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A new relaxed inertial method for solving monotone inclusion problem

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Abstract

We propose a relaxed inertial-Tseng method for solving monotone inclusion problem in a real Hilbert. The newly proposed inertial condition is relaxed and easy to implement, and its upper bound is easy to obtain unlike many conditions in the literature. We obtain a weak convergence result of the proposed method under weaker conditions and we present a series of numerical computations involving our newly suggested method, compared with established methods in the literature. In addition, the theoretical result is applied to image restoration problems. Notably, our findings enhance, broaden, and advance various preexisting outcomes available in scholarly discourse.

Keywords Relaxed inertial term · Monotone inclusion problem · Weak convergence · Inertial condition

Mathematics Subject Classification 47H06 · 47H09 · 47J05 · 47J25

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1 Introduction

In this paper, let H be a real Hilbert space and $T : H \to H$ be an operator. Then T is called

(a) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|.$$

(b) monotone if

$$\langle Tx - Ty, x - y\rangle \ge 0, \quad \forall, x, y \in H;$$

(c) inverse strongly monotone if there exists $L > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge L\|Tx - Ty\|^2 \ \forall, x, y \in H.$$

For a set-valued operator $B : H \to 2^H$, the graph of B is the set $Gra(B) := \{(x, y) \in H \times H : y \in Bx\}$. The operator B is called monotone if for all

$$(x, y), (u, v) \in Gra(B), \langle x - u, y - v\rangle \ge 0,$$

and maximal monotone if $Gra(B)$ is not properly contained in the graph of any other monotone operator. In other words, B is called a maximal monotone operator if for $(x, u) \in H \times H$, we have that $\langle u - v, x - y\rangle \ge 0$ for all $(y, v) \in Gra(B)$ implies $u \in Bx$. The resolvent of B is the operator $J_\lambda^B : H \to 2^H$ (with parameter $\lambda > 0$) defined by

$$J_\lambda^B(x) = (I + \lambda B)^{-1}(x).$$

If B is maximal monotone, then it follows from [10, Proposition 23.10] and [12, Remark 2.7 (ii)] that J_λ^B is single-valued and everywhere defined on H (i.e., the domain of J_λ^B is H). If A is single-valued, then $(I - \lambda A)$ is also single-valued for any $\lambda > 0$. Thus, the composition $J_\lambda^B(I - \lambda A)$ is also single-valued. Furthermore, we have that, for any $x \in H$, there exists a unique $z \in H$ such that $z = (I - \lambda A)(x)$. This implies that $(I - \lambda A)$ is everywhere defined on H. Hence, the composition $J_\lambda^B(I - \lambda A)$ is also everywhere defined on H.
Let $A : H \to H$ be a Lipschitz continuous monotone operator and $B : H \to 2^H$ be a maximal monotone operator. The Monotone Inclusion Problem (MIP) is defined as follows:

$$\text{Find } x \in H \text{ such that } 0 \in (A + B)x. \tag{1.1}$$

We denote the solution set of (1.1) by Ω. The MIP plays a central role in numerous crucial applied mathematics ideas, including convex minimization, split feasibility, fixed point, saddle point, variational inequality, and equilibrium problems. It simulates a wide range of problems in multiple practical sciences and engineering fields, including signal processing, optimal control, image reconstruction, statistical learning, machine learning, quantum physics, filtration theory, and others (see, for example, [2, 5, 14, 17, 22, 24, 25] and the references therein). Many scholars have examined the MIP and developed several methods for finding its solutions. In particular, the forward-backward splitting method (FBSM) was introduced and studied by the authors in [22, 28]. The iterative method is given as

$$x_{n+1} = J_\lambda^B(x_n - \lambda Ax_n) \ \forall n \in \mathbb{N}, \tag{1.2}$$

where $\lambda \in (0, \frac{2}{L})$. The sequence generated by the iterative method (1.2) converges weakly to a point in Ω under the assumption that A is $\frac{1}{L}$-inverse strongly monotone. This inverse strongly monotonicity assumption is the major drawback of this method.

To overcome this drawback, Tseng in [33] introduced and studied the forward-backward-forward splitting method (FBFSM). The iterative method is given as:

$$\begin{cases} y_n &= J_\lambda^B(x_n - \lambda A x_n), \\ x_{n+1} &= y_n - \lambda(Ay_n - Ax_n)\ \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\lambda \in (0, \frac{1}{L})$. Tseng [33] established the weak convergence of method (1.3) to a point in Ω under the assumptions that A is a Lipschitz continuous monotone operator and B is maximal monotone. A Lipschitz continuous monotone operator is much more general than an inverse strongly monotone operator.

The concept of inertial extrapolation was first introduced and studied in 1964 by Polyak in [29] as a technique for accelerating the process of solving the smooth convex minimization problem. The inertial technique which is based upon a discrete analogue of a second-order dissipative dynamical system, is also known for its efficiency in improving the convergence speed of iterative algorithms. It originates from the heavy ball method of the second order dynamical system for minimizing a smooth convex function f:

$$\frac{d^2x}{dt^2}(t) + \gamma \frac{dx}{dt}(t) + \nabla f(x(t)) = 0,$$

where $\gamma > 0$ is a damping or friction parameter and ∇f is the gradient of f. Alvarez and Attouch [1] extended Polyak's heavy ball method to the setting of a general maximal monotone operator. Since then, researchers have used this technique to improve the rate of convergence of different iterative processes. Also, numerous authors have improved, expanded and generalized the inertial extrapolation method since its inception, see [3–6, 11, 20, 23, 31, 34] and the references therein. On the other hand, relaxation techniques have proven to be a good technique for better rate of convergence. It is well-known that combining the inertial and the relaxation techniques improves and gives a better rate of convergence when compared with just relaxation technique or the inertial technique [8, 18, 19]. In recent times, notable inertial conditions have been proposed; they include Moudafi [26]:

$$y_n := x_n + \theta_n(x_n - x_{n-1}),$$

where the inertial factor θ_n adheres to

$$0 \le \theta_n \le \theta < 1, \quad \text{for all } n \ge 1, \quad \sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty. \tag{1.4}$$

It is easy to see that the above condition requires the knowledge of the iterates x_n and x_{n-1}, which are a priori unknown. However, practicality is achieved by utilizing a suitable online rule: $0 \le \theta_n \le \bar{\theta}_n$, where $\bar{\theta}_n$ is determined by

$$\bar{\theta}_n = \begin{cases} \min\{\epsilon_n \|x_n - x_{n-1}\|^{-1}, \theta\} & \text{if } x_n \ne x_{n-1}, \\ \theta & \text{otherwise}, \end{cases} \tag{1.5}$$

with $\sum_{n=1}^{\infty} \epsilon_n < \infty$ and $\theta \in [0, 1)$. Note that the online rule (1.5) also depends on the knowledge of the iterates. It was also considered in [30, 34].

In [11], a condition for the inertial factor θ_n was introduced and studied as follows:

$$0 = \theta_1 \le \theta_n \le \theta_{n+1} \le \theta < 1, \quad \text{for all } n \ge 1, \tag{1.6}$$

$$\alpha > \frac{\theta^2(1+\theta)+\theta\sigma}{1-\theta^2}, \quad 0 < \liminf_{n\to\infty} \tau_n \leq \tau_n \leq \frac{\alpha(1-\theta^2)-\theta^2-\theta^3-\theta\sigma]}{\alpha[1+\theta(1+\theta)+\alpha\theta+\sigma]}, \tag{1.7}$$

where $\sigma, \alpha > 0$. Unlike (1.4) and (1.5), condition (1.7) does not rely on information from the iterates but on the relaxation factor τ_n and other known parameters. However, obtaining an upper bound for the inertial sequence remains complex even with the knowledge of τ_n. For more details, see [15, Remark 3.2].

