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The constitution of school geometry in the Mathematics National Curriculum Statement and two Grade 10 geometry textbooks in South Africa

Lynn Bowie

A thesis submitted to the Wits School of Education, Faculty of Humanities, University of the Witwatersrand in fulfilment of the requirements for the degree of Doctor of Philosophy

Johannesburg, 2013

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DECLARATION

I declare that this thesis is my own unaided work. It is being submitted for the degree of Doctor of Philosophy at the University of Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Lynn Heather Bowie

24th day of July 2013

ABSTRACT

The National Curriculum Statement for Mathematics for grades 10 – 12 (MNCS) was the first major revision of the South African school mathematics curriculum for learners in those grades post apartheid. Thus the MNCS was created at a time when there was considerable pressure for the curriculum to play a role in redressing the social wrongs of apartheid and to provide access to mathematics for all learners in the country. In this study I analysed the MNCS and the geometry chapters in two popular textbooks aligned with the curriculum in order to examine what was constituted as school geometry in these documents. I interviewed members of the committee that created the MNCS, geometry consultants whose advice was sought on the curriculum and the textbook authors in order to understand the factors and constraints that shaped the production of school geometry in these texts. A modification of Bernstein's pedagogical device was used to frame the study and Kuzniak and Houdement's notion of the paradigms of geometry was used to gain insight into the nature of the geometry produced. This study shows how different areas in the field of production (mathematics, mathematics education and the general regulative discourse) and from the field of reproduction interact in the field of recontextualisation, mutually influence each other and are each recontextualised into the pedagogical discourse of the curriculum and textbooks.

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LIST OF ACRONYMS AND ABBREVIATIONS

Acronyms and abbreviations relating to the South African curriculum

C2005: Curriculum 2005 was the first major revision of the curriculum implemented in South Africa post apartheid.

OBE: Outcomes Based Education is the philosophy on which C2005 was based.

RNCS: The Revised National Curriculum Statement was produced to replace C2005 after a review of the difficulties experienced implementing C2005. It set out the curriculum for grades 0 – 9.

NCS: The National Curriculum Statement was implemented in 2006 and set out the curriculum for grades 10 – 12 in South Africa.

MNCS: The Mathematics National Curriculum Statement is the section of the NCS that deals specifically with the subject, Mathematics. This is the curriculum that is the subject of the research in this thesis.

LO: Learning Outcome. The content of the MNCS was organised into learning outcomes.

CAPS: Curriculum and Policy Statements is the curriculum that will replace the NCS in grade 12 in 2014.

GET band: The General Education and Training band refers to grades 0 – 9.

FET band: The Further Education and Training band refers to grades 10 – 12.

A brief note about the South African school curriculum

In South African there is a single national curriculum created for each subject that schools need to follow. Learners are examined at the end of grade 12 in a set of school-leaving examinations that are set and marked centrally. These school-leaving examinations are popularly known as "the matric exams". Commercial publishers produce textbooks in South Africa but the textbooks need to be approved by the national Department of Basic Education[1] for use in schools.

[1] The South African Department of Education split into the Department of Basic Education and

Abbreviations I use for the textbooks referred to in this thesis

CM: Classroom Mathematics

OTM: On Track with Mathematics

Acronyms relating to the theoretical work I use in the thesis

ORF: The Official Recontextualising Field

PRF: The Pedagogic Recontextualsing Field

GRD: General Regulative Discourse.

GI: Geometry I

GII: Geometry II

GIII: Geometry III

GWS: Geometric Work Space

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1 Introduction

1.1 The context

The first national democratic elections in South Africa in 1994 heralded a period of large-scale social and political change. ANC policy from this period promoted the importance of values such as equality, democracy and redress within education. (Cross, Mungadi, & Rouhani, 2002). Given the role that education had played in supporting apartheid there was a clear imperative to overhaul the education system. Despite some efforts at reform by the apartheid state (Vithal & Volmink, 2005), curricula were still perceived as rooted in apartheid ideology and so reform or indeed radical change in the curriculum was seen as important in the restructuring of education to meet the needs and values of the new democratic South Africa.

The process of curriculum reconstruction that faced the new South African government was not an easy one. It needed both to counter the effects of apartheid, promoting values of social justice and giving voice to the previously marginalized, and at the same time be a vehicle for promoting economic growth allowing South Africa to compete in a global market. This saw the introduction of the National Qualifications Framework (NQF) and outcomes based education (OBE) as the overarching framework and philosophy for education in South Africa (Allais, 2006). With this framework and philosophy the intention was to "loosen three sets of boundaries – between the academic and the everyday, between education and training, and between the different component contents of the academic curriculum, at both tertiary and school level" (2005, p284). Furthermore, there was a strong push to "loosen social relations in the classroom and promote greater learner participation"(Ensor & Galant, 2005, p284). At school level this was given life in the form of what came to be known as Curriculum 2005 (C2005). This was implemented in grade 1 in 1998 and was scheduled to be progressively phased in so that it would cover all the grades by 2005 (Harley & Wedekind, 2004). However in light of the difficulties experienced

in the implementation of C2005 the then new Minister of Education, Kader Asmal, constituted a Review Committee in 2000 which led to the production of the Revised National Curriculum Statement (RNCS) for grades 0 – 9 in 2002. This was followed by the production of the National Curriculum Statement (NCS) for grades 10 – 12 in 2003. Although the Review Committee criticized a number of the essential features of OBE (Chisholm, 2005), OBE survived as the guiding philosophy of curriculum reform.

It has been argued that OBE has not been well understood or, in fact, implemented in South Africa (Spady, 2008) and been conflated with notions from the dominant discourse of education reform (Jansen, 1998). Harley and Wedekind (2004) argue that this was exacerbated by inadequate training about OBE and thus "complex issues of pedagogy with major implications for teachers' personal and professional identity were reduced to simplistic dichotomies" (p200). This, they argue, saw the old undesirable "teacher-centred" education being replaced with "learner-centred" education and the old prescription of content replaced with outcomes. The new curriculum was thus seen as one that prioritized learner-centred education with an active learner and teacher as facilitator, group work and the importance of process and thinking skills and application of knowledge above content for its own sake (Chisholm & Leyendecker, 2008; Engelbrecht & Harding, 2008; Harley & Wedekind, 2004; Jansen, 1998). Although much of this resonates with approaches to education internationally (Chisholm, 2005) in South Africa they represented a rupture with the past and took on a strong moral and political undertone as they were deemed part of the project of reversing the negative effects of apartheid and building a new democracy.

Mathematics as a school subject has had a particularly contentious and politicized history in South Africa. The infamous quote by one of the architects of apartheid, Dr H.F. Verwoerd, who asked "What is the use of teaching the Bantu mathematics when he cannot use it in practice?" (HoA debates volume 78 as quoted in Vithal & Volmink, 2005), inextricably linked mathematics under apartheid with inequality and injustice. This inequality and injustice was realized

in the extremely poor performance of black South Africans in the school-leaving mathematics examinations and their consequent lack of access to scientific and technological careers. It is thus unsurprising that in the broad movement of opposition to apartheid, special interest groups around mathematics education were formed. From within these groups an approach known as People's Mathematics was espoused. Although People's Mathematics was in part used as a consciousness-raising tool in the fight against apartheid, post-apartheid it left strong remnants of an ideal of mathematics education that could be anti-racist, promote equity, social justice and democracy.

As a mathematics student and later a student teacher I was deeply involved in opposition to apartheid and thus well-enmeshed in the debates around People's Mathematics and aware of the stark inequalities in South African schooling. As an educator in the early 90s I was steeped in discussions around learner-centredness and constructivism as a theory of learning. During the early stages of post-apartheid curriculum restructuring in South Africa I was based in the Mathematics Department of a South African university and so observed and discussed the process and its products with colleagues who were mathematicians. The presence of these three identities (anti-apartheid activist, educator and mathematician) in my life at the time of curriculum restructuring raised questions for me about how school mathematics is created. In the process of curriculum restructuring there were three powerful forces vying for attention: the need to redress the inequalities of apartheid which had the strength of a moral imperative, the presence of mathematics as a strong vertical discourse, and an education theory that put the learner at the centre. The question for me then became what would come to be constituted as school mathematics in these circumstances and how the competing demands would come to shape this. As I embarked on my PhD study in 2006 I was in a new role as teacher educator. In this role one of the key questions for me became how this mathematics came to be constituted in two key artefacts that teachers would need to work with: the curriculum and textbooks.

At the beginning of 2012 a revised version of the RNCS and NCS was introduced into schools called the Curriculum and Policy Statements (CAPS) and it will be phased in to all the grades over the next 3 years. The CAPS revision process was introduced as the result of a review of the implementation of the NCS and RNCS. A statement from the Minister of Education that "OBE is dead" accompanied these changes (Motshekga, 2009). Whilst clearly the bulk of the work on this research project had been done by this stage and thus these events did not impact the design of the research project it is important to note here that the curriculum and textbooks discussed in this report have been superseded by a new version of the curriculum. However I do not feel that this negates the importance of the research project for two reasons. Firstly the time of transition from apartheid in South Africa was unique and thus provides an opportunity to examine curriculum change that happened rapidly and radically. Secondly the necessity for revision after a relatively short period of time and the reversal of allegiance to the main theoretical underpinning (OBE) requires that we understand that original curriculum change and the type of challenges it produced carefully – even if it is only to help us understand what went wrong.

The significant changes that occurred in the post-apartheid South African curriculum landscape have been the subject of considerable research scrutiny. A large percentage of this research has analysed the politics, philosophy and systemic constraints on the curriculum change process at a broad level. There have been a few studies which have looked specifically at the Mathematics curriculum (Graven, 2002; Mhlolo, 2011; Mwakapenda, 2008; Parker, 2006; Umalusi et al., 2010). Since the first school-leaving national examinations based on the new curriculum were sat in 2008 there have been a number of studies looking at the cognitive demand of the mathematics examinations (Mhlolo, 2011; Umalusi, 2009) and the subsequent performance of those learners in university study related to mathematics or science (Hunt, Ntuli, Rankin, Scher, & Sabastio, 2011; Nel & Kistner, 2009; Potgieter & Davidowitz, 2010). Each of these studies provides interesting insights into the process and products of curriculum reform in South Africa and some look particularly at aspects of mathematics. However, I

wanted to look in detail at what all this might mean for the type of mathematics a schoolteacher would be offered as the mathematics required for the classroom.

Although the rhetoric around C2005 initially discouraged reliance on textbooks (Department of Basic Education, 2009) the review of C2005 specifically refuted this claim and underscored the importance of textbooks and other learner teacher support material in the classroom (Department of Education, 2000). The centrality of textbooks in mathematics classrooms internationally has been well-reported (Ball & Cohen, 1996; Collopy, 2003; Dede, 2006; Haggarty & Pepin, 2002; Lianghuo & Yan, 2000; Nicol & Crespo, 2006; Pepin & Haggarty, 2007) and for many teachers they are the main guide as to what to teach in the classroom and the main source for classroom tasks. Throughout the various incarnations of the new curriculum, textbooks in South Africa have had to be submitted to the Department of Education (at some stages this was done at national level at other provincial level) and have been vetted for compliance to the curriculum. Despite this vetting process a brief look at the textbooks produced shows there was room for different interpretations of the curriculum. For these reasons a deeper analysis of the curriculum together with the textbooks was needed in order to provide insight into the type of mathematics that was constituted.

Because I wanted to look in detail at what was constituted as school mathematics and at the relationship between the social, educational and mathematical forces that were at play, I recognized that I would need to narrow my scope. The curriculum divides schooling into two bands: the first band, the General Education and Training Band (GET), encompasses grades 0 – 9, and the second band, the Further Education and Training Band (FET), encompasses grades 10 – 12. I chose to focus on the FET band as it is at this point that learners in South Africa are allowed to choose whether or not to study mathematics as a subject.[2] I

[2] In the FET band learners have to choose Mathematics or Mathematical Literacy as subjects. Mathematical Literacy involves a study of mathematics in everyday life and is not viewed as a path to further study in Mathematics.

chose, in particular, to focus on grade 10 as this marked the transition point from the broader more general studies of the GET band towards the specialization of the FET band.

I chose to focus on the content area of geometry for three reasons. The first is that geometry is seen as a particularly difficult aspect of the school curriculum and many teachers avoid teaching it. Anecdotal evidence suggests that in South Africa in many schools, particularly in disadvantaged schools, geometry is not being taught. Research studies in South Africa suggest that both learners and teachers achievement in geometry, as judged by the van Hiele levels of geometric thought they have attained, is very low (Bennie, 1998; Feza & Webb, 2005). The difficulty that teachers have with geometry is acknowledged in the government's curriculum documents. The Subject Assessment Guidelines for Mathematics (South African Department of Education, 2005), an adjunct document to the National Curriculum Statement for Mathematics for the FET band (MNCS) (South African Department of Education, 2003), released by government in 2005 designated many of the assessment standards dealing with Euclidean geometry as optional. This has meant these assessment standards were not examined in the core mathematics examinations at the end of grade 12 in 2008 and will not be examined until the first examination of the CAPS curriculum in 2014. The Subject Assessment Guidelines document states that the purpose of making some of the assessment standards optional was to allow teachers the time to develop their capacity to teach that content. The fact that a large section of the geometry curriculum was deemed to be too difficult for teachers to teach underscored the importance of investigating what had been constituted as geometry in the curriculum and what kind of challenges this posed for teachers.

The second reason for a focus on geometry arises out of the nature of geometry itself. To quote Dick Tahta "There is no doubt then that geometry is important. But what it actually is and what part it should play in mathematical education are notorious and perennially difficult questions" (Tahta, 1980, p3). Geometry, in contrast to many other branches of mathematics, is linked to physical objects and their representations (Niss, 1998) and so developed out of an understanding

of physical space. Frege, in his work on the foundations of mathematics, argued that arithmetic and analysis were founded in logic, but Euclidean geometry was not as it "rested instead on a primitive intuition of Euclidean space" (Tymoczko, 1985). At the same time, Euclid's elements are seen as "a prototype for a rational systematization of all fields of knowledge" (Mammana & Villani, 1998). Thus geometry occupies a role that is at once concrete and practical and yet also highly abstract and theoretical. In the study of geometry, objects are seemingly familiar and concrete, but are in fact unfamiliar and "unseeable". For example, even young children are able to identify and recognise a square, however the abstract geometric object "a square" is not something we can touch or hold. As Pimm says "Any physical or drawn square is not a square" (1995, p57). This tension between the two almost contradictory faces of geometry makes teaching in geometry particularly interesting.

In addition the questioning of the parallel postulate led to the development of non-Euclidean geometries in the 19th century (Hansen, 1998) and the last 100 years has seen a tremendous growth in new knowledge in geometry (Malkevitch, 1998). These issues highlight the breadth of the field of geometry, and the question of what is and what could be constituted as school geometry becomes interesting.

The third reason is that in school mathematics geometry has often been the one place where proof has been taught. Analysing how proof and the related ideas of definitions, axioms and deductive systems are presented in school is an important part of seeing what view of mathematical practices is offered to learners. A look at a few of the mathematics textbooks for the last three years of schooling in South Africa from the 1980s show that the format of definition-theorem-proof was the order of the day. Although this follows a typical pattern of presenting mathematical products, it has been much criticized as an approach to teaching geometry.

Thus I anticipated seeing significant changes in the approach to geometry in the new curriculum and textbooks, and to see tensions between the socio-political, educational and mathematical demands that help shape the approach.

1.2 The research questions

The work of Basil Bernstein speaks strongly to issues related to the nature of school knowledge and a number of his constructs have been brought to bear in analyzing curriculum change in South Africa. His notions of the official recontextualising field and pedagogic recontextualising field point to arenas in which knowledge is recontextualised to create school knowledge. I wanted to examine the forces at play in these arenas in order to understand what came to be constituted as school geometry. To do this I needed to find ways to open up the mathematics whilst at the same time keeping a strong awareness of the social and political forces at play. This I have done through creating a framework drawn from Basil Bernstein's work that delineates the forces at play and provides the overarching guide for the study. In addition I made the deliberate decision to restrict my focus to the curriculum and textbooks and not to look at how it is enacted in the classroom. I am well aware that in doing this I place a clear limitation on the study. The research literature shows that one cannot assume that what is in the textbook and/or curriculum is enacted in the classroom. I make no claims of that nature. I chose textbooks which were not only popular but also whose authors were deeply involved in the mathematics education community and so interacted with debates that surrounded the curriculum. This meant that in looking at those textbooks and the curriculum and discussing their construction with these authors I could tap into experiences of individuals who were deeply mired in the maelstrom of competing forces.

As with any recontextualisation of mathematics for the classroom I anticipated that in the curriculum and textbooks there would be tensions and choices made in the face of those tensions. It thus seemed highly unlikely that either the curriculum or textbooks would constitute a type of geometrical knowledge that a teacher should simply reproduce uncritically in their classrooms. I was not

interested in judging how well the textbooks aligned with the curriculum, or in evaluating the textbooks and curriculum as good or bad. My interest was to understand the kind of geometry that was constituted by the curriculum and textbooks, to understand the forces that played a role in this process and to understand some of the challenges that this geometry would produce for teachers.

Thus my research questions are:

1. What geometrical knowledge is constituted by the MNCS and two key South African grade 10 mathematics textbooks and how is it so constituted?

2. What factors and constraints shape what is constituted as geometrical knowledge for teaching through the curriculum and textbooks?

In order to answer these questions I did an in-depth analysis of the MNCS. I chose two key South African mathematics textbooks and analysed the geometry chapters in these textbooks. I identified key players in the official recontextualising field and interviewed them about the way in which the curriculum was constructed and the factors that influenced the choices that were made in this process. I also interviewed the authors of the textbooks as agents in the pedagogic recontextualising field and discussed with them the factors that shaped their choices in writing the textbooks.

1.3 Outline of chapters

The following is a brief outline of chapters in this dissertation

1.3.1 Chapter 1: Introduction

Chapter one introduces the reader to the context of curriculum change in South Africa and argues that the effect on school geometry of the confluence of forces that were at play in the first round of curriculum change after the end of apartheid are particularly interesting to study.

1.3.2 Chapter 2: Literature review and theoretical framework

This chapter starts by exploring Bernstein's notion of the pedagogic device as a means to discuss the way in which knowledge is recontextualised in the process of its conversion to pedagogic communication (Bernstein, 1996). Through an analysis of the literature written about the pedagogic device I develop a model which is an adaptation of the pedagogic device that I use to frame my study. This model points to the various components of the field of production (work from the academic fields of mathematics, mathematics education and the general discourse) that I need to draw on in order to understand some of the processes in the field of recontextualisation. The model also suggests an ongoing interaction between the fields of production, recontextualisation and reproduction which produce the pedagogic discourse of the mathematics curriculum and textbooks. This model provides the framework I used in this research.

In the next sections of the chapter I review the research literature related to various aspects of my study that include

> Curriculum: Here I pay particular attention to some of the international trends in recent mathematics education reform.
> Textbook analysis: Here I initially provide a broad overview of the kind of work that has been done on textbook analysis, and then I hone it down to discuss work that is closely related to my study. In this section I look particularly at work that discusses how power and social relations play out in school textbooks and at work that looks at the interplay between the mathematical and pedagogical layers of textbooks.
> Geometry: Here I discuss the van Hiele theory of thinking in geometry as it was particularly influential in the development of the geometry strand of the MNCS. I then highlight some key issues in the learning and teaching of geometry identified in the research literature. Finally I discuss the three paradigms of geometry proposed by Kuzniak and Houdement

(2002) and argue that these paradigms provide a useful way of analyzing the geometry sections of the curriculum and textbooks.

1.3.3 Chapter 3: Research design and methodology

In this chapter I describe how, following Brown and Dowling (1998), I developed a language of description through an ongoing dialogic conversation between the theoretical and empirical fields. I describe the key elements of the theoretical field as being the adaption of Bernstein's pedagogic device which provides an overall framing of the study, the use and development of the four orientations to mathematics that Graven (2002) identified in her analysis of the South African curriculum for grades R – 9 which helped structure the analysis. In relation to the empirical field, I outline the texts from the official recontextualising field I analysed and discuss which agents in the official recontextualising field I chose to interview. I motivate my selections of textbooks from the pedagogic recontexualising field and the agents in the pedagogic recontextualising field I chose to interview. I discuss my ethical considerations using the framework suggested by Stutchbury and Fox (2009). I explain how the analysis of the data was done. In particular I provide an overview of the curriculum document and show how this was divided up for analysis. I also discuss how, adapting the TIMSS textbook analysis framework (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002), I created and categorised blocks in the textbooks which became the unit of analysis in the textbook analysis. Finally I briefly discuss issues of validity in the research.

1.3.4 Chapters 4 – 7: Analysis

In order to structure the analysis of the curriculum, textbooks and interviews with agents of the official and pedagogic recontextualising fields I used the four orientations to mathematics identified by Graven (2002) and elaborated by Parker (2006) in their studies of the South African mathematics curriculum. These four orientations, described in brief, are:

1. Mathematics is important for critical democratic citizenship.

2. Mathematics is relevant and practical.

3. Mathematics is an induction for learners into what it means to be a mathematician.

4. Mathematics is a language with conventions, skills and algorithms that must be learnt.

I analysed the curriculum, textbooks and interviews in relation to each of these orientations and these analyses form the basis of each of the next four chapters. However it proved more worthwhile to analyse orientation three and four together and thus the discussion of these orientations occur together in chapter six. Thus chapter four contains the analysis of the curriculum, textbooks and interviews in relation to orientation one, chapter five contains the analysis in relation to orientation two and chapter six the analysis in relation to orientations three and four. Each of these chapters starts with a statement of orientation as provided originally by Graven (2002). Graven's statements are not sufficiently operationalized for analysis. Thus drawing on work from the field of production together with initial engagement with the empirical field I provide a discussion of key elements of the orientation and then produce a refined version of the orientation for my use. This version of the orientation is then applied to the analysis of the curriculum and followed by a discussion of issues arising from the analysis of the curriculum. The textbooks are then analysed in turn using the refined version of the orientation and any further development on it required by the issues that emerged from the curriculum analysis. Finally instances where the interviewees mentioned aspects of the orientation or issues that emerged from the analysis of the curriculum and textbook are discussed.

The only deviation from this format is in the discussion of orientation 3 and 4. It became apparent after reporting the curriculum analysis in relation to orientation 3 and 4 as described above that accessing the mathematical practices (orientation 3) and mathematical content (orientation 4) in geometry would not be possible without using a description of the nature of geometry specifically. Thus the textbook analysis related to orientation 3 and 4 was done using

Houdement and Kuzniak's (2003) paradigms of geometry. This is reported separately in chapter 7.

1.3.5 Chapter 8: Discussion and conclusion

In this final chapter I reflect on the study as a whole. I discuss my findings in relation to the theoretical work I used to frame the study and illustrate how my work enhances our understanding of the pedagogical device. I also summarise how Kuzniak's notion of the Geometric Work Space (GWS) has been useful in understanding the nature of geometry in the curriculum and textbooks.

2 Literature review and theoretical framework

2.1 Bernstein and educational change

The work of Basil Bernstein has been noted as being particularly useful for providing tools with which to analyse educational, and specifically curriculum, change. (Bernstein & Solomon, 1999; Singh, 1997). In the context of South Africa, the extent and nature of change in the post-apartheid educational arena has been examined by many researchers using Bernstein's theory and concepts. This has been done in looking at various aspects of education and schooling (Ensor, 2004; Muller, 2000; Taylor, 1999) and specifically in the context of mathematics education (Graven, 2002; Naidoo & Parker, 2005; Parker, 2006; Parker, Davis, & Adler, 2005). For its own salience and since it would allow my work to speak to and build on the work of these researchers I chose to use Bernstein's theory as a frame for examining a time of curriculum change in South Africa and the consequences for mathematics education in general and geometry education in particular. The power of Bernstein's work in providing a lens through which to view power relations in the formation of school knowledge meant it provided powerful ways to illuminate the forces that I was examining in my study.

Within the South African context, the MNCS was created in a time of extensive political change where there was considerable emphasis on nation-building. The education system was cited as a key component in this process and essentially needed to meet two key imperatives: to reverse the educational inequalities and legacy of apartheid education, and to prepare South Africa for economic competitiveness. Wong and Apple (2002) state that Bernstein's notion of the pedagogic device can be a useful tool to analyse educational change in relation to state formation. They argue that Bernstein developed this tool to counteract what he believed was an incorrect assumption, that pedagogy simply reproduces external power relations. They state that "This is the case because, rather than assuming a correspondence between the political and educational fields, the idea of the pedagogic device directs us both to anatomize the specific rules, practices,

and social relations regulating pedagogic transmission and to examine their effects on the production and reproduction of consciousness in schools" (p183). In what follows below I outline Bernstein's notion of the pedagogic device and discuss how it frames my study.

2.2 The pedagogic device as framework for my study

Bernstein uses the notion of the pedagogic device to describe "the relay or ensemble of rules or procedures via which knowledge (intellectual, practical, expressive, official or local knowledge) is converted into pedagogic communication" (Singh, 2002, p573).

The pedagogic device provides "the generative principles of the privileging texts of school knowledge" (Singh, 2002, p573) through three rules that are related hierarchically to each other: the distributive rules, the recontextualising rules and the evaluative rules. The distributive rules distinguish between two different classes of knowledge, the estoric and the mundane (Bernstein, 1996) and regulate who gets access to what knowledge. The distributive rules produce three fields: the fields of production, recontextualisation and reproduction through which pedagogic discourse is produced (Ensor, 2004). The field of production is the place where new knowledge is created. This would typically be in universities, through research papers and in laboratories (Maton & Muller, 2007). In the field of recontextualisation selections are made from knowledge produced in the field of production which is then transformed to produce 'educational' knowledge. Typical sites for the field of recontextualisation would be curriculum documents and textbooks. The field of reproduction, typified by the classroom, is where the 'educational' knowledge is transmitted and acquired (Maton & Muller, 2007). The recontextualising rules select knowledge from the field of production and transform it to produce pedagogic discourse. The evaluative rules serve to sanction what counts as legitimate knowledge in pedagogic practice.

The arena of the pedagogic device is summarised by Maton and Muller (2007) as shown in table 1 below. Here they identify the rules which regulate each field, the kinds of symbolic structures in which knowledge is presented, the principle types of knowledge and the typical sites in which that knowledge exists.

Field of practice	**Production**	**Recontextualisation**	**Reproduction**
Form of regulation	distributive rules	recontextualizing rules	evaluative rules
Kinds of symbolic structure	knowledge structure	curriculum	pedagogy and evaluation
Principal types	hierarchical and horizontal knowledge structures	collection and integrated curricular codes	visible and invisible pedagogies
Typical sites	research papers, conferences, laboratories	curriculum policy, textbooks, learning aids	classrooms and examinations

Table 1: The arena of the pedagogic device (Maton & Muller, 2007, p18)

In relation to my concerns about what has been constituted as school geometry and how it has been so constituted, the pedagogic device would suggest that the geometry described in the curriculum and school textbooks would be a recontextualisation of geometry that has been produced in universities and the academy more generally. This geometry would then be "reproduced" in classrooms.

The pedagogic device thus seemed an appropriate frame for my study. As the field of recontextualisation is the place where the knowledge is pedagogised it made sense to make this the key field on which to focus.

Bernstein (1996) describes the field of recontextualisation as being constituted by two subfields: the official recontextualising field (ORF) and the pedagogic recontexualising field (PRF). The ORF is dominated by the state and its agents (Bernstein, 1996). The PRF is made up of university departments of education, research in education and educational publishers. Bernstein argues that it is important to interrogate the nature of the relationship between the ORF and PRF. Ensor (2004) points out that the relative autonomy of the PRF from the ORF varies across countries and this constrains or enables the work of agents in the PRF. In the South African context the key document pertinent to my study embodying the policy of the ORF is the MNCS, and textbooks represent concrete products of the PRF. I thus decided that a way to investigate what was happening in the process of recontextualisation was to analyse the curriculum and textbooks, and the relationship between them, and to investigate the reported factors that influenced their production.

My choice to work with Bernstein's pedagogic device is that it is explicitly set up to examine both the "what" and the "how". It is specifically cognisant of the socio-political factors that impact school knowledge and yet does not deny the role of the discipline. However my use of the pedagogic device is tempered by two factors: the first practical and the second theoretical.

On a practical level Bernstein's theory does not give discipline-specific tools with which to provide an in-depth analysis of the nature of the mathematics in the curriculum or textbooks. For example, in order to look in detail at the way in which mathematical modelling or the idea of a mathematical definition is dealt with in the textbooks I could not find sufficient purchase on the ideas using Bernstein's notions alone. Dowling (1998, p84) argues, in addition, that "The pedagogic device is a very high level theoretical object and we must descend through multiple layers of theory before we ever get to something that we might validly refer to an empirical text". The limitations of the device in terms of discipline-specific tools and its position as a high level theoretical object meant that I needed to construct the intermediary layers by turning to mathematics and mathematics education literature. As will be described below, this recourse to

these two disciplines is suggested by the pedagogic device itself as they are two key areas of the field of production that are recontextualised in the production of the pedagogic discourse I will be analysing.

On a theoretical level, although the pedagogic device is described as recontextualising knowledge from the field of production to produce pedagogic discourse, it is not entirely clear what knowledge from the field of production is implied and I question whether this recontextualisation happens in a straightforward unidirectional manner. Thus it was important for me to look carefully at how Bernsteinian scholars (and their critics) talk about recontextualisation and to pay particular attention to what knowledge they refer to as being recontextualised in the pedagogic device. Doing this would provide me with a stronger model of the device that would provide a framework for the organisation of my data. In what follows below I discuss the work I drew on to provide deeper insights into recontextualisation and draw on these to present the model I have used to frame my study.

One clear implication in much of the writing using the pedagogic device is that school mathematics is a recontextualised version of the academic discipline of mathematics. For example, Morgan, Tsatsaroni and Lerman (2002, p448) state that school mathematics is "a pedagogic discourse formed through the recontextualising of the specialised discourse of mathematics". Many Bernsteinian theorists are unequivocal that Bernstein asserts the recontextualisation is a selection from the field of production (Apple, 2002; Ensor, 2004). Bernstein himself, using school physics as an example, states clearly that a selection is made from "the totality of practices which is called physics in the field of production of physics" (Bernstein, 1996, p48). However the notion of recontextualisation is not precisely defined by Bernstein or those who have used him. Using the example of school physics Bernstein states:

> There is selection in how physics is to be related to other subjects, and in its sequencing and pacing…But these sections (sic) cannot be derived from the

> logic of the discourse of physics or its various activities in the field of the production of discourse (Bernstein, 1996, p49).

My concern about these ideas is echoed by Dowling (2009, p83) who states that it is "questionable whether the 'field of production' is the only or even the dominant object of the gazes of recontextualising agents". Singh (2002), who has used the pedagogic device as a theoretical framework in empirical studies, points out the enormous growth in esoteric knowledge and its increasing complexity and specialisation have implications for both what knowledge is available for recontextualisation and who does the recontextualisation in the creation of pedagogic discourse. She argues that this has increased the requirements to be met for entry into a specialised knowledge domain and that it means that it is more challenging to pedagogise this highly specialised knowledge. In addition she suggests that the producers of this knowledge do not have the time or resources to do this pedagogising and so this work is increasingly undertaken within what she terms "agencies of recontextualisation". Taken together, the work of Dowling and Singh cited here suggest a need to look very carefully at who is doing the recontextualisation and to problematise the level of access they have to the field of production. The disjuncture between school mathematics and the mathematics that is currently being produced in universities and research institutions mitigates against being able to view school mathematics as a simple recontextualisation of the discipline of mathematics.

Chevallard's (1992) notion of didactic transposition, which is similar to Bernstein's recontextualisation, has come in for similar criticism. For example, Love and Pimm (1996, p374) call it an attempt "to formalise this naïve conception of relatedness and one-way dependence between 'educational' and 'scholastic' mathematical writings". Citing Freudenthal, they argue that for many pieces of school mathematics it is difficult (perhaps impossible) to identify the particular part of scholarly mathematics from which it has supposedly been transposed. Sierpinska (1995b), in her review of the book "Didactics of mathematics as a scientific discipline", comments on the ways in which the authors of different chapters deal with what is termed the "preparation of

mathematics for students". She notes that for one of the authors, Fey, the selection of mathematics to be taught is based on finding mathematics that is worthwhile teaching and that this might not be based in the scholarly discipline. In constrast another author, Artigue, uses the notion of the didactic transposition about which Sierpinska (1995b, p113) notes "whose emotive meaning is colored by the expression of concern about the risk of 'deforming' or 'corrupting' (p 28) the scholarly mathematics in the process of its transposition into a school subject". Thus from within the mathematics education research literature there is not a straightforward acceptance of school mathematics as a recontextualised form of scholarly mathematics.

In turning back to the work of Bernstein and scholars who have drawn on his work I looked to see if they included domains other than the academic discipline in question as being part of what is drawn on in the recontextualising process. Although there are differences between scholars in terms of how they discuss these ideas, there are two key domains that Bernsteinian scholars seem to talk about as specific elements of the field of production that are recontextualised or depicted as key influences on the recontextualising process. These are discussed below.

1. General regulative discourse (GRD)

Ferreira, Morais and Neves (2008) and Neves and Morais (2001) both talk about the general regulative discourse (GRD) as representing the dominant principles of society. They state that, according to Bernstein, the GRD is produced by the State influenced by the fields of economy, symbolic control and international concerns. They argue that the GRD is recontextualised in the ORF to produce official pedagogic discourse which takes the physical form of curriculum documents. They argue that this recontextualising process is also influenced by the fields of economy, symbolic control (in the form of education) and international concerns. In addition they suggest that within the PRF a further recontextualisation takes place as the ideologies and beliefs of agents in this field (textbook authors, for example) might differ from those promoted in the

curriculum documents. They argue that, according to Bernstein, "a pedagogic device which offers greater recontextualising possibilities through a greater number of fields and contexts involved" allows for a greater space for change (Neves & Morais, 2001, p226).

2. Educational knowledge produced in the intellectual field of education

Ferreira, Neves and Morais (2008) also talk about the selection of knowledge used to create pedagogic discourse being made on the basis of "educational knowledge produced in the intellectual field of education" (p3).

This is echoed by Bernstein who states that "Finally, the recontextualising principle not only recontextualises the *what* of pedagogic discourse, what discourse is to become subject and content of pedagogic practice. It also recontextualises the *how*; that is *the theory of instruction*" (Bernstein, 1996, p49).

This suggests that knowledge produced in the field of education research, and in my case in the field of mathematics education research, is also subject to recontextualisation for incorporation into pedagogic discourse.

This indicates that in using the pedagogic device as a tool for analysing the transformations in geometry knowledge, it will be important to use a broad notion of the field of production and to incorporate scholarly mathematics, mathematics education research and the general regulative discourse in South Africa at the time of the changes in curriculum into the scope of the analysis. Looking at the selections made from these domains and how the interplay between these domains shapes the form in which the selections are pedagogised will provide the overarching framework for the analysis.

A further concern I have about some readings of the pedagogic device is the underlying implications that it operates in a single direction from the field of production, through the field of recontextualisation to the field of reproduction.

Love and Pimm (1996) raise a similar objection to the unidirectionality of the didactic transposition which could be applied to the pedagogic device. They

argue that the relationship between school mathematics and scholarly mathematics is far more complex than that. They note that school mathematics is, for most mathematicians, their first experience of mathematics and that some recreational and school mathematics have become the object of research in scholarly mathematics. For me, the unidirectionality is problematic on two levels. Firstly my experience of working in mathematics education in a number of roles in South Africa over many years suggest there is an enormous schism between mathematicians (working in the field of production) and mathematics educators and teachers (working in the fields of recontextualisation and reproduction). Although mathematics educators clearly need to have some background in mathematics the extent of this will vary. Many mathematics educators in South Africa will not have exposure to recent mathematical work in the field of production and some mathematics educators will only have one year of university-level mathematics. My observations suggest to me that many of the agents working in the fields of recontextualisation and reproduction have little direct knowledge of the field of production and, I hypothesise, tend to recontextualise their own experience. By this I mean that mathematics educators tend to work within the paradigm of mathematics education and draw on work from the field of production of that domain. In addition, if part of the recontextualisation involves a theory of instruction then surely the direct experience of the classroom, existing curricula and textbooks must be part of the knowledge that is drawn on in the recontextualising process – and it must feed back into the field of production of mathematics education knowledge.

These concerns about the pedagogic device have led me to develop a modified model of the device that I will both use to frame my exploration of my data and empirically test in the analysis of my data. This model is not a statement of what is, but a model, derived from theory (along with criticisms of that theory) that points to sources to look at.

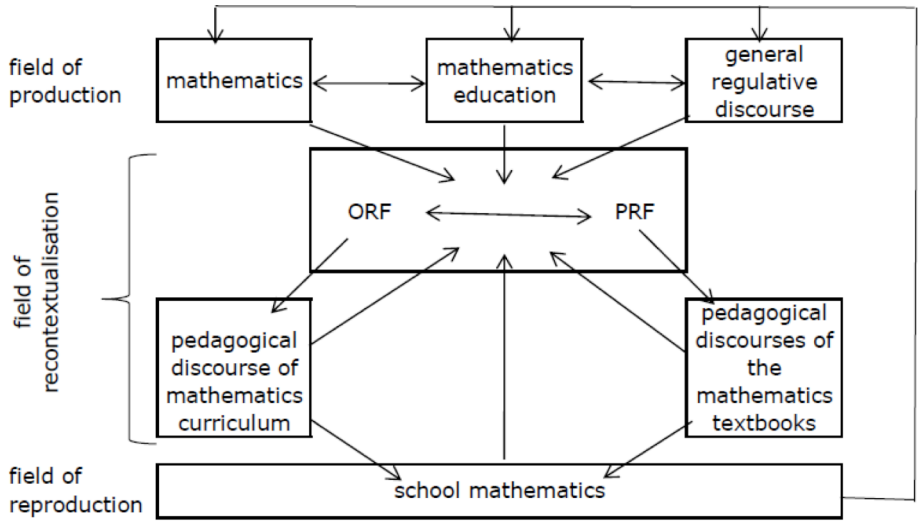


Figure 1: My model of the pedagogic device

This diagram indicates the following:

> The field of production from which the pedagogic discourse of the curriculum and textbooks has been recontextualised contains the 3 domains discussed above: mathematics, mathematics education and the general regulative discourse.
>
> Selections from these are made by agents in the ORF and PRF to produce the pedagogical discourse of the mathematics curriculum and the mathematics textbooks.
>
> These are in turn recontextualised into the field of reproduction.
>
> The arrows indicate that the direction of influence.

One issue that needs to be noted is that this diagram cannot capture temporality. For example, one could argue that the pedagogical discourse of current mathematics textbooks have an effect on the ORF and PRF that in turn produces an effect on the new mathematics textbooks that are produced. The influence in both directions (from discourse of textbooks to ORF and PRF, and from ORF and PRF to the discourse of textbooks) is indicated by the arrows that are present in this diagram, but the temporality is not apparent. I highlight temporality because it has significant implications. For example, the field of production we are talking about might not be the current field of production. Much of the mathematics that

is studied at school level is over a century old: what is being pedagogised is not from the current field of production, and this needs to be taken into consideration in the analysis.

2.3 Literature review

The framing of the study indicated that the curriculum and textbooks would be the two main artefacts to be analysed. This in turn necessitated a review of relevant literature dealing specifically with current trends in curricula and textbooks and in the way in which they have been analysed. In addition, my desire to look specifically at the nature of geometry in the curriculum and textbooks required that I explore some of the key issues in the learning and teaching of geometry present in the literature.

2.3.1 Curriculum

The field of curriculum studies is extensive and thus a comprehensive review of all the literature from within this field is beyond the scope of this study. However, my reading in this field highlighted three key areas of research that are pertinent to my research. These are:

1. Research and literature related to analysis of curriculum content. This provided an overview of the place of geometry in the curriculum in South Africa and internationally.

2. Literature which discusses educational reform and, in particular, mathematics education reform. This indicated some of the international trends that were part of the GRD in education at the time the MNCS was developed.

3. Work by Graven (2002) and Parker (2006) identifying four orientations to mathematics in the South African curriculum. This work, and the four orientations, proved particularly useful in illuminating aspects of the curriculum and textbooks and thus were used to structure the analysis in my research.

Each of these areas of research are discussed in more depth below.

