Spectral Properties Of Fourth Order Boundary Value Problems with Eigenvalue Parameter Dependent And Periodic Boundary Conditions

by

Boitumelo Moletsane

WITS
UNIVERSITY

A THESIS SUBMITTED IN FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

SCHOOL OF MATHEMATICS
UNIVERSITY OF THE WITWATERSRAND
JOHANNESBURG, SOUTH AFRICA,
21 JANUARY 2019

Declaration

I declare that this thesis is my own work. It is being submitted for the degree of PhD at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Boitumelo Moletsane

Dated the twenty-first day of January 2019, in Johannesburg, South Africa.

Dedication

This work is dedicated to everyone and everything that make my life everyday. The taxi drivers on my commute, my dad, mom, brothers, friends, supervisors and faces of strangers and passersby that make my daily walk interesting.

Acknowledgments

My gratitude goes to Prof Manfred Möller and Dr Bertin Zinsou without whom this work would not have been possible. Thank you.

Abstract

IN THIS THESIS, we give first order asymptotics of eigenvalues of quadratic pencils presenting a fourth order differential equation together a mixture of boundary conditions that depend on the eigenvalue parameter and are periodic or anti-periodic. The non-self-adjoint quadratic pencils have the two constant coefficient operators and the differential operator all self-adjoint. For the same differential equation and the same set of boundary conditions where the only difference is that the boundary conditions which are periodic are replaced with anti-periodic one, the zeros of their characterisitic determinants are interlaced. Thus, the eigenvalues of their quadratic pencils with periodic and anti-periodic boundary conditions, respectively, are interlaced and lie in the first and third quadrant of the complex plane. In both cases the periodic and anti-periodic boundary conditions do not depend on the eigenvalue parameter.

Contents

Declaration **i**

Dedication **ii**

Acknowledgments **iii**

Abstract **iv**

0 Introduction **1**

1 Preliminaries **9**
- 1.1 Definitions and properties for characterisation of self-adjoint problems 10
- 1.2 Definitions and properties for Birkhoff regular problems and their eigenvalue expansion 15

2 Periodic and Antiperiodic Boundary Conditions **32**
- 2.1 A vibrating string with periodic boundary conditions 33
- 2.2 Periodic and a single eigenvalue dependent boundary condition . . 42
- 2.3 Further examples of self-adjoint operators with periodic and a single eigenvalue dependent boundary conditions 47

3 Asymptotics of eigenvalues **54**
- 3.1 Birkhoff Regularity 55
- 3.2 Asymptotic expansions of eigenvalues when $g = 0$ 61
- 3.3 Asymptotic expansions of eigenvalues when $g \neq 0$ 81

Appendix A Appendix **92**

0

Introduction

Joseph Liouville pioneered the theory of Sturm-Liouville differential equations through his study of the second order differential equation of the form

$$\frac{d^2z}{dx^2} + [\rho^2 + g(x)]z = 0 \tag{0.0.1}$$

when ρ is real in 1837. A generalisation of the differential equation (0.0.1) in the form

$$\frac{d^nz}{dx^n} + \rho a_{n-1}(x,\rho)\frac{d^{n-1}z}{dx^{n-1}} + \ldots + \rho^n a_0(x,\rho)z = 0 \tag{0.0.2}$$

where $|\rho|$ is large was studied for its asymptotic character of its solutions by George D. Birkhoff. The functions $a_i(x,\rho)$ are analytic in the complex parameter ρ and have derivatives of all orders in the real variable x. Birkhoff in his paper [2] proves asymptotic properties on a region $\theta \leq \arg \rho \leq \psi$, where he gives determinantal inequalities, involving the leading coefficients in the boundary forms, from which asymptotic eigenfunction estimates and an expansion theorem follow. Birkhoff gives an expansion theorem and an estimate for the Green's function of the class of regular boundary conditions which are those boundary conditions that satisfy some determinantal inequalities. Stone [17] shows that Birkhoff expansion is in a sense equivalent to the Fourier expansion. Salaff [16] shows that when n in (0.0.2) is even, self-adjoint boundary conditions meet determinantal inequalities which means that he shows that self-adjointness implies Birkhoff reguarity for even order operators. Tamarkin and Stone establish the equiconvergence theorem showing that an eigenfunction series of a Birkhoff-regular operator behaves similarly to a trigonometric one on every compact subinterval K of the open interval $(0, 1)$. On the entire interval $[0, 1]$, Minkin in [6] shows equiconvergence for even order differential operators from which investigations of boundary value problems dependent on a spectral parameter or with the Stieltjies integral in the boundary conditions branched off.

A differential equation of the form (0.0.2) together with its boundary conditions can be written as quadratic operator pencil. Quadratic operator pencil, quadratic operator polynomial and quadratic eigenvalue problems(QEP) are synonyms for an operator of the form

$$Q(\lambda) = \lambda^2 M + \lambda K + A, \tag{0.0.3}$$

which is a matrix polynomial of degree 2 in the scalar λ. However, since we study eigenvalue expansions of the problem (0.0.3) in this document, A is a differential operator on a Hilbert space $L_2(0, a) \oplus \mathbb{C}$. QEP is currently receiving attention because of extensive application in dynamical analysis of mechanical systems in acoustics and linear stability of flows in fluid mechanics. The aim when solving a quadratic pencil is to find scalars λ and non-zero vectors that satisfy

$$Q(\lambda)x = 0.$$

Vector x is a right eigenvectors corresponding to the scalar λ. A major complication to solving QEP is that there is no simple canonical form analogous to the Schur form for the standard eigenvalue problem or the generalised Schur form for the generalised eigenvalue problem to this nonlinear eigenvalue problem. Properties of coefficient matrices correspond to particular spectral properties. For example, if M, K and A are all real then the eigenvalues are real or come in pairs $(\lambda, \bar{\lambda})$ and if x is a right eigenvector corresponding to λ then so is $\bar{x}$ corresponding to $\bar{\lambda}$. Another example is when M and A are real symmetric and positive definite, $K = -K^T$ then the eigenvalues are purely imaginary.

Numerical methods are classed into those that solve the quadratic operator pencils directly and those that work with the linearised form and compute its generalised Schur decomposition or a simple form to compute eigenvalues and eigenvectors. Most numerical methods that deal with QEP are variants of Newton's method, which involve a discretisation of the space on which the problem is solved which can only involve a finite number of solutions. These Newton's variants compute one eigenpair at a time and converge as long as the starting guess is close enough to the solution, but in practice even with good initial guesses there is no guarantee that the method will converge to the desired eigenvalue. When

M, K and A are symmetric, $Q(\lambda)$ is linearised into $S - \lambda T$, (S, T) is symmetric and B is definite, a Cholesky factorisation $B = QQ^{\intercal}$ can be computed, with Q lower triangular and reduces the symmetric generalised eigenvalue problem to a standard eigenvalue problem which can be solved with a symmetric QR algorithms. With $Q(\lambda)$ which usually comes from gyroscopic systems where M and A are symmetric and K is skew symmetric, linearizations of $Q(\lambda)$ where S and T are Hamiltonian or skew Hamiltonian a structure preserving algorithm for real Hamiltonian matrices as the one developed by [1] Benner, Mehrmann and Xu can be used instead of the QZ algorithm. There are several methods and algorithms that use linearisations which can be compared for stability, computational effeciency and memory using metrics like condition numbers. In the problem where a band travelling at speed ν between two fixed points and the band's transverse displacement with no external excitation force is described by the nondimensional equation

$$\left[\frac{\partial^4}{\partial x^4} + (\kappa\nu^2 - \tau) - \frac{\partial^2}{\partial x^2} + 2\nu\frac{\partial^2}{\partial x \partial t} + \frac{\partial^2}{\partial t^2}\right] u(x,t) = 0, \ x \in [0,1], \qquad (0.0.4)$$

where τ is the dimensionless tension and $\kappa \in [0,1]$ is a constant depending on the pulley mounting system of the band. Substituting an appropriate separation of variable approximation of the solution, the equation becomes a second order differential equation. Companion linearisation and QZ algorithm implemeted in Matlab 6 polyeig showed that eigenvalues with particular parameters of ν, κ and τ do not have Hamiltonian structure and some have positive real part suggesting incorrectly that the system is unstable as proved by Van Loan [4]. Given information about $||M||$, $||K||$ and $||A||$ and the structure of ε and χ in $(S - \lambda T)\varepsilon$, $\chi^*(S - \lambda T)$, it is possible to compare the condition numbers of different linearisations and identify which formulations are preferred for large and small eigenvalues, respectively. This result has practical relevance, as in applications it is often only eigenpairs corresponding to small and large eigenvalues that are of interest. Currently, there are no numerical methods that tackle QEP directly and compute all eigenpairs.

We aim to solve directly a quadratic operator polynomial where a homogenous string has at least one of the boundary conditions being periodic. A problem of small transversal vibrations across a homogenous string was studied by Pivovarchik and Van der Mee [15] and Möller and Pivovarchik [8]. In the former the transversal vibrations are described by a second order partial differential equation while in the latter, by a fourth order partial differential equation. A damping coefficient is present at the right endpoint in both instances, the string is fixed at the left end point and through a separation of variables subtitution, we have an eigenvalue dependent boundary condition in one of the boundary conditions. These two problems result in second and fourth order differential equations, respectively, which together with their boundary conditions are boundary value problems with spectral representation as a quadratic operator polynomial of the form

$$L(\lambda, \alpha) = \lambda^2 M - i\alpha\lambda K - A. \tag{0.0.5}$$

Pivovarchik and Van der Mee [15] study the inverse problem of constructing the real potential from its eigenvalues which are zeros of a sine-type entire function where $\alpha > 0$ and $\alpha \neq 1$. It is the paper of Pivovarchik and Möller [8], where they studied the location of the spectrum, algebraic and geometric multiplicities of the eigenvalues and characterised the spectrum of a compact peturbation of $L_0 = L(0, 0)$ by a smooth real function g that initiated the work of Möller and

Zinsou [11], [13], [12], [14], [10] and [18]. Möller and Zinsou [11] studied (0.0.5) a fourth order differential equation for which the coefficient matrices M, K and A are self-adjoint and found four sets of boundary conditions. These cases, both boundary conditions at the right endpoint were dependent of the eigenvalue parameter and only one depended on the eigenvalue parameter at the left endpoint. The eigenvalues of their operator where found as $\lambda_{-k} = -\overline{\lambda}_k$ for some $k \geq k_0$, for some positive integer k_0 which are termed $*$-alternating in some texts and may be viewed as generalisations of symplectic and Hamiltonian matrices.

We consider a special case of (0.0.2) with $n = 4$, represented by the differential equation

$$y^{(4)}(\lambda, x) - (gy')'(\lambda, x) - \lambda^2 y(\lambda, x) = 0. \tag{0.0.6}$$

The boundary conditions considered are such that one is separated, another is dependent on the eigenvalue parameter and the remaining two, which have periodicity are considered for both the periodic and anti-periodic cases, which we indicate by $\epsilon = \pm 1$. We initially find the eigenvalues of the boundary value problem for the simplified case where $g = 0$ in (0.0.6). An exponential multiple of the characteristic determinant is written as a sum of three terms. On a sector $\frac{5\pi}{8} < \arg(\omega_j - \omega_0) \leq \pi$, this characteristic determinant is dominated by one of the terms whose zeros are determined. We consider small circles around each zero of the dominant term and find that each circle contains a unique zero of the dominant term, thus, by Rouché's theorem each circle contains a unique zero of the characteristic determinant.

We then consider square annulus on the complex plane centered at the origin whose side of the inner square is very large. Again, through application of

Rouché's theorem we find that the number of zeros inside each square is the same as the number of zeros of the dominant term in the characteristic determinant. Hence, inside the square annulus, all the zeros of the characteristic determinant are accounted for. In order to count the number of zeros of the characteristic determinant, we consider a product of the characteristic determinants of the two boundary value problems presenting the periodic and anti-periodic cases and through a factorisation into three factors we are able to enumerate the number of zeros of the characteristic determinant.

The document is organised into four chapters. Chapter 0 is the Introduction, which gives history and a broad overview of quadratic operator pencils. Chapter 1 gives definitions, lemmas, propositions and theorems required to define concepts and ideas communicated in the entire document. Chapter 2 is organised into three sections which are published in [7]. In Section 2.1, we write the fourth order partial differential equation as an ordinary differential equation by assuming that solutions to the differential equation will be superpositions of standing waves. The ordinary differential equation is then written in terms of its quasi-derivatives and the Lagrange's identity is integrated to obtain the Green's formula. Using Matlab, we implement [9, Theorem 10.3.5] to find boundary conditions to the ordinary differential equation such that at least one of the boundary condition is periodic. We verify in Section 2.2 that the two boundary value problems with boundary conditions obtained in Section 2.1, fulfil the criteria that the differential operator in the spectral representation of the boundary value problems is self-adjoint and prove Theorem 2.2.4. Note that Theorem 2.2.4 states equivalent statements for boundary conditions of the fourth order differential equation where we have one boundary condition dependent on the eigenvalue parameter and at least one periodic boundary condition. We characterise boundary conditions that fulfil our criteria in Theorem 2.3.1 in Section 2.3. Chapter 3, also has three sections. We prove in Section 3.1 that the problem (3.2.1)–(3.2.5) is Birkhoff regular for $\epsilon_1\epsilon_2 = 1$. In Section 3.2, we provide the asymptotics of the eigenvalues for $g = 0$ while in Section 3.3, we derive the first three terms of the eigenvalue expansions.

1

Preliminaries

DEFINITIONS, LEMMAS, PROPOSITIONS AND THEOREMS required to define concepts and ideas communicated in this entire document are listed in this chapter. Efforts have been made to keep these definitions, lemmas, propositions and theorems in the order in which they appear in the document. However, deviations from this order may exist because sometimes it is easier to immediately follow with a related concept.

One of the reasons for the huge development of the theory of classical Lebesgue and Sobolev space L_p and W_p^n where $(1 \leq p \leq \infty)$ and $n \in \mathbb{N}$ is that many materials can be modelled with sufficient accuracy using these function spaces. Throughout this thesis we work in a Sobolev space. We follow notation used in

[9] and [5]. In addition, we modify results which are in the form of propositions, lemmas, theorems and corollaries in [9] and [5] that are applicable to the particular problem under consideration in this thesis.

We give in Section 1.1 definitions and properties required for the characterisation of self-adjointness of the boundary value problems under consideration. In Section 1.2, we give definitions and properties for a boundary value problem to be Birkhoff regular and definitions and properties of eigenvalue expansions.

1.1 Definitions and properties for characterisation of self-adjoint problems

A Sobolev space is defined as

$$W_2^m(a,b) := \{g \in L_2(a,b) : \forall j \in \{1,\dots,m\}\ g^{(j)} \in L_2(0,a)\},$$

where $-\infty < a, b < \infty$ and $m \in \mathbb{N}$.

Let $n = 2k$ where $k \in \mathbb{N}$. We consider an nth ($n \in \mathbb{N}\backslash\{0\}$) order differential expression ℓ of the form

$$\ell y = \sum_{m=0}^{k} (g_m y^{(m)})^{(m)} \tag{1.1.1}$$

on the interval $[a,b]$, where $g_m \in W_2^m(a,b)$, $m = 0,\dots,k$, are real valued functions and $|g_k(x)| > \varepsilon$ for some $\varepsilon > 0$ and $x \in [a,b]$. The differential expression ℓy is well defined for $y \in W_2^n(a,b)$ in which case $\ell y \in L_2(a,b)$. The operator L_0 defined by

$$D(L_0) = W_2^n(0,a), \quad L_0 y = \ell y, \quad y \in W_2^n(a,b), \tag{1.1.2}$$

is called the maximal operator associated with the differential expression ℓ on

$[a, b]$.

Definition 1.1.1. Let $y \in W_2^n(a,b)$. For $j = 0, \ldots, n$ the jth quasi-derivative of y denoted $y^{[j]}$, is recursively defined by

$$\begin{aligned} y^{[j]} &= y^{(j)} \quad \text{for} \quad j = 0, \ldots, k-1, \\ y^{[k]} &= g_k y^{(k)}, \\ y^{[j]} &= (y^{[j-1]})' + g_{n-j} y^{(n-j)} \quad \text{for} \quad j = k+1, \ldots, n. \end{aligned}$$

The quasi-derivatives depend on the differential expression (1.1.1). They are convenient for the formulation of the Lagrange identity when dealing with differential operators which have fairly general coefficients.
Let

$$Y = \begin{pmatrix} y \\ c \end{pmatrix}, \quad Z = \begin{pmatrix} z \\ d \end{pmatrix}, \quad W = \begin{pmatrix} w \\ e \end{pmatrix} \tag{1.1.3}$$

be elements of the Hilbert space $L_2(a,b) \oplus \mathbb{C}$, $y, z, w \in W_2^n(a,b)$.
A formulation of the Lagrange identity and Green's formula is quoted below from [9, Theorem 10.2.3].

Theorem 1.1.2. *For a differential expression ℓ and $y, z \in W_2^n(a,b)$, the Lagrange identity*

$$(\ell y)\overline{z} - y(\ell \overline{z}) = \frac{d}{dx}[y, z] \tag{1.1.4}$$

holds on $[a,b]$ almost everywhere, where

$$[y,z]=\sum_{j=1}^{k}(-1)^j\left(y^{[j-1]}\overline{z^{[n-j]}}-y^{[n-j]}\overline{z^{[j-1]}}\right) \tag{1.1.5}$$

and Green's formula

$$(\ell y,z)-(y,\ell z)=[y,z](b)-[y,z](a) \tag{1.1.6}$$

is valid, where $(\cdot,\cdot)$ is the inner product in $L_2(a,b)$.

Define the operator A in the Hilbert space $L_2(a,b)\oplus\mathbb{C}$ by

$$D(A)=\left\{Y\in W_2^n(0,a)\oplus\mathbb{C},\quad U_1\hat{Y}=0,\quad c=U_2\hat{Y}\right\}, \tag{1.1.7}$$

$$AY=\begin{pmatrix}\ell y\\ V\hat{Y}\end{pmatrix}, \tag{1.1.8}$$

where

$$\hat{Y}=\left(y(a),\dots,y^{[n-1]}(a),y(b),\dots,y^{[n-1]}(b)\right)^{\intercal} \tag{1.1.9}$$

and matrices U_1, U_2 and V are of sizes $3\times 2n$, $1\times 2n$ and $1\times 2n$, respectively. For $m\in\mathbb{N}$, define

$$\begin{cases} J_{m,0}=\left((-1)^{s-1}\delta_{s,m+1-t}\right)_{s,t=1}^{m},\ J_{m,1}=\begin{pmatrix}0 & J_{m,0}\\ -J_{m,0}^{*} & 0\end{pmatrix}, \\ J_m=\begin{pmatrix}-J_{m,1} & 0\\ 0 & J_{m,1}\end{pmatrix}. \end{cases} \tag{1.1.10}$$

Finally define

$$U_3 = \begin{pmatrix} J_2 \\ V \\ -U_2 \end{pmatrix}, \tag{1.1.11}$$

$$U = \begin{pmatrix} U_1 & 0 & 0 \\ U_2 & -I & 0 \\ V & 0 & -I \end{pmatrix}. \tag{1.1.12}$$

Before stipulating a criterion of self-adjointness, we give a proposition which states conditions under which $Z \in D(A^*)$. The modification of the proposition from [9, Proposition 10.3.3] is quoted below.

Proposition 1.1.3. *Assume that* rank $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = 4$. *Then* $Z \in D(A^*)$ *if and only if* $Z \in W_2^n(a,b) \oplus \mathbb{C}$ *and there is* $e \in \mathbb{C}$ *such that*

$$[y,z](b) - [y,z](a) + d^* V\hat{Y} - e^* U_2 \hat{Y} = 0 \tag{1.1.13}$$

for all $\hat{Y} \in N(U_1)$. *For* $Z \in D(A^*)$, e *is unique and*

$$A^* Z = \begin{pmatrix} \ell z \\ e \end{pmatrix}.$$

A criterion of self-adjointness as given in [9, Theorem 10.3.5] is quoted next.

Theorem 1.1.4. *Assume that*

$$\operatorname{rank}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = 4.$$

Then A is self-adjoint if and only if

$$U_3(N(U_1)) = R(U^*).$$

In addition to determining if A is self-adjoint, we use [9, Theorem 10.3.8] quoted below to conclude that A is bounded below.

Theorem 1.1.5. *Assume that A is self-adjoint. Then A has a compact resolvent. Assume additionally that*

(i) $(-1)^k g_k > 0$,

(ii) *each component of $U_1\hat{Y}$ either contains only quasi-derivatives $y^{[m]}$ with $m < k$ or contains only quasi-derivatives $m \geq k$,*

(iii) *each component of $U_2\hat{Y}$ either contains only quasi-derivatives $y^{[m]}$ with $m < k$ or contains only quasi-derivatives $m \geq k$,*

(iv) *for each component of $U_2\hat{Y}$ which only contains quasi-derivatives $y^{[m]}$ with $m \geq k$, the corresponding component of $V\hat{Y}$ only contains quasi-derivatives $y^{[m]}$ with $m < k$.*

Then A is bounded below.

