Nonlinear elastic waves in materials described by a subclass of implicit constitutive equations

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Offered unto the divine Lotus Feet of the Lord

Abstract

The propagation of displacement and stress waves for a subclass of implicit constitutive equations in a rectangular slab and a circular cylinder is investigated. The general class of implicit constitutive equations contain Cauchy elastic materials and hyperelastic materials as subclasses. We consider a special subclass of implicit constitutive equations where the strain is prescribed in terms of a non-invertible function of the stress. Two constitutive equations are studied. The first constitutive equation is called the power-law constitutive equation due to its analogy with the constitutive equation for a power-law fluid with exponent n in the expression for stress. This constitutive equation can describe elastic responses where the stress and linearised strain are nonlinearly related. Classical Cauchy elasticity and hyperelasticity cannot capture such a phenomenon. The second constitutive equation is called the strain-limiting constitutive equation. A feature of this constitutive equation is that it can describe materials that exhibit limiting stretch. To derive the mathematical models we assume a special semi-inverse solution where a specific form for both the displacement and stress are sought. This assumption leads to a system of nonlinear partial differential equations. The system of partial differential equations can be reduced to a single nonlinear hyperbolic partial differential equation which describes the propagation of solitary stress waves. The perturbation solutions for the system of partial differential equations describes either travelling waves or standing waves. To find travelling wave solutions for the displacement and stress in a rectangular slab we reduce the perturbation equations at each order to canonical form and solve the resulting wave equations. For the circular cylinder we could not obtain travelling wave solutions by reduction to canonical form. We find standing wave solutions for the displacement and stress in both the rectangular slab and circular cylinder. For the rectangular slab the solutions to the perturbation equations contain a secular term. However, the straightforward perturbation expansion breaks down outside the range of interest. The standing wave solution in the circular cylinder can only be solved at the zero and first order since the equations at the second order could not be solved analytically. The solutions at this order are however, sufficient to describe the physical properties of the wave. In the standing wave solutions for the displacement and the stress at each end, at the centre and surface of the cylinder, either the displacement or the stress vanish or the spatial gradients of the displacement or stress vanish. We find expressions for the speed of the solitary stress wave for both constitutive equations in both the rectangular slab and circular cylinder. The speed of propagation decreases in parts of the wave for large stress magnitudes for the power-law constitutive equation and increases in parts of the wave for large stress magnitudes for the strain-limiting constitutive equation. The solitary stress wave develops a shock front at the front of the wave for the strain-limiting constitutive equation and at the back of the wave for the power-law constitutive equation. The shock develops at the front for the strain-limiting constitutive equation at a much earlier time than at the back for the power-law constitutive equation. For the travelling wave solutions in the rectangular slab the wave front is determined from the condition that the displacement at the wave front is zero. The stress is non-zero at the wave front and propagates as a shock wave with a strong discontinuity. We find that the speed of propagation of the displacement wave and the stress wave is slower for the power-law exponent $n > 0$. Further the amplitude of the displacement waves are approximately the same for $n = 0$ and $n > 0$ while the amplitude of the stress wave is less for $n > 0$. The standing waves for both constitutive equations in both the rectangular slab and circular cylinder showed that for the power-law constitutive equation the period of oscillation remained approximately the same for $n = 0$ and $n > 0$ while it increased for $n > 0$ for the strain-limiting constitutive equation. In general the elastic response is enhanced for materials described by the power-law constitutive equation and inhibited when described by the strain-limiting constitutive equation.

Declaration

I declare that this is my own, unaided work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature. It is being submitted for the Degree of Doctor of Philosophy to the University of the Witwatersrand. It has not been submitted before for any degree or examination to any other University.

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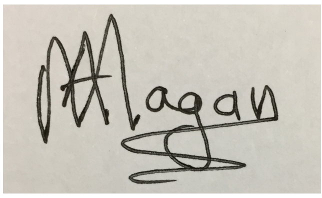

Avnish Bhowan Magan

Dedication

To my wonderful parents

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Om Namah Shivaya

I would firstly like to thank the Almighty for giving me the opportunity to pursue a Ph.D and for granting me the strength to persevere through the last three difficult years.

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Chapter 1

General introduction

1.1 Introduction

In this Chapter we discuss the applications and importance of solid mechanics and provide a brief overview of the history of the development of solid mechanics. This thesis is concerned with a new class of constitutive equations for elastic materials which are described by implicit relationships between the stress and strain. We provide a review of previously conducted research for this new class of constitutive equations.

The Chapter is organised as follows. In Section 1.2 we discuss the applications of solid mechanics. A brief history of the development of solid mechanics is given is Section 1.3 and the literature review is presented in Section 1.4. Finally, the objectives and outline of the thesis are given in Sections 1.5 and 1.6 respectively.

1.2 Applications of solid mechanics

Solid mechanics is a branch of continuum mechanics and is a field of study which has attracted attention for centuries and continues to do so. Distinguishing between a solid and a fluid has posed some difficulty. The accepted dissemblance between a solid and fluid lies primarily in whether the material under study can withstand a large shearing force over time or not. Fluids are, in a idealised situation, characterized as substances which cannot support this shearing force over long periods of time whereas materials which can are classified as solids.

When forces or torques are applied to solid bodies, these bodies will undergo a deformation. In order to study the deformations in greater detail, the concept of an elastic body needs to be defined. The most common definition of an elastic body is that when *a solid material is loaded at a sufficiently low temperature or short time scale, and with sufficiently limited stress magnitude, its deformation is fully recovered upon unloading* [1]. However, there are materials which can deform permanently. This type of deformation is called a plastic deformation or elastic-plastic deformation. This deformation results from subjecting the material to large stresses that exceed the yield stress. Viscous or creep deformation occurs when the deformation depends on the material's exposure time to the applied stress which is still below the yield stress. There are many materials which exhibit both 'solid like' and 'fluid like' properties and these materials are developed within the context of viscoelasticity. There are other materials like mud and gels which have properties common to viscoelastcity and plasticity and these materials are called viscoplastic materials.

The applications of solid mechanics are far reaching. Engineering and industry are heavily reliant on the phenomena associated with the deformations of elastic bodies. This theory is useful in studying natural phenomena as well. Seismologists are deeply interested in the propagation of seismic waves though the Earth's crust. Seismologists were able to conclude that the Earth possesses a liquid core since shear waves do not propagate through a fluid medium. The applications of finite elastic deformations are useful in the mining sector where understanding the stress behaviour at crack tips and the speed of propagation of these cracks is of importance. Another application which is gaining substantial popularity is modelling biological phenomena in terms of nonlinear elasticity. For example, the growth of a malignant and benign tumour can be modelled using thick-shell theory [2]. These varied applications have advocated a deeper study of finite elastic deformations and as such the mathematical rigour of its theory cannot be underestimated.

1.3 A brief historical review

Many great scientists have been fascinated with the mechanical behaviour of solid bodies. The roots of elasticity theory can be traced backed to Leonardo da Vinci. He was interested in understanding the tensile stress of a wire. However, the subject gained momentum with the advent of Newton's Laws of Motion. Gallileo introduced the first intuitive idea of stress. During his investigation of the behaviour of rods under tension he concluded that the load was independent of the length and proportional to the cross-sectional area [1].

The idea of a linear relationship between force and displacement was first developed by Robert Hooke in 1660. He found that the elongation of a material is directly proportional to the tensile or compression force however, it was Jakob Bernoulli who was the first to express the deformation (strain) as the force per unit area (stress). The contribution by the Bernoulli family and Leonard Euler during this period was immense. The concept of an internal tension at a given cross section along the length of a beam was one of their achievements [1].

During the early 1800's one of the most significant contributions to the field of elasticity was presented by Augustin-Louis Cauchy. He developed a framework within which the concept of stress could be generalised to three dimensions. He derived the equations of motion for a continuum and presented the components of strain in terms of the derivatives of the displacement. A key part of this contribution was that the equations transformed as tensors and as such were valid in any coordinate system. This opened up avenues for studying geometries not only restricted to the rectangular coordinate system, but also to more general coordinate systems.

The theory of beams was greatly advanced by Euler and the Bernoulli family during the 1800's. Due to its applications in structural technology, the appeal of their work became increasingly popular. The buckling of an initially straight beam under compressive loading, transverse vibrations in a beam and the proposition that the curvature of a beam was proportional to its bending moment were some of their important contributions. The theory of elasticity was further advanced when Euler introduced the concept of a strain energy per unit length and considered this function to be analogous to the potential energy of a discrete mechanical system [1].

The next phase of the development of the theory of elasticity occurred during the late 1800's where solutions to problems arising in real world applications were found. The two-dimensional theory of elasticity was modelled in the context of a continuum and not in terms of a simple particle model proposed by Navier as this model was considered to be too simple. The solution of vibrations of cylindrical and spherical geometries was of interest to seismologists who wanted to understand the vibrations of the Earth. Further, in 1863, Lord Kelvin considered the problem of a solid sphere where the solutions have applications in calculating the deformation of the Earth due to rotation and tidal waves.

The developments mentioned above fall under the umbrella of linear elasticity, where the nonlinear terms in the strain tensor are neglected. The idea of finite deformations, at this time, did not attract much attention due to the fact that most of the materials used in structural production displayed brittle failure and permanent deformation for small stresses. However, materials such as natural rubber allow for finite straining and are used extensively in the polymeric industry. It was the explosion of this industry that initiated the study of finite elastic deformations. Green, Piola and Kirchoff developed their ideas under the classical framework but a concise theory for the rubbery materials which exhibit finite straining was only developed recently by Ronald S. Rivlin during the 1940's and 1950's [1]. This theory included the development for compressible and incompressible isotropic hyperelastic materials. Popular constitutive equations, like the new-Hookean and Blatz-Ko constitutive equations, were derived from the experimental work of Rivlin and Saunders in 1951. Many important contributions have come out of the theory of nonlinear elasticity, in particular the work by Ericksen on controllable deformations for isotropic elastic materials and Truesdell's problem concerning the analytical restrictions imposed on constitutive equations.

Wave motion arises naturally out of the equations of motion for the theory of elasticity, as was shown by Cauchy, Poisson and Stokes. Two types of waves were found. The first type of wave, longitudinal or irrotational waves, are waves which propagate through an isotropic medium much faster than the second type of wave which is called a transverse or shear wave. The former wave type has the property that the particle motion propagates in the same direction as the wave whereas with the latter, the particle motion is perpendicular to wave motion. The Rayleigh wave is a special type of wave which propagates along the surface of an elastic body but decays exponentially with distance into the medium.

Further important contributions to the field of finite elasticity involved solving the stress distributions around circular and elliptic holes with the conclusion that the stress concentrations at the boundary was large in magnitude and becomes greater as the radius of curvature decreases in relation to the size of the hole. The contributions highlighted above merely serve to provide some context into the importance and effectiveness of using the theory of elasticity to model a plethora of natural and industrial phenomena.

1.4 Literature review

In order to describe the elastic response of a body, we need to firstly understand what is meant by a body being elastic and secondly to provide a mathematical framework which can be built around this definition. Different definitions of elasticity would lead to different mathematical models and so the definition that is provided is important.

The term elasticity, when being applied to the field of continuum mechanics has been used inconsistently for centuries [3]. It was initially attributed to perfect fluids and gases. The Oxford dictionary defines elasticity as *a material* *substance, solid, liquid or gaseous: that spontaneously resumes (after a longer or shorter interval) its normal bulk or shape after being contracted, dilated or distorted by an external force* [4]. Rajagopal [3] provides a detailed account of how the definition of elasticity has evolved over time and how the pioneers of the field have used this word in the past. Since there are many different interpretations of the term elasticity among scientists, providing a clear definition of what the term would mean in the context of continuum mechanics is important. The definition provided by Rajagopal [3] has allowed for a far larger class of elastic materials to be modelled. He defines the term in the following manner: *The body under consideration is said to respond like an elastic body if within an agreeable time scale of observation it does not dissipate any energy in any process to which it is subject to.* Consequently, we can see from this definition that the ability of a body to resume its normal shape after the applied loads have been removed is not the *only* characteristic by which we can define an elastic body. Further these are bodies which are incapable of dissipation. This means that the body is incapable of converting mechanical work into thermal energy. It is this restriction on the body which allows for thermodynamic consistency and this point cannot be overemphasised.

The development of linear and nonlinear elasticity has centred on Cauchy elasticity and on Green elasticity also referred to as hyperelasticity. Cauchy elastic bodies are bodies where the stress in the current configuration is determined by the state of deformation in the current configuration. In other words, to describe the state of stress we would only need to know the extent of the deformation. Since the stress is independent of the rate at which the material body deforms, it does not mean that such information is unnecessary. Rather, for a particular class of processes, the stress is independent of how fast a body deforms [5]. For Cauchy elastic materials the stress is given explicitly in terms of the deformation gradient. Hyperelastic materials form a subclass of Cauchy elastic materials. These elastic bodies are bodies where the stress is derived from a potential or stored energy associated with the body. Hyperelastic bodies have been successful in describing the elastic response of many rubber-like materials. A primary assumption that we make is that the stress and stored energy depend explicitly on the deformation gradient. In other words, as long as the body is deformed *sufficiently* slowly, the stress in the material is captured by the deformation [5]. We will later see that this is merely an idealisation and the explicit relationship between the stress and deformation gradient in the stored energy is but a subclass of a far larger class of elastic materials.

The credence of Cauchy elastic materials as being physically reasonable has been a topic of debate [3]. Rivlin [6] found that if the stress is not derived from a potential the material becomes a source of infinite energy. That is, the energy released during the unloading of an elastic material exceeds the work done during the loading process. If we consider an elastic body to be a body which does not dissipate energy during a closed cycle of deformation, then this negative work in closed cycles of deformation would produce a perpetual motion machine which is physically unreasonable. Carroll [7] proves this with a suitable example. Further, the absence of a strain-energy function has led many to question the validity of Cauchy elasticity within a thermodynamic framework. This leads us to the conclusion that for Cauchy elastic materials to be physically reasonable they *must* be hyperelastic.

As mentioned above, for Cauchy elastic and hyperelastic bodies the stress and stored energy depend explicitly on the deformation gradient. This is a very special assumption and restricts the study of elastic response to these two classes of materials. Rajagopal [3, 4, 5] has questioned whether this is the only way in which one may express the stress. An elastic material has been defined as one which is incapable of dissipating energy in whatever motion they undergo. There is a valid inquiry into whether one could have materials which are thermodynamically consistent, (incapable of dissipation), where the stress and deformation gradient are related implicitly or whether one could have a potential that depends implicitly on both the stress and the deformation gradient. As a result of this inquiry a larger class of elastic materials which are described by implicit constitutive relations has been developed [4, 5]. Further, this theory is thermodynamically consistent [8]. This framework, where elastic materials are incapable of dissipation are described by a more general class of constitutive equations and contain the classical Cauchy elastic materials and hyperelastic materials as special subclasses. Implicit theory allows for a wider class of elastic materials to be modelled and the significance of this theory is evident due to two primary avenues of interest. The first pertains to constitutive representation and the second relates the possibility of multiple solutions for the equations of motion for each constitutive equation [5]. Further work carried out on the development of implicit constitutive equations can be found in [9]

The theory of implicit constitutive equations is not new. Stokes discussed how the viscosity of a fluid may depend implicitly on the normal stress [10]. Suppose for example that we are considering an incompressible fluid. The pressure is related to the normal stress as $p = 1/3(\mathrm{tr}(\mathbf{T}))$ where $\mathbf{T}$ is the normal stress, but the viscosity depends on the pressure. This means that the viscosity is implicitly dependent on the stress thereby establishing an implicit relationship between the viscosity and stress. Further, we can trace implicit models back to the work of Maxwell for viscoelastic materials [11] and to Prandtl for the inelastic behaviour of solids [12]. For many fluid models such as the Euler and Navier-Stokes models the stress is given explicitly in terms of the kinematical variables and similarly for the classical solid models such as the linearised elastic and Neo-Hookean models. However, the investigation of implicit constitutive equations for solid materials has only recently gained attention.

Within this theory the stress and stored energy is not expressed explicitly in terms of the deformation gradient but rather these quantities, for some subclasses, are related implicitly and there are valid physical models that can be best described by such a representation. Rajagopal [3, 4, 5] provides examples where the stress and deformation gradient are implicitly related and where the stored energy depends implicitly on the stress and deformation gradient. Another example where the stored energy depends implicitly on the Piola stress and Green St. Venant strain in the context of a three dimensional thermodynamic framework was considered by Rajagopal and Srinavasa [8, 9]. Further, to validate thermodynamic consistency we have to ensure that a potential exists in order for a stress to be derived. Since these materials are incapable of dissipation, the second law of thermodynamics holds which means that a potential exists. However, in this theory, even though a potential exists the stress does not necessarily need to be derived from that potential. One can define a class of elastic bodies in terms of a Gibbs potential where the stored energy plays an insignificant role.

A detailed discussion on frame-indifference and isotropy of implicit constitutive equations can be found in [5] and so we will only provide a brief overview of these concepts. In order to study invariance properties of a material we would need to consider the invariant requirements on the constitutive quantities and balance laws [5]. Further, these requirements will be different in different domains of the state space. Rajagopal [5] argues that one can obtain constitutive equations for the stress in elasticity without the requirement of material frame-indifference and that Galilean invariance is sufficient. To justify this, consider the constitutive equation for a viscous fluid; the constitutive theories being developed within the framework of frame-indifference. However, if one were to consider all models as limits, then one can arrive at the Navier-Stokes models as the limit of a viscoelastic fluid under Galilean invariance. Another class of materials where Galilean invariance would be sufficient are models where the body could have multiple natural stress free configurations. Usually, we have an infinite set of configurations to describe the motion of the body, but for bodies with multiple stress free configurations there is a minimal set of configurations to describe elastic response [5]. For these types of materials it is necessary to have information about the kinematical quantities from a variety of configurations to describe the stress. In classical elasticity we only have one stress free configuration. Examples of materials with multiple stress free configurations and a study of their elastic response has been considered in [13]-[27]. Finally, in order to describe implicit theory completely the notion of isotropy needs to be discussed. The notion of isotropy is defined in terms of the symmetry restrictions of the material and are connected with certain implicit relationships. A more detailed explanation can be found in [5].

The assumptions made in linear elasticity are useful in describing materials in which the stress and strain are linearly related. However, there are numerous instances in literature where there is a nonlinear relationship between the stress and linearised strain. Such models do exist, however there are inconsistencies when such models are developed within the context of Cauchy elastic or hyperelastic bodies. Consider the Cauchy stress $\boldsymbol{\sigma}$ to be explicitly defined in terms of the Green St.Venant strain tensor, $\mathbf{E}$, for example. Thus the stress can be expressed as

$$\boldsymbol{\sigma} = \mathbf{Q} f(\mathbf{E}) \mathbf{Q}^T \tag{1.4.1}$$

where $\mathbf{Q}$ is an orthogonal tensor and f is a response function and $\mathbf{Q}$ is an orthogonal tensor which we find in the Polar Decomposition for the deformation gradient $\mathbf{F}$. Linearising (1.4.1) by appealing to small displacement gradients leads us to

$$\boldsymbol{\sigma} = \mathcal{L}\boldsymbol{\epsilon}, \tag{1.4.2}$$

where $\boldsymbol{\epsilon}$ is the linearised strain tensor and $\mathcal{L}$ is a fourth order strain-tensor. If we assume that $\mathcal{L}$ is invertible then $\boldsymbol{\epsilon} = \mathcal{L}^{-1}\boldsymbol{\sigma}$ is still linear. The nonlinear models, $\boldsymbol{\sigma} = F(\boldsymbol{\epsilon})$, found in literature are inconsistent in that when approximating (1.4.1) the nonlinear terms in $\boldsymbol{\epsilon}$ are retained, i.e. powers of $\left[\nabla_X u + (\nabla_X u)^T\right]$, while the nonlinearity in the Green St.Venant strain, $(\nabla_X u)^T$ and $(\nabla_X u)$ are ignored [4]. Rajagopal [5, 4] provides a consistent manner in which the linearisation of the displacement gradient will be done. This will lead to constitutive equations where the stress and linearised strain are nonlinearly related.

In this thesis we direct our attention to a special subclass of implicit constitutive equations where the strain is prescribed in terms of a non-invertible function of the stress. In general, constitutive equations of this form cannot be obtained by merely inverting the stress to obtain the strain as can be done for the class of Cauchy elastic materials. It will be shown in Chapter 3 that the general expression for the linearised strain in terms of the stretch tensor has the same form as that of a Cauchy elastic material in that the material moduli depend on the strain invariants. The difference is that the roles of the stretch tensor and the stress tensor are interchanged [3]. Elastic bodies described by these constitutive equations are neither Cauchy elastic nor hyperelastic.

The implications of this nonlinear relationship between the stress and the linearised strain has far-reaching applications. There is significant experimental work that is being carried out by Saito et al. [28], Talling et al. [29], Withey et al [30] and Zhang et al. [31] on titanium alloys and gum metal which clearly show a nonlinear relationship between the linearised strain and stress for small displacements. Such a phenomenon cannot be captured within the framework of classical Cauchy elasticity or hyperelasticity. Rajagopal [48] shows that a certain subclass of these implicit models can be used to describe the one-dimensional uniaxial experiment conducted on gum metal and titanium alloys by Saito et al. [28]. Criscione and Rajagopal [32] have also employed implicit theory to accurately describe the experimental work done by Penn on rubber [33].

Since there are numerous constitutive equations that one may work with, a larger class of elastic bodies may be studied. An interesting class of constitutive equations, one of which we will consider in this thesis, are equations which describe limiting strain responses [34]. Rajagopal [34] studied the extension, simple shear, torsion, circumferential shear and axial shear for a particular subclass of models exhibiting limiting strain. Such models have interesting applications in fracture mechanics and in particular the stresses and strains at crack-tips. Within the classical Cauchy elasticity it has been shown that the strain at the crack-tip grows like $\mathcal{O}(\frac{1}{\sqrt{r}})$ where r is the radial distance of the crack. That is, the strain becomes singular as the stress becomes very large. This contradicts the assumptions of small displacement gradients when linearising the constitutive equations. However, when the constitutive equations are described by implicit relations, for anti-plane strain problems it is found that the strain at the crack tip is bounded [35]. In [49] the response of brittle elastic materials is studied, in particular the anti-plane stress state of a plate with a V-notch for implicit constitutive equations and it is shown that in a neighbourhood of cracks and notches the strains remain small even as the stresses become large. Ortiz et al. [50] have chosen another subclass and have investigated the plane stress state of the boundary of a hole. Bulicek et al. [51] give a detailed analysis and some analytical results for boundary value problems with strain-limiting constitutive equations. Understanding the stresses and strains for polymer models where the molecules are represented by nonlinear springs and dumbells are another example where models exhibiting limiting strain may be used as these dumbells have the property of finite extensibility [8]. An interesting area of study where implicit constitutive equations are useful is investigating the elastic response of soft matter in a biological setting. Freed [36, 37] has developed membrane models of the brain using implicit theory.

In order to assess the value of such models they have been tested on various static boundary value problems. Bustamante and Rajagopal [38] considered simple boundary value problems for homogeneous and inhomogeneous stress distributions where the deformation is expressed as a nonlinear function of the stress and they found that there is a sub-domain where the gradient of the strain is far more pronounced than that for the rest of the domain. Further, biaxial tension for four different constitutive equations is also studied. Bustamante and Rajagopal [39] derived the equations for plane strain and plane stress for constitutive equations where the deformation gradient is expressed as a function of the stress and found a weak formulation for the problem. In [40] the simple shearing of an infinite slab was considered and several exact solutions were found. Rajagopal and Saravan [41] studied the inflation of a spherical annulus for compressible materials and they found that stress boundary layers are possible. Dynamic boundary value and initial value problems for implicit constitutive equations have not been studied in great detail. Kannan [42] studied the propagation of shear stress waves and displacement waves in a rectangular slab using *Comsol* which is a finite element software package. The propagation of circumferential waves for the same constitutive equation was studied by Kambapalli [43]. In both these papers it was found that a change in the wave form occurs for the power-law exponent $n > 0$ in the stress. This is interesting in that one generally associates a distortion with some kind of dissipation, but since we have defined elastic materials as materials incapable of dissipation, this distortion of the wave is completely described within the framework of implicit constitutive equations. Another interesting characteristic is the change in the amplitude of the cylindrical wave for $n = 0$ which is unlike the behaviour for a linear wave propagating in a rectangular slab.

Rajagopal and Saccomandi [52] considered the propagation of circularly polarized transverse stress waves, standing shear stress waves and oscillatory shear stress waves for viscoelastic materials described by a subclass of implicit constitutive equations. In the absence of the dissipative term in the constitutive equation, general standing wave solutions are obtained and these solutions are further generalisations of the wave solutions found in [53] for classical nonlinear isotopic elastodynamics. The mathematical reason for obtaining such solutions is discussed and it is shown that there is a connection between the linear degeneracy of the nonlinear hyperbolic equation and the possibility to find global solutions to the Cauchy problem. This idea has been extended to implicit constitutive equations and it is shown that due to the nature of the equations the existence of global solutions are possible in this theory as well [52].

The propagation of nonlinear waves in composite elastic bars has been investigated in [54]. The propagation of shock waves in elastic bars can result in cracks developing. Huang et al. [54] shows how the strength of a tensile wave can be undermined by considering material nonlinearity in an initially stress-free two-material bar. The global stability of the solutions is studied in [55] for the same wave-catching-up phenomena when the two-material bar is pre-stressed. Huang et al. [56] have extended the study on nonlinear elastic bars by considering the wave patterns for a subclass of implicit constitutive equations, in particular the one-dimensional power-law constitutive equation, when solving a Riemann problem. The results given in [56] suggest that different wave patterns occur depending on the initial data and that some of these waves are composite waves consisting of rarefaction waves and shock waves. Of particular importance is that fact that these wave patterns can be used to describe whether a material belongs to the class of implicit constitutive equations or to the class of classical constitutive equations. It is found that for small strains twelve different wave patterns occur for implicit constitutive equations while only one wave pattern is found for classical constitutive equations.

1.5 Objectives of thesis

The objective of this thesis is to study the propagation of nonlinear displacement and stress waves in a rectangular slab and a circular cylinder for a special subclass of implicit constitutive equations. We will consider two constitutive equations. The first constitutive equation that we consider is the power-law constitutive equation and the second constitutive equation is the strain-limiting constitutive equation. For the propagation of the nonlinear displacement and stress waves in a rectangular slab we look for solitary wave solutions, travelling wave solutions and standing wave solutions for both constitutive equations. Similarly, for the propagation of axial displacement and stress waves we look for solitary wave solutions and standing wave solutions.

1.6 Outline of thesis

The thesis will be structured as follows:

In Chapter 2 we provide a brief overview of the key concepts of finite elasticity and linear elasticity. We discuss the kinematics of deformation and provide definitions for the relevant strain measures. The components of the strain tensor in all coordinate systems are given explicitly. Finally, we provide a brief derivation of the balance laws and field equations and the equations of motion are given in explicit form in all coordinate systems.

In Chapter 3 the constitutive equations in Cauchy elasticity and hyperelasticity are discussed. We elaborate on the fundamental assumptions made for these materials in order to justify the forms of constitutive equations used when describing the behaviour of Cauchy elastic and hyperelastic materials. The theory of implicit constitutive equations is discussed in the final part of Chapter 3 and we provide a detailed derivation of the constitutive equations that will be used in this thesis.

In Chapter 4 we investigate the propagation of nonlinear displacement and stress waves in a rectangular slab for a special subclass of constitutive equations. The two constitutive equations that we consider are the power-law constitutive equation and the strain-limiting constitutive equation. For both constitutive equations we look for solitary wave solutions, travelling wave solutions and standing wave solutions and provide a comparison of the results.

In Chapter 5, the propagation of axial displacement and stress waves in a circular cylinder is studied. We consider the same two constitutive equations as in Chapter 4 and look for solitary wave solutions and standing wave solutions. The final part of the Chapter is a comparison of results between the two constitutive equations.

In Chapter 6 we summarise the main results and conclusions of the thesis.

We have also included two appendices. In Appendix A we present a brief overview of the main transformation laws and results used in tensor calculus in both index notation. In Appendix B we derive travelling wave solutions discuss standing wave solutions for the displacement and stress for the power-law constitutive equation using the Lindstedt-Poincaré singular perturbation method to show that the secular terms in the solutions obtained in Chapter 4 cannot be removed.

Chapter 2

Finite elasticity and linearised elasticity

2.1 Introduction

In this Chapter we will provide an overview of the concepts of finite elasticity and the assumptions made in linearised elasticity. We derive explicit forms of the strain tensor and equations of motion for all coordinate systems.

The study and description of physical phenomena at an atomic and sub-atomic level is widely investigated and has proved to be effective in describing and predicting material behaviour. However, in many instances approximating the internal molecular structure and behaviour by its overall response is more useful. Such descriptions fall under the umbrella of continuum mechanics. The field of continuum mechanics has been successfully applied to almost all fields of engineering. In particular, the theory of continuum mechanics is applicable to arbitrary materials undergoing large deformations [44]. In the first part of this chapter we discuss the key concepts considered in finite elasticity and provide derivations for the balance laws in a rectangular coordinate system as well as in more general coordinate systems.

There are many materials that are well approximated by a linear relationship between the stress and the strain. In order to provide a framework to describe such materials the kinematic quantities developed in the first part of the chapter will be linearised. That is, the nonlinear terms in the deformation gradient, Cauchy-Green stretch tensors and the Lagrangian or Green St. Venant strain tensor must be neglected. Linear elasticity is sometimes called small-strain elasticity theory and materials which exhibit this property are generally governed by Hooke's law. The second part of this chapter will briefly explore the important equations used in linear elasticity.

The Chapter is structured as follows. In Section 2.1 we discuss the kinematics of deformation. The relevant strain measures are described in Section 2.3. The Polar decomposition theorem and Cauchy's stress theorem are discussed in Section 2.4 and 2.5 respectively. The balance laws and field equations are derived in Section 2.6. Finally, we present a brief description of the key assumptions and definitions used in linear elasticity in Section 2.7.

2.2 Motion and deformation

2.2.1 Configurations and motion

In this section we begin with the kinematics of deformation. Consider the diagram below [45]

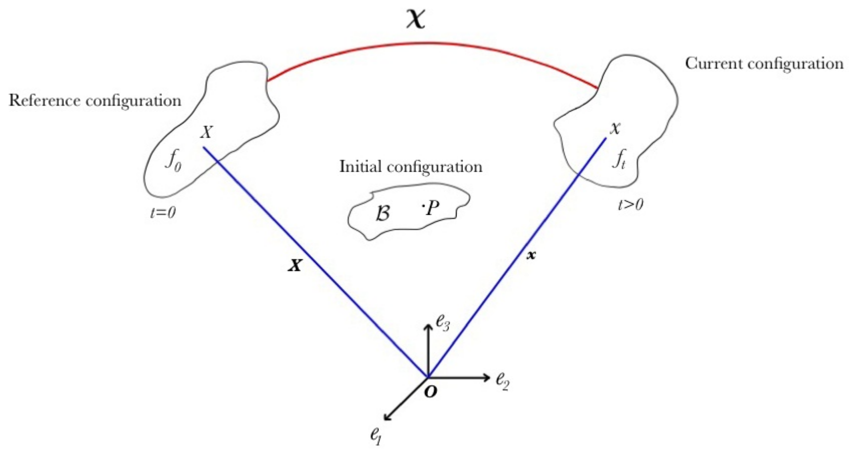

Figure 2.2.1: Configurations and motion

Suppose we have an abstract body $\mathcal{B}$ with some point $P \in \mathcal{B}$. We introduce a fixed rectangular coordinate system with unit basis vectors $\{\mathbf{e}_i\}$, $i = 1, 2, 3$ and a fixed origin $\mathbf{O}$. The continuum body $\mathcal{B}$ moves from f_0 at time $t = 0$ to f_t at time $t > 0$. The regions that this body describes during motion from f_0.......f_t are known as the configurations of the continuum body $\mathcal{B}$. The region f_0 is called the fixed reference or undeformed configuration and f_t is called the current or deformed configuration. The reference configuration corresponds to time $t = 0$ and we shall assume that the initial configuration is the reference configuration.

We define the position vector $\mathbf{X}$ of point X in the reference configuration relative to a fixed origin $\mathbf{O}$. The point X is the position of a particle occupied by $P \in \mathcal{B}$. If the continuum body moves from f_0 to f_t, we define the position vector $\mathbf{x}$ of point x in the current configuration relative to a fixed origin $\mathbf{O}$. We can write the position vectors in the reference and current configurations as

$$\mathbf{X} = X_I \mathbf{E}_I \quad \text{and} \quad \mathbf{x} = x_i \mathbf{e}_i \tag{2.2.1}$$

where X_I, $I = 1, 2, 3$ denote the material coordinates of $\mathbf{X}$ and x_i, $i = 1, 2, 3$ the spatial coordinates of $\mathbf{x}$. Throughout this thesis we will use the convention that uppercase letters refer to the reference configuration and lowercase letters to the current configuration. We note that the basis vectors in the reference configuration $\{\mathbf{E}_I\}$, $I = 1, 2, 3$ coincides with the basis vectors $\{\mathbf{e}_i\}$, $i = 1, 2, 3$ in the current configuration and these basis vectors are orthonormal. The properties of these basis vectors can be found in Appendix A: Part I.

We construct an invertible mapping between the two configurations through the vector $\boldsymbol{\chi}$. The one-to-one mapping is given by

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \quad \text{or} \quad x_i = \chi_i(X_1, X_2, X_3, t). \tag{2.2.2}$$

Since the mapping (2.2.2) is invertible we can write

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t) \quad \text{or} \quad X_I = \chi_I^{-1}(x_1, x_2, x_3, t). \tag{2.2.3}$$

The vector valued function $\boldsymbol{\chi}$ is called the motion of a continuum body $\mathcal{B}$ and describes how particles in the reference configuration move to points in the current configuration or how points in the current configuration move to points in the reference configuration. A property of this motion is that it induces a deformation (change in local shape) of a continuum body $\mathcal{B}$.

In elasticity theory it is sometimes necessary to describe the motion with respect to either the reference configuration or the current configuration. If the motion is specified with respect to the reference configuration we refer to this description as the material description or Lagrangian description. Within this description the observer will follow the particle during motion and observe the deformation. Alternately, one could describe the motion in the current configuration. The description of motion with regard to the current configuration is called the spatial description or the Eulerian description. This differs from the Lagrangian description in that one observes the changes during motion at a fixed point in space as time changes. An interesting case where the Eulerian description proves useful is in the modelling of biological systems.

2.2.2 Displacement vector

An important concept in elasticity is the definition of a displacement field. The motion $\boldsymbol{\chi}$ of a continuum body is described in terms of the displacement field. This holds true in both the Lagrangian description and the Eulerian description. The displacement field in the material description is given by

$$\mathbf{U}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X} \tag{2.2.4}$$

and in the spatial description as

$$\mathbf{u}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t) \tag{2.2.5}$$

where $\mathbf{U}(\mathbf{X},t)$ and $\mathbf{u}(\mathbf{x},t)$ are the respective displacement vectors. It is worth noting that since we have taken the initial configuration to coincide with the reference configuration, the displacements in the reference configuration disappear. Thus $\mathbf{U}(\mathbf{X},t) = \mathbf{u}(\mathbf{x},t)$.

We can also define the velocity and acceleration fields in both the material and spatial description. In the material description, the velocity and acceleration is given by

$$\mathbf{V} = \mathbf{V}(\mathbf{X},t) = \frac{\partial \boldsymbol{\chi}(\mathbf{X},t)}{\partial t}, \tag{2.2.6}$$

$$\mathbf{A} = \frac{\partial^2 \boldsymbol{\chi}(\mathbf{X},t)}{\partial t^2}. \tag{2.2.7}$$

In the spatial description these quantities are defined as

$$\mathbf{V}(\mathbf{X},t) \;=\; \mathbf{V}[\boldsymbol{\chi}^{-1}(\mathbf{x},t)] \;=\; \mathbf{v}(\mathbf{x},t), \tag{2.2.8}$$

$$\mathbf{A}(\mathbf{X},t) \;=\; \mathbf{A}[\boldsymbol{\chi}^{-1}(\mathbf{x},t)] \;=\; \mathbf{a}(\mathbf{x},t). \tag{2.2.9}$$

where we denote the components of the velocity fields and acceleration fields as V_I, A_I and v_i, a_i in the material and spatial descriptions respectively.

2.2.3 Material time derivative

The material time derivative can be computed with respect to both the Lagrangian field and the Eulerian field. The material time derivative of a smooth spatial field $\mathbf{w}(\mathbf{x},t)$ which is vector valued is given by

$$\dot{\mathbf{w}}(\mathbf{x},t) \;=\; \frac{D\mathbf{w}(\mathbf{x},t)}{Dt} \;=\; \frac{\partial \mathbf{w}(\mathbf{x},t)}{\partial t} + \mathrm{grad}\mathbf{v(x,t)w(x,t)}. \tag{2.2.10}$$

In index notation we can write (2.2.10) as

$$\dot{w}_i \;=\; \frac{Dw_i}{Dt} \;=\; \frac{\partial w_i}{\partial t} + v_j \frac{\partial w_i}{\partial x_j}. \tag{2.2.11}$$

When $w_i = v_i$ this may also be viewed as the spatial velocity and acceleration field.

2.2.4 Deformation gradient

To characterise the local change in shape of a continuum body when it moves from one configuration to another, we introduce a measure of the deformation known as the deformation gradient. Consider the mapping

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \tag{2.2.12}$$

which describes the motion from f_0 to f_t. The differential of equation (2.2.12) is

$$d\mathbf{x} = \mathbf{F}(\mathbf{X}, t)d\mathbf{X} \tag{2.2.13}$$

where the deformation gradient is defined as

$$\mathbf{F}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial \mathbf{X}} = \text{Grad}\mathbf{x}(\mathbf{X}, t). \tag{2.2.14}$$

We can express equation (2.2.14) in index notation and explicitly in terms of the basis vectors $\{\mathbf{e}_i\}$ and $\{\mathbf{E}_I\}$ as

$$F_{iI} = \frac{\partial \chi_i}{\partial X_I}, \tag{2.2.15}$$

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial X_I} \otimes \mathbf{E}_I = \frac{\partial x_j \mathbf{e}_j}{\partial X_I} \otimes \mathbf{E}_I = \frac{\partial x_j}{\partial X_I}\mathbf{e}_j \otimes \mathbf{E}_I \tag{2.2.16}$$

where

$$\text{Grad}(\cdot) = \frac{\partial(\cdot)}{\partial X_I} \otimes \mathbf{E}_I. \tag{2.2.17}$$

An interesting property of the deformation gradient is that it is a two point tensor. This means that the deformation gradient $\mathbf{F}$ will act on a vector in one configuration (reference configuration) and produce a vector in another configuration (current configuration). In other words, we investigate the change in spatial coordinates x_i relative to the material coordinates X_I.

From a physical viewpoint, the material tangent vectors (curves) are mapped into spatial tangent vectors (curves) through the deformation gradient $\mathbf{F}$. However, the unit normals $\mathbf{N}$ and $\mathbf{n}$ to the infinitesimal material surface elements $d\mathbf{S}$ and $d\mathbf{s}$ are not mapped to each other. In order to find a relationship between the unit normals $\mathbf{N}$ and $\mathbf{n}$ consider a change in the volume between the reference and current configuration given by

$$dv = J(\mathbf{X}, t)dV \tag{2.2.18}$$

where

$$J(\mathbf{X}, t) = \det\mathbf{F} > 0. \tag{2.2.19}$$

The differentials dv and dV are called material volume elements. The Jacobian J must be greater than zero. If $J < 0$ the material will penetrate itself which is physically unreasonable. Further, the restriction is strict in that the Jacobian J cannot be zero since the deformation gradient is invertible. A material is incompressible if and only if $J = 1$, while a motion in which no volume change occurs is called isochoric.

The infinitesimal volume element dv can be expressed as a dot product of the infinitesimal surface elements ds, dS in the direction of the unit normal vectors $\mathbf{n}$ and $\mathbf{N}$ and the material line elements $d\mathbf{x}$ and $d\mathbf{X}$ in the reference and current configurations respectively. That is

$$dv \;=\; ds\mathbf{n}\cdot d\mathbf{x} \;=\; JdS\mathbf{N}\cdot d\mathbf{X}. \tag{2.2.20}$$

Using the properties of a tensor and the definition of a differential for the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ we obtain

$$d\mathbf{s} = J\mathbf{F}^{-T}d\mathbf{S}. \tag{2.2.21}$$

This equation is known as Nanson's Formula. We note that when $J = 1$, we have the following equivalent statements

$$\dot{J} = 0 \quad \text{and} \quad \mathrm{div}\mathbf{v} = 0 \tag{2.2.22}$$

since $\dot{J} = J\mathrm{div}\mathbf{v}$.

2.3 Strain tensors

In this section we define strain measures relevant in the reference configuration and the corresponding strain measures in the current configuration. We will provide explicit forms of the linearised strain tensor in terms of the basis vectors and expand this expression to derive the components of the strain tensor. This will be done for rectangular Cartesian coordinates and extended to more general coordinate systems such as the cylindrical and spherical coordinate systems.

2.3.1 Material and spatial strain tensors

We first consider strain tensors in the reference configuration. We want to investigate how the change between two points in the reference configuration, say X and Y is related to its corresponding points x and y in the current or deformed configuration during motion. In other words, how the line element $d\epsilon$ in the reference configuration is mapped to the corresponding line element $\boldsymbol{\lambda}d\epsilon$ in the current configuration. The parameter $\boldsymbol{\lambda}$ is a measure of the stretch of a continuum body $\mathcal{B}$. For values of $\boldsymbol{\lambda} > 1$, the line element is said to have stretched, for $\boldsymbol{\lambda} < 1$, the line element has compressed and for $\boldsymbol{\lambda} = 1$, the line element is unstretched. We calculate the stretch vector $\boldsymbol{\lambda}$ according to

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}. \tag{2.3.1}$$

This second order tensor is called the right Cauchy-Green tensor or Green deformation tensor. It has the property of being symmetric and positive definite, i.e

$$\mathbf{C} \;=\; \mathbf{F}^T\mathbf{F} \;=\; (\mathbf{F}^T\mathbf{F})^T \;=\; \mathbf{C} \;\text{ and }\; \mathbf{u}\cdot\mathbf{C}\mathbf{u} > 0 \;\;\forall\;\; \mathbf{u} \neq 0. \tag{2.3.2}$$

Also, since the deformation gradient $\mathbf{F}$ is invertible

$$\det\mathbf{C} \;=\; (\det\mathbf{F})^2 \;=\; J^2 > 0. \tag{2.3.3}$$

We can express this second order tensor $\mathbf{C}$, which is a two point tensor since the deformation gradient tensor, $\mathbf{F}$, is a two point tensor, in index notation as

$$C_{IJ} \;=\; F_{KI}F_{KJ} \;=\; \frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J}, \tag{2.3.4}$$

and explicitly in terms of the basis vectors $\{\mathbf{e}_i\}$, $i = 1, 2, 3$ and $\{\mathbf{E}_I\}$, $I = 1, 2, 3$ as

$$\begin{aligned} \left(\frac{\partial x_k}{\partial X_I}\mathbf{E}_I \otimes \mathbf{e}_k\right)\left(\frac{\partial x_m}{\partial X_J}\mathbf{e}_m \otimes \mathbf{E}_J\right) &= \frac{\partial x_k}{\partial X_I}\frac{\partial x_m}{\partial X_J}(\mathbf{E}_I \otimes \mathbf{e}_k)(\mathbf{e}_m \otimes \mathbf{E}_J) \\ = \frac{\partial x_k}{\partial X_I}\frac{\partial x_m}{\partial X_J}(\mathbf{E}_I \otimes \mathbf{E}_J)(\mathbf{e}_k \cdot \mathbf{e}_m) &= \frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J}\mathbf{E}_I \otimes \mathbf{E}_J. \end{aligned} \tag{2.3.5}$$

We define another important strain tensor in the material coordinate system known as the Green-Lagrange strain tensor. This tensor is defined as

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}). \tag{2.3.6}$$

In index notation we write

$$E_{IJ} = \frac{1}{2}(F_{iI}F_{iJ} - \delta_{IJ}) \tag{2.3.7}$$

where δ_{IJ} is the Kronecker delta. In terms of the basis vectors $\{\mathbf{E}_I\}$ we write

$$E_{IJ} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J}\mathbf{E}_I \otimes \mathbf{E}_J - \mathbf{E}_K \otimes \mathbf{E}_K\right). \tag{2.3.8}$$

The Green-Lagrange strain tensor is derived by considering a change in squared lengths between the line elements $d\epsilon$ and $\boldsymbol{\lambda} d\epsilon$.

