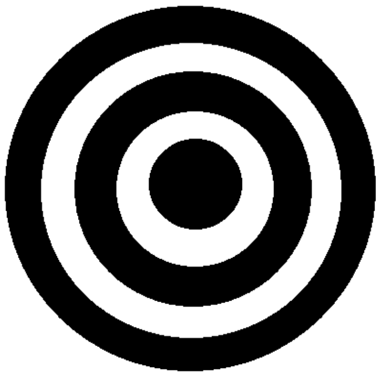


Figure 4.3: This is the LLM boundary condition corresponding to the Young diagram given in Fig 4.2. The central black disk has area equal to πr_3 and hence a radius of $\sqrt{r_3}$.

fermions (set by the area of the central disk divided by π/N) that have not been excited. The inner black annulus is some number of fermions (set by the area of this annulus divided by π/N) each excited by the same amount (set by the area of the inner white annulus divided by π/N). Finally, the outer black annulus is some number of fermions (set by the area of this annulus divided by π/N), with each again excited by the same amount (set by the area of the outer white annulus divided by π/N). The Young diagram is a picture of the same thing. Each row corresponds to a fermion. The number of boxes in any given row is equal to the amount by which this fermion is excited. Using this dictionary, we see that the central disk has an area of $\pi r_3/N$ and hence a radius squared of r_3/N. Similarly, the inner black annulus has an outer radius squared of $(r_3 + c_1 + r_2)/N$ while the outer black annulus has an outer radius squared of $(r_1 + r_2 + r_3 + c_1 + c_2)/N$. Thus, the radius (squared) on the LLM plane at which the loop is localized is set by the factor of the box divided by N.

With these basics set up, lets now recall the string theory result[4]. For well separated magnons, each magnon transforms in a definite $SU(2|2)^2$ representation. The closed string transforms as the tensor product of the individual magnon representations. To specify the representation of a magnon, we specify the central charges appearing in the $SU(2|2)^2$ algebra. Think of the LLM plane as a complex plane. Each magnon corresponds to a directed line segment on this plane. Every directed line segment is equivalent to a complex number, k with phase given by the direction of the line segment and magnitude given by the length of the line segment. The magnitude of k is the length of the line corresponding to the magnon. The energy of the magnon is determined by supersymmetry to be

$$E = \sqrt{1 + 2g^2|k|^2} = 1 + g^2|k|^2 + ... \tag{4.63}$$

For a magnon which subtends an angle θ we have $|k| = 4R^2 \sin^2 \frac{\theta}{2}$ with R the radius at which the closed string is localized. The angle θ is identified with the momentum p of the magnon[4]. As we have just discussed, $R^2 = \frac{N_{\rm eff}}{N}$ so that there is a perfect agreement between the gauge theory result (4.60) and the order g^2 contribution to the string theory result (4.63): our proposal for operators dual to localized closed string states is correct.

4.6 Another example and another method

The above results suggest that the only effect of working on an LLM background is to replace $g_{YM}^2 N \to g_{YM}^2 N_{\rm eff}$. As we have just seen, this replacement in the gauge theory reproduces the central charge predicted by the string theory analysis. In this section we want to explore what is perhaps the simplest setting in which this conclusion can be probed - simple enough that we can do it without any restricted Schur polynomial technology. This problem was considered in [115] and this section is just a quick review of those results. We will confirm our $g_{YM}^2 N \to g_{YM}^2 N_{\rm eff}$ conclusion for this simple example.

Consider a background $\chi_B(Z)$ with B a Young diagram that has N rows and M columns, with M of order N. We write

$$\chi_B(Z) = (\det(Z))^M \ . \tag{4.64}$$

The corresponding LLM boundary condition is a single black annulus hugging a central white disk of area $\pi M/N$. The simplicity of this example is due to two facts:

1. We have the well-known formula for the derivative of a determinant

$$\frac{\partial}{\partial Z^i_j}\chi_B(Z) = M(Z^{-1})^j_i \chi_B(Z)\,. \tag{4.65}$$

2. In this background, multiplying by a trace is a local operation. The background looks like a single annulus, so there is only one outer edge. As a consequence, our local loops are

$$O(\{n_k\}))\chi_B(Z) \tag{4.66}$$

Thus, we can write

$$D\chi_B(Z)O(\{n_k\})) = \chi_B(Z) D_{\rm eff} O(\{n_k\}) \tag{4.67}$$

where

$$D_{\rm eff} = \frac{1}{\chi_B(Z)} D\chi_B(Z) = D - \frac{M g_{YM}^2}{8\pi^2} {\rm Tr}\left(\left(ZYZ^{-1} + Z^{-1}YZ - 2Y\right)\frac{\partial}{\partial Y}\right) \tag{4.68}$$

It is now simple to see that

$$\begin{aligned}
& D_{\rm eff} O(\{n_k\}) \\
= \ & \frac{g_{YM}^2(N+M)}{8\pi^2}\left(2O(\{n_k\}) - O(\{n_1+1, n_2-1, \cdots, n_m\}) - O(\{n_1-1, n_2+1, \cdots, n_m\})\right) \\
+ \ & \frac{g_{YM}^2(N+M)}{8\pi^2}\left(2O(\{n_k\}) - O(\{n_1, n_2+1, n_3-1, \cdots, n_m\}) - O(\{n_1, n_2-1, n_3+1, \cdots, n_m\})\right) \\
& \vdots \\
+ \ & \frac{g_{YM}^2(N+M)}{8\pi^2}\left(2O(\{n_k\}) - O(\{n_1-1, n_2, \cdots, n_m+1\}) - O(\{n_1+1, n_2, \cdots, n_m-1\})\right)
\end{aligned} \tag{4.69}$$

Since for this B we have $N_{\rm eff} = N + M$, this is in perfect agreement with our expectations.

4.7 Excitations on the inner edge of an annulus

The loops we have constructed have been localized on the outer edge of a black disk or annulus on the LLM plane. These excitations are created by adding boxes to inwardly pointing corners of the Young diagram describing the background. It is also interesting to construct loops that are localized on the inner edge of a black annulus. The inner edges correspond to outwardly pointing corners of the Young diagram describing the background. It is not possible to add boxes at these corners; it is possible to remove boxes. Thus, it is natural to describe these excitations by removing boxes. Indeed, to orbit on the inner edge of a black ring these excitations must have negative the angular momentum carried by a KK graviton dual to Z. Thus these excitations must have an opposite $\mathcal{R}$ charge to that of Z; for each box we remove we do indeed decrease the $\mathcal{R}$ charge by 1. One could also have considered including $Z^\dagger$s in the operator, which also decrease the $\mathcal{R}$ charge by 1. However, including $Z^\dagger$s in the background also increases the dimension and takes us out of the class of $\frac{1}{2}$ BPS backgrounds. "Empty box gravitons" have been studied in [5] where they were called "countergravitons". In Appendix 4.C we have worked out enough details that we have indeed confirmed this expectation. Using the resulting loop operators we again reproduce the string theory expectations.

4.8 Conclusions

In this chapter we have constructed operators dual to closed string states probing an LLM geometry. We have not described the most general LLM geometry: we have focused on geometries arising when we colour the LLM plane with $O(1)$ concentric rings. Further, we have computed the one loop anomalous dimensions of these operators. Although our computations are quite technical the final result is remarkably simple: the action of the dilatation operator in the nontrivial geometry is given by simply scaling the N dependence in the planar result. We have only argued this at one loop, but we expect this to go through for higher loops. Indeed, at k loops, using the Casimir arguments developed in Section 4.5, we know that at most k boxes can shift position on the Young diagram and further that the background, labelled by Young diagram B, is not changed. In this case, we again recover the planar dilatation operator but with N scaled as it was at one loop level. The intuition coming from the $SU(2|2)^2$ symmetry of the problem also supports this conclusion. Indeed, from the one loop anomalous dimension we can read off the central charge of the magnon and, thanks to supersymmetry, this completely determines the energy of the magnon.

For a closed string state described by a polygon with all vertices on the outer edge of a single ring, magnon scattering is again elastic and the problem is integrable. The spectrum of anomalous dimensions and even the expressions for the exact eigenstates can all be obtained from the answers in the planar limit by simply replacing N with N_{eff}. There are also closed string states with vertices located on the outer edges of distinct rings. For these the magnon scattering problem is not elastic and the problem is not integrable. However, thanks to the $SU(2|2)^2$ symmetry, the two body S-matrix can still be determined exactly, up to a phase. It would be interesting to construct these S-matrices and verify their dynamical content. For example, the poles of these S-matrices should contain information about the boundstate spectrum of the theory[116].

Previous studies of restricted Schur polynomials have focused on operators dual to states in the trivial $\text{AdS}_5\times\text{S}^5$ background. In this chapter we have carried out a detailed study involving operators dual to states in a non-trivial geometry. What lessons have we learned from this study? Our construction of localized operators has involved a new ingredient which has not featured in any previous constructions: the Z fields belonging to the background have not been treated on the same footing as the Z fields belonging to the closed string excitation. In the $SU(2)$ sector all previous constructions of restricted Schur polynomials have made use of the subgroup $S_n \times S_m \subset S_{n+m}$. The subgroup S_n permutes indices belonging to Zs while S_m permutes indices belonging to Ys. The construction developed in this chapter makes use of the subgroup $S_{n_B} \times S_{n_C} \times S_m \subset S_{n+m}$ when constructing operators in the $SU(2)$ sector. The subgroup S_{n_B} permutes indices belonging to the Zs making up the background, S_{n_C} permutes indices

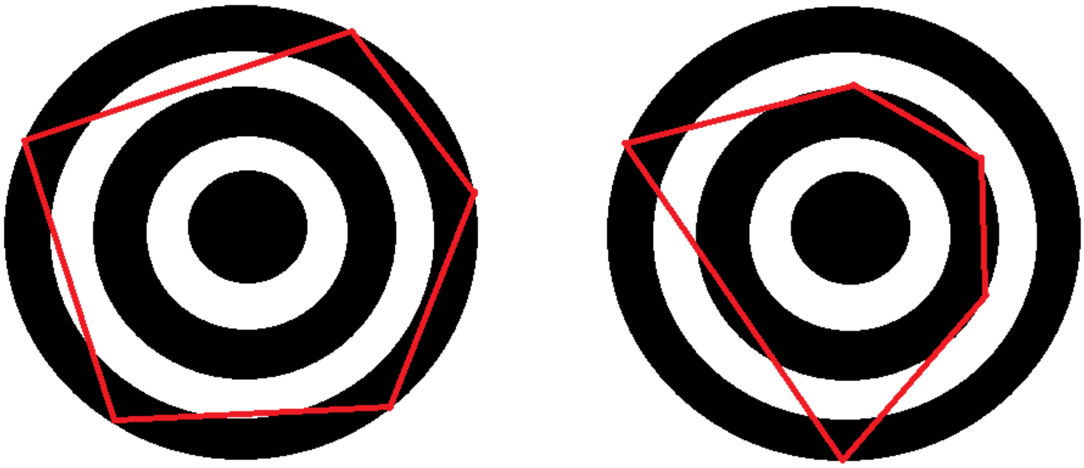


Figure 4.4: The string worldsheet on the left above corresponds to an integrable anomalous dimension problem in the dual gauge theory. The problem on the right doesn't. Conservation of momentum implies the angle swept out by the magnons is the same before and after scattering. Energy conservation is more difficult to describe geometrically. At very strong coupling it says that the sum of the lengths of the magnons scattering is the same before and after scattering. It is clear that scattering of magnons for the worldsheet on the left is elastic. For the worldsheet on the right, when a magnon with endpoints on edges of different rings scatters, the scattering is in general inelastic.

of Zs belonging to the closed string and S_m permutes indices belonging to Ys. Thus, the distinction between what is the background and what is the excitation is accomplished in the choice of the subgroup and its representations. This provides a concrete proposal for the gauge theory mechanism to distinguish between "background" and "fluctuation". As a test of this proposal, we can try to determine where this construction breaks down and see if it matches our intuition. The geometries we have studied arise from a boundary condition that consists of $O(1)$ fat black rings. This is a geometry that is smooth on the string scale and we expect back reaction by stringy probes is negligible. This is consistent with the fact that changes in the background Young diagram B are suppressed in the large N limit. We could also consider a background B described by a Young diagram with $O(N)$ corners. The boundary condition for this geometry is a set of N very thin rings, giving rise to a geometry that has nontrivial features, even on the string scale. We expect that this detailed and intricate geometry is disturbed by a stringy probe. In this case there are so many outward pointing corners, that not all of them can be well separated from the local excitation. In this case, the dilatation operator will start to mix operators with distinct background Young diagrams B, even at large N. Consequently, taking $N \to \infty$ will no longer suppress backreaction, and we see that our gauge theory mechanism to distinguish between "background" and "fluctuation" has failed precisely where we would expect it to.

Although small deformations of the $\frac{1}{2}$-BPS sector are special, this sector of the theory may provide insight into more generic situations[117]. For example, the presence of a horizon in the geometry signifies a region of high entropy - proportional to the horizon area. To obtain a large entropy we need configurations that support many almost equivalent but different excitations, that could provide the microstates responsible for this entropy. It is natural to associate this with a region on the Young diagram that has many corners, since the number of possible excitations is related to the number of representations of the subgroup we can subduce, and this is related to the number of boxes we can remove. Indeed, the typical state in the $\frac{1}{2}$-BPS sector has $O(N)$ corners[117]. For an excitation finding itself in this "corner rich" region, there are many outward pointing corners nearby the local excitation, so that it will spread over this region as time evolves. This has the potential to provide a natural field theory origin for tidal forces and even the longitudinal string spreading effects experienced at the horizon (see for example [118]). With some thought we expect that simple computations, to provide a quantitative interrogation for this identification, are possible.

It would seem to be straight forward to consider giant graviton excitations of these LLM geometries. It appears that the arguments that work for the closed string excitations will go through, without modification, for the giants. It would be interesting to check the details and verify that this is indeed the case.

Acting with $\chi_B(Z)$ on the vacuum produces the state that corresponds to the LLM background spacetime. The loop $O(\{n_k\})$ operator creates a single closed string state in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime; it may have been natural to expect that acting with $O(\{n_k\})$ on the state corresponding to the LLM geometry would produce a string probing the LLM spacetime. This is not the case: as we have seen, the product $O(\{n_k\})\chi_B(Z)$ does not produce a closed string localized on the background spacetime. However, taking a product of the representation describing the background and the representation describing the excitation does indeed give the representation relevant for describing a closed string localized on the LLM spacetime. The resulting operator involves a highly nontrivial mixing of the indices of the Z fields belonging to the background and Z fields belonging to the excitation in the operator. The simplest description of this mixing is in terms of the representations involved, so that the representation theory description of the problem furnishes a very natural language. We have also seen that the action of the dilatation operator is written in terms of the value of the factor at the location on the Young diagram R_B where the excitation is added. This factor, which is a representation theoretic quantity, is related to the radius squared at which the excitation orbits on the LLM plane, a geometric quantity. In studies of open strings attached to giant gravitons [59, 13, 14, 15], each giant graviton is described by a long row or column and each open string is associated with a box on the Young diagram. The strength of the open string end point interactions is set by the factor of this box[14, 15]. Comparisons with a magnon description again gives exact agreement, with the factors combining to exactly reproduce the magnon energy in the dual string theory description [18]. It is striking how natural the language constructed using representation theory and Young diagrams is.

Appendices to Chapter 4

4.A Local Restricted Schur Polynomials

The restricted Schur polynomials provide a complete basis for the local operators of the gauge theory. If we restrict to the $SU(2)$ sector of the theory, these polynomials are labelled by three Young diagrams and two multiplicity labels $\chi_{R,(r,s)\mu_1\mu_2}(Z,Y)$. In this study our goal has been to understand how we can describe perturbations around a nontrivial LLM spacetime geometry. Our results imply that one can write down "local restricted Schur polynomials" which provide a basis for such excitations. The background is described by a Young diagram B with order N^2 boxes. In the $SU(2)$ sector of the theory, the perturbation can again be labelled by three Young diagrams and two multiplicity labels $R,(r,s)\mu_1\mu_2$. An economical way to summarize the construction of Section 4.4 is to notice that the localized restricted Schur polynomial uses representations[8] of $S_{n_B+n_C+m}$ and of the subgroup $S_{n_B}\times S_{n_C}\times S_m$. In terms of this language, we can write the localized restricted Schur polynomial as

$$O_{\hat{T}_B,(\hat{t}_B,u)\beta\alpha}(Z,Y) = O_{T_B,(B,T,t,u)\beta\alpha}(Z,Y) \tag{4.70}$$

In the above formula the triple of Young diagrams B,t,u label an irrep of $S_{n_B}\times S_{n_C}\times S_m$. The labels $\beta\alpha$ as well as T are multiplicity labels. We have seen that taking a product of the operator describing the closed string and the operator describing the background geometry does not produce the desired closed string on the background spacetime. In the operators (4.70) there is a simple tensor product between the representation (of S_{n_B}) describing the background the representation (of $S_m\times S_{n_C}$) describing the excitation. This simple tensor product generates highly nontrivial mixing of the indices of the Z fields belonging to the background and Z fields belonging to the excitation in the operator (4.70).

A number of identities valid for the restricted Schur polynomials also hold for the local restricted Schur polynomials. The most important identity of this type is the completeness of local restricted characters in a specific background B. The usual identity for the completeness of restricted characters, obtained in [56], reads ($r\vdash n$ and $s\vdash m$)

$$\sum_{R,(r,s)\alpha\beta}\frac{d_R}{d_r d_s(n+m)!}\chi_{R,(r,s)\alpha\beta}(\tau)\chi_{R,(r,s)\beta\alpha}(\sigma)=\delta_{[\sigma][\tau]} \tag{4.71}$$

The delta function on the right hand side is 1 if we can satisfy

$$\rho\tau\rho^{-1}=\sigma \tag{4.72}$$

for some $\rho\in S_n\times S_m$. The completeness of the local restricted characters reads

$$\sum_{R,(r,s)\alpha\beta}\frac{d_R}{d_r d_s(n+m)!}\chi_{\hat{R}_B,(\hat{r}_B,s)\alpha\beta}(\tau)\chi_{\hat{R}_B,(\hat{r}_B,s)\beta\alpha}(\sigma)=d_B^2\delta_{[\sigma][\tau]} \tag{4.73}$$

where $\sigma,\tau\in S_{n_C+m}$ and the delta function on the right hand side is 1 if we can satisfy

$$\rho\tau\rho^{-1}=\sigma \tag{4.74}$$

for some $\rho\in S_{n_C}\times S_m$. Notice that in (4.73) the permutations σ,τ act only on the boxes belonging to the excitation and the orthogonality in the permutations is achieved even though we only sum over the excitation labels. It is completeness within a fixed background B.