2.3.1.1 Curriculum content analysis

The analysis of the geometry topics specified in curricula has largely taken the form of international comparisons. These kinds of comparisons are particularly useful and relevant as they allow learning between different countries and also can form the basis of important discussions about what geometry could/should be taught in school. One of the large-scale studies in this regard was undertaken in the Trends in International Mathematics and Science Study (TIMSS). Differences between countries meant that a variety of sources (curriculum documents and textbooks) were analysed to produce an extensive review of the mathematical content and performance expectations of learners at all grades across 48 different educational systems. This analysis indicated great variability in the geometry curricula of countries. At the 8th grade level the differences between countries in terms of their geometry curricula was greater than for any of the other mathematical content areas (Beaton et al., 1996). The only geometry topic designated as part of the 'world core' (i.e. present in over 70% of the countries) in the final years of schooling was 2-D coordinate geometry (Valverde et al., 2002). The finding of variability in geometry curricula was mirrored in the 2001 study by Hoyles, Kuchemann and Foxman of geometry curricula for ages 11 - 16 in 8 countries. The variation incorporated the approach taken, the ages at which various ideas are introduced, the role of ICT and the role of proof. Hoyles et al. (2003) noted that the geometry curricula of most countries appeared to be in a state of flux.

In the South African context, recent work by UMALUSI (the Council for Quality Assurance in General and Further Education and Training) and HESA (Higher Education South Africa) has shown that geometry played less of a role in the two international school-leaving qualifications (the Cambridge AS and A level qualifications and the International Baccalaureate) than it did in the full MNCS. In the two international curricula studied, geometry was included at lower levels

in the curriculum, but not in the qualifications deemed to be at a similar level to the MNCS (Umalusi et al., 2010).

In relation to my study, these international comparisons indicated that there was no international norm for what a geometry curriculum should be. This is important to note because a dominant international view of what geometry should be taught at this level could exert a powerful influence over selection of content into a curriculum. The variability between countries in terms of geometry in high school meant that at the time the MNCS was created there was no single norm to follow in terms of the specific geometry to include in the curriculum. However there was a strong reform movement in mathematics education that privileged mathematical practices over content. I describe this in greater detail below.

2.3.1.2 Reform

Although "reform" can signify an attempt to change for the better, in what I describe here I am writing specifically about a recent (beginning roughly in the 1980s) wave of changes in mathematics education that are, particularly in the USA, talked about as part of the "reform movement". In the USA this movement is perhaps best exemplified by the National Council of Teachers of Mathematics (NCTM) Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, 2000). The major shift in mathematics education described in this document is, according to Schoenfeld (2004b), a shift from 'content' to 'content and process'. Within the NCTM Principles and Standards for School Mathematics, five process standards (problem-solving, reasoning, connections, communication and representation) are outlined along with the content standards. Schoenfeld (2004a) attributes this wave of reform to a concern around global competitiveness and the need for education to provide an appropriate workforce, results from research about how and what students learn in the mathematics classroom, and a desire to make mathematics accessible and attractive to all students. The major tenets of the reform were mirrored in a number of different countries around the same time (Boaler, 2002).

Schoenfeld (2004a), in his discussion of what has come to be known as "the math wars" in the USA, points out that the conflict over NCTM Standards led to a polarization of interested parties into "reform" and "traditional" camps. Different proponents in each camp might focus on different aspects of mathematical teaching e.g. some reformists place emphasis on increased use of technology, whilst others do not. However, broadly speaking, one could say that in the reform camp one would see a focus on mathematical processes, alternatives to tests and examinations for assessment, use of problem-solving with authentic problem situations, teacher as facilitator, group work, discovery learning, learners developing their own approach to solutions and the use of calculators and computers to facilitate problem-solving. In contrast in the traditional camp one would see a view of mathematics as a fixed body of knowledge and an emphasis on the mastery of this knowledge, tests and examinations as the main form of assessment, teacher as deliverer of information, individual work and a de-emphasis on calculators and computers lest learners become reliant on them and do not develop the ability to calculate/manipulate mentally (Boaler, 2002; Romberg, 1992; Schoenfeld, 2004a; Smith, 1996; Wu, 1996). Clearly these listings are almost caricatures of the extreme poles of the two camps and, as Schoenfeld points out, there is likely to be a large middle ground that would want to incorporate elements of both the traditional and reform approach into any mathematics curriculum. However they do capture some important characteristics and tensions that have emerged in the process of this wave of reform.

Of particular interest to my study is the assertion that this wave of reform was based in and arises from findings from mathematics education research and was linked to a dominant theory of learning. In particular the rise of a constructivist perspective on teaching and learning was very influential (R. Davis, 1994; Romberg, 1992). Although constructivism can take on different guises and there is contestation between different forms of constructivism (see for example,Ernest, 1993), the central notion of learner as active participant rather than passive recipient is a common thread. A constructivist view of learning was

particularly influential in the early reforms of the South African primary mathematics curriculum (Stoker, 1993) and it, or developments from it, are likely to have played a role in the MNCS. This underscores the importance of examining the influence of relevant aspects of the field of production in mathematics education on the MNCS.

With regard to my study it will also be important to examine the influence of features of mathematics reform internationally on the South African curriculum and textbooks. Of interest is the varying characterizations of the motivations for reform. For example, Popkewitz (2004, p9) states that "Teaching reforms are characterized as bringing instructional norms into closer proximity with those found in the academic discipline of mathematics". He notes that the focus is on the process of mathematical discovery rather than on the "reconstructed logic of mathematics" (ibid, p9). However, he notes that this stress on process in mathematics "is quickly transmogrified into sociopyschological conceptions of child development" (ibid, p10). He argues that the dominance of this psychological conception has become so accepted that we no longer question it. This leads to the situation where the "lenses for 'seeing' and 'thinking' mathematics in schooling are now treated as if they were, in fact, what mathematics is" (ibid, p11). This kind of prising apart of the motivations for reform are particularly important and relevant for my study. As with any popular movement for change, it is very easy for some of the complex ideas argued for in the mathematics reform movement to be conflated or reduced to simple slogans. Is a process orientation to mathematics good because that is what mathematics is? Or because that is the best way to learn mathematics? Or simply because the active learner is the kind of child we want schooling to produce? These kinds of considerations were particularly important in my study. For example, a large portion of work in both textbooks was based on activities that the learners had to participate in. These questions underscored the importance of investigating the nature of these activities deeply instead of assuming that the presence of activities signalled a more active involvement of learners in the process of creating mathematics. Popkewitz's work in particular

suggests that the same form (e.g. the presence of activities in the textbook) can occur with many different underlying motives and different resulting effects.

This brief overview of the reform movement in mathematics education highlights some of the aspects of mathematics education reform that serve as an important background to my study. In chapters 4, 5, 6 and 7, as I analyse the influences on the MNCS and textbooks more closely, I will draw further on this field of literature and discuss aspects of it in greater detail.

The influence of the reform movement on mathematics curricula internationally signalled the importance of looking at the orientation to mathematics in the South African curriculum. For this I drew on the work of Graven, which was further elaborated by Parker, that I discuss in detail below.

2.3.1.3 Orientations to mathematics in the South African curriculum

In Graven's (2002) analysis of the original C2005 General Education and Training (GET) band for the Learning Area *Mathematical Literacy, Mathematics and Mathematical Sciences* (MLMMS)[3] she identifies four different orientations to mathematics. She identifies these as orientations in which:

> 1. Mathematics is important for critical democratic citizenship. It empowers learners to critique mathematical applications in various social, political and economic contexts. Mathematics is part of broader society and is important for all learners.
>
> 2. Mathematics is relevant and practical. It has utilitarian value and can be applied to many aspects of everyday life. Mathematics is part of broader society and is important for all learners.
>
> 3. Mathematics is an induction for learners into what it means to be a mathematician, to think mathematically and to view the world through a mathematical lens. Mathematics has its own beauty and can be explored for

[3] In Curriculum 2005 the subject *Mathematics* was replaced by the Learning Area MLMMS.

its own sake. Mathematical investigation and exploration (without necessarily utilitarian value) is emphasised. School mathematics in this sense is seen as part of a broader mathematics culture, which is produced and reproduced uncritically[4] in accordance with the norms and conventions of the broader mathematics culture.

4. Mathematics is a language with conventions, skills and algorithms that must be learnt. Many of these will not be used or applied by most learners in everyday life but are important for the FET band and for university studies in mathematics (for example, the symbols and conventions for writing exponents, factorisation of trinomials, solving Euclidean geometry riders etc.). School mathematics in this sense is seen as part of the broader mathematics culture, which is accepted and reproduced. (Graven, 2002, p50)

Although Graven identified these orientations in relation to the GET band, Parker (2006) found in her analysis of the curriculum for the FET band, the MNCS, that the description of mathematics echoes these orientations. Parker points out, however, that in the MNCS there were some differences. She suggests that orientation 2 in the MNCS is "more strongly focused on a more structured form of applied mathematics, including problem solving and mathematical modelling within different contexts including real life and other disciplines" (ibid, p63). She argues that this is an extension from the real life and local contexts focused on in the GET band. In addition, she states that orientation 3 "can be seen as expanded to include mathematics as practice, a disciplined, rigorous and systematic way of thinking about viewing and structuring the world, and communicating in the world" (ibid, p63) and that orientation 4 is expanded to "include mathematical structures as a focus of study". She adds a fifth orientation which is "mathematics as a human activity produced historically in cultural and social contexts" (ibid, p63). Parker then applies her modified version of Graven's orientations to

[4] I have used Graven's exact wording here and thus "uncritically" is taken directly from her work.

provide a count of the orientations to mathematical knowledge within the assessment standards in the MNCS.

The work of both Graven and Parker suggest that these orientations to mathematics would provide a useful lens through which to examine the curriculum and textbooks. However neither Graven nor Parker provide a sufficiently detailed or operationalized form of these orientations. Thus although, as will be discussed in chapter 3, I used these orientations as a key structure in my analysis, my starting point in using them was to discuss and elaborate each orientation in detail.

2.3.2 Textbook analysis

Although in this section I look at the literature on textbook analysis, some of what is written here will appear redundant given my description of the decision to use the theoretical framework described earlier. However despite the fact that a written report such as this is by necessity linear, the process of thinking and research that has got to me to this point has certainly not been. The choice to use a modified version of Bernstein's pedagogic device and indeed the framing of my research questions were informed by the existing literature on textbooks. In this section I thus review some of the work, thinking and process around textbook analysis that both led to my choice of framework and helped define my research questions.

As pointed out in chapter 1, a review of the literature points to the important role textbooks play in classroom practice both internationally and in South Africa. This key role of textbooks was cited as the motivator for analysing textbooks or textbook use in most of the studies reviewed. However the purpose and focus of the analyses reviewed varied considerably.

One area receiving attention was the use of textbooks by teachers and the role of textbooks in teacher learning. Perhaps the clearest theme to emerge from a review of this literature is that, although many teachers rely heavily on textbooks in their teaching, the way in which they use the textbooks does not necessarily

reflect the intentions of the textbook authors. In the South African context, with the advent of the new curricula we saw a number of textbooks which privilege an inductive modality (Z. Davis, 2001; Ensor et al., 2002), but reports of few or unintended changes in classroom practice (Chisholm, 2003; Ensor et al., 2002; Nkhoma, 2002). A number of international studies also examined the phenomenon of how teachers work with a textbook whose dominant pedagogy is at odds with theirs and ask whether it is possible for textbooks to play a role in changing classroom practice (Ball & Cohen, 1996). Remillard (2005), in particular, provides an extensive review of this field and discusses various lenses through which one can view teachers' uses of text. Amongst these lenses are those that challenge the view of teacher as simply following or subverting the text, but emphasise instead the interrelationship between teacher and textbook. Despite differences in approach, the consensus in all these studies is that there is no simple relationship between the textbook and classroom practice. From the point of view of my study, this analysis helped clarify what my study is about and what its limitations are. My study cannot speak to what is happening in classrooms in relation to geometry. My interest is in looking deeply at a step in the process of construction of school mathematics: the development of a curriculum and teaching materials. I contend that understanding this aspect is an important precursor to looking at the relationship between teacher and curriculum/textbooks. However an analysis of the relationship between teacher and curriculum/textbooks would necessarily divert the focus from an analysis of the materials themselves and provide a less full and complete picture of that first step in the process of construction, hence my decision to limit the scope of my study for the sake of providing a rich and detailed description of a part of the process of construction of school geometry in South Africa. Clearly this means that in my work I make no claims that my description of what is constructed in the textbooks implies that this is what is being enacted in South African classrooms.

In the work where the focus was on the analysis of the textbooks themselves there was also variation in both what was analysed and the tools used to do the

analysis. From the review of the literature two aspects emerged as particularly pertinent to my study. The first was the question of how power and social relations play out in school textbooks. This linked to my interest in the factors and constraints that shape what is constituted as geometric knowledge by the curriculum and textbooks. The second aspect was the question of the relationship between the mathematical and pedagogical layers of the textbooks. Some studies focused on the mathematical content of the textbook, others focused on the pedagogical style of the textbooks and others on particular mathematical practices or values. Few were explicit about focusing on the interplay between the mathematical and pedagogical layers, yet this emerged as an underlying theme in my study and became central to my thinking about my research project. In what follows I review the literature associated with each of these two aspects in more detail.

2.3.2.1 How power and social relations play out in school textbooks

Apple (1992), in his analysis of text and power, starts from the premise that the school curriculum is not neutral and that what comes to be constituted as legitimate knowledge in schools "is the result of complex power relations, struggles, and compromises among identifiable class, race, gender, and religious groups" (p4). He states that the textbook plays a major role in defining *whose* knowledge is taught. However, like Bernstein, he argues that the relationship between social power relations and school knowledge is not simple. A textbook will not be a straightforward reflection of the knowledge of dominant groups, but will also incorporate elements of the power struggles and concessions that arise during its production. In this regard he points out that as disenfranchised groups have spoken out about the need for their knowledge to be taken seriously, what has happened is that, without changing the major ideological thrust of the textbooks, some progressive items have simply been added to the textbooks.

Of interest to my study is that in South Africa we experienced a substantive shift in power relations and it was during that time that the new curriculum and textbooks were produced. As with any shift in power relations there is no simple

way to describe them nor are they unidirectional. For example, although the previously disenfranchised became the largest number of voters and the previously banned became the government, large-scale redistribution of wealth and changed views of who was seen as expert were certainly not instantaneous. The production of new curricula and textbooks in a milieu in which the shift in political power relations had opened spaces of possibility for substantive change, but where at the same time the relative sluggishness of change in social relations might have an inhibiting influence in the opened spaces, is particularly interesting to study. Apple himself points to the importance of looking at such cases. Although on one level, Apple's article appeared to underscore the relevance of my study, on another it raised important questions for me to consider. The most crucial of these was how we begin to look at "whose mathematics" is present in a textbook. In the more overtly social school subjects like history or geography it is perhaps easier to see whose histories and cultures are being privileged and whose are being neglected. But how does one begin to analyse this in terms of mathematics?

On a basic level one could take this to suggest that one should check whether the textbooks guard against being Eurocentric in the way they present the history of mathematics and whether illustrations include both genders, all races etc. and whether contexts used are accessible to all groups. However my desire is to look at the nature of mathematics at a deeper level: how the relationship between the mathematics and the social play out – not just at the illustrations that accompany the mathematics, but at the mathematics itself. The question became how to look at this and for this I turned to the work of authors who have analysed curricula and textbooks in mathematics in relation to sociological issues.

Dowling (1996) compared two textbook series from the UK – one geared for higher ability learners and the other aimed at lower ability learners. He uses the notions of the esoteric and public domains to discuss the practices in the textbooks. The public domain is the everyday in which learners would already participate. The esoteric domain, in contrast, is described by Dowling as the "as yet unknown" (p393). Dowling uses Bernstein's notion of classification to

describe these domains. Classification refers to the relationship between categories. When classification is strong each category is unique in terms of its identity and voice and in terms of the rules that govern it. When classification is weak there is less specialization of identity and voice (Bernstein, 1996). The esoteric domain is strongly classified with respect to other activities and would comprise abstract mathematical statements. The public domain, on the other hand, is weakly classified. Whereas generalising strategies (i.e. pulling together different cases via a common principle) are important in producing work in the esoteric domain, localising strategies (i.e. constructing a particular example) are used in producing work in the public domain. This means the texts within the public domain are more context-dependent. Dowling argues that the textbooks aimed at lower ability learners present mathematics as being necessary for participation in the public domain and position the learners as being on a trajectory to a non-intellectual occupation. In contrast the textbooks aimed at higher ability learners presents[5] mathematics as being able to describe the public domain, but readily shifts the focus towards the esoteric domain. He further suggests that the localised nature of the tasks in the textbooks for lower ability learners construct the learners as already competent and the pedagogical strategy as bringing together learners to share strategies. Again he contrasts this with textbooks for higher ability learners who he suggests are constructed initially as lacking in competence and in need of being inducted into the esoteric domain of mathematics by a more knowledgeable other. This work by Dowling and its further elaboration (Paul Dowling, 1998) highlighted the importance of critically examining the domains of mathematical practices in textbooks. In particular it suggests that simply identifying the presence or absence of a context is insufficient to tell us about the mathematical knowledge prioritised. The nature of the context, the way it is used and the dominant gaze (mathematical or contextual) are of crucial importance. In addition his discussion of the different

[5] Dowling does not suggest that mathematics **is** necessary for participation in the public domain or that it **can** describe the public domain. He states that the textbooks **present** it as such.

nature of the pedagogy in the two textbook series points to the second underlying theme identified (i.e the interplay between mathematical and pedagogical layers) which is discussed further in section 2.3.2.2.

In a similar vein to Dowling's detailed look at the nature of mathematics in the textbook, further insights have been provided by researchers who bring grammar to the fore. These researchers have done what can be termed as a socio-linguistic analysis of mathematics (and science) texts (Dimopoulos, Koulaidis, & Sklaveniti, 2005; B. Herbel-Eisenmann, 2007; Beth Herbel-Eisenmann & Wagner, 2007; Morgan, 1996). These analyses have used discourse analysis to examine the nature of mathematics and the position of the mathematics learner constituted by textbooks. They argue that a writer makes choices in terms of language used and these choices indicate and portray values and beliefs. Of particular interest were Morgan (1996) and Herbel-Eisenmann's (2007) discussions of their use of the work of Halliday. Both these authors draw on the three metafunctions of language Halliday proposed: the ideational, the interpersonal and the textual. The textual function describes the way in which the text is constructed to make a coherent whole. According to Herbel-Eisenmann the ideational function includes "(a) *who* is involved in doing *what* kinds of processes; and (b) the depiction or suppression of agency" (p351). Morgan suggests that an analysis of the ideational function provides an answer to the question "What is mathematics?" according to the text being analysed. This is done by analysing the types of processes depicted in the text and examining the type of participants active in those processes. Drawing again on Halliday, Morgan uses his description of six main types of processes to highlight three of these processes (material, mental and relational) that are most common and discusses the effect the predominance of each might have. She suggests that a predominance of material processes links with a view of mathematics that is constructed by doing, a predominance of mental processes link with a view of pre-existing mathematics that is discovered, and a predominance of relational processes links with a view of mathematics as a system of relationships. In addition Morgan argues that the frequent use of nominalisations in mathematical

texts hides agency. So, for example, rotations occur without any mention of a person doing the rotating. Morgan and Herbel-Eisenmann both argue that this reinforces an absolutist view of mathematics: a mathematics that exists independent of human action.

The interpersonal function examines the roles and relationship between the author and reader of the textbook. In analysing the interpersonal function in the text, Herbel-Eisenmann focused on three linguistic forms: the personal pronoun, imperatives and modality. Modality describes the degree of certainty with which ideas are expressed and the use of personal pronouns influences notions of agency and control. In relation to the use of imperatives, Herbel-Eisenmann and Wagner use Rotman's distinction between inclusive imperatives (e.g. describe and explain, which they also term 'thinker' imperatives) and exclusive imperatives (e.g. calculate, draw and fill in, which they also term 'scribbler' imperatives). They argue that the use of inclusive or thinker imperatives construct a reader who is being drawn into the community of mathematicians, whereas the use of exclusive or scribbler imperatives construct a reader who can perform those actions outside of that community.

The productive use of critical discourse analysis in analysing the nature of mathematics in texts and its increasing popularity as a tool for use by the mathematics education community led me to seriously consider its use for my analysis. However an initial attempt at using some of the tools of critical discourse analysis on the textbooks suggested that this analysis would produce little that was new. The features of mathematical texts identified by Morgan and discussed by Herbel-Eisenmann were present in both textbooks.

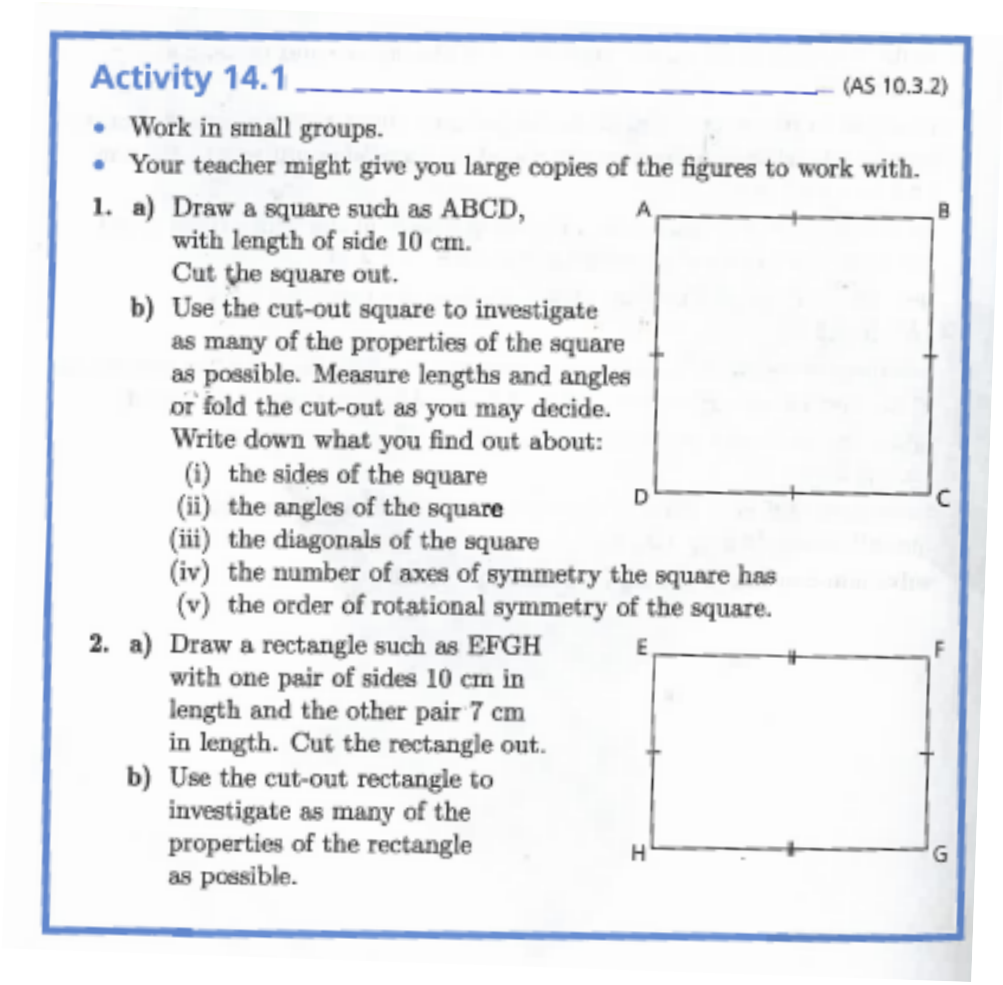
Herbel-Eisemann and Wagner's (2007) discussion of the use of different kinds of imperatives raised questions that I felt were important to consider further. They caution against drawing conclusions from a simple count of scribbler and thinker imperatives. They exemplify the issue by noting that, although the ratio of scribbler to thinker imperatives were similar in two textbooks, in one textbook the scribbler imperatives were scaffolding towards a thinker imperative whereas

in the other the thinker imperatives were independent tasks. They argue that because it is important to scribble before thinking, the textbook where the thinker imperatives were independent tasks might lose some of the inclusive effect of its thinker tasks. Without access to the textbooks they were analysing it is impossible to dispute this claim. However looking at similar tasks in the textbooks I am examining led me to believe that a tighter linking between mathematical considerations and linguistic considerations might be important. For example, the activity shown in textbook extract 1 below appears in the textbook, Classroom Mathematics Grade 10[6] (Laridon et al., 2004).

In this activity the scribbler imperatives "draw", "cut", "measure", "fold" lead to the thinker imperative "use to investigate". Herbel-Eisenmann and Wagner might suggest that the scaffolding of the scribbler imperatives enhances the inclusive effect of the thinker imperative. However this raises questions about an orientation to mathematics and the teaching and learning of mathematics. If one considers investigating as a key mathematical skill, then the textbook should enable learners to develop that skill. A key part of the skill of investigating is devising the processes to use in the investigation (e.g. folding and measuring a paper example of a square). In this example the processes are given through a series of scribbler imperatives preceding the command to investigate. One could consider this part of teaching learners how to investigate mathematical ideas (i.e. the textbook models the processes one would use to investigate to help the learners become aware of what processes can be used). However if the thinker imperatives are always scaffolded, I would argue that this indicates an assumption that learners are not able to enact those thinker imperatives independently.

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[6] Classroom Mathematics Grade 10 (Laridon et al., 2004) was one of the textbooks I analysed in this study. As I refer to it frequently throughout the report I use the acronym CM to reference it. Similarly I'll refer to the other textbook, On Track with Mathematics (Bennie, 2005a) as OTM.



Activity 14.1 (AS 10.3.2)

- Work in small groups.
- Your teacher might give you large copies of the figures to work with.

1. a) Draw a square such as ABCD, with length of side 10 cm. Cut the square out.

b) Use the cut-out square to investigate as many of the properties of the square as possible. Measure lengths and angles or fold the cut-out as you may decide. Write down what you find out about:

(i) the sides of the square
(ii) the angles of the square
(iii) the diagonals of the square
(iv) the number of axes of symmetry the square has
(v) the order of rotational symmetry of the square.

2. a) Draw a rectangle such as EFGH with one pair of sides 10 cm in length and the other pair 7 cm in length. Cut the rectangle out.

b) Use the cut-out rectangle to investigate as many of the properties of the rectangle as possible.

Textbook extract 1: Imperatives (CM p336)

There is a need, therefore, to look at the imperatives in terms of the mathematical practices they embody and to try to understand what view of mathematics and what view of learners' capabilities to engage in that mathematics they help construct. As part of the process of data analysis I experimented with doing a count of the imperatives in the textbook chapters. However, partly because of some of the issues I outline above and partly because this approach tended to provide too much detail and thus detracted from being able to focus on the mathematics itself, I chose not to pursue this in my final analysis. However I found the ideas brought up by this research useful and paid careful attention to the underlying activities the imperatives in the textbooks

were guiding learners towards in my analysis. In particular I paid attention to the scaffolding and nature of mathematical investigations.

Another avenue for textbook analysis has been that of cross-cultural studies. The biggest of these was conducted in conjunction with the TIMSS study (Valverde et al., 2002) and compared the textbooks of different countries in terms of content, structure and type of mathematical proficiency. Pepin and Haggarty (2002; 2001, 2007) provide a detailed analysis of differences between textbooks used in English, French and German classrooms. They argue that the learners in these three countries are given access to different mathematics and to different opportunities to learn that mathematics and that this is influenced by and influences the wider cultural context. They analyse a number of different dimensions in the textbooks, for example connections, context, cognitive demand and the use of multiple representations and the structure of the textbooks. Their analysis of the textbooks shows differences in terms of the nature of mathematics presented by the textbooks (pre-discovered, structured, abstract body of knowledge versus set of useful rules and procedures). Comparative work done on Japanese and UK textbooks (Fujita & Jones, 2003a) and Japanese and US textbooks (Mayer, Sims, & Tajika, 1995) shows a focus on the processes of deductive reasoning and problem-solving in the Japanese textbooks in contrast to a mastery of facts in the US textbooks and practical skills and inductive reasoning in the UK. They argue, similarly to Pepin and Haggarty, that these differences link to the cultural values of the countries and legitimise the associated educational values.

The core idea to emerge from this review of key papers in the sociological analysis of mathematics textbooks is the need to create a framework of orientations to mathematics against which the textbooks could be analysed. This reinforced the notion to use the orientations to mathematics identified by Graven (discussed in section 2.3.1.3 above) as a structure for the analysis.

2.3.2.2 Interplay between mathematical and pedagogical layers

Sierpinska (1997, p5) states that a "didactic text usually includes two types of discourse: the didactic discourse and discourse specific to the discipline being taught". She identifies the mathematical layer as being composed of definitions, theorems, examples, proofs and exercises. She identifies the didactic layer as focusing on the learner-reader's behaviour and on his/her interpretation of work from the mathematical layer. She provides two examples from the textbooks she analysed to exemplify the didactical layer. The first was a narrative that directed the learner-readers to master a particular procedure at that stage in their studies as this mastery would be important later on. This is clearly explicitly didactical. I was more interested in the second example, however, which she argues is a didactic action present in the mathematical layer. In the case she cited this was exemplified through the choice of exercises and questions to directly address known misconceptions. Sierpinska's discussion of mathematical and didactical layers in textbooks led me to question how one might identify and distinguish these layers. I was particularly interested in those instances where the didactic action was embedded in the mathematical layer and read other analyses of textbooks with this question in mind.

In Dowling's (1996) analysis of two textbooks series discussed earlier he describes tasks in the textbook geared towards lower ability learners that are set in the context of shopping. He argues that the text deliberately provides "no explicit pedagogising of method" (p405) and instead anticipates that learners' engagement with the familiar situation of shopping will bring to the fore various strategies for solution that can be discussed and evaluated in class. He then problematises the reason for inclusion of shopping in the textbook and suggests three possible scenarios: the first is that shopping provides an accessible entry point to move towards more abstract mathematics, the second is that shopping tasks might be construed as mathematical activity, and the third is that mathematics is necessary in order to shop better. He argues that it is the third scenario that provides the reason for inclusion of shopping in the textbooks for lower ability learners. In contrast the use of a shopping context in the textbooks

for higher ability learners requires learners to move out of the public domain and into the esoteric domain and privileges the learning of strategies required for moving forward in the esoteric domain.

Dowling's examples point to how a single factor (the inclusion of the shopping context) can be motivated by quite different mathematical and pedagogical goals. As stated above the use of context can allow for learners to be construed as competent and to bring their own strategies to the table, thus helping to construct a learner-centred pedagogy. Context can also be a starting point to enter the esoteric domain and thus a pedagogical strategy for rendering the esoteric domain more accessible. Modelling contexts can be seen as a mathematical skill that needs to be learnt and thus the inclusion of contexts can fulfil a mathematical goal. And, as Dowling points out, the inclusion of contexts in particular ways can have the effect of portraying mathematics as either necessary for participation in the public domain or as able to describe various aspects of the public domain. The fine-grained analysis that Dowling uses to draw out these differences suggests to me that both authors and readers might not be consciously aware of what aspects of context in the text are there for pedagogical reasons and what are there as part of the mathematical layer.

The issue of different emphases in the textbooks of different countries arose in some of the international comparisons of textbooks I reviewed (Fujita & Jones, 2002a, 2002b, 2003a; Haggarty & Pepin, 2002). A re-reading of these textbooks with notions of mathematical layer and didactical layer in mind highlighted the difficulties in thinking discretely about these layers. For example, Fujita and Jones (2003a, p1), discussing the comparative study of geometry by Hoyles, Foxman and Kuchemann, state that "the study found, a 'realistic' or practical approach is apparent in Holland, while a theoretical approach is evident and France and Japan." It is not clear whether the differences in approach stem from beliefs about **how** mathematics is best learnt or from beliefs of **what** mathematics should be learnt.

Fujita and Jones (2002a) in their discussion of Godfrey and Siddons's geometry textbook argue that the set of carefully designed experimental exercises were chosen to lead to the proof of a geometric fact, but in the process aids in the development of what Godfrey and Siddons's called the 'geometrical eye'. Godfrey and Siddons's argument is that the production of new geometry relies on a strong visual intuition (their 'geometrical eye'). I would thus argue that the structured series of experimental exercises incorporates elements from the didactical domain (i.e. belief about what is needed to teach geometry), but also elements from the mathematical domain (i.e. beliefs about what is important to know – in this case it is important to have a trained 'geometrical eye').

Haggarty and Pepin (2002) in their comparison of French and German textbooks provide the following description of mathematics in the French textbooks:

> In France, a formal view of mathematics had been traditionally held and it was still recognisable in some of the textbooks. Mathematics was regarded as a structured body of pre-discovered knowledge, perhaps less structured and with more emphasis on discovery than in Germany. Books contained many activities (which allowed pupils to discover ideas for themselves) and this was a feature of all books. It appeared that there had been an overlay of perspective from the formal to a more dynamic view of mathematics and how it is taught and learnt. (p586)

What is noticeable in this discussion is that a distinction is not made between the mathematical and pedagogical layers. This is not surprising. If we look, for example, at the work of Ernest (1991) and its elaboration in the work of Naidoo and Parker (2005) on the philosophy of mathematics, and the work of Seah and Bishop (2000) and Dede (2006) on mathematical values and mathematics educational values, we see a close link between views of mathematics and pedagogy. However I do not think this makes investigating the distinctions and interplay between the two layers unnecessary. Van Dormolen (1986) talks of a formalist view of mathematics in which deductive reasoning is prioritized. He argues that within such a view intuitive reasoning is only allowed as a teaching tool to motivate learners and to encourage them into the formal deductive

reasoning. He contrasts this with an activist view of mathematics in which "mathematics consists of being engaged in activities like generalizing, classifying, formalizing, ordering, quantifying, abstracting, exploring patterns etc" (p144) where intuitive reasoning would be considered a form of genuine mathematics.

This review of research literature with an eye on the mathematical and didactical layers has led me to believe that the interplay between the two layers is not simple. In some cases there is overlap where the view of what it takes to learn mathematics and what constitutes legitimate mathematics are consonant. In other cases, strategies for learning mathematics require activities that fall outside the canon of accepted mathematics.

In summary the literature on curriculum and textbooks strengthens my view that the creation of school mathematics is influenced by a number of different forces. These forces are sometimes mutually reinforcing but are also easily conflated and can at times be contradicting or place competing demands on the curriculum/textbook authors. This literature also highlighted the importance of understanding both the pedagogical and mathematical layers. For this reason I look in section 2.3.3 below at key literature written about geometry learning and teaching as well as at a way of understanding the nature of geometry.

2.3.3 Geometry

2.3.3.1 The van Hiele Theory

The van Hiele theory of the development of thinking in geometry has been influential in the arena of geometry teaching and learning (van Putten, Howie, & Stols, 2010) and has been used extensively in research (Jones, 1998). The theory was developed by a Dutch husband and wife team in the late 1950s (Usiskin, 1982) and is described in English in detail in a monograph issued by the National Council of Teachers of Mathematics (Fuys, Geddes, & Tischler, 1988). In their theory, the van Hieles propose that learners need to pass through five levels of geometric thinking in order. These levels are numbered and named slightly differently by different authors. Here I take the numbering and description of

each level from the monograph by (Fuys et al., 1988) and use the names as presented by Crowley (1987).

Level 0: Visualization

"The student identifies, names, compares and operates on geometric figures (e.g., triangles, angles, intersecting or parallel lines) according to their appearance" (Fuys et al., 1988, p5).

Level 1: Analysis

"The student analyses figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically (e.g., by folding measuring, using a grid or diagram)" (ibid., p5).

Level 2: Informal Deduction

"The student logically interrelates previously discovered properties/rules by giving or following informal arguments" (ibid., p5).

Level 3: Deduction

"The student proves theorems deductively and establishes interrelationships among networks of theorems" (ibid., p5).

Level 4: Rigour

"The student establishes theorems in different postulational systems and analyses/compares these systems." (ibid., p5).

There have been numerous studies internationally that have attempted to evaluate the level of geometric thinking of students using the van Hiele levels. Instruments ranging from short answer paper and pencil tests to extensive interviews have been developed to do this. The results of these tests have indicated that many high school students, even those taking or having taken geometry courses, and pre-service mathematics teachers perform below the level of formal deduction (level 3) (Burger & Shaughnessy, 1986; Mayberry,

1983; Usiskin, 1982) . These results are echoed in the South African context. For example, Feza and Webb (2005) found that none of the 30 learners they tested, who were in their last year of primary school, had attained the level of informal deduction. De Villiers (1995) argues that research experience in South Africa has shown that most children starting high school in South Africa are at level 0 or 1 and quotes the study by de Villiers and Njisane (1987) which showed that less than half the grade 12 students in their sample had attained level 1. More recent work by van Putten, Howie and Stols (2010) showed that more than half of a group of 3rd year pre-service teachers entering a geometry course at the University of Pretoria were "only efficiently functional on level 0" (p28) echoing earlier research done by van der Sandt and Niewoudt (2003) with 100 prospective teachers from the North West province.

According to the van Hiele theory one of the key reasons why learners struggle with geometry is because it is presented to them at a level beyond the level they are at (de Villiers, 2004). Because the language, symbols and network of relations are particular to each level (Fuys et al., 1988), learners will not understand work presented by the teacher at a higher level than the level they are at and the teacher will not be able to understand why they cannot understand (de Villiers, 2004). Fuys, Geddes et al. (1988) suggest there is a need to align instructional design in geometry with the van Hiele levels. The van Hiele theory has had a major impact on the design of the Space, Shape and Measurement strand in the South African mathematics curriculum (van Niekerk, 2006). De Villiers (1996, p13) argues "It seems clear that no amount of effort and fancy teaching methods at the secondary school will be successful, unless we embark on a major revision of the primary school geometry curriculum along van Hiele lines."

De Villiers has published or presented a number of papers on the teaching and learning of geometry that have strong parallels with the assessment standards given in the MNCS. These relate to the classification and definition of quadrilaterals (de Villiers, 1994, 1998), the role of definitions in geometry (de Villiers, 1996, 1998) and the use of transformation geometry throughout all

levels of the mathematics curriculum (De Villiers, 1993). This suggests that the notion of the levels of thinking as described by van Hiele and their interpretation by de Villiers have had a strong influence on the geometry curriculum in South Africa. For this reason I paid specific attention to de Villiers's work and discussed its influence on the curriculum and textbooks in interviews with him and with the textbook authors and curriculum committee members as part of my research.

In addition to the aspect of the model which characterizes student thinking, the van Hieles proposed that to move from one level to the next students needed to experience five teaching phases: Information, Guided Orientations, Explication, Free Orientation and Integration (Ding & Jones, 2007; Fuys et al., 1988). In contrast to the extensive work on the van Hiele thinking levels, there has been little work done that addresses these five teaching phases specifically (Ding & Jones, 2007). Owing to this lack of research, these five teaching phases are less fully explained and understood. In particular Clements and Battista (1992) point out that the way these phases relate to the levels of thinking is potentially confusing as some of the phases seem more appropriate to particular levels. They further point out that it is unclear how they relate to content and, in particular, whether the five phases need to be followed through linearly for each content area or whether many concepts should be treated in each phase and progression through the phases should happen in parallel for all these topics.

The van Hiele theory has been the dominant and fairly widely accepted theory characterizing students' geometry thinking, although this is not to say that there are not debates about aspects of the theory. In particular, questions have been raised about the discreteness of the levels of thinking and the uniformity of a student's van Hiele level across all concepts. However many researchers have found it productive and have provided deeper insights into or modifications of the theory (for a summary of these developments see Battista (2007)).

This centrality of the van Hiele theory as well as the influence it has had in South Africa suggests that it would be a natural choice to form part of my analytic framework. However, after careful review of the literature I decided against this.

My analytic framework needs to provide a way to analyse the nature of the geometry in the textbooks. I considered whether the van Hiele teaching phases might be useful in doing this, but two factors mitigated against it. Firstly, as described above, the phases are not sufficiently well understood and have not been at the centre of much research. Secondly, the phases describe a pedagogical approach more than they give access to the nature of the geometry. I also considered using the van Hiele levels of geometric thinking as used by Fuys, Giddes and Tischier (1988) in their analysis of American textbooks. They comment on the difficulties this posed, saying "It is not the text page which has a level of thinking, but rather the student reading it or the teacher teaching it" (p158). They dealt with this difficulty by ascribing to a page the minimum and maximum van Hiele levels of thinking that a student would need to have to deal with content on that page. Despite this potential solution I remained uncomfortable with the notion of ascribing a level of thinking to a text. Many questions can be tackled at multiple van Hiele levels of thinking. For example, deciding whether a particular geometric shape is a square or not can be answered at level 0 (i.e. yes, it looks like one) or at level 3 (i.e. with a rigorous deductive proof based on the definition of a square). In addition, my review of literature related to current trends within mathematics curricula and analyses of the South African mathematics curriculum suggested that there were emerging approaches to mathematics (such as modelling, a focus on mathematical practices etc.) that would not necessarily be adequately captured if I tied myself to the van Hiele theory of geometry.