Any $m \times n$ matrix can be written as a product of a diagonal matrix of its singular values augmented by zeros and orthogonal matrices of order m and n as stated in [3, Theorem 6.1] quoted below as

Theorem 1.1.6. *Any $m \times n$ real matrix Γ, with $m \geq n$, can be factorized as*

$$\Gamma = \Delta \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} \Theta^{\intercal}, \tag{1.1.14}$$

where $\Delta \in \mathbb{R}^{m\times m}$ and $\Theta \in \mathbb{R}^{n\times n}$ are orthogonal, and $\Sigma \in \mathbb{R}^{n\times n}$ is diagonal,

$$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{\mathrm{n}}),$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$.

1.2 Definitions and properties for Birkhoff regular problems and their eigenvalue expansion

For the definition of Birkhoff regularity, we consider a boundary eigenvalue problem for sufficiently large complex numbers λ, say $|\lambda| \geq \gamma$. We modify the first order system in [5] to fit the differential equation and boundary conditions under consideration in this document as

$$\hat{y}' - (\lambda A_1 + A_0)\hat{y} = 0, \tag{1.2.1}$$

$$\widetilde{W}^{(0)}\hat{y}(a_0) + \widetilde{W}^{(1)}\hat{y}(a_1) = 0, \tag{1.2.2}$$

where $\hat{y} \in (W_2^1(a,b))^n, \lambda \in \mathbb{C}$.

For the differential system of equations (1.2.1) we assume that the coefficient matrices A_0 and A_1 belong to $\mathrm{M}_n(L_2(a,b))$. We suppose that A_1 is a diagonal matrix function

$$A_1 = \begin{pmatrix} r_1 & & 0 & & \\ & \cdot & \cdot & & \\ & 0 & \cdot & & \\ & & \cdot & \cdot & \cdot \\ & & & & r_l \end{pmatrix}, \tag{1.2.3}$$

where l is a positive integer,

$$r_\nu \in \mathbb{C} \quad (\nu = 1, \dots, l).$$

We assume that there are distinct, nonzero numbers $\varphi_{\nu\mu} \in [0, 2\pi)$, $\nu, \mu = 1, \dots, l$ such that

$$r_\nu = |r_\nu| e^{i\varphi_\nu}, \tag{1.2.4}$$

$$r_\nu - r_\mu = |r_\nu - r_\mu| e^{i\varphi_{\nu\mu}} \quad \text{in } (a, b). \tag{1.2.5}$$

Let $\varphi_\nu := \varphi_{\nu 0} = \varphi_{0\nu} \pm \pi$ and

$$r_\nu = |r_\nu| e^{i\varphi_{\nu 0}} = | \pm i r_\nu | e^{i\varphi_{0\nu}} \quad \text{in } (a, b) \ (\nu = 1, \dots, l). \tag{1.2.6}$$

We formulate as a proposition from [5, Theorem 2.8.2].

Proposition 1.2.1. *There is a fundamental matrix function*

$$\tilde{Y}(\cdot, \lambda) = (P^{[0]} + B_0(\cdot, \lambda)) E(\cdot, \lambda) \tag{1.2.7}$$

of the differential equation (1.2.1) *having the following properties: The matrix function* $E(\cdot,\lambda)$ *belongs to* $M_n(W_2^1(a,b))$ *and*

$$E(\cdot,\lambda) = \operatorname{diag}(\mathrm{E}_0(\cdot,\lambda), \mathrm{E}_1(\cdot,\lambda), \ldots, \mathrm{E}_l(\cdot,\lambda)) \tag{1.2.8}$$

for $x \in [a,b]$ *and* $\lambda \in \mathbb{C}$, *where*

$$E_\nu(\lambda) = \exp(\lambda r_\nu (b-a)). \tag{1.2.9}$$

The matrix function $P^{[0]}$ *belongs to* $M_n(W_p^1(a,b))$ *and is a diagonal matrix according to the structure of* A_1, *i.e,*

$$P^{[0]} = \operatorname{diag}(\mathrm{P}_{11}^{[0]}, \ldots, \mathrm{P}_{ll}^{[0]}). \tag{1.2.10}$$

The diagonal elements $P_{\nu\nu}^{[0]}$ *are uniquely given as solution of the initial value problems*

$$\begin{cases} P_{\nu\nu}^{[0]\prime} = A_{0,\nu\nu} P_{\nu\nu}^{[0]}, \\ P_{\nu\nu}^{[0]}(a) = I_{n_\nu}, \end{cases} \tag{1.2.11}$$

where the $n_\nu \times n_\nu$ *matrix functions* $A_{0,\nu\nu}$ *are the diagonal elements of* A_0. *The matrix function* $B_0(\cdot,\lambda)$ *belong to* $M_n(W_2^1(a,b))$ *for* $|\lambda| \geq \gamma$ *and fulfils the estimates*

$$\begin{cases} B_0(\cdot,\lambda) = \{o(1)\}_\infty, \\ B_0(\cdot,\lambda) = \{O(\tau_2(\lambda))\}_\infty, \end{cases} \quad \text{as } \lambda \to \infty, \tag{1.2.12}$$

where

$$\tau_p(\lambda) = \max_{\substack{\nu,\mu=0 \\ \nu\neq\mu}}^{l} (1 + |\Re(\lambda e^{i\varphi_{\nu\mu}})|)^{-\frac{1}{2}}. \tag{1.2.13}$$

We set

$$\delta_\nu(\lambda) := \begin{cases} 0 \text{ if } \Re(\lambda e^{i\varphi_\nu}) < 0, \\ 1 \text{ if } \Re(\lambda e^{i\varphi_\nu}) > 0, \\ 0 \text{ if } \Re(\lambda e^{i\varphi_\nu}) = 0 \text{ and } \Im(\lambda e^{i\varphi_\nu}) > 0, \\ 1 \text{ if } \Re(\lambda e^{i\varphi_\nu}) = 0 \text{ and } \Im(\lambda e^{i\varphi_\nu}) < 0. \end{cases} \tag{1.2.14}$$

For convenience let $\delta_0(\lambda) = \delta_1(\lambda)$ and we can infer that $\Delta(-\lambda) = I_n - \Delta(\lambda)$. We define the block diagonal matrices

$$\begin{cases} \Delta(\lambda) := \operatorname{diag}(\delta_0(\lambda) I_{n_1}, \ldots, \delta_l(\lambda) I_{n_l}), \\ \Delta_0 := \operatorname{diag}(\delta_1(\lambda) I_{n_1} \ldots, \delta_l(\lambda) I_{n_l}), \end{cases} \tag{1.2.15}$$

which (by definition) reduce to

$$\Delta(\lambda) = \operatorname{diag}(\delta_1(\lambda) I_{n_1}, \ldots, \delta_l(\lambda) I_{n_l}), \tag{1.2.16}$$

and $\Delta_0 = I_n$. From the definition of the $\delta_\nu(\lambda)$ we immediately infer that

$$\Delta(-\lambda) = I_n - \Delta(\lambda). \tag{1.2.17}$$

We quote the definition of Birkhoff regularity from [5, Definitions 4.1.2 and 5.2.1].

Definition 1.2.2. The boundary eigenvalue problem (1.2.1) and (1.2.2) is called Birkhoff regular if

$$W_0^{(0)}(I_n - \Delta(\lambda))\Delta_0 + W_0^{(1)}\Delta(\lambda)\Delta_0 \tag{1.2.18}$$

is invertible for $\lambda \in \mathbb{C}\backslash\{0\}$.

We give the definition of Birkhoff regularity for a differential equation with its boundary conditions as opposed to the previous definition which applies to a boundary value problem presented as a first order system.

Let $a_j \in [a,b]$ $(j \in \mathbb{N})$ such that $a_0 = a$ and $a_1 = b$. Let

$$p_i(\cdot,\lambda) = \sum_{j=0}^{n-i} \lambda^j \pi_{n-i,j} \qquad (i = 0,\dots,n-1), \tag{1.2.19}$$

where $\pi_{n-i,j} \in L_p(a,b)$ $(i = 0,\dots,n-1, j = 0,\dots,n-i)$. We assume $\pi_{n-i,n-i} \neq 0$ for some $i \in \{0,\dots,n-1\}$. Let w_{ki} $(i,k = 1,\dots,n)$ be polynomials in λ with coefficients in $L_1(a,b)$ and $w_{ki}^{(j)}$ $(j \in \mathbb{N};\ i,k = 1,\dots,n)$ be polynomials in λ with complex coefficients.

For $\lambda \in \mathbb{C}$ and $\eta \in W_2^n(a,b)$ we consider the boundary eigenvalue problem

$$\eta^{(n)} + \sum_{j=0}^{n-1} p_j(\cdot,\lambda)\eta^{(j)} = 0, \tag{1.2.20}$$

$$\sum_{i=1}^{n}\sum_{j=0}^{1} w_{ki}^{(j)}(\lambda)\eta^{(i-1)}(a_j) = 0 \quad (k = 1,\dots,n). \tag{1.2.21}$$

The function π defined by

$$\pi(\cdot,\rho) := \rho^n + \sum_{i=0}^{n-1} \rho^i \pi_{n-i,n-i} \quad (\rho \in \mathbb{C}) \tag{1.2.22}$$

is called the characteristic function of the differential equation (1.2.20). We associate a first order system to the n-th order differential equation. This system is defined by the operator

$$T^D(\lambda)y := y' - A(\cdot,\lambda)y \quad (y \in (W_2^1(a,b))^n, \lambda \in \mathbb{C}), \tag{1.2.23}$$

where

$$A := (\delta_{i,j-1} - \delta_{i,n}p_{j-1})_{i,j=1}^n = \begin{pmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & 0 \\ 0 & & \cdot & \cdot & \\ & 0 & & 0 & \\ -p_0 & \cdot & \cdot & \cdot & -p_{n-1} \end{pmatrix}. \tag{1.2.24}$$

We assume that there are a matrix function $C(\cdot,\lambda) \in M_n(W_2^1(a,b))$ depending polynomially on λ and a positive real number γ such that

$$C(\cdot,\lambda) \text{ is invertible in } M_n(W_2^1(a,b)) \text{ if } |\lambda| \geq \gamma \tag{1.2.25}$$

and such that the equation

$$C^{-1}(\cdot,\lambda)T^D(\lambda)C(\cdot,\lambda)\hat{y} = \hat{y}' - \widetilde{A}(\cdot,\lambda)\hat{y} =: \tilde{T}^D\hat{y} \tag{1.2.26}$$

holds for $|\lambda| \geq \gamma$ and $\hat{y} \in (W_2^1(a,b))^n$, where

$$\tilde{A}(\cdot,\lambda) = \lambda A_1 + A_0 \quad (|\lambda| \geq \gamma) \tag{1.2.27}$$

and the coefficient matrices of $\tilde{A}$ fulfils the assumptions of the coefficient matrices in (1.2.1). When $A^0 = 0$ in the first order system (1.2.1) and there is a boundary condition dependent on the eigenvalue parameter λ, estimates (1.2.25)–(1.2.27) are not asymptotically constant in λ. In order for (1.2.25)–(1.2.27) to hold we require that there is an $n \times n$ matrix polynomial $C_2(\lambda)$ whose determinant is not identically zero such that the following properties hold:
There is a matrix function $W_0 \in M_n(L_1(a,b))$ such that

$$C_2^{-1}(\lambda)W(\cdot,\lambda) - W_0 = O(\lambda^{-1}) \quad \text{ in } M_n(L_1(a,b)) \text{ as } \lambda \to \infty, \tag{1.2.28}$$

and there are $n \times n$ matrices $W_0^{(j)}$ and

$$\sum_{j=0}^{1} |C_2^{-1}(\lambda)W^{(j)}(\lambda) - W_0^{(j)}| = O(\lambda^{-1}) \text{ as } \lambda \to \infty \tag{1.2.29}$$

hold. The boundary conditions (1.2.21) and a function $C(x,\lambda)$ satisfying (1.2.25)-(1.2.27) are considered with the matrix functions

$$\begin{cases} W^{(j)}(\lambda) := (w_{ki}^{(j)}(\lambda))_{k,i=1}^n C(a_j,\lambda), \\ W(\lambda) := (w_{ki}(\lambda))_{k,i=1}^n C(\lambda), \end{cases} \tag{1.2.30}$$

and set

$$\widehat{T}^R(\lambda)\hat{y} := \sum_{j=0}^{1} W^{(j)}(\lambda)\hat{y}(a_j) \qquad (\hat{y} \in W_2^1(a,b))^n. \tag{1.2.31}$$

Definition 1.2.3. The boundary eigenvalue problem (1.2.20)–(1.2.21) is called Birkhoff Regular if $\pi_{nn} \neq 0$ and if there are matrix functions $C(\cdot,\lambda)$ satisfying (1.2.25)–(1.2.27) and $C_2(\cdot,\lambda)$ satisfying (1.2.28)–(1.2.29) so that the associated boundary eigenvalue problem $\widetilde{T}^D\hat{y} = 0$, $C_2(\lambda)^{-1}\widehat{T}^R(\lambda)\hat{y} = 0$ is Birkhoff regular in the sense of Definition 1.2.2.

We quote [5, Proposition 4.1.7].

Proposition 1.2.4. *Let $l = n$ and $\Delta(\lambda) = \mathrm{diag}(\delta_1(\lambda), \ldots, \delta_\mathrm{n}(\lambda))$ as given by* (1.2.15)*. We suppose that*

$$\varphi_\nu = \frac{2\pi(\nu-1)}{n} \qquad (\nu = 1, \ldots, n).$$

i) If n is even, then the values of Δ are the diagonal matrices with $\frac{n}{2}$ consecutive ones and $\frac{n}{2}$ consecutive zeros in the diagonal in a cyclic arrangement.
ii) If n is odd, then the values of Δ are the diagonal matrices with $\frac{n+1}{2}$ consecutive ones and $\frac{n-1}{2}$ consecutives zeros in the diagonal and the diagonal matrices with $\frac{n-1}{2}$ consecutive ones and $\frac{n+1}{2}$ consecutive zeros in the diagonal, each in a cyclic arrangement.

We quote a simplified version of [5, Proposition 7.2.3].

Proposition 1.2.5. *Let $l := n$, and suppose that*

$$\pi(\cdot,\rho) = \pi_l(\cdot,\rho)$$

where

$$\pi_l(\cdot,\rho) = \rho^l + \sum_{j=1}^{l} \rho^{l-j}\pi_{j,j}. \tag{1.2.32}$$

Suppose that for all $x \in [a,b]$ *the roots of* $\pi_l(x,\rho) = 0$ *are simple and nonzero and that there is* $\kappa \in \mathbb{N}\backslash\{0\}$ *such that* $\pi_{1,1},\dots,\pi_{l,l} \in W_2^\kappa p(a,b)$. *Then there are* $r_1,\dots,r_l \in W_p^\kappa(a,b)$ *such that*

$$\pi_l(\cdot,\rho) = \Pi_{j=1}^l(\rho - r_j(x)) \tag{1.2.33}$$

holds for all $x \in [a,b]$ *and* $\rho \in \mathbb{C}$. *In addition, we have that* $r_j^{-1} \in W_2^\kappa(a,b)$ *for* $j = 1,\dots,l$.

We quote [5, Theorem 7.2.4].

Theorem 1.2.6. *Let* $l \in \{1,\dots,n\}$, *be such that* $\pi_{l,l} \neq 0$ *and* $\pi_{i,i} = 0$ *for* $i = l+1,\dots,n$. *Suppose that* $\pi_{i,i} \in L_\infty(a,b)$. *Then there is a matrix function*

$$C(x,\lambda) = \mathrm{diag}(\lambda^{\nu_1},\dots,\lambda^{\nu_n})C_1(x)$$

with $\nu_1,\dots,\nu_n \in \mathbb{Z}$ *and* $C_1 \in M_n(W_2^1(a,b))$ *such that* $\tilde{A}(\cdot,\lambda)$ *given by* (1.2.26) *has the form* (1.2.27), *where*

$$A_1 = \mathrm{diag}(0,\dots,0,\mathrm{r}_1,\dots,\mathrm{r}_l)$$

and $0 \neq r_j^{-1} \in \mathbb{C}$ *for* $j = 1,\dots,l$,
i) $\pi_{l,l}^{-1} \in L_\infty(a,b)$;
ii) $p_0(\cdot,\lambda) = \sum_{j=0}^{l} \lambda^j \pi_{n,j}$

iii) $\pi_{i,i} \in W_2^1(a,b)$ *for* $i = 1, \dots, l$

or

$l = 1$ *and* $\dfrac{\pi_{n-i+1,1}}{\pi_{l,l}} \in W_2^1(a,b)$ *for* $i = 1, \dots, n-1$*;*

iv) The zeros of $\pi_l(x, \rho)$ *are simple and different from zero for all* $x \in [a,b]$*, where* π_l *is defined in* (1.2.32).

A. If i), ii) and iv) hold and if $l = 1$ *or* $\pi_{i,l} \in W_2^1(a,b)$ *for* $i = l+1, ..., n$*, then* $\nu_i = i - 1 \quad (i = 2, \dots, n)$ *and*

$$C_1 = \begin{pmatrix} 1 & \dots & 1 \\ r_1 & \dots & r_l \\ & \vdots & r_l \\ r_1^{l-1} & \dots & r_1^{l-1} \end{pmatrix}, \tag{1.2.34}$$

where $r_1, \dots, r_l \in W_2^1(a,b)$ *are the roots of* $\pi(\cdot, \rho) = 0$ *according to Proposition 1.2.5 if* $l > 1$.

B. If i), ii) and iv) hold and if $\pi_{i,i} \in W_2^1(a,b)$ *for* $i = 1, ..., l$*, then* $\nu_i = i$ $(i = 1, \dots, n)$ *and*

$$C_1 = \begin{pmatrix} 1 & \dots & 1 \\ r_1 & \dots & r_l \\ & \vdots & r_l \\ r_1^{l-1} & \dots & r_1^{l-1} \end{pmatrix}, \tag{1.2.35}$$

where $r_1, \dots, r_l \in W_2^1(a,b)$ *are the roots of* $\pi(\cdot, \rho) = 0$ *according to Proposition 1.2.5.* We consider the differential equation

$$K\eta = \lambda^l H\eta \qquad (\eta \in W_1^n(a,b)), \tag{1.2.36}$$

where

$$K\eta = \eta^{(n)} + \sum_{i=0}^{n-1} k_i \eta^{(i)}, \tag{1.2.37}$$

$$H\eta = h_0 \eta^{(0)}, \tag{1.2.38}$$

with $k_i \in W_2^i(a,b)$ and $h_0 \in W_2^1(a,b)$. In the problem under consideration $h_0 = 1$. We associate the differential operator

$$L^D(\lambda)\eta := K\eta - \lambda^l \eta \quad (\eta \in W_2^n(a,b)), \tag{1.2.39}$$

with the differential equation (1.2.37), together with two point boundary conditions

$$L^R(\lambda)\eta := \left(\sum_{i=0}^{n-1} w_{ki}^{(0)}(\lambda)\eta^{(i-1)}(a) + \sum_{i=0}^{n-1} w_{ki}^{(1)}(\lambda)\eta^{(i-1)}(b) \right)_{k=1}^{n} = 0, \tag{1.2.40}$$

where $w_{ki}^{(j)}$ are polynomials. Let

$$W^{(j)}(\lambda) = (w_{ki}^{(j)}(\lambda))_{k,i=1}^{n} \quad (j = 0, 1). \tag{1.2.41}$$

We quote a simplified version of [5, Theorem 8.2.1].

Theorem 1.2.7. *Suppose that $l = n$ and let $k \in \mathbb{N}$. Suppose that*

α) $k_j \in L_2(a,b)$ for $j = 0, \ldots, n-1-k$ and $k_{n-1-j} \in W_2^{k-j}(a,b)$ for $j = 0, \ldots, \min\{k-1, n-1\}$

For sufficiently large λ, the differential equation $K\eta = \lambda^l \eta$ has a fundamental

system $\{\eta_1(\cdot,\lambda),\dots,\eta_n(\cdot,\lambda)\}$ *with the following properties:*
Set $\tilde{k} := k$. *Let* $\omega_j = \exp\{\frac{2\pi i(j-1)}{l}\}$ $(j = 1,\dots,l)$. *There are functions* $\varphi_r \in W_2^{k+1-r}(a,b)$, $r = 0,\dots,\tilde{k}$, *such that* φ_0 *is the solution of the initial value problem*

$$\varphi_0' - \frac{1}{l}(h_0)\varphi_0 = 0 \qquad \varphi_0(a) = 1, \tag{1.2.42}$$

$$\eta_\nu^{(\mu)}(x,\lambda) = \left[\frac{d^\mu}{dx^\mu}\right]\sum_{r=0}^{\tilde{k}}(\lambda\omega_\nu)^{-r}\varphi_r(x)e^{\lambda\omega_\nu(x-a)} + \{o(\lambda^{-\tilde{k}+\mu})\}_\infty e^{\lambda\omega_\nu(x-a)}$$
$$(\nu = 1,\dots,n; \mu = 0,\dots,n-1), \tag{1.2.43}$$

where $\left[\frac{d^\mu}{dx^\mu}\right]$ *means that we omit those terms of the Leibniz expansion which contain a function* $\varphi_r^{(j)}$ *with* $j > \tilde{k} - r$.

