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Master of Science by Dissertation:
HOLOGRAPHIC DESCRIPTIONS OF CFT SCATTERING

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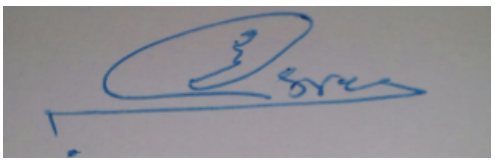
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Declaration

I declare that this dissertation is my own, unaided work. It is submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.



Esra Mohammed Shrif 14 February 2017

Abstract

The holographic computation of extremal correlators is often frustrated by divergences. The interpretation of these divergences is incomplete. The primary goal of this study is to develop a better understanding of these divergences. Towards this end, working within the AdS/CFT correspondence we review the computation of correlators. In the field theory we review well known matrix model techniques useful to study the planar limit, as well as methods exploiting group representation theory that are useful for the computation of correlators in large N but non-planar limits. On the gravity side of the correspondence, we describe in detail the computation of two- and three point functions of a scalar field coupled to gravity on the Euclidian AdS_{d+1} space, three-point functions of two giant gravitons and one pointlike graviton as well as correlators of Kaluza-Klein gravitons. A key observation of this study is that extremal correlators are mapped to scattering amplitudes of particles with parallel momenta. These are naturally accompanied by involve collinear divergences. Therefore, we suggest that the divergences in the computation of extremal correlators are linked to collinear divergences. A lot more work is needed to establish this connection.

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1. Introduction

Correlation functions are the basic observables of interest in quantum field theory (QFT) and in string theory. They are often divergent. These divergences must be interpreted before the correlation functions can be computed. Further, computing correlators is usually only possible at weak coupling. The holographic principle [1], as realized in the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [2], [3] and [4], computes correlators at strong coupling in the CFT using weakly coupled string theory. However, the weak coupling string theory computations still have divergences. The goal of this dissertation is to take the first steps towards a general understanding of the subtleties arising as a result of these divergences, in the holographic computations of extremal correlation functions. These divergences were found for extremal correlators involving giant gravitons and Kaluza-Klein gravitons.

We start this dissertation by reviewing the relevant background materials that we will use to compute correlators in matrix field theories. First we introduce, in Chapter 2, the dynamics of matrix models with one Hermitian $N \times N$ matrix. We introduce a number of different methods to compute correlators. First, the generating function for correlators is defined in the free matrix model. This can be evaluated exactly by performing a Gaussian integral and it forms the foundation for weak coupling computations. Next, we show how the Schwinger-Dyson equations can be used to compute correlators. The next thing we consider is a weak coupling expansion of correlators, using Feynman diagrams. These diagrams are called ribbon graphs in matrix models and they naturally link the matrix model with a topological structure and ultimately, with the dynamics of strings. The basic idea is simply that ribbon graphs triangulate a surface and the N dependence of the Feynman diagram is determined by a topological invariant associated with the surface, called the Euler characteristic. The connection to string theory arises because we can interpret the surface that is triangulate by the ribbon graph as the worldsheet of a string. Developing this connection, we argue that the large N limit of the matrix model corresponds to the classical limit of the string theory.

In the limit that $N \to \infty$, the only contribution to correlators of operators built using order $O(1)$ fields comes from the planar diagrams. We can also consider operators constructed with a number of fields that scale as order N as we take the limit. In this case, the number of ribbon graphs explodes and correlation functions of this class of operators are not well approximated by the planar limit. Indeed, huge combinatoric factors over power the usual $\frac{1}{N^2}$ suppression and higher genus ribbon graphs can no longer be neglected. We call this a large N but non-planar limit. To make progress in this limit, a new approach must be developed. The new approach that we pursue makes use of the group representation theory of the symmetric group, the unitary group and their relation.

In Chapter 3, we review some basic concepts and the relevant techniques from group representation theory, that will allow us to solve the problems that were encountered in large N but non-planar limits. A basis for the local gauge invariant operators of the theory are called Schur polynomials operators. These operators will dramatically simplify the computation of correlators. The construction of these operators makes use of projection operators in group theory. To develop these ideas, we describe the matrix representation of the symmetric group and of the unitary group. The extension of the theory to multi-matrix models is also possible. Towards this end, we introduce the Restricted Schur Polynomials.

In the later chapters of this dissertation, making use of the mathematical background we have developed, we study different types of correlators to get some insight into the divergences that appear

in extremal correlators. Since we are matching gauge theory computations and the string theory computations, we start by demonstrating how the correlation functions in gauge theory can be recovered from supergravity computations using holography. This is achieved in the simplest setting in chapter 4 and the basic holographic dictionary is introduced. We describe the computation of two- and three-point functions of a scalar field coupled to gravity on the Euclidian AdS_{d+1} space. We demonstrate that the computation of the two-point function from the supergravity path integral is subtle. It contains divergences that must be regularized and the answer obtained depends on the details that go into this regularization. As a clear demonstration of this point, we compute the two-point function [5], [6] in position space and in momentum space and show that one obtains different answers, because they have a different regularization prescriptions. The correct result can be determined by employing a Ward identity, which relates the two point function (which suffers from divergences) to a three point function (which does not). In this way we obtain a result from the gravitational computation that is in prefect agreement with the gauge theory result. It is pleasing that the gravity computation on the Euclidean AdS_{d+1} space recovers the spacetime dependence required by conformal invariance in the gauge theory in d-dimensions.

In Chapter 5, we consider operators that have a scaling dimension of the order of the gauge parameter N. The threepoint function of giant gravitons is computed both in the gauge theory and in the string theory. We employ Schur Polynomials to carry out the computation on the gauge theory side [7], and we consider the variation of the D3-brane action on the gravity side, following [8], [9] and [10]. Here again, we find that there are divergences in the computation of correlators using holography. An apparent mismatch between results from the gravity side and the gauge theory side was reported, but this subtlety can be traced back to the divergence that arises as a consequence of the fact that we are considering extremal correlators. We have not managed to develop an argument that employs a Ward identity in this case. However, the mismatch can be resolved by the regularization introduced by Lin [11]. This regularization procedure is motivated by showing it gives the gauge theory result. It is desirable to have a better understanding of the divergences and how they arise, since this may allow one to choose the correct regularization without comparing to the gauge theory.

In Chapter 6, we consider the three-point function of Kaluza-Klein gravitons, of the 10-dimensional supergravity. These states are dual to chiral primary operators in the field theory [12], [13]. We perform a dimensional reduction of the 10-dimensional theory using harmonic expansions. Extremal and non-extremal correlators are then computed. The computation of the extremal three-point function is carried out using two different methods. First, we perform the analytic continuation from the non-extremal correlator [12]. The result obtained from analytic continuation agrees with the field theory result, but again, we can not justify why this is the correct prescription. Second, we regulate the divergences with a spacetime cutoff [12]. The result that is obtained depends on the details of the regulator. These results demonstrate that the divergence that appears in the computation of extremal correlators is a general feature of the theory. The discussion of the associated divergences deserves a general explanation.

In Chapter 7, we explore the idea that the divergences that appear in extremal correlators are related to collinear divergences. This proposal is motivated by the connection between $\mathcal{R}$-charge of operators in gauge theory and the angular momentum of the dual particle states in string theory. This connection is potentially very useful since, as we explain, the cancellation of IR- and collinear-divergences is known to be achieved by summing over degenerate states. The same idea may be useful for understanding the divergences in extremal correlators. We conclude this work in Chapter 8.

Appendices A and B show how to derive the Ward identity used in Chapter 4 and discuss the integrals of spherical harmonics considered in Chapter 6.

2. Matrix Models in 0-dimension

We study the Anti-de Sitter/Conformal Field Theory (AdS/CFT) duality, which claims that $\mathcal{N} = 4$ super Yang-Mills (SYM) theory is equivalent to string theory on the $AdS_5 \times S^5$ background in negatively curved spacetime [2], [3], [5]. With this motivation we start by studying matrix models in zero dimensions, which provide a nice toy model for $\mathcal{N} = 4$ super Yang-Mills. In fact, many aspects that need to be realized in $\mathcal{N} = 4$ SYM theory can be formulated as questions in a zero dimensional matrix model. In this chapter we explain how to construct the correlation functions of gauge invariant operators and construct the large N limit of a matrix model [14] ,[15]. This is the key object to develop our understanding for what string theory is.

We start by discussing the path integral formalism to compute correlation functions. Then we reproduce these correlators using the Schwinger-Dyson equations to make the discussion more concrete. Next we develop the techniques which allow us to introduce Feynman diagrams to simplify the computations of correlators. These diagrams are called ribbon graphs and naturally link the matrix model with a topological structure. We will simply state the rules of these diagrams and then demonstrate their use. Further we introduce large N factorization and the 't Hooft expansion as a classical limit [16]. Finally we conclude our discussion in this chapter with a summary of our interpretation for the string theory matrix/model relation.

2.1 Correlation Functions

Correlation functions of fields are the natural objects to study in the path integral formulation. They are extremely important because they contain all the physical information about the dynamics of the system and have a simple expansion in terms of Feynman diagrams in the free theory.

To compute the correlation function it is useful to introduce the generating functional which takes the following form for a free scalar field

$$Z = \int [d\phi]\, e^{iS + \int d^4x J(x)\phi(x)}. \tag{2.1.1}$$

The functional derivative, $\frac{\delta}{\delta J(y)} J(x)$, can be defined as follows

$$\frac{\delta}{\delta J(y)} J(x) = \delta(x-y) \qquad or \qquad \frac{\delta}{\delta J(y)} \int d^4x J(x)\phi(x) = \phi(y). \tag{2.1.2}$$

This implies

$$\frac{\delta}{\delta J(y)} Z = \int [d\phi] e^{iS + \int d^4x J(x)\phi(x)} \phi(y). \tag{2.1.3}$$

Then the correlation function is simply defined as follows

$$\langle 0 |\, T\big(\phi(x_1) \dots \phi(x_n)\big)\, | 0 \rangle = \int [d\phi]\, e^{iS} \phi(x_1) \dots \phi(x_n) \tag{2.1.4}$$

$$= \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J]\big|_{J=0}. \tag{2.1.5}$$

Our goal in this section is to motivate a connection between Yang-Mills theory and string theory. We will choose the simplest setting in which this argument can be made: matrix models in zero dimension. To do this we will modify our discussion in three important ways:

i. Move to Euclidean space which replaces $iS \to -S$. With this modification (2.1.1) becomes

$$Z = \int [d\phi] e^{-S + \int d^4x J(x)\phi(x)}. \tag{2.1.6}$$

ii. Consider a matrix valued field. In this case we modify $[d\phi] \to [dM]$ where M is an $N \times N$ hermitian matrix so that $M_{ij} = M^*_{ji} \quad i, j = 1, \ldots, N$. We mean by the measure $[dM]$ that the integral is over all possible hermitian matrices. Thus, we need to integrate over all possible diagonal elements m_{ii}, giving N real integrals. We also need to integrate over the real parts $\frac{m^r_{ij}}{\sqrt{2}}$ and the imaginary parts $\frac{m^i_{ij}}{\sqrt{2}}$ of the matrix elements above the diagonal, giving $\frac{1}{2}N(N-1)$ integrals for each. In total, this is N^2 real integrals. So, we have

$$[dM] = \prod_i^N dm_{ii} \prod_{i<j}^N dm^r_{ij} dm^i_{ij} \qquad \text{and} \qquad Tr(M^2) = \sum_i M^2_{ii} + \sum_{i<j} \left((M^r_{ij})^2 + (M^i_{ij})^2\right). \tag{2.1.7}$$

iii. Move to 0-dimension so that the universe has a single event (a point in spacetime), implying that ϕ takes a single value. In this situation the path integral over a field becomes a usual integral.

With these three modifications, the generating functional for a free theory is given by the integral

$$Z[J] = \int [dM]\, e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right) + \mathrm{Tr}(JM)}. \tag{2.1.8}$$

By using this generating function and the functional derivative

$$\frac{dZ[J]}{dJ_{ij}} = \int [dM]\, e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right) + \mathrm{Tr}(JM)} M_{ji}, \tag{2.1.9}$$

the correlation function in our particular case takes the following form

$$\langle M_{ij} M_{kl} \ldots M_{xy} \rangle = \int [dM]\, e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} M_{ij} M_{kl} \ldots M_{xy} \tag{2.1.10}$$

$$= \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} \cdots \frac{d}{dJ_{yx}} Z[J]\Big|_{J=0}. \tag{2.1.11}$$

To fix the normalization of the measure, we set

$$Z[J]\big|_{J=0} = \int [dM]\, e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} = \langle 1 \rangle = 1. \tag{2.1.12}$$

We need to evaluate the generating function $Z[J]$ to compute correlators. Correlators are then obtained by taking derivatives of $Z[J]$ and setting $J = 0$. We complete the square to do this

integral

$$\begin{aligned}\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)-\mathrm{Tr}\left(JM\right) &= \frac{\omega}{2}\mathrm{Tr}\left(M^2-\frac{2}{\omega}JM\right)\\ &= \frac{\omega}{2}\mathrm{Tr}\left(M^2-\frac{JM}{\omega}-\frac{JM}{\omega}\right)\\ &= \frac{\omega}{2}\mathrm{Tr}\left(M^2-\frac{JM}{\omega}-\frac{JM}{\omega}+\frac{J^2}{\omega}-\frac{J^2}{\omega}\right)\\ &= \frac{\omega}{2}\mathrm{Tr}\left(\left[M-\frac{J}{\omega}\right]^2\right)-\frac{1}{2\omega}\mathrm{Tr}\left(J^2\right).\end{aligned}$$

The integral takes the simpler form

$$Z[J]=\int [dM]\, e^{-\frac{\omega}{2}\mathrm{Tr}\left(\left[M-\frac{J}{\omega}\right]^2\right)+\frac{1}{2\omega}\mathrm{Tr}\left(J^2\right)}.$$

We introduce a new integration variable $M'=M-\frac{J}{\omega}$, where J is a fixed constant matrix and ω is a number. Therefore $[dM]=[dM']$, and

$$\begin{aligned}Z[J] &= e^{\frac{1}{2\omega}\mathrm{Tr}\left(J^2\right)}\int \left[dM'\right] e^{-\frac{\omega}{2}\mathrm{Tr}\left(M'^2\right)}\\ \Longrightarrow Z[J] &= e^{\frac{1}{2\omega}\mathrm{Tr}\left(J^2\right)}.\end{aligned} \tag{2.1.13}$$

To compute correlators using the generating function it is useful to bear in mind following the properties of derivatives

$$\begin{aligned}\frac{d}{dJ_{ij}}e^{\mathrm{Tr}(JM)} &= \left[\frac{d}{dJ_{ij}}\mathrm{Tr}\left(JM\right)\right]e^{\mathrm{Tr}(JM)}\\ &= M_{ji}e^{\mathrm{Tr}(JM)},\end{aligned} \tag{2.1.14}$$

and

$$\frac{d}{dJ_{ij}}\mathrm{Tr}\left(J^2\right)=2J_{ji}\,. \tag{2.1.15}$$

2.2 Computation of Correlators using the Generating Function

i. $\langle M_{ij}M_{kl}\rangle$:

$$\begin{aligned}\langle M_{ij}M_{kl}\rangle &= \frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}Z[J]\Big|_{J=0}\\ &= \frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}e^{\frac{1}{2\omega}\mathrm{Tr}\left(J^2\right)}\Big|_{J=0}\\ &= \frac{d}{dJ_{ji}}\left(\frac{1}{\omega}J_{kl}e^{\frac{1}{2\omega}\mathrm{Tr}\left(J^2\right)}\right)\Big|_{J=0}\\ &= \frac{1}{\omega}\left(\delta_{il}\delta_{jk}+\frac{1}{\omega}J_{kl}J_{ij}\right)e^{\frac{1}{2\omega}\mathrm{Tr}\left(J^2\right)}\Big|_{J=0}\\ &= \frac{1}{\omega}\delta_{il}\delta_{jk}\,.\end{aligned}$$

ii. $\langle M_{ij}M_{kl}M_{mn}\rangle$:

$$\langle M_{ij}M_{kl}M_{mn}\rangle = \frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}\frac{d}{dJ_{nm}}e^{\frac{1}{2\omega}\mathrm{Tr}\left(J^2\right)}\Big|_{J=0}.$$

We use the result of the differentiations in the previous example to obtain

$$\begin{aligned}
&= \frac{1}{\omega}\frac{d}{dJ_{nm}}\left[\left(\delta_{il}\delta_{jk}+\frac{1}{\omega}J_{kl}J_{ij}\right)e^{\frac{1}{2\omega}\mathrm{Tr}\left(J^2\right)}\right]\Big|_{J=0}\\
&= \frac{1}{\omega^2}\left(\delta_{il}\delta_{jk}J_{mn}+\delta_{nk}\delta_{ml}J_{ij}+J_{kl}\delta_{ni}\delta_{mj}+\frac{1}{\omega}J_{mn}J_{kl}J_{ij}\right)e^{\frac{1}{2\omega}\mathrm{Tr}\left(J^2\right)}\Big|_{J=0}.
\end{aligned}$$

When $J=0$, all the terms vanish, so we have

$$\langle M_{ij}M_{kl}M_{mn}\rangle = 0\,. \tag{2.2.1}$$

iii. $\langle M_{ij}M_{kl}M_{mn}M_{pq}\rangle$:

By using the same procedure and the results of the previous calculations we find

$$\begin{aligned}
\langle M_{ij}M_{kl}M_{mn}M_{pq}\rangle &= \frac{1}{\omega^2}\frac{d}{dJ_{qp}}\left[\left(\delta_{il}\delta_{jk}J_{mn}+\delta_{nk}\delta_{ml}J_{ij}+J_{kl}\delta_{ni}\delta_{mj}+\frac{1}{\omega}J_{mn}J_{kl}J_{ij}\right)e^{\frac{1}{2\omega}\mathrm{Tr}\left(J^2\right)}\right]\Big|_{J=0}\\
&= \frac{1}{\omega^2}\Big(\delta_{qm}\delta_{pn}\delta_{il}\delta_{jk}+\delta_{qi}\delta_{pj}\delta_{nk}\delta_{ml}+\delta_{qk}\delta_{pl}\delta_{ni}\delta_{mj}\Big).
\end{aligned} \tag{2.2.2}$$

The basic gauge invariant observables of the zero dimensional model are given as traces of products of M. We will compute three examples of gauge invariant observables $\left\langle \mathrm{Tr}\left(M^2\right)\right\rangle$, $\left\langle \mathrm{Tr}\left(M\right)^2\right\rangle$ and $\left\langle \mathrm{Tr}\left(M^4\right)\right\rangle$ using the results found above. We have

i. $\left\langle \mathrm{Tr}\left(M\right)^2\right\rangle$:

$$\begin{aligned}
\left\langle \mathrm{Tr}\left(M\right)^2\right\rangle &= \langle M_{ii}M_{jj}\rangle = \frac{1}{\omega}\delta_{ij}\delta_{ij}\\
&= \frac{1}{\omega}\delta_{jj}\\
&= \frac{1}{\omega}N\,.
\end{aligned} \tag{2.2.3}$$

ii. $\left\langle \mathrm{Tr}\left(M^2\right)\right\rangle$:

$$\begin{aligned}
\left\langle \mathrm{Tr}\left(M^2\right)\right\rangle &= \langle M_{ij}M_{ji}\rangle\\
&= \frac{1}{\omega}\delta_{ii}\delta_{jj}\\
&= \frac{1}{\omega}N^2\,.
\end{aligned} \tag{2.2.4}$$

iii. $\langle \mathrm{Tr}\left(M^4\right)\rangle$:

$$\begin{aligned}\langle \mathrm{Tr}\left(M^4\right)\rangle &= \frac{1}{\omega^2}\left(\delta_{il}\delta_{jj}\delta_{il}\delta_{kk} + \delta_{kj}\delta_{ll}\delta_{ii}\delta_{kj} + \delta_{ij}\delta_{kl}\delta_{ki}\delta_{lj}\right)\\ &= \frac{1}{\omega^2}\left(\delta_{il}\delta_{il}N^2 + \delta_{kj}\delta_{kj}N^2 + \delta_{ij}\delta_{kl}\delta_{ki}\delta_{lj}\right)\\ &= \frac{1}{\omega^2}\left(N^3 + N^3 + \delta_{kl}\delta_{kj}\delta_{lj}\right)\\ &= \frac{1}{\omega^2}\left(2N^3 + \delta_{lj}\delta_{lj}\right)\\ &= \frac{1}{\omega^2}\left(2N^3 + N\right). \end{aligned} \tag{2.2.5}$$

In these calculations the factors of N come from performing the sum $\delta_{ii} = \sum_{i=1}^{N} = N$. We also notice that the answer for the correlator is just a polynomial in N.

2.3 Computation of Correlators using Schwinger-Dyson Equations

The Schwinger-Dyson equation is determined from the statement that the path integral measure $[dM]$ is invariant under the deformation $M'_{ij} \to M_{ij} + \delta M_{ij}$. Consider the correlator

$$\langle \mathcal{O}\rangle = \int [dM]\, e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)}\mathcal{O} = \int [dM]\, F(M), \tag{2.3.1}$$

where $\mathcal{O}$ is an arbitrary observable and

$$F(M) = e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)}\mathcal{O}. \tag{2.3.2}$$

Now, let us change variables to $M'_{ij} = M_{ij} + \delta M_{ij}$. The measure is unchanged under this transformation, so that $[dM] = [dM']$. Therefore

$$\langle \mathcal{O}\rangle = \int \left[dM'\right] F(M' - \delta M)\,. \tag{2.3.3}$$

If we take δM to be very small we only need the first order in δM when we expand $F(M' - \delta M)$ as a Taylor series. In this way we find

$$\begin{aligned}\langle \mathcal{O}\rangle &= \int \left[dM'\right]\left[F(M') - \delta M_{ij}\frac{\partial F(M')}{\partial M'_{ij}}\right]\\ &= \int \left[dM'\right] F(M') - \delta M_{ij}\int \left[dM'\right]\frac{\partial F(M')}{\partial M'_{ij}}\\ &= \langle \mathcal{O}\rangle - \delta M_{ij}\int \left[dM'\right]\frac{\partial F(M')}{\partial M'_{ij}}\\ \implies 0 &= \int \left[dM'\right]\frac{\partial F(M')}{\partial M'_{ij}}\,. \end{aligned} \tag{2.3.4}$$

The expression in (2.3.4) is the Schwinger-Dyson equation.

We will work out a few Schwinger-Dyson equations which will be useful for computing correlators. From

$$\begin{aligned}
0 &= \int [dM] \frac{\partial}{\partial M_{ij}} \left[M_{ij} e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[\frac{\partial M_{ij}}{\partial M_{ij}} - \frac{\omega}{2} \frac{\partial}{\partial M_{ij}} \mathrm{Tr}\left(M^2\right) \right] e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \\
&= \int [dM] \left[N^2 - \omega \mathrm{Tr}\left(M^2\right) \right] e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} ,
\end{aligned} \tag{2.3.5}$$

we find that $\left\langle \mathrm{Tr}\left(M^2\right) \right\rangle = \frac{1}{\omega} N^2$ in agreement with what we found in (2.2.4). We give examples of more correlators to cement the logic of this method

- $\left\langle \mathrm{Tr}\left(M^4\right) \right\rangle$

$$\begin{aligned}
0 &= \int [dM] \frac{\partial}{\partial M_{ij}} \left[(M^3)_{ij} e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[2N \mathrm{Tr}\left(M^2\right) + \mathrm{Tr}\left(M\right)^2 - \omega \mathrm{Tr}\left(M^4\right) \right] e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \\
\implies \left\langle \mathrm{Tr}\left(M^4\right) \right\rangle &= \frac{1}{\omega} \left[2N \left\langle \mathrm{Tr}\left(M^2\right) \right\rangle + \left\langle \mathrm{Tr}\left(M\right)^2 \right\rangle \right] \\
&= \frac{1}{\omega^2} [2N^3 + N] .
\end{aligned} \tag{2.3.6}$$

- $\left\langle \mathrm{Tr}\left(M^2\right) \mathrm{Tr}\left(M^2\right) \right\rangle$

$$\begin{aligned}
0 &= \int [dM] \frac{\partial}{\partial M_{ij}} \left[\mathrm{Tr}\left(M^2\right) M_{ij} e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[\mathrm{Tr}\left(M^2\right) \left(-\omega M_{ij} M_{ji} + \delta_{ii}\delta_{jj} \right) + 2 M_{ij} M_{ji} \right] e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \\
&= \int [dM] \left[-\omega \mathrm{Tr}\left(M^2\right) \mathrm{Tr}\left(M^2\right) + N^2 \mathrm{Tr}\left(M^2\right) + 2\mathrm{Tr}\left(M^2\right) \right] e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \\
\implies \left\langle \mathrm{Tr}\left(M^2\right) \mathrm{Tr}\left(M^2\right) \right\rangle &= \frac{1}{\omega} \left[N^2 \left\langle \mathrm{Tr}\left(M^2\right) \right\rangle + 2 \left\langle \mathrm{Tr}\left(M^2\right) \right\rangle \right] \\
&= \frac{1}{\omega^2} [N^4 + 2N^2] .
\end{aligned} \tag{2.3.7}$$

- $\left\langle \mathrm{Tr}\left(M^2\right) \mathrm{Tr}\left(M^4\right) \right\rangle$

$$
\begin{aligned}
0 &= \int [dM] \frac{\partial}{\partial M_{ij}} \left[\mathrm{Tr}\left(M^4\right) M_{ij} e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[\mathrm{Tr}\left(M^4\right) \left(-\omega M_{ij} M_{ji} + \delta_{ii}\delta_{jj}\right) + 4\mathrm{Tr}\left(M^4\right) \right] e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \\
&= \int [dM] \left[-\omega \mathrm{Tr}\left(M^4\right) \mathrm{Tr}\left(M^2\right) + N^2 \mathrm{Tr}\left(M^4\right) + 4\mathrm{Tr}\left(M^4\right) \right] e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \\
\implies \left\langle \mathrm{Tr}\left(M^2\right) \mathrm{Tr}\left(M^4\right) \right\rangle &= \frac{1}{\omega} \left[N^2 \left\langle \mathrm{Tr}\left(M^4\right) \right\rangle + 4 \left\langle \mathrm{Tr}\left(M^4\right) \right\rangle \right] \\
&= \frac{1}{\omega^3} [2N^5 + 9N^3 + 4N] \,. \qquad (2.3.8)
\end{aligned}
$$

- $\left\langle \mathrm{Tr}\left(M^6\right) \right\rangle$

$$
\begin{aligned}
0 &= \int [dM] \frac{\partial}{\partial M_{ij}} \left[(M^5)_{ij} e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[2N \mathrm{Tr}\left(M^4\right) + \mathrm{Tr}\left(M^2\right)^2 + 2\mathrm{Tr}\left(M\right) \mathrm{Tr}\left(M^3\right) - \omega \mathrm{Tr}\left(M^6\right) \right] e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \\
\implies \left\langle \mathrm{Tr}\left(M^6\right) \right\rangle &= \frac{1}{\omega} \left[2N \left\langle \mathrm{Tr}\left(M^4\right) \right\rangle + \left\langle \mathrm{Tr}\left(M^2\right)^2 \right\rangle + 2 \left\langle \mathrm{Tr}\left(M\right) \mathrm{Tr}\left(M^3\right) \right\rangle \right] . \qquad (2.3.9)
\end{aligned}
$$

We need to compute $\left\langle \mathrm{Tr}\left(M\right) \mathrm{Tr}\left(M^3\right) \right\rangle$

$$
\begin{aligned}
0 &= \int [dM] \frac{\partial}{\partial M_{ii}} \left[\mathrm{Tr}\left(M^3\right) e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[-\omega \mathrm{Tr}\left(M^3\right) M_{ii} + 3\mathrm{Tr}\left(M^2\right) \right] e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} \\
&= \int [dM] \left[-\omega \mathrm{Tr}\left(M^3\right) \mathrm{Tr}\left(M\right) + 3\mathrm{Tr}\left(M^2\right) \right] e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)} .
\end{aligned}
$$

$$
\implies \left\langle \mathrm{Tr}\left(M^3\right) \mathrm{Tr}\left(M\right) \right\rangle = \frac{1}{\omega} \left\langle 3\mathrm{Tr}\left(M^2\right) \right\rangle = \frac{3}{\omega^2} N^2 \,, \qquad (2.3.10)
$$

$$
\implies \left\langle \mathrm{Tr}\left(M^6\right) \right\rangle = \frac{1}{\omega^3} [5N^4 + 10N^2] \,, \qquad (2.3.11)
$$

which are the correct results.

2.4 Computation of Correlators using Ribbon Graphs

Ribbon graphs are Feynman diagrams that are used to simplify the computation of the correlators in matrix models. To draw the ribbon graphs that must summed to reproduce a given correlator, we use the following rules

- Each matrix element M_{ij} becomes a pair of dots, one labeled with i and one labeled with j.
- Place the labeled pairs of dots on a line and join pairs of pairs of dots with a pair of lines (ribbon), without crossing the line on which pairs live.
- Each ribbon edge becomes a Kronecker delta with indices given by the line's endpoint labels and each ribbon becomes $\frac{1}{\omega}$.

As an example lets repeat the computation of $\langle M_{ij}M_{kl}M_{mn}M_{pq}\rangle$ using the ribbon graph method. All possible ribbon graphs for this correlator are shown in the Figure (2.1). Summing the diagrams we find

$$\langle M_{ij}M_{kl}M_{mn}M_{pq}\rangle = \frac{1}{\omega^2}\big(\delta_{il}\delta_{jk}\delta_{mq}\delta_{np} + \delta_{qi}\delta_{pj}\delta_{nk}\delta_{ml} + \delta_{ni}\delta_{mj}\delta_{qk}\delta_{pl}\big). \tag{2.4.1}$$

This is exactly in agreement with what we found in (2.2.2).

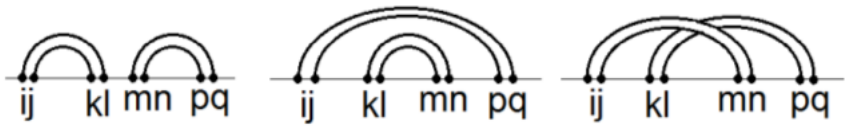


Figure 2.1: There are three ribbbon graphs contributing to (2.4.1).

For gauge invariant observables which are given by traces of products of M, the answer for the correlators are just polynomials in N. In this case we do not need to record matrix indices, so we can streamline our ribbon graph notation. In the new streamlined notation, we link dots that are labeled by the same index with a solid line. Repeated indices are summed, they don't take a value. In this situation, a closed solid loop translates into a sum over an index giving a factor of N and each ribbon gives a factor of $\frac{1}{\omega}$.

Lets reproduce the values for $\langle \mathrm{Tr}\,(M^2)\rangle$, $\left\langle \mathrm{Tr}\,(M)^2\right\rangle$ and $\langle \mathrm{Tr}\,(M^4)\rangle$ using the new streamlined ribbon graph notation. The results and the ribbon graphs of this computation of correlators are shown in the Figure (2.2).

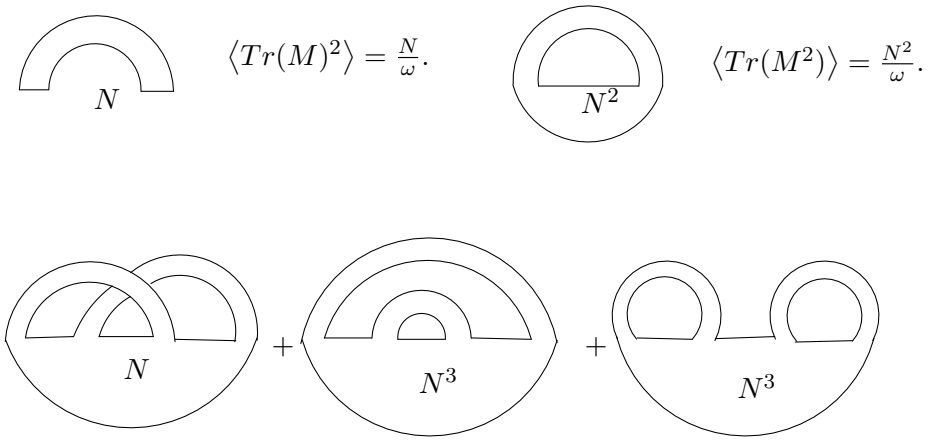
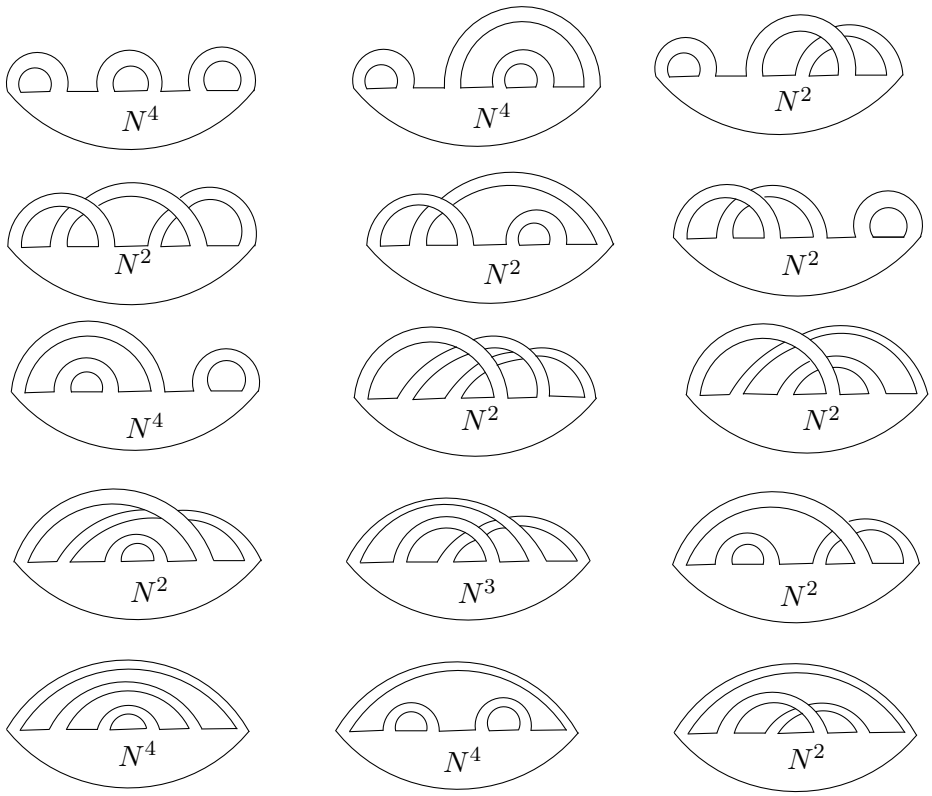


Figure 2.2: The new streamlined ribbon graph notation contributing to $\langle Tr(M)^2\rangle$, $\langle Tr(M^2)\rangle$ and $\langle Tr(M^4)\rangle$.



$$\langle Tr(M^6)\rangle = \tfrac{1}{\omega^3}(5N^4 + 10N^2).$$

Figure 2.3: The new streamlined ribbon graph notation contributing to $\langle Tr(M^6)\rangle$

2.4.1 The Interacting Theory. Our discussion so far has been for the theory with a quadratic action $\frac{1}{2}\mathrm{Tr}\left(M^2\right)$. To obtain an interacting model we can add a term quartic in M. The interacting theory is defined by

$$Z[J] = \int [dM] e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right) - g\mathrm{Tr}\left(M^4\right) + \mathrm{Tr}(JM)} \tag{2.4.2}$$

. When $g = 0$ (the free theory), we have $Z(J = 0) = 1$ and when $g \neq 0, Z(J = 0) \neq 1$. To compute the generating function (2.4.2), we can do a perturbative expansion in the parameter g. We obtain

$$\begin{aligned} Z[J] &= \int [dM] \sum_{n=0}^{\infty} \frac{1}{n!} (-g\mathrm{Tr}\left(M^4\right))^n e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right) + \mathrm{Tr}(JM)} \\ &= \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int [dM] (\mathrm{Tr}\left(M^4\right))^n e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right) + \mathrm{Tr}(JM)} \\ &= \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int [dM] \left(\frac{d}{dJ_{ab}} \frac{d}{dJ_{bc}} \frac{d}{dJ_{cd}} \frac{d}{dJ_{da}}\right)^n e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right) + \mathrm{Tr}(JM)} . \end{aligned} \tag{2.4.3}$$

To evaluate (2.4.3) we need to complete the square and change variables to do the Gaussian integral

$$\begin{aligned} Z[J] &= \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \left(\frac{d}{dJ_{ab}} \frac{d}{dJ_{bc}} \frac{d}{dJ_{cd}} \frac{d}{dJ_{da}} \right)^n e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)} \int [dM'] e^{-\frac{\omega}{2}\mathrm{Tr}(M'^2)} \\ &= \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \left(\frac{d}{dJ_{ab}} \frac{d}{dJ_{bc}} \frac{d}{dJ_{cd}} \frac{d}{dJ_{da}} \right)^n e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)} . \end{aligned} \tag{2.4.4}$$

We compute $Z(J=0)$ to order $O(g)$ using the computation of $\left\langle \mathrm{Tr}\left(M^4\right)\right\rangle$ described above.

$$\begin{aligned} Z(J=0) &= \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \left\langle (\mathrm{Tr}\left(M^4\right))^n \right\rangle \\ &= \frac{1}{\omega^2}\left[1 - g(2N^3+N)\right] + \frac{g^2}{2\omega^4}[4N^6 + 40N^4 + 61N^2] + O(g^3) . \end{aligned} \tag{2.4.5}$$

In the free theory, we normalized the generating function of correlators so that $Z(J=0)=1$. We have deserted this convention in the study of the interacting model and indeed, as (2.4.5) shows, $Z(J=0) \neq 1$. We find it convenient to change our normalization conventions, so that our new generating function $\tilde{Z}(J)$, obeys $\tilde{Z}(J=0)=1$. This is achieved by setting

$$\tilde{Z}[J] = \frac{Z[J]}{Z(J=0)} . \tag{2.4.6}$$

We will refer to $\tilde{Z}[J]$ as the normalized generating function. Now we compute the correlator $\left\langle \mathrm{Tr}\left(M^2\right)\right\rangle$ to $O(g)$ using $Z(J)$

$$\begin{aligned} \left\langle \mathrm{Tr}\left(M^2\right)\right\rangle &= \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} Z[J]\big|_{J=0} \\ &= \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} \left[\left(1 - g \frac{d}{dJ_{ab}} \frac{d}{dJ_{bc}} \frac{d}{dJ_{cd}} \frac{d}{dJ_{da}} \right) e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)} \right]_{J=0} \\ &= \left[1 - g\left(\frac{2N^3}{\omega^2} + \frac{N}{\omega^2} \right) \right] \frac{N^2}{\omega} - g\frac{8N^3}{\omega^3} - g\frac{4N}{\omega^2} \end{aligned} \tag{2.4.7}$$

Now, using the normalized generating function, we find

$$\begin{aligned} \left\langle \mathrm{Tr}\left(M^2\right)\right\rangle_{normalized} &= \frac{1}{Z(J=0)} \left(\left[1 - g\left(\frac{2N^3}{\omega^2} + \frac{N}{\omega^2} \right) \right] \frac{N^2}{\omega} - g\frac{8N^3}{\omega^3} - g\frac{4N}{\omega^2} \right) \\ &= \frac{N^2}{\omega} - g\frac{8N^3}{\omega^3} - g\frac{4N}{\omega^2} \end{aligned} \tag{2.4.8}$$

To check these answers we need to extend our ribbon graph rules.

2.4.2 The New Ribbon Graphs Rules.

- Each matrix element M_{ij} becomes a pair of dots. Indices that are summed are connected by a line.
- To compute the order g^n contribution to the correlator, include n vertices that allow four ribbons to meet at a point, as shown in the Figure below.
- Join pairs of dots, as well as the open ends of the vertices, with ribbon's without twisting the ribbon. Each ribbon contributes a facter of $\frac{1}{\omega}$.

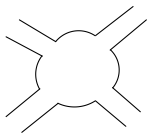


Figure 2.4: Ribbon vertex: four ribbons meet in loop.

- Each diagram becomes a power of N, with the power equal to the number of closed lines in the graph.

Lets use the above ribbon graph rules to repeat the computation of $\langle \mathrm{Tr}\left(M^2\right)\rangle$. The ribbon graphs are shown in Figure (2.5).

