ON LIE AND NOETHER SYMMETRIES OF DIFFERENTIAL EQUATIONS

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ABSTRACT

The inverse problem in the Calculus of Variations involves determining the Lagrangians, if any, associated with a given (system of) differential equation(s). One can classify Lagrangians according to the Lie algebra of symmetries of the Action integral (the Noether algebra). We give a complete classification of first-order Lagrangians defined on the line and produce results pertaining to the dimensionality of the algebra of Noether symmetries and compare and contrast these with similar results on the algebra of Lie symmetries of the corresponding Euler-Lagrange equations. It is proved that the maximum dimension of the Noether point symmetry algebra of a particle Lagrangian is five whereas it is known that the maximum dimension of the Lie algebra of the corresponding scalar second-order Euler-Lagrange equation is eight. Moreover, we show that a particle Lagrangian does not admit a maximal four-dimensional Noether point symmetry algebra and consequently a particle Lagrangian admits the maximal $r \in \{0, 1, 2, 3, 5\}$-dimensional Noether point symmetry algebra.

It is well known that an important means of analyzing differential equations lies in the knowledge of the first integrals of the equation. We deliver an algorithm for finding first integrals of partial differential equations and show how some of the symmetry properties of the first integrals help to 'further' reduce the order of the equations and sometimes completely solve the equations.

Finally, we discuss some open questions. These include the inverse problem and classification of partial differential equations. Also, there is the question of the extension of the results to 'higher' dimensions.

DECLARATION

I declare that the contents of this thesis are original except where due references have been made. It has not been submitted before for any degree to any other institution.

A H Kara

DEDICATION

To Ma

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My sincere gratitude to my supervisor, Fazal Mahomed, for introducing me to the field of Lie analysis of differential equations. It is through him that I 'saw much further' and had the opportunity of meeting many people from all over the world. He had the unenviable task of guiding me.

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CONTENTS

Chapter

INTRODUCTION

Since the publication of M. S. Lie's works (see, e.g., Lie 1891, Lie 1893 and Lie 1894) on the theory and applications of continuous groups to differential equations, a variety of avenues in the vast field of differential equations has emerged. Many of these avenues are becoming well-known and a continually increasing appeal to the results obtained are being made. These results have dealt with, inter alia, the classification and alternate methods of solutions of differential equations. Some classical works on the subject have been contributed by Dickson (1924), Ovsiannikov (1982), Ibragimov (1985), Olver (1986) and Bluman and Kumei (1989) to name just a few. Moreover, a great deal has been done by Mahomed (1986), (1989) on the classification of ordinary differential equations — some applications and extensions of these works, it is intended, will be made in the sequel. A summary and history of Lie's contribution is delivered in the references cited.

Another prominent field which is indispensable in the area of mathematical physics is the calculus of variations. A great deal has been done in this field and a variety of references are cited in the works of Sarlet and Cantrijn (1981) and Lovelock and Rund (1975). For our purposes, it is the celebrated paper of E. Noether (1918) that is most significant. Therein, Noether showed how the symmetries of a Lagrangian (more appropriately, the symmetries of an action integral) leads to conservation laws of the corresponding Euler-Lagrange equation(s). This alternate consideration of symmetries associated with a Lagrangian for a given Euler-Lagrange equation(s) has, as in the Lie case, led to new ways of studying differential equations and, in general, invariances in mathematical physics.

The kind of questions that were asked in the Lie analysis of differential equations can accordingly be rephrased in the *Noether symmetric* analysis of the said equations.

For example, to what extent can we classify Lagrangians that give rise to differential equations in the sense of utilising the symmetries of the respective Lagrangians? In short, what are the parallels, if any, between the Lie and Noether analysis of differential equations? If these parallels exist, do there exist the possibilty of relating the results in the two alternatives? For example, how does the Noether algebra relate to the algebra of Lie symmetries of the corresponding Euler-Lagrange equation in the sense of dimensionality.

Finally, and perhaps more importantly, one would be interested in the practical implications of the results. To this end, the theory will be applied to a wide range of examples involving some of the important equations in mathematical physics.

It is clear that such an undertaking is vast. In the sequel, an attempt will be made at addressing these questions, at times in a general or universal way and at other times with reference to particular cases, for example, to ordinary differential equations and not to partial differential equations and vice versa, or at times to a particular equation of importance.

In chapter 1, we present the necessary symbols and definitions and recall some relevant results that will be required in the subsequent chapters. Also, a brief survey on Lagrangians, construction of these and Euler-Lagrange equations will be made. Below, we present an outline of the second and subsequent chapters.

In chapter 2 we deal with second-order equations $E(t,q,\dot{q},\ddot{q}) = 0$ defined on a domain in the plane. Lie geometrically proved that the maximum dimension of its point symmetry algebra is eight. He showed that the maximum is attained for the simplest equation $\ddot{q} = 0$ and this was later shown to correspond to the Lie algebra $sl(3,R)$. An algebraic proof of Lie's counting theorem is presented. Furthermore, the Noether counting theorem (that the maximum dimension of the Noether algebra of a particle Lagrangian is five and corresponds to $A_{5,40}$) is proved. Then it is shown that a particle Lagrangian cannot admit a maximal four-dimensional Noether point symmetry algebra. Consequently, it is shown that a particle Lagrangian admits the maximal $r \in \{0, 1, 2, 3, 5\}$-dimensional Noether point symmetry algebra.

In chapter 3 a detailed analysis of the algebraic structure of first-order Lagrangians defined on the line is carried out. This leads to a need for the *best* Lagrangian. Definitions in this regard are made and the best Lagrangians are determined accordingly. In particular, the notion of s^r-equivalent Lagrangians is introduced.

Two examples of s^r-equivalent Lagrangians are considered in chapter 4. Here we investigate the non-linear equation $\ddot{q} + p(t)\dot{q} + r(t)q = \mu\dot{q}^2 q^{-1} + s(t)q^n$, $\mu \neq 1$. It is linearizable via a point transformation if and only if $n = \mu$ or $n = 1$. A quadratic Lagrangian $L = \frac{1}{2}U(t,q)\dot{q}^2 + V(t,q)$ is constructed. By means of a point transformation $Q = F(t,q)$ and $T = G(t,q)$, L is mapped to the simple and completely integrable Lagrangian $\bar{L} = \frac{1}{2}Q'^2$. For $n = 4\mu - 3$, the equation generates the $sl(2,R)$ algebra. Again, by a similar procedure, the quadratic Lagrangian is mapped to a representative Lagrangian $L = Q'/T + 1/(2TQ')$. Complete solutions are obtained in both cases.

In chapter 5, we continue with the classification of particle Lagrangians that give rise to second-order ordinary differential equations. Together, the chapters 3 and 5 completely classify first-order Lagrangians defined on the line.

The importance of first integrals in the analysis of differential equations is well known. However, not much has been done in the systematic determination of these for partial differential equations. In chapter 6, this is attempted by 'associating' Lie point symmetries with the first integral of the equation. Furthermore, it is shown that some proven results on symmetries associated with integrals of systems of second-order ordinary Euler-Lagrange equations have significant, previously unknown applications. A wide range of examples are considered. Moreover, it is shown that in contrast to first integrals of second-order ordinary Euler-Lagrange equations, the first integrals of (systems of) second-order partial Euler-Lagrange equations are not invariant under the prolonged Noether point symmetry.

In the conclusion, amongst other things, an attempt is made at indicating some of the open questions that may be explored.

1. INVERSE PROBLEMS AND INVARIANCE IN THE CALCULUS OF VARIATIONS

1.1. Introduction

The definitions and theory in the Lie formulation of groups and differential equations are elaborately discussed in Ovsiannikov (1982), Ibragimov (1985), Olver (1986), Bluman and Kumei (1989) and many references cited therein. For our purpose a sufficient account is given in Mahomed (1989), from which most of the notation, definitions and results are adopted.

In this chapter the Inverse Problem in the Calculus of Variations, the invariance of the funtional that gives rise to the Euler-Lagrange equations and their associated conservation laws or first integrals are briefly reviewed. Our main aim here is to provide important results which we shall use in the following chapters. Thus, we exclusively review scalar equations. However, most of the results apply to systems as well. We also introduce Definition 1.4 on Alternative Lagrangians.

1.2. Euler-Lagrange equations and the Inverse problem

Before defining our main problem in the Calculus of Variations, we introduce, briefly, the notions of multi-indices and total derivatives (see e.g., Olver 1986).

Definition 1.1. Given a real-valued function $q(t)$, $t \in \Omega \subset \mathbb{R}^m$, the k-th order partial derivative of q is defined as

$$q_J = \frac{\partial^k q}{\partial t_{j_1} \cdots \partial t_{j_k}},$$

where $J = (j_1, \ldots, j_k)$, $1 \leq j_k \leq m$. Also, the j-th *total derivative* operator is

$$D_j = \frac{\partial}{\partial t_j} + \sum_J q_{J,j} \frac{\partial}{\partial q_J},$$

where $q_{J,j} = \dfrac{\partial^{k+1} q}{\partial t_j \partial t_{j_1} \cdots \partial t_{j_k}}$ and the $\sum$ spans the multi indices of J. The J-th total derivative is given by

$$D_J = D_{j_1} \cdots D_{j_k}.$$

Now consider a *variational problem* where the extremal of the functional

$$\int L(t, q^{[n]}(t))dt \tag{1.1}$$

is desired, in which $q^{[n]}(t)$ is the tuple which contains all the derivatives of q up to order n. The solution of the above problem is related to the solutions of differential equations by way of the Euler-Lagrange operator, $\mathcal{E}$, defined below.

Definition 1.2. The ***Euler-Lagrange operator*** is defined by

$$\mathcal{E} = \sum_J (-D)_J \frac{\partial}{\partial q_J}, \tag{1.2}$$

where the $\sum$ spans the multi-indices J defined above.

Henceforth, the integrand of the funtional, L, will be referred to as the *Lagrangian.* When $m = 1$, the Euler-Lagrange equation is an ordinary differential equation of order at most $2n$.

As an example, the case $m = 2$ and $n = 1$ yields the partial differential equation

$$D_t L_{q_t} + D_x L_{q_x} - L_q = 0, \tag{1.3}$$

which in expanded form is

$$L_{q_t q_t} q_{tt} + 2L_{q_t q_x} q_{tx} + L_{q_x q_x} q_{xx} + L_{q_t q} q_t + L_{q_x q} q_x + L_{q_t t} + L_{q_x x} - L_q = 0.$$

The reader is also referred to Douglas (1941) and, recently, Olver (1986). The above discussion can be extended to higher-dimensional variational problems whose Euler-Lagrange equations form a system of differential equations — some cases of these can be found in Courant and Hibert (1953).

Theorem 1.1. If the function $q(t)$ is an extremum of the functional (1.1), then it is a solution of the Euler-Lagrange equation $\mathcal{E}(L) = 0$.

The theorem is well-known and a proof of it can be found in Courant and Hilbert (1953).

Finally, for the reader interested in the theory of Lagrangian dynamics, the notions of *convexity* and *existence* may be significant. This aspect is discussed in detail in the works of Rockafellar (1970), (1971), (1975) and Clarke (1989). The emphasis in these references, however, is towards Hamiltonian dynamics and the Lagrange-Hamilton correspondence. Furthermore, Rockafellar (1993) describes recent views on *subgradients* in which a generalization of the Euler-Lagrange equations is sought.

The *Inverse problem* in the Calculus of Variations involves determining a Lagrangian, L, corresponding to a given differential equation (if it exists). That is, given a differential equation $E(t, q^{[n]}) = 0$, find a Lagrangian, L, such that the Euler-Lagrange equation $\mathcal{E}(L) = 0$ is equivalent to the given differential equation. A stronger requirement of the inverse problem amounts to determining L such that $\mathcal{E}(L) = E$.

Olver (1986) cites the vast amount of work done on the inverse problem and the difficulties involved. Recently, active research has been done on the stronger case. For example, in the work of Olver and Shakiban (1978) and Vinogradov (1978).

In this regard it is worth noting the work of Anderson and Duchamp (1984) wherein the Lagrangians of partial differential equations of the form

$$E(t, q^{[2]}) = E_1^{ij}(t, q^{[1]})q_{i,j} + E_2(t, q^{[1]}),$$

where E_1 and E_2 are smooth functions in the arguments and the repeated indices imply summation, are discussed. The conditions under which these partial differential equations are derivable from some Lagrangian are given and, moreover, formulae for determining these are stated. A further significance of this work lies in the determination of ***variational integrating factors*** corresponding to some such partial differential equations which, at first, are not Euler-Lagrange equations. Under certain conditions one can calculate a variational integrating factor which if multiplied to the partial differential equation yields an Euler-Lagrange equation equivalent to the partial differential equation.

As an example, using the method of Anderson and Duchamp, the equation

$$E = q_{xt} + \frac{1}{2}(q_x q_t - 1) = 0$$

is not in the form of an Euler-Lagrange equation but has a variational integrating factor e^q. The equation $e^q E = 0$ has second-order Lagrangian

$$L_1 = -e^q(q_{xt} + \frac{1}{2}q_x q_t - \frac{1}{2}) - \frac{1}{2} + \frac{e^q}{q}(q_{xt} + q_x q_t) - \frac{q_{xt}}{q} - \frac{e^q q_x q_t}{q^2} + \frac{q_x q_t}{q^2}.$$

By subtracting a total derivative from L_1, the Lagrangian $L_2 = \frac{1}{2}e^q(q_x q_t + 1)$, of first order, is also a Lagrangian of the equation $e^q E = 0$.

Of particular importance in the above cited reference are Theorem 6.1 and Tables 1 and 2 in that they describe the variational principles of the equation

$$E_1 q_{xx} + E_2 q_{xt} + E_3 q_{tt} = 0,$$

where the coefficients are quadratic in q_x and q_t.

More recently, Hojman et al (1992) give an elaborate account of Lagrangians of any order for ordinary differential equations. The Lagrangians obtained are products of the form of the differential operator and a factor. Consequently, the order of the Lagrangian is that of the original equation. The paper also deals with the inverse problem and the invariances associated with the Lagrangian. It is this aspect which will be studied in detail in a different way in the following chapters.

1.3. Invariance of the Lagrangian

In this section we summarize the more important aspects of the invariance of the functional (1.1) or, equivalently, the symmetries associated with the respective Lagrangian L (in short, we shall say symmetries of the Lagrangian). There will also be a brief look at conservation laws. Some of the details and proofs have been omitted and can be found, for example, in Douglas (1941) or Olver (1986).

For real valued $q(t)$, $t \in \Omega \subset \mathbb{R}^m$, we may induce a change of variables

$$T = G(t, q)$$

and

$$Q = F(t, q).$$

A change of variables in the derivative obtained by prolongation is then, also, induced so that the functional (1.1) is then transformed into a corresponding functional $\int_{\tilde{\Omega}} \tilde{L}(T, Q^{[n]}(T))dT$, where $\tilde{\Omega} = \{T = G(t, q), t \in \Omega\}$. This will be the case if

$$L(t, q^{[n]}(t)) = \bar{L}(T, Q^{[n]}(T)) \det \mathcal{J}(t, q^{[1]}(t)) + \operatorname{Div} f, \tag{1.4}$$

by means of the prescribed change of variables, where f, the gauge term, is an m-tuple of functions of t and q, $\operatorname{Div} f = \sum_{i=1}^{m} D_i f_i$ and $\mathcal{J}$ is the Jacobian matrix with entries

$$\mathcal{J}^{ij}(t, q^{[1]}(t)) = \frac{\partial}{\partial t_j} G_i(t, q(t)) = D_j G_i(t, q(t)), \quad i, j = 1, \ldots, m.$$

The two Lagrangians, L and $\bar{L}$, in (1.4) are said to be *equivalent upto gauge*. This means that the Euler-Lagrange equations associated with L and $\bar{L}$ are equivalent to each other by the said change of variables. For strict invariance we require $f = 0$ in (1.4).

The *point* symmetries associated with a Lagrangian, $L(t, q^{[n]}(t))$, takes the form

$$G = \sum_{i=1}^{m} \xi_i(t, q) \frac{\partial}{\partial t_i} + \eta(t, q) \frac{\partial}{\partial q}, \tag{1.5}$$

defined on the space $\mathbb{R}^{m+1}$. On the prolonged space, i.e. $(t, q^{[n]})$-space, the n-th prolongation or extension of G is

$$G^{[n]} = G + \zeta_i \frac{\partial}{\partial q_i} + \cdots + \zeta_{i_1 \cdots i_n} \frac{\partial}{\partial q_{i_1 \cdots i_n}}, \tag{1.6}$$

where $\zeta_i = D_i(\eta) - q_s D_i(\xi_s)$ and $\zeta_{i_1 \cdots i_n} = D_{i_n}(\zeta_{i_1 \cdots i_{n-1}}) - q_{i_1 \cdots i_{n-1}, s} D_{i_n}(\xi_s)$. The following theorem provides a mechanism for determining the symmetries G associated with a Lagrangian L.

Theorem 1.2. G, as in (1.5) is a point symmetry of a Lagrangian L if and only if

$$G^{[n]} L + L \operatorname{Div} \xi = \operatorname{Div} f, \tag{1.7}$$

where $\operatorname{Div}\xi = \sum_{i=1}^{m} D_i\xi_i$ and $f(t,q)$ is the *gauge* term of the form $(f_1, \cdots, f_m)$.

In the absence of the gauge term f, G will be referred to as a *variational* symmetry otherwise we refer to it as a *Noether* symmetry. An important question is: How are these symmetries related to the symmetries of the corresponding Euler-Lagrange equations? To this end, we state the following theorem.

Theorem 1.3. If G is a point symmetry of L then G is a point symmetry of the Euler-Lagrange equation $\mathcal{E}(L) = 0$.

Note that the converse of the theorem is not true.

1.4. Noether's Theorem

In this section we take a brief look at a particular version of Noether's theorem (Noether 1918) dealing with the relationship between symmetries and conservation laws. This will not be a generalized version of the Noether theorem since the variations, here, do not depend on the derivatives. This is often referred to as the classical Noether theorem. An excellent account of the generalization of the classical Noether theorem in mechanics is given by Sarlet and Cantrijn (1981).

Definition 1.3. A *conservation law* of the equation $E(t, q^{[n]}) = 0$ is a tuple $I = (I_1, \cdots, I_m)$, $I_j = I_j(t, q^{[n-1]})$, which satisfies the divergence equation

$$\operatorname{Div} I = 0. \tag{1.8}$$

We frequently refer to a conservation law as a constant of the motion or first integral. Sometimes it is also referred to as an exact invariant.

Note: A *trivial* conservation law, I, is one that satisfies $\operatorname{Div} I = 0$ regardless of it being a solution of the partial differential equation. A more detailed account of trivial conservation laws is given in Olver (1986).

Theorem 1.4. Suppose G is a Noether symmetry of the Lagrangian $L(t, q^{[n]})$, $t \in \Omega \subset \mathbb{R}^m$ and q real-valued. Define the characteristic function of G by

$$C_G = \eta - \sum_{i=1}^{m} \xi_i q_i.$$

Then C_G is a characteristic of a conservation law of the Euler-Lagrange equation $\mathcal{E}(L) = 0$, i.e., there exists an m-tuple $I = (I_1, \cdots, I_m)$ such that

$$\operatorname{Div} I = C_G \mathcal{E}(L) \tag{1.9}$$

and I is a conservation law of $\mathcal{E}(L) = 0$.

(We note that

$$I = f - P - L\xi, \tag{1.10}$$

where f is the gauge term, $P = (P_1, \cdots, P_m)$ and $L\xi = (L\xi_1, \cdots, L\xi_m)$. For an explicit form of each P_j see Olver 1986).

Proof. We may rewrite (1.7) as

$$\operatorname{Div} f = G_C^{[n]}(L) + \sum_{i=1}^{m} \xi_i D_i L + L \sum_{i=1}^{m} D_i \xi_i,$$

where $G_C^{[n]} = \sum_J D_J C_G \partial/\partial q_J$. Hence

$$\operatorname{Div} f = G_C^{[n]}(L) + \operatorname{Div}(L\xi).$$

Using integration by parts we obtain

$$\operatorname{Div} f + G_C^{[n]}(L) = \sum_J D_J C_G \frac{\partial L}{\partial q_J}$$

$$= C_G \mathcal{E}(L) + \operatorname{Div} P.$$

Therefore, $\mathrm{Div}\, f - \mathrm{Div}\,(P + \xi L) = C_G \mathcal{E}(L)$ and the result follows. $\diamond$

The significance of the above theorem for the one-dimensional case is discussed in chapter 2 and, indeed, applies to the other chapters as well. We state the following result (see, e.g., Olver 1986).

Theorem 1.5. The Noether or variational symmetries of a Lagrangian form a Lie algebra.

1.5. Alternative Lagrangians and invariance

In this section, we introduce concepts which take cognisance of the ambiguities arising in the Lagrangian description of a given differential equation or system. Although the definition of alternative Lagrangians given below (Definition 1.4) is generally applicable, not much work has been done in the characterization of these for partial differential equations. Our analysis in this section is mainly concerned with one-dimensional particle dynamics — the results of which will be used in subsequent chapters.

Consider the inverse problem for the free particle equation for one-dimensional particle dynamics. It is easy to verify that the equation $\ddot{q} = 0$ has the following Lagrangians (here, $\mathcal{E}$ is as defined in the expanded form (1.2) with a single independent variable). In each case we list the corresponding Noether point symmetries. These are obtained by invoking Theorem 1.2.

$L_1 = \frac{1}{2}\dot{q}^2$:

$$G_1 = \frac{\partial}{\partial t} \qquad G_2 = t\frac{\partial}{\partial t} + \frac{1}{2}q\frac{\partial}{\partial q},$$

$$G_3 = t^2\frac{\partial}{\partial t} + tq\frac{\partial}{\partial q} \qquad G_4 = \frac{\partial}{\partial q},$$

$$G_5 = t\frac{\partial}{\partial q}.$$

$L_2 = \exp(-\dot{q})$:

$$G_1 = \frac{\partial}{\partial t} \qquad G_2 = t\frac{\partial}{\partial t} + (q+t)\frac{\partial}{\partial q},$$

$$G_3 = \frac{\partial}{\partial q}.$$

$L_3 = -\frac{1}{2}\dot{q}^2 + q\dot{q}\ln\dot{q} - q\dot{q}$:

$$G = t\frac{\partial}{\partial t}.$$

We note the non-uniqueness of the Lagrangians of the free particle equation. The symmetry properties differ significantly from one another. We shall look at this in greater detail in the following chapters.

For every one-dimensional system, the inverse problem guarantees the existence of a Lagrangian although the construction of such a Lagrangian can be highly non-trivial despite there being infinitely many. The above Lagrangians are alternative Lagrangians since the Euler-Lagrange equation corresponding to each imply any one of the other. Formally, we introduce the following definition.

Definition 1.4. Two Lagrangians L and $\tilde{L}$ are said to be *alternative* Lagrangians for a given differential equation if their respective Euler-Lagrange equations imply each other, i.e., $\mathcal{E}(L) = 0$ if and only if $\mathcal{E}(\tilde{L}) = 0$.

We now state some results pertaining to alternative Lagrangians for one-dimensional particle systems.

Theorem 1.6. If L and $\tilde{L}$ are alternative Lagrangians for a given scalar second-order ordinary differential equation, then

$$\mathcal{E}(\tilde{L}) = I(t,q,\dot{q})\mathcal{E}(L),$$

where

$$I(t,q,\dot{q}) = \tilde{L}_{\dot{q}\dot{q}}/L_{\dot{q}\dot{q}}$$

is a conservation law. Furthermore, $\tilde{L}$ is unique up to gauge.

The proof of this theorem can be found in Currie and Saletan (1966).

We now state a consequence of a significant result from the work of Sarlet (1981).

Theorem 1.7. Suppose that $G = \xi\partial/\partial t + \eta\partial/\partial q$ is a point symmetry of a given second-order equation and $I(t,q,\dot{q})$ its conservation law with the property that $G^{[1]}I = 0$. Then there exists a Lagrangian, $L(t,q,\dot{q})$, such that G becomes a Noether symmetry with respect to L and I its Noether invariant.

The significance of Theorem 1.7 will become apparent in chapters 2 and 3 when the classification of alternative Lagrangians associated with a given scalar second-order ordinary differential equation is studied.

The Lagrangian L in Theorem 1.7 can be determined in one of the following ways.