Finally, once I began my interviews with members of the curriculum committee and textbook authors, some made frequent reference to their use of van Hiele in the process of developing the geometry curriculum or textbook chapters. These discussions confirmed my belief that I should not use the van Hiele theory as a tool for analysis. Using my interpretation of the van Hiele theory as a tool to analyse someone else's interpretation of the van Hiele theory creates a circularity and thus I felt it would be better to use a different approach that

would allow me to look at the curriculum, textbooks and the influence of the van Hiele theory on these from the outside.

2.3.3.2 Some key issues in geometry teaching and learning

Once I made the decision not to use the van Hiele theory as the framework with which to view geometry, I needed to find something which would allow me to capture some of the key ideas relating to geometry teaching and learning. In my reading in the area of geometry and geometry teaching and learning one theme reoccurred regularly, albeit with slightly different emphases and varyingly nuanced descriptions. This theme, crudely put, was that of geometry as being both immensely concrete and practical as well as abstract.

Despite the fact that geometry can be used to explore physical space at school it has also been traditionally the one place in the mathematics curriculum where learners encounter proof. Herbst and Brach (2006, p74) state that "In the United States for more than a century, a course on Euclidean geometry has been justified as part of the high school curriculum primarily on the grounds that it provides a context in which students can encounter and learn the 'art of mathematical reasoning'". This notion is echoed by Galuzzi (1998) in his discussion on school geometry in Italy and implicitly in much of the research literature on the learning of proof in school mathematics which recognises that it has been largely restricted to the geometric setting (Hanna & Jahnke, 1996). Thus geometry occupies a role which is at once concrete and practical and yet also highly abstract and theoretical.

Within the van Hiele theory the notion of dealing with geometry concretely is implicit in the lower levels and the assumption is that with appropriate teaching students will be able to deal with geometry in an increasingly abstract way. Although many programmes attempt to follow this path from the concrete to the abstract, experience shows that it is not easy. Kuchemann and Hoyles (2006) in reporting on a project in which they assessed students' reasoning about proofs over a period of three years say, in relation to geometry, that their results "show not only that writing coherent geometrical explanations is problematic, but that

making progress is also problematic, most notably because students have to learn the socio-mathematical norms expected of written explanations using geometrical knowledge they have recently been taught while retaining (or further nuturing) their intuitive sense of manipulating shape and space" (p604). Fujita and Jones (2003b, p47) argue that although the dual nature of geometry as both concrete and abstract "should help teachers link mathematical theory to their pupils' lived experience, in practice for many pupils the dual nature is experienced as a gap that they find difficult to bridge."

The two sides to geometry also manifest in the way in which geometrical figures can be viewed either as a concrete representation or an abstract object. Fischbein (1993) uses the term "figural concepts" to talk about geometrical figures. The term captures the notion that any geometrical figure possesses both "conceptual and figural characters" (p139). The figural character would be the spatial representation and the conceptual character the abstract properties governed by the object's definition. Fischbein and Nachlieli (1998, p1195) argue that the highest level of geometrical reasoning is attained when "the figural and conceptual constraints are perfectly harmonized in what we have called a figural concept". However they go on to state that the harmony between the figural and conceptual constraints does not imply they should be equally weighted. In fact the conceptual constraints need to control the reasoning about the geometrical figure. They argue that it is the figural component that allows for creativity and opens new directions for investigation, but that it is the conceptual constraints that must determine the logic and rigour of the investigation. The implications of this type of idea for the teaching and learning of geometry have occupied mathematicians and educators for many years. Fujita and Jones (2003b) discuss the textbooks of Godfrey and Siddons from the first half of the 1900s as dealing directly and explicitly with this issue.

Although there seems to be widespread agreement between researchers about the importance in learning geometry of being able to deal with its concrete and abstract nature and the difficulty students experience in doing this, there is not agreement about the way in which this should be done. For example, those who

use the van Hiele theory appear to favour an extensive period of dealing with the concrete before moving toward greater abstraction, whereas Godfrey and Siddons argue that it is important to blend what they call the theoretical with experiments at all stages (Godfrey and Siddons, 1931 in Fujita & Jones, 2003b). Because the abstract-concrete duality is an important one in geometry and because there is no agreement about how it should be dealt with in the classroom, I wanted to find tools which would allow me to describe the geometry in the textbooks to be analysed in ways that related them to this duality without confusing that judgment with an implication about learners necessarily being on a particular point of a particular theorised teaching and learning path. The work of Houdement and Kuzniak (Houdement & Kuzniak, 2003) on the three paradigms of geometry emerged as particularly well-suited to this task.

2.3.3.3 The three paradigms of geometry

Houdement and Kuzniak (2003; 2002) distinguish three paradigms of geometry: Natural Geometry (Geometry I), Natural Axiomatic Geometry (Geometry II) and Formalist Axiomatic Geometry (Geometry III). In GI the objects one works with are material objects (or drawings of material objects) and one makes deductions via experiment, perception and use of instruments. The correspondence with reality is the yardstick by which deductions in this paradigm are judged. In GII one uses deduction within an axiomatic system. However the axioms are based in attempt to provide a model of the space around us. GIII is an axiomatic system where no attempt is made to maintain a link to reality. Houdement and Kuzniak (2003) state that this frame has been developed theoretically via an analysis of philosophical, mathematical and didactical texts and has been verified empirically. Kuzniak and Rauscher (2011) argue that these three paradigms help us to clarify the different meanings that can be given to the term geometry, and suggest tools of analysis that would prise apart and keep in relation the concrete and abstract duality. It is for this reason that their description of these three paradigms seemed to offer the potential to be particularly useful to my study.

Houdement and Kuzniak (2003) provide a summary of the three paradigms of geometry which has been elaborated on by Parzysz as reported by Braconne-Michoux (2011). This elaborated summary is presented below.

	Geometry I	**Geometry II**	**Geometry III**
Intuition	Sensible, linked to perception, enriched by experiment	Linked to figures	Internal to mathematics
Experience	Linked to measurable space	Linked to schemas of the reality	Logical
Deduction	Near to the real and linked to experiment	Demonstration based upon axioms	Demonstration based on a complete set of axioms
Kind of space	Intuitive and physical space	Physical and geometrical space	Abstract Euclidean space
Status of drawing	Object of study and of validation	Support for reasoning and "figural concept"	Scheme of a theoretical object, heuristic tool
Privileged aspect	Self-evidence	Properties and demonstration	Demonstration and links between objects, structure
Objects	Physical	Theoretical	Theoretical
Validations	Perceptive or by use of instruments	Deductive	Deductive

Table 2: Summary of the three paradigms of geometry (Braconne-Michoux, 2011)

So, for example, if we are asked to show the diagonals of a square are equal, working in GI it would be acceptable to draw the square and measure the diagonals. In GII, in contrast, this would not be acceptable and a deductive argument based on congruent triangles, for example, would be required.

Although the paradigms are distinct there are clearly links between them. Expert geometry users are able to move easily between the paradigms with awareness of the strengths and limitations of each. However for a student the initial move from GI to GII requires a dramatic shift in perspective of the geometric object despite the visible image not having changed at all (Houdement & Kuzniak, 2003). In addition to this difficulty it is possible for student and teacher to adopt two different paradigms and hence miscommunicate (Houdement & Kuzniak, 2003; Kuzniak & Rauscher, 2011).

At school level we work largely within GI and GII. Houdement (2007) describes the difference between the two as being evident in the objects (physical versus conceptual), techniques (material tools versus production of conjectures and logical validation) and validation mode (conformity to reality versus deduction from logical reasoning). She suggests that both paradigms are useful for solving geometric problems. Kuzniak and Rauscher (2011, p131) argue similarly and state that "Each paradigm is global and coherent enough to define and structure geometry as a discipline and to set up respective work spaces suitable for solving a wide range of problems".

Although the notions of GI and GII provide a useful lens to view the geometry in the textbooks I will analyse, there are further aspects in the development of this theory that are pertinent to understanding the human activity related to the paradigms. This aspect is elaborated through the notion of geometric work space (GWS). (Houdement, 2007; Kuzniak, 2008, 2010; Kuzniak & Rauscher, 2011). Kuzniak motivates the need for notion of GWS by noting that mathematics is a social activity performed by individuals. He argues that we need to understand how both a community of people and individual people use the geometrical

paradigms or move between the geometrical paradigms when dealing with geometric problems.

The geometric work space consists of two interacting planes.

The first plane that Kuzniak and Rausher (2011) and Kuzniak (Kuzniak, 2012) call the components or epistemological plane characterises the geometrical activity from a mathematical point of view. It consists of three elements:

> 1. The real and local space as material support with a set of concrete objects
>
> 2. Artefacts such as drawing instruments and software available
>
> 3. A theoretical system of reference consisting of definitions and properties (Kuzniak & Rauscher, 2011, p135).

The second plane is the cognitive plane which consists of 3 cognitive processes used in geometrical work

> 1. A visualisation process with regard to space representation and the material support.
>
> 2. A construction process determined by instruments used (ruler, compass etc.) and the geometric configuration
>
> 3. A discursive reasoning process that conveys argumentation and proof.
>
> (ibid, p135).

The planes are connected as follows: The real space is connected to visualization by intuition, the artefacts to construction by experiment and the theoretical model to the notion of proof by deduction.

Kuzniak thus summarises the GWS in the following diagram (Figure 2):

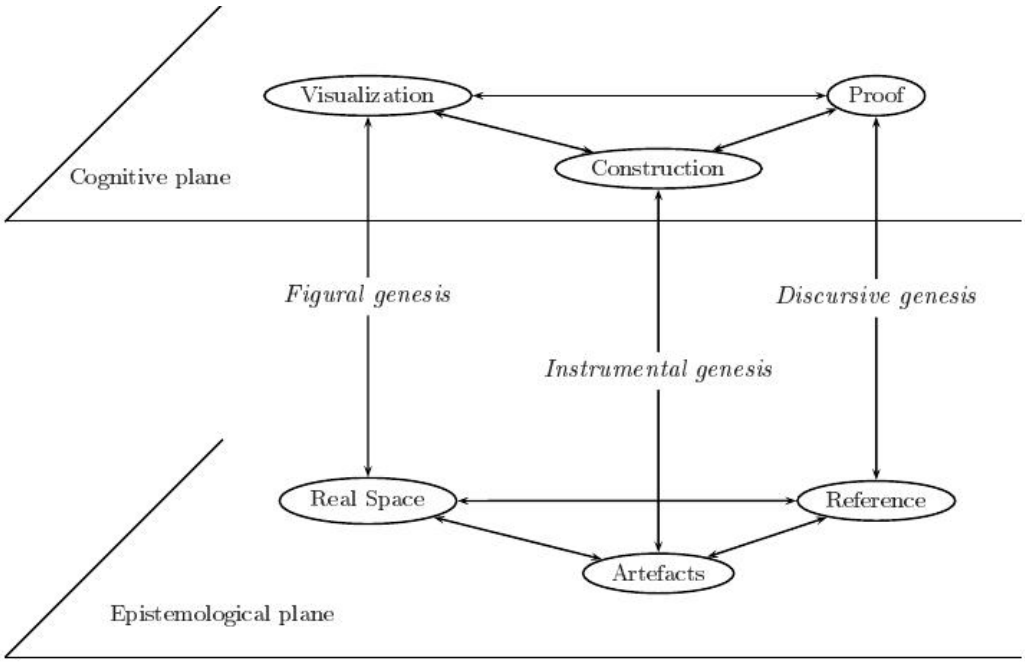


Figure 2: The Geometric Work Space (Kuzniak, 2012, p363)

Kuzniak (2008) discusses various levels of GWS and relates this notion to the didactic transposition. He talks of the reference GWS (which he also calls the intended GWS). In a particular country this GWS is that privileged by the education system and is often closely aligned with a particular paradigm. The next level would be the appropriate GWS (which he also calls the implemented GWS). This GWS takes into account the school circumstances and pedagogises the intended GWS. The final level of GWS is the personal GWS. This is the GWS activated when a person (student or teacher) is grappling with a geometric problem.

A teacher may, for example, be well aware of reference GWS that is strongly aligned to GII, but incorporates elements of GI as well as GII into the appropriate GWS to provide students with relevant pedagogical experiences. Students' personal GWS can vary and may not cohere with either the reference GWS or appropriate GWS.

My study is limited to the analysis of textual material and thus I will not be able to probe a student or teacher's personal GWS or analyse an appropriate GWS at the classroom level. However the notions of reference GWS and appropriate GWS

do provide tools with which to talk about the curriculum (a component of the reference GWS) and textbooks (which reflect their authors' depiction of an appropriate GWS).

2.3.4 How the work reviewed has framed my study

The pedagogical device provided an overarching framework for my study and thus the literature reviewed can be viewed in terms of this framework. This is summarised in table 3 below.

The field of production	Nature of mathematics/geometry Theories of how geometry is learnt Difficulties in the teaching and learning of geometry Orientations to mathematics in the school curriculum
The field of recontextualisation	The effect of power and social relations on curricula and textbooks The interplay between the mathematical and pedagogical in the constitution of school mathematical knowledge
The field of reproduction	The way in which nature of geometry and the difficulties in learning geometry have played out in the classroom

Table 3: Key aspects of literature reviewed

As my study centred on the production of texts within the field of recontextualisation I needed to find a way to organise my theoretical frame so that it provided the best tools to analyse the field of recontextualisation in depth. The literature review had revealed the interplay between mathematical and

pedagogical imperatives as important considerations in the field of recontextualisation, as well as the effect of power and social relations. The literature review also revealed that in order to consider each of these facets properly there would be a need to look at them from multiple points of view. Bringing in all the aspects the literature suggested gave rise to the need to find a systematic way to structure the theoretical field so that the various facets were illuminated. In section 3.1 below I describe in detail how I did this.

3 Research design and methodology

Brown and Dowling describe the activity of educational research as "the production of a coherent set of statements. These are established and located within explicitly stated theoretical and empirical contexts. The research process, conceived in this way, begins with vagueness and hesitance and plurality and moves towards precision and coherence" (Brown & Dowling, 1998, p137). In order to produce the ordering and coherence that Brown and Dowling refer to, Dowling (1998) outlines a general methodology that he refers to as *constructive description*. This methodology argues for the production of a language of description through "a deductive theoretical construction and inductive empirical reading" (P. Dowling, 1998, p126). In essence, the coherence is produced through an ongoing dialogic conversation between the theoretical and empirical. In my study the MNCS, the textbooks and the people involved in the production of the curriculum and textbooks constituted layers of the empirical field. An initial engagement with the theoretical field in the context of the empirical setting suggested that there would similarly need to be multiple layers in the theoretical field. The complexity of the multiple layers in both the theoretical and empirical fields and the constant interplay between the two fields as well as between the layers makes it difficult to capture the process that occurred in a written report that is linearly sequenced. It is important to point this out because aspects of the theoretical framework presented in the previous chapter are a product of the interplay I describe here. My selection of texts and my selection from those texts were shaped in interaction with the empirical field and so the work presented in chapter 2 is a first iteration of the production of a language of description which will be refined and developed through the analysis and discussion in subsequent chapters. Thus the description of the way in which work from the theoretical field interacted with the empirical field that follows appears more ordered and linear than the messy and dialogic process that actually occurred. However my intention in providing an ordered description of the layers of the theoretical field and the key ways in which they informed work in the empirical field is to provide the reader with a "route map" through the

multiple layers and complexity so that principles at play in the decisions made in the research design and data analysis and the way those were constituted to address the research questions are clear.

3.1 The theoretical field

My research questions centre around the way in which school geometry was constituted in the MNCS and the textbooks. I thus identified Bernstein's notion of the pedagogic device, which describes the way knowledge is recontextualised for pedagogic purposes, as an appropriate theoretical framing for my work. My engagement with this theoretical field together with my own initial work in the empirical field, led to a modification of the pedagogic device which I discussed in chapter 2.

This framework indicated the relationships and influences I would need to probe and thus guided the research design. The first aspect it highlights clearly was the location of the MNCS and the two textbooks in the field of recontextualisation. The influence of the MNCS on the textbooks was clearly an important factor to analyse. At the same time the framework highlights the importance of understanding how previous versions of the mathematics curriculum and textbooks played out in the field of reproduction on the MNCS and on the textbooks. This necessitated supplementing an analysis of the MNCS and textbooks as documents with interviews from agents in the ORF and PRF who could provide insight into the process of construction of these documents. Full details of the choices made in terms of who to interview are provided in section 3.2.3.

My work within the theoretical field suggested three core elements in the field of production (mathematics, mathematics education and the general regulative discourse) that are recontextualised in the process of producing school mathematics. The key challenge in the analysis was to find ways to incorporate each of these areas in an integrated manner and to find a systematic way to investigate and report this. Part of the problem was that each of these areas are

broad to review and, in the absence of some guiding system to help me focus on key aspects within each of these areas, the concern was that I would either review too broadly to be useful or alternatively end up omitting important features. Thus what I looked for was a description of mathematics curricula that had been shown to be useful and that I could see as having potential to provide a guiding system for my analysis.

As discussed in chapter 2, in my review of literature on the South African mathematics curriculum I found the work of Graven (2002) and the development of it by Parker (2006) particularly useful. This work describes the curriculum as reflecting different orientations to mathematics. The fact that it had been applied productively to an analysis of the MNCS as well as the potential for exploring each of the key aspects of my framework within the different orientations suggested these orientations could provide a guiding system for my analysis. In addition both Graven and Parker's work were informed by the notion of recontextualisation from Bernstein and thus align well with the theoretical framework of this study. Graven and Parker's work is discussed in detail in chapter 2, however to clarify my use of them I repeat the four orientations identified by Graven:

1. Mathematics is important for critical democratic citizenship.

2. Mathematics is relevant and practical.

3. Mathematics is an induction for learners into what it means to be a mathematician.

4. Mathematics is a language with conventions, skills and algorithms that must be learnt.

These four orientations, although themselves part of the theoretical field, also helped structure the way I managed the interaction between the theoretical field and the empirical field in order to produce a language of description. I took each of the orientations in turn (although, as I describe in detail in chapter 6, work in

the empirical field indicated it would be productive to look at orientation 3 and 4 in tandem) and interrogated that orientation with reference to literature relating to mathematics, mathematics education and the GRD and in relation to the empirical field. The first step in this process was an examination of the orientation together with the literature in relation to the MNCS. This step led to a refinement in the understanding of the orientation and this was used to produce analytic tools that were applied to both the MNCS and the textbooks.

3.2 The empirical field

The research questions and theoretical framework helped delineate the main elements of the empirical field as texts and agents from the ORF and PRF. The text chosen from the ORF was the curriculum document and the agents were those involved in the development of this document. The texts chosen from the PRF were two textbooks and the agents were the authors of these textbooks. Each of these components is described in greater detail below.

3.2.1 Curriculum documents

The MNCS (South African Department of Education, 2003) is an official policy document of 94 pages. It prescribes a curriculum for learners in grades 10 – 12[7]. Aligned with the MNCS are two further documents, the Subject Assessment Guidelines for Mathematics (SAGM) (South African Department of Education, 2005) and the Learning Programme Guidelines for Mathematics (LPGM) (South Africa. Department of Education, 2005). Both the LPGM and the SAGM were published after the MNCS was published and further versions of them were published in subsequent years. The LPGM provides guidance to teachers about how to create a learning programme based on the MNCS. The SAGM provides information about how learners should be assessed. In addition to this the SAGM

[7] A copy of this document can be downloaded from http://www.education.gov.za/Curriculum/CurriculumStatements/FurtherEducationandTraining/NCSSubjectStatements/tabid/246/Default.aspx

was used to designate certain content in the MNCS as core and further content as optional. As discussed in chapter 1 this was done as the result of concerns that the curriculum was overloaded and that teachers were ill-prepared to teach certain sections of the curriculum (Euclidean geometry and probability were cited in particular). Although these documents were released prior to implementation of the MNCS in schools they were released after the textbook authors had written the grade 10 books. For these reasons, although I consulted the LPGM and SAGM where necessary, I did not do an in-depth analysis of these documents.

3.2.2 Textbooks

My analysis is focused on the geometry sections of two textbooks. In deciding to analyse two textbooks I wanted to find two different recontextualisations of the MNCS not to evaluate them comparatively, but so that in contrasting the approaches I could illuminate issues that would help in answering my research questions. According to Cresswell (2012) because the purpose of qualitative research is not to generalize to a population but to develop an in-depth understanding of phenomena, the selection of the sample needs to be purposeful. Thus my choice of textbooks was not motivated by a need to select textbooks that would be representative of all mathematics textbooks written for the new curriculum. However I did want to choose textbooks that were published by well-known publishers, written by respected authors and intended for the mass market. This was to ensure the textbooks were not unusual cases (e.g. written for schools with access to technology or aimed at high achievers) and that they had credibility. For these reasons I chose the book Classroom Mathematics Grade 10 (which henceforth I refer to as CM) (Laridon et al., 2004) as the first book to analyse. This book was the bestselling grade 10 mathematics textbook in the country at the time I embarked on my study and was written by a team of authors who are (or were) schoolteachers with extensive classroom experience. A number of authors of this textbook were also involved in the curriculum process and have been in leadership positions in mathematics education organisations. The Classroom Mathematics brand is well known in South Africa.

The coordinator of the team of authors as well as a number of members of the team had been involved in writing previous editions of the book.

The second textbook I chose was On Track with Mathematics grade 10 (henceforth referred to as OTM) (Bennie, 2005b). This textbook had also sold well and was part of a well-known series. The primary author of the textbook was a teacher with extensive classroom experience who had also been involved in leadership positions in mathematics education organisations and was an active participant in commenting on drafts of the curriculum documents. The way in which the geometry sections in this textbook were organized and presented was different to Classroom Mathematics, thus providing the potential for illuminating contrasts.

3.2.3 Interviews

As discussed in section 3.1, my adaption of Bernstein's pedagogical device indicated that it would be important to conduct interviews from agents in the ORF and PRF to gain insight into aspects of the process of construction of the MNCS and the textbooks. There were a number of people involved in the construction of the MNCS and the textbooks so I needed to make decisions about who to interview.

The decision about which people involved in the textbook writing to interview was relatively straightforward as the names of the authors were available in the textbook. OTM had three authors listed. However initial enquiries indicated that one person had been responsible for the edition of OTM written to meet the MNCS and used work from the previous edition (written by the other two authors) where appropriate. I thus decided to interview the author of the current edition only. I use the code A_{OTM} for this author.

CM has nine authors. I interviewed the coordinating author who was in charge of the writing team and who wrote the chapter on Euclidean geometry in the textbook himself. I use the code A_{CM} for this author.

In addition a person who was identified as a key person to interview with regard to MNCS was also one of the senior members of the CM author team and thus I also spoke with her about the textbook. She was interviewed primarily as a member of the committee who drew up the MNCS and I use the code MC_1 to refer to her.

The process of identifying people to interview with regard to the MNCS was less straightforward. There was no list of curriculum committee members and, as it transpired, the composition of the committee changed over time. I adopted a snowball sampling approach (Cohen & Manion, 1989). I began by interviewing a person who I knew had been involved in the curriculum process and then asked him to suggest further people I should interview. I then interviewed some of the people he had suggested and they suggested further people to interview. Through this process and the discussion in the interviews I was able to piece together a picture of the way in which the team overseeing the curriculum development was structured and key processes that occurred. Ultimately this emerging picture also allowed me to decide when I had interviewed role players in all the key processes. The positions of the individuals along with a description of their role and the code used to refer to that individual throughout this dissertation are provided in table 4 below.

Curriculum process/structure	**Description**	**Code used for the person**
Ministerial Project Committee	This committee oversaw the curriculum development process for all subjects. I interviewed the person on this committee responsible for Mathematics and Mathematical Literacy. As this person was also the coordinating author of CM he has the code A_{CM} as discussed above.	A_{CM}

Mathematics curriculum team	This team was charged with the responsibility of developing the MNCS. I interviewed a member of this team who was involved from the beginning of the process. This person was also a member of the CM authoring team.	MC_1
Dealing with public comment on the MNCS	After an initial version of the MNCS was written public comment on the draft was solicited. This was then followed up and the MNCS was amended. I interviewed a member of the team who was involved in the curriculum committee at the time they were reviewing the public comment and amending the MNCS after consideration of the public comment.	MC_2
The Ministerial Review Committee	Prior to the implementation of the MNCS concerns were raised about the feasibility of the curriculum and also concern were raised with the Mathematical Literacy curriculum. The Education Minister, Naledi Pandor, established a committee to review these curricula prior to implementation. I interviewed the chair of this committee who had also been involved in writing the GET Mathematics curriculum (for grades R – 9) and thus could provide information on that curriculum.	MRC
Geometry consultants	Two academics were identified to me as geometry experts who had been consulted in the process of writing the geometry	GC_G GC_F

	sections of the curriculum. One (GC_G) had played a key role in the GET Mathematics curriculum and the other (GC_F) was closely consulted around the FET Mathematics curriculum.	

Table 4: Interviewees

The six interviews were semi-structured. For each person I had an outline of key issues to probe, but I allowed the responses of the interviewee to guide the interview. These interview guides are provided in Appendix A.

For those involved in the curriculum process my key interest was in getting insight into who the agents in the ORF were and their impression of some of the key influences on the MNCS. For those involved in the textbook writing I focused on the resources in the field of production they drew on in constructing the textbook and particularly the geometry chapters and any difficulties they had in the process.

The interviews provided information about the context in which the MNCS and the textbooks were produced. This, together with various policy documents, provides the backdrop against which the more detailed analysis of the curriculum and textbooks can be understood. The interviews were audio-recorded and these recordings were later transcribed for analysis.

3.3 Ethical considerations

It is important that research is conducted ethically. In particular, as my research involved people who were colleagues in the mathematics education community and their work products, I believed it to be particularly important to ensure that my interactions with them and subsequent reporting of the research involving them were ethical.

Stutchbury and Fox (2009) drawing on the work of Seedhouse and Flinders provide a framework which they describe as enabling one to "view each situation

from different philosophical perspectives and in doing so addresses issues about how to behave ethically, alongside methodological considerations, thus ensuring the integrity of the research" (p489). Their framework takes a broad view of ethical considerations. They describe the ethical dilemmas that face researchers as occurring both on a macro level (where considerations about gathering sufficient data to draw valid conclusions occur) and on the micro level (where considerations like the manner in which one interacts with interviewees is relevant). Their framework provides 24 issues to take into account. Each of these issues is accompanied by a possible question to help one in thinking about the issue. The issues are grouped according to four layers which have been developed out of the work of Seehouse and Flinders. They point out that not all issues raised in each of the layers will be relevant in every research project, but they suggest working systematically through the issues and revisiting the framework at various stages of the research to ensure a thorough consideration of the issues. Although I did work systematically through all questions and issues raised in their framework, I report, in summary, the ethical considerations that emerged from this process. I use the four layers they identify to structure this summary.

3.3.1 Layer 1: External/ecological

The key issue that I addressed here was in terms of what Stutchbury and Fox term the "codes of practice". Here I worked according to the code of ethics for research on human subjects as set out by the University of Witwatersrand and applied for and was granted ethics approval by the University of Witwatersand's Human Research Ethics Committee.

3.3.2 Layer 2: Consequential/utilitarian

An important ethical issue raised in this layer is the worthiness of the research project. Stutchbury and Fox suggest reflection on the benefits to individuals/organisations involved in the project as well as benefits to society more broadly. As discussed in chapter 1, the motivating factors for this research lay in a desire to contribute to an understanding of the interplay of forces that

impact on curriculum development and to understand the demands placed on teachers by the type of geometry that is constituted in the curriculum and textbooks. These research findings can be useful in informing further curriculum development, textbook creation and in thinking about the kind of preparation geometry teachers might need.

3.3.3 Layer 3: Deontological

As explained previously, incorporating interviews with people involved in the curriculum and textbooks in my research was necessary to help understand the process involved in the construction of these texts and the factors that played in the ORF and PRF. However the nature of the community of individuals involved in the curriculum construction process meant that it would not be possible to ensure the interviewees would not be identifiable to readers of the research. In addition as the analysis of the textbooks necessitated quoting extracts from the textbooks these, and hence their authors, would be easily identifiable. Thus, unlike in most education research, where research participants remain anonymous, I could not ensure this. Deontological considerations meant that I was honest with the interviewees about this and informed them (both verbally and in writing) that I could not assure them of anonymity. This was done prior to the start of the interview and interviewees were given the choice of whether or not they wished to proceed. I also disclosed fully my intended use of the interview data and explained that they would be used in my PhD dissertation and in papers prepared for education conferences or that might be published in education journals. The interviewees were also informed that they were free to withdraw from the project at any stage and that, in particular, they could to ask to see transcripts of their interview and request that part or all of the interview not be used in the research. This meant that the interviewees were fully and honestly informed about what I intended to do with the interviews and were aware that I could not assure anonymity. Although I recognized that I could not guarantee anonymity, I took the decision not to use the individuals' names in writing about the work. This was done because I felt that the positions held by

the interviewee (i.e. author of textbook or curriculum committee member etc.) were more relevant in the reporting of the research than the individuals' names.

3.3.4 Layer 4: Relational/individual

On the individual/relational level there were particular challenges. I recognized that it was likely that my analysis of the curriculum and of the textbooks might bring up concerns that could be perceived as criticisms either of the documents or of the capabilities of the individuals involved. I was fortunate that the individuals that I needed to interview are all very competent individuals with impressive credentials and work records. I personally have a great deal of respect for each of the individuals I interviewed and thus in my interactions with them there was no difficulty in conveying that respect. In my writing I've attempted to be very clear that I am not attempting to judge whether the curriculum and textbooks have "got it right" nor am I comparing the textbooks in order to conclude which is better, but that I am trying to understand how a particular confluence of political, social, mathematical and pedagogical forces shape the geometry that is made available. This framing of my writing was intended to guard against the possible perception of personal criticism. In addition, in interviewing the curriculum and textbook authors, I aimed to ensure that their voice – both in terms of their own explanations of what they were trying to do, and also in terms of their discussion of the constraints they worked under – would be represented in the research.

3.4 How the data analysis was done

3.4.1 Finding a guiding system for analysis

As described in section 3.1, Graven's four orientations to mathematics were used as a guiding system for the analysis. The first step in the analysis was to examine the MNCS in relation to each of the orientations. This involved a dialectical process of moving between the analysis of the MNCS and a review of relevant aspects indicated by the pedagogic device (e.g. research from the field of mathematics education). So, for example, the MNCS mentions indigenous

knowledge systems (South African Department of Education, 2003, p4) and this had links to work done in critical mathematics education which seemed to indicate a need to expand the ambit of orientation 1. And, for example, the MNCS talks of an exploration of alternative definitions (South African Department of Education, 2003, p32) which led me to explore more deeply the notion of mathematical definition and thus include this notion in orientation 3. As it is very difficult to reflect the dialectical process fully in a written report, what I have done for each orientation is the following:

I have provided a discussion of the orientation which draws on various relevant aspects from the field of production. Although the need to look at these was triggered in some cases by a previous iteration of analysis of the MNCS this is not reported for the sake of clarity of the report. The discussion then leads to an elaborated form of the orientation which is used to analyse how the orientation manifests in the MNCS and in the textbooks. Thus a language of description (P. Dowling, 1998) emerged through a constant movement between the theoretical and empirical fields.

In order to ensure the analysis of the curriculum and textbooks was thorough I had to develop methods of chunking the content of each of these so that I could summarise how the orientation appeared. As the MNCS and textbooks have very different forms and purposes the way in which I have done the chunking is different for each. The way in which I chose to break each of them up for analysis is described in sections 3.4.2 and 3.4.3 below.

3.4.2 How the MNCS was divided up for analysis

The indicators used to do the analysis in terms of each orientation are not discussed here as they are discussed orientation by orientation at the start of each chapter dealing with an orientation. This section simply sets up a description of the way in which information in the MNCS is presented and how I

structure and summarise my description of this in relation to the orientations to mathematics.

The format of the MNCS was prescribed so that it aligned with the format of the curriculum statements for all other school subjects. For the curriculum of each school subject chapter 1 and most of chapter 4 provide generic descriptions about the curriculum and the way it will be assessed that are common to the other subjects. The subject specific core is given in the learning outcomes and assessment standards. A learning outcome is defined as "a statement of the intended result of learning and teaching" (South African Department of Education, 2003, p7)[8] and assessment standards as "criteria that collectively describe what a learner should know and be able to demonstrate at a specific grade. They embody the knowledge, skills and values required to achieve the Learning Outcomes" (MNCS, p7). The chapters in the MNCS are as follows:

> **Chapter 1: Introducing the National Curriculum Statement**. This is a general overview of the NCS which is common to all subjects. It sets out the general principles of the NCS and does not refer to mathematics specifically.
> **Chapter 2: Mathematics.** This chapter deals specifically with Mathematics and outlines its definition, purpose, scope, educational and career links and gives a description of the learning outcomes.
> **Chapter 3: Learning outcomes, assessment stands content and contexts.** This chapter provides a detailed description of the assessment standards for each learning outcome. It also describes separately the content of each learning outcome. However these are largely a repeat of the assessment standards. A description of the contexts to be used is also provided.

[8] As I quote extensively from the MNCS, from this point on instead of using the formal reference (South African Department of Education, 2003, pX) I use (MNCS, pX). This is done in order that it is easily identifiable to the reader when I am quoting from the MNCS.

Chapter 4: Assessment. The first sections of this chapter are generic and are common to all subjects. It sets out the purpose of assessment and outlines different forms of assessment and a rating scale. The last section of the chapter outlines the competence descriptors for mathematics.

The format and layout of each of the chapters differs, and there was thus no standard way to chunk the chapters. In deciding on a way to chunk the chapters I had to bear this in mind, along with the purpose of the analysis. One of my aims in the analysis was to look at how strongly and in which chapters of the MNCS each orientation manifests, particularly in relation to the other orientations. In addition the analysis needed to elaborate and deepen the description and understanding of the orientations. The aspect relating to determining how strongly each orientation manifests was particularly difficult. The approach of some authors (Mhlolo, 2011; Parker, 2006) has been to use a quantitative representation. For example, Parker categorises each of the assessment standards according to orientation and compares the number of assessment standards in each category to draw conclusions about which orientation is dominant in the assessment standards. There is, however, a problem with this: the assessment standards are clearly not equally weighted. For example, compare the following assessment standards:

Assessment standard 11.1.1 "Understand that not all numbers are real (this requires the recognition, but not the study of non-real numbers)" (MNCS, p17).

Assessment standard 12.1.6 "Solve non-routine, unseen problems" (MNCS, p21).

Assessment standard 12.3.6 "Solve problems in two and three dimensions by constructing and interpreting geometric and trigonometric models" (MNCS, p37).

The nature of each of these assessment standards, the amount of class time required to address each and their likely weighting in assessments are very different. Thus any quantification relating to the orientations can only be read as saying that aspects of that orientation are mentioned more frequently. The quantification has to be read and understood in relation to a qualitative

judgment. What I have done is assess each chapter in terms of the way in which the chapter is set out. For each subsection of the chapter I indicate whether there are elements that are quantifiable (for example a bulleted list) and what further elements are included in the subsection. I describe this in detail below. In order to analyse how each of the orientations to mathematics manifests in the curriculum I look at each of the subsections identified in turn. I then report how many times the orientation is present in the quantifiable elements and provide a qualitative description of how the orientation manifests in that subsection. There are some elements of the subsection that are descriptive paragraphs and not conducive to quantitative description, these are therefore only analysed qualitatively.

Table 5 below provides a summary of the structure of the MNCS and the elements in it that are available to be enumerated and those that can only be discussed qualitatively. So, for example, table 5 indicates that chapter 1 of the MNCS contains five subsections entitled: *Introduction, Principles, Critical and developmental outcomes, What kind of learner is envisaged* and *What is a subject?* The subsection entitled *Introduction* consists of a paragraph only and thus will only be analysed qualitatively. However the subsection entitled *Principles* consists of a list of 9 principles and thus I will be able to enumerate how many of the 9 principles relate to each orientation. In chapter 2 we see that the subsection entitled *Purpose* will be analysed by describing the number of bullet points relating to each orientation quantitatively and by discussing the paragraph that appears in addition to those bullet points qualitatively.

Immediately after the summary table each of the chapters and subsections of the chapters is described briefly to clarify the structure of the MNCS.

Title of subsection of chapter	**Elements that can be enumerated in the analysis as well as discussed qualitatively**	**Elements that will only be analysed qualitatively**

Chapter 1: Introducing the National Curriculum Statement		
Introduction		paragraph
Principles	9 principles	
Critical and developmental outcomes	12 outcomes	
What kind of learner is envisaged?		paragraphs
What is a subject?		paragraphs
Chapter 2: Mathematics		
Definition	6 sentences	
Purpose	6 bullet points	paragraph
Scope	12 bullet points	
Educational and career links		paragraphs
Number and Number Relationships learning outcome (LO)	6 bullet points	paragraph
Functions and Algebra LO	4 bullet points	paragraph
Space, Shape and Measurement LO	9 bullet points	paragraph
Data Handling and Probability LO		paragraphs
Chapter 3: Learning outcomes, assessment standards, content and contexts		
Space, Shape and Measurement LO: Assessment standards grade 10	7 assessment standards	

Space, Shape and Measurement LO: Assessment standards grade 11	7 assessment standards	
Space, Shape and Measurement LO: Assessment standards grade 12	6 assessment standards	
Content for Number and number relations LO		paragraph
Content for Functions and Algebra LO		paragraph
Content for Space, Shape and Measurement LO		paragraph
Content for Data Handling and Probability LO		paragraph
Contexts: inclusivity, human rights and indigenous knowledge systems		paragraphs
Chapter 4: Assessment		
Competence descriptions grade 10	26 competence descriptions	
Competence descriptions grade 11	30 competence descriptions	
Competence descriptions grade 12	32 competence descriptions	

Table 5: MNCS structure summary

3.4.2.1 Description of each chapter and how it will be analysed

Table 5 provides an overview of the elements of each subsection which were analysed. This section complements this overview by providing a brief description of what each of the subsections of the curriculum is about so that the reader has an idea of the overall content and format of the curriculum.

Chapter 1: Introducing the National Curriculum Statement

This chapter was written generically for all subjects and so is not specific to mathematics. It sets out the principles of the National Curriculum and describes the structure of the document and defines particular terms used in the document. There were thus many parts of this chapter that did not relate specifically to the orientations to mathematics and were not included in the analysis. However the sections of the chapter discussed below, although not mathematics specific, do provide insight into the nature of knowledge, teaching and learning espoused in the curriculum and thus were analysed. These were:

> *Introduction:* An introductory paragraph that sets out the relationship between the curriculum and the Constitution of the Republic of South Africa.
> *Principles:* The 9 principles on which the curriculum is based.
> *Critical and developmental outcomes:* A brief discussion of outcomes-based education, the philosophy on which the curriculum was based and the seven Critical Outcomes and five Developmental Outcomes that learners are meant to attain are listed.
> *What kind of learner is envisaged?* The discussion of the type of learner that is envisaged.
> *What is a subject?* This section describes a particular view of knowledge and the structuring of knowledge domains.

Chapter 2: Mathematics

As chapter 2 of the MNCS deals specifically with Mathematics, it was important to analyse all subsections within it in relation to the orientations to mathematics. Each of these subsections is listed below:

> *Definition:* This paragraph sets out the view of the nature of the discipline of Mathematics on which the curriculum is based.
> *Purpose:* This sets out the purpose of the study of Mathematics and what it enables learners to do.

Scope: This is a set of 12 bullet points that state what learners studying mathematics will work towards being able to do.
Educational and career links: This explains the articulation of FET Mathematics with GET Mathematics and with future study or work.
Learning outcomes: The learning outcomes designate particular content areas (Learning outcome 1 is Number and Number Relationships, Learning outcome 2 is Functions and Algebra, Learning outcome 3 is Space, Shape and Measurement and Learning outcome 4 is Data Handling and Probability .) Although learning outcome 3 contains the geometric work and is thus the learning outcome relevant to the study, I analysed the orientations in all learning outcomes here to get a sense of the similarities and differences between how the orientations are reflected in each of the learning outcomes.