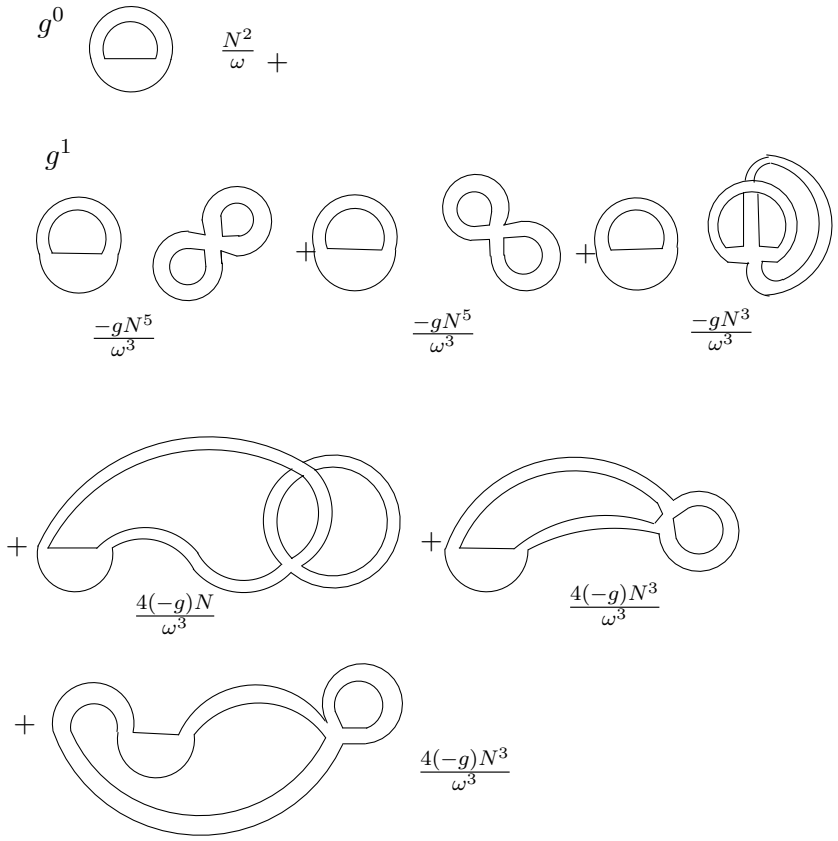


Figure 2.5: The new ribbon graphs contributing to $\langle \mathrm{Tr}\left(M^2\right)\rangle$.

To obtain the normalized correlator, you must drop all terms that have a vacuum graph component, that is, which are not connected to any pair of dots representing matrix elements. We have

$$\langle \mathrm{Tr}\left(M^2\right)\rangle_{normalized} = \frac{N^2}{\omega} - g\frac{8N^3}{\omega^3} - g\frac{4N}{\omega^2}\,. \tag{2.4.9}$$

This example shows that the ribbon graph language is useful.

2.5 Factorization and the Large N Limit as a Classical Limit

In the free theory, we have done the computation of several correlators, including

$$\begin{aligned}
\left\langle \mathrm{Tr}\left(M^2\right)\right\rangle &= \frac{1}{\omega}N^2\,,\\
\left\langle \mathrm{Tr}\left(M^4\right)\right\rangle &= \frac{1}{\omega^2}[2N^3+N]\,,\\
\left\langle \mathrm{Tr}\left(M^2\right)\mathrm{Tr}\left(M^2\right)\right\rangle &= \frac{1}{\omega^2}[N^4+2N^2]\,,\\
\left\langle \mathrm{Tr}\left(M^2\right)\mathrm{Tr}\left(M^4\right)\right\rangle &= \frac{1}{\omega^3}[2N^5+9N^3+4N]\,,\\
\left\langle \mathrm{Tr}\left(M^4\right)\mathrm{Tr}\left(M^4\right)\right\rangle &= \frac{1}{\omega^3}[4N^6+40N^4+61N^2]\,.
\end{aligned}$$

Using these results we will now point out a very important property of these correlators. Consider the $N\longrightarrow\infty$ limit. It is simple to verify that, after ignoring the sub-leading terms, we find

$$\begin{aligned}
\left\langle \mathrm{Tr}\left(M^4\right)\right\rangle &= \frac{2N^3}{\omega^2}\,,\\
\left\langle \mathrm{Tr}\left(M^2\right)\mathrm{Tr}\left(M^2\right)\right\rangle &= \frac{N^4}{\omega^2} = \left\langle \mathrm{Tr}\left(M^2\right)\right\rangle\left\langle \mathrm{Tr}\left(M^2\right)\right\rangle\,,\\
\left\langle \mathrm{Tr}\left(M^2\right)\mathrm{Tr}\left(M^4\right)\right\rangle &= \frac{2N^5}{\omega^3} = \left\langle \mathrm{Tr}\left(M^2\right)\right\rangle\left\langle \mathrm{Tr}\left(M^4\right)\right\rangle\,,\\
\left\langle \mathrm{Tr}\left(M^4\right)\mathrm{Tr}\left(M^4\right)\right\rangle &= \frac{4N^6}{\omega^4} = \left\langle \mathrm{Tr}\left(M^4\right)\right\rangle\left\langle \mathrm{Tr}\left(M^4\right)\right\rangle\,.
\end{aligned}$$

It is a general result that, at large N, we have

$$\langle\prod_i \mathrm{Tr}\left(M^{n_i}\right)\rangle = \prod_i \langle \mathrm{Tr}\left(M^{n_i}\right)\rangle\,. \tag{2.5.1}$$

This says that the expectation value of the product is the product of expectation values. This property, called factorization, is valid for the large N limit of any matrix model. This is the important property of these correlators that we wanted to point out. We will now develop a physical interpretion of factorization.

Consider a system which can be in a number of states labeled by i. The probability for the system to be in state i is μ_i, such that

$$\sum_i \mu_i = 1 \qquad \mu_i \geq 0 \quad \forall i.$$

Our theory will have a set of observables $\mathcal{O}_{\mathcal{I}}$. The value of $\mathcal{O}_{\mathcal{I}}$ in state i is denoted $\mathcal{O}_{\mathcal{I}}(i)$. The expectation value of $\mathcal{O}_I$ is given by

$$\langle\mathcal{O}_{\mathcal{I}}\rangle = \sum_i \mu_i \mathcal{O}_{\mathcal{I}}(i).$$

In terms of this language, factorization says

$$\begin{aligned}
\langle\mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_n\rangle &= \sum_i \mu_i \mathcal{O}_1(i)\mathcal{O}_2(i)\ldots\mathcal{O}_n(i)\\
&= \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle\ldots\langle\mathcal{O}_n\rangle\\
&= \sum_{i_1}\mu_{i_1}\mathcal{O}_1(i_1)\sum_{i_2}\mu_{i_2}\mathcal{O}_2(i_2)\cdots\sum_{i_n}\mu_{i_n}\mathcal{O}_n(i_n)
\end{aligned}$$

This implies

$$\mu_i = \begin{cases} 1 & for \quad i = i^* \\ 0 & for \quad i \neq i^* \end{cases}$$

Thus factorization is an indication that, in the large N limit, only a single configuration, which we call the classical configuration contributes to the path integral.

The AdS/CFT correspondence is a concrete realization of this idea. The correspondence tells us exactly what the classical configuration that determines the large N correlation functions is. The large N limit of $\mathcal{N} = 4\, SYM$ theory is given by the classical limit of IIB string theory on $AdS_5 \times S^5$. $\hbar$ of the string theory is set by $\frac{1}{N^2}$ of the matrix model.

2.6 The 't Hooft Limit

We have seen that the correlators of the free field theory exhibit factorization In this section we would like to consider the correlators of the interacting theory, to see if factorization survives when interactions are turned on.

If we look at (2.4.5), with each increasing power of g we have an increasing power of N. The expansion looks doomed. We can rescue the perturbative expansion if we consider the double scaling limit in which we scale N and g as follows

$$N \longrightarrow \infty \qquad g \longrightarrow 0 \qquad \lambda = gN = \text{ fixed} \tag{2.6.1}$$

λ, called the 't Hooft coupling, can be fixed to be small, so that we can do a perturbative expansion in λ. The general form of the expansion, for any observable $\mathcal{O}$ is

$$\langle \mathcal{O} \rangle = \sum_{n=0}^{\infty} f_n(\lambda) N^{2-2n}. \tag{2.6.2}$$

We get a contribution at every order in perturbation theory, i.e. the term of order N^2 is a non trivial function of λ. There is also a term of order N^0 and terms of order N^{-2n} with $n = 1, 2, 3, \ldots$. Ribbon diagrams have been an extremely useful tool with which we can compute correlators. The true power of ribbon graphs however, only becomes evident when we understand how they determine the N dependence we see in correlators. To determine the N dependence of each ribbon diagram, it proves useful to rescale our field variable. Instead of M, we will work with the field $M = \sqrt{N} M'$. The generating function of correlators is.

$$Z[J] = \int [dM'] e^{-N\left(\frac{\omega}{2}\mathrm{Tr}\left(M'^2\right) + \lambda \mathrm{Tr}\left(M'^4\right)\right) + \mathrm{Tr}(J'M')}. \tag{2.6.3}$$

For any given Feynman diagram, the ribbon rules that follow from this generating function have a factor of $N\lambda$ for each vertex, a factor of N^{-1} for each ribbon and a factor of N for each closed loop. We denote the numbers of ribbons, vertices and the number of closed loops by E, V and F respectively. A given Feynman diagram will contribute

$$N^{V-E+F} \lambda^V .$$

We will now give the quantity $\chi = V - E + F$ (called the Euler characteristic) a topological interpretation. In this way we will see that the ribbon graphs can be viewed as a triangulation of

a surface. Each closed loop can be viewed as the boundary of a piece of rubber. These pieces are then glued together to form the surface. For an example, see Figure (2.6).



Figure 2.6: The ribbon graph that is a triangulation of the sphere.

The pieces of rubber are the gray shaded regions in this Figure. The outer sheet may be considered an infinite sheet, so that gluing the pieces we have above produces the infinite plane. The ribbon graph therefore performs a triangulation of a two dimensional surface. For anther example see Figure (2.7).



Figure 2.7: Ribbon graph together with the surface it triangulates.

The number of closed loops in the ribbon graph is the number of faces in the triangulation. The number of ribbons in the ribbon graph is the number of edges in the triangulation that the sheets of rubber are joined on. The number of vertices in the ribbon graph is the number of vertices in the triangulation.

Every two dimensional oriented surface is topologically equivalent to a sphere with some number of holes cut out of it or some number of handles stuck onto it. For example, a torus is a sphere with one handle and a cylinder is a sphere with two holes. The Euler characteristic of the surface is

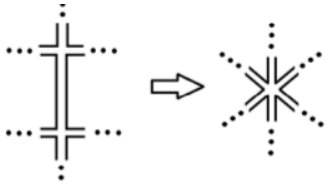
$$\chi = 2 - 2H - B\,, \tag{2.6.4}$$

where H is the number of handles glued onto the sphere and B is the number of boundaries (the holes cut out of the sphere). The quantity $F-E+V$ is a topological invariant, that is, it is invariant under smooth deformations of the two dimensional surface that the ribbon graph triangulates. To prove that we consider first a deformation that shrinks an edge to nothing, as shown in Figure (2.8a).

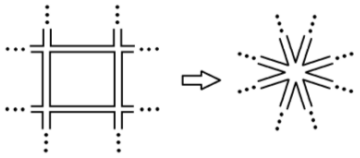
It is clear that the number of faces, edges and vertices (F, E, V) before the shrink are related to the number of faces, edges and vertices (F', E', V') after the shrink by

$$F' - E' + V' = F - (E-1) + (V-1) = F - E + V\,. \tag{2.6.5}$$

Then consider a deformation given by shrinking a face to nothing, as shown in Figure (2.8b). From this figure we see that under the operation of shrinking the face we have



(a) Shrinking an edge to nothing.



(b) Shrinking a face to nothing.

Figure 2.8: Smooth deformation in the triangulation.

$$F' - E' + V' = (F-1) - (E-4) + (V-3) = F - E + V\,. \tag{2.6.6}$$

In general assume there are m edges bounding a face. We will lose all the edges bounding the face when we shrink the face to nothing, so that $E' = E - m$ and $V' = V - m + 1$ (the m vertices between edges get grouped into one vertex). Thus,

$$F' - E' + V' = (F-1) - (E-m) + (V-m+1) = F - E + V\,. \tag{2.6.7}$$

From (2.6.5) and (2.6.7) it is easy to see that χ is invariant under shrinking an edge or shrinking a face to nothing. This would imply that χ is also invariant under growing an edge or growing a face out of nothing.

Now we introduce the ribbon diagrams which always come with a power of N^χ. In Figure (2.8), the sphere triangulated by the ribbon graph which has three closed loops ($F = 3$), one vertex ($V = 1$; the vertex is the point where 4 ribbons meet) and two edges $E = 2$. The sphere has no handles and no holes in it so that $\chi = 2$. Planar diagrams for the sphere come with N dependence $N^\chi = N^2$

If we cut a hole in the surface, we remove a face. Under this operation, F is decreased by 1 and E, V are changed by the same amount. This implies that each time we add a handle, we lose 2, because we cut two holes and glue the resulting boundaries. For the torus for example

$$\chi_{torus} = \chi_{sphere} - 2 = 0,$$

so that the diagrams that triangulate a torus multiply N^0 .

To summarize our discussion, we have developed ribbon graphs as a method to compute correlators in matrix models. We have seen that these ribbon graphs triangulate a two dimensional surface and further, that the topology of the underlying two dimensional surface determines the power of N of the ribbon graph. The fact that summing ribbon graphs clearly has something to do with summing over surfaces suggests that perhaps this matrix model is equivalent to a string theory. Further, the large N limit of the matrix model is related to the classical limit of the string theory.

3. Group Representation Theory

In the previous chapter we have reviewed matrix models and ribbon graphs. They provide useful techniques for studying correlation functions of operators built using order 1 fields. In the limit that $N \longrightarrow \infty$ the theory simplifies, because only planar diagrams need to be summed. Now however, we want to study correlators of operators built from order N fields. In this case the ribbon graph techniques are no longer useful because we need to sum much more than just the planar diagrams. Therefore, in this chapter we will explain that group representation theory can be employed to study this class of correlation functions. We review the relevant aspects of group representation theory that we will need in this study. These include the group representation theory of the symmetric group and the unitary group. We also review techniques include Schur polynomials and the restricted Schur polynomials that are labeled by irreducible representation of the symmetric group. We will see in later chapters that the language of group representation theory provides a very useful way to study giant gravitons.

3.1 Group Definition and Axioms

A group is a set $\mathcal{G}$, together with a map "$\cdot$" (called the group composition law) that combines any two elements to form another element, such that the following axioms are satisfied [17], [18]

- The group composition law is closed (*i.e.* $g_1 \cdot g_2 \in \mathcal{G} \quad \forall g_1, g_2 \in \mathcal{G}$).
- The group composition law is associative, $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3) \quad \forall g_1, g_2, g_3 \in \mathcal{G}$.
- There is an identity element $e \in \mathcal{G}$ which obeys, $e \cdot g_1 = g_1 \cdot e = g_1 \quad \forall g_1 \in \mathcal{G}$.
- For every $g \in \mathcal{G}$ there exists an inverse element of g denoted $g^{-1} \in \mathcal{G}$, such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

The number of elements in the set $\mathcal{G}$ is called the order of the group and is usually denoted $|\mathcal{G}|$. If the order of a group is finite, we say we have a finite group.

3.2 Permutations

A permutation rearranges the members of a set. The number of permutations of n distinct objects is $n!$. The set of permutations of n objects together with the composition law obtained by performing two given rearrangements in succession, is a group known as the symmetric group S_n. We can think of permutations of a set $S = \{1, 2, \cdots n\}$ as the bijections from S to itself. In this case, permutations are functions and so, can be composed with each other. We need some notation to denote the permutations which are elemenst of the group S_n. Consider S_4. One way of writing a permutation σ is

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{bmatrix} . \tag{3.2.1}$$

As a function $\sigma(1) = 3, \quad \sigma(2) = 1, \quad \sigma(3) = 2, \quad \sigma(4) = 4.$

A much better notation is given by the so called cycle notation. It expresses the permutation as a product of cycles which are the orbits of the permutation. For the example given in equation

(3.2.1) we have $\sigma = (132)(4)$. This permutation is composed of 2 orbits, each called a cycle. An orbit of length k is called a $k-$cycle. The 1-cycles are fixed points of the permutation. To simplify cycle notation, we do not write the 1-cycles explicitly.

The product of two permutations is given by using the first row of the first permutation and the second row of the "aligned" second permutation. For example

$$\sigma_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 5 & 6 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{bmatrix},$$

so that

$$\sigma_1\sigma_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 5 & 3 & 1 & 6 \end{bmatrix}. \tag{3.2.2}$$

In cycle notation $\sigma_1 = (1243)$, $\sigma_2 = (15)(24)$ and $\sigma_1\sigma_2 = (1435)$. This called the left action of a group. In the right action, the product $\sigma_1\sigma_2$ can be obtained by applying the rightmost permutation σ_2 first. Therefore, using cycle notation and the right action of group we have $\sigma_1\sigma_2 = (1523)$.

The inverse of a permutation $\sigma(x) = y$ obeys $\sigma^{-1}(y) = x$. In cycle notation, reverse the order of the elements in each cycle. Thus, $\sigma_1 = (1243)$ and $\sigma^{-1} = (3421)$.

A permutation composed of 1-cycles and a single 2-cycle is called a transposition. If the 2-cycle is of the form $(i, i+1)$, the permutation is called an adjacent transposition. Every permutation can be written as a product of transpositions. This rewriting is not unique. In fact, it is always possible to write any permutation as a product of adjacent permutations. For example, in cycle notation

$$(123) = (12)(23) = (14)(42)(23)(14) = \cdots$$

3.3 Representations of the Symmetric Group

3.3.1 Matrix Representations. A matrix representation of a symmetric group is a map from the elements of the symmetric group to some matrix algebra, such that this map preserves the structure of the group. In general, we can write

$$\Gamma(\cdot) : \mathcal{G} \longrightarrow GL(n, \mathbb{R}) \qquad \text{or} \qquad \Gamma(\cdot) : \mathcal{G} \longrightarrow GL(n, \mathbb{C})$$

where $GL(n, \mathbb{R})$ is the general linear group of $(n \times n)$ real matrices and $GL(n, \mathbb{C})$ is the general linear group of $(n \times n)$ complex matrices. The map must respect the group composition law

$$\Gamma(g_1)\Gamma(g_2) = \Gamma(g_1 \cdot g_2) \quad \forall g_1, g_2 \in \mathcal{G}\,. \tag{3.3.1}$$

In this equation, the product on the right hand side is the group composition for permutations while on the left hand side it is matrix multiplication. From (3.3.1) we can prove that the identity element of the group maps into the identity matrix. Since $e \cdot e = e$ we know that

$$\begin{aligned} \Gamma_R(e)\Gamma_R(e) =& \Gamma_R(e) \\ \Longrightarrow \Gamma_R(e)\Gamma_R(e) - \Gamma_R(e) =& \Gamma_R(e)(\Gamma_R(e) - 1) = 0 \end{aligned} \tag{3.3.2}$$

which since $\Gamma_R(e)$ is invertible, implies that $\Gamma_R(e) = 1$. Note also that

$$\Gamma_R(g)\Gamma_R(g^{-1}) = \Gamma_R(g \cdot g^{-1}) = \Gamma_R(e) = 1\,. \tag{3.3.3}$$

This implies that the matrix representing the element g^{-1} is the matrix inverse of the matrix representing g

$$\Gamma(g^{-1}) = \Gamma(g)^{-1}\,. \tag{3.3.4}$$

If $\Gamma_R(g_1) = \Gamma_R(g_2) \implies g_1 = g_2$ (i.e. the map from the abstract group to the matrix group is one-to-one), we say the representation is faithful.

3.3.2 Equivalent Representations. There are an infinite number of representations of any group $\mathcal{G}$. Given a representation, that is a set of matrices $\Gamma_R(g)$ obeying [19]

$$\Gamma_R(g_1)\Gamma_R(g_2) = \Gamma_R(g_1 \cdot g_2) \quad \forall g_1, g_2 \in \mathcal{G}\,, \tag{3.3.5}$$

we can construct another representation

$$\tilde{\Gamma}_R(g) = M^{-1}\Gamma_R(g)M \quad \forall g \in \mathcal{G}\,, \tag{3.3.6}$$

where M is any invertible matrix. Indeed, it is clear that

$$\begin{aligned}\tilde{\Gamma}_R(g_1)\tilde{\Gamma}_R(g_2) =& M^{-1}\Gamma_R(g_1)MM^{-1}\Gamma_R(g_2)M\\ =& M^{-1}\Gamma_R(g_1)\Gamma_R(g_2)M\\ =& M^{-1}\Gamma_R(g_1g_2)M\\ =& \tilde{\Gamma}_R(g_1g_2)\,.\end{aligned} \tag{3.3.7}$$

When two representations $\Gamma_R(\cdot)$ and $\tilde{\Gamma}_R(\cdot)$ are related as in (3.3.6), we say the two representations are equivalent. There are an infinite number of different invertible matrices M. This means there are an infinite number of different representations equivalent to $\Gamma_R(\cdot)$. We will only list the set of inequivalent representations of the group. For (3.3.6) to hold, it is necessary that $\Gamma_R(\cdot)$ and $\tilde{\Gamma}_R(\cdot)$ have the same eigenvalues. If $|v\rangle$ is an eigenvector of $\Gamma_R(g)$ with eigenvalue λ

$$\Gamma_R(g)\Gamma\,|v\rangle = \lambda\,|v\rangle \tag{3.3.8}$$

then

$$\tilde{\Gamma}_R(g)(M\,|v\rangle) = M\,\Gamma_R(g)\,M^{-1}M\,|v\rangle = M\,\Gamma_R(g)\,|v\rangle = \lambda(M\,|v\rangle)\,. \tag{3.3.9}$$

Thus, the eigenvalues $\Gamma_R(g)$ and $\tilde{\Gamma}_R(g)$ are equal. Further, if $\Gamma_R(g)$ has eigenvector $|v\rangle$ then $M\,|v\rangle$ is an eigenvector of $\tilde{\Gamma}_R(g)$.

If $\Gamma_R(\cdot)$ and $\tilde{\Gamma}_R(\cdot)$ are large matrices, computing their eigenvalues is not easy. For a more efficient but still sufficient condition, we consider traces of $\Gamma_R(\cdot)$. It is a simple operation to compute the trace of a matrix. Denote the d eigenvalues of $\Gamma_R(\cdot)$ by $\lambda_i \quad i = 1, \cdots d$. We can write

$$\mathrm{Tr}\,(\Gamma_R(g)^n) = \sum_{i=1}^{d} \lambda_i^n\,. \tag{3.3.10}$$

Thus, instead of computing the eigenvalues and requiring that they agree we can rather require

$$\mathrm{Tr}\big(\Gamma_R(g)^n\big) = \mathrm{Tr}\big(\tilde{\Gamma}_R(g)^n\big) \quad n = 1, \cdots, d \tag{3.3.11}$$

using the fact that we have a representation of the group and so we may freely use as the group composition law, we can write

$$\begin{aligned}\Gamma_R(g_1)\Gamma_R(g_2)\Gamma_R(g_3)\cdots =&\Gamma_R(g_1\cdot g_2\cdot g_3\cdots)\,,\\ \tilde{\Gamma}_R(g_1)\tilde{\Gamma}_R(g_2)\tilde{\Gamma}_R(g_3)\cdots =&\tilde{\Gamma}_R(g_1\cdot g_2\cdot g_3\cdots)\,.\end{aligned}$$

Since any product of matrices in the representation is another matrix in the representation

$$\mathrm{Tr}\left(\Gamma_R(g)\right)=\mathrm{Tr}\left(\tilde{\Gamma}_R(g)\right)\quad \forall g\in\mathcal{G}\,, \tag{3.3.12}$$

is a necessary and sufficient condition for two representations $\Gamma_R(\cdot)$ and $\tilde{\Gamma}_R(\cdot)$ to be equivalent. The trace of the group element is denoted

$$\chi_R(g)=\mathrm{Tr}\left(\Gamma_R(g)\right), \tag{3.3.13}$$

and is called the character of group element g in representation R. In conclusion, two representations are equivalent if they have the same characters.

We now want to figure out when two group elements will have the same characters. If two group elements are related as

$$g_1=g^{-1}g_2\,g, \tag{3.3.14}$$

it follows that

$$\Gamma_R(g_1)=\Gamma_R(g^{-1}g_2g)=\Gamma_R(g^{-1})\Gamma_R(g_2)\Gamma_R(g)=\Gamma_R(g)^{-1}\Gamma_R(g_2)\Gamma_R(g)\,. \tag{3.3.15}$$

Then

$$\begin{aligned}\chi_R(g_1)=&\mathrm{Tr}\left(\Gamma_R(g_1)\right)\\ =&\mathrm{Tr}\left(\Gamma_R(g)^{-1}\Gamma_R(g_2)\Gamma_R(g)\right)\,.\end{aligned}$$

Using cyclicity of the trace we have

$$\begin{aligned}\chi_R(g_1)=&\mathrm{Tr}\left(\Gamma_R(g_2)\Gamma_R(g)\Gamma_R(g)^{-1}\right)\\ =&\mathrm{Tr}\left(\Gamma_R(g_2)\right)\\ =&\chi_R(g_2)\,.\end{aligned} \tag{3.3.16}$$

We say that two elements g_1 and g_2 obeying (3.3.16) are conjugate to each other. The relation is written as $g_1\sim g_2\quad g_1,g_2\in\mathcal{G}$. It is simple to prove that the "conjugate to" is an equivalence relation. Indeed

$$g_1=e^{-1}g_1e \tag{3.3.17}$$

with e the identity element of the group, so that conjugate to is reflexive. Since

$$g_1=g^{-1}g_2g\quad\Longrightarrow\quad g_2=(g^{-1})^{-1}g_1g^{-1} \tag{3.3.18}$$

this shows that if g_1 is conjugate to g_2, g_2 is conjugate to g_1 so that conjugate to is symmetric. Finally, if

$$g_2=g^{-1}g_1g\qquad g_3=(\tilde{g})^{-1}g_2\tilde{g} \tag{3.3.19}$$

then we have

$$g_3=(g\tilde{g})^{-1}g_1g\tilde{g} \tag{3.3.20}$$

so that conjugate to is transitive. This proves that conjugate to is an equivalence relation. We will use "conjugate to" to partition the group into conjugacy classes. It is clear that all elements in a given conjugacy class have the same character.

3.3.3 Inequivalent Irreducible Representations. Above we found that, starting from a given representation and making use of an invertible matrix M, we easily construct another representation using (3.3.6). There is a second way that we can construct new representations out of old ones. Consider two representations: a set of $d \times d$ matrices, $\{\Gamma(g)\}$ and, $\{\tilde{\Gamma}(g)\}$ a set of $\tilde{d} \times \tilde{d}$ matrices. Using the fact that $\{\Gamma(g)\}$ and $\{\tilde{\Gamma}(g)\}$ are representations, as well as the standard rules of matrix multiplication, we can verify that ($0_{a\times b}$ is a zero matrix with a rows and b columns).

$$\Gamma'(g) = \begin{bmatrix} \Gamma(g) & 0_{d\times\tilde{d}} \\ 0_{\tilde{d}\times d} & \tilde{\Gamma}(g) \end{bmatrix} = \tilde{\Gamma}(g) \oplus \Gamma(g) \tag{3.3.21}$$

is another representation of the group.

We say that $\Gamma'(g)$ is the direct sum of $\Gamma(g)$ and $\tilde{\Gamma}(g)$. This means the matrix representations of the group elements are block diagonal. The $\Gamma'(g)$ has at least two invariant subspaces (W_1 and W_2). W_1 is denoted to be the subspace of all vectors whose last $\tilde{d}$ components vanish while W_2 is the subspace of all vectors whose first d components vanish

$$\begin{bmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \} d \\ \\ \\ \} \tilde{d} \end{matrix} \in W_1 \qquad\qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ * \\ * \\ * \end{bmatrix} \begin{matrix} \} d \\ \\ \\ \} \tilde{d} \end{matrix} \in W_2$$

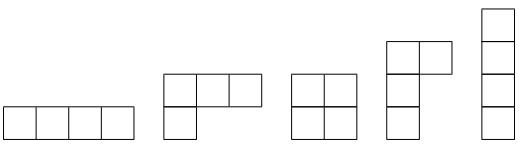
Acting on a vector in W_1 with any group element, we have

$$\Gamma'(g)\,|w_1\rangle = \begin{bmatrix} \Gamma(g) & 0_{d\times\tilde{d}} \\ 0_{\tilde{d}\times d} & \tilde{\Gamma}(g) \end{bmatrix} \begin{bmatrix} |v\rangle \\ 0_{\tilde{d}\times 1} \end{bmatrix} = \begin{bmatrix} \Gamma(g)\,|v\rangle \\ 0_{\tilde{d}\times 1} \end{bmatrix} \in W_1\,. \tag{3.3.22}$$

A similar result holds for W_2. We say that the representation $\Gamma(g)$ is reducible. Any representation that is equivalent to a block diagonal representation has at least two invariant subspaces and is called a reducible representation. An irreducible representation is one that has no non-trivial invariant subspaces under the action of the group and hence is not equivalent to a block diagonal representation. Any representation can be built up using only the inequivalent irreducible representations. For any finite group there are a finite number of them, and the number of inequivalent irreducible representations is equal to the number of conjugacy classes.

To describe the group representations of the symmetric group and give the matrices representing group elements, we will introduce Young diagrams.

3.3.4 Young Diagrams. A Young diagram is a finite collection of n boxes arranged in left-justified rows, such that each row has the same or shorter length than its predecessor. A complete set of irreducible inequivalent representations of the symmetric group S_n are labeled by all possible Young diagrams with n boxes [20]. For example, the complete set of Young diagrams for $n = 4$ is



(3.3.23)

so that S_4 has 5 inequivalent irreducible representations. Listing the number of boxes in each row of Young diagram R gives a partition of n. We write "R is a partition of n" in symbols as $R \vdash n$. The Young diagram contains the same information as that partition. The conjugate or transpose partition of R, written as R^T, can be obtained by listing the number of boxes in each column of a Young diagram or reflecting R along its main diagonal.

We can obtain labels for the elements of a complete basis of $V_R^{S_n}$ by decorating the Young diagram of a particular irreducible representation. This is called a Young-Yamanouchi symbol, which we will denote with ket $|YY\rangle$. To obtain a Young-Yamanouchi symbol, we fill boxes of the Young diagram with the integers 1, 2, 3, ..., n. Each box gets a unique integer. The entries in each row and each column are decreasing as we move to the right or bottom of the page. As an example, the irreducible representation labeled by  leads to three distinct Young-Yamanouchi symbols

$$\begin{array}{|c|c|c|}\hline 4&2&1\\\hline 3\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 4&3&1\\\hline 2\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|}\hline 4&3&2\\\hline 1\\\cline{1-1}\end{array}\,. \tag{3.3.24}$$

To determine the dimensions of a symmetric group irreducible representation labeled by the Young diagram R, we need to use a formula that makes use of the notion of a hook length. The hook length of a box x in Young diagram R is the number of boxes that are in the same row to the right of x plus the number of boxes in the same column below x plus one (for x itself). To determine the hook length we draw a line from below the bottom of R to x and then continue that line to the right until you exit R. The hook length is equal to the number of boxes this line passes through. This is illustrated in Figure (3.1).

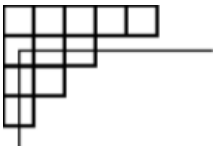


Figure 3.1: The hook associated to the first box in the second row has a length of 5 as illustrated.

Here is an example of a Young diagram with the hook lengths filled in

$$\begin{array}{|c|c|c|}\hline 5&3&1\\\hline 3&1\\\cline{1-2}1\\\cline{1-1}\end{array}\,. \tag{3.3.25}$$

The dimension of an irreducible representation labeled by Young diagram R is $n!$ divided by the product of hook lengths. We will write this as

$$d_R = \frac{n!}{\prod_{x\in R} hook(x)} = \frac{n!}{hooks_R} \tag{3.3.26}$$

where $R \vdash n$. The Young diagram in (3.3.25) labels an irreducible representation of S_6 with dimension

$$d_{\begin{array}{|c|c|c|}\hline \\\hline \\\cline{1-2}\\\cline{1-1}\end{array}} = \frac{6!}{5\cdot 3\cdot 1\cdot 3\cdot 1\cdot 1} = 16\,. \tag{3.3.27}$$

Thus, there would be 16 valid Young-Yamanouchi symbols that you could draw.

Now, we will give the matrices representing the elements of the symmetric group. It is enough to give a matrix representation of the adjacent transpositions. Towards this end we need to introduce the content of a box in a Young diagram. A box x in row i and column j of R has content $j-i$. Here is an example of a Young diagram with the content filled in

$$\begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&4\\\hline -1&0&1\\\cline{1-3} -2\\\cline{1-1}\end{array} \quad . \tag{3.3.28}$$

Given a Young-Yamanouchi symbol, each box in the Young diagram is labeled by a unique integer i with $1 \leq i \leq n$. The content of the box labeled i is c_i . Let $|R_{(i,i+1)}\rangle$ denote the Young-Yamanouchi symbol that is obtained from $|R\rangle$ by swapping the labels of boxes i and $i+1$. We only ever swap boxes whose labels differ by 1. For example,

$$|\Gamma\rangle = \begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array} \qquad |\Gamma_{(12)}\rangle = \begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array} \qquad |\Gamma_{(23)}\rangle = \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array} \quad . \tag{3.3.29}$$

Note that $|\Gamma_{(23)}\rangle$ above is not a valid Young-Yamanouchi diagram. The matrix elements of the adjacent transpositions are now specified by

$$\Gamma((i,i+1))\,|R\rangle = \frac{1}{c_i - c_{i+1}}\,|R\rangle + \sqrt{1-\frac{1}{(c_i-c_{i+1})^2}}\,|R_(c_i,c_{i+1})\rangle \ . \tag{3.3.30}$$

Any illegal Young-Yamanouchi symbols, (for example $|R_{(23)}\rangle$), in the end, will be multiplied by zero so we need not worry about this. Here is an example using this formula

$$\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12))\left|\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}\right\rangle = -\frac{1}{2}\left|\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}\right\rangle + \frac{\sqrt{3}}{2}\left|\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}\right\rangle \tag{3.3.31}$$

$$\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12))\left|\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}\right\rangle = \frac{1}{2}\left|\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}\right\rangle + \frac{\sqrt{3}}{2}\left|\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}\right\rangle . \tag{3.3.32}$$

Choosing

$$\left|\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}\right\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \qquad \left|\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}\right\rangle = \begin{bmatrix}0\\1\end{bmatrix} \tag{3.3.33}$$

we find the following matrix representation for the permutation (12)

$$\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12)) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} . \tag{3.3.34}$$

This result obeys

$$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}, \tag{3.3.35}$$

which is the statement (12)(12)=e. We have seen that, the symmetric group S_n is a group of finite order $n!$. The states in the space of a representation R can be labelled by Young-Yamanouchi symbols, which are constructed using the Young diagram R labelling the representation. Finally the Young-Yamanouchi symbol have been a very useful tool with which to construct the matrices of the representation.

3.4 Representations of the Unitary Group

Consider the group $GL(N,\mathbb{C})$, given by all complex $N \times N$ invertible matrices, with the group composition law given by matrix multiplication. The unitary group $U(N)$ is a subgroup of $GL(N,\mathbb{C})$. The restriction of any irreducible representation of $GL(N,\mathbb{C})$ to the subgroup $U(N)$ is also irreducible. Thus, the vector spaces carrying the irreducible representations of $U(N)$ can be identified with the vector spaces carrying the irreducible representations of $GL(N,\mathbb{C})$.

The irreducible representations of the unitary group $U(N)$ are labelled by Young diagrams with number of rows $\leq N$. The dimension of a $U(N)$ representation also can be computed using Young diagrams that label the irreducible representation. To write an explicit formula for the dimension of a $U(N)$ irreducible representation, we need to associate a number to each box in the Young diagram called a factor, which is given by N plus the content of the box. Here is a Young diagram with the factors filled in

$N+0$	$N+1$	$N+2$	$N+3$	$N+4$
$N-1$	$N+0$	$N+1$		
$N-2$				

.

The formula for the dimension of a $U(N)$ irreducible representation is

$$Dim_R = \frac{f_R}{\prod_{x\in R} \text{hook}(x)}, \tag{3.4.1}$$

where f_R denotes the product of the factors of Young diagram R. For example, consider the $U(3)$ representation described the Young diagram $\begin{array}{ll}\square&\square\\\square&\end{array}$, we find

$$Dim_{\begin{array}{ll}\square&\square\\\square&\end{array}} = \frac{N(N+1)(N-1)}{3\cdot 1\cdot 1} = \frac{N^3-N}{3} = 8. \tag{3.4.2}$$

States in the space of the $U(N)$ representation R can be labelled by a Gelfand-Tsetlin pattern, which is constructed out of the Young diagram R.

We will now describe how the Gelfand-Tsetlin patterns are constructed. Everything we will say about $GL(N,\mathbb{C})$ can be applied, without any change, to $U(N)$. An inequivalent irreducible representation for $GL(N,\mathbb{C})$ (which is identified by the row lengths of a Young diagram R that has number of rows $\leq N$) is given by the sequence of N integers

$$m = (m_{1N}, m_{2N}, \cdots, m_{NN}), \tag{3.4.3}$$

satisfying $m_{iN} \geq m_{i+1,N}$, for $1 \leq i \leq N-1$. The sequence (3.4.3) define the weight of the irreducible representation. The restriction of this irreducible representation onto the subgroup $GL(N-1,\mathbb{C})$ is reducible. It decomposes into a direct sum of irreducible representations with weights

$$m' = (m_{1,N-1}, m_{2,N-1}, \cdots, m_{N-1,N-1}), \tag{3.4.4}$$

satisfying the the "betweenness" conditions: $m_{kN} \geq m_{k,N-1} \geq m_{k+1,N}$ for $1 \leq k \leq N-1$. We can repeat this procedure until we get to $GL(1,\mathbb{C})$ which has one-dimensional spaces. The Gelfand-Tsetlin labelling assembles this sequence of representations of the subgroups into a Gelfand-Tsetlin

pattern. There is a unique pattern for each state in the basis of the space of the original irreducible representation of $GL(N-1,\mathbb{C})$. The pattern can be written as a triangular arrangement of integers, denoted by M, with the structure

$$M = \begin{bmatrix} m_{1N} & m_{2N} & \cdots & m_{N-1,N} & & m_{NN} \\ & m_{1,N-1} & m_{2,N-} & \cdots & m_{N-1,N-1} & \\ & & \cdots & \cdots & \cdots & \\ & & m_{12} & & m_{22} & \\ & & & m_{11} & & \end{bmatrix} . \tag{3.4.5}$$

The top row contains the weight that specifies the irreducible representation of the state and the entries of the lower rows are subject to the betweenness condition. The total number of the Gelfand-Tsetlin patterns give the dimension of the irreducible representation. Return to the example we discussed above, which considered an $U(3)$ representation, described the Young diagram $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$. We already found that this representation is 8 dimensional. Thus, there are 8 vectors in the state space of this representation. The corresponding Gelfand-Tsetlin patterns are

$$\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{bmatrix} \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{bmatrix} \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix} \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{bmatrix} \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{bmatrix}.$$

In this case the total number of these patterns is 8 which is exactly the dimension of the irreducible representation [21].

Denote the $N-$dimensional space by V_N. We are interested in taking the tensor product of n copies of V_N, denoted by $V_N^{\otimes n}$. We can define both an action of S_n on $V_N^{\otimes n}$ and an action of $U(N)$ on $V_N^{\otimes n}$. Towards this end, introduce a basis for $V^{\otimes n}$ defined by

$$V(i_1) \otimes V(i_2) \otimes \cdots \otimes V(i_n) \tag{3.4.6}$$

where $V(i) i = 1, \cdot, N$ is a set of N vectors, providing a basis for V_N. In terms of this basis, we define the action of S_n on $V_N^{\otimes n}$ as follows

$$\sigma : V(i_1) \otimes V(i_2) \otimes \cdots \otimes V(i_n) \longrightarrow V(i_{\sigma(1)}) \otimes V(i_{\sigma(2)}) \otimes \cdots \otimes V(i_{\sigma(n)}) \tag{3.4.7}$$

and we define an action of $U(N)$ on $V_N^{\otimes n}$ as

$$U : V(i_1) \otimes V(i_2) \otimes \cdots \otimes V(i_n) \longrightarrow D(U)V(i_1) \otimes D(U)V(i_2) \otimes \cdots \otimes D(U)V(i_n) \tag{3.4.8}$$

where $D(U)$ is the $N \times N$ matrix representing $U \in U(N)$ in the fundamental representation. These two actions commute

$$\begin{aligned} \sigma U\big(V(i_1) \otimes V(i_2) \otimes \cdots \otimes V(i_n)\big) &= \sigma\big(D(U)V(i_1) \otimes D(U)V(i_2) \otimes \cdots \otimes D(U)V(i_n)\big) \\ &= D(U)V(i_{\sigma(1)}) \otimes D(U)V(i_{\sigma(2)}) \otimes \cdots \otimes D(U)V(i_{\sigma(n)}) \\ &= U\big(V(i_{\sigma(1)}) \otimes V(i_{\sigma(2)}) \otimes \cdots \otimes V(i_{\sigma(n)})\big) \\ &= U\sigma\big(V(i_1) \otimes V(i_2) \otimes \cdots \otimes V(i_n)\big) . \end{aligned}$$

Since these actions commute they can be simultaneously diagonalized. This is called Schur-Weyl duality.

Our goal is to show the space $V^{\otimes n}$ reduces into subspaces labeled by Young diagrams, with each subspace carrying both an irreducible representation of the symmetric group and an irreducible representation of the unitary group, that correspond to the Young diagram labeling the subspace. In this way, every state in the space can be labeled by a symmetric group label and unitary group label. The symmetric group label is a Young-Yamanouchi symbol (YY) and the unitary group label is a Gelfand-Tsetlin pattern (GT), so that the state takes the form $\big|YY, GT\big\rangle$. Therefore, the total number of states in subspace labeled by Young diagram R is given by the total number of Young-Yamanouchi symbols times the total number of Gelfand-Tsetlin patterns. Thus the total number of states in the space is given by

$$N_{states} = \sum_R d_R Dim_R. \tag{3.4.9}$$

We will not prove that this is the case. We will be happy to verify this counting in a specific non-trivial example. Consider $V_N^{\otimes 3}$. There are three possible subspaces

$$R = \square\square\square \quad d_R = 1 \quad {}_{Dim}R = \frac{N(N+1)(N+2)}{6},$$
$$R = \begin{array}{l}\square\square\\\square\end{array} \quad d_R = 2 \quad {}_{Dim}R = \frac{N(N+1)(N-1)}{3},$$
$$R = \begin{array}{l}\square\\\square\\\square\end{array} \quad d_R = 1 \quad {}_{Dim}R = \frac{N(N-1)(N-2)}{6}.$$

Summing over all subspaces we do indeed find that the total number of states is N^3.