Similarly, we describe strain tensors in the current configuration. The first strain tensor that we define is called the left Cauchy-Green tensor $\mathbf{b}$ and is defined as

$$\mathbf{b} = \mathbf{F}\mathbf{F}^T, \tag{2.3.9}$$

and in index form

$$b_{ij} = F_{iK}F_{jK}. \tag{2.3.10}$$

We can also express the strain tensor explicitly in terms of the basis vectors as

$$\begin{aligned} \left(\frac{\partial x_i}{\partial X_I}\mathbf{e}_i \otimes \mathbf{E}_I\right)\left(\frac{\partial x_j}{\partial X_K}\mathbf{E}_K \otimes \mathbf{e}_j\right) &= \frac{\partial x_i}{\partial X_I}\frac{\partial x_j}{\partial X_K}(\mathbf{e}_i \otimes \mathbf{E}_I)(\mathbf{E}_K \otimes \mathbf{e}_j) \\ = \frac{\partial x_i}{\partial X_I}\frac{\partial x_j}{\partial X_K}(\mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{E}_I \cdot \mathbf{E}_K) &= \frac{\partial x_i}{\partial X_I}\frac{\partial x_j}{\partial X_I}\mathbf{e}_i \otimes \mathbf{e}_j. \end{aligned} \tag{2.3.11}$$

In the same way that the Green-Lagrange strain tensor is derived we can define a strain tensor by considering the squared lengths between changes in line elements when moving from the current configuration to the reference configuration. This strain tensor is called the Euler-Almansi strain tensor and is given by

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}) \tag{2.3.12}$$

or in index notation as

$$e_{ij} = \frac{1}{2}(\delta_{ij} - F^{-1}_{Ki}F^{-1}_{Kj}). \tag{2.3.13}$$

2.3.2 Displacement gradient tensor

The displacement gradient tensor is defined in the material and spatial coordinate systems as

$$\mathrm{Grad}\mathbf{U} \;=\; \mathrm{Grad}\mathbf{x} - \mathrm{Grad}\mathbf{X} \;=\; \mathbf{F}(\mathbf{X}, t) - \mathbf{I} \tag{2.3.14}$$

and

$$\mathrm{grad}\mathbf{u} \;=\; \mathrm{grad}\mathbf{x} - \mathrm{grad}\mathbf{X} \;=\; \mathbf{I} - \mathbf{F}^{-1}(\mathbf{x}, t). \tag{2.3.15}$$

We note that the Green-Lagrange strain tensor $\mathbf{E}$ and the Euler-Almansi strain tensor $\mathbf{e}$ can be expressed in terms of the displacement gradient tensor as

$$\mathbf{E} = \frac{1}{2}(\mathrm{Grad}^T\mathbf{U} + \mathrm{Grad}\mathbf{U}) + \frac{1}{2}\mathrm{Grad}^T\mathbf{U}\mathrm{Grad}\mathbf{U} \tag{2.3.16}$$

and

$$\mathbf{e} = \frac{1}{2}(\mathrm{grad}^T\mathbf{u} + \mathrm{grad}\mathbf{u}) - \frac{1}{2}\mathrm{grad}^T\mathbf{u}\mathrm{grad}\mathbf{u}. \tag{2.3.17}$$

The proof of these results is simple in that we rewrite the deformation and inverse deformation gradient from (2.3.14) and (2.3.15) in terms of the displacement vectors $\mathbf{U}$ and $\mathbf{u}$. We then substitute these expressions into the respective strain tensors to obtain the desired results.

As mentioned earlier, we will provide explicit forms (in terms of the basis vectors), for the linearised strain tensor in a rectangular Cartesian coordinate system and extend this result to a cylindrical and spherical coordinate system.

The linearised strain tensor ϵ is related to the Green-Lagrange strain tensor, Euler-Almansi and left Cauchy-Green strain tensors as

$$\mathbf{E} = \boldsymbol{\epsilon} + \mathcal{O}(\delta^2), \quad \mathbf{e} = \boldsymbol{\epsilon} + \mathcal{O}(\delta^2) \text{ and } \mathbf{b} = 1 + 2\boldsymbol{\epsilon} + \mathcal{O}(\delta^2) \tag{2.3.18}$$

where

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T\right) \tag{2.3.19}$$

by appealing to small displacement gradients and δ is the square of the displacement gradient. We defer a more detailed explanation of the above to the Subsection 3.5 on Implicit constitutive theories.

2.3.3 Dyads, the gradient operator and components of the strain tensor

We list some of the results for dyads and operators in this subsection to make the derivation of the components of the strain tensor self-contained.

A vector field $\mathbf{u}$ may be written as

$$\mathbf{u} = u_i \mathbf{e}_i \tag{2.3.20}$$

where $\{\mathbf{e}_i\}$, $i = 1, 2, 3$ is the set of basis vectors.

A tensor product or dyad of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as $\mathbf{u} \otimes \mathbf{v}$, and a dyadic is defined as a linear combination of dyads with scalar coefficients [45]. We can express a second order tensor $\mathbf{S}$ as a dyad

$$\mathbf{S} = S_{ij} \mathbf{e}_i \otimes \mathbf{e}_j. \tag{2.3.21}$$

This can be extended more generally to any tensor of order n, $n \geq 2$,

$$S_{i1i2\ldots in} \mathbf{e}_{i1} \otimes \mathbf{e}_{i2} \otimes \ldots\ldots \otimes \mathbf{e}_{in}. \tag{2.3.22}$$

The gradient ∇ is given by

$$\nabla \cdot (\cdot) \;=\; \frac{\partial(\cdot)}{\partial x_i}\mathbf{e}_i \;=\; \frac{\partial(\cdot)}{\partial x_1}\mathbf{e}_1 + \frac{\partial(\cdot)}{\partial x_2}\mathbf{e}_2 + \frac{\partial(\cdot)}{\partial x_3}\mathbf{e}_3 \tag{2.3.23}$$

where $\{\mathbf{e}_i\}$, $i = 1, 2, 3$ is the set of basis vectors in a rectangular Cartesian coordinate system.

For any scalar field ϕ,

$$\text{grad}\phi \;=\; \nabla\phi \;=\; \frac{\partial\phi}{\partial x_i}\mathbf{e}_i. \tag{2.3.24}$$

Using the results above we define the following operators

$$\nabla\cdot(\cdot) = \frac{\partial(\cdot)}{\partial x_i}\cdot\mathbf{e}_i, \quad \nabla\times(\cdot) = \mathbf{e}_i\times\frac{\partial(\cdot)}{\partial x_i}, \quad \nabla\otimes(\cdot) = \frac{\partial(\cdot)}{\partial x_i}\otimes\mathbf{e}_i. \tag{2.3.25}$$

The gradient of a vector field $\mathbf{u}$, in a rectangular Cartesian coordinate system, is

$$\text{grad }\mathbf{u} = \nabla\otimes\mathbf{u} \;=\; \frac{\partial(u_j\mathbf{e}_j)}{\partial x_i}\otimes\mathbf{e}_i \;=\; \frac{\partial u_j}{\partial x_i}\mathbf{e}_j\otimes\mathbf{e}_i. \tag{2.3.26}$$

For a second order tensor $\mathbf{S}$, the gradient can be computed as

$$\text{grad }\mathbf{S} = \nabla\otimes\mathbf{S} \;=\; \frac{\partial(S_{jk}\mathbf{e}_j\otimes\mathbf{e}_k)}{\partial x_i}\otimes\mathbf{e}_i \;=\; \frac{\partial S_{jk}}{\partial x_i}\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_i. \tag{2.3.27}$$

The gradient (2.3.23) can be transformed to a curvilinear coordinate system as

$$\nabla\cdot(\cdot) \;=\; \frac{\partial(\cdot)}{\partial\theta_i}\mathbf{g}^i \;=\; \frac{\partial(\cdot)}{\partial\theta_1}\mathbf{g}^1 + \frac{\partial(\cdot)}{\partial\theta_2}\mathbf{g}^2 + \frac{\partial(\cdot)}{\partial\theta_3}\mathbf{g}^3 \tag{2.3.28}$$

where $\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3$ are the contravariant base vectors with coordinate points $(\theta^1, \theta^2, \theta^3)$. The proof of this result can be found in Appendix A: Part II.

For a vector field $\mathbf{u}$, in curvilinear coordinates, the gradient is calculated as

$$\text{grad }\mathbf{u} = \nabla\otimes\mathbf{u} \;=\; \left(\frac{\partial\mathbf{u}}{\partial\theta^i}\right)\otimes\mathbf{g}^i \;=\; \frac{\partial}{\partial\theta^i}(u_j\mathbf{g}^j)\otimes\mathbf{g}^i. \tag{2.3.29}$$

This expression can be differentiated to obtain

$$(u_j\mathbf{g}^j)_{,i}\otimes\mathbf{g}^i \;=\; (u_{j,i}\mathbf{g}^j + u_j\mathbf{g}^j_{,i})\otimes\mathbf{g}^i \;=\; \left(\frac{\partial u_j}{\partial\theta^i} - u_k\Gamma^k_{ij}\right)\mathbf{g}^j\otimes\mathbf{g}^i \tag{2.3.30}$$

using (A.2.28). Now the components of the linearised strain tensor $\boldsymbol{\epsilon}$ can be found by considering the symmetric part of grad$\mathbf{u}$ or $\nabla\otimes\mathbf{u}$.

2.3.4 Rectangular Cartesian coordinates

We shall first derive the strain components for a rectangular Cartesian coordinate system (x, y, z).
The gradient ∇ of a vector field $\mathbf{u}$, i.e, $\nabla\otimes\mathbf{u}$ is calculated as

$$\nabla\otimes\mathbf{u} \;=\; \frac{\partial u_j}{\partial x_i}\mathbf{e}_j\otimes\mathbf{e}_i. \tag{2.3.31}$$

An important result of tensor algebra is that any tensor can be uniquely decomposed into a symmetric and skew tensor as

$$\nabla \otimes \mathbf{u} \;=\; \nabla \mathbf{u} \;=\; \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u}) + \frac{1}{2}(\nabla \mathbf{u} - \nabla^T \mathbf{u}). \tag{2.3.32}$$

We note that $\nabla \otimes \mathbf{u} = \nabla \mathbf{u}$ and $\nabla^T \mathbf{u}$ denotes the transpose of the gradient. The symmetric part of the tensor (2.3.32) will give the components of strain and is given by

$$\boldsymbol{\epsilon} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y}\right) & \dfrac{\partial u_y}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y}\right) & \dfrac{\partial u_z}{\partial z}. \end{bmatrix} \tag{2.3.33}$$

We note that

$$\mathrm{div}\mathbf{u} \;=\; \mathrm{tr}(\nabla \otimes \mathbf{u}) \;=\; \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}. \tag{2.3.34}$$

The components of the strain tensor are

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u_x}{\partial x}; \quad \epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right); \quad \epsilon_{xz} = \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right); \\ \epsilon_{yy} &= \frac{\partial u_y}{\partial y}; \quad \epsilon_{yz} = \epsilon_{zy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right); \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z}. \end{aligned} \tag{2.3.35}$$

2.3.5 Cylindrical polar coordinates

The gradient of a vector field $\mathbf{u}$ in cylindrical polar coordinates can be computed by expanding the expression

$$\nabla \otimes \mathbf{u} = \left(\frac{\partial u_j}{\partial \theta^i} - u_k \Gamma^k_{ij}\right) \mathbf{g}^j \otimes \mathbf{g}^i. \tag{2.3.36}$$

and since since any tensor can be written as the sum of a symmetric and skew symmetric tensor, using (2.3.32) we obtain the strain tensor

$$\boldsymbol{\epsilon} = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left(\dfrac{1}{r}\left(\dfrac{\partial u_\theta}{\partial \theta} - u_\theta\right) + \dfrac{\partial u_\theta}{\partial r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) \\ \dfrac{1}{2}\left(\dfrac{1}{r}\left(\dfrac{\partial u_\theta}{\partial \theta} - u_\theta\right) + \dfrac{\partial u_\theta}{\partial r}\right) & \dfrac{1}{r}\left(\dfrac{\partial u_\theta}{\partial \theta} + u_r\right) & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) & \dfrac{\partial u_z}{\partial z} \end{bmatrix}. \tag{2.3.37}$$

The corresponding components of strain in cylindrical polar coordinates are

$$\begin{aligned}
\epsilon_{rr} = \frac{\partial u_r}{\partial r}; \quad \epsilon_{r\theta} &= \epsilon_{\theta r} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right); \\
\epsilon_{rz} = \epsilon_{zr} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right); \quad \epsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right); \\
\epsilon_{z\theta} = \epsilon_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right); \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z}.
\end{aligned} \tag{2.3.38}$$

2.3.6 Spherical polar coordinates

The gradient of a vector field in spherical polar coordinates is calculated by expanding the expression

$$\nabla \otimes \mathbf{u} = \left(\frac{\partial u_j}{\partial \theta^i} - u_k \Gamma^k_{ij}\right) \mathbf{g}^j \otimes \mathbf{g}^i, \tag{2.3.39}$$

Thus the gradient of a vector field (**??**) can be written in matrix form as

$$\nabla \otimes \mathbf{u} = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial u_r}{\partial \theta} - u_\theta\right) & \left(\dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial \phi} - \dfrac{1}{r}u_\phi\right) \\ \dfrac{\partial u_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial u_\theta}{\partial \theta} + u_r\right) & \left(\dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial \phi} - \dfrac{1}{r}\cot\theta u_\phi\right) \\ \dfrac{\partial u_\phi}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_\phi}{\partial \theta} & \left(\dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial \phi} + \dfrac{1}{r}u_r + \dfrac{1}{r}\cot\theta u_\theta\right) \end{bmatrix}. \tag{2.3.40}$$

Since any tensor can be decomposed into a symmetric and skew symmetric tensor, using (2.3.32) we obtain the second order tensor for the strain for spherical polar coordinates in matrix form

$$
\begin{bmatrix}
\frac{\partial u_r}{\partial r} & \frac{1}{2}\left(\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r}\right) & \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{1}{r}u_\phi + \frac{\partial u_\phi}{\partial r}\right) \\
\frac{1}{2}\left(\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r}\right) & \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) & \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{1}{r}\cot\theta\, u_\phi + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta}\right) \\
\frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{1}{r}u_\phi + \frac{\partial u_\phi}{\partial r}\right) & \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{1}{r}\cot\theta\, u_\phi + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta}\right) & \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{r}u_r + \frac{1}{r}\cot\theta\, u_\theta
\end{bmatrix}. \tag{2.3.41}
$$

The components of strain are given by

$$
\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{r\theta} = \epsilon_{\theta r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right),
$$

$$\begin{aligned}
\epsilon_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \quad \epsilon_{r\phi} = \epsilon_{\phi r} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{1}{r}u_\phi + \frac{\partial u_\phi}{\partial r}\right), \\
\epsilon_{\theta\phi} &= \epsilon_{\phi\theta} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{1}{r}\cot\theta u_\phi + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta}\right), \\
\epsilon_{\phi\phi} &= \left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{r}u_r + \frac{1}{r}\cot\theta u_\theta\right).
\end{aligned} \tag{2.3.42}$$

2.4 Polar decomposition and strain invariants

2.4.1 Polar decomposition

The deformation $\mathbf{F}(\mathbf{X}, t)$ can be decomposed into two types of motion-pure stretch and pure rotation. A fundamental theorem which captures this is called the Polar Decomposition Theorem, where for each point $\mathbf{X} \in f_0$, the deformation gradient can be expressed as

$$\mathbf{F} = \mathbf{RU} = \mathbf{vR}, \tag{2.4.1}$$

where $\mathbf{R}$ is a proper orthogonal vector, i.e, $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ and $\det\mathbf{R} = +1$. The two point tensor $\mathbf{R}$ describes the local rotation of a continuum body and is thus called a rotation tensor. The tensors $\mathbf{U}$ and $\mathbf{v}$ are positive definite tensors and $\mathbf{U}^T = \mathbf{U}$ and $\mathbf{v}^T = \mathbf{v}$ and describes the local stretching or contraction of a continuum body in the material and spatial descriptions respectively. The tensors are related to the right Cauchy-Green stretch tensor $\mathbf{C}$ and left Cauchy-Green stretch tensor $\mathbf{b}$ by the relations [45]

$$\mathbf{U}^2 = \mathbf{UU} = \mathbf{C} \quad \text{and} \quad \mathbf{v}^2 = \mathbf{vv} = \mathbf{b}. \tag{2.4.2}$$

A rigid body rotation of a continuum body about a fixed origin is given by $\mathbf{F} = \mathbf{R}$, where $\mathbf{U} = \mathbf{v} = \mathbf{I}$. In other words, a material line element $d\mathbf{X}$ is mapped to a corresponding spatial line element $d\mathbf{x}$ through a rotation. That is

$$d\mathbf{x} = \mathbf{R}(\mathbf{X}, t)d\mathbf{X}. \tag{2.4.3}$$

Similarly, if $\mathbf{R} = \mathbf{I}$, the motion would be characterised by $\mathbf{F} = \mathbf{U} = \mathbf{v}$ which describes pure stretch [45].

Suppose a line element d$\mathbf{X}$ is mapped to d$\mathbf{x}$ though

$$d\mathbf{x} = \mathbf{RU}\, d\mathbf{X}. \tag{2.4.4}$$

This describes the continuum body being stretched by $\mathbf{U}$, followed by a pure rotation by the rotation tensor $\mathbf{R}$. Next suppose that the mapping between the line elements d$\mathbf{x}$ and d$\mathbf{X}$ is described by

$$d\mathbf{x} = \mathbf{vR}\, d\mathbf{X}. \tag{2.4.5}$$

In this case the continuum body is first rotated by a rotation tensor $\mathbf{R}$, followed by pure stretch due to tensor $\mathbf{v}$ [45].

2.4.2 Strain invariants

The right Cauchy-Green stretch tensor $\mathbf{C}$ is symmetric and positive definite so it has positive real eigenvalues. The characteristic equation for $\mathbf{C}$ is

$$\det(\mathbf{C} - \lambda \mathbf{I}) = 0 \tag{2.4.6}$$

which can be solved to give

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0, \tag{2.4.7}$$

where

$$I_1 = \mathrm{tr}\mathbf{C}, \quad I_2 = \frac{1}{2}((\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)), \quad I_3 = \det\mathbf{C}, \tag{2.4.8}$$

are the principal invariants of $\mathbf{C}$ and $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\mathbf{C}$. The principal invariants I_1, I_2, I_3 can be expressed in terms of the eigenvalues

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad I_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad I_3 = \lambda_1\lambda_2\lambda_3. \tag{2.4.9}$$

Similarly, one can obtain the eigenvalues of the left Cauchy-Green stretch tensor $\mathbf{b}$ which are the same as $\mathbf{C}$, but the principal axes, which are the normalised eigenvectors, may not coincide. These invariants are important in constitutive theory.

2.5 Stress and Cauchy's stress theorem

Consider an arbitrary region V enclosed by a surface S with unit outward normal $\mathbf{N}$. Consider also an arbitrary region v with surface s bounding v with unit outward normal $\mathbf{n}$. In order for deformation to occur, forces, temperature changes, phase changes or external or internal stimuli need to act on the boundary of the continuum body as well as on the interior of the continuum body. This leads one to the notion of stress.

We assume the existence of vector fields $\mathbf{T}(\mathbf{X}, t, \mathbf{N})$ and $\mathbf{t}(\mathbf{x}, t, \mathbf{n})$ defined over S and s respectively. The stress vector $\mathbf{T}(\mathbf{X}, t, \mathbf{N})$ is defined as the force per unit area on the surface element at $\mathbf{X}$, at time t, with unit outward normal $\mathbf{N}$ and is measured in the reference configuration. If this force is applied to the surface of the body, the vector $\mathbf{T}$ is called the first Piola-Kirchoff traction vector.

Similarly, we can define the vector $\mathbf{t}(\mathbf{x}, t, \mathbf{n})$ as the force per unit area on a surface element at $\mathbf{x}$, at time t with unit outward normal $\mathbf{n}$ and is measured in the current configuration. This vector $\mathbf{t}$, when applied to the surface s of a continuum body is called the Cauchy traction vector. It is understood that the unit normal vectors $\mathbf{N}$ and $\mathbf{n}$ point in the same direction.

2.5.1 Cauchy's stress theorem

The Cauchy traction vector $\mathbf{t}(\mathbf{x}, t, \mathbf{n})$ and the first Piola-Kirchoff traction vector $\mathbf{T}(\mathbf{X}, t, \mathbf{N})$ can be related to the Cauchy stress tensor $\boldsymbol{\sigma}(\mathbf{x}, t)$ and the first Piola stress tensor $\mathbf{P}(\mathbf{X}, t)$ respectively through Cauchy's stress theorem. This theorem states that there exists unique second-order tensor fields $\boldsymbol{\sigma}$ and $\mathbf{P}$ such that

$$\mathbf{t}(\mathbf{x}, t, \mathbf{n}) = \boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n}, \quad \mathbf{T}(\mathbf{X}, t, \mathbf{N}) = \mathbf{P}(\mathbf{X}, t)\mathbf{N}. \tag{2.5.1}$$

Further,

$$\mathbf{t}(\mathbf{x}, t, \mathbf{n}) = -\mathbf{t}(\mathbf{x}, t, -\mathbf{n}) \quad \mathbf{T}(\mathbf{X}, t, \mathbf{N}) = -\mathbf{T}(\mathbf{X}, t, -\mathbf{N}) \quad \forall\, \mathbf{n}, \mathbf{N}. \tag{2.5.2}$$

2.5.2 Components of stress

In order to find the components of the traction vectors we can resolve the vector $\mathbf{te}_i$ along $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. That is

$$\mathbf{te}_i = \boldsymbol{\sigma}\mathbf{e}_i. \tag{2.5.3}$$

Thus we have in component form

$$\begin{aligned}
\mathbf{te}_1 &= \boldsymbol{\sigma}\mathbf{e}_1 = \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3 \\
\mathbf{te}_2 &= \boldsymbol{\sigma}\mathbf{e}_2 = \sigma_{21}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{23}\mathbf{e}_3 \\
\mathbf{te}_3 &= \boldsymbol{\sigma}\mathbf{e}_3 = \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3.
\end{aligned} \tag{2.5.4}$$

These nine quantities σ_{ij} characterise the state of stress in a continuum and are called the components of the Cauchy stress vector $\boldsymbol{\sigma}$. We note that if the diagonal elements $\sigma_{11}, \sigma_{22}, \sigma_{33}$ of the Cauchy stress tensor are positive, the stress is described as a tensile stress whereas for negative values the stress is described as a compression stress.

2.6 Balance laws and field equations

In order to relate the changes in particular physical quantities due to surface flux and volume distributions, one makes use of balance laws. The balance laws are expressed mathematically as partial differential equations and hold at all points in a continuum body.

The conservation of mass in a continuum body, i.e, mass is neither created nor lost, is a fundamental concept in continuum theory and must be satisfied. We can find equations to describe the conservation of mass by relating the mass in the reference configuration to the mass in the current configuration through the continuous scalar fields ρ_0 and ρ. The quantity ρ_0 is the density (reference density) in the reference configuration and ρ is the density in the current configuration. Mathematically we express this relationship as

$$\rho_0 = \rho J(\mathbf{X}, t). \tag{2.6.1}$$

where $J(\mathbf{X}, t)$ is defined in equation (2.2.19).

In the spatial description the conservation of mass has the following equivalent forms:

$$\frac{D\rho(\mathbf{x}, t)}{Dt} + \rho(\mathbf{x}, t)\mathrm{div}\mathbf{v}(\mathbf{x}, t) = 0, \tag{2.6.2}$$

$$\frac{\partial\rho(\mathbf{x}, t)}{\partial t} + \mathrm{grad}\rho(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) + \rho(\mathbf{x}, t)\ \mathrm{div}\mathbf{v}(\mathbf{x}, t) = 0 \tag{2.6.3}$$

$$\frac{\partial(\mathbf{x}, t)}{\partial t} + \mathrm{div}(\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)) = 0. \tag{2.6.4}$$

We will state important results, without proof, in order to arrive at the equations of motion for a continuum body in both the current and reference configuration.

Theorem: Reynold's Transport Theorem

For any scalar, vector or tensor field Φ, and for any material volume V

$$\frac{D}{Dt}\int_V \rho(\mathbf{x}, t)\Phi(\mathbf{x}, t)dV = \int_V \rho\frac{D\Phi}{Dt}dV. \tag{2.6.5}$$

In order to derive the equations of motion in both the current and reference configuration the principle of linear and angular momentum in both descriptions is important.

We state the Principle of Linear Momentum as the rate of change of linear momentum of an arbitrary part P of a material body being equal to the resultant force on that arbitrary part P. Mathematically we express this as

$$\frac{D}{Dt}\int_v \rho\mathbf{v}dv = \int_s \mathbf{t}(\mathbf{x}, t, \mathbf{n})ds + \int_v \rho\mathbf{b}(\mathbf{x}, t)dv \tag{2.6.6}$$

where $\mathbf{b}(\mathbf{x}, t)$ is the body force per unit mass, $\mathbf{t}$ is the Cauchy traction vector, v is the material volume and s a closed surface bounding v.

The Principle of Angular Momentum states that the rate of change of the angular momentum of an arbitrary part P of a continuum is equal to the moment about any fixed point of the forces on P. That is

$$\frac{D}{Dt}\int_v \mathbf{x}\times\rho\mathbf{v}dv = \int_s \mathbf{x}\times\mathbf{t}(\mathbf{x},t,\mathbf{n})ds + \int_v \rho\mathbf{x}\times\mathbf{b}dv. \tag{2.6.7}$$

We can write the corresponding equation (2.6.6) and (2.6.7) in the reference configuration as

$$\frac{D}{Dt}\int_{V_0} \rho_0\mathbf{v}dV_0 = \int_{S_0} \mathbf{T}(\mathbf{X},t,\mathbf{N})dS_0 + \int_{V_0} \rho_0\mathbf{B}(\mathbf{X},t)dV_0, \tag{2.6.8}$$

$$\frac{D}{Dt}\int_{V_0} \mathbf{x}\times\rho_0\mathbf{v}dV_0 = \int_{S_0} \mathbf{x}\times\mathbf{T}(\mathbf{X},t,\mathbf{N})dS_0 + \int_{V_0} \rho_0\mathbf{x}\times\mathbf{B}_0(\mathbf{X},t)dV_0, \tag{2.6.9}$$

where ρ_0 is the reference density, $\mathbf{T}$ is the first Piola-Kirchoff traction vector, V_0 is the material volume in the reference configuration and $\mathbf{B}_0$ is the reference body force.

We are now in a position to write down the equations of motion in both the current configuration and reference configuration. Using Cauchy's stress theorem and the divergence theorem, which transforms a surface integral into a volume integral, we obtain for the Cauchy traction vector in the current configuration

$$\int_s \mathbf{t}(\mathbf{x},t,\mathbf{n}) \;=\; \int_s \boldsymbol{\sigma}(\mathbf{x},t)\mathbf{n}ds \;=\; \int_v \mathrm{div}\boldsymbol{\sigma}(\mathbf{x},t)dv. \tag{2.6.10}$$

Substituting equation (2.6.10) into the balance of linear momentum (2.6.6) we obtain

$$\int_v \left(\mathrm{div}\boldsymbol{\sigma} + \rho\mathbf{b} - \rho\frac{D\mathbf{v}}{Dt}\right)dv = 0. \tag{2.6.11}$$

We assume that the integrand is continuous. It follows that the integrand must vanish. Thus

$$\mathrm{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\dot{\mathbf{v}}, \tag{2.6.12}$$

where $\dot{\mathbf{v}} = D\mathbf{v}/Dt$. This equation is called Cauchy's first equation of motion and is defined with respect to the current configuration. Since $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{u}}$ equation (2.6.12) can be written as

$$\mathrm{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\ddot{\mathbf{u}}. \tag{2.6.13}$$

We can write the corresponding equation with respect to the reference configuration, following the same procedure as above, as

$$\mathrm{Div}\mathbf{P} + \rho_0\mathbf{B}_0 = \rho_0\dot{\mathbf{V}}, \tag{2.6.14}$$

where $\mathbf{P}$ is the first Piola stress tensor and $\dot{\mathbf{V}} = D\mathbf{V}/Dt$ is the acceleration in the reference configuration. We also take note of the following important transformation

$$\mathrm{Div}\mathbf{P} = J\mathrm{div}\boldsymbol{\sigma}. \tag{2.6.15}$$

Since the reference configuration is taken as the initial configuration, the displacement in the reference configuration vanish and $\mathbf{U}(\mathbf{X}, t) = \mathbf{u}(\mathbf{x}, t)$. Since $\mathbf{A} = \dot{\mathbf{V}} = \ddot{\mathbf{U}}$, we have $\mathbf{A} = \dot{\mathbf{V}} = \ddot{\mathbf{u}}$ where $\mathbf{A}$ is the acceleration field in the reference configuration.

It can be shown using the Principle of Angular Momentum that the Cauchy stress tensor σ is symmetric, that is

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{2.6.16}$$

This result is often referred to as Cauchy's second law of motion. The second Piola-Kirchoff stress tensor, which is defined in the reference configuraion is also symmetric. We note that in linearised elasticity these stress tensors are taken to be the same since we consider small displacements. Since the motion occurs in the current configuration we will provide explicit forms of the equations of motion in terms of the basis vectors in the current configuration. The components of the equations of motion will be derived for the rectangular Cartesian coordinate system as well as for the cylindrical and spherical coordinate systems.

2.6.1 Rectangular Cartesian coordinates

We recall that any second order tensor $\boldsymbol{\sigma}$ can be written as (2.3.21) and expanding the gradient of a second order tensor (2.3.27) by summation over repeated indices and by considering the trace of the gradient of this second order tensor we obtain the divergence of the stress defined by

$$\mathrm{div}\boldsymbol{\sigma} = \mathrm{tr}(\nabla \otimes \boldsymbol{\sigma}). \tag{2.6.17}$$

That is, after a double contraction of the basis vectors with the identity tensor we have

$$\begin{aligned} \mathrm{div}\boldsymbol{\sigma} &= \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}\right)\mathbf{e}_y \\ &+ \left(\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\mathbf{e}_z. \end{aligned} \tag{2.6.18}$$

Thus we can substitute (2.6.18) into the equation of motion (2.6.13) to obtain in component form

x-component of momentum:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = \rho \ddot{u}_x, \tag{2.6.19}$$

y-component of momentum:

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = \rho \ddot{u}_y, \tag{2.6.20}$$

z-component of momentum:

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = \rho \ddot{u}_z. \tag{2.6.21}$$

2.6.2 Cylindrical polar coordinates

In curvilinear coordinates the second order tensor may be expressed in terms of covariant and contravariant base vectors. We will write the second order tensor $\boldsymbol{\sigma}$ in terms of the contravariant base vectors,

$$\boldsymbol{\sigma} = \sigma_{ij} \mathbf{g}^i \otimes \mathbf{g}^j. \tag{2.6.22}$$

The gradient is thus given by

$$\nabla \otimes \boldsymbol{\sigma} = \frac{\partial}{\partial \theta^k} \left(\sigma_{ij} \mathbf{g}^i \otimes \mathbf{g}^j \right) \otimes \mathbf{g}^k = \left(\frac{\partial \sigma_{ij}}{\partial \theta^k} - \Gamma^l_{ki} \sigma_{lj} - \Gamma^l_{kj} \sigma_{il} \right) \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k. \tag{2.6.23}$$

Following the same procedure as for the rectangular coordinate system we obtain

$$\begin{aligned} \mathrm{div}\boldsymbol{\sigma} &= \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} \right) \mathbf{e}_r \\ &+ \left(\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} + \frac{\partial \sigma_{\theta z}}{\partial z} \right) \mathbf{e}_\theta \\ &+ \left(\frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\sigma_{zr}}{r} + \frac{\partial \sigma_{zz}}{\partial z} \right) \mathbf{e}_z. \end{aligned} \tag{2.6.24}$$

The equations of motion in component form are, after substituting (2.6.24) into (2.6.13), given by

Radial component of motion:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} + b_r = \rho \ddot{u}_r, \tag{2.6.25}$$

Circumferential component of motion:

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} + \frac{\partial \sigma_{\theta z}}{\partial z} + b_\theta = \rho \ddot{u}_\theta, \tag{2.6.26}$$

Axial component of motion:

$$\frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\sigma_{zr}}{r} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = \rho \ddot{u}_z. \tag{2.6.27}$$

2.6.3 Spherical polar coordinates

In spherical coordinates we obtain the following results

$$\begin{aligned}
\mathrm{div}\boldsymbol{\sigma} &= \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta}\cot\theta}{r}\right)\mathbf{e}_r \\
&+ \left(\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{3\sigma_{r\theta} - \sigma_{\phi\phi}\cot\theta + \sigma_{\theta\theta}\cot\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi}\right)\mathbf{e}_\theta \\
&+ \left(\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta}{r}\right)\mathbf{e}_\phi.
\end{aligned} \tag{2.6.28}$$

We can use equation (2.6.13) to deduce the equations of motion in component form

Radial component of motion:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta}\cot\theta}{r} + b_r = \rho \ddot{u}_r, \tag{2.6.29}$$

Circumferential component of motion:

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{3\sigma_{r\theta} - \sigma_{\phi\phi}\cot\theta + \sigma_{\theta\theta}\cot\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + b_\theta = \rho \ddot{u}_\theta, \tag{2.6.30}$$

Meridian component of motion:

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta}{r} + b_\phi = \rho \ddot{u}_\phi. \tag{2.6.31}$$

2.7 Linearised Elasticity

In this section we provide a very brief description of the equations of linearised elasticity.

The assumptions made in linear elasticity will allow for any solid material to modelled within this framework provided the stresses are small. Typically, this theory is useful to describe materials in which the stress and strain are linearly related and where the displacement gradients are small. Further, the material will return to its undeformed state once the forces causing the deformation are removed. This is different from plastic materials where the deformation is permanent and also different from viscoelastic materials which are materials that behave as either solids or fluids. Viscoelastic fluids display a time dependence as well. The theory of plasticity and viscoelasticity will not be discussed further as it is not within the scope of this thesis. There are two types of linear relations that can be described within the theory of linearised elasticity. The first of these linear relations is kinematic linearity where for sufficiently small displacement gradients, the stress and strain have a linear relationship. The second relationship is where the constitutive equations describing a material are linearly related. This is called material symmetry.

This leads us to one of the most important constitutive equations used in linear elasticity and that is Hooke's Law. It is found experimentally that in a one-dimensional system the stress is approximately proportional to the strain until the proportionality limit is achieved. Hooke formulated this observation in one-dimension as

$$\sigma_{11} = EE_{11} \tag{2.7.1}$$

where τ_{11} and E_{11} is the stress and strain respectively of the one dimensional system. In order to describe the stiffness of the material a constant of proportionality was introduced. This constant E is called the Young's modulus. Since Hooke's Law in linear, we can use the principle of superposition where we can assume that each stress component τ_{ik} is a linear combination of all the strain components E_{ik} and genralise his one dimensional formulation to

$$\sigma_{ik} = C_{ikrs}\epsilon_{rs}, \tag{2.7.2}$$

where C_{ikrs} is a fourth order tensor and

$$\epsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \tag{2.7.3}$$

where $\mathbf{u}$ is the displacement vector. Equation (2.7.3) is the strain tensor used in linearised Elasticity. We note that in linearised elasticity the nonlinear terms $\mathcal{O}(\frac{\partial u_i}{\partial x_j}\frac{\partial u_m}{\partial x_n})$ in the strain tensor are neglected. Equation (2.7.2) is called the

Generalised Hooke's Law.

A material is said to be elastically isotropic if the elastic properties of a material are identical in all directions. Thus for an elastically isotopic homogeneous material the Generalised Hooke's Law can be written as

$$\sigma_{ik} = \lambda\theta\delta_{ik} + 2\mu\epsilon_{ik}, \tag{2.7.4}$$

where λ and μ are the Láme constants and $\theta = \nabla \cdot \mathbf{u}$ is called the dilation. The strain tensor ϵ_{ik} is defined in (2.7.3) and δ_{ik} is the unit tensor given by (A.1.8). The parameter μ is of significance in that it describes the shear modulus which is the resistance of a material to shear.

Due to the linear relationship between the stress and strain, it is possible to write the strain in terms of the stress. This relationship is described by the Inverse Hooke's Law. It is usually expressed in terms of Young's modulus E or the Poisson ratio σ. The Poisson ration relates the lateral contraction to the longitudinal extension. The Inverse Hooke's Law is given by

$$\epsilon_{ik} = -\frac{\sigma}{E}\Theta\delta_{ik} + \frac{(1+\tau)}{E}\sigma_{ik}, \tag{2.7.5}$$

where

$$\Theta = \sigma_{kk}, \qquad E = \frac{(3\lambda + 2\mu)}{\lambda + \mu}\mu, \qquad \tau = \frac{\lambda}{2(\lambda + \mu)}. \tag{2.7.6}$$

2.7.1 Cauchy's laws of motion

We will only state Cauchy's laws of motion without proof.

For Linear elasticity the approximation is made that the equations of motion apply in the undeformed body.

- **Cauchy's first law of motion**

  $$\rho_0 \frac{Dv_i}{Dt} = \sigma_{ik,k} + \rho_0 b_i \tag{2.7.7}$$

  where ρ_0 is the density in the undeformed body and b_i is the body force.

- **Cauchy's second law of motion**

  $$\sigma_{ik} = \sigma_{ki}. \tag{2.7.8}$$

- **Navier's displacement equation of motion**

  $$\rho_0 \frac{D^2 u_i}{Dt^2} = (\lambda + \mu)\theta_{,i} + \mu\nabla^2 u_i + \rho_0 b_i, \tag{2.7.9}$$

  where ρ_0 is the density in the undeformed body and b_i is the body force per unit mass.

2.8 Conclusion

In this Chapter a summary of the fundamental concepts in Finite Elasticity was discussed. Further, explicit form of the strain tensor and equations of motion for rectangular Cartesian coordinates as well as for cylindrical and spherical polar coordinates was presented. In the final part of the chapter key definitions and useful formulae used in Linear Elasticity were presented.

Chapter 3

Constitutive equations in Cauchy elasticity, hyperelasticity and implicit constitutive equations

3.1 Introduction

We will present a brief review of Cauchy elasticity and hyperelasticity and derive a subclass of implicit constitutive equations in this Chapter. The proofs of the results which we include here will be omitted. For a more detailed account of Cauchy elasticity and hyperelasticity the reader is referred to the excellent texts by Holzapfel [45] and Green and Adkins [46]. Rajagopal [5] has developed a more general framework within which a larger class of elastic materials can be studied. In the general framework he provides implicit relationships between the stress and deformation gradient and also cases where the stored energy depends on both the stress and deformation gradient. A particular subclass of these implicit constitutive equations that we consider in this thesis are constitutive equations where the strain is prescribed as a non-invertible function of the stress. The two constitutive equations of this type that we consider are the power-law constitutive equation and the strain-limiting constitutive equation.

The Chapter is structured as follows. In Section 3.2 we discuss the subject of frame-indifference. Cauchy elastic materials and the constitutive equations used to describe such materials are given in Section 3.3. Similarly, we discuss the assumptions of hyperelasticity in Section 3.4 and provide general forms for the constitutive equation used. Finally, the theory and derivation of a subclass of implicit constitutive equations are presented in Section 3.5.

3.2 Frame-indifference

We begin our development of the constitutive equations in Cauchy elasticity and hyperelasticity with the topic of frame-indifference.

Suppose we consider a rigid body motion between the points $(\mathbf{x}_0, t_0)$, $(\mathbf{x}, t)$ and $(\hat{\mathbf{x}}_0, \hat{t}_0)$, $(\hat{\mathbf{x}}, \hat{t})$. Mathematically, we can express this mapping as

$$\hat{\mathbf{x}} - \hat{\mathrm{x}}_0 = \mathbf{Q}(t)(\mathbf{x} - \mathbf{x}_0), \tag{3.2.1}$$

where $\mathbf{Q}(t)$ is a proper orthogonal tensor since $\det\mathbf{Q} = 1$ and $\mathbf{Q}^{-1}(t) = \mathbf{Q}^T(t)$. This tensor characterises a rigid body rotation between two points. We can write equation (3.2.1) as

$$\hat{\mathbf{x}} = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}, \tag{3.2.2}$$

where

$$\mathbf{c}(t) = \hat{\mathbf{x}}_0 - \mathbf{Q}(t)\mathbf{x}_0. \tag{3.2.3}$$

In other words, the motion is now described as a rigid body rotation through $\mathbf{Q}(t)$ and a translation through vector $\mathbf{c}(t)$ between the points $(\mathbf{x}_0, t_0)$, $(\mathbf{x}, t)$ and $(\hat{\mathbf{x}}_0, \hat{t}_0)$, $(\hat{\mathbf{x}}, \hat{t})$. This transformation is called a Euclidean transformation and has important implications in the development of constitutive theories.

This leads us to the idea of objectivity or frame indifference. If a physical quantity is unchanged for a change in observer, then that physical quantity is frame indifferent. We do note however, that there are tensors which are not frame indifferent. Rajagopal [5] argues that material frame indifference is not an absolute requirement and that Galilean invariance is sufficient, especially for elastic materials. Rajagopal [5] emphasises this point in that one can arrive at an equation, for example the Navier-Stokes equation, by Galilean invariance only.

For a vector field u, tensor field A and scalar field ϕ to be frame indifferent the following requirements must be satisfied

$$\hat{\mathbf{u}}(\hat{\mathbf{x}}, \hat{t}) = \mathbf{Q}(t)\mathbf{u}(\mathbf{x}, t), \quad \hat{\mathbf{A}}(\hat{\mathbf{x}}, \hat{t}) = \mathbf{Q}(t)\mathbf{A}(\mathbf{x}, t)\mathbf{Q}^T(t), \quad \hat{\phi}(\hat{\mathbf{x}}, \hat{t}) = \phi(\mathbf{x}, t). \tag{3.2.4}$$

These relations are essential in the development of the constitutive equations used in Cauchy elasticity and hyperelasticity.

3.3 Cauchy elasticity

A Cauchy elastic material can be defined as a material where the stress in the current configuration f_t is determined by the state of deformation of the current configuration f_t. For a Cauchy elastic material the stress does not depend on the deformation path or the deformation history.

In order to describe certain properties inherent in materials we can turn to empirical data or construct mathematical generalisations to relate the macroscopic properties to the microscopic ones. Such descriptions are called constitutive equations. For a Cauchy elastic body we will consider a constitutive equation where the stress and deformation gradient are explicitly related. That is

$$\boldsymbol{\sigma} = \mathfrak{g}(\mathbf{F}), \tag{3.3.1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{F}$ the deformation gradient and $\mathfrak{g}$ the response function. Suppose we assume that the Cauchy stress tensor $\boldsymbol{\sigma}$ depends on the left Cauchy-Green stretch tensor $\mathbf{b}$

$$\boldsymbol{\sigma} = \mathfrak{g}(\mathbf{b}). \tag{3.3.2}$$

If this equation satisfies

$$\mathbf{Q}\mathfrak{g}\mathbf{Q}^T = \mathfrak{g}(\mathbf{Q}\mathbf{b}\mathbf{Q}^T) \tag{3.3.3}$$

for all proper orthogonal tensors $\mathbf{Q}$, then we say that the material is isotropic. Physically, isotropy is characterised by the property that the material exhibits no preferred direction, for example, rubber. Wood is not an isotropic material. By the Rivilin-Ericksen theorem this relation can be expressed explicitly as

$$\boldsymbol{\sigma} = \mathfrak{g}(\mathbf{b}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{b} + \alpha_2\mathbf{b}^2, \tag{3.3.4}$$

where the material moduli α_1, $i = 0, 1, 2$ depend on the invariants of the left Cauchy-Green tensor. We note that only three material parameters are necessary to completely describe the state of stress.

For an incompressible material, $J = 1$, the constitutive equation is given by

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathfrak{g}(\mathbf{F}) \tag{3.3.5}$$

where p is a pressure-like quantity and can only be determined by the boundary conditions and equilibrium conditions.

3.4 Hyperelasticity

A brief review of some of the most popular constitutive equations used in finite elasticity theory is presented.

A subclass of Cauchy elastic materials are materials wherein the stress is derived from a potential, the Helmholz free-energy function Φ, associated with the material. Such materials are referred to as hyperelastic materials or Green elastic materials. Hyperelastic models have proved to model real materials more accurately than Cauchy elastic models and recently Carroll [7] showed that Cauchy elastic bodies *must* be hyperelastic. The function $\Phi = \Phi(\mathbf{F})$ is referred to as the strain-energy function or stored-energy function.