4.B Restricted Character Identities

In this section we would like to derive the exact identity for restricted characters, that reduces to (4.18) in the planar limit. It is simple to verify that

$$-\frac{g_{YM}^2}{8\pi^2}\mathrm{Tr}\left([Y,Z]\left[\frac{d}{dY},\frac{d}{dZ}\right]\right)\mathrm{Tr}(\sigma Y^{\otimes m}Z^{\otimes n})$$

[8]Recall that there are n_B Z fields making up the background geometry and n_C+m fields in the loop.

$$= \frac{g_{YM}^2}{8\pi^2}\sum_{p=1}^{m}\sum_{q=m+1}^{m+n}\Big(\delta^{i_q}_{i_{\sigma(p)}}[Y,Z]^{i_p}_{i_{\sigma(q)}} - \delta^{i_p}_{i_{\sigma(q)}}[Y,Z]^{i_q}_{i_{\sigma(p)}}\Big)Y^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_{p-1}}_{i_{\sigma(p-1)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_m}_{i_{\sigma(m)}}$$
$$\times Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{q-1}}_{i_{\sigma(q-1)}}Z^{i_{q+1}}_{i_{\sigma(q+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}} \quad (4.75)$$

The above formula is exact - it includes much more than just the planar contractions. It is correct regardless of the way we scale m and n with N. Further, the operator $\mathrm{Tr}(\sigma Y^{\otimes m}Z^{\otimes n})$ can have any trace structure - it is not in general a single trace operator. To obtain the large N planar approximation, recall that all the N dependence in this case comes from contracting index loops and that the leading contribution comes from terms for which the dilatation operator contracts a pair of indices to produce a factor of N. Looking at the above expression, the terms that contribute in the planar approximation have $\sigma(p) = q$ or $\sigma(q) = p$. Keeping only these terms, the above expression can be written as

$$-\frac{g_{YM}^2}{8\pi^2}\mathrm{Tr}\left([Y,Z]\left[\frac{d}{dY},\frac{d}{dZ}\right]\right)\mathrm{Tr}(\sigma Y^{\otimes m}Z^{\otimes n}) = \frac{Ng_{YM}^2}{8\pi^2}\sum_{p=1}^{m}\sum_{q=m+1}^{m+n}(\delta_{q,\sigma(p)}+\delta_{p,\sigma(q)})$$
$$\times\Big(\mathrm{Tr}(\sigma Y^{\otimes m}Z^{\otimes n}) - \mathrm{Tr}\left((p,q)\sigma(p,q)Y^{\otimes m}Z^{\otimes n}\right)\Big) \quad (4.76)$$

Now, using the identity

$$\mathrm{Tr}(\sigma Y^{\otimes m}Z^{\otimes n}) = \sum_{T,(t,u)\alpha\beta}\sqrt{\frac{f_T \mathrm{hooks}_t \mathrm{hooks}_u}{\mathrm{hooks}_T}}\chi_{T,(t,u)\alpha\beta}(\sigma^{-1})O_{T,(t,u)\beta\alpha}(Z,Y) \quad (4.77)$$

and the exact action of the one loop dilatation operator, we find

$$\sum_{T,(t,u)\nu_1\nu_2}\sqrt{\frac{f_T\mathrm{hooks}_u}{\mathrm{hooks}_{T/t}}}\chi_{T,(t,u)\nu_2\nu_1}(\sigma^{-1})N_{T,(t,u)\nu_1\nu_2;R,(r,s)\mu_1\mu_2} = \frac{g_{YM}^2N}{8\pi^2}\sqrt{\frac{f_R\mathrm{hooks}_s}{\mathrm{hooks}_{R/r}}}$$
$$\times\sum_{p=1}^{m}\sum_{q=m+1}^{m+n}\left(\delta_{q,\sigma(p)}+\delta_{p,\sigma(q)}\right)\Big(\chi_{R,(r,s)\mu_2\mu_1}(\sigma^{-1}) - \chi_{R,(r,s)\mu_2\mu_1}\left((p,q)\sigma^{-1}(p,q)\right)\Big) \quad (4.78)$$

This is in complete agreement with the identity (4.18) between the restricted characters.

To go beyond the planar limit, return to (4.75). After a little work, we obtain

$$-\frac{g_{YM}^2}{8\pi^2}\mathrm{Tr}\left([Y,Z]\left[\frac{d}{dY},\frac{d}{dZ}\right]\right)\mathrm{Tr}(\sigma Y^{\otimes m}Z^{\otimes n}) = -\frac{g_{YM}^2}{8\pi^2}\sum_{p=1}^{m}\sum_{q=m+1}^{m+n}\mathrm{Tr}\left(\psi\mathbf{1}\otimes Y^{\otimes m}\otimes Z^{\otimes n}\right) \quad (4.79)$$

where ψ depends on p and q. ψ belongs to the group algebra of S_{n+m+1}, i.e. it is a linear combination of S_{n+m+1} group elements. We will use the notation ψ^{-1} to denote the element of the group algebra obtained by summing the inverse of each term summed to form ψ. Concretely if $p\neq 1$ and $q \neq m+1$

$$\begin{aligned}\psi &= (p,1)(q,m+1)\sigma(q,1,p,m+n+1,m+1)\\ &-(1,q,m+1,p)\sigma(q,m+n+1,m+1)(p,1)\\ &-(1,p)(m+1,m+n+1,q)\sigma(p,m+1,q,1)\\ &+(p,1,q,m+1,m+n+1)\sigma(m+1,q)(p,1)\end{aligned} \quad (4.80)$$

If $p = 1$ but $q \neq m+1$ we have

$$\begin{aligned}\psi &= (q,m+1)\sigma(q,1,m+n+1,m+1)\\ &-(1,q,m+1)\sigma(q,m+n+1,m+1)\\ &-(q,m+1,m+n+1)\sigma(1,m+1,q)\\ &+(q,m+1,m+n+1,1)\sigma(m+1,q)\end{aligned} \quad (4.81)$$

If $q = m+1$ but $p \neq 1$ we have

$$\psi = (p,1)\sigma(p,m+n+1,m+1,1)$$

$$
\begin{aligned}
&-(p,1,m+1)\sigma(m+n+1,m+1)(p,1)\\
&-(1,p)(m+1,m+n+1)\sigma(p,m+1,1)\\
&\qquad+(p,1,m+1,m+n+1)\sigma(p,1)
\end{aligned} \tag{4.82}
$$

Finally, if $p=1$ and $q=m+1$ we have

$$
\begin{aligned}
\psi &= \sigma(1,m+n+1,m+1)-(1,m+1)\sigma(m+n+1,m+1)\\
&-(m+1,m+n+1)\sigma(1,m+1)+(1,m+1,m+n+1)\sigma
\end{aligned} \tag{4.83}
$$

Now, on the right hand side of (4.79), replace **1** by a new field W. We will later turn W back into **1** by acting with $\mathrm{Tr}\frac{\partial}{\partial W}$. With W on the right hand side of the above expression, we can write everything in terms of restricted Schur polynomials built using 3 complex fields, W, Z and Y. These are labelled by 4 Young diagrams: $\square$ for W, $s\vdash m$ for the Y fields, $r\vdash n$ for the Z fields and $R^+\vdash m+n+1$ for the representation which mixes all three fields. When removing a single box there is no multiplicity label. Thus, we need a multiplicity for s only. In this way we obtain the following result

$$
\begin{aligned}
&-\frac{g_{YM}^2}{8\pi^2}\mathrm{Tr}\left([Y,Z]\left[\frac{d}{dY},\frac{d}{dZ}\right]\right)\mathrm{Tr}(\sigma Y^{\otimes m}Z^{\otimes n})\\
&=-\frac{g_{YM}^2}{8\pi^2}\sum_{p=1}^{m}\sum_{q=m+1}^{m+n}\sum_{R^+,r,s,\mu_1,\mu_2}\frac{d_{R^+}n!m!}{d_rd_s(n+m+1)!}\chi_{R^+,(r,s,\square)\mu_1\mu_2}\left(\psi^{-1}\right)D_W\chi_{R^+,(r,s,\square)\mu_2\mu_1}(Z,Y,W)\\
&=-\frac{g_{YM}^2}{8\pi^2}\sum_{p=1}^{m}\sum_{q=m+1}^{m+n}\sum_{R^+,r,s,\mu_1,\mu_2}\sum_{R}\frac{d_{R^+}c_{R^+R}n!m!}{d_rd_s(n+m+1)!}\chi_{R^+,(r,s,\square)\mu_1\mu_2}\left(\psi^{-1}\right)\chi_{R,(r,s)\mu_2\mu_1}(Z,Y)
\end{aligned} \tag{4.84}
$$

Converting to normalized restricted Schur polynomials the above result gives the following identity for restricted characters

$$
\begin{aligned}
&\sum_{T,(t,u)\nu_1\nu_2}\sqrt{\frac{f_T\,\mathrm{hooks}_u}{\mathrm{hooks}_{T/t}}}\chi_{T,(t,u)\nu_2\nu_1}(\sigma^{-1})N_{T,(t,u)\nu_1\nu_2;R,(r,s)\mu_1\mu_2}\\
&=-\frac{g_{YM}^2}{8\pi^2}\sum_{R^+}c_{R^+R}\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R^+}}\sqrt{\frac{f_R\,\mathrm{hooks}_s}{\mathrm{hooks}_{R/r}}}\sum_{p=1}^{m}\sum_{q=m+1}^{m+n}\chi_{R^+,(r,s,\square)\mu_2\mu_1}\left(\psi^{-1}\right)
\end{aligned} \tag{4.85}
$$

This is an exact statement and no simplification for large N has been used.

Localization of the boxes belonging to the closed string is accomplished because there is a permutation containing cycles which mix the Y fields with the Z fields. As we explained in Section 4.4, this cycle "ties" the Z and Y boxes together. Since the Y boxes are concentrated in one location on the Young diagram, the Z boxes will be concentrated there too. The original localized loop is defined using a permutation σ that is an $m+n_C$ cycle; this implies that all of the fields belonging to the local excitation will be concentrated at one location on the Young diagram. We want to explore the cycle structure of ψ^{-1} which will determine whether or not the boxes associated to the closed string in (4.85) are localized or not, i.e. if the dilatation operator acts on a localized loop operator, does it produce another localized loop operator?

ψ and ψ^{-1} have the same cycle structure, so we can focus on ψ. ψ is a sum of four terms that are conjugate to (and hence have the same cycle structure as) the following four permutations

$$
\begin{array}{ll}
\sigma(p,q)(m+n+1,q) & (p,q)\sigma(m+n+1,q)\\
(m+n+1,q)\sigma(p,q) & (m+n+1,q)(p,q)\sigma
\end{array} \tag{4.86}
$$

The permutation σ is an $m+n_C$ cycle. Permutations $(p,q)\sigma$ and $\sigma(p,q)$ correspond to a product of a k cycle with an $n_C+m+1-k$ cycle with $k=1,2,\cdots n_C+m$[9]. The value of k depends on the value

[9]The planar contribution comes from the term with $k=1$ of the form $(1)(\cdots)$ with $(\cdots)$ an n_C+m cycle.

of p and q. The sum over p and q produces $O(N)$ terms. There are only $O(1)$ terms of a specific k, so the contribution for any given fixed k can safely be neglected in the large N limit. The $k = 1$ term is enhanced with a factor of N; this is the term that dominates the planar limit. Multiplying by the two cycle $(m+n+1, q)$ does not induce any further splitting because $m+n+1$ does not appear in σ or (p, q). In Section 4.5 we argued that localization of the Y boxes after the dilatation operator acts is guaranteed. Further, localization of all but one Z box is also guaranteed. The way that this simple box can move to a distant location is if $k = 1$ and the label of the distant box sits in the 1-cycle in $(p, q)\sigma$ or $\sigma(p, q)$. As we have just explained, this term can be neglected at large N so that the dilatation operator only mixes localized loop operators.

4.B.1 Numerical Check

The formula (4.84) is rather interesting as it is the exact one loop dilatation operator written in the trace basis. Such a formula has not been written in the literature before and it will provide a useful starting point for other large N but not planar expansions. Indeed, we have used it for precisely this purpose in the LLM backgrounds. Given the potential usefulness of such a formula, we have checked it numerically, for $n = 3$ Zs and $m = 2$ Ys. In the case $m = 2$ we do not need multiplicity labels and this simplifies the computations significantly.

Evaluating restricted characters: By S_5 denote the symmetric group that permutes 1,2,3,4 and 5. By S_3 denote the symmetric group that permutes 3,4 and 5. By S_2 denote the symmetric group that permutes 1 and 2. Introduce the projectors

$$\begin{aligned}
P_s &= \frac{d_s}{2!}\sum_{\sigma\in S_2}\chi_s(\sigma)\Gamma_{R^+}(\sigma)\\
P_r &= \frac{d_r}{3!}\sum_{\sigma\in S_3}\chi_r(\sigma)\Gamma_{R^+}(\sigma)\\
P_R &= \frac{d_R}{5!}\sum_{\sigma\in S_5}\chi_R(\sigma)\Gamma_{R^+}(\sigma)
\end{aligned} \tag{4.87}$$

The restricted character can be written as

$$\chi_{R^+,(r,s,\square)}\Big(\psi\Big) = \mathrm{Tr}_{R^+}(P_sP_rP_R\Gamma_{R^+}(\psi)) \tag{4.88}$$

This can be written in terms of normal characters as

$$\chi_{R^+,(r,s,\square)}\Big(\psi\Big) = \frac{d_sd_rd_R}{n!m!(n+m)!}\sum_{\sigma_1\in S_2}\sum_{\sigma_2\in S_3}\sum_{\sigma_3\in S_5}\chi_s(\sigma_1)\chi_r(\sigma_2)\chi_R(\sigma_3)\chi_{R^+}(\sigma_1\sigma_2\sigma_3\psi) \tag{4.89}$$

where R is obtained by dropping the W box from R^+. The fact that the restricted characters can be written in terms of normal characters is a direct consequence of the fact that for the problem we are considering, there is no need for multiplicity labels. The numerical results for the restricted characters were checked by confirming the known orthogonality relation obeyed by restricted characters.

Evaluating the restricted Schur polynomials $\chi_{R,(r,s)}(Z, Y)$: The bulk of the work is in computing the restricted characters $\chi_{R,(r,s)}(\sigma)$ for $\sigma \in S_5$. We can again express this restricted character in terms of normal characters as

$$\chi_{R,(r,s,\square)}\Big(\psi\Big) = \frac{d_sd_r}{n!m!}\sum_{\sigma_1\in S_2}\sum_{\sigma_2\in S_3}\chi_s(\sigma_1)\chi_r(\sigma_2)\chi_R(\sigma_1\sigma_2\psi) \tag{4.90}$$

Evaluating the factor c_{R^+R}: This factor is equal to N plus the content of the box. The content of the box is easily coded numerically using a Jucys-Murphy element as follows[13]

$$c_{R^+R} = N + \frac{1}{d_r}\sum_{i=1}^{m+n}\chi_{R^+,(r,s,\square)}\Big(\Gamma_{R^+}(i, m+n+1)\Big) \tag{4.91}$$

Finally, some patience shows that the labels $R^+, (r, s, \square)$ run over a total of 52 values. The formula (4.84) has been verified for all possible σ and $n = 3$ and $m = 2$.

4.C Excitations with empty boxes

As explained in Section 4.7, excitations about the inner edge of a black ring are described by removing boxes from the background Young diagram B. Our initial discussion is for the geometry of Section 4.6 so that there is a single outward pointing corner. In this case we don't need to worry about ensuring our excitation is local. B has N rows and M columns and $M = O(N)$. Since each box in the Young diagram corresponds to a Z field, we can remove boxes by taking derivatives with respect to Z. The simplest example of this process is the reduction rule for Schur polynomials[119, 13], given by

$$\mathrm{Tr}\left(\frac{d}{dZ}\right)\chi_R(Z) = \sum_{R'} c_{RR'}\chi_R(Z) \tag{4.92}$$

with the sum over R' running over all R' that can be obtained by dropping a single box from R. General formulas for the action of multitraces of derivatives are worked out in [62].

4.C.1 Schur Polynomials with empty boxes

We would like to remove n Z boxes. The removed boxes are then organized with a representation $r \vdash n$ or into a trace labelled by some permutation σ. The empty Z boxes are generated by acting on some background described by Young diagram B. A very natural guess for the operator which organizes the empty boxes into a single trace described by σ is

$$(\partial_Z)^{i_1}_{i_{\sigma(1)}} \cdots (\partial_Z)^{i_n}_{i_{\sigma(n)}}\chi_B(Z) = \frac{1}{(n_B-n)!}\sum_{\rho\in S_{n_B}}\chi_B(\rho\sigma^{-1})\mathrm{Tr}(\rho\mathbf{1}^{\otimes n}Z^{\otimes n_B-n}) \tag{4.93}$$

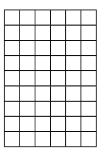
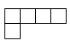
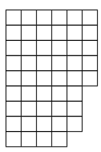
where

$$(\partial_Z)^i_j = \frac{d}{dZ^j_i} \tag{4.94}$$

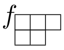
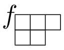
Replace the identity matrices in the above expression with a new field W and turn these back into the identity by acting with $\mathrm{Tr}\partial_W$. Then use the completeness of the restricted Schur polynomials as well as the reduction rule for the restricted Schur polynomials to obtain

$$\begin{aligned}(\partial_Z)^{i_1}_{i_{\sigma(1)}} \cdots (\partial_Z)^{i_n}_{i_{\sigma(n)}}\chi_B(Z) &= \frac{1}{n!}(\mathrm{Tr}\partial_W)^n\frac{1}{(n_B-n)!}\sum_{\rho\in S_{n_B}}\chi_B(\rho\sigma^{-1})\mathrm{Tr}(\rho\, W^{\otimes n}Z^{\otimes n_B-n})\\ &= \frac{1}{n!}(\mathrm{Tr}\partial_W)^n\frac{1}{(n_B-n)!}\sum_{\rho\in S_{n_B}}\chi_B(\rho\sigma^{-1})\sum_{R,(r,s)\alpha\beta}\frac{d_R(n_B-n)!n!}{d_rd_sn_B!}\\ &\quad\times\chi_{R,(r,s)\alpha\beta}(\rho^{-1})\chi_{R,(r,s)\beta\alpha}(Z,W)\\ &= \sum_{s\vdash n}\chi_s(\sigma)f_s(M)\chi_{B/s}(Z)\end{aligned} \tag{4.95}$$

To spell out the notation, we give an example of B, s and B/s below

$B =$  $s =$  $B/s =$  (4.96)

The notation $f_r(M)$ stands for the product of factors of Young diagram r, but with N replaced by M. Thus, for example

f $= N^2(N-1)(N+1)(N+2)$ f$(M) = M^2(M-1)(M+1)(M+2)$ (4.97)

Now consider the operator which organizes the empty boxes into some irrep $r \vdash n$. The operator which naturally achieves this is

$$\chi_{B;r}(Z,\partial_Z) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_r(\sigma)\partial Z^{i_1}_{i_{\sigma(1)}}\cdots\partial Z^{i_n}_{i_{\sigma(n)}}\chi_B(Z) \tag{4.98}$$

Using (4.95) and the orthogonality of characters, we have

$$\chi_{B;r}(Z,\partial_Z) = f_r(M)\chi_{B/r}(Z) \tag{4.99}$$

A few helpful formulas that follow from the results of this section are

$$\begin{aligned}
\langle \chi_{B,r}(Z)\chi_{B,s}(Z)^\dagger \rangle &= f_B f_r(M)\delta_{rs} \\
\chi_{B,r}(Z) &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_r(\sigma)\mathrm{Tr}(\sigma\partial_Z^{\otimes n})\chi_B(Z) \\
\mathrm{Tr}(\sigma\partial_Z^{\otimes n})\chi_B(Z) &= \sum_{r\vdash n}\chi_r(\sigma)\chi_{B,r}(Z)
\end{aligned} \tag{4.100}$$

4.C.2 Restricted Schur Polynomials with Empty Boxes

In this section the operators we study belong to the $SU(2)$ sector and are built from Z and Y fields. We would like to remove Z boxes and organize them with a representation $r \vdash n$, add Y boxes organized with a representation $s \vdash m$ and then organize the Y and Z boxes with a representation $R \vdash m+n$. The operator which naturally achieves this is

$$\chi_{B;(B/R,r,s)\alpha\beta}(Z,Y,\mathbf{1}) = \frac{1}{(n_B-n-m)!n!m!} \sum_{\sigma\in S_{n_B}} \chi_{B;(B/R,r,s)\alpha\beta}(\sigma)\mathrm{Tr}(\sigma\mathbf{1}^{\otimes n}Y^{\otimes m}Z^{\otimes n_B-n-m}) \tag{4.101}$$