3.5 Schur's Lemmas

In our application of representation theory to matrix models, we will make extensive use of the Fundamental Orthogonality Theorem. This theorem is proved using Schur's Lemmas, we state, without proof, these lemmas in this section [18], [19].

Lemma 1: A non-zero matrix which commutes with all of the matrices of an irreducible representation is a constant multiple of the unit matrix. We have

$$\Gamma_R(g)A = A\Gamma_R(g) \quad \forall g \in \mathcal{G} \quad \Longrightarrow A = \lambda \cdot \mathbb{I} \tag{3.5.1}$$

with R an irreducible representation, $\mathbb{I}$ is the unit matrix.

Lemma 2: Given two inequivalent irreducible representations R and S. The only solution to the equation

$$\Gamma_R(g)A = A\Gamma_S(g) \quad \forall g \in \mathcal{G} \qquad \text{is} \quad A = 0\,. \tag{3.5.2}$$

3.6 The Fundamental Orthogonality Relation; Character Orthogonality

Consider the quantity

$$[A(R,S,b,\alpha)]_{a\beta} = \sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(g)_{\alpha\beta}. \tag{3.6.1}$$

R and S run over the possible irreducible representations of the group. We denote the dimensions of R, S by d_R, d_S respectively. $a, b = 1, \cdots, d_R$ are indices for the representation R. $\alpha, \beta = 1, \cdots, d_S$ are indices for the representation S. A is a collection of matrices with row index a and column index β. The matrices in this collection are indexed by the values of R, S, b and α. We want to study these matrices. Notice that

$$\begin{aligned}[A(R,S,b,\alpha)]_{a\beta}\Gamma_S(g)_{\beta\gamma} &= \sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(g)_{\alpha\beta}\Gamma_S(g)_{\beta\gamma} \\ &= \sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(gg_1)_{\alpha\gamma}. \end{aligned}\tag{3.6.2}$$

The sum over g is a sum over $|\mathcal{G}|$ terms i.e. a sum over the whole group. We change summation variables in the above sum, from g to $\overline{g} = gg_1$. The terms are distinct, so that the sum over $\overline{g}$ is a sum over the whole group. Thus, we find

$$\begin{aligned}[A(R,S,b,\alpha)]_{a\beta}\Gamma_S(g)_{\beta\gamma} =& \sum_{\overline{g}\in\mathcal{G}} \Gamma_R(g_1\overline{g}^{-1})_{ab}\Gamma_S(\overline{g})_{\alpha\gamma} \\ =& \sum_{\overline{g}\in\mathcal{G}} \Gamma_R(g_1)_{ac}\Gamma_R(\overline{g}^{-1})_{cb}\Gamma_S(\overline{g})_{\alpha\gamma} \\ =& \Gamma_R(g_1)_{ac}\sum_{\overline{g}\in\mathcal{G}} \Gamma_R(\overline{g}^{-1})_{cb}\Gamma_S(\overline{g})_{\alpha\gamma} \\ =& \Gamma_R(g_1)_{ac}[A(R,S,b,\alpha)]_{c\gamma}\,. \end{aligned}$$

From Schur's Lemmas, we know that

$$[A(R,S,b,\alpha)]_{\alpha\beta} = \delta_{RS}\delta_{\alpha\beta}\lambda(b,\alpha,R) \tag{3.6.3}$$

where $\lambda(\cdot,\cdot,\cdot)$ is to be determined. To that end, return to the definition of $[A(R,S,b,\alpha)]_{\alpha\beta}$ and compute

$$\begin{aligned}\mathrm{Tr}\,(A(R,R,b,c)) =& \sum_a \sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_R(g)_{ca} = \sum_{g\in\mathcal{G}} [\Gamma_R(g)\Gamma_R(g^{-1})]_{cb} \\ =& \sum_{g\in\mathcal{G}} [\Gamma_R(gg^{-1})]_{cb} = \sum_{g\in\mathcal{G}} \delta_{cb} = |\mathcal{G}|\delta_{cb}\,. \end{aligned}$$

In terms of the form we deduced above

$$\mathrm{Tr}\,(A(R,R,b,c)) = \sum_{a=1}^{d_R} A(R,R,b,c)_{aa} = \sum_{a=1}^{d_R} \delta_{aa}\lambda(b,c,R) = d_R\lambda(b,c,R)\,. \tag{3.6.4}$$

Comparing these two expressions we have

$$\lambda(b,c,R) = \frac{|\mathcal{G}|}{d_R}\delta_{bc}\,. \tag{3.6.5}$$

Therefore,

$$\sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(g)_{\alpha\beta} = \frac{|\mathcal{G}|}{d_R}\delta_{RS}\delta_{a\beta}\delta_{b\alpha}\,. \tag{3.6.6}$$

This is the fundamental orthogonality relation [19].

We can use the fundamental orthogonality relation to define an orthogonality relation for characters. We know that $\chi_R(g) = \mathrm{Tr}\,(\Gamma_R(g))$. From (3.6.6), we find that

$$\begin{aligned}\sum_{g\in\mathcal{G}} \chi_R(g^{-1})_{aa}\chi_S(g)_{\alpha\alpha} &= \sum_a\sum_\alpha\sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{aa}\Gamma_S(g)_{\alpha\alpha}\\ &= \frac{|\mathcal{G}|}{d_R}\delta_{RS}\sum_{a,\alpha}\delta_{a\alpha}\delta_{\alpha a}\\ &= |\mathcal{G}|\delta_{RS}\,.\end{aligned} \tag{3.6.7}$$

This is the character orthogonality relation. When we apply this formula to the symmetric group, we will freely make use of the fact that $\chi_R(g^{-1}) = \chi_R(g)$.

Every representation of a finite group is equivalent to a unitary representation. This means that without any loss of generality we can assume that our representation is unitary. A unitary transformation is a transformation that preserves the inner product. Define a set of vectors $|S,\alpha,\beta\rangle_g = \Gamma_S(g)_{\alpha\beta}$, and

$$\Gamma_R(g^{-1})_{ab} = (\Gamma_R(g))^{-1}_{ab} = \Gamma^\dagger_R(g)_{ab}\,. \tag{3.6.8}$$

Then

$$\begin{aligned}\langle R,b,a|S,\alpha,\beta\rangle &= \sum_{g\in\mathcal{G}} \Gamma^\dagger_R(g)_{ab}\Gamma_S(g)_{\alpha\beta}\\ &= \sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(g)_{\alpha\beta}\\ &= \frac{|\mathcal{G}|}{d_R}\delta_{RS}\delta_{a\beta}\delta_{b\alpha}\,.\end{aligned} \tag{3.6.9}$$

This means the vectors are orthogonal. They live in a $|\mathcal{G}|$ dimensional space. Thus, we learn that $\sum_R d_R^2 \le |\mathcal{G}|$.

Finally, lets specialize the character orthogonality relation to unitary representations and to orthogonal representations:

- For a unitary representation where: $\Gamma(g^{-1}) = \Gamma(g)^{-1} = \Gamma^\dagger(g)$, the character orthogonality is given by

$$\sum_{g\in\mathcal{G}} \chi_R(g^{-1})\chi_S(g) = \delta_{RS}|\mathcal{G}|\,. \tag{3.6.10}$$

- For an orthogonal representation where: $\Gamma(g^{-1}) = \Gamma(g)^T = \Gamma(g)$, the character orthogonality is given by

$$\sum_{g\in\mathcal{G}} \chi_R(g)\chi_S(g) = \delta_{RS}|\mathcal{G}| \tag{3.6.11}$$

where T denotes the matrix transpose and † denotes conjugate transpose of the matrix.

3.7 Projection Operators

Projection operators will play a fundamental role when we construct a basis for the local gauge invariant operators. In this section we introduce the relevent projection operators. A projection operator is an operator that obeys $\hat{P}_i^2 = \hat{P}_i$. The collection of operators

$$(\hat{P}_R)^I_J = \frac{d_R}{n!}\sum_{\sigma\in S_n} \chi_R(\sigma)(\sigma)^I_J, \tag{3.7.1}$$

obey $\hat{P}_R\hat{P}_S = \delta_{RS}\hat{P}_S$ so that they are projection operators. To prove this, note that

$$\begin{aligned}\hat{P}_R\hat{P}_S =& \frac{d_R}{n!}\sum_{\sigma_1\in S_n} \chi_R(\sigma_1)\sigma_1\frac{d_S}{n!}\sum_{\sigma_2\in S_n} \chi_S(\sigma_2)\sigma_2\,,\\ =& \frac{d_R d_S}{(n!)^2}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n} \chi_S(\sigma_2)\chi_R(\sigma_1)\sigma_1\sigma_2\,.\end{aligned}$$

Now, change variables from σ_1, σ_2 to $\sigma_1, \psi = \sigma_1\cdot\sigma_2$,

$$\begin{aligned}\hat{P}_R\hat{P}_S =& \frac{d_R d_S}{(n!)^2}\sum_{\psi\in S_n}\sum_{\sigma_1\in S_n} \chi_R(\sigma_1)\chi_S(\psi\sigma_1^{-1})\psi\,,\\ =& \frac{d_R d_S}{(n!)^2}\sum_{\psi\in S_n}\sum_{\sigma_1\in S_n} \Gamma_R(\sigma_1)_{ii}\Gamma_S(\psi)_{jk}\Gamma(\sigma_1^{-1})_{kj}\psi\,.\end{aligned}$$

We use the fundamental orthogonality relation (3.6.6), to obtain

$$\hat{P}_R\hat{P}_S = \delta_{RS}\frac{d_S}{n!}\sum_{\psi\in S_n} \chi_S(\psi)\psi = \delta_{RS}\hat{P}_S. \tag{3.7.2}$$

This proves that the $\hat{P}_R$ are projectors. Then the character of an element U of the group $U(N)$, in irreducible representation R, is given by acting on $U^{\otimes n}$ with $\hat{P}_R$ and then tracing. We find

$$\begin{aligned}\chi_R(U) =& \frac{1}{d_R}\mathrm{Tr}\left(\hat{P}_R U^{\otimes n}\right)\\ =& \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\mathrm{Tr}\left(\sigma U^{\otimes n}\right)\\ =& \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)U^{i_1}_{i_{\sigma(1)}}U^{i_2}_{i_{\sigma(2)}}\cdots U^{i_n}_{i_{\sigma(n)}}\end{aligned}$$

This gives a concrete relation between the characters of the symmetric and the unitary groups. The value of the trace of the projection operator $\hat{P}_R$ is the dimension of the subspace we project to

$$\mathrm{Tr}\left(\hat{P}_R\right) = d_R Dim_R \tag{3.7.3}$$

3.8 Schur Polynomials

It is natural to obtain an operator $Z^{\otimes n} \equiv Z \otimes Z \otimes \cdots \otimes Z$, by tensoring n copies of the complex matrix Z^i_j. This operator acts on $V_N^{\otimes n}$ obtain by tensoring n copies of an N dimensional vector space V_N. Z is in the set of endomorphisms of the vector space V_N [22], [23]

$$Z : V_N \longrightarrow V_N. \tag{3.8.1}$$

We will use the the following index notations

$$(Z^{\otimes n})^I_J = Z^{i_1}_{j_1} Z^{i_2}_{j_2} \cdots Z^{i_n}_{j_n} . \tag{3.8.2}$$

For σ belonging to the symmetric group S_n we have

$$(\sigma)^I_J = (\sigma)^{i_i, i_2, \cdots i_n}_{j_1, j_2, \cdots j_n} = \delta^{i_1}_{j_\sigma(1)} \delta^{i_2}_{j_\sigma(2)} \cdots \delta^{i_n}_{j_\sigma(n)} . \tag{3.8.3}$$

Recall that the action of $\sigma \in S_n$ on $V_N^{\otimes n}$ is given by interchanging the order of the vectors in the tensor product of any given vector, as follows

$$\sigma\big(\vec{v}(i_1) \otimes \vec{v}(i_2) \otimes \cdots \otimes \vec{v}(i_n)\big) = \vec{v}(i_{\sigma(1)}) \otimes \vec{v}(i_{\sigma(2)}) \otimes \cdots \otimes \vec{v}(i_{\sigma(n)}) \tag{3.8.4}$$

Using this action, any observable can be written as a single trace in $V_N^{\otimes n}$ as follows

$$\mathrm{Tr}\left(\sigma Z^{\otimes n}\right) = (\sigma)^J_I (Z^{\otimes n})^I_J = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}} \tag{3.8.5}$$

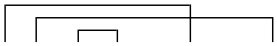
As an example, for $n = 4$ and for permutation $\sigma = (1)(2)(34)$ in cycle notation, we have

$$\mathrm{Tr}\left((1)(2)(34) Z^{\otimes n}\right) = \mathrm{Tr}\,(Z)^2\, \mathrm{Tr}\left(Z^2\right) \tag{3.8.6}$$

We can write any multi-trace operator involving n fields as a single trace in the bigger space $V_N^{\otimes n}$. We want to sum the free field Wick contractions in correlators of operators, each built with n fields. Consider the free field expectation value

$$\left\langle Z^{i_1}_{j_1} Z^{i_2}_{j_2} \cdots Z^{i_n}_{j_n} (Z^\dagger)^{k_1}_{l_1} (Z^\dagger)^{k_2}_{l_2} \cdots (Z^\dagger)^{k_n}_{l_n} \right\rangle . \tag{3.8.7}$$

There are a total of $n!$ Wick contractions that must be summed. To motivate the formula that follows, consider a total of three matrices. We can think of each contraction as a permutation. The Wick contraction below



$$Z^{i_1}_{j_1} Z^{i_2}_{j_2} Z^{i_3}_{j_3} (Z^\dagger)^{k_1}_{l_1} (Z^\dagger)^{k_2}_{l_2} (Z^\dagger)^{k_3}_{l_3} = \delta^{i_1}_{l_2} \delta^{i_2}_{l_3} \delta^{i_3}_{l_1} \delta^{k_1}_{j_3} \delta^{k_2}_{j_1} \delta^{k_3}_{j_2} .$$

corresponds to the permutation $\sigma = (123)$. By summing over all possible permutations in S_3, we will precisely sum over the complete set of Wick contractions. Notice that the right hand side is $(321)^I_L (123)^K_J$ and that the permutation (132) is the inverse of (123). It is now simple to see that

$$\left\langle Z^{i_1}_{j_1} Z^{i_2}_{j_2} \cdots Z^{i_n}_{j_n} (Z^\dagger)^{k_1}_{l_1} (Z^\dagger)^{k_2}_{l_2} \cdots (Z^\dagger)^{k_n}_{l_n} \right\rangle = \sum_{\sigma \in \mathcal{G}} (\sigma)^I_L (\sigma^{-1})^K_J . \tag{3.8.8}$$

Now, we seek to provide an exactly orthogonal basis for our operators. We emphasize the fact that all ribbon diagrams are summed, not just the planar diagrams. Recall the projection operators $\hat{P}_R$, acting on $V_N^{\otimes n}$ such that

$$\hat{P}_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma. \tag{3.8.9}$$

$$\hat{P}_R \hat{P}_S = \delta_{RS} \hat{P}_S \tag{3.8.10}$$

$$\hat{P}_R \psi = \psi \hat{P}_R \quad \forall \psi \in S_n \,. \tag{3.8.11}$$

We can check that, $\hat{P}_R^\dagger = \hat{P}_R$. In terms of these projectors introduce a set of operators, the Schur polynomials

$$\begin{aligned}
\chi_R(Z^{\otimes n}) &= \mathrm{Tr}\left(\hat{P}_R Z_N^{\otimes n}\right) = (\hat{P}_R)_I^J Z^{\otimes n}{}_J^I \\
&= \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \sigma_I^J Z^{\otimes n}{}_J^I \\
&= \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}} \\
&= \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \mathrm{Tr}\left(\sigma Z_N^{\otimes n}\right) \,.
\end{aligned} \tag{3.8.12}$$

To obtain an explicit formula for the two point correlation function of these operators, argue as follows

$$\begin{aligned}
\left\langle \chi_R(Z) \chi_S(Z)^\dagger \right\rangle &= \frac{1}{d_R d_S} (\hat{P}_R)_J^I (\hat{P}_S)_L^K \left\langle (Z^{\otimes n})_J^I (Z^{\dagger \otimes n})_L^K \right\rangle \\
&= \frac{1}{d_R d_S} \sum_{\sigma \in S_n} \mathrm{Tr}\left(\hat{P}_R \sigma^{-1} \hat{P}_S \sigma\right) \\
&= \frac{n!}{d_R d_S} \mathrm{Tr}\left(\hat{P}_R \hat{P}_S\right) = \frac{\delta_{RS} n!}{d_R d_S} \mathrm{Tr}\left(\hat{P}_R\right) \\
&= \frac{\delta_{RS} n!}{d_R} Dim_R = \delta_{RS} f_R \,.
\end{aligned} \tag{3.8.13}$$

To obtain the last line we have used the explicit expressions for d_R and Dim_R, equations (3.3.26) and (3.4.1) respectively. It is clear that, the Schur polynomials diagonalize the two point function. As an example of the application of this result is

$$\left\langle \chi_{\square\square\square}(Z) \chi^\dagger_{\square\square\square}(Z) \right\rangle = N(N+1)(N+2). \tag{3.8.14}$$

This answer is obtained by summing all ribbon graphs, not just the planar ones.

Schur polynomials are complete, we can invert the relation between traces and Schur polynomials using the character orthogonality relations (3.8.13)

$$\mathrm{Tr}\left(\sigma Z^{\otimes n}\right) = \sum_{R \vdash n} \chi_R(\sigma) \chi_R(Z) \,. \tag{3.8.15}$$

3.9 Restricted Schur Polynomials

We now consider two complex matrices, Z and Y as operators acting on N dimensional vector space V_N. Focus on the case of n Z matrices and m Y matrices [24]. We again use the compact index notation

$$(Y^{\otimes m} \otimes Z^{\otimes n})^I_J = Y^{i_1}_{j_1} Y^{i_2}_{j_2} \cdots Y^{i_m}_{j_m} Z^{i_{m+1}}_{j_{m+1}} Z^{i_{m+2}}_{j_{m+2}} \cdots Z^{i_{m+n}}_{j_{m+n}}. \tag{3.9.1}$$

Permutations again provide a consistent way to label multitrace structures of the theory. Indeed the general set of observables built out of Z and Y, with $\sigma \in S_{n+m}$ can be written as

$$\mathrm{Tr}\left(\sigma Y^{\otimes m} \otimes Z^{\otimes n}\right)^I_J = Y^{i_1}_{j_{\sigma(1)}} Y^{i_2}_{j_{\sigma(2)}} \cdots Y^{i_m}_{j_{\sigma(m)}} Z^{i_{m+1}}_{j_{\sigma(m+1)}} Z^{i_{m+2}}_{j_{\sigma(m+2)}} \cdots Z^{i_{m+n}}_{j_{\sigma(m+n)}}. \tag{3.9.2}$$

Permutations which can be related by conjugation with respect to the $S_n \times S_m$ subgroup,

$$\rho^{-1}\sigma_1\rho = \sigma_2 \qquad \rho \in S_n \times S_m, \tag{3.9.3}$$

lead to the same physical observable. To describe the multimatrix structures we need to generalize the Schur polynomials to the restricted Schur polynomials. Toward this end, we need to obtain the restricted character. We know that the characters are just the trace of the matrix representing a group element in a given irreducible representation. In this case, after restricting to a subgroup, a given irreducible representation of the group will decompose into a number of irreducible representations of the subgroup. More than one copy of a representation of the subgroup may appear, so we need to introduce a multiplicity label α and β to keep track of these copies. To label an irreducible representation of S_{n+m}, we need a Young diagram with $m+n$ boxes $R \vdash m+n$. To label an irreducible representation of $S_n \times S_m$ we need a Young diagram with m boxes for S_m and a Young diagram with n boxes for S_n. We label a representation of the subgroup $S_n \times S_m$ of the group S_{n+m} by (r,s), where $r \vdash n$ and $s \vdash m$. States in the carrier space of the subgroup $(r,s)\alpha$ are denoted by $R,(r,s)\alpha;\beta$.

To compute the restricted character $\chi_{R,(r,s)\alpha\beta}(\sigma)$ we need to take the restricted trace of the matrix representing group element σ in a given irreducible representation. We can take the trace of the row index over the subspace associated to the α-th copy of (r,s) and the column index over the subspace associated to the β-th copy of (r,s). In this way, we obtain the restricted character defined by [25]

$$\begin{aligned}\chi_{R,(r,s)\alpha\beta}(\sigma) =& \mathrm{Tr}_{R,(r,s)\alpha\beta}(\Gamma(\sigma)) \\ =& \sum_\beta \langle R,(r,s)\alpha;\beta|\Gamma(\sigma)|R,(r,s)\alpha;\beta\rangle\,.\end{aligned} \tag{3.9.4}$$

We introduce a set of operators $P_{R,(r,s)\alpha\beta}$ acting on the space V^{n+m} as follow

$$P_{R,(r,s)\alpha\beta} = \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\sigma\,. \tag{3.9.5}$$

Finally, the restricted Schur polynomial can be written as

$$\begin{aligned}\chi_{R,(r,s)\alpha\beta}(Z,Y) =& \mathrm{Tr}\left(P_{R,(r,s)\alpha\beta}Y^{\otimes m}\otimes Z^{\otimes n}\right)\\ =&\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\sigma)\mathrm{Tr}\left(\sigma Y^{\otimes m}\otimes Z^{\otimes n}\right)\\ =&\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\sigma)Y^{i_1}_{\sigma j_1}Y^{i_2}_{\sigma j_2}\cdots Y^{i_m}_{\sigma j_m}Z^{i_{m+1}}_{\sigma j_{m+1}}Z^{i_{m+2}}_{\sigma j_{m+2}}\cdots Z^{i_{m+n}}_{\sigma j_{m+n}}. \end{aligned} \tag{3.9.6}$$

The restricted Schur polynomials diagonalize the two point function in the free theory, exactly

$$\left\langle \chi_R,(r,s)\alpha\beta(Y,Z)\chi_{T,(t,u)\gamma\xi}(Y,Z)^\dagger\right\rangle \propto \delta_{RT}\delta_{rt}\delta_{su}\delta_{\beta\xi}\delta_{\alpha\gamma}. \tag{3.9.7}$$

To summarize, we have managed to give methods for computing correlation functions built from order N fields by using group theory. We have also computed exactly the two point function using Schur polynomials, i.e. all the ribbon graphs were summed. Including the non-planar ribbon graphs is summing string quantum corrections. These correlators should tell us about quantum effects in string theory. Further, we have seen that orthogonality of the Schur operators simplifies the computations.

4. Holographic Computation of CFT Correlators

Correlation functions of the fields in any quantum field theory are among the most interesting quantities we can evaluate. In this chapter we will study correlation functions of primary fields in the boundary CFT from the bulk theory in AdS. We will adopt the Euclidean formulation of quantum theory in anti-de Sitter AdS spacetime. We will consider two and three-point correlation functions. The recipe and the tools that would be needful for computing this class of correlation functions are presented in the following section [6].

4.1 The Holographic Dictionary

According to the AdS/CFT correspondence [2], [3] and [4], correlators in CFT can be computed using a dual string theory description. For every CFT operator $\mathcal{O}$ there is a corresponding gravity (or string) theory field ϕ and a corresponding boundary condition ϕ_0 [5]. The mass of a field in the gravity theory is related to the scale dimension of the corresponding operator in CFT. In the string theory side of the correspondence, the generating functional Z_{AdS} is given by

$$Z_{\text{AdS}}(\phi_0) = \int [D\phi] e^{-S_{\text{AdS}}}, \tag{4.1.1}$$

where $[D\phi]$ is the measure for the fields in string theory (there are an infinite number of them) and S_{AdS} is the classical gravity action. This integral is the path integral of string theory on spacetimes which are asymptotically $\text{AdS}_5 \times \text{S}^5$, performed with the boundary condition that $\phi \longrightarrow \phi_0$ on the boundary of AdS_5.

On the other side the generating functional for the conformal boundary of the (d+1)–dimensional AdS spacetime, Z_{CFT}, with a field theory operators $\mathcal{O}$, can be represented in terms of a source fields ϕ_0. Thehe AdS/CFT conjecture therefore equates

$$Z_{\text{CFT}}(\mathcal{O}) = \left\langle e^{\int d^d x\, \phi_0(x)\mathcal{O}(x)} \right\rangle_{\text{CFT}} = Z_{\text{AdS}}(\phi_0) \tag{4.1.2}$$

where $\langle \cdots \rangle_{\text{CFT}}$ means the path integral in the conformal field theory and ϕ_0 is the boundary value of the bulk field acting as a source for the CFT operator $\mathcal{O}$ [26]. Thus we may approximate the string generating functional $Z_{\text{AdS}}(\phi_0)$ by using the saddle point approximation. To this order only the classical contribution remains

$$Z_{\text{AdS}}(\phi_0) \approx e^{-S_{\text{AdS}}(\phi_0)}. \tag{4.1.3}$$

where $S_{\text{AdS}}(\phi_0)$ is the action for the gravity subject to the boundary condition ϕ_0 for the field ϕ. In the gravity description, we are neglecting quantum corrections. This corresponds to studying the $N \to \infty$ limit of the CFT. Finally, correlation functions of CFT primary fields can be computed by functionally differentiating $Z_{\text{AdS}}(\phi_0)$ and setting the source ϕ_0 equal to zero:

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle_{\text{CFT}} = \frac{\delta}{\delta\phi_0(x_1)} \cdots \frac{\delta}{\delta\phi_0(x_n)} Z_{\text{AdS}}(\phi_0)\Big|_{\phi_0=0}. \tag{4.1.4}$$

In order to compute these correlation functions, we need to compute the classical action in AdS as a functional of the boundary conditions. Towards this end we will start with the Eculidean action for a massive scalar field in AdS_{d+1} spacetime. Using this action, we will derive the supergravity equations of motion for a field ϕ with boundary condition ϕ_0 and solve them [6]. Finally, we will insert the solution into the action and use it to compute correlation functions in the form of (4.1.4). In this chapter and before we proceed to compute correlators we will recall some of the key features of the Euclidean AdS_{d+1} spacetime.

4.2 The Euclidean Version of AdS_{d+1} Spacetime

In this section we will start with the Euclidean AdS_{d+1} metric and obtain the induced metric in the form of the Lobaschevsky upper half-space. Then we introduce a conformal inversion, to enable us to exploit the conformal symmetry. This conformal inversion will play an important role in our method of calculating the correlators. This is partly because inversion is a conformal isometry and a symmetry of CFT_d. The Euclidean signature version of (d+1)–dimensional Anti-de Sitter spacetime, AdS_{d+1}, can be represented by taking the upper sheet $Y_{-1} > 0$ of the hyperboloid [27], [6], [5]

$$-(Y_{-1})^2 + (Y_0)^2 + \sum_{i=1}^{d}(Y_i)^2 = -\frac{1}{a^2}, \tag{4.2.1}$$

with curvature $R = -d(d+1)a^2$. The intrinsic geometry of the surface is determined by the Euclidean AdS_{d+1} metric

$$ds^2 = -(dY_{-1})^2 + (dY_0)^2 + \sum_{i=1}^{d}(dY_i)^2\,. \tag{4.2.2}$$

We may map this metric onto the upper half space H_{d+1} by changing the coordinates as

$$z_i = \frac{Y_i}{a(Y_{-1}+Y_0)} \tag{4.2.3}$$

$$z_0 = \frac{1}{a^2(Y_{-1}+Y_0)} \tag{4.2.4}$$

$$\Longrightarrow \frac{z_i}{z_0} = aY_i \quad \text{for } i \neq 0\,. \tag{4.2.5}$$

Then we differentiate the last equation and sum over i to find

$$\sum_{i=1}^{d}\big(dY_i\big)^2 = \frac{1}{a^2 z_0^2}\sum_{i=1}^{d} dz_i^2 + \frac{dz_0^2}{a^2 z_0^4}\sum_{i=1}^{d} z_i^2 - \frac{dz_0}{a^2 z_0^3} d\Big(\sum_{i=1}^{d} z_i^2\Big), \tag{4.2.6}$$

Now, we use (4.2.1) to find a relation among these different coordinates, so that we have

$$\sum_{i=1}^{d} z_i^2 = a^2 z_0^2 \sum_{i=1}^{d}(Y_i)^2 = a^2 z_0^2\big[-\frac{1}{a^2} + (Y_{-1})^2 - (Y_0)^2\big], \tag{4.2.7}$$

and

$$d\Big(\sum_{i=1}^{d} z_i^2\Big) = 2a^2 z_0^2\big[Y_{-1}dY_{-1} - Y_0 dY_0\big] + 2a^2 z_0 dz_0\big[-\frac{1}{a^2} + (Y_{-1})^2 - (Y_0)^2\big]. \tag{4.2.8}$$

We substitute these into (4.2.6) to find

$$\sum_{i=1}^{d} dY_i^2 = \frac{1}{a^2 z_0^2}\sum_{i=1}^{d} dz_i^2 - \frac{2dz_0}{z_0}\big[Y_{-1}dY_{-1} - Y_0 dY_0\big] - \frac{dz_0^2}{z_0^2}\big[-\frac{1}{a^2} + (Y_{-1})^2 - (Y_0)^2\big]. \tag{4.2.9}$$

Using (4.2.4) we find

$$\begin{aligned}
\sum_{i=1}^{d}(dY_i)^2 &= \frac{1}{a^2 z_0^2}\sum_{i=1}^{d} dz_i^2 + \frac{dz_0^2}{a^2 z_0^2} + 2\left[\frac{dY_{-1}+dY_0}{(Y_{-1}+Y_0)}\right]\big[Y_{-1}dY_{-1} - Y_0 dY_0\big] - \left[\frac{(dY_{-1}+dY_0)^2}{(Y_{-1}+Y_0)^2}\right]\big[(Y_{-1})^2 - (Y_0)^2\big] \\
&= \frac{1}{a^2 z_0^2}\sum_{\mu=0}^{d} dz_\mu^2 + \frac{1}{(Y_{-1}+Y_0)^2}\Big[\big[Y_{-1}^2 + Y_0^2 + 2Y_{-1}Y_0\big](dY_{-1})^2 - \big[Y_{-1}^2 + Y_0^2 + 2Y_{-1}Y_0\big](dY_0)^2\Big] \\
&= \frac{1}{a^2 z_0^2}\sum_{\mu=0}^{d} dz_\mu^2 + (dY_{-1})^2 - (dY_0)^2\,,
\end{aligned}$$

which brings the induced metric to the form of the Lobaschevsky upper half–space

$$ds^2 = \frac{1}{a^2 z_0^2}\Big(\sum_{\mu=0}^{d} dz_\mu^2\Big). \tag{4.2.10}$$

We set the parameter $a = 1$. Therefore, the inversion symmetry of Euclidean AdS which is an isometry of (4.2.10), can be expressed as the following coordinate transformation

$$z'_\mu = \frac{z_\mu}{z^2}, \qquad z'^2 = \frac{1}{z^2}\,. \tag{4.2.11}$$

The Jacobian associated to inversion is

$$\begin{aligned}
\frac{\partial z'_\mu}{\partial z_\nu} &= (z')^2\Big(\delta_{\mu\nu} - 2\frac{z'_\mu z'_\nu}{(z')^2}\Big) \\
&= (z')^2 J_{\mu\nu}(z') = (z')^2 J_{\mu\nu}(z)\,.
\end{aligned} \tag{4.2.12}$$

The Jacobian tensor $J_{\mu\nu}$ obeys a number of identities that will be useful for the computations in this chapter. These include the pretty property

$$J_{\mu\nu}(x-y) = J_{\mu\rho}(x')J_{\rho\sigma}(x'-y')J_{\sigma\nu}(y'), \tag{4.2.13}$$

and the orthogonality relation

$$J_{\mu\nu}(x)J_{\mu\rho}(x) = \delta_{\mu\rho}\,. \tag{4.2.14}$$

The Euclidean metric tensor $g_{\mu\nu}$, with coordinate $z = (z_0, z_\mu) = (z_0, z_1, \cdots z_d)$, can be written as $g^{\mu\nu} = z_0^{-2}\delta_{\mu\nu}$, where $\delta_{\mu\nu}$ is the Kronecker delta and $\mu, \nu = 0, 1, \cdots, d$. We define the contraction of the indices using the AdS metric tensor as *e.g.* $g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$.

4.3 The wave equation and its solution

The action for a massive scalar field in d+1 dimensional Minkowski spacetime is given by

$$S = \int d^{d+1}x\mathcal{L}(\phi, \partial_\mu\phi) \tag{4.3.1}$$

where $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2. \tag{4.3.2}$$

To find the covariant Klein-Gordon equation from this Lagrangian density, we replace the flat space metric $\eta_{\mu\nu}$ by the curved space metric $g_{\mu\nu}$. Also use the co-ordinate reparametrization invariant measure in curved spacetime. Therefore, after a Wick rotation we write the Euclidean action for a massive scalar field in a curved spacetime as follows [6]

$$S_E[\phi] = \frac{1}{2}\int d^{d+1}x\sqrt{g}\,\mathcal{L} = \frac{1}{2}\int d^{d+1}x\sqrt{g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right]. \tag{4.3.3}$$

Upon requiring that $\delta S = 0$ we obtain the Euler-Lagrange equation in the curved spacetime

$$\partial_\rho\left(\frac{\partial\sqrt{g}\mathcal{L}}{\partial(\partial_\rho\phi)}\right) - \frac{\partial\sqrt{g}\mathcal{L}}{\partial\phi} = 0\,. \tag{4.3.4}$$

Using the explicit expression for the Lagrangian density $\mathcal{L}$ we obtain

$$\begin{aligned}
\partial_\rho\left(\frac{\partial\sqrt{g}\mathcal{L}}{\partial(\partial_\rho\phi)}\right) &= \partial_\rho\left(\frac{\partial}{\partial(\partial_\rho\phi)}\frac{\sqrt{g}}{2}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right]\right)\\
&= \partial_\rho\left(\frac{\sqrt{g}}{2}g^{\mu\nu}\left[\delta_{\mu\rho}\partial_\nu\phi + \delta_{\nu\rho}\partial_\mu\phi\right]\right)\\
&= \partial_\rho\left(\frac{\sqrt{g}}{2}\left[g^{\rho\nu}\partial_\nu\phi + g^{\rho\mu}\partial_\mu\phi\right]\right)\\
&= \partial_\rho\left[\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right]\,,
\end{aligned} \tag{4.3.5}$$

and

$$\begin{aligned}
\frac{\partial\sqrt{g}\mathcal{L}}{\partial\phi} &= \frac{\partial}{\partial\phi}\frac{\sqrt{g}}{2}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right]\\
&= \sqrt{g}m^2\phi\,.
\end{aligned}$$

The equation of motion can now be written as

$$\frac{1}{\sqrt{g}}\partial_\mu\left[\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right] - m^2\phi = 0\,. \tag{4.3.6}$$

This is the Klein-Gordon equation. Now use the metric of the AdS spacetime $g_{\mu\nu} = z_0^{-2}\delta_{\mu\nu}$ and its determinant which is given by

$$g = \det(g_{\mu\nu}) = \det(z_0^{-2}\delta_{\mu\nu}) = \prod_{\mu=0}^{d} z_0^{-2}\delta_{\mu\mu} = z_0^{-2(d+1)}\,. \tag{4.3.7}$$

Applying the inversion (4.2.11), the equation of motion (4.3.6) can be written in the form

$$z_0^{d+1}\frac{\partial}{\partial z_0}\left[z_0^{-d+1}\frac{\partial}{\partial z_0}\phi(z_0,\vec{z})\right]+z_0^2\frac{\partial^2}{\partial\vec{z}^2}\phi(z_0,\vec{z})-m^2\phi(z_0,\vec{z})=0, \tag{4.3.8}$$

and the action S in the form

$$S_E=\frac{1}{2}\int\frac{d^dz\,dz_0}{z_0^{d+1}}\left[\partial_\mu\phi z_0^2\partial_\mu\phi+m^2\phi^2\right]. \tag{4.3.9}$$

Since AdS_{d+1} has a conformal boundary, a solution of the wave equation requires the specification of boundary data. Taking as an ansatz: $\phi(z_0,\vec{z})\longrightarrow z_0^{d-\Delta}\phi_0(\vec{z})$ as $z_0\longrightarrow 0$ and $\phi(z_0,\vec{z})=0$ as $z_0\longrightarrow\infty$, with Δ a constant. The roots of the indicial equation of (4.3.8) are $\Delta_\pm=\frac{1}{2}(d\pm\sqrt{d^2+4m^2})$, where the largest root is $\Delta=\Delta_+$. A Green's function solution to this boundary value problem has been constructed by Witten in [5]. This Green's function relates the field $\phi(z_0,\vec{z})$ in the bulk and the boundary configuration $\phi_0(\vec{x})$.