We denote the i-th unit vectors in $\mathbb{C}^n$ and $\mathbb{C}^l$ by e_i and ε_i. For $i \in \mathbb{Z}\backslash\{1,\dots,n\}$ or $i \in \mathbb{Z}\backslash\{1,\dots,l\}$ we set $e_i := 0$ and $\varepsilon_i := 0$, respectively. We can write the matrix $A(\cdot,\lambda^l)$ of the corresponding system as

$$A(\cdot,\lambda^l) = \left(\ \varepsilon_l a_2^\intercal + J_l + \lambda^l \varepsilon_l \varepsilon_1^\intercal\ \right)$$

where

$$a_2^\intercal := -(k_0, \ldots, , k_{n-1}), \tag{1.2.44}$$

$$J_r := \begin{pmatrix} 0 & 1 & & & & \\ & 0 & . & & 0 & \\ & & . & . & & \\ & & & . & . & \\ & 0 & & . & & 1 \\ & & & & & 0 \end{pmatrix} \in M_r(\mathbb{C}). \tag{1.2.45}$$

We set

$$\varepsilon^\intercal := \sum_{i=1}^{l} \varepsilon_i^\intercal = (1, \ldots, 1) \in \mathbb{C}^l, \tag{1.2.46}$$

$$\Omega_l := \mathrm{diag}(\omega_1, \ldots, \omega_\mathrm{l}), \tag{1.2.47}$$

$$\Xi_r(\lambda) := \mathrm{diag}(1, \lambda, \ldots, \lambda^{\mathrm{r}-1}) \in M_r(\mathbb{C}), \tag{1.2.48}$$

$$V := \sum_{i=1}^{l} \varepsilon_i \varepsilon^\intercal \Omega_l^{i-1} = \begin{pmatrix} 1 & . & . & 1 \\ \omega_1 & . & . & \omega_l \\ \vdots & & . & \vdots \\ \omega_1^{l-1} & . & . & \omega_l^{l-1} \end{pmatrix}. \tag{1.2.49}$$

If we observe that

$$\varepsilon^\intercal \Omega_l^j \varepsilon = \sum_{i=1}^{l} \omega_i^j = \begin{cases} l \;\; \text{if } j = 0 \mod (l), \\ 0 \;\; \text{if } j \neq 0 \mod (l), \end{cases} \tag{1.2.50}$$

we obtain that V is invertible with

$$V^{-1} = \frac{1}{l}\sum_{i=1}^{l} \Omega_l^{i-1}\varepsilon\varepsilon_i^{\intercal}. \tag{1.2.51}$$

We obtain that $y' - \tilde{A}y = 0$ has a fundamental system

$$\tilde{Y}(\cdot,\lambda) = \left(\sum_{r=0}^{k} \lambda^{-r}P^{[r]} + \{o(\lambda^{-k})\}_\infty\right) E(\cdot,\lambda). \tag{1.2.52}$$

if λ is sufficiently large, where $P^{[r]} \in M_n(W_2^{k+1-r}(a,b))$ and

$$E(x,\lambda) = \mathrm{diag}(1,\ldots,1,\mathrm{e}^{\lambda\omega_1(\mathrm{x}-\mathrm{a})},\ldots,\mathrm{e}^{\lambda\omega_l(\mathrm{x}-\mathrm{a})}).$$

We infer that

$$(\eta_{\mu-1,\nu}(\cdot,\lambda))_{\mu,\nu=1}^{n} := Y(\cdot,\lambda) := C(\lambda)\tilde{Y}(\cdot,\lambda) \tag{1.2.53}$$

is a fundamental matrix of $T^D(\lambda)y = 0$ if λ is sufficiently large. We set

$$\tilde{Y}(\cdot,\lambda)E(\cdot,\lambda)^{-1} =: \tilde{Q}_{22}(\cdot,\lambda) \tag{1.2.54}$$

and obtain that

$$Y(\cdot,\lambda) = \Xi_l(\lambda)V\tilde{Q}_{22}(\cdot,\lambda)E(\cdot,\lambda).$$

We set $\eta_\nu := \eta_{0,\nu}$. Then $\{\eta_1(\cdot,\lambda),\ldots,\eta_n(\cdot,\lambda)\}$ is a fundamental system of

$K\eta = \lambda^l H\eta$ and $\eta_\nu^{(\mu)} = \eta_{\mu,\nu}$ $(\nu = 1, \ldots, n;\ \mu = 0, \ldots, n-1)$. We have

$$\tilde{Q}_{ij}(\cdot, \lambda) = \sum_{r=0}^{k} \lambda^{-r} Q_{ij}^{[r]} + \{o(\lambda^{-k})\}_\infty \quad (i, j = 1, 2) \tag{1.2.55}$$

where the elements of $Q_{ij}^{[r]}$ belong to $W_2^{k+1-r}(a,b)$. We set $Q_{ij}^{[r]} := 0$ for $r < 0$ and $Q_{ij}^{[r]} := 0$ for $i \neq j$ and $r \geq 0$. Also,

$$Q_{22}^{[0]\prime} + \frac{1}{l} k_{n-1} Q_{22}^{[0]} = 0 \quad Q_{22}^{[0]}(a) = I_l, \tag{1.2.56}$$

$$\begin{cases} \Omega_l Q_{22}^{[r]} - Q_{22}^{[r]} \Omega_l = Q_{22}^{[r-1]\prime} \\ +\frac{1}{l} \sum\limits_{j=1}^{l} k_{n-1-j} \Omega_l \varepsilon \varepsilon^\intercal \Omega^{-1-j} Q_{22}^{[r-1-j]} \quad (r = 1, \ldots, k), \end{cases} \tag{1.2.57}$$

$$\begin{cases} 0 = \varepsilon_\nu^\intercal \left\{ Q_{22}^{[k]\prime} + \frac{1}{l} k_{n-1} \Omega_l \varepsilon \varepsilon^\intercal \Omega_l^{-1} Q_{22}^{[k]} \right. \\ \left. +\frac{1}{l} \sum\limits_{j=1}^{l} k_{n-1-j} \Omega_l \varepsilon \varepsilon^\intercal \Omega^{-1-j} Q_{22}^{[k-j]} \right\} \varepsilon_\nu \ (\nu = 1, \ldots, l). \end{cases} \tag{1.2.58}$$

We immediately infer that

$$\eta_\nu(x, \lambda) = \left\{ \sum_{r=0}^{k} \lambda^{-r} \varphi_{\nu r}(x) + \{o(\lambda^{-k})\}_\infty \right\} e^{\lambda \omega_\nu x}, \tag{1.2.59}$$

where $\varphi_{\nu r} \in W_2^{k+1-r}(a,b)$. If $1 \leq \nu \leq n$, and $0 \leq r \leq k$, then

$$\varphi_{\nu r} = \varepsilon_1^\intercal V Q_{22}^{[r]} \varepsilon_\nu, \tag{1.2.60}$$

leading to

$$\varphi_{\nu r} = \omega_\nu^{-r}\varphi_{1,r} \tag{1.2.61}$$

for $\nu = 1, \ldots, n$ and $r = 0, \ldots, \tilde{k}$. Hence, for $\nu = 1, \ldots, n$,

$$\eta_\nu(x, \lambda) = \left\{ \sum_{r=0}^{\tilde{k}} (\lambda\omega_\nu)^{-r}\varphi_r(x) + \{o(\lambda^{-k})\}_\infty \right\} e^{\lambda\omega_\nu x}, \tag{1.2.62}$$

where $\varphi_r := \varphi_{1,r}$, which yields

$$\varphi_0 = \varphi_{10} = \varepsilon_1^{\intercal} Q_{22}^{[0]} \varepsilon_1. \tag{1.2.63}$$

The relationship between the first order system and the nth-order differential equation is given as in [5, Proposition 6.1.2].

Proposition 1.2.8. *Let $y \in W_2^n(a, b)$, $\lambda \in \Omega$, and set*

$$\hat{y} := \begin{pmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{pmatrix}.$$

Then $\hat{y} \in (W_2^1(a,b))^n$ *and*

$$T^D(\lambda)\hat{y} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ L^D(\lambda)y \end{pmatrix}.$$

2

Periodic and Antiperiodic Boundary Conditions

Periodic boundary conditions on a homogenous beam that is compressed or stretched by a force g means that looking at the solutions of the differential equation as standing waves, we want profiles that are the same as the initial profile after some time has lapsed. We set out to find periodic and anti periodic boundary conditions for the fourth order differential equation studied by Möller, Zinsou and Pivorvarchik. They studied boundary conditions that are separated, dependent on the eigenvalue parameter and a mixture of separated and dependent on the eigenvalue parameter such that matrices M, K and A in the spectral representation

$$L(\lambda) = \lambda^2 M - i\alpha\lambda K - A \tag{2.0.1}$$

of boundary value problems are self-adjoint.

In Section 2.1, we write the fourth order partial differential equation as an ordinary differential equation by assuming that solutions to the differential equation will be superpositions of standing waves. The ordinary differential equation is then written in terms of its quasi-derivatives and the Lagrange's identity is integrated to obtain the Green's formula. Using Matlab we implement [9, Theorem 10.3.5] to find boundary conditions to the ordinary differential equation such that at least one of the boundary condition is periodic. We verify in Section 2.2 that the two boundary value problems with boundary conditions obtained in Section 2.1, fulfil the criteria that the differential operator in the spectral representaion of the boundary value problems is self-adjoint and prove Theorem 2.2.4. Theorem 2.2.4 states equivalent statements for boundary conditions of the fourth order differential equation where we have one boundary condition dependent on the eigenvalue parameter and at least one boundary condition dependent on the eigenvalue parameter. We characterise boundary conditions that fulfil our criteria in Theorem 2.3.1 in Section 2.3.

2.1 A vibrating string with periodic boundary conditions

This following sections are based on the article [7]. A fourth order partial differential equation describing small transversal vibrations of a homogeneous beam compressed or stretched by a force g can be described by

$$\frac{\partial^4}{\partial x^4}u(x,t) - \frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}u(x,t) = -\frac{\partial^2}{\partial x^2}u(x,t), \tag{2.1.1}$$

where we assume $g \in C^1[0,a]$ to be a sufficiently smooth real-valued function and $a > 0$. If $g > 0$, then the beam is stretched, and if $g < 0$, then it is compressed. Our quest is to find boundary conditions to (2.1.1) such that at least one of the boundary conditions is periodic, at least one of the boundary conditions depends on the eigenvalue parameter and all the coefficient matrices in (2.0.1) are self-adjoint.

We solve the differential equation (2.1.1) using the superposition of standing waves method, where we assume that solutions $u(x,t)$ can be written as $u(x,t) = e^{i\lambda t}y(\lambda,x)$. Then, (2.1.1) can be written as an ordinary differential equation

$$y^{(4)}(\lambda,x) - (gy')'(\lambda,x) = \lambda^2 y(\lambda,x). \tag{2.1.2}$$

We want to find boundary conditions, B_i^0 and B_i^a to (2.1.2) that satisfy a set criteria,

$$B_i^0(\lambda)\begin{pmatrix} y(0) \\ y'(0) \\ y''(0) \\ y'''(0) \end{pmatrix} + B_i^a(\lambda)\begin{pmatrix} y(a) \\ y'(a) \\ y''(a) \\ y'''(a) \end{pmatrix} = 0, \ \ i = 1,2,3,4 \ \ B_i^0, B_i^a \in \mathbb{C}^4 \tag{2.1.3}$$

and

$$y \in W_2^4(0,a) := \{y \in L_2(0,a) : \forall j \in \{1,2,3,4\}, \ y^{(j)} \in L_2(0,a)\}. \tag{2.1.4}$$

The Lagrange's identity gives boundary terms arising from integration by parts of a self-adjoint linear differential operator.

We use methods in [9], which discuss extensively the quadratic operator pencil of the form (2.0.1). We formulate our problem to implement [9, Theorem 10.3.5] to find boundary conditions which fulfil a set of criteria. The theorem specifies a criterion for self-adjointness for a class of differential operators. The differential operator in (2.0.1) is A. We would like that A is self-adjoint with added condition that at least one of the boundary conditions is periodic and at least one of the boundary conditions depends on the eigenvalue parameter. We implement [9, Theorem 10.3.5] in Matlab which gives us boundary conditions that fulfil the set of criteria.

From (1.1.1), let $n = 4$ and $4 = 2k$. We consider a fourth order differential expression of the form ℓ,

$$\ell y = \sum_{m=0}^{2} (g_m y^{(m)})^{(m)} \tag{2.1.5}$$

on an interval $[0, a]$, where $g_m \in W_2^{(m)}(0, a)$, $m = 0, 1, 2$, are real valued functions and $|g_2(x)| > \varepsilon$ for some $\varepsilon > 0$ and $x \in [0, a]$. The quasi-derivatives of the differential expression of (2.1.2) are

$$y^{[0]} = y,\ y^{[1]} = y',\ y^{[2]} = y'',\ y^{[3]} = y^{(3)} - gy',\ y^{[4]} = y^{(4)} - (gy')', \tag{2.1.6}$$

making the Lagrange's identity

$$
\begin{aligned}
(\ell y)\overline{z} - y(\ell\overline{z}) &= \frac{d}{dx}[y,z] \\
&= \frac{d}{dx}\sum_{j=1}^{2}(-1)^j\left(y^{[j-1]}\overline{z^{[4-j]}} - y^{[4-j]}\overline{z^{[j-1]}}\right) \\
&= \frac{d}{dx}\left\{-y\overline{(z^{(3)}-gz')} - (y^{(3)}-gy')\overline{z} + y'\overline{z''} - y''\overline{z'}\right\} \qquad (2.1.7)
\end{aligned}
$$

where $y, z \in W_2^4(0,a)$ which is integrated on $[0,a]$ to obtain the Green's formula

$$(\ell y, z) - (y, \ell z) = [y,z](a) - [y,z](0) \qquad (2.1.8)$$

where $(\cdot,\cdot)$ is the inner product in $L_2(0,a)$. Let U_1 be a 3×8 matrix, U_2 a 1×8 matrix and V a 1×8 matrix. Then the operator A in the Hilbert space $L_2(0,a)\oplus\mathbb{C}$ is defined by

$$D(A) = \left\{Y \in W_2^4(0,a)\oplus\mathbb{C},\ U_1\hat{Y}=0,\ c = U_2\hat{Y}\right\}, \qquad (2.1.9)$$

$$AY = \begin{pmatrix} \ell y \\ V\hat{Y} \end{pmatrix}, \qquad (2.1.10)$$

where

$$\hat{Y} = \left(y(0),\ldots,y^{[n-1]}(0), y(a),\ldots,y^{[n-1]}(a)\right)^{\mathsf{T}}. \qquad (2.1.11)$$

For $m = 2$, (1.1.10) becomes

$$\begin{cases} J_{2,0} = ((-1)^{s-1}\delta_{s,m+1-t})_{s,t=1}^{2}\,,\ J_{2,1} = \begin{pmatrix} 0 & J_{2,0} \\ -J_{2,0}^{*} & 0 \end{pmatrix}, \\ J_2 = \begin{pmatrix} -J_{2,1} & 0 \\ 0 & J_{2,1} \end{pmatrix}. \end{cases} \tag{2.1.12}$$

Finally define

$$U_3 = \begin{pmatrix} J_2 \\ V \\ -U_2 \end{pmatrix}, \tag{2.1.13}$$

$$U = \begin{pmatrix} U_1 & 0 & 0 \\ U_2 & -I & 0 \\ V & 0 & -I \end{pmatrix}. \tag{2.1.14}$$

The Matlab code is written so that we have one of the boundary conditions dependent on the eigenvalue parameter and the output is the matrix U_1 that ensures that [9, Theorem 10.3.5] is satisfied. One of the boundary conditions that fulfil the criteria is verified below. The boundary value problem with a fourth order differential equation (2.1.2) together with the following boundary conditions

$$y(\lambda, 0) - y(\lambda, a) = 0, \tag{2.1.15}$$

$$y^{[3]}(\lambda, 0) - y^{[3]}(\lambda, a) = 0, \tag{2.1.16}$$

$$y'(\lambda, 0) = 0, \tag{2.1.17}$$

$$y''(\lambda, a) + i\alpha\lambda y'(\lambda, a) = 0, \tag{2.1.18}$$

defined on the interval $[0,a]$, where $a > 0$, $\alpha > 0$ and $g \in C^1[0,a]$ initiates the study. The boundary conditions (2.1.15) and (2.1.16) are periodic, while the boundary conditions (2.1.17) and (2.1.18) are separated, the boundary condition (2.1.18) is also dependent on the eigenvalue parameter λ. The operators A, K and M are self-adjoint, M and K are bounded, K has rank 1, $M \geq 0$, $K \geq 0$, $M + K \gg 0$, $N(M) \cap N(A) = \{0\}$ and A is bounded below and has a compact resolvent.

The statements about K and M are obvious. If $(y,c)^\intercal \in N(M) \cap N(A)$ then $(y,c)^\intercal \in N(M)$ gives $y = 0$, and $(y,c)^\intercal \in D(A)$ where $c = y'(a)$ leads to $c = y'(a) = 0$. Hence $N(M) \cap N(A) = \{0\}$. We are going to use Theorem 1.1.4 to verify that A is self-adjoint. The differential expression (1.1.1) with $n = 4$, $g_0 = 0$, $g_1 = -g \in C^1[0,a]$ and $g_2 = 1$ represents (2.1.2) as

$$\ell y = (g_0 y) + (g_1 y')' + (g_2 y'')'' = y^{(4)} - (gy')' = L_0(\lambda)y. \tag{2.1.19}$$

The number of eigenvalue independent boundary conditions as given by (2.1.15)–(2.1.17) is 3, leaving only one boundary condition dependent on the eigenvalue parameter. Matrices U_1, U_2 and V given by

$$U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.1.20}$$

$$U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{2.1.21}$$

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.1.22}$$

Then the operator A can also be defined in terms of these matrices as

$$AY = \begin{pmatrix} \ell y \\ V\hat{Y} \end{pmatrix},$$

$$D(A) = \left\{ Y \in W_2^4(0,a) \oplus \mathbb{C}, \quad U_1\hat{Y} = 0, \quad c = U_2\hat{Y} \right\},$$

similar to (1.1.8) and (1.1.7). We now specify the matrices J_2, U_3 and U as

$$J_2 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.1.23}$$

$$U_3 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \tag{2.1.24}$$

and

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}. \tag{2.1.25}$$

J_2 is a 8×8 matrix, U_3 is a 10×8 matrix and U is a 5×10 matrix where I in (1.1.12) is a 1×1 matrix.

We find $N(U_1)$ and $R(U^*)$ as

$$N(U_1) = \text{span}\{e_1 + e_5, e_3, e_4 + e_8, e_6, e_7\} \subset \mathbb{C}^8 \tag{2.1.26}$$

and

$$R(U^*) = \text{span}\{e_1 - e_5, e_4 - e_8, e_2, e_6 - e_9, e_7 - e_{10}\} \subset \mathbb{C}^{10}. \tag{2.1.27}$$

Then compare $U_3(N(U_1))$ with $R(U^*)$, showing that they are equal and A is self-adjoint as postulated by Theorem 1.1.4. Lastly, A has a compact resolvent in view of Theorem 1.1.5. The coefficient of the highest derivative in the differential component of A is $g_2 = 1 > 0$ as required by Theorem 1.1.5 (i). Particular values in assumptions of Theorem 1.1.5 (ii)–(iv) for (2.1.2) and (2.1.15)–(2.1.18) are

$$U_1\hat{Y} = \begin{pmatrix} y(0) - y(a) \\ y^{[3]}(0) - y^{[3]}(a) \\ y^{[1]}(0) \end{pmatrix}, \tag{2.1.28}$$

$$U_2\hat{Y} = y^{[1]}(a), \tag{2.1.29}$$

$$V\hat{Y} = y^{[2]}(a). \tag{2.1.30}$$

The first and third component of $U_1\hat{Y}$ have quasi-derivatives of order zero and one. Hence their order given by m is less than $k = 2$, half the order of the differential equation and the second component has order three which is $m = 3 \geq k = 2$. The component of $U_2\hat{Y}$ has order one which is less than k. A does not have components of $U_2\hat{Y}$ with quasi-derivatives that are greater than k and the condition on $V\hat{Y}$ is irrelevant. Thus all the conditions of Theorem 1.1.5 are fulfilled and A is bounded below. An alternative criterion is used to show that (2.1.2) and (2.1.15)–(2.1.18) is self-adjoint. First, define

$$W = J_2 + U_2^* V - V^* U_2. \tag{2.1.31}$$

Then $W(N(U_1))$ and $R(U_1^*)$ are given by

$$W(N(U_1)) = \operatorname{span}\{e_4 - e_8, e_2, -e_1 + e_5\} \subset \mathbb{C}^{8\times 3} \tag{2.1.32}$$

and

$$R(U_1^*) = \operatorname{span}\{e_1 - e_5, e_2, e_4 - e_8\} \subset \mathbb{C}^{8\times 3}. \tag{2.1.33}$$

A comparison of $W(N(U_1))$ and $R(U_1^*)$ shows that $W(N(U_1)) = R(U_1^*)$ as required by [11, Corollary 2.5, Theorem 2.12 and Theorem 3.7].