For one-dimensional particle dynamics, (1.10) can be written as

$$I(t,q,\dot{q}) = f(t,q) - [L\xi + (\eta - \dot{q}\xi)\frac{\partial L}{\partial \dot{q}}]. \tag{1.11}$$

Then,

$$\frac{\partial I}{\partial \dot{q}} = -\frac{\partial L}{\partial \dot{q}}\xi - \frac{\partial^2 L}{\partial \dot{q}^2}(\eta - \dot{q}\xi) - \frac{\partial L}{\partial \dot{q}}(-\xi),$$

from which L is obtained by solving

$$\frac{\partial^2 L}{\partial \dot{q}^2} = -\frac{1}{\eta - \dot{q}\xi}\frac{\partial I}{\partial \dot{q}}.$$

Alternatively, we note that (1.11) is a partial differential equation in L. Rewrite it as

$$\frac{\partial L}{\partial \dot{q}} + \frac{\xi}{\eta - \dot{q}\xi}L = -\frac{I - f}{\eta - \dot{q}\xi}. \tag{1.12}$$

The integrating factor of (1.12) is $-1/(\eta - \dot{q}\xi)$ and (1.12) integrates to

$$L = \frac{f}{\xi} - (\eta - \dot{q}\xi)f_1(t,q) - (\eta - \dot{q}\xi)\int \frac{I}{(\eta - \dot{q}\xi)^2}d\dot{q},$$

where f_1 is an arbitrary function of t and q.

In the next chapter, we investigate the dimensionality of the Lie algebra of variational and Noether symmetries.

2. LIE AND NOETHER COUNTING THEOREMS

2.1. Introduction

In this chapter, amongst other things, we raise and answer a few fundamental questions. Firstly, is the following statement true? The full real Lie algebra of point symmetries of any scalar second-order ordinary differential equation is a subalgebra of $sl(3,R)$. An incomplete proof is given in Mahomed (1989). In order to prove this we require the result that the maximum dimension of the Lie algebra of point symmetries of a scalar second-order equation is eight and corresponds to $sl(3,R)$. This is popularly known as the Lie 'counting' theorem (Lie 1891). There are various proofs of this famous result (see, e.g. Dickson 1924, Anderson and Davison 1974 and González-Gascón & González-López 1983). We give an algebraic proof of the Theorem in section 2.2. Let us now return to our first question. We shall show (in section 2.2) that the full real Lie algebra of point symmetries indeed is a subalgebra of $sl(3,R)$ and we give a physical interpretation of this result. This can be linked to the symmetry breaking axiom of physics. It is interesting to note that Hermann (1976) believed that the above statement was implicitly assumed by Lie (1891) to be true (the reader is referred to Hermann 1976 for comments in this regard) in his attempt to construct a general local structure theory for second-order equations admitting groups of point symmetries. However, recently Ibragimov (1992) reviews the complex Lie algebraic classification of Lie and provides further insights into the algebraic structure of scalar second-order ordinary differential equations. Moreover, there have been studies of the Lie algebraic properties of nth ($n \geq 3$) order ordinary differential equations (see Krause and Michel 1988 and Mahomed and Leach 1990). The maximum dimension of the point symmetry algebra for higher ($n \geq 3$) order equations is $n+4$ (Lie 1893) and corresponds to $nA_1 \oplus_s gl(2,R)$ (see Krause and

Michel 1988 and Mahomed and Leach 1990). It is relevant to mention here that the full real Lie algebra of a higher order ($n \geq 3$) equation need not be a subalgebra of its maximum algebra, $nA_1 \oplus_s gl(2, R)$ (see Mahomed and Leach 1990). Thus, there is a marked difference between second and higher order equations. Secondly, we ask the question: what is the maximum dimension of the Noether point symmetry algebra (Lie algebra of Noether point symmetries) for a particle Langrangian? The answer, one would immediately get from those familiar with the classical Noether theorem for particle Lagrangians, is five. However, we have not been able to find an explicit proof of this (which we shall refer to as the Noether 'counting' theorem in analogy to the Lie 'counting' theorem) in the literature. In section 2.3 we prove that the maximum dimension of the Noether algebra is five and is attained for the simplest equation $\ddot{q} = 0$ with usual Lagrangian $L = \dot{q}^2/2$. The corresponding Lie algebra is shown to be $A_{5,40}$ using the nomenclature employed in Patera and Winternitz (1977). The knowledge of the maximum Noether algebra is necessary for the classification of first order particle Lagrangians on the line in terms of their Noether point symmetry algebras (see chapter 3).

We also prove in section 2.3 that there does not exist a maximal four-dimensional point symmetry algebra for a first-order particle Lagrangian on the line. We conclude that a first-order particle Lagrangian on the line admits the maximal $r \in \{0, 1, 2, 3, 5\}$-dimensional Noether point symmetry algebra.

Unlike for the full Lie algebra, the full Noether algebra of a first-order particle Lagrangian is not in general a subalgebra of the maximum Noether algebra $A_{5,40}$ of symmetries. We illustrate this fact in section 2.3.

Most of the work reported here can be found in Mahomed, Kara and Leach (1993a), (1993b).

2.2. Symmetry breaking and Lie's 'counting' theorem

We adopt the Mubarakzyanov classification scheme of real low-dimensional real Lie algebras given in Patera and Winternitz (1977). The canonical realizations of all the necessary three dimensional Lie algebras in terms of vector fields in the plane derived in Mahomed and Leach (1989a) are tabulated below for facilitating the proofs of the theorems that are to follow.

Table 2.1

Let $\partial/\partial t = p$ and $\partial/\partial q = r$. Then

$A_{3,1}$			
	r	p	tr
$A_{3,2}$			
$\mathcal{L}^{I}$	r	p	$tp + (t+q)r$
$\mathcal{L}^{II}$	r	tr	$p + qr$

$A_1 \oplus A_2$ $(a = 0)$, $A_{3,3}$ $(a = 1)$, $A_{3,4}$ $(a = -1)$, $A^{a}_{3,5}$ $(0 < |a| < 1)$ $a \neq 1/2$

$\mathcal{L}^{I}$	p	r	$tp + aqr$
$\mathcal{L}^{II}$	r	tr	$(1-a)tp + qr$

$A_{3,6}$ $(b = 0)$, $A^{b}_{3,7}$ $(b > 0)$

$\mathcal{L}^{I}$	p	r	$(bt+q)p + (bq-t)r$
$\mathcal{L}^{II}$	tr	r	$(1+t^2)p + (tq+bq)r$

$A_{3,8}$

$\mathcal{L}^{II}$	r	$tp + qr$	$2tqp + q^2 r$
$\mathcal{L}^{III}$	r	$tp + qr$	$2tqp + (q^2 - t^2)r$

$A_{3,9}$

$$(1+t^2)p + tqr \qquad tqp + (1+q^2)r \qquad qp - tr$$

Note that we have used $\mathcal{L}$ as a placeholder for the relevant Lie algebra(s) $A^{c\,R}_{r,j}$ ($A^{c\,R}_{r,j}$ is the jth algebra of dimension r with the superscript c, if any, indicating the parameter on which the algebra depends; also superscripts R, if any, indicating the algebra realizations or non-similar algebras). Moreover, we have also written down two sets of realizations for each of the algebras $A_{3,2}$, $A_1 \oplus A_2$, $A_{3,3}$ $A_{3,4}$ and $A^a_{3,5}$ ($0 < |a| < 1$). The algebras $3A_1$ and $A^I_{3,8}$ do not appear since they do not leave invariant any second-order equation!

There exists only one realization of the algebra $A_{3,9}$ ($so(3)$) of the vector fields defined in the plane. Thus far, there has been some confusion regarding the number of realizations of this algebra. The first realization of the algebra $A_{3,9}$ given in Mahomed and Leach (1989a) actually turns out to be $A^I_{3,8}$.

Let us recall the following important theorem which was proved in the above reference.

Theorem i. A second-order dynamical equation does not admit a real maximal $s \in \{4,5,6,7\}$-dimensional point symmetry Lie algebra.

We now give a synthesis of all real equivalence classes of second-order equations (two differential equations are said to belong to the same equivalence class if one can be transformed into the other by means of a (local) diffeomorphism), admitting real Lie algebras of point symmetries.

Second-order equations admitting the richest number (eight, see Theorem 2.1 later) of point symmetries belong to the equivalence class of the free particle equation, $\ddot{q} = 0$ and the corresponding symmetry Lie algebra is $sl(3, R)$.

There are six representatives of real equivalence classes of second-order equations admitting exactly a three dimensional algebra of point symmetries. These are listed in Table 2.2 together with their symmetry algebras. In Mahomed and Leach (1989a) five equivalence classes were given. The equivalence class of the equation associated with $A_{3,9}$ was inadvertently omitted.

Table 2.2

$A^I_{3,2}$ $\qquad \ddot{q} = A\exp(-\dot{q}) \quad \text{or} \quad t\ddot{q} = -1 + A\exp(-\dot{q})$

$A_{3,4}$ $(a = -1)$, $\quad A^a_{3,5} \quad (0 < |a| < 1)\ a \neq 1/2$

$\mathcal{L}^I$ $\qquad \ddot{q} = A\dot{q}^{\frac{a-2}{a-1}} \quad \text{or} \quad \ddot{q} = (a-1)\dot{q} + A\dot{q}^{\frac{2a-1}{a-1}}$

$A_{3,6}$ $(b = 0)$, $\quad A^b_{3,7} \quad (b > 0)$

$\mathcal{L}^I$ $\qquad \ddot{q} = A(1+\dot{q}^2)^{\frac{3}{2}} \times \exp(b\tan^{-1}\dot{q})$

$A_{3,8}$

$\mathcal{L}^{II}$ $\qquad t\ddot{q} = A\dot{q}^3 - \frac{1}{2}\dot{q}$

$\mathcal{L}^{III}$ $\qquad t\ddot{q} = \dot{q}^3 + \dot{q} + A(1+\dot{q}^2)^{\frac{3}{2}}$

$A_{3,9}$

$$\ddot{q} = A\left[\frac{1+q^2-2tq\dot{q}+\dot{q}^2(1+t^2)}{1+t^2+q^2}\right]^{\frac{3}{2}}.$$

We note the absence of the algebras $A_{3,1}$, $A_{3,2}^{II}$, $A_{3,3}$, $A_1 \oplus A_2$, $A_{3,5}^{1/2}$, $A_{3,5}^{a\,II}$, $A_{3,6}^{II}$ and $A_{3,7}^{b\,II}$ and their corresponding differential equations. The reason being that these imply linearization for the associated second-order equation and consequently eight symmetries for each (see Mahomed and Leach 1989b).

All the equations in the above table are linear or linearizable when the constant $A = 0$ (i.e., they possess eight symmetries — the $sl(3,R)$ algebra). Let us also mention that the range of the parameters in the family of algebras $A_{3,5}^a$ and $A_{3,7}^b$ are restricted to avoid double counting. Thus an equation of the form $\ddot{q} = A\dot{q}^{(a-2)/(a-1)}$ for $|a| > 1$ would have the algebra $A_{3,5}^{1/a}$ $(0 < 1/|a| < 1)$. Similarly for $A_{3,7}^b$ $(b > 0)$.

We derive the equation associated with the $A_{3,9}$ algebra since the derivation for this case differs from the others which are straightforward and reported in Mahomed Leach (1989a). This equation has been omitted in the last mentioned reference. Instead the equation given incorrectly there was $\ddot{q} = 0$. Also, see Table 8 in Ibragimov (1992).

As $A_{3,9}$ is a simple algebra, we seek invariance for a second-order equation under two of the symmetry generators. Let $X = (1+t^2)\partial/\partial t + tq\partial/\partial q$ and $Y = q\partial/\partial t - t\partial/\partial q$.

Consider an arbitrary second-order equation $\ddot{q} = E(t, q, \dot{q})$. Invariance of this equation under X gives rise to

$$\ddot{q} = (1+t^2)^{-\frac{3}{2}}\bar{E}(u,v), \tag{2.1}$$

where $u = q(1+t^2)^{-\frac{1}{2}}$ and $v = \dot{q}(1+t^2)^{\frac{1}{2}} - tq(1+t^2)^{-\frac{1}{2}})$. Further invariance under Y results in the following dual conditions on $\bar{E}$,

$$\begin{aligned} -3v\bar{E} + (1+v^2+u^2)\frac{\partial \bar{E}}{\partial v} &= 0, \\ (1+u^2)\frac{\partial \bar{E}}{\partial u} + uv\frac{\partial \bar{E}}{\partial v} &= 0. \end{aligned} \tag{2.2}$$

The solution of this system is $\bar{E} = A(1+v^2(1+u^2)^{-1})^{\frac{3}{2}}$, whence our equation for $A_{3,9}$ follows.

We now discuss the case of equations possessing a two dimensional algebra of point symmetries. Second-order equations possessing exactly two point symmetries belong to either of the equivalence classes $\ddot{q} = \bar{E}(\dot{q})$ or $t\ddot{q} = \bar{E}(\dot{q})$, where E is not a polynomial which is at most cubic in $\dot{q}$ and it is not of the form given in Table 2.2 and $\bar{E}$ is not linear in $\dot{q}$ and neither is $\bar{E}$ contained in the form $t\ddot{q} = a\dot{q}^3 + b\dot{q}^2 + (1 + b^2/3a)\dot{q} + b/3a + b^3/27a^2$, where $a(\neq 0)$ and b are constants, (see Mahomed and Leach 1989b) nor in the form given in Table 2.2.

If an equation $\ddot{q} = H(t,q,\dot{q})$ admits exactly the one-dimensional algebra A_1, then it can be reduced to an equation which is in a coordinate independent form by means of a diffeomorphism which brings the symmetry to a generator of coordinate translation. Thus, an equation with a single symmetry belongs to the equivalence class of $\ddot{q} = H(t,\dot{q})$, where $\partial/\partial q$ is the canonical realization of the algebra A_1 and H is a definite function of t and $\dot{q}$.

There are no equivalence classes of second-order equations that admit more than an eight dimensional symmetry algebra. Lie (1891) geometrically proved that the maximum dimension of the point symmetry algebra of a scalar second-order ordinary differential equation is eight. This is usually referred to as Lie's 'counting' theorem. We present a Lie algebraic proof of this theorem which has enjoyed considerable interest over the years (see e.g., Dickson 1924, Anderson and Davison 1974, Ovsiannikov 1982 and González-Gascón & González-López 1983).

Theorem 2.1. The maximum dimension of the real Lie algebra of point symmetries of a scalar second-order equation is eight and the maximum corresponds to $sl(3,R)$.

Proof. A scalar second-order differential equation cannot admit a maximal real algebra of dimension greater than eight since a Lie algebra of dimension more than eight contains a three dimensional subalgebra which can only result in either a second-order equation possessing a full Lie algebra of dimension three (equivalence classes of equations listed in Table 2.2) or a full real Lie algebra of dimension eight (equivalence class of the simplest equation, i.e., the linearizable equations) which corresponds to $sl(3,R)$. ◇

Finally, the vector fields corresponding to $A^{II}_{3,8}$ and $A^{III}_{3,8}$ are free particle symmetries (see Mahomed 1989). Thus, we have that all the realizations or non-similar algebras given in Table 2.1 are the symmetries of the free particle equation. We now have the following theorem of which an incomplete proof appears in Mahomed (1989).

Theorem 2.2. The full real Lie algebra of any second-order dynamical equation is a subalgebra of its maximum Lie algebra $sl(3,R)$.

We give a physical implication of this result. Consider a one-dimensional system governed by the dynamical equation $\ddot{q} = E(t,q,\dot{q})$. In an idealized situation ($E = 0$, free particle case), the symmetry algebra is the maximum symmetry algebra $sl(3,R)$. If we perturb the idealized state by an interaction $E(t,q,\dot{q})$ different from zero, say $E = q^2$, then the idealized symmetry is broken and the full Lie algebra of the equation $\ddot{q} = q^2$ is a subalgebra of the algebra of the projective group, $sl(3, R)$. Thus in the application to one-dimensional particle dynamics, viz., $\ddot{q} = E(t, q, \dot{q})$, one in general has symmetry breaking for different interactions E and the symmetry algebra is always a subalgebra of the maximum algebra as indeed is guaranteed by Theorem 2.1. The same does not hold for higher order equations. In order to illustrate this we consider an example. The third-order equation $3\ddot{q}^2 - 2\dot{q}q^{(3)} = 0$ has as its symmetry algebra $sl(2, R) \oplus sl(2, R)$ (see Mahomed and Leach 1990) which is not a subalgebra of the maximum algebra $3A_1 \oplus_s gl(2, R)$ of the idealized case $q^{(3)} = 0$ (the simplest third-order equation).

It is interesting to note that an analogous theorem to Theorem 2.2 does not apply in the case of Noether algebras, i.e., the full Noether algebra need not be a subalgebra of the maximum Noether algebra. This is illustrated in our remark after Theorem 2.5.

2.3. The Noether 'counting' theorem and Lagrangian equivalence

We now focus attention on the classical Noether theorem relating to a one-dimensional dynamical system. We, thus, restate and discuss the relevant equations of chapter 1. For a given particle Langrangian L, the coefficient functions ξ and η of a vector field $G = \xi\partial/\partial t + \eta\partial/\partial q$ that generates a Noether transformation (the transformation which leaves the action functional invariant up to gauge) are found by solving the *Killing* type equation (see Theorem 1.2 with $n = 1$ and also Sarlet and Cantrijn 1981)

$$G^{[1]}L + \xi^{(1)}L = f^{(1)}, \tag{2.3}$$

where f is a suitable function of t, q and $G^{[1]}$ is the first prolongation of the symmetry vector field G and is given by (1.6), i.e., $G^{[1]} = G + (\eta^{(1)} - \dot{q}\xi^{(1)})\partial/\partial\dot{q}$. In the absence of the gauge term or strict invariance, we shall refer to the transformation that leaves the action functional invariant as a variational transformation.

The classical Noether theorem (Thoerem 1.4 for a one-dimensional case) states that to each Noether transformation generated by G, there corresponds a constant of the motion $I(t, q, \dot{q})$ given by (1.11), i.e.,

$$I(t,q,\dot{q}) = f(t,q) - [L\xi + (\eta - \dot{q}\xi)\partial L/\partial\dot{q}]\,. \tag{2.4}$$

This theorem does possess the beauty of having an explicit formula (2.4) for the constants of the motion once the Noether symmetries are known. It establishes a precise correspondence between equivalence classes of point symmetries and constants of the motion. It should, however, be pointed out that such a link is strictly with respect to a fixed Lagrangian description, bearing in mind that there are ambiguities in the possible Lagrangian description of a given second-order dynamical equation as discussed in section 1.5. Therefore, in order to take full advantage of Noether's theorem for a given one-dimensional dynamical system, one needs a suitable Lagrangian that will give rise to the highest number of Noether point symmetries and so the largest number of constants of the motion that is obtainable in this way. Therefore of prime importance is the dimensionality of the Noether point symmetry algebra for a given particle Langrangian.

We at first find out the maximum dimension of the Noether point symmetry algebra for a variational problem whose Euler-Lagrange equation constitutes a scalar second-order dynamical equation. It is instructive to look firstly at the Noether point symmetries of the free particle equation with respect to the *usual* Lagangian $L = \dot{q}^2/2$. Substituting this L into the Killing equation (2.3) and performing simple manipulations we end up with, as is well known, a set of five symmetry vector fields

(or any linear combination thereof)

$$G_1 = \frac{\partial}{\partial t} \quad G_2 = t\frac{\partial}{\partial t} + \frac{1}{2}q\frac{\partial}{\partial q},$$

$$G_3 = t^2\frac{\partial}{\partial t} + tq\frac{\partial}{\partial q} \qquad G_4 = \frac{\partial}{\partial q}, \tag{2.5}$$

$$G_5 = t\frac{\partial}{\partial q}.$$

The corresponding Lie algebra is isomorphic to $A_{5,40}$ which, in a convenient basis $\{e_i : i = 1, 5\}$, has non-zero commutation relations (see Patera and Winternitz 1977)

$[e_1, e_2] = 2e_1$, $[e_1, e_3] = -e_3$, $[e_2, e_3] = 2e_3$, $[e_1, e_4] = e_5$, $[e_2, e_4] = e_4$, $[e_2, e_5] = -e_5$, $[e_3, e_5] = e_4$.

It is now opportune to re-introduce the notion of equivalence (for one-dimensional systems) by means of a point transformation of two particle Lagrangians. Two Lagrangians $L(t, q, \dot{q})$ and $\bar{L}(T, Q, dQ/dT)$ are said to be *equivalent up to gauge* (see (1.4)) if

$$L(t, q, \dot{q}) = \bar{L}(T, Q, dQ/dT)\,\frac{dT}{dt} + \frac{df}{dt}, \tag{2.6}$$

by means of the prescribed coordinate transformation $Q = F(t, q)$, $T = G(t, q)$, where f, the gauge term, is an arbitrary function of t and q. This means that the Euler-Lagrange equations associated with L and $\bar{L}$ are equivalent to each other by the said change of coordinates.

If a second-order equation in q belongs to the equivalence class of the free particle equation $Q'' = 0$ ($' = d/dT$) via the point transformation $Q = F(t, q)$, $T = G(t, q)$, then a Lagrangian for the equation in q is obtained by invoking (2.6) where $\bar{L}(T,Q,Q')$ is the Lagrangian of the transformed equation which in this case is that of the free particle equation. We take the usual Lagrangian $\bar{L} = {Q'}^2/2$. With this Lagrangian, the Noether point symmetry algebra associated with Euler-Lagrange second-order linearizable (via a point transformation) equations is a five-dimensional algebra which is isomorphic to $A_{5,40}$. The corresponding Lie group for this algebra of symmetries is $T_2 \otimes_s SL(2,R)$, i.e., the semidirect product of translations T_2 and the special linear group $SL(2,R)$. However, one should note here that there could be other five-dimensional real algebras admitted by particle Lagrangians. In the proof of Theorem 2.3 below, we implicitly assume that $A_{5,40}$ is the only such algebra and we show this to be the case in Theorem 5.3.

The question then is: what is the maximum dimension of the Noether point symmetry algebra with respect to any Lagrangian? Is it five? The answer is provided by the following theorem. In analogy to the Lie counting theorem, we refer to this theorem as the Noether counting theorem.

Theorem 2.3. The maximum dimension of the Noether point symmetry algebra for Lagrangians in one-dimensional particle dynamics is five.

Proof: The proof is by contradiction. To that end, suppose that the maximum dimension of the Noether algebra is at least six (it certainly cannot be more than eight since the associated Euler-Lagrange equation has symmetry algebra of dimension at most eight (Theorem 2.1) and the Noether algebra is a subalgebra of the full dynamical Lie algebra — see Olver 1986). If the Noether algebra is of dimension at least six then the associated Euler-Lagrange equation admits a Lie algebra of dimension at least six. However, by Theorem i, a second-order ordinary differential equation does not admit a maximal six- or seven-dimensional algebra. Hence the associated second-order equation admits the eight dimensional Lie algebra $sl(3,R)$. This in turn implies that the equation is linearizable by means of a point transformation to the simplest equation $\ddot{q}=0$ (see, eg., Sarlet et al 1987 and Ibragimov 1992). As there are ambiguities in the possible Lagrangian description of a given second-order equation, we in the first instance investigate the case for which $\ddot{q}=0$ has the *usual* Langrangian $L=\dot{q}^2/2$. In view of the discussion preceding the theorem we immediately deduce that the Noether symmetry algebra for the free particle equation with usual Lagrangian is the five-dimensional algebra $A_{5,40}$ (subalgebra of $sl(3,R)$) with corresponding Lie group $T_2 \otimes_s SL(2,R)$. Let us now consider alternative (frequently referred to as ***subordinate***, see Currie and Saletan 1966) Lagrangians for $\ddot{q}=0$. What will be the dimension of the Noether algebra? We consider

$$L' = \int^{\dot{q}} \int^{r} g(s, q-ts)ds\,dr + H\dot{q} + J\,, \tag{2.7}$$

where g is an arbitrary function of the functionally independent constants of the motion of $\ddot{q}=0$, H and J are functions of t and q such that $H_t = J_q$. The alternative Lagrangians (2.7) are constructed by solving $L'_{\dot{q}\dot{q}} = gL_{\dot{q}\dot{q}}$ where g is the constant of the motion and $L'_{\dot{q}\dot{q}}$ and $L_{\dot{q}\dot{q}}$ are the mass tensors, i.e., they are multipliers of $\ddot{q}$ in the Euler-Lagrange equations for L' and L (see Theorem 1.6) respectively. All Lagrangian descriptions of the free particle equation is of the form (2.7). Our goal is to search for the maximum number of Noether point symmetries of (2.7). Thus we need to analyse the Killing equation (2.3) with L replaced by L'. The following strategy is employed. We choose the vector field G to be

$$G = (a_1 + a_2 t + a_3 q + a_4 tq + a_5 t^2)\frac{\partial}{\partial t} + (a_4 q^2 + a_5 tq + a_6 + a_7 q + a_8 t)\frac{\partial}{\partial q}\,, \tag{2.8}$$

where the a_i's are arbitrary constants. In (2.8) we have the most general free particle symmetry, i.e., it is a linear combination of the eight independent symmetries of the free particle equation. It follows that any Noether symmetry(ies) of L' is(are) contained in (2.8). In order that G as in (2.8) be a Noether symmetry of L', the constant of the motion g has to satisfy

$$
\begin{aligned}
&(a_6 - a_1\dot{q} - a_3\dot{q}v + a_4v^2 + a_7v)\partial g/\partial v + (2a_7 - a_2 - 3a_3\dot{q} + 3a_4v)g \\
&\quad + (a_8 + a_7\dot{q} - a_2\dot{q} + a_5v - a_3\dot{q}^2 + a_4\dot{q}v)\partial g/\partial \dot{q} = 0
\end{aligned} \tag{2.9}
$$

where $v = q - t\dot{q}$. Equation (2.9) is obtained by straightforward albeit tedious manipulation invoking (2.3) with G as in (2.8) and L' as in (2.7).