The empty Z boxes are again generated by acting on the background described by Young diagram B. Thus, four irreps play a role: $s \vdash m$ organizing the Ys, $r \vdash n$ organizing the empty boxes and B/R organizing the Z fields left in the background and B the original background irrep. To spell out how the different irreps are embedded in B, remove the empty boxes first and then remove the Y boxes. This implies that there are no multiplicities for either r (thanks to fact that we are considering an outward pointing corner) or B/R (always the case); the only multiplicity label needed is for Y so the multiplicity labels take the same values as they did for the restricted Schur polynomial. Thus, at infinite N (where we don't need to worry about any possible cut off on the size of the Young diagram) the number of restricted Schur polynomials is equal to the number of restricted Schur polynomials with empty boxes. By introducing an extra field W, taking derivatives and using the reduction rule for restricted Schur polynomials, we can write

$$\begin{aligned}
\chi_{B;(B/R,r,s)\alpha\beta}(Z,Y,\mathbf{1}) &= \frac{1}{n!}\left(\mathrm{Tr}\partial_W\right)^n \chi_{B;(B/R,r,s)\alpha\beta}(Z,Y,W) \\
&= \frac{d_r}{n!} f_r(M)\chi_{B/r,(B/R,s)\alpha\beta}(Z,Y)
\end{aligned} \tag{4.102}$$

It is now straight forward to compute the two point correlation function

$$\begin{aligned}
T &= \langle \chi_{B;(B/R,r,s)\alpha\beta}(Z,Y)\chi_{B;(B/T,t,u)\gamma\delta}(Z,Y)^\dagger \rangle \\
&= \delta_{RT}\delta_{rt}\delta_{su}\delta_{\beta\delta}\delta_{\alpha\gamma}\frac{f_B f_r(M)}{(\mathrm{hooks}_r)^2}\frac{\mathrm{hooks}_{B/r}}{\mathrm{hooks}_{B/R}\mathrm{hooks}_u}
\end{aligned} \tag{4.103}$$

4.C.3 Action of the Dilatation Operator

It is rather simple to compute the action of the dilatation operator on restricted Schur polynomials with empty boxes. Indeed, the label for the empty boxes is simply a short hand and we can easily translate these labels into the usual description in terms of a representation for the Z fields and one for the Y fields. This translation is given in (4.102). This translation is nontrivial. For example, one may have expected that the operator built using no empty slots and only Ys is BPS. This is not the case. A very simple example of this is (in this example no multiplicity labels are needed)

$$D\chi_{B;(B/\Box\Box,\cdot,\Box\Box)}(Z,Y,\mathbf{1}) = \frac{g_{YM}^2 M}{4\pi^2}\chi_{B;(B/\Box\Box,\cdot,\Box\Box)}(Z,Y,\mathbf{1}) - \frac{g_{YM}^2 M}{4\pi^2}\chi_{B;(B/{\substack{\Box\\\Box}},\cdot,{\substack{\Box\\\Box}})}(Z,Y,\mathbf{1})$$

$$ \tag{4.104} $$

Our main interest is in defining localized loop operators in this background. To do this, it is more useful to use the following representation of the background

$$\chi_B(Z) = (\det(Z))^M \tag{4.105}$$

as well as the derivative formula

$$(\partial_Z)^i_j \det Z = (Z^{-1})^i_j \det Z \tag{4.106}$$

We see that each action of the derivative produces a factor of Z^{-1}. In our previous formulas, Z^{-1} did not participate. As soon as we have more than N fields, there are trace relations so that in this case there isn't a unique expression for generic multitrace operators, so this apparent discrepancy should not worry us. The natural structure for the loop operator is

$$\begin{aligned} O_B(\{n_k\}) &= M^{-n}\mathrm{Tr}\,(\partial_Z^{n_1} Y \partial_Z^{n_2} Y \cdots \partial_Z^{n_m} Y)\chi_B(Z) \\ &= \mathrm{Tr}\,(Z^{-n_1} Y Z^{-n_2} Y \cdots Z^{-n_m} Y)\chi_B(Z) \end{aligned} \tag{4.107}$$

where there are m Ys and $n = n_1 + n_2 + \cdots + n_m$ Zs is the trace. The second line above is only true in the large N limit, where we can ignore splitting of the trace. Rewrite the dilatation operator as (derivatives in D do not act on fields inside D)

$$D = \frac{g_{YM}^2}{8\pi^2}\mathrm{Tr}\left(\big[\,[Y,Z]\,,\partial_Z\big]\partial_Y\right) \tag{4.108}$$

When acting on $O_B(\{n_k\})$, D will produce m terms obtained by replacing each Y (a different one in each term) in $O_B(\{n_k\})$ with $\big[\,[Y,Z]\,,\partial_Z\big]$. After allowing all of the derivatives with respect to Z to act on $\chi_B(Z)$ we obtain (this result is not exact; it is obtained in the large N limit with $n_C + m \sim O(\sqrt{N})$)

$$\begin{aligned} &DO_B(\{n_k\}) \\ &= \frac{g_{YM}^2 M}{8\pi^2}\left(2O_B(\{n_k\}) - O_B(\{n_1+1, n_2-1, \cdots, n_m\}) - O_B(\{n_1-1, n_2+1, \cdots, n_m\})\right) \\ &+ \frac{g_{YM}^2 M}{8\pi^2}\left(2O_B(\{n_k\}) - O_B(\{n_1, n_2+1, n_3-1, \cdots, n_m\}) - O_B(\{n_1, n_2-1, n_3+1, \cdots, n_m\})\right) \\ &+ \qquad \cdots\cdots\cdots \qquad + \\ &+ \frac{g_{YM}^2 M}{8\pi^2}\left(2O_B(\{n_k\}) - O_B(\{n_1-1, n_2, \cdots, n_m+1\}) - O_B(\{n_1+1, n_2, \cdots, n_m-1\})\right) \end{aligned} \tag{4.109}$$

which matches (4.12) perfectly and confirms our identification of the loop operator. Further, we see that N is replaced by M which again reproduces the string theory expectations.

4.C.4 More general backgrounds

To construct localized restricted Schur Polynomials with empty boxes on the general multiring geometry, we need to use localized derivative operators, that only remove boxes from a single corner of B. Derivatives of this type have been defined and studied in [5]. Replacing N by the factor of the outward pointing corner again reproduces the string theory expectations.

Chapter 5

Conclusions

The results of this dissertation represent a significant advancement in the formulation of a dual CFT description to the string configurations and scattering interactions which are so elegantly captured by the geometric depictions of [4, 17]. This is further convincing evidence that the restricted Schur polynomial operators provide an exceptionally convenient basis in which eigenstates of the dilatation operator corresponding to general configurations of open strings attached to systems of giant gravitons can be constructed. We have not only reproduced the weak-coupling limit of exact results obtained by symmetry in [16] in this basis and shown that they match the expectations implied by the string theory; our results provide the field theoretic constructions dual to a more general class of string-giant configurations allowed in the dual theory. Our expressions for the energies and S-matrix elements interpolate perfectly between those for the closed string [16, 4] and the special case of open strings attached to a maximal giant graviton [17].

It is important to make some comments regarding the features of the construction which allow these powerful results to be obtained and interpreted. Of course, the underlying source of our ability to build operators which can be put in correspondence with states of the dual string theory is symmetry. An excellent example of the principles espoused in the introduction is evident in the exact matching of the symmetry algebras relevant to the descriptions of [16] and [4]. The equivalence of these two descriptions is far from obvious in every way; one is formulated as a discrete spin chain in a 4-dimensional theory without gravity, the other as a solution to the equations of motion of a string parameterized by continuous quantities and living in a 10-dimensional spacetime. The algebra obeyed in both theories is not even interpreted as describing the same group; one is associated with an $SU(2|2)$ algebra transforming fields into one another, the other with super-Poincare transformations on a $(2+1)$-dimensional spacetime[120]. Despite these apparent incompatibilities, the magnons can be formulated on either side of the correspondence as representations of the same symmetry algebra, and one can visually depict these configurations using the same diagrams with only a difference in interpretation [44].

This is perhaps now the clearest way to motivate a construction in terms of Schur polynomials - these operators are specified completely by the representation of the symmetry group under which the fields which compose them transform. The Young diagram labels which we employ in fact tell us everything about the representation of two relevant symmetry groups implementing these transformations: the $U(N)$ gauge group under which each of the fundamental fields transform, and the symmetric group which relates to the organization of the fields within the operator. The Young diagram is also a visual tool which has consistently provided intuitive physical pictures of the configuration and dynamics of the systems described, which can be mapped in a transparent way to those already obtained in the dual theory. The utility of the Young diagrams is extended by the use of Young-Yamonouchi symbols to label the states of a representation; in this basis the symmetric group transformations acquire an explicit realization in terms of the dual physical picture provided by the aforementioned map. Unitary group quantities can also be read off the Young diagram; these play a vital role in the expression of results for the anomalous dimensions of CFT operators.

The Schur-Weyl duality under which this crucial interplay, between transformations of the fields themselves and transformations under the group which organises the trace structures, is possible provided the means by which to construct the projectors appearing in our restricted Schur polynomial definitions, allowing the diagonalization of the dilatation operator when studying fundamental open string excitations in [2]. It has featured prominently, though somewhat implicitly, in the current work as well. In Chapter 3, it was noted that the action of D on the boundary magnons appearing in operators with single-trace impurities could be simply written in terms of the $U(N)$ factors. The "Fourier transform" which produces eigenstates is obtained by summing over elements of the symmetric group which organise the fields of the excitation. In Chapter 4, we have gained some further clarity on the relevance of the two groups; the dilatation action is determined entirely by a rescaling of the N dependence by a quantity proportional to a unitary group factor, while the study of symmetric group transformations provides the necessary large N suppression of terms to enforce the localization of the excitations in the dual geometry.

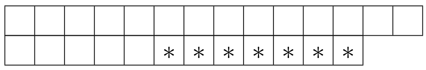
One may interpret this as evidence that the duality between the unitary and symmetric groups represents a fundamental component underpinning the AdS/CFT correspondence. The Young diagrams are an essential framework for the understanding of this conjecture; it is with this representation theoretic language that the descriptions in terms of the distinct groups can be treated simultaneously. The results presented herein suggest that reading Young diagrams as describing unitary or symmetric group representations makes contact with different aspects of the configurations in the dual string theory. This structure may aid in understanding how a description of gravity emerges in the CFT.

There is a particular large N approximation which has proven very useful in calculations involving the Schur polynomials dual to systems containing giant graviton excitations, which is implemented at the level of the representation labels and is clearly reflected in the dual theory descriptions as well. Dubbed the *displaced corners approximation*, it can be stated as the requirement that the difference in row (AdS giant) or column (sphere giant) lengths for diagrams containing $O(N)$ fields must be $O(N)$ as well. This corresponds to requiring that the giant gravitons are well separated in the geometry, and is the source of the simplifications that allowed eigenstates to be constructed in Section 3.4, as well as those providing a proof of the persisting localization of string excitations on concentric-ring LLM geometries in Section 4.5.

In the first case, simple states which could be promoted in a straightforward way to eigenstates of the dilatation operator were constructed by a "Fourier transform" which sums over all allowed open string word trace structures with the given number of magnons propagating on a lattice consisting of $J = O(\sqrt{N})$ background fields. Each term is multiplied by the phases acquired by the magnons under a series of "hops" from a set of carefully chosen initial configurations. Terms involving lattice propagation of the boundary magnons induce a change in the Young diagram specifying the representation associated to the giants; however, the interpretation advocated is one where no momentum is exchanged between the string and the giant. This interpretation is only possible with the displaced corners approximation applied, since it is realised by ignoring the effect of the additional momentum gained by the giants when expressing the dilatation action - dropping these subleading contributions amounts to ignoring backreaction of the strings on the giants, and is only achieved because adding $O(\sqrt{N})$ fields to any of the giants has a negligible effect on their separation. The resulting picture is one where the giant configuration ("the background") remains constant, so that the "Fourier transform" is in effect realised by thinking about a string lattice of fixed length J on which the magnons propagate.

This separation between background and fluctuation was articulated clearly in terms of the group theory, in the case where closed strings are attached to boundstates of $O(N)$ giants having the form necessary for identification with concentric-annuli LLM diagrams. As has consistently been the case, expressing the observation in terms of the Young diagrams realises the physics in an intuitive way. The situation studied in Chapter 4 corresponds to a closed string localized to a particular region of the background geometry, being one of the boundaries between black and white regions on the associated LLM diagram. All of the boxes corresponding to magnons on the string are localised to a particular corner of the Young diagram, and thus the thick LLM annuli ensure that the J boxes of the string lattice are localized to this same corner. Arguing in this way, it is clear that the states of the representation corresponding to the excitation are not modified by the presence of the background, and vice versa - the operator as originally constructed involves a high degree of mixing between excitation and background fields, but the use of the large N limit in the context of symmetric group transformations in Young's representation enforces strong constraints on this mixing for the backgrounds considered. The separation is understood as the fact that the system transforms in a representation of a subgroup in which the fields of the excitation are treated separately from those of the background.

The application of the displaced corners approximation in this setting illustrates an important point regarding its usage: while the approximation is stated as the requirement that the Young diagrams have a particular structure, it is in fact the action of the symmetric group transformations on states of the representation which provides the suppression of terms necessary to invoke the separation of background and fluctuation. This suggests that the displaced corners approximation may be realised in more general configurations than it would naively appear; it is not necessary for its validity that the representation involves only well separated corners on the diagram, it is also valid when corners are nearby if some of the indices labelling states in the representation are localized in such a way that they cause a separation between indices transforming under a different subgroup. That is, states taking the form (where starred boxes correspond to indices transforming under a particular subgroup)



may also admit simplifications under this approximation.

The correct results for systems of multiple closed strings connecting points on a particular ring boundary follow immediately from the arguments of Chapter 4 upon recalling that the planar dilatation operator is known to have a factorized action on a product of loops; it is also obvious how to describe multiple closed strings attached to different rings. When some of the magnons are defined as existing on distant corners, corresponding to the inclusion of open string excitations, one should expect that the Z boxes associated to the string may be allowed to populate regions surrounding any of the magnons - this expectation follows from considering a rewriting of the eigenstates of Section 3.4 in a pure representation basis. Operators describing the most general string configurations allowed on the geometries studied herein should thus be possible to construct by connecting the results of Chapters 3 and 4. Once we are able to describe systems of open strings connecting giants on the trivial geometry in the representation basis, the representation theory tells us exactly how to merge these configurations with the description of the background: attaching the sum of string-giant diagrams to a particular corner of the background diagram, we should observe a similar separation of representations into background and excitation as before, albeit with modified arguments and notation for the simplification of the loop permutation trace. The $O(N)$-box rows/columns of the giant in the $AdS_5 \times S^5$ geometry have already been seen to play the role of a background, since they are approximately unmodified by the string interactions, and will continue to play this role in more general geometries - they correspond to a modification of the background to include a number of small, well-separated corners to which only particular representation structures associated to the string can attach.

Of course, the LLM backgrounds are just an assembly of giant gravitons, with a form such that the entire string can be localized to a particular corner. This localization is captured in the definition of r_B; once this representation is given, the positions of the magnon boxes on the diagram are uniquely specified. For the giants on $AdS_5 \times S^5$, we have a number of small corners, and the labels R_1, R_2 (here we have worked in the $S_n \times (S_1)^m$ description) specify the localization of particular individual magnons to these. Consider the giant representation; there is only one corner, corresponding (to very good approximation) to the $r = 1$ boundary of the LLM plane, to which more general representations than a single column can attach - this is where the "fluff", corresponding to the Zs and bulk magnons, must live. The results of Chapter 3 can be reproduced by considering the closed string without giants on the trivial background, and scaling the N dependence separately for the term corresponding to each boundary magnon using the $U(N)$ factor of the box in R_1 or R_2 corresponding to the magnon at the relevant endpoint of the open string. The form of the operators presented in Section 3.4 suggest that the sum over representations we should consider is related to a sum over all the ways that the boxes corresponding to the excitation can be added to the giant Z diagram; $\sum_{r+(J+m)}$. This seems a natural re-expression for the sums occurring in the definition of the loop operators, in the $S_n \times (S_1)^{J+m}$ description, where the magnons can occur on different corners.

If we directly consider a rewriting of each of the terms appearing in the simple states of Section 3.4, it would appear that each term should involve a different background (giant representation), and that the sum over representations for each term would involve partitions of a different number of fields due to the shortening of the permutation associated to the string word in each term. This seems unnatural when compared to the construction and results of Chapter 4. In light of the comments regarding the realisation of displaced corners at the level of states, it may be sensible to propose that the loop operator relevant for the description of these systems will still be defined in terms of a sum over representations organising $J+m$ fields. Due to the allowed localization of the excitation Zs around the boundary magnon which sits on a distant corner, however, the states which contribute to the trace of the loop permutation need not be of a form where all of these boxes contain labels corresponding to excitation fields - some excitation field indices may occupy boxes near the boundary magnon, and some background field indices may occupy boxes on the excitation piece of the representation. This seems a natural way to capture the effect of the "transfer" of fields between string and giant in the representation basis, and clarifies the no-momentum-exchange picture of Chapter 3. Indeed, the results obtained therein could equally well have been obtained by neglecting the modifications to the giant Young diagram when constructing the operators as in (3.17). In the representation basis, it is clear that the momentum of both the giants and strings is unmodified; it is only the association of the boxes (which each carry a unit of momentum) with either background or excitation that changes.

In terms of the representation picture, ignoring back-reaction on the giants may be viewed similarly to the approximation which suppresses mixing between Zs on the excitation and Zs on the background. All the Z boxes corresponding to the string word are tied together as before, so that any states arising under the action of D which involve modifications to the background Young diagram can be ignored using our new restricted character identity. The mixing is not suppressed for the boundary magnons - they are tied to the excitation fields by the loop permutation, however, in specifying the labels R_1, R_2 with Y boxes on the distant corners, we have allowed some unsuppressed mixing between the giant fields and the boundary magnons. These boxes are well-separated from the fluff. A possible resolution may be seen in noting that the number of terms with a ψ for which a split occurs such that one of the boundary magnon indices appears in a permutation cycle with an index corresponding to an excitation Z which has been added to the same column that the boundary magnon occupies should be proportional to $\sqrt{N}$; this is perhaps expected since a similar suppression, of order $\frac{1}{\sqrt{N}}$, is used to equate the factors of boxes in $G(x)$ which differ (due to the simple state definition) by up to J when acting with D on the eigenstates of Chapter 3.

In the context of strings attached to giants, the large N limit introduces an interesting feature to the map between our Young diagrams and the LLM boundary condition diagrams. The giants' radial localization on the LLM plane is enforced by the angular momentum they carry, which is specified by the length of the row/column of the Young diagram to which they correspond. The use of the displaced corners approximation to neglect contributions due to $O(\sqrt{N})$ fluctuations of this momentum corresponds in reading the LLM description to a reduction in the resolution to which the radial position is resolved. Employing the knowledge that closed strings can be associated with operators built from $O(\sqrt{N})$ fields, this is evident also in the description of their worldsheet as joining points at the boundaries. This may suggest that the apparent necessity of a description where strings are modeled as single trace structures in the field theory specified by a permutation, and thus corresponding to a sum over operators labelled by representations, is a result of this uncertainty - we sum over all possibilities for the number of Zs associated with each of the points the magnons join to, simply because every case corresponds to the same physical configuration in the large N, asymptotic magnon limit considered.

Considering the case of a closed string with worldsheet connecting points on an LLM boundary, the duality present in the description of the background as a boundstate of sphere or AdS giants illustrates what may be an important result of the low resolution of the LLM description. When considering a point at the boundary of the white and black regions, there is no way of knowing whether the point should be considered as existing in one region or the other, since $O(\sqrt{N})$ fluctuations in either the positive or negative radial direction will not change the physical configuration. This fact is perfectly reflected in mapping to the Young diagrams; the extra boxes added to the relevant corner can be thought of as being attached to the ends of the sphere giant columns above it, or to the AdS giant rows to the left of it. In summing over all possible diagrams for the excitation, there are two classes of representation which may naturally capture which of the two regions the points occupy, if we consider the finer details of the configurations: when there are $O(\sqrt{N})$ rows and $O(1)$ columns, the large N limit should result in the suppression relevant to neglecting contributions associated with the points being in the white region; diagrams with $O(\sqrt{N})$ columns and $O(1)$ rows should describe systems with the points in the white region.

It is thus the authors opinion that descriptions in terms of operators for which all the fields transform in a definite representation may prove to remain the correct construction when subleading corrections corresponding to dynamics at the string scale are included, since in this regime applying small differences in the Z momentum on the LLM plane will produce distinct physical configurations.