Motivated by the above work and the work in [13, 15], the primary objective of this paper is to consider an inertial factor with conditions that solely depend only on the relaxation factor τ_n, where determining the upper bound of the inertial factor is relatively easy. By combining the relaxation and inertial techniques, we introduce a new approach for solving MIP (1.1) under some mild assumptions. We establish the weak convergence of the proposed method to a solution of the MIP (1.1). We provide numerical experiments and compare our method with other relevant methods in the literature.

The rest of this paper is organized as follows: In Sect. 2, we recall some useful definitions and results that are relevant to our study. In Sect. 3, we present our proposed method. In Sect. 4, we establish the weak convergence of our method and in Sect. 5, we present numerical examples of our proposed method in comparison with other methods in the literature. Lastly, in Sect. 6, we give the conclusion of this paper.

2 Preliminaries

In this section, we begin by recalling some known and useful results which are needed in the sequel. For any $x, y \in H$ and $\alpha \in \mathbb{R}$, it is well-known that

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2. \tag{2.1}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.2}$$

Lemma 2.1 *[1] Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be non-negative sequences and suppose that*

$$\alpha_{n+1} \leq \alpha_n + \beta_n(\alpha_n - \alpha_{n-1}) + \gamma_n,$$

where $0 \leq \beta_n \leq \beta < 1$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then $\lim_{n\to\infty} \alpha_n$ exists.

Lemma 2.2 *[27] Let $\mathcal{C}$ be a nonempty subset of H and let $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

(a) for each $p \in \mathcal{C}$, $\lim_{n\to\infty} \|x_n - p\|$ exists;
(b) every sequential weak cluster point of $\{x_n\}$ belongs to $\mathcal{C}$.

Then, $\{x_n\}$ converges weakly to a point in $\mathcal{C}$.

Lemma 2.3 *[21] Let H be a real Hilbert space, $A : H \to H$ be a monotone and Lipschitz continuous operator, and $B : H \to 2^H$ be a maximal monotone operator. Then the operator $(A+B) : H \to 2^H$ is maximal monotone.*

3 Proposed method

In this section, we present our proposed method for solving the MIP (1.1).

Assumption 3.1 Suppose that:

(a) $\Omega \neq \emptyset$,
(b) A is monontone and Lipschitz continuous with constant $L > 0$.
(c) B is maximal monotone on H.

Assumption 3.2

For all $n \geq 1$ and sufficiently small $\epsilon > 0$, let $0 = \theta_1 \leq \theta_n \leq \theta_{n+1}$ and:

(1) $\theta_{n+1} \leq \alpha_n$ if $\tau_n \in (0, \frac{1}{2})$, $\frac{1}{\tau_n} - \frac{1}{\tau_{n+1}} + 3 > 0$, where

$$\alpha_n := \frac{1}{2} \frac{\tau_{n+1}}{(1 - 2\tau_{n+1})} (\frac{1}{\tau_n} + \frac{1}{\tau_{n+1}} - 1 - \eta_n) \tag{3.1}$$

with

$$\eta_n := \sqrt{(\frac{1}{\tau_n} + \frac{1}{\tau_{n+1}} - 1)^2 - 4(\frac{1}{\tau_n} - 1 - \epsilon)(\frac{1}{\tau_{n+1}} - 2)}; \tag{3.2}$$

(2) $\theta_{n+1} \leq \frac{1-\epsilon}{3}$, if $\tau_n \equiv 0.5$;
(3) $\theta_{n+1} \leq \sqrt{\rho_n^2 + \phi_n} - \rho_n$ if $\tau_n \in (\frac{1}{2}, 1 - \epsilon]$, where

$$\rho_n = \frac{1}{2} \frac{\tau_{n+1}}{(2\tau_{n+1} - 1)} (\frac{1}{\tau_n} + \frac{1}{\tau_{n+1}} - 1) \tag{3.3}$$

and

$$\phi_n = \frac{\tau_{n+1}}{(2\tau_{n+1} - 1)} (\frac{1}{\tau_n} - \epsilon - 1). \tag{3.4}$$

Algorithm 3.3 *Relaxed inertial method.*

Step 0: *Choose the sequences* $\{\theta_n\}$ *and* $\{\tau_n\}$ *in* $[0, 1]$ *and* $(0, 1)$, *respectively, and let* $x_0, x_1 \in H$ *be given arbitrarily. Let* $\lambda_1 > 0$, $\mu \in (0, 1)$ *and set* $n := 1$.
Step 1: *Given the current iterates* x_{n-1} *and* x_n $(n \geq 1)$, *set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}),$$

and calculate

$$y_n = J_{\lambda_n}^B(w_n - \lambda_n A w_n).$$

If $y_n = w_n$ *stop. Otherwise go to step 2.*
Step 2: *Compute* $z_n = y_n + \lambda_n(Aw_n - Ay_n)$
and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}\right\}, & \textit{if } Aw_n \neq Ay_n \\ \lambda_n, & \textit{otherwise}. \end{cases} \tag{3.5}$$

Step 3: *Compute*

$$x_{n+1} = (1-\tau_n)w_n + \tau_n z_n.$$

Set $n := n+1$ and go back to **Step 1**.

Remark 3.4 (1) In comparison with the assumptions in [26], Assumption 3.2 does not require the knowledge of the iterates.

(2) Also, in comparison to [34], the choice of θ_n is relaxed and its upper bound is easy to obtain; once τ_n is chosen, it becomes very easy to compute θ_n. The knowledge of the upper bound simplifies parameter selection. The upper bound allows for simplified conditions on the inertial sequence, thus, reducing the complexity of choosing the inertial sequence. In other words, with the knowledge of the upper bound of the inertial sequence, it becomes easier to choose the inertial factors. This enables easier implementation since the inertial sequence can be dynamically selected without requiring information about the iterates or the complexity of the coefficient τ_n [15].

(3) By (3.5), $\lim\limits_{n\to\infty} \lambda_n = \lambda$, where $\lambda \in [\min\{\mu L^{-1}, \lambda_1\},\ \lambda_1]$.

Remark 3.5 Clearly, $1 - 2\tau_{n+1} > 0$ in Assumption 3.2 (1) since $\tau_{n+1} < \frac{1}{2}$. Similarly, $2\tau_{n+1} - 1 > 0$ in Assumption 3.2 (3). Also, we have that

$$\begin{aligned}(\frac{1}{\tau_n} + \frac{1}{\tau_{n+1}} - 1)^2 - 4(\frac{1}{\tau_{n+1}} - 2)(\frac{1}{\tau_n} - 1 - \epsilon) &= (\frac{1}{\tau_n} - \frac{1}{\tau_{n+1}})^2 + 6\frac{1}{\tau_n} \\ &\quad + 2\frac{1}{\tau_{n+1}} + 4(\frac{1}{\tau_{n+1}} - 2)\epsilon - 7 \\ &> 0. \end{aligned} \tag{3.6}$$

Thus, Assumption 3.1 is valid.

