Grade 10 mathematics teachers' discourses and approaches during algebraic functions lessons in Acornhoek, rural Mpumalanga Province, South Africa

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by

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As the candidate's supervisor, I have approved this thesis for submission.

TC. Nkambule

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29/06/2021_______________________

DECLARATION

I declare that **Grade 10 rural mathematics teachers' discourses and approaches during algebraic functions lessons in Acornhoek classrooms, Mpumalanga Province, South Africa** represents original work by the author and has not been submitted in any form to another university for assessment. Where use has been made of the work of others, this has been duly acknowledged in the text and a complete, alphabetised reference list has been provided. I fully understand that the University of the Witwatersrand will take disciplinary action against me if evidence suggests that this is not my own unaided work or that I failed to acknowledge the sources of the ideas or words in my writing.

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Hlamulo Wiseman Mbhiza

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Date

ABSTRACT

The purpose of the current study was to gain insights into the mathematical discourses and approaches of Grade 10 teachers during algebraic functions lessons within rural classrooms. The study focused on five rural teachers from the Acornhoek region of Mpumalanga Province of South Africa, to investigate their classroom discourses and approaches to algebraic functions, and the factors that influence their teaching of the topic. The teachers were purposefully and conveniently selected. In this study, all four components of algebraic functions: linear, parabolic, exponential, and hyperbolic functions were used as units of analysis to illuminate their discourses and approaches during teaching. Sfard's (2008) commognitive theory was referred to, with particular focus on characteristics of the mathematical discourse: word use, visual mediators, endorsed narratives, and routines as theoretical framing for the study. In addition to this, Scott, Mortimer and Amettler's (2011) pedagogical link-making (PLM) framework and communicative approach framework were used to identify and discuss the nuances of teachers' practices during algebraic function lessons.

The study adopted a qualitative research approach and used semi-structured interviews, non-participant classroom observations and video-stimulated recall interviews (VSRI) as methods of data generation, with an adoption of descriptive and interpretive elements of data analysis. A case study of each teacher revealed the teachers' thinking and communication about the concept of algebraic function. The main research question for the study was: "What are the rural Grade 10 teachers' discourses and approaches during algebraic functions lessons?" The findings emerging from this study are categorised into three broad themes and sub-themes. The first theme is 'Teachers' use of functions representations and their weaknesses' and has two sub-themes: 'Functions as drawing graphs: rituals to reach the end goal' and 'Multiple problems, assumed instruction for mathematical action'. This theme focuses on the teachers' use of rituals in using the different modalities of representations during teaching, which illuminate the teachers' thinking about algebraic functions to be about drawing graphs, which in turn results in the under-teaching of the related concepts as prescribed in the curriculum documents. The second theme, 'Teachers' communication about the effect of parameters' reveals teachers' use of mathematical discourses and approaches to algebraic functions to bring the notion of the effect of parameters to the fore, for students to understand the effect of changing the values of parameters on the four different families of algebraic function. This theme is divided into three sub-themes: 'Generalising from worked examples', 'The participationist approach to generalisation' and 'The use of examples: variation between parameters'. The first sub-theme addresses the nature of the classroom environment that was created by the teachers for learners to learn about the effect of parameters. The majority of the teachers employed exposition teaching strategies to talk about the effect of changing the values of parameters on the functions, without allowing learners to explore the relationships for themselves and construct mathematics meanings as advocated by the curriculum

principles (Department of Education, 2011). The second sub-theme details how one of the teachers created opportunities for learners to participate in learning, explore and observe the effect of varying the parameters and making their own meanings about the different families of functions. The third sub-theme focuses on how teachers selected and sequenced examples for the different families of algebraic functions in an attempt to illuminate the effect of varying the values of the parameters on the behaviour of the functions.

The third major theme ‘Approaches to teaching functions’ focuses on the two approaches that were predominantly used by the participants: the use of examples versus non-examples and the property-oriented approach. These approaches are discussed in relation to the identified discourses in the first two themes. The last major theme ‘Factors that shape rural teachers’ approaches and discourses’ draws mainly from the comments teachers made during VSRIs to present reasons as to how and why they taught mathematics, especially algebraic functions, the way they did. Three reasons were given for using authoritative/non-interactive communicative approaches during teaching and for the observed under-teaching of the topic. The first factor, ‘The discourse of teaching for compliance’, represents teachers’ reasons why they did not allow for participatory discourse to enable learners to be active co-constructors of mathematical knowledge as they feared that such strategies would delay them from reaching the teaching goals detailed by the subject pacesetters. The second sub-theme, ‘Teaching for assessment’ focuses on teachers’ observable actions linked to preparing learners for possible assessment questions, to enable them to answer questions correctly. During VSRI, teachers commented that they taught learners for assessment purposes such that when the learners performed well, the department would not put them under surveillance. Lastly, ‘Knowledge of Algebraic Functions and curriculum focus’ details teachers’ limited knowledge of the curriculum standards and content knowledge for algebraic functions, which resulted in the under-teaching of the topic. I believe that expanding the research locale for mathematics education in South Africa to focus more on rural areas and schools can offer insights into the nature of teaching and learning in those contexts, and can help us to configure strategies to promote effective mathematics in those areas.

Key words: Algebraic functions, discourses, approaches, rural, mathematics, teaching, mathematical discourses

ABBREVIATIONS

ANA – Annual National Assessment

CAPS – Curriculum and Assessment Policy Statement

CDA – Critical Discourse Analysis

CDE – Centre for Development and Enterprise

DBE – Department of Basic Education

FET – Further Education and Training

GDP – Gross Domestic Product

MCRRE – Ministerial Committee Report on Rural Education

MDE – Mpumalanga Department of Education

MRs – Multiple Representations

MTK – Mathematics for Teaching Knowledge

NCTM – National Council of Teachers of Mathematics

NEEDU – National Education Evaluation and Development Unit

NMF – Nelson Mandela Foundation

PLM – Pedagogical Link-Making

SACMEQ – Southern and East African Consortium for Monitoring Educational Quality

STATS SA – Statistics South Africa

TIMSS – Trends in International Mathematics and Science Study

TP – Turning Point

VSRI – Video Stimulated Recall Interviews

WITS – University of the Witwatersrand

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#RoadToPhD #FutureProfessor #RuralBoy #YouAreNotAHopelessCase

DEDICATION

I dedicate this thesis to Kwanza Mbhiza. Remember, you are not a hopeless case. Daddy loves you.

PUBLICATIONS ARISING FROM THE CURRENT STUDY

Mbhiza, H. W. (2019). Using video-stimulated recall interviews: teachers' reflections on the teaching of algebraic functions in rural classrooms. *The Independent Journal of Teaching and Learning*, *14*(2), 92-107.

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Contents

LIST OF FIGURES

LIST OF TABLES

Chapter 1

Context and background to the study: lament for rural research

The rural carriers are unloved and not getting the attention they deserve ~ Greg Gorbatenko

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1.1. My personal journey into mathematics education and research

My interest in researching mathematics education was motivated by reflecting on my mathematics learning trajectory, which was compounded by various challenges and learnable moments. I grew up in a beautiful village called Jimmy Jones in Malamulele, rural Limpopo Province of South Africa. I completed my primary and secondary education in 2005 and 2009 respectively. I viewed mathematics as the only subject that was going to help me escape the poverty I observed in my family and around the village. This was because growing up in a community where success in learning mathematics was conceived of as a 'ticket to Johannesburg', 'The City of Gold', which is considered a place where dreams are turned into reality. Although there were many occasions in which I did not understand mathematics, I had no choice but to love the only subject I considered 'my ticket' out of abject poverty. This became a normalised conception of what mathematics was in my village and I presumed this to be similar in other rural contexts.

My bitter experience with mathematics as a learner started when I was in Grade 7. I recall going to fetch my report card for Grade 7 with my mom, and along the way I told her that she should expect great results for all other subjects except for mathematics. This was because mathematics was not my favourite subject, since the teacher failed to teach the subject to enable my understanding. After receiving my progress report, my mom confirmed that I had passed all the subjects exceptionally well and failed mathematics poorly. At the time I wished mathematics was not a compulsory subject in Grade 8 and other grades moving forward, but the head of Grade 7 told me that I had no choice but to work hard in the subject because it was still going to be compulsory in Grades 8 and 9. While this made me feel dejected at the time, I was optimistic that I would have better teachers for mathematics in secondary school and thereby better experiences with the subject.

In 2005, I enrolled for my secondary education at a neighbouring village which was 4 kilometres away, because my village did not have a secondary school. I had to walk 8 kilometres daily which was challenging during rainy days as the river we used to cross would be full, and we had to swim across to get to school. This was normal for those who attended the same school before us and still is for those who came after us, making this the least of my worries. Of importance for me was to have a mathematics teacher who would help me understand the subject, thus maximising my chances of going to Johannesburg to advance my family's standard of living. However, my optimism of a better mathematics teacher in Grade 8 was shattered when I realised that the school did not have a mathematics teacher, and led to my feeling wretched at that time and failing the subject again at the end of 2005.

My desolateness was compounded by the fact that teachers at school were the only source of information and knowledge; at the time, parents in the community were uneducated. Those parents[1] who had some educational background, did not have mathematical knowledge. This pattern was repeated over a period of two years of failing mathematics, until I reached Grade 10 and I had for the first time in my schooling trajectory a mathematics teacher with knowledge to help us understand the subject. Unfortunately, in 2007 my Grade 10 mathematics teacher left the school to teach in a school located in a nearby town, leaving us without a mathematics teacher for the remainder of the year. This meant that for the entire year in Grade 11, I did not have a mathematics teacher, resulting in my peers and I having to assist each other to learn the subject until we reached Grade 12. In Grades 10 and 11 end of the year examinations, I managed to pass mathematics with 30% which is the pass percentage set by the Department of Basic Education. I knew that the 30% pass in mathematics would not put me in a competitive position for scholarships as well as university entrance. My experiences of having no mathematics teachers almost throughout my secondary school made me realise that teachers play a significant role in enabling learners' understanding of the contents of the subject matter, particularly within rural contexts which are highly characterised by parents' lower educational levels.

Furthermore, in Grade 12, I again had the privilege of having a mathematics teacher for the whole year. The teacher in Grade 12 was faced with the task of bridging the knowledge gaps in our mathematical knowledge from as far back as Grade 8, to ensure that we made links with the more advanced Grade 12 contents. Having a teacher resulted in my understanding of the

[1] Once people become educated in my village, they usually relocate to the neighbouring towns or to urban areas such as Johannesburg to either advance their academic horizons or in search of 'better' employment opportunities. Also, our teachers did not reside in the communities but travelled long distances from neighbouring towns.

mathematical contents, subsequently I passed my mathematics and other subjects well and I was able to enrol at the University of the Witwatersrand (Wits), Johannesburg to study Education, majoring in Mathematics Education. Reflecting on my unfavourable experiences of learning mathematics at school made me develop an interest in exploring the teaching and learning of mathematics within rural contexts, to gain insights into the role that teachers play in facilitating and/or constraining the development of learners' mathematics discourse.

While enrolled for my first professional degree at Wits, I was able to utilise what the academic environment presented to me to enhance my mathematics performance and address the knowledge gaps that were created by the nature of the rural education I had received. In addition to this, I became involved in the Wits Maths Connect Project led by Professor Hamsa Venkat as a research assistant. In this position, I learned about the basic skills associated with research within the mathematics education field, and the seed to become a researcher was planted. Although I learned a lot about research processes within the project, I soon realised that the majority of mathematics education research presented a contextual bias because urban and township schools were solely focused upon. In view of this, the desire of researching with rural constituencies, especially mathematics teachers and learners, developed. My personal desire is to break the intergenerational dearth of mathematics education studies located within rural contexts, and this has motivated me to pursue my postgraduate studies until post-PhD level to address the research gap and present alternative forms of knowledge through conducting research with rural teachers and learners. This chapter presents the introduction and background to the current study; problem statement; rationale of the study; the purpose and objectives, and the research questions for the study. In the current section, I have focused on my personal trajectory into mathematics education learning and research the following section presents the introduction and background of the study.

1.2. Introduction and Background

In view of my brief personal experiences of learning mathematics discussed above, a significant number of teachers and learners in rural schools across the country remain the most vulnerable. This is due to issues associated with social injustice, despite South Africa having been a democratic country for 27 years. I argue that the legacy of the apartheid education system still prevails, especially the standards of education that people in different geographic locations still receive. Msila (2007, p. 146) has stated that, "in the past, South African education system reflected the fragmented society in which it was based", as the racial discrimination reflected in all apartheid laws that favoured ‘White’, ‘Asian’, ‘Coloured’, and ‘Native’ respectively. Recently, Chirinda et al. (2021) has argued that South African learners still learn under unequal economic circumstances with existing differences regarding learning recourses. The authors attribute this to the apartheid education system that continues to impact the historically deprived schools. The origin of the social injustice in education was orchestrated by the then prime minister Verwoerd’s statement, “What is the use of teaching the Bantu child mathematics when he cannot use it in practice?” (Hirson, 1979, p. 45). The racialisation of mathematics education during apartheid had dire “implications for teacher training in Black African schools, because it perpetuated poor teaching and learning in Black African schools” (Mbhiza, 2017, pp. 3-4). The apartheid government deliberately excluded Black people from teaching and learning mathematics because it was preserved for the White learners (Soudien et al., 2019; Basebenzi, 2019). Mathematics was used as a tool for social and economic segregation, and resulted in a high shortage of qualified quality mathematics teachers in areas where the majority of the population was Black. The results of this past are noted in the continuously appalling standards of mathematics education nationally (Spaull, 2013), particularly in rural contexts.

It could be argued that Black communities were not only denied mathematics education, but the future and the limited opportunities by the apartheid government. The legacy of apartheid proves difficult to dismantle in a democratic South Africa, especially for African rural teachers and learners who continue to experience a shortage of mathematics teachers with appropriate qualifications and expertise (Adler & Venkat, 2014). Similarly, the National Education Evaluation and Development Unit (NEEDU, 2013) report reiterated that the biggest reason for poor learner performance is the lack of qualified teachers in rural schools, especially for mathematics. This statement does not overlook the outcry in township schools about insufficient mathematics teachers; however, rural schools continue to be appalling because of the difficulties in attracting and retaining quality teachers (Gardiner, 2008; Masinire, 2015). Given the above discussion, it was important to explore and understand how mathematics is taught in rural secondary schools in a democratic South Africa. This is particularly important because mathematics is considered one of the school subjects that could transform and help children to improve their standards of living and become effective citizens of their communities and the nation (Department of Basic Education (DBE), 2011; Spaull, 2013). The focus of the current study was to understand teachers’ discourses and approaches during the algebraic function lessons because it is considered one of the important topics that facilitate learners’ understanding of other topics in the school curriculum (Moeti, 2015; Mugwagwa, 2017; Viirman, 2014). A major reason for this focus is the view that the function concept is a unifying concept in mathematics (Viirman, 2014).

The standard of rural education in South Africa and in many other countries such as Ghana, Uganda, China, United States of America, Mali, and Iran, to mention just a few, faces great challenges. Some of those challenges include attracting and retaining qualified quality mathematics teachers, resulting in the difficulty of offering quality standards within rural schools (Moletsane, 2012; Masinire, 2015; Das & Samanta, 2014; Nkambule, 2017). Of interest from the above-mentioned countries is that even developed countries face similar challenges as developing countries. For example, Stelmach (2011) from the United States context stated that "teacher shortages are characterized by lack of teachers willing to work in rural schools, lack of highly qualified or certified teachers, and lack of teachers representing ethnic minority groups" (p. 36). According to Spaull (2013), "the teaching of mathematics in South African schools is amongst the worst in the world" (p. 3). Similarly, Mogari (2014) lamented this crisis, especially when compared to countries such as Zambia and Zimbabwe, as they have Gross Domestic Products (GDPs) that are not comparable to that of South Africa, yet these countries achieve more in international assessments. While Spaull and Mogari were not specific whether poor teaching competencies are prevalent in rural, farm, urban, or township schools, researchers stated that past research has shown that poor mathematics teaching has predominately been associated with rural education (Venkat et al., 2009; Moletsane, 2012; Spaull, 2013).

Regarding teachers' content knowledge and pedagogical approaches, Nkambule (2017, p. 192) argued that "teaching in rural settings ostensibly requires relevant knowledge and skills to cope with various eventualities and challenges, and teachers' ability to meet the challenges and responsibilities". Due to scarcity of rural mathematics education research in South Africa, it is unclear whether teachers within rural contexts possess the 'relevant knowledge and skills' which Nkambule (2017) viewed as necessary prerequisites for effective teaching within rural classrooms. Of importance to note is that researching with rural mathematics teachers has been consistently overlooked, and this study offers insights into teachers' discourses and approaches of teaching algebraic functions within rural mathematics classrooms.

1.2.1. *The historical development of the function concept*

I consider it important to present a brief history of the development of the function concept, in order to contextualise its importance in mathematics education and for the appropriate teaching of algebraic function. In the 18th century Gottfried Wilhelm Leibniz (1646-1716) conceptualised a function from the geometry of curves[2]. In 1692, Leibniz used the word 'function' to signify a geometric object "such as the tangent associated with a curve" (Jones, 2006, p. 3), and the "coordinates of a point on a curve or the slope of a curve" (Larson et al., 2007, p. 9). However, Leibniz did not offer a formal definition of what he meant by the term function until Johann Bernoulli (1667-1748) introduced the first formal conceptualisation of the concept in 1718, to address the non-consensual[3] operationalisation of the concept in mathematics. Bernoulli's definition is as follows: "One calls here Function of a variable a quantity composed in any manner whatever of this variable and of constants" (Kleiner, 1989, p. 284). While various authors have acknowledged that Bernoulli's definition was the beginning of the evolution of the concept (Kleiner, 1989; Jones, 2006), the definition did not explain what he meant by "composed in any manner", and resulted in Leonhard Euler's (1707-1783) improvement in 1755. He offered a more algebraic perspective of a function, using equations or formula to represent how quantities are dependent on other quantities. Table 1 depicts different definitions of function offered by different mathematicians.

Table 1

Definitions of function

Mathematician and year	Definition
Gottfried Wilhelm Leibniz in 1714	Quantities that depend on a variable.
Johann Bernoulli in 1718	Function of a variable as a quantity composed in some way of this variable and constants.
Leonhard Euler in 1755	If *x* denotes a variable quantity then all the quantities which depend on *x* in any manner whatever or are determined by it are called its functions. If some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves, then the former quantities are called functions of the latter quantities.
Peter Gustav Lejeune Dirichlet in 1829	y is a function of a variable x, defined on the interval $a < x < b$, if to every value of the variable x in this interval there corresponds a definite value y.
Bourbaki[4] in 1939	Let E and F be two sets, which may or may not be distinct. A relation between a variable element *x* of E and a variable element y of F is called a functional relation in y if, for all there exists a unique relation which is in the given relation with *x*.

[2] The dominant theory in mathematics from the Ancient Greece time until the Modern Age was Euclidean Geometry which focused on the study of lines, points and planes (Kleiner, 1989).
[3] Before Bernoulli introduced the first formal definition in mathematics, mathematicians used the term as self-explanatory, without clear conceptualisation and operationalisation.
[4] Bourbaki is not a name of an individual mathematician, but was a pseudonym a group of young French mathematicians adopted during the 1930s, officially known as the Association des collaborateurs de Nicolas Bourbaki (Association of Collaborators of Nicolas Bourbaki) (see Corry, 1992).

According to Kleiner (1989), the evolution of the function concept can be described as a tug of war[5] between geometric and algebraic approaches. Euler opined that "a quantity should be called a function only if it depends on another quantity in such a way that if the latter is changed, the former undergoes change itself" (Sfard, 1992, pp. 62-63). This definition of a function uses the notion of covariation of quantities (see Thompson, 1994, pp. 28-29), and the focus is on the relationship between "dependent and independent variables" and Euler viewed this "as a procedural concept demonstrating input-output relations" (Dede & Soybas, 2011, p. 90). The term "procedural concept" suggests that when working with the relations between variables there is an "analytic expression" containing variables and constants that governs the way a variable depends on another variable. In other words, when function is considered in terms of variables dependence, there are always established rules underpinning such relations, although Euler never referred to the notion of "rule" in his conceptualisation of the concept. What Euler's definition overlooked is the idea of 'constant functions' in which the value of the output remains the same for every input value, considering the statement "a quantity should be called a function only if it depends on another quantity in such a way that if the latter is changed, the former undergoes change itself" in his definition above (Dede & Soybas, 2011, p. 90). His approach to functions facilitated the emphasis on algorithmic dependence between variables and the use of algebraic equations as representations, and this definition is used in the South African mathematics curriculum (Kleiner, 1989).

Euler introduced the use of the notations of f and parenthesis for a function, in terms of $f(x)$ to denote the dependence between two variables (see Burton, 2003). He further developed his notion of analytic expression by defining a function as follows: "If, therefore, x denotes a variable quantity, then all quantities which depend upon x in any way or are determined by it are called functions of it" (Burton, 2003, p. 571). The functional notation is still prevalent in contemporary mathematics and its usage is also prescribed in Curriculum and Assessment Policy Statement (CAPS) (DBE, 2011). Teachers use the notation to avoid being too wordy when they pose questions during the lessons, to help the learners to focus on the input-output relationships. For instance, instead of asking learners, "What is the corresponding y value when $x = 2$?", a teacher can ask, "What is $f(2)$?" which is argued to be effective in the internalisation of functions (Thompson, 2013). That is, using the functional notation quickly

[5] As Bernoulli's definition was introduced, for example, its geometric nature fell short of expectations when it came to the algebraic definition or vice versa, and a new conceptualisation was made.

states which element of the function requires examination, for instance, find *f (2)* when $f(x) = x + 2$, is the same as saying find y when $x = 2$, for $y = x + 2$. From the classroom observations in the current study, most participants used the notation of a function and its parenthesis in terms of *f(x)* during teaching for substitution purposes.

In 1837, Peter Gustav Lejeune Dirichlet (1805-1859) defined a function in terms of one-to-one correspondence between real numbers. For him, "If a variable y is so related to a variable x that whenever a numerical value is assigned to x there is a rule according to which a unique value of y is determined, then y is said to be a function of the independent variable x" (Sierpinska, 1992, p. 46). Thus, in Dirichlet's definition the conditions for a relationship between two variables, *x* and *y*, are explicated, and of importance to note is the shift from the notion of covariation relationship in Euler's definition to a notion of correspondence relationship between such variables. What seems to be emphasised in this definition is the notion of continuity of functions, that in any relation between two variables, namely the dependent (x) and independent (y) variables, every x value has a 'unique' corresponding value y as demonstrated by the words *"*whenever a numerical value is assigned" in the above definition.

It can be discerned that Dirichlet's definition minimally enunciates the idea of covariation, which makes it an antecedent to contemporary definitions of the concept of function. Dirichlet's definition was the first to emphasise the idea of one-valuedness[6], which is regarded as one of the two essential characteristics of the contemporary concept of function together with arbitrariness[7] of functions (Even, 1990). Dirichlet's conceptualisation of function above resonates with what Carraher et al. (2008) termed the "formula-based functions" (p. 6). For these authors, formula-based functions are those that are generalisable into a rule to be used in "determining the value of the function for any input (that is, any element of the domain)" (Carraher et al., 2008, p. 6). This links closely with the notion of multiple representations, in which the above-described rules of correspondences between the values in the domain and co-domain can be formulated in algebraic notation, spoken language, tables, graphs, or some

[6] According to Even (1990), one-valuedness refers to the idea that for each element in the domain (also known as the independent variable) there is a corresponding unique element in the range of a function. This feature was not obvious during the earlier development of calculus as it is in contemporary mathematics.

[7] Arbitrariness of functions entails "both the character of the relationships between the two sets on which the function is defined and the sets themselves. In terms of the relationship between the two sets, this means that the function does not have to exhibit some regularity, be described by any specific expression or particular shaped graph. In this case two sets mean that functions do not have to be defined on any specific sets of objects; in particular, the sets do not have to be sets of numbers" (Yoon, 2007, p. 578).

combinations of such representations (Stein et al., 1990; Oehrtman et al., 2008). In section 2.4., I elaborate on the uses and difficulties associated with the notion of multiple representations, as well as the tasks teachers are faced with in the teaching and learning of functions in the school mathematics curriculum.

With the notion of one-valuedness in mind, another feature of functions that is saliently missing in Dirichlet's definition is the idea of many-to-one correspondences, in which many independent variables strictly map onto one dependent value. For example, in a parabola $y = x^2$, when $x = 2$, $y = (2)^2$ so $y = 4$. Similarly, when $x = -2$, $y = (-2)^2$ so $y = 4$. This signifies that two values of *x* (-2 and 2) strictly map onto one value of *y* (4) and Dirichlet's definition does not consider the idea of many-to-one mappings as demonstrated in the example above. Accordingly, about a century after Dirichlet's definition, Bourbaki in 1939 (see Kleiner, 1989) provided a somewhat similar but nuanced definition of the concept, which he defined in terms of a subset of Cartesian product which is the definition of function underpinning the CAPS mathematics view of the concept. This is discussed later in this chapter. Bourbaki followed the modern way of defining functions which is influenced by set-theory. He thus viewed a function in terms of morphisms, which entails the mappings between a domain *(x)* and its codomain *(y)*, in which every element in the domain corresponds to one and only one element in the codomain. As inspired by set-theory, for Bourbaki, if the domain of the function is represented by *X* and the codomain by *Y*, then the corresponding morphism can be depicted as$f : X \rightarrow Y$.

Bourbaki's definition introduced the notions of 'partial' and 'total' function, in which the former refers to a type of mapping whereby some values of the codomain may be undefined for some values of the domain. On the other hand, total functions entail the functions that are defined for every value in the domain which is a key feature for determining whether a function is continuous or discontinuous, on which previous definitions did not expound. His definition of a function is as follows:

> Let E and F be two sets, which may or may not be distinct. A relation between a variable element *x* of E and a variable element *y* of F is called a functional relation in *y* if, for all *x* in E, there exists a unique *y* in F which is in the given relation with *x*. (Kleiner, 1989, p. 299)

From the above definition, the "*x; y*" signifies that Bourbaki views a function as a set of ordered pairs, which can be represented as a co-ordinate point (*x; y*) on the Cartesian plane. The notation $f : X \rightarrow Y$ tells us that the name of the function is "*f*", and its ordered pairs are formed by an element in the domain (x) from the set *X*, and by an element in the codomain (y) from the set Y, and the arrow in the morphism above is read "is mapped to".

With all the definitions of a function presented above, authors (see, for example, Sfard, 1991; Sajka & Podchorążych, 2005; Maharaj, 2008) have posited that the following definition is predominant in school mathematics (see for example, Sfard, 1991; Sajka & Podchorążych, 2005; Maharaj, 2008): "set of ordered pairs of numbers (*x; y*) such that to each value of the first variable (*x*) there corresponds a unique value of the second variable (*y*)" (Thomas, 1972, p. 17). The detailed discussion about the historical development of function concept shows definitional changes of function in general since Leibniz's conceptualisation (Kleiner, 1989; Jones, 2006). The reviewed literature in this sub-section demonstrated that there are several accepted definitions for the concept of function, which include 'dependence relation', 'set of ordered-pairs', 'rule', and 'mapping'. With the difficulty of defining the concept of function in mind, the following sub-section focuses on the constitutive elements of a function as suggested in previous research studies.

1.2.2. The concept of algebraic functions in the South African curriculum

The knowledge of each topic teachers teach, and how they understand and use ideas related to the topic, is contained in the curriculum materials (Tarr et al., 2006; Pillay, 2014). In this section, I focus on what is given eminence for the concept of algebraic functions as well as how the topic is sequenced in CAPS. How the teachers sequence the teaching of algebraic functions is important in this study, as sequencing is linked with coherence, which is further linked with developing understanding, which starts with teaching learners. At Grade 10, CAPS describes functions as follows: "The concept of a function, where a certain quantity (output value) uniquely depends on another quantity (input value) *and the emphasis is placed on working* with relationships between[8] variables using tables, graphs, words and formulae. Convert flexibly between these representations" (DBE, 2011, p. 24, italics added). This view of function illustrates that the study of the topic is concerned with the larger interpretation of the relationship between different quantities, which is not delimited to the formal definition of the concept but includes the various ways in which one can write and describe functional relationships. Of importance to note is that in the South African mathematics curriculum a more

[8] Conceptually, a teacher needs to help learners understand that tables, graphs, words and formulae are different ways of which the same relationship is described.

formal definition of a function is only introduced when learners reach Grade 12 (the highest grade in South African secondary schooling) (see content clarification in Table 2.

The rationale for the late definition could be vested in Vinner's (1992) critique of introducing learners to functions by using the definitional method: "Before suggesting definitions to the learners, suggest examples, manipulating and other experimental opportunities as a concept definition does not guarantee understanding of the concept" (p. 196). It is however still important for teachers to know the definitions and underpinning processes because the effectiveness of teaching relies on a teacher's understanding and thinking. Similar, to Vinner (1992), Kwari (2007) suggested that learners should be given an opportunity to first explore various situations where functions occur before introducing them to the formal definition. All this depends on the teacher's classroom practices and the knowledge of the content. I acknowledge that it is the first time in Grade 10 that learners are being formally introduced to the topic of function. The statement also suggests that focusing on the specified contents can develop learners' foundational knowledge about the concept, which will in turn be helpful as learners continue to study the topic in Grades 11 and 12 as well as in their tertiary studies. Table 2 depicts the content clarification for the topic on functions at Grade 10 level.

Table 2

CAPS topic allocation in a term and content clarification at Grade 10 level (DBE, 2011, p. 24)

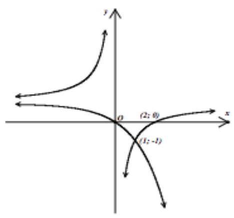
GRADE 10: TERM 2			
Weeks	**Topic**	**Curriculum statement**	**Clarification**
(4 + 1) **5**	**Functions**	1. The concept of a function, where a certain quantity (output value) uniquely depends on another quantity (input value). Work with relationships between variables using tables, graphs, words and formulae. Convert flexibly between these representations. Note: that the graph defined by $y = x$ should be known from Grade 9. 2. Point by point plotting of basic graphs defined by $y = x^2$, $y = \frac{1}{x}$ and $y = b^x$; $b > 0$ and $b \neq 1$ to discover shape, domain (input values), range (output values), asymptotes, axes of symmetry, turning points and intercepts on the axes (where applicable). 3. Investigate the effect of a and q on the graphs defined by $y = a.f(x) + q$, where $f(x) = x$, $f(x) = x^2$, $f(x) = \frac{1}{x}$ and $f(x) = b^x$, $b > 0, b \neq 1$. 4. Point by point plotting of basic graphs defined by $y = \sin\theta$, $y = \cos\theta$ and $y = \sin\theta$ for $\theta \in [0°;360°]$ 5. Study the effect of a and q on the graphs defined by: $y = a\sin\theta + q$; $y = a\cos\theta + q$; and $y = a\tan\theta + q$ where a, $q \in \mathbb{Q}$ for $\theta \in [0°;360°]$. 6. Sketch graphs, find the equations of given graphs and interpret graphs. **Note**: Sketching of the graphs must be based on the observation of number 3 and 5.	**Comments**: • A more formal definition of a function follows in Grade 12. At this level it is enough to investigate the way (unique) output values depend on how input values vary. The terms independent (input) and dependent (output) variables might be useful. • After summaries have been compiled about basic features of prescribed graphs and the effects of parameters a and q have been investigated: a: a vertical stretch (and/or a reflection about the x-axis) and q a vertical shift. The following examples might be appropriate: • Remember that graphs in some practical applications may be either discrete or continuous. **Examples:** 1. Sketched below are graphs of $f(x) = \frac{a}{x} + q$ and $g(x) = nb^x + t$. The horizontal asymptote of both graphs is the line y = 1. Determine the values of a , b , n , q and t . (C)  2. Sketch the graph defined by $y = -\sin x + \frac{1}{2}$ for $x \in [0°;360°]$ (R)

Table 2 details that the topic of functions is covered in term 2 and is allocated a maximum of five weeks to cover the first four families of functions at Grade 10 level (linear, quadratic, hyperbolic and exponential functions), plus one week to cover trigonometric functions. The importance of algebraic functions in the mathematics curriculum in South Africa is demonstrated by the amount of time allocated for teaching the topic across Grades 10 to 12, as functions have the highest number of teaching weeks of all the topics in CAPS (DBE, 2011). The emphases are on the properties[9] of each of the four families of functions in which learners are expected to investigate how different 'parameters' of a function influence the relationship between quantities. CAPS considers the investigation of how the different parameters influence the relationship between given quantities to be essential in learning the concept of function since learners are expected to make conjectures, prove them and in turn make generalisations (DBE, 2011). Further, algebraic functions in Grade 10 are introduced[10] by focusing on the ways in which input and output variables could be represented, and the linear function is introduced first focusing on drawing graphs using a table of values.

Recommendations of CAPS go further in detailing that teachers should introduce learners to the concepts of intercepts and gradient, teaching them how to use two methods for sketching linear functions: the dual intercept method and the gradient and y-intercept method (DBE, 2011). Learners are required to explore the effects of parameters 'm' (gradient) and 'q' (y-intercept) on the linear graph. The linear function section is concluded by focusing on determining the equations for lines. The quadratic function is introduced as the next family of functions and the focus is on showing learners the shape of the quadratic function. Learners are expected to complete tables of values for quadratic functions and in turn plot the ordered pairs on a Cartesian plane and explore the effects of 'a' and 'q' on the shape of a quadratic graph. Properties of a quadratic function including axis of symmetry, turning points, and domain and range are then introduced. The hyperbola is introduced next, also focusing on using the table of values to plot the graph, and learners are required to discuss the features of the function such as the asymptotes. Then, the last family of functions, the exponential function, follows next. Table 3 depicts the content specification and progression for algebraic functions in the FET

[9] Sub-section 2.5.5. details the use of the property-oriented approach to teach the function concept.
[10] While this section discussed the order and procedures of how the topic is outlined in CAPS, it is important to note that CAPS emphasises that teaching should not only focus on the "how" mathematics is done, but should equally address the "why" and "when" specific procedures are carried out.

(Further Education and Training) Phase, illustrating the components of the concept to be covered and those that should not be focused upon at each grade.

Table 3

Components of algebraic function covered in the FET[11] Phase (Umalusi, 2014)

Component of algebraic function[12]	**Covered in grade**		
	Grade 10	**Grade 11**	**Grade 12**
Informal concept of function	Yes	No	No
Formal concept of function	No	No	Yes
Convert between representations of functions as tables, graphs, words and formulae	Yes	Yes	Yes
Point by point plotting of $y = x, y = x^2$, $y = 1/x$ and $y = b^x$	Yes	No	No
Investigate the effect of a and q on graph $y = af(x) + q$ *on* $y = x$, $y = x^2$, $y = 1/x$ and $y = b^x$	Yes	No	No
Sketch graphs of form $y = af(x) + q$ where $f(x) = x$ or x^2 or $1/x$ or b^x	Yes	No	No
Find equations of given graphs of form $y = af(x) + q$ where $f(x) = x$ or x^2 or $1/x$ or b^x	Yes	No	No
Investigate effect of $p\ f(x) = a(x+p)2 + q$ on $y = x^2$, $y = 1/x$ and $y = b^x$	No	Yes	No
Sketch graphs of form $y = af(x+p) + q$ where $f(x) = x$ or x^2 or $1/x$ or b^x	No	Yes	No
Find equations of given graphs of form $y = af(x+p) + q$ where $f(x) = x$ or x^2 or $1/x$ or b^x	No	Yes	No
Investigate average rate of change between 2 points	No	Yes	No
Concept of inverse function	No	No	Yes
Restrictions on domain of function to create inverse function	No	No	Yes
Graph of inverse function	No	No	Yes
Determine and sketch the graph of the inverse of $y = ax + q$	No	No	Yes
Determine and sketch the graph of the inverse of $y = ax^2$	No	No	Yes
Determine and sketch the graph of the inverse of $y = bx$	No	No	Yes
Domain and range	Yes	Yes	Yes
Intercepts with axes	Yes	Yes	Yes
Turning points and maxima and minima	Yes	Yes	Yes
Asymptotes	Yes	Yes	Yes
Shape and symmetry	Yes	Yes	Yes
Average gradient	No	Yes	Yes
Intervals of increase and decrease	No	Yes	Yes
Discrete or continuous (given context)	Yes	Yes	Yes
Apply factor and remainder theorem to cubic polynomials	No	No	Yes

[11] South African Basic Education falls into three phases: The Foundation Phase (Grades R-3), The Intermediate Phase (Grades 4-6), The Senior Phase (Grades 7-9) and The Further Education and Training (FET) Phase (Grades 10-12).

[12] This is the intended curriculum mapping across the three grades within the FET Phase for mathematics.

The current study placed interest on the ways in which teachers teach learners how to work with the notions of independent (input) and dependent (output) variables when working with different families of functions. In addition, I paid attention to how teachers introduced learners to the basic features of prescribed graphs in CAPS, the effects of parameters *a* and *q*, as well as how they use multiple representations[13] during teaching to bring concepts such as intercepts, domain, range, minima and maxima. Thus, the crucial aspect for this study was how Grade 10 teachers introduce and communicate key ideas about the concept of function to the learners, considering that learners are introduced to algebraic functions for the first time in Grade 10. This necessitates the emphasis on the constituent parts of the function concept to ensure that learners learn, and own basic skills and knowledge associated with the concept (Sierpinska, 1992).

1.3. Defining the problem for the study

This study set out to explore and interrogate mathematics teachers' discourses and approaches teachers used during algebraic functions lessons in rural classrooms. In this section, I focus on three main identified problems for the study: the standard of mathematics education in South Africa, the deficient perspectives about rural education, and the function concept with the difficulties in teaching and learning in schools. In all three sub-sections, my argument is that expanding mathematics education research in South Africa to include rural constituencies can help us configure alternate knowledge about mathematics teaching and learning from the over-emphasised and monopolised urbanised research knowledge. Researching with rural teachers and learners remains under-researched, marginalised and 'othered' in South Africa and little is known about mathematics teachers' practices and how such practices enable and/or hinder learners' mathematics epistemological access.

1.3.1. The crisis of mathematics education in South Africa

The standard of mathematics education in South Africa has been described as being in crisis from primary to secondary schools, addressing the role of mathematics teaching, amongst other factors (Fleisch, 2008; Hoadley, 2007; Spaull, 2013). Studies conducted by Fleisch (2008), Trends in International Mathematics and Science Study (TIMSS[14], 2011); NEEDU (2012) and Southern and East African Consortium for Monitoring Educational Quality (SACMEQ, 2007)

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[13] See section 2.4. for elaborate discussion on how to work with relationships between variables using words, tables, graphs and formulae.

[14] TIMSS is a cross-country study that focuses on measuring tends in mathematics and science performance in Grade 8.

demonstrate that a raft of problems are present in mathematics teaching and learning in South African schools. For example, in SACMEQ (2007) results, South African learners were “ranked 10th of the 14 education systems[15] for reading and 8th for mathematics, behind much poorer countries such as Tanzania, Kenya and Swaziland” (Spaull, 2013, p. 4). The local Annual National Assessments (ANA) (Department of Basic Education (DBE), 2012) also illuminated learners’ poor performance in mathematics, with less than five percent of learners who achieved at least 40 percent. While the tests are for the Foundation Phase, Intermediate Phase and Senior Phase, the results suggest that teachers at Grade 10 “are expected to make up for large deficits” (CDE, 2014, p. 6), one of the reasons I decided to conduct a research with Grade 10 mathematics teachers. This lower performance was also noted from the TIMSS (2015) in which South Africa came 38th and 39th for mathematics and science respectively out of 39 countries that participated. In the 2019 TIMSS, a total of 64 countries participated in the study and South Africa came 62nd in mathematics achievements.

Although reports such as the NEEDU (2012) iterated that learners’ poor performance in mathematics was caused mainly by teachers’ poor subject matter knowledge, it is not clear whether and what percentage of rural teachers were part of the group. This would be interesting to consider, especially when the results could be understood within the context of apartheid’s well intended disadvantaging of teachers from townships, rural and farm areas, as the past appears to be haunting mathematics performance in a democratic South Africa. While I do not want to sound pessimistic, the different test results are self-explanatory for the majority of learners, which are dominated by rural learners, because South Africa is largely rural (Mbatha, 2014). Thus, it became imperative to explore and understand rural teachers’ knowledge of algebraic function, and their approaches while teaching the topic due to paucity of studies located in this context. To promote mathematics epistemological access in rural secondary classrooms, understanding the role of teaching as the practice of organising systematic learning is important (Morrow, 2007).

There is increasing local and international literature on mathematics teaching and learning generally (Wachira et al., 2013; Berger, 2013; Adler & Venkat, 2014; Decker et al., 2015; Ronda & Adler, 2017). Venkat et al. (2009) briefly mentioned that the dearth of mathematics education research “done in rural schools is problematic given that the majority of South African learners are educated in these contexts, as urban contexts continue to be explicitly and

[15] Zanzibar and Tanzania are tested separately, making 15 education systems but only 14 countries.

solely focused upon" (p. 11). It is a crisis to lack knowledge about teaching and learning aspects, especially seeing that most of the population reside in that context (Mbatha, 2014). Similarly, although not focusing on mathematics teaching and learning research, Nkambule et al. (2011, p. 341) posited that "little is known of the focus of various studies and the state of rural education and rural education research in South Africa", addressing the insufficiency of research located within rural schools. Mathematics education researchers in South Africa need to expand the scope of research to include rural education if the need to redress past injustices and ensuring social justice are seriously considered. The following sub-section focuses on research on the teaching and learning of the function concept.

1.3.2. Research on the teaching and learning of the function concept

The concept of algebraic functions has received attention within the field of mathematics education (Kazima et al., 2008; Trigueros & Martinez-Planell, 2010; Kabael, 2011). Very few protracted studies have been conducted in South Africa in respect of algebraic functions, and the existing studies were conducted with learners to explore the difficulties they experience when learning the topic. Felix Klein in 1908 viewed functions as 'the soul of mathematics', and this notion has since been discussed by various researchers (Carlson, 1998; Akkoc & Tall, 2005; Hansson, 2006; Mpofu, 2018). In relation to this, Sierpinska (1992, p. 32) stated that "functional thinking should pervade all mathematics, and at school, students should be brought up to functional thinking". This statement resonates with Eisenberg's (1992, p. 153) iteration that developing learners' sense of functions "should be one of the main goals of the school and collegiate curriculum". While the statements focus on the learners, they indirectly address the importance of teachers' content knowledge to ensure learners' development of algebraic knowledge. Lloyd et al., (2010) posited that functions are one of the key topics in secondary school mathematics because of their relatedness to other topics within the mathematics curriculum such as finance and growth, algebra and equations, as well as patterns and sequences. Thus, functions can be considered a meta-discourse of algebra internationally (Sfard, 2012).

Within the function concept in school mathematics, both the importance and problems relating to its learning have been researched and documented in mathematics education research (Moalosi, 2014; Pillay, 2013; Mpofu & Pournara, 2018). Swarthout et al., (2009) posited that functions is a very important topic in the mathematics curriculum because of the role that the topic is often seen to play as a unifying concept in mathematics. This makes it essential for learners to develop good conceptual understanding of the topic. While this is the case, Moalosi's (2014) study with Grade 11 learners demonstrated that functions is a topic that learners find difficult to understand because of the over-reliance on procedures of learning the topic. On the other hand, Denbel (2015) attributed the challenges concerning functions to the treatment of different modes of representations in isolation. In view of the above discussion, expanding the scope of mathematics education research to include researching with rural mathematics teachers could help give access to the discourses, knowledge and approaches they use while teaching algebraic functions.

1.4. Rationale of the study

The rationale to focus on Grade 10 algebraic functions stems from my informal observations as a former secondary school mathematics teacher. Over the five-year period of being a mathematics teacher, I realised that learners face difficulties in understanding the concept of algebraic functions, especially in Grade 10. The South African Department of Basic Education (DBE) curriculum planners have high regard for algebraic functions as demonstrated by the allocation of the most teaching time for the topic in Grade 10. This highlighted the great value the department has placed on the importance of algebraic functions in school mathematics. Thus, it is the difficulties associated with functions and the indispensability of functions in the curriculum that prompted me to investigate teachers' mathematical discourses and approaches during algebraic functions lessons. During my informal conversations with the learners, they often cited the teaching practices that some teachers use while teaching the topic to be a major attribute of poor conceptual understanding of functions. The informal observations and conversations resulted in research interest to explore teachers' discourses and approaches while teaching algebraic functions at Grade 10 level. In addition to the above discussion, the issue of the dearth of mathematics education research located within rural contexts and schools is further elaborated in the following section, to highlight the need to expand the research scope to include those contexts which have traditionally been marginalised in educational research in general, specifically in mathematics education research.

1.4.1. Dearth of mathematics education research located in rural schools

My interest in researching mathematics education emanates from the fact that mathematics and the teaching of mathematics have always been held in high esteem in education, as alluded to in my story at the beginning of the chapter. That is, they are inevitably important and have become the centre of conversations across different contexts, irrespective of geographic location. Despite this, the nature of mathematics teaching within rural contexts and schools remains under-researched in mathematics education research both internationally and locally (Balfour, 2012; Siyepu, 2013; Nkambule, 2017). While research on the teaching of mathematics within the South African context is popular, more work is required, particularly in rural South African contexts, because mathematics teachers in those areas have significantly different cultural environments and exposure to their urban and township counterparts which are predominately the area of focus. The research undertaken in this regard, thus, focused on teachers' discourses and approaches while teaching algebraic functions in a group of schools located in Acornhoek, rural Mpumalanga Province of South Africa, to deplore the normativity of the mathematics education research locale. The dearth of research in rural schools on how the different concepts of algebraic functions are taught coherently, connectedly and continuously influenced the conceptualisation of the current study.

In addition to the above discussion, the rationale for choosing to conduct the study with Grade 10 teachers is grounded on the fact that most of the basic mathematics terms and skills relating to functions are introduced in Grade 10 (DBE, 2011). As such, I intended to discern how teachers enable and/or constrain learners' development of mathematical discourse while learning new concepts related to the topic, considering the scarcity existing research. Regardless of the amount of research being undertaken in mathematics education, research literature focusing on mathematics teaching and learning within rural contexts, especially the teaching and learning of algebraic functions, remains imaginary in the context of South Africa. Much of the body of mathematics education research in South Africa in the past and present focuses solely on developing and improving mathematics teaching and learning in urban and township areas, leaving rural mathematics teachers' experiences of teaching the subject disregarded. According to Dickinson (2001), in the case of national research outputs, 'multi-locale research' is needed as this can enable the linking of educational events at both the micro and macro levels, especially considering that the advancement of rural education is one of the key national agendas in South Africa.

Fleisch (2008) has posited that one of the more controversial and dynamic challenges facing the South African education system in general is the average level of teachers' pedagogical competence, of which the empirical evidence in this respect remains inconclusive, particularly considering the prevailing lack of research conducted with teachers within rural schools and classrooms. Correspondingly, there are no previous studies that have been undertaken to gain insight into teachers' pedagogical approaches and discourses during teaching within rural

South African mathematics classrooms. Accordingly, the intention of the current study is to contribute to the wider research knowledge on rurality and the teaching of mathematics within rural contexts by acknowledging the importance of researching in different South African contexts, rather than reproducing knowledge from the same urban context (Nkambule et al., 2011; Mbhiza, 2017). The broader field of research focusing on rural education in general, specifically the teaching and learning of mathematics within rural classrooms, needs comprehensive research to diversify research knowledge and in turn help teachers reflect on their practices during teaching.

1.5. Purpose of the study

The purpose of this study was to explore rural Grade 10 teachers' discourses and examine the approaches during algebraic functions lessons. Teachers' mathematical utterances and classroom practices provide clear context and reflection of their mathematical discourses and approaches to teaching the topic. In this regard, both the teachers' mathematical knowledge and linguistic proficiency related to algebraic functions were inextricably aspects of the study. Teachers' mathematical discourse on algebraic functions helped to identify factors that shape the teaching of algebraic functions within rural classrooms. It was also important to explore and identify the lexicon used by teachers in expressing themselves in the context of algebraic functions, because it was in the words that teachers used that their thought processes could be used as a window to the kind of mathematics that they hold related to the topic.

1.6. Objectives of the study

The objectives of the study were as follows:

a) To describe and critically analyse Grade 10 rural mathematics teachers' discourses and approaches while teaching algebraic functions.
b) To identify and interrogate teachers' approaches during algebraic functions lessons.
c) To gain insight into teachers' use of multiple representations in the context of algebraic functions.
d) To explore and understand the nature of teachers' mathematical discourse relating to the notion of parameters of algebraic functions.
e) To examine factors that influence teachers' discourses and approaches while teaching algebraic functions within rural schools.

1.7. Research questions

Given the objectives above, the study sought to address several research questions. The main research question which informed this study was: "What are the rural Grade 10 teachers' discourses and approaches during algebraic functions lessons?" To explore the main research question, I identified the following sub-research questions:

a) What are Grade 10 rural mathematics teachers' discourses during algebraic functions lessons?
b) What approaches do Grade 10 teachers use to teach algebraic functions?
c) How do teachers use multiple representations during algebraic functions lessons?
d) How do teachers guide learners towards generality about the effect of parameters in the context of algebraic functions?
e) What are the factors influence teachers' discourses and approaches of algebraic functions within rural classrooms?

1.8. Operational definitions of concepts in the study

Table 4

The operational definitions of concepts in the study

Concept	Operational definition
Approaches	In the current study, approaches have two interrelated meanings, the first referring to the specific approaches that are commonly used to teach algebraic functions and the second operationalisation pertains to the pedagogical link-making (PLM) approaches developed by Scott, Mortimer and Ametler (2011). The success of learners' learning and understanding of the function concept is largely dependent on the teaching approaches that teachers use during teaching (Smith, 2017). Some of the approaches that have been identified by authors for teaching the concept of function and have been used in the current study as analytical tools are: the covariational, the pattern-oriented, the function machine, the word problem-based, the example and non-example, and the property-oriented approach (Tall & Bakar, 1992; Monk & Nemirowsky, 1994; DeMarois et al., 2000; Pillay, 2006; Kwari, 2007; Ronda, 2009).
Discourse	A discourse is a "special type of communication made distinct by its repertoire of admissible actions and the way those actions are paired with re-actions" (Sfard, 2008, p. 297). More descriptively, discourses refer to different types of communication distinguished by four features that are characteristic to the mathematics discourse: word use, visual mediators, endorsed narratives and routines. In this study, discourse counts as mathematical if it features words, visual mediators, endorsed narratives and routines that are characteristic to the algebraic functions discourse. Fairclough (2009) described discourse as the investigation of the relationship between the surrounding social practices and the discourse itself. In the context of this study, this means discourse as part of mathematics teaching constitutes the diverse ways of speaking and acting during the teaching of algebraic functions.
Rural	Coladarci (2007, p. 2) argued that "there is no singular or multifaceted definition (of rural) that will suffice to satisfy the research … that employ the concept". Thus, I prefer to view the term 'rural' as dynamic and shaped by a wide range of systems, specifically by individuals who live in rural areas. In this study, I view rural as referring to a social representation, a culture of life

	constructed by individuals living in those areas. Halfacree (1993) provided a comprehensive summary on the debates about the differences between viewing 'rural' as a geographic concept or as a social representation, and substantive arguments in support of viewing 'rural' as social representation.
Commognition	The theory of commognition coined by Sfard (2008) considers thinking processes as communication with one's self. Sfard views mathematical thinking in two ways that are both equivalent and complementary: cognition and communication. In this study, I use commognition to describe and analyse how the Grade 10 rural mathematics teachers think and justify mathematical concepts in algebraic functions using the four features of mathematics discourse.
Algebraic functions	The definition of function simply uses the idea of univalence that for each element in the domain there is exactly one element in the codomain, with no other required properties of the correspondence. Arbitrariness and univalence are features of the concept of function. Arbitrariness of functions refers to both the relationship between the two sets on which the function is defined and the sets themselves. The arbitrary nature of the two sets means those functions do not have to be described on any specific set of objects. Arbitrariness is closely interrelated to an analytical judgment when an instance belongs to a concept family (Even, 1990).

1.9. Structure of the thesis

This thesis comprises eleven chapters. **Chapter 1** has provided the background of the study, highlighting the importance of understanding teachers' discourses and approaches of teaching algebraic functions within rural mathematics classrooms. This chapter also presented debates about the standards of learning mathematics in rural contexts. I also elucidated on the rationale for conducting this study, the purpose, specific objectives as well as the study's research questions.

Chapter 2 presents literature that addresses the concept of algebraic functions in school mathematics. The chapter begins with a discussion of the role of Mathematics for Teaching Knowledge in mathematics teaching generally, and specifically for teaching algebraic functions. I also provide literature associated with the different approaches to the concept of functions. This chapter also reviews literature on mathematical discourse.

In **Chapter 3**, I present the theoretical framework and conceptual frameworks that are espoused in this study to critically explore teachers' discourses and approaches to teaching algebraic functions. To do this, I discuss in detail Sfard's commognitive theory and Scott, Mortimer and Amettler's (2011) concepts from the pedagogical link-making framework, expounding how a combination of these lenses enable me to understand teachers' discourses and approaches during the teaching of algebraic functions within rural classrooms.

Chapter 4 provides an overview of the nature of the current study, explicating the research-paradigm, design, methodology and methods of data generation for the study. In this chapter, I also discuss the sampling technique and the justification of the sample size. A discussion of the data analysis process is provided. In addition, the issues of credibility, transferability, dependability and confirmability are also highlighted.

Chapters 5–9 present an analysis and interpretations of selected episodes from a series of teachers' observed lessons. To ensure that I present a comprehensive analysis of the teachers' teaching in these chapters and not overlook the nuances in their teaching of algebraic functions, I present teachers' data in five separate chapters. Some of the chapters are longer due to the nature of each teacher's teaching, and thus makes the thesis longer.

Chapter 10 presents the findings and discussions from classroom observations in relation to the findings from semi-structured individual interviews and video-stimulated recall interviews, and these are discussed in view of the reviewed literature and the identified theoretical and conceptual frameworks for the study.

Chapter 11 concludes the thesis by providing a summary of the findings in relation to the reviewed literature, theoretical and conceptual frameworks, and the study's research questions. A discussion of the contributions of the study is also presented, followed by limitations of the study. The chapter ends by suggesting possibilities for future mathematics education research.

1.10. Chapter Summary

In this chapter, I presented a discussion on the background of mathematics education and mathematics education research in South Africa, arguing for the need to popularise research with rural teachers to close the existing research gap. The problem statements; the rationale for the study; the purpose of the study; research objectives and questions; as well as the possible contributions of the study were also presented. Various authors have contended that rural contexts are underrepresented, and people tend to homogenise all rural areas, and overlook thinking of the context as capable on its own. The discussions in this chapter further necessitated the need to research within rural schools with teachers and learners as I believe they can offer alternative knowledge relating to the schooling processes compared to the dominant urbanist perspectives. The following chapter provides a comprehensive critical discussion of the literature related to the focus of this study.

Chapter 2

Understanding algebraic functions: review of the literature

It is through education that a daughter of a farmworker can become a doctor, and that a son of a mineworker can become a president ~ Nelson Mandela (2013, p. 21)

2.1. Introduction

One way of interpreting Nelson Mandela's statement above is that education is one of the key attributes that gives hope to the learners in the country, especially learners from working class backgrounds. Education in general, and mathematics in particular, can empower learners to pursue careers their parents were not afforded opportunities to, because working-class learners can perform well in their studies and excel in mathematics. The fact that the parents of the learners in the current study work on farms, does not necessarily mean that they cannot give birth to children that could become doctors and leaders in the society. While this is the case, teachers are the key stakeholders in ensuring that learners learn and understand the concepts and skills that are necessary to be successful in mathematics. It is because of the different roles that teachers play to ensure successful mathematics learning, that the current study explored and interrogated Grade 10 teachers' discourses and teaching approaches of algebraic functions within rural mathematics classrooms.

This chapter begins with a discussion of the function concept in detail, followed by a review of the literature related to the learning of the function concept. Then the discussion of the use of multiple representations in the context of teaching algebraic functions is presented. I will also discuss the use of prominent approaches underpinning the teaching of algebraic functions, as the important aspect of the study. Although there are some studies (Maharaj, 2010; Malahlela, 2017; Moeti, 2015; Mugwagwa, 2017) that have been conducted in South Africa on the teaching of algebraic functions, they have overlooked the teaching of the topic in rural secondary schools, making this study significant to the contribution of the current knowledge.

2.2. Understanding the teaching of algebraic functions

In this section, the primary focus is on the algebraic functions as the core areas of mathematics education research as derived from the reviewed literature. While in Chaper 1 I discussed the historical development of the function concept, in this section I discuss the key elements and layers of the concept as covered in previous studies on algebraic functions. I start the discussion in this section with a focus on the three constitutive elements of a function, followed by a discussion on the layers of development of the function concept. An algebraic function is defined by an algorithm that provides instructions on how to determine output values from the chosen or given inputs. The information on the constitutive elements and layers of development of the different function conceptions is vital in developing an instrument for assessing what teachers make available for learners to learn at Grade 10 level.

2.2.1. The constitutive elements of a function: the importance of teaching

There are three constitutive elements of the function concept: variable magnitude; the dependence between the variables and a rule that signifies the relationship between these variables. Earlier, Anderson (1978, p. 23) stated that these refer to the "raw material, a rule or a process … and an end product". This resonates with assertions that were made by Sierpinska (1992, p. 30) that the constitutive elements of a function should be viewed as 'worlds' and the teaching of the concept should focus on three worlds: world of changes or changing objects, world of relationships and world of rules, regularity and laws. Mathematics has discourses and teachers are expected to use mathematics rules, deeds and interactions that are part of the subject's discourse and legitimise certain forms of mathematising both orally and from learners' written work (Sfard, 2008; Aineamani, 2011).

Firstly, the world of changes entails an identification of 'what' is changing in given relationships and 'how' the change is taking place. In this sense, teachers should teach learners how to work with the idea of 'transformation' in functions, and pay attention to the appearance, displacement and orientation of functions (Chimhande, 2014; Mudaly & Mpofu, 2019). For Sierpinska (1992), teachers need to emphasise to the learners the need to move from viewing X and Y as knowns and unknowns, to conceiving them as variables and constants for meaningful understanding of functions. It is essential to note that the change in functions can be viewed in two ways: numerically as magnitude changes in number operations, and graphically as transformations in terms of reflection, rotation, translation and enlargement. Considering that this is expected of teachers, it is assumed that teachers have adequate Mathematics for Teaching Knowledge (MTK) relating to the teaching of algebraic functions (Ball et al., 2008). In view of the history of training mathematics teachers in South Africa, the nature of rural teachers' MTK is unclear, especially considering the sparcity of mathematics education research located in those areas. MTK is important, especially since teachers with a stronger knowledge base are more responsive to learners' mathematical learning needs (Ball et al., 2008). In addition, when teachers possess stronger MTK, they are likely to make fewer language and mathematical errors during teaching and learning, because MTK is a precondition for effective communication during teaching. This will be seen in the presentation, analysis and interpretation of the teachers' teaching, as well as in the discussion of the key findings of the study.

Secondly, teachers should teach the learners how to observe change between the given variables and identify the relationships between them. Sierpinska (1992) also suggested that teachers should introduce functions as models of relationships drawing from real life situations, and in turn view functions as tools for representing a system in another system. The rationale

> for the motivation of mathematical concepts by using concrete examples in the teaching of mathematics stems from the commonly accepted notion that, nowadays, students are interested in the study of the subject matter if they are confident in the applicability of the material they are about to learn. (Abramovich & Leonov, 2009, p. 2)

The interest and confidence are influenced by the quality of a teacher, because learners observe and do what they observe the interlocutor does during teaching and learning in the classroom (Sfard, 2008, 2012).

In addition to the above discussion, for Sierpinska (1992), functions should be viewed "as tools of description and prediction" (p. 32) of how variables are related to each other, making functions models of patterns in real-life phenomena. The argument is that the preceding conception of function as models of real-life situations is often contrived in school textbooks across different grades into word problems (Kwari, 2007; Sherman & Gabriel, 2017). During the teaching and learning of word problems, teachers should explicitly show learners how to move from words to symbolic statements, tables to graphs systematically to express the given relationship. Some authors (Mandal & Naskar, 2021; Wang et al., 2018) have argued that the use of symbolic statements makes it easier for teachers and learners to access many characteristics of a word problem. While this could be true, shifting from a word problem to a symbolic statement requires a mastery of basic algebraic reasoning that is unknown for teachers in rural secondary schools (Blanton & Kaput, 2011; DBE, 2011), a reason it was important to observe and interact with them to gain insight into their algebraic reasoning.

Of interest in the current study is how teachers help learners navigate from the word problem to symbolic representations, and in turn do algebraic manipulations to express the given relationship(s) adequately. For learners to develop problem solving fluency, teachers should emphasise the need to interpret algebraic expressions and equations that represent real-world situations (NCTM, 2009). As noted from Euler's and Dirichlet's definitions earlier, functional relationships can be expressed in terms of covariation or by using a rule of correspondence between variables (Blanton, 2008; Bazzoni, 2015; Wilkie, 2020). In a covariation[16] relationship (see sub-section 2.5.1), when

> one quantity changes in a predictable or recognisable pattern, the other also changes, typically in a differing pattern. Thus, if one can describe how x_1 changes to x_2 and how y_1 changes to y_2 then one has described a functional relationship between *x* and *y*. (Borba & Confrey, 1996, p. 323)

When functional relationships are expressed in terms of a rule of correspondence, as coined by Dirichlet, it enables teachers and learners to describe a rule that determines the value(s) of the dependent variable (*y*) given particular values for the independent variable (*x*). Learners need to be taught that the association between the two variables can be understood as fixed points on a Cartesian plane and are usually represented by a set of ordered pairs as coined by Bourbaki in the form (*x; f*(*x*)).

Expressing relationships as 'a set of ordered pairs' is often left unnoticed as a form of representation. If teachers do not explicitly teach learners associated procedures on how to represent a function as ordered pair co-ordinates, it could result in learners not knowing how to work out a set of ordered pairs for a specific function. This could hinder their fluency in drawing graphs on the Cartesian plane. Wilkie (2020) opined that it is important for teachers to develop learners' understanding of functions through both the correspondence and covariation approaches, to elicit learners' thorough understanding of the topic and other associated sub-concepts. She further advocated for the use of the two approaches, and emphasised the need to use the covariation during the introductory stages of teaching the topic (Wilkie, 2020). Similarly, Confrey and Smith (1995, p. 78) argued that the correspondence approach is abstract and "places a heavy emphasis on stating the rule explicitly (usually algebraically) and on a directionality from *x* to *f(x)*". Although Confrey and Smith did not explicate what they mean by 'introductory stages', in the context of this study it is assumed that Grade 10 is an introductory grade for functions as sequenced within the current mathematics curriculum policy. In addition, Thompson (1994) also posited that the school mathematics curriculum should place emphasis on covariance for teaching functions because it helps learners to coordinate two quantity values as they change simultaneously. Several

[16] I provide an elaborate discussion of what the covariational approach to teaching the concept of function entails in sub-section 2.6.1.

authors have reported difficulties associated with the interpretation of the dependence between co-varying quantities (Carlson & Oehrtman, 2005; Wilkie, 2020; Digel & Roth, 2020), and in this study the interviews and classroom observations provided accounts of the teachers' physical incarnations of a function they envisaged when thinking and teaching about algebraic functions (Digel & Roth, 2020).

Thirdly, a function is considered a rule that governs the relationship between variables (Sierpinska, 1992). According to Van de Walle (2004, p. 436), a function can be viewed as a rule "that uniquely defines how the first or independent variable affects the second or dependent variable". The rules, patterns and laws refer to well-defined relationships, a reason for a strong link between this conception and the one discussed above. DeMarois and Tall (1996) argued that the development of the function concept is very complex, and that change, relationships and rules are not mutually exclusive pockets of knowledge about the concept. This means the world of changes or changing objects, world of relationships and world of rules, regularity and laws discussed above are interdependent upon each other and the topic should be taught likewise. For example, when learners are observing the change in the values of the independent variable, they should be able to observe how such change influences the values of the dependent variable to construe a rule that signifies a functional relationship. Thus, the foregoing necessitates that as teachers teach functions to their learners, all three conceptions should be developed if enabling learners' fluency in working functional problems is seriously considered. Considering the above discussion, it becomes clear that change, relationships and rules are to be seen as components of a complex association in understanding the mathematical concept of function.

2.2.2. Layers of development of the function concept

The previous discussion presented the constitutive elements of functions, which emphasised that the initial understanding of the function concept should focus on helping learners identify change and what changes. This section focuses on the stages of cognitive understanding which learners should move through to have a real understanding of the changes (Kwari, 2007; Chimhande, 2014). According to Tall and Bakar (1992), some learners hold misconceptions of the concepts of functions, such as viewing a function as a formula/an equation/an algebraic term, and that graphs should be systemic and regular amongst others. Throughout the literature relating to the teaching and learning of functions, there are four "layers of the function concept" (DeMarois & Tall, 1996; Maharaj, 2010; Chimhande et al., 2017). The four layers: Action, Process, Object and Schema indicate the level through which teachers teach algebraic functions and are not distinct but represent a dynamic continuum of abstraction (Dubisnky & Wilson, 2013). Layers of the concept of functions were first coined by Dubinsky and Harel (1992) as Action-Process-Object-Schema, the "construction in which mental actions (on objects) become repeatable processes which are encapsulated as objects" (DeMarois & Tall, 1996, p. 298). I draw particularly from DeMarois and Tall's (1996) operationalisation of the term layer, to refer to "the various levels of the depth dimension in the development via cognitive process to mental object" (p. 298). In this study, this relates to teachers' tendency to deal with conceived mathematical problems related to the function concept, by creating mental actions, processes, objects and organising them into schemas to help learners solve the problems. Below, I explain the four layers and their indicators relating specifically to the teaching of the function concept.

According to Dubisnky and Harel (1992), an action layer refers to a "form of understanding of a concept that involves a mental or physical transformation of the objects in relation to stimuli that the subject perceives as relatively external" (p. 17). In terms of teaching the concept of function, the action layer entails the process when a function is treated as a recipe[17] to apply to given numbers which remains the same for different input values, "but they must actually apply it to some number before the recipe produces anything" (Pillay, 2014, p. 31). Bayazit and Aksoy (2010, p. 149) stated that an "Action-oriented teaching is distinguished by the teacher's instructional acts which emphasize step-by-step manipulation of algorithmic procedures and engage students with the visual properties of algebraic piecewise functions". Barmby et al. (2007) strongly argued that gaining an understanding of mathematics is not about offering a right answer quickly and more easily but understanding mathematics concepts. The latter requires a connection between concepts and an understanding that can be built upon, as opposed to procedures that can be easily used but also easily forgotten. Oehrtman et al. (2008) contended that if teachers limit the learning of functions to an action view, learners "are limited to understanding only the related procedural tasks such as switching *x* and y and solving for y" (p. 157) which limits conceptual development (Maharaj, 2010; Chimhande et al., 2017). Thus, action view cannot function alone, the process layer becomes important to help learners develop mental images about the different concepts associated with algebraic functions.

The process layer entails "a form of understanding a concept that involves imagining a transformation of mental or physical objects that the subject perceives as relatively internal and totally under his or her control" (Dubinsky & Harel, 1992, p. 19). This means as individuals

[17] This relates to what Skemp (1976) termed 'instrumental understanding'.

(teachers in this study) repeat and review a particular action it may become interiorised, which refers to a mental structure that carries the same operation as the action discussed above. According to Maharaj (2014), what distinguishes a process view from an action view is that the process is embedded wholly in the mind of an individual, as individuals "think of performing a process without actually doing it" (Dubisnky & McDonald, 2001, p. 3). Relating to this study, the above discussion suggests that teachers can imagine carrying out the transformation of a function without executing each step explicitly (Reed, 2007; Dandola-Depaolo, 2011; Weyer, 2010), resulting in implicit teaching of a concept or content.

In addition to the above discussion, Thompson (1994) described a process layer of a function as instances when a learner or a teacher constructs an image of self-evaluating expressions. They can rather imagine a set of input values of a function that are mapped out to a set of output values for a given expression that defines a function. In this study, the fundamental aspect was whether and how teachers explicate to the learners the steps they do not overtly write on the board and say during teaching, that they observe common properties of specific functions and amalgamate many observations into one idea (Weyer, 2010). The process level, according to Oehrtman et al. (2008), gives a teacher and learners access to an "entire process as happening to all values at once, and is able to conceptually run through a continuum of input values while attending to the resulting impact on output values" (p. 158). Thus, teachers with a process view of the function concept teach learners how to see the change in the input variable while predicting changes in the output variable, which is commendable. Similarly, Prihandhika et al. (2020) stated that teachers and learners who hold a process conception of functions draw a function graph by plotting a few points on the Cartesian plane and construe general arguments about a function. In the South African context, the CAPS FET mathematics curriculum is framed within this conception of teaching the concept of function, as teachers and learners are required to look at the ordered pairs to generalise the relationships without having to draw arrow diagrams. Interestingly, this was also noticed in some observed lessons in the study when teachers showed learners how to determine output values from given input values, which address Even's (1990) notion of "global approach".

In terms of the object layer, Maharaj (2010, p. 57) postulated that, "If one becomes aware of a process as a totality, realizes that transformations can act on that totality and construct such transformations (explicitly or in one's imagination), resulting to the encapsulation of the process into a cognitive object". Within the function context, this conception involves the ability to utilise a function in further processes such as seeing the function in the form $f(x) = x^2 + 3$ to represent the graph of the function in the form $f(x) = x^2$ as if it were an object, and shifting the whole graph upwards three units to obtain $f(x) = x^2 + 3$ (Chimhande, 2014). Considering the graphical perspective, an object level view of function allows teachers and learners to manipulate a graph of a function, for example, shifting the graph of a function $f(x) = x^2$ two units through the y-axis in the positive direction to find the graph $f(x) = x^2 + 2$ without having to deal with the graph point by point to get the new graph.

Lastly, the schema conception of function entails a “collection of actions, processes, objects and other schemas, together with their relationships that the individual understands” in connection with the concept (Dubinsky & Harel, 1992, p. 20). In the schema-oriented teaching of function, a teacher focuses on helping learners to link the symbolic and graphic representations to “construct a precise symbolisation for the information available in the given graph and have a whole understanding of the concept of how all multiple representations of a function link together” (Chimhande, 2014, p. 59). The implication for teaching is that teachers should help learners to reach the schema level to enable them to operate at any of the four layers required by different questions related to the function concept, which facilitates effective communication of the key features of the concept. Considering the importance of linking multiple representations of the function concept at the schema level conception, in the following section I discuss the potential of using multiple representations in the teaching of functions and how this in turn facilitates learners’ understanding of the concept (Dede & Soybas, 2011; Dreher & Kuntze, 2015; Makonye, 2017).

2.3. Research on the learning of algebraic functions

The function concept is viewed as a unifying theme in the mathematics education curriculum (Steele et al., 2013) consisting of tables, graphs, symbolic equations and verbatim as representations of the concept (Chitsike, 2013; Mudaly & Mpofu, 2019). Internationally, learning the function concept with understanding is regarded to be paramount as it facilitates the understanding of other topics in the mathematics curricular (Nachlieli & Tabach, 2012). Various studies suggest that the functions concept is one of the most important topics in middle and secondary school (Dubinsky, & Wilson, 2013; Chimhande, 2014; Adu-Gyamfi, & Bossé, 2013, Mudaly & Mpofu, 2019), and that lack of functional thinking and understanding could result in no real understanding of mathematics (Nachlieli & Tabach, 2012; Adu-Gyamfi, & Bossé, 2013). With the importance of the functions concept in mind, learners in different contexts have been reported to have special difficulty understanding the notion of a functional relationship where an output variable changes in accordance with an input variable and working with different modalities of representations of functions (Nachlieli & Tabach, 2012; Adu-Gyamfi & Bossé, 2014; Dubinsky & Wilson, 2013; Chimhande, 2014; Mudaly & Mpofu, 2019).

According to Nachlieli and Tabach (2012) and the National Council of Teachers of Mathematics (NCTM) (2014), functions frustrate the learners because they are often not taught as the consolidation into one 'object'. The authors observed that the seemingly unrelated modes of functional representations: the algebraic expressions, tables and graphs make it difficult for learners to understand the concept of functions. As also observed in the current study, the difficulty of learning functions lies in the fact that teachers present functions as formulae and graphs, instead of a relationship between the two (NCTM, 2014). Earlier, Sierpinska (1994) noted that learners need to learn and know: what a function is, what are its uses, what are the different types of functions, what are the properties of different functions, what are the appropriate values for functions and so on. These aspects are part of the South African curriculum for the topic (DBE, 2011), and call for teachers to ensure that learners learn and understand them to develop functional thinking.

Literature has suggested that the properties of functions that are challenging for learners to understand include: covariation, which represents the world of changes discussed earlier in section 2.5.1.; rules of correspondence (Nachlieli & Tabach, 2012); patterns (Blanton & Kaput, 2011); and univalence (Dubinsky, & Wilson, 2013). Thus, for learners to understand the aforementioned properties of the concept of functions, teachers should ensure that learners are taught to undertake the following actions: interpretation and construction of functions in order to help them to comprehend them. Interpretation refers to "action by which student makes sense or gains meaning from a graph (or a portion of a graph), a functional equation, or a situation" (Leinhardt et al., 1990, p. 8). This statement includes but also is not limited to actions such as describing changes brought by the changes in the values of parameters in the graph or table of values as well as reading off the values of x and y from sketched graphs, for example, determining the value(s) of x for which $f(x) = g(x)$.

On the other hand, the construction action entails the process of identifying points in symbolic representations, tabular or other forms of representations of a function and plotting these to form graphical visualisers. Previous studies on the learning of algebraic functions have found that learners and many teachers often follow a route that begins from the algebraic formula and engages in mathematical conventions to move from this to the graphical mode of representation (Nachlieli & Tabach, 2012; Dreher & Kuntze, 2015; Mudaly & Mpofu, 2019). Chimhande (2014) argued that this routine does not address the flexible move between multiple representations, and therefore most aspects of knowledge and understanding of functions are overlooked. In the context of the current study, what becomes important is that learners' comprehension of these actions depends on the effectiveness of teachers' discourses and approaches during teaching. To elaborate on mathematical discourse for algebraic functions, the following section focuses on the use and benefits of multiple representations to teach functions, which links closely with the idea of visual mediators.

2.4. Mathematics as discourse

A primary premise of the current study is the view of mathematics teaching as a discursive activity (Sfard, 2012). This view sees mathematics teaching as a process of engaging in mathematical discourse, in which mathematical objects are seen to be discursively constituted. This means that, mathematical objects obtain their existence and meanings through mathematical discourse. According to Lynch and Bolyard (2012), mathematical discourse entails "the written and spoken mathematical communication that occurs in a classroom" (p. 488). Because of this, teachers need to ensure that they write and speak about mathematical ideas explicitly, to enhance learners' mathematical learning (Nguyen, & Chapin, 2019). In relation to this, Buchheister, Jackson and Taylor (2019) argues that mathematical discourse "enhances mathematical thinking and reasoning" (p. 204). Similarly, Bennett (2014) asserts that, "Discourse requires students to evaluate and interpret the perspectives, ideas, and mathematical arguments of others as well as construct valid arguments of their own" (p. 20). One way of interpreting this is that, in a mathematics classroom where there is presence of mathematical discourse, teachers allow learners to demonstrate their thought processes about mathematical ideas and discuss their mathematical thinking to enhance their understanding of the topic (Stein, 2007; Adler & Ronda, 2015).

The above discussion suggests that, mathematical discourse creates opportunities for learners to observe, explain, clarify and justify their mathematical thinking. Within the context of the current study, the view of mathematics teaching as a discursive activity implies that when teachers select and structure tasks for algebraic functions, it is important that they design tasks that enable learners to share their understanding of the topic. While this is the case, in the current study the majority of the participating teachers did not create opportunities for learners to make sense of algebraic functions in their own ways. The observations in this study are contrary to Walshaw and Anthony's (2008) suggestion that "the most effective settings provide balance between opportunities for students to benefit from teacher telling and students' involvement in discussion and debate" (p. 539). Gresham and Shannon (2017, p. 365) reported that when teachers pressed learners to "clarify their own thinking during classroom discourse ...", it resulted in learners developing mathematical ideas and improved reasoning "... when asked to conjecture, explain, and justify their solutions to others" (p. 365). Allowing learners to offer explanations for their mathematical thinking not only enables them to partake in the process of mathematical discourse, but further encourages analytical and critical thinking sought after by teachers.

In addition to the above discussion, Sfard (2008) conceptualise mathematical discourse as a specific type of communication, with its own unique key characteristics. Sfard (2008) identifies four features that show the development of mathematical discourse during teaching and learning: words, visual mediators, narratives and routines. I describe each of these features briefly, as I will elaborate on them in chapter 3. This is because I use the commognitive theory of Sfard (2008) to operationalise teachers' mathematical discourse during lessons on algebraic functions. Key words refer to mathematical words and symbols that facilitate mathematical communication. In functions discourse, key words include *range*, *domain, axis of symmetry* and *intercepts* to name just a few. According to Caspi and Sfard (2012), it is through the usage of the key words that we think and communicate mathematical ideas during teaching and learning. The usage of the key words are supported by the use of what Sfard termed visual mediators, which help participants of mathematical discourse to identify the objects they think and talk about, to co-ordinate their talk about such objects (Sfard, 2008). In the context of teaching and learning algebraic functions, visual mediators include tables, graphs and equations. The third feature of mathematical discourse according to Sfard (2008) is endorsed narratives, and are defined as rules that that have been accepted within the mathematical community, such as definitions of concepts, theorems and proofs. The fourth feature is routines, which entails the well-defined patterns of mathematical communication repeated over time. These include mathematical conventions for solving equations, simplifying expressions and drawing graphs. To elaborate on mathematical discourse for algebraic functions, the following section focuses on the use and benefits of multiple representations to teach functions, which links closely with the idea of visual mediators.

2.5. Using multiple representations to teach algebraic functions

Researchers (Duval, 2000; Dreher & Kuntze, 2015) have indicated that for learners to develop an adequate understanding of the function concept, it is advisable that teachers use all different registers (verbal, tabular, formulae, and graphical) simultaneously during teaching. Brenner et al., (1997) in their article "The role of multiple representations in learning algebra" argued that when learners are learning functions, it is important to know how to move between the verbal, symbolic, tabular, and graphical representations. While this is important, it also depends on the teacher's content knowledge or the nature of training to use different modes of representations simultaneously. Notwithstanding this, it is crucial for teachers and learners to make connections among multiple mathematical representations to develop relational understanding of mathematics concepts and use them as tools for problem solving (NCTM, 2014).

The above information highlights the importance for teachers to teach learners how to construct and navigate between multiple mathematical representations of algebraic relationships as embedded in algebraic objects, to enable learners' in-depth understanding of functions. Kaput (2018) asserted that making links between various representations provide learners with a more coherent and unified view of the concept of function, depending on the teacher's ability to use the representations to enhance teaching and learning. Elia et al. (2007, p. 538) echoed similar sentiments: "The ability to identify and represent the same concept in different representations, and flexibility in moving from one representation to another, are crucial in mathematics learning, as they allow students to see rich relationships, and develop deeper understanding of concepts". Unfortunately, it is not known how rural teachers use representations to teach algebraic functions in South Africa, which makes the current study important to contribute to the existing knowledge.

According to Isler et al. (2017), learners who have rigorous understanding of the notion of algebraic function find it easier to know which representations are most suitable for particular contexts. The learners are able to navigate[18] backwards and forwards between different modalities of representation, as illustrated in Figure 1. While the emphasis is placed on learners and learning in the above reviewed studies, teachers are tasked with ensuring that learners gain representational fluency when working with algebraic problems. For example, Eisenberg

[18] According to Brenner et al. (1997), most learners tend to solve algebraic problems well when they are only using symbolic representations, but often struggle to work with verbal representations. While this is the case, teachers play a very crucial role in facilitating that learners master how to work with multiple representations.