We have now understood clearly the relevance of the Gauss graphs, which emerged as a pictorial description of the diagonalized equations resulting from applying the dilatation operator to restricted Schur polynomials. They are, as predicted, a diagram of the fundamental open string configurations that are allowed in the dual theory. The results of [4] have taught us exactly the configuration which they represent: that of open strings with magnon excitations having zero momentum, which can be drawn on the LLM plane as line segments connecting points on different orbits, along the radial direction. The simple states of Section 3.4 utilise the asymptotic state manipulation relation of [16] to enable configurations with momentum-carrying magnons to be described in our formalism by a construction which admits interpretation as the Fourier transform of the restricted Schur polynomials - transforming to momentum space requires a sum over all the states of definite lattice occupation number, multiplied by the relevant phases. This is the origin of the diagonalization of the dilatation action on these operators; they have a definite lattice momentum, and it is this momentum in terms of which the energy is defined.

The results presented in this dissertation are expected to admit extensions in the directions considered previously in the development of the current theory. Once a calculation in the $SU(2)$ sector has been understood, it is generally straightforward to complete the construction to include all 3 bosonic fields. The inclusion of fermionic fields may present additional difficulties, but the methods of [11] should provide much of the insight necessary to achieve this extension. The existence of the $SU(2|2)$ symmetry supports the claim that the results should generalize to these sectors. We expect also that the higher loop corrections to the dilatation action on the operators constructed in this dissertation may be entirely determined by the one loop result, as seen previously in [12]. The most interesting extension, currently being pursued, is the computation of string dynamics on backgrounds where the rings are neither thick nor well-separated, where it is expected that the background will not be invariant - the strings can perturb the background. This corresponds to the study of operators labelled by Young diagrams with a large number of corners, for which the displaced corners approximation is no longer valid. It is hoped that the study of this problem will shed new light on the finer structure of the correspondence at large.

We are by now thoroughly convinced that the restricted Schur polynomial operators are a valid and powerful construction for the study of the AdS/CFT correspondence. The inspired symmetry arguments of [16] have provided the intuition necessary to understand the precise description of the impurities inserted into our operators in the dual gravitational theory. The application of this intuition in the string theory [4, 17] has provided the visual description in the dual theory which confirms the physical meaning of the Gauss graphs and their extension to include momentum-carrying, permutation-bound magnons. It is interesting, even if ultimately physically meaningless, that the diagrams resulting in this way bear so close a resemblance to atomic occupation diagrams (with strings!). The results additionally provide a further test of the map between Young diagrams and LLM diagrams, as near-BPS excitations are now present on both sides. We hope that the intuitions gained by considering the duality between the dynamics in the discrete CFT picture and the continuous string picture will aid in furthering our understanding of how spacetime itself, together with its gravitational interactions, emerges in the field theory.

Appendix A

Elementary Facts from S_n Representation Theory

The complete set of irreducible representations of S_n are uniquely labelled by Young diagrams with n boxes. From this Young diagram we can construct both a basis for the carrier space of the representation as well as the matrices representing the group elements. We will review these constructions in this Appendix. A useful reference for this material is [113].

A.1 Young-Yamonouchi Basis

A particularly convenient basis for the carrier space of an irreducible representation of the symmetric group is provided by the Young-Yamonouchi basis. The elements of this basis are labelled by numbered Young diagrams - a Young tableau. For a Young diagram with n boxes, each box in the tableau is labelled with a unique integer i with $1 \le i \le n$. In our conventions this numbering is done in such a way that if all boxes with labels less than k with $k < n$ are dropped, a valid Young diagram remains. As an example, if we consider the irreducible representation of S_4 corresponding to

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$$

then the allowed labels are

$$\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline \end{array}\,.$$

Examples of labels that are not allowed include

$$\begin{array}{|c|c|}\hline 4 & 1 \\ \hline 3 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}\,.$$

For any given Young diagram the number of valid labels is equal to the dimension of the irreducible representation and each label corresponds to a vector in the basis for the carrier space. This basis is orthonormal so that, for example

$$\left\langle \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline \end{array} \middle| \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline \end{array} \right\rangle = 1, \qquad \left\langle \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline \end{array} \middle| \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline \end{array} \right\rangle = 0\,.$$

A.2 Young's Orthogonal Representation

A rule for constructing the matrices representing the elements of the symmetric group is easily given by specifying the action of the group elements on the Young-Yamonouchi basis. The rule is only stated for "adjacent permutations" which correspond to cycles of the form $(i, i+1)$. This is enough because these adjacent permutations generate the complete group. To state the rule it is helpful to associate to each box a factor[1]. The factor of a box in the i^{th} row and the j^{th} column is given by $K - i + j$. Here K is an arbitrary integer that will not appear in any final results. We will denote the factor of the box labelled l by c_l. Let $\hat{T}$ denote a Young tableau corresponding to Young diagram T and let $\hat{T}_{ij}$ denote exactly the same tableau, but with boxes i and j swapped. The rule for the action of the group elements on the basis vectors of the carrier space is

$$\Gamma_T\left((i,i+1)\right)\left|\hat{T}\right\rangle = \frac{1}{c_i - c_{i+1}}\left|\hat{T}\right\rangle + \sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}}\left|\hat{T}_{i,i+1}\right\rangle .$$

A.3 Partially labelled Young diagrams

Consider a Young diagram containing $n+m$ boxes so that it labels an irreducible representations of S_{n+m}. We will often consider "partially labelled" Young diagrams, which are obtained by labelling m boxes. The remaining n boxes are not labelled. We only consider labellings which have the property that if all boxes with labels $\leq i$ are dropped, the remaining boxes are still arranged in a legal Young diagram. We refer to this as a "sensible labelling". What is the interpretation of these partially labelled Young diagrams? To make the discussion concrete, we will develop the discussion using an explicit example. For the example we consider take $n = m = 3$ and use the following partially labelled Young diagram

		1
	2	
3		

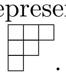
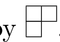
. (A.1)

If the labelling is completed, this partially labelled diagram will give rise to a number of Young tableau. For our present example two tableau are obtained

6	5	1
4	2	
3		

6	4	1
5	2	
3		

.

Each of these represents a vector in the carrier space of the S_6 irreducible representation labelled by the Young diagram . Thus, a partially labelled Young diagram stands for a collection of states. Next, note that the subspace formed by this collection of states is invariant (you don't get transformed out of the subspace) under the action of the S_3 subgroup which acts on the boxes labelled 4,5 and 6. Thus, this subspace is a representation of S_3. In fact, it is easy to see that it is the irreducible representation labelled by . This Young diagram can be obtained by dropping all the labelled boxes in (A.1). From this example we can now extract the general rule:

Key Idea: *A partially labelled Young diagram that has $n+m$ boxes, m of which are labelled, stands for a collection of states which furnish the basis for an irreducible representation of $S_n \times (S_1)^m$. The Young diagram that labels the representation of the S_n subgroup is given by dropping all labelled boxes.*

Finally, note that the only representations r that are subduced by R are those with Young diagrams that can be obtained by pulling boxes off R. This follows immediately from the well-known subduction rule for the symmetric group which states that an irreducible representation of S_n labelled by Young diagram R with n boxes will subduce all possible representations R'_i of S_{n-1}, where R'_i is obtained by removing any box of R that can be removed such the we are left with a valid Young diagram after removal. Each such irreducible representation of the subgroup is subduced once.

A.4 Simplifying Young's Orthogonal Representation

In this section we would like to consider a collection of partially labelled Young diagrams. A total of m boxes are labelled, with a unique integer i ($1 \leq i \leq m$) appearing in each box. The set of boxes to be

[1]This number is also commonly called the "weight" of the box. Here we will refer to it as the factor since we do not want to confuse it with the weight of the Gelfand-Tsetlin pattern.

removed are the same for every partially labelled Young diagram. The set of partially labelled Young diagrams we consider is given by including all possible ways in which the m boxes in the Young diagrams can sensibly be labelled. We can consider the action of the S_m subgroup which acts on the labelled boxes. This action will mix these partially labelled Young diagrams.

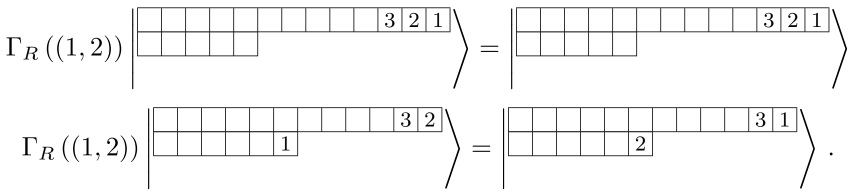
We will consider Young diagrams with p rows built out of $O(N)$ boxes. For the generic operator we consider, the difference in the length between any two rows will be $O(N)$. If we consider the case $m = \gamma N$ with $\gamma \sim O(N^0) \ll 1$, any two labelled boxes (i and j say) that are not in the same row will have factors that obey $|c_i - c_j| \sim O(N)$. Young's orthogonal representation is particularly useful because it simplifies dramatically in this situation. Indeed, if the boxes i and $i+1$ are in the same row, $i+1$ must sit in the next box to the left of i so that

$$\Gamma_R\left((i, i+1)\right)|\text{same row state}\rangle = |\text{same row state}\rangle \,. \tag{A.2}$$

The same state appears on both sides of this last equation. If i and $i+1$ are in different rows, then $c_i - c_{i+1}$ must itself be $O(N)$. In this case, at large N replace $\frac{1}{c_i - c_{i+1}} = O(b_1^{-1})$ by 0 and $\sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}} = 1 - O(b_1^{-1})$ by 1 so that

$$\Gamma_R\left((i, i+1)\right)|\text{different row state}\rangle = |\text{swapped different row state}\rangle \,. \tag{A.3}$$

The notation in this last equation is indicating two things: i and $i+1$ are in different rows and the states on the two sides of the equation differ by swapping the i and $i+1$ labels. An example illustrating these rules is



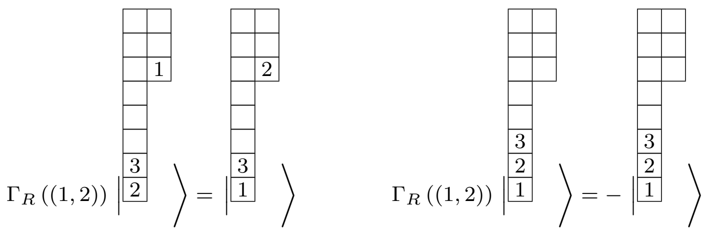
We will also consider Young diagrams with p columns built out of $O(N)$ boxes. For the generic operator we consider, the difference in the length between any two columns will be $O(N)$. Since we consider the case $m = \gamma N$ with $\gamma \sim O(N^0) \ll 1$, any two labelled boxes (i and j say) that are not in the same column will again have factors that obey $|c_i - c_j| \sim O(N)$. If the boxes i and $i+1$ are in the same column, $i+1$ must sit above i so that

$$\Gamma_R\left((i, i+1)\right)|\text{same column state}\rangle = -|\text{same column state}\rangle \,. \tag{A.4}$$

The same state appears on both sides of this last equation. If i and $i+1$ are in different columns, then $c_i - c_{i+1}$ must itself be $O(N)$. In this case, at large N again replace $\frac{1}{c_i - c_{i+1}} = O(b_1^{-1})$ by 0 and $\sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}} = 1 - O(b_1^{-1})$ by 1 so that

$$\Gamma_R\left((i, i+1)\right)|\text{different column state}\rangle = |\text{swapped different column state}\rangle \,. \tag{A.5}$$

An example illustrating these rules is:



Thus, the representations of the symmetric group simplify dramatically in this limit.

Appendix B

Clifford Algebras and Spinors

B.1 Motivation

The gamma matrices in arbitrary dimension (t, s) are defined by the property that they are the generators of the Clifford algebra $\mathrm{Cl}_{(t,s)}(\mathbb{R})$; that is, they satisfy the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_{2^{D/2}}$$

where $\eta^{\mu\nu}$ is the Minkowski metric with negative time signature. They first arose during Dirac's derivation of his famous equation, where the idea that the relativistic equation of motion for an electron should be first order in the derivatives was implemented. Dirac had hit upon the thought to take a square root of the wave operator (the Laplace operator of Minkowski space), which immediately led to the expectation that the components of the spacetime derivative should come multiplied by matrix coefficients - this further implies that the wavefunction for relativistic fermions, in contrast to the single component wavefunction of Schrödinger theory, should have multiple components in order for this to be applicable.

The reason that these matrices must be generators of the Clifford algebra can be seen by considering the equation of motion for a free electron, which, under the assumption that the equation be first order in derivatives, takes the linear form

$$(\gamma^\mu\partial_\mu - M)\psi = 0.$$

Since the expression $\gamma^\mu\partial_\mu$ was constructed as the square root of the D'Alembert operator, it follows that multiplying this equation by $(\gamma^\mu\partial_\mu + M)$ on the left must produce the Klein-Gordon equation:

$$(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - M^2) = 0 \quad \leftrightarrow \quad (\partial^2 - m^2)\psi = 0.$$

Taking $M = m$ and expanding the first term as $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \gamma^\nu\gamma^\mu\partial_\nu\partial_\mu)$, it is clear that the gamma matrices provide a solution to this identification when $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ (recall that $\partial_\mu\partial_\nu\psi = \partial_\nu\partial_\mu\psi$):

$$\frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu - m^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu - m^2 = \partial^\nu\partial_\nu - m^2.$$

Thus we have that the matrices which satisfy the requirement of reproducing the Klein-Gordon equation when the proposed linear (in the derivatives) equation is squared must satisfy the anticommutator relation

$$\{\gamma^\mu, \gamma^\nu\}^\alpha_\beta = 2\eta^{\mu\nu}\mathbb{1}^\alpha_\beta \tag{B.1}$$

where $\mathbb{1}$ denotes the identity matrix on the spinor indices. This is the defining relation for a Clifford algebra. One can verify that the simplest object satisfying the relation in D dimensions is a $2^{[D/2]}$ matrix, i.e. we can write $(\gamma^\mu)^\alpha_\beta$, $\alpha, \beta = 1..2^{[D/2]}$. The appearance of the Minkowski metric in the relation suggests a relationship between this group and the proper Lorentz group $SO(D-1, 1)$. Before making this explicit, we will discuss some important properties of the Lorentz group.

B.2 The Lorentz Group

The Lorentz group is defined as the group of linear homogeneous transformations of coordinates in D-dimensional Minkowski spacetime which preserve the Minkowski norm of any vector; that is if $x^\mu \to \Lambda^\mu_\nu x^\nu$ where Λ represents the Lorentz transformation, we require

$$x^\mu \eta_{\mu\nu} x^\nu = x'^\mu \eta_{\mu\nu} x'^\nu \Rightarrow \Lambda^\mu_\rho \eta_{\mu\nu} \Lambda^\nu_\sigma = \eta_{\rho\sigma}.$$

This property defines the Λ matrices. We can now determine the Lie algebra of the group by considering an infinitesimal transformation with small parameter ϵ; we expand the transformation as $\Lambda^\mu_\nu = \delta^\mu_\nu + \epsilon m^\mu_\nu + \cdots$, and immediately note that the definition of the Λ matrices is satisfied to first order in ϵ provided that the generators are antisymmetric, i.e. $m_{\mu\nu} = -m_{\nu\mu}$. This implies that there will be $\frac{1}{2}D(D-1)$ independent generators for the group - in 4 dimensions, these correspond to the familiar 3 rotations + 3 boosts which are implemented by the group action. The Lorentz generators are defined by requiring that they satisfy the commutator relation

$$[m_{\mu\nu}, m_{\rho\sigma}] = \eta_{\nu\rho} m_{\mu\sigma} - \eta_{\mu\rho} m_{\nu\sigma} - \eta_{\nu\sigma} m_{\mu\rho} + \eta_{\mu\sigma} m_{\nu\rho}. \tag{B.2}$$

There is an additional important property of the Lorentz group which will be useful for the purpose of describing the transformation of spinor fields, which also allows a useful labelling for the various representations of the group. One first notes that the 3 rotation and boost generators can be expressed respectively as

$$L^i = -\frac{1}{2}\epsilon^{ijk} m_{jk} \qquad , \qquad K^i = m^{0i}.$$

One can now form a *complexified* version of the Lie algebra of $SO(3,1)$ by introducing the combinations

$$J^k_\pm = \frac{1}{2}(L^k \pm iK^k).$$

Using (B.2), one finds that these generators of the complexified Lie algebra satisfy the commutation relations

$$[J^i_\pm, J^k_\pm] = \epsilon_{ijk} J^k_\pm \qquad , \qquad [J^i_\pm, J^k_\mp] = 0$$

which are exactly the commutator relations of two independent copies of the Lie algebra of $su(2)$! We have thus found that the Lorentz algebra splits up into two copies of the angular momentum algebra, for which we know the representations can be specified by a half-integer "spin". All finite dimensional irreducible representations of $so(3,1)$ can be obtained from products of two $su(2)$ representations, and therefore are specified completely by a pair of half-integer spins (j_+, j_-). The complexified Lie algebra of the Lorentz group is related to $su(2) \oplus su(2)$ - in the particular case where the fields of interest are Dirac spinors, the representation in which the fields transform is $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ - this representation is reducible, as we shall soon see.

A representation of the Lorentz group which is of particular interest is obtained by noting the appearance of the Minkowski metric in (B.1); if we define generators in terms of the generators of the Clifford algebra as

$$m^{\mu\nu} = \gamma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$$

it follows from the Clifford commutator that these generators satisfy (B.2). This is known as the Dirac representation of $SO(3,1)$ - it is interesting to note that now, since the Clifford generators can be expressed in an arbitrary basis, the commutator relations for the complexified Lorentz algebra may be satisfied in different ways depending on the choice of representation for the associated Clifford algebra.