2.2 Periodic and a single eigenvalue dependent boundary condition

Consider on the interval $[0, a]$, where $a > 0$, the differential equation (2.1.2) with boundary conditions

$$U_1\hat{Y} = 0, \tag{2.2.1}$$

$$(V + i\alpha U_2)\hat{Y} = 0, \tag{2.2.2}$$

where the matrices U_1, U_2 and V are of the following form

$$U_1 = (u^1_{i,j})^{3,8}_{i=1,j=1}, \tag{2.2.3}$$

$$U_2 = (u^2_{i,j})^{1,8}_{i=1,j=1}, \tag{2.2.4}$$

$$V = (v_{i,j})^{1,8}_{i=1,j=1}. \tag{2.2.5}$$

Consider a particular case where U_2 and V contain exactly one non-zero element such that the non-zero element of U_2 is in a different column to the non-zero element of V and the non-zero elements of U_1 are positioned such that the first column of (1.1.12) has linearly independent rows. These forms of U_1, U_2 and V ensure that each $y^{[j]}(0)$ and $y^{[j]}(a)$ in (1.1.9) occurs at most once in the boundary condition (2.2.1). The operator A in (1.1.8) is given by

$$AY = \begin{pmatrix} \ell y \\ V\hat{Y} \end{pmatrix},$$

$$D(A) = \left\{ Y \in W_2^4(0,a) \oplus \mathbb{C}, \quad U_1\hat{Y} = 0, \quad c = U_2\hat{Y} \right\}.$$

We recall that the dimension of the domain of a linear map between two spaces is given by the sum of the dimension of the null space and the rank of this linear map. In addition, two finite dimensional spaces coincide if one space is contained in the other and their dimensions are equal. A vector space $\mathbb{C}^8$ acted upon by these three matrices U_1, U_2 and V means that rank U_2 and rank V are given by

$$8 - \dim(N(U_2)) = 1 \quad \text{and} \quad 8 - \dim(N(V)) = 1,$$

respectively.

Proposition 2.2.1. *Let U_2 and V contain exactly one non-zero element such that the non-zero element in U_2 is in a different column to the non-zero element in V. Let*

$$W = J_2 + U_2^* V - V^* U_2. \tag{2.2.6}$$

Then U_2^*V and V^*U_2 are 8×8 matrices of rank *1, $U_2^*V - V^*U_2$ is an 8×8 matrix of* rank *2 and W is an 8×8 matrix of* rank *at least 6.*

Let the non-zero element of V be at $j = p$ and that of U_2 be at $j = s$, $s \neq p$.

Then

$$U_2^* V = \left((\overline{u_{ij}^2})_{i=1,j=1}^{1,8} \right)^{\mathsf{T}} (v_{ij})_{i=1,j=1}^{1,8} = (\overline{u^2}_{1j} v_{1i})_{j=1,i=1}^{8,8},$$

has exactly one non-zero element, $\overline{u^2}_{1s} v_{1p}$, at $j = s$, $i = p$. The position of the only non-zero element of $V^* U_2$ is in row p and column s, thus $U_2^* V - V^* U_2$ has rank 2. J_2 in (2.2.6) is invertible with rank 8 and $U_2^* V - V^* U_2$ has rank 2. Hence, the rank of W is at least 6.

Remark 2.2.2. Whenever $Y \in D(A)$ then $\hat{Y} \in N(U_1)$, and for every $u \in N(U_1)$ there is a $Y \in D(A)$ such that $\hat{Y} = u$.

Corollary 2.2.3. *If A is self-adjoint then* $\operatorname{rank} W = 6$ *and* $W(N(U_1)) = R(U_1^*)$. *Proposition 1.1.3 states that $Z \in D(A^*)$ if and only if $Z \in W_2^4(0,a) \oplus \mathbb{C}$ and there is $e \in \mathbb{C}$ such that*

$$[y, z](a) - [y, z](0) + d^* V \hat{Y} - e^* U_2 \hat{Y} = 0 \tag{2.2.7}$$

for all $\hat{Y} \in N(U_1)$. For $Z \in D(A^*)$, e is unique and

$$A^* Z = \begin{pmatrix} \ell z \\ e \end{pmatrix}.$$

We use (1.1.3) *for $W, Z \in D(A) = D(A^*)$ and*

$$[y, z](a) - [y, z](0) = \hat{Z}^* J_2 \hat{Y}$$

together with values of e and d as implied by (1.1.8) *and* (1.1.7) *respectively, which* *we substitute into (2.2.7) to get*

$$
\begin{aligned}
0 &= [y,z](a) - [y,z](0) + d^*V\hat{Y} - e^*U_2\hat{Y} \\
&= [y,z](a) - [y,z](0) + (U_2\hat{Z})^*V\hat{Y} - (V\hat{Z})^*U_2\hat{Y} \\
&= \hat{Z}^*J_2\hat{Y} + \hat{Z}^*U_2^*V\hat{Y} - \hat{Z}^*V^*U_2\hat{Y} \\
&= \hat{Z}^*(J_2 + U_2^*V - V^*U_2)\hat{Y} \\
&= \hat{Z}^*W\hat{Y},
\end{aligned}
$$

where $\hat{Y}$ *and* $\hat{Z}$ *are as defined in* (1.1.9). *This means that* $W\hat{Y} \perp \hat{Z}$, *i.e* $W(N(U_1)) \subset (N(U_1))^{\perp} = R(U_1^*)$. *We use this containment of* $W(N(U_1))$ *in* $R(U_1^*)$ *to compare their dimensions as*

$$
\begin{aligned}
3 = \text{rank } U_1^* &\geq \dim(W(N(U_1))) \\
&\geq \dim(N(U_1)) - (8 - \text{rank } W) \\
&= -3 + \text{rank } W.
\end{aligned}
\tag{2.2.8}
$$

Hence rank $W \leq 6$. *By Proposition 2.2.1* rank $W = 6$, *and hence all the inequalities in (2.2.8) are equalities and* $\dim(W(N(U_1))) = \dim(R(U_1^*))$ *holds. Thus* $W(N(U_1)) = R(U_1^*)$.

Theorem 2.2.4. *The following statements are equivalent*

i. A *is self-adjoint,*

ii. $U_3(N(U_1)) = R(U^*)$,

iii. $W(N(U_1)) = R(U_1^*)$.

Suppose (i) holds. Then Corollary 2.2.3 implies (iii). Suppose (iii) holds. Let $u \in N(U_1)$. Then there is $v \in D(U_1^*)$ such that $Wu = U_1^* v$ i.e

$$U_1^* v = Wu = (J_2 + U_2^* V - V^* U_2)u. \tag{2.2.9}$$

Consider

$$U_3 u = \begin{pmatrix} J_2 \\ V \\ -U_2 \end{pmatrix} u = \begin{pmatrix} J_2 u \\ Vu \\ -U_2 u \end{pmatrix}. \tag{2.2.10}$$

Let $b = -Vu$ and $c = U_2 u$ i.e $0 = Vu + b$ and $0 = U_2 u - c$ and substitute (2.2.9) below. Then

$$\begin{aligned}
\begin{pmatrix} J_2 u \\ Vu \\ -U_2 u \end{pmatrix} &= \begin{pmatrix} J_2 u + U_2^*(Vu + b) - V^*(U_2 u - c) \\ -b \\ -c \end{pmatrix} \\
&= \begin{pmatrix} (J_2 + U_2^* V - V^* U_2)u + U_2^* b + V^* c \\ -b \\ -c \end{pmatrix} \\
&= \begin{pmatrix} U_1^* v + U_2^* b + V^* c \\ -b \\ -c \end{pmatrix} \\
&= \begin{pmatrix} U_1^* & U_2^* & V^* \\ 0 & -I & 0 \\ 0 & 0 & -I \end{pmatrix} \begin{pmatrix} v \\ b \\ c \end{pmatrix} = U^* \begin{pmatrix} v \\ b \\ c \end{pmatrix}.
\end{aligned}$$

Thus $U_3(N(U_1)) \subset R(U^*)$ and $\dim(U_3(N(U_1))) \leq \text{rank } U^*$. The map $U_1 : \mathbb{C}^8 \to \mathbb{C}^3$, in (2.2.3), has $\dim(N(U_1)) = \dim(\mathbb{C}^8) - \text{rank } U_1 = 8 - 3 = 5$ as given by the rank nullity theorem. Similarly U with the first column given by (2.2.3)–(2.2.5) has rank $U = 5$ thus $\dim(N(U_1)) = \text{rank } U^*$. We then conclude that $U_3(N(U_1)) = R(U^*)$ by showing that U_3 is injective i.e 0 is the only element in $N(U_3)$. Suppose $U_3 u = 0$. Then

$$0 = U_3 u = \begin{pmatrix} J_2 u \\ V u \\ -U_2 u \end{pmatrix}, \tag{2.2.11}$$

and $J_2 u = 0$ implies $u = 0$ since J_2 is invertible. Hence (ii) follows.
Suppose that (ii) holds. Then by Theorem 1.1.4 we have (i).

2.3 Further examples of self-adjoint operators with periodic and a single eigenvalue dependent boundary conditions

Keeping with the pattern of the boundary conditions of the operator studied in [8], using the differential equation (2.1.2) and Theorem 1.1.4, we identify the boundary conditions of the self-adjoint operators under investigation as follows:

$$y^{[\beta_1]}(\lambda, 0) - \epsilon_1 y^{[\beta_1]}(\lambda, a) = 0, \tag{2.3.1}$$

$$y^{[\beta_2]}(\lambda, 0) - \epsilon_2 y^{[\beta_2]}(\lambda, a) = 0, \tag{2.3.2}$$

$$\delta y^{[\beta_3]}(\lambda, 0) + (1-\delta) y^{[\beta_3]}(\lambda, a) = 0, \tag{2.3.3}$$

$$(1-\delta)(y^{[\beta_4]}(\lambda, 0) + \epsilon_3 i\alpha\lambda y^{[\beta_5]}(\lambda, 0)) = \delta(y^{[\beta_4]}(\lambda, a) + \epsilon_3 i\alpha\lambda y^{[\beta_5]}(\lambda, a)) \tag{2.3.4}$$

where $\beta_m \in \{0,1,2,3\}$, $m = 1,2,\ldots,5$, β_m's are distinct for $m = 1,2,3$ i.e $\beta_s \neq \beta_m$ for $s \neq m$ with $s, m = 1,2,3$. $\beta_1, \beta_2, \beta_4, \beta_5$ are different from each other and $\beta_5 = \beta_4 - 1$, $\beta_1 < \beta_2$, $\epsilon_j = \pm 1$ for $j = 1,2,3$ and $\delta \in \{0,1\}$. We give necessary and sufficient conditions for which the main operator A is self-adjoint.

Theorem 2.3.1. *The quadratic operator polynomial representing the fourth order differential equation* (2.1.2) *with the boundary conditions* (2.3.1)– (2.3.4) *is self-adjoint if and only if these boundary conditions have the following structure:*

$$\epsilon_1 \epsilon_2 = 1, \tag{2.3.5}$$

$$\epsilon_3 = -1 \ \textit{for} \ \delta = 0, \tag{2.3.6}$$

$$\epsilon_3 = 1 \quad \textit{for} \ \delta = 1, \tag{2.3.7}$$

$$\beta_1 = 0, \tag{2.3.8}$$

$$\beta_2 = 3, \tag{2.3.9}$$

$$\beta_3 = 1, 2. \tag{2.3.10}$$

Consider the matrices U_1, U_2 and V of the form (2.2.3)–(2.2.5). Let the non-zero elements of U_2 and V be at $u^2_{1,2}$ and $v_{1,3}$ respectively. Using the representation of (2.3.4), these corresponds to $\beta_4 = 2$ and $\beta_5 = 1$. Let $\epsilon_3 = -1$, $\beta_1 = 0$, $\epsilon_1 = -1$, $\beta_2 = 3$, $\epsilon_2 = -1$ and $\beta_3 = 2$. Starting with this choice of U_2 and V which implies that $\delta = 0$, U_1 given by these parameters is

$$U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.3.11}$$

Then consider U_2 and V where the non-zero elements are at $u^2_{1,6}$ and $v_{1,7}$ respectively, correspond to $\beta_4 = 2$, $\beta_5 = 1$, $\delta = 1$ and $\epsilon_3 = 1$. A matrix U_1 with such periodic boundary conditions is given by

$$U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.3.12}$$

The assumption of Theorem 1.1.4 is fulfilled since rank $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = 4$ for both (2.3.11) and (2.3.12) together with their corresponding U_2's. For each U_1 we compute $U_3(N(U_1))$ and the corresponding $R(U^*)$. The result is that $U_3(N(U_1)) = R(U^*)$ for each of the two cases and any of the combination of the parameters stated. Thus the operator A for each of the 12 cases is self-adjoint. A self-adjoint quadratic operator polynomial representing the fourth order differential equation (2.1.2) with boundary conditions that satisfy (2.3.1)–(2.3.4) satisfies

$$U_3(N(U_1)) = R(U^*).$$

If we represent boundary conditions with

$$U_1 = \begin{pmatrix} a & 0 & 0 & 0 & \epsilon_1 a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & \epsilon_2 b \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.3.13}$$

$$U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \epsilon_3 d & 0 & 0 \end{pmatrix}, \tag{2.3.14}$$

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & e & 0 \end{pmatrix}, \tag{2.3.15}$$

such that (2.3.1)–(2.3.4) is satisfied. Then, using Matlab we prove that if $U_3(N(U_1)) = R(U^*)$ then the values of parameters β's, δ's and ϵ's are as given by (2.3.5)–(2.3.10).

We find an unifying structure of boundary conditions that are periodic or anti-periodic at the endpoints of the interval and have an eigenvalue parameter dependence in one of them as described by Theorem 1.1.6. The matrix U_4 defined below was decomposed into its singular values and orthogonal matrices in an effort to find a relationship in all the cases.

Define a matrix

$$U_4 := \begin{pmatrix} U_1 \\ U_2 \\ V \end{pmatrix}. \tag{2.3.16}$$

All the U_4's that result from (2.3.1)–(2.3.4) and satisfy Theorem 2.3.1 are such that each U_4's columns has at most one non-zero element and each of its rows has at least one non-zero element.

Theorem 2.3.2. *The self-adjoint quadratic operator polynomial representing the fourth order differential equation* (2.1.2) *with boundary conditions* (2.3.1)–(2.3.4) *that satisfy Theorem 2.3.1 has*

$$U_4 = \Theta \begin{pmatrix} \Sigma & 0 \end{pmatrix} \Delta^\intercal,$$

where $\Theta = I_5$, $\Sigma = \operatorname{diag}(\sqrt{2}, \sqrt{2}, 1, 1, 1)$ *and* $\Delta^\intercal \in \mathbb{R}^{8\times 8}$.

Consider (2.3.1)–(2.3.4) with $\beta_1 = 0$, $\beta_2 = 3$, $\beta_3 = 1$, $\beta_4 = 2$, $\beta_5 = 1$, $\delta = 0$ and $\epsilon_1, \epsilon_2, \epsilon_3 = 1$. This choice of parameters results in U_1, U_2 and V given in

(2.1.20)–(2.1.22). We then compute singular values of U_4 with

$$U_4^\intercal = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.3.17}$$

Then

$$U_4 U_4^\intercal = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.3.18}$$

The eigenvalues of $U_4U_4^\intercal$ are $\sigma_1 = 2$ with eigenvectors $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}^\intercal$, $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \end{pmatrix}^\intercal$ and $\sigma_2 = 1$ with eigenvectors $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix}^\intercal$, $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}^\intercal$ and $\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}^\intercal$. We construct a matrix C whose columns are the eigenvectors of $U_4U_4^\intercal$ and order these eigenvectors by the magnitude of their eigenvalues i.e $C = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{pmatrix}$. Then we implement the Gram-Schmidt orthonormalization process which in this case is $\Theta = I_{5\times5}$. We repeat the process with $U_4^\intercal U_4$ to find $\Delta^\intercal$. The eigenvalues of $U_4^\intercal U_4$ are 2, 1 and 0 with multiplicities of two, three and three respectively. We list the eigenvectors of $U_4^\intercal U_4$ as columns of $D = (d_i)_1^8$ ordered below in decreasing magnitude of their eigenvalues as

$$
\begin{aligned}
d_1 &= -\frac{1}{\sqrt{2}}(e_4 - e_8)\\
d_2 &= -\frac{1}{\sqrt{2}}(e_1 - e_5)\\
d_3 &= e_7\\
d_4 &= e_2\\
d_5 &= e_6\\
d_6 &= -\frac{1}{\sqrt{2}}(e_1 + e_5)\\
d_7 &= e_3\\
d_8 &= -\frac{1}{\sqrt{2}}(e_4 + e_8).
\end{aligned}
$$

Then project the d_i's and normalize them as before, which gives

$$
\Delta^\intercal = \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2}\\
0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
-\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}^\intercal . \tag{2.3.19}
$$

The operators have the same Θ and Σ with $\Delta^\intercal$ being the only distinguishing matrix in the decompositions of their U_4's.

where $W = J_2 + U_2^* V - V^* U_2$, and $U_3 = \begin{pmatrix} J_2 \\ V \\ -U_2 \end{pmatrix}$.

3

Asymptotics of eigenvalues

In this chapter we show that the non-self-adjoint operator $L(\lambda, \alpha)$ is Birkhoff regular. Self-adjoint boundary value problems are Birkhoff-regular and for the same differential equation, a boundary value problem is Birkhoff regular if the boundary conditions are Birkhoff regular. Birkhoff regular problems possess a number of spectral properties like an estimate of the Green function and asymptotics of eigenvalues and eigenfunctions. We consider one of the cases of self-adjoint boundary value problems in [7], defined by the differential equation

$$y^{(4)}(\lambda, x) - (gy')'(\lambda, x) = \lambda^2 y(\lambda, x), \tag{3.0.1}$$

together with the following boundary conditions

$$y(\lambda, 0) - \epsilon_1 y(\lambda, a) = 0, \tag{3.0.2}$$

$$y^{(3)}(\lambda, 0) - \epsilon_2 y^{(3)}(\lambda, a) = 0, \tag{3.0.3}$$

$$y'(\lambda, 0) = 0, \tag{3.0.4}$$

$$y''(\lambda, a) + i\alpha\lambda y'(\lambda, a) = 0. \tag{3.0.5}$$

We prove in Section 3.1 that the problem (3.2.1)–(3.2.5) is Birkhoff regular for $\epsilon_1\epsilon_2 = 1$. In Section 3.2 we provide the asymptotics of the eigenvalues for $g = 0$ while in Section 3.3 we derive the first three terms of the eigenvalue expansions.

3.1 Birkhoff Regularity

We show Birkhoff regularity of (3.2.1)–(3.2.5) using Defintion 1.2.3.