(1992, p. 76) posited that teachers' and learners' ability "to make connections among representations of the function concept is the main component of a robust understanding of the function concept". It is thus crucial that teachers teach the relationship between the representations explicitly, to enhance learners' knowledge. During the lessons, teachers can elicit learners' understanding of the concept by "designing instructional activities that are not restricted by certain types of representation but involve recognition and transformation activities of the notion in various representations" (Duval, 2002, p. 4). This needs the teacher's skills and careful planning to ensure the use of different representations during the lessons, to support knowledge building.

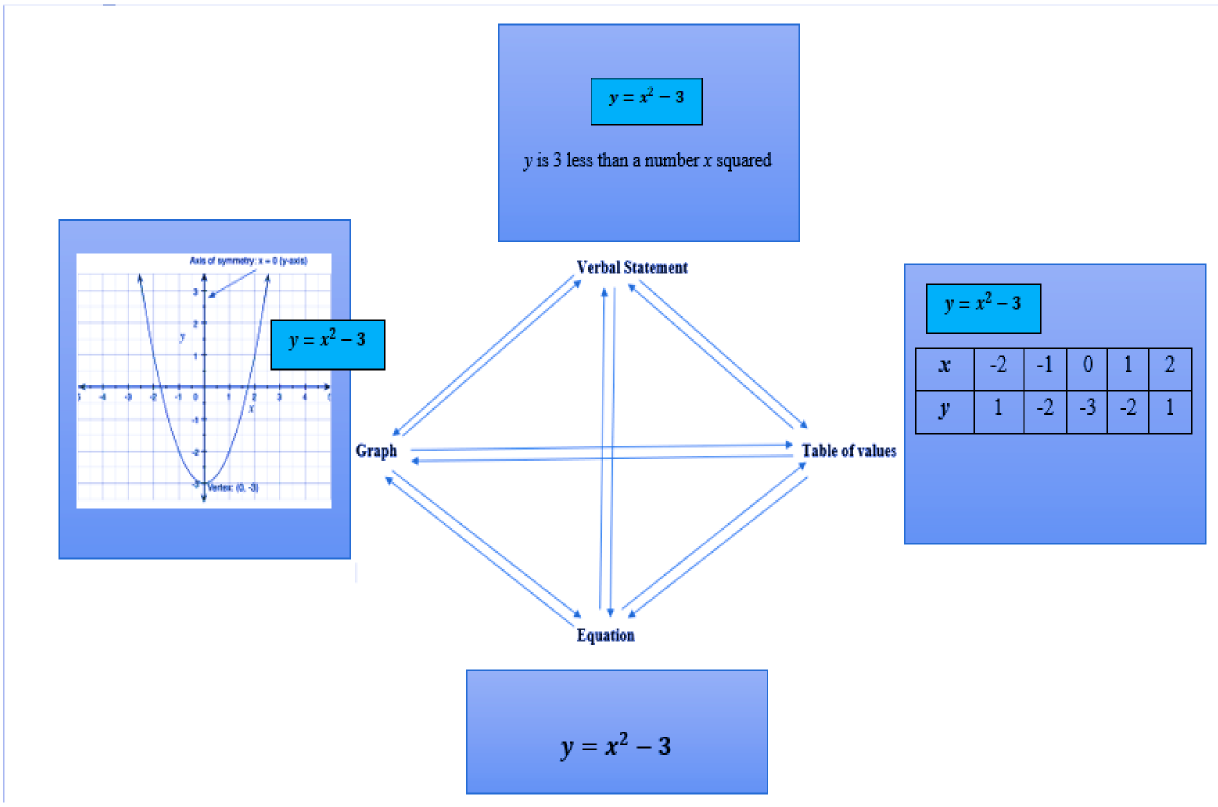
The foregoing discussion is in line with the prescriptions in the current Grades 10 to 12 South African curriculum policy (DBE, 2011) which requires learners to demonstrate the ability to work with the prescribed functions by flexibly moving between the verbal, tabular, graphic and symbolic representations. Nevertheless, Moschkovich et al.'s (2017) findings demonstrated that learners could understand a function in one representation yet express a different understanding of the same function in another representation. This addresses the significance of exposing learners to classroom activities that require translation between different representations, explicating their roles in teaching the concept as well as relationships between the representations. This discussion takes it for granted that teachers have appropriate training and content knowledge to manoeuvre between the representations, which cannot be assumed for teachers in rural schools but should be explored for information.

2.5.1. The four representations of functions and their relationship

In the context of learning and teaching the function concept, the modalities of representation have a twofold role: as different modes of expressing a function, and a way of expressing the reasoning strategies that learners use in the development of functional understanding (Chimhande, 2014). In the school curriculum, a function is depicted using different modalities of representations, sometimes graphically or alternatively as a table of values illustrating the relationship between input values and corresponding outputs (Moeti, 2015; Mpofu & Pournara, 2018). The representations discussed in this section enable learners to "organize, create, record, understand and communicate mathematical ideas" (p. 35). In Figure 1 below, Van Dyke and Craine (1997) represented diagrammatically the twelve directions for shifting across four different representations.

Figure 1

Van Dyke and Craines's twelve directions of translation (1997, p. 616)



Van Dyke and Craine's (1997) suggestion of the directions of different shifts across representations also resonate with the prescriptions made in the South African National Curriculum Statement (DBE, 2011). It is probable that when learners make references across multiple representations, they acquire knowledge about the basic structure of the domain represented. Figure 1 above depicts the twelve possible translations that teachers and learners can make when working with functional problems, which help to communicate mathematically how quantities are related in a given function. Bossé et al. (2011, p. 2) defined translation as "a process in which constructs of one mathematical representation are mapped onto those of another, *for example*, the relation expressed in a table reinterpreted using algebraic symbols" (italics added). Each of the four representations describes the way the value of one variable is determined by the value of the other variable. That is, a function $y = x^2 + 3$ expressed as an equation should be matched to a graph, a table of values or words as depicted in Figure 1 above (Mpofu, 2016).

Further, the translation between different forms of representations "have become associated with a foundational understanding of the function concept" (Eisenberg & Dreyfus, 1991, p.

87). The learning implication is that for learners to be able to solve functional problems, it is fundamental that they internalise strong links between the different representations of the function concept. Teachers, therefore, need to refrain from making strong procedural emphasis during the teaching of functions, where the learners only think of the concept in terms of procedural techniques and mastery of symbolic manipulations, because procedural understanding does not enable deep conceptual understanding of function (Oehrtman et al., 2008). To clearly understand the relationship between the four representations, Friedlander and Tabach's (2001, p. 2) explanation is precise in providing an understanding of features of functions:

> Tabular representations help in the exploration of co-variation between variables; and the creation of graphs; algebraic representations are powerful in that they provide the generalisation of the patterned relationship between variables; graphical representation is effective in providing a clear picture of the function, enabling its features (like intercepts) to be 'seen'; and verbal representation is usually linked to problem-posing and is also needed in the final interpretation of results obtained in the solution. Verbalising a given situation involves an ability to use words to accurately describe a formula, a graph or a table.

Tabach's (2001) iteration demonstrates the need for teachers to help learners represent and analyse functional relationships using verbal rules, equations, table of values and graphs as well as the need to integrate various representations of functions. While it appears like teachers are expected to do everything, it is important to also mention that teachers should give learners the opportunity to participate in the teaching and learning process. To further illustrate the information, Van Dyke and Craine's (1997) diagram in Figure 1 further suggests the importance of ensuring that learners know and understand all the different representations, since each representation emphasises and suppresses some aspects of function (Ainsworth, 2006). Without overlooking the importance of learners learning for themselves, the role of a teacher is crucial to ensure that the order of translations is clearly presented during teaching in varied ways. This is to make sure that learners see, experience, and learn all twelve translations depicted by arrows between the four representations in Figure 1 above. For example, teachers should ensure that their learners learn both tables to graphs and graphs to tables translations to see what is signified by the modes of representation (see Bossé et al., 2011). The following section focuses on different approaches to teaching the function concept as advocated in previous research.

2.6. Different approaches to the function concept

The success of learners' learning and understanding of the function concept is largely dependent on the teaching approaches that are used during the teaching and learning processes (Panaoura et al., 2015). Some of the approaches that have been identified by authors for teaching the concept of function are the covariational approach, the pattern-oriented approach, the function machine approach, the word problem-based approach, the example and non-example approach, and the property-oriented approach (Tall & Baker, 1992; Monk & Nemirowsky, 1994; DeMarois et al., 2000; Pillay, 2006; Kwari, 2007; Ronda, 2009). Of importance to note about these approaches is that while they are prevalently reported to be used by teachers during teaching of the concept, they are equally reported to be the least understood by learners (Maharaj, 2010; Siyepu, 2013). As already mentioned in different sections, learners' misunderstanding is usually linked with the teacher's content knowledge and the teaching techniques, to use Shulman's (1987) concept, the pedagogical content knowledge. The approaches are important for this study because I used them to make sense of the teachers' choices during the algebraic function lessons.

2.5.1. The covariational approach

The reasoning about the relationship between two variables, also referred to as covariational reasoning, entails the ability to analyse, judge and interpret a relationship between two variables (Doorman et al., 2012; Bazzoni, 2015; Thompson & Carlson, 2017). The covariational approach to teaching the function concept can be defined as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Zeiffler & Garfield, 2009, p. 7). This means when teachers use this approach during the function lessons, they need to emphasise the idea that the dependent and independent covary. Various researchers have suggested that teachers should introduce learners to the concept of function by using the covariational approach to emphasise the coordination of changes in variables (Köklü, 2007; Zeytun et al., 2010). This approach to teaching the function concept highlights two important tenets of the functional relationship: (1) that a function entails a relationship between quantities, which can be represented by a set of ordered pairs whose coordinates denote values of the two quantities simultaneously. (2) A covariational approach to functions involves the idea that the values of the two quantities can vary (see Saldanha & Thompson, 1998).

Thompson and Carlson (2017) have suggested that the representations that can best facilitate learners' understanding of the notion of covariation are graphs and tables, since teachers and learners can scroll through or trace the nature of correspondence between variables. This means that covariational reasoning enables learners to view tables and graphs as collection of points and, in turn, envision the collection of such points as a result of observing the two quantities whose values covary concurrently. As such, in a teaching situation, teachers need to emphasise that when drawing a graph to represent a function, every point that is plotted on a Cartesian plane represents the values of two quantities simultaneously. Consequently, lack thereof could result in learners not developing the conceptual knowledge of how two variables are related. In addition to the above discussion, Carlson et al. (2002) proposed five levels of covariational reasoning and five mental actions that characterise these levels when teachers and learners are working with functional problems. The five levels of covariational reasoning are explained as:

> Coordinating the value of one variable with change in the other, coordinating the direction of change of one variable with changes in the other variable, coordinating the amount of change of one variable with changes in the other variable, coordinating the average rate-of-change of the function with uniform increments of change in the input variable, and coordinating instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function. (Carlson et al., 2002, p. 357).

The pedagogical implication for these levels is that teachers need to explicitly teach learners that the way two variables or quantities vary is different: the independent variable varies without any influence from the other variable, but the dependent variable is "constrained" by the change in the independent variable. That is, the dependent variable varies in a particular manner informed by the way the independent variable varies, and the lack of emphasis on this could result in learners overlooking how two variables vary together and focus on a set theoretical construct (Silverman, 2006). Many studies of learners' understanding of variables are vested in Freudenthal's (1982) concern that learners cannot develop in-depth understanding of variables without first developing the idea that variables vary simultaneously (Moore & Carlson, 2012; Chimhande, 2014; Larsen et al., 2017; Thompson & Carlson, 2017).

2.5.2. The pattern-oriented approach

In mathematics, learners' cognisance of patterns is integral to the development of functional knowledge and skills. According to Beatty and Bruce (2004) "patterning activities have long been recommended as a means of supporting learners in developing an understanding of the relationships among quantities that underlie mathematical functions" (p. 1). Similarly, Du Plessis (2017, p. 31) asserts that a pattern "addresses the underlying presence of some regularity that gives meaning to the relational aspects embedded in the object of learning". These iterations suggest that patterns can enable learners to observe changes in given relationships, and identify what changes and how such changes occur (Du Plessis, 2017).

The pattern-oriented approach offers teachers an opportunity to offer learners a visual representation of the function concept if appropriate presentation of tasks is used. Thus, teachers should use appropriate discourses while linking patterns and functions during teaching, to ensure that learners are aware of the links between the two and develop the skill to observe changes in given relationships using the pattern-oriented approach (Sfard, 2008; Adler & Ronda, 2015). Van de Walle (2004, p. 441) suggested that the pattern-oriented approach to teaching the function concept can help learners to: "Identify functional relationships between variables; obtain a rule or formula, algebraic expression or equation to describe the relationships and make predictions using a rule or formula". The pedagogical implications for using this approach to develop functional understanding is that learners are required to first understand the notion of a pattern and this is dependent on a teacher's horizon content knowledge.

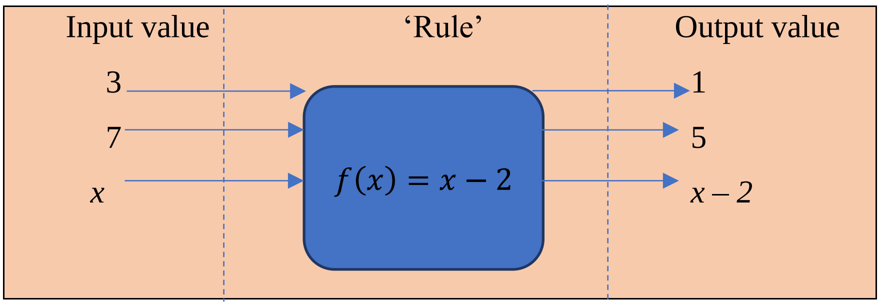
The South African CAPS mathematics curriculum (DBE, 2011) prescribes that patterns should be covered in term 1 in Grade 10 before functions are introduced, teachers should make explicit links between patterns and the function concept. This illustrates an integral part of patterns to functional understanding. Scott et al. (2011) explained that explicit link-making is important because understanding "an idea involves bringing into contact new and existing ideas" (p. 5). Accordingly, when teachers make explicit links between previously taught content and new content, they support learners "in constructing similar links for themselves" (Scott et al., 2011, p. 5). While this is the case, it is important to note that during teaching of functions, teachers must further consider that "most mathematical patterns generate numbers and learners might think that functions are sequences, yet sequences are only a special type of function" (Sierpinska, 1992, p. 218). Thus, for the pattern-approach to be effective in helping learners develop an understanding of functions, teachers are tasked with teaching learners the underlying functional relationships of a pattern during teaching.

2.5.3. The function machine approach

The third approach is defining a function as an input-output rule and, in turn, the graph and a table of the function as ordered pairs of input and output values. This approach is found in Tall et al.'s (2000, pp. 255-261) analogy of the "Function Machine" and in "Guess My Rule" games, whereby learners are expected to know how to manipulate one number to obtain another. This approach resonates with Dirichlet's conceptualisation of a function and places emphasis on both the correspondence and covariation principles of function, whereby a function is defined as a rule in which a number is related to another unique number. Kwari (2007, p. 31) postulated that, "Introducing functions using the function machine induces the idea of transforming (process) an input and returning a corresponding output". This means that if the function is applied to an input value, the result will be an output value. Figure 2 below depicts an example of a function machine:

Figure 2

An example of a function machine



In Figure 2, the number x that is used as an input value of a function represents the 'argument of a function', in the form of an 'if …, then …' statement. For example, to use the argument of 3, then $f(x) = 1$, and the argument of 7, then $f(x) = 5$. So, the output values depend upon the argument; subsequently it is also called the dependent variable and the argument is also referred to as the independent variable. In relation to the layers of development of the function concept discussed earlier in this chapter, Selden and Selden (1992) suggested that the function machine is useful in helping learners to understand the function as a process. However, it does not offer a complete notion of functions. One of the shortcomings of this approach is that it gives rise to a misconception that all functions are given by a formula, as argued by Sierpinska (1992), and thus there is a need to use a hybrid of approaches when introducing functions.

Notwithstanding the above discussion, during the teaching and learning of functions, the function machine forms a basis that enables learners to understand an intangible topic (Kwari, 2007; Bayens, 2016). In the Grade 10 South African curriculum, the task of the teacher is to help learners to know how to use this formatting technique and work towards using the function notation and the various representations, tabular, graphs, verbal without using the function machine. Tall et al. (2000) recommended the "use of function machine as a cognitive root to the development of the function concept" (p. 255). Earlier, Tall (1992, p. 497) defined a cognitive root as "an anchoring concept which the learner finds easy to comprehend yet forms a basis on which a theory may be built". This means the function machine has the potential to help a learner develop basic knowledge of functions at the beginning of the learning sequence, which can later be used tacitly as a more sophisticated understanding as the concept develops. Accordingly, within the South African mathematics school curriculum, this approach is predominately used from primary school levels, before learners reach Grade 10, to introduce them to the idea of linear relationships between variables. It therefore means that the function machine approach is encouraged in South Africa if it is used as early as primary school. It is therefore important to observe whether and how teachers use the function machine approach, if at all.

2.5.4. The word problem-based approach

Solving word problems is one of the prevalently reported main difficulties in school algebra all over the world (see for example Kwari, 2007; Jupri & Drijvers, 2016). The use of word problems to teach functions provides learners with an opportunity to use functions as models of real-life phenomena (Kwari, 2007). This requires learners to solve a word problem where they "must use text to identify missing information, construct a number sentence, and set up a calculation problem for finding the missing information" (Powell et al., 2009, p. 3). The successful meeting of the expectations relies on learners' and teachers' English fluency and the teachers' effective teaching, for learners to use text to identify missing information from word problems. Similarly, Powell et al. (2009) and Verschaffel et al. (2000, p. ix) have defined word problems as "[v]erbal descriptions of problem situations wherein one or more questions are raised, the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement". This suggests that solving word problems entails teachers' and learners' abilities to read, interpret and transform the given word problem into a symbolic form to search for manipulative or computational strategies to solve the problem (Moussa-Inaty et al., 2020). The translation from words in the context they are given to symbolic form requires knowledge of mathematical language, knowledge of the language text, and mathematical content knowledge. It thus means the approach needs a teacher that is able to communicate and reason through both spoken and written mathematical language. This discussion resonates with suggestions for the development of nexus of reading and mathematics in both reading and mathematics education research (NCTM, 2000; Slavin & Lake, 2008; Rupley et al., 2011).

A model for using reading to improve learners' mathematics learning was coined by Powell et al. (2009) and comprises seven tenets: vocabulary development, language usage, application of reading for meaning, reflection, interference, didactics in multiple meanings, and using problems closely linked to students' real-world encounters. This model suggests that when learners are presented with word problems in mathematics, the reading of the problem necessitates that they understand the meaning of the words used in a problem within the context of such word problem. It is important to seriously consider this point in the rural South African context because for teachers and learners, the school is sometimes the sole source of exposure to English, which is the language of teaching and learning (Robertson & Graven, 2020). It therefore becomes important to explore and understand how teachers communicate what is inherent in the word problems during teaching and learning in the classroom.

The task for teachers is to ensure that as learners learn the vocabulary associated with the function concept in mathematics, they also learn the meaning of new words that do not exist in their oral vocabulary and understand the alternative meanings of the words from those they are familiar with. According to various research studies on mathematical language, learners often experience difficulty with understanding word problems, especially when the language of teaching is not their first language and they are not proficient in it (Kwari, 2007, Setati et al., 2009; Sepeng, 2013; Csíkos & Szitányi, 2020). To overcome the language barrier, explicit teaching of the difference in the colloquial and formal mathematical meaning of the same word must be included. Another way is to limit the number of new words introduced in the lesson, encourage learners' use of new learned words mathematically within real-life events, and teach learners how to translate and use multiple representations (Moussa-Inaty et al., 2020).

It is worth mentioning that mathematics is taught in English in South African secondary schools, which is a second language to the large population of learners, including in Acornhoek schools. This therefore suggests a possible barrier for effective use of words mathematically within real-life experiences, since learners' everyday experiences are often expressed in their first languages which have to be translated to English at school (Phakeng & Moschkovich, 2013; Barwell et al., 2016). Sepeng (2014) conducted a study in South African urban townships with learners from multi-cultural backgrounds, exploring learners' tendencies when solving real-world problems in mathematics. The findings demonstrated that learners draw mainly from their cultural knowledge to justify their solutions to problems.

Similarly, Pimm (1991) argued that learners bring into the classroom informal mathematics language and tend to use this language to communicate their mathematics ideas and concepts. This can constrain learners' understanding of functions if teachers do not explicitly teach learners how to use words from their everyday experiences[19] mathematically (Aineamani, 2009). Thus, teachers can give learners examples that help them learn how to change the verbal form of the problems into symbolic forms, and allow learners to complete word problem functional tasks to assess their understanding or lack thereof. Krebs (2005, p. 409) opined that, "sometimes much is learned by and about students from an incomplete or incorrect solution", which means that learners' solutions provide evidence for successful learning or areas that require consolidation. This presents the opportunity for the teacher to reteach for learners' understanding. For learners to develop a good grasp of functional relationships with any other mathematical concepts, they need to learn the key properties underpinning such concepts.

2.5.5. The property-oriented approach

Another approach through which teachers can introduce learners to the function concept is referring to the properties of the concept (Nemirowsky & Rubin, 1992). A teacher can teach the linear function, which has a slope-intercept formula $\mathrm{f(x)} = \mathrm{mx} + \mathrm{c}$ by focusing on some of the defining attributes of linear functions: the y-intercept (c) and the slope (m). According to Birgin (2012, p. 142), "The slope of linear functions represents the rate of change in one variable that results from a change in the other variable". A teacher can use the slope property to teach learners how to determine if a function is constant, increasing or decreasing. The quadratic function, whose general formula is $f(x) = ax^2 + bx + c$ has the following properties: the turning point, also referred to as the vertex $(\frac{-b}{2a}; f(\frac{-b}{2a})$ and axis of symmetry at line $x = \frac{-b}{2a}$. If the value of a for a parabola is greater than zero $a > 0$, the turning point of a parabola is at minimum, the parabola opens up and the turning point is at maximum (opens down) if the value of a is less than zero $\mathrm{a} < 0$. When the above discussed properties are taught explicitly and efficiently, it can enable learners to know how to classify functions and determine how variables in given relationships are related. Yerushalmy (2000) and Chimhande (2014) have advocated a property-oriented approach to teaching and learning functions,

[19] Sepeng (2014) suggested that teachers should incorporate out-of-school real-world knowledge during formal classroom mathematics teaching in order to help learners develop mathematical reasoning skills as well as use of common sense when solving mathematical problems.

suggesting that this approach enables learners to develop an understanding of the different features of the concept and their interrelations.

Considering that the teachers in this study are Grade 10 teachers, it is important that teachers take time to explicitly teach learners about the properties underpinning each type of function, to maximise learners' development of deeper conceptual knowledge of the function concept which they will use as they continue with the topic in higher grades (Gcasamba, 2014; Moeti, 2015; Mudaly & Mpofu, 2019). To reinforce learners' understanding of the different properties for different functions, teachers are tasked with providing examples and non-examples for the learners to gain an understanding of the procedures and conceptual knowledge (Chimhande, 2014) associated with specific properties as well as how the properties are interrelated (Skemp, 1976; Kwari, 2007).

2.5.6. The example and non-example approach

The use of examples is an essential part of mathematics teaching as they are considered strong pedagogical tools for mathematics teaching (Watson & Mason, 2002; Alkan et al., 2017; Adler, 2017) because they enable learners to learn when and how to carry certain actions in their mathematisation as well as why such actions should be taken. Gökbulut and Ubuz (2013) viewed examples as effective explanations that are used to depict the general principles of concepts, depending on the skills the teacher has to achieve the effectiveness. The notion of exemplification in the teaching and learning of the function concept plays a fundamental role, as Alkan et al. (2017, p. 368) explained:

> While solving the problems about functions, students start to think with the examples in their minds rather than the definition of function and thus they may be unsuccessful … students don't learn the definition of function but they encode the first examples which match their frame of minds from the examples, representations, algebraic rules used to explain functions.

This statement demonstrates that while working with functions without an operative definition, learners respond to functional problems by resonating with the examples and explanations which teachers used during teaching. While examples play a crucial role in enabling learners' understanding of the concept, teachers need to prioritise explicit and effective explanations of the concepts, then use examples to illustrate the meanings of the concept. If there is no existing resonance between the presented problems and the previously covered examples, learners can experience confusion due to a lack of mental match. Conversely, if there is resonance, learners experience the sensation and respond to the mathematical problems positively (Tall & Bakar, 1992).

In the current study, this suggests that the type of examples teachers use to teach the concept of function play a significant role in enabling and/or constraining learners' learning of the algebraic functions. Tall and Bakar (1992, p. 13) argued that examples "lead to mental prototypes which sometimes give erroneous impressions of the general idea of a function by conflicting with the formal definition". Thus, it is important that teachers are always conscious when and how they use examples, because they reinforce meanings of concepts by making it relevant for learners to see and use in real-life situations. When teachers exemplify the function concept and/or rules associated with the concept, learners see its generality and relate to exactly what the example epitomizes (Mason & Pimm, 1984; Rissland, 1991). The teaching consequence of this, according to Mason and Pimm (1984), is that teachers should use 'generic examples' during the teaching of functions, examples that attempt to desist from emphasising the specifics of the example itself, and rather represent the general case for the concept. In relation to the above discussion, Tall and Bakar (1992, p. 13) further postulated that

> the learner cannot construct the abstract concept of function in the absence of examples of the function concept in action. Accordingly, they cannot study examples of the function concept in action without developing prototype examples having built-in limitations that do not apply to the abstract concept.

This statement suggests that examples help learners to develop the meaning of function and how it is used in different ways and contexts. During the teaching of functions, one example cannot express all the key features and meanings of the concept. Accordingly, various authors differentiated and classified examples in terms of their intended usage in different stages of a lesson, as shown in Table 5 (Polya, 1973; Michener, 1978; Bills et al., 2006).

Table 5

The classification of example types

Example type	Description	Authors
Leading example	These are the examples that are used to express the key features of the concept.	Polya (1973)
Suggestive example	The examples teachers use to elicit learners' understanding of the qualities of the concept and concurrently showing the boundaries of the concept more clearly.	Polya (1973)
Counter example	Examples used to demonstrate to the learners that a claim is false.	Polya (1973), Michener (1978), Bills et al. (2006)
Introduction example	Examples used to support the basic definitions of the concept, which help learners to create a simple perception about the concept.	Michener (1978)
Reference example	Standard examples that are frequently mentioned in development of a concept or during various stages of teaching.	Michener (1978)
Model example	Examples that are used to summarise the general case of the concept.	Michener (1978)
Generic example	Examples that show broadly the overall situation of the concept.	Bills et al. (2006)
Non-example	Examples used to highlight the features which are not part of the concept, describe the limits and conditions of rules and theorems.	Bills et al. (2006)

Table 5 above shows the different types of examples through which teachers can help learners to construct knowledge of the concepts and provide learners with a criterion by which they can recognise features of functions and be able to solve the functional tasks appropriately (Pillay, 2006). In addition to the above-mentioned examples, this study also examined whether teachers used other examples that have not been mentioned by the authors, considering the different under-researched context in South Africa. Further, while both Polya (1973) and Michener (1978) presented examples that belong to a concept, Bill et al. (2006) introduced the example types which belong to a specific concept and those which do not belong (non-examples). Alkan et al. (2017) stated that if learners "see the examples that don't belong to the concept beside the one *that* belong to the concept, they can understand better the qualities that define the concept and differentiate the taught concept from the other concept" (p. 370, italics added). It therefore means that teachers can use both examples as a way of enhancing learners' understanding, by showing them differences between aspects that belong to the concept and those that do not. Example usage is inextricably part of teaching, and in this study it further plays a significant role to bring the features of the function concept to the fore for the learners.

This study views teachers' discourses and approaches as intertwined entities of teachers' classroom practices while teaching mathematics.

2.7. Chapter Summary

This chapter engaged literature that was related to the concept of algebraic functions in CAPS. A discussion about the use of multiple representations as pedagogical tools for the concept of function was also provided, placing emphasis on the need for teachers to teach learners explicitly how to translate between representations. In addition, a discussion on the literature associated with the learning of the function concept was presented to identify the prominent difficulties and the reasons learners encounter when learning the concept. The use of multiple representations has been suggested as effective in helping learners to develop conceptual understanding of the concept. The literature relating to the approaches to teaching functions was also provided. There is a general agreement among researchers that the teaching of functions is very complex, and teachers are thus required to employ various approaches to teach the concept.

Chapter 3

Teaching functions through a commognitive lens: Theoretical and conceptual foundations for the study

"*By mathematical knowledge for teaching, we mean the mathematical knowledge needed to carry out the work of teaching mathematics. Important to note here is that our definition begins with teaching, not teachers. It is concerned with the tasks involved in teaching and the mathematical demands of these tasks*" (Ball et al., 2008, p. 395).

3.1. Introduction

The theoretical framework is one of the main parts of the research because it explains the meaning, nature, and challenges of a phenomenon, so that we may use that knowledge and understanding to act in more informed and effective ways (Adom et al., 2018). It further helps to address the questions of 'how' and 'why' by articulating the theoretical assumptions of a research. This study also answered how teachers use mathematics discourses during the lessons on algebraic function, how the topic of algebraic function is taught and why it is taught in particular ways. Various authors define a theoretical framework as the 'blueprint' or guide for an inquiry (Sinclair, 2007; Ravitch & Riggan, 2012; Grant & Osanloo, 2014), and is linked to the research problem, research design and data analysis plan. To understand teachers' discourses and the pedagogical choices and decisions during algebraic function lessons, I used Sfard's (2008) commognitive theory and Scott et al.'s (2011) Pedagogical Link-Making (PLM) frameworks. The former is the study of communication and participation in discourse(s) and the latter is used to interrogate and understand the teachers' pedagogical conceptual links they make during the algebraic function lessons.

3.2. The commognitive theory and its underpinnings

Sfard's (2008) commognitive theoretical framework is a lens to analyse and interpret teachers' communication during algebraic functions lessons, to understand the intricacies and elements of the discourses from what is or is not endorsable by the mathematics discourse community. The commognitive theoretical framework is influenced by Ludwig Wittgenstein and Lev Vygotsky who emphasised the "inseparability of thought and its expression, either verbal or not" (Sfard, 2015, p. 132). This means thinking in mathematics is a well-defined form of communication, and mathematics teaching is participating in a discourse (Roberts, 2016). The effectiveness of teachers' communication of mathematical contents during teaching depends on their content knowledge. Participants of the mathematical discourse show their internal communication through what they say, write, draw or sketch, therefore communication is seen in both talk and action (Mudaly & Mpofu, 2019). The teaching of algebraic function requires teachers to communicate different concepts, processes and rules explicitly and effectively, which is an expression of mathematics at an intrapersonal (cognition) and interpersonal (communication) level (Vygotsky, 1987). The framework helped with avoiding oversimplified views of teaching and allowed for rich descriptions and discussion of teachers' ways of teaching functions, through its focus on the contextual, cultural, dialogical, and dynamic nature of participants' discourses in mathematics. By doing so, it accounted for the differences in individual teachers' thinking and teaching methods during the lessons on algebraic functions.

According to Sfard (2008), mathematics is 'autopoietic' because it is "a system that contains the objects of talk along with the talk itself", a feature that makes school mathematics difficult to teach and learn (p. 129). Thus, familiarity with "what the discourse is all about" (Sfard, 2008, p. 130) is needed for participation in the discourse, but paradoxically this familiarity only comes through participation for mathematics teachers and learners. Even though this study focused on teachers, the nature of classroom teaching involves learners that should actively partake in the lesson for familiarity with the mathematical object and develop mathematical discourse. The word 'discourse' implies the use of words and symbols in a way that is generally endorsed by members of a community (Sfard, 2015, p. 45). Accordingly, mathematical discourse communicates mathematical ideas that are ratified by the body of theorems, proofs and laws that govern mathematics (Sfard, 2012). The unit of analysis for Sfard (2008) is discourse, which is considered a special type of communication, made distinct by its repertoire of admissible actions and the way these actions are paired with reactions.

3.2.1. Words and their uses

Word use refers to the use of mathematical vocabulary in participants' discourses, including the use of mathematical terminology such as topology and shapes. It is a critical factor in mathematical communication because word use unlocks the potential of learners and teachers to communicate effectively during teaching and learning (Sfard, 2008; Nardi et al., 2014). It further refers to ordinary everyday communication words but with unique and specific meanings in mathematics, such as differentiation, limit, open, continuous, and group (Nardi et al, 2014). It is about whether teachers are aware when they use such words during mathematics lessons and how they use and explain them to the learners during the lesson that is important for this study. The teacher's consciousness when using everyday words in mathematics is fundamental, so that learners understand the different use and meanings as compared to the everyday conception. Berger (2013) further demonstrated that certain words such as ‘equal’, ‘function’ and ‘vertical asymptote’ indicate the existence of mathematical discourse, although these words also appear in everyday discourse. In the context of this study, words such as ‘variables’, ‘graph’, ‘intercepts’, ‘coordinate’, and ‘turning point’ signify word use, whether spoken, written or in pictured form, and communicate teachers’ meanings during algebraic function lessons. Teachers’ discourses are characterised by their carefully chosen word use, because they could constrain and/or enable learners’ epistemological access to the knowledge of algebraic function. Although outdated, Wittgenstein (1953, as cited in Sfard, 2007, p. 571) posited that awareness of word use is very important, because the usage of the word constitutes its meaning that learners will repeat as their story of mathematics. While this is important, it does not overlook that sometimes teachers might forget to explain the meaning of words because of possible immersing in the teaching moment.

In this study, teachers were tasked to communicate words such as ‘function’, ‘equality’, ‘slope’, and ‘intercepts’, and to ensure that they were used uniquely within mathematics object-specific discourse that differed from how they were used in everyday discourse (Mosvold, 2016). To explain such abstract concepts, the teacher opts to use certain visual mediators to bring the underpinning mathematical meanings to the fore. Notwithstanding the importance of word use, it is also important to note that the intention of speakers is never neutral (Stubbs, 1980) but is influenced by various factors, whether consciously or unconsciously. Thus, it could happen that teachers’ communication during the lessons on algebraic function is influenced by their understanding of the curriculum standards or lack thereof, teaching for compliance and examination and lack of lesson planning. It is because of such aspects that gaining insight into teachers’ word use during algebraic function lessons in rural classrooms is important, because of the lack of research that explored classroom teaching and word use during algebraic functions lessons. The word use during teaching cannot be understood or used independently, visual mediators act to facilitate the visualisation of the mathematical meanings embedded in mathematical situations (Sfard, 2008).

3.2.2. Visual Mediators

Mathematical discourses also include certain visual mediators, which entail “visible objects such as symbols, graphs and diagrams that participants in a mathematical discourse use to identify the objects[20] of their thinking or communication and bring these objects into focus" (Berger, 2013, p. 3). The symbolic artefacts are created to communicate relationships and operations with mathematical objects. Sfard (2008) argued that visual mediators play a role of depicting the key features associated with particular mathematical objects, and help learners to develop relational connections and understanding of mathematical entities. The key features of functions in this study were the effect of parameters, domain, range, and intercepts, and teachers should use different visualisers to bring the mathematical meanings embedded in various mathematical conventions to the fore. Jones (2000, p. 1) stated that, "the intention of using such artefacts is to enable learning to take place through encounters with embodiments and representations of abstract mathematical objects". It was thus important that when I observed mathematical discourse during the algebraic function lessons, I ensured that teachers' word use was examined in relation to the carefully chosen visual mediators. This addresses the importance of lesson planning, because that is when the teacher looks at a group of learners and considers what their materials and procedures are going to be during teaching. This means that teachers need to make sense of why and how they want to use particular visual mediators as a way of determining their effectiveness in enabling learners' understanding of the subject matter contents, in this study, algebraic functions.

Sfard's visual mediators link with the notion of multiple representations (see Chapter 2) as they both focus on the modalities that are used as tools with which to think and communicate mathematical ideas in the classroom. The visual mediators are essential for teachers to use while teaching algebraic functions concepts because they encourage learners to think about the meanings of the mediators. A mathematical representation on the board plays a major role in learner understanding of the concepts, because if teachers misrepresent mathematics content visually it may result in disconnections in learner understanding. The visual mediators, according to Sfard (2008), are signifiers that mediate meaning between one entity and another for the teacher. It is important for teachers to be aware of the different visual mediators they use and interpret while explaining abstract concepts like intercepts, turning point, domain and range during the lessons, to ensure learners' understanding as they build their knowledge for higher grades.

[20] I use the word objects to mean the specific object (function) and specific (discursive practices specifically related to the notion of functions). This is purposely done, because, for example, the use of the negative sign in algebra indicates the opposite of being positive, whilst in trigonometry and vector algebra the negative indicates direction.

While Sfard (2008) considered visual mediators as the main form of medium that supports mathematical discourse, Güçler et al. (2015) suggested the acknowledgement of other modes of thinking (e.g., kinesthetics and auditory) that can play roles in mathematical learning. It may therefore be important to extend the notion of visual mediators to explore body language, gestures, and motor actions as part of mathematical communication during the lessons, because they represent thinking and action. The authors' suggestion was considered in this study because a teacher's body language or gestures are used with key words and visual mediators in the lessons, and could indicate the nature of content knowledge and confidence, or lack thereof, about the topic. This is important in particular when Sfard (2008) stated that word use and visual mediators are used to produce narratives. In this study I observed whether and how teachers used key words and visual mediators to construct endorsable narratives pertaining to algebraic function during the lessons.

3.2.3. Endorsed narratives

Narratives are defined as "a sequence of utterances, spoken or written, framed as a description of objects, of relations between objects, or of activities with or by objects" (Sfard, 2008, p. 300). Thus, narratives describe what is done with mathematical objects and activities during teaching and learning, and in turn could enable learners' conceptual understanding if they are used effectively. To indicate whether a narrative is true or false in mathematics, Sfard introduces the term "endorsed narrative" (2008, p. 134). Endorsed narratives are words used about mathematical objects that the mathematical community, including teachers as experts, upholds or agrees on, to be verifiable as truths (Sfard, 2007; 2008). While narratives are constructed by teachers using words and visual mediators, Roberts (2016) stated that researchers can also construct narratives as interpretations of what they see in the teachers' discourse. The latter pertains to how researchers interpret teachers' descriptions of, and justification for their procedures, based on their use of words and visual mediators during the lessons (Roberts, 2016). The characteristic of narratives is one of the features that I observed during the lessons to categorise teachers' discourses, as I examined whether teachers used keywords and visual mediators to construct endorsable narratives pertaining to algebraic functions.

Sfard classified endorsed narratives into two categories: object-level and meta-level narratives (Sfard, 2008). Object-level narratives result from the "growth in the number and complexity of endorsed narratives and routines" (Nardi et al., 2014, p. 300), and refer to the utterances that are made about the nature of mathematical objects. In this study the 'm' and the 'c' on the equation $y = mx + c$ represent the gradient and the *y*-intercept respectively (see Sfard, 2008). From this example, the object-level narratives focus on *what* specific features of mathematical objects signify and help in identifying such features of mathematical entities teachers can focus their explanatory talk on during teaching. The meta-level narratives, "namely express[ing] itself in the change in the metarules of the discourse" (Nardi et al., 2014, p. 300), are about *how* mathematics is done, placing focus on how specific procedures associated with a particular concept should be carried out to solve mathematical problems (Sfard, 2007; 2008). The different narratives are used during the teaching and learning of algebraic functions to introduce learners to various concepts, rules and procedures associated with the topic. Mudaly and Mpofu (2019) stated that the teaching and learning process should be more about learners exploring mathematical objects and discovering endorsed narratives by themselves, rather than a learning situation where rules and procedures of doing mathematics are often given to the learner. The learners' exploration process also relies on the communicative approach(es) the teacher used during the lesson, which was important in this study as I carefully observed and analysed teachers' practices in the videos.

Thus, making sense of formulae, use of multiple representations, various notations and rules related to the function concept is considered a major part of the current study as the components of the concept that CAPS prescribes at Grade 10 level (Sfard, 2008; DBE, 2011; Sfard, 2017). Sfard (2008) further stated that the sequence of utterances teachers make during teaching are "subject to endorsement or rejection with the help of discourse-specific substantiation procedures" (p. 134). This means that what teachers say and/or do during mathematics teaching can either be correct or incorrect, and for the latter it is important to understand the reasons and possibly identify how often they make them. Thus, teachers should make learners aware of the things they should not do while working with specific problems, since they are 'subject to rejection' within the mathematics community and labelled narrative errors. Although Sfard did not use 'narrative errors' in her writings, Brodie and Berger (2010) talked of errors as narratives that learners endorse but are not approved by experienced participants such as teachers and mathematicians. The explicit teaching helps learners know the correct[21] object-level and meta-level narratives associated with specific features of the function concept. In this study, I

[21] Endorsed narratives are also helpful in identifying erroneously endorsed narratives by learners and in turn configure corrective measures to assist learners to understand the formally endorsed narratives within the mathematics community. In this study, I paid attention to the mathematical correctness of the teachers' own narratives and whether the narratives they used during teaching were endorsed or not.

constructed both object-level and meta-level narratives based on my interpretation of what the teachers said about functional objects, as well as about classroom actions by and with objects. Further details are provided in Chapters 5 to 9 where I present and analyse teachers' teaching practices.

3.2.4. Routines

Routines are central to the discourse as they are repetitive and patterned, and Sfard (2008, p. 220) described them as "the anatomy of mathematizing". Mathematical routines are governed by the specialised mathematical words, visual mediators and endorsed narratives. This means that a routine may be a procedure that teachers follow to communicate mathematical ideas to the learners, which involves practices such as generalising and justifying mathematical narratives, which in turn makes them rule-laden (Sfard, 2008). Thus, routines are the "set of metarules that describe a repetitive discursive action" (Sfard, 2008, p. 208), where metarules "define the patterns in the activity of discursants" (Sfard, 2008, p. 201) and "guide the general course of communicational activities" (Sfard, 2008, p. 202). Routines are often tacit, and they regulate when participants perform a particular activity and how they perform it (Sfard, 2008). The term routine is broad since it can refer to many different metarules defining the discourse-specific patterns in participants' actions. For example, in my study teachers brought different narratives about the different families of functions to the fore in order for learners to learn and own them. Thus, careful analysis of these patterns was important in relation to the word use and visual mediators, to understand ways in which teachers teach learners about the appropriate conditions for taking particular actions when solving algebraic functions problems.

Similar to narratives, there are two categories of rules that govern routines within mathematics classrooms, namely the "object-level rules" and "meta-level rules" (metarules) (Sfard, 2008, p. 201). The object-level rules represent "narratives about the regularities in behaviour of the objects of the discourse" (Sfard, 2008, p. 201). These are the rules that are directly linked to the definition of the various mathematical objects such as the (arithmetic) distributive rule which states that: "$a(b + c) = ab + ac$" (Sfard, 2008, p. 202). On the other hand, metarules depict the "patterns in the activity of the discussant trying to produce and substantiate object-level narratives" (Sfard, 2008, p. 201). For example, one of the common routines in the teaching of functions is the additive inverse rule, often referred to as the "change sides, change sign" metarule (Hall, 2002, p. 12), which informs the routine in which teachers solve equations when calculating by reorganising the position of variables. Thus, routine categories enable the understanding of teachers' regularities in behaviour when teaching algebraic function, and what teachers think when solving problems on functions.

Furthermore, Sfard (2008) distinguished between three types of routines in mathematical discourse, namely: explorations, rituals and deeds, and rituals and deeds are predecessors of explorations. Ritualised routines in mathematics discourse refer to the rote enactment of memorised routines while engaging with mathematical objects (Sfard, 2008). Berger (2013, p. 3) stated that "A *ritual* is a routine whose goal is social approval that create and sustain [sic] a bond with other people", which necessitates following the same procedures as others. Rituals involve imitating routines whereby a participant in mathematics discourse aligns their mathematical activity with other people's routines. In a classroom situation, this means a teacher who follows mathematics rituals "does not act as a problem solver and has no reason of her own to engage in this kind of talk" (Sfard, 2016, p. 6).

While ritual routines help orientate learners to procedures that the interlocutor engages in, they are limited to justifying how such mathematics procedures are carried out, but not when to carry such procedures or why they work and can be argued to limit learners' conceptual understanding (Mpofu, 2018). Since learners are introduced to the algebraic function topic for the first time in Grade 10, teachers might find it necessary for learners to imitate them so they can learn how to solve problems on algebraic functions. It is important though to move beyond this level of mathematisation, especially if the goal of mathematics teaching is to enable conceptual development (Sfard, 2008). Ritualisation was the prominent routine for participants' teaching, as they introduced the functions in symbolic form by showing learners how to engage in algebraic calculations to determine the values of the independent variable, represent the values on the table of values and subsequently draw the graphs.

The deed routines entail "a set of rules for a sequence of action resulting in change objects, either primary or discursive" (Sfard, 2008, p. 241). This means that a deed is characterised by the teachers' and/or learners' engagement in practical mathematical actions, but unable to perform the same action in abstract form (Sfard, 2007; Berger, 2013). Sfard (2008) contended that to promote participationist discourse as opposed to acquisitionist, teachers should move to explorative discourse to enable learners to justify their workings and apply the taught procedures with unfamiliar mathematical objects. Sfard (2015) has defined explorations as those routines whose "goal is to produce new narratives" (p. 131) and has focused on the analysis of routines that underlie participatory regularities of discourse more than the routines that underlie the object-level narratives of mathematics (e.g., use of metaphors as a routine to talk about mathematical objects). I have acknowledged the importance of the broad description of routines in the framework, which does not prohibit but supports the analysis of metaphors (Güçler, 2013) that teachers might use to clarify concepts of algebraic function. Explorations are the most sophisticated form of routine, resulting in the production of narratives about mathematical objects that are endorsable in terms of mathematical axioms, definitions and theorems (Roberts & le Roux, 2019). The exploration routines involve the use of multiple, but corresponding means to solve mathematical problems and authenticate the routines into solving such problems (Sfard, 2008). This type of routine does not rely exclusively on situational evidences, but for what teachers together with their learners can make possible with the mathematical object, and from it.

The commognitive theory helped to highlight the significance of the underlying relationship among discursive elements, and PLM assisted with analysing the nuances of the teaching practices in relation to the discourses and approaches that were discussed in Chapter 2. The following section discusses in detail the conceptualisation and operationalisation of the PLM framework in the current study. The use of words associated with functions, interpretations of the visual mediators, the narratives teachers used and the various links between different kinds of knowledge or lack thereof determined the routines that teachers operated on in the current study.

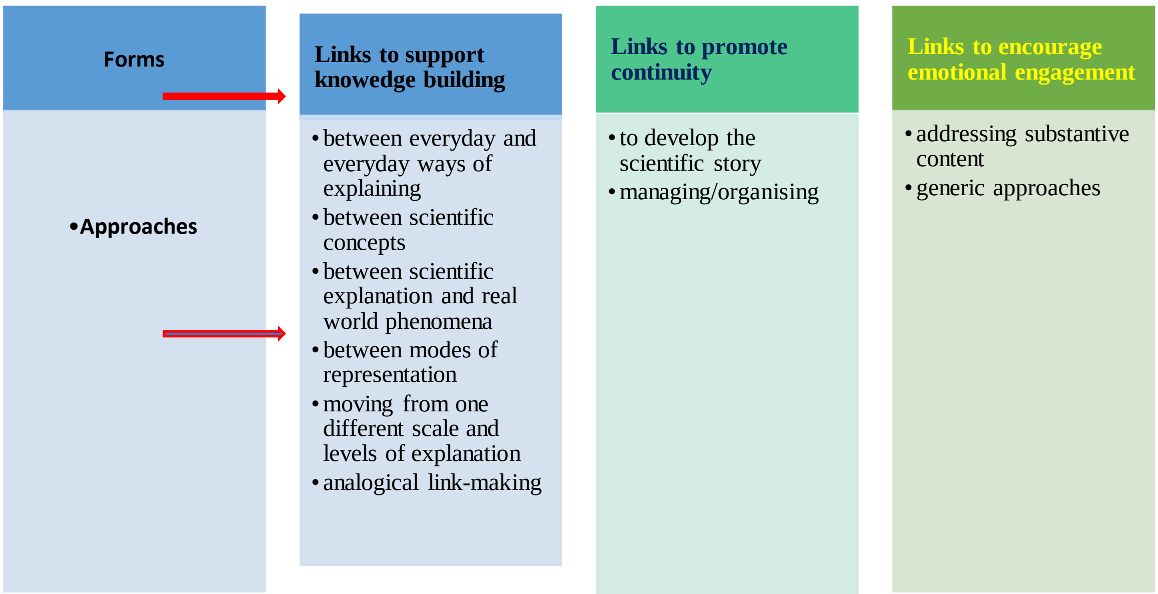
3.3. Pedagogical Link-Making in teaching algebraic functions

Teachers need to make connections between the different components of mathematical discourse to help learners move from the discourse of the interlocutor to the discourse of one-self (see Ben-Zvi & Sfard, 2007). Scott et al. (2011) developed a framework that is central to the teaching and learning processes in the classroom, which they termed pedagogical link-making (PLM). This framework was developed based on observation of teaching and learning in science classrooms but can be used to examine teachers' pedagogies in any subject-specific classroom. PLM has its origins in social constructivism and focuses on how teachers and learners make connections between different ideas in the on-going dynamic meaning-making interactions during teaching and learning (Scott et al., 2011). The PLM framework concedes that learners are not empty slates during the teaching and learning process, they bring naïve concepts into the classroom. This means that during mathematics learning, learners develop experiences based on preconceptions about the world and how the world works before they are introduced to new mathematics knowledge. Thus, the role of the teacher is to help learners link naïve concepts to new knowledge for comprehension of 'scientific concepts'. Scientific concepts in this study refer to mathematical concepts, particularly those related to the concept of algebraic functions. Scott et al. (2011, p. 6) iterated that "the term 'scientific' as used by Vygotsky is not restricted to the natural sciences, but covers all comparable communities" which includes mathematical contents.

In the context of the current study, the foregoing discussion suggests that the understanding of algebraic functions depends on the explicit links teachers make during the explanations of different concepts (Mortimer & Scott, 2003; Scott et al., 2011). There are three forms of PLM: links to support knowledge building; links to promote continuity; and links to encourage emotional engagement as depicted in Figure 3. It is important to note that when teachers are making pedagogical links between different forms of knowledge during the lessons on algebraic functions, they use approaches of teaching the topic discussed in Chapter 2.

Figure 3

Forms and approaches of pedagogical link-making



The word *approaches* in this chapter is used differently from the operationalisation of the word in Chapter 2, as for Scott et al. (2011) the word refers to the various ways in which teachers and learners make pedagogical links between different forms of knowledge. On the other hand, the approaches discussed in Chapter 2 focused on the specific commonly used strategies for teaching algebraic functions. The current study only used link-making to support knowledge building and links to promote continuity as conceptual framework, to analyse how teachers support learners' knowledge building to promote concepts continuity and understanding of the relationship between different ideas within algebraic functions. I analysed the scientific stories that teachers narrated, whether endorsed or not endorsed, the visual mediators and the routines during the lessons. The following sections provide an elaboration of each of these forms and their accompanying approaches. I highlight the operationalisation of each of the two forms of link-making in demystifying teachers' approaches during the lessons. Within the two forms of pedagogical link-making espoused for the study, some of the approaches were not dominant during the lessons.

3.3.1. *Pedagogical link-making to support knowledge building*

This form of link-making focuses on how teachers and learners make links between various kinds of knowledge to support learners' in-depth comprehension of the content subject matter (Scott et al., 2011). The diverse kinds of knowledge which teachers need to link during teaching and learning involves six approaches, as depicted in Figure 3 above, "each of which addresses making links between different kinds of knowledge" (Scott et al., 2011, p. 6). The link-making approach to support knowledge building includes: making links between everyday and scientific ways of explaining; making links between scientific concepts; making links between scientific explanations and real-world phenomena; making links between modes of representations and analogical link-making.

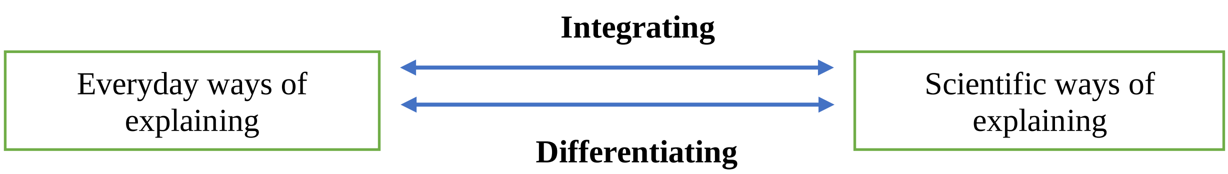
3.3.1.1. *Making links between everyday and scientific ways of explaining*

This approach focuses on making links between everyday and scientific ways of explaining, and mathematics learning inextricably includes everyday ways of thinking and talking about phenomena. According to Scott et al. (2011), this approach to link-making focuses on the links that teachers and learners make between formalised scientific views with everyday ways of explaining the phenomena. This could be linked with Sfard's word use, as the effectiveness of teachers' explanations depend on the effectiveness of their word use. The scientific way of explaining phenomena addresses the process through which learners come to "understand and to be able to use the social language of science" (Scott et al., 2011, p. 6). In algebraic function, this means the understanding and ability to use both functional skills and knowledge as well as everyday ways of explaining relationships between quantities to understand a particular situation. On the other hand, everyday ways of explaining focus on the understandings of meanings of the phenomena from everyday ways of thinking and talking about it, in turn help learners develop deeper conceptual understanding (De Jong & Luneta, 2010). For example, from an everyday way of explaining, a monthly salary is a function of the daily rate and the number of days worked. The nature of explanation during teaching is generally important and making links between scientific ways of explaining and everyday ways of explaining requires teachers to be knowledgeable about the subject matter content and its relationship with the real world of the learners (Scott et al., 2011).

According to Myhill and Brackley (2004, p. 271) learners' "prior knowledge is not simply about children's prior knowledge or facts, or children's prior social and cultural experiences, but also about cognitive connections". This suggests that teachers need to pay attention to "how one learning experience or conceptual understanding can support the development of subsequent learning" (p. 272) and allow learners to make links between scientific and everyday ways of explaining phenomena. For Scott et al. (2011), there are two ways in which teachers can make the links: integrating and differentiating, as depicted in Figure 4 below.

Figure 4

Integrating and differentiating everyday and scientific views



Scott et al. (2011) stated that where the two ways of explaining overlap, teachers should make links to integrate the two ways of explaining during teaching. For example, when teaching the notion of coordinate, a teacher can refer to coordinate systems such as seating arrangements in the classroom, rooms in a building, the positions of their homes, to sufficiently saturate the concept with the concrete (Scott et al., 2011, p. 9). Conversely, in cases where there is a difference in the ways of explaining, teachers should help learners differentiate the everyday and scientific ways of explaining phenomena. It is therefore important that teachers are aware of the different ways of explaining, as mentioned in commognitive theory, to ensure that mathematics communication during teaching and learning is effective in enabling learners' understanding of the concepts (Sfard, 2012).

3.3.1.2. Making links between scientific concepts

The second approach to support knowledge building entails making links between individual scientific concepts, recognising how concepts in the topic and across topics "fit together in an interlinking system" and are "applied in this connected form" (Scott et al., 2011, p. 8). In this approach, Scott et al. (2011) described how teachers and learners do not only have to recognise the differences and similarities between scientific and everyday ways of explaining, but also between scientific concepts that are related to each other within a certain context. Making links between scientific concepts emphasises that there is no concept that is a stand-alone or that can be considered one at a time, but concepts become useful when they are used in connected form with other scientific concepts to narrate scientific story. To exemplify the above, in teaching learners about the gradient of a straight-line graph, teachers help learners to make links between the concepts of gradient, greater than and less than to make meaningful statements or explanations about the impact of the magnitude of the gradient on the function. Thus, the task for teachers in this approach is helping learners to recognise the interrelatedness of selected concepts, to provide an explanation for any given problem that is related to the topic, for learners to see the part from the whole of the concepts and topic. This addresses Vygotsky's (1978) notion of higher order thinking, as he argues that "the development of the child's higher mental processes depends on the presence of mediating agents in the child's interaction with (*different educational*) environments" (p. 86).

Of importance for Vygotsky (1978) is for teachers to understand development as a process that is characterised by a unity of teaching and learning materials and mental aspects, of the social and the personal during the child's ascent up the stages of development. In this study, 'making links between scientific concepts' was not only used to understand *which* matrix of interrelated concepts teachers drew upon during the teaching of algebraic functions, but also *how* and *why* teachers made the links between concepts during an explanatory talk in the classroom. While making links between scientific concepts is essential for learning scientific conceptual knowledge, teachers do not make these links because they have to, but link concepts for specific purpose(s) in a specific episode of a lesson. Thus, in view of commognitive theory's postulation that thinking and communication are intertwined processes, it is also important to understand from teachers' perspectives their intended motive for drawing the links between concepts during teaching, and video-stimulated recall interviews with teachers can help to gain insight into this, as will be seen in Chapter 4.

3.3.1.3. Making links between scientific explanations and real world phenomena

The teaching of mathematics is made interesting, relevant and meaningful to learners when teachers are able to explain mathematical concepts in relation to world more beyond reallife experiences. As I have mentioned in previous chapters, teachers play crucial roles because they have to carefully select real-world phenomena that learners are familiar with, for this study in rural context, to enable learners' conceptual comprehension as well as stimulating their interest to participate in learning. According to Scott et al. (2011, p. 9), "the challenge for the teacher is to saturate scientific ideas with the concrete, in order that students can see the connections between scientific constructs and the real world". This statement suggests that rural teachers need to have mathematics teaching knowledge, general world knowledge, and know the area where the schools are located to select and use phenomena that are familiar and appropriate for the learners during the explanation process. The purpose of linking mathematical explanations with real-world phenomena is not to limit learners to context-dependent understanding but enhance their meanings beyond local contexts. Thus, although Scott et al. (2011) did not explicate the need for teachers to make use of 'local context-specific' phenomena during teaching, the context needs to be familiar to learners' lived experiences.

In selecting real world phenomena, teachers need to consider learners' familiarity with the phenomena, and this requires that teachers use the phenomena that learners have experienced or been exposed to in their local contexts. It is not meaningful for teachers to select and make use of phenomena that are elusive or foreign to the learners, because this could result in the learners' school mathematics experience being littered with 'foreign phenomena' resulting in perceived irrelevancy. However, that is not the purpose of this approach; it is to connect the content with the local experiences to show relation with the world. Corbett (2016, p. 146) has advocated for rural teachers "not to dissociate education from the ordinary life even when that life seems remote from the worlds a cosmopolitan education describes" (Corbett, 2016, p. 146). This implies that the mathematics classroom discourse in general, and specifically during the teaching of algebraic functions for this study, should be relevant and familiar to the real-world situation of the learner. In the current study, only two teachers made links between scientific explanations and real-world phenomena during teaching, for example, one teacher talked about the idea of boys and girls dating to describe the coordinate pairing between x and y, to help learners develop an understanding of what algebraic functions are about. Another teacher explained the same concept using the idea of a family, to help the learners understand the idea of a relation within mathematical discourse.

3.3.1.4. Making links between modes of representation

As discussed in Chapter 2, one representation is insufficient to help learners make meaning of the algebraic functions concepts, because mathematical meaning is usually made by an amalgamation and "extravagant use" of several modes of representation within one text (Scott et al., 2011, p. 10). There are different modes of representation that teachers can use during the teaching of algebraic function, such as table of values, graphs, coordinate pairs, formulae, function machine. I observed teachers using the symbolic and table of values to bring to the fore the mathematical substitutions and calculations to determine output values for the purpose of drawing the graphs of the functions. I noted that while Sfard (2008) considered visual mediators as the main form of medium that facilitates mathematical discourse, the theory does not explicate the intricate use of different forms of representations and how the translation between the representations supports mathematical discourse. Thus, it is important to extend the notion of visual mediators while exploring the teachers' use of mathematical discourses during algebraic functions lessons, to account for how teachers translate between representations during teaching.

In this study, the parabolic function could be expressed in terms of formulae, or through a table, or in terms of a table of values, or in terms of words. A deep understanding of parabolic functions, for instance, involves being able to "make links between these modes of representation and see how they come together to paint a full picture" (Scott et al., 2011, p. 11) of what parabolic functions are. Thus, the task for teachers is to help learners link the topic under study with these different modes of representation. Accordingly, a teacher's explanations to the links between the forms of representation play a significant role in demonstrating the idea of translating flexibly between the forms, as well as the key features that each form reveals about different families of functions. From the classroom observations, the majority of the teachers used formulae and table of values as tools in the process of plotting and drawing graphs of functions.

3.3.1.5. Moving between different scales and levels of explanation

The fifth approach to support knowledge building is moving between different scales and levels of explanation, which focuses on how teachers and learners move between explanations of concepts set at different scales of representation (Scott et al., 2011). According to Scott et al. (2011), during teaching and learning of scientific concepts, teachers need to consider three levels of representation and their meanings: symbolic; sub-microscopic and the macroscopic. The symbolic level comprises a variety of pictorial representations, computational and algebraic forms. In the teaching of algebraic functions, teachers can use algebraic equations, table of values, graphs, symbols, formulas and numbers at this level. Thus, in the context of teaching algebraic functions, this level of explanation comprises the manner in which teachers explain the different key features of the topic using a variety of representations discussed earlier in Chapter 2. The sub-microscopic comprises the abstract level of mathematisation or communicating about the behaviour of mathematical objects, which Sfard (2008, p. 172) viewed as “complex hierarchical systems of partially exchangeable symbolic artefacts”. Thus, in the teaching of algebraic functions in school curriculum, abstract discursive objects such as the effect of varying the values of parameters for different families of functions need to be explained clearly. This helps in guiding learners towards generality about the nature and effect of such parameters to develop deep understanding of the subject matter contents.

It is important to understand how abstract ideas become concrete and how the concrete assists the understanding of abstract (Basson et al., 2006). This relates to the macroscopic level of representations which demonstrate to the learners the phenomena that can be seen with their naked eyes. That is, the macroscopic includes the actual phenomena that learners experience in their daily lives; it is the level of the tangible and observable (Scott et al., 2011). This level closely links with the notion of real world phenomena discussed in 3.3.1.3 earlier, whereby teachers are expected “to saturate scientific ideas with the concrete, in order that students can see the connections between scientific constructs and real world” (Scott et al., 2011, p. 9).

3.3.1.6. Analogical link-making

The use of analogies helps learners connect mathematical knowledge presented during teaching and learning to their existing experiential knowledge and enables learners’ internalisation of mathematical knowledge and skills (Orgill, 2003; Glynn, 2007; Ünver, 2009). Although Scott et al. (2011) did not define what they mean by analogy, Gentner (1989) defined an analogy as “a mapping of knowledge from one domain (the base) into another (the target), which conveys that a system of relations that holds among the base objects also holds among the target objects” (p. 201). This suggests that teachers should help learners towards an understanding of the contents of the subject target concept, by using an analogy with a familiar case which is not far-fetched from the context. For example, while teaching learners about plotting the graph of function (target), a teacher can refer to ‘pairing up girls and boys for a dance contest’ (analogy) to explain the correspondence between the values of the dependent variable and of the independent variable. The following section focuses on the second form of pedagogical link-making.

3.4. Pedagogical link-making to promote continuity

While the previous form was about how teachers and learners make pedagogical links between different kinds of knowledge to ensure learners’ conceptual development, the second form of pedagogical link-making is link-making to promote continuity. It focuses on how teachers and learners make links between teaching and learning events that occurred in different times to help learners develop a coherent whole (Scott et al., 2011). This form suggests that significant learning does not just ‘happen’ but occurs over time as teachers teach different topics, separated in time to promote knowledge building. This links with narration, because it entails the development of a mathematical story which is influenced by mathematical word use and visual mediators. As learners learn and master new skills and concepts, promoting continuity implies that for teachers to support learners’ conceptual knowledge building, they need to plan for and build in showing concepts continuity. According to Scott et al. (2011), “teaching and learning scientific knowledge must be played out over an extended time-scale, giving rise to the need for pedagogical links to be made between teaching and learning activities enacted at different points in time” (p. 13). Considering this, if deep learning of algebraic functions is to be the outcome, teachers needed to make links between linear functions covered in Grades 8 and 9, algebra and equations, exponents and number patterns which were covered earlier in the year. There are two pedagogical link-making approaches to promote continuity in teaching and learning. These are: to develop the scientific story and to manage/organise information during transitions from one activity to another. Of importance to note is that each of these approaches has an associated time-scale, divided into three levels: *macro* (continuity links in time-scale of months or years), *meso* (continuity links in intermediate time-scale of days or weeks) and *micro* (continuity links made on different episodes of the same lesson).

3.4.1. Continuity links to develop a scientific story

In view of the commognitive theoretical framework, mathematical discourse is a special type of activity and communication of generating the story of mathematics. During teaching, teachers usually refer to what they have already introduced to the learners in lessons separated in time, which was observed in the current study during teaching and learning events in the classroom. Teachers attempted to develop a coherent story about the algebraic functions and how it relates to other topics of the subject matter, to enable learners’ development of conceptual knowledge. Scott et al. (2011, p. 15) drew on Mercer’s (2000, p. 52) idea that teachers use “basic conversational techniques for building the future on the foundations of the past”, which suggest the need for teachers to relate previously taught subject matter and experiences to a present learning situation. This is part of teaching or should be done before the new lessons are introduced or continue with the previous lesson. For Mercer (2000), teachers use two techniques to draw learners’ attention to these links: recaps and elicitations.

The former entails cases in which the teacher refers to the events or contents covered in previous lessons to introduce the present activity or topic, hence, thinking plays an important role during teaching. While this technique is familiar in teaching, the execution is important because teachers must be careful when and how they use recapping and why it is used at a particular time. Elicitation, on the other hand, focuses on teachers' classroom talk which is typically in the form of questions to obtain from learners what they have learned from previous lessons to set the scene for current or future activities (Mercer, 2000; Scott et al., 2011). In the context of teaching algebraic functions, it is important for teachers to ensure that the learners provide the maximum amount of content knowledge during teaching and learning, rather than simply being receivers of information from the teacher. When teachers create an environment in which they ask learners to describe their understanding of previously learned contents, they create learning opportunities in which learners interpret and internalise information.

In addition to the above discussion, each pedagogical approach to promote continuity in teaching has an associated time-scale. The first is a micro continuity time-scale, which refers to the links that are made by teachers and learners on a short time-scale in which they make references to different points presented in different segments of the lesson. The second time-scale is meso and focuses on continuity links that are made on an intermediate time-scale (contents covered days or weeks before). The last time-scale is macro, which involves making links on an extended time-scale, typically of months or years. According to Scott et al. (2011), these time-scales represent the idea that teachers and learners draw and build on their own and each other's ideas to develop the contents into a coherent whole and develop a mathematical story. In the teaching of algebraic functions, it is important that teachers make continuity links and allow learners to make their own links at these time-scales because such links can enable learners to talk through earlier experiences and in turn make links to newly-learned ideas, which is essential for developing conceptual understanding. A detailed analysis is made in Chapters 5 to 9, and I present each teacher's teaching in each chapter representing multiple cases. These three time-scales are important in this study, as they help illuminate the kinds of links teachers make between events separated in time to help learners make connections between mathematics concepts.

3.4.2. Continuity links to manage/organise

Scott et al. (2011) contended that teachers play a crucial role in managing/organising classroom activities with effective explanation during teaching (at the micro time-scale), to enable learners' understanding of the taught materials. Manage/organise is the second approach to promote continuity and establish the cumulative nature of teaching and learning the subject matter contents, as a way of helping learners not to experience classroom interactions as a series of disconnected events (Scott et al., 2011). It is important that teachers are able to bring the mathematical meanings associated with different mathematical processes and also be able to extract significant information from different modes of representation, to promote learners' mathematics knowledge building. This resonates with the stipulations made by the DBE (2011) that the availability and use of mathematical resources should be matched by a good understanding of how and when the resources should be used, as different resources serve different functions at different times and in different grades. The teaching of algebraic functions within rural classrooms, and the continuity links to manage/organise teaching help in understanding teachers' "transitions from one activity to another" within one observed lesson, and how teachers make use of mathematical resources to bring mathematical meanings underpinning particular activities during teaching (Scott, 2011, p. 15). Teachers in this study used this approach to make transitions between different modalities of representations of functions.

3.5. Chapter summary

The purpose of this chapter was to provide a detailed discussion of Sfard's commognitive theory with a focus on discourses, and Scott et al.'s (2011) framework of PLM that assisted with analysing the use of discourses during the teaching and the different classroom practices in the algebraic function lessons. To present the relationship between the two frameworks, teachers used words, endorsed narratives, visual mediators and routines differently, depending on their content knowledge, to support knowledge building and promote continuity during teaching in different events separated in time. The four forms of mathematical discourse from Sfard's theory helped to identify the mathematics discourses that teachers used and did not use during the lesson in algebraic function, and also to understand their reasoning during the VSR interviews. The two forms of PLM and their accompanying approaches facilitated the understanding of teachers' practices during the lessons, in particular the nuances of the different approaches that are used in different times. The first form has more approaches to understand how teachers made pedagogical links between different kinds of knowledge to support learners' knowledge building for algebraic functions. Even though the second form has two approaches, it links with the first form because when pedagogical links to promote continuity are created, they inextricably support learners' knowledge building about the concepts. Teaming up the commognitive theoretical framework and PLM enhanced the data analysis process of the current study as they enabled the descriptions of the nuances of teachers' practices during the teaching of algebraic functions. The next chapter focuses on the research processes that I engaged in to address the primary purpose of the study and answer the predetermined research questions.

Chapter 4

Researching with the marginalised: Research design and methodology

Research is to see what everybody else has seen, and to think what nobody else has thought ~
Albert Szent-Gyorgyi

4.1. Introduction

The current chapter is titled 'Researching with the marginalised', which highlights the dearth of educational research with rural mathematics teachers, and subsequently a call to popularise researching with rural communities to address the prevailing research gap. According to Leedy and Ormrod (2010), the aim of research is to discern and understand information about the subject under study, and possibly deconstruct and reconstruct the prominent theories or meta-narratives in the light of the newly discovered knowledge. The primary focus of this chapter is to outline the research processes, the journey I undertook to address the purposes of the study. This chapter discusses in detail the processes and methods I used to generate, organise and analyse the qualitative data for the study, which I gathered using non-participant classroom observations, video stimulated recall interviews (VSRIs) and semi-structured individual interviews. This chapter also describes and presents the research paradigm and design in relation to commognitive theory and pedagogical link-making frameworks, the context of the study, sampling technique and procedures, data analysis procedures, ethical considerations and methods of ensuring reliability and validity of collected data. The justifications for these approaches are explained in relation to the purpose and objectives of the study. I begin this chapter by presenting the research paradigm espoused for the current study, to shed light on my methodological choices as the two are inextricably linked (Troudi, 2010).

4.2. Paradigmatic assumptions

A researcher always has an underpinning worldview about the phenomenon being researched, which is referred to as the research paradigm. Rehman and Alharthi (2016, p. 51) have defined a research paradigm as "a basic belief system and theoretical framework with assumptions about 1) ontology, 2) epistemology, 3) methodology, 4) methods … it is our way of understanding the reality of the world and studying it". My basic belief is that there are multiple perspectives about what constitutes knowledge and reality, and people should not be studied like objects, rather researchers should get involved with the subjects and understand the phenomena in their contexts. Poststructuralism is the chosen paradigm for this study and is used to understand the teachers' discourses and approaches during the teaching of algebraic functions, which are enacted through language and actions. Before I engage in in-depth discussion of the chosen paradigm, it is important to discuss two components of a research paradigm: ontology and epistemology which subsequently shaped the methodology and methods of data generation for the study.

According to Scotland (2012, p. 9), "ontological assumptions are concerned with what constitutes reality … and researchers need to take a position regarding their perceptions of how things really are and how things really work" in the phenomena being studied. In this study, mathematics is viewed as socially constructed, and is built upon the views, actions, and interactions between teachers and learners during the learning and teaching processes. On the other hand, epistemology refers to "a way of understanding and explaining how we know what we know" (Crotty, 2003, p. 3), and refers to the relationship between the phenomena that the researcher seeks to gain insight into, and the assumptions that are made about the nature of knowledge. In this study, knowledge is taken to be "experiential, personal and subjective" (Sikes, 2004, p. 21), a reason it was important to interact with teachers to gain insight into their personal experiences about teaching mathematics, especially algebraic functions.

In view of the ontological and epistemological stances explicated above, mathematics knowledge in the current study is viewed as knowledge that is not only stored in teachers' and learners' minds, but also knowledge that is actively constructed from experience and teachers' pre-existing knowledge during learning and teaching (Major & Mangope, 2012). It is my belief that teachers' roles during mathematics teaching and learning are not to help learners across a remote and predetermined finishing line, but to work towards redefining mathematics teaching and learning as a social collaborative event where both space and time for personal sense making are created during the process. Thus, I observed their classroom actions while teaching mathematics under natural settings, conducted VSRI to identify and discuss critical incidences (Creswell, 2012), and interviewed teachers to gain some information about their biographical information and conceptions about algebraic functions.

4.2.1. The poststructuralist paradigm

Considering the brief discussion of the ontological and epistemological assumptions for the study, as mentioned earlier, this study is located within the poststructuralist research paradigm, which suggests that individuals are not the sources of their own knowledge, meanings, and actions, but rather are products of the discourses they inhabit from a particular discipline, mathematics in the current study (Foucault, 1977; Walshaw, 2007). To be well positioned within the mathematical discourse, mathematics teachers and learners need to speak the mathematical truths in a way that is recognised by others within the discipline as appropriate. Discourses are ways in which individuals constitute knowledge, truths and meaning, consciously or unconsciously (Foucault, 1972; 2000), making them not innocent or neutral explanations of the world, rather a way of "worlding", of appropriating the world through knowledge. It thus means that teachers, as individuals, play an active role in appropriating mathematics knowledge as they teach the learners. In Foucault's view[22], the term discourse refers to knowledge, what is "within the true" (Foucault, 1972, p. 224); in this study understanding what is within the true mathematics knowledge from the teachers' perspectives. Thus, interrogating and understanding the way rural teachers constitute mathematics 'truth' and meanings, and appropriating algebraic function knowledge while teaching is the focus of this study. Discourse also defines our conceptions of what we view as "normal" within our communities (Song, 2010), and the normality of mathematics knowledge during teaching and learning cannot be taken for granted. It could represent the danger of inscribing mathematics classrooms as mere sites of social reproduction and enculturation, depending on teachers' practices during the lessons.

Foucault's poststructuralist paradigm places emphasis on the role that discourse and discursive practices play in knowledge formation, by concentrating on how specific knowledge operates and the work they do in specific social practices (Foucault, 1989; Davies & Gannon, 2005). Discursive practices entail that in any social interaction, there is 'systematicity', that is, rules that govern the selection as well as the exclusion of concepts, norms and objects. Similarly, the teaching of algebraic function is governed by the selection and exclusion rules underpinning mathematics teaching in general, because not all algebraic function concepts can be taught at once, considering the continuation of the concept in Grades 11 and 12 curriculums. Foucault (1971) contends that, "we know perfectly well that we are not free to say anything, that we cannot speak of anything, when and where we like, and that just anyone, in short, cannot speak of just anything" (p. 8). This statement addresses the constraints of each field because it is governed by its rules, individual utterances and actions during social interactions. In the current

[22] Michel Foucault's early project was to denounce the transcendental status of knowledge and he showed how knowledge is formed within the interaction of plural and contingent practices within different sites. See Michel Foucault, "The Discourse on Language", in Michel Foucault, *The Archeology of Knowledge*, translated by A. Sheridan, vol. 31, no. 3, (2002), 627-629.

study, this information talks about the rules of mathematics disciplinary knowledge, which oversee what teachers can think and communicate as they teach algebraic functions (Foucault, 1994; Sfard, 2008). This discussion also resonates with two tenets of commognitive theory, ‘words and their uses’ and ‘endorsed narratives’, which allude to mathematics being governed by its unique rules and language (Sfard, 2008). The discussion also resonates with the components of PLM and the six approaches to teach algebraic functions as these illustrate when and how teachers take certain actions in their teaching of algebraic functions.

4.3. Research design and approach

Table 6 below provides a summary of the research methodology that has been used to realise the primary purpose of the study and answer the predetermined research questions. It is followed by detailed discussions of the various study processes I undertook to address the primary purpose of this study.

Table 6

Summary of the research approach and design

Research approach	**Qualitative**				
Research design	**Case study: Multi-instrumental** A case can be a unit of focus or a group of individuals that are analysed (Yazan, 2015). This case study consisted of rural Grade 10 mathematics teachers as a group, making it a multiple case study. I conducted semi-structured individual interviews with the participants, observed them in the classrooms as they taught algebraic functions and determined the nature of their discourses and approaches to teaching the topic. In addition to this, I conducted video-stimulated recall interviews with the teachers. The nature of the data I gathered in this study was qualitative (Nieuwenhuis, 2007).				
Main research question	What are the rural Grade 10 teachers' discourses and approaches during algebraic functions lessons?				
Sub-research questions	What are Grade 10 rural mathematics teachers' discourses during algebraic functions lessons?	What approaches do Grade 10 teachers use to teach algebraic functions?	How do teachers use multiple representations during algebraic functions lessons?	How do teachers guide learners towards generality about the effect of parameters in the context of algebraic functions?	What are the factors that influence teachers' discourses and approaches of algebraic functions within rural classrooms?
Objectives of the sub-research questions	To describe and critically analyse Grade 10 rural mathematics teachers' discourses while teaching algebraic functions.	To identify and interrogate teachers' approaches during algebraic functions lessons.	To gain insights into teachers' use of multiple representations in the context of algebraic functions.	To explore and understand the nature of teachers' mathematical discourses relating to the notion of parameters of algebraic functions.	To examine factors that influence teachers' discourses and approaches while teaching algebraic functions within rural schools.
Sampling technique	Within non-probability, I used purposive and convenience sampling strategies for the current study (Cohen et al., 2007).				
Participants	✓ One Grade10 mathematics teacher from each of five different school sites.				
Sources of data	✓ Semi-structured individual interviews ✓ Classroom observations ✓ Video-stimulated interviews per teacher				
Sources of data per research question	• Observations • Semi-structured interviews • Video-stimulated recall interviews	• Observations • Semi-structured interviews • Video-stimulated recall interviews	• Observations	• Observations	• Observations • Semi-structured interviews • Video-stimulated recall interviews
Data analysis	Fairclough's (1989, 1995) Critical Discourse Analysis (CDA) model[23] consists of three processes of analysis which are closely inter-related and are tied to three dimensions of discourse that are inextricably related. The three dimensions are: the object of analysis, the human processes by which the object is produced, and the socio-historical conditions which shape these processes (Janks, 2010). In this study, I used CDA to analyse semi-structured interviews, typological analysis for lesson observations and content analysis to analyse video-stimulated interviews with teachers. I used these analytical techniques to supplement Sfard's commognitive theory and Scott et al.'s (2011) pedagogical link-making to gain insight into teachers' discourses and approaches of teaching algebraic functions.				

[23] See Janks (2010) for a clear demonstration of how to use Fairclough's model of CD Analysis.

4.3.1. Research approach

The current study espoused a qualitative approach, with the acknowledgement of quantitative and mixed methods research approaches as alternatives for research. According to Flick (2018, p. 13), "qualitative research is oriented towards analysing concrete cases in their temporal and local particularity and starting from people's expressions and activities in their local contexts". This links with the research design which is the case study that focuses on the exploration of a particular phenomenon within a bounded context. The quantitative approach was not suitable for this study since the focus was on the multiplicity of teachers' meanings and experiences of teaching algebraic functions with the intent of developing a pattern of their discourses and approaches, instead of verifying theories relating to the teaching of mathematics. The mixed-methods research approach entails a procedure for generating and analysing research data using both qualitative and quantitative research approaches and methods in one inquiry, to understand a subject under scrutiny (Creswell, 2012). There was no quantitative aspect in the study, hence the focus guided the approach.