Chapter 3: Learning outcomes, assessment standards, content and contexts

This chapter contains the detail of the mathematical content and skills to be covered and thus all subsections of it were analysed. These are discussed briefly below

Assessment standards: In the first section the learning outcomes and assessment standards are specified. These give the detail of what is to be taught. There is a large degree of overlap between this section and the last section of chapter 2 (*Learning outcomes*). For this reason I only analyse the assessment standards for the Space, Shape and Measurement learning outcome as this contains the content related to geometry. The assessment standard details specific mathematical skills e.g. the first assessment standard for grade 10 learners listed under the Space, Shape and Measurement learning outcomes is that the learner is able to "Understand and determine the effect on the volume and surface area of right prisms and cylinders, of multiplying any dimension by a factor k" (MNCS, p32).

Content for the attainment of the assessment standards: This section is largely a repeat of the assessment standards. The only addition is a paragraph that describes the learning outcome. For this reason the descriptive paragraphs for each learning outcome will be analysed, but the bullet points that repeat the assessment standards will not be analysed for a second time.

Contexts: inclusivity, human rights and indigenous knowledge systems: This consists of narrative text which describes the contexts in which the content should be realised. These will be analysed and discussed in relation to the orientations.

Chapter 4: Assessment

Large parts of this chapter are generic to all subjects and describe why assessment is necessary, different types of assessment and how to collect and record information about assessment. These do not relate to the orientations to mathematics and thus were not analysed. However the last section of the chapter provides competence descriptors for mathematics. The competence descriptors describe the various competences learners are meant to attain by the end of each of the grades (grades 10, 11 and 12). There are 6 levels of competence described for each grade ranging from those for learners with outstanding achievement (level 6) to those whose achievement is inadequate (level 1). So, for example, one of the competence descriptors, for grade 10, states that by the end of grade 10 a learner with outstanding achievement will be able to "produce clear, logical, geometric and algebraic solutions of simple (though not necessarily routine) problems" (MNCS, p72).

Whilst the chunking of the MNCS for analysis is not "neat" the seeming inconsistencies were necessitated by the nature of the document itself. The structure and design of the different chapters and subsections of the chapters are very different, thus in order to provide a coherent and comprehensive analysis of the MNCS I needed to relate the chunking to the chapters and subsections as they were written and to cope with the "messiness" that resulted.

3.4.3 How the textbooks were divided up for analysis

The geometry chapters of both textbooks were easily identifiable. What required work was finding a way to consistently, coherently and comprehensively chunk the textbook chapters for analysis. Two principles guided the chunking of the textbook:

The first was a need to chunk the textbook in a way that I would be able to distinguish between sections of the textbook that were core to the main thread of development of the mathematical ideas in the textbook and those that were not.

The second was to distinguish between sections where information was provided versus sections where learners were expected to do work. This would allow me to identify whether aspects of an orientation were considered an important part of the work the learner should engage with or whether they were mentioned, but seen as not central or beyond the scope of what the learner would be expected to do.

The methodology of textbook analysis that seemed best able to capture this was that which was used for TIMSS study which compared textbooks from a number of different countries (Valverde et al., 2002). As the TIMSS study was aimed at making comparisons between textbooks on a number of different dimensions, their framework includes a categorisation scheme for mathematical content and for performance expectations. However I have not used these aspects of the TIMSS framework as they do not directly relate to my study. The part of the TIMSS methodology I have drawn on and adapted for my study is the way in which they segmented the textbook into blocks and the way in which they categorised the blocks in terms of their function. This is described further below.

The TIMSS methodology defines a lesson as "a segment of textbook material devoted to a single main science or mathematics topic and intended to

correspond to a teacher's classroom lesson on that topic taught over one to three instructional sessions" (Valverde et al., 2002, p26). They then further subdivide the lessons into blocks. They do not define exactly what constitutes a block but simply state that blocks are either narrative elements which "tell stories, state facts and principles through narration" (p141), activity elements which "prescribe a set of actions that students are intended to perform outside of the world of the textbook" (p142), question sets or worked examples. Although the TIMSS study does not define how they identified a block, I chose to use their notion of block and I chose to identify when one block stopped and another started when there was a change of function. For example, when an explanation of a concept finished and an exercise set started, one could see one block end and another block start.

I have not used the TIMSS notion of lesson. My interest is in how the textbooks have been constructed and how the textbook itself is divided into sections. I did not need a standard unit (the lesson) to use as a point of comparison. For this reason the unit of analysis in my study is a block of text. Within the TIMSS study various block types are identified according to the function of the block. I have adapted these to suit my study and the block types I have used, together with a description of each, are:

> **Chapter opener**: A block which acts as an introduction to the chapter as a whole.
> **Orienting narrative:** A narrative that explains what will be done in a section/chapter.
> **Instructional narrative:** An exposition of a piece of mathematics or statement of facts that can drawn out from an activity.
> **Summary narrative:** Material that sets out a reminder of facts learnt in previous grades or previous chapters.
> **Extra information narrative:** Information related to the topic of the textbook, but which is not part of the main narrative of the chapter. This could be a historical note or some information about how this area of mathematics has been applied, for example.

Worked example: Material that shows how to execute a particular algorithm through use of an example.

Exercise set: A set of questions designed to allow learners to practice or acquire particular skills.

Activity: Material that requires learners to actively engage with a task that allows them to uncover or deduce mathematical facts or concepts or raises further mathematical questions.

Learning reflection: A question or set of questions that asks the learners to reflect on what they have learnt or how they feel they are progressing.

Puzzle: Material that requires learners to engage with a task, but which is designated as being "for interest", "a challenge" or specifically as a puzzle.

Project: A substantive task that requires learners to do some research and/or mathematical work and to present that work in the form of an essay or presentation.

Although these categories provide a reasonable level of detail about the different kinds of blocks in the textbooks, I also wanted a broader level of characterisation of the block types into those blocks where the main purpose was to provide information to learners and those blocks where the main purpose was to provide mathematical tasks for the learners to engage in. I categorised chapter openers, orienting narratives, instructional narratives, summary narratives, extra information narratives, and worked examples as information type blocks. I categorised exercise sets, activities, learning reflections, puzzles and projects as task type blocks.

Information type blocks	**Task type blocks**
Chapter opener	Exercise set
Orienting narrative	Activity
Instructional narrative	Learning reflection
Summary narrative	Puzzle

Extra information narrative Worked example	Project

Table 6: Summary of block types

This designation of categories allowed me to create a table that summarised the geometry chapters of each textbook. The table had four columns with the following headings:

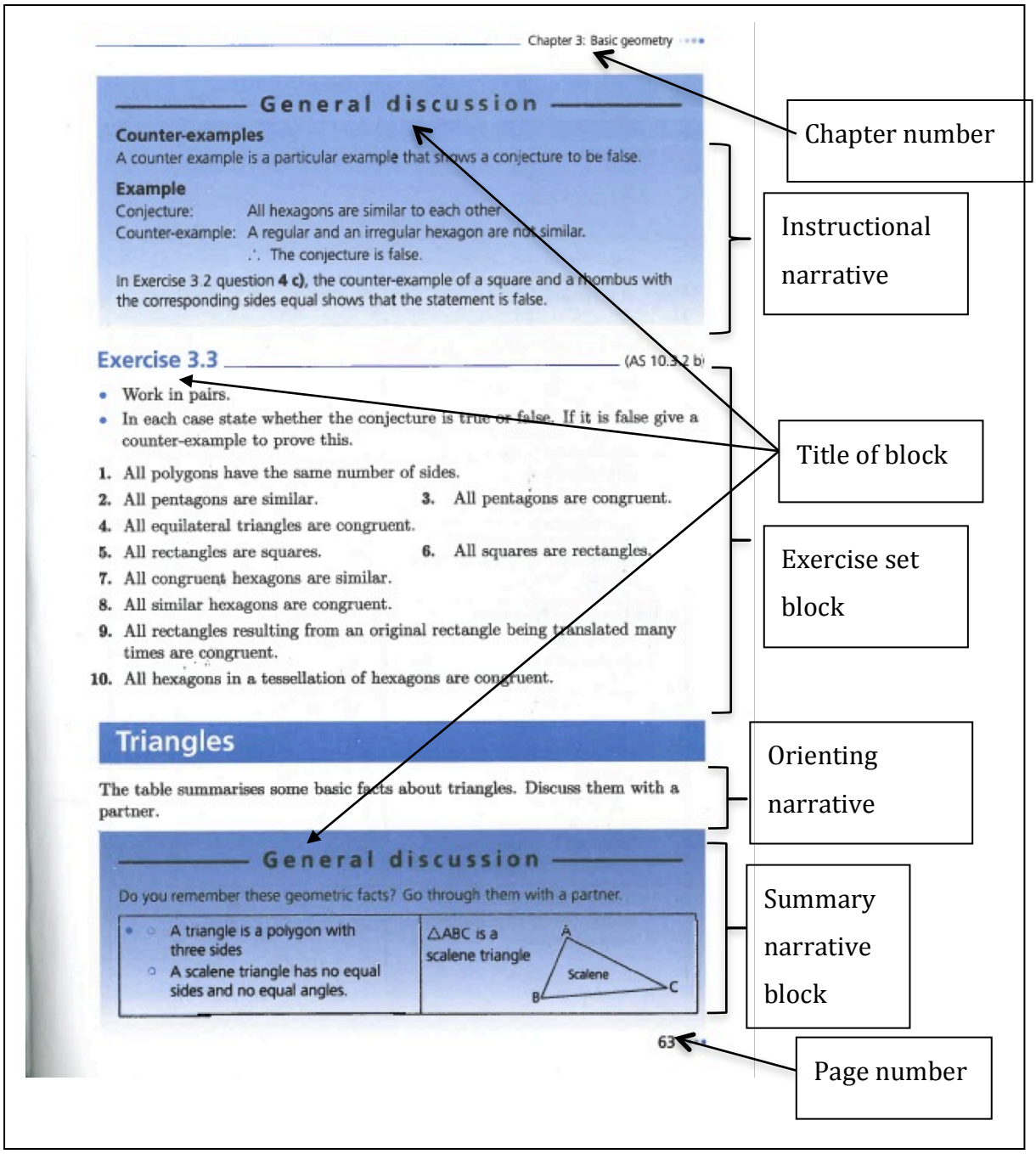
chapter: to record the chapter number in which the block appeared.

page: to record the page number on which the block started. This was simply to make it easier to identify the particular block referred to.

type: to record which of the categories of block type (as described above) the block fell into.

title: the title (if any) that the textbook provided for the block. This was recorded to make it easier to identify the particular block referred to.

In the analysis of each orientation each block was examined according to specific criteria that arose out of the elaboration of the orientation that was created in the analysis of the MNCS. These criteria will be discussed in detail in chapters 4, 5 and 7.



Textbook extract 2: Illustration of the elements of the textbook recorded in analysis (CM p63)

3.4.4 How the interviews were analysed

As discussed, the interviews were to provide insight into the themes that emerged from the analysis of the curriculum documents and textbooks. As such a content analysis was done on the interviews after the analysis of the curriculum

and textbooks. The analysis of the interviews was structured by the themes that emerged from the curriculum and textbook analysis.

3.4.5 Validity

Vailidity is largely associated with positivist research (Bush, 2007) and its use in qualitative research has been a source of considerable debate (Maxwell, 1992). This has meant some researchers question its applicability in qualitative research and propose using constructs like trustworthiness as an alternative (Lincoln & Guba, 1985). However I have found Maxwell's (1992) realist approach to validity, following many others, useful in thinking about my research. Maxwell argues that his account of validity is strongly tied to a notion of understanding and that his "typology of validity categories is also a typology of the kinds of understanding at which qualitative research aims" (p281). He distinguishes five types of validity: descriptive validity, interpretative validity, theoretical validity, generalizability and evaluative validity. I will discuss the first four of these briefly in relation to my research. I have chosen to omit evaluative validity because, as Maxwell points out, it is not central to qualitative research and my study did not deal with it explicitly.

Descriptive validity refers to whether the account is factually accurate. To ensure descriptive validity I audiorecorded all interviews and had them transcribed by a professional transcriber.

Interpretive validity relates to understanding or meaning given to ideas by the participants in the research. I have, where possible, used the interviewees' own words to convey their meaning.

Theoretical validity "refers to an account's function as an *explanation"* (ibid., p291) and is concerned with "the legitimacy of the application of a given concept or theory to established facts" (ibid, p292). In this study I worked in such a way that I moved constantly between the theoretical and empirical fields. In so doing the development of the language of description was constantly tested against the empirical evidence.

Generalizability in my study refers to the extent to which the findings could be extended to other curricula or other textbooks. I purposefully chose to study a curriculum and textbooks that were created at a time of intense political and social change, recognizing that this might make generalizability difficult. However I believed that the confluence of strong social, political and educational forces that were brought to bear on the curriculum and textbooks at that time would illuminate aspects that might be useful in understanding curriculum change and its effect on mathematics more broadly. Thus, following Maxell's argument, the generalizability in this study lies in the contribution to the development of understanding in terms of theory that can help us better explain the interaction and possible impact of various forces in the development of mathematics curricula and textbooks. I thus follow Adler (1996, p117) who suggests that generalizability is "better viewed as in the relationship between cases and theoretical assertions than from samples to populations." My goal was thus to develop a language of description that Adler (ibid.), drawing on Bernstein (1993), describes as generative of further research.

3.4.6 Limitations

As indicated and discussed earlier, only two textbooks were studied. I chose to sacrifice breadth in order to allow for greater depth of engagement with each textbook, but I recognise that this places limitations on my study.

A further clear limitation of this study is that I did not collect data from the field of reproduction and thus cannot make claims about how teachers interpret the curriculum and textbooks nor about how these are recontextualised into classrooms. The omission of classroom data from the research project was deliberate as I wanted the focus to be on the forces at play in the field of recontextualisation and to probe these in depth. However I was aware that in making this choice I would not be able to make any claims about what transpires in classrooms.

3.4.7 How the analysis chapters are structured

Chapters 4, 5, 6 and 7 all deal with the analysis of the MNCS and textbook chapters.

The original intention was that each chapter would deal with each of the orientations in turn and would take the following format:

1	Description of the orientation as originally presented by Graven.
2	Discussion of the orientation in relation to key literature and interaction with the MNCS and textbooks.
3	A revised description of the orientation in light of the discussion. This revised description forms the basis for the analysis of the MNCS and textbooks in terms of the orientation.
4	An analysis of how the orientation manifests in the MNCS detailed in terms of the chunking of the MNCS as described in section 3.4.2 above
5	An analysis of the orientation manifests in the textbooks done through analyzing each block.
6	A discussion of the themes that emerge from textbook and MNCS analysis in terms of the interviews with agents of the ORF and PRF.

Table 7: Intended format of the analysis chapters

This format worked well for orientation 1 and orientation 2 and thus chapters 4 and 5 present the analysis of these orientations in the format described in table 7.

However orientation 3 and 4, which deal with mathematical practices and mathematical content, proved to be difficult to separate. I thus chose to discuss and analyse these together. Chapter 6 thus deals with orientation 3 and 4 together and does so using steps 1 – 4 in table 7 above. However my attempt to analyse orientations 3 and 4 in the textbooks directly did not prove useful. Thus, as I describe in chapters 6 and 7, I chose to get at the manifestation of mathematical practice and mathematical content (orientations 3 and 4) by

examining in depth the way in which the geometric paradigms (as introduced by Houdement and Kuzniak (2003)) played out in the textbooks. This analysis is presented in chapter 7.

4 Analysis of orientation 1

Orientation 1 as presented in Graven (2002): *Mathematics is important for critical democratic citizenship. It empowers learners to critique mathematical applications in various social, political and economic contexts. Mathematics is part of broader society and is important for all learners.*

4.1 Discussion of orientation 1

The curriculum of which the MNCS was a part had its roots in the National Education Policy Investigation (the ANC Policy for Education and Training) and work done in the immediate post-apartheid era in South Africa and thus is underpinned by the challenges of national reconstruction and the building of an equitable and just society (Cross et al., 2002). Chisholm (2005, p84) states that "the dominant discourse with the South African educational state has been one of rights, development, social justice and nation-building." This discourse is certainly reflected within the MNCS. This discourse within the MNCS also has roots in the thinking of people and organisations who worked in opposition to apartheid mathematics education. In particular, organisations such as the National Education Coordinating Committee Mathematics Commission promoted the notion of People's Mathematics, which was a response to the context in South Africa at the time of apartheid but had much in common with the varieties of Critical Mathematics Education (Julie, 2004). Skovmose (1985) defines critical education as having three key elements: an assumption of critical competence on the part of teacher and student, a critical distance to content, and a critical engagement with social problems. Aspects of mathematics reform that emerge as having their roots in critical education include a concern with equity (Gutstein et al., 2005), ensuring previously disempowered students have access to mathematics (Gutstein, 2003) and ensuring all students are able to find themselves and their cultural heritage in mathematics (Herzig, 2005). These link to the notion of critical competence put forward by Skovmose. Skovmose's "critical distance to content" is reflected in a questioning of the nature of

mathematics as irrefutable, ready-made truth. The work of people such as Frankenstein (1993), Gutstein (2003) and Brantlinger (2011) in developing mathematics courses based around key problems in the social and political arena that influence the students' lives or reflect inequalities in terms of race, gender and class in society reflect the concern with a critical engagement with social problems.

Thus a reading of this literature together with a look at what appears in the MNCS indicates the following three core aspects of this orientation:

1. Mathematics is important and thus all learners must have access to mathematics and see themselves as being included in mathematics and its history.

2. Mathematics can be used to question and challenge social and economic injustices.

3. Mathematics is a human creation and hence mathematics and the way it is used is open to question and critique.

In what follows I will discuss each aspect in order to describe how it can be recognised:

1. Mathematics is important and thus all learners must have access to mathematics and see themselves as being included in mathematics and its history.

During the apartheid era many learners were denied access to mathematics (Vithal & Volmink, 2005) and lack of access to mathematics prevented learners from pursuing certain careers (Bopape, 1998) and from developing crucial skills required to be a self-managing individual. The denial of learning opportunities has meant that a strong push for some form of *mathematics for all* has been present in SA curriculum discourse. At the FET level the original intention was to require all learners to take a course in Mathematical Literacy which would equip them with the numerical and spatial skills required to analyse everyday situations and solve problems. Learners who intended to continue to careers

that require further study of mathematics would have studied Mathematics in addition to Mathematical Literacy at school. However practical constraints (timetabling and staffing) militated against this and the decision was made to require all learners to take some form of mathematical study at the FET level, but to offer them the option of choosing either Mathematical Literacy or Mathematics. This has meant that in terms of "access to mathematics", the school subject Mathematics at FET level has had to incorporate access to both the mathematics required for further study in higher education institutions and the kinds of mathematics required in everyday situations.

Although apartheid policy in education and the consequent under-resourcing of black schools posed a clear barrier to many learners being able to successfully study mathematics, the literature around access to mathematics has pointed to other possible reasons that certain groups of learners have not felt part of mathematics. Amongst the barriers discussed in the literature are a Eurocentric bias in the history of mathematics and type of mathematics presented to learners (Joseph, 1987); the use of contexts that are alien or less familiar to particular groups of learners (Sullivan, Zevenberg, & Mousley, 2003) and the negative stereotyping of particular groups or the absence of people of different races and genders from the texts used in classrooms (Clarkson, 1993; Sleeter & Grant, 2011). The remediation suggested to counter these barriers include a focus on ensuring that all learners can "find themselves" portrayed in a positive light in the mathematics texts used, the use of relevant and familiar contexts, a purposeful inclusion of non-European roots of mathematics (Bishop, 1988) and the inclusion of ethnomathematics (Gerdes, 1985). Based on the above, evidence of the aspect "access to mathematics" will be sought in the following ways:

a) Explicit mention of the importance of opening access to mathematics to all learners

e.g. A statement such as the following from the MNCS: "Mathematics will become a 'pump' and not a 'filter' for the learner" (MNCS, p62).

b) Discussion of barriers to learning. These could include those suggested above.

e.g. In the MNCS a statement such as the following one discussing the need to not be Eurocentric in describing Mathematics would be an example of this: "Ethnomathematics also stresses that Mathematics originated in cultures other than the Greek and that it continued to be developed in sophistication by many societies other than the European" (MNCS, p62).

c) Discussion of strategies to ensure inclusivity. These could include the types of remediation suggested above.

e.g. in the MNCS a statement such as "Another aspect of providing access and affirmation for learners of Mathematics is to look at examples of Mathematics in the variety of cultures societal practices in our country" (MNCS, p62).

2. Mathematics can be used to question and challenge social and economic injustices.

Mathematics can be used to explore a variety of different social and everyday situations. The aspect that is important in an orientation to mathematics for critical democratic citizenship is "the opportunity for an overt reflection on those issues that relate to inequality and discrimination on the basis of race, sex, social class and economic developmental level of countries" (Julie, 2004, p35). Clearly in this aspect we will be looking for evidence of mathematical modelling or applications of mathematics. In addition the contexts used will be real and relate to relevant social issues (de Lange, 1996) and some form of reflection on the issues raised will be expected (Julie, 2004).

e.g. in the MNCS it states that Mathematics will enable learners to "engage responsibly with quantitative arguments relating to local, national and global issues" (MNCS, p10).

3. Mathematics is a human creation and hence mathematics and the way it is used is open to question and critique.

This aspect is apparent on what one could term both a macro and micro level. On the macro level this relates to the nature of mathematics itself. This is a

reflection of the trend within the philosophy of mathematics away from absolutist and Platonist visions of mathematics (P. J. Davis & Hersh, 1980; Ernest, 1991; Lerman, 1990; Tymoczko, 1985) towards notions of mathematics as a product of historical and social construction. On a macro level this will be seen in accounts of history of mathematics or reflections on the nature of mathematics that show that it is contestable.

On the micro level is the issue of the way mathematics is used in modelling. Although this links strongly with the aspect "mathematics can be used to question and challenge social and economic injustices" discussed above, the emphasis here is on questioning the way mathematics is used and examining the process of mathematical modelling that occurs. Frankenstein (1993, p274) argues that the type of mathematic applications focused on in schools "impart an image of neutrality and naturalness to particular societal arrangements that obscure the class structure of our society." Julie (2004) argues that in the process of constructing a mathematical model one inevitably has to make certain assumptions or strip away certain complexities of the situation in order to enable the mathematisation. The making of these choices are rarely free of ideological considerations.

Evidence for this aspect will therefore be sought in statements that recognise mathematics and the way it is used as not being value-free and as needing to be brought into the open and probed.

e.g. in the definition of Mathematics in the MNCS it states "Mathematics is developed and contested over time through both language and symbols and by social interaction and is thus open to change"(MNCS, p9). This would be an example of something that belongs in this category.

Thus orientation one can be summarised as follows:

1. Mathematics is important and thus all learners must have access to mathematics and see themselves as being included in mathematics and its history.

2. Mathematics can be used to question and challenge social and economic injustices.

3. Mathematics is a human creation and hence mathematics and the way it is used is open to question and critique.

4.2 Orientation 1 in the MNCS

In this section I provide a discussion of how the orientation manifests in each of the subsections of the MNCS. For each chapter I first provide a summary of how the orientation manifests in each section of that chapter. In order for the table to give a quick visual indication of the prevalence of the orientation in the chapter I have shaded the row corresponding to each subsection according to how prevalent the orientation was in that subsection. The darker the shading, the more prevalent the subsection. The way in which I have judged prevalence is indicated below:

shading	**meaning**
	Orientation appears in 50% or more of the elements that were enumerated and/or was strongly present in the qualitative descriptions.
	Orientation appears in 20% – 50% of the elements that were enumerated and/or was significantly present in the qualitative descriptions.
	Orientation appears in the elements that were enumerated but in less than 20% of them and/or element is mentioned in the qualitative descriptions.
	Orientation does not appear.

Table 8: How the tables are shaded to represent the strength of the orientation

Immediately after the summary table for each chapter I provide a more detailed discussion of the way in which the orientations manifests in the chapter in order to elaborate on and justify the summary judgment of the prevalence of the orientation provided in the table.

4.2.1 Orientation 1 in chapter 1 of the MNCS

Chapter 1: Introducing the National Curriculum Statement		
Title of subsection of chapter	**Enumeration of elements (this is left blank if there were no elements that could be enumerated)**	**Discussion of elements that have been analysed qualitatively and a brief qualitative description of the elements that have been enumerated**
Principles	5 out of 9 principles	Very strong focus on opening access to education to all. Notions of the need to use knowledge critically and critique of traditional knowledge structures also included.
Critical and developmental outcomes	6 out of 12 outcomes	Focus on using knowledge critically and with social responsibility and sensitivity.
What kind of learner is envisaged		Focus on learner acting in interest of a society based on democracy and social justice.
What is a subject		Focus on blurring subject boundaries and incorporating values and non-Western

		contributions to knowledge.

Table 9: Orientation1 in chapter 1 of the MNCS

This chapter does not deal with mathematics specifically but explains the overall principles and expectations of the NCS for the FET band. The notion of learning in the FET for critical democratic citizenship is a strong thread that runs throughout this chapter. All three of the core aspects (access, questioning social and economic injustices, and questioning the discipline) of this orientation receive attention in this chapter. This chapter is not mathematics-specific, these aspects are thus seen in more general terms (e.g. access to education rather than access to mathematics).

The aspect relating to access to learning is a clear theme in this chapter. The curriculum and education is described as having an important role to play in transforming South Africa and focus is placed on redressing the educational imbalances of the past. Social transformation is one of the principles of the curriculum and is aimed at ensuring that "equal educational opportunities are provided for all sections of our population" (MNCS, p2). There is mention of the importance of sensitivity to issues of diversity and stress on inclusivity including a section on the importance of valuing indigenous knowledge systems.

The aspect *education can be used to question and challenge social and economic injustices* is also prominent throughout this chapter. Six of the twelve Critical and Developmental Outcomes allude to the idea of education for critical thinking and responsible citizenship. In addition "human rights, inclusivity, environmental and social justice" (MNCS, p1) is listed as one of the principles of the National Curriculum Statement. The type of learner that is envisaged is described as one "who will be imbued with the values and act in the interests of a society based on respect for democracy, equality, human dignity and social justice as promoted in the Constitution" (MNCS, p5).

The aspect *Mathematics is a human creation and hence mathematics and the way it is used is open to question and critique* is more difficult to recognise in a chapter

which is not subject-specific. However in chapter 1 there is a definition of what a subject[9] is:

> Historically, a subject has been defined as a specific body of academic knowledge. This understanding of a subject laid emphasis on knowledge at the expense of skills, values and attitudes. Subjects were viewed as static and unchanging, with rigid boundaries. Very often, subjects mainly emphasised Western contributions to knowledge (MNCS, p6).

The authors suggest that in the NCS this will not be the case and that subject boundaries will not be fixed: theory, skills and values will be integrated and subjects will be viewed as changing and as responding to different sources of new knowledge. Related to this notion that the traditional views of knowledge should be challenged are statements used to explain the principle "Valuing indigenous knowledge systems" (MNCS, p1) that is discussed in chapter 1. These statements argue that prior to the advent of the theory of multiple-intelligences "the Western world had only valued logical, mathematical and specific linguistic abilities and rated people as 'intelligent' only if they were adept in these ways. Now people recognised the wide diversity of knowledge systems" (MNCS, p4). They state that in the South African context, indigenous knowledge systems refer to "a body of knowledge embedded in African philosophical thinking and social practices" (MNCS, p4) and that the National Curriculum Statement has infused indigenous knowledge systems into the Subject Statements.

Thus chapter 1 contextualises the MNCS as part of a National Curriculum Statement that highly values all aspects of orientation 1, Critical Democratic Citizenship.

4.2.2 Orientation 1 in chapter 2 of the MNCS

[9] In the terminology of the NCS, Mathematics is a subject.

Chapter 2: Mathematics		
Title of subsection of chapter	**Enumeration of elements (this is left blank if there were no elements that could be enumerated)**	**Discussion of elements that have been analysed qualitatively and a brief qualitative description of the elements that have been enumerated**
Definition	2 out of 6 sentences	Contains aspects relating to mathematics as a human creation and contested.
Purpose	4 out of 6 bullet points	Focus on using mathematics critically and to address social issues
Scope	3 of 12 bullet points	Bullet points contain aspects of using mathematics to critically interrogate issues and mathematics as a human creation. Paragraph gives contexts in which maths should be embedded that are socially relevant and inclusive.
Educational and career links		No aspects reflecting orientation 1.
Learning outcome 1: Number and Number Relationships	0 out of 6 bullet points	No aspects reflecting orientation 1.
Learning outcome 2: Functions and Algebra	0 out of 4 bullet points	No aspects reflecting orientation 1

Learning outcome 3: Space, Shape and Measurement	2 out of 9 bullet points	One bullet points asks for investigation of the contested nature of geometry and the other for analysing cultural products as representations of space and shape.
Learning outcome 4: Data Handling and Probability		Mention is made of being critically aware of the abuse of data. The last paragraph proposes the use of contexts that are socially relevant and inclusive.

Table 10: Orientation 1 in chapter 2 of the MNCS

Below I elaborate on each of the subsections.

Definition. Although the definition of mathematics emphasizes its applicability to the physical and social world, no mention is made of specifically using it to transform social relations. The aspect of critical democratic citizenship that emerges in the definition is that of mathematics as a human creation, practised by all cultures, that is contested and open to change. It is spoken of as a "human activity practised by all cultures" and as "developed and contested over time" (MNCS, p9),

Purpose. In the paragraphs describing the purpose of mathematics we again see a situation where the applicability of mathematics and its ability to allow learners to "make sense of society" (MNCS, p9) are emphasized, but no notion of questioning or challenging is particularly mentioned. This stands in contrast to the bullet points in this section about what mathematics enables learners to do. In four of the six bullets the notion of critical democratic citizenship is alluded to. These are:

"identify, pose and solve problems creatively and critically", work mathematically with "sensitivity to personal and broader societal concerns", work with quantitative data to "evaluate and critique conclusions" and "engage

responsibly with quantitative arguments relating to local, national and global issues" (MNCS, p9-10).

These bullet points are essentially some of the critical and developmental outcomes rewritten in mathematical terms.

Scope. In two of the 12 bullet points suggesting what learners will work towards being able to do there is mention made of learners using mathematics to challenge social issues. These are "use Mathematics to critically investigate and monitor the financial aspects of personal and community life and political decisions" and "collect and use data to establish basic statistical and probability models, solve related problems, and critically consider representations provided or conclusions reached" (MNCS, p10). It is interesting to note here that this aspect of critical democratic citizenship is mentioned only in relation to finances and statistics and probability.

One further bullet point states "investigate historical aspects of the development and use of Mathematics in various cultures". This could be seen as part of the aspect of opening access to mathematics by countering the Eurocentric bias in the way mathematics is viewed.

At the end of the bullet points there is a paragraph which reads

> Such mathematical skills and process abilities will, where possible, be embedded in contexts that relate to HIV/AIDS, human rights, indigenous knowledge systems, and political, economic, environmental and inclusivity issues (MNCS, p11).

Educational and career links. Nothing in this section deals specifically with orientation 1.

Description of the learning outcomes. Orientation 1 is not present in either the Number and Number Relationships LO nor in the Functions and Algebra LO. In the Data Handling and Probability LO there is specific mention of being able to "become critically aware of the deliberate abuse in the way data can be

represented to support a particular viewpoint" (MNCS, p14) and a suggestion to embed the data handling material in contexts related to human rights, inclusivity, environmental and health issues. Thus the Data Handling and Probability LO foregrounds the critical democratic citizenship part of orientation 1. In the Space, Shape and Measurement LO, 2 of the 9 bullet points this could be seen to relate to this orientation. These are "analyse natural forms, cultural products and processes as representations of shape and space" and "investigate the contested nature of geometry throughout history and develop an awareness of other geometries" (MNCS, p14). In the first of these the mention of "cultural products" could be seen as part of the aspect of ensuring inclusivity. However this is not specifically stated as such. The second one is clearly about mathematics as human creation and open to change.

In summary we see that orientation 1 is present in chapter 2 of the MNCS. However the presence of orientation 1 in chapter 2 is more prevalent in the description of the definition, purpose and scope of mathematics than it is in the subsections in which the specific content is dealt with.

4.2.3 Orientation 1 in chapter 3 of the MNCS

Chapter 3: Learning outcomes, assessment standards, contents and contexts		
Title of subsection of chapter	**Enumeration of elements (this is left blank if there were no elements that could be enumerated)**	**Discussion of elements that have been analysed qualitatively and a brief qualitative description of the elements that have been enumerated**
Space, Shape and Measurement LO: Assessment standards grade 10	1 of 7 assessment standards	One assessment standard on the history of geometry and trigonometry as developed by various cultures
Space, Shape and	1 of 7 assessment	One assessment standard on the

Measurement LO: Assessment standards grade 11	standards	history of geometry and trigonometry as developed by various cultures
Space, Shape and Measurement LO: Assessment standards grade 12	1 of 6 assessment standards	One assessment standard about looking at other geometries (this was flagged in the previous section as being useful for highlighting the contested nature of geometry).
Content for Number and Number Relationships LO		No aspects reflecting orientation 1.
Content for Functions and Algebra LO		After discussing content states that the contexts should be ones that involve debates on attitudes and values.
Content for Space, Shape and Measurement LO		No aspects reflecting orientation 1.
Content for Data Handling and Probability LO		States that the content will only be meaningful for the learner if it is used to address issues of importance to the learner and society.
Contexts: inclusivity, human rights and indigenous knowledge systems		Deals with mathematics as a human creation and affirming learners by showing mathematics embedded in their cultural products. A strong focus on the

		importance of mathematics for all and opening mathematics to learners previously excluded from it.

Table 11: Orientation 1 in chapter 3 of the MNCS

Assessment standards for Space, Shape and Measurement LO. In six out of seven of the assessment standards for each grade no aspects of orientation 1 are reflected. In grades 10 and 11 the 7th assessment standard asks learners to do a project that enables them to "demonstrate an appreciation of the contributions to the history of the development and use of geometry and trigonometry by various cultures". Here the "by various cultures" could be seen as part of the aspect of ensuring inclusivity and the history of the development could be used to open mathematics up as a human creation and open to change, however this is not explicitly stated. In grade 12 the 7th assessment standard is even less explicitly related to aspects of orientation 1 in that it asks for "a basic understanding of the development and uses of geometry though history and some familiarity with other geometries" (MNCS, p37).

Content and contexts for the attainment of assessment standard. In the paragraphs describing the content for each LO, orientation 1 is not mentioned in the description of the Number and Number Relationships LO nor in the Space, Shape and Measurement LO. The description of the content for the Functions and Algebra LO and the Data Handling and Probability LO stress the importance of using relevant contexts which allow learners to engage in social issues.

The section on contexts is headed "Contexts: inclusivity, human rights and indigenous knowledge systems" (MNCS, p62). This section heading then clearly signals that the section can be expected to deal with aspects of critical democratic citizenship. The section does in fact deal with each of the three core aspects of critical democratic citizenship and most of this section is devoted to these. In terms of access to mathematics this section discusses the importance of mathematics in everyday life and for further study and states that lack of quality

education meant that the majority of South Africans were previously denied access to achieving in mathematics. The section also speaks about ensuring access to mathematics by not stereotyping, taking into account the interests of all and looking at examples of Mathematics in the "different cultures and societal practices" in South Africa (MNCS, p62). An interesting point raised in this section about overcoming barriers to learning mathematics which was not touched on in the description of the orientation is the following:

> In implementing this curriculum, it is the responsibility of the teacher to endeavour to win learners to Mathematics. This will be ensured by complying with the Assessment Standards of the subject, not formalising in the abstract prematurely but first taking care to develop understanding and process skills. (MNCS, p62)

The section on contexts also discusses mathematics as a human creation and emphasizes the importance of looking critically at how it is used in society. The section also highlights the need to be able to use mathematics to evaluate products of mathematics, and hire purchase agreements and mathematical arguments in the media are provided as examples.

In summary in chapter 3 we see that orientation 1 features very little or not at all in most of the sections except for the subsection *Contexts: inclusivity, human rights and indigenous knowledge systems* where it is the key focus.

4.2.4 Orientation 1 in chapter 4 of the MNCS

Chapter 4: Assessment		
Title of subsection of chapter	**Enumeration of elements (this is left blank if there were no elements that could be enumerated)**	**Discussion of elements that have been analysed qualitatively and a brief qualitative description of the elements that have been enumerated**

Competence descriptions level 6 (Outstanding)	0 out of 14 bullet points	
Competence descriptions level 5 (Meritorious)	0 out of 14 bullet points	
Competence descriptions level 4 (satistfactory)	0 out of 23 bullet points	
Competence descriptions level 3 (adequate)	0 out of 15 bullet points	
Competence descriptions level 2 (partial)	0 out of 15 bullet points	
Competence descriptions level 1 (inadequate)	0 out of 7 bullet points	

Table 12: Summary of orientation 1 in chapter 4 of the MNCS

Competence descriptors. None of the aspects of orientation 1 are specifically reflected in the competence descriptors which "distinguish the grade expectations of what learners must know and be able to achieve" (MNCS, p69). This is noteworthy as this suggests that orientation 1 does not form part of the competences on which learners will be assessed.

4.3 Discussion of issues arising from analysis of orientation 1 in MNCS

The analysis shows that orientation 1 features strongly in chapter 1 (which is not mathematics specific), to a lesser extent in chapter 2, and in chapter 3 and 4 it disappears almost entirely except for the section on contexts. So it is of particular note that in the assessment standards, the discussion of content and

the competence descriptors this orientation is largely absent. In a sense then one could argue that the notions of inclusivity and access, using mathematics to question and challenge and challenging and questioning mathematics are set up in chapter 1 and, to some extent, in chapter 2 and in the discussion of contexts in chapter 3 and more specifically mathematical content sections need to be read with these ideas in mind.

If we look then at how the three aspects of orientation 1 play out in the MNCS with particular reference to the Space, Shape and Measurement LO, we see the following:

Aspect 1: Access to mathematics and allowing learners to see themselves as being included in mathematics and its history.

Chapter 1 sets social transformation as one of the goals of the MNCS. It states specifically that "all South Africans have to be educationally affirmed through the recognition of their potential and the removal of artificial barriers to the attainment of qualifications" (MNCS, p2). There is a discussion in chapter one on "human rights, inclusivity, environmental and social justice" (MNCS, p4) which emphasises the need to support all learners to reach their full potential. In addition it is made clear that indigenous knowledge systems need to be valued.

However the relationship between these general aims and the mathematical content is unclear. As pointed out above, in chapters 2, 3 and 4 there is not as much emphasis on orientation 1 and so it is harder to see how access to mathematics might be achieved. In relation to the Space, Shape and Measurement LO one idea that is discussed is that of analysing "cultural products" and the development and use of geometry and trigonometry "by various cultures" could be used as strategies to make all learners feel included. This is amplified in the section "Contexts: inclusivity, human rights and indigenous knowledge systems" where it is suggested that one can provide "access and affirmation for learners of Mathematics" (MNCS, p62) by including cultural and societal practices. The mention of examples such as Ndebele murals, Zulu beadwork, art and architecture indicates possibilities for geometry.

There is also some tension between some of the ideas presented to promote inclusivity and access in chapter 1 (which is general, and not mathematics specific) and the more mathematics-specific chapters. In chapter 1 the explanation about indigenous knowledge systems suggests that up until the 1960s, when the theory of multiple intelligences showed that there were many ways to make sense of the world, "the Western world had only valued logical, mathematical and specific linguistic abilities and rated people as 'intelligent' only if they were adept in these ways" (MNCS, p4). It goes on to talk about valuing diverse knowledge systems and states that South Africa in particular IKS refers to "a body of knowledge embedded in African philosophical thinking and social practices that have evolved over thousands of years" (MNCS, p4). In some sense this can be seen to be setting IKS in South Africa up as an alternative to logical and mathematical thinking. Then in chapter 2, in the section which describes the scope of mathematics, it states that the "mathematical skills and process abilities will, where possible, be embedded in contexts that relate to ... indigenous knowledge systems" (MNCS, p11).

Aspect 2: Mathematics can be used to question and challenge social and economic injustices.

Although this aspect is mentioned in a general sense in several places, when specific examples are provided in relational to mathematical content they focus on financial issues and statistics, probability models and quantitative arguments. This aspect is thus not given any exemplification in the Space, Shape and Measurement LO.

