M.Sc (Actuarial Science) Research Report
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Title:

Optimal behaviour for bonus-hunger in motor vehicle Bonus-malus systems

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OPTIMAL BEHAVIOUR FOR BONUS-HUNGER IN MOTOR-VEHICLE BONUS-MALUS SYSTEMS

Key words: Bonus-hunger, Bonus Malus Systems (BMSs), No Claim Discount Systems (NCDSs), infinite time horizon, two-year time horizon, optimal, retention limit or rule.

ABSTRACT

In the context of a motor vehicle BMS, a situation does arise when, following an accident, it is in the best financial interest of the policyholders not to submit the claim but to pay it themselves. This is due to the fact, that the present value of the increased premiums and possibly increased self-paid claims a policyholder will pay, due to claiming and incurring a financial penalty, compared to not claiming and hence not incurring the premium penalty, up to some time horizon: is greater than the present claim amount the policyholder faces. This desire to avoid the financial penalty, but rather to save your bonus when it is in your financial interest, is called the hunger for bonus, or bonus-hunger. This paper shows theoretically and with two case studies what optimal behaviour for bonus-hunger for a policyholder constitutes using different claim size distributions, and specifically focuses this using an infinite and a two-year time horizon. The optimal retention rule for bonus hunger is numerically and graphically shown under which bonus-hunger dictates that policyholders pay a claim themselves. These case studies are based on the actual premiums and BMS that a large South African insurer is currently using. These two case studies are also compared to each other as a check on the modeling. The optimal retention rule using a two-year time horizon is compared to the optimal retention rule for an infinite time horizon under each case study to demonstrate the influence of the time horizon. The sensitivity of the optimal retention rule to changes in the discount rate and the annual claim frequency is explored and graphically shown. The dynamic programming used to solve a system of simultaneous equations so as to derive the optimal retention rule for bonus hunger is explained so as to give the reader an insight into the stability of the system and the mathematical iterative steps needed to solve the system dynamically. This paper also shows what the

stationary distribution is under each case study and what the resultant expected stationary profit to an insurer is which takes bonus-hunger into account.

1. INTRODUCTION

Bonus Malus Systems (BMSs) as they are known in Europe, Asia and some Latin American countries or No Claim Discount Systems (NCDSs) as they are known in South Africa are prospective experience rating systems.

In the context of a general insurance company, writing motor-vehicle business, a BMS is a method of charging a premium that is directly and only affected by a policyholders' claim experience over a defined past period (typically one year). Hence the reference to *experience* rating.

BMSs were first introduced in Europe in the early 1960s and the very first ASTIN Colloquium, held in France, in 1959, was exclusively devoted to BMSs.

BMSs are popular in motor-vehicle insurance, for it is known (for instance, Lemaire, 1985), that the best predictor of the number of claims a policyholder will make in the future, is not rating factors like driver-age, motor-vehicle specifications or location of residence, but a driver's past claim record.

A BMS is a prospective experience rating system and can be used in any line of general insurance business. This research focuses just on one line of business, namely motor vehicle for ease of illustration of what the optimal behaviour for bonus hunger is for a policyholder insuring their motor vehicle. The optimal behaviour is modeled and the optimal retention rule is shown numerically and graphically for a policyholder insuring their motor vehicle with a large South African insurer.

A policyholder entering a BMS is aware of his premiums, the number of bonus-malus categories and the rules for moving in the BMS. In the event of a submitted claim in a year of insurance a policyholder is normally penalized by moving down in the levels of the BMS and pays a higher premium the following year and in the event of no submitted claims during a year of insurance a policyholder is normally rewarded by moving up in the levels of the BMS and pays a lower premium the following year.

Hence, not submitting any claim in a year of insurance means that the future premium at renewal will be less. However, a situation can arise, where a policyholder does have an accident in a year of cover, but it is still in the policyholder's financial interest not to submit a claim. This is due to the fact that the financial penalty incurred: for claiming and hence paying an increased premium the next year, compared to not claiming and hence paying a reduced premium the next year, together with the comparative subsequent future of the policyholders fortunes thereafter up to some time horizon, allowing for discounting, is greater than the actual claim incurred due to the accident. This phenomenon of not submitting a claim, even though one has occurred, because it is in your financial interest to do so, is called the hunger for bonus, or bonus-hunger.

The aim of this research paper is to theoretically illustrate and computationally determine what the maximum claim amount, called a retention limit or rule, of a policyholder is, for which a claim won't be submitted due to bonus-hunger, for each category in a BMS. The BMS chosen in the modelling is that currently used by a large South African general insurance company, named Santam. A second BMS is also constructed that mimics the BMS used by Santam so as to put a check on the modeling. In modelling what the retention rule is for a policyholder with Santam, this paper assumes that at most one claim can occur in a period of insurance cover, so as to simplify the modelling. The theory presented that shows how to determine the optimal retention rule, however does not have any restriction as to the how many claims a policyholder can have in a period of insurance cover.

The decision problem can be formulated using an infinite time horizon or a finite time horizon. This paper considers both and in the latter case considers a two-year time horizon. The term time horizon refers to a point in time and not to a period over time.

2. THEORETICAL FORMULATION OF A BMS

The theoretical formulation of a BMS is needed and shown, (found in Lemaire 1998) so as to simplify and illustrate what the optimal retention rule is under an infinite and two-year time horizon.

A general insurance company uses a BMS when:

- The policyholders are partitioned into a finite number of classes (levels or categories), denoted C(i) or simply i (i=1,2,...,s), so that the annual insurance premium depends only on the specific class (level or category) the policyholder finds themselves in
- New policyholders with an insurer, start in a specific class, C(i,0)
- A policyholder's class or level for the next period (normally a year) of cover is only determined by the previous year's class and the number of claims reported during the previous cover period.

To adequately work with a BMS, you need:

- The premium scale $\bar{b}$ = (b(1),b(2),...,b(s))
- The starting class, C(i,0)
- The transition rules that govern the movement of a policyholder from one class to another, which generally depend on the number of claims they reported in the previous year of cover.

The transition rules can be seen as a transformation T(k), such that T(k,i)=j if the policyholder moves from class C(i) to class C(j) when k claims have been reported.

T(k) can be written in the form of a matrix

$T(k) = (tij^{(k)})$, where $tij^{(k)} = 1$ if T(k,i)=j and 0 otherwise.

The probability $p_{ij}(\lambda)$ of a policyholder moving from C(i) to C(j) in a year, where λ is a parameter associated with the policyholder, like claim frequency, is:

$$p_{ij}(\lambda) = \sum_{k=0}^{\infty} p_k(\lambda)\, t_{ij}^{(k)}$$

Where $p_k(\lambda)$ is the probability that a policyholder with a claim frequency λ has k accidents in a year.

Also $\sum_{j=1}^{s} p_{ij}(\lambda) = 1$ as the policyholder must make some movement in classes at the end of the year; all $p_{ij}(\lambda) \geq 0$.

The matrix: $M(\lambda) = [p_{ij}(\lambda)] = \sum_{k=0}^{\infty} p_k(\lambda)\, T(k)$ is the transition matrix of the first-order Markov chain that mathematically represents the BMS.

3. OPTIMAL BEHAVIOUR OR RULE FOR BONUS-HUNGER WITH AN INFINITE TIME HORIZON

Lemaire (1998) approaches the problem as follows:

Let the vector $\overline{x} = (x_1, x_2, \ldots, x_s)$ be a potential strategy for the optimal retention limits (rule), using an infinite time horizon. Hence, accidents with claim amounts less than x_i, given you are in class i, are paid by the policyholder, whereas claims above this limit are submitted to the general insurer.

Imagine what happens when a policyholder incurs an accident of claim amount x, as time t ($0 \leq t \leq 1$). This paper assumes the period of insurance cover is 1 year, which is normal practice in personal lines motor vehicle insurance. This paper also assumes that a policyholder remains insured up to the time horizon.

Let f(x) be the density function of a random variable X representing the cost of a motor-vehicle insurance claim. For a given retention limit strategy or rule, represented with vector $\overline{x}$, the probability p_i of a claim not being submitted by a policyholder in class C_i is:

$$p_i = P(X \leq x_i) = \int_0^{x_i} f(x)\,dx.$$

The probability $\overline{p}_k^{\,i}(\lambda)$ of submitting k claims during a year equals:

$$\overline{p}_k^{\,i}(\lambda) = \sum_{h=k}^{\infty} p_h(\lambda) \binom{h}{k} (1-p_i)^k \, p_i^{h-k}$$

The expectation of the number of reported claim in a year of cover is:

$$\overline{\lambda}_i = \sum_{k=0}^{\infty} k \, \overline{p}_k^{\,i}(\lambda)$$

The expected cost of a non-submitted claim in a year of cover, given the policyholder has had an accident, is:

$$E^i(X) = \frac{1}{p_i} \int_0^{x_i} x\, f(x)\, dx$$

Assuming that the number of claims and the claim amount are independent, a policyholder pays on average, for non-submitted claims:

$E^i(X)\,(\lambda - \overline{\lambda}_i)$, where λ = expected claim frequency in a year, like in the Poisson (λ) distribution.

The expectation of the total cost for a year of cover for a policyholder is:

$$E(x_i) = b_i + v^{1/2}\, E^i(X)\,(\lambda - \overline{\lambda}_i),$$

Where v is the discount factor (v = 1/(1+annual effective interest rate)) and assuming all claims occur in the middle of the year.

Let the vector $\overline{w}(\lambda) = [w(1,\lambda), w(2,\lambda), \ldots, w(s,\lambda)]$, represent the present value of all the future payments a policyholder will make, on the assumption that they always remain insured. I.e. $w(1,\lambda)$ refers to the present value for a policyholder who is presently in class 1 and $w(s,\lambda)$ for a policyholder currently in class s. Now:

$$w_i(\lambda) = E(x_i) + v \sum_{k=0}^{\infty} \overline{p}_k^{\,i}(\lambda)\, w(T(k,i),\lambda).$$

Hence we have s equations with s unknown $w(i,\lambda)$, with $\overline{p}_k^{\,i}(\lambda)$ implicitly dependent on x_i. There is a unique solution to this system of equations (see Lemaire 1998).

3.1 ALTERNATIVE APPROACH TO THE OPTIMAL RETENTION RULE WITH AN INFINITE HORIZON

There is also another way to approach the problem of bonus-hunger using an infinite horizon.

When a policyholder incurs a claim of amount x, they have two alternatives:

- They don't submit the claim and pay it themselves.

- The present value at time t of all the future total costs they pay themselves is then:

$$v^{-t}\ E(x_i) + x + v^{1-t} \sum_{k=0}^{\infty} \overline{p}\,^{i}_{k}(\lambda\ (1\text{-}t))\, w(T(k+m,i), \lambda)$$

Where m is the number of claims already submitted during the year of cover,

OR

- They submit the claim

- The present value at time t of all the future total costs they pay themselves is then:

$$v^{-t}\ E(x_i) + 0 + v^{1-t} \sum_{k=0}^{\infty} \overline{p}\,^{i}_{k}(\lambda\ (1\text{-}t))\, w(T(k+m+1,i), \lambda)$$

The optimal retention limit (rule) x_i , under an infinite horizon, is the claim cost x for which the two courses of action are equivalent: i.e. simplified:

$$x_i = v^{1-t} \sum_{k=0}^{\infty} \overline{p}\,^{i}_{k}(\lambda\ (1\text{-}t))\, \{ w(T(k+m+1,i), \lambda) - w(T(k+m,i), \lambda) \}$$

There is thus, a set of s equations with s unknown x_i's. Note the $\overline{p}^{\,i}_{k}(\lambda\ (1\text{-}t))$ implicitly incorporates x_i. There is thus an alternative way to derive $\overline{x}$, but it assumes you know the vector $\overline{w}(\lambda)$.

Hence in the first method one can derive $\overline{w}(\lambda)$ from $\overline{x}(\lambda)$ and in the second method one can derive $\overline{x}(\lambda)$ from $\overline{w}(\lambda)$. Put together, the 2 methods provide 2s equations with 2s unknowns. The solution to these equations, produce, the optimal retention limits (rule) $\overline{x}^{*} = (x_1^*,\ x_2^*, \ldots,\ x_s^*)$ and its associated $\overline{w}^{*}(\lambda)$. These 2s equations are best solved numerically by computer (dynamic) programming using successive approximations (see Lemaire 1998). The theoretical formulation for the optimal retention rule is clearly shown by Lemaire in 1977 and again in 1998.

$\overline{x}^{*}$ is hence a function of : class C(i), discount rate v, claim frequency λ, the time of the accident t and the number of claims m already reported. However the influence of t on x_i is quite small in comparison with C_i, v and λ (see Lemaire 1998). By setting t=0 (and hence m=0), the calculations are simplified and $\overline{x}^{*}$ is hardly changed (see Lemaire 1998). This paper follows this line of thought and in the modelling of the optimal retention rule assumes that claims occur at the beginning of the year of cover, i.e. that t=0.

This paper assumes in the modelling that at most 1 claim can occur and hence the 2s equations to solve in the modelling reduce to:

$x_i = v^{*}\ \{ w(T(1,i), \lambda) - w(T(0,i), \lambda) \}$ s equations

$w_i(\lambda) = E(x_i) + v * \sum_{k=0}^{1} \overline{p}^{\,i}_{k}(\lambda)\, w(T(k,i), \lambda).$ Further s equations

4. OPTIMAL BEHAVIOUR OR RULE FOR BONUS-HUNGER WITH A TWO-YEAR TIME HORIZON

Instead of looking at the present value of all outgo (premiums and self paid claims) that a policyholder will incur, up to an infinite horizon, one can simply look at the effect of only using a two-year time horizon. However this paper discards the influence of self-paid claims up to the two-year time horizon, which is viewed as small, and considers only the policyholder's premium outgo up to the two-year time horizon.

Definition of two-year time horizon: It refers to the date of the second future annual insurance premium to be paid to the insurance company by the policyholder, following the time at which the policyholder incurred an accident. This paper assumes that policyholders are insured up to the time horizon.

Again, following an accident of amount x in a year of cover, the policyholder is faced with two choices:

- To not submit the claim and pay it themselves

- The present value at time t of the claim x and the two subsequent annual future premiums that a policyholder will pay is then:

$$x + v^{1-t} \sum_{k=0}^{\infty} \bar{p}_k^{\,i}(\lambda\,(1-t))\, b(T(k+m,i)) + v^{2-t} \sum_{r=0}^{\infty} \bar{p}_r^{\,T(k+m,i)}(\lambda)\, b(T(r,T(k+m,i)))$$

Where m is the number of claims already submitted during the year of cover,

OR

- To submit the claim

❖ The present value at time t of the two annual subsequent future premiums is then:

$$0 \quad + v^{1-t} \sum_{k=0}^{\infty} \bar{p}\,{}^{i}_{k}(\lambda\ (1-t))\ b(T(k+m+1,i))$$

$$+ v^{2-t} \sum_{r=0}^{\infty} \bar{p}\,{}^{T(k+m+1,i)}_{r}(\lambda)\ b(T(r,T(k+m+1,i)))$$

The retention limit or rule x_i is the claim cost x for which the two courses of action are equivalent: i.e. :

$$x_i = v^{1-t} \sum_{k=0}^{\infty} \bar{p}\,{}^{i}_{k}(\lambda\ (1-t))\ b(T(k+m+1,i))$$

$$+ v^{2-t.} \sum_{r=0}^{\infty} \bar{p}\,{}^{T(k+m+1,i)}_{r}(\lambda)\ b(T(r,T(k+m+1,i)))$$

$$- v^{1-t} \sum_{k=0}^{\infty} \bar{p}\,{}^{i}_{k}(\lambda\ (1-t))\ b(T(k+m,i))$$

$$- v^{2-t} \sum_{r=0}^{\infty} \bar{p}\,{}^{T(k+m,i)}_{r}(\lambda)\ b(T(r,T(k+m,i)))$$

Simplified

$$x_i = v^{1-t} \{ \sum_{k=0}^{\infty} \bar{p}\,{}^{i}_{k}(\lambda\ (1-t))\ (b(T(k+m+1,i)) - b(T(k+m,i))) \}$$

$$+$$

$$v^{2-t} \{ \sum_{r=0}^{\infty} (\bar{p}\,{}^{T(k+m+1,i)}_{r}(\lambda)\ b(T(r,T(k+m+1,i))) - \bar{p}\,{}^{T(k+m,i)}_{r}(\lambda)\ b(T(r,T(k+m,i)))) \}$$

Hence, the retention limit (rule) x_i in the case studies (as with the infinite time horizon set t=0, m=0 and incorporate that at most 1 claim can occur in a period of insurance cover), satisfies the equation given by:

$$x_i = \{v^* b(T(1,i)) + v^2 \sum_{r=0}^{1} \bar{p}_r^{T(1,i)}(\lambda)\, b(T(r,T(1,i)))\}$$

$$- \{v^* b(T(0,i)) + v^2 \sum_{r=0}^{1} \bar{p}_r^{T(0,i)}(\lambda)\, b(T(r,T(0,i)))\}$$

Simplified:

$$x_i = v^*\{b(T(1,i)) - b(T(0,i))\}$$

$$+ \{ v^2 \sum_{r=0}^{1} [\bar{p}_r^{T(1,i)}(\lambda)\, b(T(r,T(1,i))) - \bar{p}_r^{T(0,i)}(\lambda)\, b(T(r,T(0,i)))]\}$$

Consequently, there are s equations and s unknowns, as $\bar{p}_r^{T(1,i)}(\lambda)$ and $\bar{p}_r^{T(0,i)}(\lambda)$ implicitly incorporate x_i. These s equations are also solved using successive approximations, but because only premiums are considered and that there are only s equations as opposed to 2s, converge much quicker to a unique solution $\bar{x}^{*2} = (x_1^{*2}, x_2^{*2}, \ldots, x_s^{*2})$ – the two year horizon optimal retention limits or rule -, than in the infinite time horizon case.

The effect of changing the discount rate and the probability of an accident on the two-year optimal retention limits or rule was also examined.

5. TWO CASE STUDIES

The South African general insurance company, Santam, has 6 levels in its BMS and this paper models the optimal retention limit under a two year and an infinite time horizon for a policyholder currently insured with Santam under motor vehicle insurance. To put a check on the results this paper also mimics the 6 level BMS Santam uses, with a 5 level BMS that acts in a very similar way (by similar premiums in each category and the same transition rules). The 6 level BMS's results, under an infinite and two-year time horizon, are compared to the 5 level BMS, so as to show the consistency in the modelling results. Under each case study, the optimal retention rule $\overline{x}^*$ and its associated $\overline{w}^*(\lambda)$, are shown numerically and graphically.

The actual quote obtained from Santam together with verification from them as to how their BMS works is shown in the Appendix.