Thus, the qualitative approach was appropriate because its goal "is to understand the situation under investigation primarily from the participants' not the researcher's perspective" (Hancock & Algozzine, 2016, p. 8). This means to understand mathematics teachers' experiences I had to immerse myself in their everyday teaching lives, to understand the teaching of mathematics as experienced by the teachers and what they do to enable learners' understanding of mathematics within rural settings (Hardré, 2011; Smith, 2013). Regarding a researcher's presence within a context being researched, Marshall and Rossman (1995, p. 59) mentioned that "whether that presence is sustained and intensive, as in long-term ethnographies, or whether relatively brief but personal," as in classroom observations, of importance is to gain in-depth understanding of the phenomenon under study.

In this study, I observed the teaching of algebraic functions to gain understanding of teachers' mathematical discourses and approaches during the algebraic function lessons, and conducted VSRIs to seek teachers' reasons for their classroom utterances and observable actions. I also had conversations with teachers about their personal experiences and knowledge of teaching algebraic functions by means of semi-structured individual interviews. The qualitative approach is seen as flexible, and this enabled me to have 'open-ended' conversations with each teacher during both video-stimulated recall interviews and semi-structured individual interviews pertaining to their discourses and approaches which were observed during lessons on algebraic functions (Denzin & Lincoln, 2003; Creswell, 2012) as will be discussed in section 4.7.

4.3.2. Research design

The current study was designed as a multiple case study (Stakes, 1995; Guba & Lincoln, 2005; Simons, 2009) to interrogate Grade 10 rural teachers' discourses and approaches during algebraic function lessons in five Acornhoek school sites in Acornhoek, Mpumalanga Province. Creswell (2013) described research design as a blueprint or plan for conducting the research, which entails a detailed plan according to which the research is carried out. Stakes (2005) distinguished between three types of case study designs in qualitative research: intrinsic, instrumental and multiple case study designs. While the intrinsic case study design entails gaining in-depth understanding of a specific case, the instrumental case study design is when "a particular case is examined mainly to provide insight into an issue" rather than a case itself (Stakes, 2005, in Denzin & Lincoln, 2005, p. 137). For this study, a multiple case study was defined as an instrumental study extended to five different teachers as cases, from five different school sites to explore the discourses and approaches of teaching algebraic functions within rural classrooms (Stakes, 2005, in Denzin & Lincoln, 2005, pp. 136-138). The teachers further served as instruments for understanding teachers' discourses and approaches while teaching algebraic functions within rural mathematics classrooms.

Simons (2009) defined multiple case study design as "an in-depth exploration from multiple perspectives of the complexity and uniqueness of a particular institution or system in a 'real-life' context" (p. 21). Merriam (1998) postulated that "reality is not an objective entity; rather, there are multiple interpretations of reality", acknowledging the "embeddedness of social truths" in case study research design (Merriam, 1998, p. 22). In the current study, teachers' discourses and interpretations of algebraic functions and its teaching and learning were viewed as subjective, multiple, dynamic and complex. I further considered various generative mechanisms that were at play in shaping their classroom discourses and approaches to teaching the topic. The discussion on the research paradigm and design introduced the qualitative research approach, which further linked with the chosen theoretical framework and analytical framework. In the following section, I introduce the context in which this study was conducted.

4.4. Introducing the context of the study

The current study was conducted in five rural Acornhoek secondary schools, with five Grade 10 mathematics teachers. According to Statistics South Africa (Stats SA, 2011), Acornhoek is classified as a rural area in the Greenvalley circuit, located in the Bushbuckridge[24] region of Mpumalanga Province of South Africa. All five schools that form part of the study fall under the Bohlabela District and the prominent languages in this region are Xitsonga and Sepedi, but also have some people who speak Sepulana (mixture of Sepedi and Xitsonga). As discussed in Chapter 1, Acornhoek is one of the regions that represents the complexity and difficulty of classifying whether a place is deep rural or semi-rural. Since I started researching within Acornhoek schools, I have received mixed responses from the participants about the classification of the region (see Mbhiza, 2017). While I acknowledge that the formal classification by Statistics South Africa (Stats SA, 2011) views the region as a rural area, it is equally important to understand the way rural constituencies view and classify their context. The Nelson Mandela Foundation (NMF, 2005) stated that "the universal framework employed by the government and policy documents is insufficiently sensitive to the specific conditions and needs of the rural poor" (p. 139). It is therefore important to understand how rural communities perceive their regions, and not solely rely on policy definitions that overlook the lived experiences of rural constituencies. This statement does not overlook the general guide of the government policy, rather it considers the views of the participants.

The participants in the current study also presented mixed responses about the classification of Acornhoek, perceiving the region as either semi-rural or deep-rural, depending on individuals' proximity to various amenities such as shopping centres and healthcare facilities, to name just a few (Mbhiza, 2017). Of interest is that participants who classified the region as deep-rural based this on the dominance of the residents that work on the farms to sustain themselves and their families, poor transportation services and being isolated from the national and provincial government offices. In line with this, Pateman (2011) posited that the type of work residents do can also be used to classify a region as either rural or urban. The teachers also postulated that the region is a deep-rural area because the schools in which they teach are in historically dilapidated buildings (see Images 1 and 2) with poor sanitation. In addition, learners still walk long distances to schools (Image 4), due to the lack of transport services and that they receive very limited attention from the national government officials[25] (see Gardiner, 2008).

[24] In 2001, Statistics South Africa made provincial boundary changes in South African provinces, accordingly "Bushbuck Ridge municipality was a cross boundary municipality between Limpopo and Mpumalanga and has now been allocated in full to the Mpumalanga Province" (Stats SA, 2011, p. 11).

[25] Some participants iterated that the Minister of Basic Education may not even know the location and conditions of their schools since she is mainly based in the urban areas (i.e. Gauteng Province), and due to their area's

Image 1: Falling ceiling

Image 2: Mould inside the classroom

Image 3: Residents at a communal tap

Image 4: Learners walking to school

Image 5: Overview of one of the villages

Image 6: The contrasting houses

Image 7: Newly built Acornhoek mall

Image 8: Newly built McDonald's

Photograph 1: *Some distinctive characteristics of Acornhoek*

isolation from metropolitan cities, the ministers and other government officials never come to their contexts to understand the conditions in which they live, teach and learn within the schools.

4.5. Research sampling and participants

The selection of research participants plays a key role in ensuring thorough understanding of a phenomenon being studied, since the researcher must choose specific participants who are knowledgeable about what is focused upon in a study (Creswell, 2007; Sargeant, 2012). The following sub-sections detail the selection criteria of schools and teachers, and the sample for the study.

4.5.1. The selection of schools

In this study I used purposive and convenience sampling techniques to select the participating schools. Purposive sampling seeks cases that are rich in information which can be studied in great depth relating to issues of central importance to the focus of the study (Cohen et al., 2007). The schools in which this study was located were selected purposively on the basis that four of the schools are classified as underperforming and un(der)resourced, whereas the fifth school is a newly built school with adequate resources and is reported to attract and retain 'quality' qualified teachers, thereby performing well in Grade 12 examinations[26]. All the schools were conveniently selected based on the previously established partnerships with the Wits School of Education Rural programme for preservice teaching practicum, and the research project. In addition to the above information, the five schools are within a 4 kilometres radius of each other and were selected based on convenience to travel between them daily. The small distance between the schools helped in cases where one teacher cancelled daily schedule(s) due to other commitments, and I could easily drive to another school where another participant was available and in cases where meetings with teachers were scheduled adjacent to each other.

4.5.2. Participating teachers

The selection of the teachers was also purposive. I used this technique to select teachers that possessed "particular characteristics being sought after", to make sure they could engage with the research questions (Cohen et al., 2007, p. 115). The current study acknowledged that the teachers needed to have knowledge and experience of teaching Grade 10 mathematics within rural schools. Accordingly, for the current study, five teachers were selected from five different school sites, and the selection of the participants was not linear but involved complex processes. One of the reasons for having five teachers was based on the reality that each

[26] Grade 12 is the highest grade in South African basic education, and passing the grade's examination enables learners to access various post-secondary school opportunities. The opportunities include admission to internships, learnerships, access to entry-level employment, as well as admission to higher education institutions.

secondary school had only one Grade 10 mathematics teacher. I initially assumed that the selection of the sample was going to be easier, because of the relationships that were built with the teachers when I was involved in the larger research project from 2015 until 2017. I also presumed that my 'familiarity' with the schools and the villages where the schools are located would make my data generation process smoother. However, some events occurred while at the research context which required alterations to the initially selected sample.

Before the study commenced, I approached six schools and six teachers from the group of schools that participated in the project to inform them about the nature of the current study and asked whether all were willing to participate. However, two of the six teachers in Bash and Ritlhaveta could not continue their participation when I started with the data generation process because they were moved from teaching Grade 10 to Grade 11 and from Grade 10 to Grade 12 respectively. I required teachers who were teaching Grade 10 mathematics in the year of the data collection process; thus, for the purpose of classroom observations, the two teachers were purposively ineligible to participate in the study. The two teachers recommended their colleagues who had taken over their Grade 10 classes. Table 7 below depicts the selection of teachers for the study and how I came to have five participating teachers from five different school sites.

Table 7

Difficulty in selecting and retaining some participants

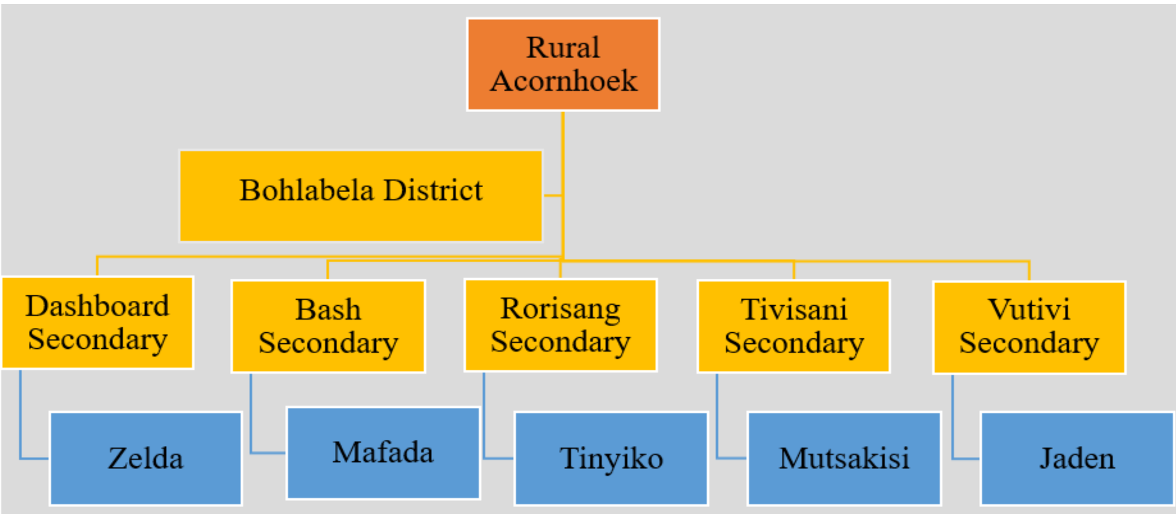
<table>
<tr><th>School name[27]</th><th>Participants at the beginning of the study</th><th>Continued or discontinued</th><th colspan="2">Participants that were cases for this study</th></tr>
<tr><td>Dashboard</td><td>Zelda</td><td>Continued</td><td colspan="2">Zelda</td></tr>
<tr><td>Bash</td><td>Khatisa</td><td>Discontinued with a referral</td><td colspan="2">Mafada</td></tr>
<tr><td>Rosisang</td><td>Tinyiko</td><td>Continued</td><td colspan="2">Tinyiko</td></tr>
<tr><td>Tivisani</td><td>Mutsakisi</td><td>Continued</td><td colspan="2">Mutsakisi</td></tr>
<tr><td>Vutivi</td><td>Jaden</td><td>Continued</td><td colspan="2">Jaden</td></tr>
<tr><td>Ritlhaveta</td><td>Thomas</td><td>Discontinued with a referral, but the new participant was never available.</td><td>Risenga</td><td>(no participant)</td></tr>
</table>

[27] All school and participants' names are pseudonyms

Furthermore, while Risenga had agreed to replace Thomas and participate in the study, he was never available for either interviews or observations because he had other commitments. Every time I scheduled an appointment with Risenga he would agree over the phone that he would be available, but each time I visited the school for our scheduled appointments he was never available. This addresses the complexity of recruiting and retaining research participants, even when the researcher is familiar with the context of the study and the potential research participants (Archibald & Munce, 2015). Accordingly, I removed Risenga from the sample leading to having only five teachers from five schools participating in the study. Figure 5 below represents the final sampling structure for the study after replacements and withdrawals from initially selected teachers as detailed above.

Figure 5

Outline of the final study sample



In view of the need for a bounded context in case study research, as discussed earlier, the schools and teachers were selected within different areas of the same district in Acornhoek as depicted in Figure 5. Notwithstanding the sampling strategy and sampling criteria as well as some complexity in selecting and retaining research participants for the study discussed above, Table 8 presents teachers' biographical information.

Table 8

Participants' biographical information

Pseudonym	Gender	Mathematics Education qualifications	Number of years teaching	Institution trained at to become a teacher
Zelda	Female	Bachelor of Education	5 years	University of North West, South Africa
Mafada	Male	Honours in Mathematics Education	20 years[28]	Giyani College of Education, South Africa
Tinyiko	Female	Bachelor of Education	5 years	University of Venda, South Africa
Mutsakisi	Female	Bachelor of Education	30 years	University of Zimbabwe
Jaden	Male	Bachelor of Education	17 years	College of Education in India

As depicted in Table 8, the current study comprised of three female teachers and two male teachers. The teachers' years of teaching experience ranged between 5-30 years and are all qualified mathematics teachers. Jaden and Mafada attended teacher training colleges in India and South Africa respectively, whereas Zelda, Mutsakisi and Tinyiko obtained their teaching qualifications from universities indicated in Table 8. It was interesting to research with this diverse group as their varied experiences and positionality pertaining to teaching in general and mathematics teaching in particular in one way or another shaped their discourses and approaches of algebraic functions. The following section presents the methods of data generation.

4.6. Research process: detailing methods of data generation

In qualitative research, researchers create data from sampled data sources such as human participants, electronic media, and documents, to list just a few. The goal for selecting specific research methods for a study is to ensure suitability, to address the objectives of the study and to answer the predetermined research questions. I used three data generation tools to gain insight into teachers' discourses and approaches during algebraic functions lessons. These were classroom observations, VSRI, and individual semi-structured interviews, which brought the versatility of the data, maximised its credibility and in turn enabled me to answer the predetermined research questions and address the primary objectives for the study. In Chapters

[28] While Mafada has 20 years teaching experience, he stated that he sees himself as a beginner mathematics teacher because it was his first time teaching mathematics in 2018 since his teaching career started. Although he is a qualified mathematics teacher, he has been teaching Physical Sciences throughout the years.

5, 6, 7, 8 and 9, the data gathered from the interviews and VSRIs are presented in quotes of each teachers' excerpts and the data from classroom observations are presented as episodes of the observed lessons.

4.6.1. Unstructured non-participant classroom observations

Since this study focused on teaching approaches and teachers' discourses while teaching algebraic functions, it was therefore important to observe how teachers taught the different concepts of algebraic functions. According to Johnson and Christensen (2014, p. 206), observation refers to "the watching of behavioural patterns of people in certain situations to obtain information about the phenomenon of interest". This suggests that observation can be used to generate depth understanding of the nature of events or activities which participants take part in (see also Guthrie, 2011).

Classroom observations helped to gain insight into teachers' classroom performances, to see what they were doing and saying during the lessons, rather than what they said they were doing or their descriptions of their classroom practices during interviews. Using an unstructured observation technique in the study allowed me to "postpone definitions and structures until a pattern emerged", out of the conditions of teachers' teaching within the classrooms that I observed (Bell, 2005, p. 185). Cohen et al. (2007) suggested that unstructured observations enable the researcher to "review the observational data before suggesting an explanation for phenomena" (p. 397). Thus, instead of imposing predetermined collections of notions onto rural teachers' mathematics teaching, the trends and patterns reliably emerged out of how teachers acted, what they said during teaching and how they interacted with learners, mathematical contents and other physical artefacts in the classrooms.

In relation to my participation as the researcher during teaching and learning processes, I adopted a non-participant observation approach, which Walliman (2016) has described as an observation approach in which the researcher assumes detachment with the intention of being ignored by the individuals they are observing. While I adopted a passive, non-intrusive role to ensure detachment in all the classroom observations, the teachers and learners noted my presence in the classrooms with a video recorder. This influenced some teachers' and learners' behaviours, even without uttering a word. I did not actively participate in classroom activities by involving myself during teaching such as interjecting. Instead, I focused on what I saw and heard during teaching and learning and subsequently made interpretations and conclusions about teachers' discourses and approaches to teaching algebraic functions. While this is true, some of the teachers and their learners did not ignore my presence as suggested by Walliman's (2016) perspective of what non-participation means while observing individuals engaging in an activity. Accordingly, it is my contention that my body as the researcher, whether it was speaking or quiet during lessons, was disruptive considering that the classroom owners recognised a foreign body to which they attached specific interpretations, expectations and meanings about what an observer 'wanted to see' during teaching and learning, resulting in "reactivity[29]" (Cohen et al., 2011).

Even though I remained a 'passive non-interactive participant' during these lessons, when teachers drew me into classroom discussions, as in Zelda's and Mafada's classes, my non-participatory[30] observation became disrupted. By passive non-interactive participant here I mean my classroom level of participation in which I planned to be a non-participant observer but teachers spoke about my presence and the presence of the video recorder in the classroom. While this happened in multiple lessons, I did not take any active part in the discussions. This reinforces Check and Schuh's (2011) and Hopkin's (2017) iteration that the researcher's presence has the potential to alter the situation and dynamics, considering that it is unusual for an individual to observe and record teaching and learning processes in the classroom setting, therefore those being observed might behave differently. I further used video recordings to ensure that I looked back at different episodes during teaching, and for VSRIs with teachers. Videotaping the lessons allowed me to replay the videos several times to make sense of the teachers' pedagogical actions during teaching. In the following sub-section, I discuss how the videotapes were used in post-observation conversations, referred to as VSRIs with the teachers in this study. Table 9 presents the summary of the number of lessons I observed for each teacher and the foci of the observed lessons.

[29] According to Cohen et al. (2011, p. 473), reactivity is when research participants alter their behaviour, perhaps to impress researchers.

[30] Green and Thorogood (2014, p. 155) viewed non-participant observation as a technique to observing individuals interact "in which the researcher is present to collect the data but does not interact with the participants".

Table 9

Summary of the observed lessons

Teacher	Number of lessons	Summary of observed lessons
Mafada	5	– *Introducing parabolic functions* – *Drawing graphs* – *Parabolic functions and drawing graphs*
Mutsakisi	6	– *Introducing linear functions* – *Parabolic (quadratic) functions* – *Drawing parabola*
Zelda	7	– *Parabola and graphing* – *Completing tables of values and drawing graphs* – *Interpretations of graphs (parabola) – focusing on parameters a and q* – *Introducing hyperbola – focus on parameter q*
Tinyiko	4	– *Parabola (dual intercept method)* – *Drawing graphs of hyperbolic functions* – *Completing tables of values and drawing graphs of different families of functions*
Jaden	5	– *Relations* – *Introduction to functions* – *Algebraic calculations and conventions (determining output values)* – *Drawing graphs of linear functions* – *Interpreting the effect of parameter a on linear functions*

4.6.2. Video-Stimulated Recall Interviews

It was important for me to make sense of teachers' discourses and approaches during the lessons, and VSRI gave teachers the opportunity to reflect on their choices and usage of particular approaches and inhibiting certain discourses during the teaching of algebraic functions. McMillan and Schumacher (2010, p. 322) stated that qualitative researchers seek to "reconstruct reality from the standpoint of participant perspectives, as the participants they are studying see it". This can be interpreted to mean that the primary goal of any qualitative research is to gain insight into how different individuals construct meaning about the same phenomenon. The other benefit of using VSRI was that it enabled teachers in this study to question their own teaching, which in turn positions them better to interrogate and problematise the taken-for-granted nature of their classroom work. This allowed teachers to configure alternative courses of action and utterances during teaching in the classroom that challenged and modified their pedagogical practices.

According to Paskins et al. (2017, p. 1), VSRI "is a method whereby researchers show research participants a video of their own behavior to prompt and enhance their recall and interpretation after the event". This method was helpful in gaining insight into teachers' knowledge and reasons for teaching algebraic functions using specific approach(es). Authors have proposed many ways of using stimulated recall interviews in research (Lyle, 2003; Hatch & Grossman, 2009), and Lyle (2003) presented two ways of conducting stimulated recall in research: structured time-sampling of a particular videotaped period and the identification of critical incidents by the researcher, the participant, or both. In this study, I identified and selected lesson episodes that represented some issues that were observed during teaching for each teacher, for instance, the way teachers introduced their lessons, and reasons for choosing to structure their lessons the way they did. Although authors and previous researchers (see Yinger, 1986; Reitano, 2006) who used VSRIs have not explicated that these interviews also have degrees of structure: structured, semi-structured and unstructured video-stimulated interviews, I used the semi-structured VSRI technique in this study. That is, the teachers and I predetermined incidents that we wanted to converse about from video-recorded lessons, but we did not limit our conversations to these incidents. I tailored subsequent questions based on the information provided by the teachers, and they also asked questions and elaborated where I sought further clarity.

During observations, there were incidences that stood out for me which I saw a need to have conversations about with the teachers. Thus, I noted these in my research journal with specific time stamps. Photograph 2 depicts some of the episodes that were selected during classroom observations, which I referred to in seeking elaborations from the teachers. Even though observations were not structured (refer to the nature of observations discussed earlier above), the notes I took during observations in each lesson focused mainly on three ‘observational themes’ although these were not consciously predetermined: interactional, mathematical correctness and cases where teachers did not provide elucidations on some workings while teaching learners how to work with certain problems. In addition, I also gave the teachers the video recordings of their classroom teaching to watch on their own and identify incidences that stood out for them. When we met for the VSRI, the teachers also reflected on their selected episodes. Giving the teachers their video recordings allowed them to make their own decisions about what they wanted to focus on, thereby giving them control over when to stop the tape, describe their classroom actions and decisions at that time as well as the alternatives they had considered during teaching.

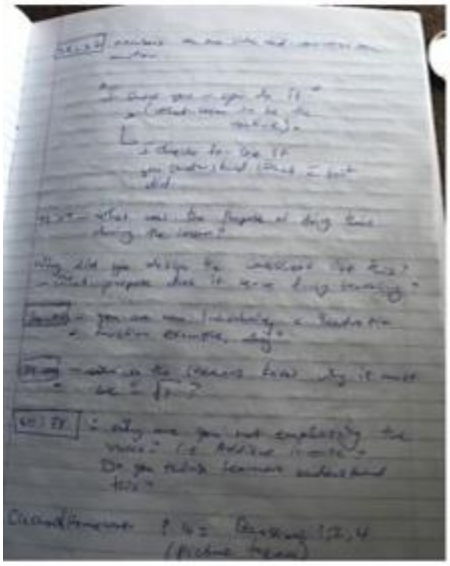


Image 9: Lesson episode from Jaden's observation

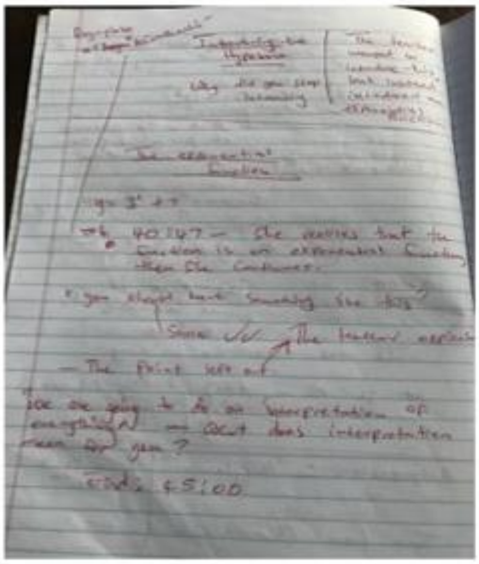


Image 9: Lesson episode from Tinyiko's observation

Photograph 2: *Lesson episodes from research journal observation notes for VSRI*

This links well with my research paradigm which advocates for the notion of power relations, in which power is shared between researchers and participants. However, none of the teachers brought any notes on their identified incidences to the VSRIs, but they commented that they had watched the videos and identified incidences during lessons that they wanted to discuss and would stop the video at any time they wanted to discuss. While the initial plan was that the participants and I would work with the pre-selected episodes, as described above, the reflective conversations were not only limited to the pre-selected episodes. We continued to identify other critical incidences while watching the videos together during the interviews, and we stopped the recordings at any time for comments, explanations and/or questions. It is important to note that the 'stimulation' was dialogic, that is, the videos were not only limited to helping the participants to recall why they acted in a certain way in the original event, but also helped me to ask questions I may not have thought of without the cues provided by the video. The reflective interviews helped me to crosscheck what mathematics teachers perceived about mathematics teaching (from semi-structured individual interviews), against what they did during teaching in the classrooms, and this subsequently assisted in minimising teachers' superficial self-representations during interviews by directly confronting them with their actual actions of classroom practice (see Reitano, 2006).

The prominent criticism for using stimulated recall interviews is that the participant may not recall the thinking that occurred in the process of the original event (Gass, 2001). While this might be the case, Yinger (1986) has posited that teachers are able to report on the aspects they did not notice as the original events occurred, and contends that having videotaped events becomes a "luxury of meta-analysis and reflection that was most likely to be absent in the original event" (p. 271). The video recordings provided the teachers with more cues than in the original episodes[31], such as words, mannerisms and/or expressions they used during the teaching of algebraic functions. Thus, critical incidents were used to enable the participants to provide elaborations for their own interpretations of the videotaped events, as discussed earlier above (Yinger, 1986; Gass, 2001). The VSRI method not only supplemented the information participants provided during interviews and observed behaviour during classroom observations, it added value in terms of its educational potential for the teachers. This is because the use of this method resulted in some participants reporting significant changes in their teaching approaches and discourses of mathematics. This echoes Reitano's (2006) postulation that "videotapes allow the teacher to examine their mental models in situ, study changes to their schemas during and after teaching episodes and formulate new teaching models as a result" (p. 3).

4.6.3. Semi-structured individual interviews

Having formal or informal conversations with teachers is one of the best ways to understand what it means to teach within particular contexts, listening to their teaching challenges and successes, and understanding the different meanings of teaching and learning mathematics. According to Alshenqeeti (2014, p. 40), a qualitative interview is "an extendable conversation between partners that aims at having an 'in-depth information' about a certain topic or subject, and through which a phenomenon could be interpreted in terms of the meanings interviewees bring to it". Accordingly, I used interviews to understand Grade 10 mathematics teachers' experiences of teaching algebraic functions within rural mathematics classrooms as well as their biographical information, which offered insight into why teachers taught the topic the way they did.

In the current study I used semi-structured interviews because I wanted the interviewees to have shared control over the conversations, which resonates with my ontological and epistemological standpoints. The semi-structured interview technique gave me the freedom to

[31] Viewing videos during reflective conversations allow teachers to see their own idiosyncrasies that might impact their teaching, which they could be unaware of during teaching. For instance, gestures, body language, facial expressions, the rate and pitch of their voices as well as phrases or words that were overly repetitive could be identified.

probe the teachers to elaborate on their utterances where I needed more clarity. At the same time, this method also allowed the teachers the freedom to express their views about the topic in their own terms, as well as pose questions seeking clarity where they did not understand the questions. Alshenqeeti (2014) suggested that semi-structured interviews are widely used in qualitative research because of the flexibility they allow during conversations with participants. Although I had a set of guiding questions, teachers' responses afforded me the flexibility to pose subsequent questions adding to the initially drafted questions (Appendix 3). Consider the following extract from a semi-structured interview with Tinyiko for example:

Table 10

Extract from Tinyiko's semi-structured interview

Interviewer	Interviewee
So, you are saying that the table method is the one that is best in teaching functions?	Not only that, but I am saying that the table method is easy, there is no way the learner will fail to draw the graph that is required, but let method is easy because it's simple, so because of time, in the exam, let method is easy because you just let and you get the intercepts and then you are done.
You are talking of functions as drawing graphs, do you think that's all there is to functions, it's about drawing graphs?	No!
What else is there?	Functions, they also remind them about how to factorise, and they did factorisation before, so now when we introduce them to functions, they now repeat what they have done before.
And, what does algebraic function mean to you, when we speak of algebraic function?	What do you want me to answer?
I don't know (I giggle) I mean, when you read on this topic, what does it mean for you? Algebraic function as a topic, what does it mean for you?	Err, I think algebraic functions are functions that are being hated by everyone because they will tell you solve for x.

During the conversation with Tinyiko, she outlined the way she approaches the teaching of algebraic function, and I tailored subsequent questions by using phrases such as, "What else is there?". Where she needed clarity on the questions, she also asked questions such as, "What do you want me to answer?" While my follow-up questions were seeking further elucidation for Tinyiko's responses, her questions were seeking clarity about the contents of the questions. This addresses the importance of paying attention to teachers' responses and the manner of talking about the teaching of algebraic functions which resulted in changing some wording and order of questions[32]. Adhabi and Anozie (2017, p. 89) posited that the implementation of semi-structured interviews is "dependent on how the interviewee responds to the question or topics

[32] See Appendices 3 and 2, the former is the original interview schedule and the latter an actual interview where the prompts and rephrasing of questions are also included.

laid across by the researcher". Due to the flexibility of semi-structured interviews, the teachers were able to express themselves openly and freely and talked in detail about their teaching of algebraic functions.

The shortest time spent in an interview in the current study was 34 minutes and 26 seconds (Mafada) and the longest time spent was 1 hour, 10 minutes and 8 seconds (Jaden). This resonates with the nature of semi-structured interviews, that some conversations are longer than others, depending on the information provided by the participants and need for probing determined by the researcher. Through using the semi-structured interview technique, I was able to follow relevant lines of enquiry during conversations that strayed from the guide where it was appropriate to do so for each teacher (Appendix 2). In line with this, Irvine (2011, p. 207) stated that it is expected that there be a "substantial variation in the duration of interviews" because some participants may be more outspoken than others. Table 11 illustrates the time taken to complete each individual interview.

Table 11

Time taken to complete each semi-structured interview

Participant's name	Time taken
Zelda	44 minutes, 12 seconds
Mutsakisi	52 minutes
Mafada	34 minutes, 26 seconds
Tinyiko	1 hour, 1 minute and 32 seconds
Jaden	1 hour, 10 minutes and 8 seconds

In view of the various contextual challenges I encountered in the research field, as described earlier above, it is worth mentioning that the semi-structured interview with Zelda was conducted at the venue where I stayed. This was suggested by her, because it would be easier to drive to come and meet me since she lived close by. To ensure that all information from all interviews were captured, I used an audiotape to get detailed responses from the participants with their consent (see Appendix 6). Audiotaping the conversations with teachers provided me the opportunity to review and replay the interviews at a later stage and identify important information that may have been missed during the interview, which would maximise the credibility of the study. In the following section, I discuss how the data from semi-structured interviews, classroom observations and VSRI was analysed to understand rural Acornhoek mathematics teachers' discourses and approaches while teaching algebraic function.

4.7. Data organisation and analysis

Data organisation and analysis refers to the processes through which researchers bring order, create structure and meaning to the data generated for a study (Marshall & Rossman, 1999). Considering that qualitative research often results in voluminous raw data gathered from the study (Creswell, 2007), the semi-structured individual interview transcripts, the classroom observation transcripts and the VSRI transcripts provided a mass of information in the current study. In view of this, De Vos (2005, p. 339) noted that "the researcher brings structure and order to the vast amount of data collected, and looks for patterns in the data in order to make sense of it, leading to interpretation and meaning-making". In the following sub-sections, I provide thick descriptions of the processes I followed to organise the data and analyse it to address the objectives of the study and answer the research questions.

4.7.1. Organisation of data: Transcription process

In this study, each set of data for each teacher was transcribed verbatim to ensure that I captured the meanings as well as recorded the context in which these were created in relation to the teaching of algebraic functions. Sfard (2008) posited that it is very important that during the transcription process participants' utterances are reported as spoken by the participants, entailing a principle of verbal fidelity to minimise the loss of meaning. All recorded data were transcribed verbatim, and where teachers' code-switched to Xitsonga or Sepedi during interviews and observations, I translated these into English after each utterance to enable readers that are unfamiliar with these languages to understand the meanings of responses. The code-switching was not a challenge for me because I am fluent in both languages. I provide an example of how each data set was transcribed in the appendices[33] section of this study.

The transcription of classroom observations did not only document what teachers said during teaching, but also what they did and images of their board work. The reason is that teachers' observable actions (what is done) and their utterances (what is said) during teaching forms a vital part of data analysis in a commognitive research, as these help to discern the participant's thinking and communication about mathematical knowledge (Sfard, 2008). The following section details how the data from the three methods of data generation were analysed to provide

[33] Appendix 2 is one of the transcripts of semi-structured interviews, appendix 1 an example of the transcription of the classroom observations and appendix 4 is one of the video-stimulated recall interview transcripts.

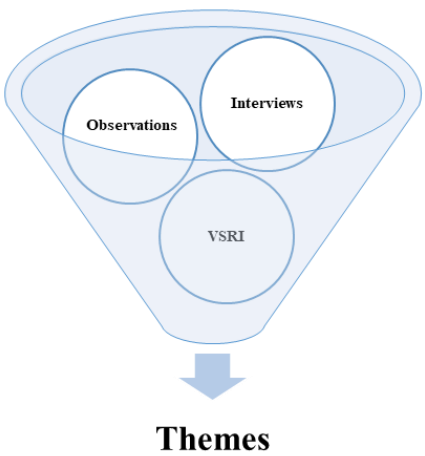
a nuanced understanding of teachers' discourses and approaches of algebraic functions within rural classrooms.

4.7.2. Data analysis

The analysis of the current data was an ongoing inductive process of categorising the different data sets to explore patterns and relationships. The data analysis process in the current study was complex and involved many stages in order to derive the key findings. The analysis process involved three data sets: non-participatory classroom observations, VSRI's and semi-structured interviews. The analysis processes focused on each of the three data sets separately and ensured that thick descriptions were achieved before looking for patterns across the data sets. As a way of simplifying this process, the data analysis approach for the classroom observations will be detailed first, followed by the analysis of the VSRIs and semi-structured interviews. The reason for analysing data from classroom observations first was to ensure that I identified prevalent emerging discourses and approaches of algebraic functions teachers used during teaching. The reason for presenting the analysis of semi-structured interviews and VSRI data was that both teachers' comments in the two methods provide complementary information about why they taught in particular ways. The example of the analysis processes can be found in Appendices 1, 2, 4, 12 and 13 which represent samples of memoed interview transcription, classroom observation transcription and VRSI. Specifically, the analysis process in this study involved identifying patterns and themes, categorising all data under the codes presented in Table 15 and combining themes throughout the responses and teachers' observable actions. This was done through collaboration between the three data generation methods in this study, which is demonstrated in Figure 6.

Figure 6

Collaboration between methods to develop themes



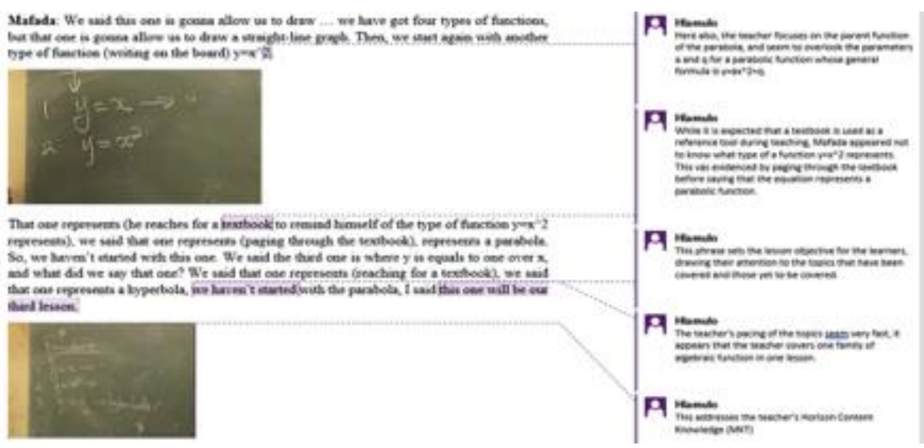
While I discuss the analytical processes for the different sources of data in different sections, they should be read relationally as they complement each other in the generation of themes and sub-themes for the current study.

4.7.2.1. Analysis of classroom observations

I watched the video-recorded lessons of each teacher numerous times, and the process was iterative as it included re-watching videotaped lessons, transcribing each lesson carefully, summarising each lesson, as well as skimming, reading and re-reading of transcripts to identify the discourses and approaches of algebraic functions. I therefore read and re-read the classroom observation transcripts, making notes in the margins of each transcript document about the teachers' mathematical communication for the purpose of working with a more clearly observable, delimited aspect of teachers' classroom practice while teaching algebraic functions. Creswell (2012, p. 243) referred to this process of analysis as 'horizontalization of data' which involves obtaining "a general sense of the ideas, noting ideas and thinking about the general organisation of the data". Figure 7 below depicts how horizontalisation was carried out in the current study. The horizontalisation of classroom observation transcripts involved labelling and categorising teachers' utterances and observable actions during teaching, and the example of this process is depicted below (and is also found in Appendix 13). This represents the analytic and reflective memos I wrote in the margins of the transcripts as well as highlighting the words and narratives that relate to the teachers' ideology about mathematics teaching, their knowledge of algebraic functions as well as their discourses while teaching the topic.

Figure 7

Horizontalisation of Mafada's lesson 1 transcript



Mafada: We said this one is gonna allow us to draw .... we have got four types of functions, but that one is gonna allow us to draw a straight-line graph. Then, we start again with another type of function (writing on the board) y=x^2

That one represents (he reaches for a textbook to remind himself of the type of function y=x^2 represents), we said that one represents (paging through the textbook), represents a parabola. So, we haven't started with this one. We said the third one is where y is equals to one over x, and what did we say that one? We said that one represents (reaching for a textbook), we said that one represents a hyperbola, we haven't started with the parabola, I said this one will be our third lesson.

Here also, the teacher focuses on the parent function of the parabola, and seem to overlook the parameters a and q for a parabolic function whose general formula is y=ax^2+q.

While it is expected that a textbook is used as a reference tool during teaching, Mafada appeared not to know what type of a function y=x^2 represents. This was evidenced by paging through the textbook before saying that the equation represents a parabolic function.

This phrase sets the lesson objective for the learners, drawing their attention to the topics that have been covered and those yet to be covered.

The teacher's pacing of the topics seem very fast, it appears that the teacher covers one family of algebraic function in one lesson.

This addresses the teacher's Horizon Content Knowledge (HCK)

Following from this process, I organised all the classroom observation data, I chunked lessons into episodes which were determined by a change in the format of the lesson, for example, when a new worked example was introduced as well as by the task set during teaching. I carried this process through for the purpose of developing initial codes which I used to categorise and compare the evidence gathered from different observed lessons for each teacher and across participants. Table 12 is an example of how the lessons were chunked into episodes, with the time stamps to show where episodes begin and where they end. The observable actions entail the teachers' classroom pedagogical actions during teaching.

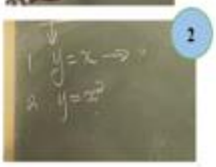
Table 12

Mafada's lesson 2 chunked into a series of episodes

Episode	Time	Observable action
1	00:00 – 01:22	Introducing learners to the general equation for a parabolic function (introduction
2	01:22 – 05:04	Exemplifying the equation $y = ax^2 + q$ with $y = -2x^2 - 3$ to explain the form of the equation in terms of the difference between positive and negative values of a and q (parameters).
3	05:06 – 09:17	**Changing representations** Table Graph Table to Graph (plotting)
4	09:17 – 28:55	Context (parameters): formulae New example in formulae form ($y = 2x^2$) Scaling of the graph Table Calculating and substitutions Plotting the function $y = 2x^2$ on the same set of axes as $y = x^2$ Introducing an example $y = 4x^2$ in equation form (*asking learners to make a conjecture about the increase of a*)
5	28:55 – 45:10	Explaining the effect of the parameter a on the function $y = ax^2 + q$ The effect of the sign of parameter a when it is negative on the function $y = ax^2 + q$ Introducing an example $y = -2x^2$ in equation form Calculating and substituting on the table to represent the function $y = -2x^2$ Plotting the graph of the function $y = -2x^2$ (*asking learners to make a conjecture about the effect of the sign of a on the shape of the graph*) Basic algebra (introducing inequalities) Giving learners a task (two functions $y = -4x^2$ and $y = 4x^2$)

After the lessons were chunked into episodes, I used typological analysis with the aim of developing sets of categories that differ from each other. According to Hatch (2002), typologies are generated from existing theories, common sense or research objectives and research questions for the purpose of reducing ambiguity when classifying research data. An example of this process is displayed in Table 13 below and is also found in the analysis chapters, where I provide thick descriptions of the teaching in each lesson. What was said and done during teaching is used to populate the table, for example, consider the keyword “function”. The iconic visual mediator indicates that the functions are introduced in symbolic form, and the routines indicate the rituals the teacher engaged in to explain different aspects of the topic.

Table 13
An example of typological analysis of Mafada’s lesson

Lesson time taken	Episodes and observable actions	Image of ‘board work’	Visual Mediator (see images in the previous column)	Words and their uses	Endorsed narratives	Routines
00:00 – 10: 07	1, introducing the four families of functions and showing learners what a coordinate is in the form of coordinate pairs (x, y).	 	Symbolic mediators: written functions are: $y = x$; $y = x^2$; $y = \frac{1}{x}$; $y = x^3$ Iconic: sketching a graph to depict a coordinate	Functions; Parabola; Hyperbola; Straight-line Graph; Variable; Coordinates; Dependent variable; Independent variable	Object-level narratives, identifying the different families of functions (Mafada does not identify the key features of mathematical entities pertaining to each family of functions).	Clarifying Ritual to find a coordinate Ritual to sketch a graph

This phase of analysis involved extracting significant teachers’ statements during teaching, as well as classroom observable actions and displaying them in poster charts for the purpose of grouping participants’ evidence with reference to discourses of algebraic functions and approaches of teaching the topic. I used the communicative approach framework (Mortimer & Scott, 2003; Scott et al., 2011) to further make sense of the different teachers’ pedagogical stances as they teach algebraic functions in the classroom. According to Scott at al. (2011), the “concept of communicative approach draws attention to the different pedagogical stances taken by the teacher as they interact with a class of students” (p. 19). This means that communicative approaches refer to the degree of boundary created by teachers on the pattern of interaction with their learners during teaching and learning. The nature of communication between teachers and learners during teaching and learning in the classroom has been categorised into four classes of communicative approaches: interactive/dialogic; non-interactive/dialogic; interactive/authoritative and non-interactive/authoritative (Scott & Mortimer, 2003; Scott et al., 2011). These classes distinguish between the non-interactive and interactive patterned talk during teaching and learning. Table 14 below depicts the four classes of communicative approaches as interactive/dialogic; non-interactive/dialogic; interactive/authoritative and non-interactive/authoritative.

Table 14

Four classes of communicative approaches (Adapted from Scott et al., 2011, p. 19)

Communicative approach	Description
Interactive/dialogic	*Both the teacher and learners collectively explore different points of view*
Non-interactive/dialogic	*The teacher reviews different points of view, highlighting both differences and similarities*
Interactive/authoritative	*The teacher is reviewing and pulling together different ideas presented by the learners to reach a particular point of view*
Non-interactive/authoritative	*The teacher does not allow learners to present their ideas, they present only one specific point of view*

Photograph 3 below demonstrates the representation of the data in poster charts to see the emerging patterns in the teachers' classroom discourses and approaches while teaching algebraic functions. At this stage, I synchronised data from all three methods in order to understand the relationship between what teachers said and did during classroom observations and what they said during semi-structured interviews and VSRI.

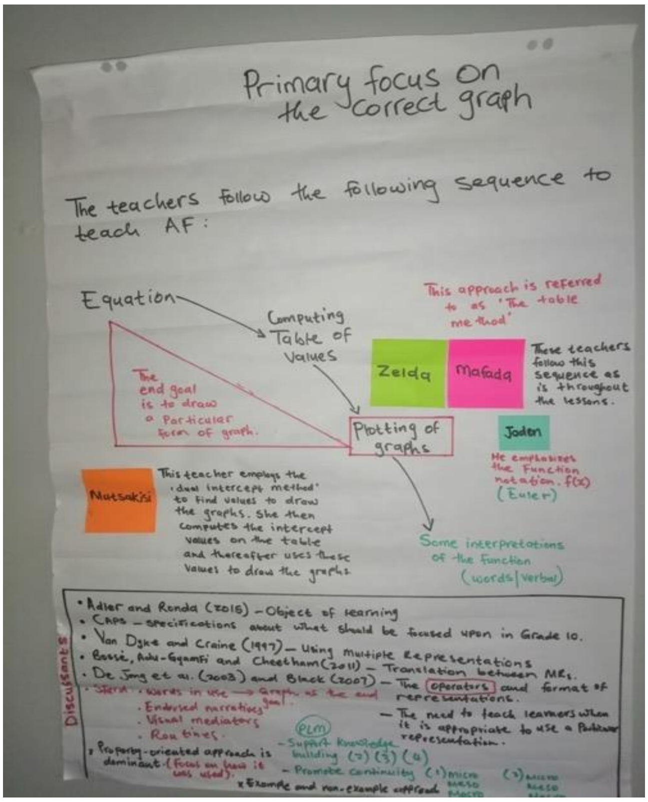


Image 11: The first category for teaching focus

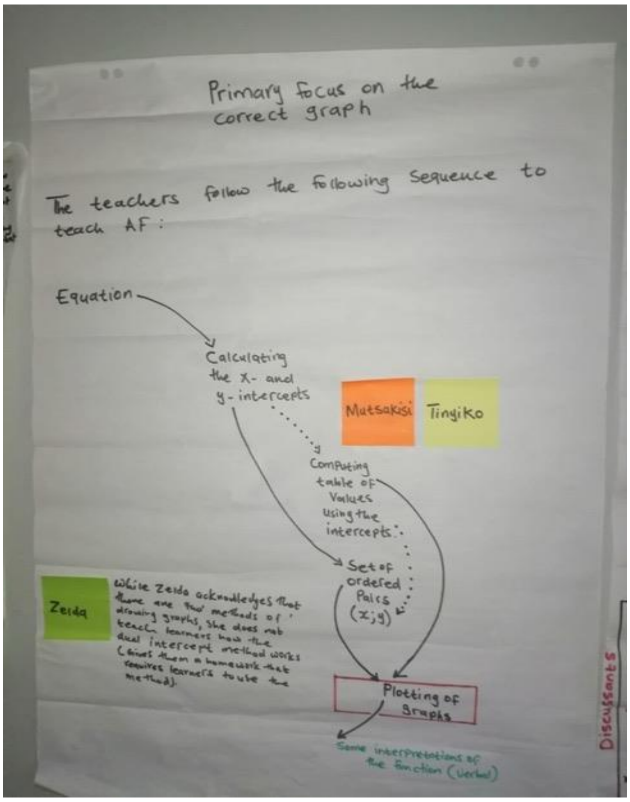


Image 12: The second category for teaching focus

Photograph 3: *Poster charts involving cross analysis of different lessons*

The above process allowed me to decide which elements of the observed lessons addressed the main objectives of the study and answered the research questions and the other information was left uncoded, which is referred to as 'dross' (Morse & Field, 1996) since it does not particularly provide evidence for the study focus. Table 15 below provides an overview of the initial codes emanating from the analysis of classroom observations which was further refined as I continued to analyse the interview and VSRI data as well as re-engaging with and reanalysing the classroom observations.

Table 15

Initial codes from the analysis of classroom observations

Focusing on drawing graphs	Mathematical errors
Mathematical calculations	Worked examples
Changeover between representations	Learner engagement
Functions are introduced in symbolic form	Object of learning
Knowledge of the parameters	Teacher roles
Physical classroom environment	

The following section details the analysis process I engaged in for the semi-structured interview data.

4.7.2.2. Analysis of semi-structured interviews and VSRI

Same as with the analysis of the observations, after the transcription of the individual interviews and VSRIs, I used the horizontalisation process to segment teachers' statements into categories. This involved going through each teacher's interview transcript, making notes in the margin as illustrated in Appendix 13 and selecting the relevant parts from the raw data that best addressed the objectives of the study. To help create a comprehensive picture of the segments of data that linked with the objectives, I used poster charts to represent teachers' statements and their embedded meanings as depicted in Photograph 4 below. Critical Discourse Analysis (CDA) foregrounds the dialectical relationship between society and language, because society is shaped through language and equally language is influenced by society (Fairclough, 1989). This means various objects of knowledge, situations and people's identities are inextricably embedded in discourse, which links with Sfard's argument that communication has unique power to shape our actions and accumulate achievements (Sfard, 2008, p. 55). Considering this, CDA was relevant for the current study because it seeks to uncover and understand teachers' thinking and experiences which are explained through language, and represents mathematics discourses and approaches of teaching algebraic functions. In the current study, I used Fairclough's textual analysis model to analyse transcribed interviews, to explore the relationship between what teachers said, how they said it and why they said it because it was located in the wider social context.

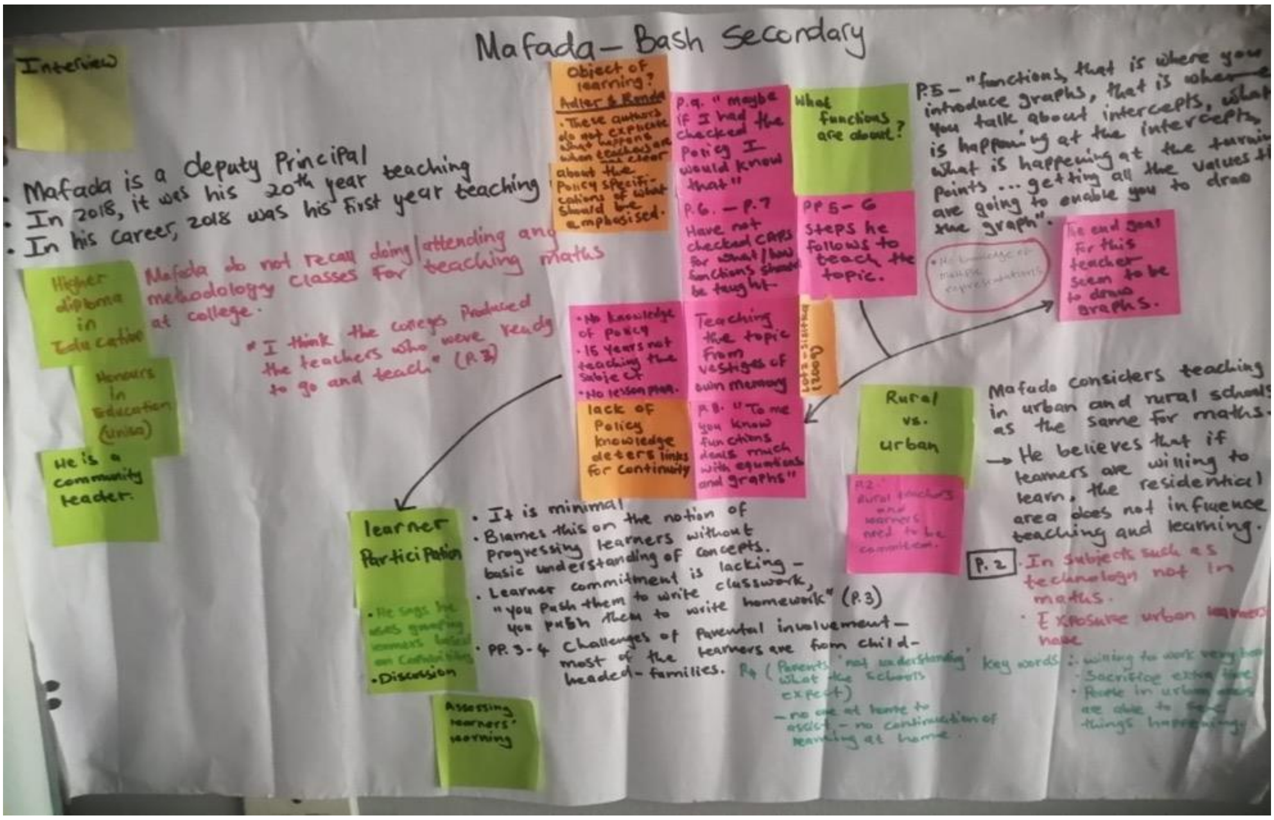


Image 13: An example of segmenting teachers' interview responses

Photograph 4: *Poster chart showing semi-structured interview analysis*

While both the commognitive theory and PLM acknowledges the centrality of social interactions during teaching and learning processes, Fairclough's (1989) CDA model moves beyond to explicitly analyse the language in use and unearth hidden meanings. I specifically used Fairclough's (1989, 1995) three interrelated processes of analysis which are: textual analysis, processing analysis and social analysis. Textual analysis refers to the segmentation and unearthing of the meanings of "the written or spoken language produced in a discursive event" (Fairclough, 1993, p. 138), which was how teachers talked about algebraic functions and its teaching and learning in this study. The processing analysis entails the process through which researchers engage in the "process of production, interpretation, distribution, and consumption. This process is concerned with how people interpret and reproduce or transform texts" (Rogers et al., 2005, p. 371). In this study, I used both textual and processing analysis to analyse meanings from the linguistic features in each teacher's comments from both the semi-structured interview and VSRI and the effect it had on the text in relation to the observed classroom behaviours. In particular, in this study, I was interested in how the teachers represent mathematical objects (by nouns, pronouns and articles), how teachers talk about the activities identified as critical incidences (by verbs) and how teachers represented time in their commentaries (by adverbs). In addition, I used the sentence-level analysis to ask questions about the teaching of mathematics generally, and specifically the teaching of algebraic functions within rural classrooms: What representations of algebraic functions' objects and activities are given significance? What textual action is given significance? When questions about presences and absences surface, what other accounts of mathematics discourses and approaches of algebraic functions are possible?

Furthermore, my classroom observation data analysis was done with Sfard's theory as the lens, Scott et al.'s (2011) (PLM) framework, the six approaches of teaching algebraic functions and Scott and Mortimer's (2003) communicative approach. The theoretical framework used for analysis was detailed in Chapter 3 and the detailed analyses of the lessons is provided in Chapters 5 to 9 of this thesis. However, in this process I not only sought to confirm the components of mathematical discourse as suggested by Sfard, the PLM framework, communicative approach framework and previous research findings, but also remained open to the emergence of findings which were not accounted for by the frameworks and/or differed with the findings from previous studies. This resonates with Robson and McCartan's (2011, p. 468) iteration that in qualitative analysis the researcher is required to maintain "clear thinking" about the phenomenon under scrutiny throughout the analysis processes. In the current study, I interpreted and revealed inherent meanings embedded in teachers' ways of talking about the teaching of algebraic functions in a rural context, before teaching (individual semi-structured interviews) and after teaching (VSRI). For example, the teachers' ways of talking about the district subject advisors unearthed the manner in which the surveillance impacted their classroom practices.

The third dimension, social analysis, focuses on "issues of power – power being a construct that is realised through interdiscursivity and hegemony" (Rogers et al., 2005, p. 371). This dimension focuses on the "signifiers that make up the text, the specific linguistic selections, their juxtapositioning, their sequencing, and layout" (Janks, 2010, p. 1). I used interdiscursive analysis to analyse my description of the rural context and mathematics community, that is, what discourses, styles and genres were drawn on to see whether these were consistent or contradictory. Due to the density of the semi-structured interview data and VSR interview data, sampling within the information provided by the participants was done to primarily focus only on teachers' comments relating to the major initial codes from the analysis of classroom observations presented earlier. I paid careful attention to the selection of words teachers used to talk about the teaching of algebraic functions before and after teaching, because the "choice of language interlocutors make reflects their intentions, ideology, and thought" about the topic and its teaching (Rahimi & Riasati, 2011, p. 107). The participants' responses about their teaching of algebraic functions could provide more information about their perceptions and experiences of teaching mathematics within rural schools as well as revealing factors that shaped their discourses and approaches during teaching, than simply conveying what they said at surface value. Teaming commognitive theory, PLM and Critical Discourse Analysis enabled me to engage in in-depth analysis to unearth hidden meanings from selected extracts from interview texts (Fairclough, 2009; Janks, 2010).

4.8. Ethical considerations

During our interactions with research participants and their environment, ethical issues arise, so as a researcher I must always take heed of the ethical component in research. This suggests that researchers must make ethical considerations of their choices pertaining to their interactions with the selected participants for their studies and be sensitive to how such choices could potentially affect the research participants (Anney, 2014). I was also obliged to avoid acts which could be ethically impermissible or infringe the rights of the teachers who participated in the study. For example, during the data generation process I had to constantly remind myself that relationships and interactions can be driven by 'power[34]' which could hamper my ethical judgements. When having conversations with the teachers and they uttered things that did not resonate with my ontological and epistemological positions, such as saying that some learners are 'stupid', I had to respect the participant(s) and be thoughtful of how to tailor subsequent questions. Accordingly, I made sure that I followed the situated ethics throughout the data collection process. Situated ethics "requires the researcher to reflect on his or her actions and understand what sense they make in each context" (Roulet et al., 2017, p.

[34] Teachers see researchers to be more knowledgeable about the contents of the subject matter because researchers work in higher education institutions. It was important to position myself as researching 'with' teachers instead of researching 'on' teachers.

28). In this section, I have discussed the various ethical considerations that I observed from the conceptualisation stage until the final stages of the study.

4.8.1. Access to the schools and classrooms

In order to gain access to the schools and work with the selected teachers, I applied for ethical clearance from the Mpumalanga Department of Education (MDE) and my proposal was approved (See Appendix 10). I also applied for ethical clearance from the Ethics Committee of the University of the Witwatersrand, School of Education and ethics approval was granted (Appendix 11). Morgans and Allen (2015, p. 1) stated that “The process of gaining ethics approval is an essential step in conducting research for the Ethics committees to assess research in the context of many ethical concepts, and address common ethical issues encountered in research, such as potential participant coercing, informed consent”. In addition to sending ethical application to the above-mentioned institutions, I also wrote a letter to the school principals detailing the nature of the study (Appendix 5). The preamble letter and letter of consent were also given to the selected teachers before the commencement of the study, detailing the purpose of the study and the nature of activities to be done during the study (Appendix 6). The teachers were assured that their classroom routines would not be advantaged or disadvantaged in any way, since the nature of my observation was ‘non-participatory’ and that they could ask me to leave the class at any time without any penalty.

4.8.2. The protection of teachers’ and their schools’ identities

Since participating teachers divulged information that was private and which might not be known by other people, such as their learners, school leaders, and their colleagues, I assured them that throughout all the writings and presentations resulting from this study I would disguise their identities. Babbie and Mouton (2007) asserted that, “The clearest concern in the protection of the subject’s interest and wellbeing is the protection of their identity” (p. 523). Before the study commenced, all teachers were assured of confidentiality by using pseudonyms in this thesis. In relation to watching and analysis of the recorded videos, I only did this in private spaces such as my office, my study room at home and in secured boardrooms when working with my supervisors. This was to ensure that no one else viewed the participants as I continued to watch the videos as I feared that their true identities could be exposed, thereby compromising confidentiality. In addition, the selected five schools were also given pseudonyms in order to conceal their true names, as stated earlier in section 4.6. (Scott & Morrison, 2005). It was equally important to ensure that the individuals who had access to the data (i.e. my supervisor) also maintained confidentiality.

Moreover, I used a personal computer that is password protected for entering and storing all the audiotapes, videotapes and transcripts. Access to all the documents is restricted to only my supervisor and me. The hard copies of data are kept in a lockable locker at my office at the Wits School of Education, and no one else other than me had access to this space for the duration of the study. I assured the participants that papers and other data files would be destroyed after five years, following the end of the study.

4.8.3. Informed consent

As mentioned in the above discussion, I hand delivered the preamble letter and letter of consent to the principals detailing the nature of the study. In addition to this, I also went through the consent form with each of the five participating teachers and written consent was granted before conducting the interviews, observations and video-stimulated interviews (see Appendix 6). This study took the overt ethical research approach rather than the covert, because the covert research design involves a level of deception in that my identity as the researcher would need to be obscured from the teachers (Van Deventer, 2007). On the contrary, the overt ethical research design which was espoused in this study involved the processes through which the researcher openly informed the participants about the nature of the study, the specific reasons for conducting the study and how the information obtained from the participants was going to be used by the researcher and for what purpose (Scott & Morrison, 2005, p. 87). Thus, before I conducted individual interviews, classroom observations, and video-stimulated recall interviews, I informed the teachers about the primary focus of the study and the nature of the methods used to generate data. As part of important information, I made teachers aware that the research was primarily for my doctoral study, however, that the information they provided would also be used for local and international conference presentations and publications of journal articles in order to contribute to the literature on mathematics teaching research in rural contexts.

4.8.4. Right to withdraw

Another ethical consideration I made during course of the study was the participants' right to withdraw their participation in the study. Before conducting the interviews and classroom observations, each participant was informed that their participation in the study was purely voluntary, and that they had a choice to withdraw from participating in the study at any time and for whatever reason without any penalty (Babbie & Mouton, 2007). Thus, only two teachers withdrew their participation, as detailed in section 4.6.

4.9. Methods of ensuring reliability and validity of collected data

This section presents the processes I followed to maximise the trustworthiness of the generated data and findings of the study. Loh (2013) asserted that it is important for a researcher to observe the criteria ensuring the reliability and validity of the findings, as this is a prerequisite "for the research to be accepted into the pantheon of knowledge and to be received as suitable for use in various means and ways" (p. 4). Lincoln and Guba (1985) have four strategies for the establishment and maximisation of trustworthiness in qualitative research: credibility, transferability, dependability and confirmability.

4.9.1. Credibility

Anney (2014) has defined credibility as "the confidence that can be placed in the truth of the research findings" (p. 276). It means the researcher is tasked with ensuring that the representations of the participants and research context, as well as the interpretations of the information provided by the participants is parallel with what they uttered and how they acted during data generation. To maximise the credibility of the current study I used triangulation, semi-structured interviews, classroom observations and video-stimulated recall interviews, prolonged engagement and member checks with the teachers. The use of the three data generation tools study promoted a comprehensive understanding of teachers' discourses and their approaches during algebraic functions lessons, because the data analysis was in depth. I also used member checking to enhance the rigour of the study, which was done in two stages. Firstly, during video-stimulated recall interviews, as described earlier above, and secondly, I went back[35] to the teachers who provided me with the information to verify my interpretations of their responses and actions. This was to check whether the teachers considered that their words and actions during interviews, observations and video-stimulated interviews matched what they meant to utter as well as allowing them to interpret their actions during classroom teaching to check if they correlated with mine.

4.9.2. Transferability

Babbie and Mouton (2007) have defined transferability as "the extent to which the findings can be applied in other contexts or with other respondents" (p. 277). The strategies for ensuring transferability included providing "detailed descriptions of data" and using purposive sampling (Babbie & Mouton, 2007, p. 277). As such, in this thesis, I provided a detailed account of

[35] In August 2018 and August 2019, I conducted member checks with the teachers for the set of transcribed data.

descriptive data, which includes the nature of the schools, the participants, sample size, sample strategy, interview and observation procedures as well as excerpts from both individual semi-structured interviews and VSRIs. In addition to this, I used the purposive research sampling technique because it promotes the variety of information provided by the participants about the teaching of algebraic functions within rural classrooms.

4.9.3. Dependability

In this study, my supervisor acted as an inquiry auditor considering rural education and mathematics expertise. Accordingly, my supervisor examined all the data that I collected, my analysis and interpretations of the raw data as well as the recommendations I suggested in this thesis, to determine whether there were correlations or not and she confirmed that there were connections and coherence. According to Moon et al., (2016, p. 2), "Dependability refers to the consistency and reliability of the research findings and the degree to which research procedures are documented, allowing someone outside the research to follow, audit, and critique the research process". In terms of the documentation of the research procedures, I provided a detailed account of the methodological processes that I had followed from the commencement of the study until the end, to allow the reader to "assess the extent to which appropriate research practices have been followed" (Moon et al., 2016, p. 2). In addition to the above discussion, the research design for the study pursued an audit trail providing detailed and transparent methodological processes. In relevant appendices of this study, I provide access to all processes I engaged in; data generation methods, raw data, analysed data, decisions as well as how I gained access to the schools and participants.

4.9.4. Confirmability

The final strategy to ensure a study's trustworthiness is the confirmability of the findings, which entails the correspondences between the researcher's descriptions and interpretations of the information provided by the participants and the actual utterances and actions that participants made during data generation (Babbie & Mouton, 2007, p. 279). In this study, I used a reflexive journal, audiotaped all the interviews and video recorded all the classroom observations in order for my supervisor to listen to the audio clips and watch the videos to ensure intersubjectivity between my interpretations of data and the actual teachers' statements and actions during interviews and observations respectively.

For the period spent in the five schools, I kept a reflexive journal which detailed all the events that took place in the research field and included the 'ah' events that arose during data generation. In the reflective journal, I transparently described "the records of the research path" (Korstjens & Moser, 2018, p. 121) that I undertook to address the purpose of the study. In addition to keeping the reflective journal, the use of an audiotape during interviews helped me to listen to the clips multiple times to ensure that the transcripts corresponded with the exact information provided by the participants, which ensured that I did not write "figments" (Korstjens & Moser, 2018, p. 279) of my own imagination as the researcher.

4.10. Chapter summary

In this chapter, I described in depth the process and methods that I used to collect, sort and analyse the qualitative data I gathered by means of semi-structured individual interviews, non-participant classroom observations and VSRIs. This description included explanations and justifications for the choices of research design, research approach, research methodology, selection of participants and data analysis techniques and processes for this study. This chapter also detailed the challenges that I encountered during the process of data generation, which shaped the nature of data generated for the study. I also discussed the ethical considerations for this study and how I maximised the quality and trustworthiness of the research findings.

Chapter 5

The imagined, hidden, the seen and the heard:

Data presentation and analysis

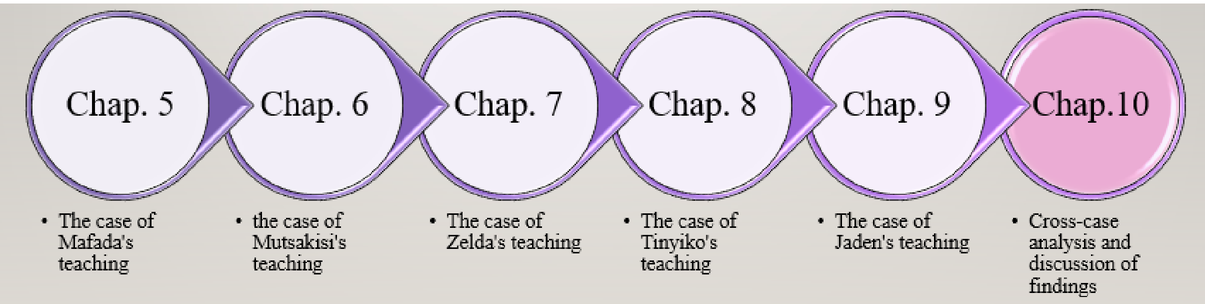
To be interested in equity and social justice but not in rural education research is social injustice (absurd) ~ my own statement

5.1. Introducing the data analysis chapters

The current chapter and the next four chapters provide a comprehensive presentation, analysis and interpretations of the five teachers' cases of the algebraic functions lessons in Grade 10 rural mathematics classrooms. The cases are presented individually and relationally, to ensure that teachers' significant teaching episodes are analysed and interpreted in detail. Chapters 5-9 present data analysis of Mafada's, Mutsakisi's, Zelda's, Tinyiko's and Jaden's teaching, respectively. Figure 8 below illustrates the chapter map and the structure of the five chapters.

Figure 8

Data presentation, analysis and findings chapters' map



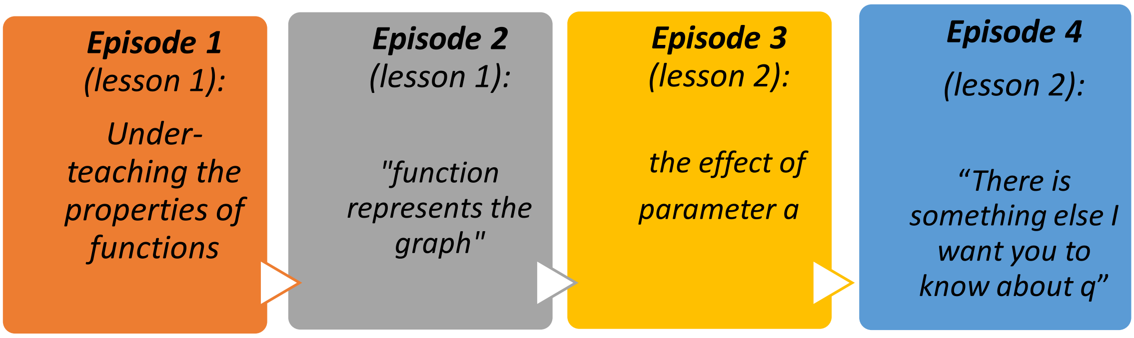
In these chapters, episodes extracted from different observed lessons are presented in separate sub-sections in each teacher's case analysis, to show the nuances within and across lessons. An episode is defined as a non-structured event extracted from the three different methods of data generation, to capture teachers' classroom practices and the thinking about the effectiveness or lack thereof of such practices. I also addressed the complexities of teachers' discourses and approaches during the algebraic functions lessons. Of importance to note is that classroom observations and video-stimulated recall interviews were dominant in the current study, and semi-structured interviews were used for the teachers' biographical information and their experiences of teaching algebraic functions at Grade 10. In the presentation of all cases, I use images for demonstration and for the readers to understand the practices that happened during algebraic functions lessons. The first case is Mafada's episodes, the reflective comments from VSRI, and relevant responses during semi-structured individual interviews.

5.2. Data presentation and analysis - The case of Mafada's teaching

The four episodes were selected from Mafada's teaching sequence of two lessons. The episodes were selected based on the relevance of information that address the research questions. Figure 9 below represents the selected episodes from Mafada's teaching and are interpreted and discussed.

Figure 9

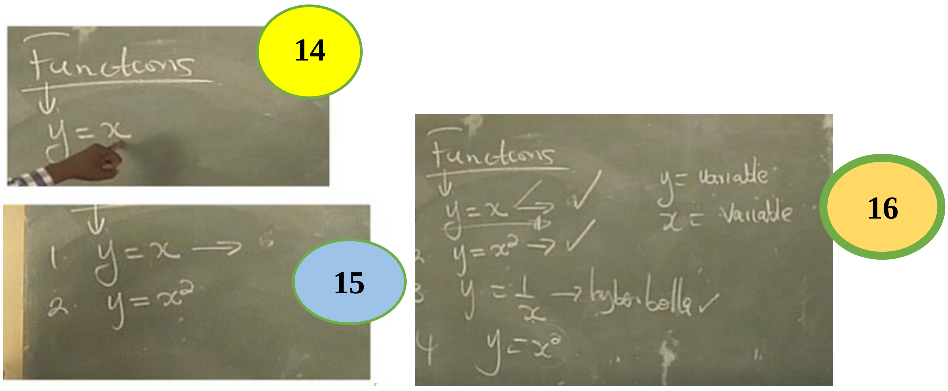
Mafada's selected episodes from two lessons



5.2.1. *Episode 1 (lesson 1): Under-teaching the properties of functions*

Episode 1 was extracted from the first lesson which introduced the concept of parabolic functions (Appendix 1). Mafada introduced the lessons by informing learners about the structure of algebraic functions, foregrounding the linear function, parabolic function, hyperbolic function and exponential function, as the four families of function in Grade 10. He wrote each representation (excerpt and images 1 to 3 below) on the chalkboard while talking, which represents the object of learning that entails the starting point of a lesson for the current lesson (Adler & Ronda, 2015; Adler, 2017).

> *As covered in the last lesson, the equation* $y = x$ *is a function because there is only one value of* x*, we said this one is gonna allow us to draw ... we have got four types of functions, but that one* (image 1) *is gonna allow us to draw a straight-line graph. Then, we start again with another type of function (writing on the board)* $y = x^2$.



Images 14 to 16: *Mafada's board work (lesson introduction)*

The writing of the four different equations (image 16) resonates with the property-oriented approach to teaching the concept of algebraic functions, and making links between mathematical concepts, which draws learners' attention to the differences in the structural appearance of the equations that represent the four families of algebraic functions mentioned above. While Mafada wrote and verbalised the four equations, as evidenced by the excerpt earlier above, he did not provide the descriptions of the different properties for each function to highlight the differences between the functions. In particular, Mafada did not explain the differences between linear functions that were covered in the previous lesson, and the parabolic functions that was the object of the current lesson. It is important for the teacher to make links between the previous and the current lesson, so that learners do not see mathematics lessons as a series of disconnected events. In addition, Mafada did not use mathematical words and narratives to discern the key properties of different functions, especially the effect of different parameters on the dissimilar families of functions. As mentioned in Chapter 2, clear understanding of the nature of parameters for different families of functions enables learners to observe what changes in given relationships and how the changes occur, to understand both the displacement and orientation of functions (Mudaly & Mpofu, 2019). Authors (Nemirowsky & Rubin, 1992; Monk & Nemirowsky, 1994) have posited that an understanding of the different properties for different families of functions can enable learners to classify different functions.

Mafada wrote 'functions' as the object of learning (images 14 and 16 above) instead of specifically focusing on parabolic functions as the current lesson, because he had already introduced the topic and one family of functions in the previous lesson. The words "*Then, we start again*" in the above quotation demonstrates that Mafada is done with the linear function in the form y = x, and is now moving to a new sub-topic of parabolic functions. There was an attempt to develop a scientific story to promote continuity using meso scale (Scott et al., 2011), even though Mafada did not review events from the lesson(s) he taught on linear functions to set the scene for parabolic functions. To prevent learners' confusion, it is expected that teachers make the object of learning clearer, which is parabolic functions, and Mafada focused on doing mathematics related to the topic on the board, with the end goal of plotting the graph of the parabola $y = x^2$. There was no coherent continuation with the topic to help learners link the previous with the current knowledge and develop the understanding of the topic early, as an important aspect in mathematics concept building.

In order to understand the intended object of learning for the lesson, I asked Mafada about the purpose of lesson 1 during VSRI:

> *The purpose of the lesson was to show progression from straight-line graph, to err parabolic graphs, so the procedures like I indicated earlier on, that we need first to have a table as a similarity from straight-line graphs, then we came up with the table, then we plotted the graph, we plotted the points and joined the graph. So here was for them to see the different shapes of the graphs.*

This response from VSRI suggests that for Mafada, the teaching of functions focused on developing learners' procedural fluency to draw graphs by means of following a series of steps. The algebraic formulae and table of values were tools to draw the graphs of functions, rather than translating between the representations as discussed in Chapter 2. This information does not ignore the importance of teaching the drawing of graphs, the concern is the overlooked idea of translating between the different forms of representations that discern the mathematical meanings and help learners develop functional thinking. The above extract represents the teacher's understanding of how the topic should be taught, drawing of the graphs, and the limited knowledge of mathematical representations and their role in the teaching and learning of algebraic functions. The reiteration during the semi-structured individual interview testifies to the above interpretation:

> *… in functions that is where you introduce your graphs, that is where you talk about the intercepts, what is happening at the turning points, the difference between intercepts and turning point, you know, getting all the values that are going to enable you to plot the graph.* (Mafada, Interview)

While in this statement Mafada identified two keywords related to parabolic functions: turning point and intercepts, he did not use the words or engage in mathematical processes during the lesson to demonstrate their significance in understanding what parabolic functions are about. Instead, he used the graphical approach in which he used the equation and table of values to draw the graphs, without using intercepts and turning points to engage in the action of interpreting the key features for parabolic functions. Since Mafada did not introduce the function using the table of values, and the graphical representation, I argue that he overlooked that these forms of representations are visual mediators and depict particular relationships. That is, Mafada did not teach learners the idea that each form of representation, for example, a table of values describes how values of one variable are determined by the values of another.

Another important aspect I observed during this lesson was Mafada's insufficient knowledge about the type of algebraic function equation $\mathrm{y} = \mathrm{x}^2$ (image 16), as evidenced by perusing through the textbook for about 43 seconds before stating that the equation represents a parabolic function. While this might also represent the lack of lesson planning to ensure that the teacher coherently and effectively organises the content in the lesson, it further addresses the importance of appropriate mathematical content knowledge. According to Mohammad (2019, p. 25), "Not only is it important that teachers have a solid understanding of the material they teach, it is also important for them to research the topic to find misconceptions and weaknesses before teaching their students". The lack of lesson planning means that Mafada missed the opportunity to enhance his knowledge of the topic by engaging with other textbooks, to see how else to teach the topic. Considering that Mafada has not been teaching mathematics for 20 years, he used the vestiges of his knowledge, irrespective of the changed curriculum. The use of one's knowledge clashed with the knowledge in the current curriculum (CAPS), which further addresses the importance of teacher's knowledge with the policy and lesson planning for coherence teaching and learning. Thus, Mafada did not create opportunities for the learners to explore the changes that are brought by the effect of different parameters for different functions.

From the four equations he wrote on the board (image 16), they all focus primarily on the parent functions of each of the families of functions, rather than on the general equations for each. Instead of writing the equation of the linear function as $y = ax + q$, for instance, he wrote $y = q$ (image 16), and instead of writing the equation of the parabolic function as $y = ax^2 + q$ he wrote $\mathrm{y} = \mathrm{x}^2$. While it is not wrong that the equation $\mathrm{y} = \mathrm{x}^2$ represents the parabolic function, however, Mafada overlooked the importance of introducing learners to the effect of parameters a and q in a parabolic function. According to the Curriculum Assessment and Policy Statement (CAPS) (DBE, 2011, p. 24), teachers should help learners "Investigate the effect of a and q on the graphs defined by $y = a.f(x) + q$". Mafada's focus on specific examples and not explicating the general equations of the functions for learners to see the structural appearance of the parabola, represents the expected curriculum knowledge gap for the different families of functions. This is concerning, when the following statement from a VSRI is seriously considered: "*Remember last week I said I did not look at CAPS … check the policy so that I can also see what the government require me to cover"* hence the teaching from the vestiges of own memory. While the curriculum is a guide for teachers, the above response illustrates that Mafada did not check the content specifications from the curriculum document, to understand the delimitations for the function topic in Grade 10. Thus, Mafada's actions are interpreted as one of the factors that shaped the teaching of the lesson.

The teacher further wrote the equation of the fourth family of functions, exponential functions, wrong even after using the textbook, which affected the effective and comprehensive teaching of the topic (see the fourth equation in image 16). The fourth function was supposed to be the equation of the exponential function as evidenced by his utterance "*The fourth one is gonna be err, an exponential function neh*[36] (checking the textbook)*, about the exponential function, what did we talk about?* (paging through the textbook) *We said y is equals to x squared. So, this one is gonna be our exponential function, for our fourth lesson*". This excerpt illustrates that Mafada repeated the equation of the parabolic function $y = x^2$ to be an algebraic representation of an exponential function, resulting in a mismatch between word use and visual mediator written on the board. The teacher's mathematical incorrectness cannot be taken for granted because learners write and learn mathematical words and their meanings, as well as the visual mediators as teachers use them during the lesson. I therefore argue that the teacher's practices indicate insufficient knowledge of the topic, as he used the textbook to confirm wrong information.