The bulk-to-boundary Green's function associated to the Klein-Gordon equation is given by

$$K_\Delta(z_0,\vec{z},\vec{x})=C_\Delta\left(\frac{z_0}{z_0^2+(\vec{z}-\vec{x})^2}\right)^\Delta,\qquad C_\Delta=\frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}, \tag{4.3.10}$$

for $\Delta>\frac{d}{2}$ [28] [29]. This solution has the correct singular behaviour as $z_0\longrightarrow 0$, viz:

$$z_0^{\Delta-d}K_\Delta(z_0,\vec{z},\vec{x})\longrightarrow\delta(\vec{z}-\vec{x}). \tag{4.3.11}$$

To prove this we will first need to demonstrate that

$$\lim_{z_0\to 0}z_0^{\Delta-d}K_\Delta(z_0,\vec{z},\vec{x})=\begin{cases}\infty & \text{if } \vec{z}=\vec{x}\\ 0 & \text{otherwise}\end{cases}. \tag{4.3.12}$$

This follows by noting that

- For $\vec{z}=\vec{x}$:

$$\lim_{z_0\to 0}z_0^{\Delta-d}K_\Delta(z_0,\vec{z},\vec{x})=\lim_{z_0\to 0}\frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\frac{1}{z_0^d}=\infty. \tag{4.3.13}$$

- For $\vec{z}\neq\vec{x}$:

$$\lim_{z_0\to 0}z_0^{\Delta-d}K_\Delta(z_0,\vec{z},\vec{x})=\lim_{z_0\to 0}\frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\frac{z_0^{2\Delta-d}}{[z_0^2-(\vec{z}-\vec{x})^2]^\Delta}=0. \tag{4.3.14}$$

To compete the demonstration of (4.3.11) we need to show that

$$\int d\vec{z}z_0^{\Delta-d}K_\Delta(z_0,\vec{z},\vec{x})=1. \tag{4.3.15}$$

Our starting point is to shift the integration variable in (4.3.15)

$$\vec{z}\longrightarrow\vec{z}'=\vec{z}-\vec{x}. \tag{4.3.16}$$

The measure is invariant so that we obtain

$$\int d\vec{z} z_0^{\Delta-d} K_\Delta(z_0,\vec{z},\vec{x}) = C_\Delta \int d\vec{z} \frac{z_0^{2\Delta-d}}{(z_0^2+\vec{z}^2)^\Delta}. \tag{4.3.17}$$

To evaluate this integral change from Cartesian co-ordinates into hyper-spherical coordinates. In d-dimensions, we write

$$z^2 = z_1^2 + z_2^2 + \cdots + z_d^2 = r^2, \tag{4.3.18}$$
$$d\vec{z} = dz_1 dz_2 \cdots dz_d = d\Omega_{d-1} r^{d-1} dr, \tag{4.3.19}$$

where $d\Omega_{d-1}$ contains the angular measure. Integration over the angles gives

$$\int d\Omega_{d-1} = \frac{d\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}. \tag{4.3.20}$$

After the change of coordinates

$$\int d\vec{z} z_0^{\Delta-d} K_\Delta(z_0,\vec{z},\vec{x}) = C_\Delta \int_{\Omega_{d-1}} d\Omega_{d-1} \int_0^{+\infty} dr z_0^{2\Delta-d} \frac{r^{d-1}}{(r^2+z_0^2)^\Delta}. \tag{4.3.21}$$

Now, we again change variables from r to $u = \frac{r^2}{r^2+z_0^2}$, to find

$$r = \sqrt{\frac{z_0^2 u}{1-u}} \qquad \Rightarrow \quad dr = du \frac{(r^2+z_0^2)^2}{2r z_0^2}. \tag{4.3.22}$$

Upon substitution we get

$$\int d\vec{z} z_0^{\Delta-d} K_\Delta(z_0,\vec{z},\vec{x}) = C_\Delta \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \cdot \frac{1}{2} \int_0^1 du \quad u^{\frac{d}{2}-1}(1-u)^{\Delta-\frac{d}{2}-1} \tag{4.3.23}$$
$$= C_\Delta \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{\Gamma(\frac{d}{2})\Gamma(\Delta-\frac{d}{2})}{\Gamma(\Delta)}. \tag{4.3.24}$$

Finally, insert the value of C_Δ into this equation to find $\int d\vec{z} z_0^{\Delta-d} K_\Delta(z_0,\vec{z},\vec{x}) = 1$. In the last step we have used the integral representation of the beta function and the connection between the beta function and the gamma function, as follows

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dt t^{a-1}(1-t)^{v-1}. \tag{4.3.25}$$

The solution to (4.3.8) with the boundary condition $\phi(z_0,\vec{z}) \longrightarrow \phi_0(\vec{x})$ as $z_0 \longrightarrow 0$ can be written in terms of the bulk-to-boundary Green's function follows

$$\phi(z_0,\vec{z}) = \int d^d x K_\Delta(z_0,\vec{z},\vec{x}) \phi_0(\vec{x}) \tag{4.3.26}$$
$$= \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})} \int d^d x \left(\frac{z_0}{z_0^2+(\vec{z}-\vec{x})^2}\right)^\Delta \phi_0(\vec{x}). \tag{4.3.27}$$

To verify this, rearrange the wave equation (4.3.8) as follows

$$(-d+1)z_0\frac{\partial}{\partial z_0}\phi(z_0,\vec{z})+z_0^2\frac{\partial^2}{\partial z_0^2}\phi(z_0,\vec{z})+z_0^2\frac{\partial^2}{\partial \vec{z}^2}\phi(z_0,\vec{z})-m^2\phi(z_0,\vec{z})=0, \tag{4.3.28}$$

and use the expression (4.3.26). The LHS of (4.3.28) is

$$\int d^dx\left[(-d+1)z_0\frac{\partial}{\partial z_0}K_\Delta(z_0,\vec{z},\vec{x})+z_0^2\frac{\partial^2}{\partial z_0^2}K_\Delta(z_0,\vec{z},\vec{x})+z_0^2\frac{\partial^2}{\partial \vec{z}^2}K_\Delta(z_0,\vec{z},\vec{x})-m^2K_\Delta(z_0,\vec{z},\vec{x})\right]\phi_0(\vec{x})\,. \tag{4.3.29}$$

We need to evaluate the following derivatives

$$\frac{\partial}{\partial z_0}K_\Delta(z_0,\vec{z},\vec{x})=K_\Delta(z_0,\vec{z},\vec{x})\left[\frac{\Delta\left[(\vec{z}-\vec{x})^2-z_0^2\right]}{z_0\left[z_0^2+(\vec{z}-\vec{x})^2\right]}\right]. \tag{4.3.30}$$

$$\frac{\partial^2}{\partial z_0^2}K_\Delta(z_0,\vec{z},\vec{x})=K_\Delta(z_0,\vec{z},\vec{x})\left[\frac{\Delta\left[z_0^4-4z_0^2(\vec{z}-\vec{x})^2-(\vec{z}-\vec{x})^4\right]+\Delta^2\left[z_0^4-2z_0^2(\vec{z}-\vec{x})^2+(\vec{z}-\vec{x})^4\right]}{z_0^2\left[z_0^2+(\vec{z}-\vec{x})^2\right]^2}\right]. \tag{4.3.31}$$

$$\frac{\partial^2}{\partial z_i^2}K_\Delta(z_0,\vec{z},\vec{x})=K_\Delta(z_0,\vec{z},\vec{x})\left[\frac{\Delta\left[-2z_0^2-2(\vec{z}-\vec{x})^2+4(z_i-x_i)^2\right]+\Delta^2\left[4(z_i-x_i)^2\right]}{\left[z_0^2+(\vec{z}-\vec{x})^2\right]^2}\right].$$

$$\begin{aligned}\frac{\partial^2}{\partial \vec{z}^2}K_\Delta(z_0,\vec{z},\vec{x})&=\sum_{i=1}^{d}\frac{\partial^2}{\partial z_i^2}K_\Delta(z_0,\vec{z},\vec{x}) \qquad (4.3.32)\\ &=K_\Delta(z_0,\vec{z},\vec{x})\left[\frac{\Delta\left[-2d(z_0^2+(\vec{z}-\vec{x})^2)+4(\vec{z}-\vec{x})^2\right]+\Delta^2\left[4(z_i-x_i)^2\right]}{\left[z_0^2+(\vec{z}-\vec{x})^2\right]^2}\right]. \qquad (4.3.33)\end{aligned}$$

Upon substitution, (4.3.29) becomes

$$\begin{aligned}\int d^dx\frac{K_\Delta(z_0,\vec{z},\vec{x})}{\left[z_0^2+(\vec{z}-\vec{x})^2\right]^2}\Big[&z_0^4\left[-\Delta(-d+1)_\Delta+\Delta^2-2d\Delta-m^2\right]+z_0^2(\vec{z}-\vec{x})^2\left[2\Delta^2-2d\Delta-2m^2\right]\\&+(\vec{z}-\vec{x})^4\left[\Delta(-d+1)-\Delta+\Delta^2-m^2\right]\Big]\phi_0(\vec{x})\,.\end{aligned} \tag{4.3.34}$$

These coefficients all vanish when we replace $m^2=\Delta^2-d\Delta$. This concludes the demonstration that (4.3.26) is the solution of the wave equation (4.3.8).

4.4 The two-point Function

In this section we follow the procedure developed in [5] and [6]. We will explicitly calculate the two-point correlation function $\langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle$, for a CFT$_d$ scalar operator of dimension Δ. We will compute this correlator using two different methods. The first method is a direct computation in position space, while the second method is an indirect computation in Fourier space. The reason why we compute the two-point function twice and obtain different results is that both the kinetic and the mass terms produce divergent integrals. These combine in the form $\infty-\infty$, if one uses the Feynman parameter method. We will decide which one is the correct result using the Ward identity for R-symmetry that relates the two-point correlator with the 3-point correlator. This is a good example for how subtleties arise in the computation of correlators.

4.4.1 The 2-point Correlation Function in Position Space. As we have reviewed above, AdS/CFT correspondence implies that we can compute the two-point correlation function in CFT as follows

$$\left\langle \mathcal{O}(\vec{X})\mathcal{O}(\vec{Y})\right\rangle = \frac{\delta}{\delta\phi_0(\vec{X})}\frac{\delta}{\delta\phi_0(\vec{Y})}e^{-S(\phi_0)}\bigg|_{\phi_0=0}.$$

This computation uses the holographic dictionary summarized in (4.1.3). Note that from now on we will use z to denote $z_0, \vec{z}$. With $\phi(z_0,\vec{z}) = \int d^dx K_\Delta(z,\vec{x})\phi_0(\vec{x})$ we have

$$\begin{aligned} S_E[\phi_0] =& \frac{1}{2}\int \frac{d^dz\, dz_0}{z_0^{d+1}}\left[\partial_\mu\left(\int d^dx K_\Delta(z,\vec{x})\phi_0(\vec{x})\right) z_0^2\partial_\mu\left(\int d^dy K_\Delta(z,\vec{y})\phi_0(\vec{y})\right)\right. \\ &\left. + m^2\partial_\mu\int d^dx K_\Delta(z,\vec{x})\phi_0(\vec{x})\int d^dy K_\Delta(z,\vec{y})\phi_0(\vec{y})\right]. \end{aligned} \tag{4.4.1}$$

To obtain the two point correlator we need to compute the following derivatives

$$\frac{\delta}{\delta\phi_0(\vec{X})}\frac{\delta}{\delta\phi_0(\vec{Y})}e^{-S(\phi_0)} = \left[\frac{\delta}{\delta\phi_0(\vec{Y})}S(\phi_0)\frac{\delta}{\delta\phi_0(\vec{X})}S(\phi_0) - \frac{\delta}{\delta\phi_0(\vec{X})}\frac{\delta}{\delta\phi_0(\vec{Y})}S(\phi_0)\right]e^{-S(\phi_0)}, \tag{4.4.2}$$

and

$$\begin{aligned} \frac{\delta}{\delta\phi_0(\vec{Y})}S(\phi_0) =& \frac{1}{2}\int \frac{d^dz\, dz_0}{z_0^{d+1}}\left[\partial_\mu K_\Delta(z,\vec{Y})\left(z_0^2\partial_\mu\int d^dy K_\Delta(z,\vec{y})\phi_0(\vec{y}) + \partial_\mu z_0^2\int d^dx K_\Delta(z,\vec{x})\phi_0(\vec{x})\right)\right. \\ &\left. + m^2 K_\Delta(z,\vec{Y})\left(\int d^dy K_\Delta(z,\vec{y})\phi_0(\vec{y}) + \int d^dx K_\Delta(z,\vec{x})\phi_0(\vec{x})\right)\right]. \end{aligned}$$

It is clear that when we set $\phi_0 = 0$, the first term of the RHS of equation (4.4.2) vanishes. Therefore, we only need to consider the first term

$$\frac{\delta}{\delta\phi_0(\vec{X})}\frac{\delta}{\delta\phi_0(\vec{Y})}S(\phi_0) = \frac{1}{2}\int\frac{d^dz\, dz_0}{z_0^{d+1}}\left[2\partial_\mu K_\Delta(z,\vec{X})z_0^2\partial_\mu K_\Delta(z,\vec{Y}) + 2m^2 K_\Delta(z,\vec{X})K_\Delta(z,\vec{Y})\right].$$

Making use of this result the 2-point function can be written as

$$\begin{aligned} \langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle &= -\int\frac{d^dz\, dz_0}{z_0^{d+1}}\left(\partial_\mu K_\Delta(z,\vec{x})z_0^2\partial_\mu K_\Delta(z,\vec{y}) + m^2 K_\Delta(z,\vec{x})K_\Delta(z,\vec{y})\right). \\ &= -\int d^dz\, dz_0 z_0^{-d-1}\left[\partial_0 K_\Delta(z,\vec{x})z_0^2\partial_0 K_\Delta(z,\vec{y})\right. \\ &\quad\left. + \sum_{i=1}^{d}\partial_i K_\Delta(z,\vec{x})z_0^2\partial_i K_\Delta(z,\vec{y}) + m^2 K_\Delta(z,\vec{x})K_\Delta(z,\vec{y})\right]. \end{aligned} \tag{4.4.3}$$

To evaluate the two contributions to the action perform an integration by parts, to obtain

$$\begin{aligned} &\int d^dz\, dz_0 z_0^{-d-1}\partial_0 K_\Delta(z,\vec{x})z_0^2\partial_0 K_\Delta(z,\vec{y}) \\ &= \int d^dz\left[K_\Delta(z,\vec{x})z_0^{-d+1}\partial_0 K_\Delta(z,\vec{y})\right]_{z_0=\epsilon\to 0}^{\infty} - \int d^dz\, dz_0 K_\Delta(z,\vec{x})\partial_0\left[z_0^{-d+1}\partial_0 K_\Delta(z,\vec{y})\right], \end{aligned} \tag{4.4.4}$$

and

$$\begin{aligned}
&\int d^d z\, dz_0 z_0^{-d-1} \partial_i K_\Delta(z,\vec{x}) z_0^2 \partial_i K_\Delta(z,\vec{y}) \\
&= \int d^{d-1} z\, dz_0 \left[K_\Delta(z,\vec{x}) z_0^{-d+1} \partial_i K_\Delta(z,\vec{y}) \right]_{z_i=-\infty}^{+\infty} - \int d^d z\, dz_0 z_0^{-d+1} K_\Delta(z,\vec{x}) \partial_i \partial_i K_\Delta(z,\vec{y}) \\
&= -\int d^d z\, dz_0 K_\Delta(z,\vec{x}) z_0^{-d+1} \partial_i \partial_i K_\Delta(z,\vec{y}). \qquad (4.4.5)
\end{aligned}$$

Inserting (4.4.4) and (4.4.5) into (4.4.3), we get

$$\begin{aligned}
&\langle \mathcal{O}(\vec{x}) \mathcal{O}(\vec{y}) \rangle \\
&= -\int d^d z \left[K_\Delta(z,\vec{x}) z_0^{-d+1} \partial_0 K_\Delta(z,\vec{y}) \right]_{z_0=\epsilon\to 0}^{\infty} \qquad (4.4.6) \\
&\quad - \int d^d z\, dz_0 K_\Delta(z,\vec{x}) \left(-\partial_0 \left[z_0^{-d+1} \partial_0 K_\Delta(z,\vec{y}) \right] - \sum_{i=1}^{d} z_0^{-d+1} \partial_i \partial_i K_\Delta(z,\vec{y}) + m^2 z_0^{-d-1} K_\Delta(z,\vec{y}) \right) \\
&= \lim_{\epsilon\to 0} \int d^d z \epsilon^{1-d} K_\Delta(\epsilon,\vec{z},\vec{x}) \frac{\partial}{\partial z_0} K_\Delta(z_0,\vec{z},\vec{y}) \Big|_{z_0=\epsilon} \\
&\quad - \int d^d z\, dz_0\, z_0^{-d-1} K_\Delta(z,\vec{x}) \left(-z_0^{d+1} \frac{\partial}{\partial z_0} \left[z_0^{-d+1} \frac{\partial}{\partial z_0} K_\Delta(z,\vec{y}) \right] - z_0^2 \frac{\partial^2}{\partial \vec{z}^2} K_\Delta(z,\vec{y}) + m^2 K_\Delta(z,\vec{y}) \right).
\end{aligned}$$

Using the free equation of motion obeyed by K_Δ we obtain

$$\begin{aligned}
\langle \mathcal{O}(\vec{x}) \mathcal{O}(\vec{y}) \rangle &= \lim_{\epsilon\to 0} \int d^d z \epsilon^{1-d} K_\Delta(\epsilon,\vec{z},\vec{x}) \frac{\partial}{\partial z_0} K_\Delta(z_0,\vec{z},\vec{y}) \Big|_{z_0=\epsilon} \\
&= \frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}} \Gamma(\Delta-\frac{d}{2})} \lim_{\epsilon\to 0} \int d^d z \epsilon^{1-d} K_\Delta(\epsilon,\vec{z},\vec{x}) \left(\frac{(\vec{z}-\vec{y})^2 - \epsilon^2}{[\epsilon^2 + (\vec{z}-\vec{y})^2]^2} \right) \left(\frac{\epsilon}{\epsilon^2 + (\vec{z}-\vec{y})^2} \right)^{\Delta-1} \\
&= \frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}} \Gamma(\Delta-\frac{d}{2})} \lim_{\epsilon\to 0} \int d^d z \epsilon^{\Delta-d} K_\Delta(\epsilon,\vec{z},\vec{x}) \left(\frac{(\vec{z}-\vec{y})^2 - \epsilon^2}{[\epsilon^2 + (\vec{z}-\vec{y})^2]^2} \right) \left(\frac{1}{\epsilon^2 + (\vec{z}-\vec{y})^2} \right)^{\Delta-1}.
\end{aligned}$$

Finally, use the fact that $z_0^{\Delta-d} K_\Delta(z_0,\vec{z},\vec{x}) \longrightarrow \delta(\vec{z}-\vec{x})$ as $z_0 \longrightarrow 0$, to find

$$\begin{aligned}
\langle \mathcal{O}(\vec{x}) \mathcal{O}(\vec{y}) \rangle &= \frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}} \Gamma(\Delta-\frac{d}{2})} \lim_{\epsilon\to 0} \int d^d z \delta(\vec{z}-\vec{x}) \left(\frac{(\vec{z}-\vec{y})^2 - \epsilon^2}{[\epsilon^2 + (\vec{z}-\vec{y})^2]^2} \right) \left(\frac{1}{\epsilon^2 + (\vec{z}-\vec{y})^2} \right)^{\Delta-1} \\
&= \frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}} \Gamma(\Delta-\frac{d}{2})} \lim_{\epsilon\to 0} \left(\frac{(\vec{x}-\vec{y})^2 - \epsilon^2}{[\epsilon^2 + (\vec{x}-\vec{y})^2]^2} \right) \left(\frac{1}{\epsilon^2 + (\vec{x}-\vec{y})^2} \right)^{\Delta-1} \\
&= \frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}} \Gamma(\Delta-\frac{d}{2})} \frac{1}{|\vec{x}-\vec{y}|^{2\Delta}}. \qquad (4.4.7)
\end{aligned}$$

This is precisely of the form expected to describe the two-point function of scalar primary operators of dimension Δ in CFT.

4.4.2 The 2-point Correlation Function in Momentum Space. As presented in the Appendix of [6], we first, Fourier transform the variable $\vec{z}$ in (4.3.8). Our conventions are specified by

$$\phi(z_0,\vec{z}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int d\vec{k}\, e^{i\vec{k}\cdot\vec{z}} \phi(z_0,\vec{k}). \qquad (4.4.8)$$

Inserting this into the free wave equation (4.3.8) we obtain

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{z}}\left[z_0^{d+1}\frac{\partial}{\partial z_0}z_0^{-d+1}\frac{\partial}{\partial z_0}\phi(z_0,\vec{k})-z_0^2k^2\phi(z_0,\vec{k})-m^2\phi(z_0,\vec{k})\right]=0\,.$$

This implies

$$z_0^{d+1}\frac{\partial}{\partial z_0}\left[z_0^{-d+1}\frac{\partial}{\partial z_0}\phi(z_0,\vec{k})\right]-(z_0^2k^2+m^2)\phi(z_0,\vec{k})=0. \tag{4.4.9}$$

We will argue that the solution of this equation is given by

$$\phi(z_0,\vec{k})=z_0^{\frac{d}{2}}F_\nu(ik\,z_0), \tag{4.4.10}$$

where F_ν is the solution of the Bessel equation of index

$$\nu=\sqrt{\frac{d^2}{4}+m^2}=\Delta-\frac{d}{2}. \tag{4.4.11}$$

We can rewrite (4.4.9) as follows

$$z_0^{d+1}\left[(-d+1)z_0^{-d}\frac{\partial}{\partial z_0}\phi(z_0,\vec{k})+z_0^{-d+1}\frac{\partial^2}{\partial z_0^2}\phi(z_0,\vec{k})\right]-(z_0^2k^2+m^2)\phi(z_0,\vec{k})=0. \tag{4.4.12}$$

To verify that we have a solution, we first compute

$$\frac{\partial}{\partial z_0}\phi(z_0,\vec{k})=\frac{d}{2}z_0^{\frac{d}{2}-1}F_\nu(ik\,z_0)+ik\,z_0^{\frac{d}{2}}\frac{\partial}{\partial(ikz_0)}F_\nu(ik\,z_0) \tag{4.4.13}$$

$$\begin{aligned}\frac{\partial^2}{\partial(ikz_0)^2}\phi(z_0,\vec{k})=&\frac{d}{2}\left(\frac{d}{2}-1\right)z_0^{\frac{d}{2}-2}F_\nu(ik\,z_0)+2ik\frac{d}{2}z_0^{\frac{d}{2}-1}\frac{\partial}{\partial(ikz_0)}F_\nu(ik\,z_0)\\&+(ik)^2\,z_0^{\frac{d}{2}}\frac{\partial^2}{\partial(ikz_0)^2}F_\nu(ik\,z_0).\end{aligned} \tag{4.4.14}$$

Therefore (4.4.12) becomes

$$\begin{aligned}&z_0^{d+1}\Bigg((-d+1)z_0^{-d}\left[\frac{d}{2}z_0^{\frac{d}{2}-1}F_\nu(ik\,z_0)+ik\,z_0^{\frac{d}{2}}\frac{\partial}{\partial(ikz_0)}F_\nu(ik\,z_0)\right]\\&+z_0^{-d+1}\left[\frac{d}{2}\left(\frac{d}{2}-1\right)z_0^{\frac{d}{2}-2}F_\nu(ik\,z_0)+2ik\frac{d}{2}z_0^{\frac{d}{2}-1}\frac{\partial}{\partial(ikz_0)}F_\nu(ik\,z_0)\right.\\&\left.+\,(ik)^2\,z_0^{\frac{d}{2}}\frac{\partial^2}{\partial(ikz_0)^2}F_\nu(ik\,z_0)\right]\Bigg)-(z_0^2k^2+m^2)z_0^{\frac{d}{2}}F_\nu(ik\,z_0),\end{aligned} \tag{4.4.15}$$

which gives

$$\begin{aligned}&\frac{d}{2}(-d+1)z_0^{\frac{d}{2}}F_\nu(ik\,z_0)+ik\,(-d+1)z_0^{\frac{d}{2}+1}\frac{\partial}{\partial(ikz_0)}F_\nu(ik\,z_0)\\&+\left[\frac{d}{2}\left(\frac{d}{2}-1\right)z_0^{\frac{d}{2}}F_\nu(ik\,z_0)+ik\,d\,z_0^{\frac{d}{2}+1}\frac{\partial}{\partial(ikz_0)}F_\nu(ik\,z_0)\right.\\&\left.+\,(ik)^2\,z_0^{\frac{d}{2}+2}\frac{\partial^2}{\partial(ikz_0)^2}F_\nu(ik\,z_0)\right]-(z_0^2k^2+m^2)z_0^{\frac{d}{2}}F_\nu(ik\,z_0).\end{aligned} \tag{4.4.16}$$

Finally, regroup all of the terms in the expression above and factor out $z_0^{\frac{d}{2}}$ to get

$$(ikz_0)^2\frac{\partial^2}{\partial(ikz_0)^2}F_\nu(ik\,z_0)+(ikz_0)\frac{\partial}{\partial(ikz_0)}F_\nu(ik\,z_0)+\left[(ikz_0)^2-\left(\frac{d^2}{4}+m^2\right)\right]F_\nu(ik\,z_0)\,. \quad (4.4.17)$$

Using the Bessel equation of index $\nu=\sqrt{\frac{d^2}{4}+m^2}=\Delta-\frac{d}{2}$, we see that the above expression vanishes which completes the demonstration that (4.4.10) does indeed solve the equation of motion in momentum space. Now, writing the action (4.3.9) in terms of Fourier components, we have

$$\begin{aligned}S&=\frac{1}{2(2\pi)^d}\int d\vec{k}\,d\vec{k}'\frac{d^dz\,dz_0}{z_0^{d+1}}e^{i(\vec{k}+\vec{k}')\cdot\vec{z}}\left[z_0^2\frac{\partial}{\partial z_0}\phi(z_0,\vec{k})\frac{\partial}{\partial z_0}\phi(z_0,\vec{k}')+(-z_0^2k_ik_i'+m^2)\phi(z_0,\vec{k})\phi(z_0,\vec{k}')\right]\\&=\frac{1}{2}\int dz_0\,d\vec{k}\,d\vec{k}'z_0^{-d+1}\delta(\vec{k}+\vec{k}')\left[\frac{\partial}{\partial z_0}\phi(z_0,\vec{k})\frac{\partial}{\partial z_0}\phi(z_0,\vec{k}')+\left(k^2+\frac{m^2}{z_0^2}\right)\phi(z_0,\vec{k})\phi(z_0,\vec{k}')\right].\end{aligned}$$

Integrate by parts with respect to z_0 and use the fact that we have a solution of the wave equation to obtain

$$S=\frac{1}{2}\int d\vec{k}\,d\vec{k}'\delta(\vec{k}+\vec{k}')\lim_{z_0\to\epsilon}z_0^{-d+1}\left[\phi(z_0,\vec{k})\frac{\partial}{\partial z_0}\phi(z_0,\vec{k}')\right]. \quad (4.4.18)$$

We can write the solution to the wave equation (4.4.17) as

$$\phi(z_0,\vec{k})=K^\epsilon(z_0,\vec{k})\phi_p(\vec{k}), \quad (4.4.19)$$

which is the one-shell bulk field, with $\phi_p(\vec{k})=\phi(\epsilon,\vec{k})$, where the boundary to the bulk propagator obeys

$$\lim_{z_0\longrightarrow\epsilon}K^\epsilon(z_0,\vec{k})=1,\qquad\lim_{z_0\longrightarrow\infty}K^\epsilon(z_0,\vec{k})=0. \quad (4.4.20)$$

We need to evaluate the action on a solution to the equation of motion. We choose the solution $z_0^{d/2}\mathcal{K}_\nu(kz_0)$, where $k=|\vec{k}|$ and $\nu=\Delta-d/2$. This solution decreases exponentially as $z_0\longrightarrow\infty$ and behaves as $z_0^{d-\Delta}$ as $z_0\longrightarrow 0$. The other possible solution $z_0^{d/2}I_\nu(kz_0)$ behaves as z^Δ at the boundary, and we discard it because it increases exponentially as $z_0\longrightarrow\infty$. We therefore write

$$K^\epsilon(z_0,\vec{k})=(\frac{z_0}{\epsilon})^{d/2}\frac{\mathcal{K}_\nu(kz_0)}{\mathcal{K}_\nu(k\epsilon)}. \quad (4.4.21)$$

The two-point correlation function in Fourier space is given by

$$\left\langle\mathcal{O}(\vec{k})\mathcal{O}(\vec{k}')\right\rangle=-\epsilon^{-d+1}\delta(\vec{k}+\vec{k}')\lim_{z_0\to\epsilon}\partial_{z_0}K^\epsilon(z_0,\vec{k}), \quad (4.4.22)$$

where $\mathcal{K}_\nu$ is the modified Bessel function. For non-integer ν, $\mathcal{K}_\nu$ has the following expansion

$$\mathcal{K}_\nu(kz_0)=(kz_0)^{-\nu}(a_0+a_1(kz_0)^2+\cdots)+(kz_0)^\nu(b_0+b_1(kz_0)^2+\cdots) \quad (4.4.23)$$

$$a_0=2^{\nu-1}\Gamma(\nu)\qquad b_0=-2^{-\nu-1}\Gamma(1-\nu)/\nu. \quad (4.4.24)$$

Then (4.4.22) gives

$$\begin{aligned}\left\langle \mathcal{O}(\vec{k})\mathcal{O}(\vec{k'})\right\rangle &= -\epsilon^{-d+1}\delta(\vec{k}+\vec{k'})\lim_{z_0\to\epsilon}\frac{\partial}{\partial z_0}z_0^{\frac{d}{2}}\frac{\left[1+\frac{a_1}{a_0}(kz_0)^2+\cdots\right]+\frac{b_0}{a_0}(kz_0)^{2\nu}\left[1+\frac{b_1}{b_0}(kz_0)^2+\cdots\right]}{\left[1+\frac{a_1}{a_0}(k\epsilon)^2+\cdots\right]+\frac{b_0}{a_0}(k\epsilon)^{2\nu}\left[1+\frac{b_1}{b_0}(k\epsilon)^2\right]\cdots}\\ &= -\epsilon^{-d+1}\delta(\vec{k}+\vec{k'})\lim_{z_0\to\epsilon}z_0^{\frac{d}{2}-1}\frac{\left[\frac{d}{2}-\nu+c_1(kz_0)^2+\cdots\right]+\frac{b_0}{a_0}(kz_0)^{2\nu}\left[\frac{d}{2}+\nu+c_2(kz_0)^2+\cdots\right]}{\left[1+d_1(k\epsilon)^2+\cdots\right]+\frac{b_0}{a_0}(k\epsilon)^{2\nu}\left[1+d_2(k\epsilon)^2\right]\cdots}\\ &= \epsilon^{-d}\left[\left(\frac{d}{2}-\nu+c_3(k\epsilon)^2+\cdots\right)+\frac{b_0}{a_0}(k\epsilon)^{2\nu}\left(2\nu+d_3(k\epsilon)^2+\cdots\right)\right]\end{aligned}$$

Here in the last line we have dropped terms that are positive integer powers of $(kz_0)^2$. These terms produce contact terms in the correlator after transforming to $x-$space, and since we only consider correlators of separated operators, they will not contribute. Thus, we find

$$\begin{aligned}\left\langle \mathcal{O}(\vec{k})\mathcal{O}(\vec{k'})\right\rangle &= -\epsilon^{-d}(k\epsilon)^{2\nu}\delta(\vec{k}+\vec{k'})2\nu\frac{b_0}{a_0}\\ &= \epsilon^{2(\Delta-d)}\delta(\vec{k}+\vec{k'})\left(\frac{k}{2}\right)^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}(2\nu)\,.\end{aligned}\tag{4.4.25}$$

where we have only kept the leading terms as $\epsilon\longrightarrow 0$. To return to position space, we need to apply the inverse Fourier transform to this equation. We have

$$\begin{aligned}\langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle &= \frac{1}{(2\pi)^d}\int d\vec{k}\int d\vec{k'}e^{i\vec{k}\vec{x}}e^{i\vec{k'}\vec{y}}\left\langle\mathcal{O}(\vec{k})\mathcal{O}(\vec{k'})\right\rangle\\ &= \epsilon^{2(\Delta-d)}(2\nu)2^{-2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\frac{1}{(2\pi)^d}\int d\vec{k}e^{i\vec{k}(\vec{x}-\vec{y})}k^{2\nu}\,.\end{aligned}$$

Now we use the fact that $\nu=\Delta-d/2$, and the power transform law

$$\frac{1}{(2\pi)^d}\int d\vec{k}e^{i\vec{k}\vec{X}}k^{2\nu}\;=\;\frac{2^{2\nu}}{\pi^{d/2}}\frac{\Gamma(\nu+\frac{d}{2})}{\Gamma(-\nu)}\frac{1}{|\vec{X}|^{2\nu+d}}\,,\tag{4.4.26}$$

to obtain the 2–point correlation function in the final following form

$$\langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle=\frac{1}{\pi^{d/2}}\epsilon^{2(\Delta-d)}\frac{(2\Delta-d)}{\Delta}\frac{\Gamma(\Delta+1)}{\Gamma(\Delta-\frac{d}{2})}\frac{1}{|\vec{x}-\vec{y}|^{2\Delta}}.\tag{4.4.27}$$

The power of ϵ indicates the rate of growth of this correlation function as we approach the boundary of the AdS spacetime. The value of the 2–point correlation functions in the momentum space is exactly a factor of $\frac{2\Delta-d}{\Delta}$ times that of (4.4.7) in the position space. This discrepancy between the two results is a clear indication that the computation of the two-point function is subtle.

4.5 Correct Value of the Two-point Correlation Functions

Our discussion is restricted to the AdS_{d+1} background, and has presented two ways of calculating the two-point correlation functions. The two methods give different values for the correlator. We see that the computation of two-point correlation functions is subtle because it is necessary to

introduce a cutoff near the boundary [6]. We will decide what the correct result for the two-point function is by making sure it obeys the Ward identity for the three–point correlator of conserved current $J_i^a(\vec{z})$ with scalar operators $\mathcal{O}_\Delta(\vec{x})$ and $\mathcal{O}_\Delta(\vec{y})$, where Δ is the scale dimension. The Ward identity relates the three and two point correlation functions as follows

$$\left\langle \partial_i J_i^a(z)\phi^I(x)\phi^J(y)\right\rangle = (T^a)^{kI}\delta(x-z)\left\langle \phi^K(x)\phi^J(y)\right\rangle + (T^a)^{kJ}\delta(y-z)\left\langle \phi^I(x)\phi^K(y)\right\rangle . \quad (4.5.1)$$

For the three–point correlation function $\left\langle J_i^a(x)\mathcal{O}^I(x)\mathcal{O}^J(y)\right\rangle$ there is a unique CFT form, for every dimension d, given by

$$\begin{aligned}&\left\langle J_i^a\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\right\rangle \\ &\quad = -\xi(d-2)(T^a)^{IJ}\frac{1}{(x-y)^{2\delta-d+2}}\frac{1}{(x-y)^{d-2}(y-z)^{d-2}}\left[\frac{(x-z)_i}{(x-z)^2}-\frac{(y-z)_i}{(y-z)^2}\right],\end{aligned} \quad (4.5.2)$$

where ξ is a constant and $(T^a)^{IJ}$ are the Lie algebra generators. The generators are imaginary antisymmetric matrices.

In order to use the Ward identity (4.5.21), we need to obtain the flavor current three–point correlation function $\left\langle J_i^a(x)\mathcal{O}^I(x)\mathcal{O}^J(y)\right\rangle$ from the gravity side of the correspondence. To do so we extend the Euclidean AdS_{d+1} action (4.3.3), to include the gauge fields $A_\mu^a(x_0,\vec{x})$ which are the sources of the conserved flavor currents $J_i^a(\vec{x})$. The gauge–invariant extension of the action (4.3.3) is

$$S[\phi^I,A_\mu^a] = \frac{1}{2}\int d^dzdz_0\sqrt{g}\left[g^{\mu\nu}D_\mu\phi^I D_\nu\phi^I + m^2\phi^I\phi^I\right] \quad (4.5.3)$$

$$D_\mu = \partial_\mu\phi^I - iA_\mu^a(T^a)^{IJ}\phi^J , \quad (4.5.4)$$

where $(T^a)^{IJ}$ are the generators of the vector representation of the $SO(6)$ flavor group. To evaluate flavor-current correlators in the boundary CFT from the gravvity side, we need a Green's function $G_{\mu i}(z,\vec{x})$ to construct the gauge field $A_\mu^a(z)$ from its boundary value $A_\mu^a(\vec{x})$. The relevant Green's function is

$$\mathcal{G}_{\mu i}(z,\vec{x}) = C^d\frac{z_0^{d-2}}{[z_0^2+(\vec{z}-\vec{x})^2]^{d-1}}J_{\mu i}(z-\vec{x}) \quad (4.5.5)$$

$$= C^d\left(\frac{z_0^{d-2}}{z_0^2+(\vec{z}-\vec{x})^2}\right)^{d-2}\partial_\mu\left(\frac{(z-\vec{x})_i}{z_0^2+(\vec{z}-\vec{x})^2}\right) , \quad (4.5.6)$$

where C^d is normalization constant which is determined to be

$$C^d = \frac{\Gamma(d)}{2\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})} . \quad (4.5.7)$$

The flavor-current correlator in the boundary CFT obtained from the AdS supergravity action is given by

$$\left\langle J_i^a\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\right\rangle = \frac{\delta}{\delta A_i^a(\vec{z})}\frac{\delta}{\delta\phi_0^I(\vec{x})}\frac{\delta}{\delta\phi_0^J(\vec{y})}e^{-S}\Big|_{A_i^a=\phi_0=0}, \quad (4.5.8)$$

where $\phi^I(\vec{x})$ is a real scalar field corresponding to $\mathcal{O}^I(\vec{x})$, and

$$A_\mu^a(z) = \int d^dxG_{\mu i}(z,\vec{x})A_i^a(\vec{x}). \quad (4.5.9)$$

The cubic vertex of the action (4.5.3) leads to the AdS integral representation for the flavor-current correlator in the gauge theory. The result is

$$\left\langle J_i^a \mathcal{O}^I(\vec{x}) \mathcal{O}^J(\vec{y}) \right\rangle = (T^a)^{IJ} \int \frac{d^d\omega d\omega_0}{\omega_0^{d+1}} G_{\mu i}(\omega, \vec{z}) \omega_0^2 \Big(K_\Delta(\omega, \vec{x}) \frac{\partial}{\partial \omega_\mu} K_\Delta(\omega, \vec{y}) - K_\Delta(\omega, \vec{y}) \frac{\partial}{\partial \omega_\mu} K_\Delta(\omega, \vec{x}) \Big). \tag{4.5.10}$$

To evaluate this integral, set $\vec{z} = 0$, and apply the inversion (4.2.11). This gives

$$\begin{aligned} \left\langle J_i^a \mathcal{O}^I(\vec{x}) \mathcal{O}^J(\vec{y}) \right\rangle &= C^d (T^a)^{IJ} \vec{x}'^{2\Delta} \vec{y}'^{2\Delta} \int \frac{d^d\omega' d\omega_0'}{\omega_0'^{d+1}} \omega_0'^{d-1} \frac{\omega_0'}{\omega'^2} J_{\mu i}(\omega') \\ &\times \Big(K_\Delta(\omega', \vec{x}') \frac{\partial \omega_\rho'}{\partial \omega_\mu} \frac{\partial}{\partial \omega_\rho'} K_\Delta(\omega', \vec{y}') - K_\Delta(\omega', \vec{y}') \frac{\partial \omega_\rho'}{\partial \omega_\mu} \frac{\partial}{\partial \omega_\rho'} K_\Delta(\omega', \vec{x}') \Big) . \end{aligned} \tag{4.5.11}$$

Using (4.2.12) and then (4.2.14) we find

$$\begin{aligned} &\left\langle J_i^a \mathcal{O}^I(\vec{x}) \mathcal{O}^J(\vec{y}) \right\rangle \\ &= \frac{C^d (T^a)^{IJ}}{\vec{x}^{2\Delta} \vec{y}^{2\Delta}} \int \frac{d^d\omega' d\omega_0'}{\omega_0'} \left(K_\Delta(\omega', \vec{x}') \frac{\partial}{\partial \omega_i'} K_\Delta(\omega', \vec{y}') - K_\Delta(\omega', \vec{y}') \frac{\partial}{\partial \omega_i'} K_\Delta(\omega', \vec{x}') \right) . \end{aligned}$$

Integrate by parts and make use of the following identity

$$\frac{\partial}{\partial z_i} K_\Delta(z, \vec{x}) = -\frac{\partial}{\partial x_i} K_\Delta(z, \vec{x}), \tag{4.5.12}$$

to obtain

$$\begin{aligned} \left\langle J_i^a \mathcal{O}^I(\vec{x}) \mathcal{O}^J(\vec{y}) \right\rangle &= 2 \frac{C^d (T^a)^{IJ}}{|\vec{x}|^{2\Delta} |\vec{y}|^{2\Delta}} \frac{\partial}{\partial x_i'} \int \frac{d^d\omega' d\omega_0'}{\omega_0'} K_\Delta(\omega', \vec{x}') K_\Delta(\omega', \vec{y}') \\ &= 2 \frac{C^d C_\Delta^2 (T^a)^{IJ}}{|\vec{x}|^{2\Delta} |\vec{y}|^{2\Delta}} \frac{\partial}{\partial x_i'} \int d^d\omega' d\omega_0' \frac{\omega_0'^{2\Delta-1}}{[\omega_0'^2 + (\vec{\omega'} - \vec{x}')]^{2\Delta} [\omega_0'^2 + (\vec{\omega'} - \vec{y}')]^{2\Delta}} \\ &= 2 \frac{C^d C_\Delta^2 (T^a)^{IJ}}{|\vec{x}|^{2\Delta} |\vec{y}|^{2\Delta}} I(\Delta, d) \frac{\partial}{\partial x_i'} |\vec{x} - \vec{y}|^{2\Delta - d} \\ &= 2 \frac{C^d C_\Delta^2 (T^a)^{IJ}}{|\vec{x}|^{2\Delta} |\vec{y}|^{2\Delta}} I(\Delta, d)(2\Delta - d) |\vec{x} - \vec{y}|^{2\Delta - d - 2} (x_i - y_i) . \end{aligned} \tag{4.5.13}$$

In the last line we have used the Feynman parameter representation of the integral, which is defined as

$$\int d^d z dz_0 \frac{z_0^a}{[z_0^2 - (\vec{z} - \vec{x})^2]^b [z_0^2 - (\vec{z} - \vec{y})^2]^c} = I[a, b, c, d] |\vec{x} - \vec{y}|^{1 + a + d - 2b - 2c} , \tag{4.5.14}$$

$$I[a, b, c, d] = \frac{\pi^{d/2} \Gamma(\frac{a}{2} + \frac{1}{2}) \Gamma(b + c - \frac{d}{2} - \frac{a}{2} - \frac{1}{2})}{2\Gamma(b)\Gamma(c)} \frac{\Gamma(\frac{1}{2} + \frac{d}{2} + \frac{a}{2} - b) \Gamma(\frac{1}{2} + \frac{d}{2} + \frac{a}{2} - c)}{\Gamma(1 + a + d - b - c)}. \tag{4.5.15}$$

Now use the following identity

$$\frac{1}{(\vec{x'} - \vec{y'})^2} = \frac{\vec{x}^2 \vec{y}^2}{(\vec{x} - \vec{y})^2} , \tag{4.5.16}$$

to obtain

$$\left\langle J_i^a \mathcal{O}^I(\vec{x}) \mathcal{O}^J(\vec{y}) \right\rangle = 2C^d C_\Delta^2 (T^a)^{IJ} I(\Delta, d) \left(\frac{1}{\vec{x}^{d-2} \vec{y}^{d-2} (\vec{x}-\vec{y})^{2\Delta-d+2}} \right) \left(\frac{x_i}{\vec{x}} - \frac{y_i}{\vec{y}} \right) . \tag{4.5.17}$$

Now substitute the constants into this expression and perform the following coordinate transformations

$$\vec{x} \longrightarrow \vec{x} - \vec{z}, \qquad \vec{y} \longrightarrow \vec{y} - \vec{z} \tag{4.5.18}$$

to find

$$\begin{aligned} \left\langle J_i^a \mathcal{O}^I(\vec{x}) \mathcal{O}^J(\vec{y}) \right\rangle = & \frac{(\Delta - \frac{d}{2})\Gamma(\frac{d}{2})\Gamma(\Delta)}{\pi^d (d-2)\Gamma(\Delta - \frac{d}{2})} (T^a)^{IJ} \\ & \times \frac{1}{(\vec{x}-\vec{y})^{2\Delta-d+2}} \frac{1}{(\vec{x}-\vec{z})^{d-2}(\vec{y}-\vec{z})^{d-2}} \left[\frac{(x-z)_i}{(\vec{x}-\vec{z})} - \frac{(y-z)_i}{(\vec{y}-\vec{z})} \right] . \end{aligned} \tag{4.5.19}$$

Comparing this result with the CFT_d form (4.5.2) gives

$$\xi = \frac{(\Delta - \frac{d}{2})\Gamma(\frac{d}{2})\Gamma(\Delta)}{\pi^d (d-2)\Gamma(\Delta - \frac{d}{2})} . \tag{4.5.20}$$

This correlator satisfies the Ward identity which relates it to the two-point function $\left\langle \mathcal{O}^I(x) \mathcal{O}^J(y) \right\rangle$

$$\begin{aligned} \frac{\partial}{\partial z_i} \left\langle \partial_i J_i^a(z) \phi^I(x) \phi^J(y) \right\rangle &= \xi \frac{(d-2) 2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} (T^a)^{IJ} \left(\delta^d(x-z) - \delta(y-z) \right) \frac{1}{(x-y)^{2\Delta}} \\ &= \delta^d(x-z)(T^a)^{IK} \left\langle \mathcal{O}^K(x) \mathcal{O}^J(y) \right\rangle + \delta^d(y-z)(T^a)^{JK} \left\langle \mathcal{O}^I(x) \mathcal{O}^K(y) \right\rangle . \end{aligned} \tag{4.5.21}$$

The value of the 2–point function (4.4.27) is consistent with this Ward identity.