5.1 SPECIFICATIONS FOR THE 6 LEVEL BMS (NAMED BMS-1)

- classes, i.e. s=6
- Claims occur at the beginning of the year of cover (i.e. t=0)
- Interest rate =10% per annum, which implies v is 1/1.1≈ 0.91
- Probability of one accident in a year = 10%
- Probability of two or more accidents in a year = 0%; hence at most one claim can be submitted each year
- Full Premium, b_1 = R 1248.79
- Premium discount for level 1 is 0%, level 2 is 6.66%, level 3 is 13.31%, level 4 is 19.98%, level 5 is 26.63% and level 6 is 33.29%
- Hence b_2=R1165.66, b_3=R1082.53, b_4=R999.30, b_5=R916.26 and b_6=R833.13
- Claim distributions assumed are: Negative Exponential, Uniform, Pareto, Lognormal and the Gamma.
- The expected claim under all these distributions is R7500, with various variances
- Transition rules are Classified as those used by the South African general insurer Santam,
- i.e. in the event of a claim free year the policyholder moves up one class (to a maximum of class 6)
- And in the event of a submitted claim, the policyholder moves down two class (to a minimum of class 1)

5.2 SPECIFICATIONS FOR THE 5 LEVEL BMS (NAMED BMS-2)

- 5 classes, i.e. s=5
- Claims occur at the beginning of the year of cover (i.e. t=0)
- Interest rate =10% per annum, which implies v is 1/1.1≈ 0.91
- Probability of one accident in a year = 10%
- Probability of two or more accidents in a year = 0%; hence at most one claim can be submitted each year
- Full Premium, b_1 = R 1248.79

- Premium discount for level 1 is 0%, level 2 is 13.31%, level 3 is 19.98%, level 4 is 26.63%, level 5 is 33.29%
- Hence b_2=R1082.53, b_3=R999.30, b_4=R916.26 and b_5=R833.13
- Claim distributions assumed are: Negative Exponential, Uniform, Pareto, Lognormal and the Gamma.
- The expected claim under all these distributions is R7500, with various variances
- Transition rules are Classified as those used by the South African general insurer Santam,
- i.e. in the event of a claim free year the policyholder moves up one class (to a maximum of class 6)
- And in the event of a submitted claim, the policyholder moves down two class (to a minimum of class 1)

6. SENSITIVITY TESTING

Under each BMS (6 level and 5 level), the effect of changing the discount rate as well as the probability of an accident is explored and graphically demonstrated.

7. DESCRIPTION OF MODELLING

7.1 OVERVIEW OF MODELLING USING MICROSOFT'S EXCEL PACKAGE

The past literature (Kolderman & Volgenant (1985), Lemaire 1998), refers to the use of a form of dynamic programming or iterative procedure to solve the system of equations shown for the optimal retention rules, but seldom explains the theoretical motivation or stability of the dynamic programming nor gives a step by step process to follow to solve the system of equations. This research paper wanted to explore the dynamic programming and give the reader insight to what is involved in systematically solving the system of equations.

In total what is necessary is to explicitly determine what each step in the process is of taking the initial estimates or guesses of the final optimal retention limits or rules and refining them systematically to closer approximations to the unique solution. A unique set of solutions always exits (see Lemaire 1998). However the system needs to be stable.

This further or better approximation goes through the same process or steps to produce a further refinement, so as to get closer to the unique solution, and this process is repeated again and again. This is thus an iterative process, very much in nature to the well-known Newton Raphson methodology. Hence when the approximation-vector converges to a set of solutions that are accurate to within the bounds set by the user, the process rests and the unique optimal retention rules or limits are found. What however must be avoided are iterative loops or a divergent and hence unstable system.

This paper found that by analyzing the system of equations to be solved that there was at least one way to proceed to solve the system iteratively, but that further iterative guesses had to be chosen as an average of the original guess and a refined guess so as to avoid iterative loops, which did occur if this averaging procedure was not followed.

The rest of section 7 of this research report explains the modelling and procedure followed for determining the optimal retention rules for both BMS-1 and BMS-2 under a two-year and infinite time horizon.

7.1.2 GENERAL PARAMETERS USED FOR THE MODELLING

The parameters needed for the claim size distributions, claim number distribution, discount rate, premium scale have already been set from the actual data of a South African general insurance company.

TIME HORIZONS
1 TWO-YEAR
2 INFINITE

	BMS-1 (6 LEVEL)
DISCOUNT	(0%,6.66%,13.31%,19.98%,26.63%33.29%)
PREMIUMS	(1248.79,1165.66,1082.53,999.30,916.26,833.13)
CLAIM-SIZE DISTRIBUTIONS	NEGATIVE EXPONENTIAL (MEAN R7500, STD R 7500) UNIFORM (MEAN R7500, STD R 4330) PARETO (MEAN R7500, STD R 9682) LOGNORMAL (MEAN R7500, STD R 11180) GAMMA (MEAN R 7500, STD R 9682)

	BMS-2 (5 LEVEL)
DISCOUNT	(0%,13.31%,19.98%,26.63%33.29%)
PREMIUMS	(1248.79,1082.53,999.30,916.26,833.13)
CLAIM-SIZE DISTRIBUTIONS	NEGATIVE EXPONENTIAL (MEAN R7500, STD R 7500) UNIFORM (MEAN R7500, STD R 4330) PARETO (MEAN R7500, STD R 9682) LOGNORMAL (MEAN R7500, STD R 11180)

	GAMMA (MEAN R 7500, STD R 9682)

CONSTRAINTS
MINIMUM GUESS= 0.01
MAXIMUM GUESS= VARIES BUT SET HIGH ENOUGH
ACCURACY REQUIRED= WITHIN 0.01% OF TRUE SOLUTION

CASE STUDY	**INITIAL GUESSES**
BMS-1, TWO YEAR HORIZON	$\overline{x}$ =ALL R 100
BMS-2, TW0 YEAR HORIZON	$\overline{x}$ = ALL R100
BMS-1, INFINITE HORIZON	$\overline{x}$ =ALL R400
BMS-2, INFINITE HORIZON	$\overline{x}$ =ALL R400

7.2 STATIONARY DISTRIBUTIONS

There is standard literature available concerning stationary distributions of Markov chains and this paper simply calculated what these distributions were. Under the assumptions that the insurer had a stationary insured population the expected profit for the insurer was calculated from the expected premiums and expected claims outgo (which took bonus hunger into account). These results are shown next where the probability of an accident is set at 10% and the interest rate is set at 10%.

For the interested reader, the stationary distribution $\overline{\pi} = (\pi_1, \pi_2, \ldots, \pi_s)$, is the steady state that is reached by the transition matrix $M(\lambda)$ used to represent the BMS. Hence in matrix notation, $\overline{\pi} = \overline{\pi}\ M(\lambda)$ .The stationary distribution is naturally dependent on the number of classes in the BMS as well as the transition rules. The solution to a BMS with s classes, rests on having s-1 linearly independent equations as well as an sth equation $\sum_{i=0}^{s} \pi_i = 1$. Solving these s equations simultaneously yields the stationary distribution.

For BMS-1 we have that

Let:

$A = p_{05}/(1-p_{06})$

$B = p_{04}$

$C = 1/(A*B)-p_{16}$

$D = C/p_{03} - p_{15}/A$

$E = (D-p_{14}/(A*B))/p_{01})$

$\pi_6 = (E + D/p_{02} + C/p_{03} + 1/(A*B) + 1/A + 1)^{(-1)}$

$\pi_5 = \pi_6 / A$

$\pi_4 = \pi_6 / (A*B)$

$\pi_3 = \pi_6 * C/p_{03}$

$\pi_2 = \pi_6 * D/p_{02}$

$\pi_1 = \pi_6 * E$

For BMS-2 we have that

Let:

$A = p_{04}/(1-p_{15})$
$B = 1/p_{03}$
$C = (B - A*p_{15})/p_{02}$
$D = (C - p_{14})/p_{01}$

$\pi_4 = (1 + A + B + C + D)^{(-1)}$

$\pi_5 = \pi_4 * A$

$\pi_3 = \pi_4 * B$

$\pi_2 = \pi_4 * C$

$\pi_1 = \pi_4 * D$

7.2.1 BMS-1, TWO YEAR TIME HORIZON AND STATIONARY EXPECTED PROFIT

TWO YEAR HORIZON: BMS-1, 6 LEVEL BMS

	TIME OF ACCIDENT					
Beginning of the year	YES					
Interest rate	10.00%					
P(Accident)	10%					
Full Premium	1,248.79					
CONVERGE (number of iterations)	4	4	4	4	4	4
	x1	x2	x3	x4	x5	x6
starting value	100	100	100	100	100	100

	NegExp	**Uniform**	**Pareto**	**Lognormal**	**Gamma**	**Average**
x1	137.78	137.59	137.86	137.48	138.44	137.68
x2	275.88	275.40	276.11	275.23	277.35	275.66
x3	420.20	419.71	420.43	419.54	421.67	419.97
x4	426.47	426.23	426.58	426.15	427.18	426.36
x5	370.23	370.57	370.06	370.65	369.33	370.05
x6	225.91	226.27	225.74	226.33	225.01	225.73
Stationary: Proportion in Class 1	0.2%	0.2%	0.2%	0.2%	0.1%	0.2%
Stationary: Proportion in Class 2	1.2%	1.2%	1.1%	1.3%	1.0%	1.2%
Stationary: Proportion in Class 3	1.9%	1.9%	1.8%	2.0%	1.6%	1.8%
Stationary: Proportion in Class 4	9.3%	9.5%	9.3%	9.6%	8.7%	9.3%
Stationary: Proportion in Class 5	8.5%	8.6%	8.4%	8.7%	8.0%	8.4%
Stationary: Proportion in Class 6	78.9%	78.5%	79.1%	78.3%	80.7%	79.1%
Premium in Class 1	1,248.79	1,248.79	1,248.79	1,248.79	1,248.79	1,248.79
Premium in Class 2	1,165.66	1,165.66	1,165.66	1,165.66	1,165.66	1,165.66
Premium in Class 3	1,082.53	1,082.53	1,082.53	1,082.53	1,082.53	1,082.53
Premium in Class 4	999.30	999.30	999.30	999.30	999.30	999.30
Premium in Class 5	916.26	916.26	916.26	916.26	916.26	916.26
Premium in Class 6	833.13	833.13	833.13	833.13	833.13	833.13
Discount for Class 1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Discount for Class 2	6.66%	6.66%	6.66%	6.66%	6.66%	6.66%
Discount for Class 3	13.31%	13.31%	13.31%	13.31%	13.31%	13.31%
Discount for Class 4	19.98%	19.98%	19.98%	19.98%	19.98%	19.98%
Discount for Class 5	26.63%	26.63%	26.63%	26.63%	26.63%	26.63%
Discount for Class 6	33.29%	33.29%	33.29%	33.29%	33.29%	33.29%
Average Stationary Premium	865.07	865.86	864.69	866.35	861.90	864.38
E(Stationary Claims insurer pays)	749.53	749.75	749.42	749.86	748.93	749.41
E(Stationary Claims p/h pays)	0.47	0.25	0.58	0.14	1.07	0.36
E(Total Stationary Claims)	750.00	750.00	750.00	750.00	750.00	750.00
E(Stationary profit for the insurer)	115.54	116.11	115.27	116.49	112.97	114.98
E(Stationary outgo for the p/h)	865.54	866.11	865.27	866.49	862.97	864.74

Parameters	**NegExp**	**Uniform**	**Pareto**	**Lognormal**	**Gamma**	**Average**
E(Claims)	7,500	7,500	7,500	7,500	7,500	7,500
Var(Claims)	56,250,000	18,750,000	93,750,000	125,000,000	93,750,000	
Std Deviation(Claims)	7,500	4,330	9,682	11,180	9,682	
parameter-1	0.013%	15000	5.00	8.34	0.60	
parameter-2			30,000	1.08	0.01%	

Minimum retention value approximation	0.01
Accuracy/Tolerance required	0.01%
Maximum retention value approximation	3000
Guess	50%
Modified	50%

Average*: For smoothing sake the average is taken as the average of the claim size distributions bar the highest value.

7.2.2 BMS-2, TWO YEAR TIME HORIZON AND STATIONARY EXPECTED PROFIT

TWO YEAR HORIZON: BMS-2, 5 LEVEL BMS

	TIME OF ACCIDENT					
Beginning of the year	YES					
Interest rate	10.00%					
P(Accident)	10%					
Full Premium	1,248.79					
CONVERGE (number of iterations)	4	4	4	4	5	4
	x1	x2	x3	x4	x5	
starting value	100	100	100	100	100	

	NegExp	**Uniform**	**Pareto**	**Lognormal**	**Gamma**	**Average***
x1	213.61	213.61	213.74	213.24	214.41	213.55
x2	351.87	351.87	352.19	351.13	353.56	351.77
x3	502.15	502.15	502.29	501.76	502.97	502.09
x4	370.42	370.42	370.29	370.77	369.65	370.19
x5	232.15	232.15	231.84	232.88	230.50	231.66
Stationary: Proportion in Class 1	1.2%	1.2%	1.1%	1.3%	1.0%	1.1%
Stationary: Proportion in Class 2	1.9%	1.9%	1.8%	2.0%	1.6%	1.8%
Stationary: Proportion in Class 3	9.4%	9.4%	9.3%	9.6%	8.7%	9.3%
Stationary: Proportion in Class 4	8.5%	8.5%	8.4%	8.7%	8.0%	8.4%
Stationary: Proportion in Class 5	79.1%	79.1%	79.3%	78.5%	80.9%	79.4%
Premium in Class 1	1248.79	1248.79	1248.79	1248.79	1248.79	1248.79
Premium in Class 2	1082.53	1082.53	1082.53	1082.53	1082.53	1082.53
Premium in Class 3	999.30	999.30	999.30	999.30	999.30	999.30
Premium in Class 4	916.26	916.26	916.26	916.26	916.26	916.26
Premium in Class 5	833.13	833.13	833.13	833.13	833.13	833.13
Discount for Class 1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Discount for Class 2	13.31%	13.31%	13.31%	13.31%	13.31%	13.31%
Discount for Class 3	19.98%	19.98%	19.98%	19.98%	19.98%	19.98%
Discount for Class 4	26.63%	26.63%	26.63%	26.63%	26.63%	26.63%
Discount for Class 5	33.29%	33.29%	33.29%	33.29%	33.29%	33.29%
Average Stationary Premium	865.21	865.21	864.81	866.51	862.00	864.31
E(Stationary Claims insurer pays)	749.48	749.48	749.36	749.83	748.85	749.29
E(Stationary Claims p/h pays)	0.52	0.52	0.64	0.17	1.15	0.46
E(Total Stationary Claims)	750.00	750.00	750.00	750.00	750.00	750.00
E(Stationary profit for the insurer)	115.73	115.73	115.45	116.68	113.14	115.01
E(Stationary outgo for the p/h)	865.73	865.73	865.45	866.68	863.14	864.77

Parameters	**NegExp**	**Uniform**	**Pareto**	**Lognormal**	**Gamma**	**Average**
E(Claims)	7500	7500	7500	7500	7500	7500
Var(Claims)	56,250,000	18,750,000	93,750,000	125,000,000	93,750,000	
Std Deviation(Claims)	7500	4330	9682	11180	9682	
parameter-1	0.013%	15000	5.00	8.34	0.60	
parameter-2			30000	1.08	0.01%	

Minimum retention value approximation	0.01
Accuracy/Tolerance required	0.01%
Maximum retention value approximation	3000
Guess	50%
Modified	50%

Average*: For smoothing sake the average is taken as the average of the claim size distributions bar the highest value.

7.2.3 BMS-1, INFINITE TIME HORIZON AND STATIONARY EXPECTED PROFIT

INFINITE TIME HORIZON: BMS-1, 6 LEVEL BMS

	TIME OF ACCIDENT					
Beginning of the Year	YES					
Interest rate	10.00%					
P(Accident)	10%					
Full Premium	1,248.79					
CONVERGE (number of iterations)	111	111	108	111	104	109
	x1	x2	x3	x4	x5	x6
starting value	400	400	400	400	400	400

	NegExp	**Uniform**	**Pareto**	**Lognormal**	**Gamma**	**Average***
x1	307.31	306.79	307.56	306.71	308.73	307.09
x2	588.66	588.42	588.76	588.52	588.91	588.59
x3	837.54	838.87	836.90	839.03	833.53	836.71
x4	709.25	712.39	707.77	712.85	700.11	707.38
x5	524.88	528.46	523.19	528.98	514.58	522.78
x6	275.99	278.01	275.05	278.48	269.97	274.75
V1	10,687	10,696	10,683	10,699	10,654	10,680
V2	10,349	10,358	10,345	10,362	10,314	10,342
V3	10,040	10,049	10,035	10,052	10,006	10,032
V4	9,766	9,773	9,762	9,777	9,737	9,760
V5	9,569	9,575	9,566	9,578	9,544	9,564
V6	9,462	9,467	9,460	9,470	9,440	9,457
Stationary: Proportion in Class 1	0.2%	0.2%	0.2%	0.2%	0.1%	0.2%
Stationary: Proportion in Class 2	1.1%	1.2%	1.1%	1.2%	0.9%	1.1%
Stationary: Proportion in Class 3	1.8%	1.9%	1.7%	1.9%	1.5%	1.8%
Stationary: Proportion in Class 4	9.3%	9.4%	9.2%	9.5%	8.6%	9.2%
Stationary: Proportion in Class 5	8.4%	8.5%	8.4%	8.6%	7.9%	8.4%
Stationary: Proportion in Class 6	79.2%	78.7%	79.5%	78.5%	81.1%	79.4%
Premium in Class 1	1,248.79	1,248.79	1,248.79	1,248.79	1,248.79	1,248.79
Premium in Class 2	1,165.66	1,165.66	1,165.66	1,165.66	1,165.66	1,165.66
Premium in Class 3	1,082.53	1,082.53	1,082.53	1,082.53	1,082.53	1,082.53
Premium in Class 4	999.30	999.30	999.30	999.30	999.30	999.30
Premium in Class 5	916.26	916.26	916.26	916.26	916.26	916.26
Premium in Class 6	833.13	833.13	833.13	833.13	833.13	833.13
Discount for Class 1	0%	0%	0%	0%	0%	0%
Discount for Class 2	7%	7%	7%	7%	7%	7%
Discount for Class 3	13%	13%	13%	13%	13%	13%
Discount for Class 4	20%	20%	20%	20%	20%	20%
Discount for Class 5	27%	27%	27%	27%	27%	27%
Discount for Class 6	33%	33%	33%	33%	33%	33%
Average Stationary Premium	864.40	865.49	863.89	865.80	860.98	863.69
E(Stationary Claims insurer pays)	749.06	749.50	748.87	749.48	748.28	748.93
E(Stationary Claims p/h pays)	0.94	0.50	1.13	0.52	1.72	0.77
E(Total Stationary Claims)	750.00	750.00	750.00	750.00	750.00	750.00
E(Stationary profit for the insurer)	115.34	115.99	115.03	116.31	112.69	114.77
E(Stationary outgo for the p/h)	865.34	865.99	865.03	866.31	862.69	864.46

Parameters	**NegExp**	**Uniform**	**Pareto**	**Lognormal**	**Gamma**	**Average**
E(Claims)	7,500	7,500	7,500	7,500	7,500	7,500
Var(Claims)	56,250,000	18,750,000	93,750,000	125,000,000	93,750,000	
Std Deviation(Claims)	7,500	4,330	9,682	11,180	9,682	
parameter-1	0.013%	15000	5.00	8.34	0.60	
parameter-2			30,000	1.08	0.01%	

Minimum retention value approximation	50
Accuracy/Tolerance required	0.01%
Maximum retention value approximation	90000
Guess	50%
Modified	50%

Average*: For smoothing sake the average is taken as the average of the claim size distributions bar the highest value.