4 Convergence analysis

Lemma 4.1 *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3 under Assumption 3.1 and Assumption 3.2. Then $\lim\limits_{n\to\infty} \|x_n - p\|$ exists for all $p \in \Omega$.*

Proof Let $p \in \Omega$. To achieve the above result, we divide our proofs into claims.
Claim 1: We claim that

$$\Psi_{n+1} \leq \Psi_n - \epsilon\|x_{n+1} - x_n\|,$$

where $\Psi_n = \|x_n - p\|^2 - \theta_n\|x_{n-1} - p\|^2 + \psi_n\|x_n - x_{n-1}\|^2$ and $\psi_n = (1+\theta_n)\theta_n + \frac{1}{\tau_n}(1-\tau_n)(1-\theta_n)\theta_n$. Using the definition of y_n, we get $(w_n - \lambda_n A w_n) \in (y_n + \lambda_n B y_n)$. Thus there exists $u_n \in By_n$ such that

$$\frac{1}{\lambda_n}(w_n - y_n - \lambda_n A w_n) = u_n. \tag{4.1}$$

We also have that $0 \in (A+B)p$ and $Ay_n + u_n \in (A+B)y_n$. Thus by the monotonicity of $(A+B)$, we obtain

$$\langle Ay_n + \frac{1}{\lambda_n}(w_n - y_n) - Aw_n, y_n - p\rangle \geq 0,$$

this implies that

$$\langle \lambda_n(Ay_n - Aw_n) - (y_n - w_n), y_n - p\rangle \geq 0. \tag{4.2}$$

Also, since $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \lambda_{n+1}$ (by Remark 3.4), we have

$$\lim_{n\to\infty} \left(1 - \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2}\right) = 1 - \mu^2 > 0.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2} > 0 \ \forall n \geq n_0. \tag{4.3}$$

Now, using (2.1), (3.5), (4.2) and (4.3), we get

$$\begin{aligned}
\|z_n - p\|^2 &= \|y_n + \lambda_n(Aw_n - Ay_n) - p\|^2\\
&= \|y_n - p\|^2 - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle + \lambda_n^2\|Ay_n - Aw_n\|^2\\
&\leq \|y_n - p\|^2 - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle + \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2\\
&= \|y_n - w_n + w_n - p\|^2 - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle + \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2\\
&= \|y_n - w_n\|^2 + 2\langle y_n - w_n, w_n - p\rangle + \|w_n - p\|^2 - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle\\
&\quad + \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2\\
&= \|y_n - w_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle + 2\langle y_n - w_n, y_n - p\rangle + \|w_n - p\|^2\\
&\quad - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle + \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2\\
&= \|w_n - p\|^2 - \|y_n - w_n\|^2 + 2\langle y_n - w_n, y_n - p\rangle - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle\\
&\quad + \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2\\
&= \|w_n - p\|^2 - 2\langle \lambda_n(Ay_n - Aw_n) - (y_n - w_n), y_n - p\rangle - (1 - \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|y_n - w_n\|^2\\
&\leq \|w_n - p\|^2 - (1 - \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|y_n - w_n\|^2\\
&\leq \|w_n - p\|^2 \ \forall n \geq n_0.
\end{aligned} \tag{4.4}$$

By (2.2) and (4.4), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1 - \tau_n)w_n + \tau_n z_n - p\|^2\\
&= \|(1 - \tau_n)(w_n - p) + \tau_n(z_n - p)\|^2\\
&= (1 - \tau_n)\|w_n - p\|^2 + \tau_n\|z_n - p\|^2 - \tau_n(1 - \tau_n)\|z_n - w_n\|^2\\
&\leq (1 - \tau_n)\|w_n - p\|^2 + \tau_n\|w_n - p\|^2 - \tau_n(1 - \tau_n)\|z_n - w_n\|^2\\
&= \|w_n - p\|^2 - \tau_n(1 - \tau_n)\|z_n - w_n\|^2.
\end{aligned} \tag{4.5}$$

Using (2.2), we get

$$
\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|(1+\theta_n)(x_n - p) - \theta_n(x_{n-1} - p)\|^2 \\
&= (1+\theta_n)\|x_n - p\|^2 - \theta_n\|x_{n-1} - p\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2. \qquad (4.6)
\end{aligned}
$$

From $x_{n+1} = (1-\tau_n)w_n + \tau_n z_n$, we obtain

$$
\|z_n - w_n\|^2 = \frac{1}{\tau_n^2}\|x_{n+1} - w_n\|^2.
$$

Hence, we have

$$
\begin{aligned}
\|z_n - w_n\|^2 &= \frac{1}{\tau_n^2}\|x_{n+1} - (x_n + \theta_n(x_n - x_{n-1}))\|^2 \\
&= \frac{1}{\tau_n^2}\|x_{n+1} - x_n - (x_n - x_{n-1}) + (1-\theta_n)(x_n - x_{n-1})\|^2 \\
&= \frac{1}{\tau_n^2}\|x_{n+1} - 2x_n + x_{n-1}\|^2 + \frac{1}{\tau_n^2}(1-\theta_n)^2\|x_n - x_{n-1}\|^2 \\
&\quad + 2\frac{1}{\tau_n^2}(1-\theta_n)\langle x_{n+1} - 2x_n + x_{n-1}, x_n - x_{n-1}\rangle \\
&= \frac{1}{\tau_n^2}\|x_{n+1} - 2x_n + x_{n-1}\|^2 + \frac{1}{\tau_n^2}(1-\theta_n)^2\|x_n - x_{n-1}\|^2 \\
&\quad + \frac{1}{\tau_n^2}(1-\theta_n)[\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 - \|x_{n+1} - 2x_n + x_{n-1}\|^2] \\
&= \frac{1}{\tau_n^2}\theta_n\|x_{n+1} - 2x_n + x_{n-1}\|^2 + \frac{1}{\tau_n^2}(1-\theta_n)^2\|x_n - x_{n-1}\|^2 \\
&\quad + \frac{1}{\tau_n^2}(1-\theta_n)[\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2] \\
&\geq \frac{1}{\tau_n^2}(1-\theta_n)^2\|x_n - x_{n-1}\|^2 + \frac{1}{\tau_n^2}(1-\theta_n)[\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2]. \qquad (4.7)
\end{aligned}
$$

Putting (4.7),(4.6) into (4.5), we get

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1+\theta_n)\|x_n - p\|^2 - \theta_n\|x_{n-1} - p\|^2 + (1+\theta_n)\theta_n\|x_n - x_{n-1}\|^2 \\
&\quad - \tau_n(1-\tau_n)\left(\frac{1}{\tau_n^2}(1-\theta_n)^2\|x_n - x_{n-1}\|^2 + \frac{1}{\tau_n^2}(1-\theta_n)[\|x_{n+1} - x_n\|^2 - \|x_n - x_{x-1}\|^2]\right) \\
&= \|x_n - p\|^2 + \theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2) \\
&\quad - \frac{1}{\tau_n}(1-\tau_n)(1-\theta_n)\|x_{n+1} - x_n\|^2 + \psi_n\|x_n - x_{n-1}\|^2, \qquad (4.8)
\end{aligned}
$$

where $\psi_n = (1+\theta_n)\theta_n + \frac{1}{\tau_n}(1-\tau_n)(1-\theta_n)\theta_n$. This implies that

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 - \|x_n - p\|^2 - \theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2) - \psi_n\|x_n - x_{n-1}\|^2 + \psi_{n+1}\|x_{n+1} - x_n\|^2 \\
&\quad \leq -(\frac{1}{\tau_n}(1-\tau_n)(1-\theta_n) - \psi_{n+1})\|x_{n+1} - x_n\|^2. \qquad (4.9)
\end{aligned}
$$