For a given g which determines L', what is the maximum number of Noether symmetries (of the form (2.8))? To answer this we investigate (2.9). In the case when g is a nonzero constant, (2.9) becomes

$$(2a_7 - a_2 - 3a_3\dot{q} + 3a_4v)g = 0$$

whence $a_7 = a_2/2$, $a_3 = 0$ and $a_4 = 0$. Hence L' with g a constant has five symmetries which constitute the symmetry group $T_2 \otimes_s sl(2, R)$. What happens when g is not a constant? What will be the maximum number of Noether symmetries? We answer these by resorting to the classification of vector fields in the plane given in Table 2.1. To this end, without loss of generality (after all we are interested in the case when L' admits the maximum number of symmetries), we assume that L' admits the symmetry $\partial/\partial q$. Then (2.9) becomes

$$
\begin{aligned}
&(a_5 + a_4\dot{q})\frac{dg}{d\dot{q}} + 3a_4g = 0 \\
&(a_8 + a_7\dot{q} - a_2\dot{q} - a_3\dot{q}^2)\frac{dg}{d\dot{q}} + (2a_7 - a_2 - 3a_3\dot{q})g = 0
\end{aligned} \tag{2.10}
$$

which, in effect, also means that L' admits $\partial/\partial t$. Thus L' with g satisfying (2.10) admits two symmetries. Let us further suppose a third symmetry for L'. Then this symmetry together with $\partial/\partial t$ and $\partial/\partial q$ generates a three-dimensional algebra $A^{c,R}_{3,j}$

$j \neq 8, 9$, given in Table 2.1. We list the constants of the motion g for which L' admits exactly three symmetries which generates the solvable algebra ($A \neq 0$ is a constant in each case):

$$
\begin{aligned}
A^I_{3,2} \quad & g = A\exp(-\dot{q}), \\
A^I_{3,4} \quad & g = A\dot{q}^{-3/2}, \\
A^{a\,I}_{3,5}\,(a = \frac{1+p}{2+p}) \quad & g = A\dot{q}^p \quad p \neq 0, -2, -3, \\
A_1 \oplus A^I_2 \quad & g = A\dot{q}^{-1}, \\
A^I_{3,6} \quad & g = A(1+\dot{q}^2)^{-3/2}, \\
A^{b\,I}_{3,7} \quad & g = A(1+\dot{q}^2)^{-3/2}\exp(b\arctan\dot{q}).
\end{aligned}
\tag{2.11}
$$

We now treat the case when L' admits exactly the semisimple algebras $A^{III}_{3,8}$ given in (2.2) and $A_{3,9}$ of Table 2.1. The g's and the corresponding algebras are ($A \neq 0$ is a constant in each case):

$$
\begin{aligned}
A^{III}_{3,8} \quad & g = A(q - t\dot{q} + \dot{q}^2/2)^{-3/2}, \\
A_{3,9} \quad & g = A[1 + (q - t\dot{q})^2 + \dot{q}^2]^{-3/2}.
\end{aligned}
\tag{2.12}
$$

The first integrals g in (2.12) can easily be derived by invoking (2.9) with the generators of $A^{III}_{3,8}$ and $A_{3,9}$ in Table 2.1, respectively. It is simple to verify that $A^{II}_{3,2}$, $A_{3,3}$, $A^{a\,II}_{3,5}$ and $A^{b\,II}_{3,7}$ are not admitted by L'. The remaining algebra realizations $A_{3,1}$, $A^{II}_{3,8}$, $A^{II}_{3,4}$, $A^{1/2\,I}_{3,5}$ and $A^{II}_{3,6}$ imply five symmetries for L' with g a nonzero constant in each case. It follows that L' cannot admit a Noether algebra of dimension greater than five since an algebra of dimension more than five contains a three-dimensional subalgebra which according to the previous argumentation can only imply either exactly three Noether symmetries or five Noether symmetries for

L'. For example, L' cannot admit the six-dimensional simple algebra $so(3,1)$ since it contains $so(3)$ (i.e., $A_{3,9}$) which by (2.12b) can only imply three Noether symmetries and no more. Consequently we deduce that the Noether point symmetry algebra for the free particle equation with alternative Langrangian L' is at most five-dimensional. Thus for all the cases, the dimension of the Noether algebra is at most five. This contradicts our assumption and hence proves the theorem. $\diamond$

The above was an algebraic proof. It is worth pursuing a geometric proof of the Noether counting theorem.

Remark: It is not only when g is a nozero constant that L' admits five Noether symmetries. The same situation arises if

$$g = (b_1\dot{q} + b_2 v + b_3)^{-3} \tag{2.13}$$

where the b_i's are constants and $v = q - t\dot{q}$. It is straightforward to verify (2.13). Indeed, one need only apply the projective transformation (which leaves the free particle equation invariant) to the free particle equation, viz.,

$$(t,q) \mapsto \left(T = \frac{at + bq + c}{\alpha t + \beta q + \gamma},\ Q = \frac{dt + eq + f}{\alpha t + \beta q + \gamma}\right)$$

applied to $Q'' = \mathcal{J}(dt/dT)^3\ddot{q} = 0$ where $' = d/dT$ and $\mathcal{J}$ is the Jacobian. One then finds that the most general coefficient function of $\ddot{q}$ which has functional dependency in $\dot{q}$ and v is of the form (2.13).

We note that in the absence of the gauge term, i.e., in the case of strict invariance of the action functional, the variational algebra is a subalgebra of the Noether algebra. The variational symmetry algebra of the usual Lagrangian $L = \frac{1}{2}\dot{q}^2$ of the free particle equation is the three-dimensional algebra $A^{\frac{1}{2}}_{3,5}$. The corresponding Lie group is $T_2 \otimes_s D$ which is the semidirect product of translations T_2 and dilation D. Hence we have the following corollary.

Corollary 2.1. The maximum dimension of the variational point symmetry algebra for Lagrangians in one-dimensional particle dynamics is three.

In view of the above discussion and corollary, one has to bear in mind that all the symmetries and therefore all the first integrals may not be determined in the absence of a gauge term. This is not to say that they all will be determined using gauge variance since there are alternative Lagrangians for a given problem that may result in fewer (incomplete) symmetries.

Theorem 2.4. There does not exist a maximal four-dimensional Noether point symmetry algebra for one-dimensional particle Lagrangians.

Proof: The proof follows from the proof of Theorem 2.3. From the argumentation there, it can be deduced that L' in (2.7) does not admit exactly a four-dimensional Noether point symmetry algebra. This in turn means that there does not exist a Lagrangian, for a second-order dynamical equation which is linearizable by means of a point transformation, which possesses a maximal four-dimensional algebra. Hence there does not exist a Lagrangian which gives rise to a maximal four-dimensional Noether algebra since its associated Euler-Lagrange equation cannot by Theorem i possess a maximal four-dimensional algebra. $\diamond$

An immediate consequence of the preceeding theorems and related discussions is the following interesting result:

Theorem 2.5. A first-order particle Lagrangian on the line admits a maximal $r \in \{0, 1, 2, 3, 5\}$ dimensional Noether point symmetry algebra.

As a final remark we point out that the full Noether algebra of a particle Lagrangian is not in general a subalgebra of the maximal Noether algebra $A_{5,40}$ of point symmetries. We show this by investigating the subalgebras of $A_{5,40}$. The only three-dimensional subalgebras of $A_{5,40}$ are $A_{3,1}$, $A_{3,5}^{\frac{1}{2}}$ and $A_{3,8}$ ($sl(2,R)$) which in effect means that the other three-dimensional algebras listed in Table 2.1 are not Noether subalgebras.

In chapter 3, we study maximally symmetric Lagrangians and show the connections with the Lie equivalence problem.

3. NOETHER EQUIVALENCE PROBLEM FOR PARTICLE LAGRANGIANS

3.1. Introduction

In particle dynamics one is sometimes confronted with a complicated Lagrangian. If this Lagrangian admits a Noether algebra or is induced into admitting a Noether algebra (see chapter 4 and Kara and Mahomed 1992), then it can be simplified, by means of a diffeomorphism of the plane, to a simpler representative particle Lagrangian. The representative particle Lagrangians for one-dimensional particle systems which admit Noether algebras are obtained in section 3.2. The solution of this problem gives the complete relationship between the Lie equivalence problem and the Noether equivalence problem for one-dimensional particle systems.

We, in fact, show that if a second-order dynamical equation admits the maximal r, where $r \in \{1,2,3\}$, dimensional point symmetry Lie algebra $\mathcal{A}$, then there exists a particle Lagrangian which admits $\mathcal{A}$ as its Noether algebra provided $\mathcal{A}$ is the real algebra $A^I_{3,4}$, $A^I_{3,6}$, $A^{II}_{3,8}$, $A^{III}_{3,8}$ or $A_{3,9}$. Thus, excluding $A^I_{3,2}$, $A^{a\,I}_{3,5}$ $(0 < |a| < 1)$ and $A^{b\,I}_{3,7}$ $(b > 0)$, the dynamical Lie algebras and the Noether algebras are, respectively, isomorphic to each other. However, if a dynamical equation admits $S\mathcal{A} \in \{A^I_{3,2}, A^{a\,I}_{3,5}, A^{b\,I}_{3,7}\}$ as its maximal Lie algebra, then the Noether algebra is at most two-dimensional with respect to any Lagrangian representation, i.e., it is a proper subalgebra of the dynamical algebra $S\mathcal{A}$.

The other situation arises when a dynamical equation admits the eight-dimensional algebra $sl(3,R)$. The Noether algebra of point symmetries for this case is at most the five-dimensional algebra $A_{5,40}$ which is a proper subalgebra of $sl(3,R)$. Here we have that the deficiency in the number of Noether point symmetries is at least three.

The gauge invariance of the action integral $\int_{t_0}^{t_1} L(t, q, \dot{q})dt$ only results in one of 0, 1, 2, 3 or 5 Noether point symmetries (see the previous chapter and also Mahomed, Kara and Leach 1993a). Although this most certainly is a vital and interesting result, one nonetheless still has to contend with the difficulty that a given second-order dynamical equation can always be cast as an Euler-Lagrange equation for an infinitude of Lagrangians all of which need not necessarily having the same number of Noether point symmetries. We have already seen this occurring for the simplest linear second-order equation.

Depending upon the form of the Lagrangian (see, e.g., in the proof of Theorem 2.3) one may end up with exactly one of 0, 1, 2, 3 or 5 Noether point symmetries. For instance, if $L = \dot{q}^2/2$ (termed the *usual* Lagrangian since it is the kinetic energy of the particle), then the Killing equation yields five Noether symmetries and hence Noether's theorem gives five constants of the motion (two of which are functionally independent) whereas with respect to the somewhat *unusual* (not obvious or not dictated by physical consideration) Lagrangian $L = -t\dot{q}^2/2 + q\dot{q}\ln\dot{q} - q\dot{q}$ there arises only one point symmetry and thus one constant of the motion. This is a disturbing feature and the dynamical problem is not completely solved in the latter case. It is not so much that Noether's theorem fails but it is to do with the alternative Lagrangian description for a given dynamical system. Therefore one needs to take extra care in the selection of a Lagrangian, assuming that there is a choice, before subjecting it to Noether's theorem. This is especially true for nonlinear systems. However, it should be pointed out that for many problems of physical interest, linear systems as well as some nonlinear systems, the Lagrangians that arise do result in the maximum number of Noether point symmetries that can arise for the said system. These Lagrangians were termed as *usual* Lagrangians presumably because they were constructed in a straightforward and obvious manner from physical considerations. For our purposes, we refer to a Lagrangian as being *maximally symmetric Lagrangian* if it is of a 'simple' form and gives rise to the maximum number of Noether point symmetries that can occur for any Lagrangian description of the given dynamical system. A *usual* Lagrangian for a given dynamical problem need not be maximally symmetric. For example, the dynamical equation $\ddot{q} = q^2 \exp t$ has *usual* Lagrangian $L = \frac{1}{2}\dot{q}^2 + q^3 \exp t/3$. But this Lagrangian is not maximally symmetric since it results in no Noether symmetries although there does exist a Lagrangian for the equation that has one symmetry.

There is some freedom of what a 'simple' form is and can be dictated by the physical problem at hand, e.g., the Lagrangian could be in terms of the kinetic and potential energies of the particle like $L = \frac{1}{2}A(t,q)\dot{q}^2 + B(t,q)$, where A and B are dependent on the problem at hand (see the next chapter and Kara and Mahomed 1992).

In readiness for the next section we give the following definition.

Definition 3.1. A particle Lagrangian having $r \in \{0,1,2,3,5\}$ Noether point symmetry(ies) is said to be s^r-*equivalent* if it can be transformed via a point transformation by formula (2.6), to a maximally symmetric Lagrangian having $r \in \{0,1,2,3,5\}$ Noether symmetry(ies).

An example of an s^5-equivalent Lagrangian corresponding to the free particle equation $\ddot{q} = 0$ is $L = 1/\dot{q}^3$. This Lagrangian can be mapped into the usual Lagrangian $L = \dot{q}^2/2$ via a point transformation using (2.6). Note that $L = \dot{q}\ln\dot{q} - \dot{q}$ is not s^5-equivalent since it gives rise to three Noether symmetries whereas a maximally symmetric Lagrangian for the same problem (free particle) gives rise to five Noether symmetries. Thus, it is important to have representations of s^r-equivalent, where $r \in \{1,2,3,5\}$, Lagrangians since they reveal the most amount of information about the dynamical problem.

This chapter is a sequel to chapter 2 and the work reported here can be found in Kara, Mahomed and Leach (1993).

3.2. Representative Lagrangians

As pointed out in chapter 1, there has been many contributions to the inverse problem of Lagrangian mechanics and the calculus of variations, i.e., the problem of constructing (if possible) a Lagrangian for a dynamical system. Here we concern ourselves with the construction of representatives of equivalence classes of s^r-equivalent Lagrangians. Two Lagrangians are said to belong to the same equivalence class if one can be transformed into the other by means of a point transformation using formula (2.6).

Consider a one-dimensional real variational problem (i.e., $n = 1$ in (1.1)) in which the extremal of the functional

$$\int_{t_0}^{t_1} L(t, q, \dot{q})\, dt \tag{3.1}$$

is required, where L is a smooth function of its real arguments with $L_{\dot{q}\dot{q}} \neq 0$ in its domain of definition. The associated Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{3.2}$$

which is a second-order equation and can be written in the normal form $\ddot{q} = N(t, q, \dot{q})$ where N is given by $N = (L_q - L_{t\dot{q}} - \dot{q}L_{q\dot{q}})/L_{\dot{q}\dot{q}}$. The second-order Euler-Lagrange equation (3.2) can have maximal 0,1,2,3 or 8 real point symmetry algebras (see Mahomed and Leach 1989a). In chapter 2 and Mahomed, Kara and Leach (1993a), we gave a synthesis of real equivalence classes of dynamical equations that admit point symmetry. In the case of equations which possess no symmetry (0 symmetry case) the Lie algebraic classification does not allow us to write a representative equation and so too we do not have an s^0-equivalent Lagrangian for the zero symmetry case.

We obtain Lagrangian representations of the representative dynamical equations presented in the previous chapter. To be precise, we construct s^r-equivalent Lagrangians for each of the equivalence classes of dynamical equations that admit 1,2,3 or 8 point symmetries. This in effect sets up the connection between the Lagrange equivalence problem with which we are concerned here and the Lie equivalence problem which was dealt with in Mahomed and Leach (1989a). We are in actual fact forcing the solution of our Lagrange equivalence problem from the solution of the Lie equivalence problem. This is facilitated by using the notion of s^r-equivalent Lagrangian since our basic concern is in obtaining first integrals via the Noether classical theorem more than just presenting the complete solution to the Lagrange equivalence problem. The Lagrange equivalence problem is attempted in Kamran and Olver (1989) which utilises the Cartan equivalence method.

We begin with the case when the dynamical equation $\ddot{q} = \mathcal{H}(t, q, \dot{q})$ admits the maximal one-dimensional algebra A_1. As is well known, the canonical form of the equation is

$$\ddot{q} = \mathcal{H}(t, \dot{q}) \tag{3.3}$$

where $\partial/\partial q$ is the canonical realization of A_1 and $\mathcal{H}$ is a definite function of t and $\dot{q}$. An s^1-equivalent Lagrangian for (3.3) is of the form

$$L = \int^{\dot{q}} F(t, x)\, dx \tag{3.4}$$

where $F(t, \dot{q})$ is a first integral of (3.3).

If instead we had an autonomous equation with time-translation $\partial/\partial t$ as its symmetry, then the Lagrangian for such an equation is of the form (Sarlet 1981)

$$L = \dot{q}\int^{\dot{q}} y^{-2}F(q,y)\,dy \tag{3.5}$$

where $F(q,\dot{q})$ is the time-independent first integral of the autonomous equation. Using formula (2.6) one can easily verify that (3.4) and (3.5) are equivalent Lagrangians. Hence (3.5) is also s^1-equivalent.

We now turn our attention to the case when a dynamical equation admits a maximal two-dimensional algebra. The algebras can be either $2A_1$ (Abelian) or A_2 and the canonical form of the equations are

$$\ddot{q} = \mathcal{H}(\dot{q}) \quad \text{or} \quad t\ddot{q} = \mathcal{H}(\dot{q}), \tag{3.6}$$

with restrictions on $\mathcal{H}$ (see Mahomed and Leach 1989a). The canonical realizations of $2A_1$ is $2A_1^I$: $\{\partial/\partial t, \partial/\partial q\}$ (the realization $2A_1^{II}$ yields a linear form) and for A_2 is $A_2^I : \{\partial/\partial q, t\partial/\partial t + q\partial/\partial q\}$ (the realization A_2^{II} yields a linear form). We construct s^2-equivalent Lagrangians for both the forms (3.6a) and (3.6b). In the Euler-Lagrange equation for (3.6a) we can without loss of generality take $\partial L/\partial q = 0$ and then solve the resulting linear first-order partial differential equation which in turn implies that a Lagrangian be of the form

$$L = \int^{\dot{q}}\left(\int^{r}\frac{ds}{\mathcal{H}(s)}\right)dr - t\dot{q}\,. \tag{3.7}$$

More generally, one can write

$$L = \int^{\dot{q}}\left(\int^{r}\frac{ds}{\mathcal{H}(s)}\right)dr + b(t,q)\dot{q} + c(t,q),$$

provided $c_q = b_t + 1$. One has freedom to choose $b = -t$ and $c = 0$ and thus (3.7). We do the same for (3.6b) and a Lagrangian is

$$L = \frac{1}{t}\int^{\dot{q}}\left(\exp\int^{r}\frac{ds}{\mathcal{H}(s)}\right)dr\,. \tag{3.8}$$

Here too one can add $b(t,q)\dot{q}+c(t,q)$ provided $c_q=b_t$. It is easy to verify by using the Killing equation (2.3) that $2A_1^I$ is the Noether symmetry algebra for (3.7) and A_2^I is the Noether symmetry algebra for (3.8). Hence (3.7) and (3.8) are s^2-equivalent Lagrangians and their Noether algebras are the same as their respective dynamical algebras. Applying Noether's theorem to (3.7), with the gauge function f linear in q, we obtain the functionally independent first integrals

$$
\begin{aligned}
I_1 &= \int^{\dot{q}} \frac{dr}{\mathcal{H}(r)} - t \\
I_2 &= q + \int^{\dot{q}} \int^{r} \frac{dw}{\mathcal{H}(w)} dr - \dot{q} \int^{\dot{q}} \frac{dr}{\mathcal{H}(r)}
\end{aligned}
\tag{3.9}
$$

corresponding to the symmetries $\partial/\partial q$ and $\partial/\partial t$ respectively. In a similar manner we obtain the functionally independent first integrals associated with (3.8), viz.,

$$
\begin{aligned}
I_1 &= t^{-1} \exp \int^{\dot{q}} \frac{dr}{\mathcal{H}(r)}, \\
I_2 &= \int^{\dot{q}} (\exp \int^{w} \frac{dr}{\mathcal{H}(r)}) dw + (q - t\dot{q}) t^{-1} \exp \int^{\dot{q}} \frac{dr}{\mathcal{H}(r)}.
\end{aligned}
\tag{3.10}
$$

These correspond to the symmetries $\partial/\partial q$ and $t\partial/\partial t + q\partial/\partial q$. It should be mentioned, however, that the first integrals (3.9) and (3.10) could equally well be derived by solution of the dual conditions $G^{[1]} I(t,q,\dot{q}) = 0$ and $dI(t,q,\dot{q})/dt = 0$ where $G \in 2A_1^I$ $(G \in A_2^I)$. This is without regard to the knowledge of a Lagrangian. The solution of these linear partial differential equations is more difficult or sometimes too difficult to obtain.

The Lagrangian formulation of dynamical equations that admit three-dimensional algebras is a difficult problem to say the least. There are six real representatives of equivalence classes of second-order equations which admit a maximal three-dimensional real algebra (see Mahomed, Kara and Leach 1993a). Moreover, there arises exceptional cases when dealing with the Lagrangian formulation corresponding to the family of algebras $A^{a\,I}_{3,5}$ $(0 < |a| < 1,\ a \neq 1/2)$, $A^{I}_{3,2}$ and $A^{b\,I}_{3,7}$ $(b > 0)$. So we firstly construct an s^r-equivalent, $r \in \{1,2,3\}$, Lagrangian for the canonical equation

$$\ddot{q} = \dot{q}^{\frac{a-2}{a-1}} \quad a \neq 0, 1, \frac{1}{2}, 2 \tag{3.11}$$

which admits $A^{a\,I}_{3,5}$ $(0 < |a| < 1,\ a \neq \frac{1}{2})$ and $A^{I}_{3,4}$ if $a = -1$. We also treat the algebra $A^{I}_{3,4}$ since it arises naturally in this setting. For $|a| > 1$, equation (3.11) admits $A^{1/a\,I}_{3,5}$ $(0 < 1/|a| < 1)$.

We return to a result discussed briefly in chapter 1 where we stated that given a symmetry Y of a second-order dynamical equation, one can find a Lagrangian L for the equation such that Y is a Noether symmetry with respect to L. The symmetry in question could be a generalised symmetry (see Sarlet 1981). However, for our purpose, we restrict to a (prolonged) point symmetry since we are dealing with a finite dimensional Noether algebra of symmetries. A Lagrangian L follows from integrating the equation (see Sarlet 1981 and chapter 1)

$$\frac{\partial^2 L}{\partial \dot{q}^2}(\eta - \dot{q}\xi) = -\frac{\partial I}{\partial \dot{q}}, \tag{3.12}$$

where ξ and η are the coefficient functions of the given symmetry vector $G = \xi(t,q)\partial/\partial t + \eta(t,q)\partial/\partial q$ of the given second-order equation and $I(t,q,\dot{q})$ is a first integral associated with the symmetry G of the equation, i.e., I is obtained by solution of the dual conditions $G^{[1]}(I) \equiv Y(I) = 0$ and $dI/dt = 0$.