4.6 The Three-point Function

As we see above the computation of the 2–point functions in position space is delicate, because of the divergence that arises. We now apply the AdS/CFT correspondence to compute 3–point correlators in the boundary CFT_d [6]. We will evaluate the integrals in AdS_{d+1} by using a certain conformal method. The result will be needed in the supergravity calculation of the 3–point correlation functions, which is the subject of Chapter 6.

Consider three scalar fields $\phi_I(z)$, $I = 1, 2, 3$ in the supergravity theory with mass m_I and interaction vertices of the form $\mathcal{L}_1 = \phi_1 \phi_2 \phi_3$ and $\mathcal{L}_2 = \phi_1 g^{\mu\nu} \partial_\mu \phi_2 \partial_\nu \phi_3$. The corresponding three-point amplitudes are

$$A_1(\vec{x}, \vec{y}, \vec{z}) = - \int \frac{d^d w \, dw_0}{w_0^{d+1}} K_{\Delta_1}(w, \vec{x}) K_{\Delta_2}(w, \vec{y}) K_{\Delta_3}(w, \vec{z}) \tag{4.6.1}$$

$$A_2(\vec{x}, \vec{y}, \vec{z}) = - \int \frac{d^d w \, dw_0}{w_0^{d+1}} K_{\Delta_1}(w, \vec{x}) \partial_\mu K_{\Delta_2}(w, \vec{y}) w_0^2 \partial_\mu K_{\Delta_3}(w, \vec{z}), \tag{4.6.2}$$

where $K_{\Delta_I}(w,\vec{x})$ is the Green's function in (4.3.10). These correlators are conformally covariant and are of the form required by conformal symmetry

$$A_i(\vec{x},\vec{y},\vec{z}) = \frac{a_i}{|\vec{x}-\vec{y}|^{\Delta_1+\Delta_2-\Delta_3}|\vec{y}-\vec{z}|^{\Delta_2+\Delta_3-\Delta_1}|\vec{z}-\vec{x}|^{\Delta_3+\Delta_1-\Delta_2}}\,. \tag{4.6.3}$$

We need to obtain the coefficients a_1 and a_2. Use inversion $w'_\mu = \frac{w_\mu}{w^2}$ as a change of variables. Then we use the following inversion property

$$\left(\frac{w_0}{w_0^2+(\vec{w}-\vec{x})^2}\right)^\Delta = \left(\frac{w'_0}{w_0'^2+(\vec{w}'-\vec{x}')^2}\right)^\Delta |\vec{x}'|^{2\Delta}\,. \tag{4.6.4}$$

In order to simplify the integral we apply a translation to place one boundary point at the origin 0, say $\vec{z}=0$. With this choice and the inversion (4.6.9), we obtain

$$\begin{aligned}
A_1(\vec{x},\vec{y},0) &= -|\vec{x}'|^{\Delta_1}|\vec{y}'|^{\Delta_2}C_{\Delta_3}\int \frac{d^dw'\,dw'_0}{w_0'^{d+1}}K_{\Delta_1}(w',\vec{x}')K_{\Delta_2}(w',\vec{y}')w_0'^{\Delta_3}\\
&= -|\vec{x}'|^{\Delta_1}|\vec{y}'|^{\Delta_2}C_{\Delta_3}C_{\Delta_1}C_{\Delta_2}\int d^dw'\,dw_0\prime \frac{w_0'^{\Delta_3+\Delta_1+\Delta_2-d-1}}{\left[w_0'^2+(\vec{w}\prime-\vec{x}\prime)^2\right]^{\Delta_1}\left[w_0'^2+(\vec{w}\prime-\vec{y}\prime)^2\right]^{\Delta_2}}\\
&= -|\vec{x}'|^{\Delta_1}|\vec{y}'|^{\Delta_2}C_{\Delta_3}C_{\Delta_1}C_{\Delta_2}I(\Delta_1,\Delta_2\Delta_3,d)|\vec{x}\prime-\vec{y}\prime|^{\Delta_3-\Delta_1-\Delta_2}\,,
\end{aligned} \tag{4.6.5}$$

where the measure in the integral is invariant under translations and inversions. We now substitute the expression for $I(\Delta_1,\Delta_2\Delta_3,d)$ and apply an inversion, under which [6], [27]

$$\frac{1}{|\vec{x}\prime-\vec{y}\prime|^2} = \frac{|\vec{x}|^2|\vec{y}|^2}{|\vec{x}-\vec{y}|^2}, \quad |\vec{x}\prime|^2 = \frac{1}{|\vec{x}|^2}, \quad |\vec{y}\prime|^2 = \frac{1}{|\vec{y}|^2}\,. \tag{4.6.6}$$

After a translation $\vec{x}\longrightarrow(\vec{x}-\vec{z})$, $\vec{y}\longrightarrow(\vec{y}-\vec{z})$, we obtain the three-point amplitude $A_1(\vec{x},\vec{y},\vec{z})$, with the coefficient a_1 as follows

$$a_1 = -\frac{\Gamma\left(\frac{1}{2}[\Delta_1+\Delta_2-\Delta_3]\right)\Gamma\left(\frac{1}{2}[\Delta_2+\Delta_3-\Delta_1]\right)\Gamma\left(\frac{1}{2}[\Delta_3+\Delta_1-\Delta_2]\right)}{2\pi^d\Gamma(\Delta_1-\frac{d}{2})\Gamma(\Delta_2-\frac{d}{2})\Gamma(\Delta_3-\frac{d}{2})}\Gamma\Big(\frac{1}{2}[\Delta_1+\Delta_2+\Delta_3-d]\Big)\,. \tag{4.6.7}$$

To compute $A_2(\vec{x},\vec{y},\vec{z})$ we can proceed similarly by knowing that $\partial_\mu K_{\Delta_2}(w,\vec{y})w_0^2\partial_\mu K_{\Delta_3}(w,\vec{z})$ is an invariant contraction under inversion. Using (4.6.9), we have

$$\begin{aligned}
\partial_\mu K_{\Delta_2}(w,\vec{y})w_0^2\partial_\mu K_{\Delta_3}(w,0) &= |\vec{y}\prime|^{2\Delta_2}\partial_{\mu\prime}K_{\Delta_2}(w\prime,\vec{y}\prime)w_0'^2\partial_{\mu\prime}K_{\Delta_3}(w\prime,0)\\
&= |\vec{y}\prime|^{2\Delta_2}C_{\Delta_2}C_{\Delta_3}\frac{\partial}{\partial w'_0}\left[\frac{w'_0}{w_0'^2+(\vec{w}\prime-\vec{y}\prime)^2}\right]^{\Delta_2} w_0'^2\frac{\partial}{\partial w'_0}w_0'^{\Delta_3}\\
&= \Delta_2\Delta_3|\vec{y}\prime|^{2\Delta_2}C_{\Delta_2}C_{\Delta_3}w_0'^{\Delta+\Delta_3}\left[\frac{1}{[w_0'^2+(\vec{w}\prime-\vec{y}\prime)^2]^{\Delta_2}}-\frac{2w'_0}{[w_0'^2+(\vec{w}\prime-\vec{y}\prime)^2]^{\Delta_2+1}}\right].
\end{aligned} \tag{4.6.8}$$

We then find two integrals of the form $I(\Delta_1,\Delta_2,\Delta_3,d)$. The result implies

$$a_2 = a_1\left[\Delta_2\Delta_3+\frac{1}{2}\big(d-\Delta_1-\Delta_2-\Delta_3\big)\big(\Delta_2+\Delta_3-\Delta_1\big)\right]. \tag{4.6.9}$$

In summary, the holographic dictionary is used to compute correlators in the strong coupling limit of the CFT from the bulk theory in AdS. We have demonstrated that the computation of the two

point function from the gravity side is subtle due to divergences. These divergences required a cut off at the boundary. Because of the divergences the result of our computation is ambiguous. We have determined the correct result using a Ward Identity. Its clear that we do not have an understanding of these divergences. If we could understand them, we might be able to explain what regularization procedure we should use.

5. Three-point Correlation Functions of Giant Gravitons

In the previous chapter, a divergence was found in the two-point function of operators built using order 1 fields. In this chapter, we will see that there are also divergences in correlators of operators with order N field. We will employ the AdS/CFT correspondence to calculate the three-point function of two giant gravitons "heavy states" and one point like graviton "light state". On the string theory side of the correspondence, we will consider either S^5 or AdS_5 giant gravitons as our heavy states, and a chiral primary operator dual to an AdS scalar field of dimension $\Delta = J << \sqrt{N}$ as the light state. On the gauge theory side, our heavy operators corresponding to the giant gravitons are Schur polynomials with dimension k of order N, and the light state is again a chiral primary operator. In the first section we introduce the operators that we will consider on the gauge theory side and compute the three-point functions using Schur polynomial techniques. Then, in section 2 we calculate the holographic three-point functions of giant graviton operators following [7]. Three–point functions on the gauge theory side of the correspondence are not in agreement with the dual string theory results. Lin [11] has presented a regularization procedure under which the string theory results are exactly match with the results from the gauge theory side. This regularization procedure will be subject of the last section of this chapter.

5.1 Gauge Theory Computation

The methods we use in this section have been reviewed in Chapter 3.

5.1.1 Correlators of Chiral Primary Operators. Let us consider the composite operator constructed using a single complex scalar field Z,

$$\mathcal{O}^J(x) = \mathrm{Tr}\left(Z^J(x)\right) . \tag{5.1.1}$$

This single trace chiral primary operator is dual to a point-like string moving along an equator of S^5 with angular momentum J. The two and three point functions of a chiral primary operator of the type $\mathrm{Tr}\left(Z^J\right)$ can be evaluated using matrix model techniques in the free theory. We calculate it using the technique presented in [30].

Consider the unitary group characters $\chi_R(Z)$ and $\chi_S(\bar{Z})$ labeled by Young diagrams R,S. Even though the matrices are complex instead of unitary, we write

$$\left\langle \chi_R(Z)\chi_S(\bar{Z})\right\rangle = \delta_{R,S}\Omega_R \tag{5.1.2}$$

where Ω_R is the product of the factors of R. Our observables $\mathrm{Tr}\left(Z^J\right)$ are a functions on the conjugacy classes of J–cycles, they are given in (3.8.14). Therefore, we can write

$$\left\langle \mathrm{Tr}Z^J\mathrm{Tr}\bar{Z}^J\right\rangle = \sum_{R\vdash J}\left(\chi_R(\sigma)\right)^2\Omega_R\,. \tag{5.1.3}$$

The J–cycle character $\chi_R(\sigma)$ is non-zero only for Young diagrams consisting of a single hook of boxes with row length k and column length $J-k$, for $k = 1,\cdots,J$. For the hooks the J–cycle

character $\chi_R(\sigma)$ is either positive or negative one. Thus, $\sum_R$ becomes a sum over the J possible hooks

$$\begin{aligned}\left\langle \text{Tr } Z^J \text{Tr } \bar{Z}^J \right\rangle &= \sum_{k=1}^{J} \prod_{i=1}^{k} (N-1+i) \prod_{m=1}^{J-k} (N-m) \\ &= \frac{1}{J+1}\left[\frac{\Gamma(N+J+1)}{\Gamma(N)} - \frac{\Gamma(N+1)}{\Gamma(N-J)}\right] \\ &= JN^J\left[1 + \binom{J+1}{4}\frac{1}{N^2} + \cdots\right]. \end{aligned} \tag{5.1.4}$$

In a similar way we find

$$\begin{aligned}\left\langle \text{Tr } Z^J \text{Tr } Z^K \text{Tr } \bar{Z}^{J+K} \right\rangle &= \left(\sum_{k=K+1}^{J+K} - \sum_{k=1}^{J}\right) \prod_{i=1}^{k} (N-1+i) \prod_{m=1}^{J-k} (N-m) \\ &= \frac{1}{J+K+1}\left[\frac{\Gamma(N+J+K+1)}{\Gamma(N)} - \frac{\Gamma(N+J+1)}{\Gamma(N-K)} + \frac{\Gamma(N+1)}{\Gamma(N-J-K)} - \frac{\Gamma(N+K+1)}{\Gamma(N-J)}\right] \\ &= N^{J+K-1}JK(J+K)\left[1 + \frac{1}{3!N^2}\binom{K+J-1}{2}\left[\binom{K}{2} + \binom{J}{2} - 1\right] + \cdots\right]. \end{aligned} \tag{5.1.5}$$

Here we ignored the dependence on the spacetime coordinates and the 't Hooft coupling constant. These are completely determined by conformal invariance and can easily be reinstated if needed. Then we get the CFT structure constant for the three-point function

$$\mathcal{C}_{J,K,K+J} = \frac{\left\langle \mathcal{O}^J \mathcal{O}^K \bar{\mathcal{O}}^{J+K} \right\rangle}{\sqrt{\left\langle \mathcal{O}^J \bar{\mathcal{O}}^J \right\rangle \left\langle \mathcal{O}^K \bar{\mathcal{O}}^K \right\rangle \left\langle \mathcal{O}^{J+K} \bar{\mathcal{O}}^{J+K} \right\rangle}} \tag{5.1.6}$$

$$= \frac{1}{N}\sqrt{JK(J+K}\left[1 + \mathcal{O}\left(\frac{1}{N^2}\right)\right]. \tag{5.1.7}$$

5.1.2 Correlators of Schur Polynomials. Consider a representation R described in terms of a Young tableau with n boxes. This Young tableau labels both a representation of $U(N)$ and a representation of S_n. The Schur Polynomial $\chi_R(Z)$ of a complex matrix Z is defined as

$$\chi_{R_n}(Z) = \frac{1}{n!}\sum_{\sigma \in S_n} \chi_{R_n}(\sigma) Z^{i_{\sigma(1)}}_{i_1} \cdots Z^{i_{\sigma(n)}}_{i_n} \tag{5.1.8}$$

where the sum is over all elements σ in the symmetric group S_n. Notice that there is no limit in which the Schur polynomial reduces to a chiral primary operator. Schur polynomials are 1/2-BPS operators in $\mathcal{N} = 4\, SYM$. They have protected two– and three–point functions [23]. The two-point function was calculated in section (3.8) and reads

$$\left\langle \chi_R(Z)\chi_S(\bar{Z}) \right\rangle = \delta_{R,S} f_R = \delta_{R,S} \prod_{i,j \in R} (N - i + j)\,. \tag{5.1.9}$$

The product $\prod_{i,j\in R}$ goes over all boxes of the Young tableau of the representation R with i indexing rows and j indexing columns. To compute the three-point correlation function we need to use the following product rule for two Schur polynomials [23] [31],

$$\chi_R(Z)\chi_S(Z) = \sum_E g(R,S;E)\chi_E(Z)\,, \tag{5.1.10}$$

where $g(R,S;T)$ is the Littlewood-Richardson coefficient, a positive integer which counts the multiplicity with which the representation T appears in the tensor product of the representations R and S. Using the product rule above we get

$$\begin{aligned}\left\langle \chi_R(Z)\chi_S(Z)\chi_T(\bar{Z})\right\rangle &= \sum_E g(R,S;E)\left\langle \chi_E(Z)\chi_T(Z)\right\rangle \\ &= \sum_E g(R,S;E)\delta_{E,T} f_E \\ &= g(R,S;T)\prod_{i,j\in T}(N-i+j)\,. \end{aligned} \tag{5.1.11}$$

The string theory duals of Schur polynomials are collections of giant gravitons, i.e. D3-branes which wrap an S^3 of either S^5 or AdS_5. Thus, it is natural to identify Schur polynomials for the antisymmetric representations with k boxes as the operator dual to a giant graviton on S^5. Another class of Schur polynomials are those corresponding to symmetric representations with k boxes. They are naturally identified as operators dual to AdS giant gravitons. The antisymmetric representations correspond to Young diagrams with one column of length k [32], while the symmetric representations correspond to to Young diagrams with one row of length k [23].

Let us denote the Schur polynomial for the antisymmetric representation with k boxes as $\chi^A_k(Z)$ and the Schur polynomial for the symmetric representation with k boxes as $\chi^S_k(Z)$. We take $k \leqslant N$ for the antisymmetric case while in the symmetric case k is unbounded. Since $g(R,S;T)=1$ for these cases as computed in [33], we find the following results

$$\left\langle \chi^S_k(\bar{Z})\chi^S_k(Z)\right\rangle = \prod_{j=1}^{k}(N-1+j)\,, \tag{5.1.12}$$

$$\left\langle \chi^A_k(\bar{Z})\chi^A_k(Z)\right\rangle = \prod_{i=1}^{k}(N-i+1)\,, \tag{5.1.13}$$

$$\left\langle \chi^S_k(\bar{Z})\chi^S_{k-j}(Z)\chi^S_j(Z)\right\rangle = \prod_{j=1}^{k}(N-1+j)\,, \tag{5.1.14}$$

$$\left\langle \chi^A_k(\bar{Z})\chi^A_{k-j}(Z)\chi^A_j(Z)\right\rangle = \prod_{i=1}^{k}(N-i+1)\,. \tag{5.1.15}$$

5.1.3 Three-point Functions of 2 Schur Polynomials and one Single Trace Operator. We study three–point correlation functions involving two massive string states, which are giant gravitons and one light string dual to a chiral primary operator of the type $\mathrm{Tr}Z^J$ [34]. In the symmetric representation the normalized three-point functions are

$$\mathcal{C}^S_{k,k-J,J} = \frac{\left\langle \chi^S_k(\bar{Z})\chi^S_{k-j}(Z)\mathrm{Tr}Z^J\right\rangle}{\sqrt{\left\langle \chi^S_k(\bar{Z})\chi^S_k(Z)\right\rangle\left\langle \chi^S_{k-j}(\bar{Z})\right\rangle\chi^S_{k-j}(Z)\left\langle \mathrm{Tr}\bar{Z}^J\mathrm{Tr}Z^J\right\rangle}}\,, \tag{5.1.16}$$

and similarly for the antisymmetric representation case $\mathcal{C}^A_{k,k-J,J}$. To calculate the expectation value in the numerator, we use (3.8.15). We have

$$\left\langle \chi^S_k(\bar{Z})\chi^S_{k-J}(Z)\mathrm{Tr}z^J\right\rangle = \sum_{R\vdash J}\chi_R(\sigma)\left\langle \chi_R(Z)\chi^S_k(\bar{Z})\chi^S_{k-J}(Z)\right\rangle , \tag{5.1.17}$$

$$\left\langle \chi^A_k(\bar{Z})\chi^A_{k-J}(Z)\mathrm{Tr}Z^J\right\rangle = \sum_{R\vdash J}\chi_R(\sigma)\left\langle \chi_R(Z)\chi^A_k(\bar{Z})\chi^A_{k-J}(Z)\right\rangle . \tag{5.1.18}$$

It is clear that, only the completely symmetric representation contributes in the first case and only the antisymmetric representation contributes in the later. The character $\chi_R(\sigma)$ can be written as follows

$$\chi^S_J(\sigma)=1, \qquad \chi^A_J(\sigma)=(-1)^{J-1}. \tag{5.1.19}$$

This implies

$$\left\langle \chi^S_J(\bar{Z})\chi^S_{k-J}(Z)\mathrm{Tr}Z^J\right\rangle = \prod_{j=1}^{k}(N-1+j), \tag{5.1.20}$$

$$\left\langle \chi^A_J(\bar{Z})\chi^A_{k-J}(Z)\mathrm{Tr}Z^J\right\rangle = (-1)^{J-1}\prod_{i=1}^{k}(N-i+1). \tag{5.1.21}$$

Using these results and the expressions (5.1.4) and (5.1.9), we find the structure constants

$$\mathcal{C}^S_{k,k-J,J} = \frac{\sqrt{\prod_{j=k-J+1}^{k}(N+j-1)}}{\sqrt{JN^J(1+c(J)\frac{1}{N^2}+\cdots)}}, \tag{5.1.22}$$

$$\mathcal{C}^A_{k,k-J,J} = (-1)^{(J-1)}\frac{\sqrt{\prod_{i=k-J+1}^{k}(N-i+1)}}{\sqrt{JN^J(1+c(J)\frac{1}{N^2}+\cdots)}}. \tag{5.1.23}$$

Now, we are interested in the situation that the chiral primary is a small operator and the Schur polynomials correspond to large Young tableaux, i.e. the limit

$$N\to\infty, \qquad k\to\infty, \qquad \frac{k}{N}\ \text{finite}, \qquad J\ll k, \tag{5.1.24}$$

and in particular $J\ll N$. In this limit we find for the structure constants

$$\mathcal{C}^S_{k,k-J,J} = \frac{1}{\sqrt{J}}\left(1+\frac{k}{N}\right)^{J/2}, \tag{5.1.25}$$

$$\mathcal{C}^A_{k,k-J,J} = (-1)^{(J-1)}\frac{1}{\sqrt{J}}\left(1-\frac{k}{N}\right)^{J/2}. \tag{5.1.26}$$

Furthermore, we note that for the antisymmetric representation we have the constraint $k\leq N$ while for the symmetric case k is unbounded.

5.2 Holographic Computation

5.2.1 Giant Graviton on S^5. For this giant graviton, the worldvolume is along $\mathbb{R}(\subset AdS_5)\times S^3(\subset S^5)$ [8][9]. The geometry of $AdS_5 \times S^5$ in global coordinates can be described as:

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\tilde{\Omega}_3^2 + d\theta + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\Omega_3^2 \tag{5.2.1}$$

where we have taken the radius of curvature of AdS_5 and S^5 equal to one. The round metric on the three sphere in the hyper-spherical coordinates given by

$$d\Omega_3^2 = d\chi_1^2 + \sin^2\chi_1 d\chi_2^2 + \sin^2\chi_1 \sin^2\chi_2 d\chi_3^2\,, \tag{5.2.2}$$

where $\chi_1,\ \chi_2$ run from 0 to π, and χ_3 runs from 0 to 2π. The D3-brane action which describes the dynamics of our giant graviton, consists of Dirac-Born-Infeld (DBI) and Wess-Zumino (WZ) terms:

$$S_{D3} = S_{DBI} + S_{WZ} \tag{5.2.3}$$

$$= -\frac{N}{2\pi^2}\int d^4\sigma(\sqrt{-g} - P[C_4]) \tag{5.2.4}$$

where $N/2\pi^2$ is the tension of the brane and, σ^a where $a = 0,\cdots,3$ label the world-volume coordinates. $g = \det(g_{ab})$ where g_{ab} is the induced world-volume metric on the D3-brane, which is defined as

$$g_{ab} = g_{\mu\nu}\partial_a x^\mu \partial_b x^\nu. \tag{5.2.5}$$

x^μ describe the embedding of the D3-brane in ten-dimensional spacetime, such that

$$x^\mu \equiv (t,\ \rho,\ \tilde{\chi}^i,\ \theta,\ \phi,\ \chi^i), \qquad \mu = 0,1,2,\cdots,9, \qquad i = 1,\cdots,3 \tag{5.2.6}$$

$P(C_4)$ denotes the pull-back of the the four-form potential C_4, which will pullback the spacetime form into the world-volume of the brane. C_4 is proportional to the volume form on the unit 3-sphere described by (5.2.2), and may be taken as

$$C_{\phi\chi_1\chi_2\chi_3} = \cos^4\theta Vol(\Omega_3)\,. \tag{5.2.7}$$

The pull-back of this four-form potential is given by

$$\begin{aligned} P(C^{(4)}) =& \frac{1}{4!}\epsilon^{\sigma^0\cdots\sigma^3}\partial_{\sigma^0}X^{\mu_0}\cdots\partial_{\sigma^3}X^{\mu_3}C^{(4)}_{\mu_0\cdots\mu_3}d\sigma^0\wedge\cdots\wedge d\sigma^3 \\ =& d^4\sigma\dot{\phi}\cos^4\theta \mathrm{Vol}(\Omega_3) \\ =& d^4\sigma\dot{\phi}\cos^4\theta\sqrt{g(\Omega_3)} \\ =& d^4\sigma\dot{\phi}\cos^4\theta\sin^2\chi_1\sin\chi_2 \end{aligned} \tag{5.2.8}$$

where χ_i are the angles covering the $S^3 \subset S^5$ and $\mathrm{Vol}(\Omega_3)$ denotes its volume element. $\epsilon^{\sigma^0\cdots\sigma^3}$ is the totally antisymmetric symbol.

Let us now specify the gauge we will use for our computation. We identify the world-volume coordinates σ^a with the appropriate space-time coordinates:

$$\sigma^0 = t, \quad \sigma^i = \chi_i, \quad i = 1,2,3. \tag{5.2.9}$$

Our ansatz for the anti-D3-brane giant graviton on $AdS_5 \times S^5$ takes the form

$$\rho = 0, \quad \phi = \phi(t). \tag{5.2.10}$$

For the particular embedding coordinates of the D3-brane in equations (5.2.9) and (5.2.10) the induced metric (5.2.5) can be written as

$$g_{ab} = \text{Diag}(-1 + \sin^2\dot{\phi}^2,\, \cos^2\theta,\, \cos^2\theta \sin^2\chi_1,\, \cos^2\theta \sin^2\chi_1 \sin^2\chi_2)\,, \tag{5.2.11}$$

and

$$g = \det(g_{ab}) = (-1 + \sin^2\theta)\cos^6\theta \sin^4\chi_1 \sin^2\chi_2\,. \tag{5.2.12}$$

Substituting (5.2.8) and (5.2.12) into the D3-brane action (5.2.4) and then integrating over the angular coordinates, yields the following action

$$\begin{aligned} S &= -\frac{N}{2\pi^2}\int_{+\infty}^{-\infty} dt \int_0^{\pi} d\chi_1 \int_0^{\pi} d\chi_2 \int_0^{2\pi} d\chi_3 \left[\sqrt{1-\dot{\phi}^2\sin^2\theta}\cos^3\theta \sin^2\chi_1 \sin\chi_2 - \cos^4\theta\, Vol(\Omega_3)\right] \\ &= -N\int dt\big[\cos^3\theta\sqrt{1-\dot{\phi}^2\sin^2\theta} - \dot{\phi}\cos^4\theta\big]. \end{aligned} \tag{5.2.13}$$

The R-charge/angular momentum conjugate to ϕ of the giant is fixed

$$k = \frac{\delta L}{\delta\dot{\phi}} = \frac{N\dot{\phi}\sin^2\theta\cos^3\theta}{\sqrt{1-\dot{\phi}^2\sin^2\theta}} + N\cos^4\,, \tag{5.2.14}$$

where L is the Lagrangian for the giant, We can invert this relation to write

$$\dot{\phi} = \frac{l - \cos^4\theta}{\sin\theta\sqrt{(l-\cos^4\theta)^2 + \sin^2\theta\cos^6\theta}}. \tag{5.2.15}$$

where $l \equiv k/N$. The action may be rewritten in terms of k, to give

$$S = N\int dt\, \frac{cos^2\theta}{\sin\theta}\frac{l-\cos^2\theta}{\sqrt{(l-\cos^4\theta)^2 + \sin^2\theta\cos^6\theta}}. \tag{5.2.16}$$

The corresponding energy is defined by

$$E = \dot{\phi}k - L = \frac{N}{\sin\theta}\sqrt{(l-\cos^4\theta)^2\sin^2\theta\cos^6\theta}\,. \tag{5.2.17}$$

It is independent of ϕ. By fixing $l = \cos^2\theta$, we find that the energy is minimized

$$\dot{\phi} = 1\,, \tag{5.2.18}$$

and the action vanishes. The energy E of the giant graviton at this minimum is

$$E_{\text{min}} = k\,. \tag{5.2.19}$$

5.2.2 Giant Graviton on AdS_5. We now consider the giant graviton with world-volume $\mathbb{R} \times S^3$ embedded in AdS_5 [10] [9]. The metric of $AdS_5 \times S^5$ can be taken as in (5.2.5). The action for the anti-D3-brane is

$$S_{D3} = -\frac{N}{2\pi^2} \int d^4\sigma(\sqrt{-g} + P[C_4]). \tag{5.2.20}$$

Here the four-form potential C_4 may be taken as

$$C_{\phi\tilde{\chi}_1\tilde{\chi}_2\tilde{\chi}_3} = -\sinh^4\rho Vol(\tilde{\Omega}_3)\,, \tag{5.2.21}$$

where $\tilde{\chi}_i$ are the angles covering the $S^3 \subset AdS_5$ and $Vol(\tilde{\Omega}_3)$ denotes its volume element. We identify the world-volume coordinates σ^a with the spacetime coordinates:

$$\sigma^0 = t, \quad \sigma^i = \tilde{\chi}_i, \quad i = 1, 2, 3. \tag{5.2.22}$$

Our ansatz for the anti-D3-brane giant graviton on $AdS_5 \times S^5$ takes the form

$$\rho = \text{constant}, \quad \phi = \phi(t), \quad \theta = \frac{\pi}{2}. \tag{5.2.23}$$

For the particular embedding of the D3-brane in equations (5.2.22) and (5.2.23) the induced metric $g_{ab} = g_{\mu\nu}\partial_a x^\mu \partial_b x^\nu$ can be written as

$$g_{ab} = \text{Diag}(-\cosh^2\rho + \dot{\phi}^2, \sinh^2\rho, \sinh^2\rho\sin^2\tilde{\chi}_1, \sinh^2\rho\sin^2\tilde{\chi}_1\sin^2\tilde{\chi}_2) \tag{5.2.24}$$

and

$$g = \det(g_{ab}) = (\cosh^2\rho + \dot{\phi}^2)\sinh^6\rho\sin^4\tilde{\chi}_1\sin^2\tilde{\chi}_6\,. \tag{5.2.25}$$

These yield the following action

$$S = \int dt\; L = -N\int dt\big[\sinh^3\rho\sqrt{\cosh^2\rho - \dot{\phi}^2} - \sinh^4\rho\big]. \tag{5.2.26}$$

The R-charge/angular momentum conjugate to ϕ of the anti-giant is fixed

$$\tilde{k} \equiv \frac{\delta L}{\delta\dot{\phi}} = \frac{N\dot{\phi}\sinh^3\rho}{\sqrt{\cosh^2\rho - \dot{\phi}^2}}, \tag{5.2.27}$$

where L is the Lagrangian for the anti-giant. We can invert this relation to write

$$\dot{\phi} = \frac{\tilde{l}\cosh\rho}{\sqrt{\sinh^6\rho + \tilde{l}}} \tag{5.2.28}$$

where $\tilde{l} = \tilde{k}/N$. The action may be rewritten in terms of $\tilde{k}$, to obtain

$$S = -N\int dt\;\sinh^4\rho\left[\cosh^2\rho\cosh\rho\sqrt{\frac{1}{\sinh^6\rho + \tilde{l}}} - 1\right]. \tag{5.2.29}$$

The corresponding energy defined by

$$E = \dot{\phi}\tilde{k} - L = N\int\left[\cosh\rho\sqrt{\sinh^6\rho + l^2} - \sinh^4\rho\right], \tag{5.2.30}$$

it is independent of ϕ. By fixing $\tilde{l} = \sinh^2\rho$, we minimize the energy,

$$\dot{\phi} = 1, \tag{5.2.31}$$

and the action vanishes. The energy E of the anti-giant graviton at this minimum is

$$E_{\text{min}} = \tilde{k} \tag{5.2.32}$$

5.2.3 The Co-ordinate System. It turns out to be convenient to use a new set of coordinates obtained by mapping the global coordinates (5.2.5) into the Poincare patch [35] [23]

$$(z,\ x^0,\ x^1) = \frac{R}{\cosh\rho\cos t - \sinh\rho\cos\psi}\,(1,\ \cosh\rho\sin t,\ \sinh\rho\sin\psi)\ . \tag{5.2.33}$$

To illustrate the change of coordinates in the simplest setting we have performed the discussion for the case of AdS_3 in which case $d\tilde{\Omega}_3^2$ simplifies to $d\psi^2$. The metric of the Poincare patch is

$$ds^2 = \frac{-(dx^0)^2 + (dx^1)^2 + dz}{z^2}. \tag{5.2.34}$$

We have $\rho = 0$ for the giant graviton path on S^5. Continuing to the Euclidean AdS, so that $t \longrightarrow t_E = -it$ and $x^0 \longrightarrow x^0_E = -ix^0$ we find

$$z = \frac{R}{\cosh t_E}, \quad x^0_E = R\tanh t_E, \quad x^1 = 0. \tag{5.2.35}$$

In the case of the AdS_5 giant graviton, use the generalization of the coordinate transformations to AdS_5

$$(z,\ x^0,\ \vec{x}) = \frac{R}{\cosh\rho\cos t - n_0\sinh\rho}\,(1,\ \cosh\rho\sin t,\ \vec{n}\sinh\rho) \tag{5.2.36}$$

where the $S^3 \subset AdS_5$ is given by the embedding coordinates $n_I = (n_0, \vec{n})$, $n_I n_I = 1$.

5.2.4 Supergravity Fluctuations. The supergravity modes that we are interested in are fluctuations of the spacetime metric and the 4-form potential, which correspond to a chiral primary operator with R-charge Δ in $\mathcal{N} = 4$ SYM [36] [37] [38]. The fluctuations are

$$\delta g_{\mu\nu} = \left[-\frac{6\Delta}{5}g_{\mu\nu} + \frac{4}{\Delta+1}\nabla_{(\mu}\nabla_{\nu)}\right]s^\Delta(X)Y_\Delta(\Omega)\,, \tag{5.2.37}$$

$$\delta g_{\alpha\beta} = 2\Delta g_{\alpha\beta}s^\Delta(X)Y_\Delta(\Omega)\,, \tag{5.2.38}$$

$$\delta C_{\mu_1\mu_2\mu_3\mu_4} = -4\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}s^\Delta(X)Y_\Delta(\Omega)\,, \tag{5.2.39}$$

$$\delta C_{\alpha_1\alpha_2\alpha_3\alpha_4} = 4\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}s^\Delta(X)Y_\Delta(\Omega)\,, \tag{5.2.40}$$

where μ, ν are indices refer to AdS_5 coordinates X, while α, β are indices refer to S^5 coordinates Ω. The $Y_\Delta(\Omega)$ represent the spherical harmonics on S^5, while $s^\Delta(X)$ represent a scalar field propagating on AdS_5 space with mass-squared=$\Delta(\Delta-4)$, where Δ labels the representation of $SO(6)$ and must be an integer greater than or equal 2.

The symmetric traceless double covariant derivative $\nabla_{(\mu}\nabla_{\nu)}$ is defined as

$$\nabla_{(\mu}\nabla_{\nu)} = \frac{1}{2}(\nabla_\mu\nabla_\nu + \nabla_\nu\nabla_\mu) - \frac{1}{5}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\nabla_\sigma \tag{5.2.41}$$

and, when acting on a scalar field

$$\nabla_\mu\nabla_\nu = \partial_\mu\partial_\nu - \Gamma^\sigma_{\mu\nu}\partial_\sigma\,. \tag{5.2.42}$$

The Christoffel symbol $\Gamma^\sigma_{\mu\nu}$ is given by

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right)\,. \tag{5.2.43}$$

The bulk to boundary propagator for s^Δ is given by

$$\sqrt{\frac{\alpha_0}{B_\Delta}}\frac{z^\Delta}{((x-x_B)^2+z^2)^\Delta} \simeq \sqrt{\frac{\alpha_0}{B_\Delta}}\frac{z^\Delta}{(x_B^2)^\Delta}. \tag{5.2.44}$$

The normalization is given by

$$\alpha_0 = \frac{\Delta-1}{2\pi^2}, \quad B_\Delta = \frac{2^{3-\Delta}N^2\Delta(\Delta-1)}{\pi^2(\Delta+1)^2}. \tag{5.2.45}$$

5.2.5 Antisymmetric Giant Graviton. The Euclidean form of the D3-brane action is given by

$$S^E_{D3} = \frac{N}{2\pi^2}\int d^4\sigma\ (\sqrt{g} - ip[C_4])\,. \tag{5.2.46}$$

Under Wick rotation, the four-form potential with legs in the AdS_5 part of the geometry transforms as $C_4^{AdS} \to -iC_4^{AdS}$ since the potential has a leg in the temporal direction, while the potential on S^5 is unaffected.