Using (4.9) and the fact that $\theta_n \le \theta_{n+1}$, we get

$$
\begin{aligned}
-(\frac{1}{\tau_n}(1-\tau_n)(1-\theta_n)-\psi_{n+1})\|x_{n+1}-x_n\|^2 \ge{}& \|x_{n+1}-p\|^2 \\
&- \|x_n-p\|^2 - \theta_n(\|x_n-p\|^2-\|x_{n-1}-p\|^2) \\
&- \psi_n\|x_n-x_{n-1}\|^2 + \psi_{n+1}\|x_{n+1}-x_n\|^2 \\
\ge{}& \|x_{n+1}-p\|^2 - \|x_n-p\| - \theta_{n+1}\|x_n-p\|^2 \\
&+ \theta_n\|x_{n-1}-p\|^2 - \psi_n\|x_n-x_{n-1}\|^2 \\
&+ \psi_{n+1}\|x_{n+1}-x_n\|^2, \qquad (4.10)
\end{aligned}
$$

which implies that

$$
\Psi_{n+1} \le \Psi_n - \left(\frac{1}{\tau_n}(1-\tau_n)(1-\theta_n)-\psi_{n+1}\right)\|x_{n+1}-x_n\|^2, \qquad (4.11)
$$

where $\Psi_n = \|x_n-p\|^2 - \theta_n\|x_{n-1}-p\|^2 + \psi_n\|x_n-x_{n-1}\|^2$.
Now, note that for $\epsilon > 0$, we have

$$
\begin{aligned}
&\frac{1}{\tau_n}(1-\tau_n)(1-\theta_n)-\psi_{n+1}-\epsilon \\
&= \frac{1}{\tau_n}(1-\tau_n)(1-\theta_n)-\theta_{n+1}-\theta_{n+1}^2+\frac{1}{\tau_{n+1}}(1-\tau_{n+1})(\theta_{n+1}-1)\theta_{n+1}-\epsilon \\
&\ge \frac{1}{\tau_n}(1-\tau_n)(1-\theta_{n+1})-\theta_{n+1}-\theta_{n+1}^2+\frac{1}{\tau_{n+1}}(1-\tau_{n+1})(\theta_{n+1}-1)\theta_{n+1}-\epsilon \\
&= -(2-\frac{1}{\tau_{n+1}})\theta_{n+1}^2 - \left(\frac{1}{\tau_n}+\frac{1}{\tau_{n+1}}-1\right)\theta_{n+1}+\frac{1}{\tau_n}-1-\epsilon. \qquad (4.12)
\end{aligned}
$$

We now consider three cases in respect to τ_n: **Case 1:** If $\tau_n \in (0, \frac{1}{2})$. Then from $\theta_{n+1} \le \frac{1}{2}\frac{\tau_{n+1}}{(1-2\tau_{n+1})}(\frac{1}{\tau_n}+\frac{1}{\tau_{n+1}}-1-\eta_n)$, we have

$$
\eta_n \le \frac{1}{\tau_n}+\frac{1}{\tau_{n+1}}-1-2(\frac{1}{\tau_{n+1}}-2)\theta_{n+1}.
$$

By Remark 3.5, we have $\eta_n > 0$. Thus,

$$
\begin{aligned}
\eta_n^2 &\le \left(\frac{1}{\tau_n}+\frac{1}{\tau_{n+1}}-1-(2(\frac{1}{\tau_{n+1}}-2)\theta_{n+1})\right)^2 \\
&= \left(\frac{1}{\tau_n}+\frac{1}{\tau_{n+1}}-1\right)^2 - 4\left(\frac{1}{\tau_n}+\frac{1}{\tau_{n+1}}-1\right)(\frac{1}{\tau_{n+1}}-2)\theta_{n+1}+4(\frac{1}{\tau_{n+1}}-2)^2\theta_{n+1}^2.
\end{aligned}
$$

That is

$$
\begin{aligned}
&\left(\tfrac{1}{\tau_n}+\tfrac{1}{\tau_{n+1}}-1\right)^2 - 4(\tfrac{1}{\tau_n}-1-\epsilon)(\tfrac{1}{\tau_{n+1}}-2) \\
&\le \left(\tfrac{1}{\tau_n}+\tfrac{1}{\tau_{n+1}}-1\right)^2 - 4(\tfrac{1}{\tau_n}+\tfrac{1}{\tau_{n+1}}-1)(\tfrac{1}{\tau_{n+1}}-2)\theta_{n+1}+4(\tfrac{1}{\tau_{n+1}}-2)^2\theta_{n+1}^2.
\end{aligned}
$$

Hence,

$$
-\left(\frac{1}{\tau_n}+\frac{1}{\tau_{n+1}}-1\right)(\frac{1}{\tau_{n+1}}-2)\theta_{n+1}+\left(\frac{1}{\tau_{n+1}}-2\right)^2\theta_{n+1}^2+(\frac{1}{\tau_n}-1-\epsilon)(\frac{1}{\tau_{n+1}}-2) \ge 0 \qquad (4.13)
$$

Since $\frac{1}{\tau_{n+1}} - 2 > 0$, we have

$$-\left(\frac{1}{\tau_n} + \frac{1}{\tau_{n+1}} - 1\right)\theta_{n+1} + \left(\frac{1}{\tau_{n+1}} - 2\right)\theta_{n+1}^2 + (\frac{1}{\tau_n} - 1 - \epsilon) \geq 0. \tag{4.14}$$

That is

$$-(2 - \frac{1}{\tau_{n+1}})\theta_{n+1}^2 - (\frac{1}{\tau_n} + \frac{1}{\tau_{n+1}} - 1)\theta_{n+1} + \frac{1}{\tau_n} - 1 - \epsilon \geq 0. \tag{4.15}$$

Using (4.15) in (4.12), we get

$$\begin{aligned}\frac{1}{\tau_n}(1-\tau_n)(1-\theta_n) - \psi_{n+1} - \epsilon &\geq -(2 - \frac{1}{\tau_{n+1}})\theta_{n+1}^2 \\ &\quad - (\frac{1}{\tau_n} + \frac{1}{\tau_{n+1}} - 1)\theta_{n+1} + \frac{1}{\tau_n} - 1 - \epsilon \geq 0,\end{aligned} \tag{4.16}$$

which implies

$$-(\frac{1}{\tau_n}(1-\tau_n)(1-\theta_n) - \psi_{n+1}) \leq -\epsilon. \tag{4.17}$$

Case 2: Suppose $\tau_n \equiv 0.5$. Then from (4.12) and Assumption 3.2 (2), we obtain

$$\frac{1}{\tau_n}(1-\tau_n)(1-\theta_n) - \psi_{n+1} - \epsilon \geq -3\theta_{n+1} + 1 - \epsilon \geq 0. \tag{4.18}$$

That is,

$$-\left(\frac{1}{\tau_n}(1-\tau_n)(1-\theta_n) - \psi_{n+1}\right) \leq -\epsilon. \tag{4.19}$$