We are now in a position to obtain an s^r-equivalent Lagrangian for equation (3.11). We shall study the situation when $a \neq -1$ since (3.11) possesses the algebra $A^{I}_{3,4}$ when $a = -1$ and this requires an allied but separate treatment. In order to construct a Lagrangian utilising (3.12) we need a first integral of (3.11) corresponding to a given symmetry. Since we have at our disposal three symmetries that constitute $A^{a\,I}_{3,5}$, we can select any of the three $\partial/\partial t$, $\partial/\partial q$ or $t\partial/\partial t + aq\partial/\partial q$ or even a linear combination thereof. Let us begin by choosing $X = t\partial/\partial t + aq\partial/\partial q$. Then $\eta - \dot{q}\xi = aq - t\dot{q}$ and the first integral associated with X satisfies

$$\frac{\partial I}{\partial \dot{q}} = \dot{q}^{\frac{2-a}{a-1}}[aq - t\dot{q}]/[t + (1-a)\dot{q}^{\frac{1}{a-1}}]^{a+1}\,. \tag{3.13}$$

Hence invoking (3.12), we deduce that a Lagrangian for (3.11) must be of the form

$$L = -\frac{1}{at}[(1-a) + t\dot{q}^{\frac{-1}{a-1}}]^{1-a} + b(t,q)\dot{q} + c(t,q)\,, \tag{3.14}$$

provided $b_t = c_q$. This Lagrangian has one other Noether symmetry $Z = \partial/\partial q$ for $a \neq -1$ as can easily be verified by the Killing equation (2.3). Thus, in all there are two Noether symmetries of (3.14) when $a \neq -1$. These two symmetries X and Z generates a proper subalgebra of $A^{a\,I}_{3,5}$. Does (3.14) qualify to be s^2-equivalent? Can one find another Lagrangian (not equivalent to (3.14)) representation of (3.11) that possesses a Noether algebra isomorphic to $A^{a\,I}_{3,5}$? The answer to the first question is yes and to the second no. To understand why this is so, let us look for another Lagrangian for (3.11). This time we choose $X' = \partial/\partial t$ as our starting symmetry. Then $\eta - \dot{q}\xi = -\dot{q}$ and the first integral associated with X' satisfies

$$\frac{\partial I}{\partial \dot{q}} = \dot{q}^{\frac{1}{a-1}}\,. \tag{3.15}$$

Now using (3.12), we arrive at a Lagrangian for (3.11) of the form

$$L = \frac{(a-1)^2}{a}\dot{q}^{\frac{a}{a-1}} + b(t,q)\dot{q} + c(t,q)\,, \tag{3.16}$$

provided $c_q = b_t + 1$. The application of Killing's equation (2.3) to (3.16) for $a \neq -1$ leads to one more symmetry $Z' = \partial/\partial q$. Since X' and Z' generate a proper subalgebra of $A^{a\,I}_{3,5}$. We could equally well have started with $\partial/\partial q$. Then one would again have ended up with (3.16) and a further symmetry $\partial/\partial t$. So no matter with which symmetry one begins, the Lagrangian that results always has at most two Noether symmetries. Thus an s^2-equivalent Lagrangian for (3.11) with $a \neq -1$ is of the form (3.14) or (3.16). Note that one has some freedom of choosing b and c in (3.14) or (3.16). This, of course, should be subject to the respective constraint equations.

We now investigate the case when (3.11) admits $A^I_{3,4}$, i.e., when $a = -1$ in (3.11). Therefore we study the equation

$$\ddot{q} = \dot{q}^{\frac{3}{2}}\,. \tag{3.17}$$

A Lagrangian for (3.17) can be obtained from (3.14) or (3.16) by setting $a = -1$. A Lagrangian so derived has Noether algebra isomorphic to the dynamical algebra $A^I_{3,4}$. It is quite staightforward to verify this. Indeed, one need only analyse the forms (3.14) and (3.16) for $a = -1$ and use appropriate choices of b and c. Consider (3.14) with $a = -1$, $b = -t$ and $c = -q - 4/t$. This leads to $L = 4\dot{q}^{\frac{1}{2}} - q$. Similarly (3.16) with $a = -1$, $b = 0$ and $c = q$ results in $L = -4\dot{q}^{\frac{1}{2}} + q$. Therefore we see that the two Lagrangians differ by sign and are trivially equivalent. Hence

$$L = -4\dot{q}^{\frac{1}{2}} + q \tag{3.18}$$

is s^3-equivalent corresponding to (3.17) with Noether algebra $A^I_{3,4}$. One could also use Killing's equation (2.3) on (3.14) or (3.16) for $a = -1$ in order to obtain the Noether algebra $A^I_{3,4}$.

It is quite interesting to note that the value $a = -1$ is special as it gives rise to a Lagrangian description (3.18) which admits a Noether algebra which is isomorphic to the dynamical algebra $A^I_{3,4}$. For other values of a ($a \neq -1$), the Noether algebra is always a proper subalgebra of $A^{a}{}^{I}_{3,5}$ as we have indeed seen for (3.14) and (3.16). One has to exercise caution as we have already encountered $A^{a}{}^{I}_{3,5}$ and $A^{I}_{3,4}$ as Noether algebras for alternative Lagrangians of the free particle equation (see Mahomed, Kara and Leach 1993a and chapter 2). However, the Lagrangians defined there were not s^3-equivalent.

Previously, in chapter 2, the Lagrangians admitting the algebras $A^{I}_{3,4}$ and $A^{a}{}^{I}_{3,5}$ were indistinguishable since they could be incorporated into one. This is also true for the Lie algebra of the dynamical equation (3.11). Now in the sense of s^r-equivalence, the picture is quite different. We no longer have $A^{a}{}^{I}_{3,5}$ but two proper subalgebras, one non-Abelian and one Abelian, both of which are associated with two s^2-equivalent Lagrangians (3.14) and (3.16) respectively. The algebra $A^{I}_{3,4}$ has one corresponding s^3-equivalent Lagrangian (3.18). Clearly the Noether symmetry properties of each of (3.14), (3.16) and (3.18) are distinct.

In order to fully describe the algebraic properties of the system governed by the dynamical equation (3.11) (or its equivalent since (3.11) is a representative of a whole class of equations which can be transformed to it) for $a \neq -1$ using a Lagrangian formulation, one requires two Lagrangians. We have presented two s^2-equivalent Lagrangians (3.14) and (3.16) that together gives the complete symmetry generators of $A^{a}{}^{I}_{3,5}$. Now, one can apply the classical Noether theorem to (3.14) and (3.16), respectively, to obtain all the independent constants of the motion or first integrals associated with all the Noether symmetries. Of course one should bear in mind that for complete integrability the full algebraic description is not necessary, but any two-dimensional subalgebra would suffice and this could be precisely the Noether algebra of (3.14) or, alternatively, (3.16). However, the following points should be borne in mind. Firstly, the two Lagrangians together give rise to an algebra which is isomorphic to the Lie dynamical algebra and this correspondence gives the relationship between the Lie and the Noether algebras associated with the two s^2-equivalent Lagrangians. Secondly, and perhaps more importantly from a physical point of view, the full algebraic description, i.e., the two Lagrangian picture allows for the complete determination of all the physical constants or integrals via Noether's theorem. We shall comment on this further when we deal with the s^5-equivalent Lagrangians of the free particle equations. Invoking the Noether invariant (2.4) with the s^2-equivalent Lagrangian (3.16) with the choices $b = 0$ and $c = q$ we obtain the integrals (the gauge function is linear in t)

$$\begin{aligned} I_1 &= \dot{q}^{\frac{a}{a-1}}(a-1)/a - q\,, \\ I_2 &= t + (1-a)\dot{q}^{\frac{1}{a-1}}, \end{aligned} \tag{3.19}$$

corresponding to the symmetries $\partial/\partial t$ and $\partial/\partial q$ respectively. Likewise for the Lagrangian (3.14) with $b = c = 0$ we obtain, associated with $\partial/\partial q$, an integral which is a function of I_2, viz., I_2^{-a}/a and corresponding to $t\partial/\partial t + aq\partial/\partial q$, the functionally dependent integral

$$I_3 = I_1 I_2^{-a}\,. \tag{3.20}$$

The algebraic properties of the equation (3.17) can be represented completely by the Noether symmetries of the s^3-equivalent Lagrangian (3.18) given by $\partial/\partial t$, $\partial/\partial q$ and $t\partial/\partial t - q\partial/\partial q$ ($A^I_{3,4}$). Hence by Noether's theorem there are three first integrals and they turn out to be precisely (3.19) and (3.20) if we set $a = -1$.

Let us construct s^r-equivalent Lagrangians (there is more than one) for the canonical equation

$$\ddot{q} = \exp(-\dot{q}), \tag{3.21}$$

which admits $A^I_{3,2}$. Once more we call on (3.12). We first of all require a first integral of (3.21) associated with a given symmetry. Since we have three symmetries that give rise to $A^I_{3,2}$ (note that the $A^{II}_{3,2}$ realization results in a linear equation which we shall deal with later) we can choose any of the three symmetries $\partial/\partial t$, $\partial/\partial q$ and $t\partial/\partial t + (t+q)\partial/\partial q$ or a linear combination thereof. We first select $X = t\partial/\partial t + (t+q)\partial/\partial q$. Then $\eta - \dot{q}\xi = t + q - t\dot{q}$ and the first integral associated with X satisfies

$$\frac{\partial I}{\partial \dot{q}} = \frac{(q - t\dot{q} + t)\exp(-\dot{q})}{[t\exp(-\dot{q}) - 1]^2}, \tag{3.22}$$

whence, by (3.12), a Lagrangian function of (3.21) has to be of the form

$$L = (\ln t)/t - \dot{q}/t - \ln[t\exp(-\dot{q}) - 1]/t + b(t,q)\dot{q} + c(t,q), \tag{3.23}$$

subject to the constraint equation $c_q = b_t + 1/t^2$. This Lagrangian (3.23) has another Noether symmetry $Z = \partial/\partial q$. The two symmetries X and Z generates a proper (non-Abelian A_2) subalgebra of $A^I_{3,2}$. Just as for the equation (3.11) ($a \neq -1$) one cannot, for the equation (3.21), obtain a Lagrangian which admits a Noether algebra of dimension greater than two. So (3.23) is s^2-equivalent. We now search for another symmetry for (3.23). We begin by selecting $X' = \partial/\partial t$ as our first symmetry. Then $\eta - \dot{q}\xi = -\dot{q}$ and the first integral linked to X' satisfies

$$\frac{\partial I}{\partial \dot{q}} = \dot{q}\exp(\dot{q}). \tag{3.24}$$

Utilising (3.12), we easily arrive at a Lagrangian for (3.21) which is of the form

$$L = \exp(\dot{q}) + b(t,q)\dot{q} + c(t,q), \tag{3.25}$$

provided $c_q = b_t + 1$. A further symmetry for (3.25) is $Z' = \partial/\partial q$. Thus, X' and Z' generate a proper subalgebra (Abelian $2A_1$) of $A^I_{3,2}$. One could equally well have started with $\partial/\partial q$. Therefore, another s^2-equivalent Lagrangian for equation (3.21) is (3.25).

One again has to be cautious as we have already encountered $A^I_{3,2}$ as a Noether algebra for an alternative Lagrangian for the linear equation (see Mahomed, Kara and Leach 1993a and chapter 2). However, as pointed out before, the Lagrangians defined in Mahomed, Kara and Leach (1993a) are not s^3-equivalent. Now, we no longer have $A^I_{3,2}$ but two proper subalgbras of $A^I_{3,2}$, $2A_1$ and A_2, both linked to two s^2-equivalent Lagrangians (3.23) and (3.25), respectively.

We have obtained two s^2-equivalent Lagrangians (3.23) and (3.25) that together gives the full set of symmetry generators of $A^I_{3,2}$. Thus, to describe completely the algebraic properties of a system governed by the dynamical equation (3.21) (or its equivalent equation) by means of a Lagrangian formulation, one needs two Lagrangians.

One can now apply the classical Noether theorem to (3.23) and (3.25), respectively, to obtain the first integrals associated with each Noether symmetry. Applying (2.4) to the s^2-equivalent Lagrangian (3.25) with the choices $b = 0$ and $c = q$ we arrive at the integrals (the gauge function is linear in t)

$$\begin{aligned} I_1 &= t - \exp(\dot{q}), \\ I_2 &= \dot{q}\exp(\dot{q}) - \exp(\dot{q}) - q, \end{aligned} \tag{3.26}$$

corresponding to the symmetries $\partial/\partial q$ and $\partial/\partial t$, respectively. Similarly, for the Lagrangian (3.23) with $b = 1/t$ and $c = -\ln t/t$ we obtain, associated with $\partial/\partial q$, an integral which is a function of I_1, viz. $-1/I_1$ and corresponding to $t\partial/\partial t + (t+q)\partial/\partial q$, the functionally dependent integral

$$I_3 = \ln I_1 + I_2/I_1. \tag{3.27}$$

We now obtain s^r-equivalent Lagrangians for the representative equation

$$\ddot{q} = (1+\dot{q}^2)^{\frac{3}{2}} \exp(b\tan^{-1}\dot{q}) \tag{3.28}$$

which admits $A^{b\,I}_{3,7}$ $(b > 0)$ and $A^{I}_{3,6}$ if $b = 0$.

We firstly study the case when (3.28) admits $A^{b\,I}_{3,7}$ and then the case $A^{I}_{3,6}$. We invoke (3.12). Thus, we require a first integral associated with a given symmetry. We may select any one of the symmetries $\partial/\partial t$, $\partial/\partial q$ or $(bt+q)\partial/\partial t + (bq-t)\partial/\partial q$ or a linear combination. We choose $X = \partial/\partial q$. Then the first integral I, associated with X satisfies

$$\frac{\partial I}{\partial \dot{q}} = \frac{1}{(1+\dot{q}^2)^{\frac{3}{2}} \exp(b \tan^{-1} \dot{q})}. \tag{3.29}$$

By invoking (3.12), we obtain a Lagrangian for (3.28) of the form

$$L = -\frac{1}{(1+b^2)} \exp(-b \tan^{-1} \dot{q})(1+\dot{q}^2)^{\frac{1}{2}} + a(t,q)\dot{q} + c(t,q), \tag{3.30}$$

provided $a_t = c_q + 1$. This Lagrangian has one more symmetry $Z = \partial/\partial t$ when $b \neq 0$. These two symmetries, X and Z, generates a proper subalgebra of $A^{b\,I}_{3,7}$. The Lagrangian (3.30) is s^2-equivalent. There does not exist another Lagrangian that has a Noether algebra isomorphic to $A^{b\,I}_{3,7}$. To see this, we obtain another Lagrangian for (3.28). We take $X' = (bt+q)\partial/\partial t + (bq-t)\partial/\partial q$ as our starting symmetry. Then we require the first integral associated with X'. The usual method of solving the dual condition $X'^{[1]}(I) = 0$ and $dI/dt = 0$ is *a priori* difficult. Therefore we employ another strategy. We firstly obtain the integrals associated with $X = \partial/\partial q$ and $Z = \partial/\partial t$. We do this by applying the classical Noether Theorem to (3.30) with the choices $a = t$ and $c = 0$ and we arrive at the integrals (the gauge functions are linear in q)

$$\begin{aligned} I_1 &= t + \frac{1}{(1+b^2)} \frac{b-\dot{q}}{(1+\dot{q}^2)^{\frac{1}{2}}} \exp(-b \tan^{-1} \dot{q}), \\ I_2 &= q + \frac{1}{(1+b^2)} \frac{1+b\dot{q}}{(1+\dot{q}^2)^{\frac{1}{2}}} \exp(-b \tan^{-1} \dot{q}), \end{aligned} \tag{3.31}$$

corresponding to the symmetries X and Z, respectively. Then we impose the requirement $X'^{[1]}(I) = 0$ where $I = \Lambda(I_1, I_2)$ for some functional form Λ. This gives

$$I = (I_1^2 + I_2^2)^{\frac{1}{2}} \exp(b \tan^{-1}(\frac{I_1}{I_2})). \tag{3.32}$$

Hence, a first integral associated with X' is (3.32). To construct a Lagrangian corresponding to I we invoke (3.12). This is done in the following manner. The integrated form of (3.12) is (1.12). In fact, (1.12) gives precisely the Noether invariant corresponding to a Noether symmetry with $f(t,q)$ being the gauge term. A further integration of (1.12) leads to

$$L = \frac{1}{\xi} f(t,q) + (-\eta + \dot{q}\xi)h(t,q) + (-\eta + \dot{q}\xi) \int \frac{I}{(-\eta + \dot{q}\xi)^2} d\dot{q}, \tag{3.33}$$

where h is some function of t and q. Whence a Lagrangian for (3.28) which admits X' is

$$L = [t(1 + b\dot{q}) + q(\dot{q} - b)] \int \frac{(I_1^2 + I_2^2)^{\frac{1}{2}} \exp(b \tan^{-1}(\frac{I_1}{I_2}))}{[t(1 + b\dot{q}) + q(\dot{q} - b)]^2} d\dot{q} + a(t,q)\dot{q} + c(t,q) \tag{3.34}$$

provided $c_q = a_t + 1$. This Lagrangian does not admit a further symmetry. Therefore, any Lagrangian representation of (3.28), whenever $b \neq 0$, has at most two Noether symmetries. An s^2-equivalent Lagrangian for (3.28) with $b \neq 0$ is of the form (3.30). The Lagrangian (3.34) is not s^2-equivalent.

We now study the situation when (3.28) admits $A^I_{3,8}$, i.e., when $b = 0$ in (3.28). Thus, we consider

$$\ddot{q} = (1 + \dot{q}^2)^{\frac{3}{2}}. \tag{3.35}$$

A Lagrangian for (3.35) is obtained from (3.30) by setting $b = 0$. This leads to, if we set $a = t$ and $c = 0$, the Lagrangian

$$L = -(1 + \dot{q}^2)^{\frac{1}{2}} + t\dot{q} \tag{3.36}$$

which is s^3-equivalent with Noether algebra $A^I_{3,b}$. For other values of b ($b \neq 0$), the Noether algebra is a proper subalgebra of $A^{b\,I}_{3,7}$ as we have seen for (3.30) and (3.34). Also, we have come across $A^I_{3,6}$ and $A^{b\,I}_{3,7}$ as the Noether algebras for alternative Lagrangians of the free particle equation. However, the Lagrangians given in Mahomed, Kara and Leach (1993a) were not s^3-equivalent. We note that the Noether symmetry properties of each of (3.30), (3.34) and (3.36) are different.

We have given two Lagrangians, viz., the s^2-equivalent Lagrangian (3.30) and the Lagrangian (3.34) that together gives the complete symmetry generators of $A^{b\,I}_{3,7}$. Hence, one requires two Lagrangians to completely describe the algebraic properties of the system governed by equation (3.28) for $b \neq 0$. The first integrals associated with the Noether symmetries are given by (3.31) and (3.32). However, for complete integrability, the full algebraic description is not necessary, but the s^2-equivalent Lagrangian (3.30) would suffice. Also pertinent here are the remarks made regarding integrability and the algebra for the equation (3.11).

The Noether symmetries $\partial/\partial t$, $\partial/\partial q$ and $q\partial/\partial t - t\partial/\partial q$ of the s^3-equivalent Lagrangian (3.36) fully describes the algebraic properties of the dynamical equation (3.35). By the Noether theorem there are three first integrals and they are (3.31), if we set $b = 0$, and

$$I_3 = \frac{t^2}{2} + \frac{q^2}{2} + \frac{q - t\dot{q}}{(1+\dot{q}^2)^{\frac{1}{2}}} \tag{3.37}$$

corresponding to $q\partial/\partial t - t\partial/\partial q$.

We now obtain s^r-equivalent Lagrangians for the equation (A is an arbitrary constant)

$$t\ddot{q} = A\dot{q}^3 - \frac{1}{2}\dot{q} \tag{3.38}$$

which admits the $A_{3,8}^{II}$ algebra of symmetries $\partial/\partial t$, $t\partial/\partial t + q\partial/\partial q$ and $2tq\partial/\partial t + q^2\partial/\partial q$. The $A_{3,8}^{I}$ symmetries do not give rise to to a second-order equation and we treat the $A_{3,8}^{III}$ case immediately after this case. One can scale A to ± 1 by means of a real scaling transformation in t of the form $t = \sqrt{\pm A}\, T$ depending on the sign of A. A Lagrangian for (3.38) can easily be constructed by utilizing (3.8). Of course it could easily be obtained by using the symmetry(ies) and invoking (3.12). If $A = 1$ we have the Lagrangian

$$L = \frac{\dot{q}}{t} + \frac{1}{2t\dot{q}}. \tag{3.39}$$

It is not hard to verify, by using the Killing equation, that $A_{3,8}^{II}$ is the Noether algebra for (3.39). Hence, (3.39) is s^3-equivalent and its Noether algebra is isomorphic to the dynamical algebra. For $A = -1$, a Lagrangian for (3.38) is

$$L = \frac{\dot{q}}{t} - \frac{1}{2t\dot{q}}. \tag{3.40}$$

The Lagrangian (3.40) is also s^3-equivalent with Noether algebra $A_{3,8}^{II}$. There is no real transformation that maps (3.39) to (3.40). However, there is an obvious complex transformation $(t, q) \mapsto (it, q)$ that maps (3.39) to (3.40). This transformation also takes (3.38) with $A = -1$ to (3.38) with $A = 1$.

Applying Noether's Theorem to (3.39) and (3.40) in turn, with the gauge function linear in t, we obtain the linearly independent first integrals (the upper sign refers to (3.39))

$$\begin{aligned} I_1 &= \frac{1}{t} \mp \frac{1}{2t\dot{q}^2}, \\ I_2 &= \pm\frac{1}{\dot{q}} + \frac{q}{t} \mp \frac{q}{2t\dot{q}^2}, \\ I_3 &= \pm 2t - \frac{q^2}{t} \mp \frac{2q}{\dot{q}} \pm \frac{q^2}{2t\dot{q}^2}, \end{aligned} \tag{3.41}$$

corresponding to the symmetries $\partial/\partial q$, $t\partial/\partial t + q\partial/\partial q$ and $2tq\partial/\partial t + q^2\partial/\partial q$, respectively.

An s^3-equivalent Lagrangian for

$$t\ddot{q} = \dot{q}^3 + \dot{q} + A(1+\dot{q}^2)^{\frac{3}{2}}, \tag{3.42}$$

where A is an arbitrary constant, which has the Lie algebra $A_{3,8}^{III}$ of symmetries $\partial/\partial q$, $t\partial/\partial t + q\partial/\partial q$ and $2tq\partial/\partial t + (q^2 - t^2)\partial/\partial q$, is

$$L = t^{-1}\sqrt{1+\dot{q}^2} + At^{-1}\dot{q}. \tag{3.43}$$

The Lagrangian (3.43) is easily obtained by using (3.8). Just as for the $A_{3,8}^{II}$ case, the Noether algebra $A_{3,8}^{III}$ is the same as the dynamical algebra. The first integrals are

$$\begin{aligned}
I_1 &= At^{-1} + t^{-1}\dot{q}(1+\dot{q}^2)^{-\frac{1}{2}}, \\
I_2 &= (1 + t^{-1}q\dot{q})(1+\dot{q}^2)^{-\frac{1}{2}} + At^{-1}q\sqrt{1+\dot{q}^2}, \\
I_3 &= -A(t + t^{-1}q^2)\sqrt{1+\dot{q}^2} + (t\dot{q} - 2q - t^{-1}q^2\dot{q})(1+\dot{q}^2)^{-\frac{1}{2}}.
\end{aligned} \tag{3.44}$$

We now determine an s^3-equivalent Lagrangian for the $A_{3,9}$ invariant equation

$$\ddot{q} = A[\frac{1+\dot{q}^2+(q-t\dot{q})^2}{1+t^2+q^2}]^{\frac{3}{2}}, \tag{3.45}$$

where A is an arbitrary constant. To use (3.12), we require a first integral of (3.45) associated with a given symmetry. We have the three symmetries $(1+t^2)\partial/\partial t + tq\partial/\partial q$, $tq\partial/\partial t + (1+q^2)\partial/\partial q$ and $q\partial/\partial t - t\partial/\partial q$ of (3.45). We choose the first symmetry and the first integral associated with this symmetry is deduced to be

$$I_1 = A\frac{u}{\sqrt{1+u^2}} + \frac{1}{\sqrt{1+u^2+v^2}}, \tag{3.46}$$

where

$$u = q(1+t^2)^{-\frac{1}{2}},$$

$$v = \dot{q}(1+t^2)^{\frac{1}{2}} - tq(1+t^2)^{-\frac{1}{2}}. \tag{3.47}$$

Invoking (3.12), a Lagrangian for (3.45) is of the form

$$\begin{aligned} L &= -\dot{q}\int (1+\dot{q}^2+(q-t\dot{q})^2)^{-\frac{3}{2}}d\dot{q} + \int \dot{q}(1+\dot{q}^2+(q-t\dot{q})^2)^{-\frac{3}{2}}d\dot{q} \\ &\quad + b(t,q)\dot{q} + c(t,q) \qquad (3.48) \\ &= \frac{1}{x^2}(1+t^2)^{-\frac{3}{2}}[(y-\dot{q})\sin w + \beta\sec w] + b(t,q)\dot{q} + c(t,q), \end{aligned}$$

where b and c satisfy $b_t = c_q + A(1+t^2+q^2)^{-\frac{3}{2}}$, $x = \sqrt{1+t^2+q^2}/(1+t^2)$, $\beta = qx$, $\tan w = (\dot{q}-y)/x$ and $y = tq/(t^2+1)$. The Noether algebra is $A_{3,9}$ and is the same as the dynamical algebra.