The DBI part is given by

$$S_{DBI} = \frac{N}{2\pi^2}\int d^4\sigma\ \sqrt{g}\,, \tag{5.2.47}$$

where $g = \det(g_{ab})$ and $g_{ab} = g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}$, where $a,b = 0,\cdots 3$ and $\mu\nu = 0,\cdots 9$. To find the variation of this action, we need to vary $\sqrt{g}$ as follows

$$\delta g = \delta\det(g_{ab}) = g\ g^{ab}\delta g_{ab}\,. \tag{5.2.48}$$

Using this we get

$$\delta\sqrt{g} = -\frac{1}{2\sqrt{g}}\delta g = \frac{1}{2}\sqrt{g}g^{ab}\delta g_{ab} \tag{5.2.49}$$

$$= \frac{1}{2}\sqrt{g}g^{ab}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}\delta g_{\mu\nu}\,. \tag{5.2.50}$$

These give the following variation

$$\delta S_{DBI} = \frac{N}{4\pi^2}\int d^4\sigma\sqrt{g}g^{ab}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}\delta g_{\mu\nu}\,. \tag{5.2.51}$$

On the path of the S^5 giant graviton we have $\rho = 0$ and $\dot\phi = 1$. Continue to Euclidean AdS, so that $t \longrightarrow t_E = -it$. The induced metric (5.2.12) becomes

$$g_{ab} = \mathrm{diag}\left(\cos^2\theta,\ \cos^2\theta,\ \cos^2\theta\sin^2\chi_1,\ \cos^2\theta\sin^2\chi_1\sin^2\chi_2\right)\,. \tag{5.2.52}$$

Inserting this into (5.2.51) we find

$$\delta S_{DBI} = \frac{N}{4\pi^2}\int d^4\sigma\sqrt{g}\left(g^{00}\left(\delta g_{tt} - \delta g_{\phi\phi}\right) + g^{11}\delta g_{\chi_1\chi_1} + g^{22}\delta g_{\chi_2\chi_2} + g^{33}\delta g_{\chi_3\chi_3}\right)Y_\Delta(\Omega)s^\Delta(X). \tag{5.2.53}$$

Using the supergravity fluctuations (5.2.37) and (5.2.38) and integrating over the angles we find

$$\begin{aligned}\delta S_{DBI} =&\frac{N}{2\pi^2}\int_{+\infty}^{-\infty} dt \int_0^{\pi} d\chi_1 \int_0^{\pi} d\chi_2 \int_0^{2\pi} d\chi_3 \ \cos^2\theta \sin^2\chi_1 \sin\chi_2 Y_\Delta(\Omega)\\ &\times\left(\left[-\frac{6\Delta}{5}g_{tt}+\frac{4}{\Delta+1}\nabla_{(t}\nabla_{t)}-2\Delta\sin^2\theta\right]+6\Delta\cos^2\theta\right)s^\Delta(X)\\ =&\frac{N}{2}\cos^2\theta\int dt\, Y_\Delta(\Omega)\left(\left[-\frac{6\Delta}{5}g_{tt}+\frac{4}{\Delta+1}\nabla_{(t}\nabla_{t)}\right]-8\Delta\sin^2\theta+6\Delta\right)s^\Delta(X).\end{aligned} \tag{5.2.54}$$

To compute the double traceless symmetric covariant derivative $\nabla_{(\mu}\nabla_{\mu)}$ we use the Euclidean metric on AdS_5 in terms of the coordinates introduced in section 5.2.3

$$ds^2_{EAdS_5}=\frac{(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2+dz^2}{z^2}\,. \tag{5.2.55}$$

From the formula (5.2.41), we have

$$\nabla_{(\mu}\nabla_{\mu)}s^\Delta=\left(\nabla_\mu\nabla_\mu-\frac{1}{5}g_{\mu\mu}g^{\sigma\sigma}\nabla_\sigma\nabla_\sigma\right)s^\Delta. \tag{5.2.56}$$

Using this metric and the fact that $s^\Delta(X)$ depends only on z, we find

$$g^{x^ix^i}\nabla_{x^i}\nabla_{x^i}s^\Delta=g^{x^ix^i}(\partial^2_{x^i}-\Gamma^z_{x^ix^i})s^\Delta=z^2\big(\partial^2_{x^i}-\frac{1}{z}\partial_z\big)s^\Delta\,,$$

where $i=0,\cdots,3$. Thus

$$g^{zz}\nabla_z\nabla_z=g^{zz}(\partial^2_z-\Gamma^z_{zz})=z^2(\partial^2_z+\frac{1}{z}\partial_z)s^\Delta.$$

With the bulk to boundary propagator (5.2.44), the double covariant derivative (5.2.56) becomes

$$\begin{aligned}\nabla_{(\mu}\nabla_{\mu)}s^\Delta&=\partial^2_\mu s^\Delta-\frac{1}{5}g_{\mu\mu}\left(-3z\partial_z+z^2\partial^2_z\right)s^\Delta\\ &=\partial^2_\mu-\frac{1}{5}(\Delta(\Delta-1)-3\Delta)s^\Delta\,.\end{aligned} \tag{5.2.57}$$

Thus

$$\nabla_{(t}\nabla_{t)}s^\Delta=\partial^2_t-\frac{1}{5}(\Delta(\Delta-1)-3\Delta)s^\Delta\,. \tag{5.2.58}$$

The variation of the DBI part of the action can be written as follows

$$\delta S_{DBI}=\frac{N}{2}\cos^2\theta\int dt\, Y_\Delta(\Omega)\left(\frac{4}{\Delta+1}\partial^2_t-\frac{2\Delta(\Delta-1)}{\Delta+1}-8\Delta\sin^2\theta+6\Delta\right)s^\Delta. \tag{5.2.59}$$

We will be interested in the spherical harmonic

$$Y_\Delta(\Omega)=\frac{\sin^\Delta\theta e^{i\Delta\phi}}{2^{\frac{\Delta}{2}}}=\frac{\sin^\Delta\theta e^{\Delta t}}{2^{\frac{\Delta}{2}}}. \tag{5.2.60}$$

Consider the Euclidean geodesic $\phi = \phi(t) = t \longrightarrow -it$, $z = \frac{R}{\cosh t}$, $x = R\tanh t$. We also replace the field s^Δ with the bulk to the boundary propagator (5.2.44), namely

$$s^\Delta \longrightarrow \frac{\Delta+1}{N\sqrt{\Delta}2^{2-\frac{\Delta}{2}}}\frac{z^\Delta}{x_B^{2\Delta}} = a_\Delta \frac{z^\Delta}{x_B^{2\Delta}}. \tag{5.2.61}$$

We finally obtain

$$\delta S_{DBI} = \frac{\cos^2\theta \sin^\Delta\theta(\Delta+1)\sqrt{\Delta}}{2}\int dt\, \frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^\Delta t}\left(2\cos^2\theta - \frac{1}{\cosh^2 t}\right), \tag{5.2.62}$$

where $\mathcal{R} = R/x_B^2$. We now turn our attention to the Wess-Zumino coupling. Because the relevant legs of C_4 are in S^5 we require only the fluctuation $\delta C_{\phi\chi_1\chi_2\chi_3}$

$$\begin{aligned}\delta C_{\phi\chi_1\chi_2\chi_3} &= 4\epsilon_{\theta\phi\chi_1\chi_2\chi_3} s^\Delta \nabla^\theta Y_\Delta(\Omega)\\ &= 2^{-\frac{\Delta}{2}+2}\epsilon_{\theta\phi\chi_1\chi_2\chi_3}\Delta s^\Delta (\sin\theta)^{\Delta-1}\cos\theta e^{\Delta t}.\end{aligned} \tag{5.2.63}$$

Therefore the variation of the Wess-Zumino part is given by

$$\begin{aligned}\delta S_{WZ} &= i\frac{N}{2\pi^2}\int d^4\sigma\, P[\delta C_4]\\ &= i\frac{N}{2\pi^2}\int_{+\infty}^{-\infty} dt\int_0^\pi d\chi_1\int_0^\pi d\chi_2\int_0^{2\pi} d\chi_3\,\sqrt{g_{\mathcal{R}\times S^3}}\; i\dot{\phi}\delta(C_{\phi\chi_1\chi_2\chi_3})\\ &= -\frac{N}{2\pi^2}\int_{+\infty}^{-\infty} dt\int_0^\pi d\chi_1\int_0^\pi d\chi_2\int_0^{2\pi} d\chi_3\;\delta C_{\phi\chi_1\chi_2\chi_3}\,\sin\theta\cos^3\theta\sin^2\chi_1\sin\chi_2\\ &= -N\int_{+\infty}^{-\infty} dt\,\delta(C_{\phi\chi_1\chi_2\chi_3})\,\sin\theta\cos^3\theta\,.\end{aligned} \tag{5.2.64}$$

Now we use (5.2.63), to obtain

$$\begin{aligned}\delta S_{WZ} &= -2^{-\frac{\Delta}{2}+2}N\Delta\int dt\, e^{\Delta t}\sin^\Delta\theta\cos^4\theta s^\Delta\\ &= -\cos^4\theta\sin^\Delta\theta(\Delta+1)\sqrt{\Delta}\int dt\,\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^\Delta t}\,.\end{aligned} \tag{5.2.65}$$

Adding the variation of the of DBI and Wess-Zumino terms we find

$$\begin{aligned}\delta S &= \frac{\cos^2\theta\sin^\Delta\theta(\Delta+1)\sqrt{\Delta}}{2}\int_{-\infty}^{\infty} dt\,\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2} t}\\ &= \frac{\cos^2\theta\sin^\Delta\theta(\Delta+1)\sqrt{\Delta}}{2}\mathcal{R}^\Delta 2^{\Delta+2}\int_{-\infty}^{\infty} dt\,\frac{1}{e^{2t}}\left(\frac{e^{2t}}{1-e^{2t}}\right)^{\Delta+2}\\ &= (2\mathcal{R})^\Delta\cos^2\theta\sin^\Delta\theta(\Delta+1)\sqrt{\Delta}\int_0^1 du\, u^\Delta\\ &= -(2\mathcal{R})^\Delta\sqrt{\Delta}\cos^2\sin^\Delta\theta.\end{aligned} \tag{5.2.66}$$

We have changed variables from t to $u = e^{2t}/(1+e^{2t})$. Recalling that, in terms of the gauge theory quantities, $\cos^2\theta = k/N$, $\Delta = J$. The structure constant of the three-point function can now be read from (5.2.66). It is

$$C^A_{k,k-J,J} = \sqrt{J}\frac{k}{N}\left(1-\frac{k}{N}\right)^{\frac{J}{2}}. \tag{5.2.67}$$

5.2.6 Symmetric Giant Graviton. We consider the Euclidean action (5.2.46), in which $t \to t_E = it$. The metric on $S^3 \subset AdS^5$ can be written as

$$d\tilde{\Omega}_3^2 = d\vartheta^2 + \cos^2\vartheta d\phi_1^2 + \sin^2\vartheta d\phi_2^2. \tag{5.2.68}$$

We can write the Euclidean AdS metric as

$$ds^2_{EAdS_5} = \cosh^\rho\, dt^2 + d\rho^2 + \sinh^2\rho(d\vartheta^2 + \cos^2\vartheta d\phi_1^2 + \sin^2\vartheta d\phi_2^2)\,. \tag{5.2.69}$$

The world-volume coordinates of the D3-brane are

$$\sigma^0 = t, \quad \sigma^1 = \vartheta, \quad \sigma^2 = \phi_1, \quad \sigma^3 = \phi_2\,. \tag{5.2.70}$$

The induced metric g_{ab} is given by

$$g_{ab} = \text{Diag}\left(\sinh^2\rho,\ \sinh^2\rho,\ \sinh^2\rho\cos^2\vartheta,\ \sinh^2\rho\sin^2\vartheta\right)\,. \tag{5.2.71}$$

From (5.2.51) and using this metric, the variation of the S_{DBI} part is

$$\begin{aligned}\text{S}_{DBI} &= \frac{N}{4\pi^2}\int d^4\sigma\sqrt{g}\left(g^{00}\left(\delta g_{tt} - \delta g_{\phi\phi}\right) + g^{11}\delta g_{\vartheta\vartheta} + g^{22}\delta g_{\phi_1\phi_1} + g^{33}\delta g_{\phi_2\phi_2}\right)Y_\Delta(\Omega)s^\Delta(X)\\ &= \frac{N}{4\pi^2}\int d^4\sigma\sqrt{g}\frac{1}{\sinh^2\rho}\left(\delta g_{tt} - \delta g_{\phi\phi} + \delta g_{\vartheta\vartheta} + \frac{1}{\cos^2\vartheta}\delta g_{\phi_1\phi_1} + \frac{1}{\sin^2\vartheta}\delta g_{\phi_2\phi_2}\right)Y_\Delta(\Omega)s^\Delta(X)\,.\end{aligned} \tag{5.2.72}$$

Since t, ϑ, ϕ_1, and ϕ_2 are sitting inside the AdS_5 spacetime, we use the fluctuations (5.2.37) and (5.2.57), to find

$$\delta g_{\mu\mu} = h_{\mu\mu} = \frac{2}{\Delta+1}\left[2\nabla_\mu\nabla_\mu - \Delta(\Delta-1)g_{\mu\mu}\right]s\,, \tag{5.2.73}$$

while ϕ is sitting inside S^5, so we use the fluctuation defined in (5.2.38), to find

$$\delta g_{\phi\phi} = 2\Delta g_{\phi\phi}s = 2\Delta s\,, \tag{5.2.74}$$

where $s = s^\Delta(x)Y_\Delta(\Omega)$. Then the variation of S_{DBI} part of the action can be written as

$$\delta S_{DBI} = \frac{N}{4\pi^2}\int_{-\infty}^{\infty}dt\int_0^\pi d\phi_1\int_0^\pi d\phi_2\int_0^{2\pi}d\vartheta\,\sinh^2\rho\cos\vartheta\sin\vartheta\left[-2\Delta s + h_{tt} + h_{\vartheta\vartheta} + \frac{h_{\phi_1\phi_1}}{\cos^2\vartheta} + \frac{h_{\phi_2\phi_2}}{\sin^2\vartheta}\right]. \tag{5.2.75}$$

The variation of the WZ part is given by

$$\begin{aligned}\delta S_{WZ} &= -i\frac{N}{2\pi^2}\int d^4\sigma\, P[\delta C_4]\\ &= -i\frac{N}{2\pi^2}\int_{+\infty}^{-\infty}dt\int_0^\pi d\phi_1\int_0^\pi d\phi_2\int_0^{2\pi}d\vartheta\ \left(-i4\nabla^\rho s^\Delta(X)Y_\Delta(\Omega)\right)\cosh\rho\sinh^3\rho\cos\vartheta\sin\vartheta\\ &= -\frac{2N}{\pi^2}\int_{-\infty}^{\infty}dt\int_0^\pi d\phi_1\int_0^\pi d\phi_2\int_0^{2\pi}d\vartheta\,\sinh^3\rho\cosh\rho\cos\vartheta\partial_\rho\, s\,.\end{aligned} \tag{5.2.76}$$

Adding the variations of the DBI and Wess-Zumino terms we find

$$\delta S = \frac{N}{4\pi^2}\int_{-\infty}^{\infty} dt \int_0^{\pi} d\phi_1 \int_0^{\pi} d\phi_2 \int_0^{2\pi} d\vartheta \ \sinh^2\rho\cos\vartheta\sin\vartheta \\ \times\left(\Big[-2\Delta s + h_{tt} + h_{\vartheta\vartheta} + \frac{h_{\phi_1\phi_1}}{\cos^2\vartheta} + \frac{h_{\phi_2\phi_2}}{\sin^2\vartheta}\Big] - 8\cosh\rho\sinh\rho\partial_\rho\right)s. \tag{5.2.77}$$

In order to simplify this equation we need to compute the following covariant derivatives, in the metric (5.2.71)

$$\begin{aligned}
\nabla_t\nabla_t s &= (\partial_t^2 + \cosh\rho\sinh\rho\partial_\rho)s\,,\\
\nabla_\vartheta\nabla_\vartheta &= (\partial_\vartheta^2 + \cosh\rho\sinh\rho\partial_\rho)s\,,\\
\nabla_{\phi_1}\nabla_{\phi_1} &= (\partial_{\phi_1}^2 + \cos^2\vartheta\cosh\rho\sinh\rho\partial_\rho - \cos\vartheta\sin\vartheta\partial_\vartheta)s\,,\\
\nabla_{\phi_2}\nabla_{\phi_2} &= (\partial_{\phi_2}^2 + \sin^2\vartheta\cosh\rho\sinh\rho\partial_\rho + \cos\vartheta\sin\vartheta\partial_\vartheta)s\,.
\end{aligned}$$

We replace the field s with the bulk-to-boundary propagator

$$s = Y_\Delta(\Omega)s^\Delta(X) \tag{5.2.78}$$

$$Y_\Delta(\Omega) = \frac{e^{\Delta t}}{2^{\frac{\Delta}{2}}}, \quad s^\Delta(X) = \frac{\Delta+1}{\sqrt{\Delta}N2^2}\mathcal{R}^\Delta z^\Delta \tag{5.2.79}$$

where $\mathcal{R} \equiv R/x_B^2$, and $z = 1/(\cosh\rho\cosh t - \cos\vartheta\sin\phi_1\sinh\rho)$. We then compute to following derivatives

$$\begin{aligned}
\partial_t\, s^\Delta &= -\Delta s^\Delta\cosh\rho\sin t z\,,\\
\partial_t^2\, s^\Delta &= \Delta s^\Delta\left((\Delta+1)(\cosh\rho\sinh t\ z)^2 - \cosh\rho\cosh t\ z\right)\,,\\
\partial_\vartheta\, s^\Delta &= -\Delta s^\Delta\sin\vartheta\sin\phi_1\sinh\rho\ z\,,\\
\partial_\vartheta^2\, s^\Delta &= \Delta s^\Delta\left((\Delta+1)(\sin\vartheta\sin\phi_1\sinh\rho\ z)^2 - \cos\vartheta\sin\phi_1\sinh\rho\ z\right)\,,\\
\partial_\rho\, s^\Delta &= \Delta s^\Delta(\cos\vartheta\sinh\phi_1\cosh\rho\ z - \sinh\rho\cosh t\ z)\\
\partial_{\phi_1}\, s^\Delta &= \Delta s^\Delta\cos\vartheta\cos\phi_1\sinh\rho\ z\,,\\
\partial_{\phi_1}^2\, s^\Delta &= \Delta s^\Delta\left((\Delta+1)(\cos\vartheta\cos\phi_1\sinh\rho\ z)^2 - \cos\vartheta\sin\phi_1\sinh\rho\ z\right)\,,\\
\partial_{\phi_2}^2\, s^\Delta &= 0\,.
\end{aligned}$$

Then we obtain

$$h_{tt} = \frac{2}{\Delta+1}\Big[2\Delta\big[(\Delta+1)(\cosh\rho\cosh t\ z)^2-(\Delta+1)(\cosh\rho\ z)^2-\cosh\rho\cosh t\ z\big] + 2\cosh\rho\sinh\rho\,\partial_\rho-\Delta(\Delta-1)\cosh^2\rho\Big]s\,, \tag{5.2.80}$$

$$h_{\vartheta\vartheta} = \frac{2}{\Delta+1}\Big[2\Delta\big[(\Delta+1)(\sin\phi_1\sinh\rho\ z)^2-(\Delta+1)(\cos\vartheta\sin\phi_1\sinh\rho\ z)^2-\cos\vartheta\sin\phi_1\sinh\rho\ z\big] + 2\cosh\rho\sinh\rho\,\partial_\rho-\Delta(\Delta-1)\sinh^2\rho\Big]s\,, \tag{5.2.81}$$

$$\frac{h_{\phi_1\phi_1}}{\cos^2\vartheta} = \frac{2}{\Delta+1}\Big[2\Delta\Big[(\Delta+1)(\cos\phi_1\sinh\rho\ z)^2-\frac{\sin\phi_1\sinh\rho\ z}{\cos\vartheta}\Big]-2\frac{\sin\vartheta}{\cos\vartheta}\,\partial_\vartheta + 2\cosh\rho\sinh\rho\,\partial_\rho-\Delta(\Delta-1)\sinh^2\rho\Big]s\,, \tag{5.2.82}$$

$$\frac{h_{\phi_2\phi_2}}{\sin^2\vartheta} = \frac{2}{\Delta+1}\Big[2\cosh\rho\sinh\rho\,\partial_\rho+2\frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta-\Delta(\Delta-1)\sinh\rho^2\Big]s\,. \tag{5.2.83}$$

Upon substitutions, and performing some algebra, we find

$$\begin{aligned}&-2\Delta s+h_{tt}+h_{\vartheta\vartheta}+\frac{h_{\phi_1\phi_1}}{\cos^2\vartheta}+\frac{h_{\phi_2\phi_2}}{\sin^2\vartheta}=\\ &\frac{2}{\Delta+1}\Big[2\Delta\big[2(\Delta-2)\cos\vartheta\sin\phi_1\sinh\rho z-(\Delta+1)z^2\big]\\ &+8\cosh\rho\sinh\rho\partial_\rho-4\Delta(\Delta-1)\sinh^2\rho\Big]s\,,\end{aligned}$$

where we have used the fact that $z = 1/(\cosh\rho\cosh t-\cos\vartheta\sin\phi_1\sinh\rho)$. Therefore, equation (5.2.77) becomes

$$\begin{aligned}\delta S = &-\frac{\sqrt{\Delta}(\Delta+1)}{4\pi^2}\int_{-\infty}^{\infty}dt\int_0^{2\pi}d\phi_1\int_0^{2\pi}d\phi_2\int_0^{2\pi}d\vartheta\\ &\times\sinh^2\rho\cos\vartheta\sin\vartheta\frac{\mathcal{R}^\Delta e^{\Delta t}}{(\cosh\rho\cosh t-\cos\vartheta\sin\phi_1\sinh\rho)^{\Delta+2}}\,.\end{aligned} \tag{5.2.84}$$

Introducing the new variable $\lambda=\cos\vartheta$, we may recast the integral as follows

$$\delta S = -\frac{\sqrt{\Delta}(\Delta+1)}{2\pi}\frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho}\int_{-\infty}^{\infty}dt\int_0^{2\pi}d\phi_1\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2}t}\int_0^1 d\lambda\,\lambda\Big[1-\frac{\lambda\sin\phi_1\tanh\rho}{\cosh t}\Big]^{-(\Delta+2)} \tag{5.2.85}$$

$$= -\frac{\sqrt{\Delta}(\Delta+1)}{2\pi}\frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho}\int_{-\infty}^{\infty}dt\int_0^{2\pi}d\phi_1\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2}t}\int_0^1 d\lambda\,.$$

Now we use the following negative binomial series

$$(1-x)^{-\alpha}=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k+1)}x^k\,, \tag{5.2.86}$$

to obtain

$$\begin{aligned}\delta S &= -\frac{\sqrt{\Delta}(\Delta+1)}{2\pi}\frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho}\int_{-\infty}^{\infty}dt\int_0^{2\pi}d\phi_1\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2}t}\int_0^1 d\lambda \\ &\quad\times\lambda\sum_{k=0}^{\infty}\left(\frac{\lambda\sin\phi_1\tanh\rho}{\cosh t}\right)^k\frac{\Gamma(\Delta+k+1)}{\Gamma(k+1)\Gamma(\Delta+2)} \\ &= -\frac{\sqrt{\Delta}}{2\pi\Gamma(\Delta+1)}\frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho}\int_{-\infty}^{\infty}dt\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2}t}\int_0^{2\pi}d\phi_1 \\ &\quad\times\sum_{k=0}^{\infty}\frac{1}{k+2}\left(\frac{\sin\phi_1\tanh\rho}{\cosh t}\right)^k\frac{\Gamma(\Delta+k+1)}{\Gamma(k+1)}.\end{aligned} \tag{5.2.87}$$

We now integrate with respect to ϕ_1 using the following integral formula

$$\int_0^{2\pi}d\theta\sin^p\theta=\begin{cases}0 & \text{if } p=2n+1\\ \frac{\pi(2n)!}{2^{2n}(n!)^2} & \text{if } p=2n\end{cases} \tag{5.2.88}$$

which gives

$$\delta S=-\frac{\sqrt{\Delta}}{2\pi\Gamma(\Delta+1)}\frac{1}{\cosh^\Delta\rho}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}dt\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2k+2}t}\frac{1}{2^{2k}}\frac{\Gamma(\Delta+2k+2)}{\Gamma(k+2)\Gamma(k+1)}\tanh^{2k+2}\rho\,. \tag{5.2.89}$$

Then, integrate with respect to t, as follows

$$\begin{aligned}\int_{-\infty}^{\infty}dt\,\frac{e^{mt}}{\cosh^{k+2n}t} &= 2^{k+2n}\int_{-\infty}^{\infty}dt\frac{e^{mt}}{(e^t+e^{-t})^{k+2n}} \\ &= 2^{k+2n}\int_{-\infty}^{\infty}dt\frac{1}{e^{-t(k+2n-m)}}\left(\frac{e^{2t}}{e^{2t}+1}\right)^{k+2n} \\ &= 2^{k+2n-1}\int_0^1 du\,u^{\frac{1}{2}(k+m)+n-1}(1-u)^{\frac{1}{2}(k-m)+n-1} \\ &= 2^{k+2n-1}\frac{\Gamma(\frac{1}{2}(k+m)+n)\Gamma(\frac{1}{2}(k-m)+n)}{\Gamma(k+2n)}.\end{aligned} \tag{5.2.90}$$

Above we have changed variables from t to u as follows

$$u=\frac{e^{2t}}{e^{2t}+1}\quad\Rightarrow e^t=\left(\frac{u}{1-u}\right)^{\frac{1}{2}},\quad dt=\frac{1}{2u(1-u)}du\,.$$

This leads to

$$\delta S=-\frac{(2\mathcal{R})^2}{\sqrt{\Delta}\Gamma(\Delta)}\frac{1}{\cosh^\Delta\rho}\sum_{k=0}^{\infty}\tanh^{2k+2}\rho\frac{\Gamma(\Delta+k+1)}{\Gamma(k+2)}. \tag{5.2.91}$$

This last expression can be simplified by noting

$$\begin{aligned}\cosh^{2\Delta}\rho &= \frac{1}{(1-\tanh^2\rho)^\Delta}=\sum_{k=0}^{\infty}\frac{\Gamma(\Delta+k)}{\Gamma(\Delta)\Gamma(k+1)}\tanh^{2k}\rho\,, \\ \Rightarrow\cosh^{2\Delta}\rho-1 &= \sum_{k=0}^{\infty}\frac{\Gamma(\Delta+k+1)}{\Gamma(\Delta)\Gamma(k+2)}\tanh^{2(k+1)}\rho\,.\end{aligned}$$

Finally, the variation of the action becomes

$$\delta S = -\frac{(2\mathcal{R})^2}{\sqrt{\Delta}}\left(\cosh^{\Delta}\rho - \cosh^{-\Delta}\rho\right). \tag{5.2.92}$$

The three-point function structure constant can now be read from this result. We find

$$C^S_{k,k-J,J} = \frac{1}{\sqrt{J}}\left(\left(1+\frac{k}{N}\right)^{J/2} - \left(1+\frac{k}{N}\right)^{-J/2}\right). \tag{5.2.93}$$

We find that the gauge and string theory results have structural similarities but do not match perfectly. In the case where the two giant gravitons were wrapping an $S^3 \subset AdS_5$, we find that at large angular momentum of the giant gravitons, i.e. $\frac{k}{N} \to \infty$, the three-point function (5.2.93) agrees with the gauge theory result. In the next section we will point out a divergence in the gravity computation. Once this subtlety is correctly dealt with, we will find complete agrment with the gauge theory.

5.3 Regularization Procedure for the 3-point Function of Giant Graviton on S^5

In this section we closely follow [11]. We compute the three-point function for the anti-symmetric giant gravitons. Start by rewriting the variations of the DBI and WZ parts of the Euclidean D-brane action in (5.2.59) and (5.2.64) as computed in [7]

$$\delta S_{DBI} = N\cos^2\theta \int dt\, Y_\Delta \left(\frac{2}{\Delta+1}(\partial_t^2 - \Delta^2) + 4\Delta\cos^2\theta\right) s^\Delta, \tag{5.3.1}$$

and

$$\delta S_{WZ} = N\cos^2\theta \int dt \left(\left(-4\sin\theta\cos\theta\partial_\theta\right) Y_\Delta s^\Delta\right). \tag{5.3.2}$$

Now we can use the expression for the spherical harmonics (5.2.60) and replace the field s^Δ with the bulk to boundary propagator (5.2.61). We also use the notation $Y_\Delta = \tilde{Y}_\Delta e^{ij\phi}$. Therefore the variations are:

$$\delta S_{DBI} = \frac{a_\Delta R^\Delta}{x_B^{2\Delta}} N\cos^2\theta\left(4\Delta\cos^2\theta\tilde{Y}_\Delta \cdot \int_{-\infty}^{+\infty} dt\, \frac{e^{jt}}{\cosh^{\Delta+2n} t}\bigg|_{n\to 0} - 2\Delta\tilde{Y}_\Delta \cdot \int_{-\infty}^{+\infty} dt \frac{e^{ji}}{\cosh^{\Delta+2} t}\right), \tag{5.3.3}$$

and

$$\delta S_{WZ} = \frac{a_\Delta R^\Delta}{x_B^{2\Delta}} N\cos^2\theta\left(-4\left(\sin\theta\cos\theta\partial_\theta\right)\tilde{Y}_\Delta\right) \cdot \int_{-\infty}^{+\infty} dt\, \frac{e^{jt}}{\cosh^{\Delta+2n} t}\bigg|_{n\to 0}. \tag{5.3.4}$$

Thus, the total variation is

$$\begin{aligned}\delta S = \frac{a_\Delta R^\Delta}{x_B^{2\Delta}} N\cos^2\theta\Bigg(&4\left(\Delta\cos^2\theta - \sin\theta\cos\theta\right)\tilde{Y}_\Delta \cdot \int_{-\infty}^{+\infty} dt\, \frac{e^{jt}}{\cosh^{\Delta+2n} t}\bigg|_{n\to 0} \\ &- 2\Delta\tilde{Y}_\Delta \cdot \int_{-\infty}^{+\infty} dt\, \frac{e^{jt}}{\cosh^{\Delta+2n} t}\bigg|_{n\to 0}\Bigg),\end{aligned} \tag{5.3.5}$$

where $\Delta = j + 2l$. For the extremal correlator

$$-(\sin\theta\cos^2\theta)\tilde{Y}\Delta = -\Delta\cos^2\theta\tilde{Y}\Delta \qquad \text{and} \qquad \int_{-\infty}^{+\infty} dt \frac{e^{jt}}{\cosh^{\Delta+2} t}\Big|_{n\to 0} \to \infty\,. \tag{5.3.6}$$

We find that the first piece of the integral in (5.3.5) has the form $0\cdot\infty$. Therefore we introduce a regularization procedure, in which we calculate for a nonexteremal correlator, and then take the limit $l \to 0$. With this regularization procedure, one can see that the piece $\Delta\cos^2\theta\tilde{Y}\Delta - (\sin\theta\cos^2\theta)\tilde{Y}\Delta$ is not completely canceled, and there is a finite piece when $l \to 0$.

We now define the spherical harmonics as introduced in [39]

$$Y_{\Delta,j}(\theta,\phi) = C(l,j)\sin^j\theta e^{i(j\phi)}{}_2F_1(-l, j+l+2, j+1; \sin^2\theta) \tag{5.3.7}$$

where $C(l,j)$ is a normalization constant which is given by

$$C(l,j) = \frac{\Gamma(j+l+1)\sqrt{(j+l+1)(l+1)}2^{-j/2}}{\Gamma(l+2)\Gamma(j+1)\sqrt{j+2l+1}2^l}\,, \tag{5.3.8}$$

and ${}_1F_2$ is the hypergeometric function which is generally defined as follows

$${}_pF_q(a_1\cdots a_p; b_1\cdots b_q; z) = \sum_{m=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}\,. \tag{5.3.9}$$

The rising factorial $(a)_n$ is defined as

$$(a)_n = \begin{cases} a(a+1)(a+2)\cdots(a+n-1) & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}\,. \tag{5.3.10}$$

Let's define ${}_2F_1$ for $|z| < 1$

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\,, \tag{5.3.11}$$

which converges if c is not a negative integer. This hypergeometric function reduces to a finite polynomial if either a or b is a nonpositive integer

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty}(-1)^n\binom{m}{n}\frac{(b)_n}{(c)_n}z^n\,. \tag{5.3.12}$$

We will need the derivative of the hypergeometric function, which is given by

$$\partial_z{}_2F_1(a,b,c;z) = \frac{ab}{c}\,{}_2F_1(a+1,b+1,c+1;z)\,. \tag{5.3.13}$$

Now let's define

$$F_{\Delta,j} = \sin^j\theta\ {}_2F_1(-l, j+l+2, j+1; \sin^2\theta)\,. \tag{5.3.14}$$

Note that

$${}_2F_1(-l, j+l+2, j+1; \sin^2\theta)|_{l=0} = 1 \tag{5.3.15}$$

and

$$\Delta \cos^2\theta F_{\Delta,j} = j\cos^2\theta F_{\Delta,j} + 2l\cos^2\theta F_{\Delta,j} \tag{5.3.16}$$

$$\begin{aligned} -\sin\theta\cos\theta\partial_\theta F_{\Delta,j} &= -\sin\theta\cos\theta\Big[j\cos\theta\sin^{j-1}\theta\ {}_2F_1(-l,j+l+2,j+1;\sin^2\theta) \\ &\quad + \sin^j\theta\ \partial_\theta\sin^2\theta\ \frac{\partial}{\partial\sin^2\theta}\ {}_2F_1(-l,j+l+2,j+1;\sin^2\theta)\Big] \\ &= -j\cos^2\theta F_{\Delta,j} + 2l\cos^2\theta\sin^{j+2}\theta\frac{j+l+2}{j+1} \\ &\quad \times {}_2F_1(1-l,j+l+3,j+2;\sin^2\theta)\,. \end{aligned} \tag{5.3.17}$$

Therefore

$$\begin{aligned} &\Delta\cos^2\theta F_{\Delta,j} - \sin\theta\cos\theta\partial_\theta F_{\Delta,j} = \\ &2l\cos^2\theta\Big(\sin^j\theta + \sin^{j+2}\theta\ \frac{j+l+2}{j+1}\ {}_2F_1(1-l,j+l+3,j+2;\sin^2\theta)\Big)\,. \end{aligned} \tag{5.3.18}$$

On the other hand if we look at the integral in (5.3.5), using the formula given in (5.2.90), we get

$$\int_{-\infty}^{+\infty} dt\ \frac{e^{jt}}{\cosh^{j+2l}t} = 2^{j+2l-1}\frac{\Gamma(j+l)}{\Gamma(j+2l)}\Gamma(l)\,. \tag{5.3.19}$$

This integral has a factor of $\Gamma(l)$, and is divergent at $l=0$, If we multiply (5.3.19) with (5.3.17) we find a factor of $l\Gamma(l)=\Gamma(l+1)$ which is equal to one at $l=0$. However we still need to evaluate the hypergeometric function in (5.3.17), in order to simplify the total variation of the D–brane action (5.3.5). We use the representation of the hypergeometric function as follows

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\ \frac{z^n}{n!}\,. \tag{5.3.20}$$

Using this formula, and replacing $\sin^2\theta$ by x, we obtain

$$\begin{aligned} &{}_2F_1(1-l,j+l+3,j+2;x) \\ &\quad = \frac{\Gamma(j+2)}{\Gamma(1-l)\Gamma(j+l+3)}\sum_{n=0}^{\infty}\frac{\Gamma(j+l+n+3)\Gamma(1-l+n)}{\Gamma(j+n+2)}\ \frac{x^n}{n!} \\ &\quad = \frac{1}{j+2}\sum_{n=0}^{\infty}(j+n+2)x^n = \frac{1}{j+2}\sum_{n=0}^{\infty}\left[(j+1)x^n + (n+1)x^2\right]. \end{aligned} \tag{5.3.21}$$

Now we need the expression of the geometric series

$$\sum_{n=0}^{m}x^n = \frac{1-x^{m+1}}{1-x}\,, \tag{5.3.22}$$

so that for $|x|<1$ we have

$$\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}\,. \tag{5.3.23}$$

This implies

$$ {}_2F_1(1-l, j+l+3, j+2; x) = \frac{1}{j+2}\left[\frac{j+1}{1-x} + \frac{1}{(1-x)^2}\right]. \tag{5.3.24} $$

Using this we find

$$ \cos^2\theta\Big(\sin^j\theta + \sin^{j+2}\theta\ \frac{j+l+2}{j+1}\ {}_2F_1(1-l, j+l+3, j+2; \sin^2\theta)\Big)\Big|_{l=0} = \frac{\sin^j\theta}{\cos^2\theta}\frac{1}{j+1}(1+j\cos^2\theta) \tag{5.3.25} $$

and the first line in (5.3.5) becomes

$$ \begin{aligned} &\frac{a_\Delta R^\Delta}{x_B^{2\Delta}} N\cos^2\theta 4\left(\Delta\cos^2\theta - \sin\theta\cos\theta\right)\tilde{Y}_\Delta \cdot \int_{-\infty}^{+\infty} dt\, \frac{e^{jt}}{\cosh^{j+2l}t}\Big|_{l\to 0} \\ &\quad = (2\mathcal{R})^\Delta \frac{1}{\sqrt{j}}\left(1+\frac{k}{N}j\right)\left(1+\frac{k}{N}\right)^{\frac{j}{2}} \end{aligned} \tag{5.3.26} $$

where $\cos^2\theta = \frac{k}{N}$. The two divergent terms in the first line of (5.3.5), therefore do not cancel completely and yield a finite piece. The other piece is

$$ \frac{a_\Delta R^\Delta}{x_B^{2\Delta}} N\cos^2\theta\left(-2\Delta\tilde{Y}_\Delta \cdot \int_{-\infty}^{+\infty} \frac{e^{jt}}{\cosh^{\Delta+2}t}\right) = (2\mathcal{R})^\Delta \frac{k}{N}\sqrt{j}\left(1-\frac{k}{N}\right)^{\frac{j}{2}}. \tag{5.3.27} $$

Adding all the pieces together from (5.3.26) and (5.3.27), we find that the three-point function structure constant is

$$ (-1)^{j-1}\left(1-\frac{k}{N}\right)^{\frac{j}{2}} \tag{5.3.28} $$

where we reinstated the factor $(-1)^j$, which depends on normalization conventions. This result exactly agrees with the gauge theory computation. Therefore the above computation indicates the mismatch in the previous discussion, and shows that the disagreement of the computation between the gauge theory and the string theory is related to the subtleties of the extremal correlators. This also shows that Schur Polynomials are indeed the operators dual to the giant gravitons.

In this chapter we have seen that divergences are also found in the case of extremal correlators containing giant gravitons. The regularization that was chosen entails the analytic continuation of the non-extremal correlators to the extremal cases. This regularization enabled us to extract the finite parts of the divergent integrals and we verified that gauge theory and gravity agree. It is however still desirable to have a deeper understanding of the origin of these divergences.

6. Correlation Functions of Kaluza-Klein Gravitons

In this chapter, we consider the three-point function of Kaluza-Klein gravitons, of the scalar gravity field dual to the chiral primary operators in gauge theory. We will find divergences in higher point functions which again suggests that these divergences are a general feature of the theory.

6.1 Supergravity computation of $\langle \mathcal{O}^{k_1}_{t,s} \mathcal{O}^{k_2}_{\phi} \mathcal{O}^{k_3}_{\phi} \rangle$

In this chapter we describe in detail the computation of the correlation function $\langle \mathcal{O}^{k_1}_{t} \mathcal{O}^{k_2}_{\phi} \mathcal{O}^{k_3}_{\phi} \rangle$ where $\mathcal{O}^k_t$ and $\mathcal{O}^k_\phi$ are the SYM operators that couple respectively to the Kaluza– Klein (KK) modes of the dilaton ϕ^k and supergravity scalar field t^k (coming from the scalar KK modes arising from the 4-form and the trace of the graviton with indices on the sphere). We will give particular emphasis to the extremal values $k_1 = k_2 + k_3$.

We use the dimensional reduction procedure to compactify the 10–dimensional supergravity action from ten to five dimensions on AdS_5. The dimensional reduction is obtained using the expansion of the fluctuations of the background fields in terms of spherical harmonics. In this computation we first expand the fields on S^5 in scalar spherical harmonics, and choose covariant gauge conditions which reduce these harmonic expansions to a very simple form. Next we insert these simple harmonic expansions into the kinetic term for the dilaton in the 10–dimensional action, thus obtaining the dimensionally reduced action. Then we manipulate the interaction Lagrangian by partial integration and use the linearized equations of motion.

We then compute 3-point correlation functions using the two methods of analytic continuation and space–time cutoff, as introduced in [12].

6.1.1 Set up. We will work with Euclidean signature. We use latin indices $i, j, k, \cdots$ for the whole 10-dimensional manifold. Indices $\mu, \nu, \lambda, \cdots$ are AdS_5 indices, while $\alpha, \beta, \gamma, \cdots$ are S^5 indices. We use G_{mn} to indicate the 10-dimensional metric, g_{mn} and h_{mn} to indicate the background metric and the fluctuation respectively. We also set the S^5 and AdS_5 scales to 1.