Case 3: Suppose $\tau_n \in (0.5, 1-\epsilon]$. Then from the condition $\theta_{n+1} \leq \sqrt{\rho_n^2 + \phi_n} - \rho_n$, we have

$$(\theta_{n+1} + \rho_n)^2 \leq \rho_n^2 + \phi_n, \tag{4.20}$$

which implies that

$$\theta_{n+1}^2 + 2\rho_n\theta_{n+1} - \phi_n \leq 0. \tag{4.21}$$

Now using (3.3) and (3.4), we have

$$\theta_{n+1}^2 + \frac{1}{2 - \tau_{n+1}^{-1}}\left(\frac{1}{\tau_n} + \frac{1}{\tau_{n+1}} - 1\right)\theta_{n+1} - \frac{1}{2 - \tau_{n+1}^{-1}}\left(\frac{1}{\tau_n} - 1 - \epsilon\right) \leq 0. \tag{4.22}$$

Thus,

$$(2 - \frac{1}{\tau_{n+1}})\theta_{n+1}^2 - \left(\frac{1}{\tau_n} + \frac{1}{\tau_{n+1}} - 1\right)\theta_{n+1} - \left(\frac{1}{\tau_n} - 1 - \epsilon\right) \leq 0, \tag{4.23}$$

which implies

$$-(2 - \frac{1}{\tau_{n+1}})\theta_{n+1}^2 - \left(\frac{1}{\tau_n} - \frac{1}{\tau_{n+1}} - 1\right)\theta_{n+1} + \left(\frac{1}{\tau_n} - 1 - \epsilon\right) \geq 0. \tag{4.24}$$

Hence, by (4.12), we have

$$-\left(\frac{1}{\tau_n}(1-\tau_n)(1-\theta_n) - \psi_{n+1}\right) \leq -\epsilon.$$

Therefore, from all the three cases, we have that

$$-\left(\frac{1}{\tau_n}(1-\tau_n)(1-\theta_n)-\psi_{n+1}\right)\leq -\epsilon. \tag{4.25}$$

Now, using this and (4.11), we have

$$\Psi_{n+1}\leq \Psi_n-\epsilon\|x_{n+1}-x_n\|^2.$$

Thus, the proof of Claim 1 is complete.
Claim 2: We claim that $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in\Omega$.
Indeed, by Claim 1, we see that $\{\Psi_n\}$ is non-increasing. Now, since

$$\Psi_n=\|x_n-p\|^2-\theta_n\|x_{n-1}-p\|^2+\psi_n\|x_n-x_{n-1}\|^2,$$

we get

$$\|x_n-p\|^2-\theta_n\|x_{n-1}-p\|^2\leq\Psi_n\leq\Psi_1.$$

This implies that

$$\begin{aligned}\|x_n-p\|^2&\leq\theta_n\|x_{n-1}-p\|^2+\Psi_1\\&\leq\theta\|x_{n-1}-p\|^2+\Psi_1 \text{ (for some } \theta<1\text{, since } \theta_n<1)\\&\leq\theta(\theta\|x_{n-2}-p\|^2+\Psi_1)+\Psi_1\\&=\theta^2\|x_{n-2}-p\|^2+\theta\Psi_1+\Psi_1\\&\vdots\\&\leq\theta^n\|x_1-p\|^2+\theta^{n-1}\Psi_1+\cdots+\theta\Psi_1+\Psi_1\\&\leq\theta^n\|x_1-p\|^2+\frac{\Psi_1}{1-\theta}.\end{aligned}\tag{4.26}$$

Hence, for $k\leq n-1$, we obtain from Claim 1 that

$$\begin{aligned}\epsilon\sum_{k=1}^{n-1}\|x_{k+1}-x_k\|^2&\leq\Psi_1-\Psi_n\\&\leq\Psi_1+\theta_n\|x_{n-1}-p\|^2\\&\leq\Psi_1+\theta^{n-1}\|x_1-p\|^2+\frac{\Psi_1}{1-\theta}.\end{aligned}\tag{4.27}$$

Thus, we have that $\sum_{k=1}^{n-1}\|x_{k+1}-x_k\|^2$ is bounded for all n. Hence,

$$\sum_{n=1}^{+\infty}\|x_{n+1}-x_n\|^2<\infty. \tag{4.28}$$

Now, from (4.8), we obtain

$$\|x_{n+1}-p\|^2\leq\|x_n-p\|^2+\theta_n(\|x_n-p\|^2-\|x_{n-1}-p\|^2)+\psi_n\|x_n-x_{n-1}\|^2.$$

From (4.28) and Lemma 2.1, we have that $\lim_{n\to\infty}\|x_n-p\|$ exists. This implies that $\{x_n\}$ is bounded. □

Theorem 4.2 *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3 under Assumption 3.1 and Assumption 3.2 such that $\liminf_{n\to\infty} \tau_n > 0$. Then $\{x_n\}$ converges weakly to $p \in \Omega$.*

Proof From (4.28), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.29}$$

Thus, we get (noting that $\theta_n \leq 1$) that

$$\|x_n - w_n\| = \theta_n \|x_n - x_{n-1}\| \leq \|x_n - x_{n-1}\| \to 0 \text{ as } n \to \infty. \tag{4.30}$$

Also, using (4.29) and (4.30), we have

$$\|x_{n+1} - w_n\| \leq \|x_{n+1} - x_n\| + \|x_n - w_n\| \to 0 \text{ as } n \to \infty. \tag{4.31}$$

Thus, we have (noting that $\liminf_{n\to\infty} \tau_n > 0$) that

$$\|z_n - w_n\| = \frac{1}{\tau_n}\|x_{n+1} - w_n\| \to 0 \text{ as } n \to \infty. \tag{4.32}$$

From (4.4), we have

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - (1 - \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|y_n - w_n\|^2, \tag{4.33}$$

which implies that

$$\begin{aligned}(1 - \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|y_n - w_n\|^2 &\leq \|w_n - p\|^2 - \|z_n - p\|^2\\ &= \Big(\|w_n - p\| + \|z_n - p\|\Big)\Big(\|w_n - p\| - \|z_n - p\|\Big)\\ &\leq \Big(\|w_n - p\| + \|z_n - p\|\Big)\|z_n - w_n\|. \end{aligned}\tag{4.34}$$

Using (4.32), and noting that $\lim_{n\to\infty}\left(1 - \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2}\right) = 1 - \mu^2 > 0$, we obtain

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{4.35}$$

Using the above limit, it is easy to see that

$$\|x_n - y_n\| \leq \|x_n - w_n\| + \|w_n - y_n\| \to 0 \text{ as } n \to \infty, \tag{4.36}$$

and

$$\|Aw_n - Ay_n\| \leq L\|w_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{4.37}$$

Now, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $x^* \in H$. From the fact that $\lim_{n\to\infty} \|y_n - x_n\| = 0$ (see (4.36)), we have that $\{y_{n_k}\}$ converges weakly to x^*. Next, we claim that $x^* \in \Omega$. Indeed, let $(u, v) \in Gra(A + B)$. Then $v - Au \in Bu$. Also, it follows from (4.1) that $\frac{1}{\lambda_{n_k}}(w_{n_k} - \lambda_{n_k} Aw_{n_k} - y_{n_k}) \in By_{n_k}$. Thus, by the monotonicity of B, we obtain

$$\langle u - y_{n_k}, v - Au - \frac{1}{\lambda_{n_k}}(w_{n_k} - \lambda_{n_k} Aw_{n_k} - y_{n_k})\rangle \geq 0. \tag{4.38}$$