7.2.4 BMS-2, INFINITE TIME HORIZON AND STATIONARY EXPECTED PROFIT

INFINITE TIME HORIZON: BMS-2, 5 LEVEL BMS

	TIME OF ACCIDENT					
Beginning of the Year	YES					
Interest rate	10.00%					
P(Accident)	10%					
Full Premium	1,248.79					
CONVERGE (number of iterations)	101	101	100	103	97	100
	x1	x2	x3	x4	x5	
starting value	400	400	400	400	400	400
	NegExp	**Uniform**	**Pareto**	**Lognormal**	**Gamma**	**Average***
x1	337.38	337.38	337.51	337.06	338.09	337.33
x2	563.46	563.46	563.40	563.72	562.74	563.26
x3	745.54	745.54	744.86	747.31	741.03	744.24
x4	503.20	503.20	502.13	506.09	496.07	501.15
x5	277.12	277.12	276.23	279.43	271.43	275.48
w1	10,388.07	10,388.07	10,384.70	10,398.69	10,359.13	10,380.00
w2	10,016.96	10,016.96	10,013.44	10,027.93	9,987.23	10,008.65
w3	9,768.27	9,768.27	9,764.96	9,778.60	9,740.12	9,760.40
w4	9,567.98	9,567.98	9,565.35	9,576.65	9,544.00	9,561.33
w5	9,463.44	9,463.44	9,461.10	9,471.23	9,441.55	9,457.38
Stationary: Proportion in Class 1	1.1%	1.1%	1.1%	1.2%	0.9%	1.1%
Stationary: Proportion in Class 2	1.8%	1.8%	1.7%	1.9%	1.5%	1.7%
Stationary: Proportion in Class 3	9.3%	9.3%	9.2%	9.6%	8.5%	9.1%
Stationary: Proportion in Class 4	8.4%	8.4%	8.4%	8.7%	7.8%	8.3%
Stationary: Proportion in Class 5	79.4%	79.4%	79.7%	78.7%	81.3%	79.7%
Premium in Class 1	1,248.79	1,248.79	1,248.79	1,248.79	1,248.79	1,248.79
Premium in Class 2	1,082.53	1,082.53	1,082.53	1,082.53	1,082.53	1,082.53
Premium in Class 3	999.30	999.30	999.30	999.30	999.30	999.30
Premium in Class 4	916.26	916.26	916.26	916.26	916.26	916.26
Premium in Class 5	833.13	833.13	833.13	833.13	833.13	833.13
Discount for Class 1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Discount for Class 2	13.31%	13.31%	13.31%	13.31%	13.31%	13.31%
Discount for Class 3	19.98%	19.98%	19.98%	19.98%	19.98%	19.98%
Discount for Class 4	26.63%	26.63%	26.63%	26.63%	26.63%	26.63%
Discount for Class 5	33.29%	33.29%	33.29%	33.29%	33.29%	33.29%
Average Stationary Premium	864.59	864.59	864.07	866.04	861.06	863.58
E(Stationary Claims insurer pays)	749.03	749.03	748.83	749.50	748.16	748.76
E(Stationary Claims p/h pays)	0.97	0.97	1.17	0.50	1.84	0.90
E(Total Stationary Claims)	750.00	750.00	750.00	750.00	750.00	750.00
E(Stationary profit for the insurer)	115.56	115.56	115.25	116.54	112.90	114.82
E(Stationary outgo for the p/h)	865.56	865.56	865.25	866.54	862.90	864.48

Parameters	**NegExp**	**Uniform**	**Pareto**	**Lognormal**	**Gamma**	**Average**
E(Claims)	7500	7500	7500	7500	7500	7500
Var(Claims)	56,250,000	18,750,000	93,750,000	125,000,000	93,750,000	
Std Deviation(Claims)	7500	4330	9682	11180	9682	
parameter-1	0.013%	15000	5.00	8.34	0.60	
parameter-2			30000	1.08	0.01%	

Minimum retention value approximation	300
Accuracy/Tolerance required	0.01%
Maximum retention value approximation	5000
Maximum cost	30000
Guess	50%
Modified	50%

Average*: For smoothing sake the average is taken as the average of the claim size distributions bar the highest value.

7.3 MODELLING WITH A TWO-YEAR TIME HORIZON

The General form of the solution to the optimal retention rule x_i with a two-year time horizon is:

$$x_i = v^*\{b(T(1,i)) - b(T(0,i))\}$$
$$+ \{ v^2 \sum_{r=0}^{1} [\bar{p}_r^{T(1,i)}(\lambda)\, b(T(r,T(1,i))) - \bar{p}_r^{T(0,i)}(\lambda)\, b(T(r,T(0,i)))]\}$$

However $\bar{p}_r^{T_{1(i)}}(\lambda)$ and $\bar{p}_r^{T_{0(i)}}$ are dependent on x_i as seen by the general definition of

$$\bar{p}_k^i(\lambda) = \sum_{h=k}^{\infty} p_h(\lambda) \binom{h}{k} (1-p_i)^k p_i^{h-k} \quad \text{where } p_i = P(X \leq x_i) = \int_0^{x_i} f(x)\,dx.$$

Thus x_i is a function of x_i.

In this research report, it is important to note that $\bar{p}_0^i$ and $\bar{p}_1^i$ have the same basic definition throughout, since this research paper assumes that there is at most one accident that can occur in a year of insurance cover. The exact definition of $\bar{p}_0^i$ and $\bar{p}_1^i$ follows on the next page.

7.3.1 METHODOLOGY USED FOR BMS-1 (6 LEVEL BMS) USING A TWO YEAR TIME HORIZON

Following from the general solution for bonus-hunger, the intermediate steps are:

$\overline{p}_0^{\,i}$ = (1 – P(Accident)) + P(Accident)*F(x_i)

$\overline{p}_1^{\,i}$ = P(Accident)*(1-F(x_i))

Definition :

- P(Accident) is the probability of 1 accident and set to 10%. At most one accident can occur in a year.
- F(x_i) is the cumulative density function and the same as p_i.

This naturally follows since

$$\overline{p}_k^{\,i}(\lambda) = \sum_{h=k}^{\infty} p_h(\lambda) \binom{h}{k} (1-p_i)^k p_i^{h-k} = \sum_{h=k}^{1} p_h(\lambda) \binom{h}{k} (1-p_i)^k p_i^{h-k}$$

and

$p_0(\lambda)$ = 1- P(Accident)

$p_1(\lambda)$ = P(Accident)

Also for BMS-1:

$Class_i$	T(0,i)	T(1,i)
1	2	1
2	3	1
3	4	1
4	5	2
5	6	3
6	6	4

Starting with a vector guess for $\overline{x}^{*2}$= (100,100,100,100,100,100,100) and using Microsoft Excel, the final solution of $\overline{x}^{*2}$ after a brief 4 iterations, where the probability of an accident is 10% and the interest rate is 10%, is as follows:

OPTIMAL RETENTION RULE	BMS-1 (6 LEVEL)
x_1	R 137.68
x_2	R 275.66
x_3	R 419.97
x_4	R 426.36
x_5	R 370.05
x_6	R 225.73

The derivation of the convergence sequence follows naturally and iteratively from following the general solution, using the transition rules and $\overline{p}_0^i$ and $\overline{p}_1^i$ as defined above.

7.3.2 METHODOLOGY USED FOR BMS-2 (5 LEVEL BMS) USING A TWO-YEAR TIME HORIZON

The methodology is the very similar to that used in BMS-1. Naturally the general form of the optimal retention rule stays the same and $\overline{p}_0^i$ and $\overline{p}_1^i$ have the same definition. The transition rules are however different and it is this difference that produces a different $\overline{x}^{*2} = (x_1^{*2}, x_2^{*2}, x_3^{*2}, x_4^{*2}, x_5^{*2})$

For BMS-2:

$Class_i$	T(0,i)	T(1,i)
1	2	1
2	3	1
3	4	1
4	5	2
5	5	3

Starting with a vector guess for $\overline{x}^{*2}$= (100,100,100,100,100) and using Microsoft Excel, the final solution of $\overline{x}^{*2}$ which converges after a brief 4 iterations, where the probability of an accident is 10% and the rate of interest is 10%, is as follows:

OPTIMAL RETENTION RULE	BMS-2 (5 LEVEL)
x_1	R 213.55
x_2	R 351.77
x_3	R 502.09
x_4	R 370.19
x_5	R 231.66

Again the derivation of the convergence sequence follows naturally and iteratively from following the general solution, using the transition rules and $\overline{p}_0^i$ and $\overline{p}_1^i$ as defined above.

7.3.3 COMPARING BMS-1 WITH BMS-2 WITH A TWO YEAR TIME HORIZON

Using the parameters that the interest rate is 10% and that the probability of an accident is 10%, the following comparison between BMS-1 (6 level) and BMS-2 (5 level) can be made:

OPTIMAL RETENTION RULE	BMS-1 (6 LEVEL)	BMS-2 (5 LEVEL)
x_1	R 137.68	R 213.55
x_2	R 275.66	R 351.77
x_3	R 419.97	R 502.09
x_4	R 426.36	R 370.19
x_5	R 370.05	R 231.66
x_6	R 225.73	

1) The optimal retention rule x_6 AND x_5 under BMS-1 are by the nature of the transition rule and the discount on the premiums in each category very similar to x_5 and x_4 in BMS-2 and the above table illustrates this

2) The optimal retention rule x_1 under BMS-1 is lower than x_1 under BMS-2 since there is less incentive not to submit a claim under BMS-1 for a policyholder in level 1 than for a policyholder in level 1 under BMS-2. A policyholder who doesn't submit a claim under BMS-1 receives a premium discount of 7% the following year, whereas a policyholder in BMS-2 receives 13% the following year and if a policyholder doesn't claim the year after it is clear that there is more discount in the premium for a policyholder in BMS-2 (20%) than for the policyholder in BMS-1(13%).

3) Other comparison between BMS-1 and BMS-2 depend on intricacies of a policyholders' fortune up to the two year time horizon and thus makes easy comparison difficult, but as the order of discount and the transition rules are similar the optimal retention rules are similar in magnitude but obviously differ somewhat is size. As there is one extra level in BMS-1 than BMS-2 it also makes sense to compare x_i under BMS-1 with x_{i-1} under BMS-2.

7.3.4 SENSITIVITY ANALYSES USING A TWO YEAR TIME HORIZON

There are two parameters that can vary under BMS-1 and BMS-2 and that is the interest rate and the probability of an accident.

The following graphs demonstrate visually what the sensitivity of the optimal retention rules are to the influence of changing these two parameters independently

However from the graphs the following observations are made:

1) The higher the interest rate the lower the optimal retention rule, all other things being constant. This also makes intuitive sense, since the premium penalty is less costly to a policyholder who can invest his free money at a higher interest rate to pay this penalty than a policyholder earning less interest on his monies.

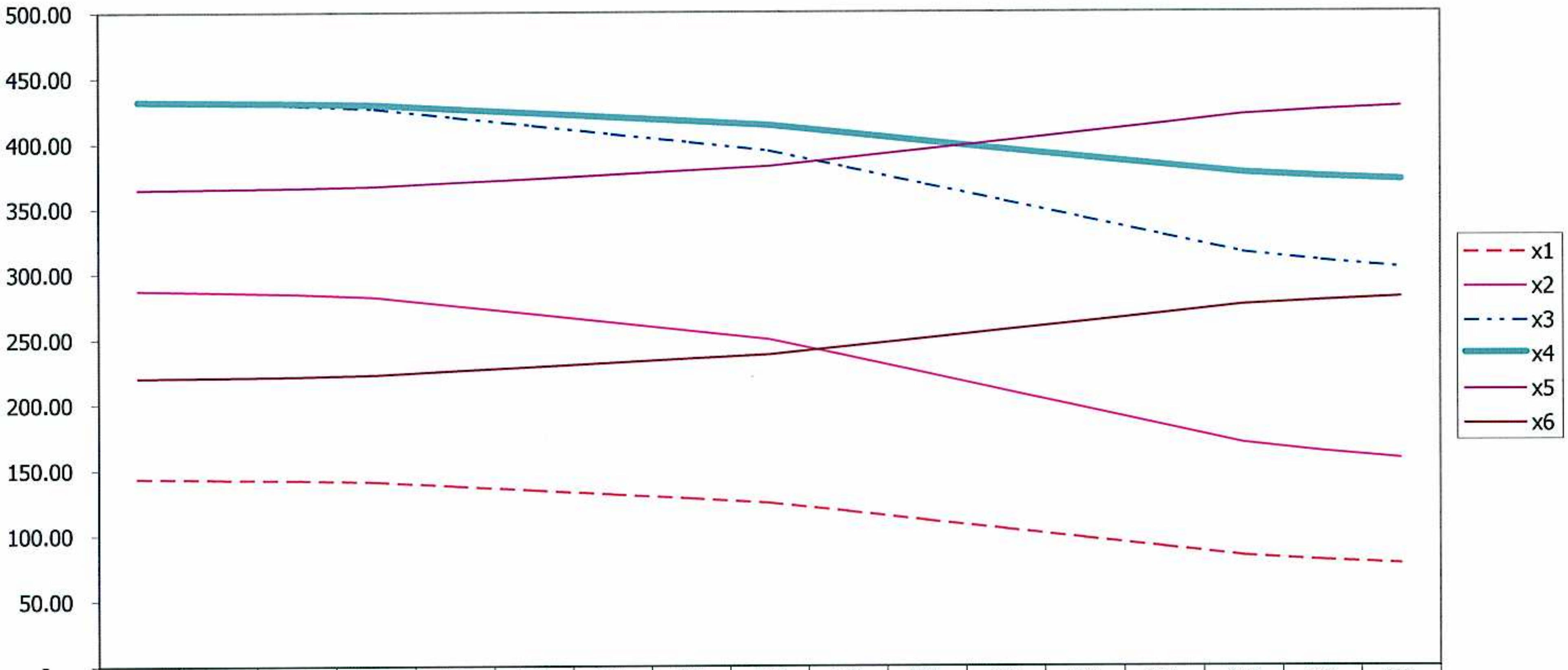
2) Under BMS-1, the higher the probability of an accident, the lower the optimal retention rule is for x_1,x_2,x_3 and x_4 (monotonically decreasing function) and the higher the optimal retention rule is for x_5 and x_6 (monotonically increasing function)

3) Under BMS-2, the higher the probability of an accident, the lower the optimal retention rule is for x_1,x_2,x_3 (monotonically decreasing function) and the higher the optimal retention rule is for x_4 and x_5 (monotonically increasing function).

4) It is clear that under BMS-1 and BMS-2 the optimal retention rule changes in the same type of manner when the probability of an accident is changed (Remember to compare x_i under BMS-1 with x_{i-1} under BMS-2). The reason the optimal retention rule increases when the probability on an accident increases for the high levels in BMS-1 and BMS-2 is that the bonus hunger is stronger for the policyholder as the probability of an accident increases whereas for the low levels in these BMS the bonus hunger is weaker as the probability of an accident increases.

7.3.4.1 BMS-1: CHANGING THE PROBABILITY OF AN ACCIDENT USING A TWO YEAR TIME HORIZON

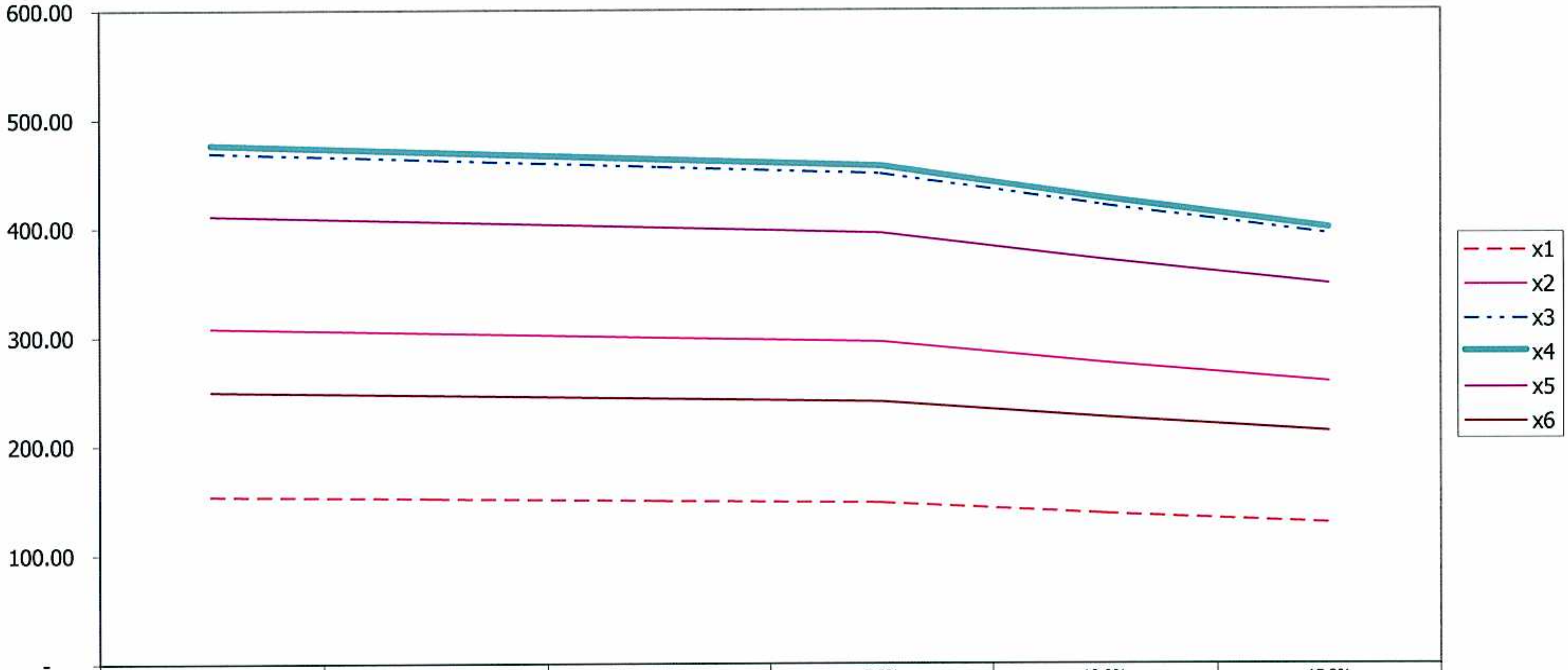
P(Acc), 2Y BMS-1 6 Level BMS



	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
x1	143.62	142.96	142.30	140.98	137.68	134.37	131.06	127.73	124.40	117.73	111.02	104.29	97.53	90.75	83.94	80.52	77.78
x2	287.34	286.04	284.74	282.15	275.66	269.15	262.63	256.10	249.55	236.43	223.25	210.03	196.76	183.45	170.08	163.39	158.02
x3	431.63	430.34	429.04	426.45	419.97	413.48	406.99	400.50	394.00	380.98	367.95	354.89	341.83	328.74	315.64	309.08	303.83
x4	432.19	431.54	430.89	429.59	426.36	423.14	419.94	416.75	413.58	407.29	401.06	394.90	388.81	382.78	376.82	373.86	371.51
x5	364.65	365.25	365.85	367.04	370.05	373.08	376.13	379.21	382.31	388.59	394.96	401.43	408.00	414.67	421.44	424.86	427.61
x6	220.36	220.95	221.55	222.74	225.73	228.74	231.76	234.80	237.84	243.99	250.19	256.44	262.75	269.12	275.54	278.77	281.37