The first Piola-Kirchhoff stress tensor $\mathbf{P}$ is related to the potential Φ through

$$\mathbf{P} = \frac{\partial \Phi(\mathbf{F})}{\partial \mathbf{F}}, \tag{3.4.1}$$

and as a result we can, by using

$$\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^T = \boldsymbol{\sigma}^T, \tag{3.4.2}$$

express the Cauchy stress tensor in terms of the function Φ as

$$\boldsymbol{\sigma} = J^{-1}\frac{\partial \Phi(\mathbf{F})}{\partial \mathbf{F}}\mathbf{F}^T = J^{-1}\left(\frac{\partial \Phi(\mathbf{F})}{\partial \mathbf{F}}\right)^T. \tag{3.4.3}$$

For isotropic hyperelastic materials we can determine constitutive equations in terms of the strain invariants. An example of this, for the right Cauchy-Green stretch tensor we have

$$\Phi = \Phi[I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C})] \tag{3.4.4}$$

which is valid for isotropic hyperelastic materials satisfying (3.3.3). We can differentiate (3.4.4) with respect to the tensor $\mathbf{C}$ using the chain rule to find a constitutive equation for hyperelastic materials in terms of $\mathbf{C}$. That is

$$\frac{\partial \Phi(\mathbf{C})}{\partial \mathbf{C}} = \frac{\partial \Phi}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{C}} + \frac{\partial \Phi}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{C}} + \frac{\partial \Phi}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{C}}, \tag{3.4.5}$$

where

$$\frac{\partial I_1}{\partial \mathbf{C}} = \mathbf{I}, \quad \frac{\partial I_2}{\partial \mathbf{C}} = I_1\mathbf{I} - \mathbf{C}, \quad \frac{\partial I_3}{\partial \mathbf{C}} = I_3\mathbf{C}. \tag{3.4.6}$$

Further, one can develop constitutive equations for the second Piola-Kirchhoff stress tensor as well as for the Cauchy stress tensor. The second Piola-Kirchhoff stress tensor is defined as

$$\mathbf{S} = -p\mathbf{F}^{-1}\mathbf{F}^{-T} + \mathbf{F}^{-1}\frac{\partial \Phi(\mathbf{F})}{\partial \mathbf{F}} = -p\mathbf{C}^{-1} + 2\frac{\partial \Phi(\mathbf{C})}{\partial \mathbf{C}}. \tag{3.4.7}$$

The second Piola-Kirchhoff stress tensor for a hyperelastic material is given by

$$\mathbf{S} = 2\frac{\partial \Phi(\mathbf{C})}{\partial \mathbf{C}} \tag{3.4.8}$$

where again Φ is given in terms of the strain invariants (3.4.4). The constitutive equation in terms of the strain invariants for the second Piola-Kirchhoff stress tensor is

$$\mathbf{S} = 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\mathbf{I} - \frac{\partial \Phi}{\partial I_2}\mathbf{C} + I_3 \frac{\partial \Phi}{\partial I_3}\mathbf{C}^{-1}\right] \tag{3.4.9}$$

and for the Cauchy stress tensor in terms of the strain invariants

$$\boldsymbol{\sigma} = 2J^{-1}\left[\left(I_2 \frac{\partial \Phi}{\partial I_2} + I_3 \frac{\partial \Phi}{\partial I_2}\right)\mathbf{I} + \frac{\partial \Phi}{\partial I_1}\mathbf{b} - I_3 \frac{\partial \Phi}{\partial I_2}\mathbf{b}^{-1}\right]. \tag{3.4.10}$$

The constraint of incompressibility is that the Jacobian, J, of the deformation gradient $\mathbf{F}$ must be unity, i.e, $J = 1$. Many materials exhibit the property of isochoric motion under finite strain. To describe this behaviour, various constitutive equations have been proposed. We state the general forms of these constitutive equations for the first and second Piola-Kirchhoff stress tensor as well as for the Cauchy stress tensor,

$$\mathbf{P} = -p\mathbf{F}^{-T} + \frac{\partial \Phi(\mathbf{F})}{\partial F}, \quad \mathbf{S} = -p\mathbf{C}^{-1} + 2\frac{\partial \Phi(\mathbf{C})}{\partial C}, \quad \boldsymbol{\sigma} = -p\mathbf{I} + \frac{\partial \Phi(\mathbf{F})}{\partial F}\mathbf{F}^{T}. \tag{3.4.11}$$

For incompressible isotropic hyperelastic materials, the strain-energy function Φ is either a function of the right Cauchy-Green stretch tensor $\mathbf{C}$ or the left Cauchy-Green stretch tensor $\mathbf{b}$ and as such may also be expressed in terms of the strain invariants. The development of constitutive equations for Cauchy elasticity and hyperelasticity can also be extended to compressible materials with the constraint that $J > 0$. This constraint ensures that the material does not penetrate itself.

In order to study the behaviour of elastic materials within the framework of hyperelasticity many forms of the strain-energy function have been proposed and tested experimentally to assess its accuracy in describing the stress response of various materials. Popular models include the Ogden, Mooney-Rivilin and Neo-Hookean models for incompressible materials and the Ogden and Blatz-Ko model for compressible materials.

3.5 Implicit constitutive theory

The aim of this section is to introduce the concept of implicit constitutive equations. We in no way intend to provide an in depth and detailed theoretical exposition of the theory, but rather to focus on how constitutive equations can be found under the umbrella of this rich theory. For a more detailed account of implicit constitutive theories, the reader is referred to the comprehensive paper by Rajagopal [5].

In this work we are considering materials which are incapable of dissipation and materials which are described by an implicit relationship between the stress and deformation gradient. Implicit constitutive equations have been studied, although not extensively, in fluid mechanics however such considerations have recently started gaining attention in solid mechanics. As alluded to earlier the importance of such classes of constitutive equations cannot be overemphasised. As an example to demonstrate the need for implicit relations consider Figure 3.5.1 below [8].

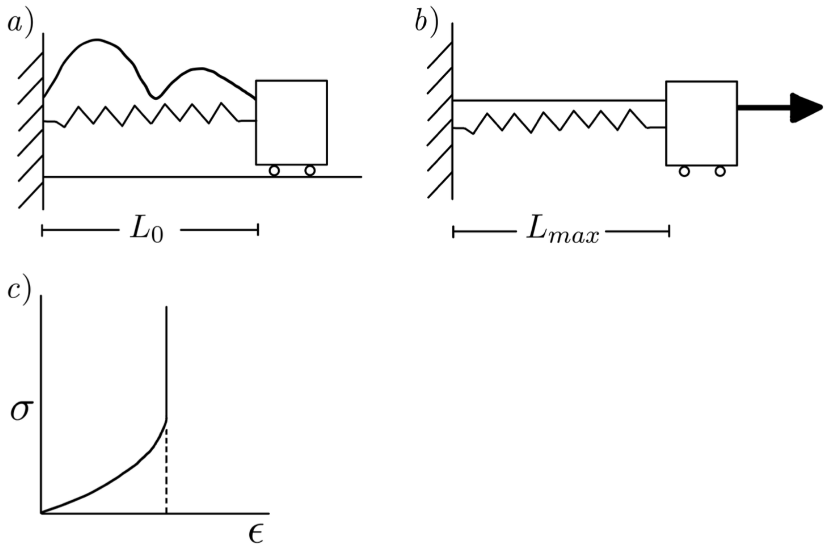

Figure 3.5.1: Mechanical analogue of a spring system in parallel with an inextensible string. (a)The unstretched spring and string. (b) Maximum extension of spring. (c) Stress versus Strain curve.

A spring in parallel with an inextensible string. The string is longer than the spring in the unloaded state and we say that the string is slack. Once the spring is loaded the string will reach a critical value when it becomes taut. This means that the extension of the spring cannot exceed a critical value. This scenario is depicted in Figure 3.5.1 (b) and (c) [8].

In the context of the traditional theories of elasticity one can see that the force cannot be expressed as a function of the stretch beyond the critical value and beyond this value we would have to define a constant strain to describe the body. Thus, in order to describe the elastic response of Figure 3.5.1 equations of the form $f(\boldsymbol{\sigma}, \mathbf{F}) = 0$ might be more useful. For a one-dimensional response, the response of the spring is reversible but one cannot derive the stress from a potential for the whole range of the response. Thus if we consider the potential, ψ, to depend on both the stress σ and strain ϵ we can describe the elastic response as [5]

$$\psi = \psi(\sigma, \epsilon) = \begin{cases} \hat{\psi}(\epsilon), & 0 \leq \sigma \leq \sigma_{\text{critical}}, \\ \psi_{\text{critical}} = constant, & \sigma > \sigma_{\text{critical}}, \end{cases} \tag{3.5.1}$$

and the constitutive equation is given by

$$\epsilon = \begin{cases} g(\sigma), & \sigma \leq \sigma_{\text{critical}} \\ \epsilon_{\text{critical}}, & \sigma > \sigma_{\text{critical}} \end{cases} \tag{3.5.2}$$

We now derive a general form for the implicit constitutive equations. We recall a few important definitions given by equations (2.2.2), (2.2.5), (2.2.14), (2.3.15) and (2.3.16) to describe the motion, displacement vector, deformation gradient and the displacement gradient tensor in both the spatial and material descriptions. The strain tensors in both descriptions are given by equations (2.3.9), (2.3.1), (2.3.6) and (2.3.12). We will pay special attention to the left Cauchy-Green tensor (2.3.9). Suppose that the Cauchy stress tensor $\boldsymbol{\sigma}$ depends on the left Cauchy-Green stress tensor $\mathbf{b}$

$$\boldsymbol{\sigma} = \mathfrak{h}(\mathbf{b}), \tag{3.5.3}$$

where $\mathfrak{h}$ is the response function. If the response function $\mathfrak{h}$ is isotropic, then by Rivlin's first representation theorem, [45], the Cauchy stress tensor may be written explicitly in terms of the stretch $\mathbf{b}$,

$$\boldsymbol{\sigma} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{b} + \alpha_2 \mathbf{b}^2, \tag{3.5.4}$$

where α_i, $i = 0, 1, 2$, are the response coefficients or material moduli. In general, for isotropic materials only three parameters are necessary to describe the stress. Further, the material moduli depend on the invariants of $\mathbf{b}$, and the invariants of $\mathbf{b}$ are defined as

$$I_1(\mathbf{b}) = \text{tr}(\mathbf{b}), \quad I_2(\mathbf{b}) = \frac{1}{2}((\text{tr}\mathbf{b})^2 - \text{tr}(\mathbf{b}^2)), \quad I_3 = \text{det}\mathbf{b}. \tag{3.5.5}$$

Rajagopal [5] has introduced the implicit constitutive relation, which we will be using,

$$f(\boldsymbol{\sigma}, \mathbf{b}) = 0. \tag{3.5.6}$$

Since f is an isotropic function it has to satisfy

$$f(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T, \mathbf{Q}\mathbf{b}\mathbf{Q}^T) = \mathbf{Q}f(\boldsymbol{\sigma}, \mathbf{b})\mathbf{Q}^T \tag{3.5.7}$$

where $\mathbf{Q}$ is orthogonal. The isotropic nature of f is defined within the framework of implicit equations and is defined in terms of the symmetry classification of materials [4]. It has been shown in [5] that f is an isotropic function so equation (3.5.6) can be written as [47]

$$\alpha_0 \mathbf{I} + \alpha_1 \boldsymbol{\sigma} + \alpha_2 \mathbf{b} + \alpha_3 \boldsymbol{\sigma}^2 + \alpha_4 \mathbf{b}^2 + \alpha_5(\boldsymbol{\sigma}\mathbf{b} + \mathbf{b}\boldsymbol{\sigma}) + \alpha_6(\boldsymbol{\sigma}^2\mathbf{b} + \mathbf{b}^2\boldsymbol{\sigma}^2)$$

$$+ \quad \alpha_7(\mathbf{b}^2\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{b}^2) + \alpha_8(\boldsymbol{\sigma}^2\mathbf{b}^2 + \mathbf{b}^2\boldsymbol{\sigma}^2) = 0 \tag{3.5.8}$$

where the material moduli α_i, $i = 0, ..., 8$ depend on

$$\rho, \ \mathrm{tr}\boldsymbol{\sigma}, \ \mathrm{tr}\mathbf{b}, \ \mathrm{tr}\boldsymbol{\sigma}^2, \ \mathrm{tr}\mathbf{b}^2, \ \mathrm{tr}\boldsymbol{\sigma}^3, \ \mathrm{tr}\mathbf{b}^3, \mathrm{tr}(\boldsymbol{\sigma}\mathbf{b}), \ \mathrm{tr}(\boldsymbol{\sigma}^2\mathbf{b}), \ \mathrm{tr}(\mathbf{b}^2\boldsymbol{\sigma}), \ \mathrm{tr}(\boldsymbol{\sigma}^2\mathbf{b}^2). \tag{3.5.9}$$

We note that equation (3.5.4) is a subclass of the implicit representation (3.5.8). We can define another subclass where the stretch is now given in terms of the stress tensor $\boldsymbol{\sigma}$, i.e,

$$\mathbf{b} = \bar{\alpha}_0\mathbf{I} + \bar{\alpha}_1\boldsymbol{\sigma} + \bar{\alpha}_2\boldsymbol{\sigma}^2. \tag{3.5.10}$$

The material moduli depend on ρ, $\mathrm{tr}\boldsymbol{\sigma}$, $\mathrm{tr}\boldsymbol{\sigma}^2$, $\mathrm{tr}\boldsymbol{\sigma}^3$. The representation (3.5.10) is not always obtainable by inverting (3.5.4). It is to these models that we direct our attention.

A further assumption is that we appeal to small displacement gradients in order to study the relationship between stress and linearised strain. Using this assumption and the relations (2.3.16), (2.3.9) and (3.5.10) we obtain the following approximations

$$\mathbf{E} = \boldsymbol{\epsilon} + \mathcal{O}(\delta^2), \quad \mathbf{e} = \boldsymbol{\epsilon} + \mathcal{O}(\delta^2), \quad \mathbf{b} = \mathbf{I} + 2\boldsymbol{\epsilon} + \mathcal{O}(\delta^2), \tag{3.5.11}$$

where

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T\right) \tag{3.5.12}$$

is the linearised strain tensor. Since the displacements are small the gradient of the displacement vector may be computed in the reference or current configuration. Since

$$\mathbf{b} = \mathbf{I} + 2\boldsymbol{\epsilon} + \mathcal{O}(\delta^2), \tag{3.5.13}$$

and using (3.5.10) we obtain the approximation

$$\boldsymbol{\epsilon} = \frac{1}{2}(\bar{\alpha}_0 - \mathbf{I}) + \frac{1}{2}\bar{\alpha}_1\boldsymbol{\sigma} + \frac{1}{2}\bar{\alpha}_2\boldsymbol{\sigma}^2. \tag{3.5.14}$$

We can write this as

$$\boldsymbol{\epsilon} = \beta_0\mathbf{I} + \beta_1\boldsymbol{\sigma} + \beta_2\boldsymbol{\sigma}^2, \tag{3.5.15}$$

where

$$\beta_0 = \frac{1}{2}(\bar{\alpha}_0 - \mathbf{I}), \quad \beta_1 = \frac{1}{2}\bar{\alpha}_1, \quad \beta_2 = \frac{1}{2}\bar{\alpha}_2. \tag{3.5.16}$$

We have the freedom to choose a subclass of models defined by (3.5.15) as a constitutive equation. The constitutive equations which we will work with are given by [42, 43, 34],

$$\boldsymbol{\epsilon} = \beta(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I} + \frac{\alpha}{2}\left(1 + \frac{\gamma}{2}(\mathrm{tr}\boldsymbol{\sigma}^2)\right)^n \boldsymbol{\sigma} \tag{3.5.17}$$

and

$$\boldsymbol{\epsilon} = \beta\left[1 - \frac{1}{1 + \dfrac{\mathrm{tr}\boldsymbol{\sigma}}{(1 + \kappa \mathrm{tr}\boldsymbol{\sigma}^2)}}\right]\mathbf{I} + \frac{\alpha}{2}\left[1 + \frac{1}{(1 + \dfrac{\gamma}{2}(\mathrm{tr}\boldsymbol{\sigma}^2))}\right]^n \boldsymbol{\sigma} \qquad (3.5.18)$$

where for both constitutive equations the constants $\beta \geq 0$, $\alpha \geq 0$, $\gamma \geq 0$, $\delta \geq 0$ and $n \geq 0$.

Constitutive equations of the form (3.5.15) provide a mathematical framework to describe experimental observations which cannot be described when using Cauchy elasticity or hyperelasticity. We have chosen two specific subclasses of equation (3.5.15) given by (3.5.17) and (3.5.18). One of the primary reasons for choosing the two models (3.5.17) and (3.5.18) is to compare the elastic response in materials described by these constitutive equations. For example, the second model (3.5.18) highlights the possibility that one can describe problems involving singularities which often lead to contradictions when using classical elasticity theory as discussed in Section 1.4. In particular, constitutive equations of this type (3.5.18) have the interesting feature of bounded strains regardless of how large the stresses (or non-dimensional stresses) become. If one considers the constitutive equation (3.5.18) and calculates the components of the strain, we find that as $\sigma \to \infty$ the strain tends to a finite (bounded) value. To develop the specific forms of the constitutive equation (3.5.15) consider again (3.5.18). Since we want the body to be unstrained when unstressed we need to ensure that the linearised strain tensor is zero when the stress is zero. That is when $\boldsymbol{\sigma} = 0, \boldsymbol{\epsilon} = 0$. If we want to describe the strain-limiting properties we would need to choose the material moduli, which depend on the invariants of the strain tensor such that the strain remains bounded as the stress becomes large. Admittedly, there are simpler constitutive models that can be used to describe strain-limiting behaviour such as the models considered in [34, 51]. However, these models do not consider the shearing parameter n and we want to investigate the effect of this shearing parameter on the wave motion. The first part of the constitutive equation (3.5.18) can be neglected for incompressible materials since the restriction that the $\mathrm{tr}(\boldsymbol{\epsilon}) = 0$ must hold. The constitutive equation (3.5.15) is quite general in that it holds for compressible materials as well. For compressible materials the $\mathrm{tr}(\boldsymbol{\epsilon}) \neq 0$. For example we could consider certain compatible stress fields and displacement fields in cylindrical polar coordinates that vary radially with non-zero components along the diagonal of the linearised strain matrix. We have not omitted this first part of the constitutive equation since it is of interest (for future work) to study wave propagation for compressible materials. The power-law constitutive equation (3.5.17) on the other hand is a model which does not exhibit strain-limiting features but rather shows that in a small displacement range there is a distinct nonlinear relationship between the linearised strain and stress. These models, unlike (3.5.18) would not be suitable to describe the stresses and strains at crack-tips or the response for brittle elastic solids. Models of this form can be viewed as a counterpart to the generaised Neo-Hookean model and it is also a generalised counterpart to the constitutive equation for a non-Newtonian power-law fluid. Equation (3.5.17) was solved numerically in [42] and we aim to provide asymptotic and perturbation methods to solve the governing equations and compare the solutions with the results for the strain limiting model.

Chapter 4

Two-dimensional nonlinear stress and displacement waves for a new class of constitutive equations

The contents of this Chapter was published in Wave Motion. The details of the publication are as follows:
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4.1 Introduction

In this Chapter we investigate the propagation of linear and nonlinear displacement and stress waves in a rectangular slab for a new class of constitutive equations. The new class of constitutive equations are a more general class of constitutive equations and are described by implicit relationships between the stress and strain. Classical Cauchy elasticy and hyperelastcity are subclasses of the implicit constitutive equations as is shown in Chapter 3. The constitutive equations that we consider in this Chapter are a special subclass of the implicit constitutive equations where there is an explicit relationship between the left Cauchy-Green stretch tensor $\mathbf{b}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ which is non-invertible. We consider two different constitutive equations. The first constitutive equation that we study is called a power-law constitutive equation (3.5.17) due to its analogy with the constitutive equation for a power-law fluid. The second constitutive equation is called a strain-limiting constitutive equation (3.5.18) because it has the property that the stresses may becomes very large while the strain does not become singular.

In this Chapter we find asymptotic solutions for the nonlinear wave equations for both the power-law constitutive equation and the strain-limiting equation. We find implicit asymptotic solutions for both constitutive equations which describe the propagation of solitary stress waves. We then find perturbation solutions for the system of partial differential equations for the displacement and stress for both constitutive equations and find travelling wave solutions and standing wave solutions. To find the perturbation solutions we consider straightforward or regular perturbation expansions for the displacement and stress. We have shown in Appendix B that singular perturbation is unsuccessful in removing the secularity however, the solutions obtained are good approximations. There are other powerful methods to find exact solutions for partial differential equations like the Lie symmetry method. We cannot use this method for the equations under study because the boundary conditions are not invariant.

The Chapter is organised as follows. In Section 4.2 we derive the mathematical model for the power-law constitutive equation. In Section 4.3 we find a solitary wave solution in implicit form for the stress. Perturbation solutions for the displacement and stress are investigated in Section 4.4 and we find travelling wave solutions and standing wave solutions. The results are discussed and analysed in Section 4.5. The second half of the Chapter for the strain-limiting constitutive equation follows the same structure. Finally, the conclusions drawn in Section 4.12.

4.2 Mathematical model for the power-law constitutive equation

In this section we derive a system of nonlinear partial differential equations which describes nonlinear wave propagation. The wave propagation will be defined on a semi-infinite domain as well as on a finite domain. The domain of interest is given by [42]

$$D = \{(X, Y, Z)| -\infty \leq X \leq \infty, 0 \leq Y \leq y_0, -\infty \leq Z \leq \infty\}, \tag{4.2.1}$$

where y_0 is the thickness of the rectangular slab. When we consider the semi-infinite domain, $y_0 \to \infty$. In classical elasticity, we usually look for a semi-inverse solution wherein we assume a specific form for the displacement field and solve for this displacement. Within this new framework, we look for a special semi-inverse solution where specific forms for both the stress field and displacement field are assumed. We solve for the stress and displacement simultaneously.

We will assume the following stress and displacement fields

$$\boldsymbol{\sigma} = \sigma(y,t)(\mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x), \quad \mathbf{u} = u(y,t)\mathbf{e}_x. \tag{4.2.2}$$

This displacement field describes transverse displacement waves The conservation of mass, where we relate the mass in the reference configuration to the mass in the current configuration through the continuous scalar fields ρ_0 and ρ is given by

$$\rho_0 = \rho \det \mathbf{F}. \tag{4.2.3}$$

For the motion assumed in (4.2.2) we find that $\det \mathbf{F} = 1$. This means that the body is undergoing isochoric deformation and by allowing for a constant density in either configuration the balance of mass is identically satisfied.

For the first problem under investigation we will consider the constitutive equation (3.5.17)

On using the equations of motion, the components of the strain tensor and the constitutive relation (3.5.17), we obtain the following dimensional system of equations

$$\begin{aligned} \frac{\partial \sigma}{\partial y} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial u}{\partial y} &= \alpha(1+\gamma\sigma^2)^n \sigma. \end{aligned} \tag{4.2.4}$$

We introduce the following non-dimensional variables

$$\bar{y} = \frac{y}{y_0}, \quad \bar{t} = \frac{t}{t_0} = \frac{t}{y_0\sqrt{\alpha\rho}}, \quad \bar{u} = \frac{u}{y_0}, \quad \bar{\sigma} = \sqrt{\gamma}\sigma. \tag{4.2.5}$$

This gives the non-dimensional system of equations

$$\frac{\partial \sigma}{\partial y} = \frac{\sqrt{\gamma}}{\alpha}\frac{\partial^2 u}{\partial t^2}, \tag{4.2.6}$$

$$\frac{\partial u}{\partial y}, = \frac{\alpha}{\sqrt{\gamma}}(1+\sigma^2)^n \sigma. \tag{4.2.7}$$

where the overhead bars have been suppressed. The non-dimensional variables have been chosen in such a way that the effect of the shearing parameter n, for small displacement gradients $\frac{\alpha}{\sqrt{\gamma}}$, can be studied. It is important to remark that for $n = 0$ the constitutive equation (3.5.17) reduces to the classical linearised elasticity model

$$\boldsymbol{\epsilon} = \beta(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I} + \alpha\boldsymbol{\sigma}. \tag{4.2.8}$$

We observe that γ does not appear in the constitutive equation (3.5.17). Using this constitutive equation we obtain the following system of equations

$$\frac{\partial \sigma}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2} \tag{4.2.9}$$

$$\frac{\partial u}{\partial y} = \alpha\sigma. \tag{4.2.10}$$

If we non-dimensionalise (4.2.9) and (4.2.10) using the dimensionless quantities (4.2.5) but with

$$\bar{\sigma} = \sigma \tag{4.2.11}$$

we obtain the dimensionless system of equations

$$\frac{\partial \sigma}{\partial y} = \frac{1}{\alpha}\frac{\partial^2 u}{\partial t^2}, \tag{4.2.12}$$

$$\frac{\partial u}{\partial y}, = \alpha\sigma. \tag{4.2.13}$$

The system of equations (4.2.6) and (4.2.7) and (4.2.9) and (4.2.10) may be written as a single nonlinear hyperbolic partial differential equation describing shear stress waves as

$$\frac{\partial^2 \sigma}{\partial y^2} = \frac{\partial^2}{\partial t^2}[(1+\sigma^2)^n \sigma] \tag{4.2.14}$$

and the system (4.2.12) and (4.2.13) reduces to the classical wave equation. We notice that when $n = 0$ α replaces $\frac{\alpha}{\sqrt{\gamma}}$. The primary reason for decoupling the system of partial differential equations (4.2.6) and (4.2.7) to obtain a single partial differential equation is to highlight the fact that solitary stress waves are possible. As a result of the higher order partial differential equation further boundary conditions and initial conditions for the stress are required and this is discussed in the next section. In the next section the solution of the system of equations (4.2.6) and (4.2.7) for the stress and displacement and the second order wave equation for the stress, (4.2.14), subject to a range of initial and boundary conditions will be investigated.

4.3 Asymptotic solutions for the power-law constitutive equation

4.3.1 Solitary wave solution

In this section we will outline the derivation of an implicit asymptotic solution of the second order wave equation (4.2.14) for the stress subject to the initial condition

$$\sigma(y,0) = \begin{cases} A\sin(2\pi y), & 0 \le y \le 1, \\ 0, & 1 < y < \infty, \end{cases} \tag{4.3.1}$$

where the constant A is prescribed and is the amplitude of the wave and the boundary conditions

$$\sigma(\infty,t)=0,\quad \frac{\partial\sigma}{\partial y}(\infty,t)=0,\quad \frac{\partial\sigma}{\partial t}(\infty,t)=0. \tag{4.3.2}$$

. This solution was first derived by Kannan et al [42] and is included here for comparison with the new solution for the strain-limiting constitutive equation derived in Section 4.7.

Let $\sigma(y,t)=\epsilon\hat{\sigma}(y,t)$, where ϵ is small. Substituting this transformation into (4.2.14) and taking the binomial expansion of $(1+\sigma^2)^n$ up to $\mathcal{O}(\epsilon^2)$ we obtain the asymptotic partial differential equation

$$\frac{\partial^2\hat{\sigma}}{\partial y^2}=\frac{\partial^2\hat{\sigma}}{\partial t^2}+n\epsilon^2\frac{\partial^2(\hat{\sigma}^3)}{\partial t^2}. \tag{4.3.3}$$

Solving equation (4.3.3) using the method of characteristic curves we obtain the implicit asymptotic solution for the stress

$$\sigma(y,t)=\begin{cases} A\sin\left[\dfrac{2\pi\left(\left(1+\frac{3n}{2}\sigma^2(y,t)\right)y-t\right)}{1+\frac{3n}{2}A^2\sin^2\left(2\pi\left(\left(1+\frac{3n}{2}\sigma^2(y,t)\right)y-t\right)\right)}\right], & t\le y\le 1+t,\\ 0, & 1+t\le y\le\infty \end{cases} \tag{4.3.4}$$

The physical interpretation of this result can be made clearer by writing it approximately as

$$\sigma(y,t)=\begin{cases} A\sin\left[2\pi\left(y-\dfrac{t}{(1+\frac{3n}{2}\sigma^2(y,t))}\right)\right], & t\le y\le 1+t,\\ 0, & 1+t\le y\le\infty. \end{cases} \tag{4.3.5}$$

Solutions (4.3.4) and (4.3.5) are exact solutions without approximation when $n=0$. An analysis and discussion of the results will be given in Section 4.5.

4.4 Perturbation solutions for the system of partial differential equations for displacement and stress

Whe $n>0$ the parameter $\alpha/\sqrt{\gamma}$ controls the magnitude of the displacement gradient and in this theory we appeal to small displacement gradients. Thus we let

$$\frac{\alpha}{\sqrt{\gamma}}=\epsilon. \tag{4.4.1}$$

When $n = 0$, in order to ensure that the small displacement gradient theory is upheld we let $\alpha = \epsilon$ for small ϵ. Thus, for $n = 0$ and $n > 0$ the system of equations (4.2.6), (4.2.7) and (4.2.12), (4.2.13) reduce to

$$\begin{aligned} \frac{\partial \sigma}{\partial y} &= \frac{1}{\epsilon}\frac{\partial^2 u}{\partial t^2}, && (4.4.2)\\ \frac{\partial u}{\partial y} &= \epsilon(1+\sigma^2)^n\sigma. && (4.4.3) \end{aligned}$$

We consider straightforward perturbation expansions of the form

$$\sigma(y,t) = \epsilon\sigma_0 + \epsilon^2\sigma_1 + \epsilon^3\sigma_2 + \mathcal{O}(\epsilon^4), \tag{4.4.4}$$

$$u(y,t) = \epsilon^2 u_0 + \epsilon^3 u_1 + \epsilon^4 u_2 + \mathcal{O}(\epsilon^5). \tag{4.4.5}$$

We note that the displacement is an order of magnitude smaller than the stress. Substituting (4.4.4) and (4.4.5) into the system (4.21) and (4.4.3) and approximating the term $(1+\sigma^2)^n$ using a binomial expansion up to order $\mathcal{O}(\epsilon^3)$, we obtain

$$\begin{aligned} \epsilon\frac{\partial \sigma_0}{\partial y} + \epsilon^2\frac{\partial \sigma_1}{\partial y} + \epsilon^3\frac{\partial \sigma_2}{\partial y} &= \epsilon\frac{\partial^2 u_0}{\partial t^2} + \epsilon^2\frac{\partial^2 u_1}{\partial t^2} + \epsilon^3\frac{\partial^2 u_2}{\partial t^2}, && (4.4.6)\\ \epsilon^2\frac{\partial u_0}{\partial y} + \epsilon^3\frac{\partial u_1}{\partial y} + \epsilon^4\frac{\partial u_2}{\partial y} &= \epsilon^2\sigma_0 + \epsilon^3\sigma_1 + \epsilon^4(\sigma_2 + n\sigma_0^3). \end{aligned}$$

Equating terms of the same order of ϵ in each equation results in the following systems of equations:

Zero order terms u_0 and σ_0

$$\begin{aligned} \epsilon : \frac{\partial \sigma_0}{\partial y} &= \frac{\partial^2 u_0}{\partial t^2}, && (4.4.7)\\ \epsilon^2 : \frac{\partial u_0}{\partial y} &= \sigma_0. && (4.4.8) \end{aligned}$$

Eliminating σ_0 from (4.4.7) and (4.4.8) gives the one-dimensional wave equation

$$\frac{\partial^2 u_0}{\partial y^2} - \frac{\partial^2 u_0}{\partial t^2} = 0. \tag{4.4.9}$$

First order terms u_1 and σ_1

$$\epsilon^2 : \frac{\partial \sigma_1}{\partial y} = \frac{\partial^2 u_1}{\partial t^2}, \tag{4.4.10}$$

$$\epsilon^3 : \frac{\partial u_1}{\partial y} = \sigma_1. \tag{4.4.11}$$

Eliminating σ_1 from (4.4.10) and (4.4.11) gives the wave equation

$$\frac{\partial^2 u_1}{\partial y^2} - \frac{\partial^2 u_1}{\partial t^2} = 0. \tag{4.4.12}$$

Second order terms u_2 and σ_2

$$\epsilon^3 : \frac{\partial \sigma_2}{\partial y} = \frac{\partial^2 u_2}{\partial t^2}, \tag{4.4.13}$$

$$\epsilon^4 : \frac{\partial u_2}{\partial y} = \sigma_2 + n\sigma_0^3. \tag{4.4.14}$$

Eliminating σ_2 from (4.4.13) and (4.4.14) gives the wave equation

$$\frac{\partial^2 u_2}{\partial y^2} - \frac{\partial^2 u_2}{\partial t^2} = 3n\sigma_0^2 \frac{\partial \sigma_0}{\partial y}. \tag{4.4.15}$$

We will now consider a travelling wave solution and a standing wave solution of the perturbation expansions. The general procedure is to solve the wave equations (4.4.9), (4.4.12) and (4.4.15) for the displacement and then to obtain the stress from (4.4.8), (4.4.11) and (4.4.14).

4.4.1 Travelling wave solution

We solve the wave equations for the displacement in the semi-infinte domain $0 \leq y \leq \infty$ subject to the boundary condition

$$u(0,t) = \begin{cases} A\sin(4\pi t), & t \geq 0, \\ 0, & t < 0 \end{cases} \tag{4.4.16}$$

and the condition that there are no incoming waves from $y = +\infty$, that is, the condition that the waves propagate only in the positive y-direction. Expanded in powers of ϵ to order ϵ^4, the boundary condition (4.4.16) is

$$\epsilon^2 u_0(0,t) + \epsilon^3 u_1(0,t) + \epsilon^4 u_2(0,t) = (\epsilon^2 A_0 + \epsilon^3 A_1 + \epsilon^4 A_2)\sin(4\pi t) \tag{4.4.17}$$

and therefore for $m = 0, 1$ and 2,

$$u_m(0,t) = \begin{cases} A_m \sin(4\pi t), & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{4.4.18}$$

There is no boundary condition on $\sigma(0,t)$ because $\sigma_0(y,t)$, $\sigma_1(y,t)$ and $\sigma_2(y,t)$ are obtained from (4.4.8), (4.4.11) and (4.4.14) once $u_0(y,t)$, $u_1(y,t)$ and $u_2(y,t)$ have been found.

Consider the zero order quantities $u_0(y,t)$ and $\sigma_0(y,t)$. Equation (4.4.9) is a hyperbolic linear second order partial differential equation with solution

$$u_0(y,t) = F_0(y-t) = \begin{cases} -A_0 \sin[4\pi(y-t)], & 0 \le y \le t, \\ 0, & y > t. \end{cases} \tag{4.4.19}$$

From (4.4.8) the stress is given by

$$\sigma_0(y,t) = \frac{\partial u_0}{\partial y} = \begin{cases} -4\pi A_0 \cos[4\pi(y-t)], & 0 \le y \le t, \\ 0, & y > t. \end{cases} \tag{4.4.20}$$

Equations (4.4.11) and (4.4.12) and the boundary conditions (4.4.18) for the first order terms $u_1(y,t)$ and $\sigma_1(y,t)$ are the same as equations (4.4.8) and (4.4.9) for the zero order terms, $u_0(y,t)$ and $\sigma_0(y,t)$. Hence

$$u_1(y,t) = \begin{cases} -A_1 \sin[4\pi(y-t)], & 0 \le y \le t, \\ 0, & y > t \end{cases} \tag{4.4.21}$$

and

$$\sigma_1(y,t) = \frac{\partial u_1}{\partial y} = \begin{cases} -4\pi A_1 \cos[4\pi(y-t)], & 0 \le y \le t, \\ 0, & y > t. \end{cases} \tag{4.4.22}$$

Finally consider the second order terms $u_2(y,t)$ and $\sigma_2(y,t)$. Substituting (4.4.20) into the wave equation (4.4.15) gives

$$\frac{\partial^2 u_2}{\partial y^2} - \frac{\partial^2 u_2}{\partial t^2} = \begin{cases} 768 n A_0^3 \pi^4 \cos^2[4\pi(y-t)] \sin[4\pi(y-t)], & 0 \le y \le t, \\ 0, & y > t. \end{cases} \tag{4.4.23}$$

Expressed in terms of the canonical variables $\xi = y-t$ and $\eta = y+t$, equation (4.4.23) reduces to

$$\frac{\partial^2 u_2}{\partial \xi \partial \eta} = \begin{cases} 192 n A_0^3 \pi^4 \cos^2[4\pi\xi] \sin[4\pi\xi], & -\eta \le \xi \le 0, \\ 0, & \xi > 0, \end{cases} \tag{4.4.24}$$

which can be integrated to give

$$u_2(\xi,\eta) = \begin{cases} -16 n A_0^3 \pi^3 \eta \cos^3[4\pi\xi] + F_2(\xi) + G_2(\eta), & -\eta \le \xi \le 0, \\ F_2(\xi) + G_2(\eta), & \xi > 0, \end{cases} \tag{4.4.25}$$

where F_2 and G_2 are functions to be determined. Transforming back to the variables y and t we obtain

$$u_2(y,t) = \begin{cases} -16nA_0^3\pi^3(y+t)\cos^3[4\pi(y-t)] + F_2(y-t) + G_2(y+t), & 0 \le y \le t, \\ F_2(y-t) + G_2(y+t), & y > t. \end{cases} \tag{4.4.26}$$

Since we are considering waves propagating in the positive y-direction we take $G_2 = 0$. In order to determine F_2 we impose the boundary condition (4.4.18) for $m = 2$. This gives

$$-16nA_0^3\pi^3 t\cos^3[4\pi t] + F_2(-t) = A_2\sin(4\pi t), \quad t \ge 0, \tag{4.4.27}$$
$$F_2(-t) = 0, \quad t < 0 \tag{4.4.28}$$

and therefore

$$F_2(w) = \begin{cases} -A_2\sin(4\pi w) - 16nA_0^3\pi^3 w\cos^3(4\pi w), & w \le 0, \\ 0, & w > 0 \end{cases} \tag{4.4.29}$$

Thus

$$u_2(y,t) = \begin{cases} -A_2\sin[4\pi(y-t)] - 32nA_0^3\pi^3 y\cos^3[4\pi(y-t)], & 0 \le y \le 0, \\ 0, & y > t. \end{cases} \tag{4.4.30}$$

From (4.4.11) and using also (4.4.20) for σ_0 we obtain for the stress field at the second order

$$\sigma_2(y,t) = \begin{cases} -4\pi A_2\cos[4\pi(y-t)] + 32n\pi^3A_0^3\cos^3[4\pi(y-t)] & \\ +384n\pi^4A_0^3 y\cos^2[4\pi(y-t)]\sin[4\pi(y-t)], & 0 \le y \le t, \\ 0, & y > t \end{cases} \tag{4.4.31}$$

By combining the zero, first and second order solutions we obtain for the displacement the asymptotic solution correct to order ϵ^4,

$$u(y,t) = \begin{cases} -\epsilon^2(A_0 + \epsilon A_1 + \epsilon^2 A_2)\sin[4\pi(y-t)] & \\ -32\epsilon^4 n\pi^3A_0^3 y\cos^3[4\pi(y-t)], & 0 \le y \le y_F(t;n), \\ 0, & y > y_F(t;n) \end{cases} \tag{4.4.32}$$

where y_F is the distance travelled by the wave front, $u = 0$, after time t. Because of the secular term in y in (4.4.32), due to the nonlinearity in the constitutive equation (3.5.17) for $n > 0$, $y_F(t;n)$ is not exactly to t. The value of $y_F(t;n)$ will be calculated in Section 4.5 by finding the zeros of $u(y,t)$. The stress correct to order ϵ^3

$$\sigma(y,t) = \begin{cases} -4\pi\epsilon(A_0 + \epsilon A_1 + \epsilon^2 A_2)\cos[4\pi(y-t)] & \\ +32\epsilon^3 n\pi^3 A_0^3 \cos^2[4\pi(y-t)](\cos[4\pi(y-t)] & \\ +12n\pi y \sin[4\pi(y-t)]), & 0 \le y \le y_F(t;n), \\ 0, & y > y_F(t;n) \end{cases} \tag{4.4.33}$$

The results will be analysed in Section 5.