A definition of the spinors can also be given in terms of the Lorentz algebra; a spinor is just an object which transforms in the representation defined above, i.e. under a Lorentz transformation generated by a parameter $\omega^{\mu\nu}$ the spinor variation is given by

$$\delta(\omega^{\mu\nu})\psi_\alpha = -\frac{1}{4}\omega^{\mu\nu}(\gamma_{\mu\nu})^\beta_\alpha \psi_\beta.$$

As always with a representation of the Lorentz group, we would like to be able to contract spinor indices in order to obtain a Lorentz scalar. Due to the spinor index structure on the gamma matrices given explicitly above, and the fact that the Hermitian conjugate includes a transposition, it is natural to write the Hermitian conjugate of the spinor with an upper index, suggesting this as a natural candidate for the "dual-spinor" which we should contract with to obtain scalars. However, when this variation is calculated, one finds

$$\delta(\omega^{\mu\nu})(\psi^\dagger)^\alpha = \frac{1}{4}(\psi^\dagger)^\beta \omega^{\mu\nu}(\gamma_\nu^\dagger \gamma_\mu^\dagger)_\beta^\alpha = -\frac{1}{4}(\psi^\dagger)^\beta \omega^{\mu\nu}(\gamma_0 \gamma_{\mu\nu}\gamma_0)_\beta^\alpha.$$

Due to the extra factors of γ_0, this does not produce a Lorentz invariant scalar under contraction of indices, necessitating the introduction of the *Dirac conjugate* of a spinor. This action is denoted by barring the spinor, and is defined as

$$\bar{\psi} = \psi^\dagger \gamma_0.$$

It is simple to check that

$$\delta(\omega^{\mu\nu})(\bar{\lambda}\psi) = \delta(\omega^{\mu\nu})(\bar{\lambda})\psi + \bar{\lambda}\delta(\omega^{\mu\nu})\psi = 0$$

and we have found a Lorentz invariant means by which to contract spinor indices. This also leads to definitions for other Lorentz covariant objects, commonly referred to as *spinor bilinears*, which transform as vectors and antisymmetric tensors under the spinor representation of the Lorentz group, and are constructed by inserting antisymmetric products of the gamma matrices between the spinors in the above expression, e.g.

$$\bar{\lambda}\gamma_\mu\psi \quad , \quad \bar{\lambda}\gamma_{\mu\nu}\psi \quad , \quad \cdots .$$

B.3 $SO(3,1) \leftrightarrow Sl(2,\mathbb{C})$ Homomorphism

Consider that, since the complexified Lie algebra of $so(3,1)$ is related to the special unitary group, the inverse should also be true - we thus now consider a complexification of the $su(2)$ group, and study its relation to the real Lie algebra of $so(3,1)$. The relevant algebra that we will take as the complexification of $su(2)$ is $sl(2,\mathbb{C})$, the group of all complex matrices with unit determinant. We start by introducing a parameterization for a general 2×2 complex Hermitian matrix:

$$\mathbf{x} = \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix}.$$

Note that $\det(\mathbf{x}) = -x^\mu \eta_{\mu\nu} x^\nu$, i.e. the determinant is negative the Minkowski norm of the 4-vector x^μ - this suggests a relation between the space of linear hermitian 2×2 matrices and 4-dimensional Minkowski space. There is indeed a homomorphism; it can be easily seen by introducing two complete sets of matrices

$$\sigma_\mu = \{\mathbb{1}_2, \sigma_i\} \qquad , \qquad \bar{\sigma}_\mu = \sigma^\mu = \{-\mathbb{1}_2, \sigma_i\}$$

where the σ_i are the usual $SU(2)$ generators, the Pauli matrices. These two sets of matrices each provide a basis for the set of complex Hermitian 2×2 matrices. Taken together, they provide a realisation of the Clifford algebra:

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2\eta_{\mu\nu}\mathbb{1}.$$

The identity here carries spinor indices taking values $1, 2$; it is the identity in the 2 dimensional representation $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$, being the representations under which the left and right handed Weyl fermions transform. The explicit form of the isomorphism is evident by realising that there is a transformation between the matrices and Minkowskian 4-vectors, given by

$$\mathbf{x} = \bar{\sigma}_\mu x^\mu \qquad , \qquad x^\mu = \frac{1}{2}\mathrm{tr}(\sigma_\mu \mathbf{x}).$$

If we now introduce a matrix $A \in SL(2,\mathbb{C})$, and consider the linear map on the matrix $\mathbf{x}$

$$\mathbf{x} \to A\mathbf{x}A^\dagger$$

we find that the 4-vectors are also linearly related under the transformation:

$$x^\mu \to x'^\mu = \frac{1}{2}\mathrm{tr}(\sigma^\mu A\bar{\sigma}_\nu A^\dagger)x^\nu = \Phi(A)^\mu_\nu x^\nu.$$

The homomorphism $Sl(2,\mathbb{C}) \leftrightarrow SO(3,1)$ is $2:1$, which is apparent since replacing $A \to -A$ in the above map does not change the result. Now, since the transformation of $\mathbf{x}$ preserves the Minkowski norm, we have

$$x^\mu \eta_{\mu\nu} x^\nu = x'^\mu \eta_{\mu\nu} x'^\nu = \Phi(A)^\mu_\rho x^\rho \eta_{\mu\nu} \Phi(A)^\nu_\lambda x^\lambda$$

and thus

$$\Phi(A)^\mu_\rho \eta_{\mu\nu} \Phi(A)^\nu_\lambda = \eta_{\rho\lambda}$$

i.e. the transformation induced on the associated 4-vector by the $SL(2,\mathcal{C})$ transformation of the matrices $\mathbf{x}$ satisfies the definition of a Lorentz transformation.

There are now two sets of matrices that can be defined in terms of the complex hermitian matrix basis elements, which also satisfy (B.2):

$$\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)$$
$$\bar{\sigma}_{\mu\nu} = \frac{1}{4}(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu). \tag{B.3}$$

These are the generators of the simplest irreducible non-trivial representation of the Lorentz algebra. We now consider the complexification of the Lorentz algebra using this representation of the generators by defining:

$$\Sigma^\pm_k = -\frac{1}{2}(\frac{1}{2}\epsilon_{ijk}\sigma_{ij} \pm i\sigma_{0k})$$
$$\bar{\Sigma}^\pm_k = -\frac{1}{2}(\frac{1}{2}\epsilon_{ijk}\bar{\sigma}_{ij} \pm i\bar{\sigma}_{0k}).$$

These matrices again satisfy the commutator relations of two independent $su(2)$ algebras, but in a special way - Σ^+_k and $\bar{\Sigma}^-_k$ both vanish, and we can thus associate the algebra generated by $\sigma_{\mu\nu}$ with the $(0,\frac{1}{2})$ representation, and that generated by $\bar{\sigma}_{\mu\nu}$ with the $(\frac{1}{2},0)$ representation of the Lorentz group.

Thus, the reduction of the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz algebra is achieved, simply by changing the representation of the Clifford algebra generators in terms of which the Lorentz generators are defined. This can be seen by considering the Weyl representation of the Clifford algebra; it can be given as

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}.$$

Considering the commutator of the matrices in this representation, we see that these gamma matrices combine the two commutators in (B.3) into a single expression - the associated Lorentz algebra thus provides an action compatible with the Weyl spinor fields, which are Dirac spinors which have been explicitly assembled from two 2-component fields each transforming under either $\sigma_{\mu\nu}$ or $\bar{\sigma}_{\mu\nu}$.

B.4 Clebsch-Gordan Coefficients

There is another useful interpretation of the complex matrix bases σ^μ and $\bar{\sigma}^\mu$ (and by combining them, also for the gamma matrices), in that they can be understood to provide the decomposition coefficients for the tensor product of two fundamental $su(2)$ irreps of opposite chirality onto the vector representation, i.e. they are the Clebsch-Gordan coefficients for the decompositions $(\frac{1}{2},0)\otimes(0,\frac{1}{2}) \to (\frac{1}{2},\frac{1}{2})$ and $(0,\frac{1}{2})\otimes(\frac{1}{2},0) \to (\frac{1}{2},\frac{1}{2})$.

To understand this, it is useful to first review the familiar decomposition of two fundamental $SU(2)$ irreps without chirality. As we have seen, Lorentz invariant combinations of spinors are obtained by taking a product involving two spinors, one of which transforms in the $(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$ irrep; the other transforms in the conjugate representation $(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})^* = (\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0})$. We will thus consider combining two spinors ψ and ψ^* which transform in the $\frac{\mathbf{1}}{\mathbf{2}}$ and $\frac{\mathbf{1}}{\mathbf{2}}^*$ irreps of $SU(2)$. It is a well-known fact that there exists only 1 irrep of $SU(2)$ of any particular dimension, so that these two representations must be related by a change of basis. Consider the generators of $SU(2)$:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad , \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Taking the complex conjugate, we have $\sigma_1^* = \sigma_1$, $\sigma_2^* = -\sigma_2$ and $\sigma_3^* = \sigma_3$, and we can define the matrix which implements the change of basis:

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad , \quad R^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Note that $\sigma_i^* = -R\sigma_i R^{-1}$. We can now write the spinors explicitly as column vectors of the spin states they contain:

$$\psi = \begin{bmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{bmatrix} \quad , \quad \psi^* = R\psi = \begin{bmatrix} |\downarrow\rangle \\ -|\uparrow\rangle \end{bmatrix}.$$

Now we recall the expression for the Clebsch-Gordan decomposition of the combination of two spin-$\frac{1}{2}$ states onto the $\mathbf{1}$ representation:

$$\begin{aligned} |\frac{1}{2}^*, m_1\rangle \otimes |\frac{1}{2}, m_2\rangle &= |\frac{1}{2}^*, m_1; \frac{1}{2}, m_2\rangle\langle\frac{1}{2}^*, m_1; \frac{1}{2}, m_2|1, (m_1 + m_2)\rangle \\ &= C^{(1,m)}_{m_1,m_2}|\frac{1}{2}^*, m_1; \frac{1}{2}, m_2\rangle. \end{aligned}$$

At this point, identifying $|\frac{1}{2}^*, m_1\rangle \leftrightarrow \psi^*_\alpha$ and $|\frac{1}{2}, m_2\rangle \leftrightarrow \psi_\beta$, we can consider the label m of the Clebsch-Gordan coefficient as an index corresponding to the states in the $\mathbf{1}$ representation of $SU(2)$, and the $m_1, m_2 \leftrightarrow \alpha, \beta$ as row and column indices of a matrix. Knowing the values of the Clebsch-Gordan coefficients for this decomposition, and remembering the transformation of spinor components under the change of basis, it is clear that the 3 matrices are

$$C^{(1,1)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad , \quad C^{(1,0)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad C^{(1,-1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is then clear how to use these matrices to obtain the explicit form of the decomposition[1](repeated indices are summed):

$$(\psi^\dagger)^{m_1}(C^m)^{m_2}_{m_1}\psi_{m_2} = |1, m\rangle.$$

Note that in this decomposition, there are only 3 possible states (as expected) of the $\mathbf{1}$ representation due to the fact that the composites with $m_1 = \pm\frac{1}{2}, m_2 = \mp\frac{1}{2}$ both contribute to the state with $m = 0$.

We now require that the vector on the right hand side of the decomposition corresponds to an actual coordinate 3-vector, i.e. we want each of the components to transform under $SO(3)$ in the same way as the coordinates x^i. Points on the plane in which a rotation takes place can be represented as complex numbers; for a rotation about the z-axis this means that we can form the combinations $x \pm iy$ of the transverse coordinates, and by the exponential map for $SO(3)$ these will be transformed under rotations by multiplying $e^{i\theta L_z}$. These combinations are then identified with states of the $SU(2)$ vector representation using the $SO(3) \leftrightarrow SU(2)$ homomorphism by noting:

$$x \to x' = \cos(\theta)x - \sin(\theta)y \quad , \quad y \to y' = \cos(\theta)y + \sin(\theta)x$$

[1]This notation is suggestive; we know due to the Lorentz homomorphism that $(\frac{1}{2}, 0)^\dagger = (0, \frac{1}{2})$ $(J^{i}_{\pm}{}^* = J^i_\mp \Rightarrow \vec{J}^\dagger_\pm = \vec{J}_\mp)$, so that the expression is schematically $\Psi^\dagger C\Psi$ - all that is missing to obtain Lorentz invariant spinor bilinears is the factor of γ^0, which arises by similar considerations for the decomposition of two Dirac spinors onto the $(0,0)$ (scalar) Lorentz representation. This will be treated shortly.

$$\Rightarrow x+iy \to x'+iy' = e^{i\theta L_z}(x+iy) = e^{i\theta}(x+iy) \quad , \quad x-iy \to x'-iy' = e^{i\theta L_z}(x+iy) = e^{-i\theta}(x+iy).$$

Of course, z is invariant under the rotation, and thus $z \to z' = e^{i\theta L_z}(z) = z$. We have now identified the states of the $SU(2)$ vector representation with their $SO(3)$ counterparts:

$$|1,1\rangle = |\uparrow\uparrow\rangle \leftrightarrow x+iy \quad , \quad |1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \leftrightarrow z \quad , \quad |1,-1\rangle = |\downarrow\downarrow\rangle \leftrightarrow x-iy.$$

This allows us to obtain the Clebsch-Gordan coefficient matrices in a form where the upper index m labelling states in the **1** representation of $SU(2)$ is exchanged for an index which labels states transforming in a representation of $SO(3)$, i.e. spatial coordinates:

$$C^1 = C^x + iC^y \quad , \quad C^{-1} = C^x - iC^y \quad , \quad C^0 = C^z.$$

Introducing a convenient normalization for the complex coordinates, $c_\pm = \frac{\sqrt{2}}{2}(x \pm iy)$, we have:

$$\Rightarrow C^x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad C^y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad , \quad C^z = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is clear that the Pauli matrices are indeed the Clebsch-Gordan coefficients for this decomposition, and as a consequence provide a means by which to obtain combinations of spinors which transform as a vector under $SO(3)$

$$(\psi^\dagger)^\alpha (\sigma^i)_\alpha^\beta \psi_\beta \leftrightarrow x^i.$$

There is, of course, another possibility for the final state when two spin-$\frac{1}{2}$ particles are combined - they can combine to form the scalar **0** representation:

$$\begin{aligned} |\frac{1}{2}^*, m_1\rangle \otimes |\frac{1}{2}, m_2\rangle &= |\frac{1}{2}^*, m_1; \frac{1}{2}, m_2\rangle\langle\frac{1}{2}^*, m_1; \frac{1}{2}, m_2|0, (m_1+m_2)\rangle \\ &= C^{(0,m)}_{m_1,m_2}|\frac{1}{2}^*, m_1; \frac{1}{2}, m_2\rangle. \end{aligned}$$

Since there is only one possible value of $m = 0$, there is only one Clebsch-Gordan matrix that can be defined, given by

$$C^{(0,m)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This decomposition allows us to obtain combinations of spinors transforming as a scalar under the action of $SO(3)$:

$$(\psi^\dagger)^\alpha (\sigma^0)_\alpha^\beta \psi_\beta \leftrightarrow \phi.$$

This reveals the nature of σ^0 as a metric on the spinor indices. As discussed in the body of this text, we can raise (lower) spinor indices using the antisymmetric symbol $\epsilon^{\alpha\beta}$ ($\epsilon_{\alpha\beta}$) - expressed as a matrix, this is exactly R (R^{-1}). Had we chosen to initiate this investigation by considering products of two spinors without conjugation, we would find that the Clebsch-Gordan coefficient matrix for decomposition onto the scalar representation can be given as R^{-1}, and the decomposition would then be $\eta^T R^{-1}\eta = |0\rangle$ - both the spinors here transform under the un-conjugated representation. We can recover the product in terms of the conjugate representation using $\eta^T = \eta^\dagger R$. This allows us to define the fundamental analogue of the Dirac adjoint; let $\bar{\eta} = \eta^\dagger R$, then, since we can relate the Clebsch-Gordan matrices for the above decomposition to those for products of two spinors without conjugation via R as $\sigma^i = R\bar{C}^i$, we have $\bar{\eta}R^{-1}\sigma^i\eta = \bar{\eta}\bar{C}^i\eta$. This construction thus provides a means by which to write down covariant spinor combinations formed from two spinors transforming with the same chirality.

The extension of these ideas to the case where we require covariance under the Lorentz group is straightforward to appreciate. We know that irreducible representations of the Lorentz group can be labelled by a set of two spins; the fundamental representation is given by either $(0, \frac{1}{2})$ or $(\frac{1}{2}, 0)$, and these two representations are related by a parity transformation, and indeed also by complex conjugation. They correspond to a product of the $\mathbf{\frac{1}{2}}$ representation of $SU(2)$ with the scalar **0** representation. The generators are given by a set consisting of the Pauli matrices with a generator for the scalar representation - this must obviously be a 2×2 identity matrix, since objects in this representation do not transform under the action of $SU(2)$:

$$\sigma^\mu = \{\mathbb{1}, \sigma^i\}.$$

We now proceed with the analysis as previously. Consider a spinor ψ which transforms in the $(\frac{1}{2}, 0)$ representation of $SU(2) \otimes SU(2)$, this representation is generated by the above set of matrices. After performing complex conjugation and changing basis using the matrix R defined previously, a set of generators is obtained for the conjugate representation:

$$(\sigma^\mu)^* = \{\mathbb{1}, -R\sigma^i R^{-1}\} = R\{\mathbb{1}, -\sigma^i\}R^{-1}.$$

Remember that this must correspond to the set of generators of the $(0, \frac{1}{2})$ representation, under which ψ^* transforms. We can recover the original set of generators by performing a parity transformation - this takes us back to the $(\frac{1}{2}, 0)$ representation. We can therefore express the combination of two spinors of opposite chirality in terms of Clebsch-Gordan coefficients for the decomposition $(\frac{1}{2}, 0)^* \otimes (\frac{1}{2}, 0) = (0, 0) \oplus (1, 0)$, where states in the $(\frac{1}{2}, 0)^*$ representation correspond to components of the transformed spinor $\psi^* = R\psi$. Since $\frac{1}{2} \otimes 0 = \frac{1}{2}$, this is simply the same analysis that was done previously; since we are now interested in obtaining Lorentz vectors which transform as $\mathbf{0} + \mathbf{1}$, we express the decomposition onto both possible resulting representations:

$$\begin{aligned}
|(\frac{1}{2}, 0)^*, m_1\rangle \otimes |(\frac{1}{2}, 0), m_2\rangle &= |(\frac{1}{2}, 0)^*, m_1; (\frac{1}{2}, 0), m_2\rangle\langle(\frac{1}{2}, 0)^*, m_1; (\frac{1}{2}, 0), m_2|(0, 0), 0\rangle \\
&+ |(\frac{1}{2}, 0)^*, m_1; (\frac{1}{2}, 0), m_2\rangle\langle(\frac{1}{2}, 0)^*, m_1; (\frac{1}{2}, 0), m_2|(1, 0), m\rangle \\
&= C^{(0,0)}_{(m_1,0),(m_2,0)}|(0, 0), 0\rangle + C^{(1,m)}_{(m_1,0),(m_2,0)}|\frac{1}{2}^*, m_1; \frac{1}{2}, m_2\rangle.
\end{aligned}$$

The coefficients for the states in the $\mathbf{1}$ representation of $SU(2)$ can be transformed to their $SO(3)$ counterparts in the same way as previously; the coefficient of the $\mathbf{0}$ representation does not require such an analysis, as it does not transform under the rotation group and thus the superscript already corresponds to a 4-vector index, transforming in the $\mathbf{0}$ irrep of $SO(3, 1) \cong SU(2) \otimes SU(2)$. At this point, it is a spatial index, which is simply not rotated by the transformation. To correct the signature such that it can be taken to correspond to the time coordinate of the $SO(3, 1)$ group, we should multiply by i. The C matrices are, explicitly:

$$C^0 = \frac{1}{i}C^t = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad , \quad C^x = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad , \quad C^y = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \quad , \quad C^z = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Clearly then, the set of Pauli matrices extended by the addition of a unit matrix provide the Clebsch-Gordan coefficients for the decomposition we consider, and allow us to produce combinations of $(\frac{1}{2}, 0)$ spinors which transform as a Lorentz vector:

$$(\psi^\dagger)^\alpha(\sigma^\mu)^\beta_\alpha\psi_\beta \leftrightarrow x^\mu.$$

We could equally well have considered a spinor η transforming in the $(0, \frac{1}{2})$ representation; then we may have defined the generators as

$$\bar{\sigma}^\mu = \{\mathbb{1}, -\sigma^i\}.$$

After complex conjugation, the matrix implementing the change of basis is

$$\bar{R} = R^{-1} = -R$$

and the spinor $\eta^* = \bar{R}\eta$ would have reversed signs for its components. This leads to the sign reversal of the coefficient matrix $\bar{C}^{(0,0)}_{(m_1,0),(m_2,0)} = -C^{(0,0)}_{(m_1,0),(m_2,0)}$, indicating that a time-reversal has occurred - this is natural, since time reversal is another means by which a parity transformation can be implemented (recalling that phases don't matter). We could thus have chosen the generators as

$$\bar{\sigma}^\mu = \{-\mathbb{1}, \sigma^i\}$$

and we see that this extended set of Pauli matrices allows for the construction of $(0, \frac{1}{2})$ spinor combinations transforming as a Lorentz vector

$$(\eta^\dagger)^\alpha (\bar{\sigma}^\mu)_\alpha^\beta \eta_\beta \leftrightarrow x^\mu.$$

Using the fact that, due to the homomorphism with the complexified Lorentz algebra, conjugation of one of the fundamental representations produces the alternate fundamental, $(j_-, j_+)^* = (j_+, j_-)$, there is another way to express the previous decompositions - as $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$:

$$\begin{aligned}
|(\frac{1}{2}, 0), m_1\rangle \otimes |(0, \frac{1}{2}), m_2\rangle &= |(\frac{1}{2}, 0), m_1; (0, \frac{1}{2}), m_2\rangle\langle(\frac{1}{2}, 0), m_1; (0, \frac{1}{2}), m_2|(\frac{1}{2}, \frac{1}{2}), (m_1, m_2)\rangle \\
&= C^{(m_1,m_2)}_{(m_1,0),(0,m_2)} |(\frac{1}{2}, 0), m_1; (0, \frac{1}{2}), m_2\rangle \\
&= C^{(m_1,m_2)}_{(m_1,0),(0,m_2)} R|(0, \frac{1}{2}), m_1; (0, \frac{1}{2}), m_2\rangle.
\end{aligned}$$

The state indices m_1, m_2 could be written as $(m_1, 0)$ and $(0, m_2)$ in the first line so as to explicitly label states of both the representations in the direct sum; we have omitted the scalar representation state index for convenience. In this case, states with $m_1 = -m_2$ no longer contribute to the same representation; they form the distinct $(\pm\frac{1}{2}, \mp\frac{1}{2})$ states of the $(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}})$ representation. This of course again leads to the appearance of a 4-vector state index on the Clebsch-Gordan coefficients:

$$C^{(m_1,m_2)}_{(m_1,0),(0,m_2)} \leftrightarrow \sigma^\mu_{\alpha\beta}.$$

When acted on by rotations, this representation transforms as $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$, in the same way as in the original analysis. The analogy with the second decomposition is obtained by reversing the order of the component representations, i.e.

$$|(0, \frac{1}{2}), m_1\rangle \otimes |(\frac{1}{2}, 0), m_2\rangle = \bar{C}^{(m_1,m_2)}_{(0,m_1),(m_2,0)} |(0, \frac{1}{2}), m_1; (\frac{1}{2}, 0), m_2\rangle.$$

$$\bar{C}^{(m_1,m_2)}_{(0,m_1),(m_2,0)} \leftrightarrow \bar{\sigma}^\mu_{\alpha\beta}.$$

The gamma matrices can be constructed from the two sets of extended Pauli matrices by recalling that the (reducible) Dirac representation of the Lorentz group corresponds to a representation of $SU(2) \oplus SU(2)$. Let's consider a Dirac spinor Ψ transforming under $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$. Clearly then, the decomposition we are interested in for the product of a spinor and its conjugate is $[(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})] \otimes [(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)] = (\frac{1}{2}, \frac{1}{2}) \oplus (\frac{1}{2}, \frac{1}{2})$. It follows from the previous observations that the gamma matrices, being that they are required to fulfil the same role as the extended Pauli matrices for the case where we work with bispinors, are given as the direct sum of the two sets σ^μ and $\bar{\sigma}^\mu$:

$$\gamma^\mu = (\sigma \oplus \bar{\sigma})^\mu = \begin{bmatrix} \sigma^\mu & 0 \\ 0 & \bar{\sigma}^\mu \end{bmatrix}.$$

This matrix satisfies $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\mathbb{1}$; it corresponds to a construction for which $\gamma^{\mu\nu}$ generates $SO(4)$ transformations. We must replace σ^0 by C^t to obtain the form which satisfies the correct relation for the construction of $SO(3, 1)$ generators, i.e. $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$. The Weyl representation is obtained by swapping the two columns in the above matrix, yielding a gamma matrix with the extended Pauli matrices in off-diagonal blocks, which satisfies this commutator relation without requiring complexification of the scalar Clebsch-Gordan coefficients.