Following [38] and [13], the fluctuations of the metric and the gravitational field are parametrized as follows

$$G_{mn} = g_{mn} + h_{mn}\,, \tag{6.1.1}$$

$$h_{\alpha\beta} = h_{(\alpha\beta)} + \frac{h_2}{5} g_{\alpha\beta}; \qquad g^{\alpha\beta} h_{(\alpha\beta)} = 0, \tag{6.1.2}$$

$$h_{\mu\nu} = h'_{\mu\nu} - \frac{h_2}{3} g_{\mu\nu}, \qquad h'_{\mu\nu} = h'_{(\mu\nu)} + \frac{h'}{5} g_{\mu\nu}; \qquad g^{\mu\nu} h'_{(\mu\nu)} = 0, \tag{6.1.3}$$

$$F = \bar{F} + \delta F, \qquad \delta F_{ijklm} = 5 \nabla_{[i} a_{jklm]}. \tag{6.1.4}$$

Here $\bar{F}$ is the background value of the F-field and we choose the Donder gauge conditions

$$\nabla^\alpha h_{\alpha\beta} = \nabla^\alpha h_{\mu\alpha} = \nabla^\alpha a_{\alpha\mu_1 m_2 m_3 m_4} = 0. \tag{6.1.5}$$

With this choice, the general expansion of the fluctuations in terms of harmonics of the sphere, are given by

$$h'_{\mu\nu} = \sum Y^k h'^k_{\mu\nu} \,, \tag{6.1.6}$$

$$h_2 = \sum Y^k h'^k_2 \,, \tag{6.1.7}$$

$$a_{\alpha_1\alpha_2\alpha_3\alpha_4} = D^\alpha \epsilon_{\alpha\alpha_1\alpha_2\alpha_3\alpha_4} b^k \,, \tag{6.1.8}$$

$$\phi = \sum Y^k \phi^k \,. \tag{6.1.9}$$

We have introduced a complete orthonormal set of scalar spherical harmonics Y^k satisfying

$$\nabla_\alpha \nabla^\alpha Y^k = -k(k+4) Y^k \,, \tag{6.1.10}$$

where k is a nonnegative integer. We are interested in the linearized field equations which imply the following three constraint equations [38]

$$0 = \left(\frac{1}{2} h'^k - \frac{8}{15} h^k_2 \right) \nabla_{(\alpha} \nabla_{\beta)} Y^k \,, \tag{6.1.11}$$

$$0 = \left[\nabla_\mu h'^{k\mu\nu} - \nabla^\nu \left(h'^k - \frac{8}{15} h^k_2 + 8 b^k \right) - \frac{8}{4!} \epsilon^{\nu\mu_1\mu_2\mu_3\mu_4} a^k_{\mu_1\mu_2\mu_3\mu_4} \right] \nabla_\alpha Y^k \,, \tag{6.1.12}$$

$$0 = \left[a^k_{\mu_1\mu_2\mu_3\mu_4} + \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5} \nabla^{\mu_5} b^k \right] \nabla_\alpha Y^k \,. \tag{6.1.13}$$

The modes h^k_2 and b^k are coupled and obey the following linearized equations of motion [13]

$$0 = \left[\nabla_m \nabla^m b^k + \left(\frac{1}{2} h'^k - \frac{4}{3} h^k_2 \right) \right] Y^k \,, \tag{6.1.14}$$

$$0 = \left[\left(\nabla_m \nabla^m - 32 \right) h^k_2 + 80 \nabla_\alpha \nabla^\alpha b^k + \nabla_\alpha \nabla^\alpha \left(h'^k - \frac{16}{15} h^k_2 \right) \right] Y^k \,. \tag{6.1.15}$$

We can rewrite (6.1.11) as follow

$$h'^k = \frac{16}{15} h^k_2, \tag{6.1.16}$$

to find

$$\nabla_m \nabla^m b - \frac{4}{5} h_2 = 0 \tag{6.1.17}$$

$$\left(\nabla_m \nabla^m b - 32 \right) h_2 + 80 \nabla_\alpha \nabla^\alpha b = 0. \tag{6.1.18}$$

Using the property of the spherical harmonics Y^k given in (6.1.10), the equations of motion for the modes h^k_2 and b^k are given by

$$\nabla_\mu \nabla^\mu b^k = k(k+4) b^k + \frac{4}{5} h^k_2 \,, \tag{6.1.19}$$

$$\nabla_\mu \nabla^\mu h^k_2 = k(k+4) \left[80 b^k + h^k_2 \right] + 32 h^k_2 \,. \tag{6.1.20}$$

These equations can easily be diagonalized. The diagonal linear combinations are

$$s^k = \frac{1}{20(k+2)} [h^k_2 - 10(k+4) b^k] \,, \tag{6.1.21}$$

$$t^k = \frac{1}{20(k+2)} [h^k_2 + 10 k b^k] \,. \tag{6.1.22}$$

Now we apply $\nabla_\mu \nabla^\mu$ on s^k and b^k. Using (6.1.20) and (6.1.19), we have

$$\begin{aligned}
\nabla_\mu \nabla^\mu s^k &= \frac{1}{20(k+2)} \Big[\nabla_\mu \nabla^\mu h_2^k - 10(k+4) \nabla_\mu \nabla^\mu b^k \Big] \\
&= \frac{1}{20(k+2)} \left(\Big[k(k+4) \big[80 b^k + h_2^k \big] + 32 h_2^k \Big] - 10(k+4) \Big[k(k+4) b^k + \frac{4}{5} h_2^k \Big] \right) \\
&= k(k-4) \frac{1}{20(k+2)} \Big[-10k(k+4) b^k + h_2^k \Big] \\
&= k(k-4) s^k \,, \qquad (6.1.23)
\end{aligned}$$

and

$$\begin{aligned}
\nabla_\mu \nabla^\mu t^k &= \frac{1}{20(k+2)} \Big[\nabla_\mu \nabla^\mu h_2^k + 10k \nabla_\mu \nabla^\mu b^k \Big] \\
&= \frac{1}{20(k+2)} \left(\Big[k(k+4) \big[80 b^k + h_2^k \big] + 32 h_2^k \Big] + 10k \Big[k(k+4) b^k + \frac{4}{5} h_2^k \Big] \right) \\
&= (k+4)(k+8) \frac{1}{20(k+2)} \Big[h_2^k + 10k b^k \Big] \\
&= (k+4)(k+8) t^k \,. \qquad (6.1.24)
\end{aligned}$$

The equations of motion (6.1.24) and (6.1.23) which follow from supergravity yield constraints between the modes defined above [13]. These constraints involve the traceless part and the trace part of the $h'_{\mu\nu}$ field are determined as follows

$$h'_{(\mu\nu)} = \nabla_{(\mu} \nabla_{\nu)} \left(\frac{2}{5(k+1)(k+3)} (h_2^k - 30 b^k) \right) \qquad (6.1.25)$$

$$h'^k = \frac{16}{15} h_2^k \,. \qquad (6.1.26)$$

6.1.2 Cubic Action. The kinetic term for the dilaton in the 10-dimensional supergravity action on $AdS_5 \times S^5$ is given by

$$S = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10} x \sqrt{G} \, \frac{1}{2} G^{mn} \, \partial_m \phi \, \partial_n \phi \,. \qquad (6.1.27)$$

We can also set

$$h_{\mu\alpha} \equiv 0 \qquad h_{(\alpha\beta)} \equiv 0. \qquad (6.1.28)$$

Then, expand the determinant of the metric G_{mn} as follows

$$G = \det(G_{mn}) = \det(g_{mn} + h_{mn}) = \det(g_{mn}) \det(1 + g^{mn} h_{mn}) \,. \qquad (6.1.29)$$

There is an identity, which states

$$\ln(\det M) = \mathrm{Tr}(\ln M) \,. \qquad (6.1.30)$$

Taking the square root of G and applying this identity, we find

$$\begin{aligned}\sqrt{G} &= \sqrt{\det g_{mn}} \exp\left[\frac{1}{2} \ln\det(1 + g^{mn} h_{mn})\right] \\ &= \sqrt{g} \exp\left[\frac{1}{2} \mathrm{Tr} \ln(1 + g^{mn} h_{mn})\right] . \end{aligned} \tag{6.1.31}$$

Now use the power series of $\ln(1+x)$ and of $\exp(x)$, to the first order, to find

$$\sqrt{G} = \sqrt{g}\Big[1 + \frac{1}{2} g^{mn} h_{mn} + \cdots\Big] . \tag{6.1.32}$$

We know that $g^{mn} h_{mn} = g^{\mu\nu} h_{\mu\nu} + g^{\alpha\beta} h_{\alpha\beta}$ and $\sqrt{g} = \sqrt{g_1}\sqrt{g_2}$, where g_1, g_2 indicate the determinant of the background metric on AdS_5 and S^5 respectively. Thus, we have

$$\begin{aligned}\sqrt{G} &= \sqrt{g_1}\sqrt{g_2}\Big[1 + \frac{1}{2} g^{\mu\nu} h_{\mu\nu} + \frac{1}{2} g^{\alpha\beta} h_{\alpha\beta} + \cdots\Big] \\ &= \sqrt{g_1}\sqrt{g_2}\Big[1 + \frac{1}{2} g^{\mu\nu}\big(h'_{(\mu\nu)} + \frac{h'_2}{5} g_{\mu\nu} - \frac{h_2}{3} g_{\mu\nu}\big) + \frac{1}{2} g^{\alpha\beta}\big(h_{(\alpha\beta)} + \frac{h_2}{5} g_{\alpha\beta}\big) + \cdots\Big] \\ &= \sqrt{g_1}\sqrt{g_2}\Big[1 - \frac{1}{3} h_2 + \frac{1}{2} h' + \cdots\Big] , \end{aligned} \tag{6.1.33}$$

where we have used (6.1.2), (6.1.3) and (6.1.28), and here d is the dimension of the boundary of AdS_{d+1}. (We will specialize to d = 4). We obtain, for the dilaton kinetic term, the expansion (to cubic order in the fluctuations)

$$\begin{aligned} S &= \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{g} \frac{1}{2}\Big(1 - \frac{1}{3} h_2 + \frac{1}{2} h'\Big)\Big(g_{mn} - h_{mn}\Big)\nabla^m \phi \nabla^n \phi \\ &= \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{g} \frac{1}{2}\Big(1 - \frac{1}{3} h_2 + \frac{1}{2} h'\Big)\Big(\nabla_\mu \phi \nabla^\mu \phi + \nabla_\alpha \phi \nabla^\alpha \phi - h_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi - h_{\alpha\beta} \nabla^\alpha \phi \nabla^\beta \phi\Big) . \end{aligned} \tag{6.1.34}$$

Again use (6.1.2) and (6.1.3), to find

$$\begin{aligned} S &= \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{g} \frac{1}{2}\Big(1 - \frac{1}{3} h_2 + \frac{1}{2} h'\Big) \\ &\quad \times \Big(\nabla_\mu \phi \nabla^\mu \phi + \nabla_\alpha \phi \nabla^\alpha \phi - h'_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi + \frac{h_2}{3} \nabla_\mu \phi \nabla^\nu \phi - \frac{h_2}{5} \nabla_\alpha \phi \nabla^\alpha \phi\Big) \\ &= \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{g}\Big[\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + \frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi \\ &\quad + \Big(\frac{1}{4} h' - \frac{4}{15} h_2\Big) \nabla_\alpha \phi \nabla^\alpha \phi + \frac{1}{4} h' \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} h'_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi\Big] . \end{aligned} \tag{6.1.35}$$

In the last line we only kept the linear terms in the fluctuations and we have used ∇ to denote the covariant derivative.

6.1.3 Dimensional Reduction. We will consider only scalar fields on the S^5, and we expand these fields in scalar spherical harmonics. The spherical harmonics are normalized, as shown in Appendix B, such that

$$\int Y^{k_1} Y^{k_2} = z(k) \delta^{k_1 k_2} , \tag{6.1.36}$$

$$\int Y^{k_1} Y^{k_2} Y^{k_3} = a(k_1, k_2, k_3) \langle C^{k_1} C^{k_2} C^{k_3} \rangle , \tag{6.1.37}$$

where

$$z(k) = \frac{1}{2^{k-1}(k+1)(k+2)}\,, \tag{6.1.38}$$

$$a(k_1,k_2,k_3) = \frac{\omega_5}{(\Sigma+2)!2^{\Sigma-1}\alpha_1!\alpha_2!\alpha_3!}\,, \tag{6.1.39}$$

with $\alpha_1 = \frac{1}{2}(k_2+k_3-k_1), \alpha_2 = \frac{1}{2}(k_1+k_3-k_2), \alpha_3 = \frac{1}{2}(k_1+k_2-k_3), \sum = \frac{1}{2}(k_1+k_2-k_3)$ and $\omega_5 = \pi^3$ is the area of a unit 5– sphere. To compute the 3–point functions $\langle \mathcal{O}^{k_1}_t \mathcal{O}^{k_2}_{\phi} \mathcal{O}^{k_3}_{\phi} \rangle$ we need to consider excitations with the fields s^k set to zero. Then, from the definition (6.1.21) of s^k and the constraints, we find

$$h^k_2 = 10(k+4)b^k\,. \tag{6.1.40}$$

From this equation and (6.1.22), we find

$$h^k_2 = 10(k+4)t^k\,. \tag{6.1.41}$$

Then we obtain

$$\begin{aligned} h'^k_{\mu\nu} &= h'^k_{(\mu\nu)} + \frac{h'^k}{5} g_{\mu\nu} \\ &= \nabla_{(\mu}\nabla_{\nu)}\Big[\frac{2}{5(k+1)(k+3)}(h^k_2 - 30b^k) + \frac{16}{15}\frac{h^k_2}{5}g_{\mu\nu}\Big] \\ &= \nabla_{(\mu}\nabla_{\nu)}\Big[\frac{2}{5(k+1)(k+3)}\Big(h^k_2 - \frac{30}{10(k+4)}h^k_2\Big)\Big] + \frac{16}{15}\frac{h^k_2}{5}g_{\mu\nu} \\ &= \Big[\frac{2}{5(k+3)(k+4)}\Big]\nabla_{(\mu}\nabla_{\nu)}h^k_2 + \frac{16}{15}\frac{h^k_2}{5}g_{\mu\nu}\,. \end{aligned} \tag{6.1.42}$$

The traceless double covariant derivative for the field h^k_2 is given by

$$\begin{aligned} \nabla_{(\mu}\nabla_{\nu)}h^k_2 &= \nabla_\mu\nabla_\nu h^k_2 - \frac{1}{5}g_{\mu\nu}\nabla_\rho\nabla^\rho h^k_2 \\ &= \nabla_\mu\nabla_\nu h^k_2 - \frac{1}{5}g_{\mu\nu}10(k+4)\nabla_\rho\nabla^\rho t^k\,. \end{aligned} \tag{6.1.43}$$

Using (6.1.24) and (6.1.41), we find

$$\begin{aligned} h'^k_{\mu\nu} &= \frac{2}{5(k+3)(k+4)}\nabla_\mu\nabla_\nu h^k_2 - \frac{2(k+8)}{25(k+3)}g_{\mu\nu}h^k_2 + \frac{16}{15}\frac{h^k_2}{5}g_{\mu\nu} \\ &= \frac{2}{5(k+3)(k+4)}\nabla_\mu\nabla_\nu h^k_2 + \frac{2}{15}\frac{k}{k+3}h^k_2 g_{\mu\nu}\,. \end{aligned} \tag{6.1.44}$$

To get the dimensionally reduced form of the action (6.1.35), we need to insert the harmonic expansions (6.1.6), (6.1.7) and (6.1.9), into the action (6.1.35) and evaluate this action, term by term.

- The first term gives

$$\begin{aligned}
&\int \mathrm{d}^{10}x \sqrt{g_1}\sqrt{g_2}\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi \\
&\qquad = \int \mathrm{d}^{10}x \sqrt{g_1}\sqrt{g_2}\frac{1}{2}\sum_{k_1 k_2} Y^{k_1}Y^{k_2}\nabla_\mu\phi^{k_1}\nabla^\mu\phi^{k_2} \\
&\qquad = \int \mathrm{d}^{5}x \sqrt{g_1}\frac{1}{2}\sum_{k_1 k_2} z(k)\delta_{k_1 k_2}\nabla_\mu\phi^{k_1}\nabla^\mu\phi^{k_2} \\
&\qquad = \int \mathrm{d}^{5}x \sqrt{g_1}\frac{z(k)}{2}\nabla_\mu\phi^{k}\nabla^\mu\phi^{k} \,. \qquad (6.1.45)
\end{aligned}$$

- The second term becomes

$$\begin{aligned}
&\int \mathrm{d}^{10}x \sqrt{g_1}\sqrt{g_2}\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi \\
&\qquad = \int \mathrm{d}^{10}x \sqrt{g_1}\sqrt{g_2}\frac{1}{2}\sum_{k_1 k_2} \nabla_\alpha Y^{k_1}\nabla^\alpha Y^{k_2}\phi^{k_1}\phi^{k_2} \qquad (6.1.46)
\end{aligned}$$

  This can be evaluated by performing an integration by parts as follows

$$\int_{S^5} \nabla_\alpha Y^{k_1}\nabla^\alpha Y^{k_2} = -\int_{S^5} Y^{k_1}\nabla_\alpha\nabla^\alpha Y^{k_2} \,, \qquad (6.1.47)$$

  and using the fact that

$$\nabla_\alpha\nabla^\alpha Y^k = -k(k+4)Y^k \,. \qquad (6.1.48)$$

  The term (6.1.46) then becomes

$$\begin{aligned}
&\int \mathrm{d}^{10}x \sqrt{g_1}\sqrt{g_2}\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi \\
&\qquad = \int \mathrm{d}^{10}x \sqrt{g_1}\sqrt{g_2}\frac{1}{2}\sum_{k_1 k_2} k_2(k_2+4)Y^{k_1}Y^{k_2}\phi^{k_1}\phi^{k_2} \\
&\qquad = \int \mathrm{d}^{5}x \sqrt{g_1}\frac{1}{2}\sum_{k_1 k_2} k_2(k_2+4)z(k)\delta_{k_1 k_2}\phi^{k_1}\phi^{k_2} \\
&\qquad = \int \mathrm{d}^{5}x \sqrt{g_1}\frac{z(k)}{2}k_2(k_2+4)\phi^{k}\phi^{k} \,. \qquad (6.1.49)
\end{aligned}$$

- The third term $\left(\frac{1}{4}h' - \frac{4}{15}h_2\right)$ vanishes by the constraint equation (6.1.26). Thus, we only have an indirect coupling to h_2 given through the excitation of the field $h'_{\mu\nu}$.

- The fourth term gives

$$\begin{aligned}
&\int \mathrm{d}^{10}x \sqrt{g_1}\sqrt{g_2}\frac{1}{4}h'\nabla_\mu\phi\nabla^\mu\phi \\
&\qquad = \int \mathrm{d}^{10}x \sqrt{g_1}\sqrt{g_2}\frac{1}{4}\sum_{k_1}\frac{16}{15}Y^{k_1}h_2^{k_1}\sum_{k_2}Y^{k_2}\nabla_\mu\phi^{k_2}\sum_{k_3}Y^{k_3}\nabla^\mu\phi^{k_3} \\
&\qquad = \int \mathrm{d}^{10}x \sqrt{g_1}\sqrt{g_2}\frac{4}{15}\sum_{k_1,k_2,k_3} Y^{k_1}Y^{k_2}Y^{k_3}10(k_1+4)t^{k_1}\nabla_\mu\phi^{k_2}\nabla^\mu\phi^{k_3} \\
&\qquad = a(k_1,k_2,k_3)\int \mathrm{d}^{5}x \sqrt{g_1}\frac{8}{3}(k_1+4)t^{k_1}\nabla_\mu\phi^{k_2}\nabla^\mu\phi^{k_3} \,. \qquad (6.1.50)
\end{aligned}$$

- The last term evaluates to

$$\begin{aligned}
&\int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\left(-\frac{1}{2}\right) h'_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi \\
&\qquad = \int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\left(-\frac{1}{2}\right)\sum_{k_1} Y^{k_1}\left[\frac{2}{5(k_1+3)(k_1+4)}\nabla_\mu\nabla_\nu h_2^{k_1} + \frac{2}{15}\frac{k_1}{k_1+3}h_2^{k_1}g_{\mu\nu}\right] \\
&\qquad\qquad \times \sum_{k_2} Y^{k_2}\nabla^\mu\phi^{k_2}\sum_{k_3} Y^{k_3}\nabla^\nu\phi^{k_3} \\
&\qquad = \int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\left(-\frac{1}{2}\right)\sum_{k_1,k_2,k_3}\left[\frac{4}{(k_1+3)}\nabla_\mu\nabla_\nu t^{k_1} + \frac{2}{15}\frac{k_1}{k_1+3}10(k_1+4)t^{k_1}g_{\mu\nu}\right] \\
&\qquad\qquad \times Y^{k_1}Y^{k_2}Y^{k_3}\nabla^\mu\phi^{k_2}\nabla^\nu\phi^{k_3} \\
&\qquad = a(k_1,k_2,k_3)\int \mathrm{d}^5x\sqrt{g_1}\left(-\frac{1}{2}\right)\left[\frac{4}{(k_1+3)}\nabla_\mu\nabla_\nu t^{k_1}\nabla^\mu\phi^{k_2}\nabla^\nu\phi^{k_3}\right. \\
&\qquad\qquad \left. + \frac{2}{15}\frac{k_1}{k_1+3}10(k_1+4)t^{k_1}\nabla^\mu\phi^{k_2}\nabla_\mu\phi^{k_3}\right] .
\end{aligned} \tag{6.1.51}$$

Combining all of these terms we find the dimensionally reduced form of the action (6.1.35) is

$$\begin{aligned}
S = \frac{1}{2\kappa_5^2}\int \mathrm{d}^5x\sqrt{g_1}\Bigg[&\frac{z(k)}{2}\left(\nabla_\mu\phi^k\nabla^\mu\phi^k - k(k+4)\phi^k\phi^k\right) \\
&+ a(k_1,k_2,k_3)\left(2\frac{(k_1+4)^2}{(k_1+3)}t^{k_1}\nabla^\mu\phi^{k_2}\nabla^\mu\phi^{k_3} - \frac{2}{k_1+3}\nabla_\mu\nabla_\nu t^{k_1}\nabla^\mu\phi^{k_2}\nabla^\nu\phi^{k_3}\right)\Bigg] .
\end{aligned} \tag{6.1.52}$$

The gravitational coupling constant in (6.1.52) is related to the SYM parameter N by

$$2\kappa_5^2 = \frac{8\pi^2}{N^2} . \tag{6.1.53}$$

6.1.4 Evaluation of the Action. The interaction Lagrangian can be manipulated by partial integration and by using the linearized equations of motion for the fields. We again evaluate the relevant terms in (6.1.52) one by one, as follows

- The first cubic term yields

$$\begin{aligned}
&\int_{AdS_5} t^{k_1}\nabla_\mu\phi^{k_2}\nabla^\mu\phi^{k_3} \\
&= \int_{(AdS_5)} t^{k_1}\frac{1}{2}\nabla^\mu\nabla_\mu(\phi^{k_2}\phi^{k_3}) - \int_{(AdS_5)}\frac{1}{2}t^{k_1}\left((\nabla^\mu\nabla_\mu\phi^{k_1})\phi^{k_2} + (\nabla^\mu\nabla_\mu\phi^{k_2})\right)\phi^{k_1} \\
&= \int_{(AdS_5)} t^{k_1}\frac{1}{2}\nabla^\mu\nabla_\mu(\phi^{k_2}\phi^{k_3}) - \int_{(AdS_5)}\frac{1}{2}t^{k_1}\left(m_\phi^2(k_2) + m_\phi^2(k_3)\right)\phi^{k_2}\phi^{k_3} .
\end{aligned} \tag{6.1.54}$$

To evaluate the first term in this equation we consider the usual upper half-plane metric for the AdS spacetime as introduced in Chapter 4. In this case, the Laplacian is defined in the coordinates z_0, z, where $i = 1 \cdots d$

$$\nabla_\mu\nabla^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu) = z^{d+1}\partial_\mu\left(z_0^{-d+1}\partial_\mu\right) . \tag{6.1.55}$$

Using this we find

$$\begin{aligned}
&\int_{(AdS_5)} t^{k_1}\frac{1}{2}\nabla^\mu\nabla_\mu(\phi^{k_2}\phi^{k_3})\\
&=\frac{1}{2}\int\frac{d^dzdz_0}{z_0^{d+1}}\; t^{k_1}z^{d+1}\partial_\mu\big(z_0^{-d+1}\partial_\mu\phi^{k_2}\phi^{k_3}\big)\\
&=\frac{1}{2}\int d^dzdz_0\; t^{k_1}\partial_\mu\big(z_0^{-d+1}\partial_\mu\phi^{k_2}\phi^{k_3}\big)\\
&=\frac{1}{2}\int d^dzdz_0\; t^{k_1}\partial_{z_0}\big(z_0^{-d+1}\partial_{z_0}\phi^{k_2}\phi^{k_3}\big)+\frac{1}{2}\int d^dzdz_0\; t^{k_1}\partial_{z_i}\big(z_0^{-d+1}\partial_{z_i}\phi^{k_2}\phi^{k_3}\big)\,.
\end{aligned}\tag{6.1.56}$$

Lets integrate, by parts, the last term in this equation to find

$$\begin{aligned}
&\frac{1}{2}\int d^dzdz_0\; t^{k_1}\partial_{z_i}\big(z_0^{-d+1}\partial_{z_i}\phi^{k_2}\phi^{k_3}\big)\\
&=\frac{1}{2}\int d^dzdz_0\big(\partial_{z_i}z_0^{-d+1}\partial z_i t^{k_1}\big)\phi^{k_2}\phi^{k_3}\\
&=\frac{1}{2}\int\nabla_\mu\nabla^\mu t^{k_1}\phi^{k_2}\phi^{k_3}-\frac{1}{2}\int_{AdS_5}z^{d+1}\partial_{z_0}\big(z_0^{-d_1}\partial_{z_0}t^{k_1}\big)\phi^{k_2}\phi^{k_3}\,.
\end{aligned}\tag{6.1.57}$$

Substituting this back into (6.1.56) we find

$$\int_{(AdS_5)} t^{k_1}\frac{1}{2}\nabla^\mu\nabla_\mu(\phi^{k_2}\phi^{k_3})=\frac{1}{2}\int_{AdS_5}\nabla_\mu\nabla^\mu t^{k_1}\phi^{k_2}\phi^{k_3}\tag{6.1.58}$$

$$\begin{aligned}
&-\frac{1}{2}\int_{AdS_5}z^{d+1}\partial_{z_0}\big(z_0^{-d_1}\partial_{z_0}t^{k_1}\big)\phi^{k_2}\phi^{k_3}+\frac{1}{2}\int d^dzdz_0\; t^{k_1}\partial_{z_0}\big(z_0^{-d+1}\partial_{z_0}\phi^{k_2}\phi^{k_3}\big)\\
&=\frac{1}{2}\int_{AdS_5}\nabla_\mu\nabla^\mu t^{k_1}\phi^{k_2}\phi^{k_3}+\frac{1}{2}\int d^dzt^{k_1}z_0^{-d+1}\partial_{z_0}\phi^{k_2}\phi^{k_3}+\frac{1}{2}\int d^dzz_0^{-d+1}\partial_{z_0}t^{k_1}\phi^{k_2}\phi^{k_3}
\end{aligned}\tag{6.1.59}$$

Using this result the first cubic term in (6.1.54) becomes

$$\begin{aligned}
\int_{AdS_5}t^{k_1}\nabla_\mu\phi^{k_2}\nabla^\mu\phi^{k_3}&=\int_{(AdS_5)}\frac{1}{2}(m_t^2(k_1)-m_\phi^2(k_2)-m_\phi^2(k_3))t^{k_1}\phi^{k_2}\phi^{k_3}\\
&+\frac{1}{2}\int_{\partial(AdS_5)}t^{k_1}\nabla_n(\phi^{k_2}\phi^{k_3})-\frac{1}{2}\int_{\partial(AdS_5)}\phi^{k_2}\phi^{k_3}\nabla_n t^{k_1}
\end{aligned}\tag{6.1.60}$$

where ∇_n indicates the outward normal derivative to the boundary and we have introduced the symbols $m_\phi^2(k)$ and $m_t^2(k)$ to denote the masses of the fields ϕ_k

$$m_\phi^2(k)\equiv k(k+4),\qquad m_t^2(k)\equiv(k+4)(k+8)\,.\tag{6.1.61}$$

The boundary integrals in the last step in (6.1.60) cannot contribute to the 3–point function if all the three points are disjoint. This is the case because

$$\int_{\partial(AdS_5)}t^{k_1}\nabla_n(\phi^{k_2}\phi^{k_3})=\int\int_{\partial(AdS_5)}t^{k_1}(x)\frac{\partial}{\partial z_n}K(z,x')\phi^{k_2}(x')\phi^{k_3}(x)dxdx'\tag{6.1.62}$$

where K is the propagator from a boundary point x' to a bulk point z, and it is assumed in the above equation that we take the limit $z\longrightarrow x$. Thus terms of the form of the boundary integrals in (6.1.60) can always be dropped.

- To analyse the second term in (6.1.52) we carry out the following steps. First define

$$P_{\mu\nu} = (\nabla_\mu \phi^{k_2} \nabla_\nu \phi^{k_3} + \nabla_\nu \phi^{k_2} \nabla_\mu \phi^{k_3}) - \frac{1}{2} g_{\mu\nu} \nabla^\lambda \phi^{k_2} \nabla_\lambda \phi^{k_3} \tag{6.1.63}$$

which satisfies the relation

$$\nabla_\mu P_{\mu\nu} = \frac{1}{2} \big(m^2_\phi(k_3) \phi^{k_3} \nabla_\nu \phi^{k_2} + m^2_\phi(k_2) \phi^{k_2} \nabla_\nu \phi^{k_3} \big) \,. \tag{6.1.64}$$

Then consider

$$\begin{aligned}
&\int_{AdS_5} \nabla_\mu \nabla_\nu t^{k_1} \nabla^\mu \phi^{k_2} \nabla^\nu \phi^{k_3} \\
&\quad = \int_{AdS_5} \nabla^\mu \nabla^\nu t^{k_1} \Big[P_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \nabla^\lambda \phi^{k_2} \nabla_\lambda \phi^{k_3} \Big] \\
&\quad = - \int_{AdS_5} \Big[\nabla^\mu t^{k_1} \nabla^\nu P_{\mu\nu} + \frac{1}{2} \nabla^\mu \nabla^\nu t^{k_1} g_{\mu\nu} \nabla^\lambda \phi^{k_2} \nabla_\lambda \phi^{k_3}) \Big] + \int_{AdS_5} \nabla^\nu \big(\nabla^\mu t^{k_1} P_{\mu\nu} \big) \\
&\quad = \frac{1}{2} \int_{AdS_5} \Big[m^2_t(k_1) t^{k_1} \nabla^\lambda \phi^{k_2} \nabla_\lambda \phi^{k_3} - \nabla^\mu t^{k_1} \big(m^2_\phi(k_3) \phi^{k_3} \nabla_\mu \phi^{k_2} + m^2_\phi(k_2) \phi^{k_2} \nabla_\mu \phi^{k_3} \big) \Big] \\
&\qquad + \int_{AdS_5} \nabla^\nu \big(\nabla^\mu t^{k_1} P_{\mu\nu} \big) \,.
\end{aligned} \tag{6.1.65}$$

The cubic couplings in the last line of this equation are total derivatives. We use the divergence theorem to perform the integral of this term. This produces the following boundary interactions

$$\begin{aligned}
&\int_{AdS_5} \nabla_\mu \nabla_\nu t^{k_1} \nabla^\mu \phi^{k_2} \nabla^\nu \phi^{k_3} \\
&\quad = \frac{1}{2} \int_{AdS_5} \Big[m^2_t(k_1) t^{k_1} \nabla^\lambda \phi^{k_2} \nabla_\lambda \phi^{k_3} - \nabla^\mu t^{k_1} \big(m^2_\phi(k_3) \phi^{k_3} \nabla_\mu \phi^{k_2} + m^2_\phi(k_2) \phi^{k_2} \nabla_\mu \phi^{k_3} \big) \Big] \\
&\qquad + \int_{\partial(AdS_5)} \nabla^\mu t^{k_1} P_{\mu n} \,.
\end{aligned} \tag{6.1.66}$$

Collecting all contributions, we find

$$\begin{aligned}
&2\kappa_5^2 S_{cubic} = \int_{AdS_5} a(k_1,k_2,k_3) t^{k_1} \phi^{k_2} \phi^{k_3} \\
&\times \Big[\frac{(k_1+4)^2}{k_1+3} \Big(m^2_t(k_1) - m^2_\phi(k_2) - m^2_\phi(k_3) \Big) + \frac{1}{2(k_1+3)} \Big(m^2_\phi(k_2) - m^2_\phi(k_3) - m^2_t(k_1) \Big) + \\
&\int_{\partial(AdS_5)} \frac{a(k_1,k_2,k_3)}{k_1+3} \Big(- \nabla_n \phi^{k_3} \nabla_\mu \phi^{k_1} \nabla^\mu \phi^{k_2} - \nabla_n \phi^{k_2} \nabla_\mu \phi^{k_1} \nabla^\mu \phi^{k_3} + \nabla_n \phi^{k_1} \nabla_\lambda \phi^{k_2} \nabla^\lambda \phi^{k_3} \Big) \Big] \,.
\end{aligned} \tag{6.1.67}$$

Using the explicit expressions for the masses of the fields (6.1.61) we finally obtain

$$\begin{aligned}
&2\kappa_5^2 S_{cubic} = -8 \frac{(\Sigma+4)\alpha_1(\alpha_2+2)(\alpha_3+2)}{k_1+3} \int_{AdS_5} a(k_1,k_2,k_3) t^{k_1} \phi^{k_2} \phi^{k_3} + \\
&\int_{\partial(AdS_5)} \frac{a(k_1,k_2,k_3)}{k_1+3} \Big(- \nabla_n \phi^{k_3} \nabla_\mu \phi^{k_1} \nabla^\mu \phi^{k_2} - \nabla_n \phi^{k_2} \nabla_\mu \phi^{k_1} \nabla^\mu \phi^{k_3} + \nabla_n \phi^{k_1} \nabla_\lambda \phi^{k_2} \nabla^\lambda \phi^{k_3} \Big) \,.
\end{aligned} \tag{6.1.68}$$

6.1.5 Three–point Function for $k_1 < k_2 + k_3$. In this case the boundary integral vanishes as we take the size of the AdS region to infinity. The contribution comes from the integral of the bulk vertex term in (6.1.68). The contribution of this term to the 3–point function is given by

$$A_i(\vec{x},\vec{y},\vec{z}) = \frac{a_i}{|\vec{x}-\vec{y}|^{\Delta_1+\Delta_2-\Delta_3}|\vec{y}-\vec{z}|^{\Delta_2+\Delta_3-\Delta_1}|\vec{z}-\vec{x}|^{\Delta_3+\Delta_1-\Delta_2}}, \tag{6.1.69}$$

with

$$a_1 = -\frac{\Gamma\left(\frac{1}{2}[\Delta_1+\Delta_2-\Delta_3]\right)\Gamma\left(\frac{1}{2}[\Delta_2+\Delta_3-\Delta_1]\right)\Gamma\left(\frac{1}{2}[\Delta_3+\Delta_1-\Delta_2]\right)}{2\pi^d\Gamma(\Delta_1-\frac{d}{2})\Gamma(\Delta_2-\frac{d}{2})\Gamma(\Delta_3-\frac{d}{2})}\Gamma\left(\frac{1}{2}[\Delta_1+\Delta_2+\Delta_3-d]\right).$$

Now use $\Delta = \frac{1}{2}(d+\sqrt{d^2+4m^2})$ and (6.1.61) to determine the conformal dimensions Δ_i of t^{k_1} , ϕ^{k_2} and ϕ^{k_3} in terms of the KK levels k_i as follows

$$\Delta_1 = k_1 + 8 \quad \Delta_2 = k_2 + 4 \quad \Delta_3 = k_3 + 4\,. \tag{6.1.70}$$

Then, the three–point function is

$$\begin{aligned}\left\langle \mathcal{O}^{k_1}_t(x_1)\mathcal{O}^{k_2}_{\phi}(x_2)\mathcal{O}^{k_3}_{\phi}(x_3)\right\rangle &= \frac{1}{2\kappa_5^2}\frac{4}{\pi^4}\frac{a(k_1,k_2,k_3)}{x_{12}^{8+2\alpha_3}x_{13}^{8+2\alpha_2}x_{23}^{2\alpha_1}} \\ &\times \frac{(\Sigma+4)(\alpha_2+2)(\alpha_3+2)}{k_1+3}\,\frac{\Gamma(\alpha_1+1)\Gamma(\alpha_3+4)\Gamma(\alpha_2+4)\Gamma(\Sigma+6)}{\Gamma(k_1+6)\Gamma(k_2+6)\Gamma(k_3+2)}\,.\end{aligned} \tag{6.1.71}$$

Although the above calculation is strictly valid only for the range $k_1 < k_2 + k_3$, we observe that there is a smooth $k_1 \longrightarrow k_2 + k_3$ limit

$$\begin{aligned}\lim_{k_1\longrightarrow k_2+k_3}\left\langle \mathcal{O}^{k_1}_t(x_1)\mathcal{O}^{k_2}_{\phi}(x_2)\mathcal{O}^{k_3}_{\phi}(x_3)\right\rangle &= \frac{1}{2\kappa_5^2}\frac{4}{\pi^4}\frac{a(k_2+k_3,k_2,k_3)}{x_{12}^{8+2\alpha_3}x_{13}^{8+2\alpha_2}} \\ &\times \frac{(k_2+3)(k_2+2)^2(k_3+3)(k_3+2)^2(k_2+k_3+4)}{(k_2+k_3+3)}\,.\end{aligned} \tag{6.1.72}$$

6.1.6 The Extremal Case $k_1 = k_2 + k_3$. In this case the coefficient of the bulk integral in (6.1.67) vanishes. The contribution comes entirely from the boundary term. We introduce a cutoff at $z_0 = \epsilon$, where z_0, z_i are the coordinates of the usual upper half-plane metric for AdS and require that bulk fields satisfy a Dirichlet boundary value problem there. This is a geometrically well defined prescription, and leads to the solution

$$t^k(z) = \int \mathrm{d}^4x K^{\epsilon}_{\Delta}(z,x)\bar{t}^k(x) \tag{6.1.73}$$

where $K^{\epsilon}_{\Delta}(z,x)$ is the Poisson/Dirichlet kernel for the cutoff space-time and $\bar{t}(x)$ is the boundary source for $t^k(z)$. A similar equation holds for $\phi^k(z)$. It is not easy to compute the 3–point function in position space. Instead, use the Fourier transform. The Fourier transform of $K^{\epsilon}_{\Delta}(z,x)$ is given by

$$K^{\epsilon}_{\Delta}(p) = \frac{z_0^{\frac{d}{2}}\mathcal{K}_{\Delta-\frac{d}{2}}(pz_0)}{\epsilon^{\frac{d}{2}}\mathcal{K}_{\Delta-\frac{d}{2}}(p\epsilon)} \tag{6.1.74}$$

where $\mathcal{K}_\nu$ is the modified Bessel function of index ν, and d is the dimension of the boundary of AdS_{d+1}. We will specialize to $d = 4$ in our final result for the correlator.

The power series expansion of the modified Bessel function for integer n has the form

$$\mathcal{K}_n(x) = (-1)^{n+1} I_n(x) \ln \frac{x}{2} + \frac{1}{2} \sum_{m=0}^{n-1} \left(\frac{x}{2}\right)^{-n+2m} \frac{(n-m-1)!}{m!} + (-1)^{n+1} \frac{1}{2} \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{m!(n+m)!} \left[2C - \sum_{k=1}^{m+n} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k} \right] \tag{6.1.75}$$

where

$$I_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+n}}{m!(n+m+1)!} \,. \tag{6.1.76}$$

Applying this we obtain

$$\begin{aligned}
&\mathcal{K}_{\Delta-\frac{d}{2}}(pz_0) \\
&\quad = (-1)^{\Delta-\frac{d}{2}+1} \sum_{m=0}^{\infty} \frac{(pz_0/2)^{2m+\Delta-\frac{d}{2}}}{m!(\Delta-\frac{d}{2}+m+1)!} \ln \frac{pz_0}{2} + \frac{1}{2} \sum_{m=0}^{\Delta-\frac{d}{2}-1} \left(\frac{pz_0}{2}\right)^{-\Delta+\frac{d}{2}+2m} \frac{(\Delta-\frac{d}{2}-m-1)!}{m!} \\
&\qquad + (-1)^{\Delta-\frac{d}{2}+1} \frac{1}{2} \sum_{m=0}^{\infty} \frac{(pz_0/2)^{\Delta-\frac{d}{2}+2m}}{m!(\Delta-\frac{d}{2}+m)!} \left[2C - \sum_{k=1}^{m+\Delta-\frac{d}{2}} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k} \right] \\
&\quad = \sum_{m=0}^{\infty} a_m (\frac{pz_0}{2})^{2m+\Delta-\frac{d}{2}} \ln \frac{pz_0}{2} + \sum_{m=0}^{\Delta-\frac{d}{2}-1} b_m \left(\frac{pz_0}{2}\right)^{-\Delta+\frac{d}{2}+2m} + \sum_{m=0}^{\infty} c_m (\frac{pz_0}{2})^{\Delta-\frac{d}{2}+2m}
\end{aligned}$$

where we introduce the coefficients

$$a_m = \frac{(-1)^{\Delta-\frac{d}{2}+1}}{m!(\Delta-\frac{d}{2}+m+1)!} \,, \tag{6.1.77}$$

$$b_m = \frac{1}{2} \frac{(\Delta-\frac{d}{2}-m-1)!}{m!} \,, \tag{6.1.78}$$

$$c_m = \frac{1}{2} \frac{(-1)^{\Delta-\frac{d}{2}+1}}{m!(\Delta-\frac{d}{2}+m)!} \left[2C - \sum_{k=1}^{m+\Delta-\frac{d}{2}} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k} \right] \,. \tag{6.1.79}$$

In terms of these coefficients

$$\mathcal{K}_{\Delta-\frac{d}{2}}(pz_0) = \left(\frac{pz_0}{2}\right)^{\Delta-\frac{d}{2}} \sum_{m=0}^{\infty} \left[a_m \ln \frac{pz_0}{2} + c_m \right] \left(\frac{pz_0}{2}\right)^{2m} + \left(\frac{pz_0}{2}\right)^{-\left(\Delta-\frac{d}{2}\right)} \sum_{m=0}^{\Delta-\frac{d}{2}-1} b_m \left(\frac{pz_0}{2}\right)^{2m} \,, \tag{6.1.80}$$

and

$$\mathcal{K}_{\Delta-\frac{d}{2}}(p\epsilon) = \left(\frac{p\epsilon}{2}\right)^{\Delta-\frac{d}{2}} \sum_{m=0}^{\infty} \left[a_m \ln \frac{p\epsilon}{2} + c_m \right] \left(\frac{p\epsilon}{2}\right)^{2m} + \left(\frac{p\epsilon}{2}\right)^{-\left(\Delta-\frac{d}{2}\right)} \sum_{m=0}^{\Delta-\frac{d}{2}-1} b_m \left(\frac{p\epsilon}{2}\right)^{2m} \,. \tag{6.1.81}$$

We see that $\mathcal{K}^{\epsilon}_{\Delta}(p)$ admits derivatives of arbitrary order with respect to z_0 which have smooth limits as we approach to the boundary at $z_0 = \epsilon$

$$z_0 \frac{\partial}{\partial z_0} K^{\epsilon}_{\Delta}(p)\Big|_{z_0=\epsilon} = \frac{\frac{d}{2} z_0^{\frac{d}{2}} \mathcal{K}_{\Delta-\frac{d}{2}}(pz_0) + z_0^{\frac{d}{2}}(pz_0) \frac{\partial}{\partial(pz_0)} \mathcal{K}_{\Delta-\frac{d}{2}}(pz_0)}{\epsilon^{\frac{d}{2}} \mathcal{K}_{\Delta-\frac{d}{2}}}\Big|_{z_0=\epsilon}$$
$$= \frac{d}{2} + \frac{z_0^{\frac{d}{2}}(pz_0) \frac{\partial}{\partial(pz_0)} \mathcal{K}_{\Delta-\frac{d}{2}}(pz_0)}{\epsilon^{\frac{d}{2}} \mathcal{K}_{\Delta-\frac{d}{2}}}\Big|_{z_0=\epsilon} . \tag{6.1.82}$$

We now need to evaluate the following derivative

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}(pz_0)} \mathcal{K}_{\Delta-\frac{d}{2}}(pz_0) \\
&= \frac{1}{2}\left(\Delta - \frac{d}{2}\right)\left[\left(\frac{pz_0}{2}\right)^{\Delta-\frac{d}{2}-1} \sum_{m=0}^{\infty}\left[a_m \ln\frac{pz_0}{2} + c_m\right]\left(\frac{pz_0}{2}\right)^{2m} - \left(\frac{pz_0}{2}\right)^{-\left(\Delta-\frac{d}{2}\right)-1} \sum_{m=0}^{\Delta-\frac{d}{2}-1} b_m \left(\frac{pz_0}{2}\right)^{2m}\right] \\
&\quad + \left(\frac{pz_0}{2}\right)^{\Delta-\frac{d}{2}-1} \sum_{m=0}^{\infty}\left[m\, a_m \ln\frac{pz_0}{2} + m\, c_m + 2a_m\right]\left(\frac{pz_0}{2}\right)^{2m} + \left(\frac{pz_0}{2}\right)^{-\left(\Delta-\frac{d}{2}\right)-1} \sum_{m=0}^{\Delta-\frac{d}{2}-1} m\, b_m \left(\frac{pz_0}{2}\right)^{2m} \\
&= \left(\frac{pz_0}{2}\right)^{\Delta-\frac{d}{2}-1} \sum_{m=0}^{\infty}\left(\left[m + \frac{1}{2}\left(\Delta - \frac{d}{2}\right)\right] a_m \ln\frac{pz_0}{2} + \left[m + \frac{1}{2}\left(\Delta - \frac{d}{2}\right)\right] c_m + 2a_m\right)\left(\frac{pz_0}{2}\right)^{2m} \\
&\quad + \left(\frac{pz_0}{2}\right)^{\Delta-\frac{d}{2}-1} \sum_{m=0}^{\infty}\left[m - \frac{1}{2}\left(\Delta - \frac{d}{2}\right)\right] b_m \left(\frac{pz_0}{2}\right)^{2m} .
\end{aligned} \tag{6.1.83}$$