4.4.2 Standing wave solution

We look for solutions of the perturbation equations (4.4.9), (4.4.12) and (4.4.15) in the finite domain, $0 \le y \le 1$, subject to the boundary conditions

$$y = 0: \quad u(0,t) = \epsilon^2 u_0(0,t) + \epsilon^3 u_1(0,t) + \epsilon^4 u_2(0,t) + \mathcal{O}(\epsilon^5) = 0, \quad t \ge 0 \tag{4.4.34}$$

$$y = 1: \quad u(1,t) = \epsilon^2 u_0(1,t) + \epsilon^3 u_1(1,t) + \epsilon^4 u_2(1,t) + \mathcal{O}(\epsilon^5) = 0, \quad t \ge 0 \tag{4.4.35}$$

and to the initial conditions

$$t = 0: \qquad u(y,0) = A\sin(\pi y), \quad 0 \le y \le 1, \tag{4.4.36}$$

$$\epsilon^2 u_0(y,0) + \epsilon^3 u_1(y,0) + \epsilon^4 u_2(y,0) + \mathcal{O}(\epsilon^5) = (\epsilon^2 A_0 + \epsilon^3 A_1 + \epsilon^4 A_2 + \mathcal{O}(\epsilon^5))\sin(\pi y), \quad 0 \le y \le 1, \tag{4.4.37}$$

$$t = 0: \quad \frac{\partial u(y,0)}{\partial t} = \epsilon^2 \frac{\partial u_0(y,0)}{\partial t} + \epsilon^3 \frac{\partial u_1(y,0)}{\partial t} + \epsilon^4 \frac{\partial u_2(y,0)}{\partial t} + \mathcal{O}(\epsilon^5) = 0, \quad 0 \le y \le 1. \tag{4.4.38}$$

Consider first the zero order equation (4.4.9) and look for a non-trivial separable solution of the form

$$u_0(y,t) = Y_0(y) T_0(t). \tag{4.4.39}$$

Substituting (4.4.39) into (4.4.9) gives the eigenvalue problems

$$\frac{d^2 Y_0}{dy^2} + \lambda_0 Y_0 = 0, \tag{4.4.40}$$

$$\frac{d^2 T_0}{dy^2} + \lambda_0 T_0 = 0, \tag{4.4.41}$$

where $-\lambda_0$ is the separation constant and λ_0 is chosen to be positive to give wave solutions. The general solutions of (4.4.40) and (4.4.41) are

$$Y_0 = a_0 \cos(\sqrt{\lambda_0} y) + b_0 \sin(\sqrt{\lambda_0} y), \tag{4.4.42}$$

$$T_0 = c_0 \cos(\sqrt{\lambda_0} t) + d_0 \sin(\sqrt{\lambda_0} t), \tag{4.4.43}$$

and therefore

$$u_0(y,t) = [a_0 \cos(\sqrt{\lambda_0} y) + b_0 \sin(\sqrt{\lambda_0} y)][c_0 \cos(\sqrt{\lambda_0} t) + d_0 \sin(\sqrt{\lambda_0} t)]. \tag{4.4.44}$$

Now from (4.4.34) and (4.4.35), $u_0(y,t)$ satisfies the boundary conditions

$$u_0(0,t) = 0, \quad u_0(1,t) = 0 \tag{4.4.45}$$

and hence

$$a_0 = 0, \quad \sqrt{\lambda_0} = m\pi, \quad m = 1, 2, 3, \cdots \tag{4.4.46}$$

Thus

$$u_{0m}(y,t) = b_{0m} \sin(m\pi y)[c_{0m} \cos(m\pi t) + d_{0m} \sin(m\pi t)], \quad m = 1, 2, 3, \cdots \tag{4.4.47}$$

and

$$u_0(y,t) = \sum_{m=1}^{\infty} b_{0m} \sin(m\pi y)[c_{0m} \cos(m\pi t) + d_{0m} \sin(m\pi t)]. \tag{4.4.48}$$

But from (4.4.38), $u_0(y,t)$ satisfies the initial condition

$$u_0(y,t) = A_0 \sin(\pi y). \tag{4.4.49}$$

Hence (4.4.48) at $t = 0$ is

$$A_0 \sin(\pi y) = \sum_{m=1}^{\infty} b_{0m} c_{0m} \sin(m\pi y). \tag{4.4.50}$$

Clearly,

$$b_{01} c_{01} = A_0, \quad b_{0m} = 0, \ \ m \geq 2 \tag{4.4.51}$$

and therefore

$$u_0(y,t) = A_0 \sin(\pi y) \left[\cos(\pi t) + \frac{d_{01}}{c_{01}} \sin(\pi t) \right]. \tag{4.4.52}$$

From (4.4.39), $u_0(y,t)$ also satisfies the initial condition

$$\frac{\partial u_0}{\partial t}(y,0) = 0 \tag{4.4.53}$$

and therefore $d_{01} = 0$. Thus

$$u_0(y,t) = A_0 \sin(\pi y)\cos(\pi t). \tag{4.4.54}$$

Using the identity

$$2\sin A \cos B = \sin(A+B) + \sin(A-B) \tag{4.4.55}$$

equation (4.4.54) can be written equivalently as

$$u_0(y,t) = \frac{A_0}{2}[\sin(\pi(y+t)) + \sin(\pi(y-t))], \quad 0 \le y \le 1, \quad t \ge 0. \tag{4.4.56}$$

The zero order solution (4.4.56) represents a linear superposition of two waves, one travelling in the negative y-direction and one travelling in the positive y-direction, both with speed unity. From (4.4.8) and (4.4.54)

$$\sigma_0(y,t) = \frac{\partial u_0}{\partial y}(y,t) = \pi A_0 \cos(\pi y)\cos(\pi t), \quad 0 \le y \le 1, \quad t \ge 0, \tag{4.4.57}$$

or equivalently using (4.4.56),

$$\sigma_0(y,t) = \frac{\pi A_0}{2}[\cos(\pi(y+t)) + \cos(\pi(y-t))], \quad 0 \le y \le 1, \quad t \ge 0. \tag{4.4.58}$$

Consider next the first order wave equation (4.4.12) which is subject to the boundary conditions

$$u_1(0,t) = 0, \quad u_1(1,t) = 0 \tag{4.4.59}$$

and to the initial conditions

$$u_1(y,0) = A_1 \sin(\pi y), \quad \frac{\partial u_1}{\partial t}(y,0) = 0, \quad 0 \le y \le 1. \tag{4.4.60}$$

Since the partial differential equation (4.4.12) and the boundary conditions and initial conditions are the same as for the zero order problem the first order solution is

$$u_1(y,t) = A_1 \sin(\pi y)\cos(\pi t) \tag{4.4.61}$$

and from (4.4.11),

$$\sigma_1(y,t) = \pi A_1 \cos(\pi y)\cos(\pi t). \tag{4.4.62}$$

Using the identity (4.4.55) the first order solutions (4.4.61) and (4.4.62) can be written equivalently as

$$u_1(y,t) = \frac{A_1}{2}[\sin(\pi(y+t)) + \sin(\pi(y-t))], \quad 0 \le y \le 1, \quad t \ge 0, \tag{4.4.63}$$

$$\sigma_1(y,t) = \frac{\pi A_1}{2}[\cos(\pi(y+t)) + \cos(\pi(y-t))], \quad 0 \le y \le 1, \quad t \ge 0. \tag{4.4.64}$$

Finally, consider the second order wave equation (4.4.15) subject to the boundary conditions

$$u_2(0,t) = 0, \quad u_2(1,t) = 0 \tag{4.4.65}$$

and to the initial conditions

$$u_2(y,0) = A_2 \sin(\pi y), \quad \frac{\partial u_2}{\partial t}(y,0) = 0, \;\; 0 \le y \le 1. \tag{4.4.66}$$

Substituting (4.4.57) into the wave equation (4.4.14) gives

$$\frac{\partial^2 u_2}{\partial y^2} - \frac{\partial^2 u_2}{\partial t^2} = -3n\pi^4 A_0^3 \cos^3(\pi t)\cos^2(\pi y)\sin(\pi y), \tag{4.4.67}$$

which with the aid of (4.4.55) can be rewritten as

$$\frac{\partial^2 u_2}{\partial y^2} - \frac{\partial^2 u_2}{\partial t^2} = -\frac{3}{4} n\pi^4 A_0^3 \cos^3(\pi t)[\sin(\pi y) + \sin(3\pi y)]. \tag{4.4.68}$$

Unlike the zero and first order partial differential equations, (4.4.9) and (4.4.12), the partial differential equation (4.4.68) does not admit a separable solution of the form

$$u_2(y,t) = Y_2(y)T_2(t). \tag{4.4.69}$$

Guided by the boundary conditions (4.4.65) and the right hand side of (4.4.68) we look for a solution of the form

$$u_2(y,t) = \sum_{m=1}^{\infty} T_m(t)\sin(m\pi y). \tag{4.4.70}$$

Now, (4.4.70) is a solution of the partial differential equation (4.4.68) provided

$$\sum_{m=1}^{\infty}\left[\frac{d^2T_m}{dt^2} + m^2\pi^2 T_m(t)\right]\sin(m\pi y) = \frac{3}{4} n\pi^4 A_0^3 \cos^3(\pi t)[\sin(\pi y) + \sin(3\pi y)]. \tag{4.4.71}$$

Multiplying (4.4.71) with $\sin(p\pi y)$ where $p \ge 1$ is an integer and integrating with respect to y from 0 to 1 and using the identity

$$\int_0^1 \sin(p\pi y)\sin(m\pi y)dy = \frac{1}{2}\delta_{pm}, \tag{4.4.72}$$

we obtain

$$\frac{d^2T_p}{dt^2} + p^2\pi^2 T_p(t) = \frac{3}{4} n\pi^4 A_0^3 \cos^3(\pi t)[\delta_{p1} + \delta_{p3}]. \tag{4.4.73}$$

By using the identity

$$\cos^3 A = \frac{1}{4}(3\cos A + \cos 3A), \tag{4.4.74}$$

equation (4.4.73) can be put in a suitable form for integration,

$$\frac{d^2T_p}{dt^2} + p^2\pi^2 T_p(t) = \frac{3}{16} n\pi^4 A_0^3 [3\cos(\pi t) + \cos(3\pi t)][\delta_{p1} + \delta_{p3}]. \tag{4.4.75}$$

Consider now the initial conditions on $T_p(t)$. Substituting (4.4.70) into (4.4.66) gives

$$\sum_{m=1}^{\infty} T_m(0)\sin(m\pi y) = A_2 \sin(\pi y), \quad 0 \le y \le 1 \tag{4.4.76}$$

and

$$\sum_{m=1}^{\infty} \frac{dT_m(0)}{dt} \sin(m\pi y) = 0, \quad 0 \le y \le 1. \tag{4.4.77}$$

By multiplying (4.4.76) and (4.4.77) by $\sin(p\pi y)$ and integrating with respect to y from $y = 0$ to $y = 1$ gives

$$T_p(0) = A_2 \delta_{p1}, \quad \frac{dT_p(0)}{dt} = 0, \quad p \ge 1. \tag{4.4.78}$$

From (4.4.75) and (4.4.78) we obtain the following system of ordinary differential equations and initial conditions for $T_m(t)$, $m \ge 1$:

$$m = 1: \quad \frac{d^2T_1}{dt^2} + \pi^2 T_1 = \frac{3}{16} n\pi^4 A_0^3 [3\cos(\pi t) + \cos(3\pi t)], \tag{4.4.79}$$

$$T_1(0) = A_2, \quad \frac{dT_1(0)}{dt} = 0; \tag{4.4.80}$$

$$m = 3: \quad \frac{d^2T_3}{dt^2} + 9\pi^2 T_3 = \frac{3}{16} n\pi^4 A_0^3 [3\cos(\pi t) + \cos(3\pi t)], \tag{4.4.81}$$

$$T_3(0) = 0, \quad \frac{dT_3(0)}{dt} = 0. \tag{4.4.82}$$

$$m = 2, \ \ m \ge 4: \quad \frac{d^2T_m}{dt^2} + m^2\pi^2 T_m = 0, \tag{4.4.83}$$

$$T_m(0) = 0, \quad \frac{dT_m(0)}{dt} = 0; \tag{4.4.84}$$

The solution of the differential equation (4.4.79) for $T_1(t)$ subject to the initial conditions (4.4.80) is

$$T_1(t) = A_2\cos(\pi t) + \frac{3}{128}n\pi^2 A_0^3[\cos(\pi t) - \cos(3\pi t) + 12\pi t\sin(\pi t)], \quad (4.4.85)$$

while the solution of the differential equation (4.4.81) for T_3 subject to the initial conditions (4.4.82) is

$$T_3(t) = \frac{9}{128}n\pi^2 A_0^3[\cos(\pi t) - \cos(3\pi t) + \frac{4}{9}\pi t\sin(3\pi t)]. \quad (4.4.86)$$

The solution of the differential equation (4.4.83) subject to the initial conditions (4.4.84) is

$$T_m(t) = 0, \quad m = 2,\ m \geq 4, \quad (4.4.87)$$

and therefore from (4.4.70),

$$u_2(y,t) = T_1(t)\sin(\pi y) + T_3(t)\sin(3\pi y), \quad 0 \leq y \leq 1. \quad (4.4.88)$$

With the aid of the identity (4.4.55) and the identity

$$2\sin A\sin B = \cos(A-B) - \cos(A+B), \quad (4.4.89)$$

equation (4.4.88) can be written as

$$u_2(y,t) = \frac{A_2}{2}[\sin(\pi(y+t)) + \sin(\pi(y-t))] + \frac{3}{256}n\pi^2 A_0^3 U_2(y,t), \quad (4.4.90)$$

where

$$\begin{aligned} U_2(y,t) = {} & \sin(\pi(y+t)) + \sin(\pi(y-t)) - (\sin(\pi(y+3t)) + \sin(\pi(y-3t))) \\ & + 12\pi t(\cos(\pi(y-t)) - \cos(\pi(y+t))) \\ & + 3(\sin(\pi(3y+t)) + \sin(\pi(3y-t))) - 3(\sin(3\pi(y+t)) + \sin(3\pi(y-t))) \\ & + \frac{4}{3}\pi t(\cos(3\pi(y-t)) - \cos(3\pi(y+t))). \end{aligned} \quad (4.4.91)$$

From (4.4.7) and using (4.4.58) for σ_0 and (4.4.90) for $u_2(y,t)$ it can be verified that

$$\sigma_2(y,t) = \frac{\pi}{2}A_2[\cos(\pi(y+t)) + \cos(\pi(y-t))] + \frac{3}{256}n\pi^3 A_0^3 S_2(y,t), \quad (4.4.92)$$

where

$$\begin{aligned} S_2(y,t) = {} & -23(\cos(\pi(y+t)) + \cos(\pi(y-t))) - 9(\cos(\pi(y+3t)) + \cos(\pi(y-3t))) \\ & + 12\pi t(-\sin(\pi(y-t)) + \sin(\pi(y+t))) + \cos(\pi(3y+t)) \\ & + \cos(\pi(3y-t)) - \frac{35}{3}(\cos(3\pi(y+t)) + \cos(3\pi(y-t))) \\ & + 4\pi t(-\sin(3\pi(y-t)) + \sin(3\pi(y+t))). \end{aligned} \quad (4.4.93)$$

By combining the zero, first and second order solutions for the displacement, given by (4.4.56), (4.4.63) and (4.4.90), we obtain from (4.4.5) correct to order ϵ^4,

$$u(y,t) = \frac{1}{2}[\epsilon^2 A_0 + \epsilon^3 A_1 + \epsilon^4 A_2][\sin(\pi(y+t)) + \sin(\pi(y-t))] + \frac{3}{256}\epsilon^4 n\pi^2 A_0^3 U_2(y,t), \tag{4.4.94}$$

where $U_2(y,t)$ is given by (4.4.91). The stress is given by (4.4.4) and using (4.4.58) for σ_0, (4.4.64) for σ_1 and (4.4.92) for σ_2 we obtain correct to order ϵ^3,

$$\sigma(y,t) = \frac{\pi}{2}[\epsilon A_0 + \epsilon^2 A_1 + \epsilon^3 A_2][\cos(\pi(y+t)) + \cos(\pi(y-t))] + \frac{3}{256}\epsilon^3 n\pi^3 A_0^3 S_2(y,t), \tag{4.4.95}$$

where $S_2(y,t)$ is given by (4.4.93).
The results for the stress and displacement will be discussed in the next section.

4.5 Discussion of results for power-law constitutive equation

4.5.1 Solitary wave solution for shear stress

The implicit asymptotic solution (4.3.4) for the power-law constitutive equation will now be discussed.

Figure 4.5.1 shows the stress solitary wave at time $t = 0$ for $n = 0$ and $n = 0.3$. A distortion in the linear sine wave for $n = 0$ is apparent when compared to the nonlinear wave for $n > 0$. Figures 4.5.2 and 4.5.3 show the propagation of the solitary stress wave at times $t = 0.1$ and $t = 1$ respectively. We find that as time evolves, for a fixed n, the wave steepens at the back. For a fixed value of time the steeping increases as n increases. A shock front forms at the back of the stress wave after a sufficiently large time. From the approximate solution (4.3.5) we see that an estimate of the speed of the stress wave is

$$C(n,\sigma) = \frac{1}{\left(1 + \dfrac{3n}{2}\sigma^2(y,t)\right)}. \tag{4.5.1}$$

The wave speed decreases as both n and σ increase. An estimate of the time the shock front forms can be obtained by differentiating (4.3.5) with respect to y. This gives

$$\frac{\partial\sigma}{\partial y} = \frac{2\pi A\cos\phi}{[1 - 3\pi n A^2 \sin 2\phi\; C^2(n,\sigma)t]}, \tag{4.5.2}$$

where

$$\phi = 2\pi(y - C(n,\sigma)t). \tag{4.5.3}$$

Thus

$$\left|\frac{\partial \sigma}{\partial y}\right| \to \infty \quad \text{as} \quad t \to t_s = \frac{1}{3\pi n A^2 C^2(n,\sigma)\sin 2\phi}, \tag{4.5.4}$$

provided $t_s > 0$. Now $t_s > 0$ for $0 \leq \phi \leq \frac{\pi}{2}$ and for $\pi \leq \phi \leq \frac{3\pi}{2}$. For $0 \leq \phi \leq \frac{\pi}{2}$, $\frac{\partial \sigma}{\partial y} > 0$ while for $\pi \leq \phi \leq \frac{3\pi}{2}$, $\frac{\partial \sigma}{\partial y} < 0$. In both cases the stress wave steepens at the back in agreement with Figure 3 and the shock front forms at the back of the stress wave.

We can also study the dependence of the estimate of the breaking time on n and we find that

$$t_s \sim \frac{1}{nC^2(n,\sigma)}. \tag{4.5.5}$$

When the value of n increases the time at which the shock will develop occurs earlier than for a lower value of n. The values that are chosen for n are the values that were used in [42] which was the first paper to provide a framework for the propagation of stress and displacement waves experimentally. That paper described the work by Saito [28] on titanium alloys and gum metal. Hence we have used the same values for illustration. Other positive values of n may also be used. The times that are chosen are again chosen to show that after some time a distortion in the wave form will occur. If we are to consider the same value of n for larger times a shock will develop after approximately time (4.5.4). If we fix the time and increase the value of n a shock front will develop earlier.

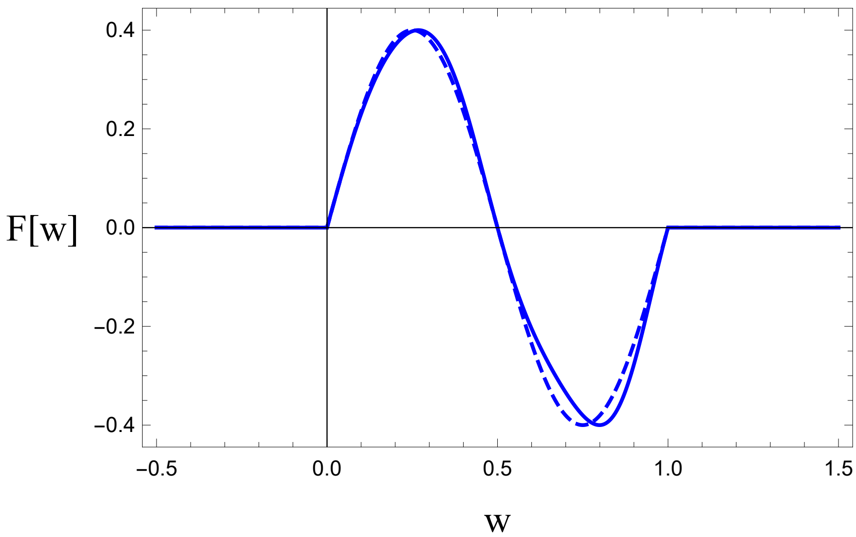

Figure 4.5.1: Initial profile of the solitary stress wave for $n = 0$ (− − −−) and $n = 0.3$ (——) with $A_0 = 0.4$.

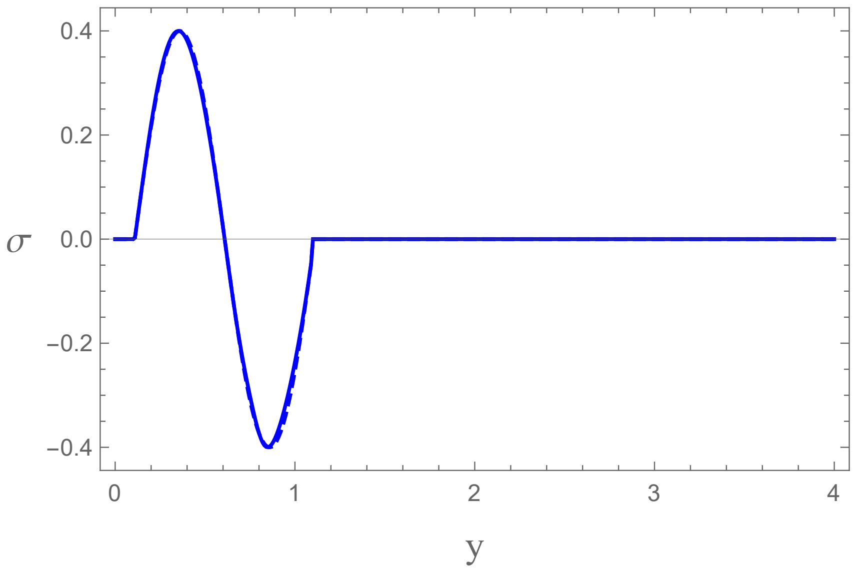

Figure 4.5.2: Propagation of the solitary stress wave for $n = 0$ (− − −−) and $n = 0.3$ (——) at time $t = 0.1$ with $A_0 = 0.4$.

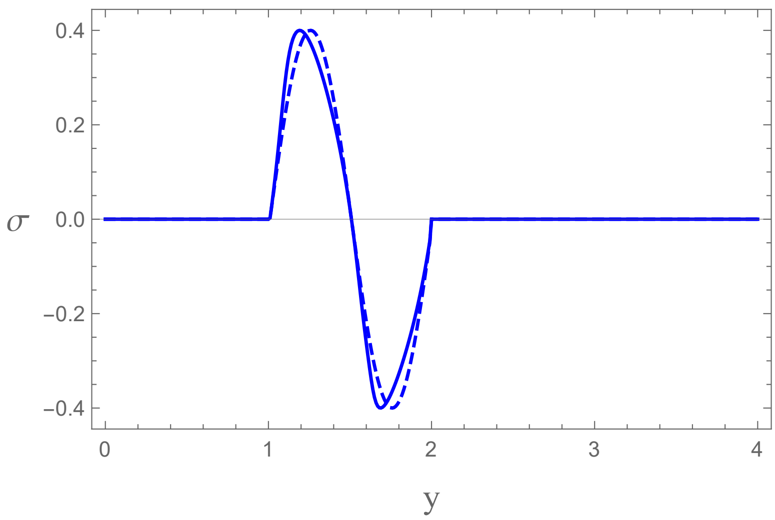

Figure 4.5.3: Propagation of the solitary stress wave for $n = 0$ (− − −−) and $n = 0.3$ (——) at time $t = 1$ with $A_0 = 0.4$.

4.6 Perturbation solutions for displacement and shear stress

In this section we present an analysis of the asymptotic solutions for the travelling wave and standing wave for the power-law constitutive equation. We choose $A_1 = 0$ and $A_2 = 0$ for both asymptotic solutions.

4.6.1 Travelling wave solution for displacement and stress

The straightforward perturbation expansion for the displacement, (4.4.32) correct to order $\mathcal{O}(\epsilon^4)$ and the stress, (4.4.33) correct to order $\mathcal{O}(\epsilon^3)$ each contain a secular term in y at the highest order in ϵ for $n > 0$. The problem is therefore a singular perturbation problem. We applied the Lindstedt-Poincaré method but the singular perturbation method was not successful in removing the secular term. This is due to the fact that the solution for the displacement at the third order requires the value of the stress, σ_0, from the lowest order solution which does not contain any parameters that can be used to eliminate the secular terms.

The position of the wave front at time t, $y_F(t; n)$, is found from the conditions that the amplitude of the displacement vanish at the wave front. From (4.4.32) for $n = 0$,

$$u(y, t) = -\epsilon^2 A_0 \sin[4\pi(y - t)] + \mathcal{O}(\epsilon^5), \tag{4.6.1}$$

which describes a travelling wave with unit velocity. The wave front is at $y = t$ so that $y_F(t; n) = t$. For $n > 0$ the position of the wave front is obtained by using (4.4.32) to calculate, at a given time t, the solution for y of $u(y, t) = 0$ closest to $y = t$. The results for $n = 0$ and $n = 0.3$ for times $t = 0.5$ and $t = 0.8$ are given in Table 4.6.1. These values for n and for t are again chosen for comparison with the results in [42]. Other larger values may also be used and it can be seen that larger values of n result is slower wave speeds.

t	n=0	n=0.3
0.5	y_F=0.5	y_F=0.4828
0.8	y_F=0.8	y_F=0.7745

Table 4.6.1: Position of the wave front for the travelling wave, $y_F(t; n)$, for $\epsilon = 0.1$ and $A_0 = 0.4$.

The travelling wave solutions for the displacement and stress, (4.4.32) and (4.4.33), are plotted in Figures 4.6.1 and 4.6.2 for $n = 0$ and $n = 0.3$ at times $t = 0.5$ and $t = 0.8$. The speed of propagation of the nonlinear wave for $n = 0.3$ is less than that of the linear wave for $n = 0$. The maximum amplitude of the displacement wave for $n = 0$ and $n = 0.3$ are approximately the same. For the stress wave the maximum amplitude for $n = 0.3$ is greater than for $n = 0$ and the difference in the amplitude increases as the wave propagates. The wave profile for positive values of the stress (tensile stress) becomes distorted. The stress at the wave front is discontinuous because the spatial gradient of the displacement is discontinuous at the wave front. The stress wave therefore propagates as a shock wave with a discontinuity in the stress at the wave front.

Further, the value of the wave front for $n > 0.3$ will show that as n increases the speed of the wave decreases.

When $A_1 = A_2 = 0$ the straightforward perturbation expansion for the displacement wave, (4.4.32), breaks down due to the secular term when the magnitude of the order ϵ^4 term exceeds $\epsilon^2 A_0$, that is, for

$$y \gtrsim \frac{1}{32n\pi^3 A_0^2 \epsilon^2} \approxeq 2.1, \tag{4.6.2}$$

with parameter values $n = 0.3$, $A_0 = 0.4$ and $\epsilon = 0.1$ as in Figures 4.6.1 and 4.6.2. The straightforward perturbation expansion for the stress wave, (4.4.33), breaks down for

$$y \gtrsim \frac{1}{96n^2\pi^3 A_0^2 \epsilon^2} \approxeq 2.33, \tag{4.6.3}$$

for the same parameter values. These values of y are outside the range, $0 \leq y \leq 0.8$, considered in Figures 4.6.1 and 4.6.2. The positions y where the perturbation expansions for the displacement (4.4.32) and stress (4.4.33) depend on n like $y \gtrsim 1/n$ and $1/n^2$ respectively which means that as n increases the perturbation solutions for the displacement and stress break down for smaller values of y.

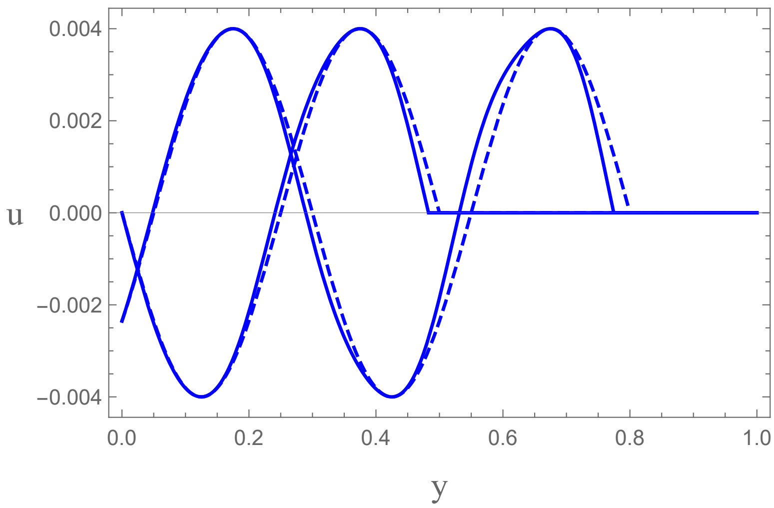

Figure 4.6.1: Travelling wave for the displacement for $n = 0$ (− − −−) and $n = 0.3$ (——) at $t = 0.5$ and $t = 0.8$ with $A_0 = 0.4$ and $\epsilon = 0.1$.

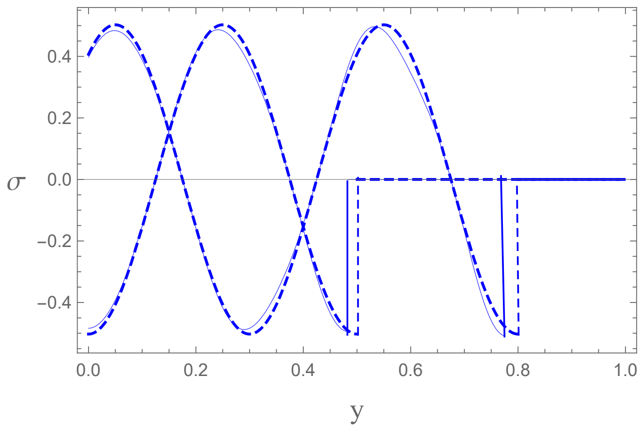

Figure 4.6.2: Travelling wave for the stress for $n = 0$ (− − −−) and $n = 0.3$ (——) at $t = 0.5$ and $t = 0.8$ with $A_0 = 0.4$ and $\epsilon = 0.1$.

4.6.2 Standing wave solution for displacement and stress

The straightforward perturbation expansions for $n > 0$ for the displacement, (4.4.94), and for the stress (4.4.95), contain secular terms in t at order ϵ^4 and at order ϵ^3, respectively. These secular terms cannot be removed by using singular perturbation methods such as the Linstedt-Poincaré method. When $A_1 = A_2 = 0$, the expansions (4.4.94) and (4.4.95) both break down for

$$t \gtrsim \frac{32}{9n\pi^3 A_0^2 \epsilon^2} \approxeq 7.17 \tag{4.6.4}$$

when evaluated for $n = 10$, $A_0 = 0.4$ and $\epsilon = 0.1$. This value of t is outside the range $0 \leq t \leq 0.8$ used in Figures 6 and 7. The large value, $n = 10$, is considered because the distortion is small and is not clearly illustrated in graphs with small values of n. This large value of n falls outside the range of values of n used in experiments but is considered here for illustration.

The solutions (4.4.94) and (4.4.95) for $u(y, t)$ and $\sigma(y, t)$ are in the form of waves propagating in the positive and negative direction between $y = 0$ and $y = 1$. Equations (4.4.94) and (4.4.95) can be rewritten in the form of standing waves in the range $0 \leq y \leq 1$ as

$$u(y, t) = (\epsilon^2 A_0 + \epsilon^3 A_1 + \epsilon^4 A_2)\sin(\pi y)\cos(\pi t) + \frac{3}{256}\epsilon^4 n\pi^2 A_0^3 U_2(y, t), \tag{4.6.5}$$

where

$$\begin{aligned} U_2 = {} & 2\sin(\pi y)[\cos(\pi t) - \cos(3\pi t) + 12\pi t\sin(\pi t)] \\ & + 6\sin(3\pi y)[\cos(\pi t) - \cos(3\pi t) + 4\pi t\sin(3\pi t)] \end{aligned} \tag{4.6.6}$$

and

$$\sigma(y,t) = \pi(\epsilon A_0 + \epsilon^2 A_1 + \epsilon^3 A_2)\cos(\pi y)\cos(\pi t) + \frac{3}{256}\epsilon^3 n\pi^3 A_0^3 S_2(y,t), \quad (4.6.7)$$

where

$$\begin{aligned} S_2 &= 2\cos(\pi y)[-23\cos(\pi t) - 9\cos(3\pi t) + 12\pi t\sin(\pi t)] \\ &\quad + 2\cos(3\pi y)\left[\cos(\pi t) - \frac{35}{3}\cos(3\pi t) + 4\pi t\sin(3\pi t)\right]. \end{aligned} \quad (4.6.8)$$

We are considering $A_1 = A_2 = 0$. Consider first the displacement. When $n = 0$, the solution (4.6.5) consists of a standing wave of amplitude $\epsilon^2 A_0$ with form $\sin(\pi y)$ for $0 \leq y \leq 1$ and oscillating with period $T = 2$. When $n > 0$ there is in addition a superposition of standing waves, with amplitude of order ϵ^4, of the form $\sin(\pi y)$ and $\sin(3\pi y)$ and oscillating with period $T = 2$ and $T = 2/3$. The displacement wave (4.6.5) is plotted in Figure 4.6.3 where the solutions for $n = 0$ and $n = 10$ are compared for $0 \leq t \leq 1$ which is the first half-cycle of the displacement wave for $n = 0$. We see that for the time range considered the differences between the two oscillations is small. The difference will be most significant when the order terms ϵ^2 vanish, which occurs at $t = 0.5$ in Figure 4.6.3, leaving only the superposition of standing waves of order ϵ^4. The difference between the two oscillations vanishes at $t = 0$ and $t = 1$.

Consider next the stress wave. When $n = 0$ the solution (4.6.7) consists of a standing wave of amplitude $\epsilon\pi A_0$ of the form $\cos(\pi y)$ for $0 \leq y \leq 1$ and oscillating with a period $T = 2$. The stress wave for $n > 0$ has in addition a linear superposition of standing waves of amplitude order ϵ^3 and are of the form $\cos(\pi y)$ and $\cos(3\pi y)$ with periods of oscillation $T = 2$ and $T = 2/3$. The solutions for the stress wave for $n = 0$ and $n = 0.3$ are compared for $0 \leq t \leq 1$ in Figure 4.6.4. The differences are small and are most significant at $t = 1/2$ when the terms of order ϵ vanish.

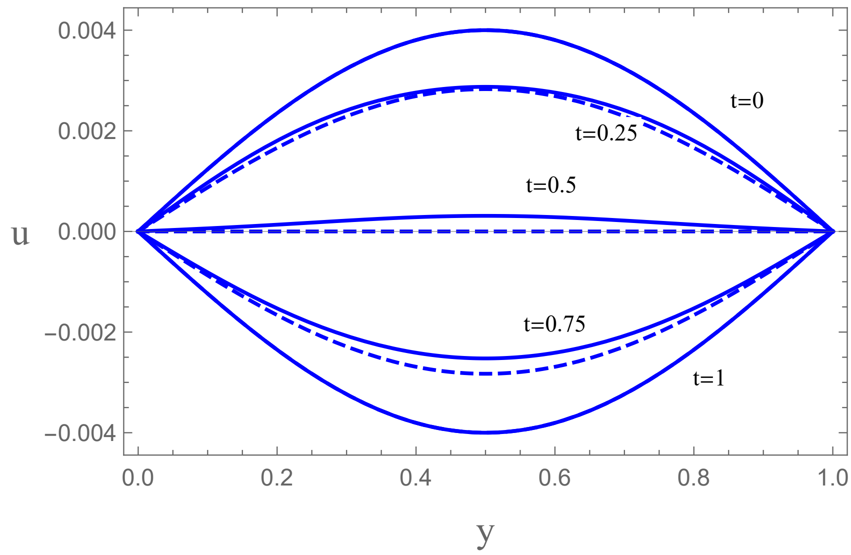

Figure 4.6.3: Standing wave for the displacement for $n = 0$ (− − −−) and $n = 10$ (——) with $A_0 = 0.4$ and $\epsilon = 0.1$ at times $t = 0, 0.25, 0.5, 0.5, 0.75$, and 1.

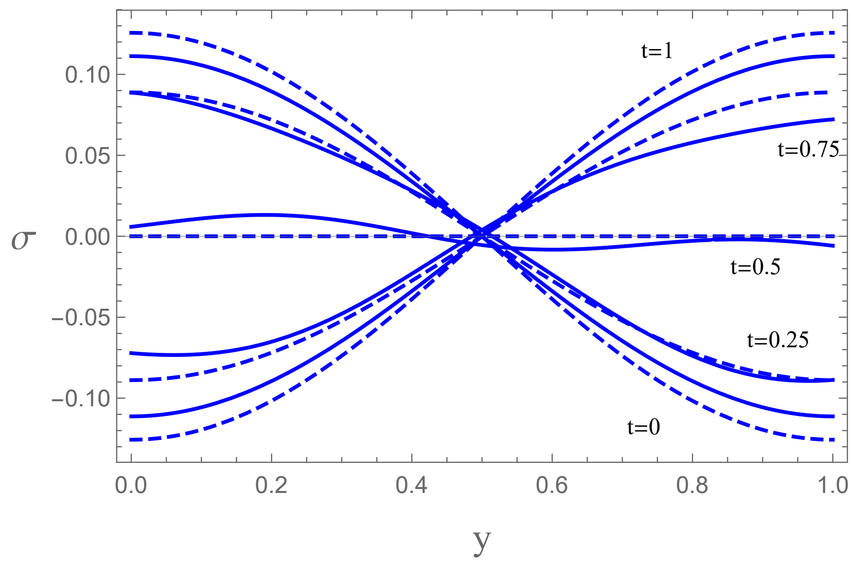

Figure 4.6.4: Standing wave for the stress for $n = 0$ (− − −−) and $n = 10$ (——) with $A_0 = 0.4$ and $\epsilon = 0.1$ at times $t = 0, 0.25, 0.5, 0.75$ and 1.

4.7 Mathematical model for the strain limiting constitutive equation

In this section we will derive a system of nonlinear partial differential equations to study wave propagation in an semi-infinite domain and in a finite domain using the strain-limiting constitutive equation (3.5.18). This constitutive equation has been studied in [34] to describe strain limiting responses for static boundary value problems.

The domain of interest is defined by (4.2.1) and we consider the stress and displacement fields given by (4.2.2)

Using the equations of motion, the components of the strain tensor and the constitutive equation (3.5.18) we obtain the following dimensional system of equations

$$\begin{aligned} \frac{\partial \sigma}{\partial y} &= \rho \frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial u}{\partial y} &= \alpha \left[1 + \frac{1}{(1+\gamma\sigma^2)} \right]^n \sigma. \end{aligned} \tag{4.7.1}$$

Introducing the following non-dimensional variables

$$\bar{y} = \frac{y}{y_0}, \quad \bar{t} = \frac{t}{t_0} = \frac{t}{y_0\sqrt{\alpha\rho}}, \quad \bar{u} = \frac{u}{y_0}, \quad \bar{\sigma} = \sqrt{\gamma}\sigma, \tag{4.7.2}$$

we obtain the non-dimensional system of equations

$$\frac{\partial \sigma}{\partial y} = \frac{\sqrt{\gamma}}{\alpha} \frac{\partial^2 u}{\partial t^2}, \tag{4.7.3}$$

$$\frac{\partial u}{\partial y} = \frac{\alpha}{\sqrt{\gamma}} \left[1 + \frac{1}{(1+\sigma^2)} \right]^n \sigma, \tag{4.7.4}$$

where the overhead bars have been suppressed. The same analysis can be performed for $n = 0$ as was done for the power-law constitutive equation.

This system of equations may be written as a single nonlinear hyperbolic partial differential equation which describes shear stress waves

$$\frac{\partial^2 \sigma}{\partial y^2} = \frac{\partial^2}{\partial t^2} \left[\left(1 + \frac{1}{(1+\sigma^2)} \right)^n \sigma \right]. \tag{4.7.5}$$

The solution of the system of equations (4.7.3) and (4.7.4) and of the second order wave equation (4.7.5) for the same initial and boundary conditions used for the power-law constitutive equation will be investigated in the next section.

4.8 Asymptotic solutions for the strain limiting constitutive equation

4.8.1 Solitary wave solution

In this section we will outline the derivation of an implicit asymptotic solution of the second order wave equation (4.7.5) for the stress subject to the initial condition (4.3.1).

Let $\sigma(y,t) = \epsilon\hat{\sigma}(y,t)$, where ϵ is small. Substituting this transformation into (??) and taking the binomial expansion of $\left[1 + \dfrac{1}{(1+\sigma^2)}\right]^n$ up to $\mathcal{O}(\epsilon^2)$ we obtain the following asymptotic partial differential equation

$$\frac{\partial^2 \hat{\sigma}}{\partial y^2} = 2^n \frac{\partial^2 \hat{\sigma}}{\partial t^2} - 2^{n-1} n \epsilon^2 \frac{\partial^2 \hat{\sigma}^3}{\partial t^2}. \tag{4.8.1}$$

Let

$$\bar{y} = 2^{\frac{n}{2}} y, \quad \bar{\epsilon} = \frac{\epsilon}{\sqrt{2}}. \tag{4.8.2}$$

Substituting (4.8.2) into (4.8.1) we obtain the partial differential equation

$$\frac{\partial^2 \hat{\sigma}}{\partial \bar{y}^2} = \frac{\partial^2 \hat{\sigma}}{\partial t^2} - n\bar{\epsilon}^2 \frac{\partial^2 \hat{\sigma}^3}{\partial t^2}. \tag{4.8.3}$$

The only difference between (4.8.3) and (4.3.3) is the difference in sign of the last term. We now introduce the transformations

$$Y = \bar{\epsilon}^2 \bar{y}, \quad \tau = t - \bar{y}. \tag{4.8.4}$$

Now, the stress is defined in terms of the transformed variables as

$$\hat{\sigma}(y,t) = \sigma^\star(Y(\bar{y}), \tau(t,\bar{y})). \tag{4.8.5}$$

Substituting (4.8.4) and (4.8.5) into the partial differential equation (4.8.3) gives

$$-2\frac{\partial^2 \sigma^\star}{\partial \tau \partial Y} + \bar{\epsilon}^2 \frac{\partial^2 \sigma^\star}{\partial Y^2} = -n \frac{\partial^2 (\sigma^\star)^3}{\partial \tau^2}. \tag{4.8.6}$$

Integrating (4.8.6) with respect to τ yields

$$-2\frac{\partial \sigma^\star}{\partial Y} + \bar{\epsilon}^2 \int^\tau \frac{\partial^2 \sigma^\star}{\partial Y^2} d\tau = -n \frac{\partial (\sigma^\star)^3}{\partial \tau} + G(Y). \tag{4.8.7}$$

where $G(Y)$ is an arbitrary function of Y. By imposing the boundary conditions (4.3.2), $G(Y) = 0$ Neglecting terms $\mathcal{O}(\epsilon^2)$ in (4.8.7) equation gives the quasi-linear first order partial differential equation

$$-2\frac{\partial \sigma^\star}{\partial Y} + 3n(\sigma^\star)^2 \frac{\partial \sigma^\star}{\partial \tau} = 0. \tag{4.8.8}$$

The differential equations of the characteristic curves of (4.8.8) are

$$-\frac{dY}{2} = \frac{d\tau}{3n(\sigma^\star)^2} = \frac{d\sigma^\star}{0}. \tag{4.8.9}$$

From the last term in (4.8.9) one invariant along a characteristic curve is

$$\sigma^\star = k_1, \tag{4.8.10}$$

where k_1 is a constant. We then solve the differential equations formed by the first pair of terms in (4.8.9) to yield

$$\frac{3n}{2}(\sigma^\star)^2 Y + \tau = k_2, \tag{4.8.11}$$

where k_2 is a constant. The general solution of (4.8.8) is

$$k_1 = F(-k_2) \tag{4.8.12}$$

where F is an arbitrary function and it is convenient to work with $-k_2$ instead of k_2. Hence

$$\sigma^\star(Y,\tau) = F\left[-\left(\frac{3n}{2}(\sigma^\star)^2 Y + \tau\right)\right]. \tag{4.8.13}$$

From (4.8.4) we find that the solution (4.8.13) can be written as

$$\hat{\sigma}(\bar{y},t) = F\left[\left(1 - \frac{3n}{2}\bar{\epsilon}^2\hat{\sigma}^2\right)\bar{y} - t\right]. \tag{4.8.14}$$

Now, using (4.8.2) and the fact that $\sigma = \epsilon\hat{\sigma}$, we obtain the solution

$$\begin{aligned} \sigma(y,t) &= \epsilon F\left[2^{\frac{n}{2}}\left(1 - \frac{3n}{4}\sigma^2(y,t)\right)y - t\right] \\ &= \tilde{F}\left[2^{\frac{n}{2}}\left(1 - \frac{3n}{4}\sigma^2(y,t)\right)y - t\right]. \end{aligned} \tag{4.8.15}$$

The function $\tilde{F}$ can be determined from the harmonic initial condition (4.3.1). We have,

$$\sigma(y,0) = \tilde{F}\left[2^{\frac{n}{2}}\left(1 - \frac{3n}{4}\sigma^2(y,0)\right)y\right]. \tag{4.8.16}$$

Therefore, using (4.3.1)

$$\tilde{F} = \left[2^{\frac{n}{2}}\left(1 - \frac{3n}{4}A^2\sin^2(2\pi y)\right)y\right] = A\sin(2\pi y), \quad 0 \le y \le 1, \tag{4.8.17}$$

$$\tilde{F}[2^{\frac{n}{2}}y] = 0, \quad 1 < y < \infty \tag{4.8.18}$$

Let

$$w = \begin{cases} 2^{\frac{n}{2}} \left(1 - \dfrac{3n}{4} A^2 \sin^2(2\pi y)\right) y, & 0 \leq y \leq 1, \\ 2^{\frac{n}{2}} y, & 1 < y < \infty. \end{cases} \tag{4.8.19}$$

Equation (4.8.19) has to be solved for y in terms of w. We proceed as in Subsection 4.1. Since $\sin^2(2\pi y) \leq 1$, a first approximation is $y = 2^{-\frac{n}{2}} w$. Iterating using this result the next approximation is

$$y = \frac{2^{-\frac{n}{2}} w}{1 - \dfrac{3n}{4} A^2 \sin^2(2^{1-\frac{n}{2}} \pi w)}, \quad 0 \leq y \leq 1. \tag{4.8.20}$$

We will use (4.8.20) to express y in terms of w. A similar approximation is not required in determining the limits of the range $0 \leq y \leq 1$ because the stress $\sigma(y, 0)$ vanishes at the wave front and at the end of the solitary wave. Equations (4.8.17) and (4.8.18) become

$$\tilde{F}(w) = \begin{cases} A \sin \left[\dfrac{2^{1-\frac{n}{2}} \pi w}{1 - \dfrac{3n}{4} A^2 \sin^2(2^{1-\frac{n}{2}} \pi w)} \right], & 0 \leq w \leq 1, \\ 0, & 1 \leq w \leq \infty. \end{cases} \tag{4.8.21}$$

In order to obtain $\sigma(y, t)$ given by (4.8.16) and (4.8.21) we set

$$w = 2^{\frac{n}{2}} \left(1 - \frac{3n}{4} \sigma^2(y, t)\right) y - t \tag{4.8.22}$$

while to obtain the limits of the range $0 \leq w \leq 2^{\frac{n}{2}}$ we set $w = 2^{\frac{n}{2}} y - t$ because $\sigma(y, t) = 0$ at each end of the solitary wave. Thus

$$\sigma(y, t) = \begin{cases} A \sin \left[\dfrac{2^{1-\frac{n}{2}} \pi \left(2^{\frac{n}{2}} \left(1 - \dfrac{3n}{4} \sigma^2(y, t)\right) y - t\right)}{1 - \dfrac{3n}{4} A^2 \sin^2 \left(2^{1-\frac{n}{2}} \pi \left(2^{\frac{n}{2}} \left(1 - \dfrac{3n}{2} \sigma^2(y, t)\right) y - t\right)\right)} \right], & 2^{-\frac{n}{2}} t \leq y \leq 2^{-\frac{n}{2}} (1 + t), \\ 0, & 2^{-\frac{n}{2}} (1 + t) \leq y \leq \infty. \end{cases} \tag{4.8.23}$$

To interpret the solution physically, we write the stress, (4.8.23), approximately as

$$\sigma(y, t) = A \sin \left[2\pi \left(y - \frac{t}{2^{\frac{n}{2}} (1 - \frac{3n}{4} \sigma^2(y, t))} \right) \right]. \tag{4.8.24}$$

An analysis and discussion of the results will be given in Section 8.