From the VSRI I realised that the absence in mathematics teaching influenced the teacher's actions during the lessons, as the following responses illustrate:

> *I am teaching this subject for the first time after 20 years, so I am teaching the learners the calculation skills … the learners will know how to find values of y and steps to draw the graphs, that is mostly examined.*

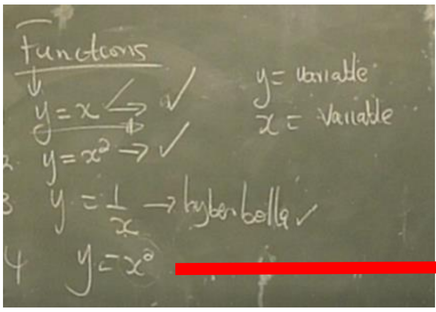
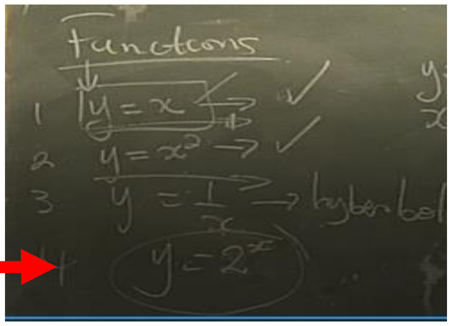
While the teacher was honest, this response highlights the continuing challenges of insufficient proficient mathematics teachers in South Africa, which has an impact on learners' understanding of mathematics knowledge and performance. Figure 10 below shows the changes that Mafada made after reflecting on what he had written on the board. This event addresses the need for teachers to ensure that they use correct mathematical expressions during

[36] 'Neh' is the word that Mafada uses to emphasise particular contents to the learners.

the lessons, in particular when the approach is teacher centred. Sapire et al. (2016, p. 13) stated that, “Precision in mathematical expression (both verbal and written) is vital since it supports a deeper understanding of the concepts under discussion” Thus, Mafada’s imprecise written and verbal mathematical language “provides further evidence *that he is* not coping with cognitive load”, and this is the impediment of effective communication during teaching (Sapiere et al., 2016, p. 13, italics added).

Figure 10

Changes made on what is written during the lesson

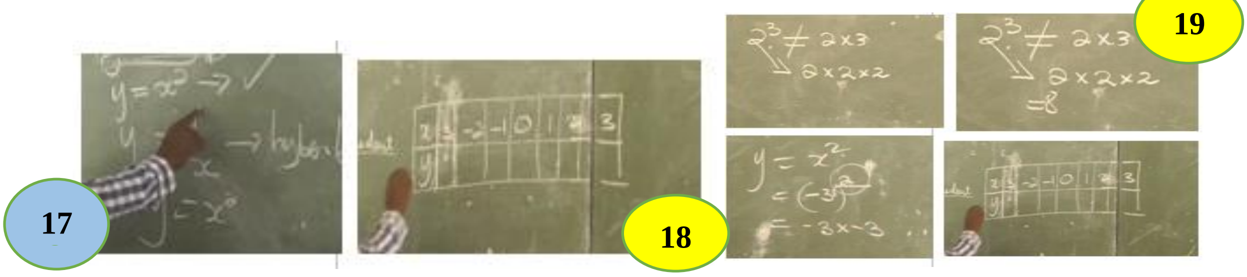
The first equations written	Changes of number 4 after reflecting
	

When I probed Mafada during the VSRI about the mistakes during the lesson, he said, *“Well I browsed quickly for that lesson because I did not prepare for it”*, irrespective of the curricular changes and absence from teaching mathematics. The purpose of lesson planning is to transform, structure and ensure sequencing of the contents, to help teachers become more effective during teaching, by providing a detailed outline to follow in the classroom. Nesari and Heidari (2014) defined a lesson plan as “a unit in which it is a sequence of correlated lessons around a particular theme or it can be specified as a systematic record of a teacher’s thoughts about what will be covered during a lesson” (p. 25). Lesson planning assists teachers to not only rely on the vestiges of their memory. As Lotz-Sisitka (2009) posited, vestiges of teacher memory are “linked to the past curriculum or their own life world experience” (p. 63). It then means that Mafada relied on the teaching of the topic from Curriculum 2005 (C2005), and the way he was taught in the teacher training college. Below I present episode 2 which continues from episode 1 the same day.

5.2.2. Episode 2 (lesson 1): ‘… function represents the graph’

In this episode, Mafada continued from the previous episode, and introduced the parabolic function in the form $y = x^2$ and changing representations for the function. He said: “*We are* *going to this second function, we are going to do the same like we did with this one (pointing at* $y = x$*), so let us continue from that one”* (pointing at the table drawn in image 18 below). The intention of the above statement was to develop a scientific story by referring back to the routine he introduced on linear functions. However, Mafada merely mentioned what he did previously without making explicit content linkages, to demonstrate the continuation of the content. Scott et al. (2011, p. 33) has argued that “the teacher should make the link available for the students to understand the links for themselves as they actively make connections between their own ideas”. Accordingly, the teaching of algebraic functions relies on the teacher’s expertise to guide learners’ understanding of the current topics, in relation to other subject matter that was addressed at other times on micro, meso or macro timescales, which Mafada overlooked. Mafada continued by pointing at the equation $y = x^2$ (image 17) accompanied by the following utterance: “*That’s an equation, it’s a function, that function represents the graph; let us now see the shape that this function is giving us*” to bring parabolic functions into focus. This statement is not endorsed mathematically because graphs are representations of functions, and an algebraic equation he was referring to as representing the graph is an equivalent form of representation for function (Van Dyke et al., 1997).

As mentioned earlier, I argue that Mafada’s classroom talk was a result of conceiving the end goal of teaching the topic as drawing the graphs, which results in viewing the algebraic equation as a mathematical object to be used for utility in the process of drawing graphs. Kalchman and Koedinger (2005) asserted that effective teaching of algebraic functions “is not just about developing students’ facility with performing various procedures, such as finding the value of *y* given *x* or creating a graph given an equation” (p. 353). It is also important to ensure that learners understand that tables, formulae and graphs are depictive and texts are descriptive of the mathematical relationships represented. The images below demonstrate what Mafada wrote on the board when he introduced the parabolic functions, representing the shift from the symbolic representation to the table of values.



Images 17, 18 and 19: *Introducing the parabola* $y = x^2$

In images 18 and 19, Mafada's aim was to show learners how to assign values of the independent variable on the table of values, and highlight the mathematical convention of input values versus output values through the use of an example vs non-example approach. He demonstrated to the learners the change in representations, from an algebraic representation of the parabolic function $y = x^2$ to the table of values. The shift from the algebraic form to the tabular represents the substitution of specific values of *x* into the equation to generate the corresponding values of *y* in the table (image 18). During the explanation process, Mafada did not mention to the learners that a table of values depicted a relationship between variables *x* and *y*, to ensure that they develop an understanding that each form of representation signifies particular features of the functions concept. Of interest is that the equivalence between the table of values to other representations that were used during the lesson were not evident, considering that features of the table such as (0; 0) represents both an *x*-intercept and *y*-intercept or the *x* and *y* values increases were not discussed. Accordingly, the table of values was not used as a signifier of a function, which could result in learners viewing the table of values as content under the functions concept rather than a form of representation that reveals particular features of different families of functions.

Instead, the dominant routine in this episode showed Mafada using a table of values exclusively to draw a graph. I observed this view of algebraic functions during the lesson as shown in image 20 below.

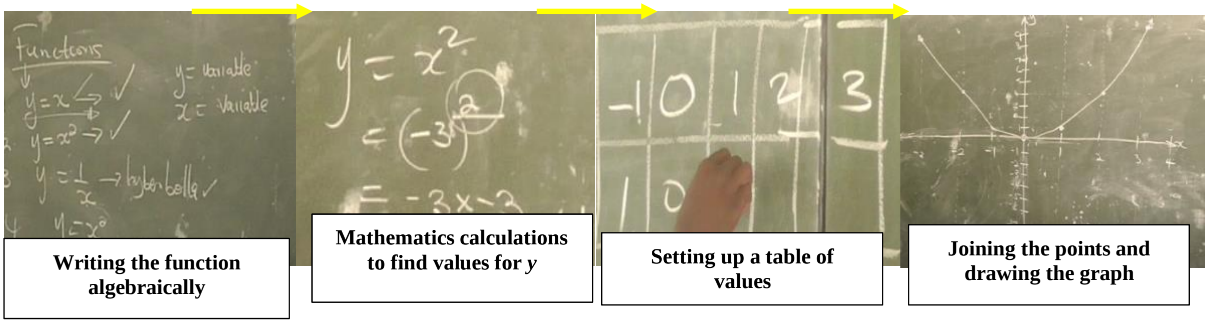


Image 20: *Steps taken to draw the graph*

Image 20 shows that for Mafada the graphs functioned as a mnemonic, that is, as a source of association with what needed to be done or said about a given function rather than treating the different forms of representations as information equivalent in discerning key properties of the concept. The arrows demonstrate that Mafada changed representations in a linear manner without illustrating the notion of flexible translation between them, and without the explanation of the features of function for each representation. Explanation in mathematics is important as a tool that is used by teachers for ‘giving a sense’ to the specific object of communication in order to manifest comprehension.

Considering the information, I argue that Mafada viewed the expression x^2 as merely producing a result of calculating, resulting in seeing the function $y = x^2$ as a recipe to apply to numbers, then remaining unchanged across numbers. Given the equation $y = x^2$; Mafada seemed to interpret the equation as a formula used to determine an answer for a specific *x*-value by squaring the number. This is concerning for Oehrtman et al. (2008, p. 157) because when learners are taught how to compute the input and output pairs one at a time, as Mafada did, their understanding could be “limited to understanding only the related procedural tasks” such as solving for y, overlooking the notion of covariance between two variables. Mafada could have used the covariational approach as another approach to assist with teaching the relationship between the variables. In commognitive terms, the creating of the object square (*x)* is fundamental to the project of advancing, and ensuring communicational effectiveness of mathematics discourse during teaching (Sfard, 2012).

To continue with the lesson, Mafada asked learners: “*So mo nimi vutisa, why hiku this is a function? (So, let me ask you, why do we say this is a function?)* (pointing at $y = x^2$*). Why hiku i function leyi? (Why do we say this is a function?)*”. Instead of waiting for learners’ response(s), Mafada went on to expound to the learners why $y = x^2$ is a function. He said:

> *For every x value* (he writes on the board), *we have only one y-value, that’s why we say it’s a function. Waswi vona nkarhi wun’wana (you see sometimes) … we deal with equations; we deal with different equations. Perhaps, two x values aniri (isn’t), in the equation. So lani (here), for every x value we have only y value, so* (he looks at the camera anxiously) *that’s why we say that’s our function.*

Mafada did not provide a learning opportunity for learners to verbalise their understanding of the contents they have just learned, to understand the learners’ “points of view raised in earlier lessons and further developing them” (Scott et al., 2011, p. 14). This, according to Scott et al., represents the non-interactive/authoritative communicative approach, because Mafada did not encourage the learners to articulate their points of view about the concept. In addition, Mafada did not create an environment where he could collect learners’ ideas to check for understanding. Instead, Mafada explained to the learners that the shape of the parabolic graph is not always going to give the shape given by the function $y = x^2$, which did not enable learners’ exploration of what brings about the changes in the shapes of the graphs. The following section focuses on the analysis and interpretations of episode 3 from lesson 2.

5.2.3. Episode 3 (lesson 2): ***The effect of parameter a***

In the lesson represented by episode 3, Mafada started the lesson by moving around the classroom, monitoring and evaluating learners' responses to written work that he had given as homework in the previous lesson. In so doing, he identified errors that some learners had made when they used the equation $y = -4x^2$ to substitute and calculate the corresponding *y*-values for given *x*-values on the table of values, and drawing the graph of the function. Mafada used the non-interactive/authoritative communicational approach to engage with the error that learners made, and the explanation is presented below:

> *The mistake that I have realised is that we have minus four x squared* (see image21), *I don't know how you have done it because you ended up getting positive sixteen. Remember what we said here, it is minus four times x squared, so, but when we multiply we don't have to put the multiplication sign, but this is minus four multiplied by x squared. So now, when we substitute after assigning the values of x, when we substitute you must first substitute there to say minus four multiplied by minus 2, this is the number we have assigned and this becomes minus four multiplied by, this one is gonna be negative 2 multiplied by negative 2 which is positive four. Aniri (isn't) we know when we multiply those signs, negative and negative we get positive. So, this becomes multiplied by four and the answer becomes negative times the negative that is here, you are going to get a negative number, then you say four multiplied by four and you get sixteen. When you get positive sixteen there, obviously it is going to affect your graph because the point is now opposite.*

This utterance illustrates that Mafada used an authoritative position to explain the process of squaring numbers, without creating a dialogic space for learners to demonstrate their engagement and understanding, or lack therefore, with the homework task. The lack of dialogic communication was concerning, especially since he had already taught learners how to engage with similar mathematical calculations the previous day. Image 21 below depicts the mathematical calculations that Mafada did on the board to address learners' calculation errors that he identified while checking learners' workings.

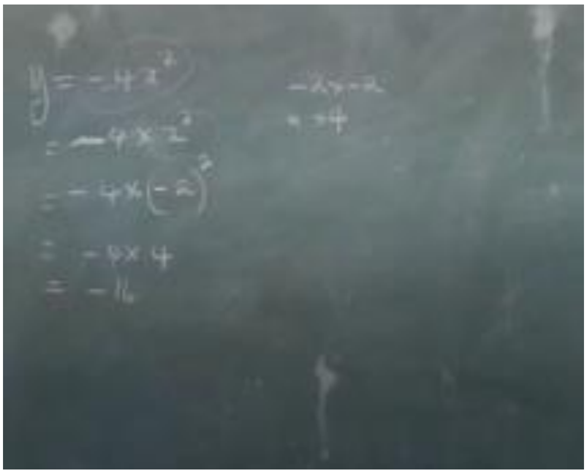


Image 21: *Mathematical calculations for squaring numbers*

As visible in image 21, Mafada used the example versus non-example approach to mediate learners' learning of the mathematical substitution and calculations symbolically, pertaining to the product of a squared number and a negative number. The role of the example versus non-example approach was to allow learners to see the archetype on how to compute mathematical calculations. Thus, writing the equation $\mathrm{y} = \text{-}4\mathrm{x}^2$ as $\mathrm{y} = \text{-}4 \times \mathrm{x}^2$ acted as a symbolic mediator for learners to see that the coefficient denotes multiplication as well as the order of operations in which the learners should carry algebraic manipulations by first squaring the substituted value then multiplying by the coefficient. From the commognitive theoretical position, the explanation that $y = -4x^2$ can be written as $\mathrm{y} = \text{-}4 \times \mathrm{x}^2$ is illustrative that Mafada's teaching is at the object level.

Tabach and Nachlieli (2016, p. 302) stated that object level learning and teaching "involves expanding the existing discourse of the participants, that is, getting to know better existing mathematical objects". It can be said that the demonstration of the calculation process on the board created opportunities to increase the germane load of the learners through setting up the stage to follow up on the committed errors, especially when having a negative coefficient. In relation to this event, Sapire et al. (2016) noted, "Teachers come across errors not only in tests but also in their mathematics classrooms virtually every day. When they respond to learners' errors in their classrooms, during or after teaching, teachers are actively carrying out formative assessment" (p. 1). Mafada identified learners' errors the following day, because of homework, and addressed their errors using the non-dialogic/authoritative approach. I argue that this approach constrained the opportunities for learners to develop a mathematical story, and promote deep knowledge and understanding.

I noticed that Mafada's correctness of mathematical calculations continued to be influenced by the production of the correct graph. The following object-level narrative accompanied his explanation to ensure the correct calculations: "*When you get +16, obviously it's gonna affect the graph ... you will end up getting the wrong table which will lead you to having the wrong graph*". The statement "*you will end up getting the wrong table which will lead you to having the wrong graph*" in the above extract resonates with the steps underpinning Mafada's ritualistic discourse and example versus non-example approach of teaching algebraic functions discerned earlier in episode 1. The exclusive use of algebraic equations and table of values for utility in the drawing of graphs was evident in his statement during VSRI:

No, what I did was for me to go and write that equation. Err, after writing the equation, you start explaining to say, on this equation, you have got two unknowns for now, let's say for example, it is y is equals to 2x, x is not known, the y is not known, but then, what you need to find first, or what you need to assume, it will be x because y is the subject of the formula, x you can assign, maybe any value between negative 2 and positive 2, in a tabular form. Then you start to substitute those values that you have assigned, assumed for x and now you are getting the y. Then you take all the points that are in the table, those ones are not graph, but they are just points, then you need to join all those points to make a graph. Then after having a graph, if ever somebody will ask, then we will be able to tell, we will be able to talk because everything is on the graph.

The response indicates that, for Mafada, the ultimate goal of working with different functions is to draw the graphs of such functions, and for him the graphical representation enables learners to interpret the functions as demonstrated by the words *"we will be able to talk*". Thus, it is the perception that the graph is the definitive modality of representation for functions that influenced Mafada's use of rituals to show learners the rote steps to draw graphs, overlooking the essence of multiplicity of representations of the concept.

To continue managing and organising the content in this episode, Mafada used two approaches to teaching algebraic functions: the example versus non-example approach and property-oriented approach to offer an explanation about the effect of increasing and decreasing the values of parameter *a* on parabolic functions. He used the prototypical examples $y = x^2$, $y = 2x^2$, $y = \frac{1}{4}x^2$ as start-up examples to bring the effect of *a*, which is one of the properties of parabolic functions, into focus. Consider the following excerpt:

So we have seen from the example that on this one, our graph cuts through the origin, it was like that, hatwanana? (am I clear?). But when we increased the value there, our graph became like that, hatwanana? (am I clear?) It was closing to the y-axis. And we concluded that because when we make it two we are increasing. And we said if we were to decrease it and perhaps make it $y = \frac{1}{4}x^2$*, we made our a to be a quarter, meaning you can have quarter, you can have one, you can have 1 or 2, meaning that these numbers going that side are increasing* (drawing an arrow from left to right, see image 22 below). *So, it shows that* $\frac{1}{4}$ *is less than* $\frac{1}{2}$*, hatwanana? (am I clear?) So, and one over two is less than one and one is less than two. So now, we assume that if we make our a to be half, it means we have decreased it from one neh. So it means that our graph is gonna open much and gets closer to the x-axis, that can be one over four, that can be one, that can be two. As a increases your graph is closing, it's coming closer to the y-axis. So, that's one of the things that we investigated about a, about how does a affect your graph.*

The extract illustrates that Mafada introduced three functions ($y = x^2$, $y = 2x^2$ and $y = \frac{1}{4}x^2$) and compared the functions in terms of the effect of parameter *a* on the behaviour of the functions (image 22). Image 22 below depicts what Mafada wrote on the board as he continued with his explanations.

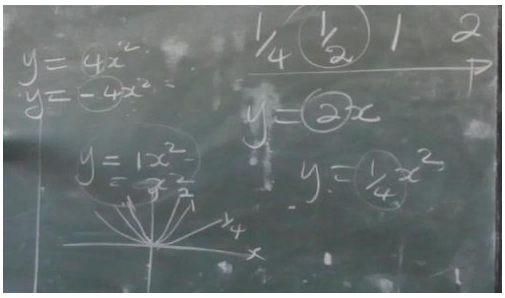


Image 22: *Demonstration of effect of parameter a*

The analysis of the above extract alongside the iconic and symbolic mediators in image 22 indicates that Mafada used the three examples to endorse a narrative about the effect of parameter *a* on parabolic functions. The endorsed narrative was represented by this part of the explanation:

> "*So now, we assume that if we make our a to be half, it means we have decreased it from one neh. So it means that our graph is gonna open much and gets closer to the x-axis, that can be one over four, that can be one, that can be two, as a increases your graph is closing, it's coming closer to the y-axis*".

The explanation created an opportunity for the teacher to bring the critical feature of parabolic functions into focus, and created a learning opportunity that guided learners towards generality about the effect of parameter *a* on the behaviour of the function. According to Moeti (2015, p. 61), during the teaching of quadratic functions, the sequencing of examples "moves from a parent function $\mathrm{f(x) = x^2}$ where simple example is taken to complex ones". The arrow that Mafada drew across the numbers in image 22 acted as an iconic visual mediator for learners to see and make sense of the increase versus decrease concepts, to associate the numbers, the words and the endorsed narratives about parameter *a*.

While the variation in the value of *a* (1, 2 and $\frac{1}{4}$) in three functions could be viewed as increasing the level of complexity and understanding of vertical stretch, Mafada did not engage learners in explorative routines to understand what varying values of parameter *a* changes on the behaviour of the functions. This is to help learners move towards generality about the effect of parameters, as one of the core curriculum standards in CAPS (DBE, 2011). It could be argued that the graphical sketch in image 22 acted as an iconic visual mediator about the effect of parameter *a*. However, the lack of the accompanying process of sketching and drawing the actual graphs to allow the learners to see exactly what is changing, limited opportunities for knowledge building. The current episode focused on Mafada's discourses and approaches during the lesson on working with parameter *a*; the following episode presents the teaching about the effect of parameter q on parabolic functions.

5.2.4. *Episode 4 (lesson 2):* "*There is something else I want you to know about q*"

This episode introduced a new property of parabolic functions: the effect of parameter q, which continues from the previous episode. Mafada used the property-oriented approach to introduce one of the properties (effect of parameter *q*) of functions, which is associated with the world of changes that the parameter brings to parabolic functions. The overall aim of the episode was to develop a mathematical story about the effect of parameter q on parabolas, as well as introducing learners to the dual intercept method.

The episode began with the teacher writing the function $y = x^2 - 1$ on the board (image 23), and asked learners to identify the values of a and q for the function. There was a change of approach in this episode, as Mafada used the interactive/authoritative communicative approach to draw learners' attention to the effect of changing the values of parameter q. The nature of interaction between the teacher and learners involved the process whereby the teacher asked questions and learners gave short answers rather than engaging in comprehensive conversations about mathematical objects. Consider the extract below in relation to images 23, 24 and 25.

> 1 Mafada: *I want you to tell me there, what is the value of a and what is the value of q? In that function, what is the value of a, what is the value of q?* (no response for about 15 seconds) *Huh! the value of a?*
> 4 Learners*: Is one.* (chorusing response)
> 5 Mafada: *Is one! One! And the value of q? It's?*
> 6 **Learner**: *It's negative one*
> 7 **Mafada**: *Negative one!*

This exchange is illustrative of a distinctive initiation-response-feedback pattern of communicative approach, which is used to frame the concept for learners: the learners respond and the teacher provides formative feedback. Although this approach has been criticised as being more about learners stating what the teacher expects to hear than really communicating, it can provide a useful framework for creating meaningful communication about concepts in a controlled form (Jaeger & Adair, 2019). From the above exchange, Mafada evaluated learners' understanding of the values of a and q in terms of structural appearance and the correct identification of the values of the parameters was symbolically mediated by the general algebraic equation for parabolic functions in image 24. It acted as a prototype for learners to structurally associate the values of a and q in $y = x^2 - 1$ in relation to their positions in the general equation, thereby facilitating the identification of the values with ease (image 24).

Image 23: *Calling for identification of a and q.*

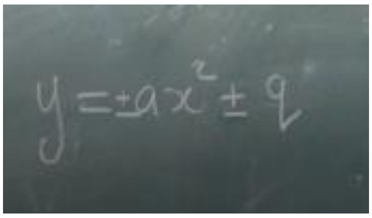


Image 24: *General equation of parabolic functions.*

Image 25: *Values of a and q identified*

Mafada continued to support learners' knowledge building by making links between modes of representations, using the prototypic symbolic mediator (the general equation in image 24) and the specific example of parabolic function (image 25). The pedagogical activity of writing the general equation of parabolic functions mediated learners' structural association and mapping between the two equations, thereby enabling them to correctly identify the values for the two parameters (Sfard, 2012; Scott et al., 2011). In this activity, it was great to observe the interplay between verbal and visual symbolic language Mafada used in the ongoing dialogue, to identify the parameters of parabolic functions. Thus, the introduction of the general equation for parabolic functions (image 24) visually mediated the identification of the values of the parameter *a* and *1* for the example in image 25.

From the above action, Mafada said he wanted to draw the graph of the function $y = x^2 - 1$ and the graph of a function with the value of the constant being +1 to illustrate the behaviour of the functions graphically, to discern the effect of the sign of parameter q. He said: "*So we want to plot the graph of that function. So, we also want to draw a function that will show us if that one is positive*". While this intent could have been a good opportunity for learners to observe the effect of parameter q while parameter a remained invariant, Mafada did not draw the graph of $y = x^2 - 1$ alongside the graph of $y = x^2$ to guide learners about the effect of q. Instead, he only drew the graph of $y = x^2 - 1$ and the graph of another function $y = 3x^2 - 3$.

The teacher explained this pedagogical incoherence during VSRI: "*You see during teaching, sometimes you forget what you have said at the beginning, so I just introduced the other equation*".

This statement reveals that the teacher forgot to make continuity links to establish the "cumulative nature of teaching and learning and counteracting the tendency for students to experience schooling as a series of isolated, disconnected events" (Scott et al., 2011, p. 16). At the same time, the teacher addressed the complexity of teaching generally, and mathematics specifically, because of the full repertoire enacting the intended knowledge during teaching. Notwithstanding this, there was limited opportunity for learners to see, visually, what happens when the value of parameter q is either positive or negative. The change of parameters of a and q simultaneously in the equation $y = 3x^2 - 3$ presents a visual mediation constraint for learners to experience the effect of varying each parameter on parabolic functions. I discuss the pedagogical and epistemological implications for Mafada's pedagogical actions in Chapter 10.

To continue with the lesson, Mafada drew a sketch of a decreasing parabola and marked the two *x*-intercepts and the *y*-intercept/turning point A, B and C respectively in image 26 below. The excerpts below present an object-level narrative regarding the *x*-intercept and should be read concurrently with image 26 to get an understanding of the teacher's discourse(s) and approach(es):

> *Along the x-axis, everywhere on the x-axis the value of x is zero, marha (but) besides this point* (pointing at the vertex) *we can have a number as a value of x, but for y it's always zero for the value of y. That's why, I am coming to it neh, when you want to get the x-intercepts, what is happening to y, and when we want to get the y-intercepts what is happening to x, hatwanana? (am I clear?).*

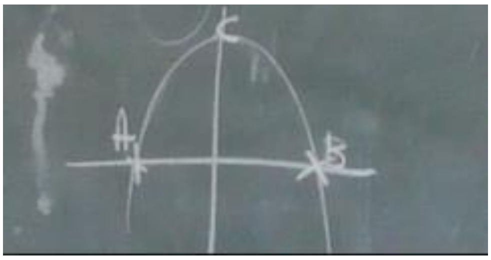


Image 26: *Parabola showing intercepts*

The sketch acted as a visual mediator and coordinated Mafada's explanation to the notion of intercepts, and used the property-oriented approach, because he was explaining the features related to intercepts. The narratives "*everywhere on the x-axis the value **of x is zero***" and "*we can have a number as a value of x, **but for y it's always zero** for the value of y*" presents a contradiction. The former narrative is mathematically incorrect since it is the *y*-values for each and every point along the *x*-axis that are 0, and the latter narrative is mathematically endorsed. Accordingly, the teacher's word use resulted in the creation of a competing narrative about the intercepts, which is worrying because learners learn the mathematics language from what the interlocutor says (Sfard, 2008). Mafada's mathematical discourse thus addresses the need for a teacher to ensure that thinking and communication (verbal and written) are aligned during teaching, because the words were not used correctly during the explanatory process of relational concepts. It is due to such mistake that Van Manen (2008, p. 7) talked about the importance of contemporaneous reflection by the teacher, to ensure that they "stop and think" about what is said and/or written on the board for the correct communication of the content. One way of interpreting this practice in commognitive theory is that incorrect use of words results in incorrect endorsement of narratives by the teacher.

The episode continued with Mafada demonstrating to the learners how to find the coordinates of the *x*-intercept by calculating the value of *x* when $y = 0$ and, similarly, the coordinates of the *y*-intercept. He got (1, 0) and (-1, 0) for the *x*-intercepts and (0, -1) for the *y*-intercept, respectively as presented below in images 27 and 28.

Image 27: *Calculation of the x-intercept*

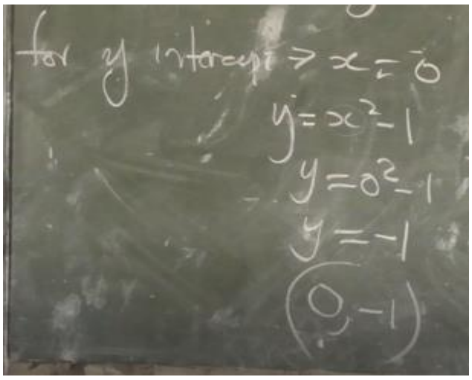


Image 28: *Calculation of the y-intercept*

The above processes represent the process called transposition, whereby an equation is being solved. While this was a good opportunity for Mafada to demonstrate symbolically what *x*- and *y*-intercepts mean to the learners, he did not explain why for *x*-intercept we let $y = 0$ and for the *y*-intercept we let $x = 0$. The explanation is important to link the mathematical calculations for intercepts and the endorsed narratives about the two properties, to facilitate learners' mathematics knowledge building. Mafada missed the opportunity to offer endorsed narratives about the intercepts (Sfard, 2008). From this, I argue that Mafada took it for granted that using visualisers (algebraic steps and coordinates) without verbalisers (the words and endorsed narratives related to the calculations and coordinates) can constrain the building of learners' intellectual quality. In this study, intellectual quality referred to teaching strategy focused on ensuring that learners develop deep understanding of essential, substantive skills, ideas and concepts related to algebraic functions. According to Sfard (2012), it is important that learners are given the opportunities to critically examine the visualisers in order to create their own mathematical meanings related to mathematical objects. Mafada did not offer a frame of cognitive links between the narratives he verbalised earlier about the intercepts, to create continuity links at the micro time-scale and the relational identifiers (mathematics calculations and the coordinates presented) (Sfard, 2012). At this point of the lesson, Mafada drew the graphical sketch of the function $y = x^2 - 1$ (image 29 below) using the intercepts calculated in images 27 and 28 and explained to the learners that parameter *q* in the equation $y = \mp ax^2 \mp q$ represents a turning point (see images 30 and 31). He further demonstrated this through the introduction of another example, $y = 3x^2 - 3$.

Image 29: *Graphical sketch of the function* $y = x^2 - 1$

Image 30: *q as a turning point*

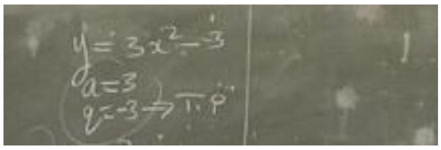


Image 31: -*3 as turning point*

Mafada stated that: "*There is something else I want you to know about q, q is always our turning point* (writing the equation $y = \mp ax^2 \mp q$ on the chalkboard in image 30, *this q is the turning point of the graph*". This statement was accompanied by the marking of *q* in image 30 and the marking of -3 in image 31 as T.P, signifying that the teacher views the numerical value for *q* to represent the turning point. Of concern is that the above object-level narrative and the symbolic mediation are not mathematically endorsed, since the value of *q* always represents the *y*-coordinate of the turning point rather than a point itself, as suggested by Mafada's statement in the above narrative (Sfard, 2008). Thus, Mafada's usage of the word turning point can be interpreted as viewing a turning point as a value rather than a coordinate pair, which could result in learners' incorrect understanding of the turning point concept or the association between the symbolic mediator and the graphical representation of the function. Mafada attempted to link the symbolic representation and the graphical representation through the introduction of the keyword turning point, saming[37] the value of q and the turning point (Sfard, 2008), which was partially incorrect as the value of parameter q only represents the y-coordinate of the T.P.

Mafada continued by focusing on the symbolic mediator, $y = 3x^2 - 3$, with an interpretive elucidation relating to the variation between the two functions $y = 3x^2 - 3$ and $y = x^2 - 1$. He said, "*You see what I have done here, I have increased this value* (pointing at the constant -3), *instead of it being minus one it is now negative three. Now let us see what happens when the number becomes bigger to the shape of the graph*". This object-level narrative is erroneous because the words "*increased*" and "*bigger*" in the above statement are used to denote that -3 is bigger than -1. This verbal mistake has implications for the interpretation of the effect of increasing and/or decreasing the value of parameter q, because learners might only consider how big the number is and overlook the sign, thereby making wrong conjectures and generalisations about the parameter. At this point of the lesson, Mafada iterated that learners must sketch the graph of the function represented by $y = 3x^2 - 3$ on the same set of axes as that of the function in the form $y = x^2 - 1$ in the previous example. He said: "*You must come and insert that graph here, so that we can compare*", which signifies that the purpose of introducing the new example is to enable learners to observe the effect of changing the value of q. He again used the dual-intercept method of substituting in the equation $y = 3x^2 - 3$ 0 for both x and y to get the x-intercepts and the y-intercepts. He got (1; 0) and (-1; 0) for the x-intercepts and (0; -1), respectively as visible in image 32 below.

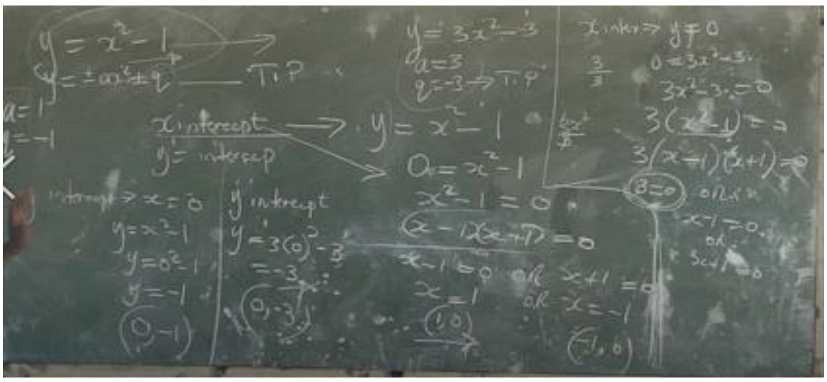


Image 32: *Calculation of the intercepts for* $y = 3x^2 - 3$

[37] Saming refers to "assigning one signifier (giving one name) to a number of things previously not considered 'the same'" (Viirman, 2014, p. 302).

Of interest to note is that once the mathematical calculations were done, Mafada did not communicate the differences and similarities between the two functions nor did he allow learners to compare the two functions from their algebraic forms. Instead, the teacher plotted the graph of $y = 3x^2 - 3$ on the same set of axes as the graph of $y = x^2 - 1$ as he indicated earlier (see image 33 below). The simultaneous change of parameters in the example $y = 3x^2 - 3$ does not enable representational linkages with the graph, for learners to visualise 'which 3' is making the graph to shift downward by 3 units from point A to point B in image 33, to develop the discourse of effect of parameters on the functions.

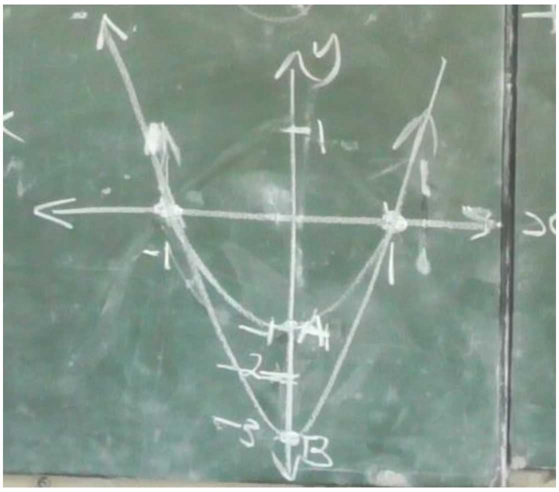


Image 33: *Graphs of* $y = x^2 - 1$ *and* $y = 3x^2 - 3$

Following the drawing of the graph of $y = 3x^2 - 3$, Mafada said:

> *And here we said our q, let us talk about our q. Our q on the first one it is negative one; our q on the second one is negative three. If you look here at point A and point B, hatwanana? (am I clear?), here we said our q is negative one and here we said our q is negative three* (image 33)*, so that is the turning point, and that's what I wanted you to see*".

While the words "*let us talk about our q*" suggest a collective discourse, Mafada took a non-interactive/authoritative communicative approach throughout the presentation and marked the turning points of the two curves A and B to mediate learners' links between the symbolic representations. In this way an approach to pedagogical link-making was developed by the teacher: support knowledge building. Mafada made links between symbolical and graphical representations using the property-oriented approach, as he brought one property of parabolic functions, turning point into focus (Kwari, 2007; Chimhande, 2014). However, the teacher did not engage in the action of interpretation to guide and ensure that learners saw the changes brought by the -3 on the graph. In addition, the teacher did not use endorsed narratives about the changes on the graphs brought by the changes in the values of parameter q, but assumed that the labelling of the turning points automatically mediated learners' visualisation of the effect of changing the values (Lo & Marton, 2012). Thus, the lack of interpretive elucidations about the effect of changing the values of the parameter resulted in the lack of endorsed narratives, which in the context of algebraic functions are the preconditions for learners to make conjectures and generalisations for each family of functions (Marton, 2012; Sfard, 2012).

5.3. Summary and conclusion regarding Mafada's observed episodes

The four episodes in this chapter foregrounded Mafada's different discourses and approaches of teaching algebraic functions, as summarised in Table 16. The dominant discourses for Mafada were the development of rules for substitution and calculating corresponding y values for the purpose of drawing or sketching graphs of given functions. This was done through ritualistic discourse that was linked with non-interactive/authoritative communicative approach as the teacher created limited chances for learners to engage in their own mathematical reasoning about the mathematical objects (Sfard, 2008; 2012). It was evident from the analysis that Mafada had challenges with explaining the key properties of different families of functions and bringing the effect of parameters to the fore. Questions about approaches that emerge from Mafada's teaching across the episodes are about teachers' decisions relating to entry, explanations, sequencing, examples and representations that were used during teaching, to eventually enable learners' development of mathematical discourse.

What is starkly evident in Mafada's teaching across the four episodes is the prevailing pattern of communicative approach that "is more concerned with the teacher telling than with acknowledging or utilising children's experiential or cognitive prior knowledge" (Myhill & Brackely, 2004, p. 268). The statement does not overlook that the teacher was introducing the topic to the learner, which requires teacher's engagement; however, active learner-teacher interaction during lessons plays a pivotal role in enabling learners' epistemological access. Mafada focused exclusively on developing computational skills as well as quick recall of facts pertaining to algebraic functions. Across the episodes, he never attempted to explain what connected the algebraic representation, for example, to the graphs that were drawn and the computed tables of values, thereby limiting learners' learning of how to translate between the representations and gain understanding of the information signified by each modality of representation. The following chapter focuses on the case of Mutsakisi's teaching.

Table 16 *Summary of Mafada's teaching episodes*

<table>
<tr><th colspan="5">Sfard's commognitive theory</th></tr>
<tr><th>Episodes and observable actions</th><th>Visual Mediator

(the images presented in-text also represent iconic and/or symbolic mediators)</th><th>Words used</th><th>Endorsed narratives</th><th>Routines</th></tr>
<tr><td>**1**. Introducing the four families of functions and showing learners what a coordinate is in the form of coordinate pairs (x, y).
Explaining the mathematical convention that x-values represent the independent variable and y-values represent dependent variables. He showed the learners how to assign values of the independent variable on the table of values, stressing mathematical convention of input versus output values. Explaining how to plot points on the Cartesian plane using the linear function in the form $y = x$.</td><td>Symbolic mediators: written functions are: $y = x$; $y = x^2$; $y = \frac{1}{x}$; $y = x^2$
Iconic: sketching a graph to depict a coordinate

Symbolic: written function is: $y = 2x$.

Coordinate pair in the form (*x*, *y*)</td><td>Functions; Parabola; Hyperbola; Straight-line Graph;Variable;Coordinates; Dependent variable; Independent variable
Point x-coordinate
y-coordinate</td><td>Object-level narratives: *"We said y is a variable* (writing on the board), *we said y is a variable, we said x is also a variable"*</td><td>Clarifying
Ritual to find a coordinate
Ritual to sketch a graph
Ritual to compute a coordinate pair on a Cartesian plane.</td></tr>
<tr><td>**2**. Introduction of the parabolic function in the form $y = x^2$. Demonstrating the change in representations, from algebraic representation of the parabolic function $y = x^2$ to the table of values. Substituting specific values of x into the equation to generate corresponding values of y in the table. Revising mathematical rules associated with performing arithmetic calculations. Asking, then telling learners why $y = x^2$ is a function.</td><td>Symbolic: using the function $y = x^2$ to compute the table of values

Iconic: table of values</td><td>Function; x-value; y-value</td><td>Describing why the given function is a function "*for x value we have only one y value"*</td><td>Rituals to determine the output values for given input values, completing the table of values and plotting and drawing the parabola.</td></tr>
<tr><td>**3.** Summarising the steps needed to draw graphs of functions. Illustrating to the learners that the shape of the parabolic function does not always give the shape given by the function $y = x^2$.</td><td>Iconic: graph of a function $y = x^2$ and sketches of 'other' parabolic functions.</td><td>Plot; Hyperbola; subject of the formula; sign of a function; face up</td><td>Meta-level narrative: "*I said we first set up the table neh, because without setting up the table, we will not be able to proceed, then the second one we said, you must use that function that you are given aniri (isn't), we substitute by the values of x (pointing at the table) that we have set up them at the table*"

Object-level narrative: "*Those things*" (referring to arrows showing continuation of graphs).This is not an endorsed narrative</td><td>Memorisation ritual on how to draw the graphs of parabolic functions</td></tr>
<tr><td>**4.** Demonstrating to learners how to determine the intercepts for $y = 3x^2 - 3$ and $y = x^2 - 1$ and in turn sketching their graphs.</td><td>Symbolic mediators: written functions are: $y = x^2-1$; $y = \pm ax^2 \pm q$ and $y = 3x^2-3$

Iconic: sketching a graph of a parabola
Iconic: The graphs of $y = 3x^2-3$ and $y = x^2-1$.</td><td>Value of *q*; value of *a*; positive; value of x; value of y; y-intercepts; negative; x-axis; point; compare; turning point</td><td>Object-level narrative about the effect of parameter *q* "*And here we said our q, let us talk about our q, our q on the first one it is negative one; our q on the second one is negative three. If you look here at point A and point B, hatwanana? (am I clear?), here we said our q is negative one and here we said our q is negative three, so that is the turning point, and that's what I wanted you to see*".</td><td>Rituals to determine the x- and y-intercepts

Exploration of the effect parameter *q*</td></tr>
</table>

Approaches of algebraic functions used
Across the episodes, Mafada used the example versus non-example approach and the property-oriented approach to demonstrate the process of mathematical substitutions, calculations and drawing graphs for selected examples of algebraic functions.
Scott et al.'s pedagogical link-making and communicative approaches
The first point to make is that Mafada takes a non-interactive/authoritative communicative approach throughout, in acting as the information giver during the teaching and learning of algebraic functions. All of the writings on the board are made by the teacher, without allowing learners opportunities to make evaluative comments. The teacher does not ask the learners for points of elaboration about the nature of algebraic functions covered. The pedagogical activities across the episodes revealed that two approaches to pedagogical link-making were developed by the teacher. These were: • Support knowledge building: making links between modes of representation – the teachers used the different representation modalities for the functions concept, but the equations and table of values were used as mere tools to draw the graphs and not to demonstrate particular information about the functions concept. • Promote continuity: managing/organising (micro scale) – this was used when the teacher made transitions from one representation to another, for example, "We are going to use those values to draw the graph".

Chapter 6

The case of Mutsakisi's discourses and the approaches

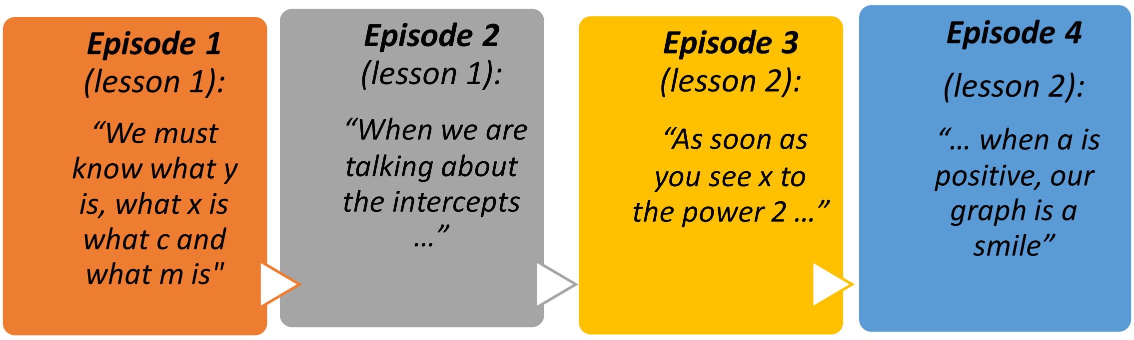
"Algebra is the backbone of mathematics, if a learner does not understand algebra, it means that he cannot understand any mathematics" (Mutsakisi, semi-structured interview).

6.1. Introduction

Mutsakisi's statement above relates to the often-cited importance of algebraic functions in research literature, as they are a prominent feature in school mathematics (Nagle & Moore-Russo, 2014; Cho & Nagle, 2017). The learners' effective initiation into mathematical discourse, especially the concept of functions, is very important to facilitate their understanding of other topics in school mathematics. This chapter present Mutsakisi's analysis and interpretations of the three observed lessons, together with the information from the VSRIs and semi-structured interviews. Figure 11 below depicts Mutsakisi's object of learning for the two selected lessons I observed and video recorded.

Figure 11

Mutsakisi's selected episodes from two lessons



6.1.1. *Episode 1 (lesson 1): "We must know what y is, what x is what c and what m is"*

Mutsakisi began the lesson by writing the following on the board: "Drawing the straight-line graph using the dual intercept method" for learners to be clear of the lesson's focus (see image 34).

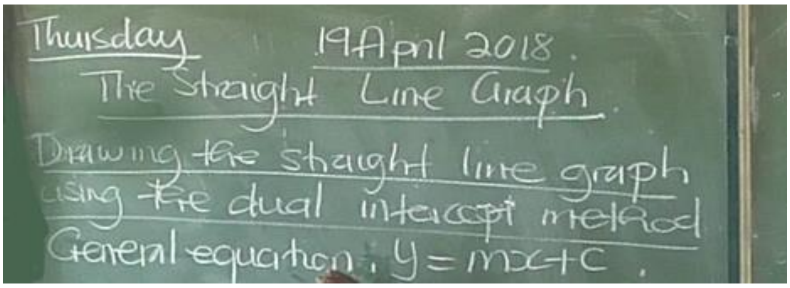


Image 34: *The focus of the lesson*

After writing the topic on the board, I thought Mutsakisi was going to review the relevant contents that were covered in the previous grades on linear graphs, to set the scene for the current lesson and the topic. Scott et al. (2011) mentioned the importance of making meso-continuity links to develop the mathematical story and show learners the relationship between the knowledge from the previous grade and the continuation of the knowledge in the current grade. Mutsakisi wrote the symbolic representation of the general equation of linear functions in the form $y = mx + c$ and drew learners' attention to what each letter in the equation represents (image 34). to re-iterate the importance of understanding the meanings of the letters in the equations (Kwari, 2007). She said: "*We must know what y is, what x is what c and what m is akere (isn't)?*". This epitomises the property-oriented approach because Mutsakisi drew learners' attention to the features of linear functions, and further introduced four concepts: independent variable, dependent variable, gradient and y-intercept, (image 35 above) linking the symbolic mediator with the concepts to promote in-depth understanding of concepts.

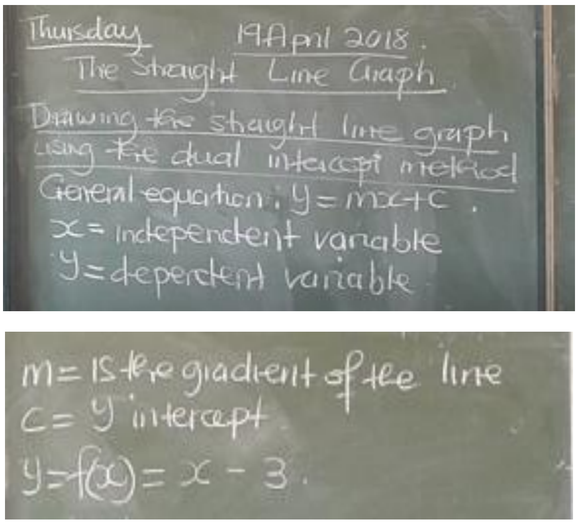


Image 35: *Unpacking the symbolic mediator*

Without overlooking the approach, she showed learners what each letter in the general equation of linear functions denoted without explicating the mathematical meanings of the words and their relationship. When mathematics words are used without accompanying meanings, as in Mutsakisi's case, the level of discourse is classified as operational word use and can constrain learners' conceptual rigour as well as the ability to make links between different concepts (Sfard, 2008; Viirman, 2013). In Moalosi's (2014, p. 295) terms, Mutsakisi's observable action could be associated with the idea of privileged features, as she focused on what 'y', 'm', 'x' and 'c' represents in the equation $\mathrm{y} = \mathrm{mx} + \mathrm{c}$. To understand the reason behind the opening statement, during the VSRI Mutsakisi explained that learners should know "*the concept, the definition and the different terms used … because you cannot use something which you do not understand*". Although Mutsakisi acknowledged the need for learners to know the concepts and their definitions to ensure conceptual understanding of the function concept, she treated the four concepts outlined above as self-explanatory while using them as stand-alone rather than in relation to the action.

In this instance, Mutsakisi introduced the function notation stating that "*y and f(x) can be used interchangeably, we can use y or we can use f(x)*", which was used with the "attitude that '*f(x)*' is an unnecessarily complicated way to say *y*" (Thompson & Milner, 2018, p. 3). This narrative is endorsed within the mathematics community and relates to Euler's conceptualisation of the function concept whereby the notation denotes the dependence between two variables (Burton, 2003). This teaching was commendable and Mutsakisi further introduced the linear function $y = f(x) = x - 3$ in its algebraic form and asked learners to identify the values of c and m using the general equation in the form $\mathrm{y} = \mathrm{mx} + \mathrm{c}$ as a prototype, based on structural equivalence between the two equations. This is an important question and it could be argued that the teacher was giving learners the opportunity to participate in the learning process. However, Mutsakisi did not give learners the chance to answer the question, instead she said: "*Before you go on to draw the graph, you must know that which you are drawing*". From my observation, I argue that the teacher was rushing to draw the graph as the following extract illustrates:

1 **Mutsakisi:** *Which values are we given in that equation y is equals to f of x equals to x minus 3?*
3 **Learners:** (no response).
4 **Mutsakisi:** *Which is the value of c?*
5 **Learners:** (no response).
6 **Mutsakisi:** *On the graph y is equals to f of x equals to x minus 3, what is c?*
7 **Learners:** *Minus 3.*

From this extract, the words "*on the graph*" (line 6) reveals that Mutsakisi views an algebraic equation as a graph and treats the objective of teaching and learning the function concept as drawing of graphs. She did not see the equation as a representational form denoting a relationship between variables. Of concern is that this way of teaching advocated the memorising of mathematical calculation steps to draw the graph, rather than the in-depth tacit knowledge that comes from experiencing what each form of representation illustrates about functions. I thus argue that Mutsakisi struggled to make links between modalities of representation to help learners understand what each form of representation signifies, in terms of the properties of each family of functions (Scott et al., 2011). During semi-structured interview, Mutsakisi iterated that the graphic representation is best for teaching learners about relationships between two variables. She said:

> *If a learner is drawing a graph, it helps him to know that now I am given this equation, what type of a graph am I expected to know … you can be given a set of ordered pairs, they can represent a graph, you can have err a graph itself, a drawn graph, or you can have the equation that represents a graph.*

This statement illustrates that Mutsakisi considers that functions are (or can always be represented as) graphs. As discussed in Chapter 2, functions can have many representations and graphs are just one of them (NCTM, 2014; Dreher et al., 2016). In relation to this, Sfard (1992) and Reed (2007) noted that learners thought that the 'function is its representation', possibly because of the way they have been taught, which appears to be Mutsakisi's view of what the topic is about. Scott et al. (2011) also talked about the need to enable learners' "ability to make links between different modalities of representation" (p. 10) as a prerequisite for developing deep conceptual understanding.

To continue with the identification of the values of *c* and *m* (micro continuity link), the teacher further asked learners to identify and verbalise the value of *m* as visible in the following extract:

8 **Mutsakisi:** *c is equals to minus 3 akere (isn't), what is our m?*
9 **Some learners:** *m is equals to x*
10 **Mutsakisi:** (showing shock in her face) *m is equals to x?*
11 **Learner:** *m is equals to x minus 3.*
12 **Mutsakisi:** (expressing shock) *x is equals to x minus 3? What is the value of m? err* (she points at a learner by name)
14 **Learner:** *1*
15 **Mutsakisi:** *1! We said if there is no value before x, if there is no coefficient akere (isn't), the value we just put there is 1.*

Of interest with this guess until the correct answer action is that learners appeared not to know how to identify the values from the equation, at least as a class. This teaching practice did not create an environment for substantive engagement, which entails creating opportunities to talk about concepts and problems of the subject for the purpose of using the opportunity to clarify learners' misunderstandings (Killen, 2015). Mutsakisi's level of discursive routine is at the object level and is concerned with learners' prototypic identification of m and c for given functions, using the algebraic equation in the slope-intercept form $\mathrm{y} = \mathrm{mx} + \mathrm{c}$ as an archetype. The teacher expected learners to identify and verbalise the parametric values on the basis of the prototypic mathematical structure being given, to coordinate their thinking about the examples of linear functions that are introduced (see lines 10 and 12). I noticed a mismatch between the expectation and the learners' ability to use the archetype to identify the values of m and c for the specific equations, which could be linked to the teacher's insufficient link making of the concepts at micro time-scale. Rider (2004) argued that to enable learners' conceptual knowledge, linear functions should ideally be taught with a method embracing Multiple Representations (MRs) and providing links between these representations. However, in this episode, Mutsakisi overlooked the notion of translating between multiple representations to show learners how a function is represented in different forms, to ensure that they develop conceptual understanding relating to the notion of parameters.

I noticed during classroom observation in the same episode that Mutsakisi used the slope property to determine if the function is increasing or decreasing, and used the words "*slant to the right*" and "*slant to the left*" to denote the increasing and decreasing functions when *m* is positive and negative, respectively. According to Chimhande (2014, p. 41), "a linear function whose general form is $y = mx + b$ can be taught by referring to the properties of its graph namely the slope (*m*) and y-intercept (b)". Thus, Mutsakisi did not provide learners with different functions varying the value of *m* to create opportunities for them to explore the effect of the parameter before generalising the nature of the parameter. The teacher's routine is classified as ritualised memorisation routines and used the authoritative/interactive communicative approach to talk about the effect of changing the value of parameter *m* (Sfard, 2008; Scott et al., 2011). The relationship between the ritualised memorisation routine and authoritative/interactive communicative approach in Mutsakisi's teaching is unsurprising because she viewed herself as the sole source of mathematical knowledge during teaching.

6.1.2. Episode 2 (lesson 1): "When we are talking about the intercepts …"

This episode continues from the previous one, and Mutsakisi used the property-oriented approach to introduce the notion of determining the graph interception on the axes, with a new keyword, 'intercept'. She gave the following narrative relating to intercepts: "*When we are talking about the intercepts, we are talking about the point where our graph touches the line. So, when* *we are talking about the x-intercept, we are taking about where our graph touches x*" (she writes on the board, in image 36).

Image 36: *Defining the x-intercept*

The choice of words "*where our graph touches x*" denotes mathematical understanding of the concept, however, the lack of explication that the *x* she is referring to is the *x*-axis could lead learners to confuse the axis from a value of *x* in a coordinate pair when communicating about the notion of intercepts. It is crucial that teachers do not take for granted the use of words and assume that learners know the specific object they are referring to during teaching, to ensure that they develop correct mental images of mathematical objects. This is especially important since Mutsakisi asked the learners to give the definition of the *y*-intercept using the definition of the *x*-intercept she wrote on the board, suggesting juxtaposing the two concepts for learners to recognise the differences between them.

I did consider that the teacher was creating the opportunities for learners to make micro continuity links and used the given narrative about the *x*-intercept to construct an alternative narrative about the *y*-intercept. It is however worrying that the teacher makes assumptions that learners will be able to give the definition without being taught what the *y*-intercept entails, especially since the topic and its concepts were introduced in Grade 10. Consequently, learners failed to provide the definition of the *y*-intercept since they had not been explicitly taught how to use the word in the current lesson, other than when they had to juxtapose the word to *x*-intercept. It was only when learners were unable to provide the definition of *y*-intercept that Mutsakisi wrote the definition of the *y*-intercept on the board (image 37). In mathematics it is important that teachers explain and/or define the meaning of some concepts before they teach, because definitions set up the meaning-relation between mathematical words and related objects and facilitate the act of assigning objects to specific names (Tall, 1992).

Image 37: *Defining the y-intercept*

Mutsakisi further engaged in a ritual routine on how to scale the Cartesian plane using a ruler, to ensure that the graph looks good. After drawing the Cartesian plan on the board, Mutsakisi introduced the function $y = f(x) = x - 1$ in its symbolic form, and informed the learners that instead of writing $y = f(x) = x - 1$ they can "*simply*" write $y = x - 1$. The writing of the function in terms of $f(x)$ was to produce a compression, whereby the symbol is used to reduce communication by saying less (Caspi & Sfard, 2012). However, the introduction of the notation $f(x)$ did not serve the purpose of illustrating the relationship between the 'function argument' and the output values. Instead, Mutsakisi emphasised that the equation would be used to draw the graph: "*We want to use this equation to draw the graph*". Consider the following extract from classroom observation:

> *m is equals to 1, m is greater than zero, in other words it means the value of m is positive … right, this has got a meaning, it has a very special impact on the graph that you are going to draw, akere, to the graph that you are going to draw.* (writing on the board as she continues speaking) *If m is positive, the graph slants to your right, which means the graph that we are going to have will slant to your right, are we together? The graph will slant to the right because m is positive, akere, the general formula says y is equals to mx minus …*

Mutsakisi's discourse can be classified as 'literate mathematics memorisation rituals' because the teacher provided the interpretation of the symbolic visual mediator for the linear function using endorsed narratives without allowing learners to create mathematical meanings for themselves (Flesher, 2003; Sfard, 2008). Scott et al. (2011, p. 33) argued that it is important for "the teacher to make the link available and [*sic*] the students so that they come to understand the links for themselves as they actively make connections between their own ideas". In the above statement, two keywords are introduced, 'greater' and 'positive', to coordinate her explanation about the behaviour of the graph when the value of *m* is greater than 0.

I noticed that the nature of the gradient was not explained in terms of the covariational relationship between *x* and *y* variables, to demonstrate to the learners what influences the gradient to be either positive or negative. Instead, Mutsakisi made a generalisation that if the value of *m* is positive, then the graph will slant to the right. While the above statement is mathematically endorsed, of concern is that Mutsakisi did not allow learners to investigate the influence of the value and sign of the gradient, while she was teaching and during the task. The consequence of this is the missed learning opportunities for learners to make conjectures, test and subsequently generalise the parametric influence of *m* as enshrined in the curriculum standards (DBE, 2011). The lack of opportunities for learners to observe the effect of parameter *m* constrained the process of making their own conjectures and proving them, which is a pre-requisite of functional understanding according to CAPS (Sfard, 2008; DBE, 2011). "Generalisation" was written on the board, with a visual mediator depicting the meaning of the word slant as visible in image 38. It is however important to mention that the teacher's use of the word 'slant' while linking the symbolic representation and the graphical, created the links between the algebraic and graphical representations to support learners' knowledge building about the effect of parameter *m* (gradient) on linear functions.

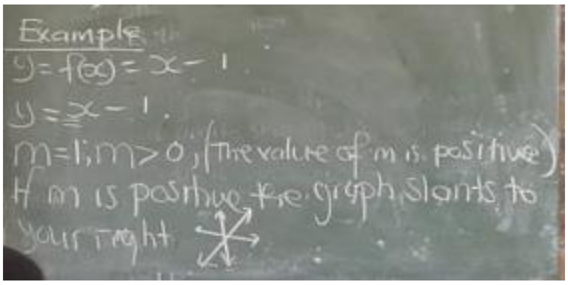


Image 38: *The effect of parameter m*

As presented in image 38, Mutsakisi returned to the function $y = x - 1$ and used the dual-intercept to calculate the *x*-intercepts and the *y*-intercepts for the purpose of sketching the graph of the function. Through engaging in the process of mathematical substitution and calculation, Mutsakisi demonstrated to the learners how to calculate the intercepts and she obtained point (0, -1) for the *y*-intercept (see image 39). Mutsakisi used the interactive/dialogic communicative approach to engage in confirmatory exchanges about the process of solving for *x* in the equation. The extract below illustrates this:

17 **Mutsakisi:** *Let y equals to zero, great! y-intercept, x-intercept, let y equals to zero. So, what should I write here?*
19 **Learner 1:** *y …*
20 **Mutsakisi:** *y is equals to x minus one. And then?*
21 **Learner 2:** *Zero!*
22 **Mutsakisi:** *Zero akere!*
23 **Learners:** (chorusing) *Yes.*
24 **Mutsakisi:** *Zero equals to x minus one. Therefore, what is x?* (she points to another learner).
25 **Learner:** *We transpose minus one to the other side.*
26 **Mutsakisi:** *We transpose minus one to the other side, and it becomes? It becomes?*
27 **Learners:** *Positive one.*

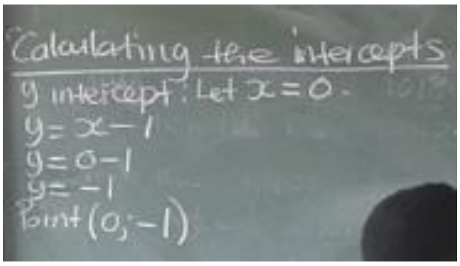


Image 39: *Calculating the y-intercept*

In this way, Mutsakisi was creating micro continuity links for the learners to recall and verbalise the procedure for determining the intercepts, representing elicitations "to obtain from students information gained in the past classroom activity which is relevant to current or future activity" (Mercer, 2000, p. 52). In the above exchange, a colloquial word 'transpose' is used to denote a ritualised routine of solving equations (line 26), and the way it has been used in the above dialogue is limited. Mutsakisi's usage of the word transpose had no accompanying justification as to why $0 = x - 1$ was transformed to be $x = 1$, as visible in image 40 below. According to Venkat and Adler (2014, p. 4), the word transpose should not be used "with no accompanying mathematical justification (e.g. we subtract 6 from both sides of the equation), because the underlying principles or properties like maintaining equivalence are never made explicit". Especially if Adler and Venkat's (2014) argument is considered that despite the common use of the word transpose in our mathematics classrooms, it is a colloquial word and it should not be used exclusively to describe algebraic manipulations of equations.

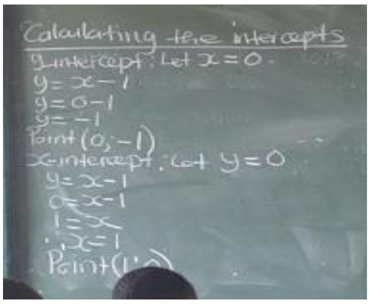


Image 40: *Calculating the x-intercept*

After the explanation in image 40, she drew a table of values to represent the intercepts that were calculated, thereby making links between the two modalities of representation, table of values and the calculated intercepts (see image 41).

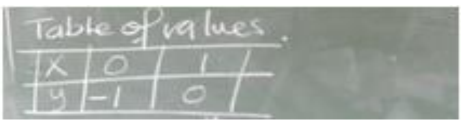


Image 41: *Representing the intercepts in a table of values*

Representing the intercepts in the table of values reveals Mutsakisi's ritualisation that the drawing of graphs of functions should be linked to the tabular representation. It is advisable that the teacher use the calculated intercepts to demonstrate to the learners how to use the coordinate pairs to sketch the graph, without the use of the table to develop procedural fluency for the dual-intercept method. Subsequently, Mutsakisi showed learners how to draw a Cartesian Plane, emphasising the need to use a ruler: "*You must use a ruler to measure the spaces*" which addresses the ritual on how to scale the graph to ensure that the graphs are drawn to scale. She then used the points for the *x*- and the *y*-intercept to sketch the graph of the function $y = x - 1$ (image 42).

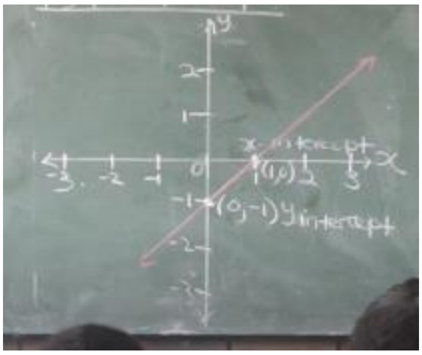


Image 42: *The graph of* y = x-1

Mutsakisi concluded this lesson by giving learners a function y = x -3 which is the same function she used as an example in episode 1, making links between the tabular representation and the graphical mode of representation to support knowledge building (Scott et al., 2011). She instructed the learners to draw the graph of the function to complete as homework. From the observed actions in the above episodes, I noticed that Mutsakisi's teaching is guided by the principle '*how*' of the routines, with the assumption that the '*when'* will take care of itself' (Sfard, 2012), considering the prominence of rituals to draw the graphs from algebraic equations. The following sub-section focuses on the analysis of Mutsakisi's episode 3, which I extracted from the second observed lesson, which occurred the day after the current lesson.

6.1.3. Episode 3 (lesson 2): "As soon as you see x to the power 2 …"

Like all the lessons observed for Mutsakisi, this lesson began with her writing the object of learning on the board (image 43). She commenced the episode with the following narrative:

> *We want to have a look at the quadratic functions. Last time we discussed about linear functions akere (isn't), there is a difference between a quadratic function and a linear function. Let me start by writing these equations y equals to x, y equals to x squared. What kind of a graph does this equation* (pointing at $y = x$) *produce?.*

The purpose of this statement was to draw learners' attention to the differences between the two classes of functions, linear and quadratic functions.

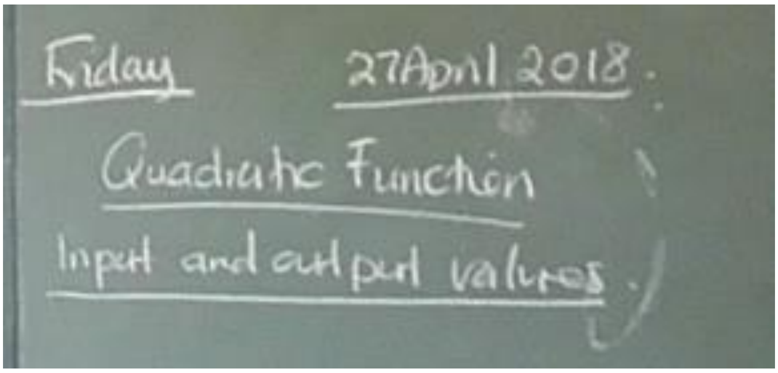


Image 43: *The focus of the lesson*

Mutsakisi asked the learners whether the "*equation y equals to x will produce a straight line or a curve?*" as a way of making links between linear and parabolic graphs. Scott et al. (2011) called this meso continuity because the teacher had already taught learners about linear functions in previous lessons. She further recapped what was taught in the previous lessons on drawing the graphs of linear functions and made links between the linear functions and parabolic functions, promoting the concept continuity in her teaching. The two equations written on the board (image 44) acted as symbolic mediators in facilitating Mutsakisi's classroom talk about the differences between the two classes of functions. Mutsakisi further wrote 1 as an exponent of x on the function $y = x$ and traced exponent 2 on the function $\mathrm{y} = \mathrm{x}^2$ (visible in image 44) to explain to the learners the differences between the two functions based on the structural appearance of the equations. By doing this, the teacher offered learners an object-level narrative in which the differences relating to the nature of two classes of functions were presented. By 'structural appearance' I mean the difference between the two classes of functions in terms of their object level features from the symbolic mediators used ($y = x$ and $y = x^2$). Mutsakisi said: "Q*uadratic functions have a power of 2 whereas linear functions have a power of 1*". In this instance, the teacher's discourse is classified as mathematical memorisation since she used a formal mathematical narrative to show the learners that linear functions are first-degree functions, whereas quadratic functions are second-degree functions as varied by the values of the exponents 1 and 2 correspondingly. This plays an important role in developing learners' mental images of the differences between the linear and parabolic functions (Tall, 1992).

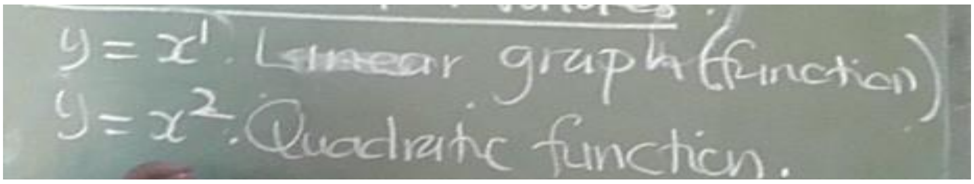


Image 44: *Differentiating between linear and quadratic functions*

To further demonstrate the differences between the two classes of functions, Mutsakisi used the example vs non-example approach and property-oriented approach to introduce a linear function $y = 2x + 3$ and a quadratic function $y = x^2 + 2$ in symbolic form to further demonstrate the differences between the classes of functions in terms of their structural appearances (image 45) (Chimhande, 2014). This pedagogic activity can be interpreted as making links to promote continuity at meso time-scale, as the teacher made links between linear functions (covered in days) and the current activity (parabolic functions) to enable learners to see the differences between the families of functions (Scott et al., 2011). From the commognitive theoretical frame, the teacher's juxtaposition of the two classes of functions can be viewed as symbolic visual mediation to enable learners to see the structural differences between the equations of the two classes of functions (Sfard, 2012).

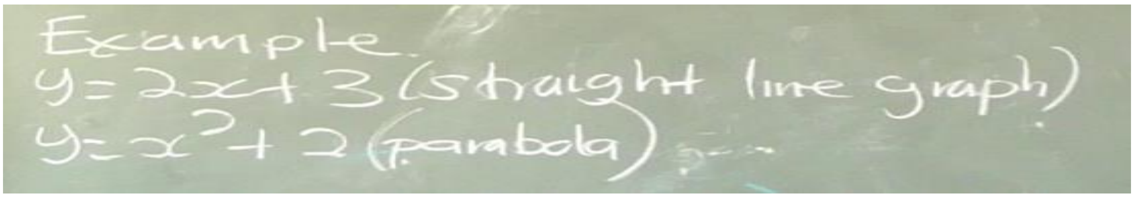


Image 45: *Differentiating linear and quadratic functions*

After writing the two functions on the board, she said: "*As soon as you see x to the power 2, you must know that the graph I am going to get is a parabola*", drawing a sketch of what a parabola looks like (image 46). She used this as a visual mediator to coordinate the talk about the association between a quadratic equation and a parabolic graph. As further illustrated by her utterance while concluding the episode: "*Before you go on to draw the graph, we must know how to find the input and the output values of a parabola*". Thus, for Mutsakisi, finding the output values for given functions was a precondition for plotting the points on the Cartesian plane and the sketching of the graph. According to Bayazit and Aksoy (2010, p. 149), this way of teaching algebraic functions is characterised by "the teacher's instructional acts which emphasize step-by-step manipulation of algorithmic procedures and engage students with the visual properties of algebraic piecewise functions". While Mutsakisi's emphasis that learners need to *"know how to find the input and the output values"* can be interpreted as acknowledging the importance of other aspects of the topic, the process of determining the output values appeals to ritualised mathematisation and acts as a prerequisite for drawing the graph.

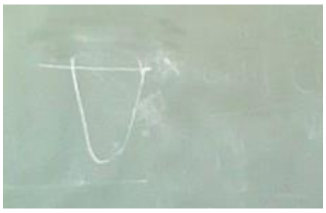


Image 46: *Sketch depicting the shape of a parabola*

Mutsakisi directed learners' attention to the activity on the board in which learners were expected to compute the arguments of 1, 2 and -3 on the functions $y = x^2$ and $y = x^2 + 1$. Mutsakisi substituted -1 on the equation $y = f(x) = x^2$ to demonstrate to the learners how to substitute and calculate the output value (see image 47).

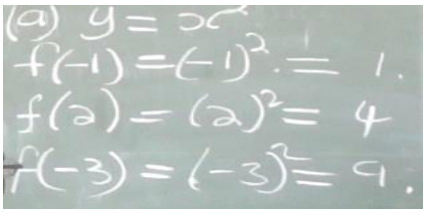


Image 47: *Calculating output values*

Interestingly, Mutsakisi did not show learners the steps to squaring the input values, instead she substituted the *x*-values into the equation and wrote the output values. When I asked the reason for not showing learners how to calculate the values during VSRI, she posited that "*these learners have done basic calculations in Grades 8 and 9, there is no need to repeat, they should know how to*" and as a result the teacher did not re-teach this knowledge.

The reflective comment suggests that the teacher has certain knowledge expectations from the learners, and that algebraic manipulations should be too basic for the learners at a Grade 10 level. While learners might have been introduced to particular knowledge in the previous grade, it does not mean that the teacher in the new grade should not reiterate that information. Recapping helps teachers to set the scene for current activities by using "basic conversational techniques for building the future on the foundations of the past" (Mercer, 2000, p. 52). Mutsakisi's assumption that learners should know how to calculate the values, shows that she thinks her learners have a greater capacity to transfer their learning from previous grades to the new contexts. Accordingly, the teacher overlooked the need to keep her eyes both on the individual learners in her classroom as well as on the community created by her and the learners.

In this instance, Mutsakisi introduced the words ‘domain’ and the ‘range’ which were used synonymously with the words ‘input values’ and ‘output values’ respectively. This synonymity in word use is evidenced by the statement Mutsakisi uttered in this part of the lesson: “*The values that we put into the equation, they are the input values, they are the domain, outputs are the values that we get, and that output is the y-value, which is our range*”. This statement demonstrates that for Mutsakisi, the words ‘domain’ and ‘range’ are synonymous to the words ‘input’ and ‘output’ correspondingly. The usage of the words synonymously presents conflicting messages about their meanings and do not enable learners to understand the definitions of the four concepts more deeply. This synonymity was also observed in how the teacher presented the domain and the range for the function $y = x^2$ as visible in image 48.

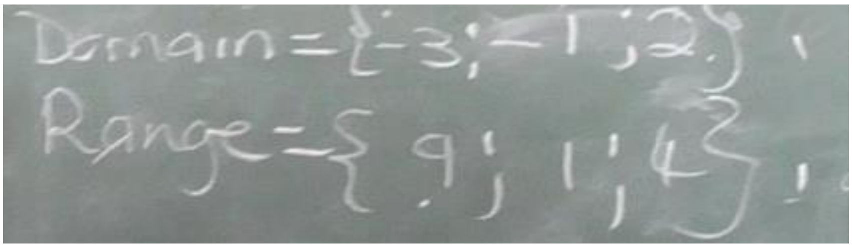


Image 48: *The range and domain for* $y = x^2$

Of concern with Mutsakisi’s use of the words interchangeably is that the word domain does not enable learners to learn that the domain refers to the set of all possible *x*-values for a given function which will output real *y*-values. Equally, the usage of the word range in this episode does not allow learners to learn that a range for a given function entails the set of all possible output values. Interestingly, during the semi-structured interview Mutsakisi mentioned that: “*Mathematics teaching is, in actual fact, mathematics is a language, so mathematics teaching involves teaching the learners the language of mathematics, understanding the concepts is the most key concepts in the teaching of mathematics*”. While she views mathematics as a language and mathematics teaching as a process of teaching learners the language of mathematics, her teaching does not engage in more formal definitions of the concepts. Instead she mentions concepts without explicating delimitations of their uses and connections with other concepts within the topic, thus limiting learners’ mathematics language development.

The above analysis addresses a broader perspective of the relationship between spontaneous word use and mathematical word use during teaching. That is, the lack of everyday word use such as '*all possible*' when describing the words domain and range results in the deficit usage of the words, which could lead to learners' limited conceptual understanding of the concepts. Sfard's commognitive theory does not account for how non-mathematical words such as 'all possible' in definitions of mathematical words such as domain and range, give identity to a teacher's mathematical language during teaching. While the teaching or learning of mathematics is about the mastery of formal mathematical words, it is also important to consider that there are words in mathematical definitions that signify holistic thinking about the meaning of mathematical words. This discussion relates to Gyllenpalm et al.'s (2010) utterance that "the meaning of a word is its use and function in a specific activity" (p. 1155). Thus, there is a need to encourage teachers to take more cognisance of the functional value of the non-mathematical words in mathematical definitions.

6.1.4. Episode 4 (lesson 2): Drawing the parabola

In episode 4, Mutsakisi wrote 'Drawing the parabola' on the board to signal a shift in the object of learning (image 49) and introduced the parabolic functions $y = ax^2 + q$ as a symbolic mediator, emphasising "*that is the general equation of the parabola that we want to draw. There are two ways of drawing a parabola, the first way is drawing the table of values*". This statement clearly stated the purpose of the lesson, considering that the previous episodes focused on determining the output values for given input values, preparing learners for this goal.

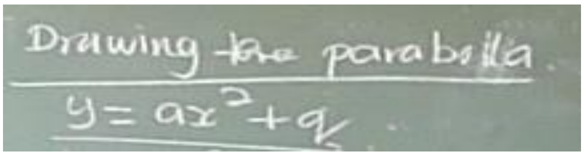


Image 49: *The general equation for parabolic functions*

Mutsakisi then introduced a function $y = 2x^2 + 1$ in symbolic form and asked the learners to engage in the process of finding output values for the 3 values of *x* on the table of values she drew on the board (image 50). The purpose of this activity was to allow learners to make links between the various knowledge the teacher has introduced, for in-depth comprehension of the contents using the interactive/authoritative approach (Scott et al., 2011). Chimhande (2014) described that working with symbolic representation can involve an operational approach to the function concept, the process that requires algebraic manipulation by translating the symbolic representation into a graphical form. In this case, in a move from a symbolic representation to a graphical one, Mutsakisi was encouraging learners to make links between modes of representation and allowed them to engage in mathematical processes to sketch the graph of $y = 2x^2 + 1$. The teacher recorded the obtained ordered pairs in a table of values and plotted the points and drew the graph (image 51), thereby making links between different modalities of representations.

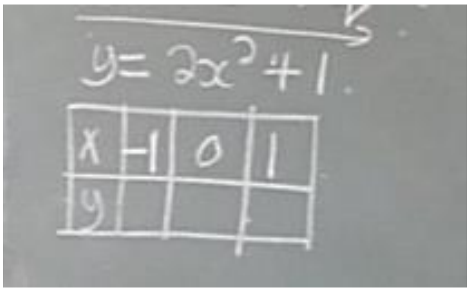


Image 50: *Completing the table for* $y = 2x^2 + 1$

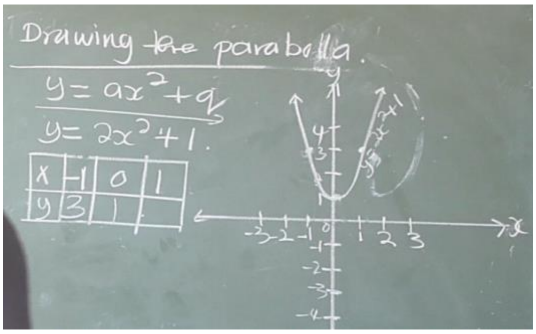


Image 51: *Completing the table of values and drawing the parabola*

For understanding the function concept and its structures, Even (1998) suggested that flexibility is needed for translating from symbolic representations to graphical ones and back, to ensure that learners understand the information equivalence in different modalities. Without overlooking the above work, I noticed that Mutsakisi used the algebraic equation as a recipe to generate output values. She did not perform a backward translation from the graphical representation to an algebraic formula, thereby treating the sketching of the graph as the end-goal of teaching and learning the topic. Even though sketching graphs might have been the main focus from the beginning of the lesson, the teacher did recognise the importance of other procedures to teach learners as a way of inducting them to mathematics ways of thinking, talking and doing mathematical processes. From the two representations and the equation, Mutsakisi continued to identify the critical points from the graph: turning point and *y*-intercept. In describing a turning point, Mutsakisi stated that "*We can see that this graph here turns, so where it turns we call it turning point, it turns.*" This statement only offers a colloquial meaning of turning point, which is not sufficient considering Moeti's explanation that a turning point "is the point where the graph changes from being increasing to decreasing or *vice versa*" (Moeti, 2015, p.

12). Effective explanation of the concepts meaning is important. I then argue that Mutsakisi deprived learners detailed meaning of the concept for understanding and future use. The implication is that there is no opportunity for learners to explore and describe what has changed in terms of the effect brought by parameters in the function, and has used the literate memorisation ritualisation (Sfard, 2008).