7.3.4.2 BMS-1: CHANGING THE INTEREST RATE USING A TWO YEAR TIME HORIZON

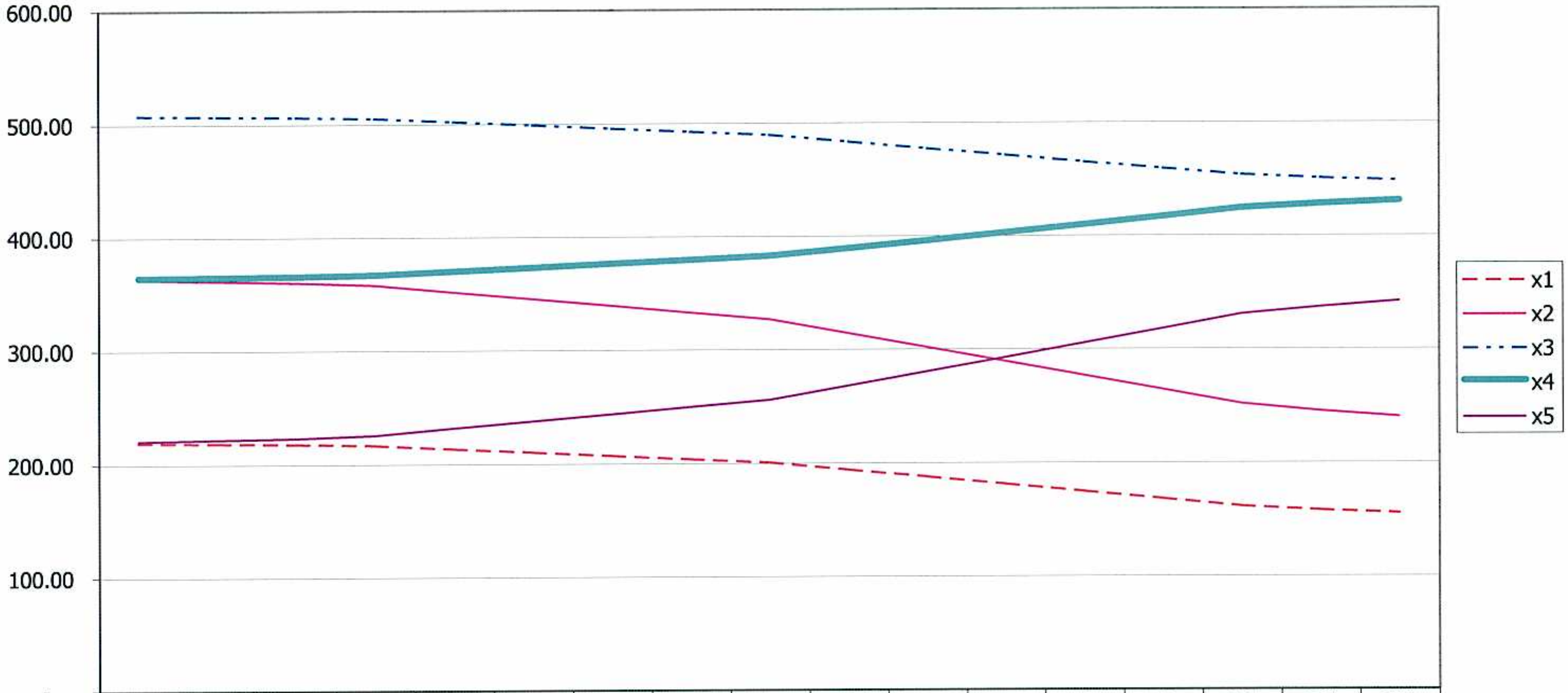
I-effect, 2Y BMS-1 6 Level BMS



	2.0%	3.0%	4.0%	5.0%	10.0%	15.0%
x1	153.77	151.58	149.44	147.36	137.68	129.09
x2	307.93	303.53	299.24	295.06	275.66	258.44
x3	469.37	462.63	456.07	449.67	419.97	393.63
x4	476.74	469.87	463.17	456.65	426.36	399.50
x5	411.09	405.51	400.07	394.76	370.05	348.04
x6	249.65	246.41	243.24	240.15	225.73	212.86

7.3.4.3 BMS-2: CHANGING THE PROBABILITY OF AN ACCIDENT USING A TWO YEAR TIME HORIZON

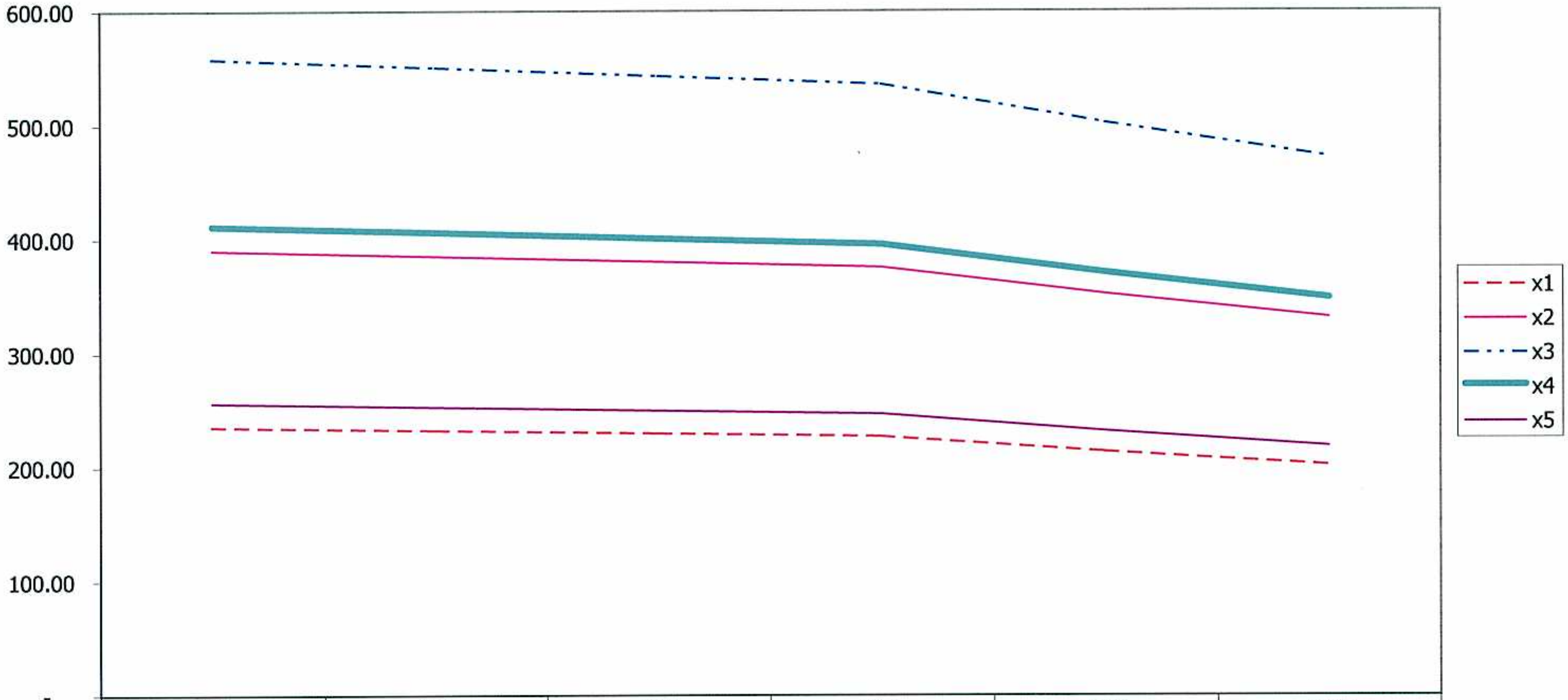
P(Acc), 2Y BMS-2 5 Level BMS



	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
x1	219.29	218.66	218.02	216.75	213.55	210.34	207.13	203.90	200.67	194.16	187.62	181.03	174.41	167.74	161.04	157.68	154.98
x2	362.98	361.73	360.49	358.00	351.77	345.53	339.29	333.04	326.79	314.26	301.72	289.15	276.57	263.97	251.34	245.03	239.97
x3	507.78	507.15	506.51	505.25	502.09	498.95	495.82	492.70	489.60	483.43	477.32	471.27	465.28	459.34	453.47	450.55	448.23
x4	364.67	365.27	365.88	367.11	370.19	373.31	376.46	379.64	382.86	389.39	396.05	402.85	409.78	416.85	424.06	427.71	430.66
x5	220.95	222.14	223.32	225.70	231.66	237.65	243.66	249.70	255.76	267.96	280.26	292.66	305.16	317.76	330.45	336.84	341.96

7.3.4.4 BMS-2: CHANGING THE INTEREST RATE WITH A TWO-YEAR TIME HORIZON

I-effect, 2Y BMS-2 5 Level BMS



	2.0%	3.0%	4.0%	5.0%	10.0%	15.0%
x1	235.65	232.65	229.73	226.87	213.55	201.64
x2	390.12	384.90	379.82	374.86	351.77	331.20
x3	558.45	550.78	543.30	536.01	502.09	471.92
x4	411.29	405.70	400.25	394.93	370.19	348.16
x5	256.45	253.09	249.81	246.60	231.66	218.32

7.4 MODELLING WITH AN INFINITE TIME HORIZON

This paper assumes that at most one claim can occur in a year of cover and that claims occur at the beginning of the year. The following 2s equations need to be simultaneously solved:

$x_i = v^* \{ w(T(1,i), \lambda) - w(T(0,i), \lambda) \}$ s equations

$w_i(\lambda) = E(x_i) + v * \sum_{k=0}^{1} \overline{p}_k^i(\lambda)\, w(T(k,i), \lambda).$ A further s equations

From these equations it is clear that x_i depends on $w_i(\lambda)$ and that $w_i(\lambda)$ depends on x_i.

In the two case studies using an infinite horizon $\overline{p}_0^i$ and $\overline{p}_1^i$ have the same form, i.e.:

$\overline{p}_0^i = (1 - P(\text{Accident})) + P(\text{Accident})*F(x_i)$

$\overline{p}_1^i = P(\text{Accident})*(1-F(x_i))$

$E(x_i)$ which has the form $E(x_i) = b_i + v^{1/2} E^i(X) (\lambda - \overline{\lambda}_i)$, can be simplified to:

$E(x_i) = b_i + v^{1/2}*P(\text{Accident})* \int_0^{x_i} x\, f(x)\, dx$

Since:

$\overline{\lambda}_i = \sum_{k=0}^{\infty} k\, \overline{p}_k^i(\lambda) = \sum_{k=0}^{1} k\, \overline{p}_k^i(\lambda) = \overline{p}_1^i(\lambda) = p_1(\lambda)(1-p_i)^1 = P(\text{Accident})*(1-F(x_i))$

$\lambda = P(\text{Accident})$

$(\lambda - \overline{\lambda}_i) = P(\text{Accident})*F(xi) = P(\text{Accident})*p_i$

and

$$E^i(X) = \frac{1}{p_i} \int_0^{x_i} x\, f(x)\, dx$$

This simplification is important since in the modeling $w_i(\lambda)$ depends on $E(x_i)$.

7.4.1 DESCRIPTION OF METHOD TO SOLVE FOR THE OPTIMAL RETENTION LIMIT USING AN INFINITE TIME HORIZON

By starting with an initial guess vector $\overline{x}$ to the true solution $\overline{x}^*$, one can find for each class i:

- $E(x_i)$, with $E(x_i) = b_i + v^{1/2} * P(\text{Accident}) * \int_0^{x_i} x\, f(x)\, dx$
- $\overline{p}_0^{\,i} = (1 - P(\text{Accident})) + P(\text{Accident}) * F(x_i)$
- $\overline{p}_1^{\,i} = P(\text{Accident}) * (1 - F(x_i))$

Now, since the transition rules are also known

$$w_i(\lambda) = E(x_i) + v * \sum_{k=0}^{1} \overline{p}_k^{\,i}(\lambda)\, w(T(k,i), \lambda).$$

is a set of s equations, which can be solved by an iterative process. By simplifying what each $w_i(\lambda)$ is in terms of the set of $w_i(\lambda)$'s, and noting that $E(x_i)$, $\overline{p}_0^{\,i}$ and $\overline{p}_1^{\,i}$ are treated effectively as constants in the iterative process, it is then just a matter of initially starting one of the $w_i(\lambda)$'s, called original says, of with a guess, and determining the other $w_i(\lambda)$'s from the original and the $w_i(\lambda)$'s solved further in the process. However the original $w_i(\lambda)$ needs to be purposefully chosen in such a way that the other $w_i(\lambda)$'s can all be iteratively determined from it by an appropriate mathematical manipulation. When all the other $w_i(\lambda)$'s have been determined through an appropriate iterative method from the original, then the original $w_i(\lambda)$ is recalculated from the other $w_i(\lambda)$'s .

However, this original $w_i(\lambda)$ is best taken as an average of the original guess and the recalculated guess of its value, so as to avoid iterative loops and assists the dynamic programming method of getting closer to the true value of $w_i(\lambda)$'s. Also to improve the iteratively determined $w_i(\lambda)$'s from the original the author found recalculating them again from the recalculated original and taking an average between the first determined $w_i(\lambda)$'s and the recalculated $w_i(\lambda)$'s delivered a stable iterative process that converged steadily to the true optimal retention rules and the associated cost-vector. Not applying an averaging procedure in the modeling did lead to iterative loops and hence to an unstable system.

Hence from starting off with a guess-vector of x_i's one produces a recalculated guess cost-vector of the $w_i(\lambda)$'s. But since x_i's actually depend on the $w_i(\lambda)$'s, one can then determine a new vector-set of x_i's from the calculated $w_i(\lambda)$'s, to be used iteratively again as the new values to calculate the following $w_i(\lambda)$'s. This process is repeated until it converges to the unique optimal retention rule vector $\overline{x}^{*} = (x_1^{*}, x_2^{*}, \ldots, x_s^{*})$ and its associated cost-vector $\overline{w}^{*}(\lambda)$.

The mathematical manipulation and iterative process for BMS-1 (6 level), to determine the cost-vector $w(\lambda)$ is as follows:

1) Guess W_2 and call it say $W_2(1)$
2) $W_1(1)= (E(x_1)+v^*\overline{p}^{1}_{0}*W_2(1))/(1-v^*\overline{p}^{1}_{1})$
3) $W_3(1)= (W_2 - E(x_2) - v^*\overline{p}^{2}_{1} * W_1(1))/(v^*\overline{p}^{2}_{0})$
4) $W_4(1)= (W_3 - E(x_3) - v^*\overline{p}^{3}_{1} * W_1(1))/(v^*\overline{p}^{3}_{0})$
5) $W_6(1) = (E(x_6) + v^*\overline{p}^{6}_{1} * W_4(1))/(1-v^*\overline{p}^{6}_{0})$
6) $W_5(1) = E(x_5) + v^*\overline{p}^{5}_{0} *W_6(1)+v^*\overline{p}^{5}_{1} *W_3(1)$
7) $W_4(2) = E(x_4) + v^*\overline{p}^{4}_{0} *W_5(1)+v^*\overline{p}^{4}_{1} *W_2(1)$
8) $W_4(*) = 0.5*(W_4(2) + W_4(1))$
9) $W_3(2) = E(x_3) + v^*\overline{p}^{3}_{0} *W_4(*)+v^*\overline{p}^{3}_{1} *W_1(1)$
10) $W_3(*) = 0.5*(W_3(2) + W_3(1))$
11) $W_2(2) = E(x_2) + v^*\overline{p}^{2}_{0} *W_3(*)+v^*\overline{p}^{2}_{1} *W_1(1)$
12) $W_2(*) = 0.5*(W_2(2) + W_2(1))$...Recalculated original guess
13) $W_6(2) = (E(x_6) + v^*\overline{p}^{6}_{1} * W_4(*))/(1-v^*\overline{p}^{6}_{0})$
14) $W_6(*) = 0.5*(W_6(2) + W_6(1))$
15) $W_1(2)= (E(x_1)+v^*\overline{p}^{1}_{0} *W_2(*))/(1-v^*\overline{p}^{1}_{1})$
16) $W_1(*) = 0.5*(W_1(2) + W_1(1))$
17) $W_5(2) = E(x_5) + v^*\overline{p}^{5}_{0} *W_6(*)+v^*\overline{p}^{5}_{1} *W_3(*)$

18) $W_5(*) = 0.5*(W_5(2) + W_5(1))$

For BMS-2 (5 Level), the iterative process is as follows:

1) Guess W_2 and call it say $W_2(1)$
2) $W_1(1)= (E(x_1)+v*\overline{p}\,^{1}_{0} *W_2(1))/(1-v*\overline{p}\,^{1}_{1})$
3) $W_3(1)= (W_2 - E(x_2) - v*\overline{p}\,^{2}_{1} * W_1(1))/(v*\overline{p}\,^{2}_{0})$
4) $W_5(1) = (E(x_5) + v*\overline{p}\,^{5}_{1} * W_3(1))/(1-v*\overline{p}\,^{5}_{0})$
5) $W_4 (1) = E(x_4) + v*\overline{p}\,^{4}_{0} *W_5(1)+v*\overline{p}\,^{4}_{1} *W_2(1)$
6) $W_3(2)= E(x_3) + v*\overline{p}\,^{3}_{0} *W_4(1)+v*\overline{p}\,^{3}_{1} *W_1 (1)$
7) $W_3(*) = 0.5*(W_3(2) + W_3(1))$
8) $W_2(2) = E(x_2) + v*\overline{p}\,^{2}_{0} *W_3(*)+v*\overline{p}\,^{2}_{1} *W_1(1)$
9) $W_2(*) = 0.5*(W_2(2) + W_2(1))$ ...Recalculated original guess
10) $W_1(2)= (E(x_1)+v*\overline{p}\,^{1}_{0} *W_2(*)/(1-v*\overline{p}\,^{1}_{1})$
11) $W_1(*) = 0. .5*(W_1(2) + W_1(1))$
12) $W_4 (2) = E(x_4) + v*\overline{p}\,^{4}_{0} *W_5(1)+v*\overline{p}\,^{4}_{1} *W_2(*)$
13) $W_4(*) = 0.5*(W_4(2) + W_4(1))$
14) $W_5(2) = (E(x_5) + v*\overline{p}\,^{5}_{1} * W_3(*)/(1-v*\overline{p}\,^{5}_{0})$
15) $W_5(*) = 0.5*(W_5(2) + W_5(1))$

7.4.2 RESULTS FOR BMS-1 (6 LEVEL BMS) USING AN INFINITE TIME HORIZON

The description of BMS-1 has already been given.

Starting with a vector guess for $\bar{x}^{*2}$= (400,400,400,400,400,400,400) for BMS-1 and using Microsoft Excel, the final solution of $\bar{x}^{*2}$ after 110 iterations, where the probability of an accident is 10% and the interest rate is 10%, is as follows:

OPTIMAL RETENTION RULE	BMS-1 (6 LEVEL)
x_1	R 307.09
x_2	R 588.59
x_3	R 836.71
x_4	R 707.38
x_5	R 522.78
x_6	R 274.75

The associated cost vector $w_i(\lambda)$ is as follows:

COST VECTOR	BMS-1 (6 LEVEL)
w_1	R 10 373.77
w_2	R 10344.88
w_3	R 10 035.55
w_4	R 9 710.38
w_5	R 9 563.52
w_6	R 7 091.82

7.4.3 RESULTS FOR BMS-2 (5 LEVEL BMS) USING AN INFINITE TIME HORIZON

The description of BMS-2 has already been given.