Aspect 3: Mathematics is a human creation and hence mathematics and the way it is used is open to question and critique.

The Space, Shape and Measurement LO is the one learning outcome where mention is specifically made of the contested nature of mathematics and where the study of mathematics as a human creation is required. However this is relegated to a separate assessment standard (one in each grade) and is to be done through a project. The grade 12 assessment standard reads:

> Demonstrate a basic understanding of the development and uses of geometry through history and some familiarity with other geometries (e.g. spherical geometry, taxi-cab geometry, and fractals). (MNCS, p37)

Being able to translate this into investigating "the contested nature of geometry throughout history" (MNCS, p14) requires a sophisticated understanding of mathematics.

For the most part the assessment standards present the mathematics that the learners need to learn as an established body of facts which stands in tension with the espoused desire to present mathematics as open to question and critique.

The analysis of the MNCS shows a strong discourse of equality, inclusivity and human rights: this is not transparent and there is little exemplification or explication of how it should play out in the mathematical content. Thus the textbook authors had a strong imperative to include this orientation in their work, but little guidance as to how to do it. Thus the way in which to incorporate orientation 1 was open to interpretation. In the next section I analyse how it manifested in the textbooks.

4.4 Orientation 1 in the textbooks

The textbook chapters were analysed block by block using the division into blocks described in chapter 3. At any place where an element of one of the 3 aspects of orientation 1 was present this was recorded in the table. The 3 aspects are numbered as on p96 and their descriptions are taken as provided on p92-p96.

4.4.1 Orientation 1 in CM

The detailed block-by-block analysis is provided in Appendix B and summarised below.

Out of a total of 135 blocks in CM	aspect 1 of orientation 1: Maths for all	aspect 2 of orientation 1: Maths can be used to challenge injustice	aspect 3 of orientation 1: Maths is a human creation	Total orientation 1
number of blocks	5	1	1	7

Table 13: Summary of orientation 1 in CM

4.4.1.1 Discussion of orientation 1 in CM

Indicators for orientation 1 could only be found in 7 of the 135 blocks.

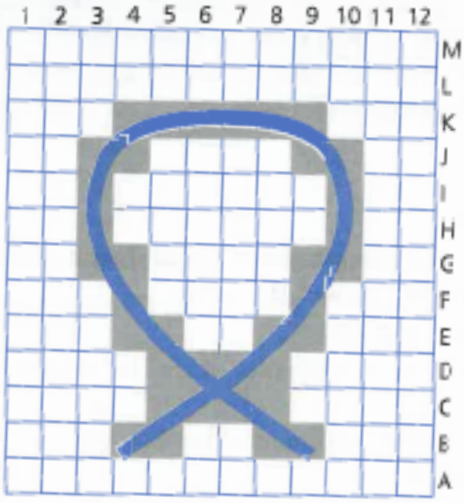
Of these 7 instances, 5 made reference to or were illustrated with pictures of South African cultural products and cultural practices and thus related to aspect 1. These could be seen as attempting to counter a Eurocentric bias and allow learners to find themselves and their culture in the textbook. However only 1 of these 5 required that learners actually do some mathematical work with the cultural product. The cultural products are thus not integral to the mathematical development in the textbook.

Only one block was classified as relating to aspect 2 and this is illustrated in textbook extract 3 below. The classification as belonging to aspect 2 was made on the basis of the fact that the use of the HIV/AIDS context suggested an attempt to reflect the curriculum concern with relevant social contexts.

The diagram below is the plan of a seating arrangement in a school hall during an HIV/AIDS awareness event. The shaded seats must be occupied by learners with red T-shirts and the rest by learners with white T-shirts.

a) How many learners in all are required to have a T-shirt (any colour)?

b) Write out the numbers of the seats where learners with red T-shirts must be seated.



Textbook extract 3: Contextual reference to HIV/AIDS (CM, p84)

This question appeared in an exercise in a chapter on linear functions and analytical geometry in a section entitled "The midpoint of a line segment". The grid system used is different to the Cartesian plane which is used for the analytic geometry in the rest of the chapter, and the question does not refer to line segments or midpoints. The question requires learners to count (or calculate) the number of squares in the grid and name the shaded grid squares. The mathematical demands of the question are thus low and not central to the mathematical development in the chapter. In addition, although the HIV/AIDS context is used it could be omitted entirely without changing the demands of the question because the model (the grid with shaded squares) has been provided for the learners.

Aspect 3 of orientation 1 appeared only in a project that asked learners to research the biography and contribution to mathematics of Euclid and Pythagoras. Although this clearly opens up the notion of human agency in

mathematics it does not move towards a discussion of the contested nature of mathematics as envisaged by the definition of mathematics given in the MNCS. This again reflects one of the tensions identified in the curriculum analysis. This tension was seen in the description of mathematics as a human creation open to critique standing in contrast with a reference to an established body of mathematical knowledge in the assessment standards.

Thus orientation 1 in the geometry chapters CM is peripheral to the mathematical development of the chapters. This reflects key issues and tensions identified in the curriculum analysis. The curriculum analysis suggested that, although orientation 1 featured strongly in the earlier more general chapters of the curriculum and the value of indigenous knowledge systems is stressed, very little guidance is given as to how these can be enacted in the context of the mathematical content and skills described in the learning outcomes and assessment standards.

4.4.2 Orientation 1 in OTM

The detailed block-by-block analysis is provided in Appendix B and summarised below.

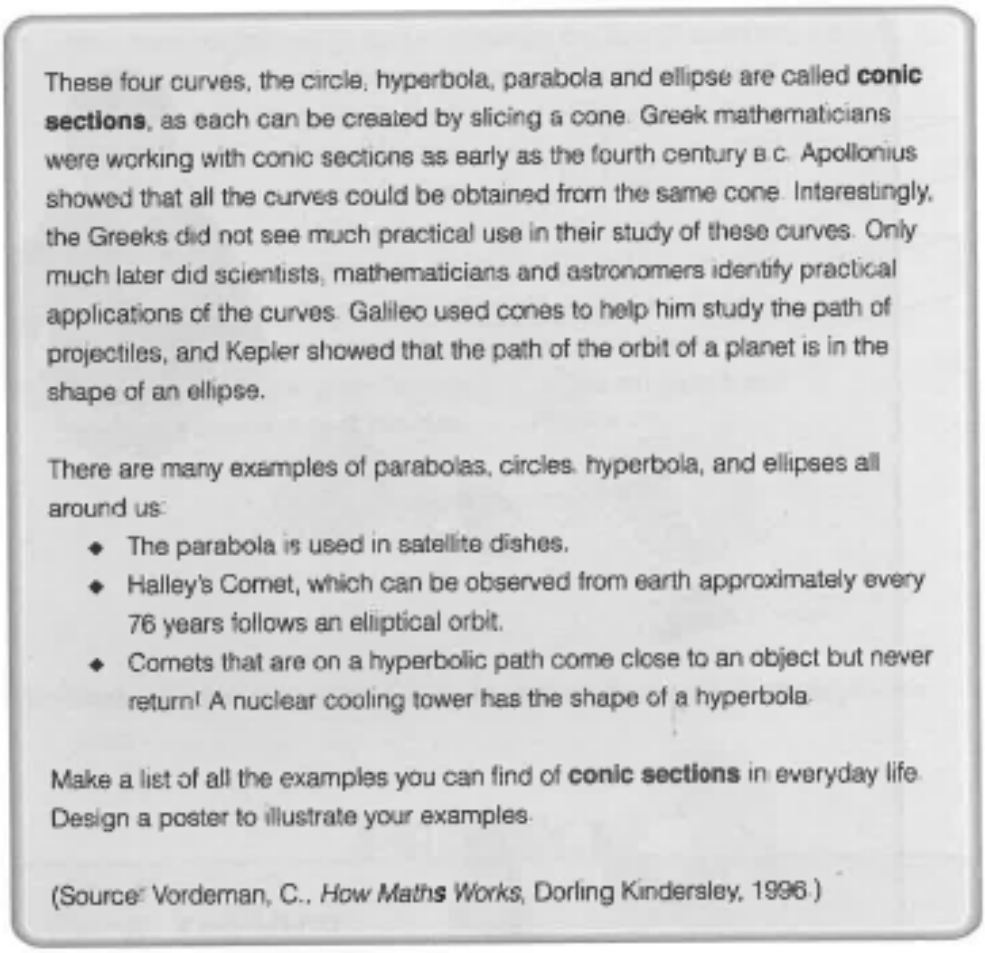
Out of a total of 118 blocks in OTM	**aspect 1 of orientation 1:** **Maths for all**	**aspect 2 of orientation 1:** **Maths can be used to challenge injustice**	**aspect 3 of orientation 1:** **Maths is a human creation**	**Total orientation 1**
number of blocks	4	0	5	8 [10]

[10] One block contained both aspect 1 and aspect 3 so the total number of blocks is 8.

Table 14: Summary of orientation 1 in OTM

4.4.2.1 Discussion of orientation 1 in OTM:

Indicators for orientation 1 were found in eight of the 118 blocks. Five blocks were categorised as displaying aspect 3 because they contained a discussion of the historical aspects of mathematics. These historical snippets, although not central to the work of the chapter, do link the mathematics that learners are engaged with with a historical account of that work and provide a sense of human agency in the development of mathematics. For example, in a project on conic sections learners are provided the information shown in textbook extract 4 below.



These four curves, the circle, hyperbola, parabola and ellipse are called **conic sections**, as each can be created by slicing a cone. Greek mathematicians were working with conic sections as early as the fourth century B.C. Apollonius showed that all the curves could be obtained from the same cone. Interestingly, the Greeks did not see much practical use in their study of these curves. Only much later did scientists, mathematicians and astronomers identify practical applications of the curves. Galileo used cones to help him study the path of projectiles, and Kepler showed that the path of the orbit of a planet is in the shape of an ellipse.

There are many examples of parabolas, circles, hyperbola, and ellipses all around us:

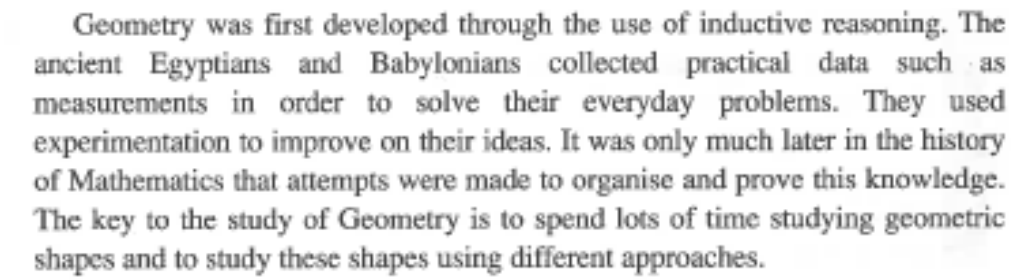
- The parabola is used in satellite dishes.
- Halley's Comet, which can be observed from earth approximately every 76 years follows an elliptical orbit.
- Comets that are on a hyperbolic path come close to an object but never return! A nuclear cooling tower has the shape of a hyperbola.

Make a list of all the examples you can find of **conic sections** in everyday life. Design a poster to illustrate your examples.

(Source: Vordeman, C., *How Maths Works*, Dorling Kindersley, 1996.)

Textbook extract 4: Historical account (OTM, p208)

The two chapters of the textbook dealing with Euclidean geometry are divided into a chapter on inductive reasoning and a chapter on deductive reasoning. The historical accounts provided in these chapters mirror this division and argue that the original development of geometry occurred through inductive reasoning followed later by the introduction of proof and organisation. An extract from the chapter opener showing this historical account is shown in textbook extract 5 below.



Geometry was first developed through the use of inductive reasoning. The ancient Egyptians and Babylonians collected practical data such as measurements in order to solve their everyday problems. They used experimentation to improve on their ideas. It was only much later in the history of Mathematics that attempts were made to organise and prove this knowledge. The key to the study of Geometry is to spend lots of time studying geometric shapes and to study these shapes using different approaches.

Textbook extract 5: Further historical account (OTM, p209)

Thus we see the nature of the discussion of the historical aspects provides a sense of mathematics as developing and changing over time and as being a product of human creativity.

There is only one instance in these historical accounts in which some notion of mathematics being open to critique is posited. This is shown in textbook extract 6. This account opens the possibility of debating the foundations of an established axiomatic system and raises the notion of other geometries.

Euclid's Axioms 1 to 4 were accepted by mathematicians, but many argued that Axiom 5 did not fit in. They spent years trying to deduce the statement in 5 from the other four axioms. These attempts failed, so they started trying other options to deduce it. What if they changed Axiom 5? This approach led to two completely new systems of Geometry. The first system is called **spherical Geometry** – think of doing Geometry on a sphere, rather than a flat surface. This Geometry has some interesting features. For example, try drawing a triangle on a round ball or a balloon. Then investigate the sum of the angles of your triangle.

The second type of Geometry is called **hyperbolic Geometry** – this is rather like doing Geometry on a hemisphere!

The development of these "new" Geometries had a great impact on the ideas and beliefs about Mathematics. The ideas of Euclid had influenced the study of Geometry and other branches of Mathematics for so long. Now some mathematicians were saying that there are other types of Geometry!

Textbook extract 6: Historical account on different types of geometry (OTM, p240)

Thus we see in the historical accounts an introduction to some reflection on the nature of mathematics.

The three further instances where an indicator of orientation 1 appeared were instances where reference was made to cultural products or practices. In two of these the cultural product is used as an illustration only and is not central to the mathematical work of the chapter. The third instance is a project on architecture in Africa. The project asks learners to look at both the cultural and mathematical reasons for the use of circular houses in Africa. Learners are charged with the task of finding out about the cultural practices relating to circular houses, about the Great Zimbabwe ruins and about how circular houses are built. This means that within the project the cultural aspects are taken seriously. However they are asked in separate questions to the core mathematical task of the project which is to investigate whether a circle has the largest area of a circle, square and rectangle of the same perimeter and to prove that if a circle and square both have perimeter $2\pi r$, the area of the circle will be larger than that of the square. It is interesting also to note that although the core mathematical question is ostensibly about showing that a circular shape for a house is the most

economical use of building material, the first stage of modelling has been done for the learner and the question is reduced to the mathematical task described above.

In this textbook we again see the difficulty in integrating this orientation with the core mathematical work of the chapter. Although historical aspects are woven through the mathematical development these focus primarily on Greek mathematics, and African mathematics[11] does not feature strongly. This orientation appears primarily in blocks outside of the key development of the chapter (i.e. chapter openers, extra-information narratives and projects).

The two key chapters related to Euclidean geometry in OTM are built around a strong notion of mathematical reasoning as moving from inductive to deductive reasoning. In describing the move from investigating in the first of these chapters to showing why in second, the textbook says "The path you have followed when studying Geometry is much like that used by all good mathematicians. They investigate and experiment until they are convinced of a result, that is, they use inductive reasoning. Explaining 'why' is the final step in a process" (OTM, p226). The deductive process described in the second of the chapters focuses on the building of a traditional mathematical deductive system and the strong use of logical proof. This type of thinking is described in the section on "valuing indigenous knowledge systems" in the MNCS as only one in a diverse range of ways of thinking. Thus we see that the textbook's focus on guiding learners through a particular style of reasoning precludes it from addressing the other types of thinking the curriculum mentions as important.

This textbook does manage to convey the notion of human agency in the creation of geometry through the discussion of the roles of various people in its history. However the sense of mathematics as open to critique and change is represented largely in the discussion of other geometries described in textbook extract 6. In

[11] See, for example, the work of George Joseph (2010).

this we see a description of the core of the challenge to mathematical thought occurring in terms of an attempt to establish the axioms on which the system of geometry was based. Some of this argument relates to what Houdement and Kuzniak (2003) would term as the geometry III paradigm. This idea, that one can create a different geometry, is not a simple one. Understanding how this implies understanding that mathematics is open to change requires a sophisticated ability to be able to understand axiomatic systems. The "facts" deduced in an axiomatic system remain facts, but it is possible to create alternative axiomatic systems that produce different facts. These subtleties are alluded to in textbook extract 6 above, but require learners to move outside of the paradigms of geometry they are used to working in order to understand them fully .The brevity of this extra-information narrative makes this unlikely.

4.5 Interviews

The curriculum analysis demonstrated that the issues of redress, inclusivity, transformation and democratic practices, which were very much part of the GRD at the time the curriculum was constructed, were prominent in the opening general chapters of the curriculum. However within the detailed description of the mathematical content in the curriculum this orientation was far less visible and this is reflected in the textbooks where we see this orientation appearing in only a few blocks and these were often not central to the main mathematical development of the chapters.

In the interviews this orientation was not a major focus of discussion for any of the interviewees although aspects 1 and 3 did come up over the course of the interviews.

Three interviewees talked about using cultural products or allowing learners to find themselves in the textbooks. One of the authors of CM talked about linking transformation geometry with cultural objects.

> MC_1: I set a number of wallpaper patterns and all those patterns – wallpapers and designs – are transformations. So it's linking up culture and

> design and kind of, which was another brief that we had – to try and integrate subjects and subject areas and that kind of thing. So I thought it was a really nice opportunity to do some of that.

However, although the author sees it as "nice opportunity" to integrate across subject areas, her expressed motivation is not about opening access to learners but relates instead to fulfilling a "brief that we had".

The author of OTM mentions exploring ethnomathematics. However her mention of ethnomathematics does not occur in the context of a discussion about opening access to mathematics, but in the context of a discussion around her frustration with the work in transformation geometry not going anywhere. She talks of exploring ethnomathematics as a way to see if she could find ways to apply some of the reasoning developed in transformation geometry.

> A_{OTM}: I did at the time try and explore some of the ethnomaths stuff that had been done, David, what's his name[12] and I looked I mean these were from, what's his name? The Mozambican Paulus Gerdes. I mean I looked at one of his books so, I did try and actually explore ways to try and bridge that, but I never got very far.

Thus we see that although the authors expressed interest including cultural products in their textbooks they did not link this to the notion of inclusivity.

A third interviewee, talking about the way in which textbooks were judged as suitable for use in South African schools by the Department of Education expressed a cynical view:

> MRC: The screening process has very, very, very little to do with the quality of syllabus interpretation. It has much more to do with the number of boys and girls in the book and the number of, you know, this or that PC[13] contexts.

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[12] Here she is referring to David Mogari who has done work on ethnomathematics in South Africa.

[13] The interviewee uses "PC" as an abbreviation for "politically correct".

Three interviewees spoke about issues that related to mathematics as a human creation.

The author of OTM did not directly address the nature of mathematics, but rather talked about the need to present mathematics as something that the students could be part of creating rather than as a finished product. This happened in response to a question about how she had changed the previous edition of the textbook (which was called *Just Mathematics*).

> A_{OTM}: I think what I tried to get away from here, now that I look back I don't know if it was I, I think sometimes you can go too far in the one direction, was to try and get away from those kind of information stuff, where you giving students you know, sort of information boxes where you giving them the stuff and say 'this is how it's done', worked examples, now let's go and practice kind of thing, you know. I've never felt comfortable with books that do that. I think I've always felt that a lot of that is the role of the teacher. So to me these books are not the kind of books that students are easily going to be able to study on their own to be able to do that whereas a book like Just Maths would be easier for a student who is teaching themselves the mathematics, to do that, whereas this, I think, needs a teacher you know, to sort of mediate the investigations and stuff because I haven't got boxes when say this, this is how you do it and, I mean, there are worked examples but not, not nearly as many as those. I think Just Maths presents maths as some sort of finished product and I wanted to try and get away from that to try and involve the students more in the mathematics. That, probably was the biggest change that I wanted to make.

One of the geometry consultants echoed a similar sentiment:

> GC_F: The problem I think that most textbook authors are that they want to include all the answers in the textbook – in the actual text – and I think that is wrong because then inevitably you are forced towards having to become more formal and to fall into the trap of writing the mathematics as a finished product. And then it defeats the purpose really of the activity because if you have the answers the kids are just going to page over anyway.

These comments can be construed as making comment about the nature of mathematics, but they also resonate with approaches to pedagogy that place the active learner constructing their own knowledge at the heart of the learning process. This type of pedagogy has been proposed by a number of different educationalists over time (de Villiers, 1998; Freudenthal, 1973; von Glasersfeld, 1991). The active learner has been a key notion of the ideas put forward by the reform movement in mathematics education (discussed in chapter 2). Proponents of this approach provide both ontological and pedagogical arguments for adopting it. For some its importance lies in giving learners experience of the process of creating mathematics and in coming to understand that mathematics is something constructed by humans and open to change. For others its strength is that it creates a better opportunity for learning. These proponents would argue that being actively involved in creating or discovering the mathematics enables learners to come to understand the mathematics better.

The two quotes above could be seen as aligning with either of these approaches. However, in one further discussion, the geometry consultant explicitly talked about the importance of recognizing geometry as a product of human creation that is open to different views.

> GC_F: Talking about non-Euclidean geometries: But that fell by the wayside, which is a pity. I think it would also teach them something about the nature of mathematics as being propositional. Mathematics is not absolute.

He goes on to say

> GC_F: So I mean there are lots of differences and I think that works why we had it in there they should do something like that so they can learn that geometry's propositional, it depends on the axioms, on the surface if you like that you choose to work with. It's not absolute, which I think, educationally think is important, philosophically it is important. Leonard who was, his lifelong passion has been teaching spherical geometry, he says you also teach someone about life because it is like looking at things from a different perspective. A triangle is different from the perspective of a plane as it is from on a sphere or, if you're working in hyperbolic geometry it's

> different again. And it teaches students that when they communicate with other people that people may look at it from a different perspective too, so it teaches tolerance from looking at it from one of the axioms that the other person's coming to the idea … When I talk about a triangle it may mean something different to me than it did because the axioms are different. One has to talk about that. The same thing when we talk about democracy. My understanding of democracy may be different from yours.

The notion that exposure to non-Euclidean geometries opens up the issue of the nature of mathematics was also reflected by A_{CM} who felt their inclusion was important to expose learners to alternative ways of seeing geometry.

The existence of so few discussions in the interview of issues relating to orientation 1 did not mean that the interviewees were not concerned about issues of equity and access. Discussions around learners' exclusion from mathematics and ways to remediate this related more to pedagogical approaches and systemic changes than to tying the content to social and cultural issues.

One clear trend that emerged from the discussions was a recognition that geometry had been perceived as being particularly difficult in the old curriculum. MRC and A_{CM} talked of the experience of finding a huge number of examination scripts in the national school leaving examinations where answers to the geometry questions were simply not attempted. This issue was clearly of concern to the committee who developed the curriculum and MC_1 reports that they attempted to deal with this by reducing the complexity of the type of questions intended to be dealt with in geometry and paying attention to the approach suggested by the van Hiele theory:

> MC_1: Our intention was that probably towards the end of the old curriculum riders were becoming increasingly difficult in the Higher Grade papers especially, and putting people off, you know. A lot of people developed a fear of Maths, or a dislike of Maths because they couldn't work these really hard riders out, didn't really know where to start and that kind of thing. And so, although it was a difficult thing to write into the paper, our hope was that it would be kept reasonably simple, that we wouldn't go – the examiners and

textbook writers and so on – wouldn't go overboard with excessively complex riders and so on.

In other words lower down we look at the properties of quadrilaterals and their diagonals and different ways of defining various quadrilaterals and other polygons and then similarity in Grade 11. But the whole idea is that theorems and axioms and corollaries and all that kind of thing we only kind of tackled in Grade 12. So we felt that, you know, perhaps that the old system rushed van Hiele's levels too quickly and that kids didn't get a chance, especially as it's acknowledged that they develop at different rates. You know, some kids were getting forced in Grade 9 into formal proof of congruence and stuff – that was probably too early, you know…

And that they got put off this whole idea of proof because proofs were something which you had to kind of learn off by heart or you had to follow this pattern. So and so and so and so, 1, 2, 3 – therefore, you know, it was kind of, it wasn't really proof, but it was a kind of rigour that was insisted on just too early.

The recognition of learners' difficulties with geometry and the importance of the van Hiele approach also informed the work of the author of OTM:

A_{OTM}: But, generally my philosophy also, I think it's also based on, I think my experience of teaching Geometry, where a lot of students just felt totally excluded because they couldn't make any sense of anything that had sort of deductive reasoning. They couldn't work out where these things were coming from and why it was structured like it was. And I remember, in my teaching trying to explain it to the students, but I don't think I ever had any success. So, I think that, sort of my personal experience of that teaching and also, I think, a lot from my work at Malati[14] where we did a lot on van Hiele levels. So, I think from the research that we did at Malati that show that

[14] Malati (the Mathematics Learning and Teaching Initiative) was a research and development project that ran between 1996 – 1999 and developed and trialled mathematics teaching material in South Africa.

> students who are in grade 10, were certainly, or according to that theory, were not ready to be doing formal deduction were being expected to do in the curriculum. So I think that shaped my thinking in terms of having that first chapter the inductive. I mean I think there are two things there, the one was for me more to get, sort of the structure around a sort of lower van Hiele level to give them experiences with the shape and space and also again to try to make them feel that they were included, in some way, and they could make decisions.

In addition to discussion of changes in approaches to the teaching and learning of geometry that would help, the author of OTM also spoke of the importance of making the language accessible for all learners.

> A_{OTM}: The other thing that happened with this book, which I think is extremely useful, which was we had a complete language edit, because there was a concern about the second language students accessing all the text. I think the writing changed quite a lot. It was incredibly, incredibly detailed. A language edit on word order and just changing the sentence order around just to make them more accessible. I learnt a huge amount it was very, very useful.

We thus see there was clear awareness and concern with making geometry accessible to all learners and attempts at pedagogical strategies to facilitate this. The interviews reflected a concern with the notion that learners had previously been denied access to mathematics. However what also emerged from some of the interviews was a concern that, despite these efforts, geometry would still remain a barrier to equity and access. These discussions centred around the decision of the Ministerial Review Committee to make Euclidean geometry optional. The Ministerial Review Committee was instituted by the Minister of Education after the MNCS and related grade 10 textbooks had been written but before it had been implemented in schools. The brief of the Review Committee was to review whether the decision to not have higher grade and standard grade mathematics was correct and whether implementing Mathematics and

Mathematical Literacy instead was correct[15]. There were also concerns raised about whether the MNCS was overloaded. This was discussed by the Ministerial Review Committee and more broadly. The justification for the decision to make Euclidean geometry optional was motivated by two other interviewees in terms of issues of equity and access:

> MRC: There our real finding was that we were really happy with the loss of standard grade and the motivation for that really is because it's an equity issue and although in white schools and just to characterise it as crudely as that, in white schools, standard grade was not necessarily abused, in black schools it was horribly abused because it was a way of getting children out of the class of teachers who couldn't teach higher grade. And de facto your black schools didn't teach higher grade and they didn't teach geometry. I mean you only have to ask anyone who's ever marked second paper[16] they love the second paper from those kind of schools because there is no geometry to mark. So our argument was that if you insisted on geometry being in then those kids would all end up in Maths Literacy and the problem is we want black children in Mathematics. I mean just to call spades shovels that's what we want, we need. So the idea was to say OK, the department has not done enough training of teachers, so in reality a lot of teachers can't teach this, this new offered curriculum. Two components the core – this is compulsory and the optional which should now be optional – exactly that – and we'll examine it, there'll be a third paper, it will be examined, you know, it will be reported on your certificate and all that and the real logic was that the kids at the black schools would now do the core while the department

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[15] In the previous curriculum learners had a choice of whether to do Mathematics or not in grades 10 –12. Those who chose Mathematics could choose to do it at one of two levels (higher grade or standard grade). At the time the MNCS was introduced all learners had to either do Mathematics (which was offered on one level only) or Mathematical Literacy (which is a subject aimed at giving learners competence in the basic mathematical skills required in everyday life).

[16] The school leaving examination for Mathematics consists of two examination papers. Geometry is examined in the second paper.

> spent time upskilling their teachers to offer the thing. Now, tragically, of course, it hasn't done any of the upskilling so that hasn't happened.
>
> MC_2: And there was a real sense also in that geometry discussion was that, you know, part of this was about transformation – not Transformation … transformation of the country, of the education system. So there was an aspect that, you know, we were looking at saying "This is a section that's holding back learners from accessing higher education and so forth and so forth, and certainly there wasn't a sense that they needed it to do their BComs[17] and some of their technikon things. So there was a real sense that that was a reasonable thing to leave out.

The decision to make Euclidean geometry optional was raised as an issue of concern by the other interviewees for reasons that ranged from its pleasure as an intellectual challenge, to its importance in developing deductive reasoning and its applicability to real world situations. Some expressed a degree of cynicism about the decision, talking of it as politically motivated.

> GC_F: At a high level they just decided and quite frankly their decision to me seems to be based purely politically on the fact that kids are doing badly in geometry and one way of improving the pass rate in Matric is by kicking geometry out.

These discussions highlight two issues: firstly that the socio-political considerations and classroom experience of teaching a particular topic have an impact on how geometry in the curriculum and classroom is constituted, and secondly that in the consideration of opening access to mathematics, Euclidean geometry was identified as a stumbling block. This led to it being made optional. However prior to that pedagogical strategies which included actively involving the learner in the construction of mathematics and attention to the van Hiele levels of development were promoted as ways of making better learning of geometry more possible. The way in which these pedagogical strategies played

[17] A BCom is an undergraduate degree in commerce.

out and some of the issues that arose in relation to this are discussed in detail in chapter 7.

4.6 Conclusions

The interviews reflect an awareness of the difficulties faced in mathematics education in South Africa and a concern with making mathematics more accessible to all South Africans. The interviewees' discussions indicated that geometry was seen as a particularly difficult area of the curriculum for many learners. The kind of strategies the interviewees reflected on for opening access to mathematics centred on systemic solutions (e.g. making geometry optional while training teachers) and pedagogical solutions. The concern reflected by the interviewees indicate that the notions of redress of apartheid inequalities and the building of a new South Africa, which were prominent notions in the GRD at the time, were prevalent considerations in the recontextualising field. The opening chapters of the MNCS echo these considerations.

However we saw in the analysis of the MNCS and textbooks, the aspects of orientation 1 relating to mathematics as being used to question social injustices and as itself being open to question and critique did not feature prominently in the sections dealing with mathematics in the MNCS and were not part of the mathematical development in the textbooks. The interviewees' comments on these aspects reflected either a fairly cynical view of being required to be "politically correct" or a difficulty in finding a way to integrate these ideas with the work required for coherent mathematical development.

5 Analysis of orientation 2

Orientation 2 as presented in Graven (2002): *Mathematics is relevant and practical. It has utilitarian value and can be applied to many aspects of everyday life. Mathematics is part of broader society and is important for all learners.*

5.1 Discussion of orientation 2

Within this orientation Graven includes "mathematics is part of broader society and is important for all learners". This has been dealt with under orientation 1 in the aspect "access to mathematics". Because of this I will omit this aspect for orientation 2 and focus on the notion of mathematics as applicable.

In this orientation Graven talks of relevance, the utilitarian value of mathematics and application to everyday life. Parker (2006, p63) suggests that in the FET band there is a focus on a "more structured form of applied mathematics, including problem solving and mathematical modelling". Parker's elaboration suggests that we should amend the orientation to read: *Mathematics is relevant and practical. It has utilitarian value and can be applied to many aspects of everyday life and in other disciplines. Mathematical modelling is an important tool that can be used in the exploration of contexts.*

In essence then this orientation encompasses the incorporation of extra-mathematical contexts in the mathematics classroom. However there are a variety of motivations one can have for incorporating these extra-mathematical contexts (de Lange, 1996; Venkat, Bowie, & Graven, 2009) and different levels of formality, complexity and authenticity argued for as a result. In what follows I extract from a review of the literature on contextualised mathematics some of the key distinctions that have been made. These distinctions will be used to highlight different possible aspects of this orientation. In the analysis of the curriculum document attention will be paid to how the various aspects of this orientation are manifested.

Niss, Blum and Galbraith (2007) point to a duality which they claim exists at upper secondary and tertiary level. They label the two poles of this duality as "applications and modelling for the learning of mathematics" and "learning mathematics for applications and modelling". Julie (2002; Julie & Mudaly, 2007) makes a similar distinction in contrasting the "model-as-vehicle" with the "model-construction" approach.

If we look at the "applications and modelling for the learning of mathematics" or "model-as-vehicle" approach, the primary motivation for including contextual problems is to enable or enhance mathematics learning. Venkat, Bowie and Graven (2009), in a review of published South African research on contextualised mathematics between 2000 and 2006, identified two broad camps within this approach. The first of these advocated for the inclusion of modelling and applications to demonstrate to learners that mathematics is used beyond the classroom, to interest learners through the use of relevant contexts and thus to stimulate learners to study mathematics. The second, which has strong roots in the Realistic Mathematics Education movement, advocates for the use of experientially real starting points in the learning of mathematical content to provide learners with handles to engage with the more abstract mathematical concepts. In contrast "learning mathematics for applications and modelling" or the "model-construction" approach focuses on developing in learners the "ability to bring mathematics to bear outside itself" (Niss et al., 2007, p6). Thus it will be important to draw out the relative emphasis on "model-as-vehicle" versus "model construction" in the curriculum and textbook analysis.

Many authors have described the process of mathematical modelling as a cycle (for example, Blomhoj & Jensen, 2007; Kaiser, 2005; Lesh & Lehrer, 2003). Although the details and the manner in which the cycle is described vary, the cycles basically all encompass some work in creating a real world model from a real world problem, a move from the real world situation to a mathematical model, work within the mathematical realm, a move back interpreting the mathematical results in the real world situation, a process of reflection on the goodness of fit which then can lead to a refined mathematical model and a

continuation of the cycle. Maaß (2006) citing Blomhøj and Jensen (2003), talks of two different methods of teaching modelling competencies the "holistic approach", where learners need to go through the whole modelling process, and the "atomistic approach". In the "atomistic approach" learners in the mathematics classroom work only with mathematizing, the mathematical model and interpreting the mathematics as these relate directly to the mathematics. They argue that both approaches are important in developing modelling competencies. Clearly it is also possible to "atomise" further. For example, one could ask learners to work simply with an already constructed mathematical model or ask them to interpret a mathematical model. This suggests that in the analysis it might be important to look at the approach to modelling and contextual problems in the curriculum and textbooks and to see what aspects of the modelling cycle are emphasized.

In addition, contextually based tasks in mathematics clearly encompass a broader range of tasks than full modelling tasks. Niss, Blum and Galbraith (2007) distinguish three common types of problems that appear in curricula that are sometimes given the heading "modelling and application problems". These are:

a) Word problems: Niss et al. (ibid., p11) claim that word problems "are nothing more than a 'dressing up' of a purely mathematical problem in words referring to a segment of the real" and that solving the problem "only consists of this undressing, the use of mathematics, and a straight forward interpretation" (ibid., p12). Although Niss et al. argue that many word problems are problematic in terms of the learning they promote, they suggest that good word problems can offer practice of the activities required at the solution and interpretation stages of the modelling cycle.

b) Standard applications: A typical example of these would be straightforward maximisation problems using calculus (e.g. find the dimensions of the square box with smallest surface area, given its volume). Here the appropriate model to use is immediately available and the process of translating from context to

mathematics is reasonably straightforward. Niss et al. argue that because of this only a portion of the modelling cycle is needed.

c) Modelling problems: Here the complete modelling cycle is required.

Maaß (2006, p114), drawing on the work of Kaiser, differentiates between "simple word problems, embedding mathematical tasks into everyday language, illustration of mathematical concepts (e.g. the use of temperatures to introduce negative figures), applying mathematical standard routines (application of well-known algorithms to solve reality related problems) and modelling, i.e. complex problem-solving processes."

A full understanding of this orientation in the curriculum and textbooks thus needs to examine the kind of contextual problems that are emphasised and the extent of mathematical modelling that is suggested.

An issue that emerges in relation to this section from a look at the curriculum itself is the question of whether to include the idea of problem-solving in this orientation. Throughout the curriculum mention is made of problem-solving in ways that do not specify whether the problem is a real-life problem or a mathematical problem. In addition the notion of what constitutes a problem is not clear. For example, in some instances certain people might refer to the question "solve $x^2+3x+2=0$" as a problem and in other instances "problems" are considered as questions where there is no direct path to a solution and where substantial work is required to reach a solution (Schoenfeld, 1992). In my analysis I will attribute statements that talk clearly about problem-solving in real-life contexts to this orientation. In addition I will separately examine all the instances in which problem-solving is mentioned and discuss those which imply or could imply real-world applications.

Thus a review of the literature suggested orientation two be summarised as follows:

Mathematics is relevant and practical. It has utilitarian value and can be applied to many aspects of everyday life and in other disciplines. Mathematical modelling is an important tool that can be used in the exploration of contexts.

This orientation has two aspects:

a) Applications and modelling for the learning of mathematics.

This can be evidenced through

i) Applications and modelling as a motivation for learning mathematics

ii) Applications and modelling as a conceptual aid to learning mathematics

b) Learning mathematics for applications and modelling

5.2 Orientation 2 in the MNCS

In this section I discuss how orientation 2 manifests in the MNCS. I again provide summaries of this for each chapter and use shading (in the same way as described in the previous chapter on p97) to provide an indication of the strength of the orientation in each subsection of each chapter.

5.2.1 Orientation 2 in chapter 1 of the MNCS

Chapter 1: Introducing the National Curriculum Statement		
Title of subsection of chapter	**Enumeration of elements (this is left blank if there were no elements that could be enumerated)**	**Discussion of elements that have been analysed quantitatively and a brief qualitative description of the elements that have been enumerated)**
Principles	1 out of 9 principles	Integration and applied competence is a principle

Critical and developmental outcomes	4 out of 12 outcomes	Problem-solving and responsible use of knowledge discussed
What kind of learner is envisaged		Orientation 2 not mentioned
What is a subject		Orientation 2 not mentioned

Table 15: Summary of orientation 2 in chapter 1 of the MNCS

This chapter is not mathematics specific and thus clearly does not mention mathematical modelling per se. However within the chapter there is discussion about the importance of ensuring that school knowledge is applicable. The chapter lists "integration and applied competence" as one of the principles of the NCS. In the discussion of this principle the MNCS states that integration needs to happen both within and across subjects. Applied competence is discussed as requiring "integration of knowledge and skills across subjects and terrains of practice" and as integrating "practical, foundational and reflective competences" (MNCS, p3). Chapter 1 thus clearly indicates that an important aspect of learning mathematics is the ability to apply what is learnt outside of the discipline.

The Critical and Developmental Outcomes talk of problem-solving and using science and technology effectively, and make it clear that learners should be able to operate in the world responsibly. These all reinforce the notion that the application of abstract knowledge is clearly something required by the NCS.

5.2.2 Orientation 2 in chapter 2 of the MNCS

Chapter 2: Mathematics		
Title of subsection of chapter	**Enumeration of elements (this is left blank if there were no elements**	**Discussion of elements that have been analysed quantitatively and a brief qualitative description of the elements that**

	that could be enumerated)	**have been enumerated)**
definition	2 out of 6 sentences	States clearly that mathematics can be used to explore problems that exist in the real-world as well as those within mathematics.
Purpose	4 out of 6 bullet points	Emphasises modelling as an important purpose of mathematics, but also links modelling to deepening learners' understanding of mathematics.
scope	7 out of 12 bullet points	Strong links to applications with particular emphasis on applications in socially relevant situations.
educational and career links		Strong emphasis on applicability of mathematics and discussion of mathematical modelling as a particular extension focus in FET band.
Number and Number Relationships LO	4 out of 6 bullet points	Clear focus with topics relating to growth and decay and financial situations mentioned
Functions and Algebra LO	1 out of 4 bullet points	Although only mentioned in one of the 4 bullet points, the bullet point that mentions mathematical modelling is described in detail thus indicating it is an important

		focus.
Space, Shape and Measurement LO	3 out of 9 bullet points	Analysis of natural forms and cultural products mentioned along with the need to make links and solve problems.
Data Handling and Probability LO		Data handling and probability described as being studied in relation to their use and applicability. Again socially relevant contexts are prioritised.

Table 16: Orientation 2 in chapter 2 of the MNCS

Definition. The first sentence of the definition is "Mathematics enables creative and logical reasoning about problems in the physical and social world and in the context of Mathematics itself" (MNCS, p9). Thus the definition puts forward a view of mathematics that is entirely consistent with the "Learning mathematics for applications and modelling" aspect of orientation 2 whilst acknowledging the importance of the pursuit of mathematical knowledge for its own sake. "Learning mathematics for applications and modelling" is further emphasised in the definition by the statement "Mathematical problem solving enables us to understand the world and make use of that understanding in our daily life" (MNCS, p9).