The first integrals corresponding to the respective Noether symmetries $tq\partial/\partial t + (1+q^2)\partial/\partial q$ and $q\partial/\partial t - t\partial/\partial q$ are

$$\begin{aligned} I_2 &= \frac{At}{\sqrt{1+t^2+q^2}} - \frac{\dot{q}}{\sqrt{1+\dot{q}^2+(q-t\dot{q})^2}}, \\ I_3 &= \frac{A}{\sqrt{1+t^2+q^2}} + \int \frac{t+q\dot{q}}{(1+\dot{q}^2+(q-t\dot{q})^2)^{\frac{3}{2}}}d\dot{q} \qquad (3.49) \\ &= \frac{A}{\sqrt{1+t^2+q^2}} + \frac{(t^2+1)^{-\frac{3}{2}}}{x^2}[\alpha\sin w + \beta\sec w], \end{aligned}$$

where x, y, β and w are as defined above and $\alpha = t+qy$.

Finally, we obtain an s^5-equivalent Lagrangian for the equation

$$\ddot{q} = 0 \tag{3.50}$$

which is a representative equation for equations linearizable by means of a point transformation. Equation (3.50) admits eight point symmetries which constitute the Lie algebra $sl(3, R)$. However, any Lagrangian representation of (3.50) possesses at most five Noether symmetries since the maximum dimension of the Noether algebra for any particle Lagrangian is five. It is easy to see that an s^5-equivalent Lagrangian for (3.50) is the *usual* Lagrangian

$$L = \frac{1}{2}\dot{q}^2 \tag{3.51}$$

and the Noether symmetries $\partial/\partial t$, $2t\partial/\partial t + q\partial/\partial q$, $t^2\partial/\partial t + tq\partial/\partial q$, $\partial/\partial q$ and $t\partial/\partial q$ generate the Noether algebra $A_{5,40}$ (see chapter 2). We note that the Noether algebra $A_{5,40}$ of symmetries is a proper subalgebra of $sl(3, R)$ of symmetries of $\ddot{q} = 0$.

An alternative Lagrangian for (3.50) is

$$L = q^2\dot{q}\ln\dot{q} - q^2\dot{q} - tq\dot{q}^2 + \frac{1}{6}t^2\dot{q}^3 + a\dot{q} + b, \tag{3.52}$$

where the functions a and b are related by $a_t = b_q$. The Noether symmetries (3.52) are $q\partial/\partial t$, $tq\partial/\partial t + q^2\partial/\partial q$ and $t\partial/\partial t$. These symmetries generate the algebra $A_1 \oplus A_2$. The Noether symmetries of the two Lagrangians (3.51) and (3.52) together give the complete symmetry generators of $sl(3, R)$. In the solution of the simple harmonic oscillator (Wulfman and Wybourne 1976) the two Lagrangian picture is essential in fully describing the algebra and the periodicity of the motion of the oscillator. The complete algebraic description, i.e., the two Lagrangian picture provides for the complete determination of all the physical constants.

3.3. Discussion

We have shown, inter alia, that the dynamical Lie algebra and Noether algebra are not, in general, isomorphic in the case of three-dimensional real algebras. However, for the algebras $A^{I}_{3,4}$, $A^{I}_{3,6}$, $A^{II}_{3,8}$, $A^{III}_{3,8}$ and $A_{3,9}$ they coincide.

The results of chapter 2 and this chapter provides a partial solution to the difficult problem of the determination of all canonical forms for particle Lagrangians. Thus, as a final remark, we give a synthesis of the canonical forms of particle Lagrangians that admit a three-dimensional real Noether algebra of point symmetries. These are defined by (2.11), (2.12), (3.18), (3.36), (3.39), (3.40), (3.43) and (3.46). Hence, in all there are fourteen canonical forms. An interesting feature is that a pair of Lagrangians may admit similar Noether algebras but give rise to Euler-Lagrange equations that are inequivalent — one linearizable and the other not.

Clearly, there is only one canonical form of particle Lagrangians that admit five Noether point symmetries. The canonical forms corresponding to the one and two symmetry cases still requires further investigation. This is done at the end of chapter 5. In chapter 4, we consider two examples of s^r-equivalent Lagrangians, viz., s^3-equivalent and s^5-equivalent Lagrangians.

4. EQUIVALENT LAGRANGIANS AND THE $sl(2,R)$ AND $sl(3,R)$ SOLUTIONS OF $\ddot{q}+p(t)\dot{q}+r(t)q=\mu\dot{q}^2q^{-1}+s(t)q^n$

4.1. Introduction

In this chapter, we consider two examples of s^r-equivalent Lagrangians.

Ranganathan (1988), (1989) makes an elaborate discussion on the integrability of the differential equation

$$\ddot{q}+p(t)\dot{q}+r(t)q=\mu\dot{q}^2q^{-1}+s(t)q^n \tag{4.1}$$

(where p, r and s are arbitrary analytic functions of t, and μ is an arbitrary constant, with $\mu \neq 1$ and $n \neq 1$) and provides conditions under which solutions involving two, one or no arbitrary constants exist. Furthermore, he draws a fair contrast between the conditions under which he obtains solutions and those of Reid (1973), Lewis (1967), Pinney (1950) and Thomas (1952). Equation (4.1) is also discussed in Ibragimov (1992) and Mellin et al (1993) for $\mu = 0$. Here, however, the treatment is quite different and relies on the notion of equivalent Lagrangians. Equation (4.1) is often considered in its canonical form $\ddot{q} = s(t)q^n$. However, here we retain the other terms as there is no difference in its treatment. Moreover, the general solution of (4.1), when it admits $sl(2,R)$ (the 'physical' representation, viz., $A_{3,8}^{II}$) or $sl(3,R)$ (linearizable by point transformation) can be written in terms of the independent solutions of the associated linear homogeneous equation (4.45) or (4.31), i.e., we have a non-linear superposition principle which is usually discussed for the Ermakov-Pinney equation (see Ibragimov 1992).

In this chapter, we determine the solutions of (4.1) by first inducing the algebra it admits and then by constructing an equivalent Lagrangian for the differential equation and mapping it to a representative Lagrangian of the transformed equation. The constructed Lagrangian is imposed with a structure that includes 'potential and kinetic energies'. Although the Noether symmetries of the resultant Lagrangian (obtained through the Killing-type equation) is useful but it is more complicated in determining solutions. So we, instead, utilise the Lagrangian to work through a methodology that provides point transformations that transforms (4.1) to the representative of the class to which it belongs from which the complete solution is easily obtainable and the properties of (4.1) becomes transparent.

It will be shown that (4.1) admits the algebras $sl(3,R)$ (linearizability) and $sl(2,R)$ (non-linearizability) for the cases $n = \mu, 1$ and $n = 4\mu - 3$ respectively. For these cases, complete solutions are obtained (cf. the refs. cited above) using equivalent Lagrangians.

A variety of examples, including the generalized Emden equation, are discussed. The results obtained here can also be found in Kara and Mahomed (1992).

A brief summary of the methodology is given below.

If a differential equation is linearizable by means of a point transformation we can find a regular point transformation $T = G(t,q)$ and $Q = F(t,q)$ for suitable functions F and G such that it transforms to $Q'' = 0$, where $' = d/dT$, for which the corresponding Lagrangian is the usual one $\bar{L} = \frac{1}{2}Q'^2$. The point transformation is obtained by constructing a Lagrangian, L, equivalent to $\bar{L}$, i.e., by invoking the equivalence relation (2.6) in expanded form

$$L(t,q,\dot{q}) = \bar{L}(T,Q,Q')\frac{dT}{dt} + f_t + f_q\dot{q}, \qquad (4.2)$$

where f is a gauge function of t and q. In particular, we impose the form

$$L(t,q,\dot{q}) = \frac{1}{2}U(t,q)\dot{q}^2 + V(t,q), \tag{4.3}$$

for functions U and V of t and q such that (4.3) is a Lagrangian for (4.1). Thus, by inserting (4.3) into (4.2) and the Lagrangian of the transformed equation, viz., $\bar{L}$, into (4.2), we can find a point transformation that reduces the orginal equation (4.1) to obvious integrable form by equating the coefficients of $\dot{q}$ on both sides of (4.2).

The same methodology is adopted when the equation is not linearizable by means of a point transformation but admits the $sl(2,R)$ algebra (in fact the algebra $A_{3,8}^{II}$ but we will not make the distinction here). The construction of the point transformation $T = G(t,q)$ and $Q = F(t,q)$ is based on the representative completely integrable differential equation $TQ'' = Q'^3 - \frac{1}{2}Q'$ for which a representative integrable Lagrangian is $\bar{L} = \dfrac{Q'}{T} + \dfrac{1}{2TQ'}$ (see equation (3.39)). Here again we utilise (4.2) and (4.3).

4.2. Equivalent Lagrangians and point transformations

Lagrangians $L(t,q,\dot{q})$ and $\bar{L}(T,Q,Q')$ are equivalent by means of a point transformation $T = G(t,q)$ and $Q = F(t,q)$ if (4.2) holds for some gauge function f. The equivalent Lagrangian we construct has the form (4.3) and the associated Euler-Lagrange equation is given by

$$\ddot{q} + \frac{U_q}{2U}\dot{q}^2 + \frac{U_t}{U}\dot{q} - \frac{V_q}{U} = 0 \tag{4.4}$$

which has the same form as (4.1). We shall use the following notation. Let

$$M = \text{coefficient of } \dot{q}^2 = \frac{1}{2}\frac{U_q}{U}.$$

Thus,

$$U = N(t) \exp \int 2M\, dq. \tag{4.5}$$

Then,

$$\text{the coefficient of } \dot{q} = \frac{U_t}{U} = \frac{N_t}{N} + \int M_t\, dq = \frac{N_t}{N}.$$

Here $M_t = 0$ since the coefficient of $\dot{q}^2$ in (4.1) is a function of q only. Also, let

$$R = \text{coefficient of } \dot{q}^0.$$

That is,

$$R = -\frac{V_q}{U}. \tag{4.6}$$

4.2.1 *$sl(3,R)$ algebra case.*

If $\bar{L} = \frac{1}{2}Q'^2$, then (4.2) becomes

$$L = \frac{\frac{1}{2}F_t^2 + F_t F_q \dot{q} + \frac{1}{2}F_q^2 \dot{q}^2}{G_t + G_q \dot{q}} + f_t + f_q \dot{q}. \tag{4.7}$$

If L assumes the form in (4.3), then by comparing coefficients of $\dot{q}$ on both sides of (4.2) (without loss of generality we set $G_q = 0$) it can be shown that F and G of the point transformation $T = G(t,q)$ and $Q = F(t,q)$ are given by

$$G = \int a^2(t)\, dt, \tag{4.8}$$

$$F = a(t) \int \sqrt{U(t,q)}\, dq + b(t), \tag{4.9}$$

where a and b are functions of t to be determined, subject to

$$\frac{F_t F_q}{a^2} + f_q = 0, \tag{4.10}$$

$$\frac{1}{2}\frac{F_t^2}{a^2} + f_t = V(t,q). \tag{4.11}$$

4.2.2 *$sl(2,R)$ algebra case.*

If a second-order ordinary differential equation is not linearizable and generates the $sl(2,R)$ algebra, we choose the representative Lagrangian $\bar{L} = \frac{Q'}{T} + \frac{1}{2TQ'}$ which has associated Euler-Lagrange equation $TQ'' = Q'^3 - \frac{1}{2}Q'$ which admits the $sl(2,R)$ algebra. Then (4.2) has the form

$$L = \frac{F_t}{G} + \frac{F_q\dot{q}}{G} + \frac{1}{2G}\frac{G_t^2 + 2G_tG_q\dot{q} + G_q^2\dot{q}^2}{F_t + F_q\dot{q}} + f_t + f_q\dot{q}. \tag{4.12}$$

If we again impose the form (4.3) for L, the transformations are obtained by the following equations and corresponding conditions. (Here, without loss of generality we take $F_q = 0$.)

$$F = \int a^2(t)dt, \tag{4.13}$$

$$G = \frac{1}{4}a^2\left(\int \sqrt{U}\,dq\right)^2 + a(t)b(t)\int \sqrt{U}\,dq + b^2.$$

It can be shown that the above transformation will still hold if $b(t) = 0$ so that without loss of generality we set $b(t) = 0$, therefore

$$G = \frac{1}{4}a^2\left(\int \sqrt{U}\,dq\right)^2, \tag{4.14}$$

and we also have

$$\frac{G_tG_q}{a^2G} + f_q = 0, \tag{4.15}$$

$$\frac{a^2}{G} + \frac{G_t^2}{2Ga^2} + f_t = V(t,q). \tag{4.16}$$

4.3. Main Results

We now construct the point transformations that provide complete solutions to (4.1) when it admits $sl(3,R)$ and $sl(2,R)$ algebras, respectively. The general solution for the $sl(3,R)$ case is obtained from $Q = u_1T + v_1$, where u_1 and v_1 are arbitrary constants, once a transformation $Q = F(t,q)$ and $T = G(t,q)$ is known. Likewise, the general solution to (4.1) for the $sl(2,R)$ case is obtained from $(u_1Q + v_1)^2 = 1 - Tu_1$.

4.3.1 *$sl(3,R)$ solutions*

Firstly, we recall the following conditions on the linearizability of second-order differential equations from Mahomed and Leach (1989b) (see also Ibragimov 1992). A second order differential equation of the form

$$\ddot{q} = \mathcal{B}(t,q)\dot{q}^2 + \mathcal{C}(t,q)\dot{q} + \mathcal{D}(t,q), \tag{4.17}$$

where $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are functions of t and q, is linearizable by means of a point transformation if and only if the following conditions hold,

$$\mathcal{C}_{qq} + \mathcal{B}\mathcal{C}_q - 2\mathcal{B}_{tq} = 0,$$

$$-3\mathcal{B}_q\mathcal{D} + \mathcal{B}_{tt} - 2\mathcal{C}_{tq} - 3\mathcal{B}\mathcal{D}_q + 3\mathcal{D}_{qq} + 2\mathcal{C}\mathcal{C}_q - \mathcal{C}\mathcal{B}_t = 0. \tag{4.18}$$

The conditions on the linearizability of (4.1) is provided by the following theorem.

Theorem 4.1. Equation (4.1) is linearizable by means of a point transformation if and only if $\mu = n$ or $n = 1$ (equivalently $s = 0$).

Proof. In (4.17), $\mathcal{B}(t,q) = \mu q^{-1}$, $\mathcal{C}(t,q) = p(t)$ and $\mathcal{D}(t,q) = s(t)q^n - r(t)q$. The first condition in (4.18) holds trivially. The left hand side of the second condition in (4.18) simplifies to $3s[n^2 - (\mu+1)n + \mu]q^{n-2}$. Thus, linearizability follows if and only if $n = \mu$ or $n = 1$ (equivalently $s = 0$). ◇

We look at the cases $n = \mu(\neq 1)$ and $n = 1$ separately.

With $n = \mu(\neq 1)$ (and, hence, linearizability), (4.1) may be written in the form of (4.4) as

$$\ddot{q} - \frac{n}{q}\dot{q}^2 + p(t)\dot{q} + r(t)q - s(t)q^n = 0. \tag{4.19}$$

If we let $E_p(t) = \exp\int_{t_0}^{t} p(s)\,ds$, then comparing (4.19) with (4.4), we have $\dfrac{N_t}{N} = p(t)$. Thus,

$$N = E_p. \tag{4.20}$$

Hence, by (4.5), $U = E_p \exp\int 2(\frac{-n}{q})\,dq$ and so

$$U = E_p q^{-2n}. \tag{4.21}$$

By (4.6), $V_q = -UR = -U[r(t)q - s(t)q^n]$. Therefore,

$$V = -E_p r\frac{q^{-2n+2}}{-2n+2} + E_p s(t)\frac{q^{-n+1}}{-n+1}, \qquad n \neq 1. \tag{4.22}$$

The form of $G(t,q)$ is given by (4.8). By (4.9) and (4.21) we have

$$F = aE_p^{\frac{1}{2}}\frac{q^{-n+1}}{-n+1} + b(t). \tag{4.23}$$

Thus,

$$F_q = aE_p^{\frac{1}{2}} q^{-n} \tag{4.24}$$

and

$$F_t = \dot{a}E_p^{\frac{1}{2}}\frac{q^{-n+1}}{-n+1} + apE_p^{\frac{1}{2}}\frac{q^{-n+1}}{2(-n+1)} + \dot{b}. \tag{4.25}$$

Therefore, by (4.10)

$$f_q = \frac{\dot{a}}{a}E_p\frac{q^{-2n+1}}{n-1} + pE_p\frac{q^{-2n+1}}{2(n-1)} - \frac{\dot{b}q^{-n}E_p^{\frac{1}{2}}}{a} \tag{4.26}$$

and

$$f = \frac{\dot{a}}{a} E_p \frac{q^{-2n+2}}{(n-1)(2-2n)} \text{[illegible]} E_p \frac{q^{-2n+2}}{2(n-1)(2-2n)} + \frac{\dot{b} q^{1-n} E_p^{\frac{1}{2}}}{a(n-1)} + c(t) \tag{4.27}$$

and f_t may be obtained from (4.27). Substituting (4.23)—(4.25) and f_t into (4.11), we obtain the following equation

$$\begin{aligned}
&[(\frac{\dot{a}}{a})^2(1-n)^2 + \frac{p^2}{8} + \frac{\dot{a}}{2a}p - \frac{1}{2}(\frac{\dot{a}}{a})^{\cdot} \\
&- \frac{1}{2}\frac{\dot{a}}{a}p - \frac{1}{4}\dot{p} - \frac{1}{4}p^2 + \frac{1}{2}r(1-n)]E_p \frac{q^{2-2n}}{(1-n)^2} \\
&+ [\frac{1}{2}p\frac{\dot{b}}{a} + \frac{\dot{b}}{a}\frac{\dot{a}}{a} - (\frac{\dot{b}}{a})^{\cdot} - p\frac{\dot{b}}{2a} - sE_p^{\frac{1}{2}}]E_p^{\frac{1}{2}} \frac{q^{1-n}}{1-n} \\
&+ [\frac{1}{2}(\frac{\dot{b}}{a})^2 + \dot{c}] = 0 .
\end{aligned}$$

Equating the coefficients of q^{2-2n}, q^{1-n} and q^0 to zero (note $n \neq 1$) the following equations are obtained:

$$(\frac{\dot{a}}{a})^2 - (\frac{\dot{a}}{a})^{\cdot} - \frac{\dot{p}}{2} - \frac{p^2}{4} + r(1-n) = 0, \tag{4.28}$$

$$(\frac{\dot{a}}{a})(\frac{\dot{b}}{a}) - (\frac{\dot{b}}{a})^{\cdot} - sE_p^{\frac{1}{2}} = 0, \tag{4.29}$$

$$\frac{1}{2}(\frac{\dot{b}}{a})^2 + \dot{c} = 0. \tag{4.30}$$

Equation (4.28), which is a Riccati first-order equation in $\dot{a}/a$, can be written as

$$\ddot{u} + \beta(t)u = 0, \tag{4.31}$$

where $u = 1/a$ and $\beta(t) = -\dot{p}/2 - p^2/4 + r(1-n)$. From (4.29), we have

$$b = -\int a^2 \int \frac{1}{a} sE_p^{\frac{1}{2}} dt\, dt. \tag{4.32}$$

We now deal with the case $n = 1$ (equivalently $s = 0$). The equations of the transformation obtained here are similar to those obtained above. We omit the details and state the relevant equations below.

$$U = E_p q^{-2\mu}. \tag{4.33}$$

$$V = -E_p r \frac{q^{-2\mu+2}}{-2\mu+2}. \tag{4.34}$$

$$G = \int a^2 dt. \tag{4.35}$$

$$F = aE_p^{\frac{1}{2}} \frac{q^{-\mu+1}}{-\mu+1} + b(t). \tag{4.36}$$

Here $a(t)$ and $b(t)$ are arbitrary functions of t where $a(t)$ is again obtained by (4.33) and

$$b(t) = -k \int a^2 dt, \tag{4.37}$$

for some constant k.

The method presented here generalizes the solutions given in Theorems 4c–4f and Theorem 5b–5d in Ranganathan (1988) as well as equation (3.12) in Ranganathan (1989). Moreover, we give necessary and sufficient conditions for complete integrability since (4.1) is linearizable for $n = \mu$ and $n = 1$. Also, the solution of (4.1), when it is linearizable, is given in terms of the solution of (4.31).

Remark: F and G obtained in the above analysis, define a whole class of point transformations. However, to obtain complete solutions for (4.1), when it admits $sl(3, R)$ algebra, we need only one such transformation. This will become clearer when we deal with examples in a later section.

4.3.2 $sl(2, R)$ *solutions*

If $\mu \neq n$, $(n \neq 1)$, we find the relation between μ and n for which (4.1) admits the $sl(2,R)$ algebra. We achieve this by applying the results of section 4.2.2 to (4.1). Comparing (4.1) and (4.[illegible]), we have $U = E_p(t)q^{-2\mu}$. By (4.6), $V_q = -U$, $R = -U(rq - sq^n)$. Thus,

$$V = -E_p[\frac{rq^{2-2\mu}}{2(1-\mu)} - \frac{sq^{n-2\mu+1}}{n-2\mu+1}]. \tag{4.38}$$

F is given by (4.13) and

$$G = \frac{a^2 E_p q^{2(1-\mu)}}{(1-\mu)^2}. \tag{4.39}$$

Thus,

$$G_q = \frac{a^2 E_p q^{1-2\mu}}{2(1-\mu)} \tag{4.40}$$

and

$$G_t = \frac{a\dot{a}E_p q^{2-2\mu}}{2(1-\mu)^2} + \frac{a^2 E_p p q^{-2\mu+2}}{4(1-\mu)^2}. \tag{4.41}$$

Therefore, by (4.15)

$$f_q = -\frac{\dot{a}}{a}E_p\frac{q^{1-2\mu}}{1-\mu} - E_p p\frac{q^{1-2\mu}}{2(1-\mu)} \tag{4.42}$$

and

$$f = -\frac{1}{2}\frac{\dot{a}}{a}E_p\frac{q^{2-2\mu}}{(1-\mu)^2} - E_p p\frac{q^{2-2\mu}}{4(1-\mu)^2} + c(t). \tag{4.43}$$

We note that f_t may be obtained from (4.43). Substituting (4.38), (4.39), (4.41) and f_t into (4.16), we get the following equation.

$$\begin{aligned}
&[\frac{1}{2}(\frac{\dot{a}}{a})^2 + \frac{p^2}{8} + \frac{1}{2}(\frac{\dot{a}}{a})p(1-\mu) - \frac{1}{2}(\frac{\ddot{a}}{a}) \\
&- \frac{1}{2}(\frac{\dot{a}}{a})p - \frac{p^2}{4} - \frac{p}{4} + \frac{r(1-\mu)}{2}]E_p\frac{q^{2-2\mu}}{(1-\mu)^2} \\
&+ \frac{4(1-\mu)^2}{E_p}q^{2\mu-2} - E_n s\frac{q^{n-2\mu+1}}{n-2\mu+1} + \dot{c} = 0.
\end{aligned}$$

Equating the coefficients of $q^{2-2\mu}$ yields the first-order Riccati equation in $\dot{a}/a$

$$4(\frac{\dot{a}}{a})^2 - 4p\mu(\frac{\dot{a}}{a}) - 4(\frac{\dot{a}}{a})^{\cdot} - 2\dot{p} - p^2 + 4r(1-\mu) = 0. \tag{4.44}$$

By letting $a = 1/u$, (4.44) becomes the second-order linear equation

$$\ddot{u} + p(t)\mu\dot{u} + \beta(t)u = 0, \tag{4.45}$$

where $\beta(t) = 4[-2\dot{p} - p^2 + 4r(1-\mu)]$. Since $n \neq 1$, we are forced to impose the relation $n - 2\mu + 1 = 2\mu - 2$ (i.e., $n = 4\mu - 3$) and then by equating the coefficients of $q^{2\mu-2}$ we get

$$\frac{4(1-\mu)^2}{E_p} - \frac{E_p s}{2(\mu-1)} = 0. \tag{4.46}$$

By comparing the coefficients of q^0 we derive

$$\dot{c} = 0. \tag{4.47}$$

Equation (4.46) clearly implies that

$$s = -\frac{8(1-\mu)^3}{E_p^2}. \tag{4.48}$$

The method presented here generalizes the solutions given in Theorem 5a in Ranganathan (1988). Moreover, we give necessary and sufficient conditions for complete integrability since (4.1) has $sl(2,R)$ algebra for $n = 4\mu - 3$. Here, too, we have a form of 'linearization'.