From the above lesson focus, she continued and wrote the coordinate of the turning point (0, 1) and the coordinate of the y-intercept (0, 1) and thereafter moved to the coefficient of x squared. She said: "*a is equals to two which means a is greater than zero meaning kuri (that) a is positive* (see image 52). *Now, with the parabola, if a is positive the graph faces up, so the graph smile* (drawing the emoji as visible in image 52, *when a is positive, our graph is a smile*". Referring to an increasing parabola as 'a smile' and the emoji provides a visual cue about what happens when the parameter a in each function is greater than 0, the graph faces up and produces a smile-like object. The teacher used what learners are familiar with to explain the shapes of the graphs brought by the effect of the value of parameter a, which is making links between scientific explanations and real world phenomena, as a way of simplifying the content and making the content relevant (Scott et al., 2011).

Image 52: *Using the colloquial word 'smile' to denote increasing parabola*

I noticed, however, that Mutsakisi did not give learners the opportunity to explore that the minimum and maximum turning point depends on the sign of the parameter a in the parabolic function. The words "*if a is positive the graph faces up*" is a generalisation statement, and Mutsakisi presented it before allowing learners to observe more than one graph with varying values of parameter a. It is important for a teacher to make conjectures before generalising the effect of the parameter on the behaviour of the function, as recommended by the curriculum policy (DBE, 2011). This helps learners to explore and construct their own meanings about the changes brought by varying the parameter and in turn develops conceptual understanding. Mutsakisi concluded the lesson by writing the three parabolic functions $y = x^2$, $y = 2x^2$ and $y = 3x^2$ in symbolic form on the board, asking learners to draw the graphs of the functions on the same set of axes. This was done to give learners the opportunity to show their understanding of what was done during teaching, to explore the effect of varying the values of parameter *a* on the parabola.

6.2. Summary and conclusion regarding Mutsakisi's observed episodes

This chapter provided a presentation, analysis and interpretations of Mutsakisi's observed lessons, her comments from VSRIs and semi-structured interviews, to unearth Mutsakisi's discourses and approaches of teaching algebraic functions. The observable action that is prevalent across Mutsakisi's lessons was drawing the graphs of functions, which was characterised by Mutsakisi demonstrating the drawing of graphs of functions. That is, functions given in symbolic form were represented in table of values, whereby the teacher demonstrated to the learners the algebraic calculations to find *y* values for given *x* values. For all the examples Mutsakisi introduced during teaching, she recorded the *y* values in a table of values, and then plotted the coordinate pairs on Cartesian planes and drew the graphs. In terms of the approaches of teaching algebraic functions, the property-oriented approach and the example vs non-example approach were predominantly used during the lessons, alongside Scott et al., (2011) interactive/authoritative communicative approach to teach learners about the concepts. Table 17 is a summary of Mutsakisi's episodes and outlines the observable actions during teaching.

Table 17 *Summary of Mutsakisi's teaching episodes*

Sfard's commognitive theory				
Episodes and observable actions	**Visual Mediator** *(the images presented in-text also represent iconic and/or symbolic mediators)*	**Words used**	**Endorsed narratives**	**Routines**
1. Using the general equation for linear functions in the form $y = mx + c$ as an archetype to symbolically mediate learners' identification of what the letters in the equation represents.	Symbolic mediators: written functions are: $y = mx + c$; y = x-3 and $f(x) = x - 3$	Variable; Dependent variable; Independent variable; *x*-coordinate; *y*-intercept; gradient;	Object-level narratives, identifying what y, m, x and c from the equation $y = mx + c$ Describing the use of the notation *f(x) "y and f(x) can be used interchangeably, we can use y or we can use f(x)"*	Clarifying Memorisation to identify the values of m and c from linear equations.
2. using the notion of intercepts to draw the graph of the function in the form $y = x - 1$.	Symbolic: using the function f(x) = x-1 to compute the table of values Iconic: table of values and graphic visual mediator Symbolic: (0, 1) – y-intercept Symbolic: $0 = x - 1$ – determining the x-intercept	Intercepts; positive; *x*-intercepts; slants; greater than; y-intercepts; transpose	Describing the notion of intercepts: "*when we are talking about the intercepts, we are talking about the point where our graph touches the line*". The effect of parameter m on linear graphs: "*If m is positive, the graph slants to your right, which means the graph that we are going to have will slant to your right*"	Ritual to complete the table of values from an algebraic equation; Ritual to find the *y*-intercept and *x*-intercept for linear functions; Ritual for plotting the straight line graph.
3. Introduction of a new family of functions (parabolic) to juxtapose the structural differences between linear and parabolic functions in terms of their symbolic appearances.	Symbolic mediators: written functions are: $y = x$; $y = x^2$; $y = 2x + 3$; $y = x^2 + 2$; $y = x^2 + 1$ Iconic: table of values	Quadratic functions; straight line; linear functions; output values; domain; input values; range; x-values; y-values	Distinguishing linear and parabolic functions in terms of their symbolic representations: "*Quadratic functions have a power of 2 whereas linear functions have a power of 1*" Saming domain with input values and output values with range: "*The values that we put into the equation, they are the input values, they are the domain, outputs are the values that we get, and that output is the y-value, which is our range*". This is not an endorsed narrative	Ritual to complete the table of values from an algebraic equation.

4. Using the parabolic function in the form $y = 2x^2 + 1$ to show learners how to complete the table of values and in turn draw the graph. From the graph, she identified the turning point and the y-intercept.	Symbolic mediators: written functions are: $y = ax^2 + q$; $y = 2x^2 + 1$, $y = x^2$ $y = 2x^2$ and $y = 3x^2$	Turning points; greater than; positive; smile; faces up	Narrative about turning point Describing the effect of parameter a: *"Now, with the parabola, if a is positive the graph faces up"*	Ritual to complete the table of values from an algebraic equation. Ritual for plotting the graph of a parabola..

Approaches of algebraic functions used

Mutsakisi used two approaches of teaching algebraic functions across the selected episodes. These were the example versus non-example approach and the property-oriented approach. She used the property-oriented approach to guide learners to identify what the letters on the general equations of the functions represent. The example versus non-example approach was used to orient learners to the mathematical conventions such as calculating output values for given input values, completing the table of values and sketching the graphs.

Scott et al.'s pedagogical link-making and communicative approaches

Across the episodes, Mutsakisi took an interactive/authoritative approach to demonstrate various mathematical conventions which included substituting, calculating and completing the table of values for the different examples she used in the lessons. Four approaches to pedagogical link-making were developed across the episodes by the teacher. These were:

- Support knowledge building: making links between mathematical concepts
- Promote continuity: making links between different modes of representation
- Promote continuity: developing the mathematical story (micro)
- Promote continuity: managing/organising (micro)

Chapter 7

Zelda's case of teaching algebraic functions

"*I am not good at all, I am not satisfied with what I am doing when it comes to this chapter, this function thing*" (Zelda, Semi-structured interview)

7.1. Introduction

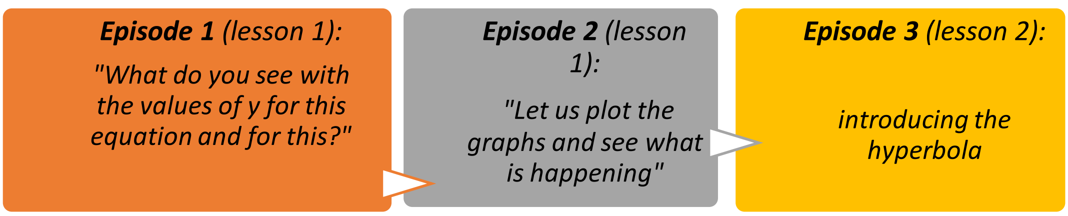
The above statement from the semi-structured interview suggests that Zelda does not feel confident to teach algebraic functions, because she is "*not good at all*" with the content knowledge to ensure learners' understanding of the topic. After saying "*I am not satisfied ... this function thing*", I became interested to observe her lessons because of the strong self-described (assessed) language. From the biographical information, Zelda is relatively new in the profession with 5 years' experience. She further stated that she always consults with her colleagues, particularly her principal before teaching the topic. She said:

> *I am really not good with functions, so I always go and ask my principal, we discuss. I have one problem ... they will show you this is how we do it, straight-line graph, it's either your graph will be increasing or decreasing, but those things nje (that's it), you will be memorising. It is what you must give to the learners, you see, you must memorise to make learners memorise.*

While consultation is not problematic, this statement demonstrates that Zelda's content knowledge for this section is limited, a reason she is not satisfied with her teaching of algebraic functions. She indicated the use of memorisation to remember the necessary procedures while teaching the different concepts of the topic, and consequently thinks that learners should be taught to rote learn the topic. Notwithstanding this, from the classroom observations, Zelda's mathematical discourses and approaches to teaching algebraic functions demonstrated the ability to enable learners' epistemological access to some concepts of algebraic functions. Figure 12 below represents the selected episodes from Zelda's observed lessons.

Figure 12

Zelda's selected episodes from two lessons



7.1.1. *Episode 1 (lesson 1): "What do you see with the values of y for this equation and for this?"*

The first episode of this lesson began with Zelda asking learners to recall what was introduced in the previous lessons to connect with the current lesson and develop the mathematical story (Scott et al., 2011). She used question/answer teaching strategy, which is linked with the property-oriented approach to encourage learners' participation in the lesson and thinking about the distinguishing features of the different functions. Consider the following exchange:

Zelda: *Before we were disturbed by the strike, what is the name of the graph that we were doing?*
Learner: *Parabola*
Zelda: *Parabola neh! Before parabola graph, we have dealt with what?*
Learners: *Straight line graphs*
Zelda: *Straight line neh. Okay, let us do corrections. We have two methods neh, we did table and after that we did what?*
Learners: (chorusing) *Intercept method*
Zelda: *Okay, let us start with the table one.*

The purpose of this dialogue was to check whether learners still remembered the main aspect of the lesson before there was a community strike (the past four days). This resonates with Scott et al.'s (2011) idea of teaching being cumulative at meso time-scale, to ensure that learners make links between previous lessons and the current lesson. The above exchange also reveals that Zelda views the teaching of functions as focusing on the drawing of graphs, as evidenced by her question "*what is the name of the graph we were doing*?" (line 1). Zelda's developing meaningful new teacher induction and the introduction of the lesson suggest that she treats mathematics teaching and learning as a collective patterned activity (Sfard, 2008) and interactive/dialogic communicative approach (Scott et al., 2011) for learners' active participation during the lesson. She used the ritual routines to draw the graphs of functions: table method and dual intercept method, and introduced three parabolic functions in a table of values for learners to engage with in image 53 below:

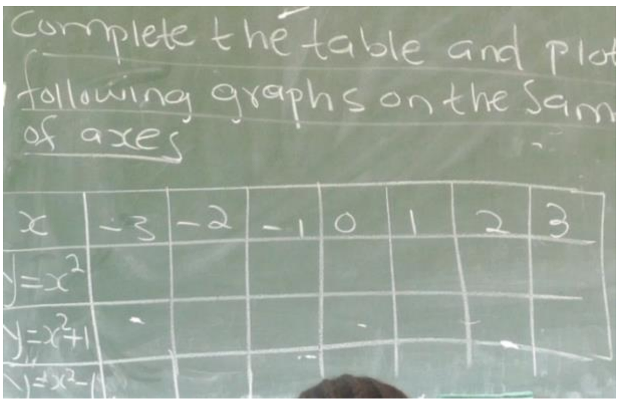
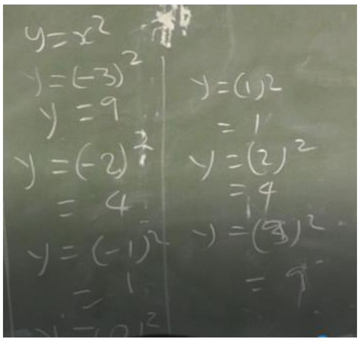


Image 53: *Three functions in tabular form*

Zelda foregrounded the idea of varying the values of parameter q and engaged in the mathematical convention process of substitution and calculation of *y* values for given values of *x* (image 54(a)). She uttered the following statement: "*Let us substitute, we said these values represents x, meaning kuri (that) in these equations where there is x we substitute these neh, okay let us substitute and get answers*". Zelda continued with the process of substituting the given *x* values on the board, calculated and substituted the values of *y* on the table for the function $y = x^2$. The calculation process is shown in image 54(a) and the completion of the table of values in image 54(b).



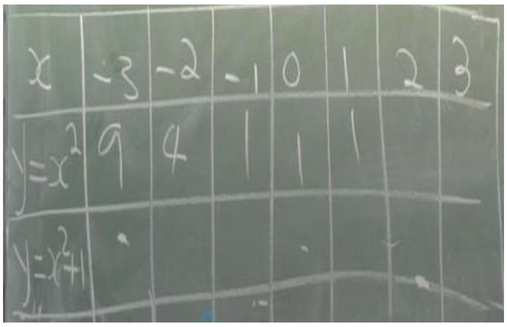


Image 54(a): *Substitution process* **Image 54(b):** *Completing the table of values*

The observable action of taking the calculated *y* values from algebraic calculation and representing them on the table of values denotes the development of a link to a new mode of representation for the functions (Scott et al., 2011). The teacher's engagement with this process was to continue with the procedures to show learners the mathematical conventions needed to translate from the symbolic representation to tabular representation. I noticed that in image 54(b) Zelda made an error when she put the value of *y* for when *x* is equals to 0, she substituted the value of *y* to be 1 instead of 0, despite her substitution and calculation process as well as her verbal utterances being correct. This represents a commognitive disjuncture, when what was said did not match the written information on the board (Sfard, 2008), and learners noticed the teacher's error on the table of values. It was interesting that even though Zelda's utterances were correct, the substitution in the table of values was incorrect, which could be linked to the lack of confidence with the content knowledge and the memorisation of the information and not the procedure. Zelda corrected the mistake (image 55 below), which can be seen as 'corrigibility' as the learners mentioned the error and the teacher made the correction accordingly (Ben-Yahuda et al., 2005). According to Ben-Yahuda et al. (2005), corrigibility routines involve the process of checking one's assertions on different mathematical narratives, either verbalised or depicted on the board. This observable action reinforces Sfard's notion of learning as participation and Loughran's (2010) comment that teachers are also learners as they teach (2008; 2012).

After the teacher's correction, the learners verbalised the *y* values for the other two functions $y = x^2 + 1$ and $y = x^2 - 1$, and Zelda completed the table of values. Image 55 below represents the completed table of values for all three functions, and acts as an iconic visual mediator in the subsequent exchange between the teacher and the learners as will be seen below.

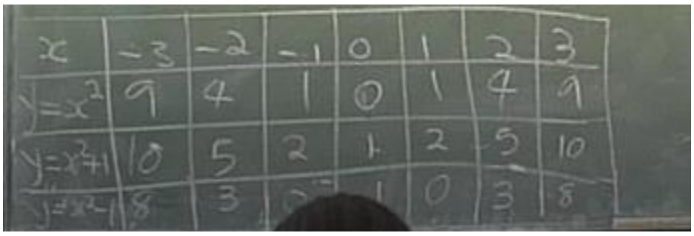


Image 55: *Completed table of values for the three functions*

Zelda continued to interpret the behaviour of the functions $y = x^2 + 1$ and $y = x^2 - 1$ in relation to the function $\mathrm{y} = \mathrm{x}^2$. This process aimed to reveal the effect of varying the value of the constant to the learners, focusing on the effect of the signs of the constant values. Computing the values of *y* for the three functions on the same table of values presented an iconic mediator for making links between the given functions to develop a mathematical story about parameter *q* (Sfard, 2008; Scott et al., 2011). This was facilitated by allowing learners to explore the variation of the values in the table for the three functions, as well as making conjectures and generalisations. It therefore means that even though Zelda did some calculations and substitution for the learners, she believes in giving learners the opportunity to participate in the learning process to build their knowledge and enhance their understanding.

The following excerpt represents the confirmatory exchanges between the teacher and the learners, as they engaged in interpretation elaborations about the relationship between the three given functions:

> **Zelda***: Let me ask you a question, maybe someone will come up with something. Looking at the table, from the original graph this one, the values of y neh, what do you see with the values of y for this equation and for this? What is it that you see with these values when you compare them with the original graph?* (the learners are quiet for 45 seconds). *There is nothing that you are seeing here?*
>
> **Learner***: The second one is adding 1 from the original one and the second one is subtracting 1 from the original one.*

17 **Zelda***: He is saying this one is adding 1 from the original and the second one is subtracting 1 from the original. Another question that I want to ask you, you are saying we are adding 1* (pointing to the values of y for the second function), *meaning kuri (that) our values are increasing or decreasing?*

21 **Learners***: Increasing*

22 **Zelda***: Are increasing neh! Okay, here are?* (pointing to the third function).

23 **Learners***: Decreasing.*

24 **Zelda***: Is it because of that negative and positive?*

25 **Learners***: Yes.*

26 **Zelda***: Okay, let us plot the graph and see what is happening.*

Before I engaged with the dialogue, I noticed that Zelda did not confirm whether the answer was correct or not in line 15 and rather asked another question (line 17), which is concerning since she is introducing the effect of two changes made on the function $y = x^2$. The teacher's confirmation on the correctness of the answers given by the learners is important to provide learners with formative feedback during teaching. That is, Zelda should have offered an elicitation narrative to guide learners to make mathematical meanings pertaining to the effect of changing the values of parameter q. Notwithstanding this, from the question and answer session, Zelda's verbal utterances can be viewed as exploration routines. She asked learners to give reasons about the effect of the parameter, rather than focusing only on the process of calculation and completion of the table of values or authoritatively talking about the variation. This is commendable because Zelda created learning opportunities for learners to observe the changes brought by the changes in the values of the parameters and allowed them to construct their mathematical meanings and understanding, which Mudaly and Mpofu (2019) suggested is effective in facilitating learners' conceptual development.

Furthermore, the questions, "*What do you see with the values of y for this equation and for this? What is it that you see with these values when you compare them with the original graph?"* in lines 11-13 signify a discursive call for learners to think and communicate about what they have observed to be the relationship between the three functions in the table of values, which resonate with the interactive/dialogic communicative approach (Scott et al., 2011). These questions also acted as prompts to promote continuity at micro level through developing a mathematical story about the changes brought by varying the values of *q*. The discourse in line 20 represents the word-use of increasing and decreasing, which facilitated learners' thinking about the values of *y* in relation to the signs of the values of the constant (Sfard, 2008). Thus, the table of values in this instance was used as an equivalent form of representation to iconically mediate learners' thinking and Zelda's verbalisation about the relationship between the given functions. This observable action is contrary to the use of a table of values as just a tool for the generation of values to be used in the drawing of graphs of the functions as observed in other teachers' teaching of algebraic functions in the current study.

Notwithstanding the above analysis and interpretation, it is also important to note that Zelda used the table of values partly in the exploration of the effect of changing the values of parameter q on parabolic functions. Although the teacher used the words 'increase' and 'decrease', she did not link the variation between the values for the different functions in the table of values with the critical feature of 'vertical shift' brought by parameter q from the table of values. This oversight might result in learners only describing the nature of parameters from a symbolic representation, but fails to create mathematical meaning(s) of what the table of values signifies in terms of the relationship between two variables, representing the covariational approach to teaching algebraic functions. For example, the teacher could have demonstrated that when the value of parameter q is changed to +1, and the value of x is -3, the corresponding y-value changes to 10 as compared to it being 9 in the parent function. When the value of parameter q is changed to -1, and the value of x is -3, the corresponding y-value changes to 8 as compared to it being 9 in the parent function. Thus, Zelda could have used the table of values to illustrate the idea of 'vertical shift' from each column on the table when the value of q is either positive or negative.

The above interpretation does not overlook Zelda's prioritisation of learners' participation in some interpretive processes about the relationship between the functions, with the aid of the table of values to discern the effect of changing the value of the constant, which is one of the goals of the curriculum (DBE, 2011). However, I observed the teacher's limited content knowledge when she could have maximised the chances for learners to fully generalise the effect of the parameter from the table of values. The episode below concentrates on plotting and drawing the graphs of the three functions in the table of values (image 55), thereby making links between different modalities of representation to show learners the effect of varying the values of parameter q for parabolic functions, which is supporting learners' knowledge building about such nature of variation (Scott et al., 2011).

7.1.2. Episode 2 (lesson 1): "Let us plot the graphs and see what is happening"

In episode 2, Zelda demonstrated to the learners the routine of translating from the table of values to a graphical representation, to support knowledge building through different modes of representations. Zelda stated: "*Let us plot the graphs and see what is happening*". The choice of words "*and see what is happening*" demonstrates that Zelda considers the graphical representation as a visual aid for seeing how a function behaves, and reveals something which the tabular representation does not. Zelda's typee of routines can be classified as 'explorative' because she examined the effect of parameter *q*, made the conjectures and generalised the effect of changing values of the parameter on the function (Sfard, 2008).

During VSRI, she commented that with the graphical representation, learners could best see the effect of changing the value of the constant: "*With the graph the learners are able to see the points moving, they are able to see how changing the value on the equation affects the graph*". This statement illustrates that Zelda views the graphical representation to be an effective form of representation to teach learners about the effect of the parameters, and this could be further linked with the ongoing development of the function concept in the learners' minds which involves intellectual activity with regard to modes of representation. The approach is helpful to maintain the development of the mathematical story, to help learners follow how the translation between different representations fits into the wider algebraic functions curriculum (Scott et al., 2011).

She then drew the Cartesian plane and engaged in object-level narratives about the processes that needed to be done to complete the graphs of the given functions:

27 **Zelda**: *When I check neh*[38]*, the y values, the highest value there is 9, should I use the scale of 1, 2, 3, 4, 5, 6?*
29 **Learners**: (chorusing) *No!*
30 **Zelda**: *It is very long neh, we can use 2, 4, 6 or 3, 6, 9, it is up to you, as long as your graph is drawn to scale.* (she uses the scale 2, 4, 6, 8, 10). *And then from here, what do we do?*
32 **Learners**: (chorusing) *We plot the graph*
33 **Zelda**: *How? What do I do? I plot neh!*
34 **Learners**: (chorusing) *Yes!*
35 **Zelda**: *How do I plot?*
36 **Learner 1**: *You go back to the table!*
37 **Zelda**: *I go back to the table? And then I look at what?*
38 **Learners**: *x*
39 **Zelda**: *x values neh, and then?*
40 **Learner 2**: *You go back to the table and get the x values and the y values.*
41 **Zelda**: *I take the y values and the x values?*
42 **Learner 2**: *Yes*
43 **Learner 3:** *You go on your x-axis, and you check your x value. If your x value is negative 3, and then you go to y values by looking at graph one, and when you and when you find positive 9, you make a point.* (Zelda plots the point with a smile).

[38] Neh as also used by Mafada is used as a confirmatory word for mathematical steps to be followed in working with algebraic functions.

The use of chorusing and individual responses was interesting to watch because the parts where learners chorused included a set of yes / no questions, as well as where the teacher expected learners to give a one-word answer can be considered 'safetime' for all learners to participate. On the other hand, the individual responses illustrate that the questions that the teacher asked in those instances signaled a level of understanding, as learners had to provide elaborate statements relating to the changes they observed. This exchange further illustrates the observable action of drawing the graphs of the functions, which includes the process of scaling the axes as well as plotting the points on the Cartesian plane with the learners.

From the observation, it appeared that the teacher was using team work deliberately to learn the topic with the learners. I noticed this in the well-structured questions "*what do we do*?" (line 29) "*What do I do*" (line 34) and "*How do I plot*" (line 35) to represent a shift in the teaching and learning turn-taking, at the same time encouraging active involvement. Additionally, I argue that Zelda used the dialogue as a form of assessment to check how much learners remembered and understood from the previous lessons and the current lesson of plotting the points on the Cartesian plane. Thus, she did not take it for granted that the learners knew the procedures associated with plotting, and the narrative about going back to the table demonstrated the flexible connection between the graphical and tabular representations. This was essential to help learners to understand the world of changes as discussed in Chapter 2. From this episode, Zelda has promoted concept continuity at meso-level to support learners' knowledge building of the relationship between tabular and graphical representations (Scott et al., 2011).

Zelda further illustrated the process of plotting by drawing visible dashed lines on the Cartesian plane, as seen in images 56(a) and 56(b) below, which acted as an iconic visual mediator to help learners learn the rituals for plotting the points on the Cartesian plane.

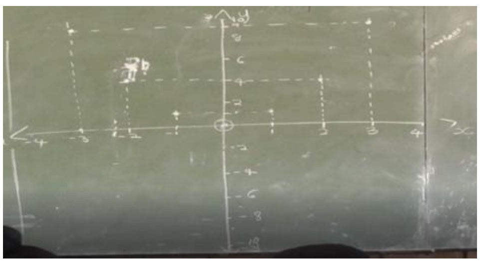


Image 56(a): *Plotting of points for* $y = x^2$

Image 56(b): *Graph of* $y = x^2$

From this process, Zelda focused on $y = x^2$ and asked the learners about the next step after joining the points as represented in the following short exchange:

46 **Zelda:** *And then, am I done?*
47 **Learners:** (chorusing) *No!*
48 **Zelda:** *What do I do next?*
49 **Learners:** (chorusing*) You name the graph.*
50 **Zelda:** (writes the name of the graph). *And you must erase these [dashed lines] so that my graph looks neat.*

This exchange shows that Zelda expected the learners to learn the steps involved in drawing graphs of functions, considering her question "*am I done*?" (line 46) which aimed at prompting learners to recall and verbalise the subsequent step after joining the points. Notwithstanding the teaching practice, I argue that Zelda is teaching learners to memorise the steps because she is teaching them as stand-alone information, which means teaching the knowledge as isolated information. Interestingly, even though Zelda used memorisation, she used interactive/dialogic to share the knowledge with the learners as they collectively talked about the graphical representation, "posing genuine questions and offering, listening to, and working on" learners' verbalisations (Scott et al., 2011, p. 19). The idea of asking learners to verbalise the series of steps needed to complete the task was also notable in the earlier exchange in this episode when Zelda asked the learners to verbalise what should be done next, further demonstrating that her level of discourse was on explorative routines as highlighted earlier above (Sfard, 2008).

To continue and introduce another graph, Zelda repeated the steps she used to draw the graph of $y = x^2$ to plot the points and draw the graph of the second function $y = x^2 + 1$. She said: "*Now I am going to use a different colour to draw the second graph so that you can see the difference*", which is important for learners to see the differences between the graphs of functions. She asked learners whether they understood why she was drawing the vertical and horizontal dashes to form a point, and one learner said they wanted to find where the *x* values and *y* values met. I interpreted this practice to mean that the teacher wanted to make sure that learners were not left behind, as she held their hands when she moved with the lesson. It means Zelda created opportunities for learners to verbalise their understanding and meanings of mathematical objects (Scott et al., 2011). Zelda used the ordered pairs on the table of values to plot the points and draw the graph of the function $y = x^2 - 1$. Of importance to note is that the teacher did not provide the interpretations of the functions from the symbolic mediators but focused on drawing the graphs of the functions, and used the graphical representations to offer interpretive elaborations. The ability to engage in algebraic symbolism and interpretations is linked to good understanding of the represented family of functions, especially to the need to develop learners' meanings and understanding of symbols involved in the equations. The three drawn graphs in image 57 below acted as iconic visual mediators for learners to observe the effect of changing the value of the constant on parabolic functions, and became a teaching tool for Zelda to offer detailed explanation about the effect of the parameter (Sfard, 2012). It was interesting to watch the teacher using different colours to represent different graphs and clarifying the changes of values, using different modes of representation (symbolic and graphic) to support learners' knowledge building.

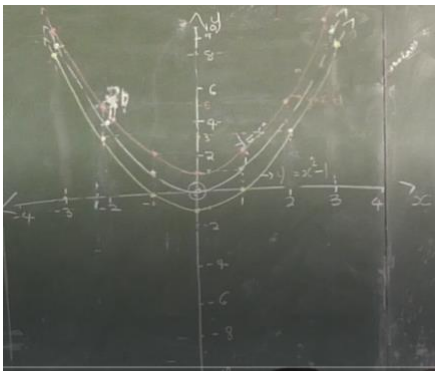


Image 57: *Three graphs for given functions*

After drawing the third graph, Zelda said:

> "*When I was introducing parabolic graphs, I gave you the standard equation for parabola neh, I said y is equals to ax squared, y is equals to ax squared plus minus q, but I didn't tell you much about a and q neh* (writing the two equations in image 58 below). *Let us look at the graphs and we will talk about a and q*".

This statement represents the meso continuity link-making as the teacher connected different concepts and continuation from the previous explanation, to help learners see that concepts and knowledge are always related and continuous. Zelda introduced the general equation of parabolic functions (image 58) using the property-oriented approach that teaches learners about the specific properties underpinning the parabolic functions, as the teacher used it to link symbolical and graphical representations to bring the changes brought by the effect of parameters to the fore (Scott et al., 2011). The writing of the equation $y = x^2 \pm q$ (image 58) acted as a symbolic mediator for learners to see that in the three functions that were introduced earlier, the values of q were the ones that were varied and this was used in the generalisation narratives that Zelda offered.

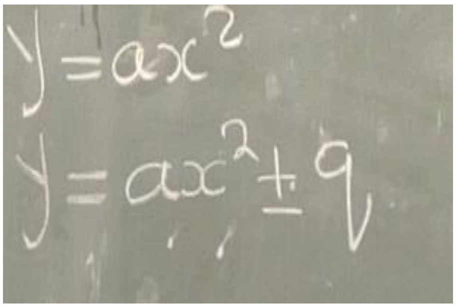


Image 58: *The general equation of a parabola*

In this part of the episode, Zelda offered generalisation statements about the effects of *a* and *q* on the graphs of the functions, but before she re-focused on the three graphs (image 57), she offered the following explanatory talk relating to *a* and *q*:

> *a determines the shape of the graph and then q is the vertical shift of the graph. I said a determines the shape akere (isn't)? while q is the vertical shift of the graph. Now, let us check something here from our three graphs. I remember when I was introducing parabolic graphs, I drew this one and this one* (writing parabolic functions $y = x^2$ and $y = -x^2$ in symbolic form as visible in image 59 *in the same Cartesian plane and then the shape of the graphs were not the same; the other one was like this and the other one was like this* (drawing sketches as visible in image 59). *We had two different shapes, mountain shape and cup shape, so we need to know when we have this and when do we have a cup shape.*

The explanation worked as the demonstration for the learners to see how the sign of the value of a in the general formula $y = x^2 \pm q$ affected the direction in which the graph faced. To do this, Zelda drew sketches (image 59) to show and explain to the learners that when the sign of the parameter *a* was negative, the graph of the function always face downward and when the sign of parameter *a* was positive, the graph always face upward. In mathematics it is always important that the teacher's word-use links with the visual mediators to show the direction of the graph, because learners also rely on this relationship to make sense of the topic. Zelda carefully used this relationship to present the links between two modalities of representations, the symbolic and graphical representation, to help learners observe the effect of varying parameter *a* on a parabola in terms of the sign of the function.

Image 59: *Two functions and their sketches*

In the earlier excerpt above, Zelda used four keywords to talk about the effect of parameters a and q on the graphs of functions: shape, vertical shift, mountain shape and cup shape. She used the word 'shape', 'mountain shape' and 'cup shape' colloquially to speak about the direction the graph faced when the sign of the parameter a was either positive or negative. The word vertical shift was used to demonstrate to the learners that the value of q made the graph shift upward or downward when the sign of the value of q was positive or negative respectively. Zelda's mathematical discourse in this instance relates to the notion of critical features of the graphs, as Zelda offered interpretive elaborations about the effect of parameters a and q on the graphs of the functions. Her ability to move from the algebraic equation and graphs and then to words simultaneously, again, demonstrated exploratory routines because she could relate various modalities of representation of the parabola (Ben-Yahuda et al., 2005).

For furher interpretative elaborations, Zelda used another key word to focus learners' attention on the downward and upward shifts of the graphs of $y = x^2 + 1$ and $y = x^2 - 1$ because of the values of q. She linked the word 'shift', and the new word 'turning point', to denote that the turning point is a point of reference in observing the vertical shift of the parabolic functions. She taught this using the property-oriented approach to help learners observe the changes on the graph that are brought by varying the values of parameter q. Image 60 below depicts what Zelda wrote on the board, which was used to support learners' observations about the idea of a turning point and its relation to the notion of vertical shift on the graph.

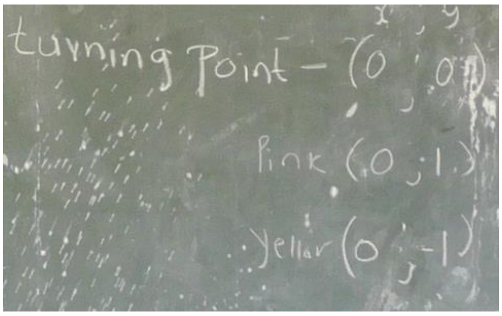


Image 60: *Turning points for the three graphs*

The coordinate pairs for the turning points for the three graphs $y = x^2$, $y = x^2 + 1$ and $y = x^2 - 1$ respectively in image 60 acted as an iconic visual mediator for learners to see the relational links between the value of q in the equations and the turning points on the graph. Zelda used different colours to represent the different turning points to mediate learners' visualisation of the changes brought about by the changes in the values of q in the three functions. She also engaged with the learners dialogically about the meaning of turning points in image 60:

52 **Zelda**: *Let us go back to the three graphs* (image 57). *What can you say about the effect of a? What can you see from the graphs?*

54 **Learner**: *They are all facing up.*

55 **Zelda**: *They are all facing up, okay, alright good! And then this one* (pointing to the graph of $y = x^2$*), maybe I need to talk about some few terms there. The first graph that we did, we never had this term, turning point, because the graph was straight. Now that we are doing this one, you see that the graph is turning neh. So laha yi jikaka kona (where it turns) we call it a turning point. Let's look at the first graph, yi tsema (it cuts) at what point?*

60 **Learners**: (chorusing) *Zero.*

61 **Zelda**: *And then, the pink one?*

62 **Learners**: (chorusing*) One!*

64 **Zelda**: *And then the yellow one?*

65 **Learners**: (chorusing) *Negative one.*

66 **Zelda**: *We are talking about turning points, what do you know about points?*

67 **Learner**: *A point has an x value and a y value.*

68 **Zelda**: *We are saying that in a point we have x and y. The turning point is where the graph turns ka (at) point ya (for) y, so meaning turning point is also a y value. I also want to talk about this q. q determines, that is why I have written plus or minus for you, it determines err, the vertical shift of the graph.*

It was interesting that while the teacher was explaining, she also realised that a new concept was not explained as part of the information. This means she was careful and conscious when she was talking, to ensure that learners were not lost in the process of explanation. This detailed and explicit teaching approach might have been influenced by her declared limited knowledge about this topic, and makes her not take anything for granted, and instead use colours to make sure that learners understand. Zelda's questions: "*What can you say about the effect of a?* and *What can you see from the graphs?* (lines 67-68), demonstrate that she expects the learners to observe the graphs of the given functions and make generalisation statements about the effect of the parameters. The nature of explanation I noted from the statement "*turning point is where the graph turns ka (at) point ya (for) y, so meaning turning point is also a y value*" (lines 67-68) reveals that Zelda was 'saming' turning point with a *y*-value, overlooking that the turning point is not a value but a point which comprises an *x*-coordinate and a *y*-coordinate. Zelda revoiced the learner's iteration that a point refers to *x* and *y* values, instead of using the formal mathematical name coordinates. Within the mathematics community, the term *values*[39] refers to something different from *coordinates*, hence I classified her use of words as 'non-mathematical' (Sfard,

[39] The word value refers to an ordinate either on the horizontal (x) or vertical (y) plane, whereas a coordinate refers to an ordered pair in the form (*x, y*).

2008). The following section focuses on an episode that was selected from Zelda's lesson 2, continuing with the topic and introduced the hyperbola.

7.1.3. Episode 3 (lesson 2): Introducing the hyperbola

This episode started with Zelda introducing two hyperbolic functions in a table of values, $y = \frac{1}{x}$ and $y = -\frac{1}{x}$. Even though Zelda did not verbally announce or write the object of learning for the current lesson on the board, the introduced family of functions in the table of values and the observable actions underpinning the lesson suggested that the lesson was to demonstrate the drawing of hyperbolic functions. To understand this action, Zelda explained this during VSRI: "*isn't I have taught the learners how to draw the graphs for linear and parabola, so here I am introducing them to drawing the hyperbola graph*". The teacher assumed that learners would read and understand the focus of the lesson by using the tables of values, which continued from the previous lessons. The South African curriculum and textbooks do not offer a definition for hyperbola, they simply express the general formula as $y = \frac{a}{x}$ where a is the constant of the variation (DBE, 2011). Consider the tabular representation in image 61 that Zelda used to compare the two functions $y = \frac{1}{x}$ and $y = -\frac{1}{x}$.

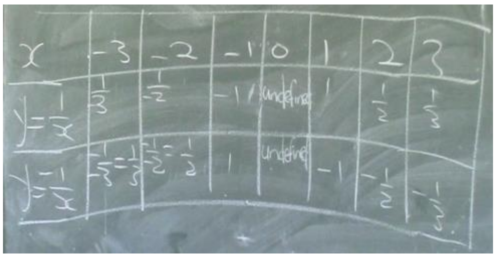


Image 61: *Completed table of values*

Zelda got the learners to verbalise the *y*-values for a given set of *x*-values as she recorded the ordered pairs in the table of values (image 61). Zelda's use of the table of values demonstrated the process of plotting the points on the Cartesian plane, and sketching of the graphs in image 64 below. She used this to highlight the links between the tabular and graphical representations, to help learners develop the understanding of hyperbolic functions. For both functions, while substituting the *y* values for $x = 0$, Zelda told the learners that: "*Undefined or maths error means that there is no relationship between x and y, there is no meaning*". This explanation showed Zelda's limited knowledge about the topic because the word 'undefined' was used to suggest that there was no relationship between the variables and that it had no meaning. Mathematically, undefined denotes a mathematical situation whereby a number is not in the domain of a function. During VSRI, Zelda commented that "*Undefined means that the graph will never touch the line, it's the asymptote*", suggesting that Zelda was making links between mathematical concepts: undefined and asymptote in terms of the line that the graph of the function approaches as x tends to $-\infty$ or $+\infty$. Of importance is that during teaching, the usage of the word 'undefined' was non-mathematical and could constrain learners' understanding of what the notion of maths error signifies in terms of the relationship between the quantities. Thus, the explanation that Zelda offered during VSRI could have helped learners to develop the understanding of the asymptote concept.

After substituting the x values to determine the corresponding y values, Zelda said: "*and then after that* (completing the table of values) *what do we do? We have err x values and the y values for the two equations, what is it that we should do there? We plot neh!*". This statement represents the ritual routine for extracting the coordinates from the table of values and plotting these on the Cartesian plane. Zelda continued to use interactive/dialogic to include learners to talk about the steps that should be followed to move from table of values to plotting the points on the Cartesian plane:

72 **Zelda***: How do I plot?* (no response) *How do I plot?* (no response) *How do I plot, yes!*
73 **Learner 1***: You must change the values of y on the table to be decimals*
74 **Zelda***: Why?*
75 **Learner 2***: Because on the Cartesian plane, you cannot find one over three.*
76 **Zelda***: Not that you cannot find it, it is not easy neh. But if you take these to decimal it is much easier. What is one over three when you convert it to decimal?*
78 **Learners***: Zero comma three.*
79 **Zelda**: (she writes the y values in decimal form as visible in image 62).

From the above extract, Zelda's classroom talk aimed at engaging learners to verbalise the ritual for plotting and sketching the graphs of hyperbolic functions. This practice helped the teacher to correct the learner's word use and understanding, using the word problem-based approach to ensure that the learner was aware of the correct mathematics expression. This further provided the teacher with the opportunity to focus on the need to write the fraction values of y on the table of values as decimals, to ensure that 'it is easier to locate the points on the Cartesian plane' (line 76). Image 62 below depicts the converted values of y from fraction form to decimals. This meant that she used the error she made as the opportunity to correct her understanding, which might be crucial later.

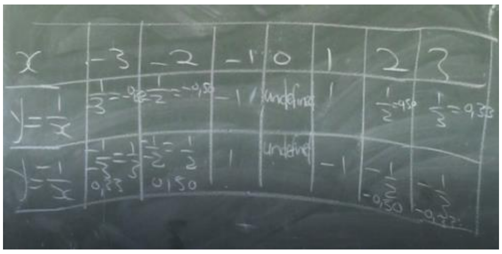


Image 62: *Table of values with decimal y values*

Zelda used the table of values as an iconic visual mediator to illustrate to the learners the association between the values of *x* and values of *y* for the two functions, to facilitate the plotting of the points on the Cartesian plane. After the table of values was completed, she asked the learners to observe the relationship between *x* and *y*, which is represented in the extract relating to the covariational relationship between *x* and *y* values on the table:

80 **Zelda**: *Before I go to the Cartesian plane I forgot something, let's look at the table. As the x values are increasing, what is happening? Let's start with this one (*pointing at the y values for $y = \frac{1}{x}$ *), what is happening here? Let us check the table, as these x values increases, what is happening with these y values?*

84 **Learners**: (chorusing) *Increasing!*

85 **Zelda**: *It is also increasing neh! And then, this one?* (pointing at y values for $y = -\frac{1}{x}$ *).*

86 **Learners**: (chorusing) *Decreasing neh!*

87 **Zelda**: *Let us put this in our heads. With the first one, if x is increasing, even the y is going to increase. This one if x is increasing, this one is decreasing* (pointing at y values for $y = -\frac{1}{x}$ *), we will come back and speak about this. Remember yesterday I gave you a general form where I talked about y equals to a over x or y equals to negative a over x* (writing the equations on the board as visible in image 64*, we will see the graph and talk akere (isn't).*

Image 63: *The general equations for hyperbolic functions*

The use of the words 'increasing' and 'decreasing' in lines 80-11, 84 and 86-87 in the above exchange offered cues for associating the direct proportionality and inverse proportionality between *x* and *y* respectively with the positive and negative signs on the 'general equations' in image 64 representing the property-oriented approach (Kwari, 2007). Thus, Zelda used the equations in image 64 in relation to the values in the table of values to syntactically mediate learners' thinking about the relationship between variables (Sfard, 2008; 2012). The generalisation statement "*if x is increasing, even the y is going to increase. This one if x is increasing, this one is decreasing*" (lines 86-87) resonates with the covariational approach to teaching algebraic functions, as the teacher focused on demonstrating to the learners how the variable quantities covary. Borba and Confrey (1996, p. 323) described the covariational approach to entail the focus on the relationship between variables in which "one quantity changes in a predictable or recognisable pattern, the other also changes, typically in a differing pattern".

Zelda's question in the above extract "*as these x values increases, what is happening with these y values?*" (lines 11-12) calls for learners to observe and describe the manner in which x_1 changes to x_2 and how $y_1$changes to y_2 in order to describe the functional relationship between the variables, paying attention to how the negative and positive signs affect the behaviour of the functions to facilitate generality. In addition, the utterance "*we will see the graph and talk akere (isn't)*" (line 20) in the above extract denotes that Zelda views the graphical representation as a better visual aid to demonstrate the effect of varying the sign of the function. In the next observable action Zelda plotted and drew the graphs of the functions $y = \frac{1}{x}$ and $y = -\frac{1}{x}$, starting with the graph of $y = \frac{1}{x}$ in image 64 below.

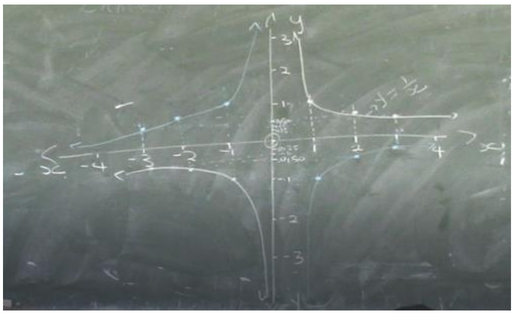


Image 64: *Graphs*[40] *of* $y = \frac{1}{x}$ *and* $y = -\frac{1}{x}$

The above diagram shows Zelda's ability to sketch and draw the hyperbola, showing some key features such as the general shape and the asymptotes. Interestingly, when it came to the narrative about the point where *x* = 0, Zelda reiterated that "*We do not plot anything there, we just skip because there is no relationship between x and y, we can't have anything*". From this statement, Zelda's routines are ritualised because, although she does not offer an endorsed narrative relating to the notion of asymptote, she is able to sketch and draw a hyperbola. To offer an explanatory talk about the asymptote, Zelda said:

> *Our graph cannot pass that line and it cannot pass that line* (pointing at the *y* and *x* axes). *This graph, this curve, it mustn't touch any line. If it touches the line it is not correct, but it must be close to the lines*.

[40] To help coordinate her talk about the two graphs, Zelda used two different colours for the two functions.

This statement relates to the use of routines, and I classify Zelda's mathematical discourse as applicability routine because she was able to use the table of values to sketch and draw the graph, even though she did not explain why the curves should not touch the axes for the two functions. Zelda's difficulty in offering a literate mathematical interpretation about why the curves should touch the axes could be insufficient knowledge with the topic and contents to help learners memorise the contents. The utterance "*If it touches the line it is not correct, but it must be close to the lines*", could be classified as memorisation based on visual aids, particularly the graphs, because the statement calls on learners to pay attention to the features which makes the graph right or wrong and they should learn through rote memorisation.

To continue presenting interpretive elaborations about the shape of the drawn graphs in relation to the variation of the values of *a*. Consider the following extract:

92 **Zelda***: Now, let us talk, we talk about the white and the blue. In Grade 9, I remember teaching you about the Cartesian plane, I said* (drawing a sketch of a Cartesian plane), *this is quadrant one, this is quadrant two, third quadrant and fourth quadrant. Let us check the white arcs. The white arcs are in which quadrant? First and third quadrant. Okay let us talk now, going back to our a, I want to talk about when a is greater than zero and when a is less than zero* (she writes a greater zero and a less than zero as visible in image 65). *What is happening when a is greater than zero? Greater than zero is one up.*

Image 65: *a greater than and a less than 0*

99 **Learner 1***: If our a is greater than zero, it increases from third quadrant to the first quadrant.*
101 **Zelda***: It increases from third quadrant to first? Is that correct? Who else can tell us what to say?*
103 **Learner 2***: If a is greater zero, the graph shifts vertical upward.*
104 **Zelda***: The graph shifts vertically upward, okay. Let us check this one. Our equation is one over x, the value of a there is positive one, so it means that if your a is greater than zero, we are going to have our arcs in the first and the third quadrants. Our arcs will be in the first and third quadrants, it's when a is greater than zero. What is happening when our a is negative? If our a is less than zero, what is happening?*
109 **Learner 3***: The arcs will be in the second and fourth quadrant.*
110 **Zelda***: Our arcs will be in the second and fourth quadrant.*

Rather than providing an authoritative account of the effect of parameter *a,* Zelda took an interactive/dialogic communicative approach to offer object-level narratives about the two functions, reminding learners about the Cartesian plane quadrants which she taught these learners in Grade 9 (lines 92-93). In interactive/dialogic communicative approach, Zelda used the macro time-scale to show learners the knowledge continuity and guide learners to observe and verbalise the changes brought by the varying of values of parameter *a* on the position of the graphs on the Cartesian plane (Scott et al., 2011). Her utterances aimed at explaining to the learners that when the value of parameter *a* is positive, the graphs of given functions will be in the first and third quadrants and when the value of the parameter *a* is negative, the graphs will be in the second and fourth quadrants to guide learners towards generality about the effect of parameter *a* for hyperbolas. As such, I see the inequality representation in image 65 as a useful way of mediating learners' thinking about the nature of parameter *a* and its effect on the parabola either when it is negative or positive. In line 98, the statement "*Greater than zero is one up*" is mathematically incorrect, because there are fractions between 0 and 1 that satisfies the statement 'greater than zero'. While this was the case, it is important to say that Zelda maintained the use of explorative routines to guide learners towards generality about the effect of parameter *a*, considering her insistence on substantiation (*What is happening when our a is negative? If our a is less than zero, what is happening?*) in lines 107-108. To conclude the lesson, Zelda gave learners the class activity in image 66 which was later given as a homework because the period had ended.

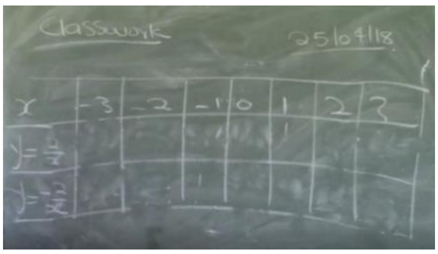


Image 66: *Learners' activity*

As visible in image 66, Zelda did not write the instructions for the completion of the task she gave to the learners, again, assuming that learners knew what they were expected to do when the functions were presented in table of values. During VSRI she commented that "*isn't we have been completing the table and drawing the graphs, that's what I want them to do here*", suggesting that learners were expected to follow the established rituals introduced during teaching to complete the table of values and subsequently draw the graphs of the two functions in the table of values. It is important for teachers to give learners clear instructions on what they are expected to do with mathematical objects, to ensure that learners become aware of the key mathematical features they are expected to explore and learn. The new variation that is introduced in the activity is the change in the magnitude for the value of the parameter *a* to be

2 and -2 and the teacher could have specified to the learners that she expected them to explore the effect of varying the parameter in terms of its sign and magnitude as a way of framing learners' thinking.

7.2. Summary and conclusion regarding Zelda's observed episodes

From Zelda's analysis and interpretations of the two lessons, the dominant discourses were mathematical conversion of substitution and calculation of values of y for the functions. She also used the covariational approach and property-oriented approach to guide learners to explore the effect of parameters on the functions. Zelda further engaged her learners in interpretations of the given functions using an interactive/dialogic communicative approach, highlighting the critical global features for the families of functions she focused on in each lesson. In terms of discursive routines, Zelda's teaching appeals to the use of explorative routines to help learners to observe some critical features for each family of functions focusing mainly on applicability routines. Table 18 is a summary of Zelda's episodes and outlines the observable actions during teaching.

Table 18 *Summary of Zelda's teaching episodes*

Sfard's commognitive theory				
Episodes and observable actions	**Visual Mediator** *(the images presented in-text also represent iconic and/or symbolic mediators)*	**Words used**	**Endorsed narratives**	**Routines**
1. Introducing the functions: $y = x^2$; $y = x^2 + 1$ and $y = x^2 - 1$ in the table of values and engaging in the process of substituting and calculating the output values and completing the table of values.	Iconic visual mediators: functions are: $y = x^2$; $y = x^2 + 1$ and $y = x^2 - 1$ depicted in the same table of values. Symbolic syntactic mediator: substitution and calculation process for output values.	Same axes; compare; values of y	(none)	Ritual to substitute and calculate the values of y for chosen values of x and in turn completion of table of values.
2. Exploration of the effect of varying the values of parameter *a* on parabolic functions, both with the aid of the table of values and the graphical representation. The effect of parameter *a* is also explored in terms of the notion of turning point.	Iconic: graphs of the three functions $y = x^2$; $y = x^2 + 1$ and $y = x^2 - 1$ on the same set of axes Symbolic: $\mathrm{y = ax^2}$; $y = ax^2 \pm q$ Iconic: The sketches of the graphs for $\mathrm{y = x^2}$; $\mathrm{y = -x^2}$ to juxtapose the direction of the graphs as a result of the sign of the value of *a* Symbolic: representing the coordinates of the turning points for the functions $y = x^2$; $y = x^2 + 1$ and $y = x^2 - 1$ in the form (0, 0); (0, 1) and (0, -1) respectively	Plot; highest value; y-values; x-values; standard equation; Cartesian plane; cup shape; mountain shape; turning point	The effects of parameters *a* and *q* on the parabola: *a determines the shape of the graph and then q is the vertical shift of the graph. I said a determines the shape akere (isn't)? while q is the vertical shift of the graph.* Saming turning point and y-value:"*Turning point is where the graph turns ka (at) point ya (for) y, so meaning turning point is also a y value*" This is mathematically incorrect.	Exploration of the effect of parameter *a* in terms of its sign from the table of values and from graphical visual mediators.
3. Introduction of a new family of functions (hyperbolic functions) using two examples: $y = \frac{1}{x}$ and $y = -\frac{1}{x}$ presented in the same table of values. The teacher and the learners complete the table of values and subsequently draw the graphs of the two functions on the same set of axes. The	Iconic visual mediators: functions are: $y = \frac{1}{x}$ and $y = -\frac{1}{x}$ depicted in the same table of values. Iconic: graphs of the two functions $y = \frac{1}{x}$ and $y = -\frac{1}{x}$ on the same set of axes to	Undefined; Cartesian plane; increasing; decreasing	Describing the notion of asymptote: "*If it touches the line it is not correct, but it must be close to the lines*"; " The effect of parameter m on hyperbola: "*The arcs will be in the second and fourth quadrant"* Direct and inverse relationship: "*If x is increasing, even the y is going to increase. This one if x is increasing, this one is decreasing*"	Exploration of the effect of parameter *a* in terms of its sign from the table of values and from graphical visual mediators.

teacher guides learners to generalise the effect of varying the sign of parameter a.	juxtapose the direction of the graphs as a result of the sign of the value of *a* Symbolic: using inequalities $a < 0$ and $a > 0$ Iconic visual mediators: functions are: $y = \frac{2}{x}$ and $y = -\frac{2}{x}$ depicted in the same table of values.		"*greater than zero is one up*"	

Approaches of algebraic functions used

Across the three selected episodes, Zelda used three approaches to teach parabolic and hyperbolic functions. These are: example versus non-example, property-oriented approach, and the covariational approach to bring to the fore the effect of different parameters on the two families of functions she taught. Through the use of the interactive communicative approach, the teacher used the abovementioned approaches to guide learners towards generality about the effect of changing values of parameters on the parabolic and the hyperbolic functions.

Scott et al.'s pedagogical link-making and communicative approaches

Rather than simply providing an authoritative account of the effect of parameters on the different functions, the teacher framed the classroom discussion in terms of the action of interpretations from different views represented by the students. The different points of view about the effect of varying parameters on the functions were brought together. Across Zelda's presented episodes, four approaches to pedagogical link-making were used. These were to:

- promote continuity: developing the mathematical story (micro)
- promote continuity: developing the mathematical story (meso)
- support knowledge building: making links between modes of representation
- promote continuity: manage/organise (micro)

Chapter 8

Data presentation and analysis – The case of Tinyiko's teaching

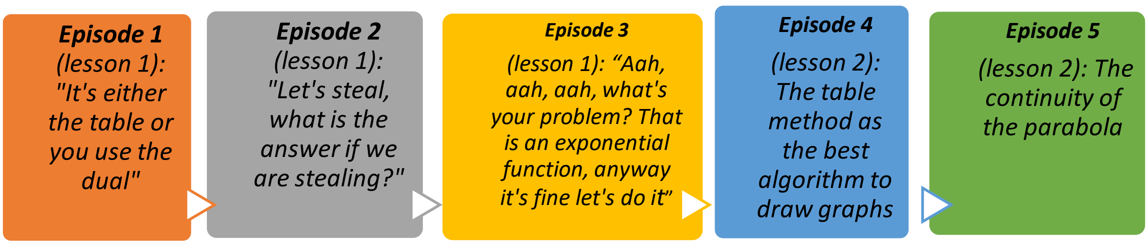
"When we draw the parabola now, it is gonna be different to the one we drew on linear" (Tinyiko, observation 1, episode 1).

8.1. Introduction

This chapter presents, analyses and interprets Tinyiko's six episodes extracted from the two lessons that I observed and recorded. I further draw from the information she provided during semi-structured interviews and VSRI to gain comprehensive understanding of Tinyiko's teaching of algebraic functions. Figure 13 below represents the foci of the two lessons that I observed.

Figure 13

Tinyiko's selected episodes from two lessons



As mentioned in Chapter 4, Tinyiko is the only teacher amongst the five participating teachers who uses technology (smartboard and projector) to teach algebraic functions. Her school has teaching and learning materials that are often considered to represent urban schools, disproving the deficit discourse about rural schools as under-resourced, backward, and disadvantaged (Mbhiza, 2019).

8.1.1. Episode 1 (lesson 1): "It's either the table or you use the dual"

Tinyiko started lesson 1 by briefly recapping the previous lessons on linear function to set the scene for a new family of functions, parabolic functions, representing the meso continuity link (Scott et al., 2011). This is important because it creates learning opportunity for learners to understand the distinguishing features between linear functions and parabolic functions and construct a coherent mathematical story of functions. To do this, she wrote the words linear on the board stating that "*remember that produces a straight-line*" (writing on the board str. for straight as visible in image 67) "*and the equation for linear is y equals to mx plus c*" (image 68).

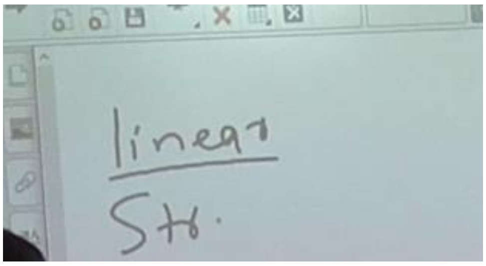


Image 67: *Writing str. for straight*

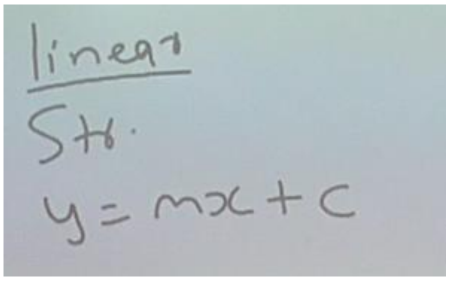


Image 68: *Recap on linear functions*

She then introduced the object of learning for the current lesson stating that, "*When we draw the parabola now, it is gonna be different to the one we drew on linear*". This statement illuminated that Tinyiko was introducing a new family of functions, to make the links between different families of functions. This was important to guide the learners to notice the distinguishing components and create linkages between the current and the previous lesson. To introduce parabolic functions, Tinyiko used the property-oriented approach to differentiate between linear and parabolic functions in terms of their symbolic representations (appearances). She wrote $y = ax^2 + c$ (image 68) which represented a symbolic visual mediator accompanied by the following statement: "*Now, there is a general formula that we know, y equals to ax squared plus c instead of mx plus c, now we have ax squared plus c*" which continues to show the differences between the linear and parabolic functions.

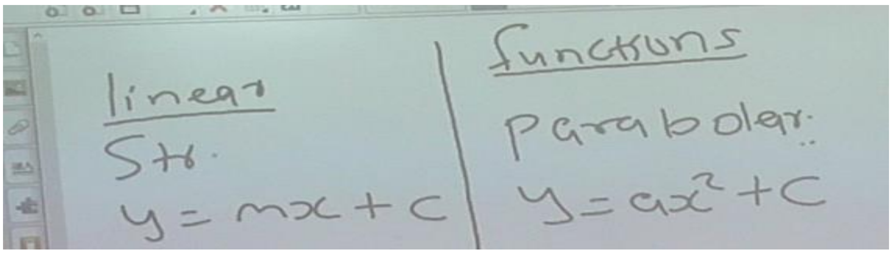


Image 69: *Symbolic mediators comparing linear and parabolic functions*

She was reinforcing for the learners to always remember the differences relationally and observe the structural variation between the two equations representing the two families of functions, and helped learners to develop a mathematical story for algebraic functions (Scott et al., 2011). This action relates to Ling Lo's (2012) iteration that teachers should assist learners to develop 'powerful ways of seeing', to enable independence in dealing with new problems in the future. From image 69 Tinyiko associated the word functions only with the parabolic functions as depicted later in image 72, which could create an impression for the learners that linear (functions) do not form part of the broader algebraic functions topic. Learners are usually guided by what teachers say and write during teaching and learning process (Sfard, 2012; Berger, 2013), making it important for Tinyiko to be aware of what she said and wrote. Below is a confirmatory exchange that Tinyiko and the learners engaged in:

Tinyiko*: You still remember that in linear graphs, we had three methods that we were using, there is the one that we said the dual intercept method. Do you still remember that one? And we also have, the other one was what?*

Learners*:* (chorusing) *The table.*

Tinyiko*: The table* (she writes on the board*). And the other one is?*

Learners*:* (chorusing) *The gradient.*

Tinyiko*: So, on the straight line, it was too much, when we come to parabola the only methods that we can use to calculate or to determine the intercepts or the turning point or the what of the parabola it's only the dual and the table, we are no longer going to use the gradient. Can you have the gradient of the curve?*

Learners*: No*

Tinyiko*: Anyone who is too clever, who can calculate the gradient of a curve? I know you guys are smarter than me. No one neh!*

This exchange focused on the metarules of discursive actions that should be performed one after another and introduced learners to the specific applicability routines for the two families of functions that the teacher presented. Tinyiko involved learners during the lessons as a way of assessing their knowledge and understanding about the previous lesson, which Scott et al. (2011) called meso-continuity links as demonstrated by the words "*You still remember that in linear graphs*" (Scott et al., 2011). What is interesting is that she explained to the learners the applicability conditions relating to *when* the 'three methods for drawing graphs' can be used (Sfard, 2008). Tinyiko expounded to the learners that the only methods that were going to be used for drawing the graphs of parabolic functions were the table and dual intercept methods, and explained why the gradient method was not applicable for such class of functions, thereby setting the scene for concepts delimitations. According to Sfard (2008), applicability conditions are "rules that delineate, usually in a nondeterministic way, the circumstances in which the routine course of action is likely to be evoked by the person" (p. 209).

From the previous conversation, Tinyiko introduced the parabolic function defined by $y = 2x^2$ accompanied by the following exchange between her and the learners:

14 **Tinyiko**: *If for instance you are given y equals to two x squared and I say draw a graph of that one, in other words, when I give you this there is an addition of zero* (see image 57 for symbolic mediator), *what is the y-intercept?*
17 **Learners**: (chorusing) *Zero!*
18 **Tinyiko**: *The y-intercept is zero. Why do you say zero? I said you can only use what? The table and the dual. By the way, how does the dual work?*
20 **Learner**: *We let x be zero.*
21 **Tinyiko**: *We said let x be equals to zero because you want to find the what?*
22 **Learners**: (chorusing) *To find the y-intercept.*
23 **Tinyiko:** *And let's remember that the x-intercepts are also the output values. What about the y-intercepts?*
25 **Learners:** *The outputs.*

The questions '*what is the y-intercept?*' (line 16), '*because you want to find the what?*' (line 21), '*I said you can only use what?*' (line 23), '*by the way, how does the dual work?*' (line 19) and '*What about the y-intercepts?*' (lines 23-24), all represent an elicitations technique at meso scale to check whether learners gained information from the previous lessons. The conversation above served as an example of mathematical communication where the teacher used the words *y*-intercept and *x*-intercepts as if they refer to outputs and inputs based on their relatedness, which Sfard called 'saming'. I noticed though that Tinyiko overlooked the idea that *y* values are the output values and the *x* values are the input values, but the notion of *x*-intercept entails a zero of a function where an input value produces an output of 0. Also, using the word *y*-intercept to signify synonymity with output values does not explain to the learners that a '*y*-intercept is a point where the input value is 0' on a given function, which also addresses the commognitive construct of saming. Furthermore, the statement "*The y-intercept is zero*" in line 18 of the exchange reveals that intercepts are treated as a numerical value[41] rather than coordinate pairs. This cannot be left unproblematised, considering that what teachers say and do during teaching shape learners' development of correct mathematical word use, to talk effectively about mathematical entities. According to Sfard (2019, p. 1), "it is a common lore that teachers bear the main responsibility for what the students learn or fail to learn", suggesting their influence with regard to learners' understanding or lack thereof for knowledge.

Tinyiko returned to the function $y = 2x^2$ in symbolic form and engaged in algebraic calculations to determine the *y*-intercept and the *x*-intercept for the function. She said "*If we let x be equals to 0 on this equation that I have, let's find the y-intercept. You say 2 into 0 plus 0, then the*

[41] This observable action was frequent also across the other episodes in this lesson (see the beginning of episode 3 for instance) and in lesson 2.

y-intercept is equals to 0. And now, we let y equals to 0, because we want to find the x-intercept[42] ” (see image 70).

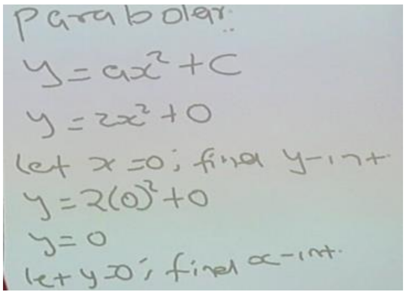


Image 70: *Determining the intercepts for* $y = 2x^2$

In view of the above extract and the calculations in image 70, it is notable that Tinyiko was again saming two words which do not represent the same mathematical entity, treating intercepts as numerical values rather than representing them as coordinate pairs. From the commognitive theory the teacher’s discourse can be classified as ‘non-mathematical rituals’, because she referred to intercepts as values. Of concern is that this could result in learners not developing a sense of covariational relations between quantities, which is important for the learners to develop in-depth understanding of functions (Kwari, 2007; Chimhande, 2014). In the process of calculating the intercepts, Tinyiko used the authoritative/non-interactive communicative approach (Scott et al., 2011) to demonstrate the process that showed learners all the steps they would follow to complete the tasks. While it is expected for the teacher to show learners the steps, especially when they are introduced to the topic, of concern is that Tinyiko did not allow learners to demonstrate their own understanding of the concepts, which is essential for conceptual development.

After sketching the graph (image 71), Tinyiko guided learners about generalising the effect of parameter a on the parabola in terms of its sign, she said: “*Because you were given the function as* ax^2 *the coefficient of your x squared is positive, it simply tells you that your graph will go up*”. This process was a way of making links between the graphical iconic mediator and verbalisation statements, to bring the effect of parameter *a* on the parabola into focus (Sfard, 2008; Scott et al., 2008). This statement calls for learners’ attention to associate the iconic mediator for the increasing parabola that was sketched on the board (image 71), with the sign of the parameter a to discern the effect of the sign on the graphs of parabolic functions.

[42] The x-intercept was also written as just 0.

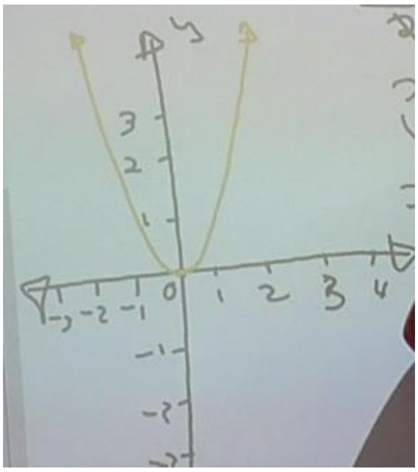


Image 71: *Parabola sketch for* $y = 2x^2$

While Tinyiko brought the effect of parameter *a* into focus and made generalisations using ritual routines (Sfard, 2008), my concern is that she did not give learners the opportunity to observe and make their own conjectures as advocated by DBE (2011). The teacher's discourse did not allow learners to engage in the activity of "conjecture-test-evaluation" (Ärlebäck & Frejd, 2013, p. 3), to enable them to explore the effect of the parameters and construct their own mathematical statements and prove them to reach generality about the effect of *a*.

To verify to the learners that the sketch of the parabola in image 72 is correct, Tinyiko proceeded to compute the table of values and asked the learners to verbalise the associated *y* values for given *x* values while completing the table. The ordered pairs recorded on the table of values were plotted on the same axes as a sketch she drew using the dual intercept method detailed earlier above. The reflective conversation during VSRI illuminated that Tinyiko used the 'table method' in conjunction with the dual intercept method to explain to learners that the graph will indeed face up. The following excerpt demonstrates this:

26 **Interviewer***: So, what was the rationale for moving from the dual intercept method to the table method?*
28 **Tinyiko***: You didn't hear that?*
29 **Interviewer***: No*
30 **Tinyiko***: … isn't I told them that the graph will go up because we are given y is equals to a x squared and the coefficient of x squared is positive, so the graph will face up. There are those who will cram that should be the case even when you are given y equals to negative a x squared. So, the reason for the table method is to assure everyone that whenever it's positive the graph will face up.*

Tinyiko's utterances in this excerpt reveal that, for her, the table ritual routine can facilitate learners' visualisations relating to how the graph came to have a particular shape or why it faces a certain direction through point-by-point plotting. Thus, calling the observable action of calculating and substituting into play was to provide an empirical justification, to show that indeed the graph of the function $y = 2x^2$ faces up as evidenced by the statement "*the reason for the table method is to ensure everyone that whenever it's positive the graph will face up*" (lines 33-34). It is commendable what Tinyiko did and the above statement demonstrates that she made links between the two modalities of representation: graphical and table of values, to mediate learners' procedural learning pertaining to the process of plotting and sketching the graph, as evidenced by the use of the word 'ensure' above. According to Chimhande (2014), the point-by-point plotting fits the action level of a function as teachers and learners that operate in this level "see a function as a machine and understand that some value is put into the machine and the machine churns out a value" (p. 241). Tinyiko's mathematical discourse appeals to a mathematical process known as substitution, either by means of using the dual intercept method or computing the ordered pairs on the table of values for the purpose of drawing the graphs for the functions.

8.1.2. Episode 2 (lesson 1): "Let's steal, what is the answer if we are stealing?"

This episode commenced with Tinyiko introducing a parabolic function $y = 3x^2 + 4$ and asked learners to 'do something' to the object: "*Can you do that one? Let's do it. You use any other method you want, it's either the table or you use the dual.*" This suggested that learners should use either one of the two endorsed methods to draw the graph of the given function. This observable action relates to Sfard's (2008) argument that "according to the universally enacted rules of school discourses, once a 'technique' is demonstrated, it becomes the default choice for tasks that immediately follow" (p. 215). Similarly, Tinyiko did not specify exactly what the learners were expected to do after determining the *y* values using either the table or dual intercept method, which resulted in the process of drawing the graph of the given function as a default choice. Tinyiko then drew the table as shown in image 72 and asked the learners to verbalise corresponding values of *y* for a set of values of *x* in the table. This activity was done to demonstrate to the learners that indeed the sketch she drew in image 71 was correct, as it would be given by the coordinate pairs computed in the table of values. The teacher's actions can be interpreted as making links between the tabular and graphical representations, to support the development of learners' knowledge of the parabola shape (Scott et al., 2011).

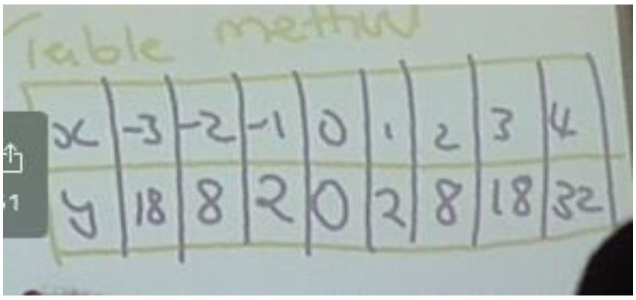


Image 72: *Table of values for* $y = 2x^2$

She further made links between the symbolic and graphical modalities, stating:

> *The reason why whenever we substitute these numbers, whether we substitute the negative or a positive we always get a positive answer, it is because of this x squared, because you all know that a negative one multiplied by negative one will always give you a positive one*.

This statement represents the object-level interpretation of the behaviour of the function in terms of what happens when input values are squared (Sfard, 2008). Tinyiko continued to use the dual intercept method to determine the *x*- and *y*-intercepts, to link the tabular and symbolic representations to enable learners to see how both methods worked in drawing the parabolic graphs. Image 73 below depicts the calculation processes she engaged with and acted as a visual tool to aid in the narration and analysis that follows:

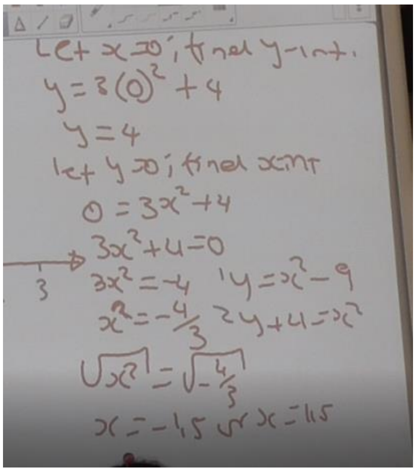


Image 73: *Tinyiko's calculations of the intercepts for* $y = 3x^2 + 4$

From the calculations, two things were notable: writing the intercepts as values (y = 4 and x = -1.5 or x = 1.5) and the mathematical incorrectness in calculating the x-intercept. Tinyiko 'calculated' the square root of $-\frac{4}{3}$ and verbalized that learners should ignore the negative sign and engage in a 'non-mathematical routine' she called 'stealing', to ensure that the calculator did not yield a 'math error' answer. This was concerning as it sets a wrong precedence that mathematical objects can be manipulated to give 'expected results', and learners might engender such attitude and overlook the need for mathematical correctness in their algorithms. Tinyiko's algorithm for identifying the intercepts is presented in image 73 above, and she described the process in the following extract:

> 35 **Tinyiko**: *Your calculator when you punch the square root of negative 4 over 3, what does it say? Math error! How do we steal again? We forget about the negative and use only the numbers and find the answer. Let's steal. What is the answer if we are stealing?*
>
> 38 **Learners**: (chorusing) *One comma five!*
>
> 39 **Tinyiko**: *But we are doing it professionally, x equals to negative one comma five or x equals to one comma five.*

While it is clear from the calculations that the teacher understood the rules (*for y-intercept let x = 0 and for x-intercept let y = 0*) and the substitution was done correctly (image 74, the problem arose when she realized that -$\frac{4}{3}$ is a complex number and square rooting the number yields an undefined result. Ignoring the negative sign could be interpreted as a commognitive conflict in the process of working with intercepts. Sfard (2007) defined commognitive conflict as the phenomenon that takes place when conflicting narratives come from incommensurable discourses. In this case, Tinyiko's narrative about 'stealing' (line 37) conflicted with the endorsed narrative of square rooting complex numbers, thus manipulating the square root of $-\frac{4}{3}$ to be -1.5 or 1.5. It is expected that the teacher would have realised the mistake or that her discourse of 'stealing' was not mathematically endorsed, and should have used corrigibility routines by evaluating whether her answers made mathematical sense (Sfard, 2008). I classify Tinyiko's routines as 'non-mathematical rituals' because they did not lead to mathematically endorsed narratives. The error can also be interpreted to represent the teacher's intentionality in the selection and use of examples, specifically the failure to offer a mathematical explanation about what the result of squaring a negative number would mean in the context of calculating intercepts. Tinyiko did not anticipate that the calculation of the *x*-intercepts for the function in the form $y = 3x^2 + 4$ would result in a complex number, thereby making the *x*-intercepts to be undefined, resulting in the non-mathematical routine of 'stealing' for the purpose of continuing the communication about mathematical objects.

The selection and use of the example occurred impetuously as a consequence of an ongoing discourse in the classroom (Venkat & Adler, 2012). After the calculations were done, Tinyiko used only the *y*-intercept to sketch the graph of the function (image 74), suggesting that at an intrapersonal communication level, Tinyiko was aware that the notion of stealing is non-mathematical.

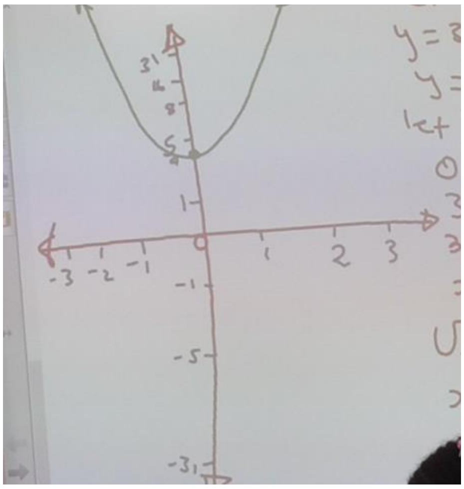


Image 74: *The parabola sketch for* $y = 3x^2 + 4$

What was interesting is that she did not[43] use the calculated intercepts in the process of sketching the graph. This analysis serves as an example of mathematical teaching where a teacher espouses an example without thinking about the characteristics of the example, and what they want to demonstrate about some features of the function concept. A further interpretation of what happened in this observable action is that the use of an unplanned example constrained the use of correct mathematisation in calculating and communication about the *x*-intercepts for the given function. The VSRI comment about this incident revealed that Tinyiko clearly understood in her mind that the idea of 'stealing' is mathematically incorrect, however she did not want learners to be aware that she did not know what to do or say with or from the mathematical object. The information below illustrates the point:

41 **Tinyiko**: *Actually this sum, just came from my brain, this is not what I wanted to write, I think my, the sum that was in my brain it was 3 x squared minus 9, that was the sum that was in my brain and the 4 came out and to reverse it or to reverse it, shhhh, uhm!*

44 **Interviewer**: *But why didn't you just erase it after realising that it won't work? Because now you are introducing something called professional stealing* (she laughs).

46 **Tinyiko**: *Mr Wiseman, you know what we do as teachers, after you have faulted, you have a way to erase that in the kids' brain, that's why I didn't use the values to plot the graph. I didn't want learners to think I was confused, so I had to continue with that example.*

[43] Commognitively, it can be said that Tinyiko engaged in the process of intrapersonal communication and realised that the x-coordinates for the x-intercepts did not match the imagery of the parabola that should be produced by the function in the form , thereby deciding to disregard using the calculated values. This relates closely with Lavie et al.'s (2018) argument that "a person can have a lengthy episode of communicating with herself mathematically while trying to solve a problem" (p. 12).

In relation to the idea of 'stealing' discerned above, one could argue that ignoring the sign for $-\frac{4}{3}$ was to ensure that there is an answer to the given question, in this regard a numerical answer, which was Tinyiko's way of engaging in corrigibility, but resulted in incorrect mathematical narratives. What Tinyiko overlooked in this instance is the idea that the function $y = 3x^2 + 4$ cannot have *x*-intercepts because the vertex of the parabola is at point (0; 4) and considering that this is an increasing function, the graph will not cut through the *x*-axis. It is also significant that throughout the exchange with learners during teaching and her utterance "*I didn't want learners to think I was confused*" in the VSRI exchange (lines 47-48), Tinyiko aimed at establishing 'the answer' as an attempt to recycle some previously encountered routines of square rooting, thereby resulting in incorrect mathematisation. It unfortunate that Tinyiko thought this way, as if it is unexpected for teachers to make mistakes or not know something, resulting in continuing with the wrong calculations. This is concerning for learners' development of the correct mathematical discourse in general, specifically the discourse of algebraic functions. Lavie et al. (2019) said that "the source of an individual's routine is in what other, more experienced performers (in this case the teacher) are doing, we all end up acting in similar, compatible ways" (p. 156). Thus, the teacher's level of mathematical discourse can be classified as ritualised routines to ensure that there is a numerical answer that comes from the mathematical calculations.

8.1.3. Episode 3 (lesson 1): " Aah, what's your problem? That is an exponential graph, anyway it's fine, let's do it".

In episode 3, Tinyiko wrote on the board that the new class of functions she wanted to focus on was the hyperbolic functions as depicted in image 77. In so doing, she offered the following explanation:

> *Now, there is the general formula of the one, we call it hyperbola, it says f at x. Now you understand that when we say f at x you understand we mean y right! f at x is equals to a over x plus q. f at x is equals to a over x plus q, that is the general. This one, hyperbola graph, yes it does have the x-intercept and it also has the y-intercept, we can use the dual and the table method. But there is a unique thing that we must always have is the asymptote, asymptote meaning 'asim-touch', we don't touch, the line which this graph will never touch. We have two asymptotes, the x asymptote and the y asymptote.*

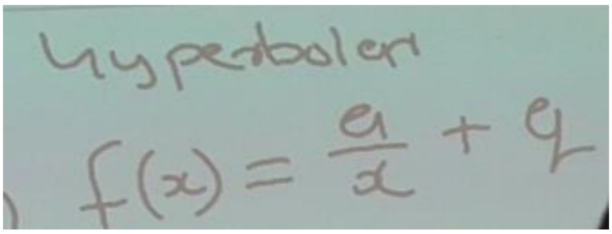


Image 77: *Intended focus of the segment of the lesson*

This statement introduced hyperbolic functions as a new object of learning, and Tinyiko made up an informal word ‘asim-touch’ to simplify the mathematical meaning for the word asymptote on the hyperbolic graph with the potential difficulties associated with the meaning of the word asymptote. Thus, the teacher’s discourse can be classified as non-mathematical because the asymptotes do not block the graph, but it is the behaviour of the graphs of the given functions that results in some graphs having asymptotes (Sfard, 2008; Denbel, 2015). The use of a concept in this way could constrain learners’ understanding of the formal definition of the concept asymptote, considering that in some hyperbolic functions there is an intersection between the Cartesian plane and the asymptotes. Tinyiko’s use of the word asymptote makes her routine ‘ritualised’ because it is based primarily on what is seen on the graph rather than on mathematical reasons embedded in the symbolic representation, which is important to help learners create mathematical conceptual meanings. Therefore, the notion of ‘asim-touch’ is limiting, because it is not true for all functions since the definition would not be true when learners are exposed to functions whose asymptotes intersect (Kuptsov, 2001; Mpofu & Pournara, 2018).