Purpose. The sentiments expressed in the first two paragraphs of this section mirror what is said in the first sentence of the definition i.e. mathematics is important for applicability as well as an important discipline in its own right. The third paragraph states that "Mathematical competence provides access to rewarding activity and contributes to personal, social, scientific and economic development" (MNCS, p9) and thus school mathematics is of great interest to a number of stakeholders. Four of the six bullet points about what mathematics enables learners to do clearly relate to orientation 2, and the fourth and final

paragraph in this section is focused on the application of mathematics. This paragraph begins “An important purpose of Mathematics in the Further Education and Training band is the establishment of proper connections between Mathematics as a discipline and the application of Mathematics in real-world contexts” (MNCS, p10). Thus the purpose section of the MNCS reflects orientation 2 strongly.

The MNCS talks of modelling and applications of mathematics as being for making “sense of society”, allowing “personal development”, “social, scientific and economic development”, engagement “with local, national and global issues”, solving “real-world problems” and work in the “physical, social and management sciences” (MNCS, p9- p10). The scope of the contexts suggested by the MNCS is thus wide and certainly extends beyond contexts that would be in the everyday experience of the learners.

Of particular interest is the discussion about mathematical modelling and applications given in the last paragraph. The first sentence (quoted above) clearly prioritises applications and asks that “proper” connections be made between maths and real-world contexts. Although no elaboration of what is meant by “proper” is given, the wording seem to indicates a desire for problem solving in authentic real-world contexts rather than “applying” mathematics through fictional word sums.

The second sentence in the final paragraph reads “Mathematical modelling provides learners with the means to analyse and describe their world mathematically, and so allows learners to deepen their understanding of Mathematics while adding to their mathematical tools for solving real world problems” (MNCS, p10). This sentence is the first time the aspect “Applications and modelling for the learning of mathematics” appears.

Scope. The scope section consists of 12 bullet points about what learners will work towards being able to do. Seven of these mention some notion of learning mathematics for applications and modelling and a further three talk of solving problems without specifying whether these are in real-life or mathematical

contexts. In addition the list of 12 bullet points is concluded with the following paragraph: “Such mathematical skills and process abilities will, where possible, be embedded in contexts that relate to HIV/AIDS, human rights, indigenous knowledge systems, and political and environmental and inclusivity issues” (MNCS, p11).

There are two points of particular interest in the scope section that are important to discuss:

The first is the way geometry is presented. Although a number of bullet points in the scope section are generic a few mention particular areas of mathematics. Geometry appears explicitly in only one of the bullet points: “describe, represent and analyse shape and space in two and three dimensions using various approaches in geometry (synthetic, analytic transformation) and trigonometry in an interrelated or connected manner” (MNCS, p10). One could argue that “shape and space” can be thought of as part of the real world and “describe, represent and analyse” as indicating the modelling process. However “space and shape” can also be thought of as mathematical notions and thus it would be possible to read the bullet point as advocating a more abstract use of geometry and trigonometry. Here the inherent duality in geometry of objects having both concrete and abstract realisations for a learner is raised which means that this sentence in the curriculum could be interpreted differently by different readers.

The second point of particular interest is the concluding paragraph of the scope section. This paragraph suggests that the mathematics learnt be embedded in contexts that relate to issues that are particular challenges in South Africa. This links back to orientation 1 and leaves the impression that a strong motivator for the inclusion of contexts in the mathematics curriculum is to enable awareness and transformation of social and environmental issues.

Educational and career links. In every paragraph in this section it is made clear that mathematics can be applied and thus will be needed in further study of a variety of different subjects at tertiary level, in dealing with financial issues and in dealing with societal issues and issues in daily life. Mathematics is described

as “being used increasingly as a tool for solving problems related to modern society” (MNCS, p11).

Of particular interest in this section is the statement on the link between the mathematics in the GET band and the mathematics in the FET band. Here the MNCS states: “The emphasis on contexts and integration within Mathematics and across the curriculum is maintained, while mathematical modelling becomes more prominent” (MNCS, p11). This statement implies a requirement in terms of modelling that goes beyond simply placing mathematics in contexts.

Description of the learning outcomes. In each of the learning outcomes contextual situations and the application of mathematics are mentioned. For example in the Number and Number Relationships learning outcome there is discussion of using mathematics to understand a number of different financial concepts (interest, hire-purchase, loans, annuities) and an instruction to explore patterns that occur in real life. The Functions and Algebra learning outcome is the learning outcome in which mathematical modelling is specifically mentioned as a focus. Here the MNCS states that learners should be able to work with mathematical models in different representations (words, table of values, graph, formula). Thus where the phrase mathematical model is mentioned it is related to the notion of a mathematical function or relation. Contexts feature prominently in the Data Handling and Probability learning outcome. As mentioned in chapter 4, the MNCS suggests that the focus should be on contexts related to human rights, inclusivity, social and environmental concerns.

The Space, Shape and Measurement learning outcome is the learning outcome in which contexts feature least prominently. However the intention that applications be included in this learning outcome is still made clear. In the paragraph introducing this learning outcome the one sentence that deals with applications directly states “Learners’ previous knowledge becomes deeper, they engage with new tools that can be used in a range of applications” (MNCS, p13). Of the 9 bullet points giving the detail of the learning outcome, two deal directly with applications. One reads “Analyse natural forms, cultural products and

processes as representations of space and shape" and the other asks for connections to other subjects "where possible" (MNCS, p14). In one further bullet point mention is made of using geometry and trigonometry to solve problems.

5.2.3 Orientation 2 in chapter 3 of the MNCS

Chapter 3: Learning outcomes, assessment standards, content and contexts		
Title of subsection of chapter	**Enumeration of elements (this is left blank if there were no elements that could be enumerated)**	**Discussion of elements that have been analysed quantitatively and a brief qualitative description of the elements that have been enumerated)**
Space, Shape and Measurement LO: Assessment standards grade 10	2 out of 7 assessment standards	Problem-solving using geometry and trigonometry. Exploring uses of geometry and trigonometry in history.
Space, Shape and Measurement LO: Assessment standards grade 11	2 out of 7 assessment standards	Problem-solving using geometry and trigonometry. Exploring uses of geometry and trigonometry in history.
Space, Shape and Measurement LO: Assessment standards grade 12	2 out of 6 assessment standards	Problem-solving using geometry and trigonometry. Exploring uses of geometry and trigonometry in history.
Content for Number and Number Relationships LO		Discusses using number to solve problems from both mathematical and real-life contexts.

Content for Functions and Algebra LO		Strong focus on functions and algebra as being integral to modelling.
Content for Space, Shape and Measurement LO		Discusses 3 broad aspects of the learning outcome, including the use of the content in contextual problems.
Content for Data Handling and Probability LO		Prioritises using the content to address contextual issues.
Contexts: inclusivity, human rights and indigenous knowledge systems		Includes both applications and modelling for the learning of mathematics and learning mathematics for application and modelling.

Table 17: Orientation 2 in chapter 3 of the MNCS

Assessment standards for learning outcome 3. Within each grade for learning outcome 3 the mathematical content (with no modelling or application mentioned) is specified in the first five assessment standards for the grade. Then in 6th assessment standard the learners are asked to

> Solve problems in two dimensions by using the trigonometric functions (sin x, cos x and tan x) in right-angled triangles and by constructing and interpreting geometric and trigonometric models (examples to include scale drawings, maps and building plans) (in grade 10).
>
> Solve problems in two dimensions by using the sine, cosine and area rules; and by constructing and interpreting geometric and trigonometric models (in grade 11).
>
> Solve problems in two and three dimensions by constructing and

> interpreting geometric and trigonometric models (in grade 12)(MNCS, p36-37).

These clearly fall within the ambit of learning mathematics for applications and modelling.

In the seventh assessment standard learners are asked to look at the use of geometry and trigonometry throughout history. This fits into the aspect "Applications and modelling as a motivation for learning mathematics".

Content and contexts for the attainment of assessment standard. In the description of content of the learning outcomes contexts and application are mentioned for all the learning outcomes, thus indicating that this is considered an important aspect across the mathematical content. However mathematical models are only specifically mentioned in the description of the Functions and Algebra learning outcome. Here it states that "Functions and algebra are integral to modelling and so to solving contextual problems" (MNCS, p48). This discussion of modelling is again linked to particular context. Here it is stated that "Human rights, health and other issues which involve debates on attitudes and values should be involved in dealing with models of relevant contexts" (MNCS, p48).

The Space, Shape and Measurement learning outcome also discusses the importance of contextual applications. The introductory paragraph has a section that stresses modelling and application. It states

> An important aspect of this Learning Outcome is the use of the content indicated in the representation of contextual problems in two and three dimensions so as to arrive at solutions through the measurement and calculation of associated values. Powerful mathematical tools which enable the investigation of space are embedded in the content (MNCS, p54).

We thus see that contextual applications are mentioned across the learning outcomes, however modelling per se seems to be viewed as part of functions and algebra and is related to being able to address social issues.

The section on Contexts is headed "Contexts: inclusivity, human rights and indigenous knowledge systems". As noted in the discussion of orientation 1, much of this section relates to access to mathematics for all, affirmation of all cultures and groups and allowing learners to see themselves in the contexts and mathematics used. There is thus a strong emphasis on the aspect "Applications and modelling for the learning of mathematics" in this section. The aspect "Learning mathematics for application and modelling" is also included in this section. This is seen in the opening sentence where its power in "concisely formulating the theoretical aspect of the sciences and in providing tools for solving problems" (MNCS, p62) is put forward in relation to both the natural, applied and social sciences. It is also seen in the discussion of the importance of mathematical literacy as the ability to use and evaluate mathematical arguments in daily life.

5.2.4 Orientation 2 in chapter 4 of the MNCS

Chapter 4: Assessment		
Title of subsection of chapter	**Enumeration of elements (this is left blank if there were no elements that could be enumerated)**	**Discussion of elements that have been analysed quantitatively and a brief qualitative description of the elements that have been enumerated)**
Competence descriptions level 6 (Outstanding)	3 out of 14 bullet points	Setting up models and using them to draw real-life conclusions.
Competence descriptions level 5 (Meritorious)	2 out of 14 bullet points	Setting up and solving mathematical models using expressions, formulae, functions and statistics.
Competence	3 out of 23 bullet	Setting up mathematical models

descriptions level 4 (satisfactory)	points	making direct use of previously learnt mathematics.
Competence descriptions level 3 (adequate)	2 out of 15 bullet points	Drawing and interpreting simple graphs and diagrams.
Competence descriptions level 2 (partial)	2 out of 15 bullet points	Trying to draw and interpret simple graphs and diagrams
Competence descriptions level 1 (inadequate)	3 out of 7 bullet points	Interpreting simple graphs and diagrams

Table 18: Orientation 2 in chapter 4 of the MNCS

Competence descriptors: Some notion of applications and modelling is given in the competence descriptors for each level of competence. The change that can be observed is that at the higher levels of competence learners are required to be able to set up models whereas at the lower levels they are required just to draw and interpret graphs. For example, a learner with outstanding achievement in grade 11 needs to be able to "set up mathematical models and draw appropriate conclusions in a manner that includes functions and techniques learned in this grade" (MNCS, p73), whereas a learner with meritorious achievement needs to be able to "set up and solve mathematical models for problems that depend on the direct use of expressions and formulae learned in this and lower grades" (MNCS, p75) and one with satisfactory achievement needs to be able to "set up mathematical models for straightforward situations that make direct use of the mathematics used in this and lower grades" (MNCS, p77). For those learners whose achievement is adequate or partial or inadequate modelling is not mentioned, but they are expected to "draw and interpret simple graphs and diagrams" (for adequate or partial achievement) (MNCS, p80) or "interpret simple diagrams and graphs" (for inadequate achievement) (MNCS, p82). Thus we see modelling is a skill that needs to be mastered by those wanting to achieve from level 4 up (i.e. above 50%). Some application of mathematics to context is

implied even for those achieving below 50% as they are required to be able to interpret graphs. Where specific mathematical content is mentioned in relation to modelling in the competence descriptors, the topics of functions, expressions and statistical techniques are raised. Thus the competence descriptors appear to prioritize algebra, functions, graphs and statistics as vehicles for applications of mathematics.

5.3 Discussion of issues arising from analysis of orientation 2 in the MNCS

This orientation is prominent throughout all the chapters of the MNCS. Chapter 1 presents "integration and applied competence" as one of the principles of the NCS. The way this is defined, together with the emphasis on problem solving and effective and responsible use of science and technology in the critical and developmental outcomes, suggest that a serious focus on applications and modelling is called for. In terms of Niss, Blum and Galbraith's (2007) distinction between the three common types of mathematical problems that are sometimes presented as modelling and application problems, it appears as if the intention in the MNCS is to work towards modelling problems rather than remain working at the level of word problems or standard applications. An analysis of the occurrences of orientation 2 in chapter 2 of the MNCS provides confirmatory evidence of this. We see this with talk of "proper connections" between mathematics and the real world, the use of the term "mathematical modelling" and the specific injunction that "The emphasis on contexts and integration within Mathematics and across the curriculum is maintained while mathematical modelling becomes more prominent" (MNCS, p11).

The way in which modelling is discussed in the assessment standards and competence descriptors shows a bias towards functions as models. This is seen in the inclusion of a specific assessment standard devoted to modelling in the Functions and Algebra learning outcome and of specific mention of models that rely on functions in the competence descriptors.

The aspect "learning mathematics for applications and modelling" predominates, but there are points where this is conflated with the "applications and modelling for the learning of mathematics" aspect and is overlain with a notion of learning for transformation and emancipation. In the places where this conflation occurs one sees a strong link to orientation 1. This orientation predominates in chapter 1 and so the notions of access to learning for all and the importance of learning for challenging social and economic injustices are set out upfront. In the sections on *Scope* and *Content and contexts for the attainment of assessment standards* we then see contexts such as "HIV/AIDS, human rights, indigenous knowledge systems, and political and environmental and inclusivity issues" (MNCS, p11) being given priority, and the notions of choice of contexts being important for ensuring all groups of learners feel able to access mathematics and are culturally affirmed are presented. Although there is nothing in this, per se, that precludes a serious look at modelling it does bring to the fore a particular ideology. Consider, for example, the difference in impression given by a rephrasing of the paragraph at the end of the *Scope* section from:

> Such mathematical skills and process abilities will, where possible, be embedded in contexts that relate to HIV/AIDS, human rights, indigenous knowledge systems, and political and environmental and inclusivity issues (MNCS, p11).

to:

> Such mathematical skills and process abilities will, where possible, be embedded in contexts that relate to computer aided design, telecommunications, medical imaging, population modelling, environmental resource management and the chemical, physical and economic sciences.

In relation to the sections of the MNCS that relate to geometry, in particular, we see that the duality of geometric objects as having both concrete and abstract realisations makes it difficult to discern when and whether mathematical modelling is being called for.

5.4 Orientation 2 in the textbooks

In examining how this orientation is manifested in the textbooks I examined all blocks where extra-mathematical contexts are used. Obviously the textbooks would be unlikely to be explicit about whether a context was being used as "application and modelling for the learning of mathematics" or "learning mathematics for applications and modelling". Thus I analysed each block according to observable criteria that I then used to build a picture of the way in which contexts were being used in the textbooks. These criteria were:

a) whether modelling or interpretation was required in the block;

b) whether mathematical work was required in the block;

c) the nature of the context.

Further explanation of these criteria and the motivation for them is given below.

The emphasis in the curriculum on full mathematical modelling, along with Niss, Blum and Galbraith's (2007) assertion that other types of contextualised problems are often given the title of modelling, indicated that it would be important to look at what components of the modelling process are required in each block. In order to do this I refer to Kaiser's (2005) diagrammatic representation of the modelling process.

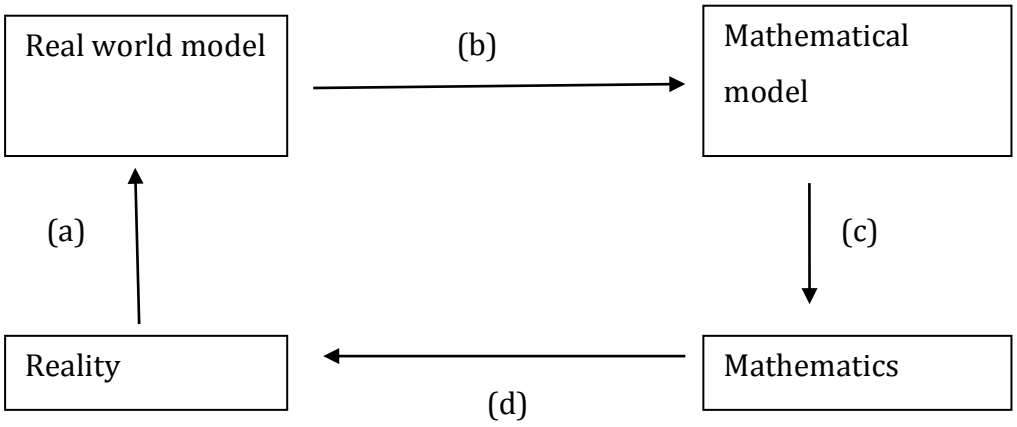


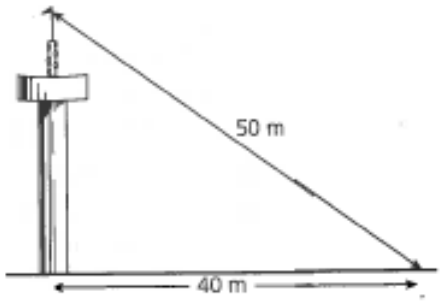
Figure 3: Kaiser's (2005) diagrammatic representation of the modelling process

The main distinction I want to make is between working with a mathematical model to get mathematical results (i.e. process c in figure 3) and modelling a real situation with mathematics or interpreting and validating the mathematical results in relation to the real situation (i.e. processes a, b and d in diagram 1). Drawing this distinction gives an indication of how much of the contextual work presented requires engaging with the connection between reality and mathematics. Thus in the analysis of the blocks in OTM and CM I indicated whether the contextual problem required modelling (process a or b in the diagram) or interpretation work (process d in the diagram) and separately indicated whether it required mathematical work (process c in the diagram).

However working with the data made it clear that it was not always simple to decide when modelling or interpretation work is required, because inevitably when a context is used some connection needs to be made between the context and the mathematics. However I did not want trivial connections to be included as modelling and interpretation work. I wanted to indicate a contextual problem as requiring modelling only if learners needed to engage with the context in a substantial way and build the mathematical model through an active engagement with the context. Similarly I wanted to indicate a contextual problem as requiring interpretation only if learners needed to reflect on what the mathematical answer they produced meant in terms of the context and

possibly reflect on the goodness of fit of the model itself. To illustrate what I mean I provide a few examples:

In one question learners are asked to describe translations they see on a playing card (such as the four of diamonds). I would argue that, despite being asked to "see" the mathematics in a real object, the learner is not asked to model the situation. The learner is told what mathematics to apply to that situation. Contextual problems often require that learners give the answers to the question in terms of the context. I did not regard this as sufficient cause to classify this as requiring interpretation. For example if the learner is given the diagram below and asked to calculate the height of the tower in metres, I did not regard providing the answer as interpretation.



Textbook extract 7: Height of the tower (CM, p67)

In the analysis of the curriculum the nature of the extra-mathematical context emerged as an important aspect to focus on. From the curriculum analysis I identified four broad categories of contexts

- social issues, inclusivity and social change (e.g. use of cultural products, indigenous knowledge systems, HIV/AIDS, political matters, economic issues, making sense of society)
- contexts linked to the physical world or to science (e.g. making sense of the physical world, environmental issues)

- financial issues in a business context (e.g. depreciation, share trading, investments)
- everyday contexts (e.g. personal finances, contexts connected to "our daily life")

Clearly there is overlap and ambiguity between these categories and so I will refine the categories here in order that they become a useful dimension along which I can categorise the contextual problems from the textbooks.

i) social issues, inclusivity and social change: This category includes contexts where there is some form of social change or awareness attached to the inclusion of the context. The indication of a social change agenda can either be found directly in the way the problem has been set up and the context is discussed or is there by virtue of the fact that the context is highlighted as one of the contexts important for the transformation of South Africa in the curriculum. These contexts include the use of South African cultural products and practices, discrimination, indigenous knowledge systems, and HIV/AIDS.

ii) contexts linked to the physical world or science: This includes contexts such as temperature; building; packaging; roads; work related to land, plants, animals etc.

iii) financial issues in a business context: These relate specifically to concepts that require a more specialist understanding of finances in a business context. For example, a question that asks how much change a teller should give a customer would not be included here, but the depreciation of a piece of equipment would be.

iv) everyday contexts: these are contexts that people might engage in during the course of the normal activities of their daily life, such as shopping, creating a personal budget, banking, making a cake, or going to the theatre or a sports event. The main criterion for inclusion here is that the context is either portrayed or can be seen as one in which the learner might feasibly personally take part.

5.4.1 Analysis of orientation 2 in CM

The detailed block-by-block analysis is provided in Appendix C and summarised below.

39 of 135 blocks (29%) contained contexts. We will call these blocks 'contextual blocks' for ease of reference. Two of these 39 contextual blocks were blocks in which modelling itself was discussed. No context was mentioned for these blocks and it was not appropriate to discuss whether mathematical work was required. Thus they were given the code n/a for the context and for mathematical work. There were two blocks where the contexts were explained in greater detail. These were given the code n/a for all the aspects except the category. Tables 19 and 20 provide a summary of the characteristics of the contextual blocks by showing the percentage of contextual blocks that fall under each heading.

	Required modelling	**Required interpretation or validation of model**	**Did not require modelling or interpretation of model**	**n/a**
Number of blocks	10	4	24	2
Percentage of the contextual blocks	26%	10%	62%	5%

Table 19: Summary of characteristics of contextual blocks in CM in relation to modelling

	Required mathematical work	**Did not require mathematical work**	**n/a**
Number of blocks	32	3	4
Percentage of the contextual blocks	82%	8%	10%

Table 20: Summary of characteristics of contextual blocks in CM in relation to mathematical work

The fact that contexts were present in over a quarter of the blocks indicates that this orientation was certainly attended to in the geometry chapters of CM. However only 36% of the contextual blocks required modelling or interpretation and the majority (62%) did not. In contrast the vast majority of contextual blocks did require mathematical work.

	Social issues, inclusivity and social change	**Physical world or science**	**Financial issues in business context**	**Everyday**	**n/a**
Number of blocks	3	10	1	25	2
Percentage of the contextual blocks	8%	26%	3%	64%	5%

Table 21: Summary of the type of contexts used in the contextual blocks in CM

Although varied contexts were used the majority fell into the category of everyday contexts with a sizeable proportion being contexts linked to the physical world. Very few contexts were linked to social issues which was the key suggested contextual setting stated in the curriculum, but considering the findings of the analysis of orientation 1 this is unsurprising.

5.4.1.1 Discussion of orientation 2 in CM

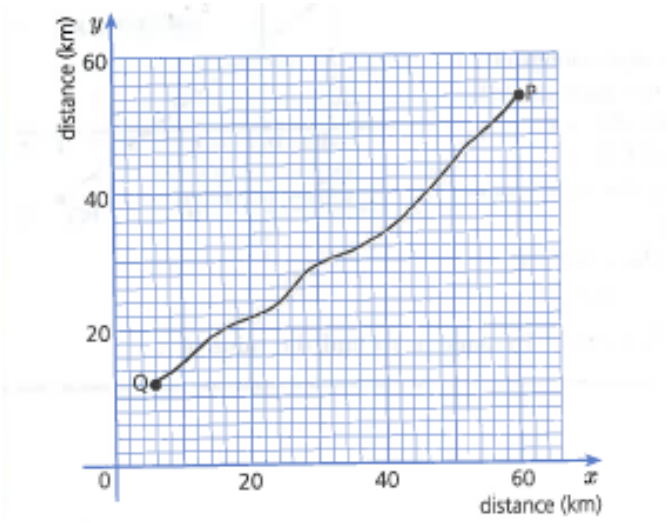
In order to get a deeper insight into the ways in which contexts were used, I looked at the blocks where no modelling or interpretation was required and then looked at those where it was. In those blocks where no modelling or interpretation was required I looked at the role the context played. In those where there was modelling or interpretation I looked at what notion of modelling was conveyed.

Use of context in contextual blocks where no mathematical modelling or interpretation required

In exploring the 24 contextual blocks where no modelling or interpretation is required three broad categories of examples were identified based on the role the context played. These are discussed and illustrated using examples in what follows.

Presenting the mathematics learnt as applicable in context

In this category I included blocks where, after a particular mathematical idea is discussed, an exercise set or activity is presented in which the mathematical idea is applied in a context. There were nine such blocks in CM. A typical example of this is the question shown in textbook extract 7 on p152 where learners, having reviewed the theorem of Pythagoras, are asked to calculate the height of the tower. A similar type of example is the application of the distance formula to finding the length of the river as shown below:



1. PQ represents a short section of the course of a river. Find the approximate length of this section.

Solutions

1. Assume that the course of the river from P to Q is a straight line: Since Q(6; 12) and P(60; 54)

$$PQ = \sqrt{(60-6)^2 + (54-12)^2}$$
$$= \sqrt{54^2 + 42^2}$$
$$= \sqrt{2\,916 + 1\,764}$$
$$PQ = 68{,}4 \text{ km}$$

The approximate length of the river along its course is 68,4 km.

Textbook extract 8: Length of the river (CM, p80-81)

Both these examples illustrate an issue that was present in six out of the nine blocks that fell into the category, which was that the application illustrated was not one that was realistic. In the tower example (textbook extract 7 on p152) one is unlikely to be able to get a physical measure of the diagonal distance, but not of the vertical height. In the river example a river is unlikely to be presented on the Cartesian plane but is more likely to be presented on a map where it can be physically measured and the real length calculated according to the scale.

In the three further blocks in this category, the contexts that were used were contextually reasonable. In one of these blocks, an extra-information narrative, no mathematical work was required and learners were simply told about linkages on steam engines. In the other two blocks the mathematical work required was trivial for a learner at grade 10 level. The first of these, set in the context of creating an HIV ribbon from a seating arrangement, required learners to count blocks on a grid and was discussed in detail in chapter 4. The second illustrates the notion of rate of change by putting values into a formula to calculate the rate (in words per minute) at which a child reads.

Thus we see that the blocks in which the mathematics learnt was presented as being applicable in context either presented a contrived context or mathematical work was absent or trivialised.

Using an everyday object as a starting point

In 11 of the 24 contextual blocks the context played the role of a starting point used to introduce a mathematical idea. In the majority of these cases an everyday

object or cultural product has been used as a starting point for discussion around transformation geometry. In these cases the context serves as an illustration of the geometric patterns or transformations. I refer to the context as a starting point because the bulk of the mathematical development happens outside of the context. For example, after transformations are introduced by getting learners to describe the transformations that can be seen on a playing card, the work is followed up by placing points on the Cartesian plane and developing formulae that describe the transformations.

Illustrating an aspect of modelling

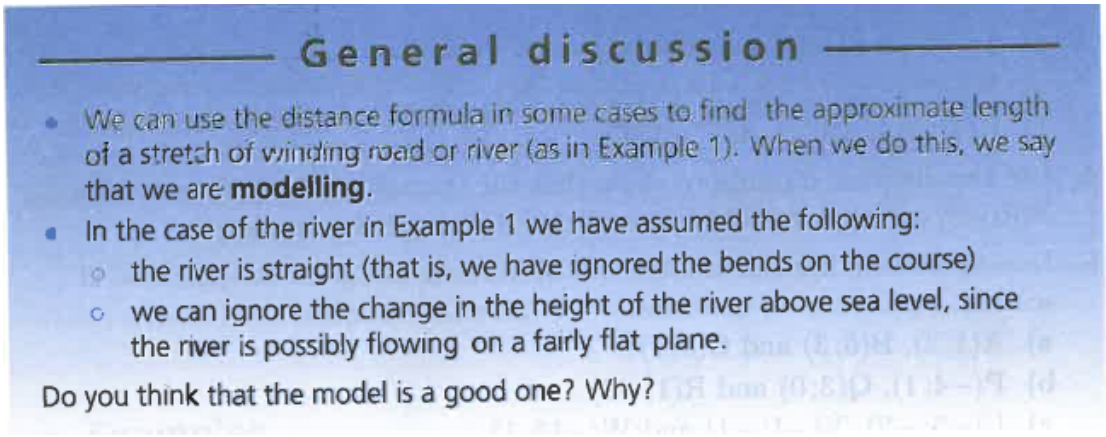
In four of the 24 contextual blocks, although the focus is on the mathematical content, an aspect of mathematical modelling is exemplified. For example, in the analytical geometry section on linear functions, learners are asked to match one of a number of graphs of linear functions with descriptions such as "A leaking container holds exactly 1,5 litres of water. The entire contents leak out in 20 minutes" (CM, p86). Although this kind of question does not require learners to do any significant modelling it does focus learners' attention on linking textual descriptions with graphical information that one could argue is a skill that will be important in modelling. Similarly the chapter on length, area and volume works with various concepts of measurement and the ratio of these measurements using nested Russian dolls as the object of exploration. Learners are asked to estimate the volume and surface area of these dolls and thus need to think about what regular shape they can use to best approximate these measures. This kind of skill clearly could be useful and applied in many real situations. However the focus of the work remains on how changes in the dimensions of the shape affect the surface area and volume and not on the process of modelling the volume and surface area of the dolls. In addition, although the dolls provide an interesting starting point for investigating how changes in dimension affect surface area and volume, it is not easy to develop a realistic context which would require one to do so with these dolls.

The notion of modelling portrayed in the blocks where modelling or interpretation is present

There were nine contextual blocks where modelling was required, three where interpretation of a model was required, and one in which both modelling and interpretations were required. These could be grouped according to the mathematical content area they related and, within each of these content areas, there was some statement in the textbook that either discussed modelling or directed the learners' attention to a particular aspect of modelling. Thus my discussion of these blocks is structured according to the content area.

The distance formula in coordinate geometry

Immediately after the worked example indicating how one might go about finding the length of a river (textbook extract 8 above), the following discussion about modelling and the model are provided.



General discussion

- We can use the distance formula in some cases to find the approximate length of a stretch of winding road or river (as in Example 1). When we do this, we say that we are **modelling**.
- In the case of the river in Example 1 we have assumed the following:
  - the river is straight (that is, we have ignored the bends on the course)
  - we can ignore the change in the height of the river above sea level, since the river is possibly flowing on a fairly flat plane.

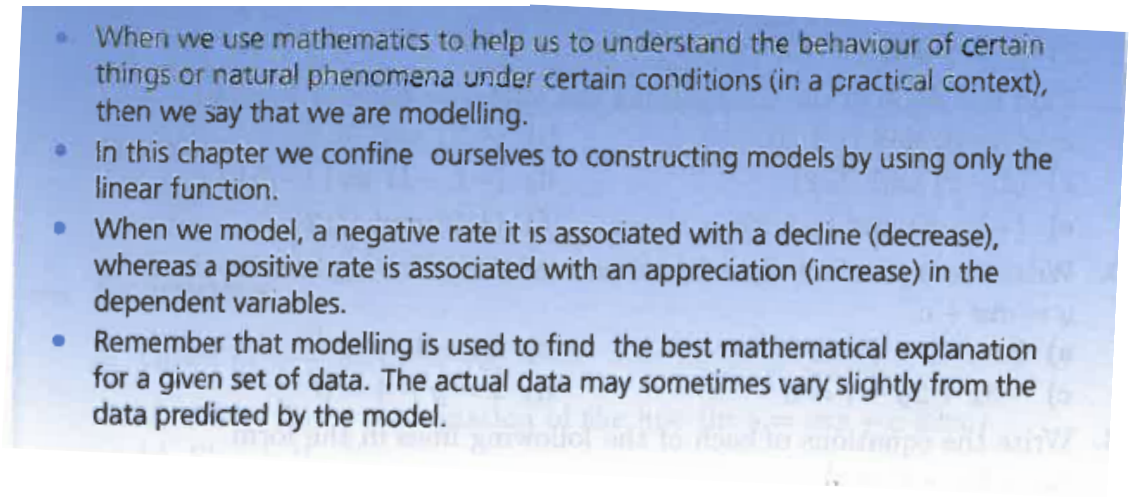
Do you think that the model is a good one? Why?

Textbook extract 9: Discussion about modelling (CM, p81)

The impression created in the general discussion is that recognising that one can use the distance formula is the crux of the modelling required in this type of example. However the decision to impose a coordinate system onto a map of the river (which is itself already a model) is the key in the modelling process here. The textbook asks learners to reflect on whether the model is a good one and highlights the fact that models are not exact replicas of the context and require making decisions about what assumptions and simplifications to make.

Linear functions as models

After introducing the notions of slope in coordinate geometry there is a section that focuses on linear functions and that looks, in particular, at linear functions as models. Here the discussion on modelling is more extensive. This discussion is shown in textbook extract 10 below. Despite this discussion the questions that learners are asked to engage with in the exercises that follow (see textbook extracts 11 and 12 below) do not require significant modelling from the contexts and in many respects the contexts are not realistic.



- When we use mathematics to help us to understand the behaviour of certain things or natural phenomena under certain conditions (in a practical context), then we say that we are modelling.
- In this chapter we confine ourselves to constructing models by using only the linear function.
- When we model, a negative rate it is associated with a decline (decrease), whereas a positive rate is associated with an appreciation (increase) in the dependent variables.
- Remember that modelling is used to find the best mathematical explanation for a given set of data. The actual data may sometimes vary slightly from the data predicted by the model.

Textbook extract 10: Instructional narrative about modelling (CM, p94)

Exercise 4.11 (AS 10.2.6)

1. A butcher was seen selling 250 g of mutton to a customer for R9. Another customer bought 5,5 kg of mutton for R198 at the same butcher. Determine:
   a) The price per kilogram of mutton at this butchery.
   b) If a graph showing the sales of mutton at this butchery is plotted, where will it cut the y-axis?
2. A greengrocer sells bananas at the low but fixed price of R3/kg. In order to ensure that he makes a reasonable profit, he adds a certain fixed amount of money to any quantity of bananas purchased. Calculate the equation the greengrocer applies to different purchases if a customer who bought 4 kg of bananas was observed paying R14.
3. A bank charges customers R2,50 for the first R100 that they withdraw from their accounts and thereafter an additional 50c for each additional R100 or part of R100.
   a) Determine the bank charges for the following withdrawals:
      (i) R150 (ii) R220 (iii) R590
   b) Model the situation by using a graph that shows the changes for withdrawing amounts up to R1 000. (**Hint:** Your graph will look stepped.)

Textbook extract 11: Exercise set on modelling (CM, p95)

4. Pierre de Fermat claimed to have solved the equation $x^n + y^n = z^n$, $n \in \mathbb{N}$, $n \geq 3$, but his solution was never found. Mathematicians tried hard to find a solution to the problem, which became known as "Fermat's Last Theorem". In 1900 Paul Wolfskehl, set a prize of 100 000 German marks, equivalent to about R14 000 000, for a successful solution. In 1997 the prize was worth only about R500 000. Model the depreciation in the value of this prize money with an equation, assuming linear depreciation. (Andrew Wiles eventually published a proof of the conjecture in 1995. Wiles accepted the prize in 1997.)



Pierre de Fermat 1601–1665

Textbook extract 12: Further work from the exercise set on modelling (CM, p95)

In question 1a, calculating the cost per kilogram does not really require modelling. The graph requested in question 1b is vaguely stated and it is not clear why one would want to consider where the graph crosses the y-axis. The scenario described in questions 3 and 4 are both unrealistic. In none of these questions is reflection on the model required. Thus although modelling is set up both in the title of this subsection ("Modelling with linear equations") and in the instructional narrative at the start of this section, the choice of examples suggest that the modelling is being done to learn linear equations rather than to give learners actual engagement with modelling.

In addition to the exercises set (textbooks extracts 11 and 12) there are two activities where learners use linear functions to model scenarios. In each of these learners are asked to engage in some reflection on the model. In the first they are told to assume that the depreciation in the value of a road grader, with a given initial price and a given book value 5 years later, can be modelled with a straight line and to find the equation and sketch its graph. They then calculate when the road grader will have a value of zero. They are asked to reflect on whether they think this is an appropriate model and justify their answer. It is not clear on what basis learners could decide on whether the model is appropriate or not. Learners are unlikely to have much experience with the value of road graders. In addition, given that the value modelled by the linear equation is based on book value it is unclear whether learners are being asked to judge whether book value reflects potential re-sale value or whether they would have the required background knowledge to make judgments about this.

In the second activity learners are asked to use two given equations to predict the winning time for a race in the most recent Olympics as given by each equation and find the difference between the predicted times given by the two equations. They are then asked to find the actual winning time for the most recent Olympics and asked to reflect on whether the two equations are good models. The solutions provided in the educator's guide suggest noting the similarity in times between the answers given by each of the equations and the actual winning time is sufficient to judge this as a good model. However the

educator's guide also points out that learners need to be aware that the model only provides reasonable predictions over a limited domain.

Thus we see that in CM a link is made between straight lines in analytical geometry and linear functions as models. The focus on modelling in relations to functions aligns with the suggestions of the MNCS. However, although modelling is ostensibly in focus it is only in one activity that learners are given the opportunity to engage meaningfully with reflections on the model itself.

Area and volume

In the chapter on length, area and volume, nested Russian dolls are used throughout as a vehicle to discuss various aspects of measurement, and, in particular to look at how volume and area change as the dimensions to change. In the first activity the learners are given the height and diameter of the dolls and asked to estimate the surface area and volume of the doll. This requires using geometric solids for which they have known formulae for surface area and volume to model the dolls. Learners are asked to reflect on how accurate they think their answers are and to consider ways to improve them. Thus learners are here asked both to create and reflect on a model. At the end of the chapter learners are asked to find some irregularly-shaped objects and estimate their surface area and volume. Thus the skill of modelling irregularly-shaped objects with known regular solids is both introduced and followed up in this chapter.

Quadrilaterals

In the chapter opener of the chapter on quadrilaterals the learners are told that, amongst other things, they will "apply the geometry you have learned to model real-life situations" (CM, p335). Within the chapter there is a subsection entitled "some applications" followed by another subsection entitled "linkages" in which the contextual blocks in this chapter are situated. All of these blocks relate to contexts in the physical world. One asks learners to consider how they could use their geometric knowledge to ensure the wood a carpenter cuts will be rectangular and the second illustrates how builders can use pegs and ropes to

mark out a circle or rectangle. The third gets learners to explore ways to create the shortest path (road) linking four vertices (cities). Although these three blocks do not require learners to engage in substantive modelling they do ask learners or show learners how the geometry they have been used can be applied to real-world situations in reasonably authentic ways.

The fourth block takes them carefully through creating a model of a car jack using cardboard and split-pins that they then use to investigate how the jack behaves by removing a particular pin and changing its position along a particular strip. They are then asked to make conjectures about the way in which a particular point and strip behave and are instructed to use a geometric model to prove the conjecture. The purpose of this kind of scaffolding can be viewed in two ways: that careful structuring is necessary if you want the learners doing the activity to work with particular mathematics and be able to come to predictable conclusions, and that for learners new to modelling this kind of scaffolding alerts them to the kind of methods one can use in modelling before they are fully immersed in the modelling process itself.

5.4.1.2 Summing up orientation 2 in CM

The majority of contexts chosen were everyday contexts and required neither substantial modelling of the context nor interpretation of the mathematics in relation to the context. This suggests that the model-as-vehicle (Julie, 2002; Julie & Mudaly, 2007) or the applications and modelling for learning mathematics approach (Niss et al., 2007) dominated. In contrast to the modelling and interpretation, mathematical work was required within most of the contextual blocks. This focus on mathematical content factored into the difficulties observed in developing modelling ideas. The structuring of chapters according to mathematical content meant contextual blocks tended to illustrate specific mathematical content. In some cases this resulted in the use of contexts being contrived or in an absence of any key modelling ideas.

Modelling or models are mentioned in a few places in these chapters in the learner book. However, as we have seen, there is no instance in which

independent and substantial mathematical modelling is required. This suggests that there is an implicit assumption underlying the work in the book that one builds ideas of mathematical modelling through practicing work on specific components of the modelling cycle or working with highly scaffolded examples of aspects of modelling. However there is no place in either the learner book or educator's guide where this is made explicit.