Using the monotonicity of A, we obtain

$$\begin{aligned}\langle u - y_{n_k}, v\rangle &\geq \langle u - y_{n_k}, Au + \frac{1}{\lambda_{n_k}}(w_{n_k} - y_{n_k}) - Aw_{n_k}\rangle \\ &= \langle u - y_{n_k}, Au - Ay_{n_k}\rangle + \langle u - y_{n_k}, Ay_{n_k} - Aw_{n_k}\rangle \\ &\quad + \frac{1}{\lambda_{n_k}}\langle u - y_{n_k}, w_{n_k} - y_{n_k}\rangle \\ &\geq \langle u - y_{n_k}, Ay_{n_k} - Aw_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle u - y_{n_k}, w_{n_k} - y_{n_k}\rangle. \end{aligned} \tag{4.39}$$

Taking limit as $k \to \infty$ in (4.39), we obtain from (4.37), (4.35) and the fact that $y_{n_k} \rightharpoonup x^*$, that $\langle u - x^*, v\rangle \geq 0$. Thus, by the maximal monotonicity of $A + B$ (see Lemma 2.3), we get that $x^* \in (A + B)^{-1}(0)$. That is, $x^* \in \Omega$. Hence, we have shown that every sequential weak cluster point of $\{x_n\}$ belongs to Ω. Also, in Lemma 4.1, we showed that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \Omega$. Therefore, by Lemma 2.2, we conclude that $\{x_n\}$ converges weakly to an element in Ω. The proof is complete. □

5 Numerical examples

In this section, we give the numerical analysis of our proposed algorithm (Algorithm 3.3) and compare it with Algorithm 3.1 in [32] (PSC (Alg. 3.1)), Algorithm 2 in [34] (VM (Alg. 2)) and Algorithm 2 in [13] (CHC (Alg. 2)).

Example 5.1 We consider the image restoration problem:

$$\min_{\hat{x}\in\mathbb{R}^m} \{||\mathcal{D}\hat{x} - \hat{c}||_2^2 + \lambda||\hat{x}||_1\}, \tag{5.1}$$

where $\lambda > 0$ (we take $\lambda = 1$), $\hat{x} \in \mathbb{R}^m$ is the original image to be restored, $\hat{c} \in \mathbb{R}^N$ is the observed image and $\mathcal{D} : \mathbb{R}^m \to \mathbb{R}^N$ is the blurring operator. The quality of the restored image is measured by

$$\mathrm{SNR} = 20 \times \log_{10}\left(\frac{\|\hat{x}\|_2}{\|\hat{x} - x^*\|_2}\right),$$

where SNR means signal-to-noise ratio, and x^* is the recovered image. The larger the SNR, the better the quality of the restored image.
For the implementation, we take $x_0 = \mathbf{0} \in \mathbb{R}^{m\times m}$ and $x_1 = \mathbf{1} \in \mathbb{R}^{m\times N}$, and use the following image found in the MATLAB Image Processing Toolbox:

(a) Tire Image of size 205×232. To create the blurred and noisy image (observed image), we use the Gaussian blur of size 9×9 and standard deviation $\sigma = 4$.
(b) Cameraman Image of size 256×256. We use the Gaussian blur of size 7×7 and standard deviation $\sigma = 4$.
(c) Medical Resonance Imaging (MRI) of size 128×128. We use the Gaussian blur of size 7×7 and standard deviation $\sigma = 4$.

Table 1 Numerical results for Example 5.1

Algorithms	Tire		Cameraman		MRI	
	CPU Time	SNR	CPU Time	SNR	CPU Time	SNR
Alg. 3.3 $\left(\tau_n = \frac{n-0.5}{2n}\right)$	2.1100	22.5103	3.9102	21.3003	1.8131	21.1304
Alg. 3.3 $\left(\tau_n = 0.5\right)$	1.2006	23.9291	3.7120	27.2914	1.2023	22.2001
Alg. 3.3 $\left(\tau_n = \frac{n-0.1}{n}\right)$	1.1318	24.8134	3.0214	28.4413	1.0019	23.5132
PSC (Alg. 3.1) in [32]	3.4218	21.3471	4.3304	23.8101	2.0124	20.6491
VM (Alg. 2) in [34]	3.1214	21.4261	4.3241	22.1921	2.0110	21.9910
CHC (Alg. 2) in [13]	3.0112	21.9999	4.0917	25.8402	1.8912	21.1887

Table 2 Numerical results for Example 5.2 $\epsilon = 10^{-7}$

Algorithms	Case 1		Case 2		Case 3	
	CPU Time	Iter	CPU Time	Iter	CPU Time	Iter
Alg. 3.3 $\left(\tau_n = \frac{n-0.5}{2n}\right)$	0.0361	148	0.0490	150	0.1524	155
Alg. 3.3 $\left(\tau_n = 0.5\right)$	0.0170	93	0.0415	96	0.1441	98
Alg. 3.3 $\left(\tau_n = \frac{n-0.1}{n}\right)$	0.0168	70	0.0283	72	0.0107	73
PSC (Alg. 3.1) in [32]	1.1812	163	1.4000	163	1.5125	164
VM (Alg. 2) in [34]	1.2100	223	1.2972	232	1.3264	230
CHC (Alg. 2) in [13]	0.0917	156	1.0219	158	1.0924	164

Example 5.2 Let $H = \mathfrak{l}_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_i, \ldots),\quad x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $||x||_{\mathfrak{l}_2} := \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}} \quad \forall x \in \mathfrak{l}_2(\mathbb{R})$. Define $A, B : \mathfrak{l}_2 \to \mathfrak{l}_2$ by

$$Ax := \left(\frac{x_1 + |x_1|}{2}, \frac{x_2 + |x_2|}{2}, \ldots, \frac{x_i + |x_i|}{2}, \ldots\right), \ \forall x \in \mathfrak{l}_2$$

and

$$Bx := (2x_1, 2x_2, \ldots, 2x_i, \ldots), \ \forall x \in \mathfrak{l}_2.$$

Then A is monotone Lipschitz continuous with Lipschitz constant $L = 1$ and B is maximal monotone.
Choose the initial points as follows:

Case 1: $x_1 = \left(\frac{5}{7}, \frac{1}{7}, \frac{1}{35}, \cdots\right)$ and $x_0 = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \cdots\right)$.

Case 2: $x_1 = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \cdots\right)$ and $x_0 = \left(\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots\right)$.

Case 3: $x_1 = \left(\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots\right)$ and $x_0 = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{10}, \cdots\right)$.