Let $\lambda = \mu^2$. From (1.2.20) and (1.2.22), $p_i(\cdot, \lambda) = \sum_{i=0}^{3} \mu^j \pi_{3-i,j}$, $(i = 0, 1, 2, 3,\ j = 0, 1, 2, 3)$, $\pi_{1,1} = \pi_{2,2} = \pi_{3,3} = 0$ and $\pi_{4,4} = -1$. The characteristic function of the differential equation (3.2.1) is $\pi(\rho) = \rho^4 - 1$. The zeros of the characteristic function are i^{k-1}, $k = 1, 2, 3, 4$. By Proposition 1.2.5, $l = 4$, matrices Δ of the boundary eigenvalue problem (3.2.1)–(3.2.5) are the following 4×4 diagonal matrices with two consecutives ones and two consecutives zeros in the diagonal in a cycle arrangement:

$$\begin{cases} \Delta_1 = \mathrm{diag}(1, 1, 0, 0), \\ \Delta_2 = \mathrm{diag}(0, 1, 1, 0), \\ \Delta_3 = \mathrm{diag}(0, 0, 1, 1), \\ \Delta_4 = \mathrm{diag}(1, 0, 0, 1). \end{cases} \tag{3.1.1}$$

With $n_0 = 0$ and $l = 4$ the matrix C_1 defined in (1.2.34) is reduced to $(i^{(k-1)(l-1)})_{k,l=1}^4$. Then from Theorem 1.2.6.A $\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 2$ and $\nu_4 = 3$. Hence, we can choose

$$C(x, \mu) = \operatorname{diag}(\mu^0, \mu^1, \mu^2, \mu^3)\, C_1(x) \tag{3.1.2}$$

with

$$C_1(x) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \tag{3.1.3}$$

defined in (1.2.34) and

$$C(x, \mu) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mu & i\mu & -\mu & -i\mu \\ \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ \mu^3 & -i\mu^3 & -\mu^3 & i\mu^3 \end{pmatrix}. \tag{3.1.4}$$

Boundary matrices associated with (3.2.1)–(3.2.5) are given by

$$W^{(0)}(\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} C(0,\mu)$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mu & i\mu & -\mu & -i\mu \\ \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (3.1.5)$$

and

$$W^{(1)}(\mu) = \begin{pmatrix} -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_2 \\ 0 & i\alpha\mu^2 & 1 & 0 \end{pmatrix} C(a,\mu)$$
$$= \begin{pmatrix} -\epsilon_1 & -\epsilon_1 & -\epsilon_1 & -\epsilon_1 \\ 0 & 0 & 0 & 0 \\ -\epsilon_2\mu^3 & \epsilon_2 i\mu^3 & \epsilon_2\mu^3 & -\epsilon_2 i\mu^3 \\ i\alpha\mu^3+\mu^2 & -\alpha\mu^3-\mu^2 & -i\alpha\mu^3+\mu^2 & \alpha\mu^3-\mu^2 \end{pmatrix}. \qquad (3.1.6)$$

We choose $C_2(x,\mu) = \mathrm{diag}(\mu^0, \mu^1, \mu^3, \mu^3)$ defined in (1.2.28) then (1.2.14) leads to

$$
\begin{aligned}
C_2(\mu)^{-1} W^{(0)}(\mu) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu^{-1} & 0 & 0 \\ 0 & 0 & \mu^{-3} & 0 \\ 0 & 0 & 0 & \mu^{-3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mu & i\mu & -\mu & -i\mu \\ \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ \frac{1}{\mu} & -\frac{1}{\mu} & \frac{1}{\mu} & -\frac{1}{\mu} \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= W_0^{(0)}(\mu) + O(\mu^{-1}),
\end{aligned} \tag{3.1.7}
$$

$$C_2(\mu)^{-1}W^{(1)}(\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu^{-1} & 0 & 0 \\ 0 & 0 & \mu^{-3} & 0 \\ 0 & 0 & 0 & \mu^{-3} \end{pmatrix}$$

$$\times \begin{pmatrix} -\epsilon_1 & -\epsilon_1 & -\epsilon_1 & -\epsilon_1 \\ 0 & 0 & 0 & 0 \\ -\epsilon_2\mu^3 & \epsilon_2 i\mu^3 & \epsilon_2\mu^3 & -\epsilon_2 i\mu^3 \\ i\alpha\mu^3+\mu^2 & -\alpha\mu^3-\mu^2 & -i\alpha\mu^3+\mu^2 & \alpha\mu^3-\mu^2 \end{pmatrix}$$

$$= \begin{pmatrix} -\epsilon_1 & -\epsilon_1 & -\epsilon_1 & -\epsilon_1 \\ 0 & 0 & 0 & 0 \\ -\epsilon_2 & i\epsilon_2 & \epsilon_2 & -i\epsilon_2 \\ i\alpha+\frac{1}{\mu} & -\alpha-\frac{1}{\mu} & -i\alpha+\frac{1}{\mu} & \alpha-\frac{1}{\mu} \end{pmatrix}$$

$$= W_0^{(1)}(\mu) + O(\mu^{-1}). \tag{3.1.8}$$

Thus $C_2(\mu)^{-1}W^{(j)}(\mu) + O(\mu^{-1})$ holds for $j = 0, 1$ and

$$|W_0^{(j)}| = \sum_{i=1}^{4} \max_{k=1}^{4} |w_{0_{ik}}^{(j)}| < \infty,$$

$$\sum_{j=0}^{1} |W_0^{(j)}| < \infty,$$

$$|C_2(\mu)^{-1}W^{(j)}(\mu) - W_0^{(j)}| = O(\mu^{-1}), \tag{3.1.9}$$

and

$$\sum_{j=0}^{1} |C_2^{-1}(\mu)W^{(j)}(\cdot,\mu) - W_0^{(j)}| = O(\mu^{-1}) \text{ as } \mu \to \infty. \tag{3.1.10}$$

The order of the differential equation (3.2.1) is 4 then the values of Δ are diagonal matrices with 2 consecutive ones and 2 consecutive zeros in the diagonal in a cyclic arrangement. The boundary value problem (3.2.1)–(3.2.5) is Birkhoff regular since

$$W_0^{(0)}\Delta_j + W_0^{(1)}(I_4 - \Delta_j) = \begin{cases} \begin{pmatrix} 1 & 1 & -\epsilon_1 & -\epsilon_1 \\ 1 & i & 0 & 0 \\ 0 & 0 & \epsilon_2 & -i\epsilon_2 \\ 0 & 0 & -i\alpha & \alpha \end{pmatrix} & j=1 \\ \begin{pmatrix} -\epsilon_1 & 1 & 1 & -\epsilon_1 \\ 0 & i & -1 & 0 \\ -\epsilon_2 & 0 & 0 & -i\epsilon_2 \\ i\alpha & 0 & 0 & \alpha \end{pmatrix} & j=2 \\ \begin{pmatrix} -\epsilon_1 & -\epsilon_1 & 1 & 1 \\ 0 & 0 & -1 & -i \\ -\epsilon_2 & i\epsilon_2 & 0 & 0 \\ i\alpha & \alpha & 0 & 0 \end{pmatrix} & j=3 \\ \begin{pmatrix} 1 & -\epsilon_1 & -\epsilon_1 & 1 \\ 1 & 0 & 0 & -i \\ 0 & i\epsilon_2 & \epsilon_2 & 0 \\ 0 & -\alpha & -i\alpha & 0 \end{pmatrix} & j=4 \end{cases} \tag{3.1.11}$$

is invertible for $\mu \in \mathbb{C}\backslash\{0\}$, Δ_j $j = 1, 2, 3, 4$ (3.1.1). Hence,

Proposition 3.1.1. *The eigenvalue problem* (3.2.1)–(3.2.5) *is Birkhoff regular for*

$\alpha > 0$.

3.2 Asymptotic expansions of eigenvalues when $g = 0$

We consider one of the cases of self-adjoint boundary value problems in [7], defined by the differential equation

$$y^{(4)}(\lambda, x) - (gy')'(\lambda, x) = \lambda^2 y(\lambda, x), \tag{3.2.1}$$

together with the following boundary conditions

$$y(\lambda, 0) - \epsilon_1 y(\lambda, a) = 0, \tag{3.2.2}$$

$$y^{(3)}(\lambda, 0) - \epsilon_2 y^{(3)}(\lambda, a) = 0, \tag{3.2.3}$$

$$y'(\lambda, 0) = 0, \tag{3.2.4}$$

$$y''(\lambda, a) + i\alpha\lambda y'(\lambda, a) = 0, \tag{3.2.5}$$

with $\epsilon_1, \epsilon_2 = \pm 1$ and $\epsilon_1 \epsilon_2 = 1$. In this section we consider $g = 0$ and put $\mu = \sqrt{\lambda}$, $\lambda \neq 0$.

A fundamental system of (3.2.1) with $g = 0$ is $\{e^{\mu x}, e^{-\mu x}, e^{i\mu x}, e^{-i\mu x}\}$. We associate (3.2.1) with a first order system, T^D and a fundamental matrix of T^D is given by

$$Z(x, \mu) = \begin{pmatrix} e^{\mu x} & e^{-\mu x} & e^{i\mu x} & e^{-i\mu x} \\ \mu e^{\mu x} & -\mu e^{-\mu x} & i\mu e^{i\mu x} & -i\mu e^{-i\mu x} \\ \mu^2 e^{\mu x} & \mu^2 e^{-\mu x} & -\mu^2 e^{i\mu x} & -\mu^2 e^{-i\mu x} \\ \mu^3 e^{\mu x} & -\mu^3 e^{-\mu x} & -i\mu^3 e^{i\mu x} & i\mu^3 e^{-i\mu x} \end{pmatrix}. \tag{3.2.6}$$

To apply Proposition 1.2.8, another matrix $Y(x,\mu)$ is a fundamental matrix if there is an invertible matrix $C \in M_4(\mathbb{C})$ for a fundamental matrix $Z(x,\mu)$ such that

$$Y(x,\mu) = Z(x,\mu)C. \tag{3.2.7}$$

It means that for $Y(0,\mu) = I_4$,

$$Z(0,\mu) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mu & -\mu & i\mu & -i\mu \\ \mu^2 & \mu^2 & -\mu^2 & -\mu^2 \\ \mu^3 & -\mu^3 & -i\mu^3 & i\mu^3 \end{pmatrix} \tag{3.2.8}$$

is a fundamental matrix and an invertible C is

$$C = \frac{1}{4}\begin{pmatrix} 1 & \frac{1}{\mu} & \frac{1}{\mu^2} & \frac{1}{\mu^3} \\ 1 & \frac{-1}{\mu} & \frac{1}{\mu^2} & -\frac{1}{\mu^3} \\ 1 & \frac{-i}{\mu} & \frac{-1}{\mu^2} & \frac{i}{\mu^3} \\ 1 & \frac{i}{\mu} & \frac{-1}{\mu^2} & \frac{-i}{\mu^3} \end{pmatrix}. \tag{3.2.9}$$

The characteristic matrix function defined in [5, (3.1.7)] of (3.2.1)–(3.2.5) is

$$M(\mu) = W^{(0)}Y(0,\mu) + W^{(1)}Y(a,\mu), \tag{3.2.10}$$

where

$$W^{(0)}(\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.2.11}$$

and

$$W^{(1)}(\mu) = \begin{pmatrix} -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_2 \\ 0 & i\alpha\mu^2 & 1 & 0 \end{pmatrix}. \tag{3.2.12}$$

It then follows that the characteristic determinant is given by

$$\begin{aligned} D(\mu) := \det M(\mu) =& \det(W^{(0)}(\mu) \cdot Z(0,\mu) + W^{(1)}(\mu) \cdot Z(a,\mu)) \det c \\ =& 2\mu^6 \Big[-4i(e^{\mu a} + e^{-\mu a} + e^{i\mu a} + e^{-i\mu a} - \epsilon_2) \\ & + i(2\epsilon_1 + \epsilon_2)(e^{(1+i)\mu a} + e^{-(1+i)\mu a} + e^{(1-i)\mu a} + e^{-(1-i)\mu a}) \\ & + \mu\alpha(4(e^{\mu a} - e^{-\mu a} - i(e^{i\mu a} - e^{-i\mu a})) \\ & - (1-i)(\epsilon_1 + \epsilon_2)(e^{(1+i)\mu a} - e^{-(1+i)\mu a}) \\ & -(1+i)(\epsilon_1 + \epsilon_2)(e^{(1-i)\mu a} - e^{-(1-i)\mu a}))\Big] \frac{-i}{16\mu^6}. \end{aligned} \tag{3.2.13}$$

We rewrite the characteristic determinant as

$$D(\mu) = \frac{-i}{8} \left(\sum_{j=0}^{8} \left(\alpha\mu C_j + B_j \right) e^{\omega_j \mu a} \right), \tag{3.2.14}$$

where

$$\begin{cases} \omega_0 = 1 - i,\ \omega_1 = 1 + i,\ \omega_2 = -1 + i,\ \omega_3 = -1 - i, \\ \omega_4 = -i,\ \omega_5 = 1,\ \omega_6 = i,\ \omega_7 = -1,\ \omega_8 = 0. \end{cases} \tag{3.2.15}$$

and

$$\begin{cases} -C_0 = C_2 = (1+i)(\epsilon_1 + \epsilon_2),\ -C_1 = C_3 = (1-i)(\epsilon_1 + \epsilon_2), \\ iC_4 = -C_5 = -iC_6 = C_7 = -4,\ C_8 = 0, \\ B_0 = B_1 = B_2 = B_3 = i(2\epsilon_1 + \epsilon_2), \\ B_4 = B_5 = B_6 = B_7 = -4i,\ B_8 = 4i\epsilon_2. \end{cases} \tag{3.2.16}$$

Let

$$\begin{aligned} \widetilde{D}_0(\mu) :=& D(\mu) e^{-\omega_0 \mu a} \\ =& \frac{-i}{8} \Bigg(\alpha\mu(-(1+i)(\epsilon_1+\epsilon_2) + 4e^{i\mu a} - (1-i)(\epsilon_1+\epsilon_2)e^{2i\mu a}) \\ & \qquad + (i(2\epsilon_1+\epsilon_2) - 4ie^{i\mu a} + i(2\epsilon_1+\epsilon_2)e^{2i\mu a}) \\ & \qquad + \sum_{\substack{j=2 \\ j\neq 5}}^{8} \left(\alpha\mu C_j + B_j\right) e^{(\omega_j - \omega_0)\mu a} \Bigg), \end{aligned} \tag{3.2.17}$$

where

$$
\begin{cases}
\omega_1 - \omega_0 = 2i,\ \omega_2 - \omega_0 = -2 + 2i,\ \omega_3 - \omega_0 = -2, \\
\omega_4 - \omega_0 = -1,\ \omega_5 - \omega_0 = i,\ \omega_6 - \omega_0 = -1 + 2i, \\
\omega_7 - \omega_0 = -2 + i,\ \omega_8 - \omega_0 = -1 + i.
\end{cases} \tag{3.2.18}
$$

Let

$$
D_0^0(\mu) := -(1+i)(\epsilon_1 + \epsilon_2) + 4e^{i\mu a} - (1-i)(\epsilon_1 + \epsilon_2)e^{2i\mu a}), \tag{3.2.19}
$$

$$
D_1^0(\mu) := i(2\epsilon_1 + \epsilon_2) - 4ie^{i\mu a} + i(2\epsilon_1 + \epsilon_2)e^{2i\mu a}, \tag{3.2.20}
$$

$$
D_2^0(\mu) := \sum_{\substack{j=2 \\ j\neq 5}}^{8} \left(\alpha\mu C_j + B_j\right) e^{(\omega_j - \omega_0)\mu a}. \tag{3.2.21}
$$

The principal values of the arguments are $\frac{5\pi}{8} < \arg(\omega_j - \omega_0) \leq \pi$ for $j = 2, 3, 4, 6, 7, 8$. The arguments of exponential terms $e^{(\omega_j - \omega_0)\mu a}$, $j = 2, 3, 4, 6, 7, 8$ can be written as

$$
\begin{aligned}
(\omega_j - \omega_0)\mu a &= |(\omega_j - \omega_0)\mu a|\, e^{i(\arg((\omega_j - \omega_0)\mu a)} \\
&= |(\omega_j - \omega_0)||\mu| a[\cos(\arg((\omega_j - \omega_0)\mu a)) + i\sin(\arg((\omega_j - \omega_0)\mu a))]
\end{aligned}
$$

and for $\arg\mu \in [-\frac{\pi}{12}, \frac{5\pi}{12}]$, $\arg((\omega_j - \omega_0)\mu a) \in (\frac{13\pi}{24}, \frac{34\pi}{24}]$,

$$
\cos(\arg((\omega_j - \omega_0)\mu a)) < \cos\frac{13\pi}{24}
$$

and $|\omega_j - \omega_0)| \geq 1,\ j = 2, 3, 4, 6, 7, 8$. Thus,

$$\begin{aligned}
|e^{(\omega_j-\omega_0)\mu a}| =& e^{\Re\{(\omega_j-\omega_0)\mu a\}} \\
=& e^{|(\omega_j-\omega_0)||\mu|a\cos(\arg((\omega_j-\omega_0)\mu a))} \\
\leq& e^{a|\mu|\cos(\arg((\omega_j-\omega_0)\mu a))} \\
\leq& e^{-a|\mu|\cos(\frac{\pi}{24})} \\
=& o(|\mu|^{-s}),\ \ s \in \mathbb{N}. \qquad (3.2.22)
\end{aligned}$$

The coefficients $\mu\alpha C_j + B_j$ of exponential terms $e^{(\omega_j-\omega_0)\mu a}$, $j = 2, 3, 4, 6, 7, 8$ can be estimated as $\mu\alpha C_j = O(|\mu|)$, $B_j = O(1)$ and $\mu\alpha C_j + B_j = O(1+|\mu|) = O(|\mu|)$. For large μ in the sector $\arg\mu \in [-\frac{\pi}{12}, \frac{5\pi}{12}]$, $(\mu\alpha C_j + B_j)e^{(\omega_j-\omega_0)\mu a} = o(|\mu|^{1-s})$ and

$$|D_2^0(\mu)| = o(|\mu|^{1-s}),\ \ s \in \mathbb{N}. \qquad (3.2.23)$$

We find the zeros of $D_0^0(\mu)$ in the sector where $\arg\mu \in [-\frac{\pi}{12}, \frac{5\pi}{12}]$. Zeros of $D_0^0(\mu)$ are given by

$$\begin{aligned}
\tilde{\mu}_k^0 =& \frac{i}{a}\ln\left|\frac{-4 \pm \sqrt{16 - 8(\epsilon_1+\epsilon_2)^2}}{-2(1-i)(\epsilon_1+\epsilon_2)}\right| \\
&+ \frac{1}{a}\left[\arg\left(\frac{-4 \pm \sqrt{16 - 8(\epsilon_1+\epsilon_2)^2}}{-2(1-i)(\epsilon_1+\epsilon_2)}\right) + 2\pi k\right],\ \ k \in \mathbb{N}. \qquad (3.2.24)
\end{aligned}$$

When $\epsilon_1 = \epsilon_2 = 1$, these zeros simplify to

$$\begin{cases}
\tilde{\mu}_{k,1}^{0,+} = (4k-3)\frac{\pi}{2a}, \\
\tilde{\mu}_{k,2}^{0,+} = 2k\frac{\pi}{a}.
\end{cases} \qquad (3.2.25)$$

and when $\epsilon_1 = \epsilon_2 = -1$, the zeros are

$$\begin{cases} \tilde{\mu}_{k,1}^{0,-} = (4k-1)\frac{\pi}{2a}, \\ \tilde{\mu}_{k,2}^{0,-} = (2k-1)\frac{\pi}{a}, \end{cases} \tag{3.2.26}$$

$k \in \mathbb{N}$.

Consider $D_{\tilde{\mu}_{k,1}}^{0,+} = \{\mu \in \mathbb{C} : |\mu - \tilde{\mu}_{k,1}^{0,+}| = \frac{\pi}{20a}\}$, which are inside the sector $\arg\mu \in [-\frac{\pi}{12}, \frac{5\pi}{12}]$. For $\mu \in D_{\tilde{\mu}_{k,1}}^{0,+}$,

$$\left|D_0^0(\mu)\right| > 0. \tag{3.2.27}$$

Therefore,

$$m_{k,1}^{0,+} := \min\left\{|D_0^0(\mu)| : \mu \in D_{\tilde{\mu}_{k,1}}^{0,+}\right\} > 0,\ \ k \in \mathbb{N}. \tag{3.2.28}$$

The number $m_{k,1}^{0,+}$ in (3.2.28) is independent of $k \in \mathbb{N}$, because $\mu \in D_{\tilde{\mu}_{k,1}}^{0,+}$ written as $\mu = \tilde{\mu}_{k,1}^{0,+} + \zeta$ means $|\zeta| = \frac{\pi}{20a}$ and therefore,

$$\begin{aligned} e^{i(\tilde{\mu}_{k,1}^{0,-}+\zeta)a} &= e^{i((4k-3)\frac{\pi}{2a}+\zeta)a} \\ &= e^{i(-\frac{3\pi}{2}+\zeta a)} = ie^{i\zeta a} \end{aligned} \tag{3.2.29}$$

and

$$\left|D_0^0(\mu)\right| = |-(1+i)(\epsilon_1+\epsilon_2) + 4ie^{i\zeta a} + (1-i)(\epsilon_1+\epsilon_2)e^{2i\zeta a}|. \tag{3.2.30}$$

Thus,

$$m_1^{0,+} := \inf_{k\in\mathbb{N}} m_{k,1}^{0,+} > 0. \tag{3.2.31}$$

Similarly, for $\mu = \tilde{\mu}_{k,1}^{0,+} + \zeta \in D_{\tilde{\mu}_{k,1}}^{0,+}$

$$|D_1^0(\mu)| =|i(2\epsilon_1 + \epsilon_2) + 4e^{i\zeta a} - i(2\epsilon_1 + \epsilon_2)e^{2i\zeta a}|. \tag{3.2.32}$$

Let

$$M_{k,1}^{0,+} = \max\left\{|D_1^0(\mu)| : \mu \in D_{\tilde{\mu}_{k,1}}^{0,+}\right\},\ k \in \mathbb{N}.$$

Since (3.2.32) is independent of $k \in \mathbb{N}$, we have

$$M_1^{0,+} := \sup_{k\in\mathbb{N}} M_{k,1}^{0,+} < \infty. \tag{3.2.33}$$

On all circles $D_{\tilde{\mu}_{k,1}}^{0,+}$, $|D_0^0(\mu)| > m_1^{0,+}$, $|D_1^0(\mu)| < M_1^{0,+}$ and (3.2.23) can be bounded by $L > o(1) = |D_2^0(\mu)|$ for $s = 1$, where $m_1^{0,+}$, $M_1^{0,+}$ and L are constants independent of k. Thus,

$$|D_0^0(\mu)| > \frac{m_1^{0,+}}{M_1^{0,+} + L}|D_1^0(\mu) + D_2^0(\mu)| \tag{3.2.34}$$

and there is a positive integer k_0 such that, for $k > k_0$ and $\mu \in D_{\tilde{\mu}_{k,1}}^{0,+}$, $\alpha|\mu| > \frac{M_1^{0,+}+L}{m_1^{0,+}}$ and (3.2.34) becomes

$$\alpha|\mu||D_0^0(\mu)| > |D_1^0(\mu) + D_2^0(\mu)|. \tag{3.2.35}$$

By Rouché's theorem we conclude that inside each circle $D^{0,+}_{\tilde{\mu}_{k,1}}$, there is a unique zero of $\widetilde{D}_0(\mu)$ which we denote by $\mu^{0,+}_{k,1}$, and therefore

$$|\mu^{0,+}_{k,1} - \tilde{\mu}^{0,+}_{k,1}| < \frac{\pi}{20a},\ k > k_0,\ k \in \mathbb{N}. \tag{3.2.36}$$