Remark: Here again F and G obtained in the above analysis, define a whole class of point transformations. However, to obtain complete solutions for (4.1), whenever it admits $sl(2,R)$ algebra, one needs only one transformation. This will become more apparent when we solve examples later.

4.4. Examples

In the following section we consider some special examples and discuss their solutions in the light of the results obtained in sections 4.2 and 4.3. These examples and alternative methods of solutions are obtainable in other papers such as Ranganathan (1988), (1989) and references therein (see also Mellin et al 1993). The method here, however, provides insight into the nature of general solutions.

4.4.1 *Linearizable*

The solution to differential equation (4.1) which admit the $sl(3,R)$ algebra is essentially given by

$$Q = u_1 T + v_1, \tag{4.49}$$

where u_1 and v_1 are arbitrary constants, if the linearizing transformation to $Q'' = 0$ is given by $T = G(t,q)$ and $Q = F(t,q)$.

Example 1

We outline the procedure to integrate

$$\ddot{q} + p(t)\dot{q} + r(t)q = n\dot{q}^2 q^{-1} + \frac{\sigma}{E_p^2} q^n, \quad \sigma \neq 0. \tag{4.50}$$

Here, $u(t)$ is given by the solution of (4.31) and by (4.32) we may find $b(t)$. The transformation functions F and G are then obtained by (4.23) and (4.8), respectively. The complete solution to (4.50) is obtained via (4.49).

Example 2

We solve

$$\ddot{q} + \frac{\alpha^2 - 1}{4(n-1)t^2} q = n\dot{q}^2 q^{-1} + \sigma t^{-(\alpha+1)/2} q^n, \quad \sigma \neq 0, \quad \alpha \neq 0, 1, \quad n \neq 1. \tag{4.51}$$

With $p(t) = 0$ and $r(t) = \dfrac{\alpha^2 - 1}{4(n-1)t^2}$, $E_p(t) = k^2$(constant). Then by equation

(4.31), $u(t) = t^{(1+\alpha)/2}$ and by (4.8) $G = -\frac{1}{\alpha}t^{-\alpha}$. Also, by (4.32), $b(t) = \frac{\sigma k}{\alpha - 1}t^{1-\alpha}$ and by (4.23), $F = \frac{k}{1-n}t^{-(1+\alpha)/2}q^{1-n} + \frac{\sigma k}{\alpha - 1}t^{1-\alpha}$. The complete solution by (4.49) is then given by

$$\frac{1}{1-n}q^{1-n} = t^{(1+\alpha)/2}[-\frac{A_1}{k\alpha}t^{-\alpha} - \frac{\sigma}{\alpha-1}t^{1-\alpha} + B_1 k],$$

where A_1 and B_1 are constants (cf. Ranganathan 1988).

Example 3

We obtain the solution for the equation with parameters k_1, k_2 and k_3, viz.,

$$\ddot{q} + k_1\dot{q} + k_3 q = -\dot{q}^2 q^{-1} + \sigma e^{k_2 t} q^{-1}, \quad \sigma \neq 0. \tag{4.52}$$

Here, $E_p = \alpha^2 e^{k_1 t}$ (α constant). By (4.31), $a(t) = e^{-kt}$ ($k = \sqrt{\frac{1}{2}({k_1}^2 + 4k_3)}$) so that, by (4.8), $G = -\frac{1}{2k}e^{-2kt}$. Hence, by (4.32), $b(t) = -\frac{\sigma\alpha}{KK^*}e^{Kt}$, where $K = k_2 + k_1/2 - k$ and $K^* = k_2 + k_1/2 + k$. The solution to (4.52) via (4.49) is

$$q^2 = \frac{u_1}{\alpha k}e^{-(k+k_1/2)t} + \frac{2v_1}{\alpha}e^{(k-k_1/2)t} + \frac{2\sigma}{KK^*}e^{k_2 t},$$

where u_1, v_1 are constants (cf. Ranganathan 1989).

Example 4

We give an outline to solve the class of equations (with parameters k_1, k_2 and k_3)

$$\ddot{q} + \frac{k_1}{t}\dot{q} + \frac{k_3}{t^2}q = -\dot{q}^2 q^{-1} + \sigma t^{k_2} q^{-1}, \quad \sigma \neq 0. \tag{4.53}$$

Here $E_p = \alpha t^{k_1}$, α constant. The form of (4.31), $\ddot{u} + (\frac{4k_3 - 2k_1 - k_1^2}{4t^2})u = 0$, is similar to that in Example 3 — the solution to (4.53) is obtained in a similar way.

4.4.2 *Nonlinearizable*

The following examples deals with differential equations which generate the $sl(2,R)$ algebra and the solutions are obtained by determining the transformation $T = G(t,q)$ and $Q = F(t,q)$ and remembering that

$$(u_1Q+v_1)^2 = 1 - Tu_1, \tag{4.54}$$

where u_1 and v_1 are arbitrary constants, is the general solution of the representative equation $TQ'' = Q'^3 - \frac{1}{2}Q'$.

Example 1

In the following differential equation $n = 4\mu - 3$ and, hence, generates the $sl(2,R)$ algebra.

$$\ddot{q} + p(t)\dot{q} + r(t)q = \frac{n+3}{4}\dot{q}^2q^{-1} + \frac{\sigma}{E_p^2}q^n, \tag{4.55}$$

where $\sigma \neq 0$. By (4.38), we firstly have the condition $\sigma = -8(\frac{1-n}{4})^3$. We need $a(t) = \dfrac{1}{u(t)}$ where $u(t)$ is obtained by the solution of (4.45). The transformation functions F and G are obtained by (4.13) and (4.39), respectively. The complete solution of (4.55) is obtained via (4.54). This example is contained in Theorem 5a in Ranganathan (1988).

Example 2

We solve the equation

$$\ddot{q} + \frac{1}{t}\dot{q} + (k^2 - \frac{m^2}{t^2})q = \sigma t^{-2}q^{-3}, \quad \sigma \neq 0. \tag{4.56}$$

Here $\mu = 0$, $n = -3$ and m a constant. Thus if $k \neq 0$, $\beta(t) = 4[\frac{1}{t^2} + \frac{4}{t^2}(k^2t^2 - m^2)]$ and (4.45) becomes $\ddot{u} + \frac{4}{t^2}(4k^2t^2 + 1 - 4m^2)u = 0$. We get $u = t^{\frac{1}{2}}[c_1J_l(4kt) +$

$c_2 Y_l(4kt)]$, which is in terms of Bessel functions, where $l = \sqrt{\frac{1}{4} - 4(1-4m^2)}$ so that $m \geq \frac{\sqrt{15}}{8}$ or $m \leq -\frac{\sqrt{15}}{8}$. Without loss of generality we set $c_1 = 1$ and $c_2 = 0$ since, as before, we require a single transformation. Thus, $a(t) = 1/[t^{\frac{1}{2}} J_l(4kt)]$ and by (4.13) $F = [\pi Y_l(4kt)]/[2(4k)^2 J_l(4kt)]$ (see Gradshteyn and Ryzhik 1980 p 679 for the integral of the Bessel function in combination with a rational function of t). Also by (4.39) $G = [\alpha^2 t q^2]/[J_l^2(4kt)]$ where $E_p = \alpha t$. By (4.48), $\sigma = -8/\alpha^2$ and the complete solution is then obtained by (4.54). The case $k = 0$ is soluble as an Euler equation in u and F and G are easily obtainable (Ranganathan 1988 does not provide a solution when $k = 0$).

Example 3

We integrate the equation for some constant ω

$$\ddot{q} = \frac{5}{4}\dot{q}^2 q^{-1} - \frac{1}{4}\omega q^2. \tag{4.57}$$

Here $\mu = \frac{5}{4}$ and $n = 2$ with $p(t) = r(t) = 0$. Thus, (4.45) becomes $\ddot{u} = 0$ so that $u = t$ is a particular solution. By (4.13), $F = -1/t$ and by (4.39) $G = [8K]/[t^2 q^{\frac{1}{2}}]$, where $E_p = K$ (K a constant). By (4.48) we get $\omega = -\frac{1}{2K^2}$. The solution of (4.57), by (4.54), is $q = -8/(\omega[\frac{1-v_1^2}{2u_1}t^2 + v_1 t - \frac{u_1}{2}]^2)$, where u_1 and v_1 are arbitrary constants (cf. Ranganathan 1988). A particular solution is given in Kamke (1971).

Example 4

The generalised Emden Equation is of the form

$$\ddot{q} + \frac{k}{t}\dot{q} = \sigma t^\omega q^n, \quad \sigma \neq 0, n \neq 1. \tag{4.58}$$

If (4.58) admits the $sl(2,R)$ algebra, then $n = -3$. Firstly, $E_p = \alpha t^k$ and by

$c_2Y_l(4kt)]$, which is in terms of Bessel functions, where $l = \sqrt{\frac{1}{4} - 4(1 - 4m^2)}$ so that $m \geq \frac{\sqrt{15}}{8}$ or $m \leq -\frac{\sqrt{15}}{8}$. Without loss of generality we set $c_1 = 1$ and $c_2 = 0$ since, as before, we require a single transformation. Thus, $a(t) = 1/[t^{\frac{1}{2}} J_l(4kt)]$ and by (4.13) $F = [\pi Y_l(4kt)]/[2(4k)^2 J_l(4kt)]$ (see Gradshteyn and Ryzhik 1980 p 679 for the integral of the Bessel function in combination with a rational function of t). Also by (4.39) $G = [\alpha^2 tq^2]/[J_l^2(4kt)]$ where $E_p = \alpha t$. By (4.48), $\sigma = -8/\alpha^2$ and the complete solution is then obtained by (4.54). The case $k = 0$ is soluble as an Euler equation in u and F and G are easily obtainable (Ranganathan 1988 does not provide a solution when $k = 0$).

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Example 4

The generalised Emden Equation is of the form

$$\ddot{q} + \frac{k}{t}\dot{q} = \sigma t^{\omega} q^n, \quad \sigma \neq 0, n \neq 1. \tag{4.58}$$

If (4.58) admits the $sl(2, R)$ algebra, then $n = -3$. Firstly, $E_p = \alpha t^k$ and by

(4.48), $-\dfrac{8}{\alpha^2 t^{2k}} = \sigma t^{\omega}$, i.e., $\omega = -2k$ and $\sigma = -8/\alpha^2$. Now (4.45) becomes $\ddot{u} + \dfrac{8k-4k^2}{t^2}u = 0$. Then $a(t) = 1/[t^{(1+\sqrt{\Delta})/2}]$, where $\Delta = 16k^2 - 32k + 1 \geq 0$. Then, $F = -1/[\sqrt{\Delta}t^{\sqrt{\Delta}}]$ and by (4.39) $G = \alpha q^2 t^{k-1-\sqrt{\Delta}}$. The complete solution is then obtained by (4.54) (cf. Theorem 5a in Ranganathan 1988). The case $n \neq -3$ does not admit $sl(3, R)$ or $sl(2, R)$. This case does not naturally arise here. We treat (4.58) in greater detail in Kara and Mahomed 1993 and as an application of Proposition 6.2 stated in chapter 6.

Example 5

We now give an outline to solve

$$\ddot{q} + \frac{k_1}{t}\dot{q} + \frac{k_3}{t^2}q = \sigma t^{k_2} q^{-1}, \quad \sigma \neq 0. \tag{4.59}$$

If $n = -3$, (4.59) admits the $sl(2, R)$ algebra. By (4.48), we have $\sigma t^{k_2} = -8/[\alpha^2 t^{k_1}]$, α a constant. Now (4.45) becomes $\ddot{u} + \frac{4}{t^2}(2k_1 - k_1^2 + 4k_3)u = 0$. The solution is obtained in a similar way to Example 4 with $\Delta = 16k_1^2 - 32k_1 - 64k_3$.

4.5. Discussion

It is well known that the solutions of differential equations can be determined by finding appropriate point transformations and mapping the differential equation to the representative differential equation. We have shown that a point transformation can be obtained by constructing a Lagrangian for the differential equation and mapping it to the equivalent Lagrangian of the representative differential equation. It is noted here that for (4.1), we constructed a Lagrangian of quadratic form -- the above analysis can be carried out for other forms of Lagrangians. In particular, the $sl(2, R)$ analysis may be performed using a Lagrangian of the form (3.40). In section 4.4 we saw that a wide variety of examples of type (4.1) can be solved when they are linearizable ($sl(3,R)$ algebra) or when they admit the $sl(2,R)$ algebra.

In each of the examples in section 4.4, we do not only have complete solutions but also a Lagrangian, of the quadratic type, equivalent to the representative Lagrangian. We note that U and V are easily obtainable from (4.20) and (4.21) or (4.32) and (4.33), respectively. The general forms of the Lagrangians are given below.

For the linearizable case of (4.1), i.e. $\mu = n$, the Lagrangian is

$$L = \frac{1}{2}E_p q^{-2n}\dot{q}^2 + E_p[s\frac{q^{-n+1}}{-n+1} - r\frac{q^{-2n+1}}{-2n+1}].$$

Since L is equivalent to $\bar{L} = \frac{1}{2}Q'^2$, it admits the maximal $A_{5,40}$ algebra and is s^5-equivalent — see the previous chapter and Kara, Mahomed and Leach (1993).

For the non-linearizable case corresponding to $sl(2,R)$ (or more correctly $A^{II}_{3,8}$) of (4.1), the Lagrangian is

$$L = \frac{1}{2}q^{-2\mu}E_p\dot{q}^2 + E_p[s\frac{q^{n-2\mu+1}}{n-2\mu+1} - r\frac{q^{2-2\mu}}{2-2\mu}].$$

In particular, if $n = 4\mu - 3$ ($sl(2,R)$ case),

$$L = \frac{1}{2}q^{-2\mu}E_p\dot{q}^2 + E_p[s-r]\frac{q^{2\mu-2}}{2\mu-2},$$

which is equivalent to $\bar{L} = \dfrac{Q'}{T} + \dfrac{1}{2TQ'}$. Hence, the algebra admitted by L is $sl(2,R)$ and is s^3-equivalent.

Here, we considered two examples of s^r-equivalent Lagrangians which provides us with an elegant way of integrating ordinary differential equations. One should also bear in mind that we used the representative Lagrangian $\bar{L} = \frac{Q'}{T} + \frac{1}{2TQ'}$ which admits the $sl(2, R)$ algebra. One could equally well carry out the computations for the representative Lagrangian $\bar{L} = \frac{Q'}{T} - \frac{1}{2TQ'}$ corresponding to the representative differential equation $TQ'' = -Q'^3 - \frac{1}{2}Q'$.

In the following chapter we present standard forms for particle Lagrangians of linear (or linearizable) scalar second-order ordinary differential equations. Moreover, a general classifiction scheme is outlined.

5. CANONICAL FORMS FOR PARTICLE LAGRANGIANS OF THE LINEAR SECOND-ORDER EQUATION

5.1. Introduction

The most elementary equivalence problem in the calculus of variations or in Lagrangian mechanics is connected to the linear second-order ordinary differential equation. The problem is to determine under what conditions two variational problems of first-order of the Euler-Lagrange linear ordinary differential equation are transformable into each other by means of a suitable point transformation.

The solution of this problem is essential for the solution of the more general equivalence problem connected to scalar second-order ordinary differential equations. We dealt with aspects of the general problem in chapter 3. The objective of this chapter is to determine all canonical forms for particle Lagrangians defined on the line associated with the Euler-Lagrange equation of a linear (equivalently, linearizable) second-order equation. After doing so we make a synthesis of all first-order Lagrangians on the line that admit symmetry algebra.

A linear second-order ordinary differential equation admits the maximum eight dimensional Lie algebra $sl(3, R)$. In contrast, a particle Lagrangian of a *linear* second-order ordinary differential equation, as we learn here, can admit any one of 1,2,3 or 5 dimensional Noether algebra of point symmetries (the Lie algebra of point symmetries of the action functional). We note here that there is no zero symmetry case.

Some of the results obtained here can be found in Mahomed, Kara and Adam (1993).

5.2. Equivalent and alternate Lagrangians

A general second-order linear equation in q

$$\ddot{q} + a(t)\dot{q} + b(t)q = c(t), \tag{5.1}$$

where a,b and c are analytic in t, is equivalent to the simplest equation $Q'' = 0$ via a point transformation. Thus, a Lagrangian for the linear equation (5.1) is obtainable by invoking (4.2) using a point change of variables where $\bar{L}(T, Q, Q')$ will be a Lagrangian of the equation $Q'' = 0$. The details of the methodology can be found in chapter 4. Here we give a summary using $\bar{L} = \frac{1}{2}Q'^2$ (the usual Lagrangian of $Q'' = 0$) as our transformed Lagrangian. Firstly, we impose a quadratic Lagrangian, $L = \frac{1}{2}U(t,q)\dot{q}^2 + V(t,q)$, on (5.1). It can be shown that the transformation variables are given by

$$G = \int \alpha^2(t,q)dt\,, \qquad F = \alpha \int \sqrt{U}dq + \beta(t),$$

where α and β are functions of t to be determined. For (5.1) we get that $U = E$, where $E = \exp \int_{t_0}^{t} a(t)dt$ and $V = -\frac{1}{2}Ebq^2 + Ecq$. Thus,

$$F = \alpha E^{\frac{1}{2}} + \beta$$

and the gauge function is

$$f(t,q) = -E\frac{q^2\dot{\alpha}}{2\alpha} - aE\frac{q^2}{4} - \dot{\beta}E^{\frac{1}{2}}\frac{q}{\alpha} + \gamma(t)\,.$$

The functions α, β and γ are obtained by solving the following equations:

$$(\frac{\dot{\alpha}}{\alpha})^2 - (\frac{\dot{\alpha}}{\alpha})^{\cdot} - \frac{\dot{a}}{2} - \frac{a^2}{4} + b = 0,$$

$$(\frac{\dot{\alpha}}{\alpha})(\frac{\dot{\beta}}{\alpha}) - (\frac{\dot{\beta}}{\alpha})^{\cdot} - cE^{\frac{1}{2}} = 0,$$

$$\frac{1}{2}(\frac{\dot{\beta}}{\alpha})^2 + \dot{\gamma} = 0.$$

The first equation is a Riccati equation in $\dot{\alpha}/\alpha$ and can be written in the form

$$\ddot{\mu} + \delta(t)\mu = 0,$$

where $\mu = 1/\alpha$ and $\delta(t) = -\dot{a}/2 - a^2/4 + b$. Whence,

$$\beta = -\int \alpha^2 \int \frac{1}{\alpha} cE^{\frac{1}{2}} dt dt.$$

There are an infinitude of Lagrangians for (5.1). Our interest is in classifying them. The benefit of such a solution contributes to our knowledge of Lagrangian mechanics, symmetries and conservation laws, and indeed the inverse problem in the calculus of variations.

Among the alternative Lagrangians of (5.1) there could be Lagrangians that are equivalent to each other in the sense of (4.2). If there are such Lagrangians, how do we determine them? Moreover, there could be different equivalence classes of Lagrangians for (5.1) (we remind the reader that two Lagrangians are said to belong to the same equivalence class if one can be transformed into the other by means of a prescribed point transformation). In other words, among the alternative Lagrangians there could be Lagrangians that are not equivalent to each other. Our methodology, which involves investigating the Lie algebra of Noether symmetries of the Lagrangians, allows us to classify inequivalent Lagrangians or, stating it in another way, classify equivalence classes of Lagrangians for (5.1).

Without loss of generality, we investigate our Lagrangian equivalence problem for $\ddot{q} = 0$. That is, we consider alternative Lagrangians for this equation in order to obtain our classification of equivalent Lagrangians for linear equations. The reason being that we can map any Lagrangian of $\ddot{q}=0$ to a Lagrangian of (5.1) by means of the equivalence relation (4.2), using a point transformation. So we consider

$$L' = \int^{\dot{q}}\int^{r} g(s, q-ts)ds\,dr + H(t,q)\dot{q} + J(t,q), \tag{5.2}$$

where g is an arbitrary function of the first integrals of $\ddot{q}=0$ and $H_t = J_q$ (see chapter 2). All the Lagrangians of the equation $\ddot{q}=0$ can be obtained from (5.2). Our objective is to classify the Lagrangians contained in (5.2) by investigating the Lie algebra of Noether symmetries they admit.

5.3. Canonical forms of Particle Lagrangians for Linear equations

We firstly look at the case when (5.2) admits a maximal one-dimensional algebra A_1. The Noether symmetry for this case is (see chapter 3)

$$G' = (a_1 + a_2 t + a_3 q + a_4 tq + a_5 t^2)\frac{\partial}{\partial t} + (a_4 q^2 + a_5 tq + a_6 + a_7 q + a_8 t)\frac{\partial}{\partial q}, \tag{5.3}$$

where the a_i's are arbitrary constants. The function g satisfies

$$\begin{aligned}&(a_6 - a_1\dot{q} - a_3\dot{q}v + a_4 v^2 + a_7 v)\partial g/\partial v + (2a_7 - a_2 - 3a_3\dot{q} + 3a_4 v)g \\ &\quad + (a_8 + a_7\dot{q} - a_2\dot{q} + a_5 v - a_3\dot{q}^2 + a_4\dot{q}v)\partial g/\partial\dot{q} = 0\end{aligned} \tag{5.4}$$

where $v = q - t\dot{q}$. By setting $a_2 = a_7 = 1$ and the rest of the a_i's to zero in (5.3), we obtain the symmetry $t\partial/\partial t + q\partial/\partial q$. Thus, from (5.4) we obtain the partial differential equation (p.d.e) $v\partial g/\partial v + g = 0$. Thus, $g = \alpha(\dot{q})/v$, where α is an arbitrary function of $\dot{q}$. By choosing various combinations of the a_i's in a likewise manner, we obtain all the cases in Lie's classification of the one-dimensional algebras of projective transformations in the plane (Lie 1893). All the possible one-dimensional cases are listed below.

(i) $t\partial/\partial t + q\partial/\partial q$

(ii) $t\partial/\partial t + aq\partial/\partial q$ (a constant and $a \neq 0, 1$)

(iii) $\partial/\partial t + t\partial/\partial q$

(iv) $\partial/\partial t + q\partial/\partial q$

(v) $\partial/\partial q$.

In each of the cases we obtain g by solving the p.d.e (5.4). The case (v) yields $g = \mu(\dot{q})$ so that $\partial/\partial t$ is also a symmetry of the Lagrangian. Since we are considering the maximal one-dimensional algebras, we conclude that (v) does not give a solution for g. The forms of g in the remaining cases are (in each case α, β, γ and δ are arbitrary)

(i) $g = \alpha(\dot{q})/v$,

(ii) $g = \dot{q}^{(1-2a)/(a-1)}\delta(v\dot{q}^{a/(1-a)})$,

(iii) $g = \beta(v + \frac{1}{2}\dot{q}^2)$,

(iv) $g = \gamma(\dot{q}\exp(v/\dot{q}))/\dot{q}^2$.

We now analyse the case when (5.2) admits a maximal two-dimensional algebra. There are two algebras of of dimension two, $2A_1$ and A_2. We prove the following non-existence theorem concerning connected or proportional symmetries that generate $2A_1$ or A_2.

Theorem 5.1. A Lagrangian for a linear second-order equation does not admit a maximal two-dimensional Noether algebra of proportional symmetries.