During the computation, we make use of the following:

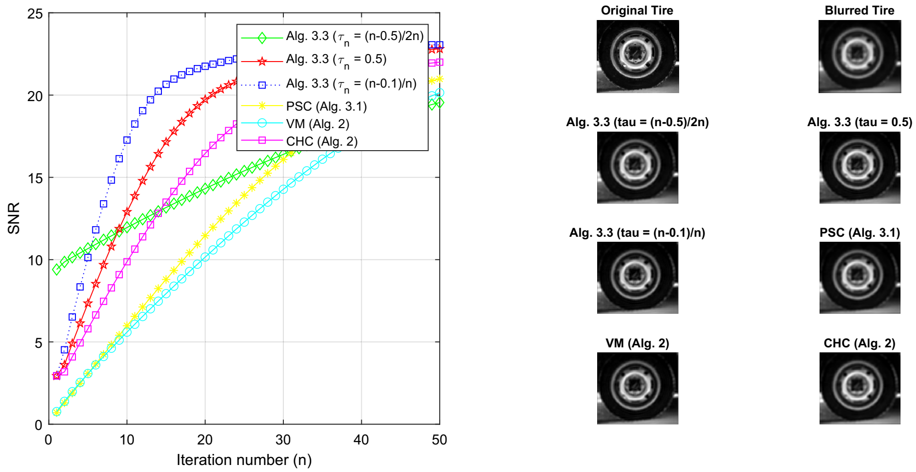

Fig. 1 Numerical results for Tire

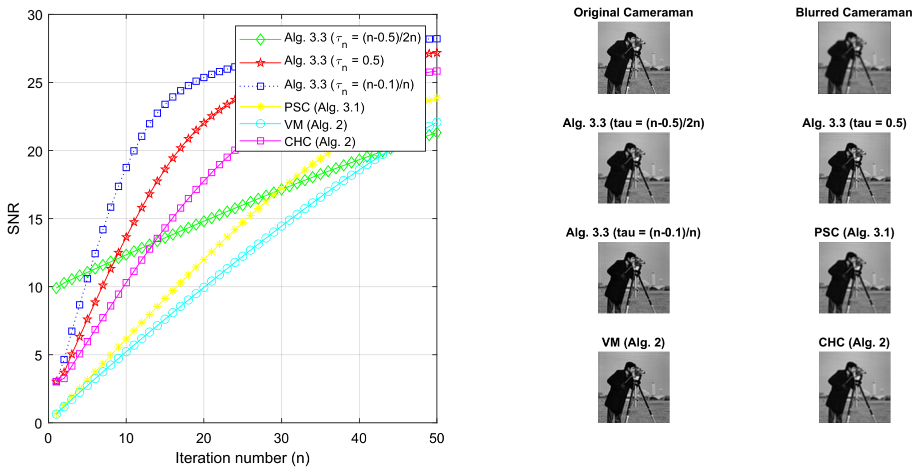

Fig. 2 Numerical results for Cameraman

- Algorithm 3.3: $\lambda_1 = 0.1$, $\mu = 0.5$, $\epsilon = 0.000001$, $\theta_n = \alpha_n$ (when $\tau_n = \frac{n-0.5}{2n}$), where α_n is as defined in (3.1), $\theta_n = \frac{1-\epsilon}{3}$ (when $\tau_n = 0.5$) and $\theta_n = \sqrt{\rho_n^2 + \phi_n} - \rho_n$ (when $\tau_n = \frac{n-0.1}{n}$), where ρ_n and ϕ_n are as defined in (3.3) and (3.4), respectively.
- PSC (Alg. 3.1) in [32]: $\xi_n = \frac{1}{10n+1}$, $\alpha_n = \frac{1}{2n+1}$ and $\gamma_n = \frac{0.9}{\beta}$ (where β is the inverse strongly monotonicity constant).
- VM (Alg. 2) in [34]: $\lambda = \frac{0.9}{L}$, $\theta = 0.9$ and $\epsilon_n = \frac{100}{(n+1)^2}$.
- CHC (Alg. 2) in [13]: $\lambda_1 = 0.1$, $\mu = 0.5$, $\theta = 0.5$ and $\alpha = 0.1$.

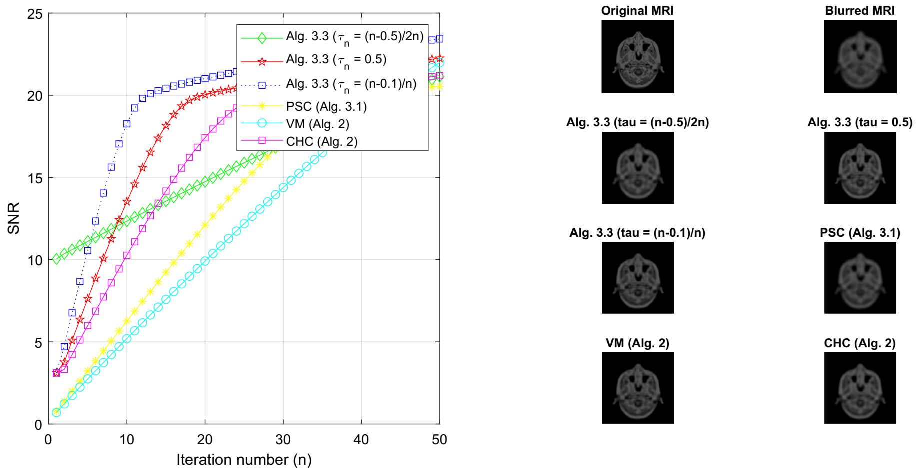

Fig. 3 Numerical results for MRI

We then use the stopping criterion; $\text{TOL}_n := \|y_n - w_n\| < \varepsilon$, where ε is the predetermined error (Tables 1, 2).

All the computations are performed using Matlab 2016 (b) which is running on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30GHz and 8.00 Gb-RAM. In the tables below, "Iter" means the number of iterations. Also, in the tables and figures, Alg. 3.3, PSC (Alg. 3.1), VM (Alg. 2) and CHC (Alg. 2) represent Algorithm 3.3, Algorithm 3.1 in [32], Algorithm 2 in [34] and Algorithm 2 in [13], respectively (Figs. 1, 2, 3, 4).

6 Conclusion

In this work, we have introduced and studied a new relaxed inertial condition for approximating a solution of the MIP (1.1) in the frame work of a real Hilbert space. We obtained the weak convergence of the proposed method under weaker conditions and we present a series of computational experiments. The experiments show that our method performs better than existing methods in the literature. In Example 5.1, we used the Signal-to-Noise Ratio (SNR) to measure the quality of the restored image. A variant of the SNR is the Peak Signal-to-Noise Ratio (PSNR) which is commonly used in digital image compression to measure the ratio between maximum possible power of a signal and the power of the noise that affects the quality of that signal. We shall employ the PSNR for this purpose in our future project. We shall also study the linear convergence of the proposed method of this paper in our future project.