Starting with a vector guess for $\bar{x}^{*2}$= (400,400,400,400,400) for BMS-2 and using Microsoft Excel, the final solution of $\bar{x}^{*2}$ which converges after 100 iterations, where the probability of an accident is 10% and the rate of interest is 10%, is as follows:

OPTIMAL RETENTION RULE	BMS-2 (5 LEVEL)
x_1	R 337.33
x_2	R 563.26
x_3	R 744.24
x_4	R 501.15
x_5	R 275.48

The associated cost vector $w_i(\lambda)$ is as follows:

COST VECTOR	BMS-2 (5 LEVEL)
w_1	R 10 380.00
w_2	R 10 008.65
w_3	R 9 760.40
w_4	R 9 561.33
w_5	R 9 457.38

7.4.4 COMPARING BMS-1 WITH BMS-2 WITH AN INFINITE TIME HORIZON

Using the parameters that the interest rate is 10% and that the probability of an accident is 10%, the following comparison between BMS-1 (6 level) and BMS-2 (5 level) can be made:

OPTIMAL RETENTION RULE	BMS-1 (6 LEVEL)	BMS-2 (5 LEVEL)
x_1	R 307.09	R 337.33
x_2	R 588.59	R 563.26
x_3	R 836.71	R 744.24
x_4	R 707.38	R 501.15
x_5	R 522.78	R 275.48
x_6	R 274.75	

1) The optimal retention rule x_6 AND x_5 under BMS-1 are by the nature of the transition rule and the discount on the premiums in each category very similar to x_5 and x_4 in BMS-2 and the above table illustrates this. The two-year time horizon comparison showed the same result.

2) The optimal retention rule x_1 under BMS-1 is slightly lower than x_1 under BMS-2. However the difference was marked using a two-year time horizon, whereas using an infinite time horizon has brought the optimal retention rule close to each other. Thus a policyholder in level 1 under BMS-1 and BMS-2 has a similar outlook for bonus hunger using an infinite time horizon even though their short-term fortunes differ as illustrated using a two-year time horizon.

3) Other comparison between BMS-1 and BMS-2 depend on intricacies of a policyholders' fortune up to the infinite time horizon and thus makes easy comparison difficult. The effect of using an infinite time horizon is that it demonstrates not only the short-term fortunes of a policyholder (like using a two-year time horizon) but also very long term fortunes. However by using a discount factor the very long-term effects are made to have small effect.

7.4.5 SENSITIVITY ANALYSES USING AN INFINITE TIME HORIZON

There are two parameters that can vary under BMS-1 and BMS-2 and that is the interest rate and the probability of an accident.

The following graphs demonstrate visually what the sensitivity of the optimal retention rules are to the influence of changing these two parameters independently

However from the graphs the following observations are made:

1) The higher the interest rate the lower the optimal retention rule, all other things being constant. This also makes intuitive sense, since the premium penalty (an "infinite"-penalty) is less costly to a policyholder who can invest his free money at a higher interest rate to pay this penalty than a policyholder earning less interest on his monies.

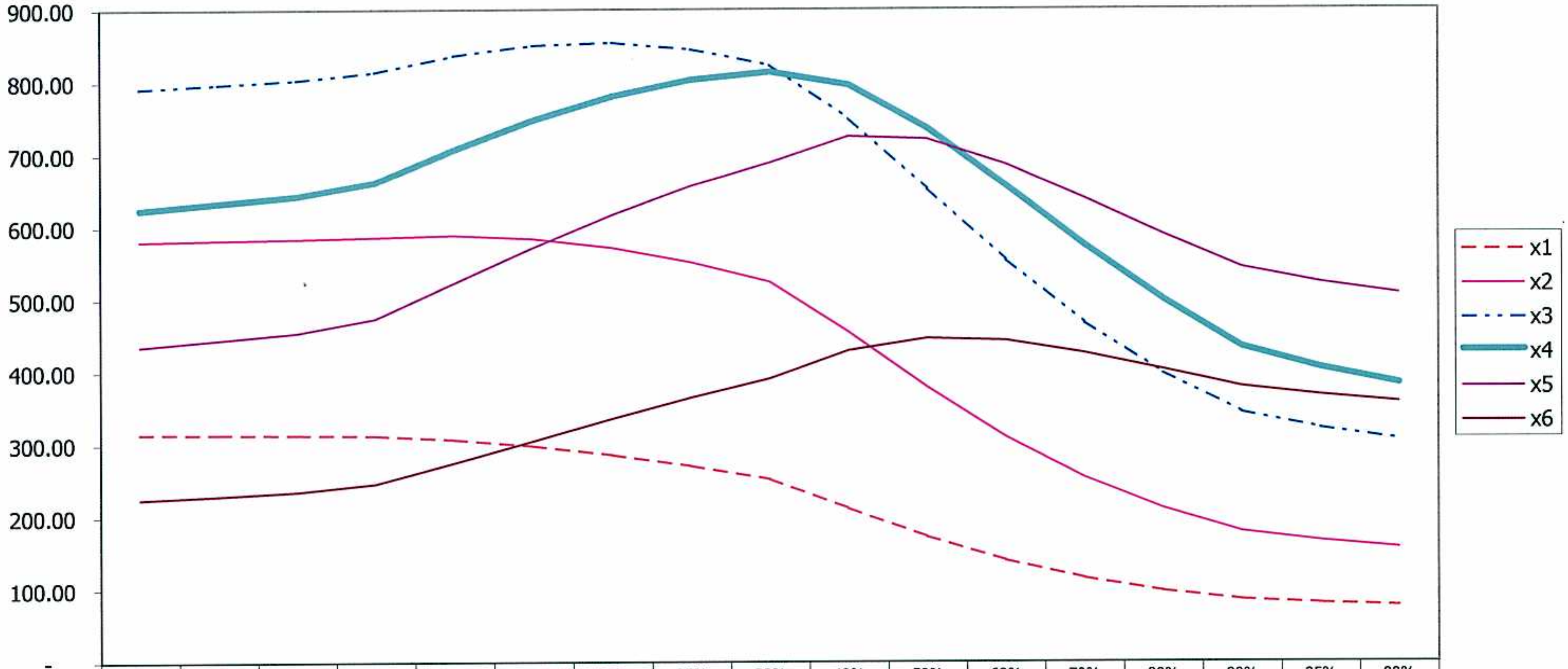
2) Under BMS-1, the effect of changing the probability of an accident has an unpredictable effect and the graphs show that seldom in there an monotonic increasing or decreasing effect (only x1 is monotonically decreasing).

3) Under BMS-2, the effect of changing the probability of an accident has an unpredictable effect and the graphs show that seldom in there an monotonic increasing or decreasing effect (only x1 is monotonically decreasing).

4) The pattern of behaviour when the probability of an accident is changed is similar between BMS-1 and BMS-2.

7.4.5.1 BMS-1: CHANGING THE PROBABILITY OF AN ACCIDENT USING AN INFINITE TIME HORIZON

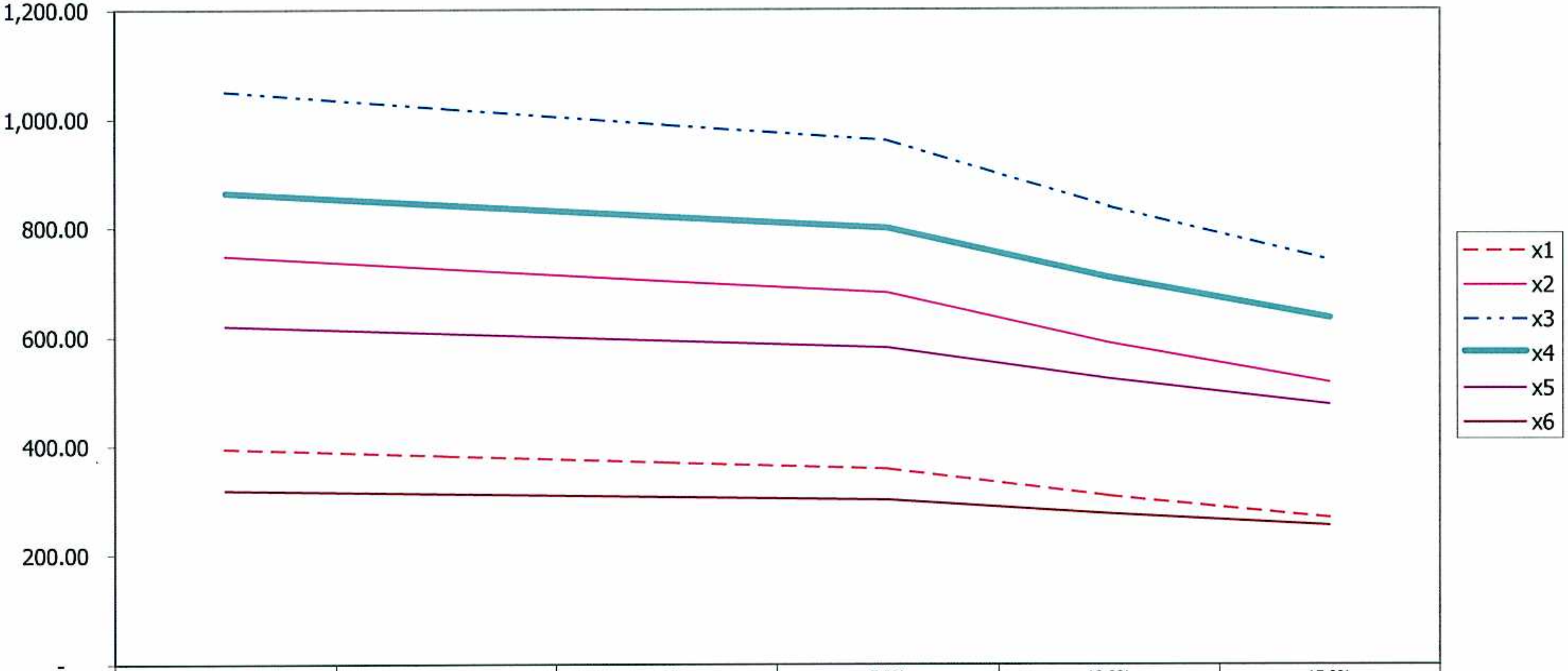
P(Acc), INF BMS-1 6 Level BMS



	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
x1	314.87	314.48	313.99	312.65	307.09	298.11	285.64	269.96	251.72	211.30	172.15	139.60	115.08	97.52	85.14	80.39	77.13
x2	580.57	582.29	583.83	586.33	588.59	583.76	571.26	551.04	523.81	454.55	378.78	309.85	253.93	211.21	179.22	166.33	157.19
x3	791.56	797.53	803.30	814.19	836.71	850.78	854.45	844.90	823.12	749.07	651.95	553.45	467.18	397.51	343.54	321.62	306.21
x4	624.06	633.64	643.19	662.10	707.38	747.63	780.18	802.74	813.70	795.54	735.79	656.16	574.07	499.00	433.95	405.11	383.62
x5	435.75	445.11	454.58	473.80	522.78	571.25	616.72	656.64	688.82	724.70	721.01	686.22	638.70	589.16	543.29	522.54	507.13
x6	224.75	229.87	235.11	245.95	274.75	305.10	335.58	364.60	390.63	428.92	445.65	442.62	425.45	402.86	378.97	367.24	358.11

7.4.5.2 BMS-1: CHANGING THE INTEREST RATE USING AN INFINITE TIME HORIZON

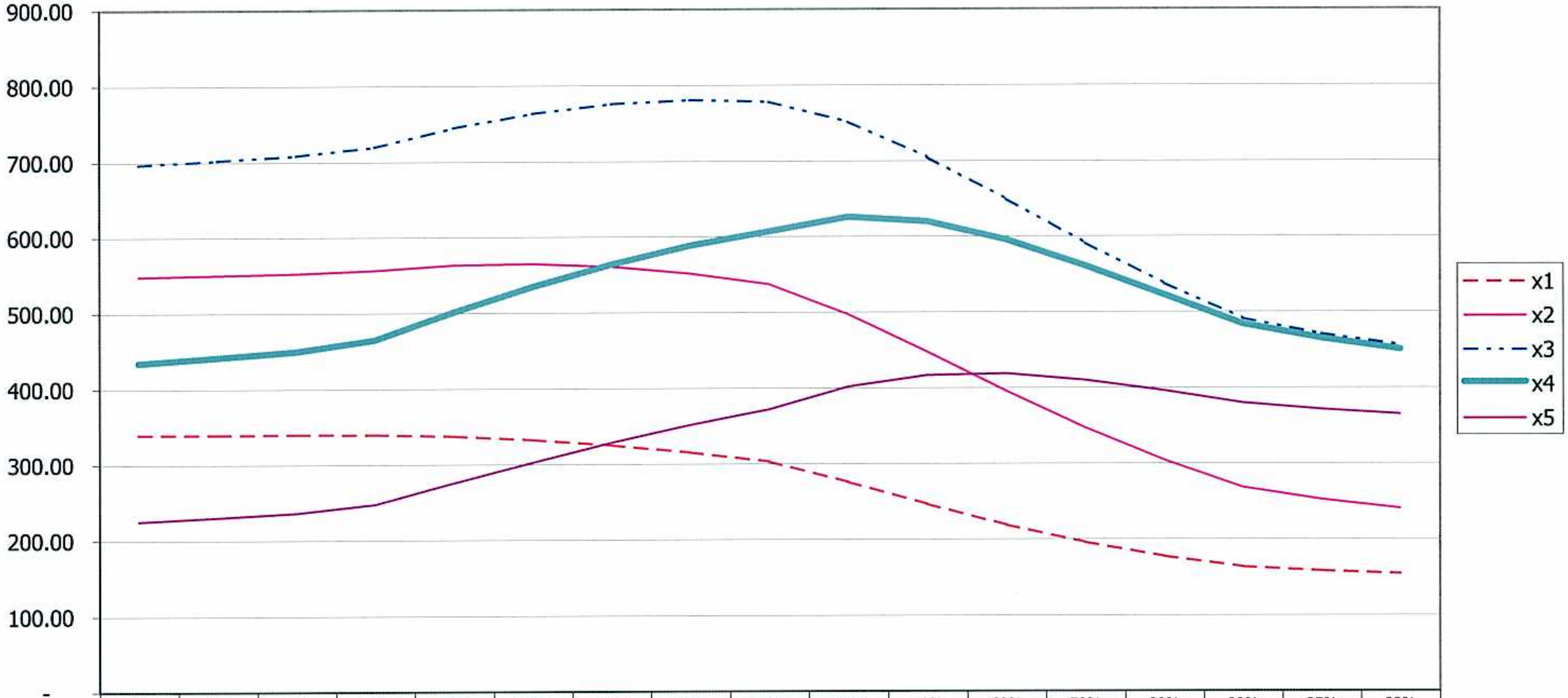
I-effect, INF BMS-1 6 Level BMS



	2.0%	3.0%	4.0%	5.0%	10.0%	15.0%
x1	395.32	382.07	369.54	357.70	307.09	267.67
x2	747.75	724.05	701.59	680.30	588.59	516.17
x3	1,050.07	1,018.46	988.45	959.95	836.71	738.85
x4	863.68	840.87	819.08	798.30	707.38	633.79
x5	620.23	606.20	592.73	579.83	522.78	475.76
x6	317.92	311.78	305.86	300.18	274.75	253.44

7.4.5.3 BMS-2: CHANGING THE PROBABILITY OF AN ACCIDENT USING AN INFINITE TIME HORIZON

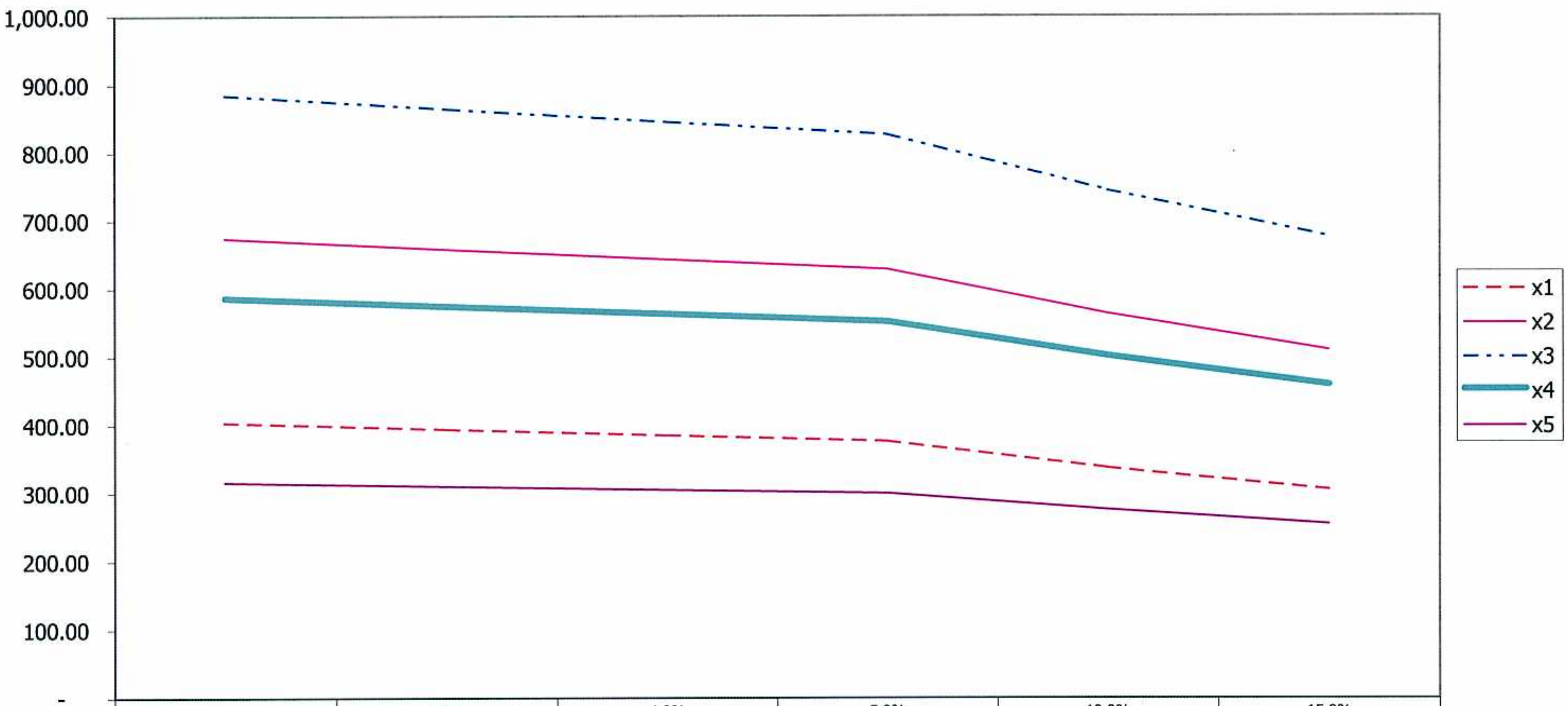
P(Acc), INF BMS-2 5 Level BMS



	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
x1	339.37	339.55	339.64	339.50	337.33	332.48	325.03	315.20	303.37	275.72	246.34	219.02	195.77	177.34	163.54	158.18	154.57
x2	548.33	550.66	552.83	556.69	563.26	564.92	560.95	551.67	537.47	497.21	447.39	395.56	346.87	304.09	267.85	252.04	240.39
x3	696.26	702.33	708.24	719.53	744.24	763.14	775.43	780.70	777.94	751.36	704.82	648.52	590.82	537.38	491.01	470.79	456.08
x4	434.23	441.92	449.57	464.71	501.15	534.60	563.90	588.11	606.60	625.43	619.58	595.34	561.02	522.59	483.86	465.08	450.46
x5	225.28	230.82	236.38	247.56	275.48	302.66	328.26	351.50	371.72	401.53	416.62	418.73	409.92	395.84	379.55	371.23	364.64