Proof. We investigate (5.2) for proportional symmetries, i.e., symmetries G_1 and G_2 where $G_2 = \rho G_1$ for some function ρ of t and q. Since the symmetries of (5.2) are of the form (5.3), for $G_2 = \rho G_1$ to hold the function ρ has to be linear or linear fractional in t and q. Accordingly we deduce the following forms for G_1 and G_2.

(i) $G_1 = (a_1 + a_2 t)\partial/\partial t + (a_6 + a_7 q)\partial/\partial q$ and $G_2 = (At + Bq)G_1$

(A and B are arbitrary constants)

(ii) $G_1 = t\partial/\partial t$ and $G_2 = \dfrac{a_1 + a_3 q}{t} G_1$

(iii) $G_1 = q\partial/\partial t$ and $G_2 = \dfrac{a_1 + a_2 t}{q} G_1$.

The generators G_1 and G_2 contained in each of (i), (ii) and (iii) will be Noether symmetries of (5.2) if the function g satifies

(i) $g = \dfrac{(A + B\dot{q})^{-2}}{a + b\dot{q} + cv}$, $\quad a$, b and c are constants,

(ii) $g = \dfrac{\dot{q}^{-1}}{(a + bv)^2}$, $\quad a$ and b are constants,

(iii) $g = \dfrac{\dot{q}^{-1}}{(a\dot{q} + bv)^2}$, $\quad a$ and b are constants,

repectively. The above forms for g are obtained by solving (5.4) for G_1 and G_2 in turn. For example, if g as in (5.2) is inserted in (5.4), we obtain $a_5 = a_6 = a_7 = a_8 = 0$ and $a_1 b + a a_2 = 0$, where a_2, a_3 and a_4 are arbitrary. These imply three or five symmetries for (5.2) where g is of the form (iii). Likewise for (i) and (ii) we obtain three or five symmetries.

Thus, (5.2) does not admit a maximal two-dimensional Noether algebra of proportional symmetries G_1 and G_2 such that $G_2 = \rho G_1$. ◇

As an immediate consequence of Theorem 5.1, we note that a Lagrangian of any scalar second-order equation does not admit a maximal two-dimensional algebra of connected symmetries.

The situation when (5.2) admits a maximal two-dimensional Noether algebra corresponding to the unconnected case is again obtained by appropriately choosing combinations of the a_i's. We utilise the cases that correspond to the Lie (1893) classification — these are listed below. However, when solving for g using (5.4), some cases provide no solution and some (viz. (vii) and (x)) coincide with the connected case for which it was shown that no g exists.

Let $p = \partial/\partial t$ and $r = \partial/\partial q$. Then

(i) r and $p + tr$

(ii) r and $qr + p$

(iii) tr and $tp + r$

(iv) tr and $tp + aqr, \quad a \neq 0, 1$,

(v) r and tp

(vi) p and r

(vii) r and tr

(viii) tp and qr

(ix) r and $tp + qr$

(x) r and qr

(xi) $p + tr$ and $tp + 2qr$.

Only the following three cases have solutions for g (C and D are arbitrary constants).

(ii) $A_2 \quad g = C\dot{q}^{-2}$,

(vi) $2A_1 \quad g = \alpha(\dot{q})$,

(viii) $2A_1 \quad g = D/(v\dot{q})$.

The function α, not a constant, is arbitrary but is not of the form (i) to (vi) for three-dimensional algebras given below.

In Mahomed and Leach (1989a) the real three-dimensional algebras admitted by second-order ordinary differential equations were classified. The possible three-dimensional real algebras admitted by a Lagrangian of a linear second-order ordinary differential equations can be deduced from the proof of Theorem 2.3. By the solution of the p.d.e (5.4), each of the following cases arise providing a solution for g. The following is a consequence of Theorem 2.3 but is reproduced here for completeness.

(i) $A^I_{3,2}\quad g = A\exp(-\dot{q})$,

(ii) $A^I_{3,4}\quad g = A\dot{q}^{-3/2}$,

(iii) $A^{a\,I}_{3,5}\,(a = \frac{1+s}{2+s})\quad g = A\dot{q}^s\quad s \neq 0, -2, -3$,

(iv) $A_1 \oplus A^I_2\quad g = A\dot{q}^{-1}$,

(v) $A^I_{3,6}\quad g = A(1+\dot{q}^2)^{-3/2}$,

(vi) $A^{b\,I}_{3,7}\quad g = A(1+\dot{q}^2)^{-3/2}\exp(b\arctan\dot{q})$,

(vii) $A^{III}_{3,8}\quad g = A(q - t\dot{q} + \dot{q}^2/2)^{-3/2}$,

(viii) $A_{3,9}\quad g = A[1+(q-t\dot{q})^2+\dot{q}^2]^{-3/2}$.

We now prove the result pertaining to four-dimensional algebras invoking the Lie forms of vector fields which provides us with an alternative but elegant way of proving Theorem 2.4 for particle Lagrangians of linear equations. The proof relies only on real four-dimensional algebras.

Theorem 5.2. A particle Lagrangian for a linear second-order ordinary differential equation does not admit a maximal four-dimensional Noether algebra.

The proof of this result is obtained by analysing the following Lie forms of real vector fields in the plane and invoking (5.4).

(i) p, r, tr and $atp + qr$

(ii) p, r, tr and $tp + 2qr$

(iii) p, r, tr and tp

(iv) tp, qp, tr and qr

(v) p, r, tp and qr

(vi) r, qr, tr and tp.

It is easy to verify that no g arises in each case. An alternative and more general proof appears in chapter 2 and in Mahomed, Kara and Leach (1993a).

Finally, we state the theorem relating to the case when (5.2) admits the maximal five-dimensional algebra. This is referred to as the Noether 'counting' theorem (see also chapter 2).

Theorem 5.3. The maximum dimension of the Noether point symmetry algebra of a particle Lagrangian corresponding to a linear second-order equation is five.

The proof of it follows by analysing the five-dimensional algebra realizations given below and invoking (5.4).

(i) p, r, tr, $tp - qr$ and qr

(ii) tr, $tp - qr$, qp, $t^2p + tqr$ and $tqp + q^2r$

(iii) p, r, tp, tr and qr

Only in case (ii) a g of the form $g = \mathcal{A}/v^3$, where $\mathcal{A}$ is a constant, arises. Indeed, the 'counting' theorem is proved in chapter 2 in a more general setting. However, the proof there presupposes the above Theorem 5.3.

We now make a final remark regarding the situation when the Lagrangian $L = \frac{1}{2}\dot{q}^2$ is form invariant under a point transformation, i.e., by invoking (4.2), we determine the change of variables $Q = F(t, q)$ and $T = G(t, q)$ such that L is mapped to $\bar{L} = \frac{1}{2}Q'^2$. The functions F and G are obtained by the method outlined in section 5.2. They are

$$G = \int_{t_0}^{t} \alpha^2(s)\, ds \quad \text{and} \quad F = \alpha q + \beta$$

and the gauge function is $f = -\frac{\dot{\alpha}}{\alpha}\frac{q^2}{2} - \frac{\dot{\beta}}{\alpha}q$. By further manipulation using the equations for α and β given in section 5.2, we arrive at the most general projective point transformation for the invariance of $L = \dot{q}^2/2$, viz.,

$$Q = \frac{A_1(q - B_0 A_1)}{t + A_0} + B_1 \quad \text{and} \quad T = -\frac{A_1^2}{t + A_0} + B_2,$$

where the $A_i's$ and $B_i's$ are five arbitrary constants (cf. Aguirre and Krause 1991 who obtain six). This is a subgroup of the eight dimensional projective group $SL(3, R)$. Hence, the corresponding Noether algebra is five-dimensional which is maximal. By interepreting the constants as five essential continuous parameters, one obtains a realization of a five-dimensional group and the following symmetry generators or a linear combination results

$G_1 = \partial/\partial t$

$G_2 = \partial/\delta q$

$G_3 = t\partial/\partial q$

$G_4 = t^2\partial/\partial t + tq\partial/\partial q$

$G_5 = 2t\partial/\partial t + q\partial/\partial q$

The above symmetry generators correspond to the case for g a constant in (5.2). This case is equivalent, by means of (4.2), to the Lie form (ii) in Theorem 5.3 by the change of coordinates $T = -q/t$, $Q = 1/t$. It is also relevant to mention here that the most general g that gives rise to five symmetries is of the form $g = (b_1\dot{q} + b_2 v + b_3)^{-3}$ where the b_i's are constant (see chapter 2).

5.4. Discussion

This chapter in conjunction with chapters 2 and 3 completely classify first-order Lagrangians associated with scalar second-order ordinary differential equations. We summarize these below. For the case in which the Euler-Lagrange equation admits the $sl(3, R)$ algebra, we list the functions g from which the Lagrangian is obtained from (5.2) subject to the condition $H_t = J_q$ (note that $v = q - t\dot{q}$). For the other cases we list the Lagrangian, L.

Table 5.1

Let $\partial/\partial t = p$ and $\partial/\partial q = r$. Then

Dim	*Symmetry*	*Lagrangian/g*
1	r	$\int^{\dot{q}} F(t,x)\,dx \quad (\dot{F} = 0)$
	$tp + qr$	$g = \alpha(\dot{q})/v$
	$tp + aqr \quad a \neq 0, 1$	$g = \dot{q}^{(1-2a)/(a-1)}\alpha(v\dot{q}^{a/(1-a)})$
	$p + tr$	$g = \alpha(v + \frac{1}{2}\dot{q}^2)$
	$p + qr$	$g = \alpha(\dot{q}\exp(v/\dot{q}))/\dot{q}^2$
2	p, r	$\int^{\dot{q}}(\int^{r} ds/\mathcal{H}(s))dr - t\dot{q}$
	$r, tp + qr$	$\frac{1}{t}\int^{\dot{q}}(\exp\int^{r} ds/\mathcal{H}(s))dr$
	$r, qr + p$	$g = A\dot{q}^{-2}$
	p, r	$g = \alpha(\dot{q})$
	tp, qr	$g = A/(v\dot{q})$

3	$p, r, tp - qr$	$-4\dot{q}^{\frac{1}{2}} + q$
	$p, r, qp - tr$	$-(1+\dot{q}^2)^{\frac{3}{2}} + t\dot{q}$
	$p, tp + qr, 2tqp + q^2 r$	$\dot{q}/t + 1/(2t\dot{q})$ *or* $\dot{q}/t - 1/(2t\dot{q})$
	$r, tp + qr, 2tqp + (q^2 - t^2)r$	$t^{-1}\sqrt{1+\dot{q}^2} + At^{-1}\dot{q}$
	$(1+t^2)p + tqr, tqp + (1+q^2)r, qp - tr$	**
	$r, p, tp + (t+q)r$	$g = A\exp(-\dot{q})$
	$p, r, tp - qr$	$g = A\dot{q}^{-\frac{3}{2}}$
	$p, r, tp + aqr (a = \frac{1+s}{2+s})$	$g = A\dot{q}^s \quad s \neq 0, -2, -3$
	p, r, tp	$g = A\dot{q}^{-1}$
	$p, r, qp - tr$	$g = A(1+\dot{q}^2)^{-\frac{3}{2}}$
	$p, r, (bt+q)p + (bq - t)r, b > 0$	$g = A(1+\dot{q}^2)^{-\frac{3}{2}}\exp(b\arctan\dot{q})$
	$r, tp + qr, 2tqp + (q^2 - t^2)r$	$g = A(q - t\dot{q} + \dot{q}^2/2)^{-\frac{3}{2}}$
	$(1+t^2)p + tqr, tqp + (1+q^2)r, qp - tr$	$g = A[1 + (q - t\dot{q})^2 + \dot{q}^2]^{-\frac{3}{2}}$
5	$p, r, tr, t^2 p + tqr, 2tp + qr$	$\frac{1}{2}\dot{q}^2$

**: here the Lagrangian is of the form

$$L = \frac{1}{x^2}(1+t^2)^{-\frac{3}{2}}[(y - \dot{q})\sin\omega + \beta\sec\omega] + b(t,q)\dot{q} + c(t,\dot{q}),$$

where b and c satisfy $b_t = c_q + A(1+t^2+q^2)^{-\frac{3}{2}}$ and $x = \sqrt{1+t^2+q^2}/(1+t^2)$, $\beta = qx$, $\tan\omega = (\dot{q} - y)/x$ and $y = tq/(1+t^2)$.

We note that in the above table, A is a non-zero constant and α an arbitrary non-zero function of its argument in each case. Also, Table 5.1 presents a real classification of particle Lagrangians. In the complex classification there are seven non-isomorphic three-dimensional algebras and one no longer requires the fifth, eleventh and thirteenth cases of dimension three in Table 5.1.

Some remarks are now in order. Firstly, and this is an important point peculiar to Noether algebras, two Noether algebras can be similar but their respective Lagrangians need not be equivalent to each other. This can be easily be observed from Table 5.1. To illustrate this, we consider the algebra of symmetries p, r, $tp-qr$. This representation implies two Lagrangians (see Table 5.1) which are not equivalent to each other by means of a point transformation. Secondly, we make the point that the dimensionality of the Noether algebra of a Lagrangian associated with a linear or linearizable equation need not be maximal (i.e., five-dimensional). This was apparent in our discussion on alternative Lagrangians.

6. REDUCTION USING LIE AND NOETHER SYMMETRIES ASSOCIATED WITH FIRST INTEGRALS

6.1. Introduction

Sarlet and Cantrijn (1981) proved a theorem that the Noether point symmetry, G, and corresponding Noether first integral, I, for systems of second-order ordinary Euler-Lagrange equations satisfy the property $G^{[1]}I = 0$. It appears that the importance of this theorem has not been realized since the literature does not reflect applications of it.

Furthermore, whilst the significance of first integrals of differential equations is generally accepted, not much has been done in the literature towards determining first integrals of partial differential equations.

In section 6.2, we formulate a relationship between the symmetries of differential equations and symmetries of the corresponding constants of the motion (conservation laws). We show how this relationship can be used to reduce the order of equation and in some cases completely solve the differential equation.

In section 6.3, we briefly discuss some work by Bluman *et al* (1988) dealing with 'new symmetries' and more recently in Bluman (1993) wherein potential symmetries and linearization of partial differential equations are studied. In particular, we highlight the significance of conserved forms of differential equations and later reflect its relevance to the work done here.

In section 6.4, we present some examples applying the results obtained in the previous sections. Some of the material in this chapter can be found in Kara, Mahomed and Adam (1993).

Firstly, we use the following definition in conjunction with Definition 1.3 to determine first integrals.

Definition 6.1. A point symmetry, G, of a given kth-order differential equation $E(t, q^{[k]}) = 0$ is said to be ***associated*** with a first integral, I, if

$$G^{[k-1]} I_j = 0, \quad 1 \leq j \leq m. \tag{6.1}$$

This definition is motivated by the works of Sarlet and Cantrijn (1981) on second-order ordinary Euler-Lagrange equations and is an analogue. However, we shall show in the next section that this property need not hold (see Theorem 6.1) so (6.1) is really a strict invariance. Notwithstanding this, we apply this property in subsection 6.4.2 to some familiar partial differential equations to obtain first integrals.

6.2. Conservation laws, reduction of order and Noether's Theorem

We now relate the above ideas to Noether's theorem (see section 1.4) by constructing a relationship between the symmetries of Lagrangians and the symmetries associated with the respective conservation laws. A significant result, which is hardly mentioned in the literature, will be stated. A version of it appears in Sarlet and Cantrijn (1981) and is stated with particular reference to second-order ordinary differential equations. Here we present a more general statement. This result is not only significant in the theoretical sense but, as will be expounded in section 6.4, can be used in an elegant way to completely solve certain Euler-Lagrange equations. In this section we restrict the discussion to second-order equations.

Firstly, by Noether's theorem, to each symmetry, G, corresponding to a given Lagrangian, L, we recall that a first integral of the respective Euler-Lagrange equation is given by

$$I_j = f_j - P_j - L\xi_j.$$

In particular, for first-order Lagrangians, we have $P_j = (\eta - \sum_{i=1}^m \xi_i q_i)\partial L/\partial q_j$.

Theorem 6.1. The Noether symmetry, G, of a first-order Lagrangian, $L(t, q^{[1]})$, and conservation law $I = (I_1, \cdots, I_m)$, satisfy the following property:

$$G^{[1]}(I_j) = S_j^1 - S_j^2 - S_j^3, \quad 1 \leq j \leq m,$$

where

$$S_j^1 = \sum_{i\neq j}^m \xi_i \frac{\partial f_j}{\partial t_i} - \xi_j \sum_{i\neq j}^m D_i f_i + \frac{\partial f_j}{\partial q} \sum_{i\neq j}^m \xi_i q_i,$$

$$S_j^2 = L[\sum_{i\neq j}^m \xi_i \frac{\partial \xi_j}{\partial t_i} - \xi_j \sum_{i\neq j}^m D_i \xi_i + \frac{\partial \xi_j}{\partial q} \sum_{i\neq j}^m \xi_i q_i],$$

$$S_j^3 = \frac{\partial L}{\partial q_j}[G^{[1]}(\eta - \sum_{i=1}^m \xi_i q_i) - (\eta - \sum_{i=1}^m \xi_i q_i)\operatorname{Div}\xi] - (\eta - \sum_{i=1}^m \xi_i q_i)\sum_{i=1}^m \frac{\partial L}{\partial q_i}\frac{\partial \zeta_i}{\partial q_j}.$$

Proof. For this case $P_j = \partial L/\partial q_j(\eta - \sum_{i=1}^m \xi_i q_i)$. Then

$$G^{[1]}(I_j) = G(f_j) - G^{[1]}(\eta - \sum_{i=1}^m \xi_i q_i)\frac{\partial L}{\partial q_j} - (\eta - \sum_{i=1}^m \xi_i q_i) G^{[1]}(\frac{\partial L}{\partial q_j})$$
$$- G^{[1]}(L)\xi_j - LG^{[1]}(\xi_j).$$

By (1.7) and the result

$$G^{[1]}(\frac{\partial L}{\partial q_j}) = \frac{\partial}{\partial q_j}(G^{[1]}(L)) - \sum_{i=1}^m \frac{\partial L}{\partial q_i}\frac{\partial \zeta_i}{\partial q_j},$$

we get that

$$G^{[1]}(I_j) = G(f_j) - G^{[1]}(\eta - \sum_{i=1}^{m} \xi_i q_i)\frac{\partial L}{\partial q_j}$$

$$- (\eta - \sum_{i=1}^{m} \xi_i q_i)\left\{ \frac{\partial}{\partial q_j}[\sum_{i=j}^{m} D_i f_i - L\sum_{i=j}^{m} D_i \xi_i] - \sum_{i=1}^{m} \frac{\partial L}{\partial q_i}\frac{\partial \zeta_i}{\partial q_j} \right\}$$

$$- \left(\sum_{i=j}^{m} D_i f_i - L \sum_{i=j}^{m} D_i \xi_i \right) \xi_j L G(\xi_j).$$

By expanding and simplifying we obtain the required result. ◇

Proposition 6.1. If G is a symmetry of the system of equations $I_j = B_j$, $1 \leq j \leq m$, $B_j = B_j(t, q)$, then

$$\sum_{i=1}^{m} \xi_i \frac{\partial B_j}{\partial t_i} + \eta \frac{\partial B_j}{\partial q} = S_j^1 - S_j^2 - S_j^3.$$

Proof. If G is a symmetry of the system $I_j = B_j$, then

$$0 = G^{[1]}(I_j - B_j) = G^{[1]}(I_j) - G^{[1]}(B_j).$$

The result follows by Theorem 6.1. ◇

Remark: Corresponding to an Euler-Lagrange equation

$$E(t, q^{[k]}) = 0 \tag{6.2a}$$

we have a system of $(k-1)$th-order partial differential equations

$$I_j = B_j, \quad 1 \leq j \leq m, \tag{6.2b}$$

where the B_j's are functions of the variables t and q. This reduced system has a readymade symmetry G if it is a symmetry of (6.2a). Then G may be used to reduce (6.2b) and, hence, further used to reduce (6.2a).

Proposition 6.2. For $m = 1$, $G^{[1]}(I) = 0$ and G is a point symmetry for the reduced first-order equation $I(t, q, \dot{q}) = c$, where c is an arbitrary constant.

Proof. The first result follows easily by noting that S_1^1, S_1^2 and S_1^3 in Theorem 6.1 vanish. The second part follows since for c a constant
$G^{[1]}(I - c) = G^{[1]}(I) - G^{[1]}(c) = G^{[1]}(I) = 0.$ ◇

We remind the reader that the results here also applies to systems of equations and it is not difficult to carry this over. As mentioned earlier, Sarlet and Cantrijn (1981) presented a result equivalent to the first part of Proposition 6.2 for systems, viz., that $G^{[1]}I = 0$, where I is a Noether invariant. Also, previously, Lutzky (1979) proved this result for scalar second-order ordinary differential equations. We use Definition 6.1 and the above propositions in section 6.4.

6.3. Potential symmetries and partial differential equations in conserved form

It has been shown in Bluman (1993) that the solutions q of a partial differential equation (6.2a) are embedded in the solutions $(q, V_1, \cdots, V_{m-1})$, $V_j = V_j(t)$, of a $(k-1)$th-order system of partial differential equations

$$\mathcal{I}_1(t, q^{[k-1]}) = \frac{\partial V_1}{\partial t_2}$$

$$\mathcal{I}_j(t, q^{[k-1]}) = (-1)^{j-1}[\frac{\partial V_j}{\partial t_{j+1}} + \frac{\partial V_{j-1}}{\partial t_{j-1}}], \quad 1 < j < m$$

$$\mathcal{I}_m(t, q^{[k-1]}) = (-1)^{m-1} \frac{\partial V_{m-1}}{\partial t_{m-1}}, \tag{6.3}$$

where the auxiliary system (6.3) is obtained by introducing potentials $(V_1, \cdots, V_{m-1})$. This is deduced by first writing (6.2a) in conserved form, viz.,

$$\mathrm{Div}\,\mathcal{I} = 0, \tag{6.4}$$

where $\mathcal{I} = (\mathcal{I}_1, \cdots, \mathcal{I}_m)$. In the reference cited above, Bluman utilizes the results to linearize certain partial differential equations. It is frequently not obvious or a straightforward matter to write a partial differential equation in conserved form. In the next section, we invoke Definition 6.1 to obtain conserved forms for some equations of mathematical physics. However, before we consider partial differential equations, we embark on the conceptually simpler ordinary differential equations.

6.4. Examples

6.4.1. *Reduction of second-order ordinary differential equations*

6.4.1.1. In chapter 4 the solutions of some classes of non-linear second-order ordinary differential equations were discussed. In particular, when these equations generate the Lie algebras $sl(2,R)$ and $sl(3,R)$. The solutions were obtained by employing the method of 'Equivalent Lagrangians'. A particular case of these equations is the Emden-Fowler equation (4.58) for which solutions for some cases were obtained. In Kara and Mahomed (1993), equation (4.58) was investigated by employing the Lie method. It was mentioned therein that the method was tedious and cumbersome.

We analyse (4.58), using the results obtained in section 6.2. Firstly, it is easily verifiable that a Lagrangian for (4.58) is

$$L = \frac{1}{2} t^k \dot{q}^2 + \frac{\sigma}{n+1} t^{\omega+k} q^{n+1}, \quad n \neq -1. \tag{6.5}$$

The symmetries of (6.5) are obtained by substituting L into the Killing-type equation (1.7). For $n \neq 2$ and $n \neq -1$, we get the following cases. (The cases $n = 2$ and $n = -1$ can be analysed in a similar way. The Lagrangian for $n = -1$ is $L = \sigma t^{\omega+k} \ln q + \frac{1}{2} t^k \dot{q}^2$.)