4.9 Perturbation solutions for the first order system for displacement and stress

We require the displacement gradient to be small and so we let

$$\frac{\alpha}{\sqrt{\gamma}} = \epsilon. \tag{4.9.1}$$

Equations (4.7.3) and (4.7.4) become

$$\frac{\partial \sigma}{\partial y} = \frac{1}{\epsilon}\frac{\partial^2 u}{\partial t^2}, \tag{4.9.2}$$

$$\frac{\partial u}{\partial y} = \epsilon\left[1 + \frac{1}{(1+\sigma^2)}\right]^n \sigma. \tag{4.9.3}$$

We consider straightforward perturbation expansions of the form (4.4.5) and (4.4.5). Substituting (4.4.4) and (4.4.4) into the system (4.9.2) and (4.9.3) and approximating the term $\left[1 + \frac{1}{(1+\sigma^2)}\right]^n$ using a binomial expansion up to order $\mathcal{O}(\epsilon^3)$, we obtain

$$\epsilon\frac{\partial \sigma_0}{\partial y} + \epsilon^2\frac{\partial \sigma_1}{\partial y} + \epsilon^3\frac{\partial \sigma_2}{\partial y} = \epsilon\frac{\partial^2 u_0}{\partial t^2} + \epsilon^2\frac{\partial^2 u_1}{\partial t^2} + \epsilon^3\frac{\partial^2 u_2}{\partial t^2}, \tag{4.9.4}$$

$$\begin{aligned}\epsilon^2\frac{\partial u_0}{\partial y} + \epsilon^3\frac{\partial u_1}{\partial y} + \epsilon^4\frac{\partial u_2}{\partial y} &= \epsilon^2 2^n\sigma_0 + \epsilon^3 2^n\sigma_1 \\ &+ \epsilon^4 2^n(\sigma_2 - \frac{n}{2}\sigma_0^3).\end{aligned} \tag{4.9.5}$$

Equating terms of the same order of ϵ in each equation results in the following systems of equations:

Zero order terms u_0 and σ_0:

$$\epsilon : \frac{\partial \sigma_0}{\partial y} = \frac{\partial^2 u_0}{\partial t^2}, \tag{4.9.6}$$

$$\epsilon^2 : \frac{\partial u_0}{\partial y} = 2^n\sigma_0. \tag{4.9.7}$$

Eliminating σ_0 from (4.9.6) and (4.9.7) gives the wave equation

$$\frac{\partial^2 u_0}{\partial y^2} - 2^n\frac{\partial^2 u_0}{\partial t^2} = 0. \tag{4.9.8}$$

First order terms u_1 and σ_1:

$$\epsilon^2 : \frac{\partial \sigma_1}{\partial y} = \frac{\partial^2 u_1}{\partial t^2}, \tag{4.9.9}$$

$$\epsilon^3 : \frac{\partial u_1}{\partial y} \quad = \quad 2^n \sigma_1. \tag{4.9.10}$$

Eliminating σ_1 from (4.9.9) and (4.9.10) gives the wave equation

$$\frac{\partial^2 u_1}{\partial y^2} - 2^n \frac{\partial^2 u_1}{\partial t^2} = 0. \tag{4.9.11}$$

Second order terms u_2 and σ_2:

$$\epsilon^3 : \frac{\partial \sigma_2}{\partial y} \quad = \quad \frac{\partial^2 u_2}{\partial t^2}, \tag{4.9.12}$$

$$\epsilon^4 : \frac{\partial u_2}{\partial y} \quad = \quad 2^n \sigma_2 - 2^{n-1} n \sigma_0^3. \tag{4.9.13}$$

Eliminating σ_2 from (4.9.12) and (4.9.13) gives

$$\frac{\partial^2 u_2}{\partial y^2} - 2^n \frac{\partial^2 u_2}{\partial t^2} = -2^{n-1} n \frac{\partial}{\partial y}(\sigma_0^3). \tag{4.9.14}$$

We will consider a travelling wave solution and a standing wave solution to the perturbation equations. As with the power-law constitutive equation, the general procedure is to solve the wave equations (4.9.8), (4.9.11) and (4.9.14) for the displacement and then to find the corresponding stress field from equations (4.9.7), (4.9.10) and (4.9.13).

4.9.1 Travelling wave solution

We solve the wave equations for the displacement subject to the boundary condition (4.4.16) and use the condition that there are no incoming waves from $y = +\infty$.

We first consider the zero order quantities $u_0(y,t)$ and $\sigma_0(y,t)$ first. The hyperbolic linear partial differential equation (4.9.8) has solution for the displacement

$$u_0(y,t) = \begin{cases} -A_0 \sin[2^{\frac{4+n}{2}} \pi (y - 2^{-\frac{n}{2}} t)], & 0 \le y \le 2^{-\frac{n}{2}} t, \\ 0, & y > 2^{-\frac{n}{2}} t. \end{cases} \tag{4.9.15}$$

From (4.9.7) the stress is given by

$$\sigma_0(y,t) = 2^{-n} \frac{\partial u_0}{\partial y} = \begin{cases} -2^{\frac{4-n}{2}} \pi A_0 \cos[2^{\frac{4+n}{2}} \pi (y - 2^{-\frac{n}{2}} t)], & 0 \le y \le 2^{-\frac{n}{2}} t, \\ 0, & y > 2^{-\frac{n}{2}} t. \end{cases} \tag{4.9.16}$$

Equations (4.9.11) and (4.9.10) and the boundary conditions (4.4.18) for the first order terms $u_1(y,t)$ and $\sigma_1(y,t)$ are the same as equations (4.9.8) and (4.9.7) and the boundary conditions for the zero order terms, $u_0(y,t)$ and $\sigma_0(y,t)$. Hence

$$u_1(y,t) = \begin{cases} -A_1 \sin[2^{\frac{4+n}{2}}\pi(y - 2^{-\frac{n}{2}}t)], & 0 \le y \le 2^{-\frac{n}{2}}t, \\ 0, & y > 2^{-\frac{n}{2}}t. \end{cases} \tag{4.9.17}$$

From (4.9.10) the stress is

$$\sigma_1(y,t) = \frac{\partial u_1}{\partial y} = \begin{cases} -2^{\frac{4-n}{2}} A_1 \pi \cos[2^{\frac{4+n}{2}}\pi(y - 2^{-\frac{n}{2}}t)], & 0 \le y \le 2^{-\frac{n}{2}}t, \\ 0, & y > 2^{-\frac{n}{2}}t. \end{cases} \tag{4.9.18}$$

Finally consider the second order terms $u_2(y,t)$ and $\sigma_2(y,t)$. Substituting (4.9.16) into the wave equation (4.9.14) gives

$$\frac{\partial^2 u_2}{\partial y^2} - 2^{-n}\frac{\partial^2 u_2}{\partial t^2} = \begin{cases} -3 \cdot 2^7 n\pi^4 A_0^3 \cos^2[2^{\frac{4+n}{2}}\pi(y - 2^{-\frac{n}{2}}t)]\times & \\ \sin[2^{\frac{4+n}{2}}\pi(y - 2^{-\frac{n}{2}}t)], & 0 \le y \le 2^{-\frac{n}{2}}t, \\ 0, & y > 2^{-\frac{n}{2}}t. \end{cases} \tag{4.9.19}$$

Expressed in terms of the canonical variables ξ and η, equation (4.9.19) becomes

$$\frac{\partial^2 u_2}{\partial \xi \partial \eta} = \begin{cases} -3 \cdot 2^5 n\pi^4 A_0^3 \cos^2[2^{\frac{4+n}{2}}\pi\xi] \sin[2^{\frac{4+n}{2}}\pi\xi], & -\eta \le \xi \le 0, \\ 0, & \xi > 0. \end{cases} \tag{4.9.20}$$

Integrating (4.9.20) with respect to ξ and η and transforming back in the original variables y and t and imposing the boundary condition (4.4.18) for $m = 2$ gives the second order solution for the displacement

$$u_2(y,t) = \begin{cases} -A_2 \sin[2^{\frac{4+n}{2}}\pi(y - 2^{-\frac{n}{2}}t)] & \\ +2^{4-\frac{n}{2}} n\pi^3 A_0^3 y \cos^3[2^{\frac{4+n}{2}}\pi(y - 2^{-\frac{n}{2}}t)], & 0 \le y \le 2^{-\frac{n}{2}}t, \\ 0, & y > 2^{-\frac{n}{2}}t. \end{cases} \tag{4.9.21}$$

From (4.9.13) and also using (4.9.16) for σ_0 we obtain the stress field at the second order

$$\sigma_2(y,t) \tag{4.9.22}$$

$$= \begin{cases} -2^{2-\frac{n}{2}}\pi A_2 \cos[2^{\frac{4+n}{2}}\pi(y - 2^{-\frac{n}{2}}t)] & \\ -3 \cdot 2^{6-n} n\pi^4 A_0^3 y \cos^2[2^{\frac{4+n}{2}}\pi(y - 2^{-\frac{n}{2}}t)] \sin[2^{\frac{4+n}{2}}\pi(y - 2^{-\frac{n}{2}}t)] & \\ -2^{4-\frac{3n}{2}} n\pi^3 A_0^3 \cos^3[2^{\frac{4+n}{2}}\pi(y - 2^{-\frac{n}{2}}t)], & 0 \le y \le 2^{-\frac{n}{2}}t, \\ 0, & y > 2^{-\frac{n}{2}}t. \end{cases}$$

By combining the zero, first and second order solutions we obtain for the displacement the asymptotic solution correct to order $\mathcal{O}(\epsilon^4)$:

$$u(y,t) = \begin{cases} -\epsilon^2(A_0+\epsilon A_1+\epsilon^2 A_2)\sin[2^{\frac{4+n}{2}}\pi(y-2^{-\frac{n}{2}}t)] & \\ +\epsilon^4 2^{4-\frac{n}{2}} n\pi^3 A_0^3 y\cos^3[2^{\frac{4+n}{2}}\pi(y-2^{-\frac{n}{2}}t)], & 0\le y\le y_F(t;n), \\ 0, & y > y_F(t;n) \end{cases} \tag{4.9.23}$$

where $y_F(t;n)$ is the distance travelled by the wave front, $u=0$, after time t. The value of $y_F(t;n)$ will be calculated in Section 8 by investigating the zeros of $u(y,t)$. For the stress correct to order ϵ^3. The stress correct to order ϵ^3,

$$\sigma(y,t) \tag{4.9.24}$$
$$= \begin{cases} \epsilon 2^{\frac{4-n}{2}}\pi(A_0+\epsilon A_1+\epsilon^2 A_2)\cos[2^{\frac{4+n}{2}}\pi(y-2^{-\frac{n}{2}}t)] & \\ -\epsilon^3 2^{4-\frac{3n}{2}} n\pi^3 A_0^3\Big[\cos^3[2^{\frac{4+n}{2}}\pi(y-2^{-\frac{n}{2}}t)] & \\ +2^{\frac{4+n}{2}}3\pi y\cos^2[2^{\frac{4+n}{2}}\pi(y-2^{-\frac{n}{2}}t)]\sin[2^{\frac{4+n}{2}}\pi(y-2^{-\frac{n}{2}}t)]\Big], & 0\le y\le y_F(t;n), \\ 0, & y > y_F(t;n). \end{cases}$$

The results will be discussed in section 8.

4.9.2 Standing wave solution

We look for solutions of the perturbation equations (4.9.8), (4.9.11) and (4.9.14) in the finite domain, $0\le y\le 1$, subject to the boundary conditions (4.4.34) and (4.4.35) and the initial conditions (4.66) and (4.4.39). Consider first the zero order wave equation (4.9.8). If we make the transformation

$$\bar{t} = 2^{-\frac{n}{2}}t, \tag{4.9.25}$$

equation (4.9.8) reduces to the wave equation with speed unity,

$$\frac{\partial^2 u_0}{\partial y^2} - \frac{\partial^2 u_0}{\partial \bar{t}^2}. \tag{4.9.26}$$

The boundary and initial conditions remain satisfied when expressed in terms of $\bar{t}$. Therefore, using (4.4.56) for u_0 with t replaced by $\bar{t}$ we obtain for the displacement at the zero order

$$u_0(y,\bar{t}) = \frac{A_0}{2}[\sin(\pi(y+\bar{t}))+\sin(\pi(y-\bar{t}))],\ \ 0\le y\le 1,\ \ \bar{t}\ge 0. \tag{4.9.27}$$

Using (4.9.7) and (4.9.27) the stress field at the zero order is

$$\sigma_0(y,\bar{t}) = \frac{\pi A_0}{2^{n+1}}[\cos(\pi(y+\bar{t}))+\cos(\pi(y-\bar{t}))],\ \ 0\le y\le 1,\ \ \bar{t}\ge 0. \tag{4.9.28}$$

The first order wave equation (4.9.11) and the boundary conditions (4.4.59) and initial conditions (4.4.60) are the same as for the zero order problem. Hence the first order solution for the displacement and the stress field is

$$u_1(y,\bar{t}) = \frac{A_1}{2}[\sin(\pi(y+\bar{t})) + \sin(\pi(y-\bar{t}))], \quad 0 \le y \le 1, \quad \bar{t} \ge 0. \tag{4.9.29}$$

$$\sigma_1(y,\bar{t}) = \frac{\pi A_1}{2^{n+1}}[\cos(\pi(y+\bar{t})) + \cos(\pi(y-\bar{t}))], \quad 0 \le y \le 1, \quad \bar{t} \ge 0. \tag{4.9.30}$$

Finally, consider the second order wave equation (4.9.14) subject to the boundary conditions (4.4.65) and the initial conditions (4.4.66). Eliminating σ_2 gives the wave equation

$$\frac{\partial^2 u_2}{\partial y^2} - 2^n \frac{\partial^2 u_2}{\partial t^2} = -2^{n-1} n \frac{\partial}{\partial y}(\sigma_0^3). \tag{4.9.31}$$

In order to make use of the results in subsection 4.2.2 we denote by $\hat{u}$ and $\hat{\sigma}$ the displacement and stress for a standing wave with power-law constitutive equation. Then

$$\sigma_0 = \frac{1}{2^n}\hat{\sigma}_0, \tag{4.9.32}$$

and using (4.9.25) equation (4.9.14) becomes

$$\frac{\partial^2 u_2}{\partial y^2} - \frac{\partial^2 u_2}{\partial \bar{t}^2} = -\frac{n}{2^{2n+1}} \frac{\partial}{\partial y}(\hat{\sigma}_0^3). \tag{4.9.33}$$

By letting

$$\bar{u}_2 = -2^{2n+1} u_2, \tag{4.9.34}$$

equation (4.9.33) can be rewritten as

$$\frac{\partial^2 \bar{u}_2}{\partial y^2} - \frac{\partial^2 \bar{u}_2}{\partial \bar{t}^2} = n \frac{\partial}{\partial y}(\hat{\sigma}_0^3). \tag{4.9.35}$$

which has the same form as (4.4.15) with u_2 replaced by $\bar{u}_2$ and t replaced by $\bar{t}$. Further, the initial condition (4.4.66) can be transformed to

$$\bar{u}_2(y,0) = \bar{A}_2 \sin \pi y, \quad 0 \le y \le 1, \tag{4.9.36}$$

where

$$\bar{A}_2 = -2^{2n+1} A_2. \tag{4.9.37}$$

Thus, using (4.4.90), we obtain for the displacement at the second order

$$u_2(y,\bar{t}) = \frac{A_2}{2}[\sin(\pi(y+\bar{t})) + \sin(\pi(y-\bar{t}))] - \frac{3}{256}\frac{n\pi^2 A_0^3}{2^{2n+1}} U_2(y,\bar{t}), \tag{4.9.38}$$

where $U_2(y,\bar{t})$ is given in (4.4.91) with t replaced by $\bar{t}$. Now $\sigma_2(y,t)$ is obtained from (4.9.13) and using (**??**) it can be verified that

$$\sigma_2(y,t) = \frac{\pi}{2^{n+1}} A_2[\cos(\pi(y+\bar{t})) + \cos(\pi(y-\bar{t}))] - \frac{3}{256}\frac{n\pi^3 A_0^3}{2^{3n+1}} S_2(y,\bar{t}), \quad (4.9.39)$$

where $S_2(y,\bar{t})$ is given in (4.4.93) with t replaced by $\bar{t}$.

By combining the zero, first and second order solutions for the displacement, given by (4.9.27), (4.9.29) and (4.9.38), we obtain from (4.4.5) correct to order ϵ^4,

$$u(y,\bar{t}) = \frac{1}{2}[\epsilon^2 A_0 + \epsilon^3 A_1 + \epsilon^4 A_2][\sin(\pi(y+\bar{t})) + \sin(\pi(y-\bar{t}))] - \frac{3}{256}\frac{\epsilon^4 n\pi^2 A_0^3}{2^{2n+1}} U_2(y,\bar{t}), \quad (4.9.40)$$

where $U_2(y,\bar{t})$ is given by (4.4.91) and $\bar{t}$ is given by (4.9.25). The stress is given by (4.4.4) and using (4.9.28) for σ_0, (4.9.30) for σ_1 and (4.9.39) for σ_2 we obtain correct to order ϵ^3,

$$\sigma(y,\bar{t}) = \frac{\pi}{2^{n+1}}[\epsilon A_0 + \epsilon^2 A_1 + \epsilon^3 A_2][\cos(\pi(y+\bar{t})) + \cos(\pi(y-\bar{t}))] - \frac{3}{256}\frac{\epsilon^3 n\pi^3 A_0^3}{2^{3n+1}} S_2(y,\bar{t}), \quad (4.9.41)$$

where $S_2(y,\bar{t})$ is given by (4.4.93) and $\bar{t}$ is given by (4.9.25).

The results for the displacement and stress will be discussed in the next section.

4.10 Discussion of results for strain limiting constitutive equation

4.10.1 Solitary wave solution for shear stress

The implicit asymptotic solution for the stress (4.8.23) will now be discussed.

Figure 4.10.1 shows the initial profile of the solitary stress wave for $n = 0$ and $n = 0.2$. The initial condition (4.3.1) was not exactly satisfied for $n > 0$ because of the approximation made in going from (4.8.19) to (4.8.20). An expanded scale for w was used in Figure 8 to show more clearly the approximation made in the initial profile for $n > 0$.

Figures 4.10.2 and 4.10.3 show the propagation of the solitary stress wave for $n = 0$ and $n = 0.2$ at times $t = 0.1$ and $t = 1$ respectively. The values of n are again chosen for comparison with the results in [42]. Larger values of n

may also be used and these values will cause the wave to steepen much faster than for lower values of n. From the constitutive equation (3.5.18) we can see that the shearing increases as n is increased. We see that for $n > 0$ the wave speed is decreased and that as time evolves for a fixed $n > 0$ the wave steepens at the front of the wave. Hence after a sufficiently large time a shock front will form at the front of the wave. An estimate of the time that the shock will form can be obtained by differentiating the approximate solution (4.8.24) with respect to y:

$$\frac{\partial \sigma}{\partial y} = \frac{2\pi A \cos\phi}{[1 + 3\pi n 2^{\frac{n}{2}-1} A^2 \sin 2\phi \, C^2(n,\sigma) t]} \tag{4.10.1}$$

where

$$\phi = 2\pi(y - C(n,\sigma)t) \tag{4.10.2}$$

and

$$C(n,\sigma) = \frac{1}{2^{\frac{n}{2}}\left(1 - \frac{3n}{4}\sigma^2(y,t)\right)} \tag{4.10.3}$$

is an approximate expression for the speed of the solitary wave. Hence

$$\left|\frac{\partial \sigma}{\partial y}\right| \to \infty \quad \text{as} \quad t \to t_s = -\frac{1}{3\pi n 2^{\frac{n}{2}-1} A^2 C^2(n,\sigma) \sin 2\phi} \tag{4.10.4}$$

provided $t_s > 0$. Now $t_s > 0$ for $\frac{\pi}{2} \leq \phi \leq \pi$ and for $\frac{3\pi}{2} \leq \phi \leq 2\pi$. From (4.10.1), $\frac{\partial \sigma}{\partial y} < 0$ for $\frac{\pi}{2} \leq \phi \leq \pi$ and $\frac{\partial \sigma}{\partial y} > 0$ for $\frac{3\pi}{2} \leq \phi \leq 2\pi$. In both cases the wave steepens at the front which is in agreement with Figure 4.10.3.

We find that the dependence of the breaking time on the shearing parameter n is given by

$$t_s \sim \frac{1}{2^{\frac{n}{2}-1} n C(n,\sigma)} \tag{4.10.5}$$

and so the same analyses that applies to the power-law constitutive equation applies here as well.

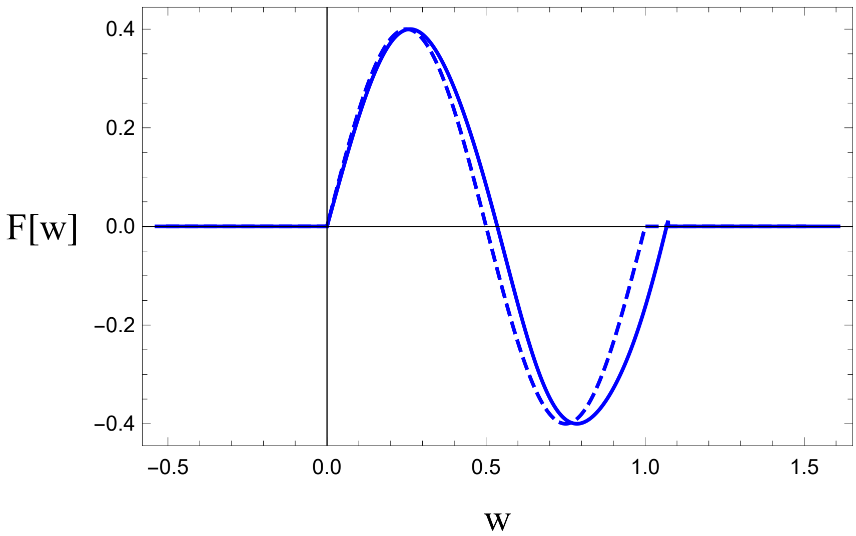

Figure 4.10.1: Initial profile of the solitary stress wave for $n = 0$ (− − −−) and $n = 0.2$ (——) with $A_0 = 0.4$.

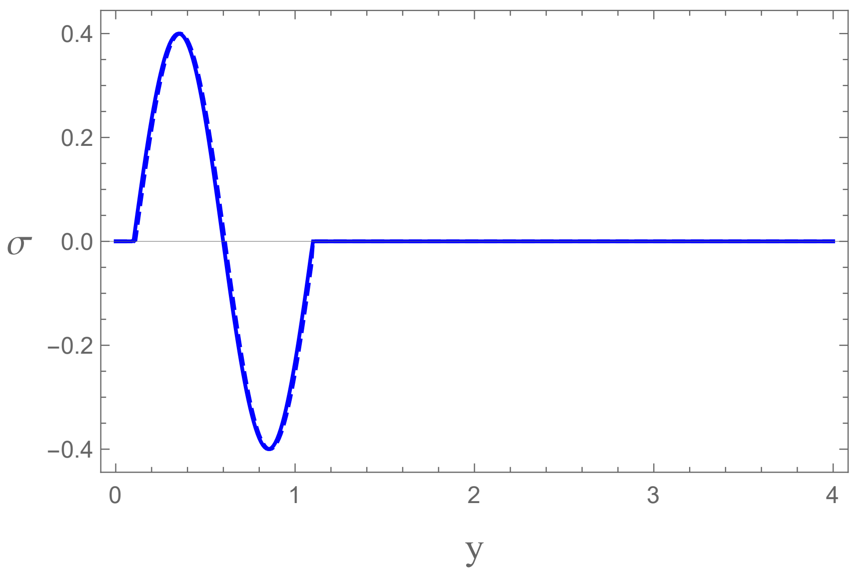

Figure 4.10.2: Propagation of the solitary stress wave for $n = 0$ (− − −−) and $n = 0.2$ (——) at time $t = 0.1$ with $A_0 = 0.4$.

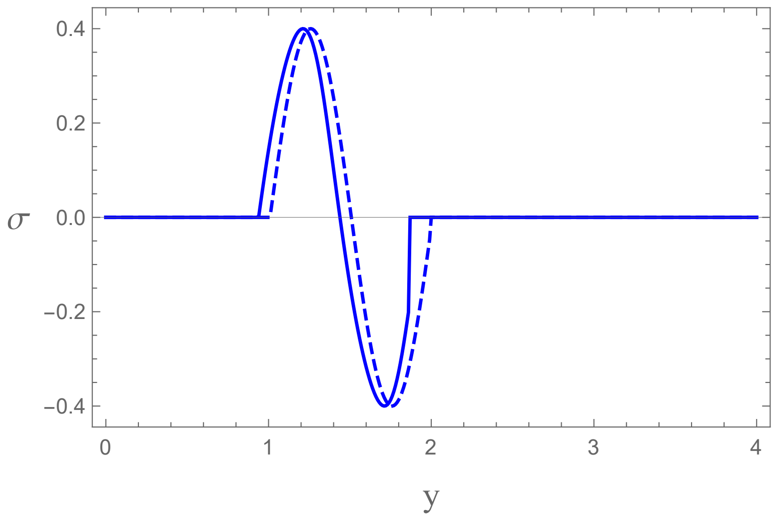

Figure 4.10.3: Propagation of the solitary stress wave for $n = 0$ (− − −−) and $n = 0.2$ (——) at time $t = 1$ with $A_0 = 0.4$.

4.11 Perturbation solutions for displacement and shear stress

The asymptotic solution for the strain-limiting constitutive equation are analysed in this section. We choose $A_1 = 0$ and $A_2 = 0$ for both the travelling wave solution and the standing wave solution.

4.11.1 Travelling wave solution for displacement and stress

The straightforward perturbation expansion for the displacement, (4.9.23), correct to order ϵ^4 and the stress (4.9.24) correct to order ϵ^3 each contain a secular term in y at the highest order in ϵ for $n > 0$. The Linstedt-Poincaré method was not successful in removing the secular term in either the displacement or the stress fields.

The position of the wave front, $y = y_F(t; n)$, is determined by the condition that the amplitude of the displacement vanish at the wave front. From (4.9.23) when $n = 0$,

$$u(y, t) = -\epsilon^2 A_0 \sin[4\pi(y - t)] + \mathcal{O}(\epsilon^5), \tag{4.11.1}$$

and the wave front is at $y = t$ so that $y_F(t; 0) = t$. For $n > 0$ the lowest order approximation is $y_F(t; n) = 2^{-\frac{n}{2}}t$. The position of the wave front is obtained by calculating the zeros of $u(y, t)$ given by (4.9.23) closest to $2^{-\frac{n}{2}}t$. The results for $n = 0$ and $n = 0.3$ are presented in Table 4.11.1.

t	n=0	$2^{-\frac{n}{2}}t$ (n=0.3)	n=0.3
0.5	y_F=0.5	0.451	y_F=0.458
0.8	y_F=0.8	0.721	y_F=0.732

Table 4.11.1: Position of the wave front for the travelling wave, $y_F(t;n)$, for $\epsilon = 0.1$ and $A_0 = 0.4$.

The travelling wave solutions, (4.9.23) for the displacement and (4.9.24) for the stress, are plotted in Figures 4.11.1 and 4.11.2 for $n = 0$ and $n = 0.3$ at times $t = 0.5$ and $t = 0.8$ for $A_0 = 0.4$ and $\epsilon = 0.1$. We see that the speed of propagation of the nonlinear wave for $n > 0$ is less than the speed of propagation of the linear wave for $n = 0$. For the displacement wave the maximum amplitude for $n = 0.3$ is approximately the same as the amplitude for $n = 0$. For the stress wave the maximum amplitude for $n = 0.3$ is less than that for the linear wave $n = 0$. Since the stress at the wave front is non-zero the stress wave propagates as a shock wave with a discontinuity in the stress at the wave front.

When $A_1 = A_2 = 0$, the straightforward perturbation expansion (4.9.23) breaks down for

$$y \gtrsim \frac{2^{\frac{n}{2}}}{16n\pi^3 A_0^2 \epsilon^2} \approxeq 4.66, \tag{4.11.2}$$

evaluated for parameter values $n = 0.3$, $A_0 = 0.4$ and $\epsilon = 0.1$. The straightforward perturbation expansion for the stress, (4.9.24), breaks down for

$$y \gtrsim \frac{2^{\frac{n}{2}}}{48n\pi^3 A_0^2 \epsilon^2} \approxeq 1.55 \tag{4.11.3}$$

for the same parameter values. These values of y are outside the range $0 \leq y \leq 1$ considered in Figures 4.11.1 and 4.11.2. Again we find that as the value of n increase the perturbation solution will break down for smaller values of y.

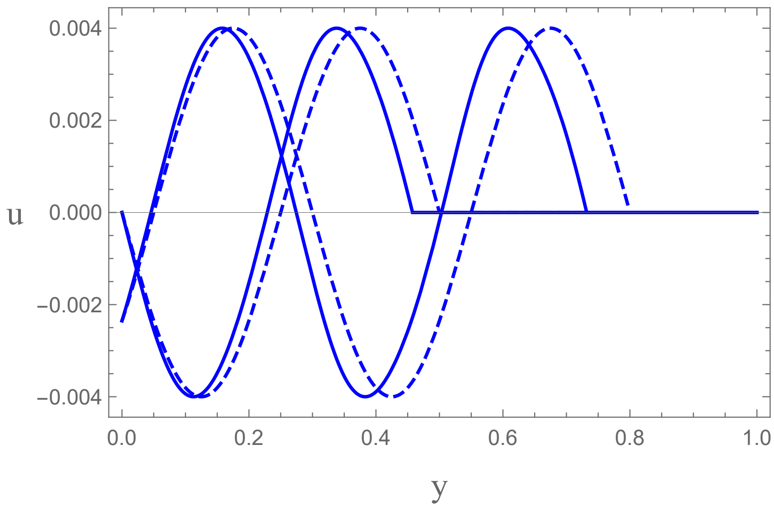

Figure 4.11.1: Travelling wave for the displacement for $n = 0$ (− − −−) and $n = 0.3$ (——) at times $t = 0.5$ and $t = 0.8$ with $A_0 = 0.4$ and $\epsilon = 0.1$.

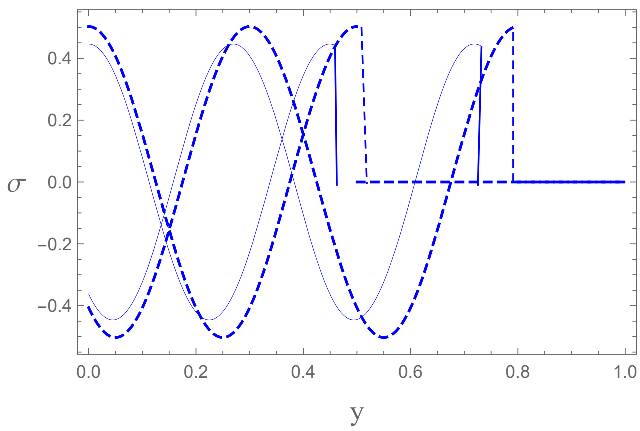

Figure 4.11.2: Travelling wave for the stress for $n = 0$ (− − −−) and $n = 0.3$ (——) at times $t = 0.5$ and $t = 0.8$ with $A_0 = 0.4$ and $\epsilon = 0.1$.

4.11.2 Standing wave solution for displacement and stress

For $n > 0$ the straightforward perturbation expansions, (4.9.40) for the displacement and (4.9.41) for the stress contain secular terms in t at order ϵ^4 and order ϵ^3 respectively, which cannot be removed using singular perturbation methods such as the Lindstedt-Poincaré method. When $A_1 = A_2 = 0$ the expansions (4.9.40) and (4.9.41) both break down for

$$t \gtrsim \frac{2^n\, 64}{9n\pi^3 A_0^2 \epsilon^2} \approxeq 724, \tag{4.11.4}$$

evaluated for $n = 0.3$, $A_0 = 0.4$ and $\epsilon = 0.1$. This value of t is well outside the range $0 \leq t \leq 1$ used in Figures 13 and 14.

The solution for the displacement wave, (4.9.40), and the stress wave, (4.9.41) can be rewritten in the form of a superposition of standing waves as

$$u(y,\bar{t}) = (\epsilon^2 A_0 + \epsilon^3 A_1 + \epsilon^4 A_2)\sin(\pi y)\cos(\pi \bar{t}) - \frac{3}{256}\frac{\epsilon^4 n\pi^2 A_0^3}{2^{2n+1}} U_2(y,\bar{t}), \quad (4.11.5)$$

where $U_2(y,\bar{t})$ is given by (4.6.2) with t replaced by $\bar{t}$ and

$$\sigma(y,\bar{t}) = \frac{\pi}{2^n}(\epsilon A_0 + \epsilon^2 A_1 + \epsilon^3 A_2)\cos(\pi y)\cos(\pi \bar{t}) - \frac{3}{256}\frac{\epsilon^3 n\pi^3 A_0^3}{2^{3n+1}} S_2(y,\bar{t}), \quad (4.11.6)$$

where $S_2(y,\bar{t})$ is given by (4.6.2) with t replaced by $\bar{t}$ where $\bar{t} = 2^{-\frac{n}{2}}t$. We are considering $A_1 = A_2 = 0$. There are two differences between (4.11.5) and (4.11.6) for the strain-limiting constitutive equation and (4.6.5) and (4.6.7) for the power-law constitutive equation. The scaled time $\bar{t}$ replaces t for $n > 0$ and the sign of the terms of order ϵ^4 in (4.11.5) and order ϵ^3 in (4.11.6) are opposite to the sign of the corresponding terms in (4.6.5) and (4.6.7). Although the displacement waves for $n = 0$ and $n = 0.3$ are the same for $t = 0$, they are different for $t = 1$, unlike the standing waves for the power-law constitutive equation.

Consider first the displacement wave. When $n = 0$, we have $\bar{t} = t$ and (4.11.5) reduces to a standing wave of amplitude $\epsilon^2 A_0$ with form $\sin(\pi y)$ and period of oscillation $T = 2$ as for the power-law constitutive equation. When $n > 0$ the period of oscillation of the terms of order ϵ^2 increases to $T = 2^{1+\frac{n}{2}}$ and there is a superposition of standing waves with amplitude of order ϵ^4 of the form $\sin(\pi y)$ and $\sin(3\pi y)$ with period of oscillation $T = 2^{1+\frac{n}{2}}$ and $T = \frac{2^{1+\frac{n}{2}}}{3}$. In Figure 4.11.3 the displacement waves for $n = 0$ and $n = 0.3$ are compared for $0 \leq t \leq 1$. The differences are small although larger than for the power-law constitutive equation because of the change of period for $n > 0$.

Consider next the stress wave. When $n = 0$ we again have $\bar{t} = t$ and (4.11.6) is a standing wave of amplitude $\epsilon\pi A_0$ of the form $\cos(\pi y)$ for $0 \leq y \leq 1$ and period $T = 2$ as for the power-law constitutive equation. For $n > 0$ the period of oscillation of the terms of order ϵ increases to $T = 2^{1+\frac{n}{2}}$ and there is a linear superposition of standing waves of amplitude order ϵ^3 and of the form $\cos(\pi y)$ and $\cos(3\pi y)$ with period $T = 2^{1+\frac{n}{2}}$ and $T = \frac{2^{1+\frac{n}{2}}}{3}$. In Figure 4.11.4 the stress waves for $n = 0$ and $n = 0.3$ are compared for $0 \leq t \leq 1$. The differences are larger than for the power-law constitutive equation, again because of the change of period of oscillation for $n > 0$.

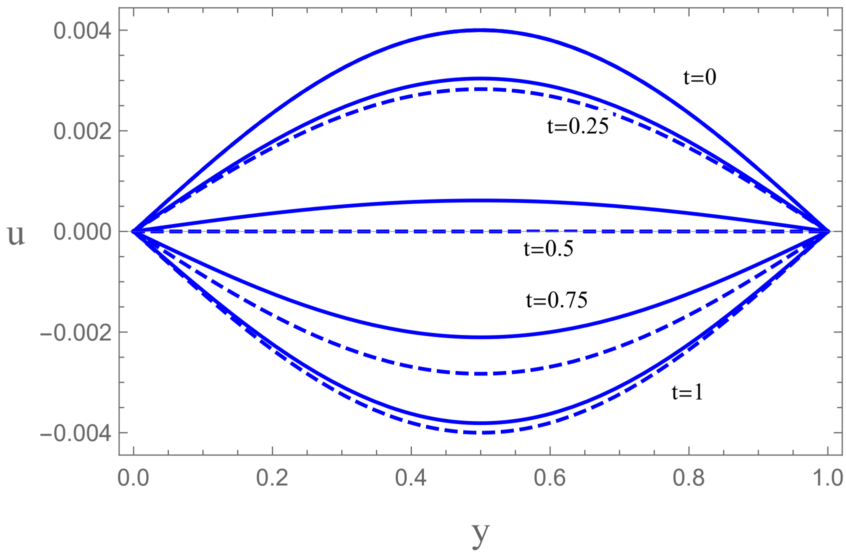

Figure 4.11.3: Standing wave for the displacement for $n = 0$ (− − −−) and $n = 1$ (——) with $A_0 = 0.4$ and $\epsilon = 0.1$ at times $t = 0, 0.25, 0.5, 0.75$ and 1.

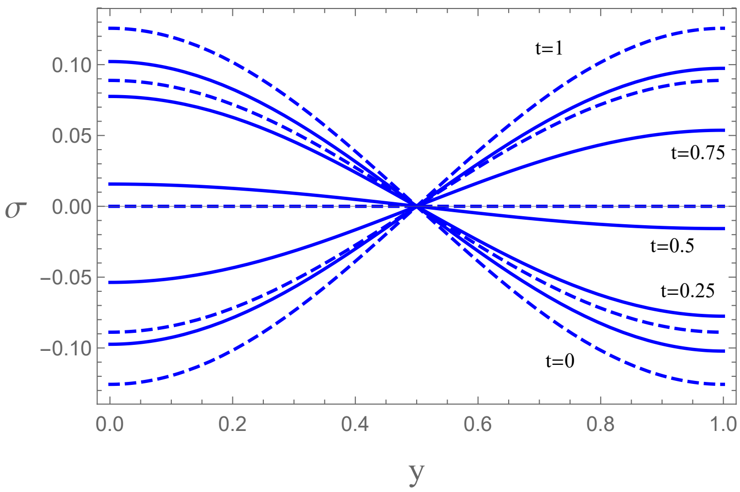

Figure 4.11.4: Standing wave for the stress for $n = 0$ (− − −−) and $n = 0.3$ (——) with $A_0 = 0.4$ and $\epsilon = 0.1$ at times $t = 0, 0.25, 0.5, 0.75$ and 1.

4.12 Conclusions and comparison of results

We investigated the propagation of shear stress waves and displacement waves described by a new class of constitutive equations. A feature of this class of constitutive equations is that the relationship between the stress and linearised strain is nonlinear. Such a phenomenon cannot be captured within the classical theory of Cauchy elasticity or hyperelasticity.

We considered two constitutive equations. The first constitutive equation is referred to as a power-law constitutive equation because of its analogy with the power-law constitutive equation for a non-Newtonian fluid. However, in elasticity the kinematical variables are interchanged. The second constitutive equation is referred to as a strain limiting constitutive equation and has application in the study of fracture mechanics. For each constitutive equation we found solitary wave solutions, travelling wave solutions and standing wave solutions.

To find solitary wave solutions for each constitutive equation we reduced the system of partial differential equations to a nonlinear second order hyperbolic partial differential equation which describes nonlinear shear stress waves. For the power-law constitutive equation the speed of the solitary wave is approximately

$$C(n,\sigma) = \frac{1}{\left(1 + \dfrac{3n}{2}\sigma^2(y,t)\right)}. \tag{4.12.1}$$

As the magnitude of the stress $\sigma(y,t)$ increases the speed of parts of the solitary wave with larger stress magnitudes decreases, the wave steepens at the back and a shock front forms at the back of the solitary wave. This occurs at approximately time t_s where

$$t_s = \frac{1}{3\pi n A^2 C^2(n,\sigma)\sin 2\phi} = \frac{1}{3\pi n A^2 \sin 2\phi}\left(1 + \frac{3n}{2}\sigma^2(y,t)\right)^2, \tag{4.12.2}$$

where $\sin 2\phi > 0$.
For the strain-limiting constitutive equation the speed of the solitary wave is approximately

$$C(n,\sigma) = \frac{1}{2^{\frac{n}{2}}\left(1 - \dfrac{3n}{4}\sigma^2(y,t)\right)^2}. \tag{4.12.3}$$

As the magnitude of the stress increases the speed of parts of the solitary wave with larger stress magnitude increases, the wave steepens at the front and a shock wave forms at the front of the solitary wave. The shock wave forms at approximately time

$$t_s = -\frac{1}{3\pi 2^{\frac{n)}{2}-1} n A^2 C^2(n,\sigma)\sin 2\phi} = -\frac{2^{\frac{n}{2}+1}}{3\pi A^2 \sin 2\phi}\left(1 - \frac{3n}{2}\sigma^2(y,t)\right) \tag{4.12.4}$$

where $\sin 2\phi < 0$. Comparing the times t_s given by (4.12.2) and (4.12.4) we see that the shock front will generally form earlier for the strain-limiting constitutive equation.

Rajagopal and Saccomandi [52] have shown that even for implicit constitutive equations the linear degeneracy of the equations allows for the existence and possibility of global solutions to a Cauchy problem. We have also shown that such solutions are possible by deriving equation (4.8.16). Specific forms of these unknown functions are obtained in this paper.

We then considered perturbation solutions for the system of partial differential equations for the displacement and stress, and found travelling wave solutions and standing wave solutions for the displacement and stress.

For the travelling wave solutions, the straightforward perturbation expansions for the displacement fields and stress fields for both constitutive equations contained a secular term in y, which could not be removed by using singular perturbation methods such as the Lindstedt-Poincaré method. Despite the secular term in each solution, it was found that the perturbation expansion breaks down only for values of y outside of the ranges of y considered. The straightforward perturbation solution is therefore sufficient to investigate the physical properties of the displacement and stress waves. The wave front of the travelling waves was determined from the condition that the displacement vanish at the wave front. Since the stress is non-zero at the wave front the stress wave propagates as a shock wave with a finite discontinuity in the stress at the wave front. For the linear wave with $n = 0$ the wave propagates with unit speed for both constitutive equations. For the nonlinear wave for $n > 0$ the propagation speed is slower, just below unit speed, for the power-law constitutive equation and approximately equal to $2^{-\frac{n}{2}}$ for the strain-limiting constitutive equation. For both constitutive equations the amplitude of the displacement wave is approximately the same for $n = 0$ and $n > 0$. For the stress wave for both constitutive equations the amplitude for $n = 0.3$ is less than for $n = 0$. The change in the stress wave for $n > 0$ reflects the deformation taking place in the nonlinear wave.

For standing waves the straightforward perturbation solution for the displacement and stress for both constitutive equations contained a secular term in t which could not be removed by standard singular perturbation methods. Since the perturbation expansions break down only for values of time outside the ranges considered they could be used to investigate the physical properties of the waves. For $n = 0$ and for both constitutive equations the period of oscillation of the standing waves for the displacement and stress was $T = 2$. For $n > 0$, for the power-law constitutive equation the period of oscillation of the waves remained approximately $T \simeq 2$, while for the strain-limiting constitutive equation the period increased to $T \simeq 2^{1+\frac{n}{2}}$.

The dependence of the solution on n shows that in general as we increase the value of n the speed of the wave decreases. The distortion in the wave is more pronounced when compared with $n = 0$ and when compared with a smaller value of $n > 0$. For larger values of n the steepening of the wave increases as the wave propagates into the material, as for smaller values of n, and the same characteristics are observed if we fix n and consider a later time. The perturbation solutions also break down for smaller values of y and t as n increases.

The analysis of wave motion in elastic materials with power-law and strain-limiting constitutive equations has shown that these materials have different elastic properties. In general we found that in elastic materials described by the power-law constitutive equation the elastic response is enhanced while in elastic materials with the strain-limiting constitutive equation the elastic response is restricted. For the solitary wave the shock forms at the front of the wave for the strain-limiting constitutive equation and after a shorter time than for the power-law constitutive equation for which it forms at the back of the wave. For the travelling wave, the wave speed for the strain-limiting constitutive equation is reduced to a greater extent than for the power-law constitutive equation. The amplitude of the stress for both constitutive equations is reduced when compared with the solution for $n = 0$. For the standing wave, for $n > 0$ the period of oscillation of the power-law wave was approximately the same as for the linear wave for $n = 0$, while for the strain-limiting material the period of oscillation increased.

Chapter 5

Axial stress waves and displacement waves in a circular cylinder for a new class of constitutive equations

5.1 Introduction

As was done in Chapter 4, we will consider a particular subclass of constitutive equations where the strain is prescribed in terms of a non-invertible function of the stress. The power-law constitutive equation and the strain-limiting constitutive equation are the two constitutive equations that will be considered.

Non-linear wave equations described by such constitutive equations have not been studied in much detail. The propagation of shear displacement and stress waves in a rectangular slab for a power-law constitutive equation was first studied by Kannan et al [42] and numerical solutions were obtained. In Chapter 4 we found solitary wave solutions, travelling wave solutions and standing wave solutions for the power-law constitutive equation and the strain-limiting constitutive equation. The propagation of circumferential stress and displacement waves in a nonlinear cylindrical annulus was studied by Kambapalli [43] and the numerical results were obtained using the finite element software package *Comsol*.

In this Chapter we find solitary wave solutions and standing wave solutions for both constitutive equations. To find the standing wave solutions we consider straightforward perturbation expansions for the stress and displacement. The perturbation equations at the second order for the displacement could not be solved for either constitutive equation. Further, we looked for travelling wave solutions by reducing the perturbation equations at each order to canonical form and used the Riemann function to solve for the displacement however, the boundary conditions could not satisfied. Finally, methods like Lie group analysis could not be used to solve the equations at each order exactly as the boundary conditions are not invariant.

The Chapter is organised as follows. In Section 5.2 the mathematical model for the power-law constitutive equation is derived. In Section 5.3 we find an implicit asymptotic solution for the stress. Perturbation solutions for the displacement and stress in the form of standing waves are sought in Section 5.4 and the results are discussed and analysed in Section 5.5. The second half of the Chapter investigated the propagation of axial displacement and stress waves for the strain-limiting constitutive equation and is structured in the same way as the first half of the Chapter.

5.2 Mathematical model for the power-law constitutive equation

In this section we derive a system of nonlinear partial differential equations which describe the propagation of axial displacement waves and stress waves in a circular cylinder. The domain of interest is

$$D = \{(R, \Theta, Z) | 0 \le R \le R_0, 0 \le \Theta \le 2\pi, -\infty \le Z \le \infty\}, \tag{5.2.1}$$

where R_0 is the radius of the cylinder. In this work we study the propagation of axial waves.