B.5 General Representations of the Clifford Algebra

Continuing the discussion surrounding representations of the Clifford algebra, we first note that one can impose additional hermiticity constraints on the gamma matrices, provided they respect the anticommutator relation. In the metric signature chosen here, we can have:

$$(\gamma^0)^\dagger = -\gamma^0 \quad , \quad (\gamma^i)^\dagger = \gamma^i.$$

This can be concisely restated for general spacetime index μ as [2]

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0.$$

There is a fundamental theorem due to Pauli which states that if two sets of D matrices, each being of dimension $2^{D/2} \times 2^{D/2}$, satisfy the defining property for generators of the associated Clifford algebra then there must exist a similarity transformation which relates them, i.e. if $\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1}$ and $\{\gamma'^a, \gamma'^b\} = 2\eta^{ab}\mathbb{1}$, then there exists a matrix S for which

$$\gamma'^a = S\gamma^a S^{-1}.$$

One can check that γ'^a still satisfies the anticommutation relation. We consider only Hermitian representations of the Clifford algebra, as reflected by the hermiticity properties given above, and thus the matrix S must be unitary since

$$(\gamma'^a)^\dagger = (S\gamma^a S^{-1})^\dagger = (S^{-1})^\dagger(\gamma^a)^\dagger S^\dagger = (S^{-1})^\dagger\gamma^0\gamma^a\gamma^0 S^\dagger = \gamma^0 S\gamma^a S^{-1}\gamma^0 = \gamma^0\gamma'^a\gamma^0$$

only if $S^\dagger = S^{-1}$.

This theorem allows us to express the Dirac equation in terms of an arbitrary basis of the Clifford algebra, while maintaining Lorentz invariance of any conditions we impose on the spinors. This is most useful in the context of complex conjugation - if we were to impose a reality condition on a spinor ψ as $\psi = \psi^*$ and subsequently apply a Lorentz transformation under a complex representation of the Lorentz group, there is no reason to expect that this condition would still hold after the transformation; the Lorentz action can introduce new complex factors into the spinor, which spoil its reality:

$$\psi = \psi^* \to \Lambda\psi = \Lambda^*\psi^*.$$

Due to the fact that we can write the Lorentz generators in terms of products of gamma matrices as $\Lambda^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$, we may circumvent this by choosing a representation of the Clifford algebra in which the generators are all pure imaginary, thus rendering the Lorentz generators real; $\Lambda = \Lambda^*$. With this choice of basis the reality condition $\psi = \psi^*$ is Lorentz invariant.

However, the physics of any problem involving this construction is required to be invariant under a change of basis of the Clifford algebra, and it is undesirable to require a specific choice of basis in calculation - the model should be constructed in a basis-independent way in order to be considered general. Pauli's fundamental theorem provides another means by which Lorentz invariance of spinor conditions can be imposed. We can define an operation which is analogous to complex conjugation, known as the *charge conjugation* of the spinor, by recognising that the complex conjugate of the gamma matrices, γ^*_μ, as well as its negative both satisfy the same anticommutation relation as the matrix itself. We therefore have that the conjugate of the gamma matrices in any basis can be given as

$$\pm(\gamma^\mu)^* = B_\pm\gamma^\mu B_\pm^{-1}.$$

In even dimensions, both B_+ and B_- must exist, since there are two representations differing only in their chirality, whereas for odd dimensions there is only one possible representation and no notion of chirality - thus, only one of the conjugation matrices are permissible. This is related to the fact that the chirality matrix defined in even dimensions is actually one of the generators when working in one higher dimension. The *charge conjugation matrix* C is defined by applying this same theorem to the transpose of the gamma matrices; that is

$$(\gamma^\mu)^T = -C_\pm\gamma^\mu C_\pm^{-1}.$$

[2]Note that this is valid for Lorentzian gamma matrices only; that is, for the generators of Clifford algebras $\mathrm{Cl}_{(1,d)}(\mathbb{R})$. In the general case with t time directions, we define $A = (-1)^{\frac{1}{4}t(t+1)}\gamma^1\gamma^2\cdots\gamma^t$, which satisfies

$$(\gamma^\mu)^\dagger = (-1)^t A\gamma^\mu A^{-1}.$$

It is clear that this reproduces the Lorentzian relation for $t = 1$, since we have $A = i\gamma^0$.

Let us now consider the complex conjugation of the LHS of the free Dirac equation:

$$\begin{aligned}
(\gamma^\mu \partial_\mu - M)^* \psi^* &= (\gamma^{\mu *}\partial_\mu - M)\psi^* \\
&= (B_\pm \gamma^\mu B_\pm^{-1} \partial_\mu - M)\psi^* \\
\Rightarrow (\gamma^\mu \partial_\mu - M) B_\pm^{-1} \psi^* = 0 &\leftrightarrow (\gamma^\mu \partial_\mu - M)\psi^c = 0.
\end{aligned}$$

Where we have defined the *charge conjugate* spinor ψ^c. There is in fact a second way in which this can be expressed, by using the fact that the transpose of the hermitian conjugate is equivalent to taking the complex conjugate:

$$\begin{aligned}
(\gamma^\mu \partial_\mu - M)^{T\dagger} \psi^* &= (-C_\pm \gamma^\mu C_\pm^{-1} \partial_\mu - M)^\dagger \psi^* \\
&= (-\gamma^0 C_\pm \gamma^\mu C_\pm^{-1} \gamma^0 \partial_\mu - M)\psi^* \\
&= (\gamma^\mu \partial_\mu - M) C_\pm^{-1} \gamma^0 \psi^* \\
\Rightarrow (\gamma^\mu \partial_\mu - M) C_\pm^{-1} \gamma^0 \psi^* &= (\gamma^\mu \partial_\mu - M) C_\pm^{-1} \bar{\psi}^T = 0 \leftrightarrow (\gamma^\mu \partial_\mu - M)\psi^c = 0.
\end{aligned}$$

Comparing these two expressions, we see that $C_\pm = B_\pm \gamma^0$ and further we can define the charge conjugate of the spinor as

$$\psi^c = C_\pm^{-1} \gamma^0 \psi^* = C_\pm^{-1} \bar{\psi}^T.$$

This makes it obvious how to obtain the charge conjugate of the adjoint spinor; we have

$$(\psi^c)^\dagger \gamma^0 = \psi^T (-\gamma^0) C_\pm \gamma^0 = \psi^T C_\pm.$$

This demonstrates how Pauli's theorem allows a basis-independent description; instead of requiring a specific basis of the Clifford algebra for Lorentz invariance of the reality condition to hold, the conjugated spinors are transformed using the unitary matrix which implements this change of basis to a form in which the Lorentz invariance of the condition is automatic:

$$\begin{aligned}
\psi^c \to \Lambda^{\mu\nu} \psi^c &= B_\pm \Lambda^{\mu\nu *} \psi^* \\
&= \frac{1}{4} B_\pm^{-1} (\gamma^{\mu *} \gamma^{\nu *} - \gamma^{\nu *} \gamma^{\mu *}) \psi^* \\
&= \frac{1}{4} B_\pm^{-1} B_\pm (\gamma^\mu B_\pm^{-1} B_\pm \gamma^\nu - \gamma^\nu B_\pm^{-1} B_\pm \gamma^\mu) B_\pm^{-1} \psi^* \\
&= \Lambda^{\mu\nu} \psi^c \\
\Rightarrow \psi^c = \psi &\to \Lambda^{\mu\nu} \psi^c = \Lambda^{\mu\nu} \psi.
\end{aligned}$$

Thus, if ψ is a solution to the Dirac equation, then the charge conjugated spinor ψ^c is as well, and is imbued with the additional property of providing a Lorentz invariant means by which to impose reality conditions on the field.

We mentioned that the requirement of matrix derivative coefficients in the relativistic wave equation implies that the wave function must have multiple components - this is the origin of the use of spinors to describe relativistic fermionic wavefunctions. In Pauli theory, which was an earlier attempt to describe half-integer spin particles, the formalism required that in $D = 4$ the spinors had two components; under Dirac theory, we find that in this case the spinors have four components. The existence of the additional components can be understood by noting that there is a natural way to lift an even dimensional Clifford algebra in D dimensions to a Clifford algebra in $D + 1$ dimensions by introducing an additional gamma matrix $\gamma_* = c\gamma_0\gamma_1 \cdots \gamma_{D-1}$. This additional matrix anticommutes with all the others, and is additionally Hermitian. The matrix will thus admit a basis of eigenvectors with eigenvalues $= \pm 1$; the sign determines the *chirality* (or *handedness*) of the eigenvector - this implies that in even dimensions, the $2^{D/2}$-component spinors can be decomposed into two $2^{D/2-1}$ component spinors which differ only by their chirality, that is

$$\psi = \psi_L + \psi_R \quad , \quad \gamma_* \psi_L = \psi_L \quad , \quad \gamma_* \psi_R = -\psi_R.$$

Spinors having definite eigenvalue under the chirality matrix γ_* are called Weyl spinors - the four-component Dirac spinors can thus be interpreted as an assembly of two Weyl spinors of opposite chirality.

The Weyl spinors are obtained from the Dirac spinor by application of the chiral projectors, defined in terms of the chirality matrix as

$$P_L = \frac{1}{2}(\mathbf{1} + \gamma_*) \qquad , \qquad P_R = \frac{1}{2}(\mathbf{1} - \gamma_*).$$

Considering the action of P_L on the Dirac field $\Psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$

$$P_L \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix} = \frac{1}{2}(\begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix} + \begin{bmatrix} \psi_L \\ -\psi_R \end{bmatrix}) = \begin{bmatrix} \psi_L \\ 0 \end{bmatrix}$$

it is obvious why these definitions for the projectors are valid. By recalling that a left-handed particle is related to a right-handed particle by a parity transformation, we are aware that this condition reduces the number of independent spinor components by half.

There is another special case of the spinors which arises in certain dimensions, where the gamma matrices can be defined to be purely imaginary-valued. As discussed, this in turn allows one to define spinors which are purely real, so that they will be related to their complex conjugate, thus also reducing the number of independent components by half. This condition is expressed in terms of the charge conjugate as $\psi^c = \psi \Rightarrow \psi^* = B\psi = C\gamma^0\psi$.

In 2 mod 8 dimensions (which includes the dimensionality of interest to us, $D = 10$) one can in fact have these two conditions simultaneously, in which case we have Majorana-Weyl spinors. These will have a definite eigenvalue under the chirality matrix, and are related to their complex conjugate:

$$\gamma_* \psi_{MW} = \pm\psi_{MW} \qquad , \qquad \psi_{MW} = C\psi^*_{MW}.$$

There is another spinor transformation which allows us to construct Lorentz invariant quantities, this is known as the *Majorana conjugate* and is defined by

$$\bar{\psi} = \psi^T C$$

where C is the charge conjugation matrix. In the case that we work with Majorana spinors, this is equivalent to the Dirac conjugate - this is another way of expressing the Majorana condition.

The bilinears are of primary importance in the application of this theory, for exactly the reason that they are Lorentz invariant, and thus provide natural candidates for inclusion in the Lagrangian of any theory containing relativistic fermions which we wish to formulate. It is important also, for example in determining the hermiticity of such a Lagrangian, that we are able to perform complex conjugation of the Lorentz invariant forms. The standard procedure which one may be inclined to follow (of simply applying the complex conjugation to the product) is unwieldy since it requires detailed analysis of the hermiticity of both the gammas and charge conjugation matrices. It is more convenient to work with the action of charge conjugation, which is implemented using the matrix B introduced previously. Any quantity for which all the spinor indices are contracted (as is the case with the bilinears) permits the use of charge conjugation as an equivalent operation to complex conjugation, with the added benefit of being simpler to implement on these physically relevant combinations. Drawing on the previous discussion surrounding charge conjugation and Lorentz invariance, this should be obvious - the spinor bilinears are Lorentz invariant, and complex conjugation should be expected to be equivalent to charge conjugation when applied to these forms.

We have already given the relation between the complex and charge conjugates of the spinors; we will also need to express the complex conjugate of the gamma matrices in terms of this operation in order to analyse the bilinear conjugation. The charge conjugate of an arbitrary $2^m \times 2^m$ matrix is given by $(M)^c = B^{-1}M^*B$, leading to the charge conjugation property of the gamma matrices:

$$(\gamma_\mu)^c = B^{-1}\gamma^*_\mu B = B^{-1}B\gamma_\mu B^{-1}B = \gamma_\mu.$$

Using these relations, one can derive a rule for charge conjugation of general spinor bilinears:

$$(\bar{\chi}M\lambda)^* = (\bar{\chi}M\lambda)^c = (-t_0 t_1)\bar{\chi}^c M^c \lambda^c.$$

Note that the barred spinor here is the Majorana conjugate, and that for $D = 10$, $-(t_0 t_1) = 1$; the form of the final result is convenient since there are no swaps of the fermion fields. This also immediately implies, for Majorana (and hence also Majorana-Weyl) spinors where $\psi^c = \psi$, $(\bar{\chi}\gamma^{\mu_1 \cdot\cdot}\lambda)^* = \bar{\chi}\gamma^{\mu_1 \cdot\cdot}\lambda$.

Appendix C

Gauss Operators in $S_n \times (S_1)^m$ Description

C.1 Decomposition in terms of $S_n \times (S_1)^m$ Projectors - Examples

C.1.1 Gauss Operators for $H = (S_1)^3$

Operators corresponding to permutations in the double coset $H \backslash S_m / H$ are given in terms of linear combinations of restricted Schur polynomials labelled by $S_n \times S_m$ irreps as[8]:

$$O_{R,r}(\sigma) = \frac{|H|}{\sqrt{m!}} \sum_{j,k} \sum_{s \vdash m} \sum_{\mu_1,\mu_2} \sqrt{d_s} \Gamma^{(s)}_{jk}(\sigma) B^{s \to 1_H}_{j\mu_1} B^{s \to 1_H}_{k\mu_2} O_{R,(r,s)\mu_1\mu_2}. \tag{C.1}$$

Recalling that $d_s = \frac{m!}{\mathrm{Hooks_s}}$ and implementing the approximation $\frac{Hooks_r}{Hooks_R} \approx 1$, it is in fact slightly easier to write this in terms of unnormalized operators $\chi_{R,(r,s)\mu_1\mu_2}$:

$$O_{R,r}(\sigma) = |H| \sum_{j,k} \sum_{s \vdash m} \sum_{\mu_1,\mu_2} \Gamma^{(s)}_{jk}(\sigma) B^{s \to 1_H}_{j\mu_1} B^{s \to 1_H}_{k\mu_2} \chi_{R,(r,s)\mu_1\mu_2}.$$

Using this definition, the linear combinations of Restricted Schurs in this case for each of the permutations in the double coset are easily found to be:

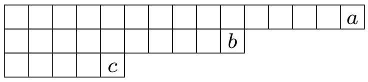
$$\begin{aligned}
O_{R,r}(\mathbb{1}) &= O_1 + O_2 + O_3 + O_6 \\
O_{R,r}((12)) &= O_1 - O_2 + \frac{1}{2}O_3 + \frac{\sqrt{3}}{2}O_4 + \frac{\sqrt{3}}{2}O_5 - \frac{1}{2}O_6 \\
O_{R,r}((23)) &= O_1 - O_2 - O_3 + O_6 \\
O_{R,r}((13)) &= O_1 - O_2 + \frac{1}{2}O_3 - \frac{\sqrt{3}}{2}O_4 - \frac{\sqrt{3}}{2}O_5 - \frac{1}{2}O_6 \\
O_{R,r}((123)) &= O_1 + O_2 - \frac{1}{2}O_3 + \frac{\sqrt{3}}{2}O_4 - \frac{\sqrt{3}}{2}O_5 - \frac{1}{2}O_6 \\
O_{R,r}((321)) &= O_1 + O_2 - \frac{1}{2}O_3 - \frac{\sqrt{3}}{2}O_4 + \frac{\sqrt{3}}{2}O_5 - \frac{1}{2}O_6
\end{aligned} \tag{C.2}$$

where

$$O_1 = \chi_{R,(r,\yng(3))}(Z,Y) \qquad O_2 = \chi_{R,(r,\yng(1,1,1))}(Z,Y) \qquad O_3 = \chi_{R,(r,\yng(2,1))11}(Z,Y)$$

$$O_4 = \chi_{R,(r,\yng(2,1))12}(Z,Y) \qquad O_5 = \chi_{R,(r,\yng(2,1))21}(Z,Y) \qquad O_6 = \chi_{R,(r,\yng(2,1))22}(Z,Y).$$

The Clebsch-Gordan coefficients which provide the transformation between the $S_n \times (S_1)^m$ states in terms of which we would now like to write these operators, and states of $U(3)$ which are Schur-Weyl dual to the $S_n \times S_m$ states were given in [2]. Using these results, it is straightforward to expand the states appearing in the projectors for each of the restricted Schurs above in terms of linear combinations of $S_n \times (S_1)^m$ states, with the coefficients of each being given by the Clebsch-Gordan coefficient for the corresponding $U(3)$ Gelfand-Tsetlin pattern.