To exemplify the hyperbolic functions, Tinyiko introduced the function $y = 3^x + 2$ which is a wrong example for the family of functions in focus. She engaged in numerical calculations to determine the *y*-coordinates for when $x = -3$ and $x = -2$, which she computed on the table of values (image 77). It was only when Tinyiko proceeded to draw the Cartesian plan for the graph that she realised that the example she had selected was for a different class of functions rather than hyperbolic functions. She shockingly said: “*Aah, aah, aah, what’s your problem? That is an exponential graph, anyway it’s fine, let’s do it*”. This pedagogical action addresses two things: contemporaneous reflection during teaching and use of unplanned examples (van Manen, 2008; Pillay, 2013). Instead of correcting the mistake she made in the example to prevent incoherence in her teaching, she continued as if everything was fine, while the focus of the lesson was ignored. It could be said that the reason why Tinyiko did not correct the mistake was that she did not want the learners to think that she did not know the content, like she said in an earlier analysis.

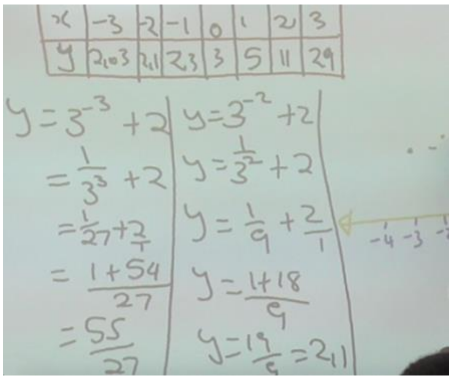


Image 78: *Mathematical calculation to complete the table of values*

The utterance "*It's fine, let's do it*" above reveals that Tinyiko was concerned with teaching the skills of drawing the graphs of functions, even if the processes did not elucidate the critical features for specific classes of functions she introduced as the object of learning. Secondly, the use of unplanned examples was noticeable in this instance, as also revealed in the observable actions earlier in episode 2 with the discourse of 'stealing'. One could argue that the examples Tinyiko used in this lesson were spontaneous not carefully planned, as the use of the examples suggests that she selected the examples while teaching rather than having carefully thought them through (see Pillay, 2013). These observable actions contradicted her utterances relating to the importance of lesson preparation during semi-structured interviews: "*Being a maths teacher, you need to be someone who is, you should always be prepared, and always support the learners that you are teaching, and preparation is a key*". Tinyiko's selection of examples and use demonstrated a lack of preparation albeit her emphasis that "*preparation is key*" in the above statement.

Tinyiko further used the coordinate pairs in the table of values to plot the points of the Cartesian plane and draw the graph. As visible in image 79, the graph she produced on the board was not drawn to scale, resulting in the points on the graph not joining efficiently and others being left out. This was concerning because the teacher did not demonstrate the scaling skills needed for drawing graphs, which may result in learners not paying attention to the importance of drawing proper graphs. The drawing of the graphs to scale is an important element in mathematics, considering that in tests and examinations marks are allocated for the correct points on the graph.

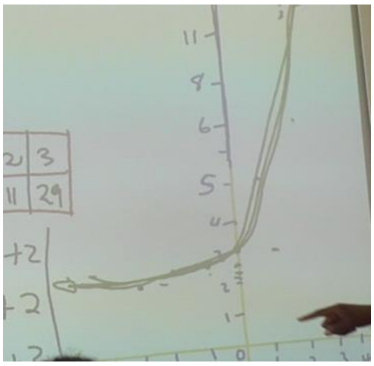


Image 79: *Points left out because of scaling*

Tinyiko's assertions about different properties of the different families of functions are based on empirical evidence instead of being a result of mathematical explorations to help learners make their own meanings of the mathematical objects. The following sub-section presents descriptions and analysis of Tinyiko's teaching in episode 4 extracted from the second observed lesson.

8.1.4. Episode 4 (lesson 2): "So, the x-intercept is also zero"

Tinyiko began this lesson by introducing the general formula for linear functions: $y = mx + q$, which acted as a symbolic tool for classroom talk. For Tinyiko to direct her learners' attention to the focus of the current lesson and the examples, she verbalised the key features of linear functions and brought the focus of the lesson to the learners' attention. She said:

> *The general equation for linear graphs is y is equals to mx plus q, whereby q is our y-intercept and we can easily get our x-intercepts right! And we know what m stands for, the gradient. In order for us to draw linear functions, we are supposed to use either the dual intercept method, the table method or the gradient method. And you know that the table method is always best for getting the answer. Then now, we go to err what we call parabola.*

This statement suggests that Tinyiko's key objective for teaching algebraic functions is helping learners to identify some of the names of the global properties for functions and follow the established rituals to draw the graphs representing the functions. Tinyiko's mathematisation was about learners' manipulation of mathematical signifiers, particularly algebraic symbols. The message about the role of the table of values as a memory prompt comes into focus again, as Tinyiko prefers to use the table method to determine the values of *y*. This analysis is also reinforced by the utterance she made during the semi-structured interview:

> *The reason why I am introducing them to the table method is because when you are given any function in the world, when you use the table method, there is no way your graph will not come out, there is no way!*

Indeed, Tinyiko's way of introducing tasks to the learners consistently implied that the focus is to get a right graph, of which the table of values is the tool to facilitate the correct drawing of graphs. That is, like observable actions of episode 1, Tinyiko's utterances suggest that she uses the table of values merely as a tool for generating correct values of *y*, which will ensure that the graph that is drawn is correct instead of using the table of values as a signifier of relationships between dependent and independent variables.

In this instance, Tinyiko used the property-oriented approach to introduce the general formula for parabolic functions in the form $y = ax^2 + q$, stating that "*It is a curve graph, either it is a cup or a curve or a cave or concave, depending on what we are given*". The latter quotation shows the learners that the graph of the parabola will have different shapes based on the sign of the value of a on the function. To exemplify this family of functions, she introduced a function $y = 3x^2$ in symbolic form, substituted and calculated the *y*-intercept and the *x*-intercepts as presented in image 80.

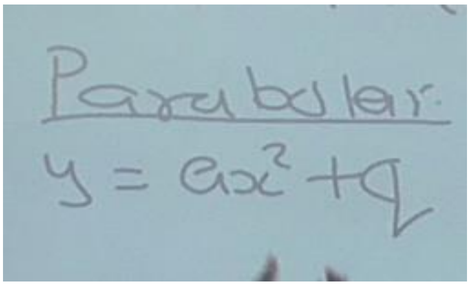


Image 80: *Introducing parabolic functions*

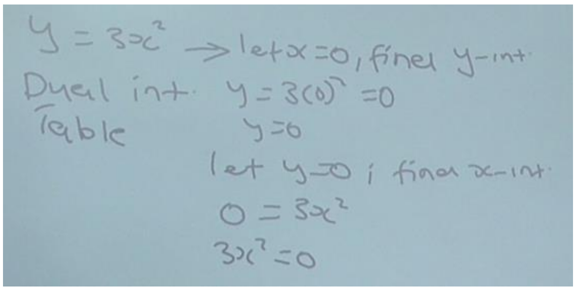


Image 81: *Intercepts for* $y = 3x^2$

Again, in this episode, Tinyiko's use of the intercept concept does not reflect the notion of ordered pairs, because she treated the *y*- and *x*-intercepts as numerical values in the form *y* = and *x* = as visible in image 81 and her verbalisation of the process. That is, after substituting 0 for *x* to calculate the *y*-intercept she said: "*y is equals to zero, meaning that our y-intercept is equals to zero*" and after engaging in algebraic calculations to determine the *x*-intercepts, she said "*So, the x-intercept is also zero*". These iterations overlook the idea that the intercepts entail that an element of one set is associated with a unique element of another set, especially considering the choice of words "*is equals to zero*" and "*is also zero*" in the above statements.

8.1.5. Episode 5 (lesson 2): The continuity of the parabola

This episode continues from episode 4 and the teacher introduced the function $y + 4 = x^2$ in symbolic form stating, "*Let's try that one*". The statement "*Let's try that one*" resulted in the rapid transformation of the equation into an explicit form $y = x^2 - 4$ and subsequently used the dual intercept method to determine the intercepts. After completing the calculation process, she plotted the intercepts on the Cartesian plane and joined the points to draw the graph[44] (see image 82:.

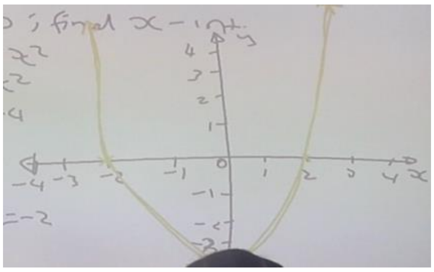


Image 82: *Graph of* $y + 4 = x^2$

Interestingly, for this part of the lesson Tinyiko presented an endorsed narrative about the continuity of the graph using visual tool (the graph drawn on the board) and she said: "*The reason of the arrow is that it shows that this thing is still continuing*". While this narrative addresses the component of continuity which forms part of global features of the parabola, the statement did not highlight how the mappings between the variables resulted in the drawn parabola being continuous. Tinyiko's statement demonstrated that the graph, albeit referred to as a demonstrative pronoun "this thing", functioned merely as a mnemonic in her teaching rather than a representation of another entity. As a mathematics teacher, I was disappointed to hear Tinyiko refer to the parabola as a thing rather than using the proper word, because it could be interpreted as not taking mathematics seriously. The teacher's action above suggests that she was 'cutting corners' for not explicitly explaining the key features relating to a parabola.

Tinyiko asked the learners the following question: "*Can I give you something to try?*" and subsequently introduced another function $y + 16 = x^2$ in symbolic form. The teacher gave learners an activity to engage with the knowledge that had been taught, to assess whether and how they would use what they had been taught on their own. The utterance relates to the notion of 'task situation', which denotes any context in which interlocutors consider themselves "bound to act – to do something" (Lavie et al., 2019, p. 7). It is expected that a teacher gives learners activities as a way of determining their teaching and learners' understanding or lack thereof of the knowledge that has been taught. Consider the following exchange:

49 **Learner***: Ma'am, you are saying the little arrows at the end of the curve mean that the graph is continuing?*
51 **Tinyiko***: Yes*

[44] This observable action reinforces the previous argument about the ritualised nature of Tinyiko's discourse.

52 **Learner***: So, what happens when they want us to draw a function that does not continue?*
53 **Tinyiko***: There is no way in this grade you can be asked to do that. In this grade on a parabola, maybe they will say there is a restriction, not now!*

Tinyiko's justification about the continuity of the parabola is solely related to the curriculum content delimitations for Grade 10 level (line 53), which could be interpreted as limiting learners' thinking to what is examined at the current grade. The explanation did not focus on the association between the variables to denote the notion of the behaviour of the functions, or how two variables are related in a given function. The learner's question was a call for explorative routines to be put in place for learners to compute different values and observe the behaviour of the function to understand what the notion of continuity entails. It was interesting that the teacher did not explain to the learners the aspects of continuity and discontinuity of functions, especially considering that the learner was inquisitive about such mathematical concepts.

Another learner asked Tinyiko to clarify why the drawn graph for the function $y = x^2 - 4$ continues after $x = -2$ and $x = 2$. Before answering the question, Tinyiko asked the whole class to provide the justification why the graph could not end at the specified *x*-values and one of the learners said: "*The graph cannot end at -2 and 2 because when we use the table, there are other x-values which will make the graph to continue"*. This statement illustrates that the learners are inhabiting the discourse of associating the table of values as a primary aid in the drawing of the graphs, which alludes to the action conception of functions (Even, 1998). It should equally be noted that the learner's explanation above offers the correct mathematical justification for the continuance of the graph beyond the *x*-intercepts, which could be argued to mean that through Tinyiko's rituals to draw the graph, learners are gaining some correct mathematical discourse about functions (Sfard, 2008). That is, to ensure that the learners were able to draw the correct graph of the function, Tinyiko emphasised that learners should always opt to use the table method as it provides empirical evidence for the correspondences between values of *x* and values of *y.* To elaborate on the above learner's response, Tinyiko offered the following utterance to justify why the graph cannot end at the *x*-intercepts:

> *Do you remember the domain and the range, the input and the output? The inputs are the values of x and the output are the values of y. In other words, if you end at these points* (pointing to the x-intercepts)*, you are saying that if we are using the table method, it only works when we substitute 0, -1, and -3, it can't take anything from -3; the output that you will get from -3 will be undefined.*

Although the teacher did not engage in the mathematical processes of determining domain and range for the graphs to illustrate the notion of continuity of the graphs, the above explanation demonstrated the idea of the function values becoming undefined. After the narration, Tinyiko asked the learners to verbalise the intercepts and she wrote $y = -16, x = 4$ *or* $x = -4$, again overlooking that the intercepts should be represented as coordinate pairs in the form (x , y) to denote the association between the variables of the function in focus. She then plotted the intercepts on the Cartesian plane and joined the points to sketch the graph. In relation to her emphasis in the previous example that learners should always put the arrows around at the edges of the graph to show that the graph continues to negative infinity and to positive infinity, she did not put the arrows on the current parabola in image 83. This represents incoherence in developing a mathematics story about the need to put arrows to indicate the continuity of the parabola.

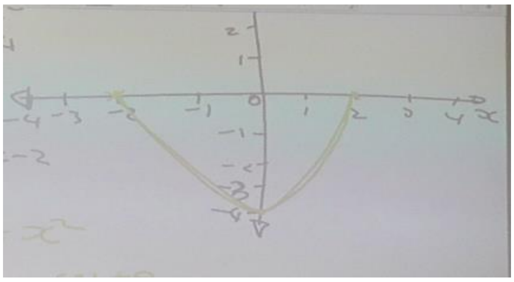


Image 83: *Continuity of the graph overlooked*

The next observable action in the current episode saw Tinyiko introduce the function $y + 4 = -x^2 + 9$, accompanied by the instructional utterance "*What if you are given y is equals to negative x squared plus nine? Draw that one*". The purpose of this task was to direct learners' attention to use the predominant rituals to draw the graph of the function as a way of practising and demonstrating their understanding of the taught mathematical processes. The assumption was that the learners would be able to perform the prerequisite steps for drawing the graph because the current task situation linked back to precedents – the past routines which she interpreted as sufficiently similar to the present. It was commendable that in this part of the lesson the teacher asked the learners a question focused on object-level narrative, using the graphs on the board (image 83) as aids for her questioning: "*What makes them to be different? On this one, the coefficient of x squared is negative, that is why it is facing down and the first one the coefficient of x squared was positive, that's why the graph is facing up*". This utterance aimed at explaining to the learners the effect of the parameter a on the parabola in terms of the sign of the coefficient of x squared, even though the learners were not given an opportunity to answer the question. The graphs were used as visual mediators to draw learners' attention to the difference in the directions the parabola faces when the value of *a* is positive and when it is negative. Of importance to note from the above utterance is that the question "*What makes them to be different?"* resonates closely with explorative routines since it calls learners to consider the object-level differences between the parameters of the given functions to make generalisation statements, however, the teacher's immediate 'self-answer' made the routine fall into the ritual category of routinisation (Sfard, 2015).

8.2. Summary and conclusion regarding Tinyiko's observed episodes

The presentation, analysis and interpretation of Tinyiko's pedagogical actions in all the episodes reveals the prominent use of the rituals to draw or sketch the graphs of given functions in symbolic form. This makes the observable action of drawing or sketching the graphs the end-goal of Tinyiko's teaching of functions. Focusing on the how of the routines resulted in Tinyiko's discourse of rituals rather than explorations, as she emphasised the following of rules without explication and understanding their applicability. Also, what was notable in Tinyiko's teaching across the five episodes was that she prefers a table of values routine as an aid for getting the correct shape of the graphs, in her words, the usage of the table of values helps facilitate the correct drawing or sketching of the graphs since it helps in mitigating mistakes.

In some episodes, there were also some instances where Tinyiko could not engage in correct mathematical discourse to offer classroom explanatory talk about the nature of the relationship between the variables of given functions, especially in cases where the calculations yielded an undefined result for *x*-intercepts. These instances were interpreted as commognitive conflict as the teacher was faced with a situation in which she could not communicate mathematically what the undefined result in the calculations signified in terms of the intercepts with the axes. In terms of the teaching approach for function topic, Tinyiko's teaching can be categorised as the graph-oriented approach to teaching the concept; even the examples are introduced for learners to draw different graphs. The pedagogical and learning implications for using this approach to teach algebraic functions will be discussed in Chapter 10. Table 19 is a summary of Tinyiko's episodes and outlines the observable actions during teaching. The following chapter provides an analysis and interpretations of the last case: Jaden's teaching of algebraic functions.

Table 19 *Summary of Tinyiko's teaching episodes*

Sfard's commognitive theory				
Episodes and observable actions	**Visual Mediator** *(the images presented in-text also represent iconic and/or symbolic mediators)*	**Words used**	**Endorsed narratives**	**Routines**
1. Recapping on the features of linear functions to set the scene for parabolic functions. Substitution and calculations and completion of table of values to draw the parabolas of given functions.	Symbolic mediators: $y = x^2$; $y = x^2 + 1$; $y = x^2 - 1$ Iconic visual mediators: function in the form $y = x^2$ depicted in the table of values.	Parabola; linear; functions; straight; linear graphs; dual intercept method; intercepts; gradient; turning point; y-intercept; output values; x-intercepts; positive; face up; table method	The effect of parameter *a:* "*because you were given the function as* $y = x^2 + 1$ *the coefficient of your x squared is positive, it simply tells you that your graph will go up*"	Clarifying
2. Showing learners how to use the dual-intercept method and table method to determine the output values for chosen inputs. Engaging in non-mathematical memorisation of 'stealing' procedure to determine the intercepts.	Symbolic syntactic mediators: $y = x^2$-1; $y = x^2$; $y = x^2 + 1$; $y = ax^2 \pm q$ and $y = ax^2$ to perform mathematical calculations. Iconic visual mediators: graphical representations for $y = x^2$ and $y = -x^2$	Substitute; formula; stealing; x-intercepts	The object-level narrative about calculations: "*the reason why whenever we substitute these numbers, whether we substitute the negative or a positive we always get a positive answer*"	Rituals to use the dual-intercept method and table method to determine the output values and drawing graphs of functions.
3. The teacher intended to introduce hyperbolic functions, but the example she introduced was for exponential functions. Although she realised this after engaging in mathematical calculations, she continued performing rituals to demonstrate to the learners how to substitute and calculate for output values.	Symbolic syntactic mediators: $y = x^2$; $y = x^2 + 1$ Iconic visual mediators: table of values and graphical representations for $y = \frac{1}{x}$	General formula; hyperbola graph; x-intercepts; y-intercept; exponential graph; asymptote; asim-touch; *x*-asymptote; *y*-asymptote	Describing the notion of asymptote: "*But there is a unique thing that we must always have is the asymptote, asymptote meaning 'asim-touch', we don't touch, the line which this graph will never touch. We have two asymptotes, the x asymptote and the y asymptote*".	Rituals to use the dual-intercept method and table method to determine the output values and drawing the graph of the function.

4. Substitution and calculation of intercepts for $y = -\frac{1}{x}$.	Symbolic syntactic mediators: $y = -\frac{1}{x}$; $y = \frac{1}{x}$	General equation; linear graphs; parabola; cup; cave; concave; gradient; table method; y-intercept; x-intercept	The object-level narratives about intercepts: "*y is equals to zero, meaning that our y-intercept is equals to zero*" ; "*So, the x-intercept is also zero*" – treating intercepts as numerical values instead of coordinate pairs".	Rituals to use the dual-intercept method to determine the output values.
5. Using two examples $y = -\frac{1}{x}$; $y = \frac{1}{x}$ to generalise the effect of changing the sign of parameter *a*.	Symbolic syntactic mediators: $y = -\frac{1}{x}$; Iconic visual mediators: graphical representations: $y = -\frac{1}{x}$; $y = mx + c$	Arrow; continuing; inputs; domain; range; points; facing up; output; positive; negative	The effect of parameter *a:* "*What makes them to be different? On this one, the coefficient of x squared is negative, that is why it is facing down and the first one the coefficient of x squared was positive, that's why the graph is facing up*"	Memorisation about the effect of parameter *a* in terms of its sign from graphical visual mediators.

Approaches of algebraic functions used

Across the three selected episodes, Tinyiko used two approaches to teach parabolic and hyperbolic functions. These are: example versus non-example and property-oriented approach. Through the use of the interactive/authoritative communicative approach, the teacher used the abovementioned approaches to show learners how to engage in mathematical conventions of substitution and calculation to determine the output values for specific input values, complete the table of values and sketching the graphs.

Scott et al.'s pedagogical link-making and communicative approaches

Tinyiko took an interactive/authoritative approach when teaching learners about the substitution and calculation processes, completing table of values and sketching the graphs of functions. Three approaches to pedagogical link-making were developed by the teacher. These approaches were:

- Promote continuity: developing a mathematical story (micro)
- Promote continuity: developing a mathematical story (meso)
- Support knowledge building: making links between modes of representation

Chapter 9

Data presentation and analysis 5 – The case of Jaden's teaching

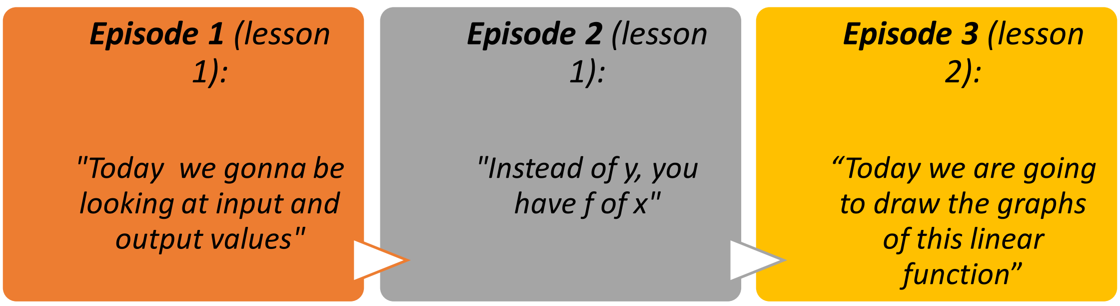
"*What I am seeing here is that learners are facing some difficulties in their maths, so I think that those problems come from the lower grades, so they lack basic concepts, even when they are given simple questions like 2+5, they want a calculator to add*" ~ (*Jaden, Semi-structured Interview*).

9.1. Introduction

The above statement from the semi-structured interview with Jaden demonstrates that the learners have challenges with the basic mathematical knowledge and skills, and this experience influenced his teaching approach to only give learners the information. The current chapter provides an analysis and interpretations of Jaden's teaching in two of the lessons I observed, and also draws on the data from semi-structured interviews and VSRI to offer further evidence of the discourses and approaches during the algebraic functions lessons. Figure 14 below represents the foci of the two lessons:

Figure 14

Jaden's selected episodes from two lessons



9.1.1. *Episode 1 (lesson 1): "Today we gonna be looking at input and output values"*

Jaden started this lesson by writing the object of learning on the board (image 84) and said, "*Today we gonna be looking at input and output values*", focusing learners on the relationship between two variables. Jaden's drawing of a function machine to demonstrate the functional relationship between the value of the input (*x*) and output value (*y*) (image 84) suggested the use of the function machine approach. This approach brings to the fore the notion of controlling the inputs and operations of the function machine to produce the values of the outputs. The function machine acted as an iconic visual mediator, to show learners the association between *x* and *y*. Mpofu (2016, p. 12) stated that the use of iconic mediators "help the participants to identify the objects of their talk and coordinate their talk", and Jaden asked learners to provide an algebraic formula which represented the association between the variables in image 84. The purpose was to help learners to develop an understanding of how the rule underpinning the association between output values and input values is determined.

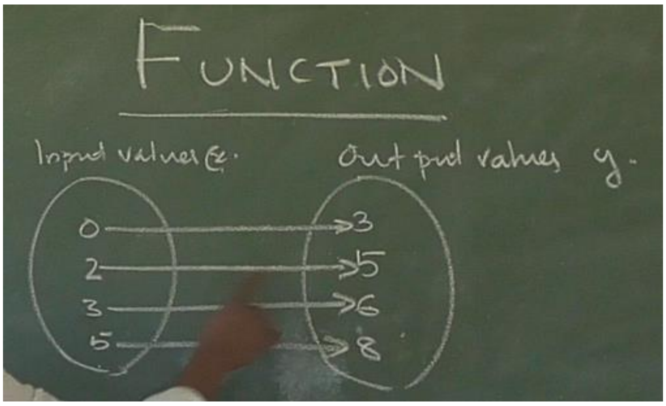


Image 84: *Jaden's function machine*

Jaden further asked learners to verbally describe the relationship between *x* and *y* on the function machine, to check whether learners still remembered what was done before. The learners' response on the relationship between the two variables: "*Three is added to the values of x to get the values of y*" illustrated their understanding of *covariation* between the two variables. It could be said that the learners transferred their knowledge of patterns and relationships to determine the rule underpinning the relationship between the variables, given that the teacher had not introduced the notion of covariation to the learners. According to DBE (2011), teachers should engage learners in the process of interpreting the relationship between two variables, to denote to the learners the covariational nature of the relationship between the variables. Thus, Jaden used the function machine approach (image 84) and covariational approach implicitly, to introduce learners to the idea of how two quantities covary. When probed about the use of this approach during VSRI, Jaden mentioned that: "*The function machine helps learners to see how the output and input values are related, so that it becomes easier for them to determine the formula that represents the function*". This statement closely links with the view that a function machine acts as a visual mediator that reveals the mapping between two variables (Bayens, 2016). This resonates closely with Tall et al.'s (2000) and Kwari's (2007) suggestion that a function machine acts as a cognitive root during teaching and learning, to ensure that learners develop conceptual understanding and gain an understanding of an intangible topic.

To teach the table completion task, Jaden used the pattern-oriented approach, as it required learners to determine the rule governing the relationship between the quantities and consequently determine the 'general formula' that represented the terms in the progression. The following instruction illustrates this: "*Copy and complete the tables given below then write the rule that represents the relationship between x and y in the form* $y = \ldots$*"*. The teacher was creating links between modes of representations, between the symbolic and the function machine in image 84 as learners were expected to deduce a symbolic rule from the presented iconic representation which forms part of interpretation activity (Scott et al., 2011). According to Leshota (2015), interpretation activities involve generalising on the behaviour of the functions, and are important in determining the nature of relationship between quantities. In this case, Jaden prompted learners to verbalise the coordinate pairs in the table of values, as visible in the following excerpt:

1 **Jaden***: What is the first value of x?*
2 **Learners***: One*
3 **Jaden***: What is the corresponding y-value? It's five, x is two, what is the corresponding y-value?*
5 **Learners***: It is six.*
6 **Jaden***: x is three, the corresponding y-value is?*
7 **Learners***: Seven*

In the above exchange, Jaden semiotically mediated learners' visualisation of the association between the values of *x* and *y*, by providing them with the first coordinate pair through asking a question and answering it (line 3). While the teacher asked questions that could enable learners to observe the nature of the relationship between the variables and make their own meanings, I argue that answering the questions himself limited the productivity of this process. This question-self-response strategy can be interpreted as a way of leading learners to structurally view the association between *x* and *y* on the table of values. Of concern to note is that the teacher took away opportunities for learners to make sense of functions and learn to reason mathematically as well as to connect mathematics ideas and the application of procedures. The observation further reveals the authoritative nature of Jaden's communicative approach during teaching, which can be argued to limit learners' intellectual thought processes and meaning making (Scott et al., 2011). From the observable action the teacher offered learners the cues to view the correspondence nature of the two variables, especially when considering the way that the teacher represented the coordinate pairs as visible in image 85 below:

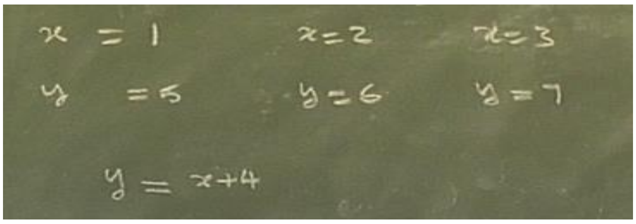


Image 85: *Ordered pairs for the function and the equation for the function*

In this observable action, the teacher took the values and showed the learners what the association between x and y entailed, appealing to the notion of 'when x is …, then y is …'. Without overlooking that the instruction prompted learners to represent the function in the form $y = \ldots$, Jaden did not use the words 'independent' and 'dependent' variable, since he did not explain to the learners why the relationship between the two variables should be expressed in the form $y = \ldots$. This could potentially lead learners to view and express the relationship in terms of x, therefore in turn getting the algebraic representation for the relationship to be $x = y - 4$. Thus, I argue that in the explanation of the ritual routine of determining the equations that represent the function, the teacher did not expound on the applicability of the routines, which could have been facilitated by the use of the two words 'independent' and 'dependent' variables (Sfard, 2008; Mpofu, 2018).

In relation to the above ordered pairs for the function, out of interest I asked Jaden the reason for representing the ordered pairs as seen in image 85 and he commented:

> *It becomes easier for the learners to see how x and y are related. Remember I told you the other day* [semi-structure interview day] *that our learners are struggling, so writing the values like this helps them see the relation faster, as I wrote x and y for every value and that is what they are expected to do in tests and examination. So I must always teach in a way that will make it easier for them to answer the exam questions*.

While it appears like the teacher was making the learning of calculation of output values easier by organising the content in this way, the above response reveals that Jaden was making it easy for learners to see how to answer questions during assessments rather than for them to understand the knowledge. While it could be argued that we teach for this end goal, the main purpose is to promote learners' understanding, to own the knowledge rather than regurgitate the information and the processes to reproduce in the exam situations. What emerged as the lesson continued was that Jaden also used the example versus non-example approach to show learners the mathematical conventions of substitution and calculation of the values of *y* for the given values of *x*. At this point of the lesson, he moved to the worksheet to introduce the example on how to determine missing y-values, which also appeals to the pattern-oriented approach (see image 86).

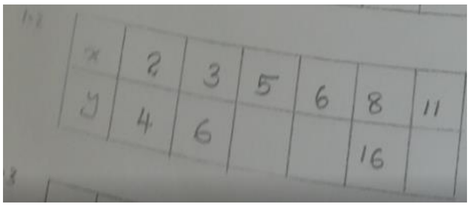


Image 86: *A table of values showing a function*

Jaden reminded learners that the equation $y = x.2$ is the same as $y = 2x$. Given the explanation of the action, Jaden wanted learners to be aware of the structural equivalence between the two equations, as a way of facilitating the substitution and calculation processes. Once this was done, he wanted the learners to use the ‘formula’ to calculate the value of *y* when *x* is 5, which is the ritual for ‘finding the missing terms’ in the topic of sequences and series. The notion of missing values relates to the recommendation of CAPS (DoE, 2011) which asks learners to make conjectures about the behaviour of given functions to generalise. Another way of interpreting the missing values of *y* in the table of values is that it offers learners the learning opportunities to interpret and construe critical predictions from functional relationships.

Jaden introduced another functional relationship in the table of values into a mapping, then into an algebraic equation in image 87 below. Of interest in this example is that some learners verbalised “*x is equals to*” when they were asked to provide the algebraic representation of the function, reinforcing my contention that there was no explanation for why the function is expressed in terms of *y*. Jaden continued using image 87 and asked learners to verbally determine the formula that represented the relationship between *x* values and *y* values. Learners were unable to determine the algebraic representation for the relationship, and the teacher probed, especially when the choice of words “this is linear pattern” is considered, and focused on the relationship between the values of *y*. This represents the pattern-oriented approach, because Jaden used the idea of a linear progression to determine the missing values of y. He said: “*five, eight, eleven, fourteen; what is the difference of eight and five?* (the learners responded 3), *eleven and eight?* (the learners again said 3), *fourteen and eleven?* (the learners again said 3). *So, this is linear pattern neh, so tell me what the relation is*”. From this statement Jaden treated the *y* values individually without demonstrating the relationship between the two variables. I therefore argue that the use of the pattern-oriented approach to discern the relationship between *x* and *y* proved to be difficult for Jaden, because he neither provided an explanation nor asked the learners to consider the covariational relationship between the two variables.

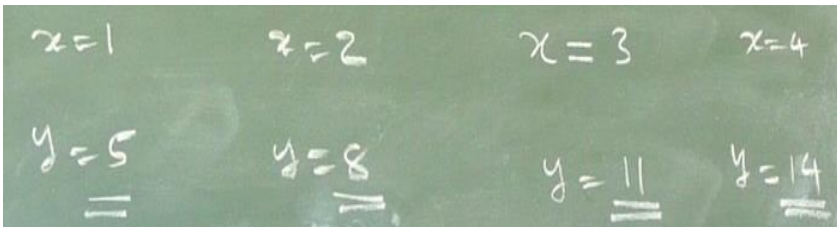


Image 87: *Mapping input and output values*

Instead of showing the learners or asking them how the equation was determined, Jaden engaged in mental substitutions and calculations to show the learners by inspection that the equation indeed represented the relationship.

9.1.2. Episode 2 (lesson 1): "Instead of y, you have that f of x"

In this episode Jaden introduced the function $y = 3x - 1$ symbolically and focused on the substitution and calculations process on the board (image 88). The purpose was to enable learners to learn and master the substitution and calculation skills of output values. Jaden used image 88 to represent the symbolic syntactic mediator, because he did not expound the rationale for engaging in the observable action of substitution and calculations to the learners. While it is acceptable for a teacher to use symbolic representations to demonstrate mathematical conventions, it is also important to verbally explain to the learners the inherent nature of the relationship between variables to help learners to develop the relational understanding of such variables. To address the relationship between mathematical and/or non-mathematical words in image 88 below, I expected the teacher to say '**when** *x* is equals to …' should be used in conjunction with the statement '… **then** *y* is equals to …' to signify that statement *x* is the premise of the implication and *y* is the conclusion or generalisation. The omission of the word 'then' in Jaden's verbalisation of the implication statement might constrain mathematical meaning(s) if used in a literate manner.

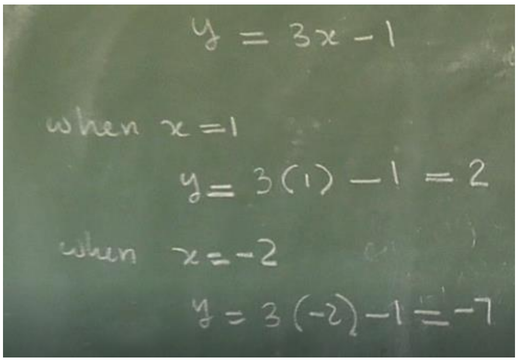


Image 88: *Jaden's substitution and calculations*

From the substitution and calculation processes Jaden introduced the functional notation in the form stating that,

> *If I ask you a question like this, instead of y, you have that f of x, function of x, f of x is equals to two x minus four, if I ask you a question like this, determine the value of f of 1 (f1), f of zero f(0), f of minus three f(3), how do we do that question?*

The teacher's approach is questionable, especially if his reflective comment during VSRI is seriously considered: "*to ensure that in tests and examination, learners are able to recognise that the f(x) denotes the same meaning as y. The learners will not struggle to substitute you see*". One way of interpreting the teacher's approach is that he is asking learners questions to prepare them for examinations, not necessarily as the continuous assessment approach during the lesson to evaluate learners' understanding or lack thereof of the content.

He further showed learners that the functional relationship between the variables can be represented as ordered pair coordinates using the pair (1; -2) to illustrate the idea as can be seen in image 89.

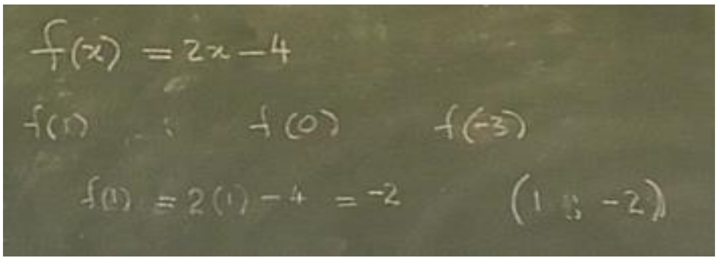


Image 89: *Algebraic calculations and coordinate pair*

Although Jaden introduced a new routine of representing the functional relationship between variables which is considered literate mathematical (DoE, 2011), he did not demonstrate to the learners why the value of *x* was written first and the value of *y* was written last in the brackets. It means he overlooked the need for justification routines to expound to the learners that the coordinate pair should be written in the form (*x*; *y*). It is crucial to demonstrate to the learners the nature of mathematical objects to ensure that they understand both the form and content in such objects. This observable action addresses a situation whereby a teacher represents functional relationships in a literate mathematical way, and does not offer mathematical reasoning for why things are or should be the way they are. Following from the teaching, Jaden asked learners to complete questions 2.1. and 2.2. which were related to the observable actions in image 89 above. For the first two questions, the learners were expected to determine *f* (2) and *f* (-2) as visible in image 90 below. This activity could be interpreted to mean that the teacher wanted learners to reproduce the information he showed, without allowing learners' participation in a lesson for their understanding. This statement does not overlook the importance of using activity to determine learners' understanding or lack thereof of the taught knowledge.

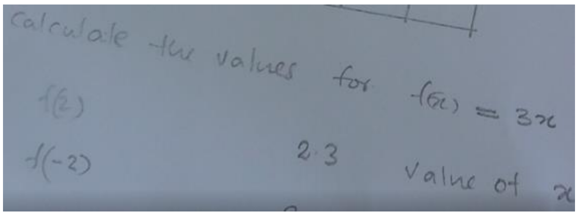


Image 90: *Questions to find f (2) and f (-2)*

However, the observable actions of showing and telling learners the ritual for completing the task and subsequently giving them a chance to reproduce the ritual, resonates with the conception of learning as memorising content. It can be said that, for Jaden, it is important for learners to absorb the unit specific content for the purpose of reproducing during examination. The idea of teaching for memorisation and mimicking was further demonstrated by Jaden's action after realising that he did not exemplify cases whereby learners are expected to calculate the value(s) of x when given $f(x)$ in questions 2.3. and 2.4., as represented in image 92. This possibly means that the teacher did not plan the lesson before teaching, to ensure that such knowledge gaps were identified and addressed before teaching. The reason is because he returned to the function and showed learners how to substitute and do algebraic calculations to find the value of x where $f(x)$ is -2, to ensure that learners have mental images for the routine when completing the task in image 91.

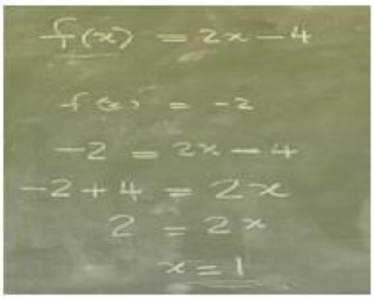


Image 91: *Algebraic calculations to find the value of x*

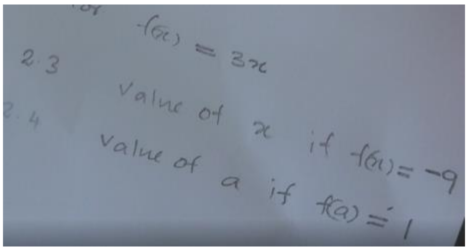


Image 92: *Questions to find values of x when* $f\ (x) = -9$ *and* $f\ (x)\ = 1$

In the next observable action, Jaden wrote the words ‘Graphs, Linear function and Line’ presented in image 93 These words were not explained throughout the lesson to understand the purpose of writing them in relation to the previous information, instead the teacher returned to the worksheet and engaged in substitution and algebraic calculations to answer question 2.4. in image 92. This could cause incoherence in the lesson and learners might also be confused about the significance of the introduced words and how they feature in the broader scope of the topic.

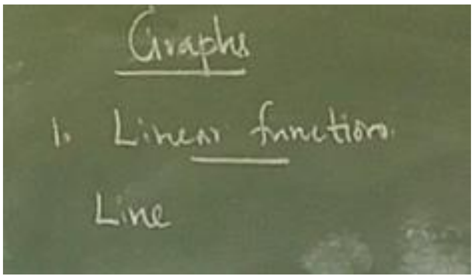


Image 93: *Word written, nothing said*

In view of the three words in image 93, a close reading of the VSRI transcript reveals that Jaden’s view of the curriculum expectation relating to the teaching of algebraic functions entails teaching learners how to engage in algebraic calculations and draw graphs of functions. This was the reason he wrote the words to prompt learners that the graphs produced by the examples he introduced are linear/lines. He said:

> *Because I am following the curriculum neh, so I have the pacesetter and everything is there, when I wrote these words here I wanted to explain how we draw the graphs of linear functions because the curriculum say we should do that and it’s part of the examination you see. Learners should know how to calculate the values and draw graphs, but I realised that I was moving away from what I had planned to teach, completing the work I planned to teach.*

The ritual routines Jaden engaged with were to ensure that learners could memorise and reproduce homogenous products during examinations and could be interpreted as an appeal to rote learning or memorisation. What was interesting was that when Jaden realised that he was deviating from his planned object of learning, which was teaching learners how to substitute and calculate values of y for given values of x, he engaged in instantaneous reflection and reverted to the worksheet that he had planned to complete with the learners. In this instance, Jaden showed the learners the ritual of substituting into equations in terms of function notation, as visible in image 94 below.

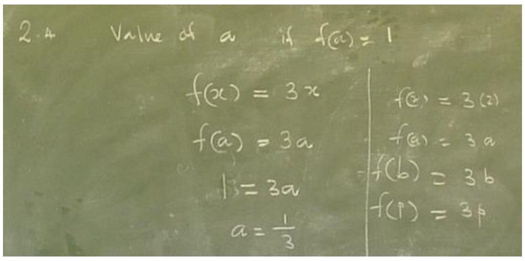


Image 94: *Jaden's algebraic calculations*

The algebraic calculations in image 95 acted as a symbolic syntactic mediator for learners to see how to substitute and use the function notation and its parenthesis in terms of *f(x)* to calculate the input and output values. Thompson (2013) argued that the use of the function notation is effective in the learning of functions as it prompts learners to focus on the element of the function requiring examination, and in the above example determining the value of *a* if *f(a) = 1* . Jaden's observable actions in this episode called for the realisation of the similarity of the substitution and calculation processes. This was to enable learners to engage in metathinking (reflection) on discursive events he introduced on the board, considering the repeated substitutions on the right side of the vertical line drawn in image 94. Although this might be the case, he used the non-interactive/authoritative approach, which is influenced by what he said in the extract at the beginning of the chapter. Jaden further introduced the function $f(x) = x^2 - 1$ and showed learners how to determine the value of *a* when *f(a)* = 0. Below (images 95 and 96) is the example of how he used two alternative algorithms to illustrate the necessary steps to complete the task. From the observation, the teacher introduced algorithm 2 (image 94) to possibly enhance learners' understanding of solving equations through the use of different algorithms.

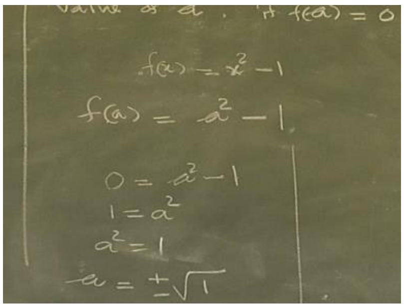


Image 95: *Algorithm 1 to find values of a*

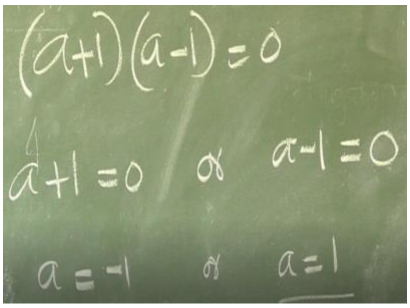


Image 96: *Algorithm 2 to find values of a*

The appeal to rote memorisation in Jaden's teaching is further evidenced by his observable actions when he first introduced a prototypic example in image 97 to demonstrate to the learners how to engage in substitutions and calculations as required in the activity given in image 92. Providing an example for every activity can be interpreted as an expectation that learners should mimic and reproduce what the teacher did in the examples (Sfard, 2012). As stated earlier, Jaden's pedagogical actions seem to be rooted on the belief that his learners lack basic mathematical knowledge and skills. Thus, his exposition strategy could be interpreted as a mitigation technique to ensure that his learners master the skills for substitution and calculation.

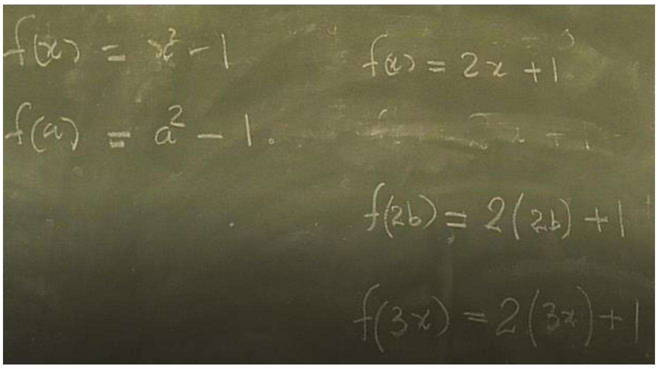


Image 97: *Prototypic example for the activity*

In overview, Jaden's focus in the current lesson was teaching learners how to generate algebraic equations that represent the functional relationships given in tabular form, and showing them how to substitute and calculate. What was interesting across the five episodes and sub-episodes was the example-activity sequencing Jaden used. That is, before learners could answer the questions in the worksheet Jaden prepared, he provided them with prototypic examples for them to refer to when answering the questions, which means Jaden expected the learners to learn through memorisation.

9.1.3. *Episode 3 (lesson 2): Drawing the graphs for linear function*

The current episode started with Jaden writing "Linear functions" on the board and stating that "*Today we are going to draw the graphs of this linear function*". This statement prepared learners for the new lesson which continued from the previous one, where Jaden demonstrated the steps needed to draw the graphs of linear functions. It is important for a teacher to clearly state the focus of the lesson, to prepare learners for the object of learning (Adler & Ronda, 2014). Jaden introduced three examples of linear functions in symbolic form as visible in image 99 below, which Polya (1973) coined as leading examples. The equations were used as examples to express the key features of linear functions, and Jaden explained that why the three equations exemplify linear functions "*is because the value of the exponent is one. It is also a linear function because when we draw the graphs of these functions, we get a line*" (drawing a sketch of a linear graph as can be seen in image 99 below).

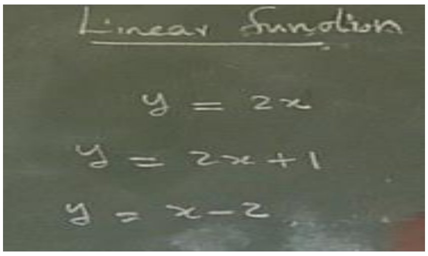


Image 98: *Three examples of linear function.*

Image 99: *Sketch of linear functions*

The above statement and the sketch called for learners' attention to the global features of linear functions, both from the symbolic and graphical representations. Thus, the three equations written on the board acted as a symbolic mediator, and the sketch represented an iconic visual mediator for learners to see the links between the symbolic representations and the type of graph they produce. In addition to the three linear functions in image 98, Jaden instructed the learners to draw the graph of the function on the graph paper he provided with a domain *x* is the element of {-1, 0, 1, 2}. The instruction is presented in image 100 below:

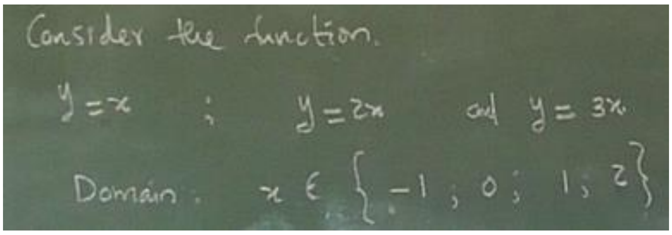


Image 100: *First activity for learners to do*

Jaden took it for granted that learners had a similar understanding of the meaning of 'domain' in the instruction, because he did not explain the concept before using it in the instruction. Again, considering that learners are formally introduced to the discourse of algebraic functions in Grade 10, the meaning of word use should be explained explicitly, to ensure that learners gain conceptual understanding (Sfard, 2008). In addition, I noticed incoherence in the teacher's teaching, as he did not allow learners to complete the task he set, instead he introduced the table of values (image 102) and told the learners that "*When you substitute negative one here, you gonna get the same value, negative one. When you substitute zero, zero! So, draw that table and complete for other values*". This situation was not an isolated event as Jaden predominately did the mathematics for the learners, without giving them a chance to be active participants in the co-construction of mathematical meanings. This involved making links between different representations, as Jaden required learners to translate from the algebraic representation into tabular representation (Scott et al., 2011). Thus, learners had to recall and apply the routines of substitution and calculation that had dominated Jaden's teaching from the first lesson. It could further be argued that the teacher wanted to see whether learners could use the knowledge he had introduced to engage with the task.

The above-stated interruptions by Jaden were further observed when a shift from table of values to drawing the graph on the Cartesian plane was made. Although Jaden had given learners an opportunity to complete the table of values, he did not allow learners to finish the task, instead he completed the table of values on the board (image 101) and used a meter ruler to draw a Cartesian plane, as depicted in image 102.

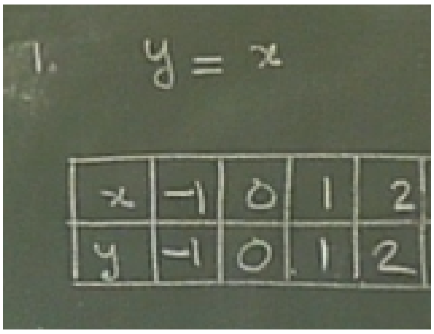


Image 101: *Table of values for* $y = x$

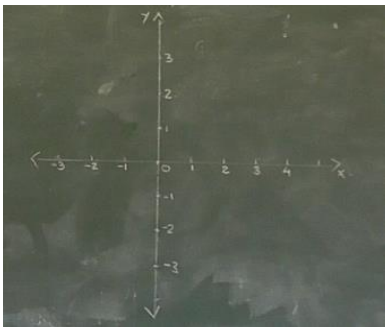


Image 102: *Cartesian plane drawn*

Jaden moved on to show the learners how to represent the ordered pairs associated with the linear function as points on the Cartesian plane. This led to the drawing of the graph, in image 103 below, and along with the table of values in image 101, which acted as iconic visual mediators for learners to see the mappings between the two variables. It is concerning that when tasks were introduced for learners to engage with, the teacher always interrupted and offered solutions, without even asking the learners to offer their own explanations of the content to demonstrate the nature and the meaning making of the internalised contents that were covered in the lesson. Thus, the teacher used the non-interactive/authoritative approach throughout the activities without allowing learners opportunities to do the mathematics for themselves (Scott et al., 2011). In the next observable action, Jaden introduced the function, and completed the table of values, then moved to plotting and sketching the graph on the same set of axes using two different colours for variation as depicted in image 104 below. The use of the different chalk colours was a mediational aid to ensure that learners could distinguish the two graphs, observe their behaviours and subsequently construe conjectures pertaining to the value of the coefficient of x.

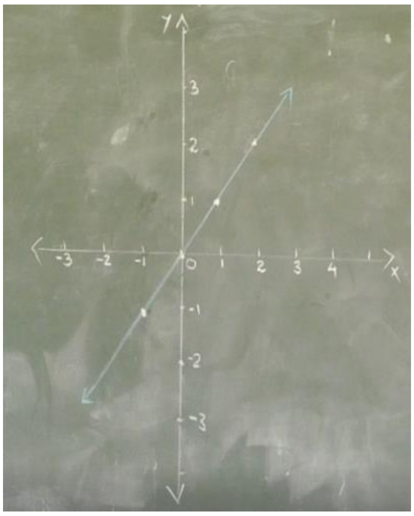


Image 103: *Graph of* $\mathrm{y} = \mathrm{x}$

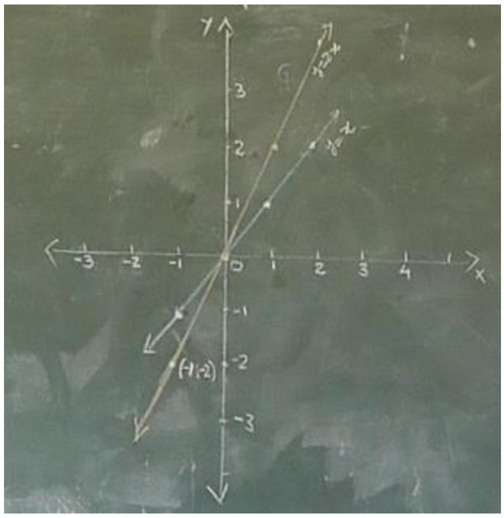


Image 104: *Graphs of* $\mathrm{y} = \mathrm{x}$ *and* $y = 2x$

Jaden then asked the learners to make a conjecture about the position where they thought the graph of $y = 3x$ would be in relation to the two graphs. The two graphs in image 104 acted as iconic mediators to prompt learners to consider the behaviour of the functions when the coefficient of *x* increases, which called on the learners to make conjecture statements about the steepness of the graphs. However, Jaden did not prompt learners to verbalise their observational statements about the effect of varying the coefficient of *x*, to demonstrate their meanings about the effect and for him to obtain feedback about the quality of his learners' understanding. Interestingly, whilst learners were still engaging with the task, Jaden drew the sketch of a line alongside the graph of $y = 2x$ as visible in image 105 below, to demonstrate that when the value of the coefficient of *x* increases, the graph gets closer to the *y*-axis. This is concerning because Jaden did not engage with the learners to check their positionality in terms of what they were observing about the effect of the changes in the values of a, instead he presented the generalisation statements to the learners.

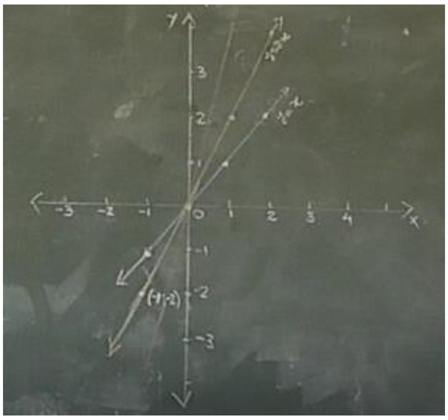


Image 105: *Conjecture for the graph of* $y = 3x$

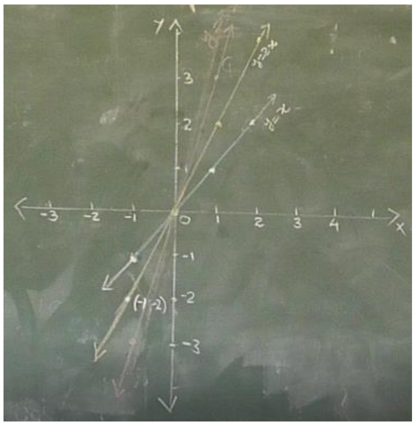


Image 106: *Graph of* $y = 3x$ *drawn*

For the function $y = 3x$, Jaden did not follow the ritual routine of completing the table of values before drawing the graph, as he did with the other two graphs he had drawn. His non-usage of the table of values as an iconic mediator to draw the graph of the function could be linked to the assumption about learners' familiarity with the discourse of drawing graphs. His comment during VSRI demonstrates this: "*I have done the first two questions, so I was giving the learners a chance to do the other question on their own so that I can see that they understand what I have been showing them*". This statement, in relation to the observable actions, explains Jaden's belief that for learners to become familiar with what the discourse is about, it is essential to participate in the discourse through task completion. This reinforces Sfard's iteration that discourse "familiarity can only emerge from this participation", which in this case is the drawing of the graphs (Sfard, 2008, p. 130). While this is the case, Jaden did not allow learners to finish the tasks, as he interfered with their work.

Jaden further asked the learners to use the graphs depicted in image 106 to make observation statements about the effect of increasing the value of the coefficient of *x*. He asked the learners to make conjectures about where the graphs of functions would be positioned in the Cartesian plane, in relation to the previously drawn graphs. He presented the following narrative:

> *The graphs of y equals to 4x and y equals to 5x would be between the y-axis and the graph of y equals to 3x because the value is increasing. The graph of y equals to 3x is more steeper than 2x, that number 1, that number 2, that number 3* (pointing to the coefficients of *x* in the three functions)*, it is because of that number that when it is increasing, the graphs are coming closer to the x-axis. This is called the gradient; it is called the gradient of this line*.

Even though Jaden used this statement to present an endorsed narrative about the effect of increasing the value of the coefficient, in terms of the graphs leaning more towards the *y*-axis, it would have been advantageous if the teacher also allowed learners to construct and verbalise the narratives for themselves. This was to ensure that learners made their own meanings of the effect of varying parameter a. The explanation of concepts should not only end with the teacher demonstrating to the learners, and using activities, that were not completed in this case, to find out learners' understanding or lack thereof. In terms of the usage of the words "steeper" and "gradient", the statement did not denote the meanings of the words, especially since Jaden did not focus on the distance between the points to illustrate what the gradient values did to the distance between the points on the graph. The commognitive framework does not expound on such cases where a statement is mathematically endorsed, but the use of some words in the utterance do not provide mathematical meanings. A further commognitive interpretation of the above statement, in relation to the graphs drawn, is that the graphs were used as visual mediators to coordinate the teacher's talk about the effect of increasing the values of the parameter. Also, the statement offered the learners an interpretive elaboration about the behaviours of linear functions, especially considering Jaden's call for learners to extrapolate where the graphs of $y = 3x$, $y = 4x$ and $y = 5x$ would be positioned in relation to the previously drawn graphs.

As a way of checking learners' understanding, Jaden asked the learners to draw graphs of three functions as part of homework (image 107). He instructed the learners to use the same steps used during teaching to draw the graphs of the functions on the same set of axes.

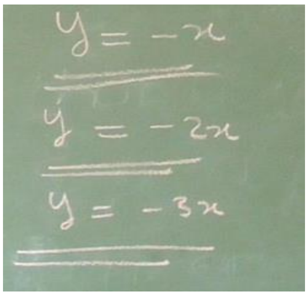


Image 107: *Three functions for homework*

Jaden prompted the learners to focus on the effect of the negative signs on the graphs of the functions: "*You can see that these equations are the same as the ones we just did, the only difference is that these ones have negative signs. Draw those graphs and see what the negative signs do*". These observable actions resonate closely with the curriculum expectations (CAPS) that learners should be introduced to the notion of interpretation of the behaviour of given functions. Jaden first met this curriculum expectation by varying the values of the coefficient of *x* in terms of the increase and in the activity given to the learners for homework, by introducing the negative coefficient of *x.*

9.2. Summary and conclusion regarding Jaden's observed episodes

In overview, from the two lessons analysed in this chapter, the observable routines from Jaden's teaching were rituals to translate the functions presented in symbolic form into the table of values and drawing of graphs. Across all Jaden's episodes, the teaching was dominated by his explanatory talk without providing learners with the learning opportunities to create mathematical meanings for themselves during the lessons. I stated that this teaching practice appears to be influenced by his assumption that his learners lack basic mathematical skills and knowledge. The analysis of Jaden's teaching also discerned that he views mathematical teaching as exposition for memorisation to answer examination questions and mathematical learning as memorisation and mimicking, as demonstrated by his actions of providing learners with an archetype example for each problem he gave them to solve.

The functions teaching approaches that were used in the two lessons include: the example vs non-example approach, the property-oriented approach and the pattern-oriented approach. In episode 3, Jaden used verbal representation in which he offered learners interpretive elaborations relating to the behaviour of functions in graphical form, making his routines more exploratory in guiding learners to generalise the effect of parameter *m* on the graphs of linear functions. Table 20 is a summary of Jaden's episodes and outlines the observable actions during teaching. The next chapter presents the findings of the study and a critical discussion of the findings to highlight to the reader the major themes and sub-themes of the current study.

Table 20 *Summary of Jaden's teaching episodes*

Sfard's commognitive theory				
Episodes and observable actions	**Visual Mediator** *(the images presented in-text also represent iconic and/or symbolic mediators)*	**Words used**	**Endorsed narratives**	**Routines**
1. Using the function machine approach to demonstrate to the learners the ritual to substitute and calculate output values. Determining the 'rules' for given relations using the patterns-oriented approach.	Iconic visual mediators: Using the function machine to calculate the output values. Iconic visual mediator: Using table of values to determine missing values and rules for given relations.	Output values; input values; function; difference; values of x; relation; values of y; corresponding	Meta level narrative about the rule underpinning the relation: "*three is added to the values of x to get the values of y*"	Ritual to substitute and calculate the values of y for chosen values of x using the function machine.
2. Teaching learners how to substitute and calculate the output values using the function notation in the form *f(x)*.	Symbolic: Using the examples of functions: $y = 3x - 1$; $f(x) = 2x - 4$; $f(x) = 3x$; $f(x) = x^2 - 1$; $y = 2x$; $y = x$ and $y = 2x + 1$ to demonstrate to the learners how to use the function notation to determine the output values.	*f* of; substitute; value; linear function; line	Meta level narrative about how the function notation is used: "*instead of y, you have that f of x, function of x*"	Clarifying Ritual to demonstrate to the learners how to use the function machine to determine the output values.
3. Using the examples of linear functions $y = x$; $y = 2x$ and $y = 3x$ to teach learners how to substitute and calculate output values, complete the table of values and draw the graphs. Once the graphs are drawn, Jaden engages in the action of interpretation, exploring the effect of varying the value of parameter *a* for linear functions. Providing learners with two examples ($y = 4x$ and $y = 5x$) asking them to make conjectures about where they think the graphs would be positioned compared to the other three.	Iconic visual mediators: functions $y = x$; $y = 2x$ and $y = 3x$ depicted in the same table of values and graphs drawn in the same set of axes. Symbolic: $y = x$; $y = 2x + 1$; $y = x - 1$; $y = 2x$; $y = 3x$; $y = 4x$ and $y = 5x$; $y = -x$; $y = -2x$; $y = -3x$	Graph; linear function; exponent; line; substitute; increasing; steeper; gradient; y-axis; x-axis; negative signs; domain	Object level narrative about the relationship between the symbolic mediators and the graphical mediator: "*The reason why these functions are linear functions is because the value of the exponent is one. It is also a linear function because when we draw the graphs of these functions, we get a line*" Object level narrative about the effect of parameter m on linear functions: "*The graph of y equals to 3x is more steeper than 2x, that number 1, that number 2, that number 3* (pointing to the coefficients of *x* in the three functions), *it is because of that number that when it is increasing, the graphs are coming closer to the x-axis".*	Exploration of the effect of parameter *a* in terms of its magnitude from graphical visual mediators.

Approaches of algebraic functions used
In the selected episodes I presented in this chapter, Jaden used five approaches of teaching algebraic functions. These were: • Function machine approach – in episode 1 to demonstrate to the learners the process of mathematical calculations and substitution • Example versus non-example approach – throughout the episodes to demonstrate how to work with examples of the different families of functions • Property-oriented approach – across the episodes when bringing the idea of effect of changing the values of the parameters on the functions • Pattern-oriented approach – in episode 1, to illustrate to the learners how to determine the values of the dependent variable • Covariational approach – in doing the process of substitution and calculation to show learners how the input and output values covary.
Scott et al.'s pedagogical link-making and communicative approaches
Jaden took a non-interactive/authoritative approach in teaching learners about the substitution and calculation processes and bringing the effect of changing values of parameters to the fore. Four approaches to pedagogical link-making were used by the teacher. These approaches were: • Promote continuity: developing a mathematical story (micro) • Support knowledge building: making links between scientific concepts • Support knowledge building: making links between modes of representation • Promote continuity: managing/organising (micro scale)

Chapter 10

Findings and discussions: the normalised, the emergent and the expected

Every child should have the opportunity to receive a quality education ~ Bill Frist

10.1. Introduction

The previous five chapters focused on the analysis and presentation of the data that was obtained by means of classroom observations, VSRIs and semi-structured interviews. This chapter presents and discusses findings on the discourses and approaches that emerged from the participants' classroom observable actions, reflective comments during VSRI with respect to their teaching of algebraic functions in rural classrooms and their responses during semi-structured individual interviews. Sfard's commognitive theoretical framework (2008), the six approaches of algebraic functions that were discussed in Chapter 2 and Scott et al.'s (2011) pedagogical link-making to support knowledge building and to promote continuity within PLM framework, underpinned my thinking as I observed and analysed teachers' observable actions and illuminated their classroom practices. Before I engage in in-depth discussions of the teachers' discourses and approaches while teaching algebraic functions within rural classrooms, I present a summary of the major themes and sub-themes that emerged from the analysis which I presented in Chapters 5 to 9.

Table 21

Themes and sub-themes for the study

Themes	Sub-themes
Teachers' use of functions representations and their weaknesses	• *Functions as drawing graphs: rituals to reach the end goal* • *The situations in which learners were immersed: lack of instructions*
Teachers' communication about the effect of parameters	• *Generalisation for learners using worked out examples* • *The participationist approach to generalisation* • *The use of examples: variation between parameters*
Approaches to teaching functions	• *Property-oriented approach* • *Example versus non-example approach*
Factors that shape rural teachers' approaches and discourses	• *The discourse of teaching for compliance* • *Teaching for assessment* • *Knowledge of algebraic functions and curriculum focus*

10.2. Teachers' use of functions representations and their weaknesses

This theme focuses on how the teachers used the different modalities of representations of functions during teaching. While teachers introduced different representations, which included formulae, table of values, and graphs, they used them to demonstrate rote procedures for determining the output values for given input values. Of the five participating teachers, three did not explain explicitly to the learners how and why these procedures work. The literature on functions representations has suggested that to understand the concept in the way a mathematician does, is to be able to juggle with its verbal, tabular, symbolic, and graphical aspects, applying with ease whichever is most appropriate in the moment and flexibly translating among them (Dreher & Kuntze, 2015; Mpofu & Pournara, 2018; Mudaly & Mpofu, 2019). Teachers' observable actions, reflective comments during VSRIs and discussions during interviews show that the use of the algebraic equation and table of values demonstrate a changeover between these representations and produced a graphical representation from the generated values. The changeover between representations means instances whereby teachers use the symbolic and tabular representations as tools for drawing the graphs, without explaining what properties of functions each signify or visualise.

According to Chimhande (2014) and Rau (2016), teachers and learners must see these forms of representation as 'informationally equivalent', to demonstrate a deeper conceptual teaching and understanding of the algebraic function concept. Teachers treated the symbolic representation and the table of values as tools for utility in the process of drawing the graphs of functions, rather than equivalent forms of representation. Furthermore, teachers demonstrated limited flexibility in using different modalities of representations across the different representations and did not explicate the notion of the dependence relationship between two sets of independent and dependent variables. In this regard, Cilliano (2021) posited that "over-relying on algebraic forms of functions can also lead to a purely procedural understanding, with students focusing on following the steps in algorithms to solve for ordered pairs, or find numeric values without linking the functional relationship to its context". In this study, teachers' over-reliance on algebraic forms of functions resulted in their teaching being about finding ordered pairs, completing tables of values and drawing graphs, without demonstrating the links between given functional relationships and their contexts. It is important that teachers guide learners to observe the relationship between two variables, to ensure that they develop understanding of what changes in a particular relationship as well as how two variables covary. I argue that teachers offered incomplete explanations about the information that each form of representation reveal about functions properties, and this resulted in ritual teaching to draw graphs of functions. This kind of teaching is disadvantageous for learners because they might think of the concept in terms of procedural mastery of symbolic manipulations, which limits their ability to "organize, create, record, understand and communicate mathematical ideas" (Mpofu & Pournara, 2018, p. 35).

The teachers overlooked how each mode of representation encodes and depicts information, and how the different modalities relate to the concept they have represented and the relational links between them. As mentioned in Chapter 2, an in-depth understanding and teaching of different modalities of representations can help learners to express a function in different ways, and express the reasoning strategies that learners use in the development of functional understanding (Chimhande, 2014). Thus, teachers' limited information of and the challenges with using modes of representations during algebraic function lessons constrained the development of insights into understanding the facets of the functions concept. Viirman (2014, p.19) has suggested that "understanding the concept *of functions* requires the grasp of the different representations (*algebraic, graphic and tabular*) and the interplay between them" (italics added). It was clear from the findings that teachers demonstrated ritualisation routine, which did not adequately and effectively reflect the expected teaching of the interplay between the different representations as suggested by Viirman (2014) and prescribed in CAPS. Especially when the expected competency to "convert flexibly between representations of functions as tables, graphs, words and formulae" (Umalusi, 2014, p. 49) is important to communicate the information about the key features of the functions concept. This theme has two sub-themes: 'functions as drawing graphs: rituals to reach the end-goal' and 'the situation in which learners were immersed: lack of instructions'.

10.2.1. Functions as drawing graphs: rituals to reach the end goal

The analysis of Mafada's, Mutsakisi's, and Tinyiko's observable actions reveals that their reasoning about the use of representations was limited to the rituals of following a series of steps to draw the graphs and using them as separate entities. The National Council of Teachers of Mathematics has emphasised that teachers should teach learners how to "translate among tabular, symbolic, and graphical representations of functions" (NCTM, 1989, p. 154) to develop conceptual understanding. The observed teaching shows that teachers treated the produced graphical representation as a property of the different families of functions and did not teach learners the critical features that the graph and other modalities of representation bring to the fore. It is important that teachers teach the critical features of the different families of functions explicitly, to enable learners to experience how changes in one variable have a corresponding effect on another. According to Chimhande (2014), the inability to flexibly move among different representations of functions can result in conceptual gaps, as learners will not develop an understanding of what each modality of representation signifies. Similarly, Walde (2017, p. 2) argued that "to be able to link the different representations of function is probably the most important node in the network of students' understanding of the concept of function". Table 22 below represents the three teachers' utterances that appeal to the ritual of drawing graphs, the narratives and word use, which reveal that they view the object of learning for algebraic functions as graphing skills and knowledge.

Table 22

Teachers' word use and narratives [45]

Teacher's name	Evidence – What is said? (word use and narratives)
Mafada	1) "to enable you to plot the graph" (semi-structured interview). 2) "function represents the graph" (episode 2, lesson 1) 3) "Here was for them to see the different shapes of the graphs" (episode 1, lesson 1, VSRI). 4) "getting all the values that are going to enable you to plot the graph" (episode 1, lesson 1, interview). 5) "I think we are using the table to get the coordinates in order to draw the graph, that's what the textbook say".
Mutsakisi	6) "on the graph y is equals to f of x equals to x minus 3, what is c?" (episode 1, lesson 1, line 5). 7) "you can have the equation that represents a graph" (episode 1, lesson 1, semi-structured interview). 8) "we want to use this equation to draw the graph" (episode 2, lesson 1).
Tinyiko	9) "we need to interpret this like we did with the linear graphs" (episode 2, lesson 1) 10) "the reason why I am introducing them to the table method is because when you are given any function in the world, when you use the table method there is no way your graph will not come out" (semi-structured interview). 11) "A graph and a formula, it simply tells you, in order for a graph to come out a certain formula was used, a formula says being me like this, there is something that you can draw from me" "(semi-structured interview).

The findings above represent two types of narratives from the teachers' classroom discourse: statements which refer to the equations and table of values as tools for the process of drawing graphs, and statements which reveal that teachers' understanding of functions was merely the production of correct shapes for the graphs. Chimhande (2014) argued that teachers should

[45] The statements in this table were purposefully selected from longer excerpts presented in the data presentation and analysis chapters, as the fragments are representative of the focus of the current sub-theme.

teach learners the properties of each setting and representatives, in terms of what is signified by each form of representation, instead of teaching functions in terms of formulas and graphing. Mafada, Mutsakisi and Tinyiko used the equations and the graphs as if they were labels for the different functions they had introduced in different lessons, rather than representational means for expressing the concepts and tools for proving (Bloch, 2003). It is important for teachers to explain the information that each form of representation depicts, to ensure that learners understand the different properties of functions that are signified by each form. Instead, teachers were concerned with the question: 'How do I proceed?' from one representation to the other until the graphs of given functions were drawn, rather than the question: 'What is it that I want to get?' from each signifier about the nature of the given functions (Lavie et al., 2019). This discussion links with Thompson's (1994) criticism of the generally accepted meanings of representations:

> the idea of multiple representations has not been carefully thought out, and the primary construct needing explication is the very idea of a representation. ...the core concept of 'function' is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produce a subjective sense of invariance. (p. 39)

The implications for teaching functions as the drawing of the graphs is that learners might only understand algebraic functions as 'graphing', a topic to be learned in isolation and the modalities of representation as tools to facilitate the graphing process. Earlier, Thompson (1994) posited that when teachers do not facilitate learners' understanding that each representation presents particular information about the function concept and that the different modes should be used relationally, "learners will see each representation as a 'topic' to be learned in isolation of the others" (1994, p. 23). Since the different representations signify the same function, it becomes important for teachers to help learners identify and analyse the connections among them. In addition, the quality of mathematical knowledge of underlying ideas of the functions introduced during teaching, is intertwined with teachers' and learners' ability to translate from one modality of representation to another (Moussa-Inaty et al., 2020).

A critical analysis of the teachers' narratives demonstrates that they have limited skills in linking the different modes of representations, because they were not aware that it is only in the integration of and translating flexibly between different modalities of representation that the whole concept of algebraic functions exists. Graf et al. (2018, p. 2) claimed that the teaching of function "tends to be highly procedural and focuses solely on the mechanics of graphing and algebraic manipulations", which resonate with ritual routines. Sfard (2008) contended that rituals concern performing particular procedures on mathematical objects and not about knowing, which means the understanding of functions as drawing graphs is not about knowing but performing because "there is no room for a substantiation narrative" (p. 246). In addition, the teachers' observable actions "relate to the computational aspects associated with functions, such as arithmetic process or an input-output 'function machine'" (Aylon et al., 2017, p. 3), because they treated the computational formula as a necessary condition for a function. Their teaching revealed that they see formulas, table of values and graphs as things in themselves, instead of representations of other entities. I therefore argue that the teachers' ritualisation did not support learners' mathematical knowledge building, especially making links between the different modes of representation, because of the overemphasis on performative actions for learners to learn the procedures to work with functions blindly.

At the meta-level narratives all three teachers spoke of functions as objects to perform step-by-step algebraic manipulation to obtain coordinate pairs to draw the graphs. The problem with the step-by-step algebraic manipulation without offering explanations of the information embedded in different forms of representations is that, while characterisation of algebraic functions through rule and generation of graphs are done, teachers did not make the important features of the global function explicit during teaching processes. I noticed that teachers used action-oriented teaching, which is characterised by "the teacher's instructional acts which emphasize step-by-step manipulation of algorithmic procedures" to draw the graphs of functions (Boyazit & Aksoy, 2010, p. 149). For effective teaching of algebraic functions, it is essential that teachers focus on objectification (conceptual understanding in acquisition of algebraic functions concepts), rather than just ritualised routines (computational imitation). Sfard (2012) stated that, when teachers focus primarily on computations rather than objectification, learners will apply uncritical and unthoughtful imitation of procedures with little or no awareness for the reasons of such computations. This subsequently constrains learners' opportunities for conceptual understanding.

I noticed during Mafada's, Tinyiko's and Mutsakisi's lessons that they dominantly used ritual routines to teach learners mathematical substitutions and calculations. They overlooked the importance of encouraging learners' participation needed to develop deep knowledge and critical thinking as a precondition of objectification, particularly, after they had introduced and explained the concepts and procedures to the learners, as a way of inducting learners to the new topic. To this effect, Nachlieli and Tabach (2012, p. 17) asserted that "participation in discourse is a precondition for the objectification of functions", which is important for learners to make their own mathematical meanings and promote their confidence to engage with algebraic functions processes.

The teachers further presented functions as instructions to engage in mathematical calculations from one numerical set to another. The teachers' observable actions were problematic because such teaching practices do not allow learners opportunities to observe changes between variables, to link these changes and look for relationships, which is the essential curriculum objective for the topic (DoE, 2011; Sfard, 1991). The statements conveyed the assumed association of symbolic representations of given functions with the graphical representation for all the functions introduced during teaching. These findings demonstrate that teachers used mainly the operational discourse where the established steps were followed to complete all the tasks associated with functions, resulting in graphs becoming the immediate realisation of the functions presented in symbolic form. According to Sfard (1991) and Maharaj (2008), the operational discourse in the teaching and learning of functions limits learners' ability to identify the unchanging and changing quantities in given relationships as well as skills to determine the effect of change for one variable over others.

The three teachers' focus during the lessons reveals that the precise and accurate performance of the ritual to draw the graphs of functions was the only requirement for the topic. For example, the use of words "*function represents the graph*" (Mafada), "*on the graph y is equals to f of x equals to x minus 3, what is c*?" (Mutsakisi) and "*you can have the equation that represents a graph*" (Mutsakisi) signifies that the equation which is referred to as function in the statements represents the graph, suggesting that the function is its mode of representation. The teachers' utterances denote that the equation is a representative of the graph rather than equivalent forms of representations of the same function as suggested by previous research (Chimhande, 2014; Rau, 2016; Isler et al., 2017). Considering the teachers' narratives and the use of the words 'function' and 'graph' in the selected excerpts, I argue that they disregarded the mathematical knowledge related to the properties of functions. This way of teaching algebraic functions has ignored the process of making flexible links between different forms of representation, including tables, graphs, formulae and words to promote concepts' continuity and support learners' knowledge building (Scott et al., 2011). There is also a high level of correctability relating to arithmetic operations to ensure that 'correct graphs' are produced, but not relating to the structure of their solutions and discernment of critical features of functions.

The three teachers did not address that each visual mediator stands for a referent to depict the relationship between two variables in the mathematical domain, considering the usage of the equations and table of values during teaching. Instead, the way teachers used the symbolic and tabular representations while drawing graphs justifies that routines are ritualised (Sfard, 2008), as they engaged in mathematical substitutions and calculations across different modes of representations. From the classroom observations, it might be possible that teachers overlooked the role visual mediators play while teaching algebraic functions, especially in helping learners to move towards objectification of the function concept, because their teaching focuses on procedural computations to draw the graphs and their observable actions did not facilitate visual understanding (Nachlieli & Tabach, 2012). Rau and Matthews (2017, p. 4) defined visual understanding as "the ability to make sense of a visual representation by mapping its visual features to information relevant to understanding the target domain". Teachers used different representations without explaining the mathematical meanings of the concept conveyed by each representation. Similar findings were reported by Martinho and Viseu (2019) in their study of Portuguese pre-service teachers' conceptions of the function concept. The authors highlighted that one of the trends involved pre-service teachers identifying functions with their representations, such as a diagram or a Cartesian graph. In addition, teachers did not offer interpretive elaborations relating to the different properties of the concept that the graphs brought to the fore for different families of algebraic functions. This further indicates that the focus of the three teachers' teaching was to ritualise learners to know the steps to reach the end-goal, which was to draw the graphs.

Mafada, Tinyiko and Mutsakisi made explicit reference to adhering to the use of algebraic symbols in exams and within the textbooks. The type of visual mediators the three teachers used resonates with those found in Viirman's (2014) study of seven university mathematics teachers' discourse on functions, which entailed the use of symbolic representation to introduce the functions, and the graphs being drawn for the sake of completing the 'series of steps' for teaching the concept. Leshota (2015) noted that the use of step-by-step represent a procedural process, I argue that too much use of this strategy during the teaching of algebraic functions limits learners' conceptual understanding and does not promote the development of specific skills stipulated in CAPS. The CAPS document states that the teaching and learning of functions at Grade 10 level should focus on investigating the effect of a and q on the graphs defined by $y = a.f(x) + q$ and work with relationships between the variables through using tables, graphs, words and formulae and converting flexibly between these modalities of representations. Thus, it is important for teachers to help learners identify the changing and unchanging quantities in a relationship and enable them to determine the effect of the change of a particular quantity over others, which was unnoticed in this study when teachers only focused on procedural processes to draw graphs. Ronda (2009) posited that the study of algebraic functions can be regarded as the study of relationships between quantities and their properties (Ronda, 2009).