Using these results (6.1.82) becomes

$$z_0 \frac{\partial}{\partial z_0} K^{\epsilon}_{\Delta}(p)\Big|_{z_0=\epsilon} = \frac{d}{2} + \frac{2\left[\left(\frac{pz_0}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)} \sum_{\infty}^{m=0} A_m \left(\frac{pz_0}{2}\right)^{2m} + \sum_{m=0}^{\Delta+\frac{d}{2}-1} B_m \left(\frac{pz_0}{2}\right)^{2m}\right]}{\left(\frac{pz_0}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)} \sum_{\infty}^{m=0} C_m \left(\frac{pz_0}{2}\right)^{2m} + \sum_{m=0}^{\Delta+\frac{d}{2}-1} D_m \left(\frac{pz_0}{2}\right)^{2m}} \tag{6.1.84}$$

where we have again introduced a set of coefficients

$$A_m = \left[m + \frac{1}{2}\left(\Delta - \frac{d}{2}\right)\right] a_m \ln\frac{pz_0}{2} + \left[m + \frac{1}{2}\left(\Delta - \frac{d}{2}\right)\right] c_m + 2a_m , \tag{6.1.85}$$

$$B_m = \left[m - \frac{1}{2}\left(\Delta - \frac{d}{2}\right)\right] b_m , \tag{6.1.86}$$

$$C_m = \left(\frac{p\epsilon}{2}\right)^{2\left(\Delta-\frac{d}{2}\right)} \left[a_m \ln\frac{p\epsilon}{2} + c_m\right] , \tag{6.1.87}$$

$$D_m = b_m . \tag{6.1.88}$$

Consider the expansion

$$
\begin{aligned}
& z_0 \frac{\partial}{\partial z_0} K^{\epsilon}_{\Delta}(p) \Big|_{z_0=\epsilon} \\
&= \frac{d}{2} + 2 \left[\frac{\left(\frac{pz_0}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)} \left(A_0 + A_1 \left(\frac{pz_0}{2}\right)^{2m} + \cdots\right) + \left(B_0 + B_1 \left(\frac{pz_0}{2}\right)^{2m} + \cdots\right)}{\left(\frac{pz_0}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)} \left(C_0 + C_1 \left(\frac{pz_0}{2}\right)^{2m} + \cdots\right) + \left(D_0 + D_1 \left(\frac{pz_0}{2}\right)^{2m} + \cdots\right)} \right] \Bigg|_{z_0=\epsilon} \\
&= \frac{d}{2} + 2\frac{B_0}{D_0} \Bigg[1 + \frac{B_1}{B_0} \left(\frac{pz_0}{2}\right) + \cdots + \left(\frac{pz_0}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)} \left(\frac{A_0}{B_0} + \frac{A_1}{B_0} \left(\frac{pz_0}{2}\right) + \cdots\right) \\
&\qquad + 1 - \frac{D_1}{D_0} \left(\frac{pz_0}{2}\right) - \cdots - \left(\frac{pz_0}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)} \left(\frac{C_0}{D_0} + \frac{C_1}{D_0} \left(\frac{pz_0}{2}\right) + \cdots\right) \Bigg] \Bigg|_{z_0=\epsilon}
\end{aligned}
$$

and keep only the dominant terms

$$
\begin{aligned}
z_0 \frac{\partial}{\partial z_0} K^{\epsilon}_{\Delta}(p) &= \frac{d}{2} + 2\frac{B_0}{D_0} + \cdots + \left(\frac{p\epsilon}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)} \frac{A_0}{D_0} + \cdots \\
&= \frac{d}{2} + (-\Delta + \frac{d}{2}) + \cdots - 2 \left(\frac{p\epsilon}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)} \frac{(-1)^{\Delta-\frac{d}{2}} \ln \frac{p\epsilon}{2}}{\frac{1}{2}\Gamma\left(\Delta - \frac{d}{2}\right)^2} + \cdots \\
&= (d-\Delta) + \cdots a_{\Delta} (p\epsilon)^{2\left(\Delta-\frac{d}{2}\right)} \ln(p\epsilon) + \cdots
\end{aligned} \tag{6.1.89}
$$

with

$$
a_{\Delta} = \frac{(-1)^{\Delta-\frac{d}{2}+1}}{2^{2\left(\Delta-\frac{d}{2}\right)-2}\Gamma\left(\Delta - \frac{d}{2}\right)^2} \tag{6.1.90}
$$

where the first $\cdots$ refer to positive integer powers of p and the second $\cdots$ to terms containing $\ln(p\epsilon)$ times higher powers of p. We recall that $a_{\Delta}(p\epsilon)^{2\left(\Delta-\frac{d}{2}\right)}$ is the Fourier transform of the correctly normalized expression for the 2–point function

$$
\frac{1}{(x-y)^{2\Delta}} \frac{(2\Delta - d)\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2})} . \tag{6.1.91}
$$

The relevant term in the product of three propagators is

$$
\begin{aligned}
& D_n K^{\epsilon}_{\Delta_2+\Delta_3}(p_1) D_n K^{\epsilon}_{\Delta_2}(p_2) D_n K^{\epsilon}_{\Delta_3}(p_3) \\
&= \left[(d - \Delta_2 - \Delta_3) + \cdots a_{\Delta_2+\Delta_3} (p_1\epsilon)^{2\left(\Delta_2+\Delta_3-\frac{d}{2}\right)} \ln(p_1) + \cdots\right] \\
&\quad \times \left[(d-\Delta_2) + \cdots a_{\Delta_2} (p_2\epsilon)^{2\left(\Delta_2-\frac{d}{2}\right)} \ln(p_2) + \cdots\right] \left[(d-\Delta_3) + \cdots a_{\Delta_3} (p_3\epsilon)^{2\left(\Delta_3-\frac{d}{2}\right)} \ln(p_3) + \cdots\right] \\
&= \cdots + \epsilon^{2\Delta_2+2\Delta_3-2d} (d - \Delta_2 - \Delta_3) a_{\Delta_2} a_{\Delta_3} p_2^{2\left(\Delta_2-\frac{d}{2}\right)} \ln(p_2) p_3^{2\left(\Delta_3-\frac{d}{2}\right)} \ln(p_3) + \cdots
\end{aligned} \tag{6.1.92}
$$

From (6.1.91) and (6.1.92), we finally get the 3–point function

$$
\begin{aligned}
\left\langle \mathcal{O}^{k_1}_t(x_1) \mathcal{O}^{k_2}_{\phi}(x_2) \mathcal{O}^{k_3}_{\phi}(x_3) \right\rangle &= -\frac{1}{2\kappa_5^2} \frac{a(k_1,k_2,k_3)}{k_1+3} (d - \Delta_2 - \Delta_3) \\
&\left(\frac{1}{|\vec{x}_1 - \vec{x}_2|^{2\Delta_2}} \frac{(2\Delta_2 - d)\Gamma(\Delta_2)}{\pi^{\frac{d}{2}}\Gamma(\Delta_2 - \frac{d}{2})} \right) \left(\frac{1}{|\vec{x}_1 - \vec{x}_3|^{2\Delta_3}} \frac{(2\Delta_3 - d)\Gamma(\Delta_3)}{\pi^{\frac{d}{2}}\Gamma(\Delta_3 - \frac{d}{2})} \right) .
\end{aligned} \tag{6.1.93}
$$

Now replace k_1 by k_2+k_3 and use the formula (6.1.70) for the Δ_i. With $d=4$, we obtain

$$\left\langle \mathcal{O}_t^{k_2+k_3}(x_1)\mathcal{O}_\phi^{k_2}(x_2)\mathcal{O}_\phi^{k_3}(x_3)\right\rangle = \frac{1}{2\kappa_5^2}\frac{4}{\pi^4}\frac{a(k_2+k_3,k_2,k_3)}{x_{12}^{8+2k_2}x_{13}^{8+2k_3}} \times \frac{(k_2+3)(k_2+2)^2(k_3+3)(k_3+2)^2(k_2+k_3+4)}{(k_2+k_3+3)}\,. \tag{6.1.94}$$

Comparison with the expression (6.1.72) shows exact agreement.

6.2 The Gauge Theory Correlators in the Large N Limit

We consider $\mathcal{N}=4$ super Yang-Mills theory with the gauge group $U(N)$. We use the conventions of [13]. Denote the single trace chiral primary operator CPO of SYM_4 by

$$\mathcal{O}^I = C^I_{i_1\cdots i_k}\,\mathrm{Tr}\left(\phi^{i_1}\cdots\phi^{i_k}\right) \tag{6.2.1}$$

where $i_1,\cdots i_k$ are $SO(6)$ vector indices and ϕ^i are six $N\times N$ hermitian traceless matrices transforming in the adjoint of $U(N)$. The trace in this formula is over the $U(N)$ indices and $C^I_{i_1\cdots i_k}$ are totally symmetric traceless rank k tensors of $SO(6)$. The traceless symmetric tensors $C^I_{i_1,\cdots,k_i}$ are related to the spherical harmonics (see Appendix C).

The normalized action is given by

$$S=\int d^4x\,\frac{1}{2g_{YM}^2}\mathrm{Tr}\left(F^2\right)+\cdots=-\int d^4x\,\frac{1}{4g_{YM}^2}F^a_{\mu\nu}F^{a\mu\nu}+\cdots\,. \tag{6.2.2}$$

where $\mu=0,\cdots,3$ and $F^a_{\mu\nu}$ are the field strength of the gauge field A^a_μ. The Yang-Mills coupling g_{YM} the string coupling g_s are related by $g_{YM}^2=4\pi g_s$. In this theory, the propagator is given by

$$\left\langle \phi^i_a(x_1)\phi^j_b(x_2)\right\rangle = \frac{g_{YM}^2\delta_{ab}\delta^{ij}}{(2\pi)^2|x-y|^2}, \tag{6.2.3}$$

where $a,b,\cdots$ are $U(N)$ color indices.

The correlators are computed by performing Wick contractions. In the large N limit we need only sum planar diagrams. For the correlator of a product of two traces, planar diagrams correspond to contracting i's and j's in the same cyclic order, which gives

$$\left\langle \mathrm{Tr}\left(\phi^{i_1}(x_1)\cdots\phi^{i_k}(x_1)\right)\mathrm{Tr}\left(\phi^{j_1}(x_2)\cdots\phi^{j_k}(x_2)\right)\right\rangle = \frac{N^k g_{YM}^{2k}\left(\delta^{i_1j_1}\delta^{i_2j_2}\cdots\delta^{i_kj_k}+\text{cyclic}\right)}{(2\pi)^{2k}|x-y|^{2k}}. \tag{6.2.4}$$

Let $\mathcal{O}^{I_1}$ and $\mathcal{O}^{I_2}$ be two CPO specified by the tensors $C^{I_1}_{i_1\cdots i_k}$ and $C^{I_2}_{j_1\cdots j_k}$. The correlator of the product of these two operators is nonzero only if their scaling dimensions are equal. For $k_1=k_2=k$, we have

$$\begin{aligned}
&\left\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\right\rangle \\
&= C^{I_1}_{i_1\cdots i_k}C^{I_2}_{j_1\cdots j_k}\left\langle \mathrm{Tr}\left(\phi^{i_1}(x_1)\cdots\phi^{i_k}(x_1)\right)\mathrm{Tr}\left(\phi^{j_1}(x_2)\cdots\phi^{j_k}(x_2)\right)\right\rangle \\
&= C^{I_1}_{i_1\cdots i_k}C^{I_2}_{j_1\cdots j_k}\frac{N^k g_{YM}^{2k}\left(\delta^{i_1j_1}\delta^{i_2j_2}\cdots\delta^{i_kj_k}+\text{cyclic}\right)}{(2\pi)^{2k}|x-y|^{2k}} \\
&= C^{I_1}_{i_1\cdots i_k}C^{I_2}_{i_1\cdots i_k}\frac{N^k g_{YM}^{2k}\,k}{(2\pi)^{2k}|x-y|^{2k}}\,.
\end{aligned} \tag{6.2.5}$$

Using the orthonormality of the $\mathcal{C}$ coefficients we have

$$\left\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\right\rangle = \left\langle C^{I_1}C^{I_2}\right\rangle \frac{\lambda^k\, k}{(2\pi)^{2k}|x-y|^{2k}}\,,$$

where the t'Hooft coupling is given by $\lambda = Ng_{YM}^2$. The factor k denotes the number of the possible cyclic permutations with k pairs of indices.

The 3-point function of CPOs specified by three tensors of rank k_1, k_2 and k_3 is given by

$$\begin{aligned}
&\left\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\mathcal{O}^{I_2}(x_3)\right\rangle = \\
&C^{I_1}_{i_1\cdots i_{k_1}} C^{I_2}_{j_1\cdots j_{k_2}} C^{I_3}_{l_1\cdots l_{k_3}} \left\langle \mathrm{Tr}\left(\phi^{i_1}(x_1)\cdots\phi^{i_{k_1}}(x_1)\right)\mathrm{Tr}\left(\phi^{j_1}(x_2)\cdots\phi^{j_{k_2}}(x_2)\right)\mathrm{Tr}\left(\phi^{l_1}(x_3)\cdots\phi^{l_{k_3}}(x_3)\right)\right\rangle \\
&= \left\langle C^{I_1}C^{I_2}C^{I_3}\right\rangle \frac{\lambda^{\Sigma/2}\, k_1k_2k_3}{N(2\pi)^{\Sigma}|x-y|^{2\alpha_3}|y-z|^{2\alpha_1}|z-x|^{2\alpha_2}}\,. \qquad (6.2.6)
\end{aligned}$$

where $\Sigma = k_1 + k_2 + k_3$. The factor $k_1k_2k_3$ is the number of planar diagrams, and

$$\left\langle C^{I_1}C^{I_2}C^{I_3}\right\rangle = C^{I_1}_{i_1\cdots i_{\alpha_3} j_1\cdots j_{\alpha_2}} C^{I_2}_{i_1\cdots i_{\alpha_3} l_1\cdots l_{\alpha_1}} C^{I_3}_{j_1\cdots j_{\alpha_2} l_1\cdots l_{\alpha_1}}\,. \qquad (6.2.7)$$

We rescale the CPOs $\mathcal{O}^I = \mathcal{O}^I \frac{(2\pi)^k}{\sqrt{\lambda\, k}}$, such that the normalized 2-point functions are

$$\left\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\right\rangle = \left\langle C^{I_1}C^{I_2}\right\rangle \frac{1}{x_{12}^{2k}}\,. \qquad (6.2.8)$$

The 3-point function is given by

$$\left\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\mathcal{O}^{I_2}(x_3)\right\rangle = \left\langle C^{I_1}C^{I_2}C^{I_3}\right\rangle \frac{\sqrt{k_1k_2k_3}}{N x_{12}^{2\alpha_3} x_{23}^{2\alpha_1} x_{31}^{2\alpha_2}}. \qquad (6.2.9)$$

In summary, we ave computed the extremal three-point function in the gravity theory using to methods. Both result were in agreement. We also fond that the gravity and CFT results have the same conformal structure but do not agree. We have also demonstrated that higher point functions are subject to the same subtley known for 2-point functions, i.e. they are also divergent. Again a deeper understanding of the origin of these divergences is missing.

7. Some Remarks on Extremal correlators

The main goal of this chapter is to consider the divergences that appear in the computation of extremal correlators. By exploring the connection between extremal correlators and the dual gravity description of these correlators, we will see a possible connection between the divergences in extremal correlators to divergences in collinear amplitudes, in quantum field theory. This is useful because collinear divergences have been studied and are well understood. The remarks we make in this chapter are preliminary since we have not had time to pursue them to their logical conclusion.

7.1 R-charge and Angular Momentum

In this section we want to review the connection between R-charge and and angular momenta. This connection explains why we expect that extremal correlators are related to amplitudes involving collinear particles, that is, particles with momenta that are parallel. One of the basic features the AdS/CFT correspondence, is the fact that global symmetries in the CFT are dual to isometries in the dual gravity. This is an extremely important aspect of the AdS/CFT correspondence because it allows us to match the conserved charges of the two dual theories. Since these conserved charges can be used to label states, this allows us to match states between the two sides. Of course, if there are degenerate values for the charges, we don't quite get a 1-to-1 mapping. As an example of this matching, the $SO(4,2)$ conformal symmetry of the CFT maps to the isometry group of the AdS_5 spacetime. The matching of most interest to us connects the $SU(4)$ R-symmetry of the CFT with the isometry group of the S^5 in the dual gravity. More specifically, the $SU(4)$ is a double cover of the special orthogonal group SO(6), which is an isometry of S^5. This means that $SU(4)$ has spinor representations but $SO(6)$ does not. Thus, when there are fermions in the dual gravity, we should really use $SU(4)$ as the symmetry group and not $SO(6)$. The conserved charge associated with the isometry of the sphere is angular momentum. We conclude that the R-charge of an operator in the CFT maps to angular momentum of a state in the dual string theory. This correspondence puts the modes arising from an expansion in spherical harmonics on the S^5 into one-to-one correspondence with operators of definite R-charge. Of course, if there are degeneracies this will complicate things. A nice example of this matching is provided by the map chiral primary operators on the gauge theory side and the modes coming from an expansion in terms of spherical harmonics, from a Klein-Kaluza reduction of the gravity theory. This correspondence between chiral primary operators and spherical harmonics was already used in Chapters 5 and 6 of this dissertation, where we computed correlators involving giant gravitons and Klein-Kaluza gravitons.

With this picture in mind, we can now explain why we think extremal correlators are linked to the scattering of particle states, when the particles in the state have parallel momenta. First, we can define three complex matrices using the six scalars ϕ_i, with $i = 1, \cdots 6$, of the SYM theory, transforming in the vector representation of the $SO(6)$ R symmetry group. These matrices, from which chiral primary operators can be formed, are given by

$$\begin{aligned} Z &= \phi_1 + i\phi_2, \qquad Z^\dagger = \phi_1 - i\phi_2, \\ Y &= \phi_3 + i\phi_4, \qquad Y^\dagger = \phi_3 - i\phi_4 \\ X &= \phi_5 + i\phi_6, \qquad X^\dagger = \phi_5 - i\phi_6 \end{aligned} \tag{7.1.1}$$

The generators of $SO(6)$ are the angular momenta. We can label the generators of $SO(6)$ as L_{ij} with $i, j = 1, 2, \cdots, 6$. Recall that the generator $L_{ij} = -L_{ji}$ performs an infinitesimal rotation in

the ij-plane. These generators close the $SO(6)$ Lie algebra

$$\big[L_{ij}, L_{kl}\big] = i\delta_{jk}L_{il} - i\delta_{ik}L_{jl} - i\delta_{jl}L_{ik} + i\delta_{il}L_{jk} \tag{7.1.2}$$

There is a basis of states in which L_{12}, L_{34} and L_{56} are simultaneously diagonal. Indeed, this follows because

$$\big[L_{12}, L_{34}] = 0 = [L_{12}, L_{56}] = [L_{34}, L_{56}] \tag{7.1.3}$$

In what follows, we collect these three quantum numbers into a vector $\vec{L} = (L_{12}, L_{34}, L_{56})$. We will now compute the $\vec{L}$ quantum numbers of X, Y and Z. Under a rotation in the 12-plane we have $\phi_i \to \phi_i'$ where

$$\begin{array}{rclrcl} \phi_1' &=& \cos\theta\,\phi_1 + \sin\theta\phi_2, & \phi_2' &=& \cos\theta\,\phi_2 - \sin\theta\phi_1, \\ \phi_3' &=& \phi_3, & \phi_4' &=& \phi_4, \\ \phi_5' &=& \phi_5, & \phi_6' &=& \phi_6\,. \end{array} \tag{7.1.4}$$

Consequently, we find

$$Z = \phi_1 + i\phi_2 \to Z' = \phi_1' + i\phi_2' = e^{-i\theta}Z, \qquad Y \to Y' = Y \qquad X \to X' = X \tag{7.1.5}$$

Similarly, under a rotation in the 34-plane we have

$$Z' = Z, \qquad Y' = e^{-i\theta}Y, \qquad X' = X, \tag{7.1.6}$$

and under a rotation in the 56-plane we have

$$Z' = Z, \qquad Y' = Y, \qquad X' = e^{-i\theta}X. \tag{7.1.7}$$

From this we read off $\vec{L} = (1,0,0)$ for Z, $\vec{L} = (0,1,0)$ for Y and $\vec{L} = (0,0,1)$ for X.

We now come the main observation of our analysis. An operator such as $\mathrm{Tr}(Z^{n_1})\mathrm{Tr}(Y^{n_2})\mathrm{Tr}(X^{n_3})$ is mapped to a three-particle state with the momenta of the three particles given by $(n_1,0,0)$,$(0,n_2,0)$ and $(0,0,n_3)$. The three particles are clearly moving in different directions. The multi-trace operator of the form $\mathrm{Tr}(Z^{n_1})\mathrm{Tr}(Z^{n_2})\mathrm{Tr}(Z^{n_3})$, which participates in extremal correlators, corresponds to a three-particle state with momenta, $(n_1,0,0)$, $(n_2,0,0)$ and $(n_3,0,0)$. Thus, the particles in this state are moving collinearly. Clearly then, extremal correlators, for example

$$\Big\langle \mathrm{Tr}(Z^{n_1})\mathrm{Tr}(Z^{n_2})\mathrm{Tr}(Z^{n_3})\mathrm{Tr}(Z^{\dagger\, n_1+n_2+n_3}) \Big\rangle \tag{7.1.8}$$

map to processes with collinear particles. Non-extremal correlators

$$\Big\langle \mathrm{Tr}(Z^{n_1})\mathrm{Tr}(Y^{n_2})\mathrm{Tr}(X^{n_3})\mathrm{Tr}(Z^{\dagger n_1}Y^{\dagger n_2}X^{\dagger n_3}) \Big\rangle \tag{7.1.9}$$

are not related to states with collinear amplitudes. Having establish this connection, we will now explore properties of collinear amplitudes in QFT. In particular, we want to understand how the divergences in collinear amplitudes are treated.

7.2 Collinear and Soft Divergences

We have just argued that extremal correlators are closely related to collinear amplitudes. These amplitudes are associated to processes where the momentum of the particles that are being described are parallel. In fact, it is well known that there are divergences associated with collinear amplitudes. This suggests the very attractive possibility that perhaps the divergences associated with extremal correlators can be interpreted as collinear singularities.

Collinear singularities are usually accompanied by soft or IR divergences. In this section our goal is to give a brief description of these divergences, how they are interpreted and how they are cured. We hope that a similar approach to extremal correlators will resolve the subtleties we are studying in this dissertation.

Feynman diagrams are known to produce various divergent (infinite) expressions. What do they mean? Here, it is important to note that there are different sources of divergences and their meaning is very different for the different sources. "Infrared divergences" is the name for the infinities that emerge because we have to integrate over arbitrarily long-wavelength (or low-energy) virtual particles (or quanta). They are produced when we send the minimum allowed momentum or energy of virtual particles to zero. In this case, loop diagrams are infinite. What does it mean? To answer this question it is helpful to start by recalling that ultraviolet (short-distance or high-energy) divergences usually imply that a quantum field theory is incomplete and should be thought of as a limit of a more accurate theory. Infrared divergences are quite different. The asymmetry between the two kinds of divergences arises because physics at long distances is derived from the physics at short distances.

How should we interpret infrared divergences? It is important to recognize that quantum field theory has two kinds of questions (with a whole continuum in between): questions that are are directly linked to the results of measurements, that are easily interpreted experimentally as well as questions that are natural and simple from a theoretical viewpoint, questions that are connected with fundamental concepts and quantities in the theory. It is the second type of question that suggests calculations that may produce infrared divergences.

A classic example in which IR divergences enter is in Quantum Electrodynamics, when we calculate the cross section for the scattering of two charged particles. Perturbative quantum field theory will produce a cross section that is a Taylor expansion in the fine-structure constant (or the electric charge). The first term in this expansion comes from a tree diagram. The particles simply exchange one virtual photon and it reproduces the predictions you could make using classical physics. Loop corrections provide quantum corrections to the classical prediction. Already the one-loop corrections suffers from infrared divergences. The amplitude includes a term proportional to $\ln(E_{min})$ where E_{min} is the minimum allowed energy of a virtual photon in the loop. We should set this limit, E_{min}, to zero which produces a divergence. Remarkably, this divergences is canceled if we do the computation carefully.

The quantity that will be compared with experiment is the cross section of an observable process, which is obtained by squaring the amplitude the Feynman diagram computes. The squared amplitude, $\left[\text{finite} + \ln(E_{min})\right]^2$, will produce terms of the form $(\text{finite}^2 + 2\text{finite}\ln(E_{min}) + \cdots)$. The dots contain higher powers of the fine-structure constant. To see how the term proportional to $\ln(E_{min})$ be canceled, note that a real experiment can't observe photons of arbitrarily low energies. We must actually compute the inclusive cross section in which we allow an arbitrary number of low energy photons, that are invisible to the experiment's detectors. These photons have such a

low energy that they can never be observed by an detector. Thus, we must include diagrams with extra external low-energy (soft) photons. It's so soft that your device cannot see it. This extra diagram is also infrared divergent but, remarkably the sum is finite. This method of dealing with soft divergences is called the Bloch and Nordsieck Theorem [40]. It says that as long as you sum over degenerate final states (this means you sum over the final state plus other versions of it in which soft photons are included), the answer you get for any physical quantity is free of infrared divergences.

The collinear divergences are a new effect, in which infinities are produced as a consequence of the fact that the momenta of particles in the amplitude are parallel. The treatment of these divergences is very similar to the treatment of soft divergences. Again by summing the correct classes of Feynman diagrams one obtains a result that is free of any divergences. The theorem stating the result is due to Kinoshita [41], Lee and Nauenberg [42]. The theorem proves that if you sum over both degenerate final and initial states, the answer you get for any physical quantity is free of infrared divergences.

To summarize the discussion, all the divergences ultimately cancel as long as you properly calculate quantities that can be observed. In the cases we considered here, there are many degenerate states that can't be resolved and so to get a physical quantity you must sum over these degenerate possibilities.

7.3 Extremal Correlators and Collinear Divergences

To summarize our discussion in this chapter, we have established a close correspondence between collinear amplitudes and extremal correlators. Further, we have reviewed the fact that collinear amplitudes display extra divergences, intimately related to the fact that the particles participating are collinear. This immediately suggests that the singularities present in extremal correlators may be related to collinear singularities. Assuming this is the case, what do we learn about the divergences in extremal correlators?

The divergences in collinear amplitudes are eliminated once one sums over degenerate initial and final states. One possible approach to the divergences in extremal correlators would entail exploring precisely what degenerate states are relevant for the correlator, and then exploring the sum over these. What effect would these sums have? An aspect that one would need to explore here would be the existence or nonexistence of threshold bound states, which would definitely have implications for the precise nature of the sets of degenerate states. Sadly, due to a lack of time, these interesting questions must be left for the future.

8. Conclusions

The dissertation started by motivating the AdS/CFT correspondence by exploring the planar limit of matrix models in Chapter 2. This analysis was all in a large N limit that is correctly captured by summing the planar diagrams. In Chapter 3 we have extended this analysis to explore the correlation functions of operators with a dimension of order N in the matrix model. For these operators, the large N limit is more complicated and non-planar diagrams need to be summed. This was achieved by employing techniques that exploit group representation theory, effectively allowing a study of finite N effects in the CFT. These operators are constructed by building certain projection operators and their properties are connected properties of these projectors. In Chapter 4, holographic methods for computing correlation functions, in the strong coupling and large N limit of the conformal field theory, have been introduced. As has been reviewed, there are divergences that appear in the holographic evaluation of correlators, for the case of two point functions. These require the introduction of a regulator and the value of the correlators is sensitive to the details of this regulator.

For extremal correlators the divergences are more severe, as has been review in Chapter 5 and 6. As reviewed in these chapters, the correct values for the extremal correlators are obtained by performing an analytic continuation of the non-extremal correlators to the extremal case. In this way it is possible to obtain a perfect match of the gauge theory and gravity results. This is however, far from a satisfactory understanding of the physics that is involved in these divergences. Indeed, we have not understood the origin of these divergences and without this, it is impossible to motivate the analytic continuation that has been used. The key question we have explored in this MSc is an attempt to develop an understanding of these divergences in order that we can provide a complete understanding of these divergences.

In Chapter 7, exploiting the identification of R-charge in the CFT with angular momentum in the string theory, we have suggested that extremal correlators are mapped to amplitudes involving particles with parallel momenta. It is well known that collinear particles give rise to divergences so it is somewhat natural to identify these divergences with the divergences in extremal correlators. The Kinoshita-Lee-Nauenberg Theorem [41], [42] states that all collinear divergences are removed by summing over degenerate initial and final states. Our identification suggests that, perhaps, by summing over degenerate initial and final states we can remove the divergences that appear in the extremal correlators. Future work should explore these preliminary ideas to establish if a correspondence between the divergences that appear in extremal correlators and collinear divergences does in fact exist.

Appendix A. Ward Identity

Consider the two-point function

$$\left\langle \phi^I(x)\phi^J(y)\right\rangle = \int \mathcal{D}\phi\, e^{iS[\phi]}\, \phi^I(x)\phi^J(y)\,. \tag{A.0.1}$$

Under a field transformation we have

$$0 = \int \mathcal{D}\phi\, e^{iS[\phi]}\phi^I(x)\phi^J(y) - \int \mathcal{D}\phi'\, e^{iS[\phi']}\phi'^I(x)\phi'^J(y) \tag{A.0.2}$$

where we have used primes to denote quantities after transformation. Explicitly, the transformation is infintesimally given by

$$\phi^I \mapsto \phi'^I = \phi^I + \delta\phi^I \tag{A.0.3}$$

$$S[\phi^I] \mapsto S[\phi'^I] = S[\phi^I] + \delta S[\phi^I] \tag{A.0.4}$$

Assuming that the integration measure is invariant, i.e $\mathcal{D}\phi = \mathcal{D}\phi'$, we can write

$$0 = \int \mathcal{D}\phi\, e^{iS[\phi]}\left[\phi^I(x)\phi^J(y) - e^{\delta S[\phi]}\big(\phi^I(x)+\delta\phi^I(x)\big)\big(\phi^J(y)+\delta\phi^J(y)\big)\right]. \tag{A.0.5}$$

Now we expand $\exp(\delta S[\phi])$ to the first order to obtain

$$0 = \int \mathcal{D}\phi\, e^{iS[\phi]}\Big[i\delta S[\phi]\phi^I(x)\phi^J(y) + \delta\phi^I(x)\phi^J(y) + \phi^I(x)\delta\phi^J(y)\Big]. \tag{A.0.6}$$

We are interested in the conformal Ward identity, so we perform the global symmetry transformation, with a local parameter $\epsilon_a(x)$. The fact that this is symmetry when $\epsilon_a(x)$ is a constant implies that

$$\delta S[\phi] = \int d^dx\, \partial_i\epsilon_a(x) J_i^a(x) = -\int d^dx\, \epsilon_a(x)\partial_i J_i^a(x)\,, \tag{A.0.7}$$

and

$$\delta\phi(x) = i\epsilon_a(x)(T^a)^{kI}\phi^I(x)\,, \tag{A.0.8}$$

where $J_i^a(x)$ are the Noether currents, and T^a are the generators of the transformation. Therefore we can write (A.0.6) as

$$\left\langle \int d^dw\, \epsilon_a(w)\partial_i J_i^a(x)\phi^I(x)\phi^J(y)\right\rangle = (T^a)^{kI}\left\langle\epsilon_a(x)\phi^K(x)\phi^J(y)\right\rangle + (T^a)^{kJ}\left\langle\phi^I(x)\epsilon_a(y)\phi^K(y)\right\rangle. \tag{A.0.9}$$

For $\epsilon_a(x) = \alpha_a(x)\delta(x-z)$,

$$\left\langle\partial_i J_i^a(z)\phi^I(x)\phi^J(y)\right\rangle = (T^a)^{kI}\delta(x-z)\left\langle\phi^K(x)\phi^J(y)\right\rangle + (T^a)^{kJ}\delta(y-z)\left\langle\phi^I(x)\phi^K(y)\right\rangle. \tag{A.0.10}$$

These are precisely the Ward identities that relate the three and two point correlation functions.

Appendix B. Spherical Harmonics on the Five-sphere

B.1 Integrals of polynomials on the five-sphere

Describe the 5−sphere of radius ρ by $S^n \equiv \{\vec{y}^2 = \rho^2 | \vec{x} \in \mathcal{R}^6\}$. Let $\vec{x} = (x^1, \cdots, x^6)$ be coordinates on $\mathbb{R}^6$. let $d\Omega_5$ be a volume form of the 5-sphere. The area of the unit 5-sphere is

$$\omega_5 = \int_{S^5} d\Omega_5 = \int d\theta_1 \cdots d\theta_5 \sqrt{\det g_{S^5}} = \pi^3 . , \tag{B.1.1}$$

where θ_n parametrize the 5-dimensional sphere as follows

$$\begin{aligned} x^1 &= \cos\theta_1 \,, \\ x^2 &= \sin\theta_1 \cos\theta_2 \,, \\ &\vdots \\ x^5 &= \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \cos\theta_5 \,, \\ x^6 &= \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \sin\theta_5 \,, \end{aligned}$$

with $\theta_1, \cdots, \theta_4 \in [0, \pi]$ and $\theta_5 \in [0, 2\pi]$. These coordinates satisfy

$$\sum_{k=1}^{6} (x^k)^2 = 1 \,. \tag{B.1.2}$$

The metric on the 5-sphere is induced from the metric of the 6-dimensional Euclidean space. It is given by

$$ds^2_{S^n} = \frac{\partial x^\mu}{\partial \theta_i} \frac{\partial x^\mu}{\partial \theta_j} d\theta_i d\theta_j = g_{ij} d\theta_i d\theta_j \tag{B.1.3}$$

such that

$$g_{ij} = \mathrm{diag}(1, \sin^2\theta_1, \sin^2\theta_1 \sin^2\theta_2, \cdots, \sin^2\theta_1 \sin^2\theta_2 \cdots \sin^2\theta_5) \tag{B.1.4}$$

We want to point out that any rotations are symmetries of the sphere. The measure $d\Omega_5$ in (B.1.1) is invariant under these transformations. Any permutation of the coordinates $(x^1, \cdots, x^6)$ also leaves $d\Omega_5$ invariant. Using the following formula [13]

$$\int_{S^5} d\Omega_5 x^{i_1} \cdots x^{i_{2m}} = \frac{\delta^{2m}}{\delta J_{i_1} \cdots \delta J_{i_{2m}}} \int_{S^5} d\Omega_5 \, e^{J \cdot x} \,, \tag{B.1.5}$$

one can use recursion to prove

$$\frac{1}{\Omega_5} \int_{S^5} d\Omega_5 x^{i_1} \cdots x^{i_{2m}} = \frac{1}{2^{m-1}(m+2)!} [\text{all possible contractions}] \,, \tag{B.1.6}$$

where all possible contractions means $\delta^{i_1 i_2}$ for $m = 1$, $\delta^{i_1 i_2}\delta^{i_3 i_4} + \delta^{i_1 i_3}\delta^{i_2 i_4} + \delta^{i_1 i_4}\delta^{i_2 i_3}$ for $m = 2$ and analogous objects for higher m.

B.2 Integrals of Spherical Harmonics on S^5

The spherical harmonics restricted to a unit n-sphere are homogeneous harmonic polynomials of the $(n+1)$-dimensional Euclidean space. Thus defined by any polynomial of the form

$$Y^{I,k} = C^I_{i_1\cdots i_k} x^{i1}\cdots x^{i_k} \tag{B.2.1}$$

where $C^I_{i_1\cdots i_k}$ is a traceless symmetric tensor of rank k, defines a spherical harmonic on the n-sphere. We need to evaluate the integral of the product of two or three spherical harmonics over S^5. First we evaluate the integral of Y^{I_1,k_1} and Y^{I_3,k_2} as follows

$$\int_{S^5} d\Omega_5\, Y^{I_1,k_1} Y^{I_2,k_2} = C^{I_1}_{i_1\cdots i_{k_1}} C^{I_2}_{j_1\cdots j_{k_2}} \int_{S^5} d\Omega_5\, x^{i_1}\cdots x^{i_{k_1}} x^{j_1}\cdots x^{j_{k_1}}. \tag{B.2.2}$$

We perform this integral using (B.1.6). We obtain zero if the two tensors are not of the same rank i.e. if $k_1 \neq k_2$. This is because the contractions of two indices belonging to the same tensor will vanish. Thus, we only consider the case where $k_1 = k_2 = k$. In this case we have

$$\int_{S^5} d\Omega_5\, Y^{I_1,k} Y^{I_2,k} = C^{I_1}_{i_1\cdots i_k} C^{I_2}_{j_1\cdots j_k} \int_{S^5} d\Omega_5\, x^{i_1}\cdots x^{i_k} x^{j_1}\cdots x^{j_k} \tag{B.2.3}$$

$$= C^{I_1}_{i_1\cdots i_k} C^{I_2}_{j_1\cdots j_k} \frac{1}{2^{k-1}(k+2)!}\Omega_5 \big[\text{all possible contractions of } i_1,\cdots,i_k,j_1,\cdots,j_k\big] \tag{B.2.4}$$

$$= \big\langle C^{I_1} C^{I_2}\big\rangle \frac{1}{2^{k-1}(k+1)(k+2)}\Omega_5, \tag{B.2.5}$$

where we have used the fact that the tensors C^{I_1} and C^{I_2} are symmetric and traceless. We have introduced the notation

$$\big\langle C^{I_1} C^{I_2}\big\rangle = C^{I_1}_{i_1\cdots i_k} C^{I_2}_{i_1\cdots i_k}. \tag{B.2.6}$$

In the same fashion we evaluate the integral of three spherical harmonics Y^{I_1,k_1}, Y^{I_2,k_2} and Y^{I_3,k_3}. We find

$$\int_{S^5} d\Omega_5\, Y^{I_1,k_1} Y^{I_2,k_2} Y^{I_3,k_3} = C^{I_1}_{i_1\cdots i_{k_2}} C^{I_2}_{j_1\cdots j_{k_2}} C^{I_3}_{l_1\cdots l_{k_3}} \int_{S^5} d\Omega_5\, x^{i_1}\cdots x^{i_{k_1}} x^{j_1}\cdots x^{j_{k_2}} x^{l_1}\cdots x^{l_{k_3}}. \tag{B.2.7}$$

We will use (B.1.6) with m defined by $\Sigma = k_1 + k_2 + k_3 = 2m$. Thus, we have

$$\int_{S^5} d\Omega_5\, Y^{I_1,k_1} Y^{I_2,k_2} Y^{I_3,k_3}$$
$$= C^{I_1}_{i_1\cdots i_{k_1}} C^{I_2}_{j_1\cdots j_{k_2}} C^{I_3}_{l_1\cdots l_{k_3}} \frac{1}{2^{\frac{1}{2}\Sigma-1}\left(\frac{1}{2}\Sigma+2\right)!}\big[\text{all possible constractions of } i_1\cdots i_{k_2} j_1\cdots j_{k_2} j_l\cdots l_{k_3}\big]. \tag{B.2.8}$$

Let n_{pq} be the number of indices of the tensor C^{I_p} of rank k_p contracted with the indices of C^{I_q} of rank k_q. These numbers satisfy the relations

$$n_{pq} = n_{qp}\,, \tag{B.2.9}$$

$$n_{pp} = 0\,, \tag{B.2.10}$$

$$n_{p1} + n_{p2} + \cdots = k_p. \tag{B.2.11}$$

For three tensors, we have the following constraints

$$n_{12} + n_{13} = k_1, \qquad n_{12} + n_{23} = k_2, \qquad n_{13} + n_{23} = k_3\,. \tag{B.2.12}$$

These equations have a unique solution given by

$$n_{12} = n_{21} = \frac{1}{2}(k_1 + k_2 - k_3) = \alpha_3\,, \tag{B.2.13}$$

$$n_{13} = n_{31} = \frac{1}{2}(k_3 + k_1 - k_2) = \alpha_2\,, \tag{B.2.14}$$

$$n_{23} = n_{32} = \frac{1}{2}(k_2 + k_3 - k_1) = \alpha_1\,. \tag{B.2.15}$$

The value $\langle C^{I_1} C^{I_2} C^{I_3} \rangle$ denotes the unique $SO(6)$ scalar contraction of three tensors, defined by

$$\langle C^{I_1} C^{I_2} C^{I_3} \rangle = C^{I_1}_{i_1 \cdots i_{\alpha_3} j_1 \cdots j_{\alpha_2}} C^{I_2}_{i_1 \cdots i_{\alpha_3} l_1 \cdots l_{\alpha_1}} C^{I_3}_{j_1 \cdots j_{\alpha_2} l_1 \cdots l_{\alpha_1}}\,. \tag{B.2.16}$$

Therefore, the integral of three spherical harmonics is given by

$$\int_{S^5} d\Omega_5\; Y^{I_1,k_1} Y^{I_2,k_2} Y^{I_3,k_3} = \langle C^{I_1} C^{I_1} C^{I_1} \rangle\, \frac{k_1! k_2! k_3!}{\alpha_1! \alpha_2! \alpha_3!}\, \frac{1}{2^{\frac{1}{2}\Sigma - 1}\, \left(\frac{1}{2}\Sigma + 2\right)!}\, \Omega_5. \tag{B.2.17}$$

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