7.4.5.4 BMS-2: CHANGING THE INTEREST RATE USING AN INFINITE TIME HORIZON

I-effect, INF BMS-2 5 Level BMS



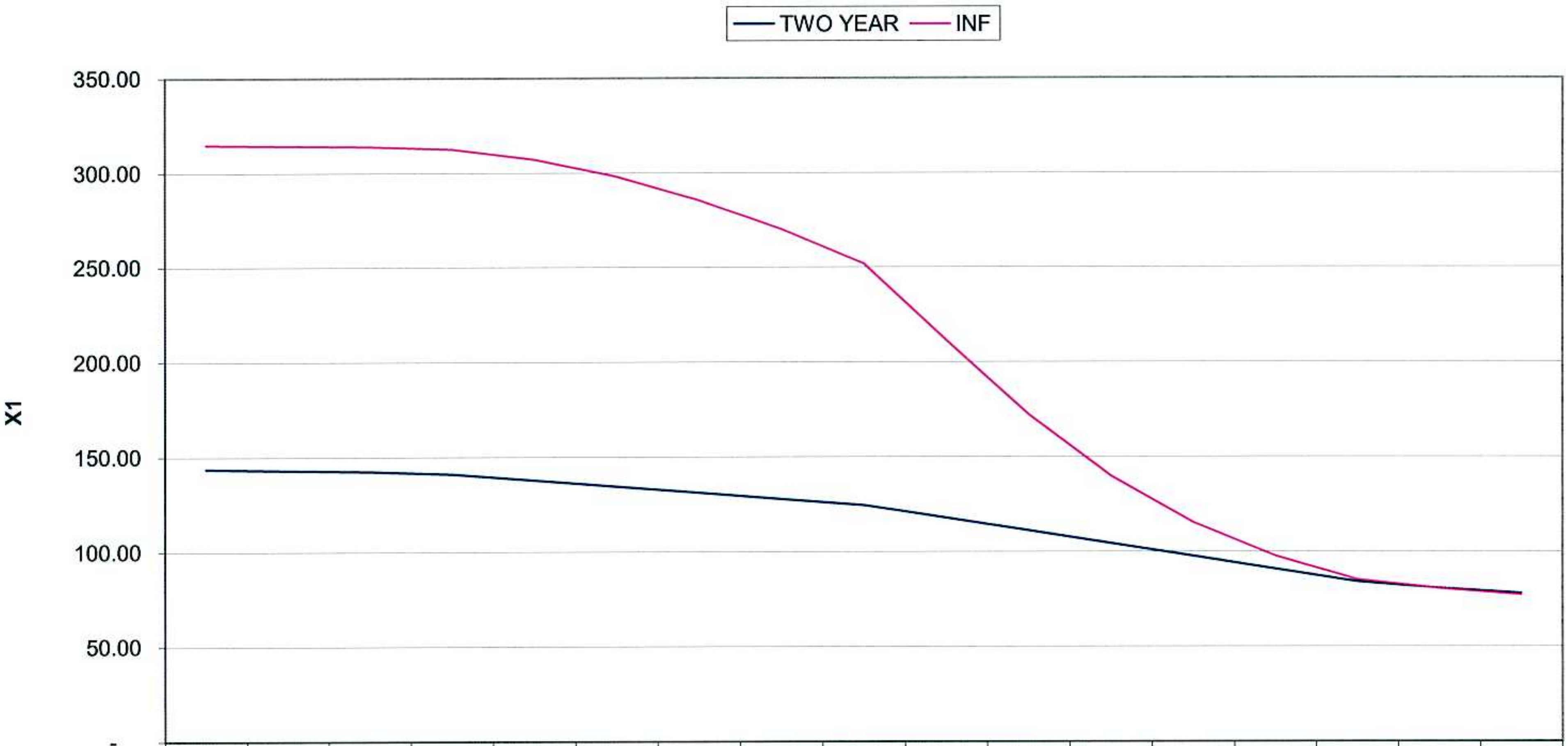
	2.0%	3.0%	4.0%	5.0%	10.0%	15.0%
x1	403.90	394.26	385.05	376.23	337.33	305.42
x2	674.13	658.15	642.85	628.18	563.26	509.76
x3	884.75	864.57	845.22	826.67	744.24	675.96
x4	586.51	574.36	562.68	551.45	501.15	458.96
x5	316.35	310.59	305.03	299.67	275.48	254.92

7.5 COMPARING THE OPTIMAL RETENTION RULE USING A TWO YEAR TIME HORIZON WITH THAT OF USING AN INFINITE TIME HORIZON

The following graphs demonstrate the comparison of the optimal retention rules of using a two-year time horizon with that of using an infinite time horizon. As the probability of an accident increases to close to 100% the optimal retention rule for each category in the BMS's converge mostly to each other's values. The optimal retention rule, where the probability of an accident is changed, using a two year time horizon often has a linear pattern whereas the optimal retention rule using the infinite time horizon has a non linear patters

7.5.1 GRAPH FOR BMS-1: X1 (CATEGORY 1)

X1 TWO VS INF

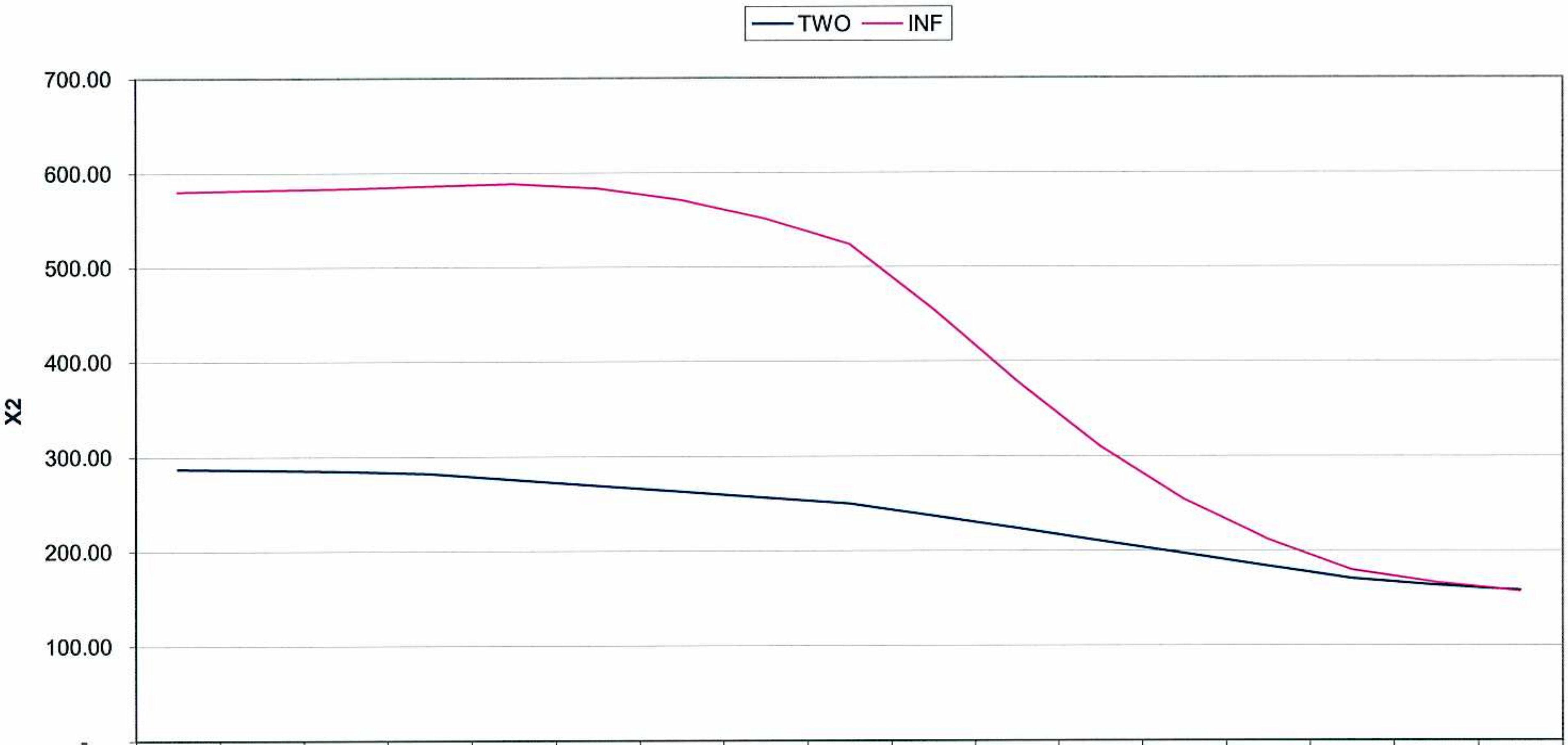


	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
TWO YEAR	143.62	142.96	142.30	140.98	137.68	134.37	131.06	127.73	124.40	117.73	111.02	104.29	97.53	90.75	83.94	80.52	77.78
INF	314.87	314.48	313.99	312.65	307.09	298.11	285.64	269.96	251.72	211.30	172.15	139.60	115.08	97.52	85.14	80.39	77.13

PROBABILITY OF AN ACCIDENT

7.5.2 GRAPH FOR BMS-1: X2 (CATEGORY 2)

X2 TWO VS INF

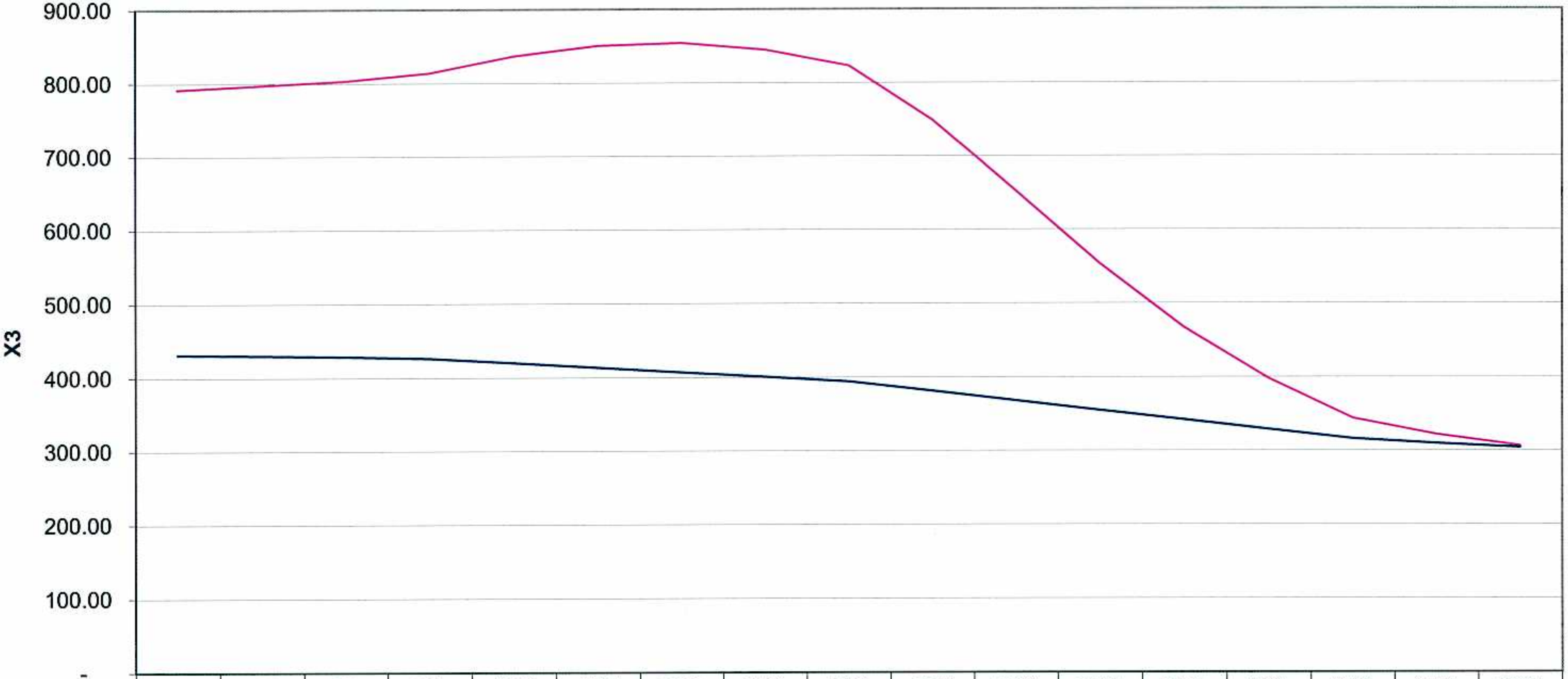


	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
TWO	287.34	286.04	284.74	282.15	275.66	269.15	262.63	256.10	249.55	236.43	223.25	210.03	196.76	183.45	170.08	163.39	158.02
INF	580.57	582.29	583.83	586.33	588.59	583.76	571.26	551.04	523.81	454.55	378.78	309.85	253.93	211.21	179.22	166.33	157.19

PROBABILITY OF AN ACCIDENT

7.5.3 GRAPH FOR BMS-1: X3 (CATEGORY 3)

X3 TWO VS INF

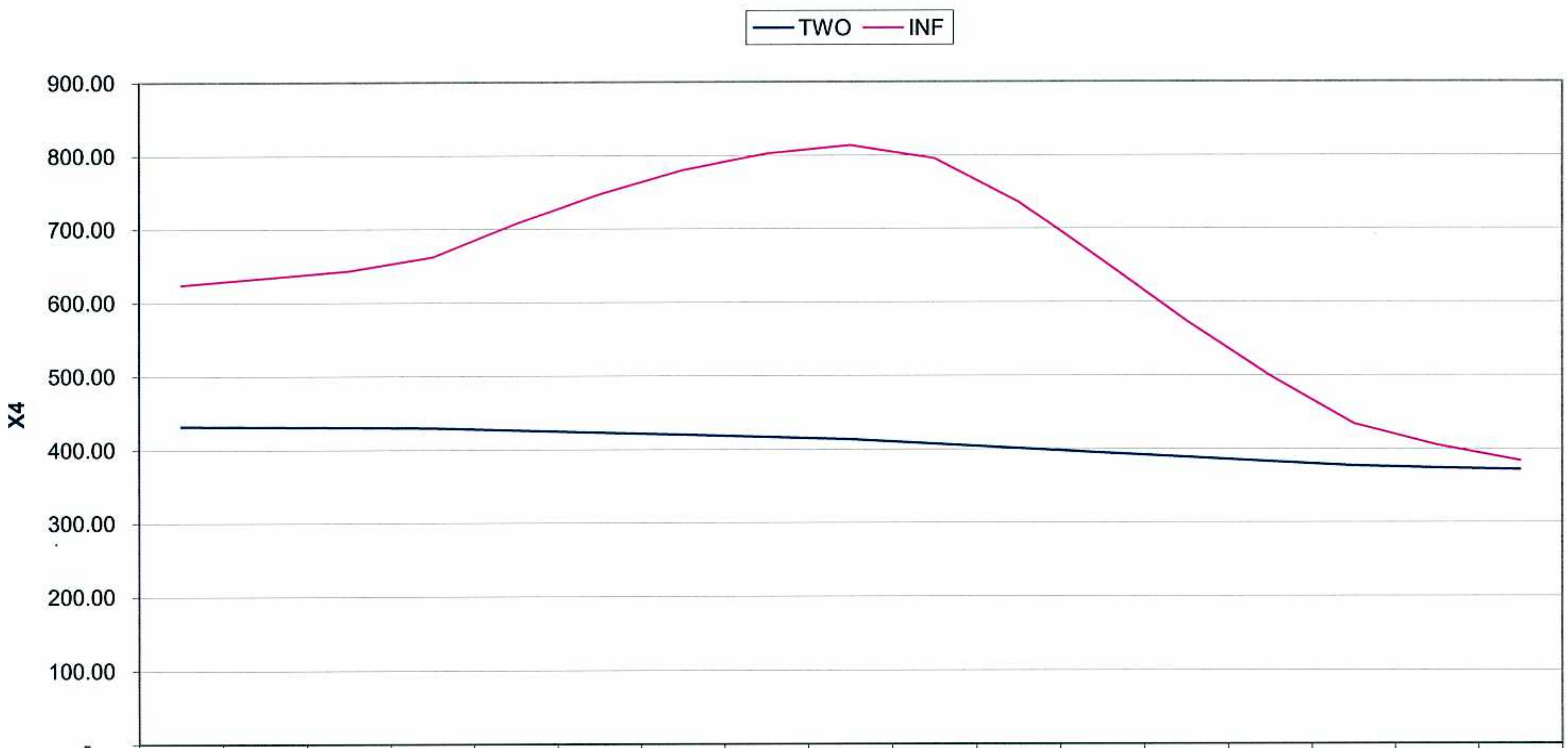


	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
TWO	431.63	430.34	429.04	426.45	419.97	413.48	406.99	400.50	394.00	380.98	367.95	354.89	341.83	328.74	315.64	309.08	303.83
INF	791.56	797.53	803.30	814.19	836.71	850.78	854.45	844.90	823.12	749.07	651.95	553.45	467.18	397.51	343.54	321.62	306.21