Consider circles $D^{0,+}_{k,1,\epsilon} := \{\mu \in \mathbb{C} : |\mu - \tilde{\mu}^{0,+}_{k,1}| = \epsilon\}$, $0 < \epsilon < \frac{\pi}{20a}$. For each ϵ we can find constants as in (3.2.31) and (3.2.33), respectively, that correspond to circles $D^{0,+}_{k,1,\epsilon}$ and are independent of k. Denote these constants as $m^{0,+}_{1,\epsilon}$ and $M^{0,+}_{1,\epsilon}$. We can find a positive integer $k_\epsilon > k_0$ such that for $k \geq k_\epsilon$, $L_\epsilon > o(1) = |D^0_2(\mu)|$ and $\mu \in D^{0,-}_{k,1,\epsilon}$, $\alpha|\mu| > \frac{M^{0,+}_{1,\epsilon}+L_\epsilon}{m^{0,+}_{1,\epsilon}}$. For each ϵ and $\mu \in D^{0,+}_{k,1,\epsilon}$, arguments that lead to (3.2.34) hold and

$$\alpha|\mu||D^0_0(\mu)| > |D^0_1(\mu) + D^0_2(\mu)| \tag{3.2.37}$$

is upheld. Again, by Rouché's theorem we conclude that inside each circle $D^{0,+}_{k,1,\epsilon}$, $\widetilde{D}_0(\mu)$ has a unique zero. The circle $D^{0,+}_{k,1,\epsilon} \subseteq D^{0,+}_{\tilde{\mu}_{k,1}}$ has the same center as $D^{0,+}_{\tilde{\mu}_{k,1}}$, which is a zero of $D^0_0(\mu)$. Since these concentric circles both contain a unique zero of $\widetilde{D}_0(\mu)$, we can conclude that these two circles contain exactly one zero, which is inside $D^{0,+}_{k,1,\epsilon}$, and therefore,

$$|\mu^{0,+}_{k,1} - \tilde{\mu}^{0,+}_{k,1}| < \epsilon,\ k > k_0,\ k \in \mathbb{N}. \tag{3.2.38}$$

Thus,

$$\mu^{0,+}_{k,1} = (4k-3)\frac{\pi}{2a} + o(1), \quad k > k_0,\ k \in \mathbb{N} \tag{3.2.39}$$

in the sector $\arg\mu \in [-\frac{\pi}{12}, \frac{5\pi}{12}]$. Another sequence of zeros of $D^0_0(\mu)$ when $\epsilon_1 =$

$\epsilon_2 = 1$ is $\tilde{\mu}_{k,2}^{0,+}$, $k \in \mathbb{N}$, inside the sector $\arg \mu \in [-\frac{\pi}{12}, \frac{5\pi}{12}]$. Let $D^{0,+}_{\tilde{\mu}_{k,2}} = \{\mu \in \mathbb{C} : |\mu - \tilde{\mu}_{k,2}^{0,+}| = \frac{\pi}{20a}\}$. Then $\mu \in D^{0,+}_{\tilde{\mu}_{k,2}}$ written as $\mu = \tilde{\mu}_{k,2}^{0,+} + \zeta$ means $|\zeta| = \frac{\pi}{20a}$ and therefore,

$$\begin{aligned} e^{i(\tilde{\mu}_{k,2}^{0,+}+\zeta)a} &= e^{i(2k\frac{\pi}{a}+\zeta)a} \\ &= e^{i(2k\pi+\zeta a)} = e^{i\zeta a}. \end{aligned} \tag{3.2.40}$$

An estimate similar to (3.2.35) can be obtained for $\mu \in D^{0,+}_{\tilde{\mu}_{k,2}}$ and $k > k_0$ for some positive integer k_0. The same reasoning as above is applied to circles $D^{0,+}_{\tilde{\mu}_{k,2}}$ redefined with radii of ϵ. We repeat this argument again when $\epsilon_1 = \epsilon_2 = -1$.

Proposition 3.2.1. *For $g = 0$ there exists a positive integer k_0 such that zeros of $D(\mu)$ are*

$$\begin{cases} \mu_{k,1}^{0,+} = (4k-3)\frac{\pi}{2a} + o(1), \\ \mu_{k,2}^{0,+} = 2k\frac{\pi}{a} + o(1), \end{cases} \tag{3.2.41}$$

when $\epsilon_1 = \epsilon_2 = 1$ and when $\epsilon_1 = \epsilon_2 = -1$, they are

$$\begin{cases} \mu_{k,1}^{0,-} = (4k-1)\frac{\pi}{2a} + o(1), \\ \mu_{k,2}^{0,-} = (2k-1)\frac{\pi}{a} + o(1), \end{cases} \tag{3.2.42}$$

$k > k_0$, $k \in \mathbb{N}$.

We use properties of $D(\mu)$ to locate its other zeros using zeros already found in

Proposition 3.2.1. Let

$$D_c(\mu) = -\frac{i}{8}\mu\alpha\sum_{j=0}^{7} C_j e^{\omega_j \mu a} \tag{3.2.43}$$

and

$$D_b(\mu) = -\frac{i}{8}\sum_{j=0}^{8} B_j e^{\omega_j \mu a}. \tag{3.2.44}$$

From (3.2.14), $D(\mu)$ can be written as

$$D(\mu) = D_c(\mu) + D_b(\mu). \tag{3.2.45}$$

The terms of the characteristic determinant, $D(\mu)$, satisfies

$$\begin{aligned} D_c(i\mu) &= -\frac{i}{8}\mu\alpha\sum_{j=0}^{7} iC_j e^{\omega_j i\mu a} \\ &= -\left(-\frac{i}{8}\right)\mu\alpha\sum_{j=0}^{7} \overline{C_j} e^{\overline{\omega_j}\mu a} = -\overline{D_c(\overline{\mu})}, \end{aligned} \tag{3.2.46}$$

and

$$D_b(i\mu) = -\frac{i}{8}\sum_{j=0}^{8} B_j e^{\omega_j i\mu a} = -\overline{D_b(\overline{\mu})}. \tag{3.2.47}$$

Thus,

$$D(i\mu) = -\overline{D(\overline{\mu})}. \tag{3.2.48}$$

Thus, μ is a zero of $D(\mu)$ if and only if $-\mu$ is a zero of $D(\mu)$ and $i\mu$ is a zero of

$D(\mu)$ if and only if $\overline{\mu}$ is a zero of $D(\mu)$. For $g = 0$ and $k > k_0$, zeros of $D(\mu)$ which are the eigenvalues of M are given below.

Proposition 3.2.2. *For $g = 0$ there exists a positive integer k_0 such that zeros of $D(\mu)$ are*

$$\begin{cases} \mu_{k,1}^{j,0,+} = i^j(4k-3)\frac{\pi}{2a} + o(1), \\ \mu_{k,2}^{j,0,+} = i^j 2k\frac{\pi}{a} + o(1), \end{cases} \tag{3.2.49}$$

when $\epsilon_1 = \epsilon_2 = 1$ and

$$\begin{cases} \mu_{k,1}^{j,0,-} = i^j(4k-1)\frac{\pi}{2a} + o(1), \\ \mu_{k,2}^{j,0,-} = i^j(2k-1)\frac{\pi}{a} + o(1), \end{cases} \tag{3.2.50}$$

when $\epsilon_1 = \epsilon_2 = -1$,
$k \in \mathbb{N}$, $k > k_0$ and $j = 0, 1, 2, 3$.

We have already considered zeros of the exponential sum $D(\mu)$ inside the sector $\arg\mu \in [-\frac{\pi}{12}, \frac{5\pi}{12}]$ and it is our intention to enumerate its zeros inside big circles centred at the origin. We rewrite $D(\mu)$ as

$$D(\mu) = \frac{-i}{8}\left(\mu\alpha\sum_{j=0}^{7} C_j e^{\omega_j\mu a} + \sum_{j=0}^{8} B_j e^{\omega_j\mu a}\right). \tag{3.2.51}$$

Let $\Gamma_k := \{\mu \in \mathbb{C} : \max((\Re\mu)^2, (\Im\mu)^2) = R_k^2\}$ where $R_k = 2(k+\frac{1}{3})\frac{\pi}{a}$, $k \in \mathbb{N}$ and $S_j := \{\mu \in \mathbb{C}\backslash\{0\} : \arg\mu \in [-\frac{\pi}{12} - j\frac{\pi}{2}, \frac{5\pi}{12} - j\frac{\pi}{2}]\}$, $j = 0, 1, 2, 3$. We rewrite (3.2.17) as

$$\begin{aligned}D(\mu)e^{-\omega_0\mu a} =&\frac{-i}{8}\left(\mu\alpha\sum_{j=0}^{7}C_je^{(\omega_j-\omega_0)\mu a}+\sum_{j=0}^{8}B_je^{(\omega_j-\omega_0)\mu a}\right)\\ =&\left(\alpha\mu D_0^0(\mu)+\alpha\mu\sum_{\substack{j=2\\ j\neq 5}}^{7}C_je^{(\omega_j-\omega_0)\mu a}+D_1^0(\mu)+\sum_{\substack{j=2\\ j\neq 5}}^{8}B_je^{(\omega_j-\omega_0)\mu a}\right).\end{aligned}\tag{3.2.52}$$

Let $\mu a = x+iy$ where $x=(2k+\frac{2}{3})\pi$. Then

$$\begin{aligned}e^{i\mu a} &= e^{i(x+iy)}\\ &= e^{(-y+ix)}\\ &= e^{-y}e^{ix}\\ &= e^{-y}e^{i(2k+\frac{2}{3})\pi}\\ &= e^{-y}e^{i\frac{2}{3}\pi}\end{aligned}$$

and

$$e^{2i\mu a} = e^{-2y}e^{i\frac{4}{3}\pi}.$$

Consider $D_0^0(\mu)$ when $\epsilon_1\epsilon_2=1$. The zeros of $D_0^0(\mu)$ are given in (3.2.25), and μ is not a zero of $D_0^0(\mu)$ when $\mu a = x+iy$ and $x=(2k+\frac{2}{3})\pi$. Along the line $x=(2k+\frac{2}{3})\pi$,

$$\begin{aligned}|D_0^0(\mu)| =&|-(1+i)(\epsilon_1+\epsilon_2)+4e^{i\mu a}-(1-i)(\epsilon_1+\epsilon_2)e^{2i\mu a}|\\ =&|\pm 2(1+i)-4e^{-y}e^{i\frac{2}{3}\pi}\pm 2(1-i)e^{-2y}e^{i\frac{4}{3}\pi}|\end{aligned}\tag{3.2.53}$$

$$\begin{aligned}|D_1^0(\mu)| =& |i(2\epsilon_1+\epsilon_2) - 4ie^{i\mu a} + i(2\epsilon_1+\epsilon_2)e^{2i\mu a}| \\ =& | \pm 3i + 4ie^{-y}e^{i\frac{2}{3}\pi} \pm 3ie^{-2y}e^{i\frac{4}{3}\pi}|\end{aligned} \tag{3.2.54}$$

Therefore,

$$\begin{aligned}\lim_{y\to\infty} |D_0^0(\mu)| &= 2\sqrt{2}, \\ \lim_{y\to-\infty} |D_0^0(\mu)| &= \infty, \\ \lim_{y\to\infty} \frac{|D_1^0(\mu)|}{|D_0^0(\mu)|} &= \frac{3}{2\sqrt{2}}, \\ \lim_{y\to-\infty} \frac{|D_1^0(\mu)|}{|D_0^0(\mu)|} &= \frac{3}{2\sqrt{2}}\end{aligned} \tag{3.2.55}$$

and

$$\inf\left\{|D_0^0(\mu)| : \mu a = x+iy,\ x = \left(2k+\frac{2}{3}\right)\pi,\ y\in\mathbb{R},\ k\in\mathbb{N}\right\} =: m_{C_L} > 0. \tag{3.2.56}$$

Limits in (3.2.55) imply that there is a constant $m_{C_U} > 0$ such that

$$\frac{|D_1^0(\mu)|}{|D_0^0(\mu)|} < m_{C_U} \tag{3.2.57}$$

and

$$|D_0^0(\mu)| \geq m_{C_L}, \tag{3.2.58}$$

for all $\mu a = x+iy,\ x = (2k+\frac{2}{3})\pi,\ y\in\mathbb{R},\ k\in\mathbb{N}$. Consider $\mu a = x+iy$, where

$x \in \mathbb{R}$ and $y = (2k + \frac{2}{3})\pi$, $k \in \mathbb{N}$. Then,

$$\begin{aligned} |D_0^0(\mu)| &= |-(1+i)(\epsilon_1+\epsilon_2) + 4e^{i\mu a} - (1-i)(\epsilon_1+\epsilon_2)e^{2i\mu a}| \\ &= |\pm 2(1+i) + 4e^{-(2k+\frac{2}{3})\pi}e^{ix} \pm 2(1-i)e^{-(4k+\frac{4}{3})\pi}e^{2ix}| \\ &\geq 2\sqrt{2} - 4e^{-(2k+\frac{2}{3})\pi} - 2\sqrt{2}e^{-(4k+\frac{4}{3})\pi} > 2 \end{aligned} \tag{3.2.59}$$

and

$$\begin{aligned} |D_1^0(\mu)| &= |i(2\epsilon_1+\epsilon_2) - 4ie^{i\mu a} + i(2\epsilon_1+\epsilon_2)e^{2i\mu a}| \\ &= |\pm 3i - 4ie^{-(2k+\frac{2}{3})\pi}e^{ix} \pm 3ie^{-(4k+\frac{4}{3})\pi}e^{2ix}| \\ &\leq 3 + 4e^{-(2k+\frac{2}{3})\pi} + 3e^{-(4k+\frac{4}{3})\pi} < 4. \end{aligned} \tag{3.2.60}$$

Thus,

$$\frac{|D_1^0(\mu)|}{|D_0^0(\mu)|} < 2 \tag{3.2.61}$$

for all $\mu a = x + iy,\ x \in \mathbb{R},\ y = (2k + \frac{2}{3})\pi,\ k \in \mathbb{N}$. Let $m_C := \min\{m_{C_U}, 2\}$. Then for $\mu \in \Gamma_k \bigcap S_0$

$$\frac{|D_1^0(\mu)|}{|D_0^0(\mu)|} < m_C. \tag{3.2.62}$$

Let $\tau_0(\mu) = \sum\limits_{\substack{j=2\\ j\neq 5}}^{7} C_j e^{(\omega_j-\omega_0)\mu a}$ and $\tau_1(\mu) = \sum\limits_{\substack{j=2\\ j\neq 5}}^{8} B_j e^{(\omega_j-\omega_0)\mu a}$. For large $\mu \in S_0$, $\tau_0 \sim \tau_1 = o(|\mu|^{-s})$, $s \in \mathbb{N}$ by (3.2.22) i.e for each $\varepsilon > 0$ there exists $k_1(\varepsilon)$ such that for

$k \geq k_1(\varepsilon)$ and for each $\mu \in \Gamma_k \bigcap S_0$, $|\tau_0(\mu)| < \varepsilon$, $|\tau_1(\mu)| < \varepsilon$. Thus,

$$\left| D_0^0(\mu) + \tau_0(\mu) \right| \geq |D_0^0(\mu)| \left(1 - \frac{\varepsilon}{|D_0^0(\mu)|} \right). \tag{3.2.63}$$

Let $m_B := \min\{2, m_{C_L}\} > 0$. From (3.2.58) and (3.2.59), $|D_0^0(\mu)| \geq m_B$. We suppose that $\varepsilon < m_B$, then $(1 - \varepsilon |D_0^0(\mu)|^{-1})^{-1} \leq G := \left(1 - \frac{\varepsilon}{m_B}\right)^{-1} > 0$. Also,

$$\begin{aligned}
\left| D_1^0(\mu) + \tau_1(\mu) \right| &< |D_1^0(\mu)| + \varepsilon \\
&< m_C |D_0^0(\mu)| + \varepsilon \\
&\leq \left(1 - \frac{\varepsilon}{|D_0^0(\mu)|} \right)^{-1} m_C \left| D_0^0(\mu) + \tau_0(\mu) \right| + \varepsilon \\
&= \left(\left(1 - \frac{\varepsilon}{|D_0^0(\mu)|} \right)^{-1} m_C \right. \\
&\qquad \left. + \frac{\varepsilon}{|D_0^0(\mu) + \tau_0(\mu)|} \right) \left| D_0^0(\mu) + \tau_0(\mu) \right| \\
&< \left(G m_C + \frac{\varepsilon}{|D_0^0(\mu) + \tau_0(\mu)|} \right) \left| D_0^0(\mu) + \tau_0(\mu) \right|.
\end{aligned} \tag{3.2.64}$$

For each $\mu \in \Gamma_k \bigcap S_0$ and $k \geq k_1(\varepsilon)$, $|D_0^0(\mu) + \tau_0(\mu)| \geq ||D_0^0(\mu)| - |\tau_0(\mu)|| > m_B - \varepsilon$ and (3.2.64) becomes

$$\left| D_1^0(\mu) + \tau_1(\mu) \right| < \left(D m_C + \frac{\varepsilon}{m_B - \varepsilon} \right) \left| D_0^0(\mu) + \tau_0(\mu) \right|.$$

Set $M := \left(D m_C + \varepsilon (m_B - \varepsilon)^{-1} \right)$. Then $k_M \in \mathbb{N}$ is such that $\Gamma_{k_M} := \{\mu \in \mathbb{C} : \max((\Re\mu)^2, (\Im\mu)^2) = M^2\}$, and for $\mu \in \Gamma_k \bigcap S_0$ where $k \geq k_1$

$$\left| D_1^0(\mu) + \tau_1(\mu) \right| < M \left| D_0^0(\mu) + \tau_0(\mu) \right|. \tag{3.2.65}$$

For all $\mu \in \Gamma_k \bigcap S_0$ such that $k \geq k_2 = \max\{k_1(\varepsilon), k_M\}$

$$\alpha|\mu| \left|D_0^0(\mu) + \tau_0(\mu)\right| > \left|D_1^0(\mu) + \tau_1(\mu)\right|. \tag{3.2.66}$$

Hence, for all $\mu \in \Gamma_k \bigcap S_0$ such that $k \geq k_2$,

$$\begin{aligned} \alpha|\mu| \left| D_0^0(\mu) + \sum_{\substack{j=2 \\ j\neq 5}}^{7} C_j e^{(\omega_j - \omega_0)\mu a} \right| |e^{\omega_0 \mu a}| > \left| D_1^0(\mu) + \sum_{\substack{j=2 \\ j\neq 5}}^{7} B_j e^{(\omega_j - \omega_0)\mu a} \right| |e^{\omega_0 \mu a}| \\ \Leftrightarrow \left| \mu\alpha \sum_{j=0}^{7} C_j e^{\omega_j \mu a} \right| > \left| \sum_{j=0}^{8} B_j e^{\omega_j \mu a} \right| \\ \Leftrightarrow |D_c(\mu)| > |D_b(\mu)|. \end{aligned} \tag{3.2.67}$$

The square Γ_k is invariant under rotation by $\frac{\pi}{2}$. From the symmetry in (3.2.46) and (3.2.47), a rotation of the function $D(\mu)$ by $\frac{\pi}{2}$ preserves the domination of $D_c(\mu)$ on $D_b(\mu)$. Thus,

$$|D_c(\mu)| > |D_b(\mu)| \tag{3.2.68}$$

on $\Gamma_k \bigcap S_j$, $k \geq k_2$ and $j = 0, 1, 2, 3$. Hence, by Rouché's theorem we conclude that inside Γ_k the number of zeros of $D(\mu)$ is the same as the number of zeros of $D_c(\mu)$.