The above discussion reinforces various scholars' findings that the relations between modes of representations in mathematics and their referents are often opaque, resulting in difficulties learning with them (Viirman, 2014; Rau, 2016; Walde, 2017). Moalosi (2014) posited that the teaching and learning of functions should focus on the relationship between variables rather than the process of calculating the output values. Even though it is acceptable for teachers to teach learners mathematical computational skills, a sole focus on this without interpretive elaboration of the mathematical calculations to bring mathematical meanings to the fore can result in rote memorisation of the procedures, emanating from absorbing ritualised routines (Chimhande, 2014; Mudaly & Mpofu, 2019). From the observations of Mafada's, Tinyiko's and Mutsakisi's teachings, there were no interpretations nor were learners allowed opportunities to interpret the mediators, resulting in ritualistic practise. That is, the teachers overlooked the 'action of interpretation' of the different families of functions (Bell & Janvier, 1981). By action of interpretation, I mean the action by which learners and teachers contrive meaning or gain meaning from the multiple representations that are used during teaching and learning (a graph, a functional equation, table or a situation) (Leinhardt et al., 1990). I further noticed from Mafada's, Mutsakisi's and Tinyiko's observed lessons that they did not allow learners to interpolate and/or extrapolate the pattern from the behaviour of the functions, and the continuation[46] of the graph. According to Kwari (2007), allowing learners to interpret the behaviour of different functions contributes towards their development and understanding of algebraic functions.

The consequences of disallowing learners to construct and verbalise their interpretations address the lack of knowing the necessary conditions for a relation to be a function, as learners

[46] see Leinhardt et al., 1990 for elaborate discussion on actions that relate to functions.

continue to learn the topic in further grades. These findings resonate with Leshota's (2015) notion of procedural process, as the teaching of functions focuses on showing the learners the procedure or set of steps involved in the drawing of graphs. In the current study, the teachers' general feature of their teaching practices involved the use of running commentary about the functions; that is, they focused on verbalising what they wrote on the board rather than talking about the behaviour of the different functions they introduced during teaching. The teachers' discourses thus reinforced Sfard's (2008) argument that "When the talk about processes is replaced with the talk about objects, many different forms of actions become tied to the same noun, *in the current study the noun is the drawing of graphs*" (p. 62, italics added) she termed 'consequential omission'. The resulting consequential omission in the current study is the lack of emphasis on moving flexibly among the representations and helping learners understand how they relate to each other as stipulated in our curriculum approach (DoE, 2011). The teachers in this sub-theme mistook a signifier (graph) for the signified, overlooking the discernment of critical features for different families of functions across multiple representations (Sfard, 2008).

Furthermore, teachers did not teach learners about concepts such as minimum, turning point, increase and decrease, domain and range as the key components stipulated within the CAPS curriculum. These are important because an understanding of these properties can enable learners to classify different families of functions and establish relational links between different representations (Monk & Nemirowsky, 1994; Chimhande, 2014). The learners' lack of access to the discourse of translation between representations of functions can constrain their mathematical communication and participation in mathematics discourse (see Gucler, 2015). In essence, I argue that the ritualisation discourse which aimed to ensure that learners knew the step-by-step memorisation of how to draw graphs of functions resulted in a lack of productive and intellectual interpersonal communication by the teacher during the teaching and learning of algebraic functions in this study. The classroom patterns of communication became limited and uncritical two-way interactions between teachers and learners, because learners only answered a word of teachers' questions in relation to confirmation of understanding of steps. Other prominent teachers' utterances focused on mapping the input and output values as they engaged in mathematical calculations, substitution process or while sketching the graphs. The following sub-theme focuses on the assumptions that teachers made about lack of written and/or verbal instructions to guide learners in the process of mathematisation about examples of functions presented during teaching.

10.2.2. The situation in which learners were immersed: lack of effective instructions

While the choice of pertinent forms of representations for the functions concept is necessary to enable learners' understanding of the concept, it is not sufficient; the situation in which learners are immersed during the lessons is essential to promote their understanding. By 'situation', I am referring to the types of problems learners are led to solve in the classroom, and instructions entail directions about what and how learners should solve problems, as well as questions pertaining to the key aspects of the topic which learners need to pay attention to or determine. This sub-theme is important because it illustrates teachers' assumptions about the rituals learners are expected to perform whenever they are presented with functions in formulae form, and the lack of guiding questions to bring particular features of the concept to the fore. Delastri and Muskar (2018) argued that teachers tend to focus on procedural rules and algorithmic skills during the teaching of the functions concept. The authors acknowledged the importance of procedural direction and algorithmic skills during lessons; however, they further argued that "those skills do not help students understand *the function* concept in various situations meaningfully" (Delastri & Muskar, 2018, p. 2). The analysis of the different tasks in the previous five chapters indicated lack of (accompanying) effective instructions or guiding questions (e.g., "compare the objects or functions", "solve the problem", "find the intercepts with axes") in the type of problems that were introduced to the learners. Teachers' classroom practices revealed that the assumed general instruction for different functions teachers introduced were mathematical calculations and drawing of graphs which, according to Delastri and Muskar (2018), do not promote conceptual understanding.

The lack of effective instructions in this study resulted in the teaching of functions being about the successful completion of recognisable procedural skills rather than about the extension of the procedures to investigate or generate evolving complexity of the global features of the functions concept. Aguilar et al. (2017) advocated that learners should be asked to "graph and explore how the coefficients of a function relate to translations, stretches and reflections of its graph" (p. 3778). In view of this, teachers in the current study introduced different functions in symbolic form and did not offer instructions for what they wanted to bring to the fore during teaching. Instead, teachers engaged in the process of mathematical calculations and drew graphs. There were no questions framed to enable learners to explore and create meanings of the translations, stretches and reflections after graphs were drawn. While instructions are generally important in the teaching profession, the nature of instructions teachers use is more crucial, as they can either enable or constrain learners' access to the key elements of the topic.

Similarly, Killen (2015) argued that the “challenge for teachers is to structure learning experiences that will systematically lead learners to new levels of understanding” (p. 42), which in this study is linked to the lack of instructions underpinning the different examples teachers used which constrained systematicity in their teaching practices. The Curriculum Assessment Policy Statement (DoE, 2011) requires teachers to continuously assess learners throughout the teaching process and select tasks that promote learners’ understanding of the contents. While teachers introduced various examples during the teaching of algebraic functions, they were not accompanied by clear instructions on what to do with given functions.

The analysis of the teachers’ verbal behaviours indicates an assumption they hold about the completion of tasks, that learners are aware of the expected action to draw the graphs whenever a function is presented in symbolic form. Of the five teachers, Mafada, Mutsakisi and Tinyiko introduced the functions in symbolic form, engaged in substitution and calculation of y-values for given x-values, as well as the drawing of the graphs of the functions. All these processes happened without giving learners instructions, neither verbal nor written, about the features of the function concept the learners were apt to pay attention to. Therefore, the lack of instructions for working with functions in symbolic form could be related to the notion of a closed task (Yeo, 2017) in which the goal and the answer are closed. Thus, the goal with the examples and tasks that teachers introduced during teaching was the production of the correct graph for a given function, overlooking the need to use questions that address specific components of the topic and concepts. Dahal et al. (2019) stipulated that “Questioning has been used to provide a variety of situations that increase student involvement, regulate classroom processes, focus attention on a particular issue or concept, structure a task in order to maximize learning and understanding, assess students’ prior and current knowledge” (p. 122). The teachers did not use the tasks to maximise learning and promote learners’ deep learning and understanding of mathematics concepts and knowledge about algebraic function, thus enforcing intellectual quality (Ahmad et al., 2012).

The findings reveal that Mafada, Mutsakisi and Tinyiko did not use questions and instructions to scaffold learners’ investigation of the behaviour of the functions during teaching, and encourage learners to configure conjectures about the effect of different parameters and test them. Teachers focused primarily on ensuring learners’ development of process skills associated with drawing graphs (Lee & Kinzie, 2012). Of course, when learners are given a mathematical object without clear instructions on what to do with them, the instinctive assumption is to do what the teacher has done with previous similar objects. Accordingly, the lack of instructions when functions were introduced in symbolic form also address the ritual routine, as the teachers did not give learners clear instructions to engage with the task to support knowledge building.

I argue that the discourses of recognition, recall and association were the foci of mathematisation and not mathematical reasoning in the teachers' teaching (Essak, 2016). Sfard and Linchevski (1994, p. 192) contended that "algebraic symbols do not speak for themselves. What one actually sees in them depends on the requirements of the specific problem to which they are applied". Thus, effective instructions were important to stimulate critical thinking about the topic and its concepts, and to build learners' independent meaning making processes. I further argue that the lack of effective instructions in this study resulted in the presentation of knowledge in an uncritical way, as teachers did not create opportunities for argumentative discussion of the information discernible from given examples. Instead, the teaching practice and questions influenced learners to accept the information that was provided by the teachers instantly. This resonates with the ritualisation discourse and the authoritative/non-interactive communicative approach (Scott et al., 2011).

To a certain extent, Jaden attempted to use some instructions in the worksheet he designed for the learners in episode 2, which demonstrated an attempt to draw learners' attention to specific mathematical principles as well as processes. There were written instructions for what learners were supposed to do with the equations by using words such as "*Determine the value of ...*" and "*consider the function ...*" which were written on the board. However, in episode 3 Jaden's subsequent observable actions revealed the lack of instructions for learners to continue with the task, and they merely engaged in the process of substituting and calculating y-values for specific values of x. There were no further instructions to focus learners' attention to specific features of the concept as suggested by Dahal et al. (2019) earlier, above. The lack of specific instructions about the various components of the function concept could constrain teachers' interpretive elaborations, since there was no organising principle to facilitate their classroom talk and/or foster learners' thinking and communication about the different aspects of the topic. In terms of PLM, a non-explication of the instructions for the examples done by the teachers and tasks presented to the learners is a factor that can hinder the notion of supporting knowledge building. I noticed that Scott et al. (2011) did not view this as one of the approaches to facilitate learners' knowledge building.

Thus, the different tasks teachers used in the classroom demanded low cognitive engagement, as they emphasised using memorised routines or formulas to solve problems (Anderson & Krathwohl, 2001). To use Bloom's taxonomy, the tasks were in the first stage of learning (remembering knowledge), which deprived learners' development of mathematics intellectual quality (Killen, 2015). They further required learners' proficiency in computational procedures to draw graphs, and the closed methods was the prominent norm for teacher-learner interaction. The results support the findings of a study by Lee and Kinzie (2012) among secondary school mathematics teachers, that most of the teachers used questions which constrained learners' responses such as test questions, and yes-no questions. The following section focuses on the second theme that emerged from the analysis of the classroom observations about the critical global features associated with the function concept.

10.3. Teachers' communication about the effect of parameters

This theme addresses teachers' selection of examples during algebraic function lessons, and whether and how they facilitated and/or constrained the learning of algebraic functions' critical features during teaching. According to Renkl (2017), it is important that teachers choose appropriate examples to facilitate and deepen the learning and understanding of the concepts and knowledge for the content. The theme focuses specifically on how teachers worked with the examples during the lessons to help learners understand the critical features for linear functions, parabolic functions, hyperbolic functions and exponential functions. The South African CAPS curriculum recommends that teachers teach critical features for the different families of algebraic functions, such as the effect of different parameters, domain, range, intercepts, and turning points (DBE, 2011). The curriculum further asks teachers to provide learners with opportunities to make conjectures, and prove them, to formulate generalisations, especially with the effect of different parameters for different functions.

The theme comprises three sub-themes: 'Generalising for learners', 'the participationist approach to generalisation' and 'The use of examples as symbolic mediators: variation between parameters'. 'Teachers learning algebraic functions for the learners' addresses the limited opportunities for learners to observe the behaviour of mathematical objects to make their own meanings of the topic and generalisations about the effect of different parameters. 'The participationist approach to generalisation' highlights how Zelda moved more towards the participationist approach to help learners to observe the effect of different parameters for the different families of functions. 'The use of examples as symbolic mediators: variation between parameters' is about the systematicity or lack thereof in the selection and use of examples to guide learners to see the effect of variation in the parameters on the behaviour of the functions. The reason I named the following sub-theme 'Generalising for learners using worked-out examples' is vested on the perspective that teachers' generalisation from worked examples about the nature of different families of functions did not elicit learners' understanding of functions, as teachers did not give learners opportunities to make meanings of mathematical processes, specifically the effect of different parameters.

10.3.1. Generalising for learners using worked-out examples

While it is expected that teachers dominate the lesson when they introduce a new topic and explain concepts, it is also important to give learners the opportunity to learn the content as the lessons continue, for their own understanding. In this study, four out of five teachers used worked-out examples, in which the different mathematical procedures being applied were performed by the teachers and generalisations about functions were verbalised for the learners throughout the lessons. In this sub-theme, generalising for learners means teaching situations in which teachers did not allow learners to observe the behaviour of functions, make conjectures and prove such conjectures for themselves to make generalisations, instead teachers verbalised the generalisation statements for learners. The findings indicate that of the five teachers, Mafada, Tinyiko, Mutsakisi and Jaden provided limited opportunities for learners to use their cognitive skills to critically engage with the maths content and concepts. Only Zelda guided learners to construct and verbalise their observation about the effect of different parameters.

Mafada, Tinyiko, Mutsakisi and Jaden used 'transmission style' to narrate the procedures that learners must memorise, and the effect of different parameters for different families of functions (i.e. generalisations). This teaching process is linked with Freire's (1973) banking model of education, where learners are perceived as empty vessels and the teacher does the learning by giving information. The teachers overlooked that the interpretation of mathematical ideas should be "taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas" (Cobb et al., 2001, p. 126). The four teachers generalised the effect of the parameters from new examples without allowing learners the opportunities to experiment with different parameters and formulate interpretations and generalisations for themselves. The teachers treated the teaching of algebraic functions as being about giving learners mathematical facts, as teachers asked and answered questions without giving learners enough time to think and engage. There were no opportunities to answer the questions to demonstrate their own understanding or lack thereof, for the teacher to be aware of the areas where learners had challenges. Freire (1973, p. 80) argued that in this transmission style, “the students are not called to know, but to memorise the contents narrated by the teacher”. This is unfortunate because the teaching environment does not encourage learners to observe the behaviour of mathematical objects and create their own meanings with peers and learners do not feel at ease in a classroom setting (Sullivan, 2011).

The following excerpts are examples of the four teachers’ generalisation statements relating to the effect of different parameters of algebraic functions and the numbering will be used in subsequent discussion of the narratives:

1 “*now we have seen that when the sign is positive, where is our graph facing? Our graph is facing up*” (Mafada, episode 3, lesson 2).

2 “*… if a is positive the graph faces up, so the graph smile, when a is positive, our graph is a smile*” (Mutsakisi, episode 4, lesson 2).

3 “*because you were given the function as* ax^2*, the coefficient of your x squared is positive, it simply tells you that your graph will go up*” (Tinyiko, episode 1, lesson 1).

4 “*… it is because of that number that when it is increasing, the graphs are coming closer to the x-axis*” (Jaden episode 3, lesson 2).

The above narratives demonstrate that teachers were *telling* learners what the effect of the different parameters are instead of *teaching* them (Hawes, 2004). According to Hawes (2004) and Slunt and Giancarlo (2004), telling entails the teachers feeding learners with the information relating to the effect of parameters on the different families of functions, without learners necessarily understanding what the teacher is doing and why. The four teachers did not help learners understand the inherent principles of what they are to do, why they should use a specific procedure and how they should make meaning of the topic irrespective of the questions. Schoenfeld (2012, p. 7) stated that it is important for teachers to enable learners’ “predilection to explore, to model, to look for structure, to make connections, to abstract, to generalize, and to prove”, for the learners to co-construct mathematical meanings for themselves.

If the purpose is to ensure learners’ conceptual understanding, it is crucial that teachers create opportunities for learners to observe the effect of parameters and construct conjectures and prove them for themselves. The purpose is to teach with the learners to build the knowledge they will need for progression and understanding. Hawes (2004, p. 47) posited that “Rather than telling students what we know, we need to ask questions that lead students along the path of learning". In mathematics, the teaching of the variation between variables and the changes brought by such variations should be the chance to allow learners to experiment with open-ended, reflexive epistemological questions such as 'What has changed?', and 'What effect do the changes have on the function?' to enable their analytical and critical skills about functions. Predominantly, Mafada, Mutsakisi, Tinyiko and Jaden used generalising of the effect of parameters from only a few examples and without creating opportunities for the learners to verbalise their own conjectures and generalisations, which resonates closely with the notion of telling. Of concern is that teachers did not provide "opportunities for learners to explore, reflect upon and share their developing understanding" (Killen, 2015, p. 47) about the effect of the parameters. Thus, the teachers' teaching did not allow learners to engage in meaningful mathematical learning that supports progression towards formal knowledge as set out in CAPS (Taber, 2010); instead they allowed partial engagement during teaching. The teachers did not allow learners to engage in comprehensive conversations about the behaviour of different functions, to construct their mathematical meanings from what they observed changing in different functions.

The teachers predominantly asked 'confirmatory' questions, for learners to confirm the teachers' statements and information, without full understanding of what they meant. Thus, unsurprisingly, teachers used the authoritative communicative approach, which forced them to teach for their own learning rather than introducing and dialoguing the knowledge with learners for epistemological access. Teachers' generalisation for the learners was earlier problematised by Lockhart (2002) that mathematics education at school level is nightmarish and destroying children's "natural curiosity and love of pattern-making", because this teaching strategy only focuses on the exercise of given algorithms and given rules, which does not enable conceptual understanding (p. 2). Similarly, Govender (2010, p. 1) contended that if teachers are "not in the habit of getting their learners to experiment, make conjectures, express and justify their own justifications, then mathematical thinking becomes the worst casualty of our mathematics classroom". Mafada, Mutsakisi, Tinyiko and Jaden did not allow their learners to experiment, make conjectures and test such conjectures on their own to make generalisations; instead they narrated facts to the learners about the effects of parameters. Accordingly, I argue that mathematics learning was for the teachers rather than learners, considering their dominant teaching practices.

Allowing learners to make their own observational statements about mathematical objects, test such observations and make generalisations is one of the key principles endorsed by the CAPS curriculum. Abramovich et al. (2019) asserted that learning mathematics is a complex endeavour that involves developing new ideas while transforming one's ways of doing, thinking, and being. The four teachers in this study limited learners' observation, thinking, and practice of mathematical processes as a way of developing their identities as mathematics learners. The CAPS document (DBE, 2011) does not explicate how mathematics should be taught in schools but provides details on the contents that should be taught within a grade, for learners to "identify and solve problems and make decisions using critical and creative thinking" (DBE, 2011, p. 5). The envisaged teacher-learner interactions in the CAPS policy suggests that teachers should provide learners with the opportunity to develop the ability to be methodical, to generalize, make conjectures and try to justify or prove them" (DBE, 2011, p. 8). It is therefore important that teachers prepare learners as early as grade 10, where they are introduced to algebraic functions, to be independent thinkers to develop their self-confidence. I therefore argue that the four teachers did not give learners such opportunities, as suggested by the curriculum, to allow for the creation of their own mathematical meanings and understandings.

Learners were given limited opportunities to make meanings about the effect of parameters on different families of functions, engage in problem solving, independent thinking or making sense of the effect of varying the values of the parameters on their own. Anthony and Walshaw (2008, p. 202) posited that "classroom cultures that provide opportunities for mathematical argumentation on a regular basis, lead to enhanced learners' mathematical understanding". Considering that classroom talk and questioning were 'confirmation–based', it means learners' understanding of algebraic function was limited to what the teachers said and wrote on the board during teaching, and not learners' interpretations and understanding of algebraic functions. Mortimer and Scott (2003) argued that the teaching purpose in mathematics involves guiding learners to work with mathematics meanings and supporting internalisation. Similarly, NCTM (1991) postulated that instead of doing all the talking, modelling, and explaining alone, teachers must encourage and expect students to do so, and teachers must do more listening and students more reasoning (p. 36). On the contrary, in this study four teachers did less listening and more modelling and explanation for the learners, which means they were "filling and storing the deposits" (Freire, 1973, p. 72) information passively. It is disadvantageous for the learners not to construct meanings for themselves, especially if education is a process through which children develop a sense of judgement and develop knowledge and skills that enable them to meet the needs of the society (DBE, 2011).

I also observed that teachers monopolised learning, which was observed in their completion of all examples on the board to illustrate procedures of how to draw graphs of functions, and in some instances mathematical principles to make generalisation statements without creating a dialogic space for learners to demonstrate their understanding or lack thereof. The use of the chalkboard was the opportunity for the teachers to encourage learners' participation after they have made introductions, because what teachers write on the chalkboard provides learners with visual reinforcement and enables them to coherently follow the presentation of content. Earlier Watson and Mason (2002, p. 378) stated the need "for pupils to have several examples from which to get a general sense of what is being taught". Equally, previous studies (Marton, 2015; Kullberg & Skodras, 2018) argued that the use of few examples or single examples are insufficient to help learners generalise from, since the use of several examples are necessary to get a sense of a concept. Thus, making generalisation statements from worked examples without providing adequate opportunities for learners to investigate the effect of parameters of the functions before teacher's generalisations, might constrain learners' understanding of the nature and effect that each parameter has on the functions. The teachers' teaching methods are inconsistent with Killen's (2015) argument that "it is essential to structure learning environments and activities to help learners construct knowledge rather than just absorb it" (p. 47). Accordingly, the classroom environments in this study discouraged effective learning and promoted passive learning, as teachers dominantly taught for their learning and not necessarily learners' development of intellectual quality through deep understanding of algebraic functions skills and processes, which includes the generalisation of the effect of different parameters (Kwari, 2007; Chimhande, 2014).

To use Mortimer and Scott's (2003) concept of the 'authoritative/non-interactive' approach, teachers presented mathematical contents as an objectified body of knowledge, without encouraging learners' active participation in the learning of algebraic function as a new topic. Mafada's, Tinyiko's, Jaden's and Mutsakisi's teaching reinforces Lockhart's (2002) claim that "the art of proof has been replaced by a rigid step-by-step pattern of uninspired formal deductions", in which the teacher is at the centre of teaching and learning processes (p. 22). As mentioned earlier, while it is acceptable for the teacher to introduce a new topic and concepts to the learners, it is further important to allow learners to learn for themselves by allowing them to engage with new mathematics meanings. Instead, the four teachers fell into the trap that Freudenthal (1973) has termed 'anti-didactical inversion', as they taught learners only the final, polished mathematical product by providing them with generalisation statements relating to the nature of functions without allowing them to explore the nature of different functions.

10.3.2. The participationist approach to generalisation

In contrast to the four teachers in the above sub-theme, Zelda used a more participationist discourse and approaches during the lessons, which is seen as "patterned collective doings" as learners are active participants in the mathematisation processes (Sfard, 2013, p. 6). As mentioned earlier, even though learners are introduced to the concept of algebraic functions in Grade 10, it is still crucial that they experience instances where they are allowed to make their own observations, conjectures and generalisations about functions. Zelda's interactions with the learners helped them to exhibit functional thinking and make narratives about the behaviour of given functions. For example, in episode 3, specifically in the exchange on page 164, she asked learners to observe and verbalise the effect of changing values of parameter *a* on the hyperbola. According to Molefe and Brodie (2010, p. 9), "creating room for learners to explain themselves is a practice that goes with reform approaches", and Zelda allowed learners to make conjectures and generalisations about the effect of parameters on different functions. Across Zelda's lessons, the participationist approach for generalisation was evident: she allowed learners to observe and interpret the relationship between quantities during the completion of the table of values for three functions and compare the functions in graphical form.

Zelda's classroom observable actions demonstrated that she views mathematical meaning as "a product of social processes, in particular as a product of social interactions" (Voigt, 1994, p. 276). The 'participationist pedagogy' in Zelda's classroom presents learners with affordances to make sense of reasoning and internalise mathematical meanings as they can observe the changes for themselves and are encouraged to utter their observational statements and move towards generality. As suggested by Scott at al. (2011), drawing from the knowledge of the Cartesian plane recapitulated the knowledge covered in the previous grade, since this had not been discussed in mathematics class during this unit and grade. This further shows that the teacher capitalised on opportunities to make connections between different parts of mathematics. Accordingly, the instances of generalisation in Zelda's teaching highlights that there is the presence of mathematically productive interactions in her teaching, as well as the presence of mathematical activities, including analysing and justifying that links with explorative routines (Sfard, 2008).

What sets Zelda's teaching apart from the other teachers is the thoughtful participation that made learners respond positively, her prompting about some critical features, and trying to help learners make sense of their routines using endorsed narratives about the effect of parameters on different functions. The nuances that emerged between Zelda's and the other teachers' generalisation discourse are a possible lever to shifting learners towards explorative discourse. The behaviour of the learners in Zelda's classroom can be classified as explorative because she allowed learners to observe, create patterns about the system of variation and the changes each brought, and to co-construct their own mathematical narratives from such observations about the nature of parameters. It follows from the Vygotskian constructivist point of view and the commognitive theoretical view (Sfard, 2012), that the teachers' role is not to transfer their understanding of mathematical concepts related to algebraic functions to their learners. Instead, teachers should create teaching and learning situations for learners to construct their interpretations about the effect of parameters for themselves. This is not to posit that the learners are expected to discover all, or even most, of the functional knowledge for themselves during teaching and learning. Rather, teachers should structure classroom activities to provide learners with the knowledge to work with different concepts in algebraic functions. In relation to this, Friere (1973) argued that "narration (with the teacher as narrator) leads the students to memorize mechanically the narrated content. Worse yet, it turns them into 'containers,' into 'receptacles' to be 'filled' by the teacher" (pp. 71-72). The following sub-theme focuses on the way the teachers used the symbolic mediators (i.e. the functions presented in symbolic form), focusing primarily on the patterns of variation between the parameters of the different functions that teachers used to exemplify such families of functions.

10.3.3. The use of examples: variation between parameters

This sub-theme is about the system of variation that teachers used in their examples to bring the notion of effect of parameters into focus. The development of the parameters discourse within the function concept depends on the content that teachers make available to the learners to learn. Also, the teaching approaches teachers used for the notion of parameters to the learners as well as how they selected and varied examples play a major role in developing learners' thinking about the effect of different parameters on the behaviour of the functions. The focus here is on the role of examples and how they were sequenced to enable or constrain systematic learning, as symbolic mediators to support learners' knowledge building. There are two categories in the teachers' systems of variation of parameters: teachers who sequenced the examples showing the effect of one parameter while keeping the other invariant (Jaden, Mutsakisi and Zelda), and teachers whose set of examples in the lessons simultaneously varied both parameters (Mafada and Tinyiko). My argument is that the latter does not create learning opportunities for learners to observe the effect of a specific parameter on the functions.

Teasing out how the teachers selected and sequenced a set of examples in each lesson, enabled a view of whether and how the examples accumulated to bring the object of learning in different lessons into focus for learners. The process also helped me to understand whether there was movement to achieve generality which is one of the curriculum objectives for Grade 10 level in South Africa (DBE, 2011; Adler & Ronda, 2017). The set of examples teachers introduced across the different lessons can be described in terms of the following patterns of invariance and variation: in all the examples we find an equation in the form "$y = f(x)$", together with an equation in the form "$y = a.f(x) + q$" as stipulated in CAPS. According to Resnick (1997), mathematics is "a science of pattern" in which there is an emerging invariant structure when a phenomenon is undergoing variation.

For this sub-theme, the above statement means that teachers' system of examples and their sequencing reveals whether or not there was movement towards generality relating to the parameters of algebraic functions. This relates to curriculum statement 3 for functions which expects learners to "Investigate the effect of a and q on the graphs defined by $y = a.f(x) + q$" (DBE, 2011, p. 24). This curriculum principle envisages that teachers vary parameter a while keeping q invariant or varying parameter q while keeping a invariant to ensure that learners make conjectures, prove them and construe generalisations relating to the effect of each parameter where $f(x)$ is defined by the following functions: x; x^2; $\frac{1}{x}$; and b^x. It is discernible that in the examples that Mafada and Tinyiko used in their lessons, they did not use patterns in which they vary one parameter while keeping the other one invariant. Table 23 below illustrates the examples that the two teachers used and their sequences in selected lessons:

Table 23

Sequences of examples in Mafada's and Tinyiko's lessons

Teacher	Examples and their sequence
Mafada	$y = x^2$; $y = -2x^2 - 5$; $y = 2x^2 + 3$; $y = -4x^2$; $y = 2x^2$; $y = \frac{1}{4}x^2$; $y = x^2 - 1$; $y = 3x^2 - 3$
Tinyiko	$y = 3x^2$; $y + 4 = x^2$; $y + 16 = x^2$; $y = 3^x$; $y = 3^x + 2$

While Mafada's and Tinyiko's sequencing of the examples moved from simplicity to complexity, as presented in Table 23, the system of variation did not create opportunities for learners to observe what was changing because teachers did not vary one parameter while keeping the other invariant. On this Moeti (2015, p. 61) stated that during the teaching of quadratic functions, the sequencing of examples "moves from a parent function $f(x) = x^2$ where simple example is taken to complex ones". For Mafada and Tinyiko, the lack of invariance-variance relationship to bring the world of changes to the fore in the sets of examples did not allow for systematic comparison of the different families of functions in terms of the effect of changing the values of a and q. According to Martensson (2019, p. 7), "rather than telling the students the critical aspects, the teacher must structure the critical aspects in terms of variation and invariance", to ensure that the effect of parameters is discerned and discriminated across examples.

I argue that Mafada's and Tinyiko's examples across their lessons have constrained the discernment of the meaning and structure of the parameters of functions because there was no systematicity in the explanation of what varies and what remains the same between two parameters. That is, the set system of examples the teachers used did not demonstrate knowledge of what changes, what stays invariant and what the underlying meanings behind varying parameters a and q are. According to Marton (2015), during teaching and learning, variation is a necessary component to enable learners to notice what they are expected to learn. The discernment of critical features related to the concept of algebraic functions did not occur since there was no systematicity in terms of varying one parameter while the other remained invariant in the three teachers' lessons. I therefore argue that the variation of one parameter while the other parameter remains invariant is a precondition for learners to develop a sense of structure and meanings of the parameters, to see what is changing and what remains unchanged and the related effects on different families of functions (Martensson, 2019; Al-Murani et al., 2019).

In addition, the patterns of variation in Mafada's and Tinyiko's examples are contrary to Lueng's (2012, p. 434) postulation that "Invariants are critical features that define or generalise a phenomenon … for a major aim of mathematical activity is to separate out invariant patterns while different mathematical entities are varying, and subsequently to generalise". The ways Mafada and Tinyiko varied the parameters during teaching did not bring about the discernment of structure in working with the different families of algebraic functions as well as generality about the effect of the parameters *a* and *q* as per curriculum standards. This was lacking what Sfard termed 'interpretive elaboration' because they did not offer learners elucidations about the behaviour of given functions when some variation was introduced on the parameters of functions (Sfard, 2008; 2012). This lack of interpretive elaborations and intellectual discussions with the learners about the effect of the parameters indicate that the teachers did not create a teaching and learning environment that facilitates learners' deep understanding of algebraic functions. The lack of focus on the systematicity of variation for parameters *a* and *q* for the different functions that were introduced during teaching, constrain the visualisation of the effect of changing the values for different parameters on the functions. I argue that using parameters simultaneously without first exploring the effect of each parameter whilst the other remains invariant, constrain learners' shrewdness of the effects of the parameters.

The above discussion resonates with Mason's (2002) argument that worked-out examples might constrain learners' ability to generalise the nature of mathematical objects and the nature and effect of the parameters on different functions, as the teachers primarily focused on ensuring that learners recognise the syntactical template of the symbolic representation for functions. From Mafada's and Tinyiko's teaching approaches, it could be argued that learners could not notice what stayed the same and what varied, resulting in learners' inability to associate the patterns of variation with the different representations as well as the word use and narratives that go with them. Al-Murani et al. (2019, p. 8) argued that learners' conceptualisation about the function concept "depends on discerning common and differing features among examples and experiences, generalising from these according to the scope of examples that are presented, and fusing these features into a concept". It is arguable that varying the two parameters simultaneously without first varying one while the other remains invariant makes it difficult for learners to experience the difference of their effect on the functions. Sifting out invariants in the parameters during the teaching of functions is an essence of experiencing the depth of the topic, and in turn develops conceptual understanding (Chimhande, 2014; Moeti, 2015).

Table 24

Sequences of examples in Zelda's, Jaden's and Mutsakisi's lessons

Teacher	Examples and their sequence
Zelda	$y = x^2$; $y = x^2 + 1$; $y = x^2 - 1$; $y = -x^2$
Jaden	$y = 2x$; $y = 2x + 1$; $y = x - 2$; $y = x$; $y = 2x$; y = 3x; $y = 4x$; y = 5x; $y = -x$; $y = -2x$; y = -3x
Mutsakisi	$y = x$; $y = x^2$; $y = 2x + 3$; $y = x^2 + 2$; $y = x^2 + 1$

Zelda's, Jaden's and Mutsakisi's patterns of variation in the selected lessons as presented in Table 24 demonstrated systematicity in terms of varying one parameter while the other stayed invariant to guide the learners about the effect of the parameter in focus. Although the degree of interpretive action differed for the three teachers, their selection and sequencing of examples demonstrated some intentionality to help learners move towards generality about the effect of the parameters on the different families of functions. Zelda, Jaden and Mutsakisi created opportunities through the system of variation and sequencing of examples to bring the idea of 'transformation' in functions, attention to the appearance (structure), displacement and orientation of functions to the fore (Sierpinska, 1992; Chimhande, 2014; Mudaly & Mpofu, 2019). This allowed learners to observe the changes brought about by changing values of one parameter while the other remained invariant and explained the generalisations about such effects. According to Ling Lo (2012, p. 29), "The learning of an object is not possible if we cannot first discern the object from its context". The example sequences mediated the identification of 'what' is changing in given relationships, 'how' the change is taking place as well as how the changes were linked to the different parameters. The following section focuses on the approaches to algebraic functions that teachers used during teaching.

10.4. Approaches to teaching functions

From the analysis of the selected episodes, I have identified two approaches that were predominantly used by the participants: the use of examples versus non-examples and the property-oriented approach (see Tables 16–20). Other approaches were also identified during the lessons, including the covariational approach, function machine approach, pattern-oriented approach and the covariational approach, and are discussed alongside the two dominant approaches.

10.4.1. Property-oriented approach

As discussed in Chapter 2, a property-oriented view of function focuses on the learning opportunities that teachers created for gradual awareness of specific functional properties, and enables the ability to recognise, analyse and interpret different functions by identifying the absence and/or presence of specific properties. In this study, the teachers' classroom practices revealed some elements of this approach through the examples such as the different symbolic representations of different families of functions. However, they did not create comprehensive learning opportunities for the functional properties linked to the specific classes of functions, and properties that generalise classes of functions, such as symmetry and intercepts (Moeti, 2015; Mpofu & Mudaly, 2020). The focus on calculation procedures across a series of examples and non-examples during classroom observations overlooked verbal and graphic descriptions of the key properties for different families of functions, which resulted in most properties of the topic not being brought to the fore. In Mafada's, Mutsakisi's and Tinyiko's teaching, the observations revealed that due to the focus on only algebraic or computational manipulations to draw the graphs of functions, compounded with the lack of interpretations of the inherent properties of functions once the graphs were drawn, properties of different families of functions were not made discernible. For example, the idea that a parabolic function could be seen as a continuous function possessing only one extrema, at most two zeros, and which is symmetrical about a vertical line was left unexplored and unexplained.

Although the exemplars teachers used during teaching revealed some properties about different functions, such as structural differences between classes of functions, most critical features were left unexplored and unexplained. For Mafada, Tinyiko and Mutsakisi specifically, there were limited opportunities created for learners to observe characteristics of different families of functions that were invariant across different examples they used during teaching, resulting in superficial teaching of different observable characteristics for different functions. There were no descriptions of all the properties that different families of functions must possess. From the observed lessons, Mafada, Mutsakisi and Tinyiko used the property-oriented approach to demonstrate the process of mathematical substitutions, calculations and drawing graphs for selected examples of algebraic functions. Of concern is that the use of these processes was limited to the recognition and repeating of mathematical procedures, which is problematic because doing so resulted in teachers overlooking the characteristics associated with various objects satisfying the definition of function (Doorman et al., 2012; Alkan et al., 2017).

Mafada, Tinyiko and Mutsakisi selected examples to demonstrate the particular instances of mathematical calculations and drawing graphs, and did not draw learners' attention to the key properties for each family of functions, such as domain, range, symmetry and intercepts, because they focused on mathematical calculations and drawing graphs which limited the effective use of examples to bring particular properties of the concept to the fore. In essence, the procedures the three teachers performed on different functions they introduced during teaching, did not create a learning environment that promotes an understanding of functional properties such as concavity, zeroes, intercepts, and asymptotes. This could be attributed to the lack of comprehensive conversations with learners during teaching and learning, to unpack and interpret the growth properties embedded in each example they used. In relation to this, previous studies have demonstrated that when teachers do not engage in the process of interpreting the nature of different functions, learners' reification of the notion of functions as mathematical objects having or not having particular properties is constrained (Sfard, 2012; Moeti, 2015). According to Kwari (2007) and Chimhande (2014), a teacher makes use of a property-oriented view in analysing the nature of specific examples of functions and their key features.

A more in-depth conceptual understanding of the function concept, which Sfard (2008, 2012) referred to as an object, with a set of properties related to each family of algebraic functions was not prominent in the teachers' teaching of functions, especially for Mafada, Tinyiko and Mutsakisi. A critical analysis of the three teachers' lessons unearthed an understanding that their knowledge structures for functions tend to be compartmentalised, especially the knowledge structures relating to the representations of the concept, which in turn prevented the three teachers from presenting interpretations of the relationships or allowing learners to present their own. Accordingly, I argue that two experiences to establish learners' learning of various properties of functions were not created during teaching in the three teachers' teaching. First, the lack of interpretations relating to the equivalence of procedures performed by teachers in different notational systems constrained the effective teaching of different properties. For instance, Mafada, Mutsakisi and Tinyiko did not demonstrate to the learners the process of solving $f\ (x)\ =\ 0$ symbolically and determining x-intercepts graphically to help learners develop this awareness. Second, the teachers also did not create adequate opportunities for learners to cultivate the ability to generalise the procedures and effect of parameters for different families of functions as discussed earlier above.

Zelda and Jaden were the only teachers that used the covariational approach alongside the property-oriented approach to interpret covariational relationships between variables, to emphasise the changes brought about by the changes in the values of parameters. While the previous research I have reviewed (see Kwari, 2007; Chimhande, 2014; Moeti, 2015) did not offer accounts about the simultaneous use of approaches during mathematics teaching, I argue that the simultaneous usage of the property-oriented approach and the covariational approach in Zelda's and Jaden's lessons facilitated teachers' interpretative elaborations about the effect of the parameters. This combination enabled an environment for understanding the way the independent and dependent variables change. Zelda and Jaden used the notion of covariance to discern the functional properties (effect of parameter a, effect of parameter q, etc.) that helped define the nature and behaviour of the functions under investigation. The analysing, comprehension and interpretation of the relationships between changing quantities was brought about by the use of the covariational approach. In both the teachers' lessons, after they produced graphical descriptions of the functions, Zelda and Jaden analysed the functional situations and moved in the direction of using covariation to reason about the mathematical relationships between quantities. The interpretation of the effect of different parameters revealed an understanding that examples should be accompanied by exploration of the key features for the topic to help learners develop conceptual understanding. This extended the exemplification process to include interpretive engagement, which entails the comparisons of the different features in different examples to reveal the general aspects of the topic which learners should be guided to learn. The following section focuses on how the teachers used the example versus non-example approach during the lessons.

10.4.2. Example versus non-example approach

The example versus non-example approach dominated teachers' lessons on algebraic functions. As highlighted in Chapter 2, examples provide insight into the nature of mathematics topics through their utilisation in complex tasks to help learners understand the methods, demonstrate relationships, and in explanations of the key components of mathematical concepts. In view of the preceding themes, the majority of the teachers in the current study used the example versus non-example approach to demonstrate the application of procedures to draw the graphs, with limited examples and non-examples of concepts. The analysis of teachers' teaching in Chapters 5 to 9 revealed that Mafada, Tinyiko and Mutsakisi used worked-out examples to demonstrate the application of mathematical calculations and drawing graphs. The procedures needed to draw graphs of functions were solely performed by the teachers, often accompanied by commentary about the applicability of the routines. Of concern is that the teachers' exemplification did not create opportunities for learners to engage in 'exercises', whereby tasks are set for learners to complete. The lack of such opportunities constrained the learning affordances for learners to develop their explanatory communication (Adler, 2017), as discussed in previous themes.

In relation to the above discussion, Bergsten et al. (2016, p. 570) stated that "as is the case in many countries in the world, mathematics teaching at the upper secondary level often has an emphasis on procedural skills rather than conceptual understanding". This statement explains why the participants successfully completed mathematical calculations and drew the graphs in their worked examples, but did not engage learners in the analysis and interpretations of the different functional relationships to enable learners to learn different properties related to the concept. While the three teachers introduced examples that belong to particular families of functions, the lack of analysis, learner explanations, interpretations and descriptions of key features of different functions in the three teachers' lessons led to insufficient awareness of adequate functional properties as prescribed by the curriculum (DBE, 2011). The essence of giving examples is that learners view them as generic, and internalise them as archetypes or templates so they develop general tools for solving problems for different classes of functions. Unfortunately, in the current study, Mafada's, Tinyiko's and Mutsakisi's use of examples during teaching was reduced to mere demonstrations or sequence of actions for mathematical calculations, in contrast to opportunities for learners to experience the mathematisation of functional situations as a practice or investigation (Cai & Ding, 2017). While it is useful for teachers to demonstrate the mathematical calculations and skills to draw graphs while teaching functions, the lack of interpretations of the concepts' special features resulted in the relevant and irrelevant features of the concept not being brought to the fore.

On the other hand, Zelda and Jaden made strides to use examples as exercises as they allowed learners to investigate the behaviours of different functions, asked them to verbalise their findings and allowed learners to express different solutions and different opinions, thereby creating opportunities for learners to develop awareness of mathematical communication. For example, asking learners to observe the effect of changing values of the parameters a and q revealed that Zelda and Jaden viewed learners' verbalisations as not only a determinant of functional understanding but also as a means to develop understanding. As discussed in previous themes, the two teachers' use of the example versus non-example approach differed from the other three, as Zelda and Jaden provided guidance to elicit learner explanations and asked learners prompting questions to guide them to see the essence of the function concept. The examples the two teachers used had the potential to provide the learners with opportunities to experience and understand the explanatory nature of examples (Farrell 2013)—in terms of parameters, to experience the effect of changing the values of parameters on the graphs of functions. The presence of explorations and interpretations of the special features of the concept in the two teachers' lessons resulted in the key properties of the concept being brought to the fore. The following section focuses on the factors that reinforce teachers' discourses and approaches of teaching mathematics in general, specifically algebraic functions. The discussion allows for understanding the factors that are at play in shaping and reinforcing teachers' practices during teaching and draws mainly from the information teachers provided during VSRI and semi-structured individual interviews.

10.5. Factors that shape rural teachers' discourses and approaches

The analysis of the VSRIs and semi-structured interviews, in relation to the observable actions during the lessons, revealed three factors that shape and reinforce teachers' teaching. The factors include 'the discourse of teaching for compliance', 'teaching for assessment' and 'knowledge of algebraic functions and curriculum focus'. The factors discussed herein are the key underlying reasons shaping teachers' mathematical discourses and teaching approaches of algebraic functions.

10.5.1. The discourse of teaching for compliance

This sub-theme addresses teachers' urgency to complete the prescribed contents within the specified times, to ensure that their teaching pace is aligned with the pacesetter[47] that is closely monitored by the district subject advisors. From the analysis of classroom practices, the VSRI, and the semi-structured interviews, the teachers' discourses and approaches were influenced by compliance to the system, which needed teachers to complete topics within specified times. Mashele (2018) reported similar findings that teachers feel like they have restricted autonomy in their classrooms as they must follow scripted lessons and must always be compliant with the policymakers' demands without questioning. This resulted in teachers rushing to complete the contents by doing all the talking, demonstrations and answering their own questions, rather than encouraging learners' participation during the lessons to learning. This further influenced

[47] The pacesetters are provided by the department and details the specific dates particular contents should be covered and examined.

the non-interactive/authoritative approach and learners' passive learning. Teaching for compliance and examination in mathematics has been mentioned before in other studies (Kirkpatrick & Zang, 2011; Tatana, 2014; Mashele, 2018) as a key hindrance for teachers to engage learners in critical thinking about mathematical concepts, to enable ownership of the skills and knowledge of algebraic functions for their understanding.

It was therefore problematic to observe and hear teachers mentioning the rush to complete the contents because learners were conditioned to rely on teachers for information. This is against the preferred post-apartheid South African learner-centred approach and curriculum goals, which advocate for imaginative, creative, and critical thinking future mathematicians. In relation to this, Skovsmose (2011) described similar teaching practices as an 'exercise paradigm', which is typified by the teacher being at the centre of teaching and learning, demonstrating mathematical procedures, followed by learners practising the same procedure repeatedly with identical closed questions. From the observed lessons, teachers gravitated towards using the rituals teaching because they wanted to cover the examined content, resulting in "depositing" their mathematics knowledge to the learners. There were limited opportunities to share and interchange ideas between teachers and learners, to build mutual understanding and co-creation of mathematics knowledge and understanding (Freire, 1972).

As briefly mentioned above, the discourse of teaching for compliance was also influenced by the visits of district officials, who were also under pressure to prioritise the completion of the syllabus at the expense of learners' understanding. Thus, the teaching of algebraic functions predominantly became a practice of simple exchange of ideas to be consumed by learners in a non-participatory environment. According to Skerritt (2020, p. 1), "teachers in many schools in many education systems are now being watched in various ways and by various people" and this results in teachers teaching for compliance. I argue that learners did not participate actively in the building of mathematics knowledge and experimentation with mathematics knowledge for deep understanding and self-meaning making, because of the rush to comply with the department of education officials. The teachers did not see district officials as subject advisors to assist with the improvement of content and teaching practices to improve learners' learning, but as surveillance officers concerned only with enforcing content coverage compliance (Mavuso & Moyo, 2014). While the findings suggest that teachers did not have a voice and had limited choice on how to teach algebraic function because of the pressure, they still had "the capacity to exercise control over the nature and quality" of their teaching to ensure that learners learn and own the knowledge and skills effectively (Bandura, 2001, p. 1). The information from the VSRIs demonstrate that teachers' key focus was to align their content coverage with the pacesetter to ensure that when the district advisors conduct monitoring and evaluation they are pleased or else "*heads must roll*" (Mafada, Mutsakisi, and Jaden). The following responses illustrate the point:

> *I have a pacesetter; our time is too limited … I haven't finished the specified work and our CAs (Curriculum Advisors) are coming, they are checking. I must follow the pacesetter.* (Jaden)

> *I understand that functions require much time for learners to understand, but there is not much one can do because we are rushing to finish the curriculum in time or else heads must roll. The district officials want to see proof that you have covered the topics on time as specified in the pacesetter they give us.* (Mutsakisi)

It is unfortunate that teachers have to develop strategies such as corrections and revisions as the quickest way to give learners answers or correct information to memorise in order to save teaching time to meet the department's expectations (Mangwende & Maharaj, 2018; Yurekli et al, 2020). There is also a feeling of disempowerment in this discourse because, to a certain extent, teachers appeared incapacitated to make thoughtful decisions about what works best for their learners, since the focus was placed on meeting the pacesetter's expectations. According to Mashele (2018, p. 112), the surveillance and narrowing of the curriculum "takes away professionalisation from teachers by denying them the opportunity to apply their professional thinking capability in making pedagogical decisions in their classrooms to enhance their pedagogies". Thus, teachers' fear of falling behind schedule could be attributed to the limited opportunities they created for their learners to learn for their understanding, as they developed their identity as mathematics learners.

While teachers acknowledged that functions require more time for learners to understand because of its complexity, they could not pace their lessons to support the learners' cognitive development. Teaching to meet curriculum expectations was about the production of evidence for the officials to ascertain that the prescribed contents have been covered under time constraints, especially if the choice of words "*we are rushing*", "*see proof*", "*specified*" and "*they give us*" are considered. The issue of surveillance is also dominant in the teachers' comments about why they teach the topic the way they do (Page, 2017). The dominance of surveillance reported by the teachers makes them to "work within a constant state of 'inspection readiness'" (Perryman et al., 2018), to ensure that when the inspection occurs they have proof that they have covered the contents. I argue that, while the district officials do not necessarily dictate how teachers should teach and engage with the learners during lessons, they indirectly influence teachers' pace when teaching, to ensure compliance. Even if teachers did not want to teach using the exposition technique, as demonstrated by Jaden's and Mafada's comments, teachers were still limited because at the back of their minds there was a need to comply due to surveillance. The way Jaden, Mafada and Mutsakisi described their experiences with the subject advisors is very much akin to the notion of power over the teachers. The emphasis seems to be on keeping the teachers from thoughtfully using pedagogical approaches and discourses they might otherwise want to use and limiting their ability to teach in ways they believe would enable their learners' deep learning and understanding, as suggested by the choice of words "*heads must roll*".

The inherent discerned 'iron fist' approach used by the subject advisors grounds the teachers to use ritual routines in which learners engage in mindless mimicking, as well as teachers doing generalisation for the learners. For the two teachers, giving learners time to engage in functional thinking and engage actively during the teaching of algebraic functions would derail the progress of content coverage, resulting in punitive measures[48] from the subject advisor (Page, 2017; Monahan & Torres, 2020). Undoubtedly, teachers could not focus on enabling learners' effective participation during mathematics teaching, even though some teachers tried to use dialogues, since the condition discussed herein did not promote pedagogical thinking about what works best for their classrooms or learners (see Tatana, 2014). The pressure from the subject advisors is concerning, considering that the specific aims of mathematics education is to produce a learner that is creative, innovative and a critical thinker.

10.5.2. Teaching for assessment

The previous sub-theme focused on curriculum contents coverage to ensure compliance; the current sub-theme addresses the teaching of algebraic functions to ensure that learners are familiar with steps that will enable them to answer questions that might appear in tests and examinations. Julie et al. (2019, p. 179) defined examination-driven teaching as "teaching the content of previous examinations and/or anticipated questions that might crop up in an upcoming examination of the subject". This teaching approach focuses on the mastery of procedures that are going to help learners answer examination questions correctly, but does not guarantee learners' in-depth development of conceptual understanding (Okitowamba et al., 2018). This was noticed in Mafada's and Jaden's reflective comments that examinations played

[48] While the teachers were not forthcoming about the nature of punishment the district officials apply should they be found to be lagging with content coverage, the words "*or else heads must roll*" suggests that there are some threats that come from the officials.

a fundamental role in the constitution of valued and legitimate school mathematics knowledge. The teachers focused on teaching learners to follow particular steps that would help them to engage with and pass the tests and examinations, considering the 'politicisation' of examination results. For example, Mafada's response "*teaching learners calculation skills to answer the questions during examinations and test ... making sure that they are able to answer the questions, they will always know the steps and when they pass the department is happy"*, illustrate this pedagogical method, its difficulty and that teachers did not to think about the implications of such teaching for the learners. Julie (2013, p. 4) argued that teaching for examination "fragments knowledge, focuses on low level content which frequently becomes the only content learners are exposed to, leads to a loss of disciplinary coherence, mitigates against flexible knowing", which means learners' coherent conceptual development is compromised.

Similar to Mafada's statement, Jaden said:

> *so I am teaching the learners the calculation skills which will help them to answer the questions during examinations and test, ..., that is mostly examined. ... Remember education is very political, if you don't produce the results, the district will breathe down your throat … I must always teach in a way that will make it easier for them to answer the exam questions and when they pass, the district will be off our back you see, whereas if they fail the district officials will visit your school every day.*

The teachers' utterances demonstrate that the teaching is complicated by meeting performance based teaching, as the mention of the 'exam expectations' unearth that his focus is also on examined curriculum to ensure that the 'district is off their backs'. The words "*they will always know the steps*", "*when they pass*" and not "*if they pass*" could be linked with the drilling practice that the teachers used in their lessons to ensure that learners pass. In particular, that they did not allow learners to participate in learning and make mathematical meanings for themselves, but solved the mathematical problems for the learners. In the above extract, Jaden and Mafada stressed the fact that the district educational authorities prioritised teachers producing high pass rates in the exam results, making teaching for memorisation relevant with the hope that some learners would develop understanding. Accordingly, the teaching was more ritualistic as the need to meet the accountability demands, which led to the exam-inclined teaching approaches in the classrooms. This resonates with Sahlberg's (2010) argument that the focus on exam-driven teaching has taken teachers' focus away from providing learners with basic knowledge of the subject matter to focusing on accountability in terms of producing good pass rates to avoid punitive measures from the educational authorities. The fear of not meeting the pacesetter's expectations influenced the dominant use of ritual routines did not give opportunities to demonstrate their understanding or lack thereof. Consequently, teaching focused on whether learners could engage with the procedures to answer the questions correctly and pass the examination (Barnes, 2010; Umugiraneza et al., 2017; Maddock & Maroun, 2018; Mpho, 2018), again with the hope for understanding. It is undeniable that one of the purposes of teaching is for the learners to pass the examinations and tests; however, encouraging learners to make meaning of the nature of algebraic functions is part of teaching.

What Mafada and Jaden talked about in the narratives is a practice that is referred to as 'curriculum narrowing' in which teachers respond to accountability pressures from educational authorities by teaching only the content that is most likely to be examined. These findings confirm Felabella's (2014) argument that the demand for accountability alters school life in very complex ways that in turn affect the teaching profession and the work ethics. In the current study, the teachers abandoned their code of ethics, which emphasises that a teacher "acknowledges the uniqueness, individuality, and specific needs of each learner, guiding and encouraging each to realise his or her potentialities" (South African Council for Educators, 2000, p. 4). Mafada's and Jaden's cases reinforce Bishop et al.'s (1993, p. 11) iterations that "examinations operationalise the significant components of the intended mathematics curriculum, so they tend to determine the implemented curriculum" as teachers often resort to drilling practice to make learners 'ready' for examinations. Without overlooking the teachers' reasons, their teaching was unproductive in helping learners to develop conceptual understanding. This reinforces Julie's (2013) argument that "the intended and interpreted curricula provide only boundaries of content to be dealt with, but the implemented curriculum is heavily driven by the examined curriculum" (p. 6). In the current study this resulted in the lack of teaching for deep knowledge development. In view of the teachers' utterances above, the examined curriculum drives what is taught regardless of the curriculum-specific aims and skills.

According to Boaler (2016), the teaching of mathematics should draw upon rich activities, which are characterised by high intellectual demand, instead of resorting to the use of rote memorisation, so that it can inculcate learners' positive attitudes towards mathematics. Even though it could be possible that teachers' methods of teaching could be influenced by the pressure from the district officials, I argue that they could have used the activities they had designed to challenge learners' thinking and promote meaning making. Unfortunately, due to the pacesetters from the department, teachers did not design their activities and relied on the packaged information. As I mentioned earlier, teachers are incapacitated if they are given packaged information, as this takes away their authority to practise what they have been trained to do as professionals. The Department of Education and Training in New South Wales (NSW DET, 2003, p. 10) stated that “high quality student outcomes result if learning is focused on intellectual work that is challenging, centred on significant concepts and ideas, and requires substantial cognitive and academic engagement with deep knowledge”. In view of this, it is concerning that the district officials put pressure on teachers to rush through the content coverage as reported by the teachers, which constrains deep knowledge. I then argue that the politicisation of education disables learners’ mathematics intellectual quality, considering Killen’s (2015, p. 71) contention that “approaches to teaching that emphasise intellectual quality will not involve learners in simply memorising information and then regurgitating it in examinations”. The following section focuses on the third factor that reinforces teachers’ classroom discourses and approaches while teaching algebraic functions, which is concerned with teachers’ knowledge of curriculum content specifications and delimitations relating to the concept of functions.

10.5.3. Knowledge of algebraic functions and curriculum focus

Becoming more informed or having a lack of information about the curriculum content specifications and delimitations for topics across different grades has effects on teachers’ teaching of algebraic functions, as it does with other topics and subjects (Molefe & Brodie, 2010; Adler, 2017). Even though the department gave teachers lesson plans, it was expected that they engage with the knowledge, plan how to present the content coherently, and provide reasons for teaching the topic in a particular way. This section presents the argument that mathematics teachers must know the content in depth, which is outlined in the national curriculum (CAPS). In relation to the observed lessons, during VSRIs and semi-structured individual interviews teachers reflected on what learners needed to learn in algebraic functions, trying to identify the critical aspects relating to topics and skills they are expected to teach at Grade 10 level. Their descriptions of the curriculum standards on the topic, their word use, the visual mediators, narratives and routines during teaching revealed their limited content knowledge and the curriculum knowledge in general, specifically the contents specifications for functions. This sub-theme addresses the relationship that ought to exist between teachers’ curricular knowledge, content knowledge and their teaching practices during algebraic functions lessons. Some teachers’ statements revealed the lack of curriculum reading to understand the contents specifications and delimitations for the topic as well as general misunderstanding of what functions are about. The responses below are illustrative:

> *I might not be hundred percent about all the sections that are phased in and those that are phased out (in the curriculum) … I haven't checked the policy*. (Mafada, Semi-structured interview)
>
> *I am not good at all, when it comes to this this function thing! I don't know what the curriculum expects me to teach, I don't even have a curriculum document. I just use the textbooks to check the topics; it is what you must give to learners you see*. (Zelda, semi-structured interview)
>
> *Functions, they also remind them about how to factorise, and they did factorisation before, so now when we introduce them to functions, they now repeat what they have done before … Err, … they will tell you solve for x. … Sometimes, we work with an expression only, the same expression can be added equals to zero, then now it becomes an equation, then now they need to know the difference between an expression and an equation*. (Tinyiko, semi-structured interview)

When teachers do not read the curriculum documents to understand the content specification, it could result in under- or over-teaching of the topics within a grade. The teachers' statements above demonstrate the lack of adequate curriculum knowledge of the topic (Du Plessis & Marais, 2015). This closely resonates with Du Plessis and Letshwene's (2020, p. 80) argument that "teachers are not being properly trained to implement CAPS, nor are conditions favourable for implementation, due to insufficient resources, unqualified teachers and a lack of support from the DBE". I noted that teachers did not have lesson plans and the curriculum materials were not consulted to ensure that the teaching was aligned with the CAPS content specification and delimitation of the topic. In Mafada's statement above, for instance, his vestiges of own memory (linked to past curriculum and personal views about the topic) determined what ought to be taught, without any reference being made to the CAPS curriculum, content, structure or assessment standards (Maharajh et al., 2016).

Tinyiko's excerpt indicates that she views algebraic functions to be about solving equations, which is a component of the first topic that is covered in term 1 before algebraic functions is introduced and this view of the topic underpinned her teaching of the topic. Their teaching practices are providing opportunities for learners to gain access to some forms of mathematical knowledge, but if judged by the curricular and assessment standards detailed within the CAPS for FET mathematics for functions, it is of lower than stipulated standard. In mathematical discourse, teachers must know exactly what they are talking about, and precise mathematical conventions serve the purpose of reminding us what we are talking about as well as the mathematical meanings behind the conventions (Sfard, 2012). Lotz-Sisitka (2009) categorised similar observations in some of the case studies from rural Eastern Cape schools as ‘under-teaching’ as the teaching practices did not provide adequate access to the knowledge that the curriculum prescribes. Du Plessis and Letshwene (2020) found that South African teachers have not become skilled in the CAPS curriculum they are supposed to teach, meaning that teachers cannot teach what is expected of them. I therefore argue that the teachers’ teaching practices were not providing learners with adequate access to knowledge about algebraic functions at Grade 10 level, due to insufficient knowledge of the CAPS standards for the topic. The next chapter is the final chapter of this thesis and outlines the key findings emerging from this study and presents some recommendations linked to the research questions framing this study.

Chapter 11

Conclusions and recommendations: diversifying research locale

What we find changes who we are ~ Peter Morville

11.1. Introduction

This study explored Grade 10 rural teachers' mathematical discourses and approaches to teaching algebraic functions with five teachers at five different school sites in Acornhoek, Mpumalanga Province in South Africa. The purpose was to gain an understanding of the existing teaching practices in algebraic functions and diversify the research locale for mathematics education in South Africa, because of the lack of literature on mathematics teaching and learning within rural schools and classrooms. Considering the primary focus of this study, the main research question has been: What are the rural Grade 10 teachers' discourses and approaches during algebraic functions lessons? As discussed in Chapter 1, the dearth of research about the teaching of mathematics in South Africa in rural schools, in particular algebraic functions, resulted in the conceptualisation of this study. The purpose was to understand what teachers say and do during the teaching of the topic and its concepts, to gain insight into their content knowledge and approaches and contribute to the existing knowledge. From the reviewed and searched literature, there were no studies in South Africa that have explored the teaching of algebraic functions with the Grade 10 mathematics teachers in rural schools. The current study also examined the factors that influenced teachers' discourses and chosen approaches during the algebraic functions lessons.

To explore the discourses and approaches, I used Sfard's (2008) commognitive theory, the Scott et al's. (2011) concepts of support knowledge building and promote continuity, as well as the communicative approaches from the pedagogical link-making framework (Scott et al., 2011), as lenses to answer the research questions and analyse data. The three frameworks offered the opportunity to understand the nature of teachers' communication of mathematical ideas and enabled the examination of the ways in which teachers talked and depicted the mathematical concept of functions during the teaching process. Second, I used the six approaches of teaching algebraic functions that were discussed in Chapter 2 to examine and understand the dominant approaches that teachers used during the lessons in relation to their discourses. When discourses are linked to the link-making approaches, communicative approaches and the approaches of teaching the topic, they provide the need and the means to analyse the teachers' ways of talking and doing, as well as thinking about mathematics and its teaching and learning, specifically the teaching and learning of algebraic functions.

The main research question which informed this study was: "What are the rural Grade 10 teachers' discourses and approaches during algebraic functions lessons?" To explore the main research question, I focused on the following sub-research questions:

a) What are Grade 10 rural mathematics teachers' discourses during algebraic functions lessons?
b) What approaches do Grade 10 teachers use to teach algebraic functions?
c) How do teachers use multiple representations during algebraic functions lessons?
d) How do teachers guide learners towards generality about the effect of parameters in the context of algebraic functions?
e) What are the factors that influence teachers' discourses and approaches of algebraic functions within rural classrooms?

In this final chapter, I present an overview of the key findings from the study. The chapter begins by summarising the findings from classroom observations, semi-structured individual interviews and VSRIs to answer the research questions, as described in Chapters 1 and 4. The next section of the chapter provides a discussion of the contribution of the study and is linked to the wider research and debates on the teaching of mathematics. Then, the implications of the study and the related limitations are detailed. Finally, I conclude this chapter by discussing the possibilities for further research, recommendations pertaining to teaching as well as recommendations for teacher support and training. I propose the need for more research in other rural schools to understand the nature of mathematics teaching, focusing on Grade 10 algebraic functions.

11.2. Summary of the findings

The findings revealed that teachers' routines were generally ritualised mathematical routines where teachers taught learners how to engage in mathematical actions of substitution and calculations, computing the table of values and drawing graphs of functions. In the context where teachers used non-mathematical rituals, their thinking was often non-mathematical. I also found that the teachers had limited understanding of the curriculum standards for algebraic functions. The following sub-sections present the summaries of the key findings from this study in relation to the research questions.

11.2.1. The use of representations of functions

I investigated the teachers' mathematical discourse in the multiple modalities of representation of a function as prescribed in the CAPS curriculum of the DBE's requirements. The findings indicate that while teachers could work with the specific representations of a function individually, their teaching did not reveal inter-connectedness between the different forms of representations, and the key features of different families of functions that are discerned by each form of representation. The teachers' over-reliance on algebraic forms of functions leaned more towards procedural computations, with teachers focusing on following steps in algorithms to determine the ordered pairs or finding the numerical values, overlooking making links between the functional relationships and their contexts. Thus, in the current study, the mathematical representations predominantly became an end in itself instead of being tools to communicate ideas about functions or instruments of problem solving. In the teaching and learning of functions, the notion of flexibly translating between different forms of representation is important to help learners objectify the mathematical object of a function (Sfard, 2008; Cilliano, 2021). The teachers' discourses revealed that they understand the concept of algebraic functions to be about drawing graphs of functions, which is contrary to what the CAPS curriculum standards stipulates (DBE, 2011).

11.2.2. Variation patterns in parameters across examples

I also investigated the teachers' endorsed narratives and their communication about the critical global features, in terms of the effect of parameters for each family of algebraic functions. While there were some instances where teachers used endorsed narratives to talk about the effect of varying the parameters of functions, the actions of interpretation were limited and at times absent in the five cases, and this resulted in the teachers' routines being ritualistic. When teachers spoke of the effect of varying parameters, they exposed a gap between their mathematical discourse and the discourse of the community of mathematicians, especially the curriculum standards enshrined within the CAPS mathematics curriculum for Grade 10 which requires teachers to guide learners to explore the effect of parameters (DBE, 2011). In this study, the teachers who do not vary one parameter while keeping the other invariant in the examples did not engage in interpretive actions about the effects of the parameters on the different families of functions, relating to the lack of explorative routines. Accordingly, this results in lack of formal word use and endorsed narratives related to the concept of functions.

In view of the above, I argue that systematic variation, selection and sequencing of examples in symbolic form are the preconditions of productive communication about the behaviour of different parameters for families of functions in terms of formal words and endorsed narratives. That is, without systematic and sequential variation of parameters, teachers' communication becomes limited to rote steps to draw graphs and nothing is revealed to the learners about the effect of the parameters. The teachers who selected and sequenced the examples showing the variance-invariance patterns of working with parameters for different families of functions engaged in interpretive actions about the effect of the parameters. Accordingly, the variation patterns mediated both their communication about the effect of the parameters and created opportunities for learning for learners to learn about the notion of parameters. From the classroom observations and conversations with the teachers, I realised that word use and endorsed narratives are enabled and/or constrained by the availability and systematicity of patterned variation or lack thereof.

11.2.3. Approaches of algebraic functions

Teaching of mathematics does not mean that teachers should simply expect learners to reproduce what teachers have done during teaching through rote learning. Equally, learners often adhere strictly to the procedures demonstrated in class by the teacher. This sub-theme presented the way teachers used the example versus non-example approach during lessons and the property-oriented approach to help learners develop knowledge of properties related to the concept of functions. Zelda’s and Jaden’s interpretive elaborations of the examples and the analysis of the nature of the relationships between quantities created mathematics learning environments for learners to learn about some properties of the topic. On the other hand, Mafada’s, Tinyiko’s and Mutsakisi’s use of examples to model mathematical calculations and drawing graphs without presenting the interpretations of the relationships resulted in properties of the concept being overlooked. This demonstrated that examples teachers use during teaching act as communication tools essential to mathematical explanations, discourse and understanding of key properties of different classes of functions. Teachers commented that the use of the example versus non-example approach and non-interactive communicative approaches in their teaching aimed at establishing steps that learners should memorise in order to regurgitate during tests and examinations.

11.2.4. Factors influencing rural teachers’ discourses and approaches of functions

The overall findings indicate that teachers rushed the teaching of algebraic functions because of the pressure from the department of education, assessment practices and expectations and the insufficient curriculum policy knowledge. The pressure to be on par with the pacesetter resulted in teachers using the ritualistic routines discourse and non-interactive/authoritative approach, because they taught for themselves and not for learners' promotion of engagement to develop their knowledge and understanding of the topic and sub-concepts. Some teachers commented that creating interactive classroom environments where there was collective engagement between teachers and learners was a time waster and a hurdle for them, because of the expectations in using and being on par with the pacesetters. Accordingly, they resorted to adopting an exposition strategy and being givers of ready-made contents about the different families of functions.

The teachers' focus on preparing learners for assessments resulted in under-teaching of the topic. Consequently, teachers predominantly used rituals to ensure that the learners become familiar with the mathematical substitution and calculations as well as drawing graphs as part of the examinable parts of the topic. This limited the opportunities of all learners to explain the generalisations about key features of algebraic functions, which are dependent on explorations of aspects such as the effect of changing the values of parameters on the four families of functions.

Lastly, the findings reveal that the teachers had insufficient knowledge of the curriculum standards on algebraic functions and also possessed limited content knowledge of the topic. In addition, the teachers did not have the curriculum documents for reference and to check the contents specifications and delimitations for the grade, in relation to the pacesetter. This resulted in the utilisation of the vestiges of their own memory pertaining to what the topic was about, which was not in line with the CAPS curriculum standards. The findings illustrated that teachers' understanding of the curriculum standards for the topic and limited content knowledge resulted in under-teaching of the topic, as well as in the use of non-mathematical narratives and routines. Teachers' limited curriculum knowledge constrained the effectiveness of teachers' communication about the key features related to the functions discourse at Grade 10 level, and constrained learners' knowledge building and limited the exploration of critical features of the concept such as range, domain, and even effect of parameters. The following section presents some of the limitations of the current study.

11.3. Limitations of the study

The findings of the current study are generally limited to the five participating teachers who were selected from five schools in Acornhoek region in Mpumalanga. The study focused on the teaching practices of five teachers, which might be considered a small sample, even though I used the qualitative approach. In addition, another limitation is related to time and distance constraints which limited my flexibility to constantly access the schools and teachers. Acornhoek is 473 kilometers from Johannesburg and this resulted in limited time spent with the participants.

11.4. Recommendations

In this section, I provide the following recommendations: *recommendations pertaining to teaching, recommendations pertaining to teacher support and training* and *recommendations for future research.*

11.4.1. Recommendations for the teaching of algebraic functions

To create an environment that demands learners explore the behaviour of different families of functions, teachers need to place more emphasis on the translations between the modalities of representation of a function. Teachers should be conscious that they are not only tasked with ensuring that learners develop understanding of how each mode of representation presents and encodes information, but they are also faced with the complex task of enabling learners' understanding of how each modality of representation relates to the concept of algebraic function. I therefore recommend that teachers should avoid using mathematical mediators during teaching for the purpose of merely drawing graphs, without their explanations or interpretation of the key features of the different families of functions. I further recommend that the tasks teachers use during teaching should go beyond the algorithms to calculate the ordered pairs, drawing graphs and introducing features for different families of functions.

The recommendation is that teachers should use more explorative routines and an interactive/dialogic communicative approach to allow learners to create their own narratives about the mathematical objects and learn to use mathematical words during learning. This would ensure that learners move from the discourse of the interlocutors and develop a deeper understanding of the functions concept. The following section focuses on recommendations pertaining to teacher support and training to ensure that teachers learn and own teaching skills to meet the demands of teaching mathematics within rural classrooms.

11.4.2. Recommendations pertaining to teacher support and training

I recommend that the Department of Basic Education, either at provincial or national level, should provide teachers with materials to ensure that teachers are guided by the curriculum standards in their teaching of mathematics. It is recommended that teachers are explicitly trained on how to use the curriculum documents as resources that frame their teaching of the contents of the subject matter. Intervention programmes that aim to help teachers bridge the gaps in the content and pedagogical knowledge for algebraic functions and other mathematics topics should be developed.

In addition, the district officials need to be trained to know how to support teachers without instilling fear and vilifying teachers' job performance. Thus, I recommend that the Department of Basic Education should introduce an induction programme for subject advisers on how to support teachers for professional development rather than surveillance. Universities and education authorities should commit to training subject advisers instead of it being a political appointment. It is also recommended that educational authorities in South Africa should explain the roles subject advisers should play in monitoring and evaluation, without negatively affecting their relationships with teachers.

11.4.3. Recommendations for future research

There is a need for future research which focuses on the teaching of mathematics within rural schools, to expand the scope of the mathematics education research locale. Research in the teaching of mathematics in general, specifically the teaching of algebraic functions has not been prioritised to explore the nature of teachers' discourses and discourses at Grade 10 rural classrooms. Future research is needed in the exploration of teachers' mathematical discourses and approaches during algebraic functions lessons. The research may focus on teachers' exemplification and tasks teachers choose during algebraic functions teaching as tasks and examples determine the nature of routines teachers use as they introduce and engage in functional problem solving. Subsequent research should also focus on how teachers guide learners towards the idea of generality about the effect of changing values of parameters for the different families of algebraic functions. The study may focus on the use of different modalities of representation for algebraic functions which determine the types of routines the teachers will have as they bring the key features for the concept to the fore.

11.5. Conclusion of the study

This study has made some recommendations and created questions to answer, acknowledging the complexity and dynamic nature of rural education in South Africa and the sparse mathematics research knowledge focusing on rural mathematics teaching and learning. However, the current study hopes to have expanded new possibilities for more mathematics education research within rural areas and schools. We are yet to learn more in South African education system about and address the complex collection of issues associated with teaching and learning in rural areas, if we are to adequately address issues of social transformation, redress and equity through education as promulgated in the Constitution of the country. I was very enthused by listening to teachers during algebraic lessons and their teaching experiences of the topic.

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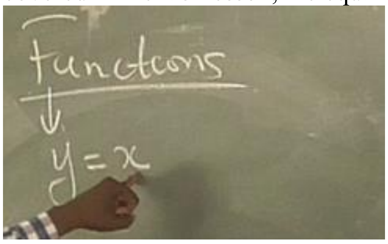
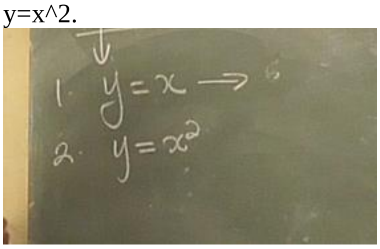
Appendices

Appendix 1: Mafada lesson 1 transcript - introducing the parabola

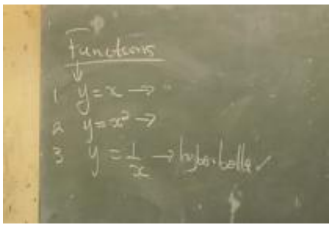
Mafada started his lesson by telling the learners that they will be learning about functions. He stated that “as covered in the last lesson, the equation y=x is a function because there is only one value of x”



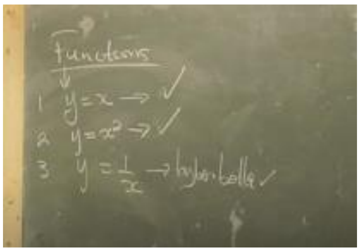
Mafada: We said this one is gonna allow us to draw … we have got four types of functions, but that one is gonna allow us to draw a straight-line graph. Then, we start again with another type of function (writing on the board) y=x^2.



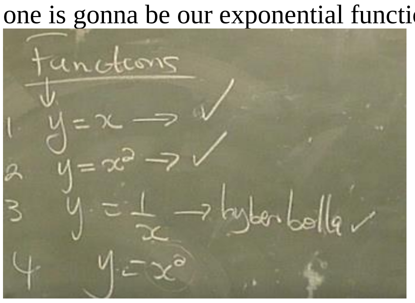
That one represents (he reaches for a textbook to remind himself of the type of function y=x^2 represents), we said that one represents (paging through the textbook), represents a parabola. So, we haven’t started with this one. We said the third one is where y is equals to one over x, and what did we say that one? We said that one represents (reaching for a textbook), we said that one represents a hyperbola, we haven’t started with the parabola, I said this one will be our third lesson.



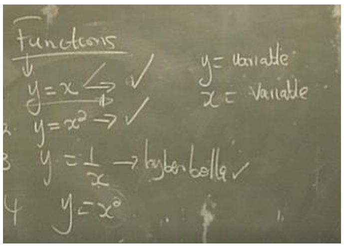
We have done this one in our first lesson (ticking off y=x on the board, we are dealing with this one, which is our second lesson.



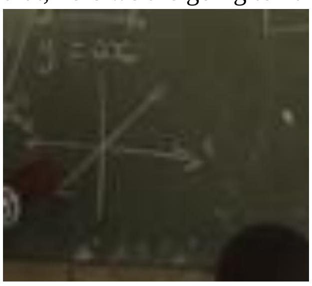
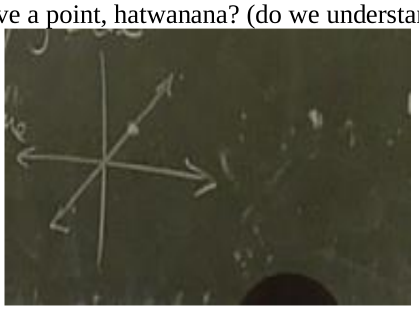
The fourth one is gonna be err, err an exponential function neh (checking the textbook), about the exponential function, what did we talk about? (paging through the textbook) we said y is equals to x squared (wrong). So, this one is gonna be our exponential function, for our fourth lesson.

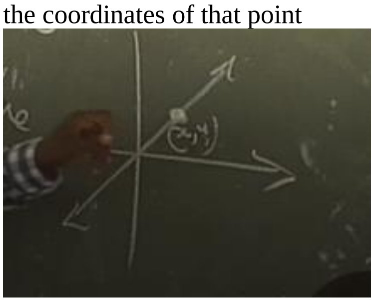


We said for us to draw that function (pointing at the equation for a linear function y=x), we said we have this variable (pointing at x), we said y is a variable (writing on the board), we said y is a variable, we said x is also a variable.



Then now, when you look at this equation (pointing at the equation of a parabola), you see now, it depends on whether x is the subject of the formula or it is not, it also depends on whether y is the subject of the formula or it is not. That's why I am saying that all these things are variables, but then they depend on their location when it comes to the function itself, that is why we have y being the dependent variable and x the independent variable. So, we said for this one (marking the equation y=x), we said for this one, y is equals to x, y is equals to two x, we said a lot of stories about those ones. We said is our equation, our function is like that (underlining y=x), it means for us to know the value of y, we want to know the value of y, we must first draw the value of x, so if we are looking at this one (marking y=x), it means we are directed by x to draw the value of y. in other words it means that, y depends from the x, we cannot draw y if we don't know x, so y depends on x. so, it means this one (pointing at and writing next to x on the equation y=x) will be our independent variable, that's what we talked about last time, that one is an independent variable because y depends from x, then y is the dependent variable. So, for us to draw that particular graph, we said we must represent here our values in the table, we must first have a table in order for us to have our coordinates, by the way, what are coordinates?

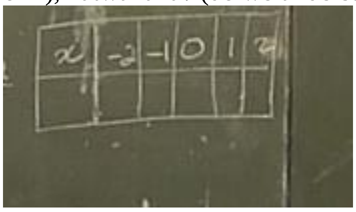

What are the coordinates? (no answer), we said if we have got a sketch graph like that, then we have a graph like that, here we are going to have a point, hatwanana? (do we understand each other?).



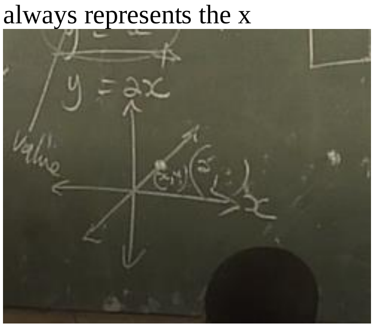
That's what we call a coordinate because we have a value of x and a value of y, so we said that point, these are the coordinates of that point


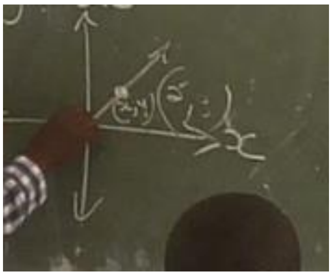
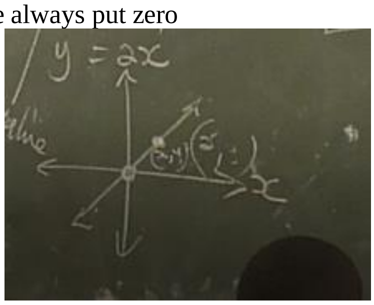
Ungeyikumi poyindi aniri (you can have a point isn't) with only one value, if you don't have a value of y. so those two points mati vona (do you see them) let's say its 2 and 3, these are the coordinates of that point, this will be the value that represents the x and this will be the value that represents the y. and we said that the one that we start with, the first one, it represents the x coordinate of the point, then the second one, it represents the y coordinate of that point, so each and every point, it has a value of x and it also has the value of y, and we must know that the first value of x … the first value is the value of x, the second value is the value of y. so we said that for us to draw this graph the one that is independent, we put it first because when it is independent, it is the one that we have assigned the values, do you remember?