$n \neq 2, n \neq -1.$

(i). $n = \frac{3+k+2\omega}{k-1}$, $k \neq 1$.

$$G = \tfrac{1}{1-k} t\partial/\partial t + \tfrac{1}{2} q\partial/\partial q$$

$$f = 0$$

$$I = t^k \dot{q} q + \tfrac{t^{k+1}}{k-1} \dot{q}^2 - \tfrac{\sigma}{k+\omega+1} t^{k+\omega+1} q^{2(k+\omega+1)/(k-1)}.$$

(ii). $n = \frac{k-3-\omega}{1-k}$, $k \neq 1$, $n \neq -3$.

$$G = \tfrac{1}{2(1-k)^2} t^{2-k}\partial/\partial t + \tfrac{1}{2(1-k)} t^{1-k} q\partial/\partial q$$

$$f = \tfrac{1}{4}$$

$$I = \tfrac{1}{4(1-k)} t^2 \dot{q}^2 - \tfrac{1}{2} t q \dot{q} + \tfrac{\sigma}{\omega+2} t^{\omega+2} q^{(\omega+2)/(k-1)}.$$

(iii). $k = -\omega/2$, $k \neq 1$.

$$G = t^k \partial/\partial t$$

$$f = 0$$

$$I = \tfrac{1}{2} t^{-\omega} \dot{q}^2 - \tfrac{\sigma}{n+1} q^{n+1}.$$

(iv). $k = 1$, $\omega = -2$.

$$G = t\partial/\partial t$$

$$f = 0$$

$$I = \tfrac{1}{2} t^2 \dot{q}^2 - \tfrac{\sigma}{n+1} q^{n+1}.$$

(v). $n=-3$, $k=-\omega/2$, $k\neq 1$.

$$G^1=\frac{2}{(2+\omega)^2}t^{2+\omega/2}\partial/\partial t+\frac{1}{2+\omega}t^{1+\omega/2}q\partial/\partial q$$

$$f^1=\tfrac{1}{4}q^2$$

$$I^1=\tfrac{1}{4}q^2-\frac{1}{(2+\omega)}t\dot q+\frac{1}{(2+\omega)^2}t^2\dot q^2+\frac{\sigma}{(2+\omega)^2}t^{\omega+2}q^{-2}.$$

$$G^2=\frac{2}{(2+\omega)}t\partial/\partial t+\tfrac{1}{2}q\partial/\partial q$$

$$f^2=0$$

$$I^2=\tfrac{1}{2}t^{-\omega/2}q\dot q-\frac{1}{(2+\omega)}t^{1-\omega/2}\dot q^2-\frac{\sigma}{(2+\omega)}t^{1+\omega/2}q^{-2}.$$

$$G^3=t^{-\omega/2}\partial/\partial t$$

$$f^3=0$$

$$I^3=\tfrac{1}{2}t^{-\omega}\dot q^2+\tfrac{\sigma}{2}q^{-2}.$$

(vi). $n=-3$, $k=1$.

$$G^1=\tfrac{1}{2}t\ln^2 t\partial/\partial t+\tfrac{1}{2}q\ln t\partial/\partial q$$

$$f^1=\tfrac{1}{4}q^2$$

$$I^1=\tfrac{1}{4}q^2-\tfrac{1}{2}tq\dot q\ln t+\tfrac{1}{4}t^2\dot q^2\ln^2 t+\tfrac{\sigma}{4}t^{\omega+2}\ln^2 tq^{-2}.$$

$$G^2=t\ln t\partial/\partial t+\tfrac{1}{2}q\partial/\partial q$$

$$f^2=0$$

$$I^2=\tfrac{1}{2}tq\dot q-\tfrac{1}{2}t^2\dot q^2\ln t-\tfrac{\sigma}{2}t^{\omega+2}\ln tq^{-2}.$$

$$G^3=t\partial/\partial t$$

$$f^3 = 0$$

$$I^3 = \tfrac{1}{2}t^2\dot{q}^2 + \tfrac{\sigma}{2}t^{\omega+2}q^{-2}.$$

In the above, the superscript does not refer to a power. We now discuss the solutions of some of the cases of (4.58), viz., (i), (ii) and (iv), using the results obtained in section 6.2. The method is the same for (iii), (iv) and (vi).

(i). According to Proposition 6.2, G is also a symmetry of the reduced first-order equation in q, i.e., $I(t, q, \dot{q}) = c$. Thus, we solve the characteristic equation associated with G, viz.,

$$\frac{1-k}{2}\frac{dt}{t} = \frac{dq}{q}$$

in order to obtain the invariant $u = t^{(k-1)/2}q$ of G. Thus, $q = ut^{(1-k)/2}$ and $\dot{q} = \dot{u}t^{(1-k)/2} + \frac{1-k}{2}ut^{(-1-k)/2}$ so that our first-order equation $I = c$, in terms of the invariant u, becomes

$$c = \frac{1-k}{4}u^2 + \frac{1}{k-1}\dot{u}^2t^2 - \frac{\sigma}{2(1+k+\omega)}u^{2(1+k+\omega)/(k-1)}$$

which is separable and can be written as

$$\frac{du}{\sqrt{U}} = \frac{dt}{t},$$

where $U = \frac{\sigma}{n+1}u^{n+1} + \frac{(k-1)^2}{4}u^2 + c^*$, for c^* a constant. The solution follows by quadrature.

(ii). Here, an invariant of G is $u = qt^{k-1}$ and the first-order equation $I = c$ can then be rewritten as

$$\frac{1}{4(1-k)}\dot{u}^2t^{4-2k} + \frac{\sigma}{\omega+2}u^{(\omega+2)/(k-1)} = c.$$

The solution follows by means of quadrature.

(v). In this case, we may choose any one of the G^i's, $1 \leq i \leq 3$. We select G^2. As before, G^2 is also a symmetry of the equation $I^2 = c$. An invariant of G^2 is $u = q/t^{(2+\omega)/4}$ so that

$$c = -\frac{1}{2+\omega}\dot{u}^2 t^2 + \frac{2+\omega}{16}u^2 - \frac{\sigma}{2+\omega}u^{-2},$$

which is variables separable.

6.4.1.2. Leach (1985) discusses the first integrals of the modified Emden equation

$$\ddot{q} + \alpha(t)\dot{q} + q^n = 0, \qquad n \neq 0. \tag{6.6}$$

These are obtained by using the Lagrangian

$$L = \frac{1}{2}A(t)\dot{q}^2 - \frac{1}{n+1}A(t)q^{n+1}, \quad n \neq -1,$$

where $A(t) = \exp(\int^t \alpha(t')dt')$ and invoking the generalized Noether's Theorem with derivative or velocity dependent symmetries.

By considering some of the cases, we demonstrate how the results in Leach (1985) can be extended to deduce solutions of (6.6).

A. A first integral corresponding to the differential equation

$$\ddot{q} + \frac{5}{t+K}\dot{q} + q^2 = 0$$

is $I = \frac{1}{2}T^6\dot{q}^2 + 2T^5 q\dot{q} + \frac{1}{3}T^6 q^3$, where $T = t + K$ (Leach 1985). A scaling symmetry of the induced equation $I = c$ is $G = T\partial/\partial T - 2q\partial/\partial q$ with invariant $u = qT^2$. Then, $\dot{q} = \dot{u}T^{-2} - 2T^{-3}u$ and the reduced equation, $I = c$, becomes

$$c = \frac{1}{2}T^2\dot{u}^2 - 2u^2 + \frac{1}{3}u^3$$

which can be rewritten as

$$T\dot{u} = \sqrt{2(c - 2u^2 - \frac{1}{3}u^3)},$$

which is variables separable.

B. A first integral for (see Leach 1985)

$$\ddot{q} - \frac{1}{t+M}\dot{q} + \frac{1}{q} = 0$$

is $I = \frac{1}{2}\dot{q} - \frac{1}{T}q\dot{q} + \log q - \log(T)$, where $T = t + m$. The reduced equation $I = c$ has symmetry $G = T\partial/\partial T + q\partial/\partial q$ which has an invariant $u = q/T$. The reduced equation is then

$$\frac{dT}{T} = \frac{(\frac{1}{2} - u)du}{c - \frac{1}{2}u + u^2 - \log u}.$$

6.4.2. *First integrals of partial differential equations*

Here we invoke Definition 6.1 to determine first integrals. This definition is a strict criterion for invariance of the associated first integral of the partial differential equation. However, this need not be the case as a first integral need not satisfy the strict property for a given symmetry (c.f. Theorem 6.1). In practice, there are usually many symmetries and Definition 6.1 which is an analogue of the ordinary differential equations case (first part of Proposition 6.2) is a very useful property as we see here in various examples.

6.4.2.1. We apply Definitions 1.2 and 6.1 to determine first integrals of the now famous Burger's equation

$$u_{xx} - uu_x - u_t = 0. \tag{6.7}$$

The equation has been discussed, amongst many others, by Bluman in the reference cited in section 6.3. Also, Nucci (1993) applies the method of 'nonclassical symmetries' to the equation.

It is clear that a symmetry of the equation is $G = \partial/\partial x$ ($G^{[1]} = \partial/\partial x$). G is associated with a first integral $I = (I_1, I_2)$ if

$$\frac{\partial}{\partial x} I_1 = 0,$$

$$\frac{\partial}{\partial x} I_2 = 0. \tag{6.8}$$

Also, invoking Definition 1.2, we have

$$\frac{d}{dt} I_1 + \frac{d}{dx} I_2 = 0. \tag{6.9}$$

These are obtained by applying (1.8) and (6.1). Equations (6.8) result in

$$I_1 = \mathcal{A}_1(t, u, u_x, u_t),$$

$$I_2 = \mathcal{A}_2(t, u, u_x, u_t), \tag{6.10}$$

where the $\mathcal{A}_i$'s are arbitrary in the arguments. Substituting (6.10) into (6.9) and Burger's equation into the resultant equation, we get the following partial differential equation

$$\begin{aligned} &\frac{\partial \mathcal{A}_1}{\partial t} + u_t \frac{\partial \mathcal{A}_1}{\partial u} + u_{tt} \frac{\partial \mathcal{A}_1}{\partial u_t} + u_{xt} \frac{\partial \mathcal{A}_1}{\partial u_x} + \\ &u_x \frac{\partial \mathcal{A}_2}{\partial u} + u u_x \frac{\partial \mathcal{A}_2}{\partial u_x} + u_t \frac{\partial \mathcal{A}_2}{\partial u_x} + u_{xt} \frac{\partial \mathcal{A}_2}{\partial u_t} = 0. \end{aligned} \tag{6.11}$$

By equating coefficients of u_{tt} and u_{xt} respectively to zero, we deduce that $\mathcal{A}_1$ is independent of u_t and that

$$\frac{\partial \mathcal{A}_1}{\partial u_x} + \frac{\partial \mathcal{A}_2}{\partial u_t} = 0.$$

Without loss of generality, we further assume that $\partial\mathcal{A}_2/\partial u_t = 0$. This implies that $\mathcal{A}_2$ is independent of u_t and $\mathcal{A}_1 = \mathcal{A}_1(t,u)$. The remaining equation becomes

$$\begin{aligned} &\frac{\partial\mathcal{A}_1}{\partial t} + u_t\frac{\partial\mathcal{A}_1}{\partial u} + \\ &u_x\frac{\partial\mathcal{A}_2}{\partial u} + uu_x\frac{\partial\mathcal{A}_2}{\partial u_x} + u_t\frac{\partial\mathcal{A}_2}{\partial u_x} = 0. \end{aligned} \tag{6.12}$$

We separate the coefficients of u_t:

$$\frac{\partial\mathcal{A}_1}{\partial u} + \frac{\partial\mathcal{A}_2}{\partial u_x} = 0.$$

Thus, $\mathcal{A}_2 = -u_x\partial\mathcal{A}_1/\partial u + C(t,u)$ for C arbitrary in its argument. Inserting $\mathcal{A}_2$ into (6.12) yields

$$\frac{\partial\mathcal{A}_1}{\partial t} + u_x(-\frac{\partial^2\mathcal{A}_1}{\partial u^2}u_x + \frac{\partial C}{\partial u}) + uu_x(-\frac{\partial\mathcal{A}_1}{\partial u}) = 0.$$

We further separate by u_x^2 to get:

$$\frac{\partial^2\mathcal{A}_1}{\partial u^2} = 0$$

so that $\mathcal{A}_1 = D(t)u + E(t)$, for some functions D and E of t. Also, from the terms involving u_x, we get

$$\frac{\partial C}{\partial u} - u\frac{\partial\mathcal{A}_1}{\partial u} = 0$$

which implies that $C = \frac{1}{2}u^2 D + F(t)$, F some function of t.

Finally, we get that

$$\dot{D}u + \dot{E} = 0.$$

Without loss of generality, we may further assume $F = 0$ and $E = 0$. The latter yields D =constant= 1, say. Thus,

$$\mathcal{A}_1 = u, \qquad \mathcal{A}_2 = -u_x + \frac{1}{2}u^2.$$

That is,

$$I_1 = u,$$

$$I_2 = -u_x + \frac{1}{2}u^2,$$

which are components of the first integral of Burger's equation.

6.4.2.2. In the following example of Fisher's equation, we show that, by applying the methodology in 6.4.2.1, we obtain a trivial conservation law corresponding to its symmetry $G = \partial/\partial x$, i.e., a symmetry of an equation need not correspond to a conservation law in the sense of Definition 6.1 (c.f. the converse statement in Vinogradov 1984 that a conservation law need not be a consequence of a symmetry).

The Fisher's equation which has symmetry $G = \partial/\partial x$ is

$$u_t = u_{xx} + u(u-1). \tag{6.13}$$

By (1.8) and (6.1), we obtain the following equations:

$$\frac{\partial}{\partial x} I_j = 0, \quad j = 1, 2,$$

$$\mathrm{Div}\,(I_1, I_2) = 0. \tag{6.14}$$

Thus,

$$I_j = \mathcal{A}_j(t, u, u_t, u_x), \quad j = 1, 2,$$

so that by (6.14), we have the following partial differential equation

$$\frac{\partial \mathcal{A}_1}{\partial t} + \frac{\partial \mathcal{A}_1}{\partial u}(u_{xx} + u(u-1)) + \frac{\partial \mathcal{A}_1}{\partial u_t}u_{tt} +$$

$$u_{xt}(\frac{\partial \mathcal{A}_1}{\partial u_x} + \frac{\partial \mathcal{A}_2}{\partial u_t}) + \frac{\partial \mathcal{A}_2}{\partial u}u_x + \frac{\partial \mathcal{A}_2}{\partial u_x}u_{xx} = 0.$$

By separating coefficients in derivatives of u, as in section 6.4.2.1, the $\mathcal{A}_j$'s have the following form

$$\mathcal{A}_1 = -\alpha(t)u_x + K_1(t),$$

$$\mathcal{A}_2 = \alpha(t)u_t + \dot{\alpha}(t)u + K_2(t),$$

where α and K_j ($j = 1, 2$) are functions of t. So, a first integral of (6.13) associated with G is

$$I_1 = -\alpha u_x + K_1,$$

$$I_2 = \alpha u_t + \dot{\alpha} u + K_2.$$

It is easily verifiable that this is a trivial conservation law.

6.4.2.3. Euler *et al* (1988) discusses the point symmetries associated with the equation

$$u_{xt} + u_x + u^2 = 0, \tag{6.15}$$

which arises in the study of Maxwellian tails. It has been shown that the symmetries of (6.15) are

$$G_1 = \frac{\partial}{\partial x},$$

$$G_2 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},$$

$$G_3 = \frac{\partial}{\partial t},$$

$$G_4 = e^t\frac{\partial}{\partial t} - e^t u\frac{\partial}{\partial u}.$$

We firstly determine the first integral $I = (I_1, I_2)$ obtained through G_2. Since $G_2^{[1]} = x\partial/\partial x - u\partial/\partial u - 2u_x\partial/\partial u_x - u_t\partial/\partial u_t$, applying Definition 6.1 yields the system

$$x\frac{\partial I_1}{\partial x} - u\frac{\partial I_1}{\partial u} - 2u_x\frac{\partial I_1}{\partial u_x} - u_t\frac{\partial I_1}{\partial u_t} = 0,$$

$$x\frac{\partial I_2}{\partial x} - u\frac{\partial I_2}{\partial u} - 2u_x\frac{\partial I_2}{\partial u_x} - u_t\frac{\partial I_2}{\partial u_t} = 0.$$

Rewriting the first partial differential equation in the characteristic form

$$\frac{dx}{x} = -\frac{du}{u} = -\frac{du_x}{2u_x} = -\frac{du_t}{u_t} = \frac{dI_1}{0},$$

yields the invariants of $G_2^{[1]}$, viz., $\alpha = u_x$, $\beta = x^2u_x$, $\gamma = xu_t$ and $\delta = I_1$. It can then be shown that

$$I_1 = h(\alpha, \beta, \gamma),$$

$$I_2 = l(\alpha, \beta, \gamma),$$

where h and l are some functions of α, β and γ. Invoking Definition 1.2 gives the partial differential equation

$$\gamma\frac{\partial h}{\partial \alpha} + \frac{\partial h}{\partial \beta}x^2(-u_x - u^2) + \frac{\partial l}{\partial \gamma}x(-u_x - u^2) + \frac{\partial h}{\partial \gamma}xu_{tt}$$

$$+ \frac{\partial l}{\partial \beta}x^2u_{xx} + \frac{\partial l}{\partial \alpha}u + \frac{\partial l}{\partial \beta}2xu_x + \frac{\partial l}{\partial \gamma}u_t + \frac{\partial l}{\partial \alpha}xu_x = 0.$$

Equating the coefficients of u_{tt} and u_{xx} to zero yields

$$h = h(\alpha, \beta),$$

$$l = l(\alpha, \beta).$$

The remaining terms in the above equation give

$$\gamma\frac{\partial h}{\partial\alpha}-\beta\frac{\partial h}{\partial\beta}-\alpha^2\frac{\partial h}{\partial\beta}=0,$$

$$-\beta\frac{\partial l}{\partial\gamma}-\alpha^2\frac{\partial l}{\partial\gamma}+\alpha\frac{\partial l}{\partial\alpha}+\gamma\frac{\partial l}{\partial\gamma}+\beta\frac{\partial l}{\partial\alpha}=0.$$

Differentiating the second equation with respect to γ yields $\partial^2 l/\partial\gamma^2=0$, so that

$$l=r(\alpha)\gamma+k(\alpha).$$

Similarly, we can show that

$$h=-r(\alpha)\beta+s(\alpha),$$

where r, k and s are functions of α. We can then choose

$$I_1=-x^2u_x,$$

$$I_2=xu_t.$$

However, the first integral I is trivial. Similarly, the first integrals obtained through G_3 and G_4 are trivial.

It can be shown that G_1 is associated with the non-trivial conservation law

$$I_1=\frac{1}{3}u^3\exp(3t),$$

$$I_2=(\frac{1}{2}u_t^2+uu_t+\frac{1}{2}u^2)\exp(3t).$$

Applying $\operatorname{Div}(I_1,I_2)=0$ yields

$$(u_{xt}+u_x+u^2)(u+u_t)\exp(3t)=0$$

and thus I is indeed a first integral.

6.4.3. *Reduction of systems of ordinary differential equations*

Gorringe and Leach (1991) determined the first integrals of the system of equations

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 - \mu r^{1-2/\alpha}, \\ \ddot{\theta} &= -2\dot{r}\dot{\theta}/r, \end{aligned} \tag{6.16}$$

where r and θ are functions of t. It was shown that (6.16) has, inter alia, the symmetry $G = t\partial/\partial t + \alpha r\partial/\partial r$ and associated first integrals

$$I = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{\alpha\mu}{2(\alpha-1)}r^{2-2/\alpha},$$

and

$$J = r^2\dot{\theta}. \tag{6.17a}$$

Since $G^{[1]}I = 0$ and $G^{[1]}J = 0$, by Proposition 6.2 carried over to systems, G is a symmetry of the system of equations

$$\begin{aligned} I(t, r, \theta, \dot{r}, \dot{\theta}) &= c_1, \\ J(t, r, \theta, \dot{r}, \dot{\theta}) &= c_2, \end{aligned} \tag{6.17b}$$

where c_1 and c_2 are constants. The invariants of G, obtained by solving the characteristic equation

$$\frac{dt}{t} = \frac{dr}{\alpha r} = \frac{d\theta}{0},$$

are $u_1 = rt^{-\alpha}$ and $u_2 = \theta$. Thus, $\dot{r} = \alpha t^{\alpha-1}u_1 + t^\alpha \dot{u}_1$ and $\dot{\theta} = \dot{u}_2$. Equation (6.17) can now be written as

$$\begin{aligned} &t^{2\alpha}\dot{u}_1^2 + 2\alpha t^{2\alpha-1}u_1\dot{u}_1 + \alpha^2 t^{2\alpha-2}u_1^2 + t^{2\alpha}u_1^2\dot{u}_2^2 + \frac{\alpha\mu}{\alpha-1}t^{2\alpha-2}u_1^{2-2/\alpha} = k, \\ &t^{2\alpha}u_1^2\dot{u}_2 = c_2, \end{aligned} \tag{6.18}$$

where k is a constant. Substituting (6.18b) into (6.18a), we obtain

$$\dot{r}^2 + c_2^2 \frac{1}{r^2} + \frac{\alpha\mu}{\alpha - 1} r^{2-2/\alpha} = k, \tag{6.19}$$

where $r = t^\alpha u_1$. We note that, purely coincidently, (6.19) is of the same form as obtained by substituting J in (6.17a) into I. By solving (6.19) in terms of r and t, we have the solution for (6.16).

6.5. Discussion

Determining the first integrals of ordinary differential equations (and systems of ordinary differential equations) have been done before (some of the references have been cited above). However, we have shown a systematic, albeit strict, way of determining these for partial differential equations when we considered invariance of the integral. Whilst the calculations may be tedious, the method is an algorithmic one and easily lends itself to the computer.

In particular and as a consequence, the results obtained by Bluman (see section 6.3) will have a wider range of application since a knowledge of first integrals of partial differential equations provides a conserved form of the partial differential equation in question (this is a sufficient requirement in Bluman's method).

Furthermore, we have shown that the result of Sarlet and Cantrijn (1981) that $G^{[1]}I = 0$ (G a Noether symmetry and I the corresponding Noether invariant) for systems of second-order ordinary differential equations has, previously unknown, applications in the reduction of the equation. However, the result is not true for partial differential equations as we have seen in Theorem 6.1. However, Definition 6.1, which was motivated by the works of Sarlet and Cantrijn (1981) on second-order ordinary differential equations, was used to analyse equations from mathematical physics.

CONCLUSION

From the above material, it is, indeed, clear that many issues need to be addressed and numerous questions can be raised — some dealing with the possibilities of further *generalizations*, some involving *extensions* and others with *converses*. In this concluding chapter, we will discuss some of these. Most certainly, the list is not exhaustive and the issues that we present below do not appear in the sequence of the preceding chapters.

In chapter 2, we presented some results dealing with the algebraic structures associated with the inverse problem in Lagrangian mechanics and compared and contrasted these with similar results corresponding to Lie algebras of differential equations. We proved that the maximum dimension of the Noether algebra for particle Lagrangians is five ($A_{5,40}$) and that particle Lagrangians do not admit a maximal four-dimensional algebra. Similar work needs to be done for higher-order Lagrangians corresponding to ordinary and partial differential equations.

In chapter 6, we develo ed an algorithm to determine first integrals for partial differential equations making use of symmetries *associated* with the required first integral. Furthermore, we saw how the notion of associated symmetries, which naturally holds for Noether invariants (first integrals) was utilised to further reduce second-order ordinary differential equations and, in some cases complete solve them. We proved that this notion does not automatically hold for partial differential equations (even second-order). However, the remarks following Proposition 6.1 suggests how a similar result may be achieved so that a vast number examples of partial diffe ial equations can be analysed using Noether's Theorem.

In chapter 1, we had a brief look at some methods and problems associated with the inverse problem in the calculus of variations. In particular, the construction of first-order Lagrangians using symmetries and associated first integrals was reviewed. Here again, a generalization of the results will be useful and some aspects of it are investigated in Mahomed and Kara (1994).

The non-uniqueness of a Lagrangian corresponding to a differential equation is well-known and was discussed in detail in chapter 3. These observations were exploited in several ways. We introduced the concepts of *equivalent*, *alternative* and *s^r-equivalent* Lagrangians. This led us to classify particle Lagrangians for second-order ordinary differential equations, firstly in chapter 3 and then in chapter 4 we completed the classification. Classification of higher-order Lagrangians still needs investigation and it should be the case that the method adopted in the preceding chapters can be extended.

Finally, we can pursue some algebraic studies. For example, we suggest here that the Lie point symmetry algebra of a differential equation can be partitioned by the Lagrangian of the differential eqaution in the following manner. If $A_1, \cdots, A_n$ are the point symmetry algebras of the Lagrangians $L_1, \ldots, L_n$ (alternative Lagrangians) corresponding to the equation $E(t, q^{[m]}(t)) = 0$, then there exist algebras $B_1, \ldots, B_k$ (with the B_i's disjoint), $k \leq n$, chosen from the A_i's such that the algebra of symmetries B_i's is isomorphic to the Lie algebra of $E(t, q^{[m]}(t)) = 0$.

As it was pointed at the outset of this chapter, many more questions can be raised. Very importantly, if the results can be extended, the scope of application is widened. Thus, we will have more tools at our disposal to tackle some of the many problems of mathematical physics.

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