In this theory, a special semi-inverse solution is sought. That is, specific forms for both the displacement and stress fields are assumed and we solve for the displacement and stress simultaneously. As a result of this assumption, we obtain a system of partial differential equations instead of a single partial differential equation to describe wave propagation.

We consider the following stress and displacement fields,

$$\boldsymbol{\sigma} = \sigma(r,t)(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r), \quad \mathbf{u} = u(r,t)\mathbf{e}_z. \tag{5.2.2}$$

From (5.2.2) we are investigating transverse displacement waves. For the the motions assumed in (5.2.2) we find that det$\mathbf{F} = 1$ and so the body is undergoing an isochoric deformation. Further, if we assume that the density is constant in both the reference and current configuration, we find that the conservation of mass is identically satisfied.

The first constitutive equation that we study is the power-law constitutive equation (3.5.17) Using the components of the strain tensor in terms of the displacement (2.3.38), the equations of motion in cylindrical coordinates (2.6.25), (2.6.26) and (2.6.27) and the constitutive equation (3.5.17) we obtain the following system of partial differential equations

$$\begin{aligned} \frac{\partial \sigma}{\partial r} + \frac{\sigma}{r} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial u}{\partial r} &= \alpha(1+\gamma\sigma^2)^n \sigma. \end{aligned} \tag{5.2.3}$$

We introduce the following non-dimensional quantities

$$\bar{r} = \frac{r}{R_0}, \quad \bar{t} = \frac{t}{R_0\sqrt{\alpha\rho}}, \quad \bar{u} = \frac{u}{R_0}, \quad \bar{\sigma} = \sqrt{\gamma}\sigma. \tag{5.2.4}$$

Substituting (5.2.4) into (5.2.3) we obtain the following dimensionless system of partial differential equations for the first constitutive equation, where we have suppressed the bars for convenience

$$\frac{\partial \sigma}{\partial r} + \frac{\sigma}{r} = \frac{\sqrt{\gamma}}{\alpha}\frac{\partial^2 u}{\partial t^2} \tag{5.2.5}$$

$$\frac{\partial u}{\partial r} = \frac{\alpha}{\sqrt{\gamma}}(1+\sigma^2)^n \sigma. \tag{5.2.6}$$

The same analysis can be carried out when $n = 0$ as was done in Chapter 4. The non-dimensional system of partial differential equations (5.2.5) and (5.2.6) can be written as a single partial differential equation in terms of the stress as

$$\frac{\partial^2 \sigma}{\partial r^2} + \frac{1}{r}\frac{\partial \sigma}{\partial r} - \frac{1}{r^2}\sigma = \frac{\partial^2}{\partial t^2}[(1+\sigma^2)^n \sigma]. \tag{5.2.7}$$

In the sections that follow the solutions to the system of equations (5.2.5) and (5.2.6) and the solution to the nonlinear hyperbolic partial differential equation (5.2.7) for the stress will be investigated for a range of initial and boundary conditions. Consider first equation (5.2.7) and the domain $0 \leq r \leq R_0$.

5.3 Asymptotic solutions for the power-law constitutive equation

5.3.1 Solitary wave solution

In this section we derive an asymptotic solution in implicit form for the second order wave equation (5.2.7) for the stress subject to the initial conditions

$$\sigma(r,0) = \begin{cases} A\sin(2\pi r), & 0 \leq r \leq 1 \\ 0, & 1 \leq r \leq R_0 \end{cases} \tag{5.3.1}$$

where A is the prescribed amplitude. The boundary condition is

$$\sigma(R_0, t) = 0. \tag{5.3.2}$$

Let $\sigma = \epsilon\hat{\sigma}(r,t)$, where $\epsilon < 1$. We substitute this approximation into (5.2.7) and take the binomial expansion of the term $(1+\sigma^2)^n$ up to the order $\mathcal{O}(\epsilon^3)$ to obtain the asymptotic partial differential equation

$$\frac{\partial^2 \hat{\sigma}}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{\sigma}}{\partial r} - \frac{1}{r^2}\hat{\sigma} = \frac{\partial^2 \hat{\sigma}}{\partial t^2} + n\epsilon^2 \frac{\partial^2 \hat{\sigma}^3}{\partial t^2}. \tag{5.3.3}$$

Introducing the new variables

$$\tau = t - r, \quad R = \epsilon^2 r, \tag{5.3.4}$$

and substituting (5.3.4) into (5.3.3) we obtain

$$-2\frac{\partial^2 \sigma^\star}{\partial \tau \partial R} + \epsilon^2 \frac{\partial^2 \sigma^\star}{\partial R^2} - \frac{1}{R}\frac{\partial \sigma^\star}{\partial \tau} + \epsilon^2 \frac{\partial}{\partial R}\left(\frac{\sigma^\star}{R}\right) = n\frac{\partial^2 (\sigma^\star)^3}{\partial \tau^2} \tag{5.3.5}$$

where we have defined the stress in terms of the new variables

$$\hat{\sigma}(r,t) = \sigma^\star(\tau(r,t), R(r)). \tag{5.3.6}$$

Integrating (5.3.5) with respect to τ and neglecting terms of order $\mathcal{O}(\epsilon^2)$ we obtain the following first order linear partial differential equation

$$2\frac{\partial \sigma^\star}{\partial R} + 3n(\sigma^\star)^2 \frac{\partial \sigma^\star}{\partial \tau} = -\frac{1}{R}\sigma^\star. \tag{5.3.7}$$

where the arbitrary integration function of R is set to zero because we are looking for a particular solution and not the general solution. The differential equations of the characteristic curves of (5.3.7) are

$$\frac{dR}{2} = \frac{d\tau}{3n(\sigma^\star)^2} = -R\frac{d\sigma^\star}{\sigma^\star}. \tag{5.3.8}$$

Unlike the solitary wave solution in a rectangular slab, $\sigma^\star$ is not a constant along a characteristic curve.

Integrating the first and the last terms gives the first integral

$$\sqrt{R}\sigma^\star = k_1. \tag{5.3.9}$$

Since k_1 is a constant along a characteristic curve we use (5.3.9) to rewrite $\sigma^\star$ in terms of R. The first pair of terms become

$$\frac{3n}{2}k_1^2 \frac{dR}{R} = d\tau \tag{5.3.10}$$

and integrating (5.3.10) we obtain after replacing k_1 using (5.3.9)

$$\frac{3n}{2}(\sigma^\star)^2 R\ln R - \tau = k_2 \tag{5.3.11}$$

where k_2 is a constant. Expressed in terms of the original variables (5.3.4) and using (5.3.6) and $\sigma = \epsilon\hat{\sigma}$, (5.3.11) becomes

$$\frac{3n}{2}\sigma^2 r\ln(\epsilon^2 r) - t + r = k_2. \tag{5.3.12}$$

Expanding (5.3.12) gives

$$\frac{3n}{2}\sigma^2 r\ln(\epsilon^2) + \frac{3n}{2}\sigma^2 r\ln r - t + r = k_2. \tag{5.3.13}$$

Using (5.3.9), equation (5.3.13) can be written as

$$\frac{3n}{2}k_1^2\ln(\epsilon^2) + \frac{3n}{2}\sigma^2 r\ln r - t + r = k_2 \tag{5.3.14}$$

where k_1 is a constant along a characteristic curve. Thus the first term of (5.3.14) is also a constant and can be incorporated into the constant k_2. Therefore we can write the first integral of the first pair of terms in (5.3.8) as

$$\frac{3n}{2}\sigma^2 r\ln r - t + r = k_2^\star. \tag{5.3.15}$$

The general solution of (5.3.8) is

$$k_1 = F(k_2^\star) \tag{5.3.16}$$

where F is an arbitrary function. Expressing the first integral (5.3.9) in terms of the original variables (5.3.4) and using the fact that $\sigma = \epsilon\hat{\sigma}$ we obtain for the stress

$$\sigma(r,t) = \frac{1}{\sqrt{r}}F\left[\left(1 + \frac{3n}{2}\sigma^2(r,t)\ln r\right)r - t\right]. \tag{5.3.17}$$

Equation (5.3.17) can also be written as $= \dfrac{1}{\sqrt{r}}F(\xi)$ where $\xi = \left(1 + \frac{3n}{2}\sigma^2(r,t)\ln r\right)r - t$. The function F can be determined using the initial condition (5.3.1)

$$F\left[\left(1 + \frac{3n}{2}A^2\sin^2(2\pi r)\ln r\right)r\right] = \sqrt{r}A\sin(2\pi r), \quad 0 \le r \le 1, \tag{5.3.18}$$

$$F[r] = 0, \quad 1 \le r \le \infty. \tag{5.3.19}$$

Let

$$w = \begin{cases} \left(1 + \dfrac{3n}{2}A^2\sin^2(2\pi r)\ln r\right)r, & 0 \le r \le 1 \\ r, & 1 \le r \le \infty. \end{cases} \tag{5.3.20}$$

Equation (5.3.20) has to be solved for r in terms of w and to do that for $0 \leq r \leq \infty$ we use $r = w$ as a first approximation since $sin^2(2\pi r)\ln r$ is small for $0 < r \leq 1$ and tends to zero as $r \to 0$. Iterating with $r = w$ the next approximation is

$$r = \frac{w}{\left(1 + \frac{3n}{2}A^2 \sin^2(2\pi w)\ln w\right)}, \quad 0 \leq r \leq 1. \tag{5.3.21}$$

We will use the approximate result (5.3.21). No approximation is needed in the limits of the range $0 \leq r \leq 1$ because the stress $\sigma(r,0)$ vanishes at the front and back of the solitary wave. Thus equations (5.3.18) and (5.3.19) become

$$F(w) \tag{5.3.22}$$

$$= \begin{cases} \left[\dfrac{w}{\left(1 + \frac{3n}{2}A^2 \sin^2(2\pi w)\ln w\right)}\right]^{\frac{1}{2}} A \sin\left[\dfrac{2\pi w}{\left(1 + \frac{3n}{2}A^2 \sin^2(2\pi w)\ln w\right)}\right], & 0 \leq w \leq 1 \\ 0, & 1 \leq w \leq \infty. \end{cases}$$

To find the solution for the stress $\sigma(r,t)$ given by (5.3.17), we set in (5.3.22)

$$w = \left(1 + \frac{3n}{2}\sigma^2(r,t)\ln r\right) r - t \tag{5.3.23}$$

and to obtain the end points of the range $0 \leq w \leq 1$ we set $w = r - t$ because the stress $\sigma(r,t) = 0$ at the wave front and end of the solitary wave. Thus

$$\sigma(r,t) \tag{5.3.24}$$

$$= \begin{cases} \left[\dfrac{(1 + \frac{3n}{2}\sigma^2(r,t)\ln r)r - t}{[1 + \frac{3n}{2}A^2 \sin^2(2\pi((1 + \frac{3n}{2}\sigma^2(r,t)\ln r)r - t))\ln((1 + \frac{3n}{2}\sigma^2(r,t)\ln r)r - t)]}\right]^{\frac{1}{2}} \times \\ A \sin\left[\dfrac{2\pi((1 + \frac{3n}{2}\sigma^2(r,t)\ln r)r - t)}{[1 + \frac{3n}{2}A^2 \sin^2(2\pi((1 + \frac{3n}{2}\sigma^2(r,t)\ln r)r - t))\ln((1 + \frac{3n}{2}\sigma^2(r,t)\ln r)r - t)]}\right], \\ \hfill t \leq r \leq 1 + t, \\ 0, \hfill 1 + t \leq r \leq \infty. \end{cases}$$

To interpret the solution physically, we can write (5.3.24) approximately as

$$\sigma(r,t) = \begin{cases} A(r - Ct)^{\frac{1}{2}} \sin[2\pi(r - Ct)], & t \leq r \leq 1 + t \\ 0, & 1 + t \leq r \leq \infty, \end{cases} \tag{5.3.25}$$

where

$$C(n,\sigma,r) = \frac{1}{1 + \frac{3n}{2}\sigma^2(r,t)\ln r}. \tag{5.3.26}$$

Since $\sigma(0,t) = 0$ this approximate solution will apply for $r \geq 0$ but will mainly be of interest for $r > 0$ as the solitary wave forms a breaking wave. Solutions (5.3.24) and (5.3.25) are exact solutions without approximation when $n = 0$. The results will be analysed and discussed in Section 5.5.1.

5.4 Perturbation solutions for the first order system for displacement and stress

Since we require the displacement gradient to be small we set

$$\frac{\alpha}{\sqrt{\gamma}} = \epsilon. \tag{5.4.1}$$

Therefore, equation and (5.2.5) and (5.2.6) become

$$\frac{\partial \sigma}{\partial r} + \frac{\sigma}{r} = \frac{1}{\epsilon}\frac{\partial^2 u}{\partial t^2} \tag{5.4.2}$$
$$\frac{\partial u}{\partial r} = \epsilon(1+\sigma^2)^n \sigma. \tag{5.4.3}$$

We consider straightforward perturbation expansions of the form

$$u(r,t) = \epsilon^2 u_0(r,t) + \epsilon^3 u_1(r,t) + \epsilon^4 u_2(r,t) + \mathcal{O}(\epsilon^5), \tag{5.4.4}$$

$$\sigma(r,t) = \epsilon\sigma_0(r,t) + \epsilon^2\sigma_1(r,t) + \epsilon^3\sigma_2(r,t) + \mathcal{O}(\epsilon^4) \tag{5.4.5}$$

Substituting (5.4.4) and (5.4.5) into the system of partial differential equations (5.4.2) and (5.4.3) and expanding the term $(1+\sigma^2)^n$ up to $\mathcal{O}(\epsilon^3)$ using the binomial theorem we obtain the following system of equations

$$\begin{aligned}
\epsilon\frac{\partial \sigma_0}{\partial r} + \epsilon^2\frac{\partial \sigma_1}{\partial r} + \epsilon^3\frac{\partial \sigma_2}{\partial r} + \frac{1}{r}(\epsilon\sigma_0 + \epsilon^2\sigma_1 + \epsilon^3\sigma_2) &= \epsilon\frac{\partial^2 u_0}{\partial t^2} + \epsilon^2\frac{\partial^2 u_1}{\partial t^2} + \epsilon^3\frac{\partial^2 u_2}{\partial t^2} \\
\epsilon^2\frac{\partial u_0}{\partial r} + \epsilon^3\frac{\partial u_1}{\partial r} + \epsilon^4\frac{\partial u_2}{\partial r} &= \epsilon^2\sigma_0 + \epsilon^3\sigma_1 + \epsilon^4(\sigma_2 + n\sigma_0^3).
\end{aligned} \tag{5.4.6}$$

Equating terms of the same order of ϵ in (5.4.6) yields a system of partial differential equations at each order:

Zero order terms u_0 and σ_0:

$$\epsilon \quad : \quad \frac{\partial \sigma_0}{\partial r} + \frac{1}{r}\sigma_0 = \frac{\partial^2 u_0}{\partial t^2} \tag{5.4.7}$$
$$\epsilon^2 \quad : \quad \frac{\partial u_0}{\partial r} = \sigma_0, \tag{5.4.8}$$

Eliminating σ_0 from (5.4.7) and (5.4.8) gives the cylindrical wave equation

$$\frac{\partial^2 u_0}{\partial r^2} - \frac{\partial^2 u_0}{\partial t^2} + \frac{1}{r}\frac{\partial u_0}{\partial r} = 0. \tag{5.4.9}$$

First order terms u_1 and σ_1:

$$\epsilon^2 \quad : \quad \frac{\partial \sigma_1}{\partial r} + \frac{1}{r}\sigma_1 = \frac{\partial^2 u_1}{\partial t^2} \tag{5.4.10}$$

$$\epsilon^3 \quad : \quad \frac{\partial u_1}{\partial r} = \sigma_1, \tag{5.4.11}$$

Eliminating σ_1 from (5.4.10) and (5.4.11) gives

$$\frac{\partial^2 u_1}{\partial r^2} - \frac{\partial^2 u_1}{\partial t^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} = 0. \tag{5.4.12}$$

Second order terms u_2 and σ_2:

$$\epsilon^3 \quad : \quad \frac{\partial \sigma_2}{\partial r} + \frac{1}{r}\sigma_2 = \frac{\partial^2 u_2}{\partial t^2} \tag{5.4.13}$$

$$\epsilon^4 \quad : \quad \frac{\partial u_2}{\partial r} = \sigma_2 + n\sigma_0^3. \tag{5.4.14}$$

Eliminating σ_2 from (5.4.13) and (5.4.14) gives the nonlinear cylindrical wave equation

$$\frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial t^2} + \frac{1}{r}\frac{\partial u}{\partial r} = n\left(\frac{1}{r}\sigma_0^3 + 3\sigma_0^2\frac{\partial \sigma_0}{\partial r}\right). \tag{5.4.15}$$

We will only consider standing wave solutions. The general procedure is to solve the wave equation (5.4.9) for the displacement and then to obtain the stress from (5.4.8). We attempted to find travelling wave solutions by reducing the wave equations at each order to canonical form and solved the canonical equations using the Riemann method. However, such solutions could not be obtained as the boundary conditions are not satisfied.

5.4.1 Standing wave solution

We look for solutions of the perturbation equation (5.4.9) subject to the boundary conditions

$$r = 1: \quad u(1,t) = \epsilon^2 u_0(1,t) + \epsilon^3 u_1(1,t) + \epsilon^4 u_2(1,t) + \mathcal{O}(\epsilon^5) = 0, \quad t \geq 0, \tag{5.4.16}$$

and to the initial conditions

$$t = 0: \quad u(r,0) = A\sin(4\pi r), \quad 0 \leq r \leq 1, \tag{5.4.17}$$

$$\epsilon^2 u_0(r,0) + \epsilon^3 u_1(r,0) + \epsilon^4 u_2(r,0) + \mathcal{O}(\epsilon^5) = (\epsilon^2 A_0 + \epsilon^3 A_1 + \epsilon^4 A_2 + \mathcal{O}(\epsilon^5))\sin(4\pi r), \quad 0 \leq r \leq 1 \tag{5.4.18}$$

$$t = 0: \quad \frac{\partial u(r,0)}{\partial t} = \epsilon^2\frac{\partial u_0(r,0)}{\partial t} + \epsilon^3\frac{\partial u_1(r,0)}{\partial t} + \epsilon^4\frac{\partial u_2(r,0)}{\partial t} + \mathcal{O}(\epsilon^5) = 0, \quad 0 \leq r \leq 1. \tag{5.4.19}$$

For the standing wave solutions considered in Chapter 4, we considered the initial condition $u(r,0) = A\sin(\pi r)$. In this Chapter the initial condition (5.4.17) is used, where the argument of the sine function is $4\pi r$, in order to study the complete period of the displacement and stress waves.

Consider the zero order equations (5.4.9) subject to the boundary conditions (5.4.16) and initial conditions (5.4.18) and (5.4.19). We look for a non-trivial solution of the form

$$u_0(r,t) = R_0(r)T_0(t). \tag{5.4.20}$$

Substituting (5.4.20) into (5.4.9) gives the eigenvalue problems

$$\frac{d^2R_0}{dr^2} + \frac{1}{r}\frac{dR_0}{dr} + \lambda_0 R_0 = 0, \tag{5.4.21}$$

$$\frac{d^2T_0}{dt^2} + \lambda_0 T_0 = 0, \tag{5.4.22}$$

where $-\lambda_0$ is the separation constant and λ_0 is chosen to be positive to admit wave solutions. Let

$$s = \sqrt{\lambda_0 r}. \tag{5.4.23}$$

Then (5.4.21) becomes

$$\frac{d^2R_0}{ds^2} + \frac{1}{s}\frac{dR_0}{ds} + R_0 = 0 \tag{5.4.24}$$

which is Bessel's equation of order 0. Thus the general solution of equation (5.4.21) is

$$R_0(r) = a_0 J_0(\sqrt{\lambda_0} r) + b_0 Y_0(\sqrt{\lambda_0} r) \tag{5.4.25}$$

where J_0 is the Bessel function of the first kind of order zero and Y_0 is the Bessel function of the second kind of order zero and a_0 and b_0 are constants to be determined. For the solution (5.4.25), $\lim_{r\to 0^+} R(r)$ is finite if and only if $b_0 = 0$. Therefore the general solution (5.4.25) becomes

$$R_0(r) = a_0 J_0(\sqrt{\lambda_0} r). \tag{5.4.26}$$

Imposing the boundary condition (5.4.16) we find that

$$\sqrt{\lambda_0} = z_m, \quad m = 1, 2, \cdots \tag{5.4.27}$$

where z_m is the m^{th} zero of the Bessel function J_0. Thus,

$$R_0(r) = J_0(z_m r), \quad m = 1, 2, \cdots \tag{5.4.28}$$

The general solution for (5.4.22) is

$$T_0(t) = c_0 \cos(\lambda_0 t) + d_0 \sin(\lambda_0 t) \tag{5.4.29}$$

and using (5.4.27) we find that

$$u_0(r,t) = \sum_{m=1}^{\infty} J_0(z_m r)[c_{0m}\cos(z_m t) + d_{0m}\sin(z_m t)]. \tag{5.4.30}$$

From (5.4.18), $u_0(r,t)$ satisfies the initial condition

$$u_0(r,0) = A_0 \sin(4\pi r). \tag{5.4.31}$$

Thus putting $t = 0$ in (5.4.30) we obtain

$$A_0 \sin(4\pi r) = \sum_{m=1}^{\infty} c_{0m} J_0(z_m r). \tag{5.4.32}$$

Multiplying both sides by $rJ_0(z_n r)$ for $n \geq 1$ and integrating between the limits 0 and 1 gives

$$A_0 \int_0^1 r\sin(4\pi r)J_0(z_n r)dr = \sum_{m=1}^{\infty} c_{0m} \int_0^1 rJ_0(z_m r)J_0(z_n r)dr. \tag{5.4.33}$$

Using the identity [57]

$$\int_0^1 rJ_0(z_n r)J_0(z_m r)dr = \frac{1}{2}[J_1(z_m)]^2\delta_{nm}, \tag{5.4.34}$$

where $J_1(z_m)$ is the Bessel function of order 1, we find that

$$c_{0m} = \frac{2A_0}{J_1^2(z_m)} \int_0^1 r\sin(4\pi r)J_0(z_m r)dr. \tag{5.4.35}$$

Also, from the derivative initial condition (5.4.19),

$$\frac{\partial u_0(r,0)}{\partial t} = 0, \tag{5.4.36}$$

But from (5.4.30)

$$\frac{\partial u_0(r,t)}{\partial t}(r,t) = \sum_{m=1}^{\infty} z_m J_0(z_m r)[-c_{0m}\sin(z_m t) + d_{0m}\cos(z_m t)] \tag{5.4.37}$$

and therefore the initial condition (5.4.36) gives

$$\sum_{m=1}^{\infty} d_{0m} z_m J_0(z_m r) = 0. \tag{5.4.38}$$

Multiplying both sides of (5.4.38) by $rJ_0(z_n r)$ for $n \geq 1$ and integrating from 0 to 1 we obtain

$$\sum_{m=1}^{\infty} d_{0m} z_m \int_0^1 r J_0(z_n r) J_0(z_m r) dr = 0 \tag{5.4.39}$$

and again using the identity (5.4.34) we find that

$$z_n J_1^2(z_n) d_{0n} = 0. \tag{5.4.40}$$

Since $z_n \neq 0$ and $J_1(z_n) \neq 0$ it follows that

$$d_{0n} = 0, \quad n \geq 1. \tag{5.4.41}$$

Thus, the displacement (5.4.30) at the zero order becomes

$$u_0(r,t) = \sum_{m=1}^{\infty} c_{0m} \cos(z_m t) J_0(z_m r). \tag{5.4.42}$$

Using (5.4.8) for the stress σ_0 at the zero order, we obtain

$$\sigma_0(r,t) = \frac{\partial u_0}{\partial r} = -\sum_{m=1}^{\infty} c_{0m} \cos(z_m t) z_m J_1(z_m r), \tag{5.4.43}$$

since

$$J_0'(z_m r) = -J_1(z_m r). \tag{5.4.44}$$

The first order cylindrical wave equation (5.4.12) has the same form as the zero order wave equation (5.4.9) and the boundary condition and initial conditions with A_0 replaced by A_1 are the same. The first order solution is therefore

$$u_1(r,t) = \sum_{m=1}^{\infty} c_{1m} \cos(z_m t) J_0(z_m r). \tag{5.4.45}$$

$$\sigma_1(r,t) = \frac{\partial u_0}{\partial r} = -\sum_{m=1}^{\infty} c_{1m} \cos(z_m t) z_m J_1(z_m r) \tag{5.4.46}$$

where c_{1m} is given by (5.4.35) with A_0 replaced by A_1.

We cannot find solutions for the displacement and stress at the second order because the equations at this order could not be solved analytically but would need to be solved numerically. Using (5.4.4) up to order ϵ^3 and (5.4.5) up to order ϵ^2 we obtain for the displacement and the stress

$$u(r,t) = \epsilon^2 u_0(r,t) + \epsilon^3 u_1(r,t) = \epsilon^2 \sum_{m=1}^{\infty} (c_{0m} + \epsilon c_{1m}) \cos(z_m t) J_0(z_m r), \tag{5.4.47}$$

$$\sigma(r,t) = \epsilon\sigma_0(r,t) + \epsilon^2\sigma_1(r,t) = -\epsilon\sum_{m=1}^{\infty}(c_{0m} + \epsilon c_{1m})\cos(z_m t) z_m J_1(z_m r), \tag{5.4.48}$$

where

$$c_{0m} + \epsilon c_{1m} = \frac{2(A_0 + \epsilon A_1)}{J_1^2(z_m)} \int_0^1 r\sin(4\pi r) J_0(z_m r)dr. \tag{5.4.49}$$

The value of the constant (5.4.49) is evaluated numerically for each value of m. It depends only on the initial condition (5.4.18).

In Chapter 4 we found travelling wave solutions and standing wave solutions to the perturbation expansions for the power-law constitutive equation and the strain-limiting constitutive equation. For the power-law constitutive equation for the standing wave solution it was found that for small values of n, the effect of the nonlinearity was almost negligible when compared to the linear wave for $n = 0$. Thus, we used a large value of n to clearly illustrate the nonlinear effects. The value $n = 10$ however, falls outside the range of physically valid values for n. We can therefore conclude that for physically valid values of n, the lowest order solutions are sufficient to describe the problem.

The results are discussed in Section 5.5.2.

5.5 Discussion of results

5.5.1 Solitary wave solution for stress

The implicit asymptotic solution for the stress (5.3.24) for the power-law constitutive equation will now be discussed. Figure 5.5.1 shows the initial profile of the solitary stress wave for $n = 0$ and $n = 0.3$. The values of n are again chosen to compare with the results in [42]. There is a slight distortion in the wave for $n > 0$ when compared to $n = 0$ for the cylindrical wave. The initial condition (5.3.1) was not exactly satisfied for $n > 0$ because of the approximation made in going from (5.3.20) to (5.3.23). The scale for w has been expanded in Figure 5.5.1 to show the approximation more clearly.

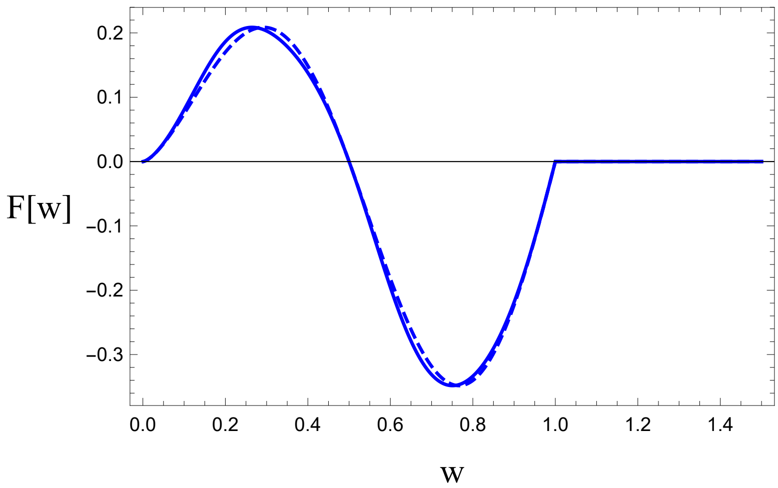

Figure 5.5.1: Initial profile of the solitary stress wave for $n = 0$ (− − −−) and $n = 0.3$ (——) with $A = 0.4$.

Figures 5.5.3 and 5.5.4 show the propagation of the solitary stress wave at time $t = 0.1$ and $t = 1.8$ respectively. It is apparent that as time evolves there is a significant distortion in the wave for $n > 0$ when compared to the linear cylindrical wave for $n = 0$.

Using (5.3.25) we can approximate the wave speed by

$$C(n, \sigma, r) = \frac{1}{\left(1 + \frac{3n}{2}\sigma^2(r, t)\ln r\right)}. \tag{5.5.1}$$

As for the two-dimensional solitary stress wave the wave speed decreases as n and σ increase. Further, we find that for a fixed $n > 0$ the wave steepens at the back. This is similar to the result for the solitary stress wave for the power-law constitutive equation studied in Chapter 4. A shock front forms at the back of the wave as the magnitude of the spatial gradient of the stress approaches infinity. We can write (5.3.25) as

$$\sigma(r, t) = \begin{cases} F(\phi), & t \leq r \leq 1 + t, \\ 0, & 1 + t \leq r \leq \infty, \end{cases} \tag{5.5.2}$$

where

$$\phi(r, t) = 2\pi(r - C(n, \sigma, r)t) \tag{5.5.3}$$

with $C(n, \sigma, r)$ given by (5.3.26) and

$$F(\phi) = \frac{A}{\sqrt{2\pi}}\phi^{\frac{1}{2}} \sin\phi. \tag{5.5.4}$$

Thus

$$\frac{\partial\sigma}{\partial r} = \frac{2\pi\left[1 + \frac{3n}{2}C^2(n,\sigma,r)\frac{F^2(\phi)}{r}t\right]\frac{dF}{d\phi}}{\left[1 - 6\pi n C^2(n,\sigma,r)\ln r F(\phi)\frac{dF}{d\phi}t\right]} \tag{5.5.5}$$

and therefore

$$\left|\frac{\partial\sigma}{\partial r}\right| \to \infty \quad \text{as} \quad t \to t_s = \frac{1}{6\pi n C^2(n,\sigma,r)\ln r F(\phi)\frac{dF}{d\phi}t}. \tag{5.5.6}$$

Now

$$F(\phi)\frac{dF}{d\phi} = \frac{A^2}{2\pi}\left[\cos\phi + \frac{1}{2\phi}\sin\phi\right]\phi\sin\phi. \tag{5.5.7}$$

Figure 5.5.2 shows that $\sigma > 0$, $\frac{\partial\sigma}{\partial r} > 0$ and $t_s(\phi) > 0$ for $0 < \phi < \phi_1$ and that $\sigma > 0$, $\frac{\partial\sigma}{\partial r} < 0$ and $t_s(\phi) > 0$ for $\pi < \phi < \phi_2$ with $\phi_1 = 1.84$ and $\phi_2 = 4.83$. The solitary wave steepens at the back in both cases which agrees with Figure 5.5.3 and the shock front forms at the back of the solitary wave.

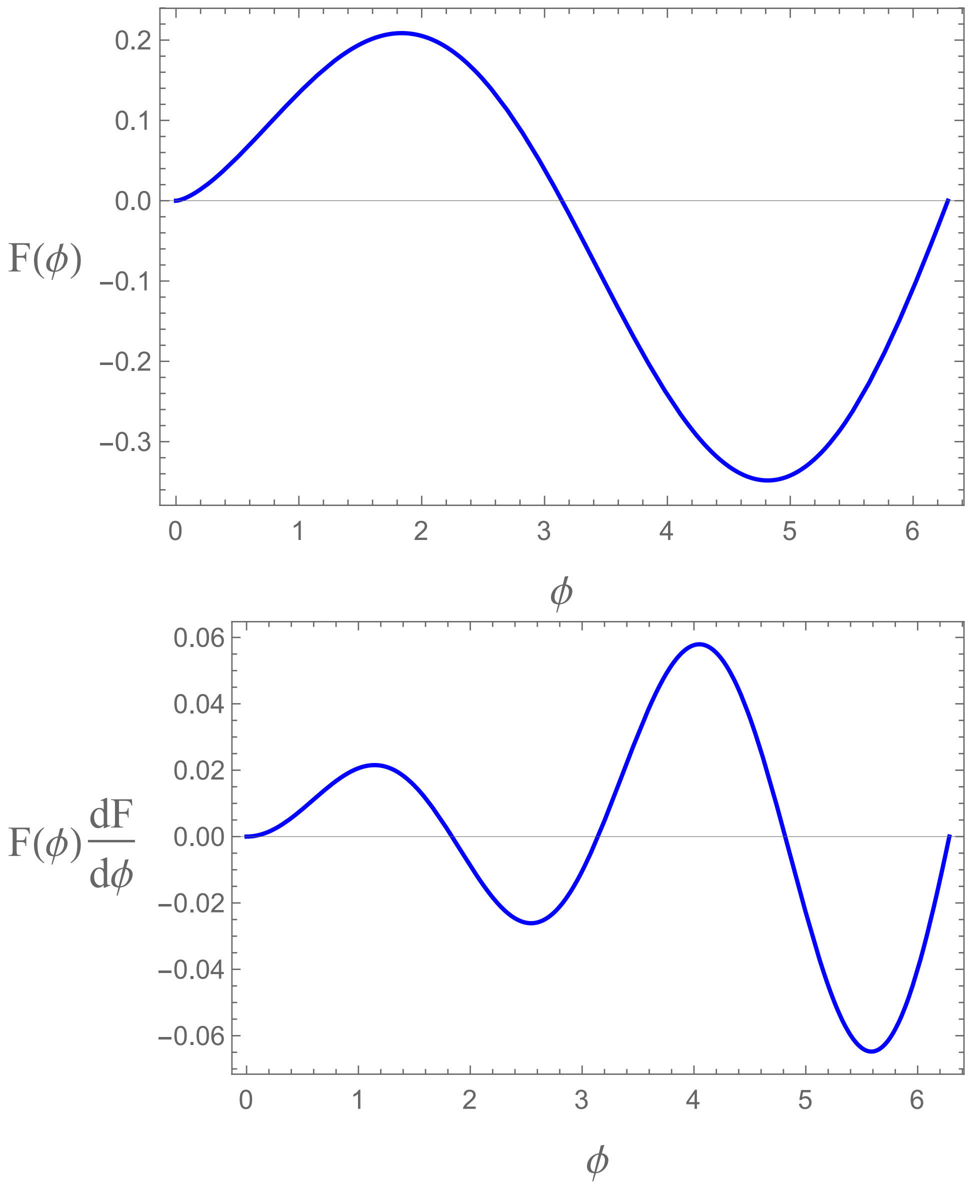

Figure 5.5.2: Graphs of $F(\phi)$ and $F(\phi)\frac{dF}{d\phi}$ given by (5.5.4) and (5.5.7) plotted against ϕ. The time $t_s > 0$ for $0 < \phi < \phi_1$ and $\pi < \phi < \phi_2$ where $\phi_1 = 1.84$ and $\phi_2 = 4.83$.

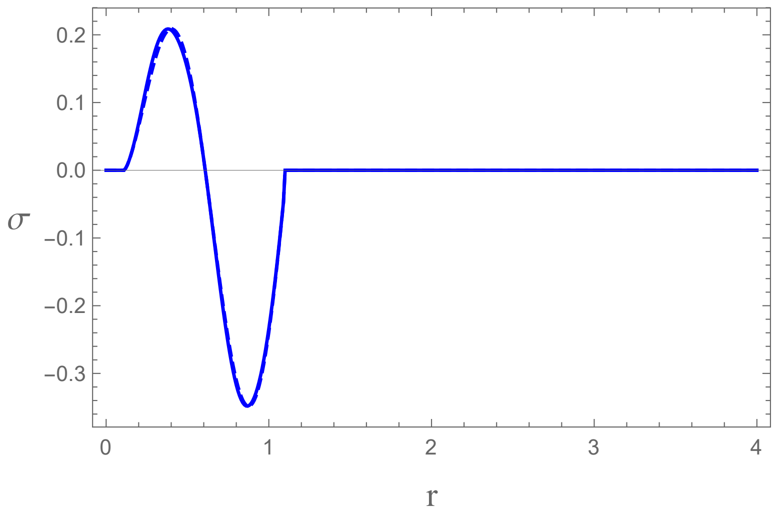

Figure 5.5.3: Propagation of the solitary stress wave for $n = 0$ (− − −−) and $n = 0.3$ (——) at time $t = 0.1$ with $A = 0.4$.

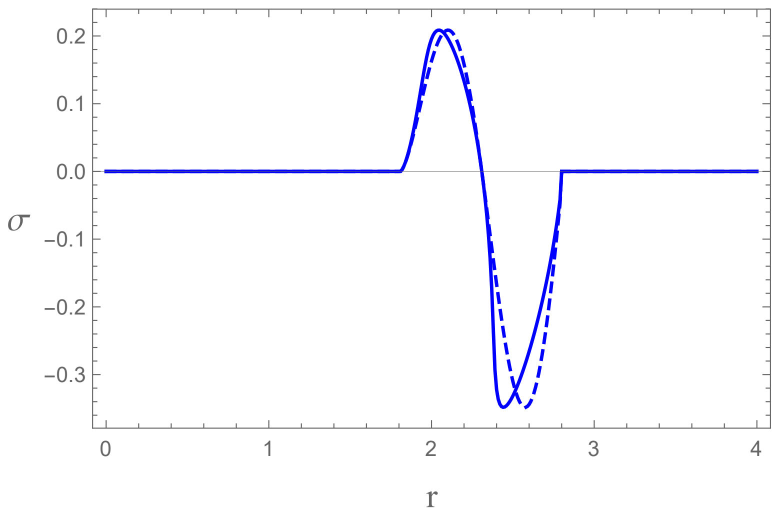

Figure 5.5.4: Propagation of the solitary stress wave for $n = 0$ (− − −−) and $n = 0.2$ (——) at time $t = 1.8$ with $A = 0.4$.

5.5.2 Standing wave solution

We now analyse the solutions (5.4.47) for the displacement field and (5.4.48) for the stress field. These solutions consist of standing waves in the circular cylinder. Solutions are only found for the zero and first order perturbation equations because the perturbation equations at the second order could not be solved exactly. As a result, we cannot study the effects of the nonlinear terms for the power-law constitutive equation. However, for experimental values of

n the zero order solutions are sufficient to describe the physical properties of the wave.

The displacement standing waves for a range of times in the interval $0 \leq t \leq 1$ are plotted in Figure 5.5.5. In the summation from $m = 1$ to $m = \infty$, only the first ten terms are used to calculate the displacement. The series is slowly converging and if sufficiently more terms are used the graph for the initial condition would be satisfied. The amplitude of the standing waves at all times considered decreases to a zero displacement at the boundary in order to satisfy the boundary condition at $r = 1$. The condition at $r = 0$ is that the displacement is finite but not necessarily zero for $t > 0$. Although the displacement does not vanish at $r = 0$, the spatial gradient of the displacement does vanish at $r = 0$ since $J_0'(z) = -J_1(z)$ and $J_1(0) = 0$. Unlike the two-dimensional standing wave solution, the displacement is not fixed at both ends. The displacement at $r = 0$ is more similar to the displacement in the two-dimensional travelling wave solution. The displacement standing wave is an infinite superposition of waves of the form $\cos(z_m t)J_0$ which oscillate with a period of $T_m = \frac{2\pi}{z_m}$ and with a frequency of $w_m = \frac{1}{T}$ where z_m is the m^{th} zero of the Bessel function $J_0(z)$.

Consider next the standing stress wave. The solution (5.4.48) is plotted for a range of times in the interval $0 \leq t \leq 1$ in Figure 5.5.6. The first ten terms in the infinite series in (5.4.48) are used to calculate the stress. The stress standing wave at the boundary $r = 1$ is non-zero because the spatial gradient of the displacement is non-zero at $r = 1$. The stress is zero at $r = 0$ because the spatial gradient of the displacement vanishes at $r = 0$ since $J_1(0) = 0$. This compares with the two-dimensional standing wave for which the stress is non-zero at both ends. The stress standing wave is an infinite superposition of waves of the form $\cos(z_m t)J_1$ which oscillate with the same period and frequency as the displacement waves.

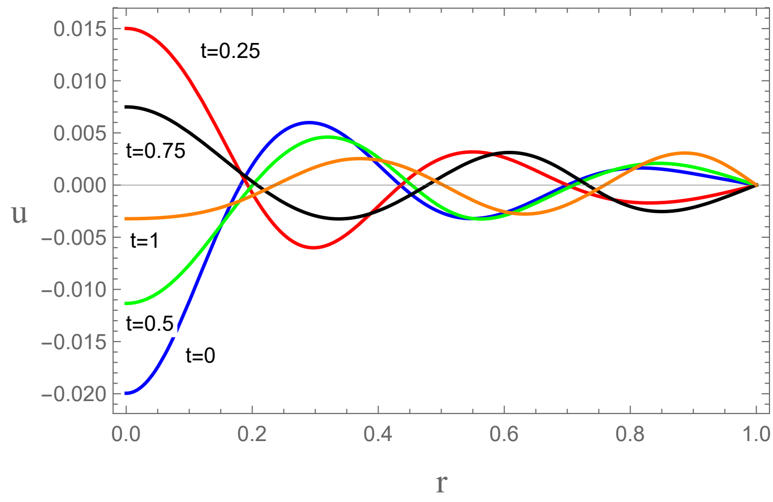

Figure 5.5.5: Propagation of the displacement standing wave at times $t = 0, 0.25, 0.5, 0.75$ and 1 with $A_0 = 0.4$ and $\epsilon = 0.1$.

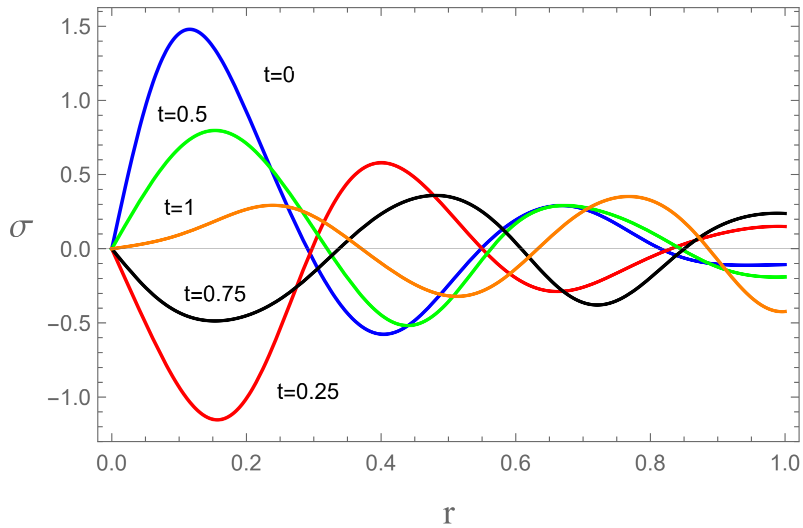

Figure 5.5.6: Propagation of the stress standing wave at times $t = 0, 0.25, 0.5, 0.75$ and 1 with $A_0 = 0.4$ and $\epsilon = 0.1$.

5.6 Mathematical model for the strain-limiting constitutive equation

In this section we investigate the propagation of axial displacement waves and stress waves in a circular cylindrical for the second constitutive equation under study given by equation (3.5.18). The domain of interest is given by (5.2.1) with stress and displacement fields given by (5.2.2) Using the components of the strain tensor in terms of the displacement, the equations of motion and the constitutive equation (3.5.18) yields the system of partial differential equations

$$\frac{\partial \sigma}{\partial r} + \frac{\sigma}{r} = \rho \frac{\partial^2 u}{\partial t^2} \tag{5.6.1}$$

$$\frac{\partial u}{\partial r} = \alpha \left[1 + \frac{1}{(1+\gamma\sigma^2)}\right]^n \sigma. \tag{5.6.2}$$

Introducing the non-dimensional transformations

$$\bar{r} = \frac{r}{R_0}, \quad \bar{t} = \frac{t}{R_0\sqrt{\alpha\rho}}, \quad \bar{u} = \frac{u}{R_0}, \quad \bar{\sigma} = \sqrt{\gamma}\sigma \tag{5.6.3}$$

we obtain the dimensionless system of partial differential equations

$$\frac{\partial \sigma}{\partial r} + \frac{\sigma}{r} = \frac{\sqrt{\gamma}}{\alpha}\frac{\partial^2 u}{\partial t^2}, \tag{5.6.4}$$

$$\frac{\partial u}{\partial r} = \frac{\alpha}{\sqrt{\gamma}} \left[1 + \frac{1}{(1+\sigma^2)}\right]^n \sigma, \tag{5.6.5}$$

where the overhead bars have been suppressed for convenience. The system of partial differential equations, (5.6.4) and (5.6.5), can be written as a single nonlinear hyperbolic partial differential equation

$$\frac{\partial^2 \sigma}{\partial r^2} + \frac{1}{r}\frac{\partial \sigma}{\partial r} - \frac{1}{r^2}\sigma = \frac{\partial^2}{\partial t^2}\left[\left(1 + \frac{1}{(1+\sigma^2)}\right)^n \sigma\right], \tag{5.6.6}$$

which describes the propagation of nonlinear axial stress waves. The solutions of the system of equations (5.6.4) and (5.6.5) and the solution to (5.6.6) subject to the same initial and boundary conditions as used for the power-law constitutive equation will be investigated in the next section. We consider first the nonlinear hyperbolic partial differential equation (5.6.6) and the domain $0 \le r \le R_0$.