$|a, b, c\rangle =$ 

.

$$|1,2,3\rangle = \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix}$$

$$|2,1,3\rangle = \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix}$$

and so on. Performing this expansion for each of the Restricted Schurs in the operators (C.2), we find that the Gauss operators can be written very simply and intuitively in terms of $S_n \times (S_1)^m$ projectors (and hence in terms of the corresponding Schur polynomials):

$$\begin{aligned}
O_{R,r}((1)(2)(3)) &= |123\rangle\langle 123| + |213\rangle\langle 213| + |312\rangle\langle 312| + |132\rangle\langle 132| + |231\rangle\langle 231| + |321\rangle\langle 321| \\
O_{R,r}((12)) &= |123\rangle\langle 132| + |132\rangle\langle 123| + |213\rangle\langle 231| + |231\rangle\langle 213| + |312\rangle\langle 321| + |321\rangle\langle 312| \\
O_{R,r}((23)) &= |123\rangle\langle 213| + |213\rangle\langle 123| + |312\rangle\langle 132| + |132\rangle\langle 312| + |231\rangle\langle 321| + |321\rangle\langle 231| \\
O_{R,r}((13)) &= |123\rangle\langle 321| + |321\rangle\langle 123| + |213\rangle\langle 312| + |312\rangle\langle 213| + |132\rangle\langle 231| + |231\rangle\langle 132| \\
O_{R,r}((123)) &= |123\rangle\langle 312| + |132\rangle\langle 213| + |231\rangle\langle 123| + |312\rangle\langle 231| + |321\rangle\langle 132| + |213\rangle\langle 321| \\
O_{R,r}((321)) &= |312\rangle\langle 123| + |213\rangle\langle 132| + |123\rangle\langle 231| + |231\rangle\langle 312| + |132\rangle\langle 321| + |321\rangle\langle 213|. \qquad \text{(C.3)}
\end{aligned}$$

Bearing in mind the interpretation of the numbering on the two $S_n \times (S_1)^m$ labels as specifying the positions of the two endpoints of each string, this is exactly the result that should be expected - each Gauss operator appears as the sum over all the states which correspond the configuration specified by the permutation, for each possible numbering of the strings in the configuration. This is sensible - the double coset symmetrises on the endpoints at each giant, and we should not expect that a particular choice of string labelling is preferred - they are, after all, indistinguishable.

The procedure of calculating the Clebsch-Gordan coefficients is a tedious one; there is in fact a simpler method by which to obtain these decompositions which was presented in the follow up article [10] to that in which the Gauss operators were first proposed. There is a simple group theoretic formula which relates the states in each description:

$$|v_s, i, j\rangle = \sum_{\sigma \in S_m} \Gamma^{(s)}_{ij}(\sigma)\sigma|v, \vec{m}\rangle. \qquad \text{(C.4)}$$

$|v_s, i, j\rangle$ corresponds to a state that will appear in the projectors in the $S_n \times S_m$ basis, which is evident in the fact that it is labelled by a representation s and a pair of multiplicity labels. The state $|v, \vec{m}\rangle$ can be represented as $|v_1^{\otimes m_1} \otimes \cdots \otimes v_g^{\otimes m_g}\rangle$, where g is the number of giants in the system being considered, v_i is the vector associated with a box being pulled from row i and $\vec{m} = [m_1, m_2, \cdots, m_g]$ encodes the number of boxes pulled from each row. This is a natural way to label $S_n \times (S_1)^m$ states in the context of strings attached to well-separated giants. The action of σ on these states is given by

$$\sigma|v, \vec{m}\rangle = \sigma|v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_m}\rangle = |v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(m)}}\rangle.$$

It is now clear how this formula can be applied to the states forming projectors in the operators of (C.2) to reproduce (C.3)

C.1.2 Gauss Operators for $H = S_1 \times S_2$

In this case, two boxes are pulled off one of the rows of the Young diagram R, and one box is pulled off another row. Associating a box appearing in each row with a state in the fundamental representation of $U(1)$ for each valid labelling of the $S_n \times (S_1)^m$ irrep as before, the Clebsch-Gordan coefficients for this decomposition

$$|a, b, c\rangle = \young(\ \ \ \ \ \ \ \ \ \ \ \ \ \ a,\ \ \ \ \ \ \ \ \ cb,\ \ \ \ ) \; .$$

can be determined by applying the same methods as before to the tensor products:

$$|1,2,3\rangle = \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix}$$

$$|2,1,3\rangle = \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix}$$

$$|3,1,2\rangle = \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} .$$

There are only 3 $S_n \times (S_1)^m$ subspaces in this case, one $U(3)$ state corresponding to the diagram $\yng(2,1)$ with the correct delta weight $\Delta = (0, 2, 1)$, and no Gelfand-Tsetlin patterns corresponding to the diagram $\yng(1,1)$ with this delta weight - there are thus only two possible restricted Schurs that can be defined, and the multiplicity indices are not used. After calculating the Clebsch-Gordan coefficients for each of the tensor products above, we indeed find that only states corresponding to two of the S_m irreps have non-zero expansions in terms of the above. These expansions are given by:

$$\left|\yng(3)\right\rangle = \begin{bmatrix} 3 & & 0 & & 0 \\ & 3 & & 0 & \\ & & 1 & & \end{bmatrix} = \frac{1}{\sqrt{3}} \left(|1,2,3\rangle + |2,1,3\rangle + |3,1,2\rangle\right)$$

$$\left|\yng(2,1), 1\right\rangle = \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{bmatrix}^{(1)} = \frac{1}{\sqrt{6}} \left(-2|1,2,3\rangle + |2,1,3\rangle + |3,1,2\rangle\right)$$

$$\left|\yng(2,1), 2\right\rangle = \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{bmatrix}^{(2)} = \frac{1}{\sqrt{2}} \left(|2,1,3\rangle - |3,1,2\rangle\right).$$

Substituting these expansions into the Gauss operators for the permutations in this double coset:

$$O_{R,r}(\mathbb{1}) = 2(\chi_{R,(r,\yng(3))} + \chi_{R,(r,\yng(2,1))})$$
$$O_{R,r}((12)) = 2(\chi_{R,(r,\yng(3))} - \frac{1}{2}\chi_{R,(r,\yng(2,1))})$$

the resulting linear combinations again follow perfectly the intuition gained by considering the associated Gauss graphs:

$$O_{R,r}(\mathbb{1}) = 2(|123\rangle\langle 123| + |213\rangle\langle 213| + |312\rangle\langle 312|)$$
$$O_{R,r}((12)) = |123\rangle\langle 213| + |213\rangle\langle 123| + |123\rangle\langle 312| + |312\rangle\langle 123| + |213\rangle\langle 312| + |312\rangle\langle 213|.$$

There are two copies of each state in the identity permutation operator due to the indistinguishability of configurations where two strings are attached at both ends to the same giant and numbered by the same two numbers. It is interesting to note:

$$O_{R,r}((23)) = 2(|123\rangle\langle 123| + |213\rangle\langle 213| + |312\rangle\langle 312|)$$
$$O_{R,r}((13)) = |123\rangle\langle 213| + |213\rangle\langle 123| + |123\rangle\langle 312| + |312\rangle\langle 123| + |213\rangle\langle 312| + |312\rangle\langle 213|.$$

These results are expected; the permutation (23) corresponds to the same Gauss graph as the identity, since the impurities with indices 2 and 3 are associated with boxes in the same row, while (13) corresponds to an equivalent configuration to (12) for the same reason (since the impurities are indistinguishable).

C.2 General Expression for Gauss operators in terms of $Sn \times (S_1)^m$ Projectors

C.2.1 Construction for $H = (S_1)^3$

In this case, it seems there is an obvious guess for the general expression. It involves only an easily constructed matrix, the indices of which correspond to the multiplicity of subduction of $S_n \times (S_1)^m$ from the S_{n+m} irrep R:

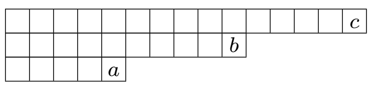
$$O_{R,r}(\sigma) = |H| \sum_{\mu_1,\mu_2} \Gamma^{(r,1_m)}_{\mu_1\mu_2}(\sigma)\chi_{R,(r,1_m)\mu_1\mu_2}(Z,Y). \tag{C.5}$$

The symbol 1_m refers to the $(S_1)^m$ irrep $\square \otimes \square \otimes \square$. The multiplicity indices run over the number of valid $S_n \times (S_1)^m$ labellings - for example, in the operator $\chi_{R,(r,1_m)12}(Z,Y)$, the restricted trace is performed using the projector $|123\rangle\langle 213|$; this is just a more compact way of labelling the operators. We could also have given both the operator and matrix indices as partially labelled Young diagrams. The convention chosen for correspondence between this numbering and that in terms of partially labelled Young diagrams is:

$$\begin{aligned}
|abc\rangle &\longleftrightarrow |\mu\rangle\\
|123\rangle &\longleftrightarrow |1\rangle\\
|213\rangle &\longleftrightarrow |2\rangle\\
|312\rangle &\longleftrightarrow |3\rangle\\
|132\rangle &\longleftrightarrow |4\rangle\\
|231\rangle &\longleftrightarrow |5\rangle\\
|321\rangle &\longleftrightarrow |6\rangle.
\end{aligned}$$

The matrix $\Gamma^{(r,1_m)}_{\mu_1\mu_2}(\sigma)$ is constructed in this case by simply using our simplified version of Young's orthogonal representation to obtain the action of σ on each of the $|\mu\rangle$. Using the convention for the association of state $|abc\rangle$ with a partially labelled Young diagram, we find that we must in fact number the giants in reverse order to the strings, and consider the permutation action of e.g. (12) as swapping the box in row 1 with the box in row 2 (this would in fact be the action of (23) in the standard use of simplified Young's rep). This can be made to directly coincide with the standard representation by any of the following convention redefinitions:

We can reverse the numbering of the partially labelled Young diagram as it read from the state:

$|a, b, c\rangle =$ 

.

We could also reverse the convention for associating $U(1)$ fundamentals with rows:

$$|1,2,3\rangle = \begin{bmatrix} 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix}.$$

Finally, we could also use the opposite convention for defining the action of the permutation on S_m irreps in the construction of the Gauss operators - this requires that the branching coefficients for each > 1 dimensional representation be permuted amongst each other

$$B_1^{\yng(2,1)} = [1,0] \quad B_2^{\yng(2,1)} = [0,1] \to B_1^{\yng(2,1)} = [0,1] \quad B_2^{\yng(2,1)} = [1,0].$$

After implementing any one of these convention changes, the Gauss operator decompositions are given as:

$$\begin{aligned}
O_{R,r}(\mathbb{1}) &= |123\rangle\langle 123| + |213\rangle\langle 213| + |312\rangle\langle 312| + |132\rangle\langle 132| + |231\rangle\langle 231| + |321\rangle\langle 321| \\
O_{R,r}((12)) &= |123\rangle\langle 213| + |213\rangle\langle 123| + |312\rangle\langle 132| + |132\rangle\langle 312| + |231\rangle\langle 321| + |321\rangle\langle 231| \\
O_{R,r}((23)) &= |123\rangle\langle 132| + |132\rangle\langle 123| + |213\rangle\langle 231| + |231\rangle\langle 213| + |312\rangle\langle 321| + |321\rangle\langle 312| \\
O_{R,r}((13)) &= |123\rangle\langle 321| + |321\rangle\langle 123| + |213\rangle\langle 312| + |312\rangle\langle 213| + |132\rangle\langle 231| + |231\rangle\langle 132| \\
O_{R,r}((123)) &= |312\rangle\langle 123| + |213\rangle\langle 132| + |123\rangle\langle 231| + |231\rangle\langle 312| + |132\rangle\langle 321| + |321\rangle\langle 213| \\
O_{R,r}((321)) &= |123\rangle\langle 312| + |132\rangle\langle 213| + |231\rangle\langle 123| + |312\rangle\langle 231| + |321\rangle\langle 132| + |213\rangle\langle 321|.
\end{aligned}$$

It is now clear how to construct $\Gamma^{(r,1_m)}_{\mu_1\mu_2}(\sigma)$ using simplified Young's representation:

$$\begin{aligned}
(12)|123\rangle &= (12)\begin{ytableau} {} & {} & {} & {} & {} & {} & {} & {} & {} & {} & {} & {} & {} & {} & 1 \\ {} & {} & {} & {} & {} & {} & {} & {} & {} & 2 \\ {} & {} & {} & {} & 3 \end{ytableau} = |213\rangle \\
(12)|213\rangle &= |123\rangle \\
(12)|312\rangle &= |132\rangle \\
(12)|132\rangle &= |312\rangle \\
(12)|231\rangle &= |321\rangle \\
(12)|321\rangle &= |231\rangle.
\end{aligned}$$

Writing this as a matrix which acts on $[|123\rangle, |213\rangle, |312\rangle, |132\rangle, |231\rangle, |321\rangle]^T = [|1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle]^T$:

$$\Gamma^{(r,1_m)}_{\mu_1\mu_2}((12)) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{\mu_1\mu_2}.$$

Clearly, substituting this into (C.5) produces the correct combination of $S_n \times (S_1)^m$ projectors for the corresponding Gauss graph. This is convenient - it suggests that there is a simple group theoretic representation of the double coset element that can be used to determine the linear combination of $S_n \times (S_1)^m$ projectors which diagonalize the dilatation operator. As we will see in the next section, however, this is not the correct representation of the coset element once multiple strings appearing on a single row are introduced - the additional symmetrization which occurs for such configurations (resulting from the S_i, $i > 1$ present in H for these cases) requires using a representation in which this is respected.

C.2.2 Construction for $H = S_1 \times S_2$

If we wish to construct a matrix with indices given as the multiplicity indices for this case, it seems that the representation used to construct the matrix must act as follows:

$$\begin{aligned}
(12)|123\rangle &= \frac{1}{2}(|213\rangle + |312\rangle) \\
(12)|213\rangle &= \frac{1}{2}(|123\rangle + |312\rangle) \\
(12)|312\rangle &= \frac{1}{2}(|123\rangle + |213\rangle).
\end{aligned}$$

Clearly, this is not just Young's representation as it acts on $S_n \times (S_1)^m$ states - it would seem that we must think of the permutation as acting on rows of the Young diagram R, so that the result of acting with a permutation produces a sum (multiplied by $\frac{1}{|H|}$) over all the valid states that can be obtained by swapping the box in row 1 with each of the boxes in row 2, and then renumbering the boxes in the same row if necessary. This statement obviously does not constitute a definition for the representation, and we must rethink how the operators are to be constructed.

This representation can be understood in terms of the full set of 6 possible labellings (i.e. including the invalid states), by identifying

$$|132\rangle = |123\rangle \qquad |231\rangle = |213\rangle \qquad |321\rangle = |312\rangle.$$

This seems sensible and intuitive - by working with $H = S_1 \times S_2$, we have symmetrized on two of the boxes - those corresponding to the second two entries in $|abc\rangle$. In this way, the Gauss operator for any H can be constructed using only Young's orthogonal representation on the full set of possible labellings, and then identifying states that are invalid under association with $(r, 1_m)$ with their valid counterparts by implementing the symmetry implied by H.

In the original construction, we built operators by restricting the S_m symmetry used to construct the restricted Schurs to the more constrained symmetry of the double coset group. In writing the double coset operator in terms of $S_n \times (S_1)^m$ operators, we have done the same thing; except this time, we enhance the symmetry of the constituent operators in order to obtain operators on the double coset. It seems logical then to consider operators of a very similar form to those used in the $S_n \times S_m$ restriction - since it can be expected that the analog of the matrix Γ is given by using Young's representation in the way described in the previous section, it should be possible to construct branching coefficients which implement the symmetry enhancement from $S_n \times (S_1)^m$ to that of the double coset.

Bearing in mind the points of the last two paragraphs, the general expression for the Gauss operators in terms of $S_n \times (S_1)^m$ operators should be expected given by:

$$O_{R,r}(\sigma) = |H| \sum_{j,k} \sum_{\mu_1,\mu_2} \Gamma^{1_m}_{jk}(\sigma) B^{1_m \to 1_H}_{j\mu_1} B^{1_m \to 1_H}_{k\mu_2} \chi_{R,(r,1_m)\mu_1\mu_2}(Z,Y). \tag{C.6}$$

Note that the reference to the Z Young diagram r has been dropped from the labels on the coset element matrix representation and branching coefficients - this is due to the fact that we should in fact begin with a representation of $(S_1)^m$, where there is no symmetrization between indices as would be implied by the $(r, 1_m)$ label. This symmetry will be implemented by the branching coefficients. $\Gamma^{1_m}(\sigma)$ is an $m!$ dimensional matrix representation of $(S_1)^m$; it is constructed in the same way as for the $H = (S_1)^3$ case. It is not immediately obvious how this definition arises without having the r label with one box being pulled from each row - in the previous case, where the r label was retained, the representation was really just simplified Young's representation, since we could indeed state that the separation between boxes being removed was of $O(N)$.

We may understand this representation of $(S_1)^m$ in terms of the vector space $V_m^{\otimes m}$; in obtaining the Clebsch-Gordan coefficients for the decomposition of the $S_n \times S_m$ states, we had already implemented the displaced corners approximation - this is what allows for the association of a box in each row with a vector, which transforms in $V_p^{\otimes m}$ under the action of a permutation in a way which commutes with the unitary group action in this space. We were then allowed to associate a particular partially labelled Young diagram with a tensor product of $U(m)$ fundamental states, and hence to compute the Clebsch-Gordan coefficients which provided the transformation to or from the fully assembled S_m irrep states.

The logic followed here has a different starting point; the group in terms of which the objects appearing in the definition of the Gauss operators are defined is only an $(S_1)^m$ (which organises the Y boxes), with no reference to the Z Young diagram which participates in the definition of the Schur polynomial operator.

It therefore does not make sense to define the vector space in the same way - the boxes appearing in a representation of this group are completely free, and there is no way to associate them to a vector based on the row from which they are pulled, as they simply do not exist in association with a larger Young diagram in this setting. There should, however, still be an association with a tensor product of unitary group states. The association should additionally be exactly the same as was used previously for $H = (S_1)^3$, with only a change in the interpretation of the meaning of each factor. We previously thought of the factor appearing in slot 1 as corresponding to the first box removed from the diagram, and the $U(3)$ state appearing there as determining the row from which the box is pulled. Now it seems we must think of this entirely in terms of the boxes, and consider attaching them to a diagram at a later stage.

This implies that we build the representation of $(S_1)^3$ under the assumption that it is subduced from a Young diagram with well-separated rows/columns; or perhaps rather with the knowledge we intend to attach the boxes to such a diagram. In [2], the simplified representation which allows for unitary group tensor product states to be taken as dual to the symmetric group states depends on the distant corners approximation, which requires that we actually have a YD with well-separated corners. This has been put in by hand under the current construction, in the sense that we must assume that the boxes are to be attached to a diagram where the displaced corners approximation applies.

Consider the Young diagram representation of $(S_1)^3$; we can number each of the boxes:

$$\boxed{1} \otimes \boxed{2} \otimes \boxed{3} \quad , \quad \boxed{2} \otimes \boxed{1} \otimes \boxed{3} \quad , \quad \cdots . \tag{C.7}$$

There are obviously $m! = 6$ possible ways of labelling the representation. Each one should correspond to a different tensor product of unitary group states, that is, the interpretation of the tensor product is such that the position in the product corresponds to the position in the tensor product of boxes labelling the representation, and the state which occupies it represents which of the numbered boxes is in that slot. We are invoking Schur-Weyl duality; each of the boxes appearing is considered to be a representation of $U(3)$, so that the numbers in the boxes reference states in the fundamental of this group. When it comes time to attach the boxes in a particular configuration to the giant Young diagram, all that is necessary to recover the previous description is to symmetrize this product in accordance with the number of boxes appearing in each row - this is exactly what the branching coefficients are responsible for. It seems that the action of the double coset element and elements of H should then be thought of as permuting the positions of the states as they appear in the tensor product.