We consider two boundary value problems, one when $\epsilon_1 = 1$ and the other when $\epsilon_1 = -1$ and their characteristic determinants. From (3.2.13), ϵ_1 and ϵ_2

only occurs in terms where $j = 0, 1, 2, 3$ in (3.2.52). For $\epsilon_1 = \pm 1$, denote

$$D_c(\mu) = \frac{-i}{8}\mu\alpha \sum_{j=0}^{7} C_j e^{\omega_j \mu a} = \frac{-i}{8}\mu\alpha \left(\epsilon_1 A(\mu) + B(\mu)\right) \tag{3.2.69}$$

where

$$\begin{aligned} A(\mu) =& \sum_{j=0}^{3} C_j e^{\omega_j \mu a} \\ =& -2(1-i)(e^{(1+i)\mu a} - e^{-(1+i)\mu a}) \\ & -2(1+i)(e^{(1-i)\mu a} - e^{-(1-i)\mu a}) \\ =& -4i\left((1-i)\sin(1+i)\mu a + (1+i)\sin(1-i)\mu a\right) \\ =& -8(\sin\mu a \cos i\mu a - i\sin i\mu a \cos\mu a), \end{aligned} \tag{3.2.70}$$

and

$$\begin{aligned} B(\mu) &= \sum_{j=4}^{7} C_j e^{\omega_j \mu a} \\ &= 4(e^{\mu a} - e^{-\mu a} - i(e^{i\mu a} - e^{-i\mu a})) \\ &= 8(\sin\mu a - i\sin i\mu a). \end{aligned} \tag{3.2.71}$$

Let $D_1(\mu)$ and $D_{-1}(\mu)$ be the characteristic determinants when $\epsilon_1 = 1$ and $\epsilon_1 = -1$, respectively. By Rouché's theorem the number of zeros of $D_1(\mu)$ and $D_{-1}(\mu)$ inside Γ_k, $k \geq k_2$, is the same as the number of zeros of $\frac{-i}{8}\mu\alpha\left(A(\mu) + B(\mu)\right)$ and $-\frac{i}{8}\mu\alpha\left(-A(\mu) + B(\mu)\right)$, respectively, because the estimate (3.2.68) holds. We now want to count the number zeros of

$$\begin{aligned}A(\mu)+B(\mu) &= \sin\mu a(1-\cosh\mu a)+\sinh\mu a(1-\cos\mu a)\\ &= 2\sin\frac{\mu a}{2}\left(\cos\frac{\mu a}{2}(1-\cosh\mu a)+\sinh\mu a\sin\frac{\mu a}{2}\right)\\ &= 4\sin\frac{\mu a}{2}\sinh\frac{\mu a}{2}\left(\cosh\frac{\mu a}{2}\sin\frac{\mu a}{2}-\sinh\frac{\mu a}{2}\cos\frac{\mu a}{2}\right) \qquad (3.2.72)\end{aligned}$$

and

$$\begin{aligned}-A(\mu)+B(\mu) &= \sin\mu a(1+\cosh\mu a)+\sinh\mu a(1+\cos\mu a)\\ &= 4\cos\frac{\mu a}{2}\cosh\frac{\mu a}{2}\left(\cosh\frac{\mu a}{2}\sin\frac{\mu a}{2}+\sinh\frac{\mu a}{2}\cos\frac{\mu a}{2}\right) \qquad (3.2.73)\end{aligned}$$

inside Γ_k, $k \geq k_2$. The location of the zeros of the factors $\cosh\frac{\mu a}{2}\sin\frac{\mu a}{2} - \sinh\frac{\mu a}{2}\cos\frac{\mu a}{2}$ and $\cosh\frac{\mu a}{2}\sin\frac{\mu a}{2}+\sinh\frac{\mu a}{2}\cos\frac{\mu a}{2}$ in (3.2.72) and (3.2.73) is given in [10, Lemma 3.1, Case 3] and [10, Lemma 3.1, Case 5], respectively. We quote the two cases of the lemma below.

Lemma 3.2.3. ***Case 3***. *Let $\phi_0(\mu) = \frac{1}{\mu}(\cosh\mu a\sin\mu a - \sinh\mu a\cos\mu a)$. The function ϕ_0 has a zero of multiplicity 2 at 0, exactly one simple zero in each interval $((m-\frac{1}{2})\frac{\pi}{a},(m+\frac{1}{2})\frac{\pi}{a})$ for positive integer m with asymptotics*

$$\widetilde{\mu}_m = (4m-3)\frac{\pi}{4a}+o(1), \quad m=1,2,\ldots,$$

and simple zeros at $-\widetilde{\mu}_m = \widetilde{\mu}_{-m}$, $i\widetilde{\mu}_m$, $-i\widetilde{\mu}_k$, for $m=1,2,\ldots,$ and no other zeros.
Case 5. *Let $\phi_0(\mu) = \mu(\cosh\mu a\sin\mu a+\sinh\mu a\cos\mu a)$. The function ϕ_0 has a zero of multiplicity 2 at 0, exactly one simple zero in each interval $((m-\frac{1}{2})\frac{\pi}{a},(m+\frac{1}{2})\frac{\pi}{a})$*

for positive integer m with asymptotics

$$\widetilde{\mu}_m = (4m-1)\frac{\pi}{4a} + o(1), \quad m = 1, 2, \ldots,$$

and simple zeros at $-\widetilde{\mu}_m = \widetilde{\mu}_{-m}$, $i\widetilde{\mu}_m$, $-i\widetilde{\mu}_m$, for $m = 1, 2, \ldots$, and no other zeros.

Let a in the Lemma above be replaced by $\frac{a}{2}$. The zeros of (3.2.72) from $\sin\frac{\mu a}{2}$ are at $\mu = 2n\frac{\pi}{a}$ and those from $\sinh\frac{\mu a}{2}$ are at $\mu = 2ni\frac{\pi}{a}$, where n is an integer, which implies that the number of zeros from $\sin\frac{\mu a}{2}$ and $\sinh\frac{\mu a}{2}$ is $2k+1$ each, inside Γ_k, $k \geq k_2$. Lemma 3.2.3 gives asymptotics of $\frac{1}{\mu}(\cosh\frac{\mu a}{2}\sin\frac{\mu a}{2} - \sinh\frac{\mu a}{2}\cos\frac{\mu a}{2})$ which coincide with the asymptotics of Proposition 3.2.2 when $\epsilon_1 = 1$. Applying the statement of Lemma 3.2.3 to the factor $\cosh\frac{\mu a}{2}\sin\frac{\mu a}{2} - \sinh\frac{\mu a}{2}\cos\frac{\mu a}{2}$, means that it contributes $4(k+1)$ nonzero zeros on the 4 axes inside Γ_k, $k \geq k_2$, a zero at 0 of multiplicity 3 and no other zeros. The zeros of $-\frac{i}{8}\mu\alpha(A(\mu)+B(\mu))$ then total to $8k+10$ inside Γ_k, $k \geq k_2$. Similarly, the zeros of (3.2.73) from $\cos\frac{\mu a}{2}$ are at $\mu = (2n+1)\frac{\pi}{a}$ and those from $\cosh\frac{\mu a}{2}$ are at $\mu = (2n+1)i\frac{\pi}{a}$, where n is an integer and there is $2k$ zeros from each of these two factors inside Γ_k, $k \geq k_2$. The factor $\mu(\cosh\mu a\sin\mu a + \sinh\mu a\cos\mu a)$, according to Lemma 3.2.3, has $4k$ nonzero zeros and the zero at 0 has multiplicity 2 giving $-\frac{i}{8}\mu\alpha(-A(\mu)+B(\mu))$ a total of $8k+2$ zeros inside Γ_k, $k \geq k_2$. Denote the number of zeros of $-\frac{i}{8}\mu\alpha(A(\mu)+B(\mu))$ and $-\frac{i}{8}\mu\alpha(-A(\mu)+B(\mu))$ inside Γ_k, $k \geq k_2$ as $\#^+\Gamma_k$ and $\#^-\Gamma_k$, respectively. As it has already been shown in (3.2.68), the number of zeros of $D_1(\mu)$ and $D_{-1}(\mu)$ inside Γ_k, $k \geq k_2$, is $\#^+\Gamma_k$ and $\#^-\Gamma_k$. Consider the square Γ_{k+1}. Then the number of zeros of $D_1(\mu)$ and $D_{-1}(\mu)$ inside Γ_{k+1}, $k \geq k_2$, is $\#^+\Gamma_{k+1}$ and $\#^-\Gamma_{k+1}$ which is $8(k+1)+10$ and $8(k+1)+2$. Thus, inside the annulus, $\Gamma_{k+1} - \Gamma_k$, $D_1(\mu)$ and $D_{-1}(\mu)$ have 8 zeros each, as their only zeros. Proposition 3.2.2 posits that for $g = 0$, there exists a positive integer k_0 such that zeros of $D(\mu)$ are as given in

(3.3.1) and (3.3.2). If we choose k_0 such that the index k of Γ_k is larger than k_0, then the number of zeros given by (3.3.1) and (3.3.2) is $8(k-k_0)$ each, missing $8k_0$ zeros of $D_1(\mu)$ and $D_{-1}(\mu)$ inside Γ_k, $k \geq k_2$.

Theorem 3.2.4. *For $g=0$ the zeros of $D(\mu)$ are*

$$\begin{cases} \mu_{k,1}^{j,+} = i^j(4k-3)\frac{\pi}{2a} + o(1), \\ \mu_{l,2}^{+} = 2l\frac{\pi}{a} + o(1), \\ \mu_{l,3}^{+} = i2l\frac{\pi}{a} + o(1) \end{cases} \tag{3.2.74}$$

when $\epsilon_1 = \epsilon_2 = 1$, and

$$\begin{cases} \mu_{m,1}^{j,-} = i^j(4m-1)\frac{\pi}{2a} + o(1), \\ \mu_{p,2}^{j,-} = i^j(2p-1)\frac{\pi}{a} + o(1), \\ \mu_{1}^{-} = \frac{-\pi}{2a} + o(1) \\ \mu_{2}^{-} = \frac{-\pi}{a} + o(1) \end{cases} \tag{3.2.75}$$

when $\epsilon_1 = \epsilon_2 = -1$, $k \in \mathbb{N} \cup \{0\}$, $l \in \mathbb{Z}$, $m, p \in \mathbb{N}$ and $j = 0, 1, 2, 3$.

3.3 Asymptotic expansions of eigenvalues when $g \neq 0$

It is easy to show that (3.2.1)–(3.2.5) is Birkhoff regular. Solutions of Birkhoff regular problems can be written as a series of the eigenvalue parameter and differentiable functions on $[0, a]$ and Birkhoff regular problems have an estimate for the Green's function. In this section we find the first few leading terms of the asymptotic expansion of the eigenvalues. We will use the results of [5, Chapter VIII] for asymptotic fundamental systems of differential equations of the form

(3.2.1).

For the case $g = 0$ we already know the asymptotic distribution of the eigenvalues, see Theorem 3.2.4. Denote the corresponding characteristic function D by D_0. Due to the Birkhoff regularity, g only influences lower order terms in D, and therefore, it follows from the estimates in [5, Appendix A.2] that, away from small disks around the zeros of D_0, $|D(\lambda) - D_0(\lambda)| < |D_0(\lambda)|$ if $|\lambda|$ is sufficiently large. The function $D(\lambda)$ is not analytic, but this estimate extends to the analytic equivalents with, e. g., a fundamental system y_j, $j = 1, \dots, 4$, with $y_j^{(m)}(0) = \delta_{j,m+1}$ for $m = 0, \dots, 3$, since these fundamental systems for general g and $g = 0$ are asymptotically close. Hence, applying Rouché's theorem both to large circles centered at zero avoiding the small disks and to the boundaries of the small discs which are sufficiently far away from 0, it follows that the eigenvalue problem for general g have the same asymptotic distribution as for $g = 0$. Hence, Theorem 3.2.4 leads to

Theorem 3.3.1. *For $g \in C^1[0, a]$, the zeros of $D(\mu)$ are*

$$\begin{cases} \mu_{k,1}^{j,+} = i^j(4k-3)\frac{\pi}{2a} + o(1), \\ \mu_{l,2}^{+} = 2l\frac{\pi}{a} + o(1), \\ \mu_{l,3}^{+} = i2l\frac{\pi}{a} + o(1) \end{cases} \tag{3.3.1}$$

when $\epsilon_1 = \epsilon_2 = 1$*, and*

$$\begin{cases} \mu_{m,1}^{j,-} = i^j(4m-1)\frac{\pi}{2a} + o(1), \\ \mu_{p,2}^{j,-} = i^j(2p-1)\frac{\pi}{a} + o(1), \\ \mu_1^- = \frac{-\pi}{2a} + o(1) \\ \mu_2^- = \frac{-\pi}{a} + o(1) \end{cases} \tag{3.3.2}$$

when $\epsilon_1 = \epsilon_2 = -1$*,* $k \in \mathbb{N} \cup \{0\}$*,* $l \in \mathbb{Z}$*,* $m, p \in \mathbb{N}$ *and* $j = 0, 1, 2, 3$*.*

Let $\lambda = \mu^2$. Then, since the coefficient of $y^{(3)}$ in (3.2.1) is zero, [5, (8.2.3)] immediately gives $\varphi_0(x) = 1$. It now follows that the characteristic function $D_g(\mu)$ of (3.2.1)–(3.2.5) is the determinant of the associated characteristic matrix given by

$$M_g(\mu) = W^{(0)}(\mu)Y(0,\mu) + W^{(1)}(\mu)Y(a,\mu)$$

and

$$M_g(\mu) = \begin{pmatrix} \delta_{1,0}(0,\mu) - \epsilon_1\delta_{1,0}(a,\mu)e^{\mu a} & \delta_{2,0}(0,\mu) - \epsilon_1\delta_{2,0}(a,\mu)e^{i\mu a} \\ \delta_{1,1}(0,\mu) & \delta_{2,1}(0,\mu) \\ \delta_{1,3}(0,\mu) - \epsilon_2\delta_{1,3}(a,\mu)e^{\mu a} & \delta_{2,3}(0,\mu) - \epsilon_2\delta_{2,3}(a,\mu)e^{i\mu a} \\ [\delta_{1,2}(a,\mu) + i\alpha\mu^2\delta_{1,1}(a,\mu)]e^{\mu a} & [\delta_{2,2}(a,\mu) + i\alpha\mu^2\delta_{2,1}(a,\mu)]e^{i\mu a} \end{pmatrix.$$
$$\begin{matrix} \delta_{3,0}(0,\mu) - \epsilon_1\delta_{3,0}(a,\mu)e^{-\mu a} & \delta_{4,0}(0,\mu) - \epsilon_1\delta_{4,0}(a,\mu)e^{-i\mu a} \\ \delta_{3,1}(0,\mu) & \delta_{4,1}(0,\mu) \\ \delta_{3,3}(0,\mu) - \epsilon_2\delta_{3,3}(a,\mu)e^{-\mu a} & \delta_{4,3}(0,\mu) - \epsilon_2\delta_{4,3}(a,\mu)e^{-i\mu a} \\ [\delta_{3,2}(a,\mu) + i\alpha\mu^2\delta_{3,1}(a,\mu)]e^{-\mu a} & [\delta_{4,2}(a,\mu) + i\alpha\mu^2\delta_{4,1}(a,\mu)]e^{-i\mu a} \end{matrix}\Bigg),$$

where $Y(x,\mu)=(\eta_j^{(q-1)}(x,\mu))_{q,j=1}^4$ is the fundamental matrix associated with the fundamental system $\{\eta_1,\eta_2,\eta_3,\eta_4\}$ and $\eta_j^{(q-1)}(x,\mu)$ are defined in (1.2.43).

The characteristic determinant is

$$\begin{aligned} D_g(\mu) &= \det(M_g(\mu)) \\ &= \sum_{j=0}^{8} \psi_j e^{\omega_j \mu a}, \end{aligned} \tag{3.3.3}$$

where ω_j are as in (3.2.15). For large μ in the sector $\arg\mu \in [-\frac{\pi}{12}, \frac{5\pi}{12}]$,

$$\begin{aligned} D_g(\mu)e^{-\omega_0\mu a} &= \psi_0 + \psi_1 e^{2i\mu a} + \psi_5 e^{i\mu a} + \sum_{\substack{j=2\\ j\neq 5}}^{8} \psi_j e^{(\omega_j-\omega_0)\mu a} \\ &= \psi_{0,2} + i\alpha\mu^2\psi_{0,1} + (\psi_{1,2} + i\alpha\mu^2\psi_{1,1})e^{2i\mu a} \\ &\quad + (\psi_{5,2} + i\alpha\mu^2\psi_{5,1})e^{i\mu a} + o(\mu^{1-s}),\ \ s\in\mathbb{N}, \end{aligned} \tag{3.3.4}$$

where

$$\begin{aligned} \psi_{0,l} = &- \epsilon_1\delta_{1,0}(a,\mu)\delta_{2,1}(0,\mu)\delta_{3,3}(0,\mu)\delta_{4,l}(a,\mu) \\ &+ \epsilon_1\delta_{1,0}(a,\mu)\delta_{3,1}(0,\mu)\delta_{2,3}(0,\mu)\delta_{4,l}(a,\mu) \\ &- \epsilon_2\delta_{2,0}(0,\mu)\delta_{3,1}(0,\mu)\delta_{1,3}(a,\mu)\delta_{4,l}(a,\mu) \\ &+ \epsilon_2\delta_{2,0}(0,\mu)\delta_{3,1}(0,\mu)\delta_{4,3}(a,\mu)\delta_{1,l}(a,\mu) \\ &+ \epsilon_2\delta_{3,0}(0,\mu)\delta_{2,1}(0,\mu)\delta_{1,3}(a,\mu)\delta_{4,l}(a,\mu) \\ &- \epsilon_2\delta_{3,0}(0,\mu)\delta_{2,1}(0,\mu)\delta_{4,3}(a,\mu)\delta_{1,l}(a,\mu) \\ &+ \epsilon_1\delta_{4,0}(a,\mu)\delta_{2,1}(0,\mu)\delta_{3,3}(0,\mu)\delta_{1,l}(a,\mu) \\ &- \epsilon_1\delta_{4,0}(a,\mu)\delta_{3,1}(0,\mu)\delta_{2,3}(0,\mu)\delta_{1,l}(a,\mu), \end{aligned}$$

$$
\begin{aligned}
\psi_{1,l} = &- \epsilon_1\delta_{1,0}(a,\mu)\delta_{3,1}(0,\mu)\delta_{4,3}(0,\mu)\delta_{2,l}(a,\mu) \\
&+ \epsilon_1\delta_{1,0}(a,\mu)\delta_{4,1}(0,\mu)\delta_{3,3}(0,\mu)\delta_{2,l}(a,\mu) \\
&+ \epsilon_1\delta_{2,0}(a,\mu)\delta_{3,1}(0,\mu)\delta_{4,3}(0,\mu)\delta_{1,l}(a,\mu) \\
&- \epsilon_1\delta_{2,0}(a,\mu)\delta_{4,1}(0,\mu)\delta_{3,3}(0,\mu)\delta_{1,l}(a,\mu) \\
&- \epsilon_2\delta_{3,0}(0,\mu)\delta_{4,1}(0,\mu)\delta_{1,3}(a,\mu)\delta_{2,l}(a,\mu) \\
&+ \epsilon_2\delta_{3,0}(0,\mu)\delta_{4,1}(0,\mu)\delta_{2,3}(a,\mu)\delta_{1,l}(a,\mu) \\
&+ \epsilon_2\delta_{4,0}(0,\mu)\delta_{3,1}(0,\mu)\delta_{1,3}(a,\mu)\delta_{2,l}(a,\mu) \\
&- \epsilon_2\delta_{4,0}(0,\mu)\delta_{3,1}(0,\mu)\delta_{2,3}(a,\mu))\delta_{1,l}(a,\mu),
\end{aligned}
$$

$$
\begin{aligned}
\psi_{5,l} = &- \epsilon_1\epsilon_2\delta_{1,0}(a,\mu)\delta_{3,1}(0,\mu)\delta_{4,l}(a,\mu)\delta_{2,3}(a,\mu) \\
&+ \epsilon_1\epsilon_2\delta_{1,0}(a,\mu)\delta_{3,1}(0,\mu)\delta_{2,l}(a,\mu)\delta_{4,3}(a,\mu) \\
&+ \epsilon_1\epsilon_2\delta_{2,0}(a,\mu)\delta_{3,1}(0,\mu)\delta_{4,l}(a,\mu)\delta_{1,3}(a,\mu) \\
&- \epsilon_1\epsilon_2\delta_{2,0}(a,\mu)\delta_{3,1}(0,\mu)\delta_{1,l}(a,\mu)\delta_{4,3}(a,\mu) \\
&- \qquad \delta_{2,0}(0,\mu)\delta_{3,1}(0,\mu)\delta_{1,l}(a,\mu)\delta_{4,3}(0,\mu) \\
&+ \qquad \delta_{2,0}(0,\mu)\delta_{4,1}(0,\mu)\delta_{1,l}(a,\mu)\delta_{3,3}(0,\mu) \\
&+ \qquad \delta_{3,0}(0,\mu)\delta_{2,1}(0,\mu)\delta_{1,l}(a,\mu)\delta_{4,3}(0,\mu) \\
&- \qquad \delta_{3,0}(0,\mu)\delta_{4,1}(0,\mu)\delta_{1,l}(a,\mu)\delta_{2,3}(0,\mu) \\
&- \qquad \delta_{4,0}(0,\mu)\delta_{2,1}(0,\mu)\delta_{1,l}(a,\mu)\delta_{3,3}(0,\mu) \\
&+ \qquad \delta_{4,0}(0,\mu)\delta_{3,1}(0,\mu)\delta_{1,l}(a,\mu)\delta_{2,3}(0,\mu) \\
&- \epsilon_1\epsilon_2\delta_{4,0}(a,\mu)\delta_{3,1}(0,\mu)\delta_{2,l}(a,\mu)\delta_{1,3}(a,\mu) \\
&+ \epsilon_1\epsilon_2\delta_{4,0}(a,\mu)\delta_{3,1}(0,\mu)\delta_{1,l}(a,\mu)\delta_{2,3}(a,\mu)
\end{aligned}
$$

and $l = 1, 2$.