5.7 Asymptotic solutions for the strain limiting constitutive equation

5.7.1 Solitary wave solution

We derive an implicit asymptotic solution for the stress field for the equation (5.6.6) subject to the initial condition (5.3.1) and boundary condition (5.3.2).

Let $\sigma = \epsilon\hat{\sigma}$, where $\epsilon \ll 1$. Substituting this expression into (5.6.6) and taking the binomial expansion of the term $\left[1 + \frac{1}{(1+\sigma^2)}\right]^n$ up to the order $\mathcal{O}(\epsilon^2)$ we obtain the asymptotic partial differential equation

$$\frac{\partial^2 \hat{\sigma}}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{\sigma}}{\partial r} - \frac{1}{r^2}\hat{\sigma} = 2^n\frac{\partial^2 \hat{\sigma}}{\partial t^2} - 2^{n-1}n\epsilon^2\frac{\partial^2 \hat{\sigma}^3}{\partial t^2}. \tag{5.7.1}$$

Let

$$\bar{r} = 2^{\frac{n}{2}} r, \quad \bar{\epsilon} = \frac{\epsilon}{\sqrt{2}}. \tag{5.7.2}$$

Substituting (5.7.2) into (5.7.1) gives

$$\frac{\partial^2 \hat{\sigma}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \hat{\sigma}}{\partial \bar{r}} - \frac{1}{\bar{r}^2} \hat{\sigma} = \frac{\partial^2 \hat{\sigma}}{\partial t^2} - n\bar{\epsilon}^2 \frac{\partial^2 \hat{\sigma}^3}{\partial t^2}. \tag{5.7.3}$$

The only difference between (5.7.3) and (5.3.3) is the difference in sign. Introducing the new variables

$$\tau = t - \bar{r}, \quad R = \bar{\epsilon}^2 \bar{r}. \tag{5.7.4}$$

The stress is defined in terms of the new variables as

$$\hat{\sigma}(r, t) = \sigma^\star(\tau(\bar{r}, t), R(\bar{r})). \tag{5.7.5}$$

Substituting (5.7.5) into (5.7.3) yields

$$-2\frac{\partial^2 \sigma^\star}{\partial \tau \partial R} + \bar{\epsilon}^2 \frac{\partial^2 \sigma^\star}{\partial R^2} - \frac{1}{R} \frac{\partial \sigma^\star}{\partial \tau} + \bar{\epsilon}^2 \frac{\partial}{\partial R} \left(\frac{\sigma^\star}{R} \right) = -n \frac{\partial^2 (\sigma^\star)^3}{\partial \tau^2} \tag{5.7.6}$$

We neglect terms $\bar{\epsilon}^2$ and then integrate with respect to τ. The arbitrary integration function of R is set to zero because we are looking for a particular solution and not the general solution. This gives the first order linear partial differential equation

$$-2\frac{\partial \sigma^\star}{\partial R} + 3n(\sigma^\star)^2 \frac{\partial \sigma^\star}{\partial \tau} = \frac{1}{R} \sigma^\star. \tag{5.7.7}$$

Equation (5.7.7) compares with (5.3.7) for the power-law constitutive equation. The differential equations of the characteristic curves of (5.7.7) are

$$\frac{dR}{-2} = \frac{d\tau}{3n(\sigma^\star)^2} = R\frac{d\sigma^\star}{\sigma^\star}. \tag{5.7.8}$$

We integrate the first and last term of (5.7.8) and obtain the first integral

$$\sqrt{R}\sigma^\star = k_1 \tag{5.7.9}$$

where k_1 is a constant. Next, replacing $\sigma^\star$ in the first pair of terms and integrating yields

$$\frac{3n}{2}(\sigma^\star)^2 R \ln R + \tau = k_2, \tag{5.7.10}$$

where k_2 is a constant.

Transform back from variables (τ, R) to $(t, \bar{r})$ using (5.7.4). Then

$$\bar{\epsilon}\sqrt{\bar{r}}\hat{\sigma} = k_1, \quad \frac{3n}{2}\bar{\epsilon}^2\hat{\sigma}^2\bar{r}\ln\bar{r} + t - \bar{r} = k_2^\star \tag{5.7.11}$$

where

$$k_2^\star = k_2 - \frac{3n}{2}k_1^2\ln\bar{\epsilon}^2. \tag{5.7.12}$$

The general solution of the partial differential equation (5.7.7) is

$$k_1 = F(k_2^\star) \tag{5.7.13}$$

and therefore

$$\hat{\sigma} = \frac{1}{\bar{\epsilon}\sqrt{\bar{r}}}F\left[\frac{3n}{2}\bar{\epsilon}^2\hat{\sigma}^2\bar{r}\ln\bar{r} + t - \bar{r}\right]. \tag{5.7.14}$$

Now, using (5.7.2) and $\sigma = \epsilon\hat{\sigma}$, equation (5.7.14) becomes

$$\sigma(r,t) = \frac{2^{\frac{2-n}{4}}}{\sqrt{r}}F\left[-2^{\frac{n}{2}}\left(1 - \frac{3n}{4}\sigma^2(\ln 2^{\frac{n}{2}} + \ln r)\right)r - t\right]. \tag{5.7.15}$$

The function F can be determined using the initial condition (5.3.1). Thus, using (5.3.1) we have

$$F\left[2^{\frac{n}{2}}\left(1 - \frac{3n}{4}A^2\sin^2(2\pi r)\ln(2^{\frac{n}{2}}r)\right)r\right] = 2^{\frac{n-2}{4}}\sqrt{r}A\sin(2\pi r), \quad 0 \le r \le 1, \tag{5.7.16}$$

$$F[2^{\frac{n}{2}}r] = 0, \quad 1 \le r \le \infty. \tag{5.7.17}$$

Let

$$w = \begin{cases} 2^{\frac{n}{2}}\left(1 - \dfrac{3n}{4}A^2\sin^2(2\pi r)\ln(2^{\frac{n}{2}}r)\right)r, & 0 \le r \le 1 \\ 2^{\frac{n}{2}}r, & 1 \le r \le \infty. \end{cases} \tag{5.7.18}$$

Equation (5.7.18) has to be solved for r in terms of w. We proceed as for the power-law constitutive equation. Since $\sin^2(2\pi r)\ln(2^{\frac{n}{2}}r)$ is small a first approximation is $r = 2^{-\frac{n}{2}}w$ for $0 \le r \le \infty$. Iterating using the result $r = 2^{-\frac{n}{2}}w$ the next approximation for $0 \le r \le 1$ is

$$r = \frac{2^{-\frac{n}{2}}w}{1 - \frac{3n}{4}A^2\sin^2(2^{1-\frac{n}{2}}\pi w)\ln w)} \tag{5.7.19}$$

We will use this result. No approximation is required in the limits of the range $0 \le r \le 1$ because the stress $\sigma(r, 0)$ vanishes at the front and back of the solitary wave. Thus equations (5.7.16) and (5.7.17) become

$$F(w) = \begin{cases} 2^{\frac{n-2}{4}} \left[\dfrac{2^{-\frac{n}{2}} w}{\left(1 - \frac{3n}{4} A^2 \sin^2(2^{1-\frac{n}{2}} \pi w) \ln w\right)} \right]^{\frac{1}{2}} & \\ A \sin \left[2\pi \left(\dfrac{2^{-\frac{n}{2}} w}{\left(1 - \frac{3n}{4} A^2 \sin^2(2^{1-\frac{n}{2}} \pi w) \ln w\right)} \right) \right], & 0 \le w \le 1 \\ 0, & 1 \le w \le \infty. \end{cases} \tag{5.7.20}$$

To find the solution for the stress $\sigma(r,t)$ given by (5.7.15), we set in (5.7.20)

$$w = 2^{\frac{n}{2}} \left(1 - \frac{3n}{4} \sigma^2(r,t) \ln(2^n r) \right) r - t \tag{5.7.21}$$

and to obtain the end points of the range $0 \le w \le 1$ we set $w = 2^{\frac{n}{2}} r - t$ because the stress $\sigma(r,t) = 0$ at the wave front and end of the solitary wave. Thus

$$\sigma(r,t) \tag{5.7.22}$$
$$= \begin{cases} 2^{\frac{n-2}{4}} \left[\dfrac{2^{-\frac{n}{2}} (2^{\frac{n}{2}} (1 - \frac{3n}{4} \sigma^2(r,t) \ln(2^n r)) r - t)}{[1 - \frac{3n}{4} A^2 \sin^2 \left(2^{1-\frac{n}{2}} \pi (2^{\frac{n}{2}} (1 - \frac{3n}{4} \sigma^2(r,t) \ln(2^n r)\right) r - t)) \ln \left(2^{\frac{n}{2}} \left(1 - \frac{3n}{2} \sigma^2(r,t) \ln(2^n r)\right) r - t \right)} \right]^{\frac{1}{2}} & \\ \times A \sin \left[2\pi \left[\dfrac{2^{-\frac{n}{2}} (2^{\frac{n}{2}} (1 - \frac{3n}{4} \sigma^2(r,t) \ln(2^n r)) r - t)}{[1 - \frac{3n}{4} A^2 \sin^2 \left(2^{1-\frac{n}{2}} \pi (2^{\frac{n}{2}} (1 - \frac{3n}{4} \sigma^2(r,t) \ln(2^n r)\right) r - t)) \ln \left(2^{\frac{n}{2}} \left(1 - \frac{3n}{2} \sigma^2(r,t) \ln(2^n r)\right) r - t \right)} \right] \right], & \\ & 2^{-\frac{n}{2}} t \le r \le 2^{-\frac{n}{2}} (1+t) \\ 0, & 2^{-\frac{n}{2}} (1+t) \le r \le \infty \end{cases}$$

To interpret the solution physically, the stress (5.7.22) can be written approximately as

$$\sigma(r,t) = \begin{cases} 2^{\frac{n-2}{4}} A (r - Ct)^{\frac{1}{2}} \sin[2\pi (r - Ct)], & 2^{-\frac{n}{2}} t \le r \le 2^{-\frac{n}{2}} (1+t) \\ 0, & 2^{-\frac{n}{2}} (1+t) \le r \le \infty \end{cases} \tag{5.7.23}$$

where

$$C(n,\sigma,r) = \frac{1}{2^{\frac{n}{2}} \left(1 - \frac{3n}{4} \sigma^2(r,t) \ln(2^{\frac{n}{2}} r)\right)}. \tag{5.7.24}$$

In order to arrive at the approximation (5.7.23) we follow the same procedure as for (4.3.5) where in the numerator of we take out the term $2^{\frac{n}{2}} \left(1 - \frac{3n}{4} \sigma^2(r,t) \ln(2^{\frac{n}{2}} r)\right)$ and use the fact that $\sigma^2(r,t)$ is zero at the wave front. We do not approximate the stress inside the bracket so that we can get some insight into the physical properties of the wave. In the denominator we consider the equation (5.7.20) and find that in the range $0 \le w \le 1$ the second term in denominator tends to zero since $w \ln w$ tends to zero for small w and also tends to zero for $w = 1$. It must be stressed that this expression is just an approximation to gain physical insight into the problem for a very complicated expression. Solutions (5.7.22) and (5.7.23) are exact solutions without approximation when $n = 0$. The results will be analysed and discussed in Section 5.9.1.

5.8 Perturbation solutions for the first order system for displacement and stress

Since we require the displacement gradient to be small we set

$$\frac{\alpha}{\sqrt{\gamma}} = \epsilon. \tag{5.8.1}$$

The system of equations (5.6.4), (5.6.5) becomes

$$\frac{\partial \sigma}{\partial r} + \frac{\sigma}{r} = \frac{1}{\epsilon}\frac{\partial^2 u}{\partial t^2}, \tag{5.8.2}$$

$$\frac{\partial u}{\partial r} = \epsilon\left[1 + \frac{1}{(1+\sigma^2)}\right]^n \sigma. \tag{5.8.3}$$

We consider straightforward perturbation expansions of the form

$$\sigma(r,t) = \epsilon\sigma_0 + \epsilon^2\sigma_1 + \mathcal{O}(\epsilon^3), \tag{5.8.4}$$

$$u(r,t) = \epsilon^2 u_0 + \epsilon^3 u_1 + \mathcal{O}(\epsilon^3). \tag{5.8.5}$$

As with the power-law constitutive equation, the equations at the second order could not be solved exactly. Despite this drawback, we can study the effects of the nonlinearity at the zero order and the first order for this constitutive equation. Substituting (5.8.4) and (5.8.5) into (5.8.2) and (5.8.3) and approximating the term $\left[1 + \frac{1}{(1+\sigma^2)}\right]^n$ using the binomial theorem up to order $\mathcal{O}(\epsilon^3)$ we obtain:

$$\epsilon\frac{\partial \sigma_0}{\partial r} + \epsilon^2\frac{\partial \sigma_1}{\partial r} = \epsilon\frac{\partial^2 u_0}{\partial t^2} + \epsilon^2\frac{\partial^2 u_1}{\partial t^2}, \tag{5.8.6}$$

$$\epsilon^2\frac{\partial u_0}{\partial r} + \epsilon^3\frac{\partial u_1}{\partial r} = \epsilon^2 2^n\sigma_0 + \epsilon^3 2^n\sigma_1. \tag{5.8.7}$$

By equating terms of the same order in ϵ in each equation the following system of equations are obtained.

Zero order terms in u_0 and σ_0:

$$\epsilon \quad : \quad \frac{\partial \sigma_0}{\partial r} + \frac{1}{r}\sigma_0 = \frac{\partial^2 u_0}{\partial t^2}, \tag{5.8.8}$$

$$\epsilon^2 \quad : \quad \frac{\partial u_0}{\partial r} = 2^n\sigma_0. \tag{5.8.9}$$

Eliminating σ_0 from (5.8.8) and (5.8.9) we obtain the cylindrical wave equation

$$\frac{\partial^2 u_0}{\partial r^2} - 2^n\frac{\partial^2 u_0}{\partial t^2} + \frac{1}{r}\frac{\partial u_0}{\partial r} = 0. \tag{5.8.10}$$

First order terms in u_0 and σ_0:

$$\epsilon^2 \quad : \quad \frac{\partial \sigma_1}{\partial r} + \frac{1}{r}\sigma_1 = \frac{\partial^2 u_1}{\partial t^2}, \tag{5.8.11}$$

$$\epsilon^3 \quad : \quad \frac{\partial u_1}{\partial r} = 2^n \sigma_1. \tag{5.8.12}$$

Eliminating σ_0 from (5.8.11) and (5.8.12) gives the cylindrical wave equation

$$\frac{\partial^2 u_1}{\partial r^2} - 2^n \frac{\partial^2 u_1}{\partial t^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} = 0. \tag{5.8.13}$$

The perturbation equations at the first order (5.8.11) to (5.8.13) are of the same form as the perturbation equations at the zero order, (5.8.8) to (5.8.10) with u_0 replaced with u_1 and σ_0 replaced with σ_1.

5.8.1 Standing wave solution

We look for solutions of the perturbation equation (5.8.10) subject to the boundary condition (5.4.16) and initial conditions (5.4.18) and (5.4.19). If we make the transformation

$$\bar{t} = 2^{-\frac{n}{2}} t, \tag{5.8.14}$$

equation (5.8.10) reduces to

$$\frac{\partial^2 u_0}{\partial r^2} - \frac{\partial^2 u_0}{\partial \bar{t}^2} + \frac{1}{r}\frac{\partial u_0}{\partial r} = 0, \tag{5.8.15}$$

which is the linear cylindrical wave equation. The only difference between (5.4.9) and (5.8.15) is that u_0 is a function of r and t in equation (5.4.9) while it is a function of r and $\bar{t}$ in equation (5.8.15). The boundary condition (5.4.16) and the initial conditions (5.4.18) and (5.4.19) remain unchanged with t replaced by $\bar{t}$. Therefore, using (5.4.42) for u_0 we obtain for the displacement at the zero order

$$u_0(r, \bar{t}) = \sum_{m=1}^{\infty} c_{0m} \cos(z_m \bar{t}) J_0(z_m r) \tag{5.8.16}$$

and using (5.8.9) and (5.8.16) the stress at the zero order is given by

$$\sigma_0(r, \bar{t}) = 2^{-n}\frac{\partial u_0}{\partial r} = -2^{-n} \sum_{m=1}^{\infty} c_{0m} \cos(z_m \bar{t}) z_m J_1(z_m r), \tag{5.8.17}$$

where $\bar{t}$ is given by (5.8.14) and c_{0m} by (5.4.35). The first order solution is

$$u_1(r,\bar{t}) = \sum_{m=1}^{\infty} c_{1m} \cos(z_m\bar{t}) J_0(z_m r), \tag{5.8.18}$$

$$\sigma_1(r,\bar{t}) = -2^{-n} \sum_{m=1}^{\infty} c_{1m} \cos(z_m\bar{t}) z_m J_1(z_m r), \tag{5.8.19}$$

where c_{1m} is given by (5.4.35) with A_0 replaced by A_1.

Thus, using (5.8.5) and (5.8.4) we obtain the perturbation solutions for the displacement and stress correct to order ϵ^3 and ϵ^2 respectively,

$$u(r,\bar{t}) = \epsilon^2 \sum_{m=1}^{\infty} (c_{0m} + \epsilon c_{1m}) \cos(z_m\bar{t}) J_0(z_m r), \tag{5.8.20}$$

$$\sigma(r,\bar{t}) = -2^{-n}\epsilon \sum_{m=1}^{\infty} (c_{0m} + \epsilon c_{1m}) \cos(z_m\bar{t}) z_m J_1(z_m r), \tag{5.8.21}$$

where $\bar{t}$ is given by (5.8.14) and c_{0m} by (5.4.35) and c_{1m} by (5.4.35) with A_0 replaced by A_1.

The results are discussed in Section 5.9.2.

5.9 Discussion of results

5.9.1 Solitary wave solution

In this section we discuss the implicit solution for the stress (5.7.22) for the strain-limiting constitutive equation. Figure 5.9.1 shows the initial profile of the solitary stress wave for $n = 0$ and $n = 0.3$. There is a significant distortion in the wave for $n > 0$ when compared to $n = 0$. The initial condition (5.3.1) was not exactly satisfied for $n > 0$ because of the approximation made in going from (5.7.18) to (5.7.21). We expanded the scale of w in Figure 5.9.1 to show the approximation more clearly.

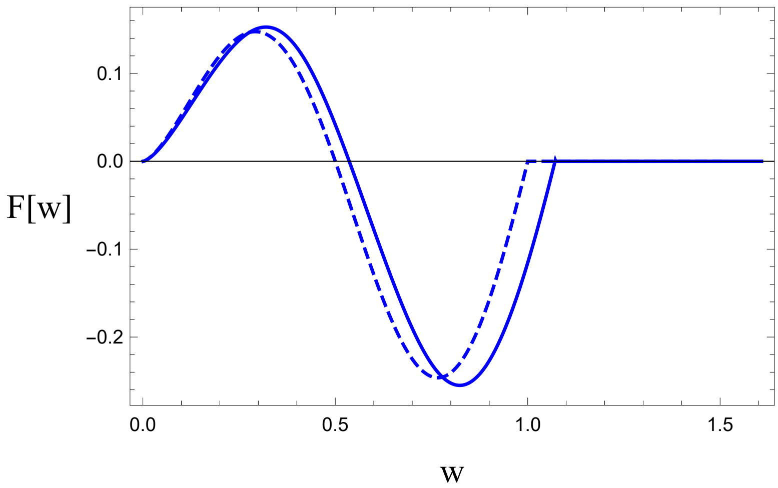

Figure 5.9.1: Initial profile of the solitary stress wave for $n = 0$ (− − −−) and $n = 0.2$ (——) with $A = 0.4$.

In Figures 5.9.2 and 5.9.3 the solitary stress wave is plotted at times $t = 0.1$ and $t = 0.5$ respectively for $n = 0$ and $n = 0.2$. For larger times there is a distortion in the wave for $n > 0$ when compared to the wave for $n = 0$. The amplitude of the wave is also affected by the nonlinearity. For positive values of the stress the amplitude increases as n increases and similarly for negative values of the stress. We can approximate the wave speed using (5.7.23) by

$$C(n, \sigma, r) = \frac{1}{2^{\frac{n}{2}}\left(1 - \frac{3n}{4}\sigma^2 \ln(2^{\frac{n}{2}} r)\right)}. \tag{5.9.1}$$

Further, we see that for a fixed n the wave steepens at the front. As the magnitude of the spatial gradient of the stress tends to infinity a shock front will form at the front of the wave. This is similar to the two-dimensional solitary stress wave for the strain-limiting constitutive equation studied in Chapter 4. We estimate the time that a shock front forms by differentiating the approximate solution (5.7.23) with respect to r. Equation (5.7.23) can be written as

$$\sigma(r, t) = \begin{cases} 2^{\frac{n-2}{4}} F(\phi), & 2^{-\frac{n}{2}} t \leq r \leq 2^{-\frac{n}{2}}(1 + t), \\ 0, & 2^{-\frac{n}{2}}(1 + t) \leq r \leq \infty \end{cases} \tag{5.9.2}$$

where

$$\phi = 2\pi(r - C(n, \sigma, r)t) \tag{5.9.3}$$

with $C(n, \sigma, r)$ given by (5.9.1) and

$$F(\phi) = \frac{A}{\sqrt{2\pi}} \phi^{\frac{1}{2}} \sin \phi. \tag{5.9.4}$$

The function $F(\phi)$ has the same form as $F(\phi)$ defined by (5.5.4) for the power-law constitutive equation. Thus

$$\frac{\partial\sigma}{\partial r} = \frac{2^{\frac{n}{4}+\frac{1}{2}}\pi[1 - 2^{\frac{n}{2}-2}3nC^2(n\sigma, r)\frac{F^2(\phi)}{r}t]\frac{dF}{d\phi}}{\left[1 + 2^{\frac{3n}{4}-\frac{1}{2}}3n\pi C^2(n, \sigma, r)\ln(2^{\frac{n}{2}})F(\phi)\frac{dF}{d\phi}t\right]} \tag{5.9.5}$$

and therefore

$$\left|\frac{\partial\sigma}{\partial r}\right| \to \infty \quad \text{as} \quad t \to t_s \tag{5.9.6}$$

where

$$t_s = -\frac{1}{2^{\frac{3n}{4}-\frac{1}{2}}3n\pi C^2(n, \sigma, r)\ln(2^{\frac{n}{2}})F(\phi)\frac{dF}{d\phi}}. \tag{5.9.7}$$

Unlike $\frac{\partial\sigma}{\partial r}$ for a power-law constitutive equation given by (5.5.5), $\frac{\partial\sigma}{\partial r}$ given by (5.9.5) can become zero in a finite time. We have

$$\frac{\partial\sigma}{\partial r} = 0 \quad \text{at} \quad t = t_0 = \frac{r}{2^{\frac{n}{2}-2}3nC^2(n, \sigma, r)F^2(\phi)}. \tag{5.9.8}$$

However, Figures 5.9.2 and 5.9.3 indicate that $t_s > t_0$. Since $F(\phi)$ has the same form for the power-law and strain-limiting constitutive equations, the graphs in Figure 5.5.4 again apply. For the strain-limiting constitutive equation however, Figure 5.5.4 shows that $\sigma > 0$, $\frac{\partial\sigma}{\partial r} < 0$ and $t_s > 0$ for $\phi_1 < \phi < \pi$ and that $\sigma < 0$, $\frac{\partial\sigma}{\partial r} > 0$ and $t_s > 0$ for $\phi_2 < \phi < 2\pi$ where $\phi_1 = 1.84$ and $\phi_2 = 4.83$. This approximate calculation supports the conclusion that the solitary stress wave steepens at the front in both cases which is in agreement with Figure 5.9.3 and that the shock wave forms at the front of the solitary stress wave for a strain-limiting constitutive equation.

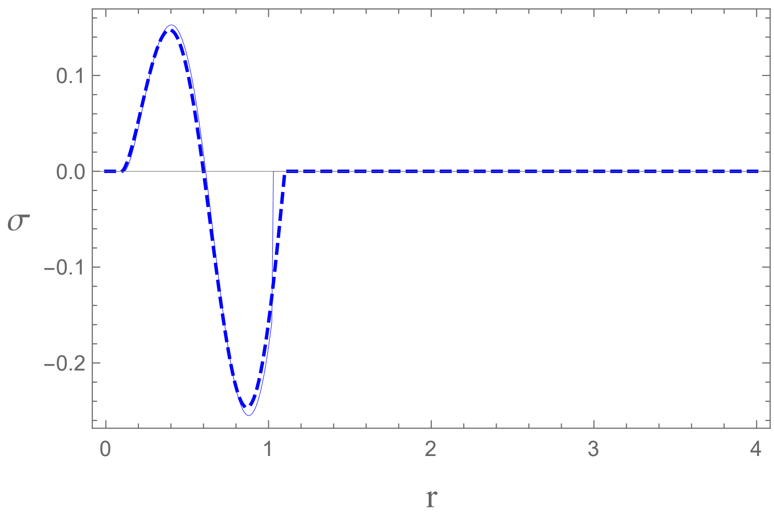

Figure 5.9.2: Propagation of the solitary stress wave for $n = 0$ (− − −−) and $n = 0.2$ (——) at time $t = 0.1$ with $A = 0.4$.

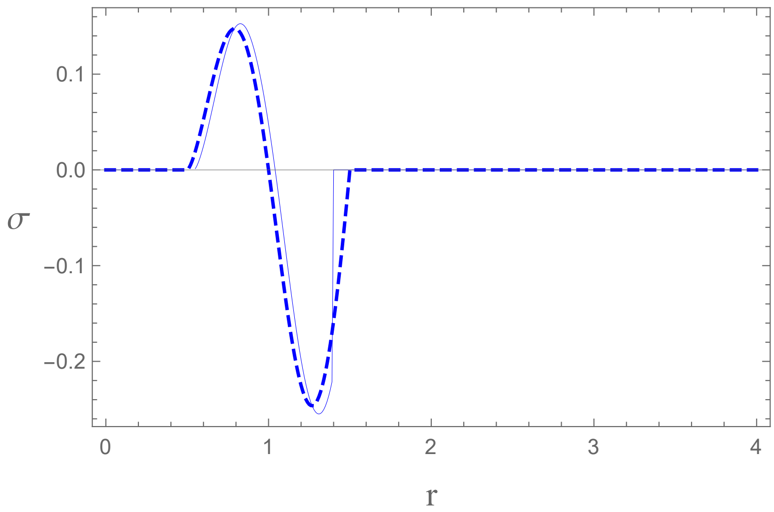

Figure 5.9.3: Propagation of the solitary stress wave for $n = 0$ (− − −−) and $n = 0.2$ (——) at time $t = 0.5$ with $A = 0.4$.

5.9.2 Standing wave solution

The solutions for the displacement field (5.8.20) and for the stress field (5.8.21) will now be discussed. We only looked for solutions of the perturbation equations at the zero and first orders because the perturbation equations at the second order cannot be solved exactly. The equations and the boundary and initial conditions at the first order are the same as at zero order. For the strain-limiting constitutive equation the effects of the nonlinear terms can be studied at the zero and first order.

Consider the displacement wave. The standing displacement waves for $n = 0$ and 0.3 are plotted for $0 \leq t \leq 1$ in Figure 5.9.4. The displacement wave is only plotted for the $m = 1$ to $m = 10$ and this is the reason why the curve for $t = 0$ does not pass through the origin. We observe a significant distortion in the standing wave for $n > 0$. At time $t = 0$ there is no distortion in the standing wave for $n = 0$ and $n > 0$. The amplitude of the displacement wave is zero at $r = 1$ for all $t \geq 0$ in order to satisfy the boundary condition at $r = 1$. The condition for the displacement to be finite at $r = 0$ gave a non-zero displacement at $r = 0$ but a zero displacement gradient at $r = 0$ since $J_0'(0) = -J_1(0)$. The axial displacement standing wave is an infinite superposition of waves of the form $\cos(z_m 2^{-\frac{n}{2}} t) J_0(z_m r)$ which oscillate with a period of $T_m = \frac{2\pi}{z_m}$ for $n = 0$ which increases to $T_m = \frac{2^{1+\frac{n}{2}}\pi}{z_m}$ for $n > 0$.

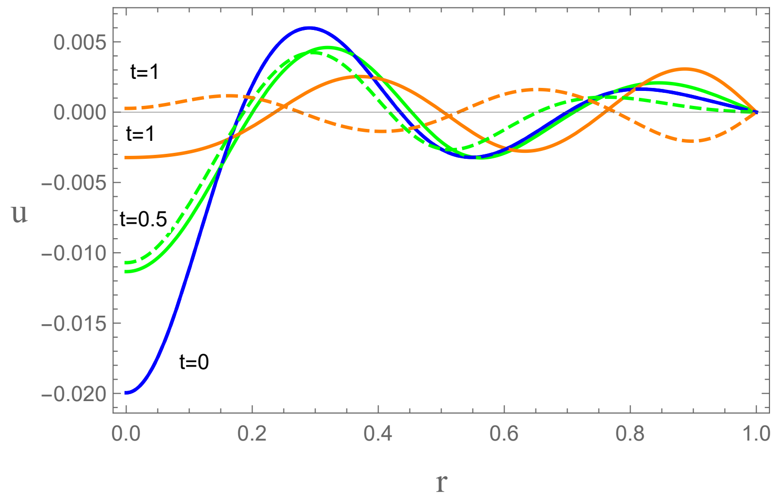

Figure 5.9.4: Propagation of the displacement wave for $n = 0$ (− − −−) and $n = 0.3$ (——) at times $t = 0, 0.5$ and 1 with $A_0 = 0.4$ and $\epsilon = 0.1$.

Consider next the standing stress waves plotted in Figure 5.9.5. The stress wave is only plotted for $m = 1$ to $m = 10$. The stress is zero at $r = 0$ because the spatial gradient of the displacement is zero at $r = 0$. The stress is non-zero at $r = 1$ because from Figure 5.9.4 the spatial gradient of the displacement is non-zero at $r = 1$. However, we see that the spatial gradient of the stress is zero at $r = 1$. The stress standing wave is an infinite superposition of waves of the form $\cos(z_m 2^{-\frac{n}{2}} t) J_1(z_m r)$ which oscillate with the same period and frequency as the displacement waves.

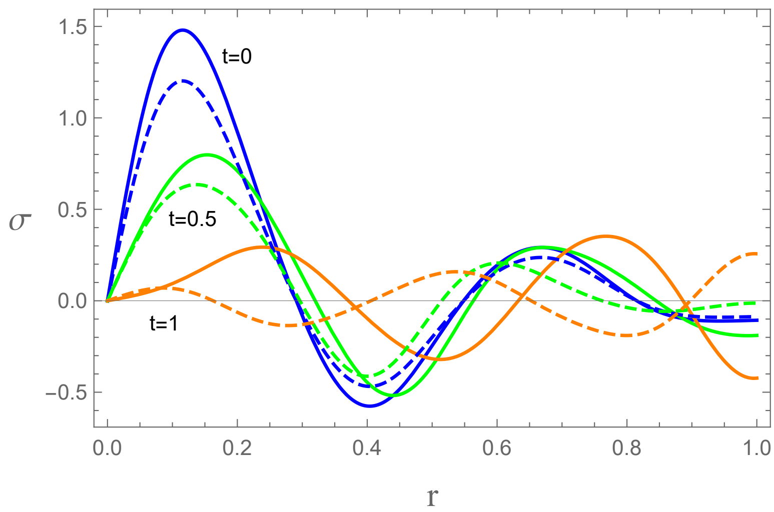

Figure 5.9.5: Propagation of the stress wave for $n = 0$ (− − −−) and $n = 0.3$ (——) at times $t = 0, 0.5$ and 1 with $A_0 = 0.4$ and $\epsilon = 0.1$.

For the displacement standing wave the spatial displacement gradient was zero at $r = 0$ and the displacement was zero at $r = 1$ while for the stress standing wave the stress was zero at $r = 0$ and the spatial stress gradient was zero at $r = 1$. This also applied for the displacement and stress standing waves for the power-law constitutive equation.

5.10 Conclusions and comparison of results

In this Chapter we investigated the propagation of axial displacement and stress waves for a new class of constitutive equations. Two different constitutive equations were considered. The first constitutive equation, the power-law constitutive equation, has the property that the relationship between the linearised strain and stress in nonlinear. This is different from the constitutive equations considered in Cauchy elasticity and hyperelasticity. In fact, Cauchy elastic and hyperelastic materials are subclasses of this more general framework. A feature of the second constitutive equation, the strain-limiting equation, is that the strain will remain bounded even for very large stresses. This has, in addition to applications in fracture mechanics, applications in the polymeric industry where materials exhibit limiting stretch. For each constitutive equation we found solitary wave solutions and standing wave solutions. We cannot obtain travelling wave solutions in a circular cylinder for these constitutive equations by reducing the perturbation equations to canonical form as was done to obtain travelling wave solutions in a rectangular slab. In this Chapter to obtain standing wave solutions we imposed the constraint that the solution to the Bessel equation must be finite at $r = 0$ and so we set the Neumann function $Y_n = 0$. However, any linear combination of the Bessel function J_n and Y_n is also a solution to the Bessel equation. The linear combinations can be expressed as

$$H_v^1(z) = J_v(z) + iY_v(z), \tag{5.10.1}$$

$$H_v^2(z) = J_v(z) - iY_v(z) \tag{5.10.2}$$

which are called cylindrical functions of the third kind or Hankel functions of the first and second kind respectively. In the same way that the exponential function is related to the sine and cosine function similarly, the Hankel functions are connected to the Bessel and Neumann functions. When we consider the real solution to the problem we obtain standing waves and when we consider the imaginary solution we obtain spreading waves. This is not considered in this thesis.

To find solitary wave solutions, the system of partial differential equations for each constitutive equation is reduced to a nonlinear hyperbolic partial differential equation for the stress. The speed of the solitary wave for the power-law constitutive equation is approximately

$$C(n,\sigma) = \frac{1}{\left(1+\frac{3n}{2}\sigma^2(r,t)\ln r\right)}. \tag{5.10.3}$$

As the magnitude of the stress $\sigma(r,t)$ increases the speed of parts of the solitary wave with large stress magnitude decreases and this is most apparent for negative values of the stress. For positive values of the stress for $n = 0$ and $n > 0$ the amplitude achieves a smaller maximum than for negative values of the stress. Further, the wave steepens at the back of the solitary wave since parts of the solitary wave with larger magnitude propagate slower and a shock front forms at the back of the solitary wave. This occurs at approximately time

$$t_s = \frac{1}{6\pi n C^2(n\sigma,r)\ln r F(\phi)\frac{dF}{d\phi}} = \frac{\left(1+\frac{3n}{2}\sigma^2(r,t)\ln r\right)}{6\pi n\ln r F(\phi)\frac{dF}{d\phi}} \tag{5.10.4}$$

where

$$F(\phi)\frac{dF}{d\phi} = \frac{A^2}{2\pi}\left[\cos\phi + \frac{1}{2\phi}\sin\phi\right]\phi\sin\phi > 0. \tag{5.10.5}$$

For the strain-limiting constitutive equation we can approximate the wave speed by

$$C(n,\sigma,r) = \frac{1}{2^{\frac{n}{2}}\left(1-\frac{3n}{4}\sigma^2(r,t)\ln(2^{\frac{n}{2}}r)\right)}. \tag{5.10.6}$$

As the magnitude of the stress $\sigma(r,t)$ increases the speed of the part of the wave with larger magnitude will increase. The amplitude of the stress wave is again larger for negative values of the stress than positive values of the stress for $n = 0$ and $n > 0$. This is in contrast with the amplitude of the solitary stress waves for the power-law constitutive equation and strain-limiting constitutive equation in a rectangular slab where we found that the amplitude of the solitary wave for positive and negative values of the stress are approximately the same. We found that for the strain-limiting constitutive equation a shock front developed at the front of the solitary wave because parts of the solitary wave with larger magnitude propagate faster. This occurs at approximately time

$$t_s = -\frac{1}{2^{\frac{3n}{2}-1}3n\pi C^2(n\sigma,r)\ln(2^{\frac{n}{2}}r)F(\phi)\frac{dF}{d\phi}} = -\frac{2^{\frac{n}{2}}\left(1-\frac{3n}{4}\sigma^2(r,t)\ln(2^{\frac{n}{2}}r)\right)}{2^{\frac{3n}{2}-1}3n\pi\ln(2^{\frac{n}{2}}r)F(\phi)\frac{dF}{d\phi}} \tag{5.10.7}$$

where

$$F(\phi)\frac{dF}{d\phi} = \frac{A^2}{2\pi}\left[\cos\phi + \frac{1}{2\phi}\sin\phi\right]\phi\sin\phi > 0. \tag{5.10.8}$$

Comparing the two times, (5.10.4) and (5.10.7), and considering the graph of $F(\phi)\frac{dF}{d\phi}$ in Figure 5.5.4 we see that a shock front will generally form earlier for the strain-limiting constitutive equation. This was also the conclusion for the two-dimensional solitary waves considered in Chapter 4. We find the same features of the n dependence on the properties of the wave as was found in Chapter 4.

For the standing wave solutions we found solutions to the zero and first order perturbation equations for the displacement and stress for both constitutive equations. The perturbation equations could not be solved exactly at the second order. The solutions at the lowest order for the power-law constitutive equation does not contain the nonlinear parameter $n > 0$ which means analysing the effects of the nonlinear term cannot be done. Nonetheless, from Chapter 4 on the propagation of standing waves in a rectangular slab up to the second order, for experimentally valid values of n there is a negligible difference between the standing waves for $n = 0$ and $n > 0$. Thus, the solution to the first order is sufficient to describe the properties of the wave. We find that the standing waves oscillate with a period of $T = \frac{2\pi}{z_m}$ for the power-law constitutive equation for both the displacement and stress. For the strain-limiting constitutive equation the standing waves oscillate with a period of $T = \frac{2\pi}{z_m}$ for $n = 0$ and this increases to $T = \frac{2^{1+\frac{n}{2}}\pi}{z_m}$ for $n > 0$ for both the displacement and stress waves.

As with the two-dimensional wave motion in a rectangular slab we found that for axial stress and displacement waves in a circular cylinder the elastic response is enhanced for elastic materials with a power-law constitutive equation and inhibited (restricted) for elastic materials with a strain-limiting constitutive equation. The solitary stress waves for a power-law constitutive equation breaks at the back of the wave and breaks at a later time than the solitary stress waves for the strain-limiting constitutive equation which develops a shock at the front of the wave. The periods of oscillation are approximately the same for $n = 0$ and $n > 0$ for the standing wave in an elastic material with a power-law constitutive equation, but increases for $n > 0$ when compared with $n = 0$ in an elastic material with a strain-limiting constitutive equation. Similar properties for the dependence of the solution on the shearing parameter n are also noted for axial waves.

The properties of the two-dimensional stress and displacement waves in a rectangular slab and the axial stress and displacement waves in a circular cylinder are qualitatively similar. For both geometries the solitary stress wave form a shock front at the back of the wave for the power-law constitutive equation and at the front of the wave for the strain-limiting constitutive equation. For standing waves the period of oscillation is approximately the same for $n = 0$

and $n > 0$ for the power-law constitutive equation but increases for $n > 0$ for the strain-limiting constitutive equation. The essential properties of the elastic waves considered do not depend on the geometry of the elastic medium.

Chapter 6

Conclusions and remarks

In this Chapter we provide an overview of the work conducted and the conclusions reached.

In Chapter 2 we gave a brief discussion on finite elasticity and linear elasticity. We discussed the kinematics of deformation and defined the relevant strain measures in both the reference configuration and current configuration. Further, the components of the strain tensor in rectangular, cylindrical polar and spherical coordinates were derived using the dyad notation. The balance laws and field equations were also derived and explicit forms of the equations of motions were given. We concluded the Chapter with a brief discussion on linear elasticity.

In Chapter 3 the development of the constitutive equations used in Cauchy elasticity and hyperelasticity is presented. Cauchy elastic materials are materials where the stress in the current configuration is determined by the state of deformation of the current configuration, and for these materials the stress is independent of the path taken during deformation and the deformation history. An important feature of these materials is that the stress is defined explicitly in terms of the deformation gradient. Hyperelastic materials are materials where the stress is derivable from a potential or stored energy associated with the body. These materials form a subclass of Cauchy elastic materials. A primary assumption made for hyperelastic materials is that the stored energy depends explicitly on the deformation gradient. This assumption however, is a very special assumption and restricts the study of deformation to these two classes of materials. There has been a valid inquiry by Rajagopal [4, 5] as to whether there are more general ways of describing the stress-strain relationship. In other words, can we have materials where the stress and deformation gradient are implicitly related and can we have materials where the stored energy depends implicitly on both the stress and the deformation gradient. This inquiry has led to the development of a far larger class of elastic materials described by implicit constitutive equations.[4, 5]. Further, it has been shown that constitutive equations of this type are thermodynamically consistent [8]. It is constitutive equations of this type that we study in this thesis. In the final part of Chapter 3 we provide a derivation of the implicit constitutive equations. The particular subclass of implicit constitutive equations that we consider are equations where the strain is prescribed in terms of a non-invertible function of the stress. The two constitutive equations that we consider are the power-law constitutive equation and the strain-limiting constitutive equation. The power-law constitutive equation is suitable to describe the nonlinear behaviour between the linearised strain and the stress. There has been significant work done on gum metal and titanium [28, 29, 30, 31] that clearly show-in the small displacement range-a nonlinear relationship between the linearised strain and stress. Such a phenomenon cannot be captured in Cauchy elasticity or hyperelasticity. A feature of the strain-limiting constitutive equation is that it allows for the stress to become very large while the strain remains bounded. Such constitutive equations have important applications in fracture mechanics where traditionally it has been found that at crack tips the strain becomes singular for large stresses [35].

The propagation of nonlinear displacement and stress waves in a rectangular slab for the power-law constitutive equation and strain-limiting equation is investigated in Chapter 4. For the derivation of the mathematical model for both constitutive equations we look for a special semi-inverse solution where specific forms of both the stress and displacement are sought. The stress and displacement are then solved simultaneously. This is unlike classical Cauchy elasticity or hyperelasticity where only a specific form for the displacement is assumed. For each constitutive equation we find solitary wave solutions, travelling wave solutions and standing wave solutions. To find solitary wave solutions we reduced the system of partial differential equations to a single nonlinear hyperbolic partial differential equation to describe the propagation of shear stress waves. We observed a significant distortion in the solitary wave for $n > 0$ when compared to the linear wave for $n = 0$. We found an expression for the approximate wave speed and this expression along with the graphs for the solitary stress wave show that for large stress magnitudes parts of the solitary wave propagate at a slower speed. As time evolves and also for larger values of n the solitary wave steepens at the back and develops a shock front after a sufficient time which we have estimated. The behaviour of the solitary stress wave for the strain-limiting constitutive equation is different in that for large stress magnitudes the speed of the wave increases in parts of the wave. A shock front also develops but develops at the front of the wave and occurs at an earlier time than for the power-law constitutive equation.

We then considered perturbation solutions for the displacement and stress for both constitutive equations. We found travelling wave solutions by reducing the perturbation equations at each order to canonical form and solved the resulting wave equations. The travelling wave solutions to the perturbation equations for the displacement and stress for both constitutive equations contained secular terms in y. Singular perturbation methods like the Lindstedt-Poincaré method are unsuccessful in removing these terms. However, the perturbation expansions break down for values of y outside the range of validity. For the travelling wave solutions the wave front is determined by using the zero displacement boundary condition. We found that the stress at the wave front is non-zero and propagates as a shock wave with a strong discontinuity. The displacement and stress waves are distorted for $n > 0$ and we found that the amplitude of the displacement waves for $n = 0$ and $n > 0$ are approximately the same while the amplitude of the stress wave is less for $n > 0$. In Appendix B we have provided the details of the Linstedt-Poincaré method when applied to the perturbation equations.

The standing wave solutions for both constitutive equations in the rectangular slab also contained a secular term in t which could not be removed using singular perturbation methods such as the Lindstedt-Poincaré method. This is discussed in Appendix B. The perturbation expansion breaks down well outside the range of validity of time t so the solutions are sufficient to study the physical properties of the standing waves. We found that the period of oscillation for the displacement waves are approximately the same for $n = 0$ and $n > 0$ while the period of oscillation for the stress wave is larger for $n > 0$.

The reason why the singular perturbation methods could not remove the secular terms in y and t may be because the perturbation expansions are not convergent.