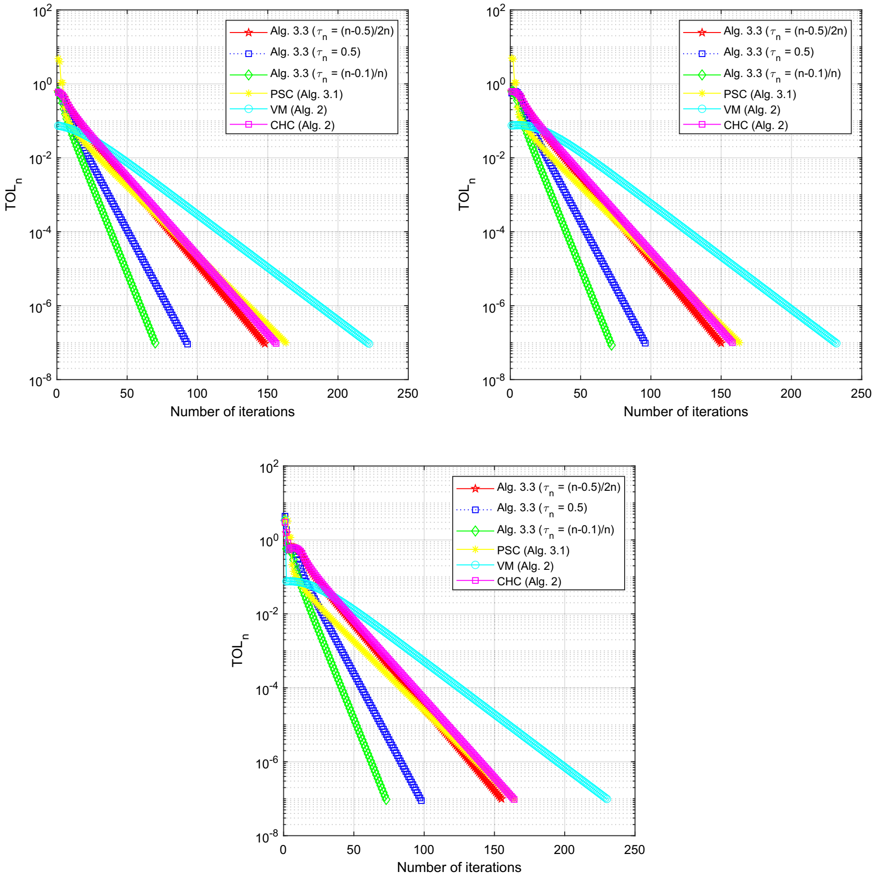

Fig. 4 The behavior of TOL_n with $\epsilon = 10^{-7}$: Top Left: **Case 1**; Top Right: **Case 2**; Bottom: **Case 3**

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Data Availability Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

References

1. Alber, Y., Ryazantseva, I.: Nonlinear Ill-Posed Problems of Monotone Type, p. xiv+410. Springer, Dordrecht (2006)
2. Al-Qurashi, M., Rashid, S., Jarad, F., Ali, E., Egami, R.H.: Dynamic prediction modelling and equilibrium stability of a fractional discrete biophysical neuron model. Results Phys. **48**, 106–405 (2023)
3. Akutsah, F., Mebawondu, A.A., Ugwunnadi, G.C., Narain, O.K.: Inertial extrapolation method with regularization for solving monotone bilevel variation inequalities and fixed point problems in real Hilbert space. J. Nonlinear Funct. Anal. **2022**, 5 (2022)
4. Akutsah, F., Mebawondu, A.A., Ugwunnadi, G.C., Pillay, P., Narain, O.K.: Inertial extrapolation method with regularization for solving a new class of bilevel problem in real Hilbert spaces. SeMA J. (2022). https://doi.org/10.1007/s40324-022-00293-2
5. Alvares, F., Attouch, H.: An inertial proximal monotone operators via discretization of a nonlinear oscillator with damping. Set Valued Anal. **9**, 3–11 (2001)
6. Anh, P.N.: Strong convergence theorems for nonexpansive mappings Ky Fan inequalities. J. Optim. Theory Appl. **154**, 303–320 (2012)
7. Anh, P.K., Anh, T.V., Muun, L.D.: On bilevel split pseudomonotone variational inequality problems with applications. Acta. Math. Vietnam. **42**, 413–429 (2017)
8. Attouch, H., Cabot, A.: Convergence of a relaxed inertial forward-backward algorithm for structured monotone inclusions. Appl. Math. Optim. **80**, 547–598 (2019)
9. Attouch, H., Goudon, X., Redont, P.: The heavy ball with friction. I. The continuous dynamical system. Commun. Contemp. Math. **21**(2), 1–34 (2000)
10. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, New York (2011)
11. Boţ, R.I., Csetnek, E.R., Hendrich, C.: Inertial Douglas-Rachford splitting for monotone inclusion problems Appl. Math. Comput. **256**(3), 472–487 (2015)
12. Censor, Y., Gibali, A., Reich, S.: A von Neumann alternating method for finding common solutions to variational inequalities. Nonlinear Anal. Theory. Meth. Appl. **75**, 4596–4603 (2012)
13. Cholamjiak, P., Hieu, D.V., Cho, Y.J.: Relaxed forward-backward splitting methods for solving Variational inclusions and applications. J. Sci. Comp. (2021). https://doi.org/10.1007/s10915-021-01608-7
14. Combettes, P.L., Wajs, V.R.: Signal recovery by proximal forward-backward splitting. Multiscale Model. Simul. **4**(4), 1168–1200 (2005)
15. Dong, Y.: New inertial factors of the Krasnosel'skiĭ-Mann iteration. Set-Valued Var. Anal. **29**, 145–161 (2021)
16. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer, Berlin (2003)
17. Heydari, M.H., Rashid, S., Jarad, F.: A numerical method for distributed-order time fractional 2D Sobolev equation. Results Phys. **45**, 106–211 (2023)
18. Iutzeler, F., Hendrickx, J.M.: A generic online acceleration scheme for optimization algorithms via relaxation and inertia. Optim. Methods Softw. **34**, 383–405 (2019)
19. Izuchukwu, C., Reich, S., Shehu, Y.: Relaxed inertial methods for solving the split monotone variational inclusion problem beyond co-coerciveness. Optimization **72**, 607–646 (2023)
20. Kesornprom, S., Cholamjiak, P.: A new relaxed inertial forward-backward-forward method for solving the convex minimization problem with applications to image inpainting. Appl. Set-Valued Anal. Optim. **5**, 439–450 (2023)
21. Lemaire, B.: Which fixed point does the iteration method select? Recent Adv. Optim. **452**, 154–157 (1997)
22. Lions, P.L., Mercier, B.: Splitting algorithms for the sum of two nonlinear operators. SIAM J. Numer. Anal. **16**, 964–979 (1979). https://doi.org/10.1137/0716071
23. Liu, L.: A stochastic projection and contraction algorithm with inertial effects for stochastic variational inequalities. J. Nonlinear Var. Anal. **7**, 995–1016 (2023)
24. Malitsky, Y., Tam, M.K.: A forward-backward splitting method for monotone inclusions without cocoercivity. SIAM J. Optim. **30**, 1451–1472 (2020)

25. Mewomo, O.T., Okeke, C.C., Ogbuisi, F.U.: Iterative solutions of split fixed point and monotone inclusion problems in Hilbert spaces. J. Appl. Numer. Optim. **5**, 271–285 (2023)
26. Moudafi, A.: Second-order differential proximal methods for equilibrium problems. J. Inequal. Pure Appl. Math. **4**(1), 1–7 (2003)
27. Opial, Z.: Weak convergence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 591–597 (1967)
28. Passty, G.B.: Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. J. Math. Anal. Appl. **72**, 383–390 (1979)
29. Polyak, B.T.: Some methods of speeding up the convergence of iteration methods. Politehn Univ. Buchar Sci. Bull. Ser. A Appl. Math. Phys. **4**(5), 1–17 (1964)
30. Rehman, H.U., Kumam, P., Shutaywi, M., Pakkaranang, N., Wairojjana, N.: An inertial extragradient method for iteratively solving equilibrium problems in real Hilbert spaces. Int. J. Comp. Math. **99**(6), 1081–1104 (2020)
31. Shehu, Y., Vuong, P.T., Zemkoho, A.: An inertial extrapolation method for convex simple bilevel optimization. Optim. Methods Softw. **36**, 1–19 (2021)
32. Peeyada, P., Suparatulatorn, R., Cholamjiak, W.: An inertial Mann forward-backward splitting algorithm of variational inclusion problems and its applications. Chaos Solit. Fract. **158**, 112048 (2022)
33. Tseng, P.: A modified forward-backward splitting method for maximal monotone mappings. SIAM J. Control Optim. **38**, 431–446 (2000)
34. Vinh, N.T., Muu, L.D.: Inertial extragradient algorithms for solving equilibrium problems. Acta Math. Vietnam **44**(3), 639–663 (2019)