The coefficient of $y^{[3]}$ in (3.2.1) is zero and application of [5, (8.2.3)] to (3.2.1) gives $\varphi_0(x) = 1$. To determine φ_1, $n_0 = 0$ and $l = 4$ according to [5, Theorem 8.1.2]. From [5, (8.2.45)]

$$\varphi_r = \varphi_{1,r} = \varepsilon_1^T V Q^{[r]} \varepsilon_1 \tag{3.3.5}$$

where ε_ν is the νth unit vector in $\mathbb{C}^4$, $V = (i^{(j-1)(k-1)})_{j,k=1}^4$, and $Q^{[r]}$ are 4×4 matrices given by [5, (8.2.28)], [5, (8.2.33)] and [5, (8.2.34)], i.e, $Q^{[0]} = I_4$,

$$\Omega_4 Q^{[1]} - Q^{[1]}\Omega_4 = Q^{[0]\prime} = 0, \tag{3.3.6}$$

$$\Omega_4 Q^{[2]} - Q^{[2]}\Omega_4 = Q^{[1]\prime} - \frac{1}{4} g \Omega_4 \varepsilon \varepsilon^T \Omega_4^{-2} Q^{[0]} = 0, \tag{3.3.7}$$

$$0 = \varepsilon_\nu^T \left(Q^{[2]\prime} + \frac{1}{4} \sum_{j=1}^{2} k_{3-} \Omega_4 \varepsilon \varepsilon^T \Omega_4^{-1-j} Q^{[2-j]} \right) \varepsilon_\nu, \qquad (\nu = 1, 2, 3, 4), \tag{3.3.8}$$

where $k_2 = -g$, $k_1 = -g'$, $\Omega_4 = \operatorname{diag}(1, \mathrm{i}, -1, -\mathrm{i})$ and $\varepsilon^T = (1, 1, 1, 1)$. A lengthy calculation gives

$$\varphi_1(x) = \frac{1}{4} \int_0^x g(t)dt. \tag{3.3.9}$$

The highest power of μ in (3.3.4) is 7. To find the value of $\tau_{k,1}$ we equate coefficeints but first we find highest powers of μ of the terms of (3.3.4),

$$
\begin{aligned}
\epsilon_1\psi_{0,2} &= \mu^6(-6i) + O(\mu^5)\\
\epsilon_1\psi_{0,1} &= \mu^5(4-4i) + \mu^4 8\varphi_1(a) + O(\mu^3)\\
\epsilon_1\psi_{1,2} &= \mu^6(-6i) + O(\mu^5)\\
\epsilon_1\psi_{1,1} &= \mu^5(-4-4i) + \mu^4(-8)\varphi_1(a) + O(\mu^3)\\
\psi_{5,2} &= \mu^6 8i + O(\mu^5)\\
\psi_{5,1} &= \mu^5 8i + \mu^4 8i\varphi_1(a) + O(\mu^3).
\end{aligned}
\tag{3.3.10}
$$

We now want to find the asymptotics

$$
\begin{aligned}
&\mu_{k,1}^{0,+} = k\frac{2\pi}{a} - \frac{3\pi}{2a} + \tau_k^{0,+}, \qquad \tau_k^{0,+} = \sum_{j=1}^{n} \tau_{k,j}^{0,+} k^{-j} + o(k^{-n}) \qquad k = 0, 1, \ldots,\\
&\mu_{l,2}^{+} = l\frac{2\pi}{a} + \tau_l^{+}, \qquad \tau_l^{+} = \sum_{j=1}^{n} \tau_{l,j}^{+} l^{-j} + o(l^{-n}) \qquad l = 0, 1, \ldots,\\
&\mu_{m,1}^{0,-} = m\frac{2\pi}{a} - \frac{\pi}{2a} + \tau_m^{0,-}, \qquad \tau_m^{0,-} = \sum_{j=1}^{n} \tau_{m,j}^{0,-} m^{-j} + o(m^{-n}) \qquad m = 1, 2, \ldots,\\
&\mu_{p,2}^{0,-} = p\frac{2\pi}{a} - \frac{\pi}{a} + \tau_p^{0,-,2}, \qquad \tau_p^{0,-,2} = \sum_{j=1}^{n} \tau_{p,j}^{0,-,2} p^{-j} + o(p^{-n}) \qquad p = 1, 2, \ldots.
\end{aligned}
\tag{3.3.11}
$$

We note that

$$\frac{1}{\mu_{k,1}^{0,+}} = \left(k\frac{2\pi}{a} - \frac{3\pi}{2a} + \tau_k^{0,+}\right)^{-1}$$
$$= \frac{a}{2\pi k}\left(1 + \frac{3}{4k} + o(k^{-1})\right),$$

$$\frac{1}{\mu_{l,2}^{+}} = \left(l\frac{2\pi}{a} + \tau_l^{+}\right)^{-1}$$
$$= \frac{a}{2\pi l}\left(1 + o(l^{-1})\right),$$

$$\frac{1}{\mu_{m,1}^{0,-}} = \left(m\frac{2\pi}{a} - \frac{\pi}{2a} + \tau_m^{0,-}\right)^{-1}$$
$$= \frac{a}{2\pi m}\left(1 + \frac{1}{4m} + o(m^{-1})\right),$$

$$\frac{1}{\mu_{p,2}^{0,-}} = \left(p\frac{2\pi}{a} - \frac{\pi}{a} + \tau_p^{0,-,2}\right)^{-1}$$
$$= \frac{a}{2\pi p}\left(1 + \frac{1}{2p} + o(p^{-1})\right), \tag{3.3.12}$$

$$
\begin{aligned}
e^{i\mu_{k,1}^{0,+}a} &= e^{i(k\frac{2\pi}{a}-\frac{3\pi}{2a}+\tau_{k,1}^{0,+}k^{-1}+o(k^{-1}))a} \\
&= ie^{iao(k^{-1})}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{ia\tau_{k,1}^{0,+}}{k}\right)^{j} \\
&= i\left(1+o(k^{-1})\right)\left(1+\frac{ia}{k}\tau_{k,1}^{0,+}+O(k^{-2})\right) \\
&= i-\frac{a}{k}\tau_{k,1}^{0,+}+o(k^{-1}), \\
e^{i\mu_{l,2}^{+}a} &= e^{i(l\frac{2\pi}{a}+\tau_{l,1}^{+}l^{-1}+o(l^{-1}))a} \\
&= 1+\frac{ia}{l}\tau_{l,1}^{+}+o(l^{-1}), \\
e^{i\mu_{m,1}^{0,-}a} &= e^{i(m\frac{2\pi}{a}-\frac{\pi}{2a}+\tau_{m,1}^{0,-}m^{-1}+o(m^{-1}))a} \\
&= -i+\frac{a}{m}\tau_{m,1}^{0,-}+o(m^{-1}), \\
e^{i\mu_{p,2}^{0,-}a} &= e^{i(p\frac{2\pi}{a}-\frac{\pi}{a}+\tau_{p,1}^{0,-,2}p^{-1}+o(p^{-1}))a} \\
&= -1-\frac{ia}{p}\tau_{p,1}^{0,-,2}+o(p^{-1})
\end{aligned}
\tag{3.3.13}
$$

and

$$
\begin{aligned}
e^{2i\mu_{k,1}^{0,+}a} &= -1-\frac{2ia}{k}\tau_{k,1}^{0,+}+o(k^{-1}), \\
e^{2i\mu_{l,2}^{+}a} &= 1+\frac{2ia}{l}\tau_{l,1}^{+}+o(l^{-1}), \\
e^{2i\mu_{m,1}^{0,-}a} &= -1-\frac{2ia}{m}\tau_{m,1}^{0,-}+o(m^{-1}), \\
e^{2i\mu_{p,2}^{0,-}a} &= 1+\frac{2ia}{p}\tau_{p,1}^{0,-,2}+o(p^{-1}).
\end{aligned}
\tag{3.3.14}
$$

For large $\mu_{k,1}^{0,+}$ in the sector $\arg\mu\in[-\frac{\pi}{12},\frac{5\pi}{12}]$,

$$
\begin{aligned}
\frac{D_g(\mu_{k,1}^{0,+})e^{-\omega_0\mu_{k,1}^{0,+}a}}{\left(\mu_{k,1}^{0,+}\right)^7} &= \epsilon_1\frac{-6i}{\mu_{k,1}^{0,+}} + i\alpha\epsilon_1\left(4-4i+\frac{8\varphi_1(a)}{\mu_{k,1}^{0,+}}\right)\\
&\quad+\left(\epsilon_1\frac{-6i}{\mu_{k,1}^{0,+}} + i\alpha\epsilon_1\left(-4-4i+\frac{-8}{\mu_{k,1}^{0,+}}\varphi_1(a)\right)\right)\left(-1-\frac{2ia}{k}\tau_{k,1}^{0,+}\right)\\
&\quad+\left(\frac{8i}{\mu_{k,1}^{0,+}} + i\alpha\left(8i+\frac{8i}{\mu_{k,1}^{0,+}}\varphi_1(a)\right)\right)\left(i-\frac{a}{k}\tau_{k,1}^{0,+}\right)+o(k^{-1})\\
&= \epsilon_1\frac{-6ia}{2\pi k} + i\alpha\epsilon_1\left(4-4i+\frac{8a}{2\pi k}\varphi_1(a)\right)\\
&\quad+\left(\epsilon_1\frac{-6ia}{2\pi k} + i\alpha\epsilon_1\left(-4-4i+\frac{-8a}{2\pi k}\varphi_1(a)\right)\right)\left(-1-\frac{2ia}{k}\tau_{k,1}^{0,+}\right)\\
&\quad+\left(\frac{8ia}{2\pi k} + i\alpha\left(8i+\frac{8ia}{2\pi k}\varphi_1(a)\right)\right)\left(i-\frac{a}{k}\tau_{k,1}^{0,+}\right)+o(k^{-1})
\end{aligned}
\tag{3.3.15}
$$

and $D_g(\mu_{k,1}^{0,+})$ in (3.3.15) is 0, meaning that the coefficient of k^{-1} must be 0. Equating the coefficient of k^{-1} to 0, we get the value of $\tau_{k,1}^{0,+}$. In a similar manner, we calculate all the other values of the asymptotic terms listed below as

$$
\begin{aligned}
\tau_{k,1}^{0,+} &= \frac{1}{2\pi}(\varphi_1(a)+\frac{i}{\alpha}),\\
\tau_{l,1}^{+} &= \frac{1}{2\pi}\frac{-i\varphi_1(a)-\frac{1}{2\alpha}}{2+i},\\
\tau_{m,1}^{0,-} &= \frac{1}{2\pi}\varphi_1(a)(-2-\frac{i}{\alpha}),\\
\tau_{p,1}^{0,-,2} &= \frac{1}{2\pi}\frac{i\varphi_1(a)-\frac{1}{2\alpha}}{2-i}.
\end{aligned}
\tag{3.3.16}
$$

Theorem 3.3.2. *For $g \in C^1[0,a]$ and $\alpha > 0$ the zeros of $D(\mu)$ are*

$$\begin{cases} \mu_{k,1}^{j,+} = i^j \left((4k-3)\frac{\pi}{2a} + k^{-1}\tau_{k,1}^{0,+}\right) + o(k^{-1}), \\ \mu_{l,2}^{+} = 2l\frac{\pi}{a} + l^{-1}\tau_{l,1}^{+} + o(l^{-1}), \\ \mu_{l,3}^{+} = i\left(2l\frac{\pi}{a} + l^{-1}\tau_{l,1}^{+}\right) + o(l^{-1}), \end{cases} \tag{3.3.17}$$

when $\epsilon_1 = \epsilon_2 = 1$, and

$$\begin{cases} \mu_{m,1}^{j,-} = i^j \left((4m-1)\frac{\pi}{2a} + m^{-1}\tau_{m,1}^{0,-}\right) + o(m^{-1}), \\ \mu_{p,2}^{j,-} = i^j \left((2p-1)\frac{\pi}{a} + p^{-1}\tau_{p,1}^{0,-,2}\right) + o(p^{-1}), \\ \mu_1^{-} = \frac{-\pi}{2a}, \\ \mu_2^{-} = \frac{-\pi}{a}, \end{cases} \tag{3.3.18}$$

when $\epsilon_1 = \epsilon_2 = -1$, $k \in \mathbb{N} \cup \{0\}$, $l \in \mathbb{Z}$, $m, p \in \mathbb{N}$, $j = 0, 1, 2, 3$ and $\tau_{k,1}^{0,+}, \tau_{l,1}^{+}, \tau_{m,1}^{0,-}, \tau_{p,1}^{0,-,2}$ are as in (3.3.16).

A

Appendix

Below are sets of Matlab code which where used to find some of the boundary conditions specified in the text. Further manual work was carried out on the output to select desired boundary conditions.

FINDINGSA.M

```
% This is the main program.
% This program uses functions : getU1_cellElements(U2,V,
    numOnes) and
% getSA(U1,U2,V,J).
clear all;
clc;
```

```
% Initialization
U1_cell_SA = cell(2,1);
j = 1;

U2 = [0 0 0 0 1 0 0 0];
V = [1 0 0 0 0 0 0 0];
J = zeros(8);
indexJ1 = [ 4 11 18 25];
indexJ2 = [40 47 54 61];

J(indexJ1) = (-1).^(indexJ1 + 1);
J(indexJ2) = (-1).^(indexJ2);

U1_cellFinal = getU1_cellElements(U2,V,6);

% No need to continue if there are no generated arrays
if isempty(U1_cellFinal{1})
    return;
end

for i = 1 : size(U1_cellFinal,1);
    [SA1,SA2,SA3] = getSA(U1_cellFinal{i},U2,V,J);
    if sum(sum(SA1)) == 0 && sum(sum(SA2)) == 0 && sum(
        sum(SA3)) == 0
```

```
            U1_cell_SA{j} = U1_cellFinal{i};
            j = j + 1;
            disp(U1_cellFinal{i});
        else
            display('No such U1 found!!!');
        end
end
```

GET$_c$*ellElements.m*

```
% This function is used to generate row vectors (containing values of 1's, -1's and 0's using some criteria).
% The criteria is that : Given 3 input arrays, A, U2, V, the generated row vectors must not have a value of 1
% or -1 in a column if a value of 1 is contained in the corresponding columns of either A, U2 or V.
% numRowOnes represents the number of +/- 1's to be placed in the generated row vectors.
% U1_cell is an array that stores the generated row vectors.

function U1_cell = get_cellElements(A,U2,V,numRowOnes)

% Initialization.
U1_cell = cell(2,1);
numColumns = size(U2,2);
```

```
U1_rowVector = zeros(1,numColumns);
l = 0;
k = 1;
i = 1;
val_1 = 1;
numOnes = 0;
if numRowOnes > 0
    % If numRowOnes = 2, the next while loop moves the
        first 1/-1 along the row vector.
    while i < (numColumns + 1)
        j = i;
        % The next while loop is responsible for moving
            the second 1/-1 along the row vector.
        while j < (numColumns + 1)
            % Checking the criteria
            if numOnes < numRowOnes && sum(A(1:end,j)) ==
                0 && sum(U2(1:end,j)) == 0 && sum(V(1:end
                ,j)) == 0
                if val_1 == 1
                    U1_rowVector(j) = 1;
                elseif val_1 == -1
                    U1_rowVector(j) = -1;
                end
                numOnes = numOnes + 1;
                if l == 0
```

```
                    i = j;
                    l = 1;
                end
            end
            % The following if statement helps to move the second 1/-1 along the row vector.
            if numOnes == numRowOnes
                U1_cell(k) = {U1_rowVector};
                if val_1 == 1
                    U1_rowVector(j) = -1;
                    k = k + 1;
                elseif val_1 == -1
                    U1_rowVector(j) = 1;
                end
                U1_cell(k) = {U1_rowVector};
                U1_rowVector(j) = 0;
                numOnes = numOnes - 1;
                k = k + 1;
            end
            j = j + 1;
        end
        numOnes = 0;
        % If numRowOnes = 1, need to break from the outer loop, since no other 1 to move.
        if numRowOnes == 1
```

```
                break;
            end
            if val_1 == 1
                val_1 = -1;
            elseif val_1 == -1
                val_1 = 1;
                U1_rowVector(i) = 0; % resetting the -1 to 0,
                    for next pattern.
                i = i + 1;
            end
        end
else
    display('numOnes value is zero, nothing to do!');
end
```

GETSA.M

```
% This function returns arrays SA1, SA2 and SA3 by using
    input arrays : U1,U2, V and J.
function [SA1,SA2,SA3] = getSA(U1,U2,V,J)
SA1 = (U1*J*U1');
SA2=(U1*J*U2');
SA3=(U1*J*V');
```

GETU1$_cellElements.m$

```
% This function is used to generate U1 arrays by using
    row vectors generated from
```

```
% 'get_cellElements(A,U2,V,numOnes)'function.
% U2 and V are input arrays, which are passed to '
    getU1_cellElements(A,U2,V,numOnes)'function.
% numOnes is the number of 1's to be used in the
    generated arrays, 1 < numOnes < 7.
% U1_cellFinal is used to store the generated arrays.

function U1_cellFinal = getU1_cellElements(U2,V,numOnes)

% Initialization.
U1_cellFinal = cell(2,1);
A = zeros(1,size(U2,2));
% This stores the number of rows to be generated.
U1_numRows = 4 - size(U2,1);
i = 1;
init_numOnes = numOnes;

if numOnes < 2 || numOnes > 6
    display('The min. number of numOnes = 2 &');
    display('The max. number of numOnes = 6');
    display('Please enter in that range!');
    return;
end

if numOnes > (U1_numRows * 2)
```

```
    display('Only a maximun of 2 ones are allowed per row');
    return;
end

if numOnes > U1_numRows
    numOnes = 2;
else if numOnes == U1_numRows
        numOnes = 1;
    else
        display('numOnes < U1_numRows (number of ones is less than number of rows');
        return;
    end
end

if i <= U1_numRows
    % Getting the first generated row vectors (first row in the generated U1 array).
    U1_cell_1 = get_cellElements(A,U2,V,numOnes);
    U1_cellFinal = U1_cell_1;

    % If there are more than one rows to be generated.
    while i < U1_numRows
        % Remaining 1's for row (i+1).
```

```
        numOnes = init_numOnes - numOnes;
        % Remaining rows thus far.
        left_U1_numRows = U1_numRows - i;
        init_numOnes = numOnes;
        if numOnes > left_U1_numRows
            numOnes = 2;
        else if numOnes == left_U1_numRows
                numOnes = 1;
            end
        end
        l = 1;
        j = 1;
        while j <= size(U1_cell_1,1)
            % Generating row vectors for each U1_cell_1 elements.
            U1_cell_2 = get_cellElements(U1_cell_1{j},U2,V,numOnes);
            k = 1;
            while k <= size(U1_cell_2,1)
                if ~(isempty(U1_cell_2{k}))
                    % Appending U1_cell_1 element with its generated row vectors.
                    U1_cellFinal{l} = [U1_cell_1{j};U1_cell_2{k}];
                    l = l + 1;
```

```
                        k = k + 1;
                    else
                        break;
                    end
                end
                j = j + 1;
            end
            U1_cell_1 = U1_cellFinal;
            i = i + 1;
        end
else
    display('No rows to work with!!!');
    return;
end
```

References

[1] P. Benner, V. Mehrmann, and H. Xu, *A numerically stable, structure preserving method for computing the eigenvalues of real hamiltonian or symplectic pencils*, Numer. Math. **78** (1998).

[2] G. D. Birkhoff, *On the asymptotic character of the solutions of certain linear differential equations containing a parameter*, Trans. Am. Math. Soc **9** (1908), 219–231.

[3] L. Eldén, *Matrix methods in data mining and pattern recognition*, SIAM (2007).

[4] C. F. Van Loan, *A symplectic method for approximating all the eigenvalues of a hamiltonian matrix*, Linear Algebra Appl. **61** (1984).

[5] R. Mennicken and M. Möller, *Non-self-adjoint boundary eigenvalue problems*, vol. 192, Elsevier, 2003.

[6] A. Minkin, *Odd and even cases of birkhoff-regularity*, Math. Nachr **174** (1995), 219–230.

[7] B. Moletsane and B. Zinsou, *Self-adjoint fourth order differential operator with eigenvalue parameter dependent and periodic boundary conditions*, Boundary Value Problems **2017:33** (2017).

[8] M. Möller and V. Pivovarchik, *Spectral properties of a fourth order differential equation*, J. Anal. App **25** (2006), 341–366.

[9] M. Möller and V. Pivovarchik, *Spectral theory of operator pencils, hermite-biehler functions, and their applications*, vol. 246, Birkhäuser, 2015.

[10] M. Möller and B. Zinsou, *Asymptotics of the eigenvalues of self-adjoint fourth order differential operators with separated eigenvalue parameter dependent boundary conditions*, Rocky Mountain J. Math.

[11] M. Möller and B. Zinsou, *Self-adjoint fourth order differential operators with eigenvalue parameter dependent boundary conditions*, Quaest. Math **34** (2011), 393–406.

[12] M. Möller and B. Zinsou, *Spectral asympotics of self-adjoint fourth order differential operators with eigenvalue parameter dependent boundary conditions*, Complex Anal. Oper. Theory **6** (2012).

[13] M. Möller and B. Zinsou, *Spectral asymptotics of self-adjoint fourth order boundary value problem with eigenvalue parameter dependent boundary conditions*, Bound. Value Probl. **2012:106** (2012).

[14] M. Möller and B. Zinsou, *Sixth order differential operators with eigenvalue dependent boundary conditions*, Appl. Anal. Discrete Math. **7** (2013), 378–389.

[15] V. Pivovarchik and C. van der Mee, *The inverse generalised regge problem.*, Inverse problems **17** (2001), 1831–1845.

[16] S. Salaff, *Regular boundary conditions for ordinary differential operators.*, AMS (1968).

[17] M. H. Stone, *A comparison of the series of fourier and birkhoff*, Trans. Amer. Math. Soc **28** (1926), 695–761.

[18] B. Zinsou, *Fourth order birkhoff regular problems with eigenvalue parameter dependent boundary conditions*, Turk J Math **40** (2016), 864–873.