PROBABILITY OF AN ACCIDENT

7.5.4 GRAPH FOR BMS-1: X4 (CATEGORY 4)

X4 TWO VS INF

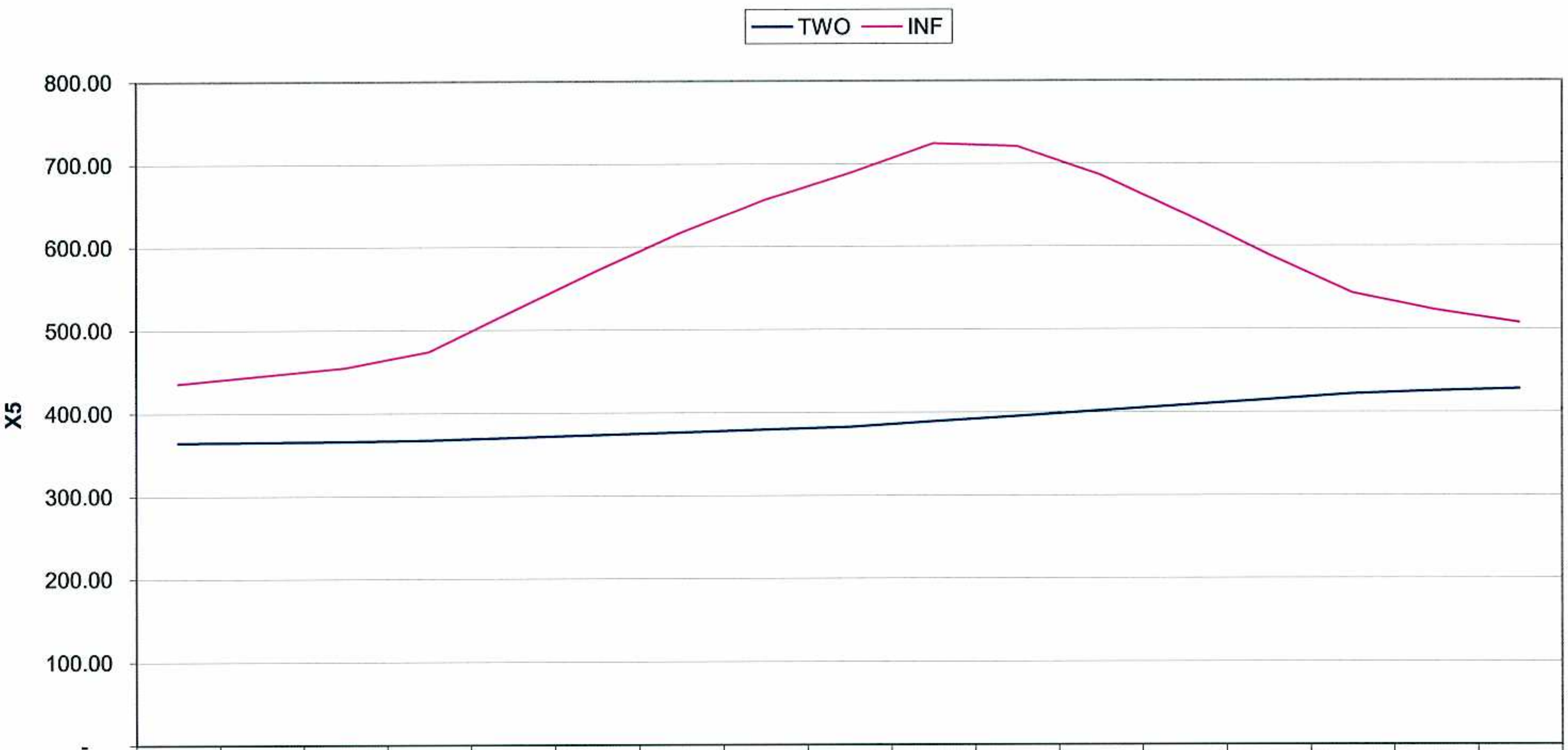


	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
TWO	432.19	431.54	430.89	429.59	426.36	423.14	419.94	416.75	413.58	407.29	401.06	394.90	388.81	382.78	376.82	373.86	371.51
INF	624.06	633.64	643.19	662.10	707.38	747.63	780.18	802.74	813.70	795.54	735.79	656.16	574.07	499.00	433.95	405.11	383.62

PROBABILITY OF AN ACCIDENT

7.5.5 GRAPH FOR BMS-1: X5 (CATEGORY 5)

X5 TWO VS INF

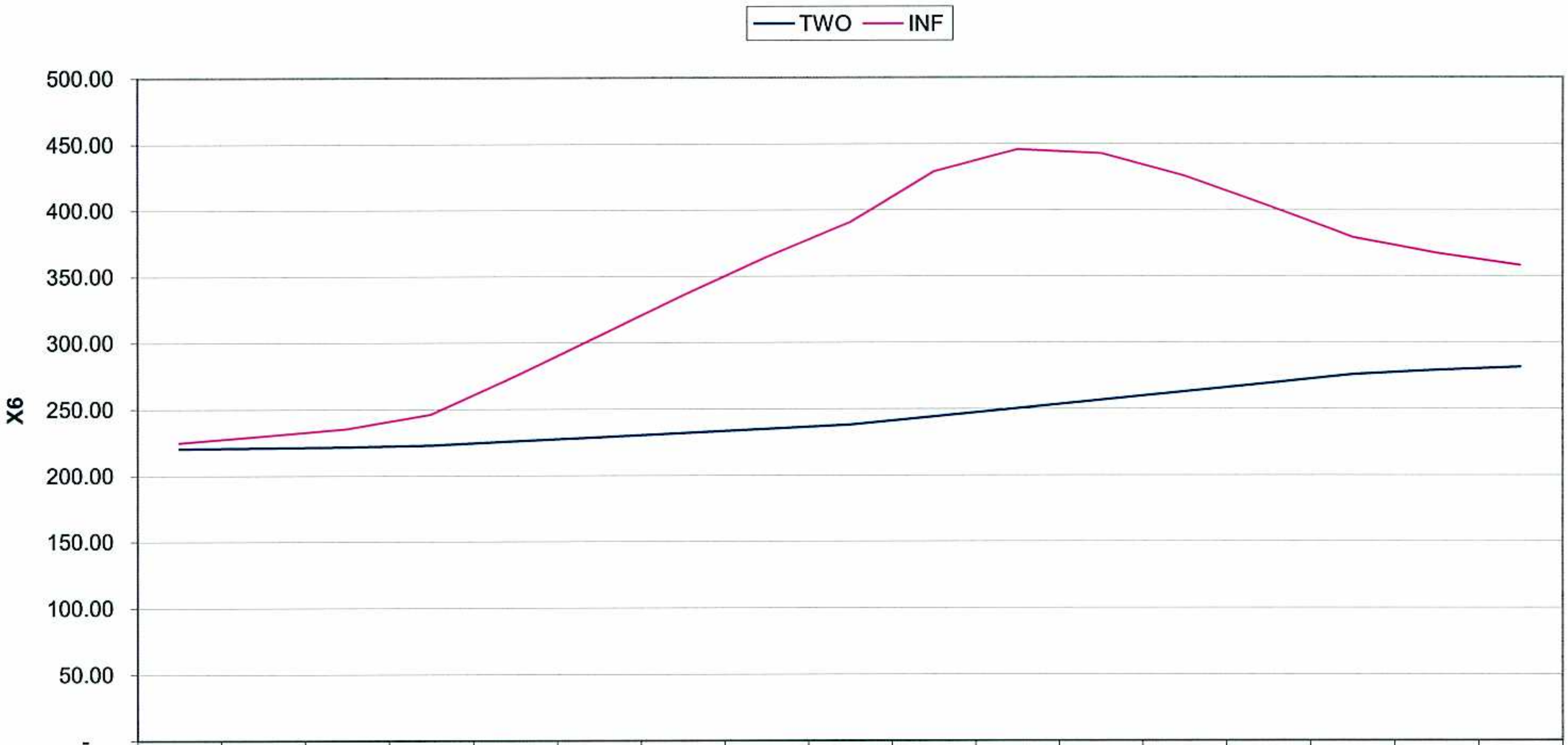


	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
TWO	364.65	365.25	365.85	367.04	370.05	373.08	376.13	379.21	382.31	388.59	394.96	401.43	408.00	414.67	421.44	424.86	427.61
INF	435.75	445.11	454.58	473.80	522.78	571.25	616.72	656.64	688.82	724.70	721.01	686.22	638.70	589.16	543.29	522.54	507.13

PROBABILITY OF AN ACCIDENT

7.5.6 GRAPH FOR BMS-2: X6 (CATEGORY 6)

X6 TWO VS INF

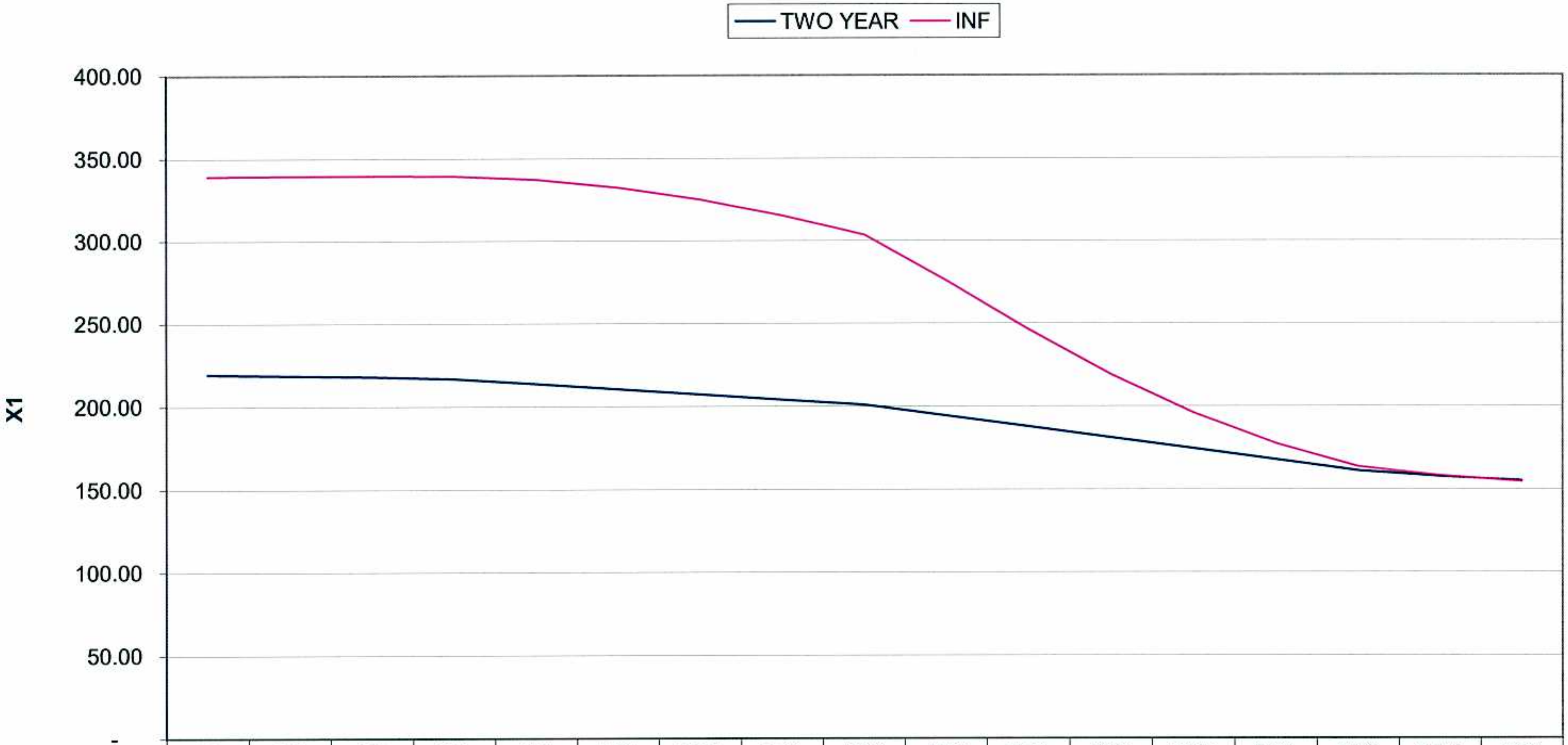


	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
TWO	220.36	220.95	221.55	222.74	225.73	228.74	231.76	234.80	237.84	243.99	250.19	256.44	262.75	269.12	275.54	278.77	281.37
INF	224.75	229.87	235.11	245.95	274.75	305.10	335.58	364.60	390.63	428.92	445.65	442.62	425.45	402.86	378.97	367.24	358.11

PROBABILITY OF AN ACCIDENT

7.5.7 GRAPH FOR BMS-2: X1 (CATEGORY 1)

X1 TWO VS INF

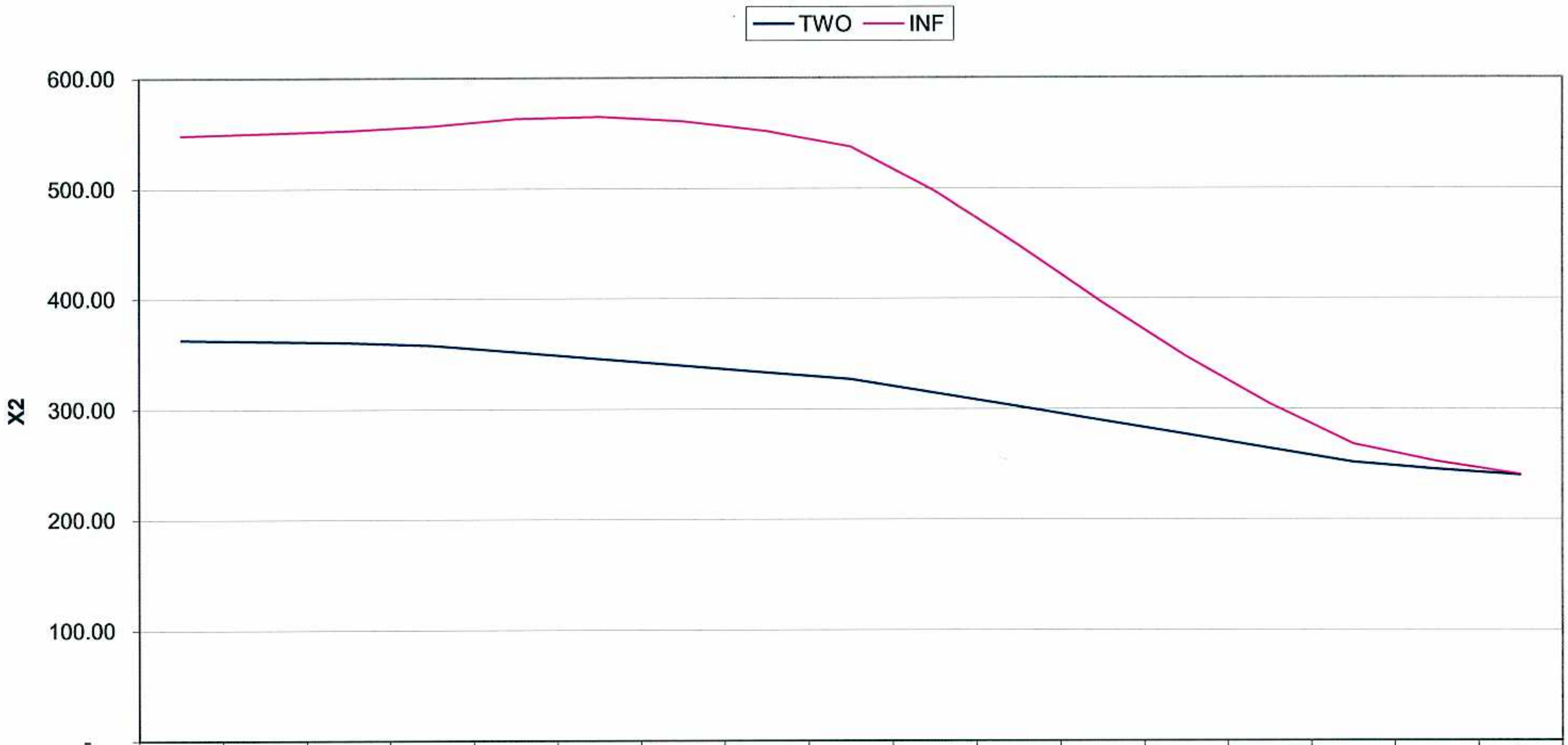


	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
TWO YEAR	219.29	218.66	218.02	216.75	213.55	210.34	207.13	203.90	200.67	194.16	187.62	181.03	174.41	167.74	161.04	157.68	154.98
INF	339.37	339.55	339.64	339.50	337.33	332.48	325.03	315.20	303.37	275.72	246.34	219.02	195.77	177.34	163.54	158.18	154.57

PROBABILITY OF AN ACCIDENT

7.5.8 GRAPH FOR BMS-2: X2 (CATEGORY 2)

X2 TWO VS INF

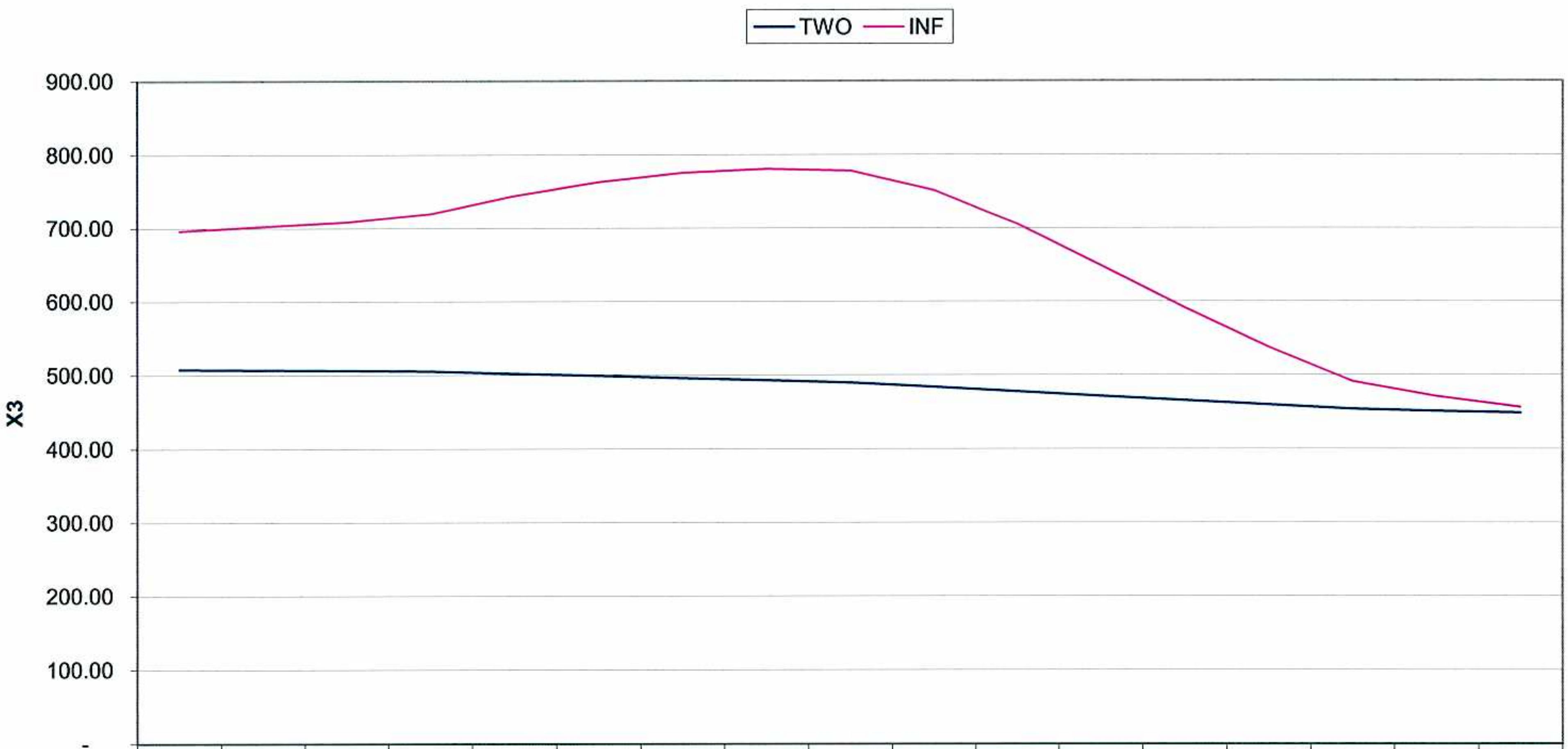


	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
TWO	362.98	361.73	360.49	358.00	351.77	345.53	339.29	333.04	326.79	314.26	301.72	289.15	276.57	263.97	251.34	245.03	239.97
INF	548.33	550.66	552.83	556.69	563.26	564.92	560.95	551.67	537.47	497.21	447.39	395.56	346.87	304.09	267.85	252.04	240.39