Applying this to the $S_2 \times S_1$ example, we have for the action of the coset element (12):

$$\Gamma^{1_m}((12)) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

The branching coefficients are determined by

$$\sum_\mu B_\mu^{1_m\to 1_H} B_\mu^{1_m\to 1_H} = \frac{1}{2}\left(\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix} + \begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{bmatrix}\right)$$
$$= \frac{1}{2}\begin{bmatrix}1&0&0&1&0&0\\0&1&0&0&1&0\\0&0&1&0&0&1\\1&0&0&1&0&0\\0&1&0&0&1&0\\0&0&1&0&0&1\end{bmatrix}$$

to be

$$B_1^{1_m\to 1_H} = \frac{1}{\sqrt{2}}[1,0,0,1,0,0]^T \qquad B_2^{1_m\to 1_H} = \frac{1}{\sqrt{2}}[0,1,0,0,1,0]^T \qquad B_3^{1_m\to 1_H} = \frac{1}{\sqrt{2}}[0,0,1,0,0,1]^T.$$

As expected, there are 3 branching coefficients, which correspond to the subduction multiplicity of $(S_1)^3$ from $(r, 1_m)$ for the case where two of the boxes are pulled off a single row. Note that the action of the permutation (23) is used when constructing the branching coefficients; the fact that the permutation used for the branching coefficient construction and that of the double coset are always different was also present when writing out the Gauss operator in terms of $S_n \times S_m$ operators. To understand the reason for taking different permutations, we must recall the role of the two groups. The double coset specifies the connections between giants, it is related to the exchanges of factors in the tensor product which produce distinct configurations by swapping factors which will not correspond to boxes in the same row. The group H is the symmetry group of the string endpoints on the giants, it is related to the exchanges of factors which produce identical configurations once the connection to the giants are included. It is therefore expected and necessary that the elements of the group H must be different to the elements of the double coset.

It is simple to check that, by inserting the above into the expression (C.6), the correct combinations of $S_n \times (S_1)^m$ operators are obtained for each of the double coset elements. The contents of this Appendix are by no means the simplest or most elegant method to implement the change of description; it should be considered as an exploration which illustrates some details of how the operators labelled by the two different sets of representations are related. The identity (C.4) provides an excellent starting point for a more refined proof of (C.6); in eq. (4.3) of [10], this relation was used to directly obtain a formula for the rewriting of the Gauss operators in terms of $S_n \times (S_1)^m$ restricted Schurs. The expression obtained in this way is given in terms of δ-functions on the permutations, which produce the same results as (C.6) but avoid the necessity of formulating arguments regarding how the string boxes exist in relation to the representations of the giants.

Appendix D

Explicit Constructions for LLM Solution (Mathematica)

4D GAMMA MATRICES:

```
σ[0] = {{1, 0}, {0, 1}};
σ[1] = {{0, 1}, {1, 0}};
σ[2] = {{0, -ⅈ}, {ⅈ, 0}};
σ[3]  = {{1, 0}, {0, -1}};
"4D WEYL REP GAMMAS:";
γ[0] = ⅈ * KroneckerProduct[σ[1], σ[0]];
γ[1] = KroneckerProduct[σ[2], σ[1]];
γ[2] = KroneckerProduct[σ[2], σ[2]];
γ[3] = KroneckerProduct[σ[2], σ[3]];
```

CHECK 4D CLIFFORD ALGEBRA:

```
met = IdentityMatrix[4];
n = IdentityMatrix[4];
n[[1, 1]] = -1;
l = IdentityMatrix[4];
For[i = 1, i ≤ 4, i++,
  For[j = 1, j ≤ 4, j++, l[[i, j]] = n[[i, j]] * IdentityMatrix[4]]];
For[i = 1, i ≤ 4, i++,
  For[j = 1, j ≤ 4, j++, m = γ[i - 1].γ[j - 1] + γ[j - 1].γ[i - 1];
   met[[i, j]] = m ]];
met // MatrixForm
met == 2 * l
```

$$\begin{pmatrix} \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \end{pmatrix}$$

```
True
```

4D Clifford Algebra Satisfied ✓✓

4D CHIRALITY MATRIX:

```
γ[5] = i * γ[0].γ[1].γ[2].γ[3];
"γ5 = "
γ[5] // MatrixForm
```

γ5 =

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

10D GAMMA MATRICES:

```
Γ[0] = KroneckerProduct[γ[0], σ[0], σ[0], σ[0]];
Γ[1] = KroneckerProduct[γ[1], σ[0], σ[0], σ[0]];
Γ[2] = KroneckerProduct[γ[2], σ[0], σ[0], σ[0]];
Γ[3] = KroneckerProduct[γ[3], σ[0], σ[0], σ[0]];
Γ[4] = KroneckerProduct[γ[5], σ[0], σ[1], σ[2]];
Γ[5] = KroneckerProduct[γ[5], σ[0], σ[2], σ[2]];
Γ[6] = KroneckerProduct[γ[5], σ[0], σ[3], σ[2]];
Γ[7] = KroneckerProduct[γ[5], σ[1], σ[0], σ[1]];
Γ[8] = KroneckerProduct[γ[5], σ[2], σ[0], σ[1]];
Γ[9] = KroneckerProduct[γ[5], σ[3], σ[0], σ[1]];

Γ[0] == Transpose[Γ[0]]
Γ[1] == Transpose[Γ[1]]
Γ[2] == - Transpose[Γ[2]]
Γ[3] == Transpose[Γ[3]]
```

```
True
False
False
False
```

THE GAMMAS CONSTRUCTED DO NOT RESPECT THE CONVENTIONS FOR SYMMETRICITY DESCRIBED IN FOOTNOTE 27 (P .34) OF LIN, LUNIN, MALDACENA (LLM) - THESE CONVENTIONS SEEM ONLY TO BE SATISFIED USING A REPRESENTATION WHICH DOES NOT DIAGONALIZE THE CHIRALITY MATRIX Γ_{11}. USING SUCH A REP COMPLICATES LATER CALCULATION; IN PARTICULAR, THE CONSTRUCTION OF THE 4D SUB-SPINORS ϵ REQUIRED TO REPRODUCE A .45 IS LESS OBVIOUS.

10D CHIRALITY MATRICES:

```
G[5] = i * Γ[0].Γ[1].Γ[2].Γ[3];
G[5] // MatrixForm
G[5].KroneckerProduct[σ[0], σ[0], σ[0], σ[0], σ[3]] // MatrixForm
Γ[11] = Γ[0].Γ[1].Γ[2].Γ[3].Γ[4].Γ[5].Γ[6].Γ[7].Γ[8].Γ[9];
Γ[11] // MatrixForm
```

$\gamma^5 =$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{pmatrix}$$

$\gamma^5\ \hat{\sigma}_3 =$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Γ_{11} =

$$\left(\begin{array}{*{32}{c}}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

$$\Gamma_{11} = \gamma^5\, \hat{\sigma}_3 \ \checkmark\,\checkmark$$

STANDARD SU(2) COMMUTATION ($[\sigma_i, \sigma_j] = 2\, i\epsilon_{ijk}\, \sigma_k$):

```
σ[1].σ[2] - σ[2].σ[1] == 2 i * σ[3] // MatrixForm
σ[1].σ[3] - σ[3].σ[1] == -2 i * σ[2] // MatrixForm
σ[2].σ[3] - σ[3].σ[2] == 2 i * σ[1] // MatrixForm
```

```
True
True
True
```

NOTE THAT THE ORDER OF THE SPHERE Γ PRODUCTS IN Γ_{11} HAS BEEN REVERSED COMPARED TO THE EXPRESSION IN LLM; THE PRODUCT OVER Γ_a APPEARS IN FRONT OF $\Gamma_{\tilde{a}}$ THERE. THIS LEADS TO $\Gamma_{11} = -\gamma^5\, \hat{\sigma}_3$, UNLESS ONE MODIFIES THE PAULI MATRICES SUCH THAT THEY SATISFY $[\sigma_a, \sigma_b] = -2\, i\epsilon_{abc}\, \sigma_c = 2\, i\epsilon_{bac}\, \sigma_c$, IN WHICH CASE $\Gamma_{11} = \gamma^5\, \hat{\sigma}_3$ WITH THE ORDER OF PRODUCTS AS IN LLM.

THIS MODIFICATION JUST CHANGES THE CORRELATION BETWEEN THE SO(4) CHIRALITY (ENCODED IN a,b) AND THE REPS OF THE SU(2) SUBGROUPS (SO(4) ↔ SU(2)×SU(2) : (+,-) ↔ $\left(\frac{1}{2}, 0\right) / \left(0, \frac{1}{2}\right)$; a=±/∓1).

THE RESULT OF THIS FOR THE GEOMETRIC MAGNON DESCRIPTION IS THAT POSITIVE ANGLES IN THE PHASE ARE ASSOCIATED WITH EITHER CLOCKWISE / COUNTERCLOCKWISE MEASUREMENT.

CHECK 10D CLIFFORD ALGEBRA:

```
met = IdentityMatrix[10];
n = IdentityMatrix[10];
n[[1, 1]] = -1;
l = IdentityMatrix[10];
For[i = 1, i <= 10, i++,
  For[j = 1, j <= 10, j++, l[[i, j]] = n[[i, j]] * IdentityMatrix[32]]];
For[i = 1, i <= 10, i++, For[j = 1, j <= 10, j++, m = Γ[i - 1].Γ[j - 1] + Γ[j - 1].Γ[i - 1];
   met[[i, j]] = m]];
met // MatrixForm;
met == 2 * l
True
```

10D CLIFFORD ALGEBRA SATISFIED ✓✓

SPINOR CONSTRUCTION:

```
Clear[i];
Ψ = ConstantArray[0, {32, 1}];
For[i = 1, i ≤ 32, i++, Ψ[[i, 1]] = ψ_i];

η = 1/2 (IdentityMatrix[32] + Γ[11]).Ψ
ϵ = ConstantArray[0, {32, 1}];
ϵ = 1/2 (IdentityMatrix[32] + G[5]).η /. ψ → ε

For[i = 8, i ≤ 32, i++, ϵ[[i, 1]] = -0];
Transpose[ϵ] // MatrixForm
```

$\eta = P_+ \Psi =$

```
{{ψ1}, {0}, {ψ3}, {0}, {ψ5}, {0}, {ψ7}, {0}, {ψ9}, {0},
 {ψ11}, {0}, {ψ13}, {0}, {ψ15}, {0}, {0}, {ψ18}, {0}, {ψ20}, {0},
 {ψ22}, {0}, {ψ24}, {0}, {ψ26}, {0}, {ψ28}, {0}, {ψ30}, {0}, {ψ32}}
```

ϵ is defined as transforming under an effectively 4D subgroup with the same Chirality as η:

```
{{ε1}, {0}, {ε3}, {0}, {ε5}, {0}, {ε7}, {0}, {ε9}, {0}, {ε11}, {0}, {ε13}, {0}, {ε15},
 {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}
```

If ϵ is Majorana-Weyl, we should take only the first half as corresponding to the independent components of the 4D spinor:

```
( ε1 0 ε3 0 ε5 0 ε7 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 )
```

CALCULATE ω_μ :

```
For[i = 0, i ≤ 9, i++,
 ω_i =
  FullSimplify[
   e^(1/2 (H+G)) *
    Transpose[ϵ].
     Transpose[
      MatrixExp[i * δ * G[5].Γ[3].KroneckerProduct[σ[0], σ[0], σ[0], σ[0], σ[1]]]].
     Γ[2].Γ[i].
     MatrixExp[i * δ * G[5].Γ[3].KroneckerProduct[σ[0], σ[0], σ[0], σ[0], σ[1]]].ϵ]]
For[i = 0, i ≤ 9, i++, Print[ω_i]]
```

ω_μ:

{{0}}

$\left\{\left\{-i\, e^{\frac{G+H}{2}} \operatorname{Cosh}[2\delta] \left(\varepsilon_1^2+\varepsilon_3^2+\varepsilon_5^2+\varepsilon_7^2\right)\right\}\right\}$

$\left\{\left\{e^{\frac{G+H}{2}} \operatorname{Cosh}[2\delta] \left(\varepsilon_1^2+\varepsilon_3^2+\varepsilon_5^2+\varepsilon_7^2\right)\right\}\right\}$

{{0}}

{{0}}

{{0}}

{{0}}

{{0}}

{{0}}

{{0}}

$$e^{\frac{G+H}{2}} \operatorname{Cosh}[2\delta] = e^{\frac{G+H}{2}} \sqrt{1+a^2 e^{-2G}} = \sqrt{e^H \left(e^G + a^2 e^{-G}\right)} = \sqrt{2 e^H \operatorname{Cosh}[G]} = h^{-1} \left(a^2 = 1\right)$$

$$-i e^{\frac{G+H}{2}} \operatorname{Cosh}[2\delta] =$$

$$-i e^{\frac{G+H}{2}} \sqrt{1+a^2 e^{-2G}} = -i a \sqrt{e^H \left(\frac{e^G}{a^2} + e^{-G}\right)} = -i a \sqrt{2 e^H \operatorname{Cosh}[G]} = -i a h^{-1} \left(a^2 = 1\right)$$

THIS REPRODUCES (A .45) OF LLM.

USING PAULI MATRICES SATISFYING THE NEGATIVE COMMUTATION RELATION (OR REVERSING THE ORDER OF $\prod \Gamma_a$ and $\prod \Gamma_{\tilde{a}}$ IN Γ_{11}), WHICH GIVE GAMMA MATRICES SATISFYING $\Gamma_{11} = -\gamma^5 \hat{\sigma}_3$, WE HAVE

$$\omega_1 = i\, e^{\frac{G+H}{2}} \operatorname{Cosh}[2\delta]\, \varepsilon_0{}^t\, \varepsilon_0$$

THE SIGN OF a SHOULD BE REVERSED IF THIS CONVENTION IS USED, WHICH RECOVERS THE ABOVE EXPRESSION.

ω COEFFICIENTS ON y=0 PLANE:

ON $y = 0$ PLANE, $e^H = 0 \Rightarrow H \to -\infty$; FOR SPHERE (BLACK) DROPLET,

$z = \frac{-1}{2} = \frac{1}{2}\tanh(G) \Rightarrow G \to -\infty$. h^{-1} on this plane is thus $= \lim_{H,G\to-\infty}\sqrt{2\,e^H\,\mathrm{Cosh}[G]} =$

$$\text{Limit}\left[\sqrt{2\frac{\text{Cosh}[-x]}{e^x}},\ x\to\infty\right]$$

1

In fact, since $\mathrm{Cosh}[\pm\infty] = +\infty$, this is the value of h^{-1} over the whole $y = 0$ plane.

$$\text{Limit}\left[\sqrt{2\frac{\text{Cosh}[x]}{e^x}},\ x\to\infty\right]$$

1

$$\omega_1 = \omega_1\ /.\ e^{\frac{G+H}{2}}\ \text{Cosh}[2\,\delta]\to a$$

$$\omega_2 = \omega_2\ /.\ e^{\frac{G+H}{2}}\ \text{Cosh}[2\,\delta]\to 1$$

$$\{\{-i\,a\,(\varepsilon_1^2+\varepsilon_3^2+\varepsilon_5^2+\varepsilon_7^2)\}\}$$

$$\{\{\varepsilon_1^2+\varepsilon_3^2+\varepsilon_5^2+\varepsilon_7^2\}\}$$

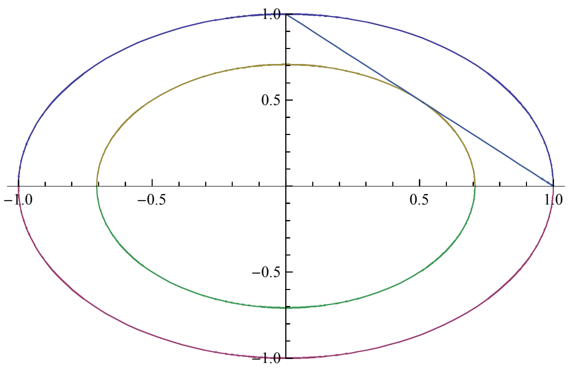
PHASE ACQUIRED BY OPEN STRING UNDER GAUGE TRANSFORMATION:

By the fact that only the ω_i are non – zero, the 2D metric in these directions is flat, and the configuration is not time dependent, the integral can be written as simply $\int\omega_1\,dx^1 + \int\omega_2\,dx^2$, where this is evaluated from the initial point to the final point of the magnon as it is drawn on the LLM 1-2 plane (More details in body of dissertation):

```
Clear[i];
Plot[{√(1 - x^2), -√(1 - x^2), √(0.7071067811865476^2 - x^2), -√(0.7071067811865476^2 - x^2),
  -x + 1}, {x, -1, 1}, PlotRange → 1]
```

$$\text{FullSimplify}\left[\int_{x_1^i}^{x_1^f}\omega_1\,dx1+\int_{x_2^i}^{x_2^f}\omega_2\,dx2\right]$$

$$\text{FullSimplify}\left[\int_{1}^{0.5}\omega_1\,dx1+\int_{0}^{0.5}\omega_2\,dx2\right]$$



$$\{\{(-i\,a\,(x_1^f-x_1^i)+x_2^f-x_2^i)\,(\varepsilon_1^2+\varepsilon_3^2+\varepsilon_5^2+\varepsilon_7^2)\}\}$$

$$\{\{(0.5+(0.+0.5\,i)\,a)\,(\varepsilon_1^2+\varepsilon_3^2+\varepsilon_5^2+\varepsilon_7^2)\}\}$$

COMPARISON TO VECTOR LENGTH IN 1-2, r-θ COORDINATES:

```
Abs[0.5 + 0.5 ⅈ]
N[Abs[0.7071067811865476 ⅇ^(ⅈ*π/4) - 1]]
```

```
0.707107
0.707107
```

$$\sqrt{(r_i - r_f)^2 + 4\, r_f\, \mathrm{Sin}^2\left[\frac{\theta}{2}\right]}\ :$$

$$\sqrt{(1 - 0.7071067811865476)^2 + 4\,(0.7071067811865476)\ \mathrm{Sin}[\pi/8]\text{^}2}$$

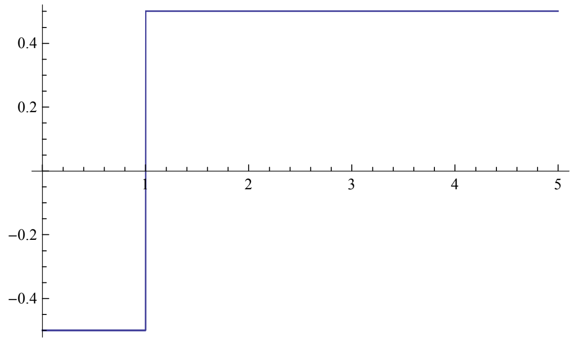
```
0.707107
```

✓✓

$AdS_5 \times S_5$ SOLUTION:

$$z[r\_,\ y\_,\ r0\_] = \frac{r^2 - r0\text{^}2 + y^2}{2\sqrt{(r^2 + r0\text{^}2 + y^2)^2 - 4\, r^2\, r0^2}};$$

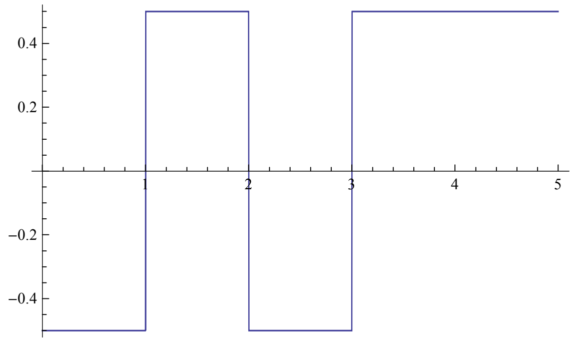
```
Plot[z[r, 0, 1], {r, 0, 5}]
```



CONCENTRIC RINGS SOLUTION:

$$z[r\_,\ y\_,\ r0\_] = \frac{r^2 - r0\text{^}2 + y^2}{2\sqrt{(r^2 + r0\text{^}2 + y^2)^2 - 4\, r^2\, r0^2}};$$

```
Plot[z[r, 0, 1] - z[r, 0, 2] + z[r, 0, 3], {r, 0, 5}]
```



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