5.4.2 Analysis of orientation 2 in OTM

The detailed block-by-block analysis is provided in Appendix C and summarised below.

11 of the 117 blocks (9%) contained contexts. This is fewer than in CM both in absolute terms and in percentage terms.

	Required modelling	**Required interpretation or validation of model**	**Did not require modelling or interpretation of model**
Number of blocks	1	0	10
Percentage of the contextual blocks	9%	0%	91%

Table 22: Summary of characteristics of contextual blocks in OTM in relation to modelling

	Required mathematical work	**Did not require mathematical work**
Number of blocks	6	5

Percentage of the contextual blocks	55%	45%

Table 23: Summary of characteristics of contextual blocks in OTM in relation to mathematical work

The fact that contexts were present in only 9% of the blocks indicates that this orientation was not a priority in the geometry chapters of OTM. Only one of the contextual blocks required modelling or interpretation, and roughly half of the contextual blocks did not require mathematical work.

	Social issues, inclusivity and social change	**Physical world or science**	**Financial issues in business context**	**Everyday**
Number of blocks	2	9	0	1
Percentage of the contextual blocks	18%	82%	3%	9%

Table 24: Summary of the type of contexts used in the contextual blocks in OTM

The vast majority of contexts used were those linked to the physical world or science. Very few contexts were linked to social issues which was the key suggested contextual setting stated in the curriculum.

5.4.2.1 Discussion of orientation 2 in OTM

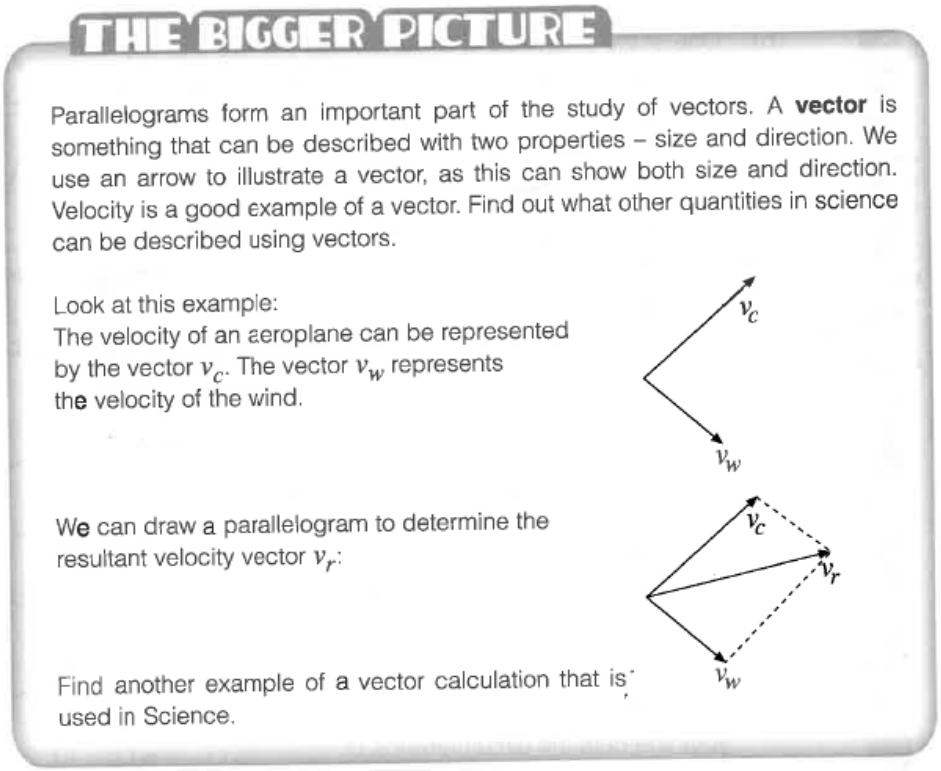
In order to get a deeper insight into the ways in which contexts were used, I looked at the blocks where no modelling or interpretation was required and then looked at those where it was. In those blocks where no modelling or interpretation was required I looked at the role the context played. In those where there was modelling or interpretation I looked at what notion of modelling was conveyed.

Use of context in contextual blocks where no mathematical modelling or interpretation was required

In exploring the ten contextual blocks where no modelling or interpretation was required in terms of the role the context played, I found only two of the three broad categories identified for CM. I discuss each of these below.

Presenting the mathematics learnt as applicable in context

Six of the blocks fell into this category. Of these five also required no mathematical work. Three of these were contained in an extra-information narrative and provided a discussion of how the work studied could be used in context. These were not central to the mathematical development of the chapters and provided only an illustration of the applicability of the mathematics learnt. For example, the work on parallelograms is followed by the extra-information narrative shown in textbook extract 13 below.



THE BIGGER PICTURE

Parallelograms form an important part of the study of vectors. A **vector** is something that can be described with two properties – size and direction. We use an arrow to illustrate a vector, as this can show both size and direction. Velocity is a good example of a vector. Find out what other quantities in science can be described using vectors.

Look at this example:
The velocity of an aeroplane can be represented by the vector v_c. The vector v_w represents the velocity of the wind.

We can draw a parallelogram to determine the resultant velocity vector v_r:

Find another example of a vector calculation that is used in Science.

Textbook extract 13: Application in extra-information narrative (OTM, p222)

The applicability of the mathematics learnt or to be learnt is also mentioned in a chapter opener and as an aside in a project. In the project, which asks learners to explore various conic sections by slicing cones, a brief narrative is provided stating that the various conic sections either occur in natural phenomena (e.g. the path of a comet) or are used in a device (e.g. a satellite dish) (see textbook extract 4 on p116). The chapter opener (see textbook extract 5 on p117) mentions that geometry was developed initially to solve everyday problems.

The one contextual block in this category where mathematical work was required is a project on architecture in Africa that has already been discussed in the analysis of orientation 1. As discussed in chapter 1, although the project is portrayed as asking learners to investigate the circular shape of African houses in relation to economical use of building material the key mathematical task is presented as follows:

Assume that a square and a circle have perimeter $2\pi r$. Use Algebra to **prove** that the area of the circle will be larger than the area of the square. (OTM, p224)

We see here that the key modelling work has already been done for the learners and the activity is reduced to a purely mathematical investigation.

The project continues with the following quote attributed to Labelle Prisin, who studied the architecture of West Africa:

> I did a simplified "engineering analysis" comparing the round and square house in the savannah, with equal volume, for bending, shear, bearing etc., given similar vertical and horizontal loading, assuming constants in humidity, temperature, and soils, vegetation and technology. Of course, the round house won by a landslide. (OTM, p224)

Learners are then asked to find out what bending, shear and bearing mean in engineering. Here again we get a sense of the possibility of applying mathematics in other disciplines and investigating substantial problems. However again these are designated as beyond the scope of this mathematics course.

Using an everyday object as a starting point

Three of the four contextual blocks that fall into this category involve designing a pizza box which is a right prism and calculating its volume and surface area. The focus is on the mathematical work of the volume and surface of a right prism and aspects of the real situation (e.g. the flaps needed to construct and close the box) are explicitly ignored. The teacher's guide is clear about this and states "Question 1(d) is a rather 'contrived' problem, as we are ignoring the flaps needed to hold the box together. Talk to learners about this: although we know what should happen in real life, sometimes we have to discard this and just focus on the Mathematics" (Bennie, 2005b, p165). Thus we see that the context provides an everyday object as a starting point for a mathematical investigation. In a later activity learners are asked to use the words translate, rotate and reflect to describe traditional patterns found in South Africa and Lesotho. Here also the focus is on developing the mathematics ideas and the context provides a starting point.

The notion of modelling portrayed in the blocks where modelling or interpretation is present

The only contextual block in which learners are required to engage in some aspect of modelling is one where they are asked to draw graphs of the height of water in a vase against time, for water flowing into different shaped vases at a constant rate. In this case one of the skills required to do the work is linking a contextual situation with a graphical one. Modelling itself is not discussed and the situation presented is contrived. However it does engage learners in thinking about the relationship between a physical situation and the graphical model that could be used to represent it.

5.4.2.2 Summing up orientation 2 in OTM

The analysis of contextual problems in the geometry chapters of OTM reveals that the major focus is on mathematical work and there is little focus on applications and modelling. Where contextual blocks exist they are used either

as illustrations of the application of mathematics or simply as starting points to move into the mathematics.

In addition we see an attempt to include substantial contexts which are linked to fairly advanced ideas in science or engineering. However where these contexts are used, learners are not required to work with the mathematics in relation to them.

5.5 Interviews

In the interviews modelling was not talked about as a major concern by any of the authors of the textbook or curriculum. However both geometry consultants mentioned the importance of modelling and expressed regret at its lack in the curriculum. They came as it from two different directions. GC_G spoke about it from the perspective of learning mathematics for applications and modelling:

> GC_G: But a whole genre of geometry that is necessary to operate the new technology, you know, the world is saying this is priority number one.
>
> I would like to really see a much more deeper, or let's say, a fundamental understanding of first and foremost the whole 3-D space thing. If you look at the materials that are written in the Dutch books, I mean they start talking about a plane, sectioning a plane, what happens in a plane, faces of objects, in 4th grade. And then you should see how they pull it through and you should see the type of generalisations they make by 11th and 12th grade and the type of reasoning that comes out of it. But you can only do it if there's a strong foundation. I mean understanding to section planes and objects in a plane and rotating them in space and combining shapes and then putting those things together and reasoning about it – that's what I would have liked to see. So my starting point would not have been necessarily… obviously 2D is very interrelated – the 2D dimensionality – and obviously I'm aware that some of your 3 dimensional work requires some fundamental maths – other mathematics that is too, not accessible at an early age, I'm aware of that, so I would have liked to see that. And you see that's supported now by the software that's available – Cabri, Sketchpad,

> Geocadabra, your CAD tools. I've worked now with a CAD tool for engineering design which utilises all of these things. Your industry today works with CAD tools being developing or designing a motor vehicle or designing an electric mixer. And all of those principles are based on three dimensional concepts of space.

GC_F spoke about it from the perspective of applications and modelling for the learning of mathematics:

> GC_F: I mean then another thing also I think in terms of transformations – I think I did mention – but I don't see a lot of evidence of that in the curriculum, is … Well not only in relation to transformations, but then is also to use a lot more practical context. You know, to start with a practical context and then to model the geometry from that. I mean traditionally geometry is still completely divorced from any real world application. You know, I mean for example this example of the perpendicular bisectors. I start with a practical context of finding a water reservoir that's the same distance from 4 villages that want water. Getting an idea of what is equidistance and then looking at 2 villages and so on. So starting with a practical context rather than, you know, just teaching, you know, a perpendicular bisector is a line that's perpendicular at the midpoint of a line segment. So what? You know, it's just something. It doesn't have any practical meaning. To also use a lot more modelling. But I don't see a lot of that either in the curriculum, ja.

Those involved in the curriculum committee alluded to applications as playing a part in the decisions around geometry, but the motivation behind it they expressed was less clear.

> MC_2: And so I think that was one of the issues, was to see the follow through of transformation into FET so teachers in GET would have a reason to teach it, you know, and where it went from there. I don't … I never got a sense that there was an underlying theoretical underpinning as to why a shift from Euclidean to transformation. We never… I remember comments about it being, you know, you could easily apply it to advertising logos. But again I got the sense that was coming from some of the NGOs, that those were little

things they were doing in GET and they thought it would be fun to continue with them into FET.

Similarity was not meant to be the nasty prove that, you know, AG x GB is equal to some bizarre thing. It was meant to be doing similarity so that there could be application of it in real life problems. That was the intention behind similarity in Grade 11. So that you wouldn't go into the heavy rigorous stuff, there'd be some simple things, but it would primarily be focused on applications – that was the discussion.

MC_1: I think to some extent just showing that there is mathematics involved, that kind of thing that people actually use Maths …

Geometry in film-making and things like that. They use transformation geometry in that kind of way.

In a number of these instances we see discussion of modelling and geometry in a way that doesn't tie in neatly with the curriculum. The discussion of GC_G proposes a vision that relies on a greater link to 3D geometry and links closely to the presence of technological tools in the classroom. Both authors and curriculum committee members were mindful of the fact that few schools would have access to these. The curriculum committee members speak of an awareness of these kinds of applications (advertising logos, film-making etc) but the manner in which they speak of it suggests they saw it as a possible illustration of an application rather than a meaningful opportunity to engage with the modelling and mathematics.

The comment by GC_F, suggesting the use of a real-world situation to provide meaning to the mathematics and that of the curriculum committee members suggesting application of known geometry are more concretely described, and these are the kind of examples that were manifested in the textbook where mathematical work was required.

The one author who did talk about modelling, the author of OTM, spoke about how her focus was on the mathematical story that made it hard to incorporate

the modelling. She then spoke about how this meant the applications got placed into the extra-information blocks.

> A_{OTM}: I didn't really go into it, I didn't develop it, use the potential that there could possibly be. I mean I sort of tried, sort of made, but they're very sort of contrived, I would think sort of looking back, contrived links, you know, this kind of thing. But that again that comes in the projects and those kind of things, it's not really sort of part of the text in other ways. I think what drove me and it's the kind of stuff that comes into your, the bigger pictures[18] I mean, that's what the feature was for. The bigger picture was to try and make links to the applications and that kind of thing. So I haven't really included that very much in the investigations and the exercises. I think what was more was driving me was the mathematical applications to the extent that we can have mathematical applications at this level rather than the practical applications.

When asked if she felt there was potential to have done more with the modelling she talks about how she updated the previous edition of the textbook (called Just Maths) to make the context more socially relevant, but did not feel she had particularly amended them to get at modelling. It is interesting to note that, although she feels that more could have been done, she ends by questioning how much this could be done if one's focus was on the development of the mathematics.

> A_{OTM}: I think, uh look, I think there's potential to do a lot better in terms of the modelling than what I've done here in terms of, the kind of sort of, when we spoke about how I used the, the Just Maths, I mean, a lot of those sort of problems, my sort of word problems and the trig word problems sort of came from there. I mean, I tried to sort of revamp them a little bit and update the contexts and have more Ja, it was sort of my way of getting at the, at the social kind of issues that should be in the curriculum. So I sort of

[18] The bigger picture was the title given to many of the extra-information narratives in this textbook. An example can be seen in textbook extract 13 on p168

> updated them in that way, but they still contrived, they're still word problems, sort of traditional school word problems although the contexts might change slightly. So, I think in that way, I think, you could do more. How much you could, if you also want to show the development the mathematics, I don't know.

Thus we see in the textbooks and in the reflections of the interviewees a struggle to find a coherent role for contextual problems and a real struggle to fit in any form of meaningful modelling.

The analysis of orientation 1 and orientation 2 show that these orientations feature strongly in the MNCS. However orientation 1 is not strongly visible in the Learning Outcomes of the MNCS where the mathematical content is discussed and we see that, although it filters into the textbooks it is not prominent, and is generally peripheral to the main development of the work in the textbooks. Orientation 2 is visible in the Learning Outcomes although less so in the Space, Shape and Measurement learning outcome. Although orientation 2 is present in both textbooks, it is similarly not prominent and, by and large, not well integrated into the main mathematical development of the chapters. It thus seemed the remaining orientations (orientations 3 and 4), which deal with mathematical content and practices, would be dominant orientations in the textbooks. Chapters 6 and 7 provide an analysis of how this manifested.

6 Analysis of orientation 3 and 4

Orientation 3 as presented in Graven (2002): *Mathematics is an induction for learners into what it means to be a mathematician, to think mathematically and to view the world through a mathematical lens. Mathematics has its own beauty and can be explored for its own sake. Mathematical investigation and exploration (without necessarily utilitarian value) is emphasised. School mathematics in this sense is seen as part of a broader mathematics culture, which is produced and reproduced uncritically in accordance with the norms and conventions of the broader mathematics culture.*

Orientation 4 as presented in Graven (2002): *Mathematics is a language with conventions, skills and algorithms that must be learnt. Many of these will not be used or applied by most learners in everyday life but are important for the FET band and for university studies in mathematics (for example, the symbols and conventions for writing exponents, factorisation of trinomials, solving Euclidean geometry riders etc.). School mathematics in this sense is seen as part of the broader mathematics culture, which is accepted and reproduced.*

6.1 Discussion of orientations 3 and 4

As discussed earlier, I needed to look at these two orientations together as they are hard to separate analytically. Both of these orientations include elements that suggest a preparation for further study: orientation 3 in that it talks of inducting learners into academic mathematics and orientation 4 in that it speaks of learning skills for university studies in mathematics. Both of these imply an existing canon of work that has norms, conventions and facts that learners need to master. However there is a distinction in that orientation 3 seems to focus more on broader mathematical processes and orientation 4 on mathematical content. This type of distinction is reflected in the literature. For example, Sierpinska and Lerman (1996, p838) state: "Put briefly, mathematics is identified either as a particular body of knowledge, a subset of which is deemed appropriate for all school students and a somewhat larger subset for those who may go into higher education in mathematical subjects, or it is identified by particular kinds of activities that are called mathematising, including modelling, pattern-recognition, generalising, proving, etc."

And Schoenfeld (2004b, p243), in talking about the goals of mathematical instruction as 'thinking mathematically', suggests that within this there has been a major shift "from 'content' to 'content and process'". Schoenfeld mentions in particular the National Council of Teachers of Mathematics' (NCTM) Principles and Standards for School Mathematics in the USA which provides five content standards and specifies five process standards: problem-solving, reasoning, connections, communication and representation.

Thinking of the distinction as 'content' and 'process' or 'body of knowledge' and 'activity' avoids the issue in orientation 3 which presupposes a shared understanding of what mathematicians do and what thinking mathematically is.

However as we see from the fact that the activities Sierpinska and Lerman identify differ from the processes the NCTM identify that there is a need to be specific about the kind of processes focused on. In addition, the processes and activities that the NCTM and Sierpinska and Lerman suggest are broad and some, for example connections and modelling, are already included in earlier orientations discussed. The processes I want to capture in orientation 3 are those that are used or have been used in the creation and organisation of the discipline of mathematics. Because the notion of "process" has been used extensively in the literature in such a way that it can bring to mind the idea of "performing mathematical algorithms", I have chosen to use the term mathematical practices instead.

Thus my adaptation of orientation 3 is:

Mathematics is created and organised using particular practices. These include investigating, conjecturing, generalising, justifying, proving, axomatising, and defining.

In order to differentiate orientation 4 from orientation 3, I have adapted it as follows:

Mathematics is a body of knowledge and learners at school need to study a subset of it. This knowledge includes specific algorithms, facts, conventions, notations and forms of representation.

6.2 Orientations 3 and 4 in the MNCS

In this section I report on both orientation 3 and 4 in the MNCS in a similar way to the way in which I did for orientation 1 and orientation 2. In order to give information about both orientations 3 and 4 for each chapter of the MNCS in one table I have inserted a column in which I summarise the quantitative count of

elements related to orientation 3 and a column that does the same for orientation 4. In the column in which I discuss elements that have been analysed qualitatively as well as giving a brief qualitative description of those that have been enumerated, I do so in relation to both orientations. In the same way as I have done previously I use shading to represent the weighting of the orientation in each subsection. Because the weighting for orientation 3 and orientation 4 can be different in different subsections I indicate the weighting of each orientation separately by shading the cell in the table in that subsection for that orientation.

6.2.1 Orientations 3 and 4 in chapter 1 of the MNCS

Chapter 1: Introducing the National Curriculum Statement			
	Orientation 3	**Orientation 4**	
Principles	0 out of 9 principles	0 out of 9 principles	As the chapter is not subject specific the discussion here is too generic to identify whether orientation 3 or 4 is being referred to.
Critical and developmental outcomes	0 out of 12 outcomes	0 out of 12 outcomes	As the chapter is not subject specific the discussion here is too generic to identify whether orientation 3 or 4 is being referred to.
What kind of learner is envisaged			As the chapter is not subject specific the discussion here is too generic to identify whether orientation 3 or 4 is being referred to.
What is a			Prioritizes skills, values and

subject			attitudes over a body of knowledge.

Table 25: Orientations 3 and 4 in chapter 1 of the MNCS

This chapter does not address the issue of mathematical practices. However in the definition of what a subject is the MNCS states "Historically, a subject has been defined as a specific body of academic knowledge. This understanding of a subject has laid emphasis on knowledge at the expense of skills, values and attitudes" (MNCS, p6) and sets up subjects in the NCS as having less defined boundaries and integrating "theory, skills and values" (MNCS, p6). However in this general introduction no details are given as to what the skills referred to might be.

6.2.2 Orientations 3 and 4 in chapter 2 of the MNCS

Chapter 2: Mathematics			
	Orientation 3	**Orientation 4**	
Definition	4 out of 6 sentences	0 out of 6 sentences	Prioritizes orientation 3 and talks about the process by which mathematics is created.
Purpose	1 out of 6 bullet points	0 out of 6 bullet points	States clearly that process skills are more important than acquisition of content knowledge for its own sake.
Scope	1 out of 12 bullet points	8 out of 12 bullet points	Content is prioritized.
Educational			Neither mathematical

and career links			practices nor mathematical content specifically mentioned in this section.
Number and Number Relationships LO	1 out of 6 bullet points	5 out of 6 bullet points	Focus on content required in number work.
Functions and Algebra LO	1 out of 4 bullet points	2 out of 4 bullet points	Focus on content required in functions and algebra, but mentions investigating.
Space, Shape and Measurement LO	5 out of 9 bullet points	4 out of 9 bullet points	Discusses exploration, conjecturing, justifying and proving.
Data Handling and Probability LO			Focuses largely on the content required stating that this content will be used to analyse real-life situations.

Table 26: Orientations 3 and 4 in chapter 2 of the MNCS

Definition. In the definition of mathematics the MNCS states that mathematics is constructed "through the establishment of ... relationships". There is mention of mathematical practices such as "logical reasoning" and "observing patterns" and that "theories of abstract relations" are created (MNCS, p9). Mathematics as a an established body of knowledge is not directly mentioned

Purpose. In this section mathematical process skills are discussed as important for an appreciation of the discipline itself and in applications. Although mathematics as a body of knowledge is mentioned as important the MNCS states in this section "Competence in mathematical process skills such as investigating,

generalising and proving is more important than the acquisition of content knowledge for its own sake" (MNCS, p9).

Scope. Of the twelve bullet points about what learners will work towards being able to do, only one deals specifically with mathematical practices. This reads "competently use mathematical process skills such as making conjectures, proving assertions and modelling situations" (MNCS, p10). In contrast 8 of the 12 bullets pointing to what learners will work towards being able to do relate to orientation 4. These include calculating, producing and using equivalents for algebraic expressions, working with functions, using trigonometry and geometry, using statistical techniques, solving problems involving sequences and series and using available technology. All these bullet points make clear the need for mastery/knowledge of particular mathematical content.

Educational and career links. Although it is made clear in this section that mathematics in the FET band will provide a route into mathematics at higher education institutions and thus into academic mathematics, neither mathematical practices nor mathematical content are mentioned specifically in this section.

Description of the learning outcomes. Five of the nine bullet points about what the study of shape and space enables learners to do fall within orientation 3. These talk about exploring relationships, conjecturing, investigating, justifying and proving and establishing properties. The Space, Shape and Measurement learning outcome is the learning outcome in which the mathematical practices feature most strongly. In the Number and Number Relationships learning outcome they are only mentioned in 1 of the 6 bullet points, and in the Functions and Algebra learning outcome in 1 of the 4 bullet points. They are not mentioned in the description of Data Handling and Probability learning outcome.

Four of the 9 bullet points about what the study of Space, Shape and Measurement enables learners to do can be seen as linking directly to orientation 4. These four bullet points are however not specific in terms of content. They read "link algebraic and geometric concepts through *analytic*

geometry", "link the use of *trigonometric relationships*[19] and *geometric properties* to solve problems", "use *construction and measurement or dynamic geometry* software for exploration and conjecture", "use *synthetic, transformation or other geometric methods* to establish *geometric properties*". In contrast we see bullet points in the Number and Number Relationships LO and Functions and Algebra LO that are far more content specific. In the Number and Number Relationships LO we have "develop the concepts of simple and compound growth and decay" or "solve problems related to arithmetic, geometric and other sequences and series, including contextual problems related to hire-purchase, bond repayments and annuities", for example. In the Functions and Algebra LO we have "considering the increasing and decreasing nature of functions, rates of change, gradient, derivative, maxima and minima", for example. Although the description of the Data Handling and Probability LO does not have bullet points it does mention specific content, e.g. measures of central tendency and measures of spread are specifically mentioned. The fact that the Space, Shape and Measurement LO mentions content at a level of generality together with the relatively strong weighting towards the orientation (in comparison to the other LOs) discussed above marks it out as different to the other LOs and certainly gives the impression of any particular content being subservient to the mathematical practices.

6.2.3 Orientations 3 and 4 in chapter 3 of the MNCS

Chapter 3: Learning outcomes, assessment standards, content and contexts			
	Orientation 3	**Orientation 4**	
Space, Shape and	3 out of 7 assessment	4 out of 7 assessment	

[19] The italics are my formatting to indicate the particular aspects of the bullet point that I see as reflecting orientation 3.

Measurement LO: Assessment standards grade 10	standards	standards	
Space, Shape and Measurement LO: Assessment standards grade 11	3 out of 7 assessment standards	5 out of 7 assessment standards	
Space, Shape and Measurement LO: Assessment standards grade 12	3 out of 6 assessment standards	4 out of 6 assessment standards	
Content for Number and Number Relationships LO			No focus on practices. States that the outcomes is about representing, calculating and working confidently with numbers.
Content for Functions and Algebra LO			Investigation is mentioned, but the focus is on analysing describing and representing functions.
Content for Space, Shape			There is a focus on investigation, development

and Measurement LO			of proof skills and understanding axiomatic systems. But the need to use measurement and calculation and to make connection between geometries is also discussed.
Content for Data Handling and Probability LO			No specific mathematical practices are discussed. Focus is on collecting and organising data to solve problems.
Contexts: inclusivity, human rights and indigenous knowledge systems			Emphasises mathematical process skills and does not emphasise mathematics as a body of knowledge.

Table 27: Orientations 3 and 4 in chapter 3 of the MNCS

Assessment standards for Space, Shape and Measurement LO Mathematical practices feature in 3 of the 7 assessment standards for the Space, Shape and Measurement LO in grades 10 and 11 and in 3 of the 6 assessment standards for this learning outcome in grade 12. Investigating, conjecturing, justifying, explaining, proving, producing counter-examples, generalising and deriving formulae are the scope of mathematical practices discussed in this learning outcome. The assessment standards for this learning outcome also give specific detail of mathematical content to be learnt. In 4 of the 7 assessment standards for grade 10, 5 of the 7 for grade 11 and 4 of the 6 for grade 12, content is given as the focus. For example, in grade 11 one part of an assessment standard reads: "Prove and use (accepting results established in earlier grades): that a line

drawn parallel to one side of a triangle divides the other two sides proportionally (the Mid-point Theorem as a special case of this theorem); that equiangular triangles are similar; that triangles with sides in proportion are similar; the Pythagorean Theorem by similar triangles" (MNCS, p33). And in grade 12, for example, one of the assessment standards states "Use a two-dimensional Cartesian co-ordinate system to derive and apply: (a) the equation of a circle (any centre); (b) the equation of a tangent to a circle given a point on the circle" (MNCS, p33).

Content and contexts for the attainment of assessment standard. The paragraphs describing each of the learning outcomes reflect a similar balance between orientation 3 and orientation 4 for each of the learning outcomes, as was displayed in the description of the learning outcomes in chapter 2 of the MNCS. This means that for each paragraph describing the learning outcomes there was an emphasis on orientation 4. Although investigation was mentioned in the Functions and Algebra learning outcome, it was only in the space, shape and orientation learning outcome that mathematical processes were clearly highlighted.

The paragraph in the Space, Shape and Measurement learning outcome discusses the use of geometry to solve contextual problems and provides tools for the investigation of space. It states "The treatment of formal Euclidean geometry is staged through three grades so as to assist in the gradual development of proof skills and an understanding of local axiomatic systems" (MNCS, p54). This indicates clearly that the process by which mathematics is established and organised is an important component of this learning outcome.

The section on contexts gives priority to orientation 1, however aspects of orientation 3 do occur here. The fact that mathematics is a human creation is emphasised and the importance of what the MNCS term mathematical process skills is discussed. This section states: "The mastery of Mathematics depends to a large extent on mathematical processes such as investigating patterns, formulating conjectures, arguing for the generality of such conjectures, and

formulating links across the domains of Mathematics to enable lateral thinking" and further states "This curriculum focuses on the development of mathematical process skills, and in so doing endeavours to unlock the power of Mathematics" (MNCS, p62). It suggests that it is important for the teacher to win learners to Mathematics and states that this can be done by "complying with the Assessment Standards of the subject, not formalising into the abstract prematurely but first taking care to develop understanding and process skills" (MNCS, p62). If one puts these together an argument emerges from this section that mathematical processes such as investigating, conjecturing and generalising are at the heart of mathematics and should be prioritised, with formalising only following after these have been worked with.

Orientation 4 is not given any specific mention in the discussion of contexts.

6.2.4 Orientations 3 and 4 in chapter 4 of the MNCS

Chapter 4: Assessment			
	Orientation 3	**Orientation 4**	
Competence descriptions level 6 (Outstanding)	6 out of 14 bullet points	7 out of 14 bullet points	Competence descriptors are split between content and practices. The mathematical practices mentioned include investigating, generalising, conjecturing, justifying and proving and understanding local axiomatic systems.
Competence descriptions level 5 (Meritorious)	7 out of 14 bullet points	9 out of 14 bullet points	Competence descriptors are split between content and practices. The mathematical practices mentioned include investigating, generalising,

			conjecturing, justifying and proving.
Competence descriptions level 4 (Satisfactory)	4 out of 23 bullet points	19 out of 23 bullet points	Competence descriptors are largely content related. The mathematical practices, investigation and justification are mentioned.
Competence descriptions level 3 (Adequate)	2 out of 15 bullet points	13 out of 15 bullet points	Competence descriptors are largely content related. The mathematical practice, investigation is mentioned.
Competence descriptions level 2 (Partial)	3 out of 15 bullet points	12 out of 15 bullet points	Competence descriptors are largely content related. The mathematical practice, investigation is mentioned.
Competence descriptions level 1 (Inadequate)	0 out of 7 bullet points	7 out of 7 bullet points	Competence descriptors are content-related.

Table 28: Orientations 3 and 4 and chapter 4 of the MNCS

Competence descriptors. At level 1 there are no competence descriptors mentioning mathematical practices. At level 2 and level 3 for grades 10 and 11 the only mathematical practice mentioned in the competence descriptors is investigation and here it is noted the learners would do the investigation "in an unstructured an arbitrary manner" (MNCS, p80). Thus it is only in grade 12 at level 3 competence that successful performance of a mathematical practice is required. Here they state that the learner will be able to do investigations "in a structured manner to arrive at conclusions." This is repeated in level 4 in grades 10 and 11. For level 4 there are also competence descriptors that involve

justifying inference, and finding errors in logical arguments. In contrast to the dearth of bullet points involving mathematical practices at the lower levels we see that about half of the bullet points for each grade at levels 5 and 6 involve mathematical practices. These involve investigations, generalisations, conjecturing, providing logical arguments and proof. For grade 12, level 6, the bullet points directly involving mathematical practices read:

> extend investigations, posing insightful questions; critically analyse and compare mathematical arguments and proof; demonstrate an understanding of proof in local axiomatic systems. (MNCS, p73)

Here we see that, at the highest level of competence, learners are expected to be able to both create and organise mathematics.

From level 4 down to level 1 the majority of bullet points for each grade for each level reflect orientation 4 and mention specific content. These include statements such as the learner can "make good estimates" (grade 10, level 4); "draw and interpret graphs and diagrams and investigate problem situations" (grade 12, level 4); "solve simple equations" (grade 11, level 3); "in a rote manner use the techniques, algorithms and formulae learned in this and lower grades (grade 12, level 2); and "interpret simple diagrams and graphs" (grade 10, level 1). In contrast, at levels 5 and 6 the bullet points are more general in nature and thus harder to see as pertaining to a specific piece of content knowledge. For example we see bullet points like "think creatively and laterally on a broad range of complex mathematical concepts" (grade 12, level 6) and "produce clear logical solutions to and proofs to routine and simple non-routine problems (grade 11, level 5). The "mathematical concepts" and "problems" referred to here will clearly be part of or rely on some mathematical knowledge base, but are not related to specific content.

6.2.5 Discussion of issues arising from analysis of orientations 3 and 4

I discuss the issues that arise from these two orientations together as the MNCS sets up a tension between them that is important to explore. In the introduction

there is a statement that implies that in the past too much emphasis has been laid on knowledge instead of skills and that the MNCS intends to remedy this. With the unfolding of the definition and purpose of mathematics in chapter 2 of the MNCS, it becomes clear that in terms of mathematics this means a prioritisation of orientation 3 over orientation 4 i.e. that mathematical practices are "more important than the acquisition of content knowledge for its own sake" (MNCS, p9). However when we get to the section of the MNCS entitled *Scope*, only one of the bullet points stating what learners will work towards being able to do deals with mathematical practices, whereas eight align with the notion of mathematics as a body of knowledge.

Throughout the various chapters mathematical practices, which are termed "mathematical process skills" in the MNCS (MNCS, p9), are described as valuable and important in and of themselves. However in chapter 3 we see, in the discussion about ensuring that all learners are able to participate in mathematics, a statement that suggests that "not formalising into the abstract prematurely, but first taking care to develop understanding and process skills" (MNCS, p62) is important. This suggests a role for mathematical practices in the learning of mathematics.

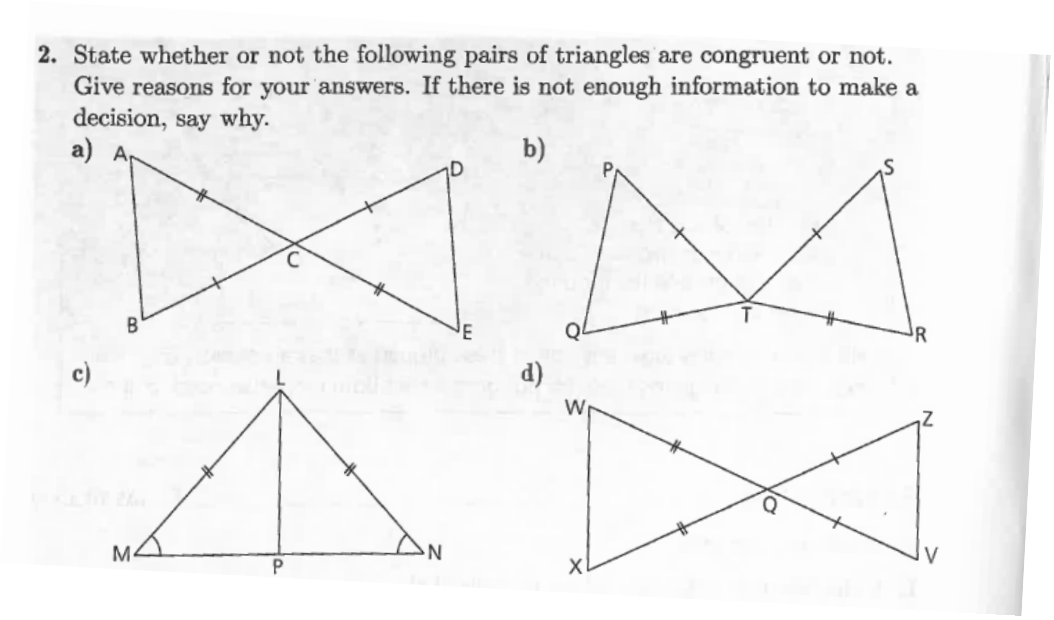
The competence descriptors are interesting in the way orientations 3 and 4 differ in their placement within these. The ability to deal with mathematical practices is placed in the higher levels (largely levels 5 and 6) of competence and largely absent in the lower levels. In contrast in the lower levels of the competence descriptors (levels 1 to 4) the majority of the description deals with ability to perform particular procedures or know particular pieces of knowledge. At the higher levels (largely levels 5 and 6) mastery of content is not directly mentioned but is implied as it would need to provide a basis for the problem-solving and creative thinking around mathematical concepts that are mentioned in their competence descriptors.

What is of particular interest to this study is the fact that orientation 3 seems to feature most strongly in the description of the Space, Shape and Measurement

learning outcome and does not feature particularly strongly in the descriptions of the other learning outcomes. In addition orientation 3 is also prominent in the assessment standards for learning outcome 3.

6.3 Analysing orientations 3 and 4 in the textbooks

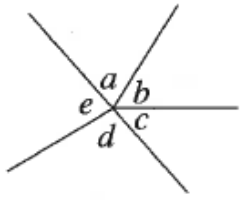
An initial attempt to decide which of orientation 3 or 4 each block in the textbook prioritised indicated that this method of analysing these two orientations would be problematic. For example the question shown in extract 14 below could be categorised as belonging to orientation 3 as learners are asked to "give reasons for your answers". On the other hand it could be categorised as belonging to orientation 4 as it asks learners to apply the cases for congruency of triangles in some specific examples.



Textbook extract 14: Question where orientation not clear (CM, p66)

A further example of the difficulty can be illustrated by looking at the block shown in extract 15 below. Again the subject matter (axioms, theorems) link directly with orientation 3 and speak to how mathematics is created and organised. However at the same time it links to orientation 4 as it provides the meaning of some standard terms in mathematics.

What do all these words mean? We use the example on page 227 to explain all these new words.

Word	Meaning	Example on page 227
Undefined term	In the modern version of Euclidean Geometry the **undefined terms** are "point", "straight line" and "plane". If you try to define these terms, you are forced to use other words which you have also not defined. Try!	
Axiom	Axioms are facts that we accept as true.	In the example the axiom we used is: If one straight line meets another, the sum of the adjacent angles formed is 180°.
Theorem	A theorem is a result we obtain from axioms or from other theorems using deductive reasoning.	In the example we used deductive reasoning to show that, if two straight lines intersect, the vertically opposite angles are equal.
CorollaryⒼ	Sometimes it is possible to logically deduce a result from an axiom without giving a logical argument. We call this a **corollary**.	If any number of straight lines are drawn from a point , then the sum of the angles around the point is 360°  $a + b + c + d + e = 360°$

Textbook extract 15: Meaning of standard terms (OTM, p228)

These difficulties essentially relate to the idea that it is in the doing of mathematics orientation 3 and 4 are inextricably intertwined. For example, when one proves (orientation 3), one has to prove something that then becomes a fact (orientation 4) in the mathematical canon. Or when one investigates (orientation 3) one has to investigate the properties of a mathematical object (orientation 4). Clearly, while at one extreme, it would be possible to just list a range of facts and be displaying work entirely in orientation 4, the majority of

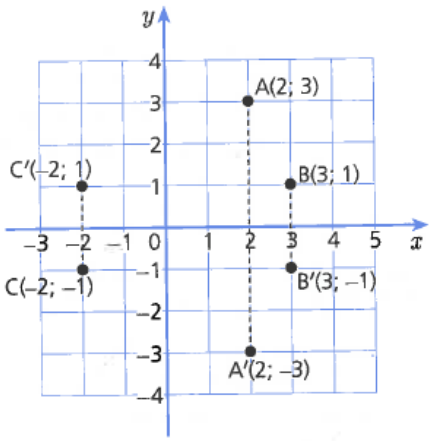
blocks in the geometry chapters in the textbooks contained work that could be described as having elements of both.

The initial attempt at analysis of block according to orientation 3 and 4 also highlighted the fact that it would be worthwhile looking at the nature of mathematical practices, the relationship between the mathematical practices and the relationship between the mathematical practices and the mathematical objects, representations and conventions they brought into being. To explain what I mean by this I use an example shown in textbook extract 16 below to illustrate.

This exercise could be classified as asking learners to investigate because the learners have to use observations from the table to make conjectures about the perpendicular distances and coordinates. However the learners are guided step by step in terms of what information to gather and then what aspects to look at and comment on. Thus the process of investigation is set up for the learners and they do not have to design the investigation themselves. The investigation starts with stating that learners are looking at reflections and then they proceed to investigate reflections to find out something that is, in fact, a defining feature of reflections (i.e. that the perpendicular distance from the line of reflection to the point is equal to the perpendicular distance from the line of reflection to the reflected point). This raises the question about what notion of reflection is used at the starting point and the intended relationship between this notion of reflection and the investigation.

Exercise 9.5 (AS 10.3.4)

1.