In Chapter 5 the propagation of axial displacement and stress waves were studied for both the power-law constitutive equation and the strain-limiting constitutive equation. The method of deriving the mathematical model is similar to the derivation in Chapter 4. For each constitutive equation we find solitary wave solutions and standing wave solutions. For the solitary wave solutions we observed similar characteristics of the physical properties of the waves to the solitary waves studied in a rectangular slab. The speed of propagation for large stress magnitudes for the power-law constitutive equation decreases in parts of the wave while it increases in parts of the solitary wave for the strain-limiting constitutive equation. A shock front develops at the front of the solitary wave for the strain-limiting constitutive equation and occurs at a much earlier time than in the solitary wave for the power-law constitutive equation which develops a shock wave at the back of the wave. The axial solitary stress wave have a distinct characteristic different from the solitary stress wave in a rectangular slab, and that is in the amplitude of the wave. For positive values of the stress the amplitude of the axial stress wave has a smaller maximum than for negative values of the stress for $n = 0$ and $n > 0$. For the two-dimensional solitary wave the amplitude of the positive and negative stress values is approximately the same.

Finally, we found standing wave solutions for both constitutive equations. In the standing wave solution for the displacement, the displacement was fixed at zero at the surface of the cylinder and from the condition that the displacement is finite at the centre of the cylinder, the spatial gradient of the displacement had to be fixed at zero. As a result in the standing wave for the stress, the stress was fixed as zero at the centre and the spatial gradient of the stress was fixed as zero at the outer surface. The solutions were found only for the perturbation equations at the zero order. The perturbation equations at the second order could not be solved analytically. The solutions for the displacement and stress at the zero order for the power-law constitutive equation are independent of the parameter n. This means that for this constitutive equation we could not study the effects of the nonlinearity at that order. However, we showed in Chapter 4 that for the standing wave solution for the power-law constitutive equation for physically valid values of n the effects of the nonlinearity is so small that it is negligible. Therefore, in order to illustrate the effect of the nonlinearity we used values of n that fall outside the range of those used in experiments. This suggests that the lowest two order solutions are sufficient to describe the problem. For the strain-limiting constitutive equation the effect of the nonlinearity can be studied at the zero and first orders and we find that for negative values of the displacement and stress the amplitude of the wave increases as n increases. We also find that the period of oscillation for the standing waves are approximately the same for $n = 0$ and $n > 0$ for the power-law constitutive equation while the period of oscillation increases for $n > 0$.

We could not obtain travelling wave solutions for the displacement and stress in a circular cylinder by reducing the perturbation equations to canonical form because the boundary conditions could not be satisfied. However, spreading waves or travelling waves are possible in a cylindrical annulus with the central region removed. The solution to the Bessel equation remains finite in the annulus. The solution would then contain the Neumann function Y_n and be expressed in terms of Hankel functions of the first and second kind. This was not considered in this thesis.

In general, an increase in the value of n for both constitutive equations for both the rectangular slab and cylindrical cylinder, resulted in a decrease in the speed of the wave. The distortion is more pronounced for an increase in the value of n. For a fixed time, a larger value of n will cause the wave to steepen at the front or back at a faster rate than a lower value for n. Further, the perturbation solutions will break down much sooner for a larger value of n.

6.1 Future work

In terms of the implicit theory there are many open questions which one can look into. An important problem is the dynamic propagation of cracks and studying the stress-strain relationships at crack tips for different assumptions on the stress. Currently, only anti-plane strain problems have been considered [35]. The strain-limiting constitutive equations are suitable to studying the stress-strain responses of brittle solids and not much attention has been given to theses problems. Of particular interest would be the propagation of waves in compressible materials for such constitutive equations. An interesting field that has only recently started gaining attention is the application of implicit constitutive equations to soft matter, in particular in a biological setting. Due to the nature of these equations, finding exact solutions are very difficult and even when asymptotic solutions are found one can only plot the solutions, especially for the solitary wave solutions, for a certain range of n. It might be a better option to solve these equations numerically and I will discuss some of the possible options below.

In future work, suitable conservative numerical schemes need to be developed. The asymptotic solutions will be useful in determining the efficacy of the numerical results. In this thesis we deal with hyperbolic partial differential equations and many hyperbolic equations can be written in conserved form. Scalar hyperbolic conservation laws can be written as [58]

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}(u) = 0, \tag{6.1.1}$$

where $u(\mathbf{x}, t)$ is the density and $\mathbf{F}$ is the flux vector. We can extend this definition to systems of homogeneous hyperbolic partial differential equations,

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = 0 \tag{6.1.2}$$

and for non-homogeneous hyperbolic conservation laws

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = S(\mathbf{U}) \tag{6.1.3}$$

where $S(\mathbf{U})$ is a source term. Writing the equation in conserved form is advantageous as many conservative schemes tend to converge to the weak solution of the problem whereas non-conservative schemes do not.

We propose to use modern shock capturing schemes as opposed to the more classical methods such as the Lax-Wendroff and its modified counterparts. Classical shock capturing schemes break down when strong shocks are present. They either admit nonlinear instability or result is oscillations around discontinuities. Further, the classical schemes may select non-physical solutions. Such a situation may arise if the entropy inequality is not satisfied.

One of the primary differences between classical shock capturing methods and modern shock capturing methods is the behaviour of numerical dissipation. The numerical dissipation is linear for the classical methods and the same amount of dissipation is present at each grid point. Modern methods on the other hand have nonlinear numerical dissipation which is different at each grid point. Further, these schemes have built in mechanisms to control the amount of numerical dissipation at each grid point.

Modern shock capturing methods are built from basic first order upwind schemes and Lax-Wendroff schemes. We propose to use higher order TVD (total variation diminishing) schemes. The total variation of a function u^n is [59]

$$\mathrm{TV}(u^n) = \sum_{j=-\infty}^{\infty} |u_{j+1}^n - u_j^n| \tag{6.1.4}$$

and a numerical scheme is said to be TVD if

$$\mathrm{TV}(u_{j+1}^n) \leq \mathrm{TV}(u_j^n). \tag{6.1.5}$$

In order to construct higher order TVD schemes we can consider approaches such as the modified flux approach, the numerical fluctuation approach or a second order extension of the Godunov scheme. Each one of these design principles differ in the construction of the scheme. For example, Godunov finds the next time step by solving the Riemann problem. The Riemann problem is an initial value problem with piecewise constant data. To obtain the second order extension one can replace the piecewise constant data with piecewise linear data. For the modified flux approach and numerical fluctuation approach one will chose a modified flux and average function respectively in order to negate oscillatory behaviour around discontinuities.

To ensure that higher order TVD methods do not result in oscillations around shocks limiters or flux limiters are introduced. These limiters impose constraints on the gradient or the flux function to ensure transitions around discontinuities [59]. We propose to study upwind TVD schemes such as the

MUSCL (monotonic upstream schemes for conservation laws) and higher order symmetric TVD schemes. There are essentially two branches of schemes that one can use to solve hyperbolic conservation laws; the TVD schemes as discussed above or the WENO (weighted essentially non-oscillatory) or ENO (essentially non-oscillatory) schemes. These schemes might be an option to consider. The main distinction between the TVD and WENO/ENO schemes depends on the spatial accuracy of the schemes. WENO/ENO schemes can maintain second order accuracy at discontinuities whereas TVD schemes reduce to first order at these points [59]. Although these schemes seem reasonable, they are built on the idea that the system of partial differential equations can be written in conserved form. The equations considered in this thesis are modelled as conservative systems. To explain this, no material is purely elastic in that it does not dissipate any energy in whatever process it is subject to however, the dissipation is small that it can be neglected allowing for the system to be modelled as a conservative one.

Alternately, we could consider finite element schemes that have been used successfully to solve wave equations such as the Galerkin method using B-splines. In particular, the B-spline Galerkin method has been extensively applied to the regularised wave equation [60, 61, 62, 63]. In [64] cubic and quadratic B-spline finite elements were used to solve Burger's equation. Garder et al. [64] finds that implementing quadratic B-splines with the Galerkin method results in a stable scheme. It is also found that the scheme possesses good conservation properties and the physical properties of the wave are well represented. The coupled KdV equation is also solved numerically using the quadratic B-spline Galerkin method [65]. The accuracy of such schemes are investigated using the L_2 and L_∞ norms and it is found that even for coupled systems of partial differential equations the error norms are small.

Appendix A

Tensor analysis

In Part I of this appendix we provide a brief summary of the transformation laws used in moving from one coordinate system to another. Further, many useful operators used in continuum mechanics will be defined. A tensor is a mathematical object which generalises the notions of vectors and matrices. Due to the complex manner in which continuum models are defined, vectors and matrices are not always suitable. However, tensors provide a framework in which these models can be described. The appeal of tensors is further strengthened in that they are independent of any coordinate system or frame of reference. For example, if the coordinates are valid in one coordinate system then they are valid in all coordinate systems. We will first consider a rectangular Cartesian coordinate system and generalise this to a curvilinear coordinate system. The results of Part I are summarised from [66].

Part I

A.1 Transformation of coordinates

Consider a rectangular Cartesian coordinate system. Let $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ be the base vectors with Cartesian coordinates $\{x_n\}$, for $n = 1, 2, 3$. The position vector $\mathbf{r}$ is given by

$$\mathbf{r} = x_n \mathbf{e}_n \tag{A.1.1}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are mutually orthogonal unit vectors and where the repeated indicies imply summation over the range of n.

The transformation laws for the coordinates $\{x_i\}$, $i = 1, 2, 3$ when base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are rotated about the origin to a new set of base vectors $\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3$ is

$$\bar{x}_i = p_{ik} x_k, \qquad x_i = p_{ki} \bar{x}_k, \tag{A.1.2}$$

where

$$p_{mn} = \text{cosine angle between } \bar{\mathbf{e}_n} \text{ and } \mathbf{e}_m \quad (A.1.3)$$
$$= \text{cosine angle between } \mathbf{e}_m \text{ and } \bar{\mathbf{e}}_n.$$

Equation (A.1.2) can be represented by a matrix,

$$\bar{\mathbf{X}} = \mathbf{P}\mathbf{X}, \qquad \mathbf{X} = \mathbf{P}^T\mathbf{X}, \quad (A.1.4)$$

where P is a proper orthogonal matrix since $det\mathbf{P} = +1$ and $\mathbf{P}^{-1} = \mathbf{P}^T$.

The cosine angles can also be expressed in terms of the Kronecker delta,

$$p_{ik}p_{jk} = \delta_{ij}, \qquad p_{ki}p_{kj} = \delta_{ij}, \quad (A.1.5)$$

where the Kronecker delta is defined as

$$\delta_{ik} = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{for } i \neq k. \end{cases} \quad (A.1.6)$$

Further, the cosine angles can be used to establish transformation laws between two coordinate systems

$$\bar{\mathbf{e}_n} = p_{nm}\mathbf{e}_m, \qquad \mathbf{e}_n = p_{mn}\bar{\mathbf{e}_m}. \quad (A.1.7)$$

We also define respectively the unit tensor and the alternating tensor as

$$\delta_{ik} = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases} \quad (A.1.8)$$

and

$$e_{ijk} = \begin{cases} 0 & \text{if any two indices are the same} \\ 1 & \text{if i, j, k is an even peremutation of 1, 2, 3} \\ -1 & \text{if i, j, k is an odd peremutation of 1, 2, 3.} \end{cases} \quad (A.1.9)$$

These tensors are useful when deriving scalar and vector products. Consider mutually orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The dot product and cross product between base vectors are given as

$$\mathbf{e}_m \cdot \mathbf{e}_n = \delta_{mn}, \qquad \mathbf{e}_m \mathbf{\,x\,} \mathbf{e}_n = e_{mnp}\mathbf{i}_p \quad (A.1.10)$$

Using these properties we can establish the dot and cross product for vectors. Suppose we have vectors

$$\mathbf{A} = A_m\mathbf{e}_m, \qquad \mathbf{B} = B_n\mathbf{e}_n \quad (A.1.11)$$

where A_m $(m = 1, 2, 3)$ and B_n $(m = 1, 2, 3)$ are the components of vectors $\mathbf{A}$ and $\mathbf{B}$ respectively, then

$$\mathbf{A} \cdot \mathbf{B} = A_n B_m, \qquad \mathbf{A} \times \mathbf{B} = e_{pmn} A_m B_n \mathbf{e}_p. \tag{A.1.12}$$

We now summarise useful operators, which are defined in a rectangular Cartesian coordinate system, that are used in continuum mechanics.

The del operator ∇ is defined as

$$\nabla = \mathbf{e}_n \frac{\partial}{\partial x_n} = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3}. \tag{A.1.13}$$

A convention that is commonly used in tensor calculus is that a partial derivative is represented by a comma. For example

$$\phi_{,i} = \frac{\partial \phi}{\partial x_i}. \tag{A.1.14}$$

If $\phi = \phi(x_1, x_2, x_3)$ is a scalar field then

$$\mathbf{grad}\phi = \nabla \phi = \mathbf{e}_n \frac{\partial \phi}{\partial x_n} = \phi_{,n} \mathbf{e}_n. \tag{A.1.15}$$

The divergence and curl of a vector $\mathbf{A}$ are given by

$$\mathbf{div}\ \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_n}{\partial x_n}, \tag{A.1.16}$$

$$\mathbf{curl}\ \mathbf{A} = \nabla \times \mathbf{A} = e_{snr} \frac{\partial A_r}{\partial x_n} \mathbf{e}_s. \tag{A.1.17}$$

Finally, we state the Divergence theorem without proof. Let V be a volume bounded by a closed surface S. Then

$$\int_V \frac{\partial T}{\partial x_i} dV = \int_S T n_i dS, \tag{A.1.18}$$

where T is any tensor field and $\mathbf{n}$ is the unit vector along the outward normal to S.

A.2 Curvilinear coordinates

We now discuss a more general coordinate system, in which it is possible through local transformations, to transform a point in a Cartesian rectangular coordinate system to its curvilinear coordinates and back if required. Cartesian coordinates are a special case of curvilinear coordinates. Since these coordinates transform as tensors they are valid in all coordinate systems.

The curvilinear coordinates θ^i, where $\theta^i = \theta^i(x^1, x^2, x^3)$, $i = 1, 2, 3$ are differentiable up to required order and possess the property of invertibility. In other words,

$$\det\left[\frac{\partial\theta^i}{\partial x^j}\right] \neq 0. \tag{A.2.1}$$

These curvilinear coordinates have a geometric significance, in that if each of the curvilinear coordinates are constant, $\theta^i(x^1, x^2, x^3) = \text{constant for } i = 1, 2, 3$, then this equation defines the equation of a coordinate surface. Naturally, one could vary the value of the constant to produce a family of coordinate surfaces. The intersection of the three coordinates surfaces at an arbitrary point are called coordinate curves.

A.2.1 Base vectors

Consider the position vector

$$\mathbf{r} = x^n \mathbf{e}_n. \tag{A.2.2}$$

The position vector is a function of the curvilinear coordinates θ^i, i.e, $\mathbf{r} = \mathbf{r}(\theta^1, \theta^2, \theta^3)$. The differential of the position vector is

$$d\mathbf{r} \;=\; dx^n \mathbf{e}_n \;=\; \frac{\partial x^n}{\partial \theta^i} d\theta^i \mathbf{e}_n \;=\; \mathbf{g}_i d\theta^i \tag{A.2.3}$$

where

$$\mathbf{g}_i = \frac{\partial x^n}{\partial \theta^i} \mathbf{e}^n \tag{A.2.4}$$

The differential can also be expressed as

$$d\mathbf{r} \;=\; dx_n \mathbf{e}^n \;=\; \frac{\partial \theta^i}{\partial x^n} d\theta^i \mathbf{e}^n \;=\; \mathbf{g}^i d\theta_i \tag{A.2.5}$$

where

$$\mathbf{g}^i = \frac{\partial \theta^i}{\partial x^n} \mathbf{e}^n \tag{A.2.6}$$

Expression (A.2.4) and (A.2.6) are called covariant and contravariant base vectors respectively. These base vectors are in general not unit base vectors or mutually orthogonal.

We can express the mutually orthogonal unit base vectors for a Cartesian coordinate system in terms of the covariant and contravariant base vectors for a curvilinear coordinate system by

$$\mathbf{e}_n = \frac{\partial \theta^i}{\partial x^n} \mathbf{g}_i, \qquad \mathbf{e}^n = \frac{\partial x^n}{\partial \theta^i} \mathbf{g}^i \tag{A.2.7}$$

For a vector $\mathbf{A} = A^n \mathbf{e}_n$, we can transform the components A^n from a Cartesian basis to a curvilinear basis $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ using the transformation law

$$\bar{\mathbf{A}}^i = \frac{\partial \theta^i}{\partial x^n} \mathbf{A}^n \tag{A.2.8}$$

where $\mathbf{A}^i(\theta^1, \theta^2, \theta^3)$ are called the contravariant components of a contravariant vector $\mathbf{A}$. Similarly, we can transform the components A_n of vector $\mathbf{A} = A_n \mathbf{e}^n$ from a Cartesian basis to a curvilinear basis $\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3$ according to the rule

$$\bar{\mathbf{A}}_i = \frac{\partial x^n}{\partial \theta^i} \mathbf{A}_n \tag{A.2.9}$$

where $A_i(\theta^1, \theta^2, \theta^3)$ are the covariant components of a covariant vector $\mathbf{A}$.

A.2.2 The metric and alternating tensors

The Kronecker delta is a Cartesian tensor of order two and is defined as

$$\delta_{ik} = \delta^{ik} = \delta^i_k = \delta^k_i = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases} \tag{A.2.10}$$

The metric tensors g_{ik}, g^{ik} and g^i_k can be defined in terms of the Kronecker deltas as follows

$$g_{ik} = \frac{\partial x^m}{\partial \theta^i} \frac{\partial x^n}{\partial \theta^k} \delta_{mn} = \sum_{m=1}^{3} \frac{\partial x^m}{\partial \theta^i} \frac{\partial x^m}{\partial \theta^k}, \tag{A.2.11}$$

$$g^{ik} = \frac{\partial \theta^i}{\partial x^m} \frac{\partial \theta^k}{\partial x^n} \delta^{mn} = \sum_{m=1}^{3} \frac{\partial \theta^i}{\partial x^m} \frac{\partial \theta^k}{\partial x^m}, \tag{A.2.12}$$

$$g^i_k = \frac{\partial \theta^i}{\partial x^m} \frac{\partial x^n}{\partial \theta^k} \delta^m_n = \frac{\partial \theta^i}{\partial x^m} \frac{\partial x^m}{\partial \theta^k} = \frac{\partial \theta^i}{\partial \theta^k} = \delta^i_k. \tag{A.2.13}$$

The tensors g_{ik}, g^{ik} and g^i_k are called the covariant, contravariant and mixed metric tensors respectively. The covariant and contravariant metric tensors are symmetric, i.e,

$$g_{ik} = g_{ki}, \qquad g^{ik} = g^{ki}. \tag{A.2.14}$$

The transformations (A.2.11), (A.2.12) and (A.2.13) are invertible and it is possible to express the Kronecker delta in terms of the metric tensors.

The discussion can be extended to transform Cartesian tensors of order three to general curvilinear coordinates. The transformation laws for the components of covariant and contravariant tensors of order three from Cartesian coordinates x^i to curvilinear coordinates θ^i are

$$\epsilon_{ijk} = \frac{\partial x^r}{\partial \theta^i} \frac{\partial x^s}{\partial \theta^j} \frac{\partial x^t}{\partial \theta^k} e_{rst}, \tag{A.2.15}$$

$$\epsilon^{ijk} = \frac{\partial\theta^i}{\partial x^r}\frac{\partial\theta^j}{\partial x^s}\frac{\partial\theta^k}{\partial x^t}e^{rst}. \tag{A.2.16}$$

The tensors ϵ_{ijk} and ϵ^{ijk} are the covariant and contravariant alternating tensors respectively and e_{rst} and e^{rst} is the permutation symbol in curvilinear coordinates and is defined as

$$e_{ijk} = e^{ijk} = \begin{cases} 0 & \text{if any two indices are the same} \\ 1 & \text{if i, j, k is an even peremutation of 1, 2, 3} \\ -1 & \text{if i, j, k is an odd peremutation of 1, 2, 3} \end{cases} \tag{A.2.17}$$

The relationship between the covariant and contravariant base vectors and the metric tensors is summarised below.

$$\mathbf{g}_r \cdot \mathbf{g}_s = g_{rs}, \tag{A.2.18}$$

$$\mathbf{g}^r \cdot \mathbf{g}^s = g^{rs}, \tag{A.2.19}$$

$$\mathbf{g}^r \cdot \mathbf{g}_s \;=\; g^r_s, \;=\; \delta^r_s \tag{A.2.20}$$

$$\mathbf{g}^r \;=\; g^{rs}\mathbf{g}_s, \qquad \mathbf{g}_r \;=\; g_{rs}\mathbf{g}^s. \tag{A.2.21}$$

The last property (A.2.21) allows for the metric tensors to be used in the lowering and raising of indicies. The lowering and raising of indicies is not restricted to the base vectors but may be used for higher order tensors.

Covariant and contravariant base vectors are useful when establishing vector identities. The covariant and contravariant base vectors possess the following useful properties. Let $g = \det[g_{ij}]$, then we have

$$\mathbf{g}_i \times \mathbf{g}_j \;=\; \epsilon_{ijk}\mathbf{g}^k \;=\; \sqrt{g}e_{ijk}\mathbf{g}^k, \tag{A.2.22}$$

$$\mathbf{g}^i \times \mathbf{g}^j \;=\; \epsilon^{ijk}\mathbf{g}_k \;=\; \frac{1}{\sqrt{g}}e^{ijk}\mathbf{g}_k, \tag{A.2.23}$$

$$\mathbf{A} \times \mathbf{B} \;=\; \epsilon_{ijk}A^iB^j\mathbf{g}^k \;=\; \epsilon^{ijk}A_iB_j\mathbf{g}_k, \tag{A.2.24}$$

$$\epsilon_{ijk} = \sqrt{g}e_{ijk}, \qquad \epsilon^{ijk} = \frac{1}{\sqrt{g}}e^{ijk}. \tag{A.2.25}$$

A.2.3 Covariant differentiation

We now investigate the differentiation of base vectors and summarise the transformation laws associated with covariant differentiation. The standard notation of a comma to represent a partial derivative still holds.

The derivatives of the base vectors $\mathbf{g}_{i,j}$ and $\mathbf{g}^i_{,j}$ can be expressed in terms of the covariant and contravariant base vectors $\mathbf{g}_i$ and $\mathbf{g}^j$ as follows

$$\mathbf{g}_{i,j} = \Gamma^k_{ij}\mathbf{g}_k \quad \text{where} \quad \Gamma^k_{ij} = \frac{\partial^2 x^n}{\partial\theta^i\partial\theta^j}\frac{\partial\theta^k}{\partial x^n}, \tag{A.2.26}$$

$$\mathbf{g}_{i,j} = \Gamma_{ijk}\mathbf{g}^k \quad \text{where} \quad \Gamma_{ijk} \;=\; g_{ks}\Gamma^s_{ij} \;=\; \frac{\partial^2 x^n}{\partial\theta^i\partial\theta^j}\frac{\partial\theta^m}{\partial\theta^k}\delta_{nm}, \tag{A.2.27}$$

$$\mathbf{g}^i_{,j} = -\Gamma^i_{jk}\mathbf{g}^k. \tag{A.2.28}$$

The quantities Γ^k_{ij} and Γ_{ijk} are called the Christoffel symbols of the first and second kind respectively. The Christoffel symbols can be expressed in terms of the derivatives of the metric tensor as

$$\Gamma_{ijk} = \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k}), \tag{A.2.29}$$

$$\Gamma^i_{jk} \;=\; g^{is}\Gamma_{jks} \;=\; \frac{1}{2}g^{is}(g_{js,k} + g_{ks,j} - g_{jk,s}), \tag{A.2.30}$$

$$\Gamma^i_{ik} \;=\; \frac{\sqrt{g}_{,k}}{\sqrt{g}} \;=\; \frac{1}{2g}g_{,k}. \tag{A.2.31}$$

In the special case when the coordinate curves are orthogonal,

$$\mathbf{g}_i \cdot \mathbf{g}_j \;=\; g_{ij} \;=\; 0 \;\text{ and }\; \mathbf{g}^i \cdot \mathbf{g}^j \;=\; g^{ij} \;=\; 0 \;\text{ if }\; i \neq j, \tag{A.2.32}$$

then

$$\Gamma_{ijk} = 0 \;\text{ and }\; \Gamma^i_{jk} = 0 \;\text{ for }\; i \neq j \neq k \neq i. \tag{A.2.33}$$

Covariant differentiation can be extended to vectors. Consider a vector

$$\mathbf{v} \;=\; v^r\mathbf{g}_r \;=\; v_r\mathbf{g}^r. \tag{A.2.34}$$

The derivative is defined by

$$\mathbf{v}_{,i} \;=\; v^r_{|i}\mathbf{g}_r \;=\; v_{r|i}\mathbf{g}^r, \tag{A.2.35}$$

where the covariant derivatives of the components of v^r and v_r are given by

$$v^r_{|i} = v^r_{,i} + \Gamma^r_{it}v^t, \tag{A.2.36}$$

$$v_{r|i} = v_{r,i} - \Gamma^t_{ri} v_t, \tag{A.2.37}$$

respectively.

We now summarise some useful operators expressed in curvilinear coordinates.

The del operator ∇ in curvilinear coordinates is defined as

$$\nabla = \mathbf{g}^i \frac{\partial}{\partial \theta^i}. \tag{A.2.38}$$

For a scalar field $\phi = \phi(\theta^1, \theta^2, \theta^3)$, the gradient of ϕ is

$$\mathbf{grad}\phi \ = \ \nabla\phi \ = \ \mathbf{g}^i \frac{\partial \phi}{\partial \theta^i}. \tag{A.2.39}$$

The divergence and curl of a vector $\mathbf{A}$ are given by

$$\mathbf{div\ A} \ = \ \nabla \cdot \mathbf{A} \ = \ A^i_{|i} \ = \ \frac{1}{\sqrt{g}} \left(\sqrt{g} A^i\right)_{,i}, \tag{A.2.40}$$

$$\mathbf{curl\ A} \ = \ \nabla \times \mathbf{A} \ = \ \epsilon^{ijk} A_{j|i} \mathbf{g}_k \ = \ \epsilon^{ijk} A_{j,i} \mathbf{g}_k. \tag{A.2.41}$$

As we have mentioned before, the base vectors $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ are not unit base vectors. In order to allow for physically meaningful components, these base vectors need to be normalised.

The magnitudes of the base vectors $\mathbf{g}_i$ and $\mathbf{g}^i$ is

$$|\mathbf{g}_i| \ = \ (\mathbf{g}_i \cdot \mathbf{g}_i)^{\frac{1}{2}} \ = \ \sqrt{g_{ii}}, \tag{A.2.42}$$

and

$$\left|\mathbf{g}^i\right| \ = \ (\mathbf{g}^i \cdot \mathbf{g}^i)^{\frac{1}{2}} \ = \ \sqrt{g^{ii}}, \tag{A.2.43}$$

We can define the unit base vectors for the covariant and contravariant base vectors as

$$\bar{\mathbf{g}}_i = \frac{\mathbf{g}_i}{\sqrt{g_{ii}}}, \qquad \bar{\mathbf{g}}^i = \frac{\mathbf{g}^i}{\sqrt{g^{ii}}}. \tag{A.2.44}$$

Finally, the physical components A^i and A_i of vector A are

$$\bar{A}^i = \sqrt{g_{ii}} A^i, \qquad \bar{A}_i = \sqrt{g^{ii}} A_i. \tag{A.2.45}$$

Appendix B

Singular perturbation

B.1 Introduction

In this Appendix we present the perturbation solutions for the stress and displacement for the power-law constitutive equation using the Lindstedt-Poincaré method which is a singular perturbation method. Singular perturbation methods have been successful in removing secular terms, however, we show that this method breaks down for the nonlinear wave equations considered in this thesis. We only show the details for the power-law constitutive equation and not for the strain-limiting equation as the details are similar.

B.2 Perturbation solutions for the first order system for displacement and stress

We consider the non-dimensional system of equations

$$\frac{\partial \sigma}{\partial y} = \frac{\sqrt{\gamma}}{\alpha}\frac{\partial^2 u}{\partial t^2}, \tag{B.2.1}$$

$$\frac{\partial u}{\partial y} = \frac{\alpha}{\sqrt{\gamma}}(1+\sigma^2)^n \sigma. \tag{B.2.2}$$

The derivation of this system of equations can be found in Chapter 4. Since we require the displacement to be small, we set

$$\frac{\alpha}{\sqrt{\gamma}} = \epsilon. \tag{B.2.3}$$

The system of equations become

$$\frac{\partial \sigma}{\partial y} = \frac{1}{\epsilon}\frac{\partial^2 u}{\partial t^2}, \tag{B.2.4}$$

$$\frac{\partial u}{\partial y} = \epsilon(1+\sigma^2)^n\sigma. \tag{B.2.5}$$

We make the following change of variables

$$\sigma(y,t) = \sigma^\star(Y,s), \quad u(y,t) = u^\star(Y,s), \tag{B.2.6}$$

where

$$Y = \beta y, \quad s = \omega t. \tag{B.2.7}$$

Substituting (B.2.6) and (B.2.7) into (B.2.5) gives

$$\beta\frac{\partial \sigma^\star}{\partial Y} = \frac{1}{\epsilon}\omega^2\frac{\partial^2 u^\star}{\partial s^2}, \tag{B.2.8}$$

$$\beta\frac{\partial u^\star}{\partial Y} = \epsilon(1+(\sigma^\star)^2)\sigma^\star. \tag{B.2.9}$$

We consider straightforward perturbation expansions of the form

$$u(y,t) = \epsilon^2 u_0(Y,s) + \epsilon^3 u_1(Y,s) + \epsilon^4 u_2(Y,s) + \mathcal{O}(\epsilon^5), \tag{B.2.10}$$

$$\sigma(y,t) = \epsilon\sigma_0(Y,s) + \epsilon^2\sigma_1(Y,s) + \epsilon^3\sigma_2(Y,s) + \mathcal{O}(\epsilon^4), \tag{B.2.11}$$

$$Y = \beta y = (1+\epsilon\beta_1+\epsilon^2\beta_2+\cdots)y, \tag{B.2.12}$$

$$s = \omega t = (1+\epsilon\omega_1+\epsilon^2\omega_2+\cdots)t. \tag{B.2.13}$$

Substituting (B.2.10), (B.2.11), (B.2.12) and (B.2.13) into the system of equations (B.2.8) and (B.2.9) and approximating the term $(1+\sigma^2)^n$ using a binomial expansion up to order $\mathcal{O}(\epsilon^3)$, we obtain the following system of equations after equating the terms of the same order of ϵ in each equation.

Zero order terms u_0 and σ_0:

$$\epsilon: \frac{\partial \sigma_0}{\partial Y} = \frac{\partial^2 u_0}{\partial s^2}, \tag{B.2.14}$$

$$\epsilon^2: \frac{\partial u_0}{\partial Y} = \sigma_0. \tag{B.2.15}$$

Eliminating σ_0 from (B.2.14) and (B.2.15) gives the homogeneous one-dimensional wave equation

$$\frac{\partial^2 u_0}{\partial Y^2} - \frac{\partial^2 u_0}{\partial s^2} = 0. \tag{B.2.16}$$

First order terms u_1 and σ_1:

$$\epsilon^2: \frac{\partial \sigma_1}{\partial Y} + \beta_1\frac{\partial \sigma_0}{\partial Y} = \frac{\partial^2 u_1}{\partial s^2} + 2\omega_1\frac{\partial^2 u_0}{\partial s^2}, \tag{B.2.17}$$

$$\epsilon^3: \frac{\partial u_1}{\partial Y} + \beta_1 \frac{\partial u_0}{\partial Y} = \sigma_1. \tag{B.2.18}$$

Eliminating σ_1 from (B.2.17) and (B.2.18) gives the wave equation

$$\frac{\partial^2 u_1}{\partial Y^2} - \frac{\partial^2 u_1}{\partial s^2} = -\beta_1 \left(\frac{\partial^2 u_0}{\partial Y^2} + \frac{\partial \sigma_0}{\partial Y} \right) + 2\omega_1 \frac{\partial^2 u_0}{\partial s^2}. \tag{B.2.19}$$

Second order terms u_2 and σ_2:

$$\begin{aligned} \epsilon^3: \frac{\partial \sigma_2}{\partial Y} + \beta_1 \frac{\partial \sigma_1}{\partial Y} + \beta_2 \frac{\partial \sigma_0}{\partial Y} &= \frac{\partial^2 u_2}{\partial s^2} + 2\omega_1 \frac{\partial^2 u_1}{\partial s^2} \\ &+ (\omega_1^2 + 2\omega_2) \frac{\partial^2 u_0}{\partial s^2}, \end{aligned} \tag{B.2.20}$$

$$\epsilon^4: \frac{\partial u_2}{\partial Y} + \beta_1 \frac{\partial u_1}{\partial Y} + \beta_2 \frac{\partial u_0}{\partial Y} = \sigma_2 + n\sigma_0^3. \tag{B.2.21}$$

Eliminating σ_2 from (B.2.20) and (B.2.21) gives the wave equation

$$\begin{aligned} \frac{\partial^2 u_2}{\partial Y^2} - \frac{\partial^2 u_2}{\partial s^2} &= -\beta_1 \left(\frac{\partial^2 u_1}{\partial Y^2} + \frac{\partial \sigma_1}{\partial Y} \right) + 3n\sigma_0^2 \frac{\partial \sigma_0}{\partial Y} \\ &- \beta_2 \left(\frac{\partial^2 u_0}{\partial Y^2} + \frac{\partial \sigma_0}{\partial Y} \right) + 2\omega_1 \frac{\partial^2 u_1}{\partial s^2} + (\omega_1^2 + 2\omega_2) \frac{\partial^2 u_0}{\partial s^2}. \end{aligned} \tag{B.2.22}$$

We will now consider a travelling wave solution and a standing wave solution to the perturbation expansions. The general procedure is to solve the wave equations (B.2.16), (B.2.19) and (B.2.22) for the displacement and then to obtain the stress from (B.2.15), (B.2.18) and (B.2.21).

B.2.1 Travelling wave solution

We solve the wave equations for the displacement subject to the boundary condition

$$u(0,t) = \begin{cases} A\sin(4\pi t), & t \geq 0 \\ 0, & t < 0. \end{cases} \tag{B.2.23}$$

We also have the condition that there are no incoming waves from $y = +\infty$ and so waves only propagate in the positive $x-$direction. Expanded in powers of ϵ to order ϵ^4 and using (B.2.13) for t, we obtain

$$\begin{aligned} &\epsilon^2 u_0(0,s) + \epsilon^3 u_1(0,s) + \epsilon^4 u_2(0,s) = (\epsilon^2 A_0 + \epsilon^3 A_1 + \epsilon^4 A_2) \sin\left[4\pi \left(\frac{s}{\omega}\right)\right] \\ = \ &(\epsilon^2 A_0 + \epsilon^3 A_1 + \epsilon^4 A_2) \sin(4\pi s(1 - \epsilon\omega_1 + (\omega_1^2 - \omega_2)\epsilon^2 + \mathcal{O}(\epsilon^3)). \end{aligned} \tag{B.2.24}$$

Consider the zero order quantities $u_0(Y,s)$ and $\sigma_0(Y,s)$. The linear wave equation (B.2.16) has canonical variables

$$\xi = Y - s, \quad \eta = Y + s, \tag{B.2.25}$$

with canonical form

$$\frac{\partial^2 u_0}{\partial \xi \partial \eta} = 0. \tag{B.2.26}$$

The general solution of (B.2.26) which when expressed in the original variables is

$$u_0(Y,s) = F_0(Y-s) + G_0(Y+s). \tag{B.2.27}$$

Since we are only considering waves propagating in the positive x-direction we take $G_0 = 0$. Therefore

$$u_0(Y,s) = F_0(Y-s). \tag{B.2.28}$$

The unknown function F_0 can be determined using the initial condition (B.2.24) for terms of order ϵ^2. Therefore we obtain for the displacement at the zero order

$$u_0(Y,s) = F_0(Y-s) = \begin{cases} -A_0 \sin[4\pi(Y-s)], & 0 \leq Y \leq s \\ 0, & Y > s. \end{cases} \tag{B.2.29}$$

From (B.2.15), the stress field is given by

$$\sigma_0(Y,s) = \begin{cases} -4\pi A_0 \cos[4\pi(Y-s)], & 0 \leq Y \leq s \\ 0, & Y > s. \end{cases} \tag{B.2.30}$$

We consider next the first order terms $u_1(Y,s)$ and $\sigma_1(Y,s)$. Substituting (B.2.29) and (B.2.30) into (B.2.19) we obtain the nonhomogeneous wave equation

$$\frac{\partial^2 u_1}{\partial Y^2} - \frac{\partial^2 u_1}{\partial s^2} = \begin{cases} 32(\omega_1 - \beta_1)\pi^2 A_0 \sin[4\pi(Y-s)], & 0 \leq Y \leq s \\ 0, & Y > s \end{cases} \tag{B.2.31}$$

Equation (B.2.37) has canonical variables $\xi = Y - s$ and $\eta = Y + s$, and equation (B.2.37) expressed in terms of these canonical variables is

$$\frac{\partial^2 u_1}{\partial \xi \partial \eta} = \begin{cases} 8(\omega_1 - \beta_1)\pi^2 A_0 \sin(4\pi\xi), & -\eta \leq \xi \leq 0 \\ 0, & \xi > 0. \end{cases} \tag{B.2.32}$$

Equation (B.2.32) can integrated twice and when transformed back to the original variables Y and s gives

$$u_1(Y,s) = \begin{cases} -2(\omega_1 - \beta_1)\pi A_0(Y+s)\cos[4\pi(Y-s)] + F_1(Y-s) & \\ +G_1(Y+s), & 0 \le Y \le s \\ 0, & Y > s \end{cases} \tag{B.2.33}$$

Since we only consider waves propagating in the positive x-direction we set $G_1 = 0$. The unknown function $F_1(Y-s)$ can be determined using the boundary condition at the first order, which when imposed on (B.2.33) gives the displacement field at the first order

$$u_1(Y,s) = \begin{cases} -4(\omega_1 - \beta_1)\pi A_0 Y\cos[4\pi(Y-s)] - A_1\sin[4\pi(Y-s)] & \\ +4\pi A_0(Y-s)\omega_1\cos[4\pi(Y-s)], & 0 \le Y \le s \\ 0, & Y > s \end{cases} \tag{B.2.34}$$

In order to remove the secular term we choose $\omega_1 = 0$ and $\beta_1 = 0$. Therefore, the displacement field at the first order becomes

$$u_1(Y,s) = \begin{cases} -A_1\sin[4\pi(Y-s)], & 0 \le Y \le s \\ 0, & Y > s \end{cases} \tag{B.2.35}$$

and from (B.2.18) using (B.2.29) for u_0 and (B.2.35) for u_1 we obtain the corresponding stress field at the first order

$$\sigma_1(Y,s) = \begin{cases} -4\pi A_1\cos[4\pi(Y-s)], & 0 \le Y \le s \\ 0, & Y > s \end{cases} \tag{B.2.36}$$

Finally, we consider the second order terms $u_2(Y,s)$ and $\sigma_2(Y,s)$. Substituting (B.2.29), (B.2.30), (B.2.35) and (B.2.36) into the wave equation (B.2.22) and using $\omega_1 = \beta_1$ we obtain the nonhomogeneous wave equation

$$\frac{\partial^2 u_2}{\partial Y^2} - \frac{\partial^2 u_2}{\partial s^2} = \begin{cases} 32(\omega_2 - \beta_2)\pi^2 A_0\sin[4\pi(Y-s)] & \\ +768n\pi^4 A_0^3\cos^2[4\pi(Y-s)]\sin[4\pi(Y-s)], & 0 \le Y \le s \\ 0, & Y > s \end{cases} \tag{B.2.37}$$

Expressed in terms of the canonical variables $\xi = Y-s$ and $\eta = Y+s$, (B.2.37) reduces to

$$\frac{\partial^2 u_2}{\partial \xi \partial \eta} = \begin{cases} 8(\omega_2 - \beta_2)\pi^2 A_0\sin(4\pi\xi) & \\ +192n\pi^4 A_0^3\cos^2[4\pi\xi]\sin[4\pi\xi], & -\eta \le \xi \le 0 \\ 0, & \xi > 0 \end{cases} \tag{B.2.38}$$

Integrating (B.2.38) with respect to the canonical variables and transforming back to variables Y and s for waves propagating in the positive x-direction only we obtain

$$u_2(Y,s) = \begin{cases} -2(\omega_2-\beta_2)(Y+s)\pi A_0\cos[4\pi(Y-s)] & \\ -16n\pi^3A_0^3(Y+s)\cos^3[4\pi(Y-s)] + F(Y-s), & 0\le Y\le s \\ 0, & Y>s \end{cases} \tag{B.2.39}$$

The unknown function $F(Y-s)$ can be determined using the boundary condition at the second order, which when imposed on (B.2.39) gives the displacement field at second order

$$u_2(Y,s) = \begin{cases} -4(\omega_2-\beta_2)Y\pi A_0\cos[4\pi(Y-s)] & \\ -32n\pi^3A_0^3Y\cos^3[4\pi(Y-s)] - A_2\sin[4\pi(Y-s)] + & \\ 4\pi A_0\omega_2(Y-s)\cos[4\pi(Y-s)], & 0\le Y\le s \\ 0, & Y>s \end{cases} \tag{B.2.40}$$

To remove the secular term Y from (B.2.40) we choose

$$\beta_2 = 0, \quad \omega_2 = 0. \tag{B.2.41}$$

Therefore, the displacement field at the second order becomes

$$u_2(Y,s) = \begin{cases} -8n\pi^3A_0^3Y\cos[12\pi(Y-s)] - 24n\pi^3A_0^3\cos[4\pi(Y-s)] & \\ -A_2\sin[4\pi(Y-s)], & 0\le Y\le s \\ 0, & Y>s \end{cases} \tag{B.2.42}$$

and using (B.2.29) fpr u_0, (B.2.30) for σ_0, (B.2.42) for u_2 along with $\beta_1 = 0$, the stress at the second order is

$$\sigma_2(Y,s) = \begin{cases} 8n\pi^3A_0^3\cos[12\pi(Y-s)] + 96n\pi^4A_0^3Y\sin[12\pi(Y-s)] & \\ +24n\pi^3A_0^3\cos[4\pi(Y-s)] + 96n\pi^4A_0^3Y\sin[4\pi(Y-s)] & \\ -4A_2\pi\cos[4\pi(Y-s)], & 0\le Y\le s \\ 0, & Y>s \end{cases} \tag{B.2.43}$$

By combining the zero, first and second order solutions we obtain for the displacement the asymptotic solution correct to order ϵ^4

$$u(y,t) = \begin{cases} -\epsilon^2(A_0+\epsilon A_1+\epsilon^2A_2)\sin[4\pi(Y-s)] & \\ -24\epsilon^4n\pi^3A_0^3Y\cos[4\pi(Y-s)] & \\ -8\epsilon^4n\pi^3A_0^3Y\cos[12\pi(Y-s)], & 0\le Y\le Y_F(t;n) \\ 0, & Y>Y_F(t;n), \end{cases} \tag{B.2.44}$$

where $Y_F(t;n)$ is the distance travelled by the wave front, $u = 0$, after time t. and for the stress correct to order ϵ^3

$$\sigma(y,t) \tag{B.2.45}$$
$$= \begin{cases} -4\epsilon\pi(A_0 + \epsilon A_1 + \epsilon^2 A_2)\cos[4\pi(Y-s)] \\ +8\epsilon^3 n\pi^3 A_0^3 \cos[12\pi(Y-s)] + 96\epsilon^3 n\pi^4 A_0^3 Y \sin[12\pi(Y-s)] \\ +24\epsilon^3 n\pi^3 A_0^3 \cos[4\pi(Y-s)] + 96\epsilon^3 n\pi^4 A_0^3 Y \sin[4\pi(Y-s)], & 0 \le Y \le Y_F(t;n) \\ 0, & Y > Y_F(t;n) \end{cases}$$

where

$$Y = \beta y = (1 + \epsilon^2\beta_2 + \mathcal{O}(\epsilon^3))y, \tag{B.2.46}$$

$$s = \omega t = (1 + \epsilon^2\beta_2 + \mathcal{O}(\epsilon^3))t \tag{B.2.47}$$

and Y_F is the range of Y and depends on the time t and the value of n. The solutions for the displacement (B.2.44) and for the stress (B.2.45) both contain the secular term Y, and the Lindstedt-Poincare method does not remove the secularity at the second order. Therefore, we simply considered the straightforward perturbation expansions in Chapter 4.

The reason that we cannot remove the secular terms at the second order is due to the fact that the second term in the wave equation (B.2.22) does not contain any parameters which can be used to remove the secular term. Another reason for this is that, that particular term is nonlinear and once the solution is written in the form of multiple angles there are no other terms which cancel with it. Since the perturbation equations for the travelling wave solutions and the standing wave solutions at each order are the same we expect that the secular terms in the standing wave solutions will not be removed.

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