PROBABILITY OF AN ACCIDENT

7.5.9 GRAPH FOR BMS-2: X3 (CATEGORY 3)

X3 TWO VS INF

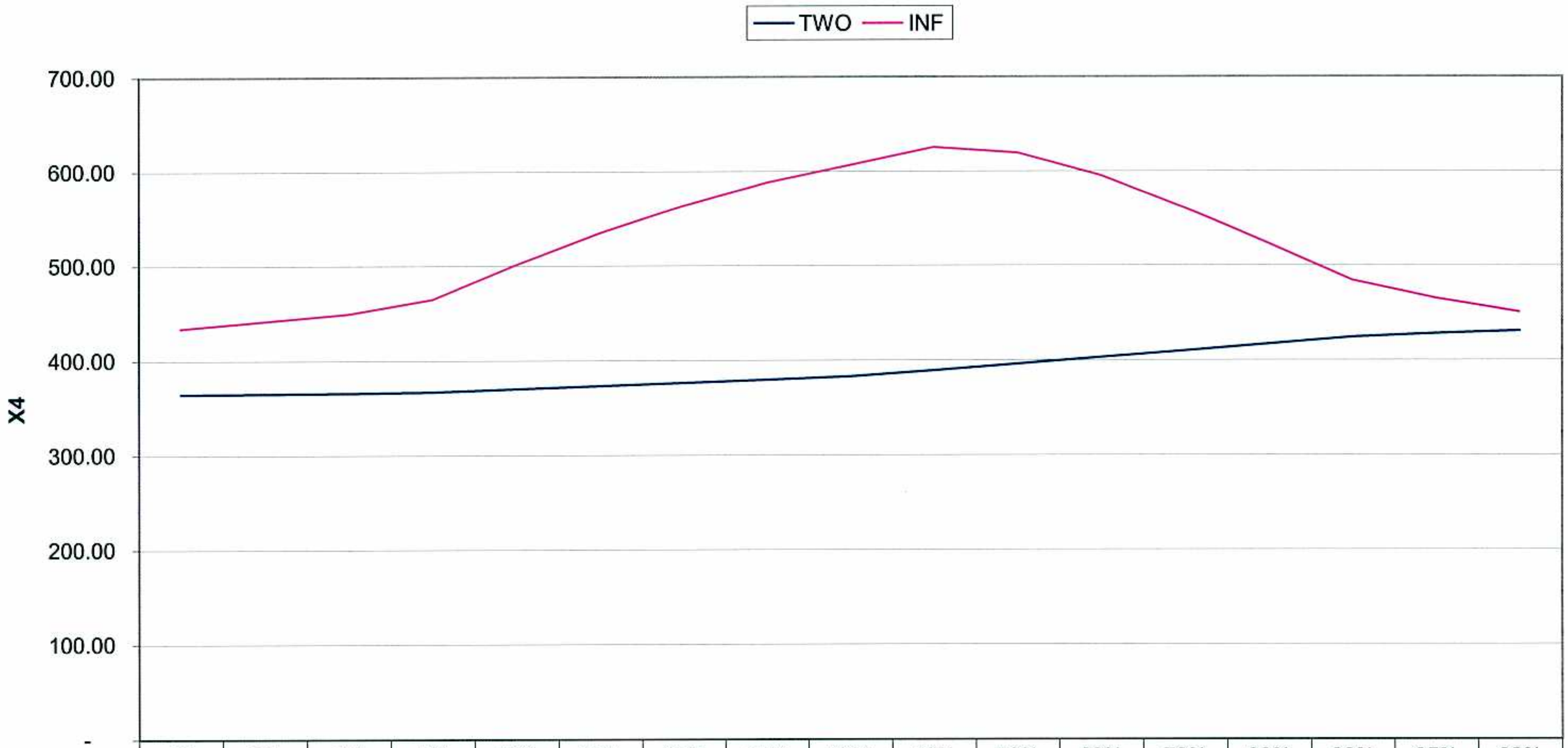


	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
TWO	507.78	507.15	506.51	505.25	502.09	498.95	495.82	492.70	489.60	483.43	477.32	471.27	465.28	459.34	453.47	450.55	448.23
INF	696.26	702.33	708.24	719.53	744.24	763.14	775.43	780.70	777.94	751.36	704.82	648.52	590.82	537.38	491.01	470.79	456.08

PROBABILITY OF AN ACCIDENT

7.5.10 GRAPH FOR BMS-2: X4 (CATEGORY 4)

X4 TWO VS INF

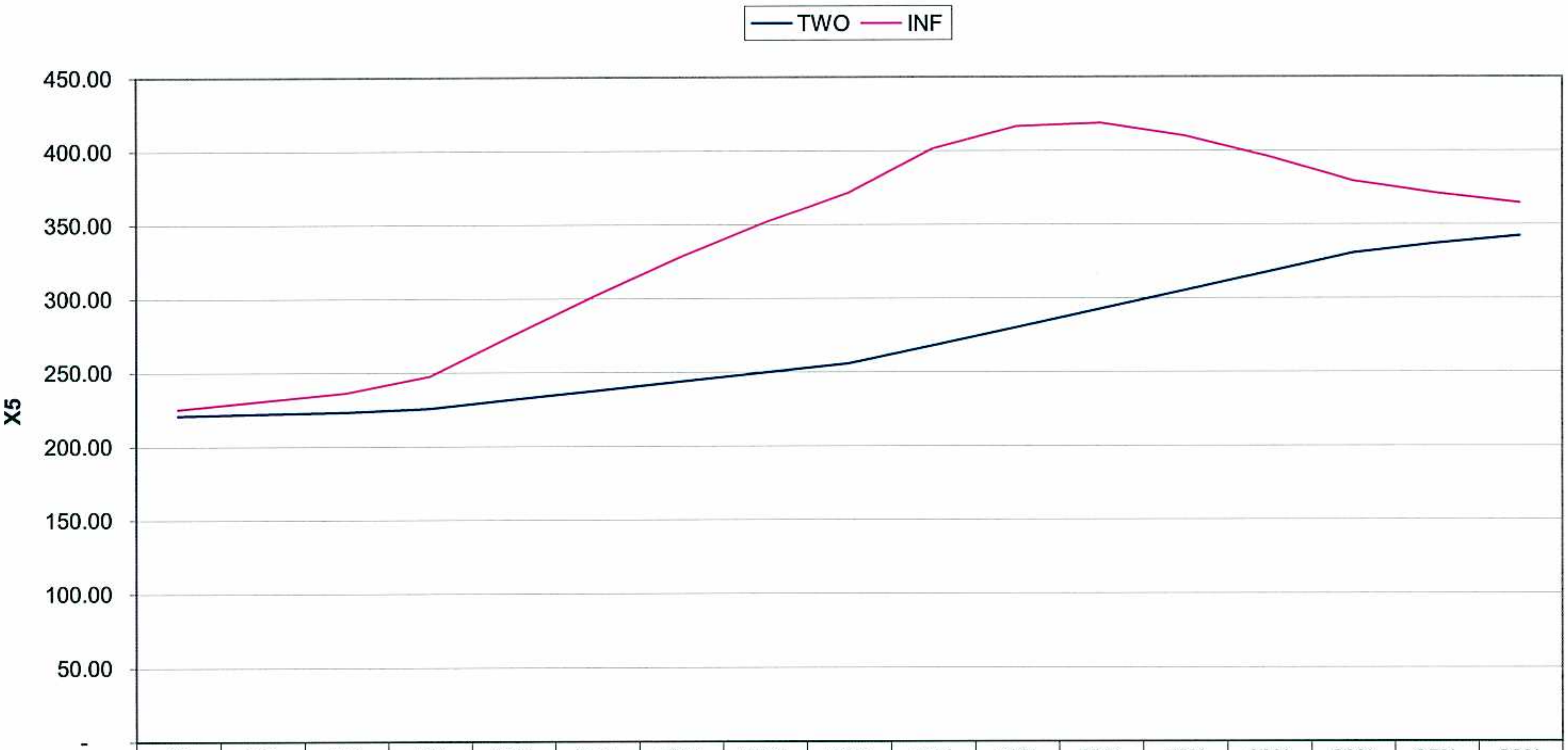


	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
TWO	364.67	365.27	365.88	367.11	370.19	373.31	376.46	379.64	382.86	389.39	396.05	402.85	409.78	416.85	424.06	427.71	430.66
INF	434.23	441.92	449.57	464.71	501.15	534.60	563.90	588.11	606.60	625.43	619.58	595.34	561.02	522.59	483.86	465.08	450.46

PROBABILITY OF AN ACCIDENT

7.5.11 GRAPH FOR BMS-2: X5 (CATEGORY 5)

X5 TWO VS INF



	1%	2%	3%	5%	10%	15%	20%	25%	30%	40%	50%	60%	70%	80%	90%	95%	99%
TWO	220.95	222.14	223.32	225.70	231.66	237.65	243.66	249.70	255.76	267.96	280.26	292.66	305.16	317.76	330.45	336.84	341.96
INF	225.28	230.82	236.38	247.56	275.48	302.66	328.26	351.50	371.72	401.53	416.62	418.73	409.92	395.84	379.55	371.23	364.64

PROBABILITY OF AN ACCIDENT

8. CONCLUSION

The optimal behaviour or rule of a policyholder for bonus-hunger is dependent on determining what the optimal retention vector or rule $\overline{x}^*$ is. This vector is dependent on the time horizon chosen.

Using an infinite time horizon:

The theory from the literature shows that one needs to solve 2s equations with 2s unknowns. Solving these equations produces a unique solution of the optimal retention rule $\overline{x}^* = (x_1^*, x_2^*, \ldots, x_s^*)$ and its associated cost vector $\overline{w}^*(\lambda)$. The vector $\overline{x}^*$ is particularly sensitive to changes in the probability of an accident and slightly sensitive to large changes in the interest rate. The graphs in the section 7 on modeling illustrate this. The optimal retention rule was hardly influenced by the choice of the claim size distribution.

Using a two-year time horizon:

This paper builds on the theory of the infinite horizon to derive the equations necessary to solve for the optimal retention rules using a two-year time horizon. The author shows that there are only s equations with s unknowns, and solving these equations by a computer iterative procedure produces the optimal retention rule vector $\overline{x}^{*2} = (x_1^{*2}, x_2^{*2}, \ldots, x_s^{*2})$. The effect of changing the interest rate has a very small effect on $\overline{x}^{*2}$. The optimal retention rules are sensitive to a change in the probability of an accident. The graphs in the section 7 on modeling illustrate this. The optimal retention rule was hardly influenced by the choice of the claim size distribution.

Determining $\overline{x}^*$ is naturally important to motor-vehicle insurance policyholders, as they can make better-informed financial decisions, especially when faced with a potential claim. With $\overline{x}^*$ known one can eliminate the rough gut-feel policyholders use when faced with a claim and need to determine whether they should claim or not. It may also be in the interest of general insurance companies to openly publish an approximation to $\overline{x}^*$, as small claims below $\overline{x}^*$ won't be submitted, which saves administration costs and time and also is in the financial interest of their policyholders.

Currently in South Africa there are no publicly available approximations to $\overline{x}^*$ for each operating general insurance company, and this paper shows that this needn't be the case. This paper also shows the method of determining the optimal retention vector or rule $\overline{x}^*$ using dynamic programming. This paper encourages South African general insurers to inform their motor-class policyholders about bonus-hunger and to find a sensible way to openly publish to the public or disclose to their policyholders the optimal retention rule $\overline{x}^*$ or a sensible range this vector $\overline{x}^*$ falls within. It is interesting to note that in South Africa some general insurers pay their policyholders between 10% and 25% of their premiums back after three to four claim free years and thus bonus-hunger is even stronger under those BMS's, which would mean that the optimal retention rule is higher in each class.

Knowing $\overline{x}^*$ is also useful when converting censored claim frequency and claim severity data into gross data, as Walhin & Paris (2000) did. Models can be fitted to the gross data, and can give the general insurance company insight into the nature of their business and claims, which they mayn't totally have had before. Knowing $\overline{x}^*$ will also clearly highlight to general insurance companies the policyholder's need for bonus-hunger and how they may think and behave when faced with an insurance claim.

In practically determining $\overline{x}^*$ and where need be $\overline{w}^*(\lambda)$ care needs to be taken that the dynamic programming employed is stable and consistent with the theory presented, and that all the parameters and steps the model takes to iteratively solve the set of simultaneous equations is correct. Furthermore the program needs to be debugged for various errors that may occur, i.e. like referencing to a wrong cell. The model ought to be well documented and user-friendly.

The output of the models should also be verifiable and reasonable. This can be done by:

- Comparing results with other internally created models (In this paper BMS-2 was a check on BMS-1)
- Applying common-sense
- Applying sound actuarial and business judgement
- Comparing results with business competitors, if available
- Consulting the relevant literature

The author of this paper found the modelling part to be very interesting and stimulating, but stresses that much care needs to be taken when doing it, as described above. The comparison of the optimal retention rule using an infinite time horizon to that of using a two-year time horizon clearly showed the influence of the time horizon on bonus-hunger and was an excellent check on the modeling. Lastly, what is also encouraging, is that there are researchers like Holtan (1994), who have tried to devise ways around the problem that bonus-hunger creates, but equally encouraging is that such new ideas are critically examined, like by Lemaire and Zi (1994).

9. LITERATURE REVIEW

Since the start of BMS in the late 1950s to date, the problem of bonus-hunger has been addressed in scientific literature. Furthermore BMS are still used today as prospective rating systems to charge insurance premiums. The main motivation for their use in motor vehicle insurance is that the best predictor in motor-vehicle insurance of the number of claims a policyholder will submit is not rating factors like driver-age, motor-vehicle specifications or location of residence but rather the driver's past claim record. A BMS is thus theoretically a simple, easy-to-understand and efficient system to manage a portfolio of policyholders by an insurer and by its nature incentivises a policyholder to not submit claims or to improve his or her driving so as to pay fewer premiums the following year and so on. The fact that the model of BMS is still in frequent use all over the world is proof to the compelling theoretical arguments for it. In practice it is found that a BMS does not discriminate adequately between good risks and poor risks. This is shown by the fact that in most BMS a policyholder (poor risk) with double the claim frequency of another policyholder (good risk) does not in the long term pay twice the insurance premium.

In South Africa in recent years there has been a refinement of the BMS system, where a policyholder receives after 3 or 4 years of claim free years, 10% to 25% of the premiums paid in those claim free years. Again the insurer is incentivising the policyholder not to claim. However with this refinement the desire for bonus hunger is increased and naturally will lead to an increase in the optimal retention rule.

However, not everybody is in agreement with the traditional BMS, where only the policyholders' past claim experience determines his future premiums. Holtan (2001a) proposes that with the modern marketing of a BMS that there is a real need in the pricing to take a person's individual characteristics into account. He thus proposes a twofold BMS, where one part is open and communicative system that the public can freely access and the other part a closed "black box" system which makes a price adjustment for each individual. The latter part, i.e. the closed "black box" system lends itself to subjectivity in the author's opinion, but the "black box" system can be refined with experience.

Furthermore, although in a traditional BMS, the problem of bonus-hunger will always arise, it is not a given that a traditional BMS will always be used. Already Holtan (2001a) has shown that one can develop a modern BMS and he also furthermore proposes a system (Holtan 1994) where the need for bonus-hunger can totally disappear. He does this by proposing that insurers change their system of pricing from a BMS to a simpler system where the insurer implements a high deductible financed by a short term loan from them on a policyholders claim. Hence claims below the deductible are not submitted and all policyholders suffer the deductible if they submit a claim above the deductible. However they are not penalized financially more than the deductible if they claim, nor is there any financial incentive not to claim (hence bonus-hunger disappears) if your claim is higher than the deductible. Holtan's idea of a high deductible didn't go unnoticed in the literature and Lemaire & Zi (1994) do criticize Holtan (1994) method, by showing that a high deductible would increase both the variability of payments and the efficiency of the rating system for most policyholders, but praise Holtan for an ingenious idea.

Frangos & Vrontos (2001) also have an alternative to the traditional BMS. They designed an optimal BMS, which takes the claim frequency, claim severity and individual characteristic of the policyholder into account. Here optimality refers to minimizing the insurer's risk. Their approach decreases the desire for bonus-hunger, as the severity of the claim is also taken into account. In their system it also pays to make small claims, which is generally discouraged in traditional BMS.

When however a traditional BMS is used or a refinement, which doesn't change the nature of the BMS that the insurer uses, then bonus hunger arises. The past literature shows that several authors approached this problem from the nineteen sixties to date. A common thread in all the past literature is that all the authors used dynamic programming to solve the system of simultaneous equations so as to determine the optimal retention rule. In the vast majority of the cases the authors assumed that a policyholder makes the decision to claim or not to claim at the beginning of the year or end of the year of insurance, but De Pril (1979) develops a continuous time model where a policyholder can immediately decide whether or not to submit a claim following an accident. However Lemaire (1998) shows that the optimal retention rule is not changes much if one assumes that a policyholder has an accident at the beginning of a year of cover and hence makes the decision to claim or not to claim at the beginning of the year of insurance cover.

Some authors used credibility theory to derive the optimal retention rule. Morlock (1984) also shows that in the special case that claims follow the negative exponential distribution that the optimal retention rules are given by the solution of some Bernoulli differential equations. Some authors just confined themselves to the negative exponential distribution for the claim distribution for simplicities sake and this research paper found that choice of claim size distribution had mostly minimal effect on the optimal retention rule.

The past literature also shows that both finite and infinite time horizons were considered. De Pril & Goovaerts (1983) give boundary limits for the optimal retention rules. Walhin & Paris (2000) determine the optimal retention rules so as to convert censored claim frequency and claim severity date into gross data. Thus bonus-hunger cannot be ignored in practice when studying claim severity and claim frequency statistics.

This paper found Bertekas as an authority on dynamic programming (dynamic programming provides a set of general methods for making sequential, interrelated decisions under uncertainty).

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11. APPENDIX

SANTAM

Quotation number	: 74717544571	Inception date	: 15/03/2004
Policy Type	: 747	Renewal date	: 01/04/2005
Branch	: 037	Other insured	: N
Agnt/Insp	: 0050/000000	ID number	: 7109225034083
Client	: NJ Strohmenger, Mr	Telephone	:
Postal address	: RANDBURG 2194		

	NCB	Insrd Amt	Premium
Vehicle 001 2004 VOLKSWAGEN POLO 1.4	0	105,210	1,246.99

VIN : Unknown
Engine Number : Unknown
Comprehensive (Private)
Voluntary excess 0
Safety measures : Immobiliser

Excesses	Claim %	Value	Min	Max
Compulsory	5	1500	0	0
Other	5	1500	0	0
Under 25	0	1000	0	0
Theft/hi-jacking	10	3000	0	0
Window	20	350	0	0
Specified articles	0	100	0	0

Legal Access	Limit of Premium indemnity
Policy premium	1,246.99
Sasria (Sasria provides cover against public disorder)	1.80
Monthly payment for policy	R 1,248.79

Endorsements

046 GENERAL EXCLUSIONS
047 SHARING OF INSURANCE INFORMATION
074 GENERAL EXCLUSION RELATING TO NUCLEAR LOSSES

(For detailed wording of the above endorsement(s) please refer to the schedule.)

This is for quotation purposes only and valid for 30 days from 15/03/2004
Quotation is subject to final approval upon accepting of Quotation.

Version 8.3.1601
() Marianna van Zyl
Date when policy summary printout was made: 16 March 2004

Neville Strohmenger

From: quote@santam.co.za
To: strohmengern@nbc.co.za
Subject: Santam Multiplex Personal

SANTAM.TXT

Premiums on different n c b's are as follows: On 1 year R1165.66. On 2 years R1082.53. On 3 years R999.30. On 4 years R916.26 and on 5 years R833.13. One claim free year earns 1 ncb and 1 claim will result in 2 years lost.

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