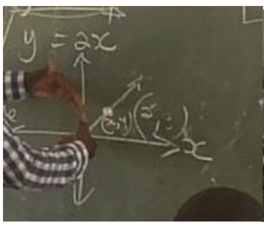
Learners: mmm (yes).

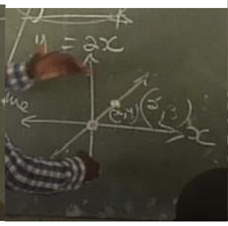
Mafada: We said we can assign the points but we must make it a point that in the middle, there must be a zero, hatwanana (do we understand each other?)

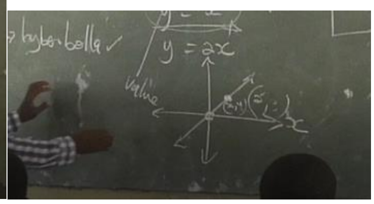
So then you can put your points, I mean you can put your values again on the other side (assigning negative values of x), hatwanana? (do we understand each other?)


We have zero, we have the positive numbers, we have got also the negative numbers. So, these numbers we are just selecting them randomly. Hoti nyika (we just give them), we just assume when x is negative two, then what would be the value of y? we assume that when x is negative one, what would be the value of y in that regard? We have completed the table, because the first thing, we need to be able to draw a table, so that we have the values of x and the values of y because on out plotting now on our (drawing arrows on the Cartesian plane on the board) graph we need to have those values. I think you still know that one that is horizontal, the line that is horizontal always represents the x


But then here, at the centre, we always put zero



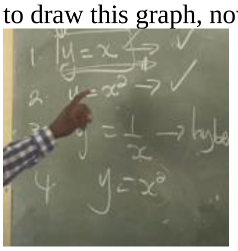
There we have zero. All the numbers that are on your left on the number-line if we can take this one out (referring to the y-plane) and we remain with this one, if we take this one out (the y-plane) and take this as a number-line, the one that is horizontal, the x-axis, it would mean that at the middle, at the centre we are going to have zero and on the numbers that will be that side, we are going to have negative numbers, I don't think we are going to talk about them now because we talked about them previously.




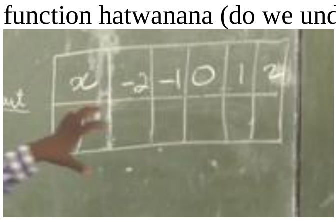
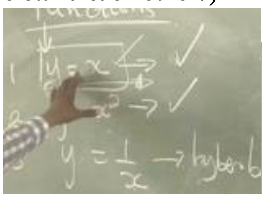


So here (pointing at y=x), we were talking about the sketch, what is meant by sketching and what is meant by drawing aniri (isn't). when you sketch the graph, we are interested in only seeing the shape, we don't necessarily have to draw it using the scale, the sketch they would want to see the shape, for example, you can see in the textbook, they have drawn something like that and they only shown you maybe the turning point there.

But, our interest on the sketch, our interest is only to see the shape of the graph, we don't necessarily have to draw the graph using the scale. Using the scale would mean kuri (that) when say this is one, we must measure it and to get to 2, there must be the same space in between, so that is why we are drawing according to the scale, the spacing for the x-axis, and also the scale must be the same for the y-axis, the negative y and the positive y. Now for use to draw this graph, now

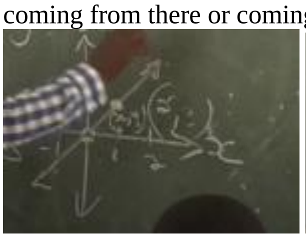


We must know both the value of x and the value of y, but then remember we said, the values of x, we just assign them, and then they allow us to calculate and get the value of y, so we have done this table to complete that function hatwanana (do we understand each other?)

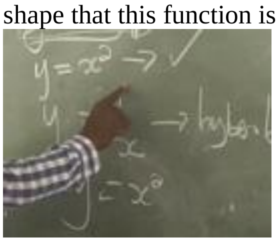




We have completed that function, now we must come to this one

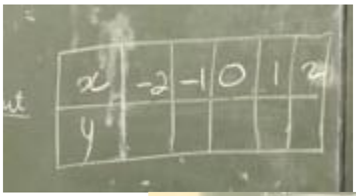
Kuri (that), this function, what type of a graph does it give us! We have seen the shape of that particular one (pointing at y=x), it was giving us a straight-line (demonstrating by hand on the air). It's either it is a straight-line coming from there or coming from there




But, what we did were all straight-line graphs, are we all together? Then now, we are working with this one (pointing at y=x^2). That's an equation, it is a function, that function represents the graph. Let us now see the shape that this function is giving us



So, what we are going to do (someone knocks) come in. (he listens to the learners who was knocking) kuna la angana textbook, classwork book ya Physics (Is there anyone who has a textbook, classwork book for Physics?) uta huma na wena ka TV (you too will appear on TV) (learners laugh, looking at the camera). Mara aswi fani, na wena utava u humile (at least you too will appear). Okay right, hatlisani min'wu pfuna a huma (please her quickly so she can excuse us) (paging through the textbook) then now, we are going to this second function (time stamp is 0:11:20), we are going to do the same like we did with this one (pointing at y=x), so let us continue from that one (pointing at the table drawn).



Here, 

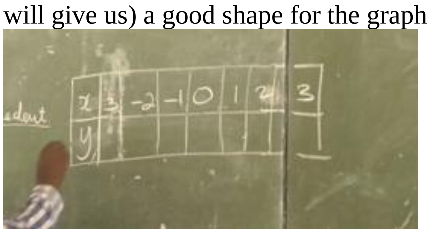
we are saying we have assigned values for x 

we have assigned values for x neh, but remember, now, you must know what is y, y is x squared. What does it mean? It means here we are going to put what, those values (pointing at the table) that we have assigned for x

aniri (isn't) laniwani (here) x is negative 2 

so it means here (pointing at x on the equation y=x^2) we are going to put negative 2 to get the value of y (pointing at the table), if we put the value of negative two here, what do we get? What is the value of y? what is our y? phela (because) I can't give you all the answers! The function is there and the value of x which you are supposed to substitute is there, what do we get for y? this isn't a new lesson, (he moves to the back of the classroom, and then he walks back to the board and notices something is wrong with one of the equations and quickly rushes to the textbook) what happened there? I wrote similar things. (paging through the textbook for about 1 minute 30 seconds) what is the value of y? (someone knocks and he doesn't pay attention to the knock at the door, but keeps on paging through the textbook) come in, (he stops and look at the learner who was knocking, and she whispers on his ear and he walks to the back and gives the learner a book).

Let me extend aniri (please) these values so that we can have a good table, ti velue leti tit ahi nyika (these values will give us) a good shape for the graph



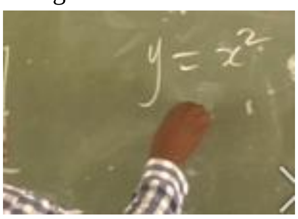
Then now, we are saying, if x is negative three, what is y?

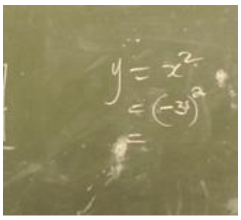
Learners: (mumbling)

Mafada: Pardon! Huh!

Learners: negative six. (14:00)

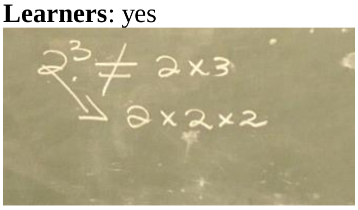
Mafada: Negative six? Let's prove it. we are saying x, y is equals to squared (writing on the board), we are saying that our x is negative three, you are going to put negative three there, then we put a square there, so is this giving us negative six?



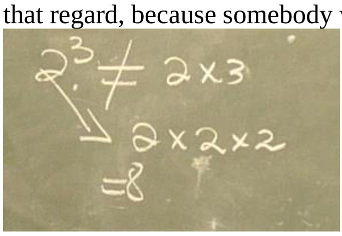


Learners: No

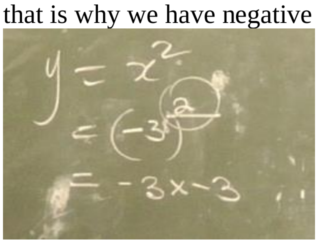
Mafada: What is it?
Some learners: Nine
Mafada: remember, I said when we were dealing with exponents, two to the exponent three, this is not the same as two times three, it is not the same! You still remember that one?
Learners: mmmm (yes)
Mafada: but I said that is the same as, what does that equates to, what does that equals to? we said here you must multiply 2 by itself three times, that is the meaning, not two times three, are we all together?
Learners: yes



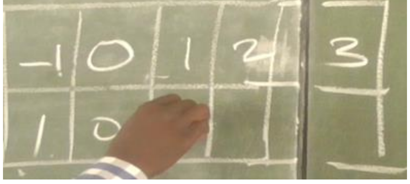
Mafada: but, what is missing is that you must multiply two by itself three times which is eight, not six as is in that regard, because somebody will say two times three and get six, which is wrong. The answer to that is eight.



Mafada: so, square means we multiple that number by itself by the exponent that is given there, here it is two, that is why we have negative three multiplied by negative three.



If it was five here, we were going to multiply all this by 5 hatwanana (am I clear). So then now, what's the answer there?
Learners: nine
Mafada: Ukona un'wani a tsalaku negative nine? (is there anyone who is writing negative nine?) positive nine akere (isn't)!
Learners: yes
Mafada: you need to first multiply the sign to give you the positive sign, then you multiply the numbers, then what is negative two squared?
Learner: positive four!
Mafada: Positive four! Negative one?
Learners: positive one.
Mafada: zero?
Learners: zero
Mafada: one?
Learners: one
Mafada: (he points at one (value of x) on the table)
Learners: one
Mafada: one?
Learners: eeh (yes)
(for the above process, Mafada was filling in the corresponding values of y on the table drawn on the board).



Mafada: you said here (pointing at negative one for x) and here also is one (pointing at positive one for x)?

Learners: yes!

Mafada: for 2?

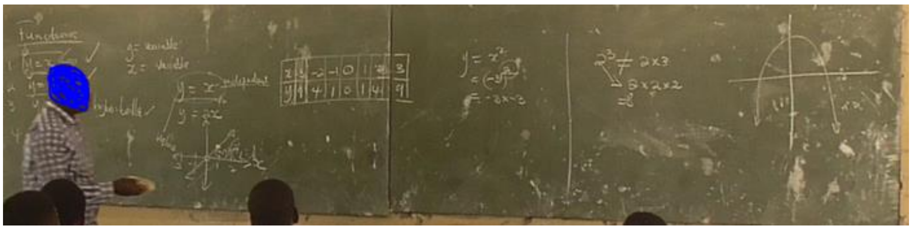
Learners: four!

Mafada: three?

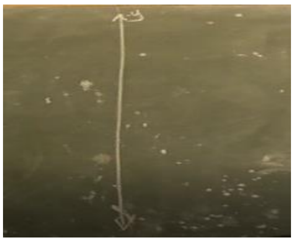
Learners: 9!

Mafada: so are we ready to plot the graph?

Learners: yes

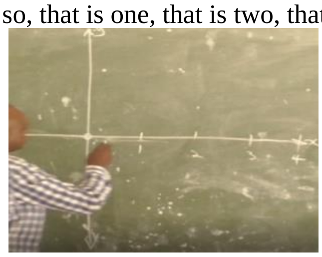


Mafada: let us draw the graph and see the shape, the shape that it's gonna give us. We said here it doesn't matter whether you start with your y-axis, as long as u kombile kuri (you have showed that) is the y.

 And then this one here (he notices that his line is not straight)

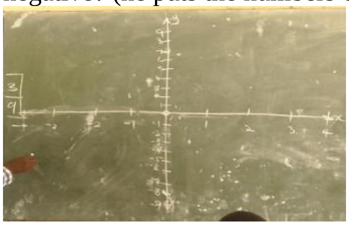
I am sorry I don't have a ruler neh, this one here, we said is the what? The x-axis neh. Then that is where we have the point zero neh. Hatwanana (am I clear)?

And we said (pointing at the table of values) that one x starts at negative three and end at positive three anir (isn't). so, that is one, that is two, that is three. You can extend by 1 and have 4.

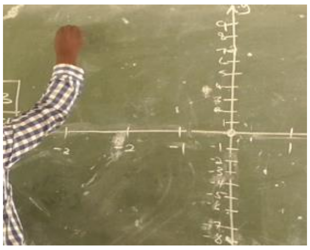
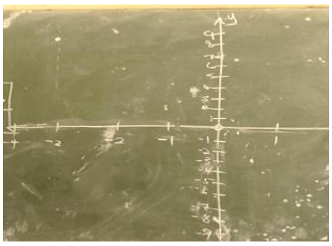


For this one we said these are our negative values, we have negative one there, negative two, negative three and that is negative four

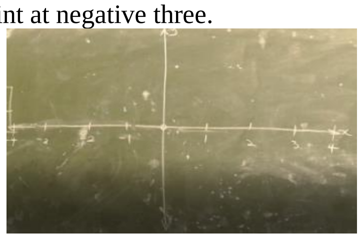
And then now here we said, we are going to have our negative (he realises that the scale he used is too big, so he erases and starts again). This one must have enough space (the negative y-axis) because it has got nine. One, two, three, four, five, six, seven, eight (he puts he scaling marks as he says this on the negative y-axis) (he realises that there is no enough space for nine, he erases the arrow at the bottom and put the scaling mark for 9). Here (the positive y-axis), we have got one, two, three, four, five, six, seven, eight, nine (he puts the scaling marks and numbers on the scale for the positive y-axis). By the way, which values of y are positive and which ones are negative? (he puts the numbers on the bottom part of the y-axis).



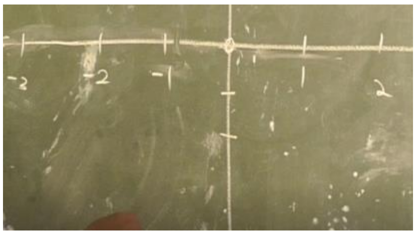
Mafada: Which part of the graph must be negative? The one that is up or the one that is below?
Learners: the one that is below
Mafada: the one that is below?
Learners: yes
Mafada: meaning these ones will be the negatives neh (putting negative signs alongside the numbers on the bottom part of y-axis). Then now, let us try to put the points that we have. We said for x is equals to negative three, x is equals to negative three is this one (placing the chalk on x is equals to negative three), I said let us have an invisible line like this (drawing dots through negative three), let us have an invisible line cutting through negative three neh. So, what is the corresponding value of, of, negative three?
Learners: nine
Mafada: negative nine or positive nine?
Learners: positive nine
Mafada: positive nine?
Learners: mmm (yes)
Mafada: and y is equals to negative nine is that one neh. We draw an invisible line and we put it here. that is the point aniri (isn't).

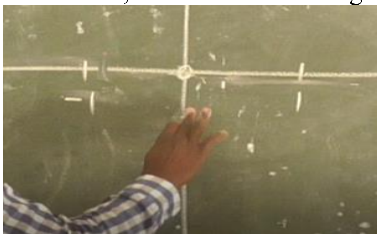
Mafada: so, that is the point, it represents x of negative three and y of nine, look, why are we not putting that point (pointing at negative three on the table) here (pointing at negative three for x on the Cartesian plane). Aniri x lani i three (isn't x here is three), so why are we not putting the point at negative three.

Mafada: and then here (pointing at the negative y-axis) we said we have negative (he begins scaling the y-axis and he looks at the table alongside),



These ones, these ones we must go far because it has got nine akere (isn't).



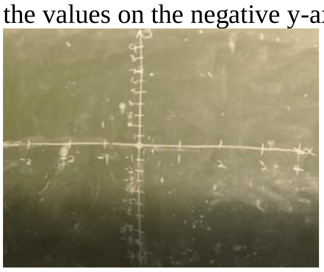
(he rescale to 'fit' the nine).

Mafada: one, two, three, four, five, six, seven, eight (he does this as he places the scaling marks on the axis) (he realises that the space he needs to make space for 8 and 9 to fit) nine!

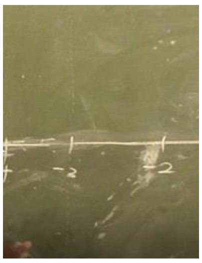
Here we have got one, two, three, four, five, six, seven, eight, nine (he says this as he puts the scaling marks on the positive y-axis together with the numbers). Which values of y must be negative and which values of y must be positive? (he puts numbers on the negative y-axis as he ask this question), the one that is at the top or the one that is below?

Learners: the one that is below.

Mafada: the one that is below. Meaning these ones should be the negatives neh (putting negative signs alongside the values on the negative y-axis).



Then now, let us start plotting the points that we have. We said for x is equals to negative three, x is equals to negative three is this one akere (isn't), this invinsible line cutting through negative three, let us have an invinsible line cutting through negative three neh.



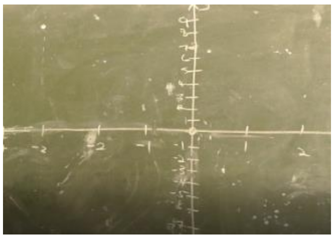
Mafada: what is the corresponding y value for x is equals to negative three?

Learners: nine

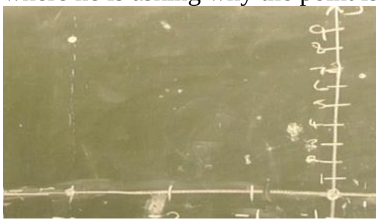
Mafada: positive nine or negative nine?

Learner: positive nine.

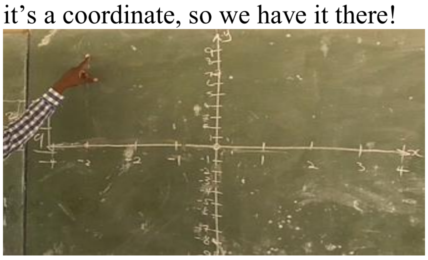
Mafada: positive nine neh, and y of positive 9 is that one neh, and we also draw an invisible line and we put it here. that is the point neh, that point represents x of negative three and y of 9.



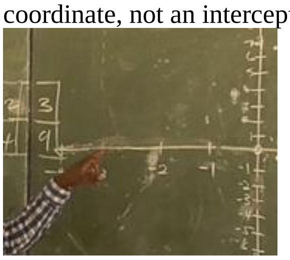
Look, why are we not putting that point here? (pointing at the x-axis, x=3), why are we not putting the point here? we have x is equals to negative three aniri (isn't), so why are we not putting the point here (plotting the point where he is asking why the point is not there) why hiyayi veka le henhla? (why do we plot it at the top?)



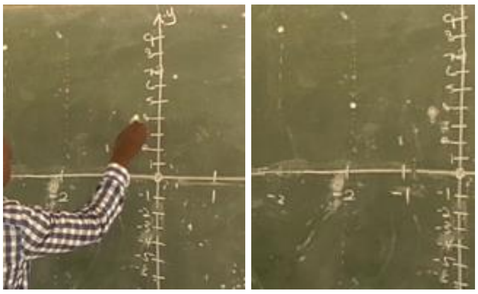
Mafada: Aniri (isn't) the x is negative three?
Learners: eeh (yes)
Mafada: why are we not putting it here and we come and put it there?
Learners: (mumbling, inaudible)
Mafada: mi vulavulela hansi (you are inaudible), vulavulela henhla uta twakala, aniri xilo lexiya xi lava na voice ya wena (speak louder so you can be audible, isn't that device should also capture your voice also).
Learners: (some giggling)
Mafada: (looking at the learner in front of him) Lesley, vulavulela ehenhla.
Lesley: because the coordinate does not have zero for y.
Mafada: okay, we are going to have a point here neh, hatwanana? (am I clear?). loko negative three ayihi nyike zero lani neh, because we said that is a coordinate, a coordinate means we put the x value with the corresponding y value, hatwanana (am I clear)? So, here the corresponding y value is nine, it is not zero, if we put a point there (pointing at x=-3, y=0) it would mean kuri (that) here hiya ka y-intercept (we are going to y-intercept), if we had our graph it was going to cut the x, and remember what we said about the x-intercept, we said when the other value has got the value that is maybe one, two or three, for us to have an intercept, it would mean the value across or the other coordinate should be zero hatwanana (am I clear)? Because here, we are not looking for the x-intercept, it's a coordinate, so we have it there!



So, this point here is x negative three and y nine, hatwanana? (am I clear?) It is the coordinate , we were going to have it come here if we were looking for the x-intercept, where our graph is supposed to cut x. we are saying here, when our x is negative three, our y is zero on that one. So, it was going to be our intercept. because it is a coordinate, not an intercept, that is our point up there.



Now, let us look at the second point. We said here, let us draw them nicely, that will be the smooth line that comes through negative two, hatwanana? (am I clear?). so, what is the coordinate for, the corresponding value for the graph, from the table?
Learners: four
Mafada: it is positive four neh, so here is our positive four. And now, we put that point there.



(Clip 2)

Mafada: This is not a graph, we are just plotting points, it will be a graph when we join all the points. For our negative one, here is our negative one. What is the y? what is the y-value?

Learners: one

Mafada: positive one, so that is out one. And then now, you have x is equals to zero, and what is our y-value there? (pointing at the number on the table)

Learners: zero

Mafada: zero! So that is the point there (pointing to the origin where he had written zero earlier in the lesson) the one that is indicated there. And now, let us look, we have our x equals to one, x is equals to one is that one there, x is equals to one is this one here (drawing a dotted line through x is equals to one) oh swa bendela? (oh its getting skewed?) (he erases the line and starts again when he realises that the line was not going to pass through x equals to 1). This is the danger of not using a ruler, even though it is not straight you know we are following the same format, from the first function to the second function, the procedure is the same, it is just that the shapes of the graphs are the ones that are different. And then now the corresponding value of y there, where x is equals to one, what is our y-value there?

Learners: one

Mafada: it's one! This one!) pointing at the y-value 1. Then now, we have got x is equals to two, what is our y? huh?

Learners: four!

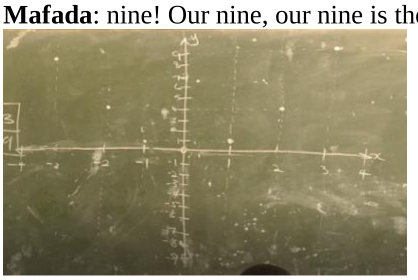
Mafada: is this four positive or negative?

Learners: positive

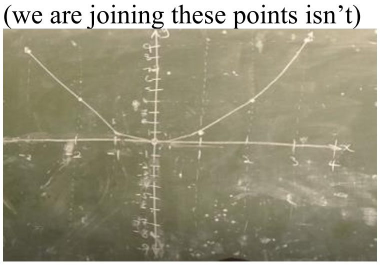
Mafada: positive and that is the four there. And we have x is equals to 9, what is the value of y there?

Learners: nine

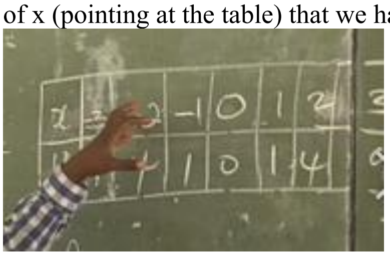
Mafada: nine! Our nine, our nine is there!

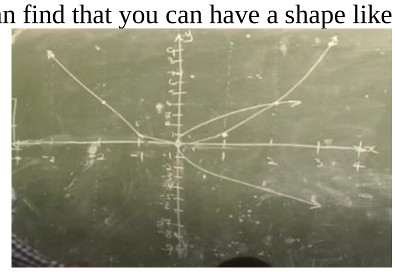
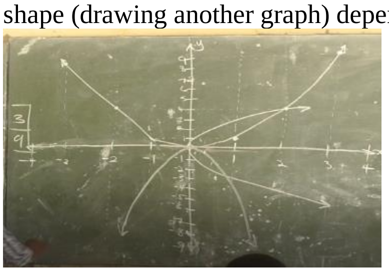
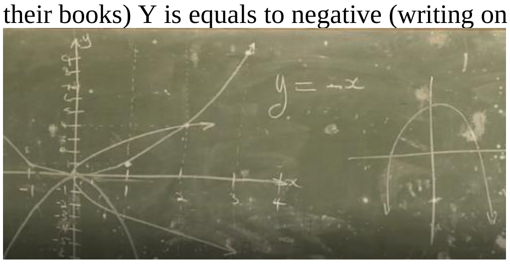


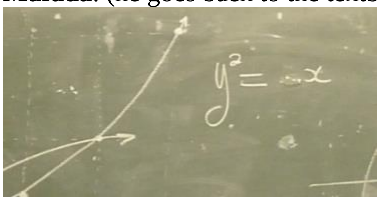
Mafada: So now we can safely draw our points which is then going to give us our graph, and then now after that we can look at the features, the features of the graph because each and every graph has got its own features. You remember when we were talking about the, err trigonometric graphs, we had range, which was our y, where it starts and where it ends, we had the period, which is where our angles were starting and where our angles were ending. We had our maximum value and minimum value and so on, but this graph does not have features like the trigonometric graphs. So now, let us draw our graph, err (he starts joining the points), hi cacela ti point leti aniri (we are joining these points isn't)

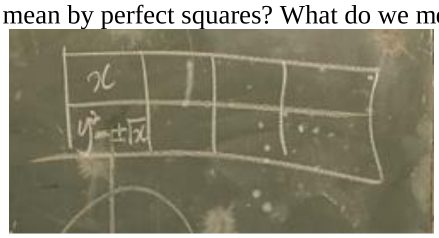


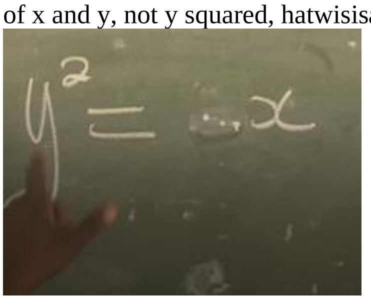
Mafada: so now, that is the shape of the graph that we got aniri (isn't), so why are we putting these things? (tracing the arrows on the edges of the curve). Kuri having the numbering this side, vo (such as) negative 4 and four, the graph can still proceed, it can still continue. Hatwanana? (am I clear?) that is why we have those things, we are not ending there, hatwanana? (am I clear?). so, is there anyone who perhaps does not understand? (busy

paging through the textbook as he ask the question (04:15) huh? So mo nimi vutisa, why hiku this is a function? (so, let me ask you, why do we say this is a function?) (pointing at y=x^2). Why hiku i function leyi? (why do we say this is a function?). For every x value (he writes on the board), we have only one y-value, that's why we say it's a function. Waswi vona nkarhi wun'wana (you see sometimes) … we deal with equations, we deal with different equations. Perhaps, two x values aniri (isn't), in the equation. So lani (here), for every x value we have only y value, so (he looks at the camera anxiously) that's why we say that's our function. So, I don't know whether there is something which you cannot interpret (he looks at the textbook), hatwanana? (am I clear?), that is what I wanted us to do for today. (He walks around a bit), who can tell me the steps before we arrive kaku dirowa (at drawing) graph? Who can do that for me first? What are the steps?
Learner: hi kuma ti coordinate (we first find the coordinates)
Mafada: I said we first set up the table neh, because without setting up the table, we will not be able to proceed, then the second one we said, you must use that function that you are given aniri (isn't), we substitute by the values of x (pointing at the table) that we have set up them at the table aniri (isn't)?


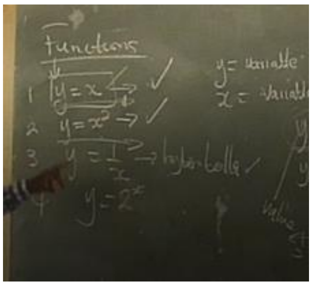
Learners: mmm (yes)
Mafada: thank you, after that you plot, you plot the points, exactly as they are in the table, then from there (looking at the textbook) you join the points, then you have drawn the graph. So, as you can see, this graph (pointing at the drawn graph) represents the hyperbola, the hyperbola is gonna have, it's not only going to have this shape, you can find that you can have a shape like this (drawing on the board).

Mafada: Aswi bohi ku shape ya wena yiva so (it's not a must that you get the shape of the graph to be exactly like this one), it depends on the function you are given, you can work and get that, hatwanana? (am I clear?), depending kaku (on) which is the subject of the formula, hatwanana? (am I clear?) you can work and get that shape (drawing another graph) depending on the sign for your function.

So, it does not necessarily mean it's gonna face up always. It can face that way (showing to the side by hands), it can face downwards, on this one it depends on what is the subject of the formula aniri (isn't) (pointing at the inverse parabola). On this one it depends on the sign of the function. So, can I give you a quicker one for you to do for me? Can I give you one for you to do? Let us, let us do this one. (for the first time learners are taking out their books) Y is equals to negative (writing on the board) err, (he moves away from the board).

(for about 25 seconds, he is paging through the textbook)

Mafada: y is equals to … (for about 20 seconds, he looks at the textbook and look at the board, then look at the textbook again).
Learner: we don't know where to begin.
Mafada: it's fine, let's try it. Or maybe nimi nika (maybe I should give you) hint?
Learners: yes
Mafada: nimi nika hint? (should I give you a hint?)
Learners: yes
Mafada: That one is gonna give us serious problems, let us choose another one (he pages through the textbook for 50 seconds). Let me say, y square is equals to x.
Learners: yhii! (complaining)
Mafada: (he goes back to the textbook)


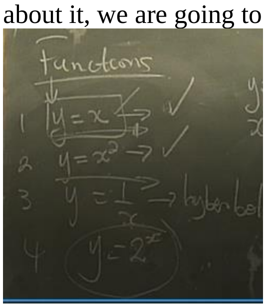
Mafada: or, is it gonna be difficult for you?
Learners: yes!
Mafada: it's fine, let's try it!
Learners: yhii!
(0:11:00-second clip)
Mafada: u teka ti values ta wena neh ta x (you take your x-value neh), hatwanana? (am I clear?) assign perfect squares, on the table. Have your table, assign perfect squares. What do we mean by perfect squares? What do we mean by perfect squares? What do we mean by perfect squares?


Is ten a perfect square?
Learners: (some learners say yes, some say no)
Mafada: why is it not a perfect square? Why is it not a perfect square? It is because when you want to find its square root neh, it not a whole number. Hatwanana? (am I clear?) So nawehe (you also) you must make your x values to be perfect squares so that when you find your values for y, because the graph that we are going to draw here is not the graph of y squared, uta fane undla yini (what you must do is) draw a graph for y. so, you must put square roots on both sides, hatwanana? For you to get y here, the graph that we are going to draw here is the graph of x and y, not y squared, hatwisisana (are we together?).


When you assign the values of x, put the ones that are perfect squares, so that when you want your y, u teka (you take) square root ka (on) both sides and get your y. (he starts looking at the textbook). Kuna swin'wana ni fanelaka ni hlamusela kwalano? (is there anything else I should explain there?) (he looks at the board and then the textbook, and he erases y=x^2 which was repeated when he was writing down the different functions at the beginning of the lesson)

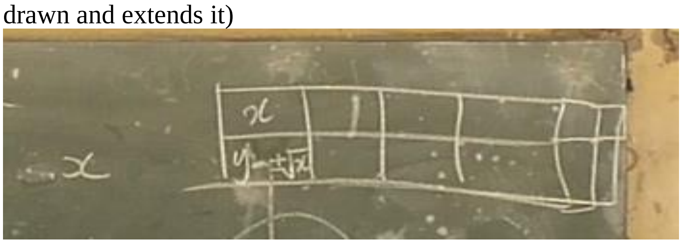


Mafada: let us correct this one neh, for the exponential one, we said initially it is x to the exponent two, it is not that one, it is two to the exponent x, that one iya (is for) parabola), the exponential one is this one, but don't worry about it, we are going to look at it.



(he goes back to looking at the textbook, he pages through the textbook).

Mafada: loyi angana problem neh, utani vita (the one who has a problem will call me) when you are busy with your tables, if you have challenges, awuse sungula he malume (you have not yet started uncle?). you can even talk to the friend next to you, ahi (it is not) test, it's just exercise. No, but you are taking longer to start, I have given you a hint, yaku (that) when you are assigning the values of x there, then you must assign perfect squares so that when you want your values of y, you are not going to draw the graph of y squared, we are going to draw the graph of y. when we put a square sign on both sides, we must get perfect numbers. I mean you must get whole numbers, not numbers that have got fractions or numbers that have got comma (he looks at the new table he has drawn and extends it)



Nimi tsalerile a bodweni (I wrote for you on the board) (he closes the textbook). We are working with what is on the board, we are working the function that is on the board. (he walks around checking what learners are doing). Why are you not talking to a person next to you, so it means un'wana na un'wana waswi tiva (each and every person knows), I said for values of x assign perfect squares! So that when you take a square root, you can have whole numbers hatwanana? (am I clear?), so start with your table. What numbers are you putting? Are those perfect squares? Okay, I want you to tell me neh, any perfect square that you know.

Learner1: four

Mafada: four is a perfect square (writing it on the board)

Learner2: sixteen

Mafada: Sixteen is a perfect square

Learner3: nine

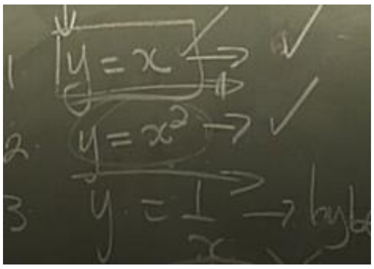
Mafada4: nine is a perfect square, and what else?

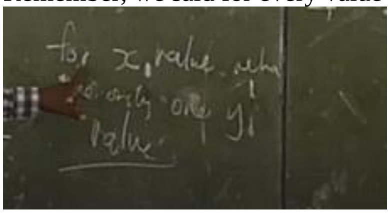
Learner5: twenty-five

Mafada: twenty-five is a perfect square, those are fine for your graph neh, kuza yi kula yi fana n aliya (for the graph to grow like that one), no we must have five, it's not five. Twenty-five is a perfect square, what else? (he takes out his phone and answers) nile klasini (I am in class). (he opens the textbook again). So, let me ask you something neh, let me ask something.

Learners: yes

Mafada: let me ask you something, we said aniri (isn't), we said this is a function (marking y=x^2), let us also look at these ones, can we also, why did we say this one is a function? Hite i function hi mhaka yini? (why did we say it's a function?) hite i function hi mhaka yini? (why did we say it's a function?)



Remember, we said for every value of x, we have only one y value.

For example, I said here, let us assign the perfect squares, if we can give an example, if anyone can ask is this a function?


Let’s say here we assign our x, it’s sixteen, hatwanana? (am I clear?) so it means y squared is sixteen. Remember what we said, the graph that you are going to draw, it’s a graph of x and y, not of x and y squared hatwanana? (am I clear?), meaning here, we are going to put the square roots and we get the square roots, so that now, our y becomes, what is the square root of sixteen?
Learners: four
Mafada: it’s four only?
Learner: yeah
Mafada: are you sure?
Learners: eeh (yes)
Mafada: let me give an example, let me give an example here. four multiplied by four, what do I get?
Learner: sixteen
Mafada: if I say negative four multiplied by negative, what do I get?
Learners: sixteen
Mafada: are you sure?
Learner: yes
Mafada: it is positive sixteen?
Learner: yes
Mafada: so, what is the square root of sixteen?
Learner: four and negative four
Mafada: very good! It is four and negative four! When you multiply these two set of numbers you get 16, meaning when we are looking for the square root of sixteen, it is four and negative four. Loko hi tsala la (when we write here) we are going to say plus or minus four, hatwanana? (am I clear?). so now, how many values of y do we have there? For one value of x, how many values of y do we have there?
Learner: two
Mafada: two?
Learner: yes
Mafada: so, can we say that one is a function?
Learners: no!
Mafada: why? Because we have two values of y. for a function, one value of x must give us one value of y. hatwanana? (am I clear?) So, that one gives us two values, does it make sense?
Learners: yes
Mafada: I want you to proceed with that one so we can quickly do the corrections. So we must make corrections before swi strike swa vanhu lava swihi kuma aniri (before these people’s strikes find us here), because you are left with only three weeks before you write exams. So the time has run, I will see you during study time, aniri (isn’t) the principal announced last week that’s on, yeah we were just disturbed by those things, meaning today we must learn after school, so can I have that period register?
Learner: se strike xa rini? (so, when is the stike?)

Mafada: ntami byela (I will tell you). Swinge humeleli namuntlha, maybe na mina ni vulavula ngopfu swita humelela (it won't happen today, maybe I talk too much, it might happen). So, ntan'wu byela ku leyi lesson angayi kombi ka TV (I will tell him not to show this lesson on TV).
Learners: haa (no!)
Mafada: yeah
Learner: swa fana angahi fodangi hina (it's the same, he didn't even capture us)
Mafada: he did, kasi n'wina ami languse camera not focusing on me?
Learners: (they giggle).
Mafada: mita switwa kwahala malayinini, vani vutisa kuri mundzuku kuna strike (you will hear from the community, they are asking me if tomorrow there is strike).
Learner1: ahiti (we are not coming)
Learner2: hiyata next week (we are going to come next week)
Mafada: so, mini vutisela yini loko miswi tiva? (so why are you asking me whereas you know?). swa strike mina aniswi tivi (I don't know anything about strike) (he smiles)
Learners: (objecting)
Learner3: kanjhani kuri wehe murhangeri (how come whereas you are the leader?)
Mafada: (he walks out).

The End

Appendix 2: Mutsakisi's semi-structured interview transcript

Interviewer (Hlamulo)	Mutsakisi
I am still struggling to pronounce your surname	Mutsakisi
Okay, I am with ma'am Mutsakisi at Lesedi, ma'am please introduce yourself in terms of what you teach and how long have you been teaching?	I am Mrs Mutsakisi, I am a mathematics teacher, I have been teaching mathematics for the past thirty years.
Wow, so there is lot of experience here	Yeah
Wow, so, where did you train to be a teacher?	I trained to be a teacher at Gweru Teachers' College and I also went to the University of Zimbabwe
And, how was it, how were you trained to be a teacher there, what are some of the approaches they used to train you to be a teacher?	They taught us err, in actual fact, there was a number of subjects which contributed to the course for teaching, they taught us the subject content, they also taught us education, they also taught us methodology, the methods that we can teach when teaching mathematics, they also, we also learnt a bit of agriculture.
Oh, okay, in terms of method, what were they actually focusing on?	They were focusing on the teaching aids, the approach you can use, we had also an audio-visual department where there was a lecturer specifically for teaching us err, how to prepare learning aids, like chats, models.
So, do you incorporate those when you are teaching here in a different context?	Yeah, I do.
Okay, how would you incorporate those audio-visuals let's say in teaching functions?	Okay, in teaching functions, let's say you get to class and you want to draw a smooth graph neh, you draw that graph on a chat and then you perforate the chat, at the back of the chat you smear some coloured chalks, when you place it on the board the graph comes out, so you don't waste time by drawing, that is one example of the techniques they taught us, that is Mr Kheshuf, an Indian guy.
And when was this?	1984 to 87
Yho, but you still remember all those techniques up to this day?	Yes, I do and I apply them.
Everyday hey?	Yes
What is your understanding of mathematics teaching?	Mathematics teaching is, in actual fact, mathematics is a language, so mathematics teaching involves teaching the learners the language of mathematics, understanding the concepts is the most key, concepts in the teaching of mathematics.
And how do you emphasise on some of the concepts that are related to function?	Err, concept like what?
That are concerned with functions or function as a topic	Okay, you mean function as a topic?
Yes	Okay, in actual fact it covers a wide range from grade one to maybe university, so you find that maybe at primary level they learn about the coordinates, they are taught how lines are drawn, by that time, they are not really into the relationships, they are not really into coordinates, but as they go up to grade 8 or 9, they are taught now how to draw a straight line graph, using the table of values as they go on to grade 10, they are, we now introduce different types of graphs, like straight line, the parabola, hyperbola and the exponential graph, also the concept of the asymptote, the range and the domain they come in.
Okay, so when you are teaching functions, how do you actually approach it?	If I am teaching functions, I know that there are key concepts in functions, there are for example, a learner must know how work with the relationship, the input and the output that is the basic. Secondly, it involves the interpretations of graphs, so the interpretations of graphs depends on the graph that you are given, the key aspect is the scale, so the learner must understand the scale in order for him or her to interpret the graph correctly, then the learners must know how to draw graphs, so graphs also should be labelled, they must know how to create the variables, the dependent and the independent variables, the variables they will use to draw a particular graph, they will also be able to read values from a graph, if you are given the graph you should be able to read the values from the graph, so to be able to identify, you must also identify the gradient of the graph, you must also know the range and the domain, if it's a parabola you must also know the turning point, the minimum value or the maximum value, depending on whether the graph is concave up or concave down, the learners must also know the terminology, so when they reach grade 12, then we add some more new concepts, the

	differentiation, we introduce the gradient of, the tangent, we also introduce the turning points, we also introduce the notion of drawing the graph from the given information, you are just given the turning points, you are just given the inputs in the table of values, you are given the gradient, then we are supposed to draw the graph from that information.
So, would you say that if a teacher is teaching in grade 10, they should have the knowledge of what is taught in grade 12?	True, because the grade 10 is the basic foundation for 11 and 12, in actual fact for the matric examination, yes, if a learner does not grasp the basic concepts, he or she will have challenges when she goes to the higher grades, for example, those smaller basic concepts like the concave up, the concave down is part of the graph, let's take for example the parabola, or even the exponential graph, when the graph is decreasing, where is the graph decreasing? We must also, the learners must also know the domain and the range, the learners must also know how to get the gradient and that is at grade 10. So, those are the basic concepts, they must know also when they are given points how do they find the equation of a parabola, so they must know the general formula for a parabola because it comes in many ways, so you expose them to those many ways so that when they see some other variables instead of the one he knows, he will be able to identify and they must know also, if we are dealing with the parabola, the highest power of x is two, so it's a square …
Okay, why do you think that is important to emphasise?	This is important because a learner might get confused, err the learners must differentiate by merely looking at the graph, the learner must know that this is a straight line, a straight line and the exponential, the highest exponent is one, this is why we say y is equals to 2x plus 3, 2x to the power 1, they must know when we go to the parabola, if I just see x power 2, I must know that I am dealing with a parabola, …
And therefore, the roots of a parabola apply now?	Yes, because it has two roots, two roots of x and then if you get to the cubic function, the same applies, it is x power 3, it has three roots, so by merely looking, we know exponential graph isn't? we give them the basic knowledge of exponents, so if they see an exponent on an equation, they must know that they are dealing with an exponential graph, if they see a value in the denominator of an exponent, they must know that it is a parabola,
A parabola?	No, not a parabola …
A linear function?	No, a hyperbola.
X at the denominator?	Yes, the basic things they must use to identify the graphs, so if a learner grasps this at grade 10, err he has no problem in … (someone asks her something about the school details, I pause the recorder).
… you were saying ma'am, you were talking about …	Because if a learner is drawing a graph, it helps him to know that now I am given this equation, what type of a graph am I expected to know. It's unlike a learner seeing an equation and putting numbers, he doesn't know what is going on, he is just confused.
Okay, please talk to be briefly about the notion of multiple representations, what do you understand by that?	Mul …?
Multiple representations? So, when you are teaching functions, the different representations themselves that you use to represent functions?	Multiple representations?
Yes	Like uhm, can you please give me an example
For example, you have been talking about err, being able to get the general formula from given values, so that's simply means that learners are now actually moving from err numeric into now identifying a symbolic representation for that relationship, so how do you actually incorporate that idea into the teaching of functions?	Okay, in actual fact, you teach all the concepts, the last concept you teach is that one for representing, okay, graph representations come in different forms.
We have graph representation …	Yes
Would you say that functions is about graph representation and that is the only type of representation that can enable learners to see this relationship?	It, it incorporates many things akere (isn't).
Such as?	Such as, you can be given a set of ordered pairs, right we have ordered pairs, they can represent a graph, you can have err a graph itself, a drawn graph, or you can have the equation that represent a graph.
So, there are only three things?	They are more, but …
For you, which ones do you actually emphasise on while teaching? is it the graph and the table, and being able to move from the table to the graph, and being able to move	Both, because they ask either way, they might start with by asking a learner to draw a graph, the what-what, they might also define that a graph with this gradient, let's a graph with a gradient -2 and they draw

from the graph to the table, what do you actually emphasise on while you are teaching?	the graph, give the domain, give the range, is the graph decreasing or increasing.
According to CAPS, which representations do they actually emphasise? Have you engaged with CAPS?	Both.
Do you know that they actually specify the types of representations within the South African mathematics curriculum? Have you looked at that?	No, I haven’t checked on it.
For example, there are words, there are going to be symbolic, there is going to be a graph, there is going to be a table …	Okay …
So, these four, so the curriculum is actually advocating for the learners to be able to work within and across these representations, so do you allow the learners to work within first, let’s say explore the relationship between two variables, only from a table or the emphasis is mainly on the graph, and if so, why do you think that is the case?	They must be able to know because sometimes the relationship is defined, let’s say the learner want to know is the graph decreasing or increasing, the values in the table, if both values are increasing, it means that the graph is increasing, if the x is increasing and the y is decreasing, it means that the graph is decreasing, but with some graphs now, it depends, for example, a parabola, it depends how they ask the question.
Okay, like the exam question?	Yes
So, would you say that you are teaching them towards the exam?	The grade 10’s I can say I am teaching them towards the exam, towards the grade 12 exam
Oh, okay, interesting, so it is not; so, do you actually focus on the explication of the concepts that you want them to focus on how they are going to be asked questions in the exam?	No! first of all I teach them content, they must understand, after understanding towards the end, I teach them how to approach the, how to answer the exam, I give them the exam structure, in some cases where it is applicable, I take questions from grade 12, those which have the questions which they can answer, because you find that some of the grade 12 questions they come from grade 10.
Okay, interesting, do you think teaching in a rural school influences the manner in which you teach mathematics?	Yes, the input by the parent is very, is minimal, but in the rural urban areas you find that the input by the parents is maximum, parents even assist their learners to, learners they have libraries, they may research, but here in the rural areas the school itself does not have a library, the community in which the school is situated does not have a library, so another thing is attitude, …
What do you mean by attitude?	The parents at home tell the learners that mathematics is difficult, then what do you expect the learner to do when she gets to school? She also thinks mathematics is difficult, yes! So, err, in order for you to change the mindset of the learner, we must change the community first because those ones they got a great influence on the learner than the teacher.
So, how do we change the perceptions of the community first before we can actually change these of our learners?	It is very difficult because we do not stay with them, even at times when you call them, some of them are not around they are at work, some work at Joburg, it means all of them don’t come and they send representatives and some of the representatives which are sent are very young, what you tell them it ends here, it is not applied at home
Oh, okay, and how would you describe your mathematics teaching methods?	Err, at times it is very difficult to assess yourself, …
Given that you have been teaching for those thirty-five years right, so, over the years, what would you say that these are my methods that I actually apply when teaching mathematics?	In actual fact, methods vary and they depend with the learners you have, the ability of the learners influences your teaching methods, because there are some learners who are self-motivated, as a teacher you just need a little push, there are some learners who are from the rural areas like ours, these learners need effort, you must put extra effort, you must teach them err a number of methods and approaches so that they can select the best on, but with the learners who are gifted, they discover methods on their own.
Okay, let’s say they are these one who are gifted in class, (**20:00**) how do you actually approach teaching them mathematics?	In most cases I give them more work, you find that at times we use our textbooks, you find that at times I prepare worksheets, sometimes they finish before the others finish, I tell them to move forward.
And the ones that are struggling, do you assume the position where you have to tell them everything first then you assess how they are actually doing?	No, it depends, they also need to discover because they are going to write the same examination, so at times I don’t, I know they are there, but I don’t treat them differently, so that they may cope because they will write the same examination, the examination is for all learners, those who are capable and those who are not capable, but in my teaching I will make sure that there are some things that I will teach much simpler concepts which will come to examination, whereby those learners who are not gifted will answer.
And, in terms of participation in class, how do you ensure that your learners are engaged in class during teaching and learning?	Different motivational methods.

Okay, and what do you mean by that?	Sometimes I give incentives, sometimes I praise them for giving the correct answer, at times I exempt them from sweeping, those who have been answering questions throughout, I just use a variety, they just come in my mind while I'm teaching, what can I do today so that they participate?
And, do you think that allows for mathematics learning, those strategies that you are using?	Yeah, they do. Because you find that even that learner who does not participate, if you promise that today who participate I will give you this, even that learner who doesn't have an answer will raise up their hands and answer.
Oh wow! So, they raise their hands with the hope that their answer will be right so that they can get their incentives?	Yes, mara (but) at times incentives are not materials, verbal incentives can help, learners must not get used to being given material incentives.
Okay, and what does algebraic function mean to you?	Err, algebra is the back bone of mathematics, if a learner does not understand algebra, it means that he cannot understand any mathematics.
I am intrigued, what do you mean by that?	I mean to say, any, any type of mathematics that you can do there is some algebra in it.
Is that why the mathematics in South Africa is weighting more on algebraic functions?	Yes …
And giving it much time, you think?	Yes, true!
And how do you ensure that there is continuity from algebraic function to other topics?	Usually the algebra is implied in all topics
So, you draw from that knowledge and incorporate it as you continue teaching?	Yes
But, were you trained to teach algebraic function in Zim?	In Zim we do everything, the syllabus is very intense, we do everything, even linear programming, Euclidean Geometry, even, we do what they call the mathematics itself …
Oh, so here in South Africa (I giggle), what are we doing?	It is also mathematics itself, but with some limitations.
What do you mean? Are we lagging behind?	The period that we learn in Zimbabwe is longer, before we go to university.
How long does it take?	Secondary school is six years.
And here we have what, five years?	Yes.
Okay, and what do they do over those five years?	They write examinations in between, in actual fact our system has got three phases of examination, certificated examinations, from grade 1 until grade 7, at grade 7 a learner writes examination, this examination helps to screen the learner into secondary education, as a result there is hard work from here in grade 1 because here (in grade 7) there are results as we have in matric.
Oh, so if a child does not pass that grade 7 exam?	They rewrite, but that's very rare for them to fail because that is where they learn all the basics for mathematics.
And then, the next examination is at what grade?	In what we call form 4 - 9, 10, 11; grade 11
In grade 11?	Yes
And again, there is a certificate?	They write a national examination, like the NCS, so at grade 7 they write an examination, like an NCS, they learn for four years, after four years they write an examination, it screens them for specialisation
Oh, so you cannot do maths if you were not good with maths?	If you did not do well at maths, yes.
So you did maths?	That's were learners begin to divert into various categories, into different fields.
Okay, that means you did well at maths because you continued with maths even in grade 12?	Yes, you get to form four and you write an examination, you are screened into, but at this level now here, there are many options, some may branch for courses, some continue with the education, you can go for nursing, you can go for, and again there is this other extra two years, you can go for nursing, but you will be taken for higher diploma because you have added these two years at school, so from here you cannot go to university, you must go now to specialise in your subject, let's say you chose physics, mathematics and biology, for these two years you will be doing mathematics, physics and biology, so this grade 12 and 13 is for specialisation, you write examination at this level.
So, oh you have grade 13 that side, that's what differs from this side?	Yes, then at grade 13, you write an examination, so these results now qualify you for, they give you an entry into university.
Okay, according to your understanding, what does parameters of function mean?	…
When we talk of parameters while teaching functions …	Parameters?
Yes	Maybe the term?
Let's say you are saying y is equals to ax squared, what is the parameter there? X and y are variables …	Yes
Then the a is the parameter …	Okay, the a is the, determines the steepness of the graph

Okay, so what do you call it while you are teaching?	Is the gradient …
Is the gradient?	Yeah!
Yes, but another word for parameter for you would be what? So, let's say I was to speak about q for example on a straight line or the c on the straight line …	Yes, the intercept
So, you call them by name specifically? you don't say this is a parameter and it affects the graph in the following manner like that?	Yes
And what are some of the features that you actually emphasise while you are teaching functions?	The features?
Yeah!	The, you find that the variables of the graph, they help the learners, for example, the, the constant value it always represents, stands for the y-intercept, always in any graph, always!
Everywhere?	Everywhere! It stands for the y-intercept and the parameter before x, it stands for the steepness of the graph and it also tells us about the direction of the graph. If the graph is, let's say we are dealing with the parabola, it tells us, if it is negative, it tells us that the graph is facing down, if it is positive it tells us that the graph is facing up, if it's a straight line it tells us that the graph slants to the right, slants to the left, if it's a hyperbola, it tells us that the first curve is on the first and third quadrant if it is negative in the second quadrant.
And, how do you approach that, let's say you are teaching the hyperbola, and you want to tell your learners about the quadrant in which the curves are going to fall under?	First of all, we draw the graph, with the given equation, then I explain the relationship of the, each value of the equation to the graph that we have drawn, now we have the equation, this is, here this value is positive, now you see this one, now we draw two graphs next to each other, side by side, another one with a positive gradient, another one with a negative gradient, then I explain the effects.
But, do you emphasise on the signs?	Yes
That negative and negative …	That if it's negative, then if it's a negative gradient the position of the graph (shows by drawing on the paper), if it's negative, this is the graph, so, I draw them separately, one there and one here, so that if there is a negative gradient, this parabola will be in the second and fourth, then that's what we discuss from what we are seeing.
And what would you say are some of the challenges that you experience when you are teaching algebraic functions?	Okay, the problem that I experience, especially with graphs, the learners at times they don't link the concepts, that even if it's a straight line, or it's a parabola, or it's a cubic function, the intercepts, the last value is the intercept in any graph and also the link of the shifts, that q is a shift either upwards or downwards in any graph, it could be the parabola, it could be the, sorry, the parabola or the straight line
And, how do you actually address those difficulties or challenges?	Right, through your teaching, for example, at the end of the, err if you are drawing, you are teaching your learners, you must have focus, I want my learners to achieve this and that and that, which apply to ABC, then at the end of the lesson you refer, or at the end of each concept you compare the two, hore (that) in a straight-line graph or the parabola, what is it that this requires.
So, you are saying that you cannot teach if you don't have the purpose of the lesson?	Obviously, you always teach for a purpose, you cannot just go into class and say today I just want to teach them, what is it that you want to achieve and what is it that is going to help the learners in class? So, as of now, err, yesterday I was thinking of a new approach, isn't as you get into the field you learn new things, it doesn't mean I still apply what I have learnt 35 years ago, I develop inside, so I was thinking, I saw that most learners in South Africa, they forget, the tendency of forgetting, even if I teach them something and I come into class after an hour and ask, they have forgotten, they forget and I don't know what is the problem.
And you call it a South African thing hey?	Isn't I have taught …
Okay, where have you taught before?	I taught in Zimbabwe,
And those problems were not prevalent?	No! the problem of forgetting there, I don't know maybe because the learners are self-motivated, they work on their own, if you teach them, they go home and practice.
And, what would you say is the motivational factor that side?	The examination at grade 7
Okay, mm, so you cannot be left behind?	Yes, akere (isn't) as soon as a learner gets into grade 1, the learner must know how to study so that she passes grade 7, so this will be a skill instilled in the earlier grades.
Okay, that you need this knowledge?	Yes

So, you cannot for…	You need this knowledge because you are going to write an exam, at the end of the seven years.
Wow, so, you must actually retain that knowledge for seven years?	Yeah, because they will be asking from the things that they did in grade 1, a mixture of questions.
And what are some technologies and resources that you use during the teaching of functions?	Err, I previously used their DVDs, from book suppliers, different book suppliers.
So, they would actually watch videos?	Yeah, the previous. I was actually doing it with the grad twelves
Okay, and would you say that was helping?	Err, yeah at some point, because the, they pay more attention because it is not a routine thing you do. So, if you are doing that, learners pay more attention
What else would you like to say about the teaching of mathematics and the teaching of algebraic functions in South Africa? Especially in grade 10?	Yeah, I say that by teaching grade 10 to 12, I realised that there is a link between concepts, it is the same concept which is just slightly developed, such that if the learner is very good at grade 10, the learner won't have problems in grade 12.
So, is it your duty as the teacher to conscientize these learners on how to make these links?	Yes, that even if you are teaching them, you must tell them that this is what you are going to meet in grade 12, so, understand it now.
Mm, wow, okay, if there is nothing else you would like to say, thank you so much for your time.	Okay, I would like to say it was good being with you, yeah and some of the things we discussed are an eye opener, as I said that you learn, you learn even if you are the deathbed …
Every time	Every time of your life. Yeah, like what I am saying, while I have not explained the new way of teaching, I was thinking about, so we, we discussed more on the issue of forgetting, I was saying I was thinking that from today, because I was thinking of how I'm going to implement this strategy, every day before I teach my learners, err, every day before I give them homework for the day, I must give some two problems for the previous work, even the work that we did in January, so that they don't forget it, they can see it as routine, because I'm sure one mistake I have been doing is that if I teach a concept in January, I would assume that they will revise on their own, but now I am think no! every time I give them home work, there must be two small problems from the work that we did before, two small problems of what we did before, two small problems and that will help, they will see it every time , it becomes routine to them, when they see this problem in the test, then they won't …
You see, I am also learning now, because we mainly think that the curriculum should be from this day to this day you must actually be covering functions, therefore the emphasis should only be on functions.	Yes, but I realised that maybe there is a small loophole here
Maybe that's what is missing hey, in our teaching?	And I must identify if I give them a test, or since we have been giving them tests from January up to now, which areas are they not doing well, those are the questions I will be giving them on a daily basis, before we start the lesson we revise, we start with the previous work, so that they become used to what is difficult to them.
Wow, and if you were to be given a chance to teach functions differently, what would you do? How would you teach it differently?	The functions?
They say, stop following this routine that you are following, think of something else that is creative that can enable learners' understanding?	In actual fact, technology is a limit, if they had, err, laptops and what-what, I would go to a package that they could draw graphs, whereby the computer would correct them, I would go for that package.
Okay, wow, did you use to use that in Zimbabwe?	No, in Zimbabwe we used that in college, because we had computers, we would do what you call PET teaching, so you teach other students.
Okay, so you would actually definitely go for technology?	Yes, the audio-visual department had all these things
So, we need schools to have the audio-visual team you say?	Err, we can have err, err maybe at the colleges where the teachers are trained, there must be an audio-visual department cause on the timetable for when we were at college, there was a slot for audio-visual, once a week, but that department was giving us a lot of work, where they would say, for every topic that you are doing now, at college, the mathematics was divided into two groups, the core mathematics and the broadened mathematics and the what we call, the …
What is the difference between the broadened and the core?	The mathematics is the mathematics that we are going to teach the learners at school, that was the mathematics, that was where we were taught how to prepare some aid and all.
And then, the other one?	The broadening was the mathematics beyond what you know, isn't you did up to form, grade 13?

Yes	Then, you go and train as a teacher for four years, during the four years you will be doing university mathematics, you have to be above your learners.
Oh oka, why do you think that is the case? Why can't I just learn what I am going to teach my learners?	You cannot only learn what you are going to teach your learner because what if there some questions that are beyond their level? There are some questions that need your knowledge. Your higher level of understanding, so that is ti create a gap.
Between you and your learners?	Between you and the learners.
So, the teacher and the learners cannot be at the same level?	Yes
But, I would assume, let's say if I am going to teach grade 10, I just need to know the content of grade 10	Yeah, but you will have challenges because at times a grade 12 learner comes to you and ask a question, how do I solve this?
So, I would be stuck …	You are a grade 10 teacher of course, but at the school where you are going there are grade 13 learners who would even come to a grade 10 teacher to ask for something.
So, if as a teacher I were to say I don't know, would it be wrong?	It won't be wrong, but as a teacher, the learners should have trust in you, because if those ones in grade 13 start to lose trust in you, obvious these ones will doubt because learners liaise …
So, that would have an impact on their learning altogether?	Yeah, on the attitude, towards you and your teaching, because if I come and ask and you don't help and I say I will go to grade 10, they will say urgh, teacher ole a tsebe selo (that teacher does not know anything), so in actual fact it is to build confidence.
So, the teacher needs confidence when they are actually teaching the learners?	Yes
So, if you don't have confidence in teaching function, would you say that will have an impact in your teaching of function?	Yes, because at times you will be stuck, you take, you don't have time, you pick an example, you want to go to class, you did not prepare for it, you must be able to teach the learners and answer the questions.
So, on preparation, how do you actually prepare for your lessons?	We have got preparation books, in which we prepare the questions that we are going to teach and try the answers because at times you find that the answers are not correctly done.
In the textbooks?	Yes
So, you have to first fix the textbook before you can actually go and teach?	As a teacher I must, in mathematics, as a mathematics teacher you must first of all, answer the questions you are going to teach, to check because you will be embarrassed in front of the learners
Okay, wow. Thank you, ma'am, thank you so much for your time	Okay, you are welcome.

Appendix 3: Semi-structured individual interview schedule

1. How long have you been teaching?
2. What mathematical background do you have?
3. Tell me about the learners you are currently teaching.
4. What does mathematics teaching and learning mean to you?
5. Do you think teaching in a rural school has an impact on your mathematics teaching?
6. What teaching approaches were used in teaching you mathematics in primary, secondary and tertiary?

a) How does the culture of your classroom enable mathematics learning?

a) What are the learner and teacher roles in your mathematics classroom during teaching and learning?

7. What does algebraic functions mean for you?
8. What approaches do you use to teach algebraic functions?
9. What does CAPS say about the teaching and learning of functions?
10. What are some of the difficulties you have in teaching algebraic functions?
11. Do you think about your questions and question asking inside and/or outside class?
12. What are curriculum delimitations for Grade 10 algebraic functions?

Appendix 4: Video-Stimulated Recall Interview Schedule

1. **Introduction**
   a) Explain to the teacher what SR interview is.
2. Focus: question asking and comments during the videotaped classroom observation (i.e. this is just a guideline for the interviewer; the interviewee will also ask questions and/or comment on any aspect they decide to).
3. Stimulated Recall Interview Rules:
   a) Either the interviewer or the participant can stop the video at anytime
   b) Distinguish between new observations and actual recall to the teacher

4. **Orient**
   a) The teacher briefly describes the purpose of the observed teaching episode
   b) Stimulated Recall seeks to address the following aspects:
      1. Teacher's perspective on what happened during the teaching episode
      2. The goal(s) the teacher aimed at achieving
      3. What prompted the teacher to act in certain ways during teaching.
      4. Teacher's perspectives on what they could have done better.

5. **Questions** to be asked each time the videotape is paused

a) Can you recall what motivated you to do this?
b) Did anything that occurred in the classroom influence your decision to teach this way/ask this question? Please explain.
c) What information did you base that decision on? (i.e. it could be a teaching approach or any form of interaction with the learners or teaching materials).
d) Was there anything else you wanted to say/do at that point but decided against?
e) Is there anything else you would like to share about this teaching episode?

6. Thank the teacher again for their time.

(The interview structure was adopted from Maloney, 2012).

Appendix 5: letter to the principals and SGB chairpersons

15 March 2018

Dear Principal and SGB Chair

My name is Hlamulo Wiseman Mbhiza; I am a Doctor of Philosophy (Education) Candidate at School of Education at the University of the Witwatersrand. I am doing research on Grade 10 rural teachers' discourses and teaching approaches of algebraic functions in Acornhoek, Mpumalanga Province.

My research involves individual semi-structured interviews, non-participatory classroom observations and video stimulated recall interviews with six teachers from six schools in Acornhoek. Video-recording and audio-recordings will be used during data collection. Both the semi-structured interviews and video stimulated interviews will take approximately 45 minutes to an hour and will take place after school hours. Your school was selected based on the relationship we have established as the Wits School of Education through the Wits School of Education Rural Education programme for teaching experience.

I am inviting your school to participate in this research and for your teachers to share their experiences, knowledge and understanding of teaching mathematics within rural classrooms. This study has four purposes. It firstly seeks to interrogate Grade 10 rural mathematics teachers' discourses of teaching algebraic functions. Secondly, to examine teachers' approaches while teaching algebraic functions in rural classrooms. Thirdly, to explore Grade 10 rural mathematics teachers' experiences of teaching algebraic functions in rural schools. Fourthly, to examine factors that shapes teachers' discourses and approaches while teaching algebraic functions in Grade 10 classrooms.

The research participants will not be disadvantaged in any way. They will be reassured that they can withdraw their permission at any time during this research for whatever reason without any consequences or penalty. There are no foreseeable risks in participating in this study. The participants will not be paid for this study.

To ensure that the participants' and your school's true identities are protected, I will use pseudonyms to conceal both the true names of the schools and the participants in all writings of the study. The information provided by the participants will be used for the thesis and journal publications both locally and internationally. All research data will be destroyed after 5 years of completion of the research.
Please let me know if you require any further information. I look forward to your response as soon as is convenient.

Yours sincerely,
Mbhiza Hlamulo Wiseman
121 Carr Street, Newtown, Johannesburg
Email: wmbhiza@gmail.com
Cell phone: 0769019192

Appendix 6: Information sheet for teachers and consent form

15 March 2018

Dear Teacher

My name is Hlamulo Wiseman Mbhiza; I am a Doctor of Philosophy (Education) Candidate at School of Education at the University of the Witwatersrand. I am doing research on Grade 10 rural teachers' discourses and teaching approaches of algebraic functions in Acornhoek, Mpumalanga Province.

My research involves individual semi-structured interviews, non-participatory classroom observations and video stimulated recall interviews with six teachers from six schools in Acornhoek. Video-recording and audio-recordings will be used during data collection. Both the semi-structured interviews and video stimulated interviews will take approximately 45 minutes to an hour and will take place after school hours.

I was wondering whether you would mind that I come and do a classroom observation while you teach your learners algebraic functions. The information will be used to understand teachers' discourses and approaches while teaching the topic and how these facilitate learners' learning.

Your class routine will not be advantaged or disadvantaged in any way. You can ask me to leave the class at any time without any penalty. There are no foreseeable risks in participating and you will not be paid for this study.

Your names and identity will be kept confidential at all times and in all academic writing about the study. Your individual privacy will be maintained in all published and written data resulting from the study. All research data will be destroyed within five years after completion of the project. Please let me know should you require any further information.

Thank you very much for your help.

Yours sincerely,

Mbhiza Hlamulo Wiseman

121 Carr Street, Newtown, Johannesburg

Email: wmbhiza@gmail.com

Cell phone: 0769019192

Teacher's Consent Form

Please fill in and return the reply slip below indicating your willingness to allow us to engage with your child as one of the participants in the study titled: **Grade10 rural mathematics teachers' discourses and teaching approaches of teaching algebraic functions in Acornhoek classrooms, Mpumalanga Province, South Africa.**

I, ____

Circle one

Permission to be videotaped

I agree that my class can be videotaped during classroom observations.	YES/NO
I know that the videotapes will be used for this project only	YES/NO

Permission to be interviewed

I agree to be interviewed for this study.	YES/NO
I know that I can stop the interview at any time and doesn't have to answer all the questions asked.	YES/NO

Informed Consent

I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I can ask the researcher to leave the classroom at any time.
- I can ask for my class not to be videotaped
- all the data collected during this study will be destroyed within five years after completion of the project.

Sign____________________________ Date__________________________

Appendix 7: information sheet for parents and consent form

15 March 2018

Dear Parent

My name is Hlamulo Wiseman Mbhiza; I am a Doctor of Philosophy (Education) Candidate at School of Education at the University of the Witwatersrand. I am doing research on Grade 10 rural teachers' discourses and teaching approaches of algebraic functions in Acornhoek, Mpumalanga Province.

My study involves coming into your child's classroom and observing their grade 10 mathematics teachers teach. During the classroom observation a video-recorder will be used to record the teacher while teaching. During observations, your child may be captured by the video-recorder, therefore I am asking for permission from you to allow me to capture your child in the classroom.

I have chosen your child's class because my study seeks to work with Grade 10 mathematics teachers to gain insight into their discourses and approaches as they teach the subject. Your child will not be disadvantaged in any way during the course of the study. He or she will be assured that she can leave the classroom during observations without any penalty. There are no foreseeable risks in participating, and your child will not be paid for the study. Since learners are not the primary participants in the study, I am going to blur their faces in the video after recording to protect their identities and throughout all the writings of the study, your child's true name will be concealed. His/her individual privacy will be maintained in all published and written data resulting from the study. All research data will be destroyed after 5 years of completion of the research.

Please let me know if you require any further information. Thank you very much for your help.

Yours sincerely,

Mbhiza Hlamulo Wiseman
121 Carr Street, Newtown, Johannesburg
Email: wmbhiza@gmail.com
Cell phone: 0769019192

Parent's Consent Form

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called : Grade 10 rural teachers' discourses and teaching approaches of algebraic functions in Acornhoek, Mpumalanga Province

I, ______________________________the parent of ____________________________

	Circle one
Permission to observe my child in class	
I agree that my child may be observed in class.	YES/NO
Permission to be videotaped	
I agree my child may be videotaped in class.	YES/NO
I know that the videotapes will be used for this project only.	YES/NO

Informed Consent

I understand that:

- my child's name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- he/she does not have to answer every question and can withdraw from the study at any time.
- he/she can ask not to be audiotaped, photographed and/or videotape
- all the data collected during this study will be destroyed within 3-5 years after completion of my project.

Sign____________________________ Date__________________________

Thank you very much for your help.

Yours sincerely,

Mbhiza Hlamulo Wiseman
121 Carr Street, Newtown, Johannesburg
Email: wmbhiza@gmail.com
Cell phone: 0769019192

Appendix 8: information sheet for learners and consent

15 March 2018

Dear Learner

My name is Hlamulo Wiseman Mbhiza; I am a Doctor of Philosophy (Education) Candidate at School of Education at the University of the Witwatersrand. I am doing research on Grade 10 rural teachers' discourses and teaching approaches of algebraic functions in Acornhoek, Mpumalanga Province.

My study involves coming into your classroom and observing your Grade 10 mathematics teachers teach. During the classroom observation a video-recorder will be used to record the teacher while teaching. During observations, you may be captured by the video-recorder, therefore I am asking for permission from you to allow me to capture you in the classroom.

I have chosen your class because my study seeks to work with Grade 10 mathematics teachers to gain insight into their discourses and approaches as they teach the subject. You will not be disadvantaged in any way during the course of the study. You may leave the classroom during observations without any penalty. There are no foreseeable risks in participating, and you will not be paid for the study.

Since learners are not the primary participants in the study, I am going to blur your faces in the video after recording to protect your identities and throughout all the writings of the study, you true name will be concealed. Your individual privacy will be maintained in all published and written data resulting from the study. All research data will be destroyed after 5 years of completion of the research.

Please let me know if you require any further information. Thank you very much for your help.

Yours sincerely,

Mbhiza Hlamulo Wiseman
121 Carr Street, Newtown, Johannesburg
Email: wmbhiza@gmail.com
Cell phone: 0769019192

Learner Consent Form

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called: Grade 10 rural teachers' discourses and teaching approaches of algebraic functions in Acornhoek, Mpumalanga Province

My name is: ____

	Circle one
Permission to observe you in class	
I agree to be observed in class.	YES/NO
Permission to be videotaped	
I agree to be videotaped in class.	YES/NO
I know that the videotapes will be used for this project only.	YES/NO

Informed Consent

I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- I can ask not to be videotape
- all the data collected during this study will be destroyed 5 years after completion of the project.

Sign_______________________________ Date____________________________

Thank you very much for your help.

Yours sincerely,

Mbhiza Hlamulo Wiseman
121 Carr Street, Newtown, Johannesburg
Email: wmbhiza@gmail.com
Cell phone: 0769019192

Appendix 9: Letter of PhD candidature

Faculty of Humanities: Education Campus

Room 208/9, Administration Block, 27 St. Andrews Road, Parktown · Tel: +27 11 717-3018 · Fax: 0865532464
E-mail: Thabo.Makuru@wits.ac.za

UNIVERSITY OF THE WITWATERSRAND JOHANNESBURG

PERSON NUMBER: 487440

16 March 2018

Mr Hlamulo Wiseman Mbhiza
Cc: Dr T Nkambule & Dr J Du Plessis

Dear Mr Mbhiza

Results for the Doctor of Philosophy in Education

I am writing to inform you that the Graduate Studies Committee of the Faculty of Humanities, acting on behalf of the Senate has considered your proposal entitled *"Grade 10 rural mathematics teachers' discourses and approaches of teaching algebraic functions in Acornhoek classrooms, Mpumalanga Province, South Africa"* and recommended that you should be admitted to candidature subject to minor corrections suggested by the readers.

Corrections should be done to the satisfaction of the supervisor.

Kindly liaise with the supervisor regarding the content of the reader's report. The report was sent to the supervisor

I confirm that **Dr T Nkambule & Dr J Du Plessis** have been appointed as your supervisors.

Your attention is drawn to the Senate's requirement that all higher degree candidates submit brief written reports on their progress to the Faculty Office once a year.

Please note that higher degree candidates are required to renew their registration in January each year.

Please keep us informed of any changes of address during the year.

Yours sincerely

Thabo Makuru
Faculty Officer
Faculty of Humanities
Education Campus
Tell: 011 717 3018

Appendix 10: Mpumalanga Department of Education Approval letter

education
MPUMALANGA PROVINCE
REPUBLIC OF SOUTH AFRICA

Building No. 5, Government Boulevard, Riverside Park, Mpumalanga Province
Private Bag X11341, Mbombela, 1200;
Tel: 013 766 5552/5115, Toll Free Line: 0800 203 116

Litiko le Temfundvo, Umnyango we Fundo | Departement van Onderwys | Ndzawulo ya Dyondzo

Mr. H.W. Mbhiza
121 Carr Street,
Newtown
JOHHANESBURG
2000

RE: APPLICATION TO CONDUCT RESEARCH: MR H.W. MBHIZA

Your application to conduct research study was received and is therefore acknowledged. The title of your study reads thus: "**Grade 10 rural mathematics teachers' discourses and approaches of teaching algebraic functions in Acornhoek classrooms, Mpumalanga Province, South Africa.**" I trust that the aims and the objectives of the study will benefit the whole department in particular the curriculum division. Your request is approved subject to you observing the provisions of the departmental research policy which is available in the departmental website. You are also requested to adhere to your University's research ethics as spelt out in your research ethics document.

In terms of the research policy, data or any research activity can only be conducted after school hours as per appointment with affected participants. You are also requested to share your findings with the relevant sections of the department so that we may consider implementing your findings if that will be in the best interest of the department. To this effect, your final approved research report (both soft and hard copy) should be submitted to the department so that your recommendations could be implemented. You may be required to prepare a presentation and present at the department's annual research dialogue.

For more information kindly liaise with the department's research unit @ 013 766 5476 or a.baloyi@education.mpu.gov.za.

The department wishes you well in this important project and pledges to give you the necessary support you may need.

MRS M.O.C MHLABANE
HEAD: EDUCATION
25, 6, 18
DATE

MPUMALANGA
THE PLACE OF THE RISING SUN

Appendix 11: Wits School of Education Ethics Approval letter

Wits School of Education

WITS UNIVERSITY

27 St Andrews Road, Parktown, Johannesburg, 2193 Private Bag 3, Wits 2050, South Africa. Tel: +27 11 717-3064 Fax: +27 11 717-3100 E-mail: enquiries@educ.wits.ac.za Website: www.wits.ac.za

07 May 2018

Student Number: 487440

Protocol Number: 2018ECE006D

Dear Hlamulo Wiseman Mbhiza

Application for Ethics Clearance: Doctor of Philosophy

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate has considered your application for ethics clearance for your proposal entitled:

Grade10 rural mathematics teachers' discourses and approaches of teaching algebraic functions in Acornhoek classrooms, Mpumalanga Province, South Africa

The committee recently met and I am pleased to inform you that **clearance was granted**. However, there were a few small issues which the committee would appreciate you attending to before embarking on your research.

The following comments were made:

- The language used in the learner and parent participation information sheets may be too "academic" and should be simplified.

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely,

MMabeta

Wits School of Education

011 717-3416

Cc Supervisor: Dr. Thabisile Nkambule

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Appendix 12: Mafada's VSRI transcript

Interviewer	Interviewee
I am with Mr Mafada at Bash secondary school, we are just going through the video that I recorded him teaching for the video stimulated interview. Sir, just to begin, please tell me about the purpose of this lesson. What was the purpose of the lesson?	The purpose of the lesson was to show progression from straight-line graph, to err parabolic graphs, so the procedures like I indicated earlier on, that we need first to have a table as a similarity from straight-line graphs, then we came up with the table, then we plotted the graph, we plotted the points and joined the graph. So here was for them to see the different shapes of the graph.
Okay, can you recall what motivated you to write these four functions here? y is equals to x, y is equals to x squared, y is equals to one over x and y is equals to x squared there?	Yeah, although this one of y is equals to x squared we have corrected it to say two to the exponent x, actually, here I was showing them the different graphs that we can have from these functions.
And what was the purpose of that, of doing that?	The purpose of that was to actually tell them that actually not all you know, same functions gives the same shape of the graph. So, we wanted also to find the different shapes of the graph, the graphs from the functions that were given or ones that will be given.
So, according to your understanding, what do you need to emphasise here when you are introducing these different functions, y equals to x which is linear, y is equals to x squared which is parabola, y equals to one over x and y equals to two to the exponent of x as you call it, what must learners get from this lesson?	Err ...
What feature of function actually are we focusing on when we look at Grade 10, what do you want them to understand there? Is it only about learners moving from this to the table to the graph?	Yeah, like I have indicated that there are features as well, yeah which perhaps maybe they need to understand, you know also this lesson I also emphasised that we are going to look at some features about where the graph starts and ends, talking about the minima and the maxima, yeah so those are also the features that we will be talking to so that they see different features. As we proceed again, there will be features like asymptotes where its line where the graph does not touches, but it comes closer to it, so those features that learners you know need to differentiate as we move from one type of function into another.
And whatelse do you think is missing there, on those equations?	What do I think is missing?
Yeah	Mmm, err for now I can't tell.
Or rather let me say, err for the linear function, what is the general linear equation for a linear function?	Err, that one is y is equals to m x plus c

Appendix 13: Horizontalisation of Mafada's lesson transcript

Time (in minutes)	Episode	Observable actions	What is said?
00:00 – 04:00	Broad introduction	 • Writing the different equations representing 4 different families of functions. • Naming of the written functions and stating the order in which he is going to teach them (Horizon Content Knowledge). • Mafada forgot the name of the function represented by $y = x^2$ and started to page through the textbook before saying that it is a parabola (Teacher Content Knowledge, lesson preparation) • He spelt hyperbola as "hyberbolla" (instantaneous reflection) • Mafada repeated the equation $y = x^2$ stated that number 4 represents the exponential function (instantaneous reflection)  • Mafada tells the learners the difference between the dependent and independent variable.	As covered in the last lesson, the equation $y = x$ is a function because there is only one value of x. We said this one is gonna allow us to draw … we have got four types of functions, but that one is gonna allow us to draw a straight-line graph. Then, we start again with another type of function (writing on the board $y = x^2$). That one represents (he reaches for a textbook to remind himself of the type of function $y = x^2$ represents), we said that one represents (paging through the textbook), represents a parabola. So, we haven't started with this one. We said the third one is where y is equals to one over x, and what did we say that one? We said that one represents (reaching for a textbook), we said that one represents a hyperbola, we haven't started with the parabola, I said this one will be our third lesson. The fourth one is gonna be err, err an exponential function neh (checking the textbook), about the exponential function, what did we talk about? (paging through the textbook) we said y is equals to x squared (wrong). So, this one is gonna be our exponential function, for our fourth lesson. We said for us to draw that function (pointing at the equation for a linear function y=x), we said we have this variable (pointing at x), we said y is a variable (writing on the board), we said y is a