In the diagram A′, B′ and C′ are the reflections in the x-axis of the points A, B and C respectively.

a) Copy and complete the following table:

Point	Perpendicular distance from the x-axis ($y = 0$)	Sign of the x-coordinate	Sign of the y-coordinate
A			
A′			
B			
B′			
C			
C′			

b) Comment on the perpendicular distances from the x-axis ($y = 0$) of each pair of points A, A′, B, B′ and C, C′.

c) Comment on the sign of the x-coordinate in each pair of points.

d) Comment on the sign of the y-coordinate in each pair of points.

e) A′ is called the reflection of A about the x-axis ($y = 0$). A′ is also called the image of A obtained by reflection in the x-axis. If $P(x; y)$ is any point and P′ is the reflection of P about the x-axis comment on:

(i) the perpendicular distance of P′ from the x-axis

(ii) the sign of the x-coordinate of P′

(iii) the sign of the y-coordinate of P′.

Textbook extract 16: Exploring reflection (CM, p199)

These types of issues indicated that simply analysing orientation 3 or 4 by categorising each block of the textbook according to the orientation would not provide the type of information that would enable me to answer my research questions. It became apparent that analysing orientation 3 and 4 and the interplay between them in the textbook chapters required a way of talking about geometric work that takes into account the nature of geometry. The work of

Houdement and Kuzniak (2003) on the paradigms of geometry answered this need most closely. In the discussion of the geometric paradigms Houdement and Kuzniak look both at the nature of processes used to create facts and the way in which geometrical objects are viewed and represented. I thus analysed the way in which the geometric paradigms were presented in the textbook in order to capture the key aspects of orientation 3 and 4 in a way that did justice to the complexity of geometric work. This analysis is discussed in detail in the next chapter.

7 Analysis of paradigms of geometry used in the MNCS and the textbooks

7.1 Introduction

Chapter 7 represents the empirical heart of the thesis. This happened in part because orientation 1 and 2 turned out to not play a significant role in the geometry chapters of the textbooks and thus examining the way in which orientations 3 and 4 manifested in the textbooks became key. In addition to this, the fact that the interplay between these orientations was difficult to pull apart necessitated creating a framework that would capture the geometry-specific nuances of these orientations. I chose to base this framework on the work of Kuzniak and his colleagues whose discussion of the geometric paradigms (GI, GII and GIII) I outlined in chapter 2. Chapter 7 is dense and lengthy. For this reason I provide an outline of the way the story unfolds in chapter 7 in order to help orientate the reader as to what is to come.

The sections of chapter 7, along with a brief description of each, are shown below:

7.2 Initial analytic framework: In this section I discuss how I initially had intended to identify, in the geometry sections of the MNCS and the chapters in the textbooks, which paradigm of geometry (GI or GII) is favoured by looking at the nature of the objects, nature of the tools and nature of deduction and validation.

7.3 Curriculum analysis: Here I analyse the geometry sections of the MNCS and show that they indicate a movement towards a GII paradigm as the learners move from grade 10 to grade 12 in the Euclidean geometry section, but with greater ambiguity in the transformation geometry and coordinate geometry sections.

7.4 Problems in using the framework for the textbook analysis: Here I indicate how the nature of the geometric work in the textbooks made it difficult to analyse the nature of deduction and validation required, the nature of the objects and the nature of the tools. I discuss how this necessitated my looking separately at the work on transformation geometry, coordinate geometry and at the work related to the study of the properties of geometric shapes[20]. For each of these I developed analytic tools drawing on the work on the paradigms of geometry and used these to analyse the textbook chapters. These are reported on separately in **7.5 Transformation geometry**, **7.6 Coordinate geometry** and **7.7 The properties of geometric shapes**.

7.8 Discussion: In this section I pull together the themes that emerged from the analysis of the textbooks and discuss them together with extracts from the interviews.

7.2 Initial analytic framework

As discussed in chapter 2, the distinction that Houdement and Kuzniak (2003) make between three paradigms of geometry provides a useful tool for analysis of the nature of geometry embodied in the curriculum and textbooks.

As mentioned by Kuzniak and Vivier (2009), geometry at school level is usually based in GI and GII and these paradigms will be the focus of my work.

Kuzniak's (2011a) notion of the geometric work space (GWS) provides further constructs that are useful in examining the nature of school geometry although, as discussed in chapter 2, because I am not looking at teachers' or learners'

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[20] In both textbooks the chapters that related to the properties of geometric shapes were largely based in Euclidean geometry. However they also included work based in coordinate geometry and transformation geometry as well as empirical explorations. These chapters deal with the properties of triangles and quadrilaterals and thus I have referred to these chapters as "The properties of geometric shapes".

interactions with the texts, I will not be able to look at the GWS in full. However the notion that a work space is constituted by the tools (e.g. drawing instruments or software) available for use and the theoretical system of reference is useful in probing the nature of geometric work a written text presents. Combining these ideas together with the ideas of GI and GII summarized in the table 2 on p54, indicated that in analysing each block of the textbook I would need to look at three things:

a) Nature of objects: I saw that this would encompass three aspects from the table above viz. kind of space, status of drawing and objects.

b) Nature of tools: this aspect of the GWS helps to confirm the nature of the objects being dealt with, and helps distinguish the role of intuition and the nature of deduction and validation.

c) The nature of deduction and validation: this aspect would distinguish whether deduction and validation were based on axioms or were linked to physical or perceptual experiment.

7.3 Analysis of MNCS in terms of paradigms of geometry

In analysing the MNCS in terms of the paradigms of geometry, it clearly made sense to look only at the sections of the curriculum that related specifically to geometry.

I analysed the description of the *Space, Shape and Measurement Learning Outcome* in the MNCS in relation to the paradigms of geometry. The description did not point specifically to either a GI or GII paradigm. The overarching statement of this learning outcome typifies this ambiguity:

The learner is able to describe, represent, analyse and explain properties of shapes in 2-dimensional and 3-dimensional space with justification. (MNCS, p13)

It is possible that the space referred to is “intuitive and physical space”, but it is equally possible that it is “physical and geometrical space”. No specific tools for

analysis are mentioned and the justification referred to could be linked to experiment or deduction based on axiom. The overarching statement thus does not indicate a clear orientation to either GI or GII.

In the paragraph that provides more detail on the Space, Shape and Measurement learning outcome there is indication of movement towards GII from the GI orientation that was prevalent in the General Education and Training band. It states that the work must build on the experiences from that band to "make formal and extended levels of knowledge accessible" and that as "learners' previous knowledge becomes deeper … they become more proficient in processes leading to proof" (MNCS, p13). Although this indicates a movement towards GII, other statements in this paragraph retain some of the ambiguity e.g. the statement that "location, visualisation and transformation" are important for attaining this learning outcome or that the learners should "engage with new tools that can be used in a range of applications" (MNCS, p13).

In the nine bullet points that follow this paragraph, six were ambiguous or could not be classified in terms of the geometric paradigm they favoured e.g. "link algebraic and geometric concepts through analytic geometry" or "use synthetic, transformation or other geometric methods to establish geometric properties" (MNCS, p14). Only one, the statement "investigate geometric properties of 2-dimensional and 3-dimensional figures in order to establish, justify and prove conjectures" (MNCS, p14) gave a clear indication of intending to work with deductive methods of validation and thus in GII. Two bullet points referred specifically to work in GI: one of these referred to the use of construction, measure and dynamic geometry software for exploration and the other to the analysis of "natural forms and cultural products and processes as representations of shape and space" (MNCS, p14).

In the detail of the assessment standards I looked at each assessment standard related to geometry separately. The results of this are summarised in the table below and discussed in detail thereafter.

Area of geometry	Grade 10 Assessment standard	Grade 11 Assessment standard	Grade 12 Assessment standard
Area and volume	Ambiguous	Ambiguous	No assessment standard on area and volume in grade 12
Euclidean geometry	1 assessment standard: ambiguous 2 assessment standards: GII	1 assessment standard: ambiguous 1 assessment standard: GII	2 assessment standards: GII
Coordinate geometry	Ambiguous	Ambiguous	Ambiguous
Transformation geometry	Ambiguous	Ambiguous	Ambiguous
Modelling with trigonometry and geometry	Ambiguous	Ambiguous	Ambiguous
History of geometry and trigonometry	Ambiguous	Ambiguous	Ambiguous

Table 29 Paradigms of geometry in MNCS

In the area of transformation geometry, coordinate geometry and area and volume it was possible to interpret the assessment standards from a GI or GII paradigm perspective. The assessment standards in these sections were all similar. An example would be:

> Investigate, generalise and apply the effect of the following transformations of the point (x; y)
> a) a translation of p units horizontally and q units vertically
> b) a reflection in the x-axis, the y-axis or the line y=x. (MNCS, p34)

It would be possible to do this using a drawing on the Cartesian plane and physically doing the transformations to conclude what the effect on any coordinates would be empirically (GI). It is also possible to generalise the effect of the transformation on the coordinates using deduction from known geometric facts and a geometric definition of the transformation (GII).

The assessment standards related to modelling and history are stated in a broad way and could be interpreted to require rigorous work in GII or could be worked on from a GI perspective.

In relation to Euclidean geometry we see some ambiguity in the assessment standards for grade 10 and 11. In grade 10 this takes the following form:

> Through investigations, produce conjectures and generalisations related to triangles, quadrilaterals and other polygons, and attempt to validate, justify, explain or prove them, using any logical method (Euclidean, co-ordinate and/or transformation). (MNCS, p32)

Here, although there is reference to proving conjectures using Euclidean methods that would imply GII, it is also suggested that other methods of justification or explanation would also be acceptable. There is no discussion of the distinction between a justification, explanations, validation and proof and so it is unclear as to what is meant by these. However it could be interpreted as allowing for validation using physical tools and experiment and would thus fall under GI.

In both grade 10 and 11 there are however assessment standards in Euclidean geometry that fall within the GII paradigm and by grade 12 all assessment standards fall within this paradigm. These assessment standards refer clearly to proof, based on previously established facts.

In addition to the learning outcomes and assessment standards which specify the geometry to be learnt, the competence descriptors indicate the attributes which will be looked at in the assessment of learners. Although these are given in general terms, they do include attributes that are relevant to this discussion. The competence descriptors state that in order to achieve a level 6 rating (the highest level) the learners must, in grades 10, 11 and 12, be able to use deductive argument. In grade 12 they need to be able to "demonstrate an understanding of proof in local axiomatic systems" (MNCS, p73). To achieve a level 5 rating in grades 11 and 12 they need to be able to produce proofs, but at the lower levels this is not explicitly mentioned. Thus the competence descriptors seem to imply a desire for an understanding of deduction based on axioms, but indicates it as a high level skill.

Thus in summary the curriculum indicates a movement towards a GII paradigm through the grades, but is not unequivocal in this regard. Although within Euclidean geometry the movement towards GII seems clear, in transformation and coordinate geometry the desired paradigm is ambiguous.

7.4 Problems in using the framework for the textbook analysis

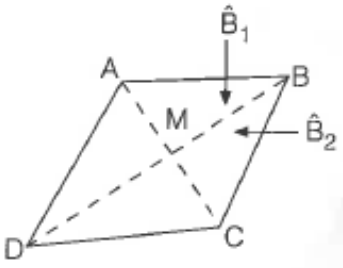
My initial attempt at analysis of the blocks in the textbook immediately threw up a number of problems with the initial analytic framework described in 7.2. I elaborate these problems below:

7.4.1 Issues relating to the nature of deduction and validation

It had originally seemed that it would be straightforward to make a distinction between validation based the use of instruments/perception (GI) or deduction (GII) and between the GI form of deriving new facts as linked to experiment and that of GII as proceeding deductively from the axioms. However, in interacting with the textbooks this was often less straightforward.

For example, the proof in textbook extract 17 below is from OTM. Although at first glance this appears to be a formal proof and thus fits into the GII paradigm, the reason why proving two corresponding sides and an included angle equal is sufficient to ensure congruency is not established in this book. This suggests it is a known fact from previous grades. However in the GET band, which encompasses the previous grades, no formal proofs are provided. Thus the cases for congruency of triangles are likely to have simply been stated or derived by experiment. Thus although the nature of validation is deduction, the deduction is not based on axioms but on facts established by experiment.

I used congruency:
I drew in the diagonals DB and AC.
In an earlier activity I used congruency
to prove $\Delta ABD \equiv \Delta CBD$
So $\hat{B}_1 = \hat{B}_2$
Then I focused on ΔABM and ΔCBM:



BM is a common side
AB = BC (sides of a kite)
$\hat{B}_1 = \hat{B}_2$ (angles of a kite, proved)
So $\Delta ABM \equiv \Delta CBM$ (SAS)
$\therefore$ AM = MC

It is not possible to prove that ΔADM and ΔABM are congruent, so $BM \neq MD$.

Textbook extract 17: Proof (OTM, p210)

In addition, as area and volume are topics in the curriculum both textbooks contained questions asking learners to calculate the area or volume of various shapes. The area and volume formulae were given. Thus these kinds of questions did not seem to rely on perception, the use of instruments or deduction.

In the work on coordinate geometry, having established the formula for the distance between two points, learners were asked to show that

If $P(x_p; y_p)$ and $Q(x_q; y_q)$ are points on a plane then

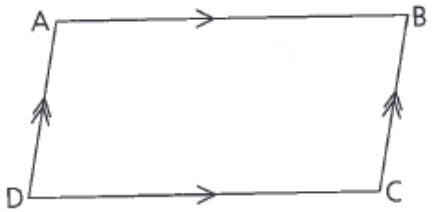
$$\sqrt{(x_q - x_p)^2 + (y_q - y_p)^2} = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}$$

This deduction can be made purely algebraically.

7.4.2 Issues relating to nature of objects:

The distinction between an object as physical (GI) or theoretical (GII) and between a drawing as the object of study and of validation (GI) and as a figural concept that acts a support for reasoning seems, and is, obvious at the extremes. For example, if I am asked to cut out a square and make deductions by folding it or I take measurements with a ruler on a drawing to make deductions it is clear that I am dealing with the object as something physical and the drawing is the object of study and of validation. Similarly if one is provided with a task such as the one shown in textbook extract 18 below, it is clear that the object is "theoretical" and the drawing is to be used to support reasoning that is based on deduction.

The quadrilateral ABCD has
AB ∥ DC and AD ∥ BC.
Prove that $\hat{A} = \hat{C}$ and $\hat{B} = \hat{D}$.



Textbook extract 18: Illustration of object as "theoretical" (CM, p342)

However in between these two extremes were a number of other objects.

For example, in CM learners are asked to state whether the conjecture "All squares are rectangles" is true or false and, if false, to provide a counter-example to prove this. On the face of it one might assume that the objects (squares, rectangles) being referred to are theoretical (i.e. controlled by a definition). However at the stage in CM where this conjecture is presented the definitions of

a square and rectangle have not been discussed. Thus the status of the objects referred to is ambiguous.

In addition objects such as the following appeared in both textbooks:

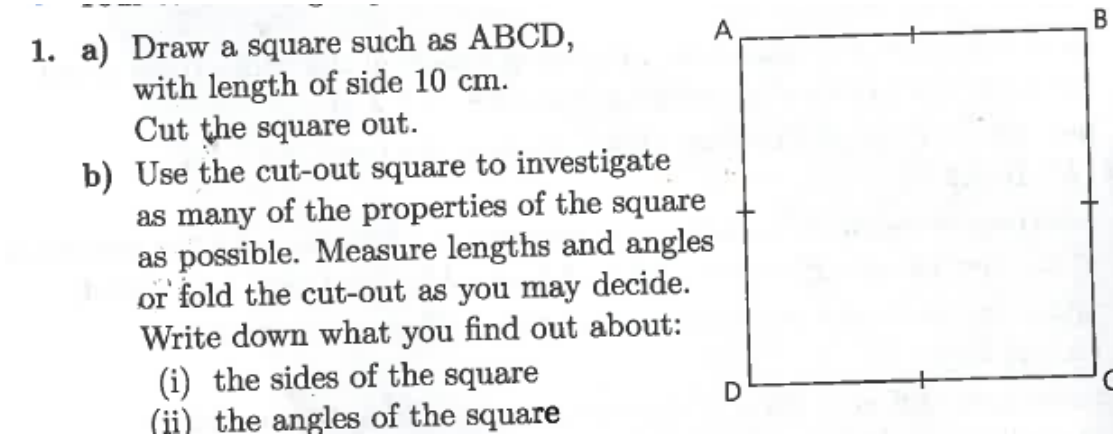
A(1; 1), B(3; 6) and C(6; 3) are points on the Cartesian plane. What type of triangle is ΔABC?

It is possible to state that it is an isosceles triangle simply by applying the distance formula, without having to draw the triangle or visualise it at all.

7.4.3 Issues relating to nature of tools:

Although drawing instruments and computer software are mentioned as examples of the tools (Kuzniak, 2011a; Kuzniak & Vivier, 2009), in analysing the blocks there seemed to be other possible tools.

One tool that became apparent in the analysis was the use of manipulation as a tool. In textbook extract 19, shown below, we see that learners are asked to cut out and fold the square to make deductions.

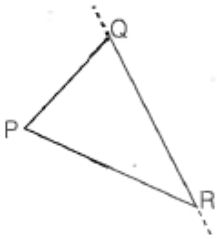

1. a) Draw a square such as ABCD, with length of side 10 cm. Cut the square out.
   b) Use the cut-out square to investigate as many of the properties of the square as possible. Measure lengths and angles or fold the cut-out as you may decide. Write down what you find out about:
      (i) the sides of the square
      (ii) the angles of the square

Textbook extract 19: Exploring properties of the square (CM, p336)

However in the activity shown in textbook extract 20 below, learners are explicitly asked to visualise the transformations, thus visualisation seemed to be a tool that was being employed to make deductions.

We can work with transformations in two ways – we can physically translate, reflect, or rotate a shape or we can visualise what will happen when we perform each of these transformations. In order to work efficiently as a mathematician, we need visual skills that will help us to predict a result. It is comforting to know that we can check our prediction by actually doing the transformation, if necessary. Practise your visual skills in this Investigation.

1. PQR is an acute-angled scalene triangle. What quadrilateral is formed if the triangle is reflected in the line QR? Explain your answer.
2. (a) What if ΔPQR is an acute-angled isosceles triangle with PQ = QR? Investigate what quadrilateral is formed by reflecting the triangle in the line QR. Explain your reasoning.

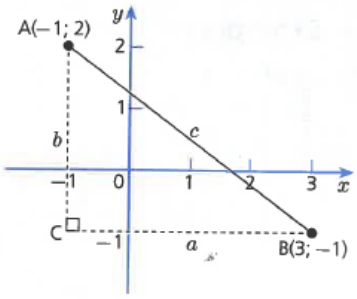


Textbook extract 20:Visualisation (OTM, p212)

The use of the Cartesian plane also raised issues relating to the nature of tools. For example, in a question like the one shown in textbook extract 21 below, to find the length of AC, one can use the markings on the Cartesian plane. So, although no physical measuring tool is used, the plane itself operates like a tool for measurement.

Activity 4.1 (AS 10.2.1)

- Work in pairs.
- You should know the Theorem of Pythagoras well before you do this activity. (See Chapter 3.)
- Let A(−1; 2) and B(3; −1) be the end points of a line segment in a plane.



1. a) Copy the graph alongside. Complete the right-angled △ABC with AB = c, AC = b and CB = a.
   b) Write down the coordinates of point C.
   c) Then find the length of:
   (i) AC (ii) BC
2. Now use the Theorem of Pythagoras to find the value of c.

Textbook extract 21: Cartesian plane as measuring tool (CM, p79)

7.4.4 Revision of the analytic framework

The issues raised above spoke to a need to operationalize the categories for classification of geometric work into GI or GII more carefully. However it also appeared that there were different issues arising in the work on coordinate and transformation geometry, Euclidean geometry and the work on area and volume. It thus seemed important to look at these different sections of geometry separately and to explore what GI and GII could look like in these sections of geometry.

In addition my initial work with both textbooks revealed that it would not be illuminating to investigate the chapters on area and volume in relation to the geometric paradigms in detail. This chapter in both textbooks essentially deals with demonstrating how to calculate the surface area and volume of various 3D shapes and exploring what happens when the dimensions are increased by a factor. Thus they are not attempting to build a geometry. Attempting to analyse these chapters in terms of geometric paradigms does not provide any greater insight. I thus do not include these chapters in the analysis.

Transformation geometry is dealt with as a separate chapter in both textbooks and is used in a very limited way in exploring the properties of geometric shapes. I thus discuss how one could identify the paradigms of geometry in relation to transformation geometry separately and have analysed the way this manifests in the textbook. This is discussed in section 7.5 below.

Coordinate geometry is used to explore the properties of geometric shapes in both textbooks. In this sense it became another tool (like measurement or deduction from axioms) in the exploration of geometric shapes. It thus needed to be incorporated as a tool into the analysis of the geometric paradigms used in the exploration of geometric shapes. However it is important to first discuss how the formulae and related facts of coordinate geometry are established. Thus section 7.6 below describes how the formulae and related facts of coordinate geometry are established in each textbook. Then, in section 7.7 the way in which these formulae and facts, along with other tools and types of reasoning (such as measurement or deductive reasoning) are used in establishing or exploring

properties of geometric shapes is discussed. Section 7.7 deals with the way in which the textbooks deal with exploring and establishing the properties of triangles and quadrilaterals. As discussed earlier these sections of the textbook relate to the parts of the MNCS that deal with the development of Euclidean geometry.

7.5 Transformation geometry

One of the complications in talking about transformation geometry is the existence of two different mathematical objects. The first types of object are shapes that are transformed. The second are the transformations themselves. Clearly the transformations can be viewed as an operation (e.g. a triangle **is reflected**), but ultimately they can be viewed as an object (e.g. a **reflection**). This duality of mathematical concepts as both process and object has been pointed out clearly by Sfard (1991) and similar ideas have been elucidated by Dubinsky in his APOS theory (Asiala et al., 1996). Within APOS theory one can know a mathematical concept as an action, a process or an object which are then organised into a schema. Hollenbrands (2003) used APOS theory to explore transformation geometry. Her work is useful in trying to think about aspects of what might constitute GI, GII or GIII in relation to transformation geometry.

Hollenbrands (2003) argues that to have an object understanding of transformations students need to view them as one-to-one, onto functions that map points in the plane to points in the plane. Transformations viewed as objects can then be composed with each other. It is then possible to look only at the functions and composition of the functions and detach them from the physical referents. Edwards (2003) talks of how this move historically, in the form of Klein's Erlangen programme, "changed the focus of geometry from specific objects with concrete, visualizable referents (points, lines, planes, etc.) to notions of invariance, group theory, and mappings" (p4-5). Clearly this view of transformation geometry can be seen as representing the GIII paradigm, where deduction is based on axioms and objects are not linked to physical drawings.

In discussing an action conception of transformation Hollenbrands gives the example of finding the image of a point P under translation by a vector as requiring substitution of the coordinates into the equations or requiring various mouse clicks in a dynamic geometry software (DGS) package to produce the result. She talks of a student as having a process conception when they do not have to physically perform the transformation to see the results of the transformation and that they would be able to identify the invariant features of a transformation. For the invariant features she gives the example of the fact that a translation preserves distances between points. It is tempting to align the action conception with GI and the process conception with GII as the first seems more "concrete" and the second a move towards abstraction. However the primary distinction between GI and GII is in relation to the physical. It thus seems that if one used coordinates in transformation geometry in the GI environment the "concreteness" would not be about substitution in a formula or mouse clicks as much as it would be about making and maintaining a strong connection between the physical movement on the plane and the resultant effect on the coordinates. In contrast in the GII environment, although one would be able to see or imagine a clear connection between the effect on coordinates and the movement in the plane this would be less tied to physical movements on the plane, would be couched in more general terms and would begin to provide a basis for deductions from axioms. The contrast can be best understood through the use of examples.

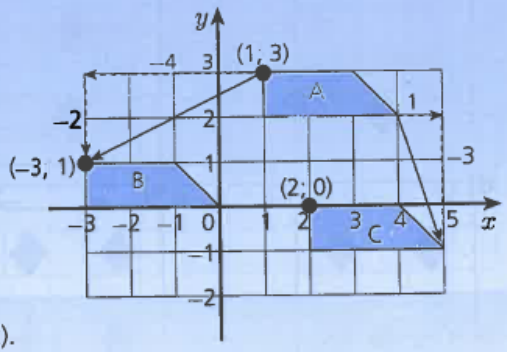
In the example shown in textbook extract 22 below, we see a very strong link being made between the physical movement on the plane and the effect on the coordinates. One could count the 4 units to the left and 2 units down on the grid to see where each point ends up. Although there is a move towards a general rule (e.g. a translation of 4 units to the left and 2 units down translates $(x;\ y)$ to $(x-4;\ y-2)$) it is still strongly tied to the physical movement. The Cartesian plane (especially if one superimposes a grid onto it) becomes a tool that can be used to provide an empirical validation (similar to drawing and measuring instruments). This block would thus be classified as being in the GI paradigm.

Translation of regions

- A translation changes the position of a region, but not its orientation, size or shape. The regions are congruent (they have not changed size of shape).
- A region is translated by "sliding" it to another position without reflecting it or rotating it.

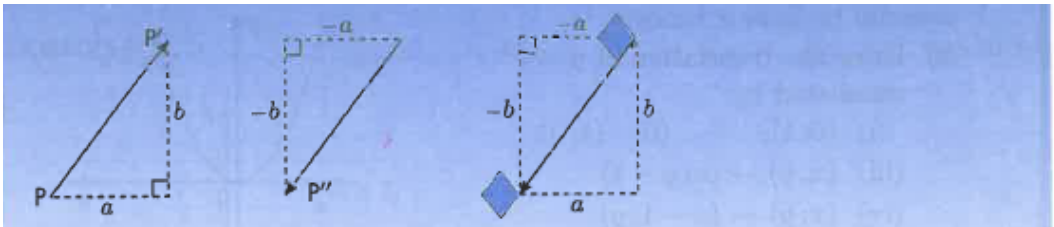
Example

- The region A is translated to the region B by the translation $(-4; -2)$ i.e. 4 units to the left and 2 units down.
- Every point $(x; y)$ in region A is translated by $(-4; -2)$ to $(x - 4; y - 2)$ in region B. B is the image of A. The translation can be represented by the rule $(x; y) \rightarrow (x - 4; y - 2)$.



Textbook extract 22: Example of GI paradigm (CM, p195)

In textbook extract 23 below, although there is still clearly a link to the physical movement, it provides a general rule for the effect on coordinates of a translation that can be used as the basis for deduction. The type of deduction that can be made using the general rule is also illustrated (i.e. that the translation $(a;\ b)$ followed by the translation $(-a;\ -b)$ gives $(x; y) \rightarrow (x + a - a; y + b - b) \rightarrow (x; y)$). This block would thus be classified as being in the GII paradigm.



- The translations $(a; b)$ and $(-a; -b)$ go in opposite directions, or $(x; y) \rightarrow (x + a; y + b)$ and $(x; y) \rightarrow (x - a; y - b)$ are in opposite directions.
- A translation $(a; b)$ followed by a translation $(-a; -b)$ of a region (or a point) takes you back to the original region (or point).
  $(x; y) \rightarrow (x + a; y + b) \rightarrow (x + a - a; y + b - b) \rightarrow (x; y)$

Textbook extract 23: Example of GII paradigm (CM, p197)

Although the above seems to distinguish between GII and GI in transformation geometry when coordinates are involved, in my interaction with the empirical field the above characterisation of GI and GII in transformation geometry could

not describe fully all the empirical cases. In the case of GI there was also the situation where transformations can be thought of using physical objects (e.g. a piece of paper) and manipulations (folding it, turning it, moving it) outside of the framework of the Cartesian plane. Working from within this paradigm we would define a transformation in terms of concrete, physical motions. For example, one might say "a reflection takes place when a shape is flipped across a line. The line acts as a mirror and the shape is reflected in that line." This would be a GI definition of a reflection.

In GII one would be looking for a definition that is strongly connected to the physical properties of the transformation, but is well-defined mathematically and thus can be used as a basis for decision-making or proof. A definition of a reflection such as the following, would be a GII definition:

A reflection in line l, r_l, is an isometry where if P is any point not on l then $r_l(P) = P'$ where l is the perpendicular bisector of PP' and if P is on l then $r_l(P) = P$.

Working from a definition of this nature one could make deductions about the properties of the transformation or use the transformation to prove geometric facts. In the same way as for the work in GI described immediately above, this does not necessarily need to be connected to the Cartesian plane.

Defining a reflection in the x-axis in terms of geometric properties (which one can do in a way similar to the above) is different in nature to defining it as the transformation $(x; y) \rightarrow (x; -y)$. Thus it is useful to make a distinction between GII work with coordinates and GII work with physical properties. Similarly, in GI, thinking of a translation as sliding an object across a flat surface (working with the physical) is different in nature to describing it as a move of 4 units to the right and 2 units up on the Cartesian plane (working with coordinates). Thus in my analysis of the transformation geometry chapters I worked with 4 categories as shown in Table 30.

Abbreviation used	Category
GIp	GI working with physical properties
GIc	GI working with coordinates
GIIp	GII working with physical properties
GIIc	GII working with coordinates

Table 30: Categories and abbreviations for transformation geometry analysis

I then analysed each block of the transformation geometry sections of the textbooks and decided which of these categories was present in each block.

In some cases work in more than one category occurred in a block. For example, textbook extract 24 below starts with a GIp definition (a translation as a "slide" to another position), then uses a specific example on the Cartesian plane to illustrate this (GIc). Thereafter the effect of a translation on the coordinates of a point are generalised and given in the form $(x; y) \rightarrow (x + a; y + b)$ which can be categorised at GIIc.

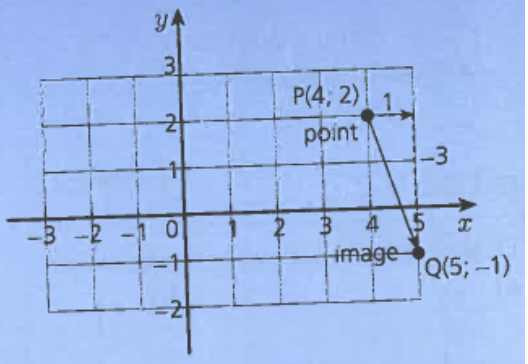
To describe situations like this in my analysis, I used commas between the categories. For example, in the table showing the full analysis of all blocks in Appendix D, a block designated as GIp, GIc, GIIc indicates that each of those categories are present in different aspects of that block.

Translation of points

- A translation changes the position of a point by "sliding" the point to another position.

Example 1

We translate the point P(4; 2) to the point Q(5; −1) by moving it 1 unit to the right and 3 units down. This is a translation of (1; −3).



- We translate the point $(x; y)$ to the point $(x + a; y + b)$ by a translation of $(a; b)$, where a is a horizontal move and b is a vertical move.

If $a > 0$ the horizontal translation is to the right.
If $a < 0$ the horizontal translation is to the left.
If $b > 0$ the vertical translation is upward.
If $b < 0$ the vertical translation is downward.

We can write the translation of $(x; y)$ by $(a; b)$ as $(x; y) \rightarrow (x + a; y + b)$

Textbook extract 24: GIp, GIc, GIIc example (CM, p193)

In contrast a block designated as GIc or GIIc indicates that there is ambiguity about which paradigm is expected. For example, the block shown in textbook extract 25 below could be done either by using known facts about the effect of reflections on coordinates or by actually drawing the points on the coordinate plane and visualising the line of reflection.

2. Give the line of reflection for the following pairs of points (if it exists).

a) A(3; 2) and A′(2; 3)
b) A(3; 2) and A′(3; −2)
c) A(3; 2) and A′(−3; 2)
d) A(3; 2) and A′(−2; −3)
e) A(3; 2) and A′(−3; −2) Discuss.

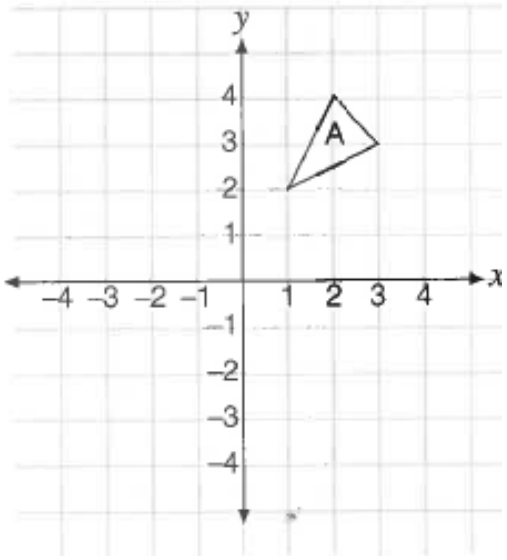
Textbook extract 25: GIc or GIIc example (CM, p203)

Finally the use of an arrow in a categorisation in the tables in Appendix D e.g. GIc → GIIc indicates that there is a deliberate movement from GIc to GIIc. For example in textbook extract 26 below we see that learners are expected to investigate the effect of reflections on the coordinates of points in the Cartesian plane by visualising the reflection of a triangle (GIc). This investigation is

intended to lead them to establish a general rule and to write it in the form $(x; y) \rightarrow (\ldots ; \ldots)$ which is in GIIc.

Make a copy of triangle A on co-ordinate axes. Label the co-ordinates of the vertices of the triangle.

1. (a) Reflect triangle A in the ***x*-axis**. Try to do the transformation mentally! Label the image triangle B.
   (b) Make a conjecture about the co-ordinates of the vertices of B.
   (c) Use some more points to test your conjecture in (b).
   (d) When you are convinced that your rule is correct, write it like this: $(x;y) \rightarrow (\ldots;\ldots)$



Textbook extract 26: GIc -> GIIc example (OTM, p245)

Certain blocks were designated as n/a as there was no paradigm to identify in these blocks. These were typically blocks such as the chapter opener or orientating narratives that indicated the content of the chapter or the following block.

7.5.1 Paradigms of geometry in transformation geometry chapter in CM

In CM there were 21 blocks in the transformation geometry chapter. The detailed analysis of these blocks is provided in Appendix D. A summary of the number of blocks in which each of the paradigms is provided in table 31 below. Note that because a block may contain more than one paradigm (as discussed above), the totals in the table do not add up to 21.

	GI	**GII**	**total**
p (physical)	10	4	14

c (coordinate)	11	7	18
Not specified[21]	0	2	2
total	21	11	

Table 31: Paradigms in transformation chapter of CM

This summary indicates that, although work in the GI paradigm predominated over work in the GII paradigm, and work in the coordinate plane predominated over work in a physical setting, all four sub-paradigms were well represented in the chapter. In addition a scan down the column in the table in Appendix D indicates that the four sub-paradigms occur throughout the chapter. This suggests that there is no clear development towards a particular paradigm in the chapter.

Because there is no clear trajectory in terms of the paradigms through the chapter, to get a more detailed idea of how the paradigms played out in the chapter I took a closer look at the following issues:

a) The starting point: what does the chapter opener suggest about the paradigms and what, if any, definitions/understanding of transformations are used as a starting point?

b) "Local" trajectories: as there is no trajectory through the chapter as a whole, are there places where there is a development from GI to GII over a smaller section?

c) Results of the movement between paradigms: what issues result from the movement between paradigms?

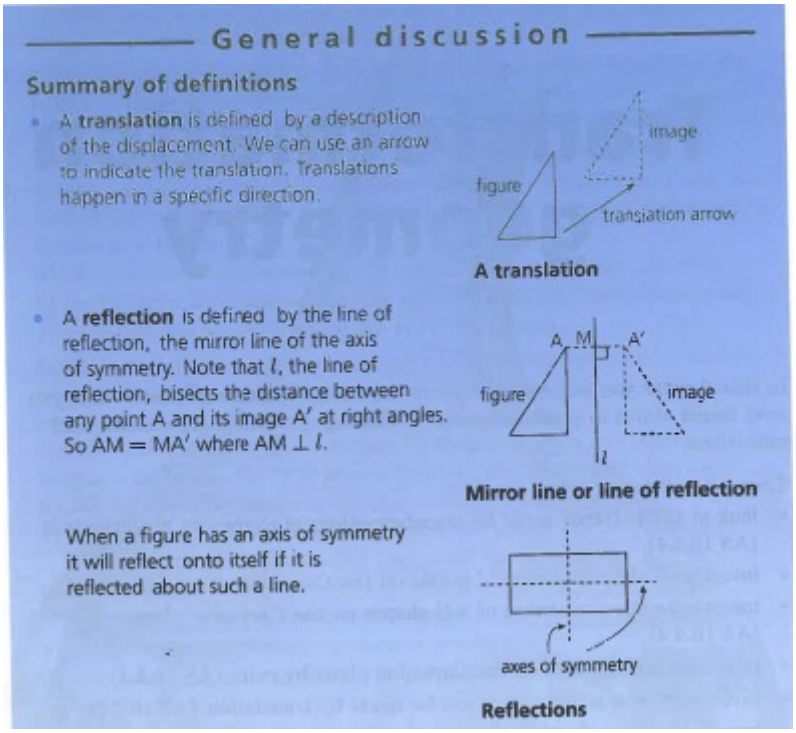
[21] In two blocks the narrative made it clear that work in the GII paradigm was expected. However it was not specified whether this would be in the coordinate plane or not. These blocks were categorized as GII.

Finally I looked through the other geometry chapters to see if transformation geometry was used in them and discuss how it manifests in these chapters.

7.5.1.1 The starting point

In CM the chapter opener has a picture depicting a transformation in an everyday object (an African calabash). There are 9 bullet points about what learners will do in the chapter. Four of these relate to transformation geometry and everyday objects e.g. "investigate how patterns on an African calabash can be derived through transformation geometry" (CM, p189), three relate specifically to transformation geometry and coordinates and two speak more generally about investigating transformations. Thus the chapter opener reflects the focus on both GI and GII and on the physical as well as the coordinate which, we see from table 41 in Appendix D, are present throughout the chapter.

In the first instructional narrative block CM introduces the transformations via definitions as shown in textbook extract 27 below.

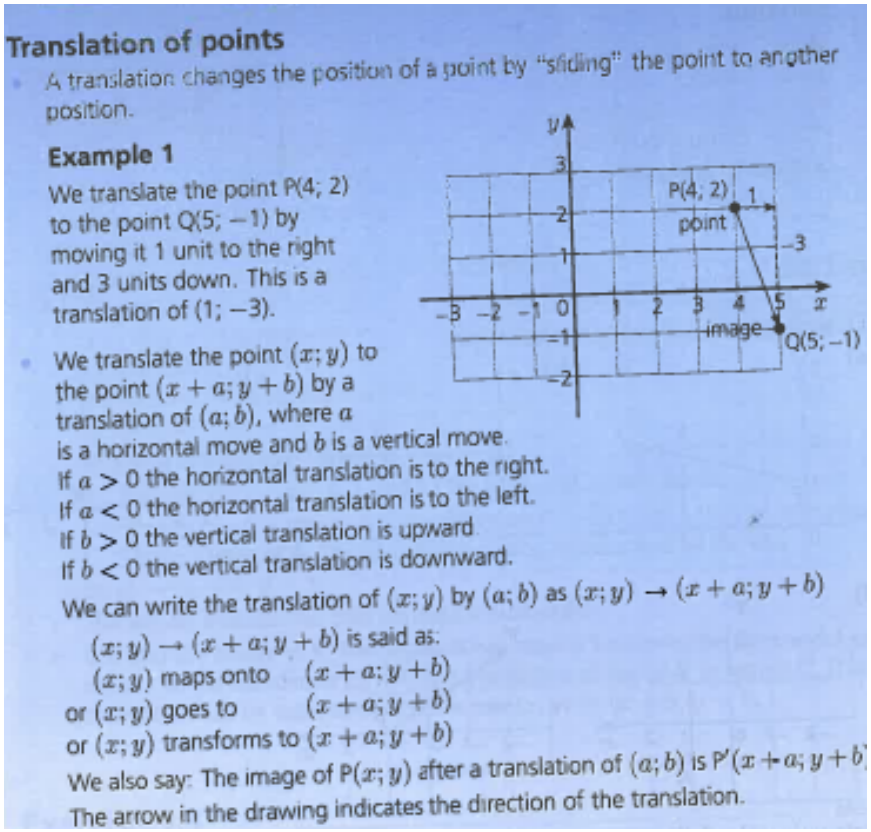


Textbook extract 27: Transformation definitions (CM, p190)

Interestingly although the definitions of a translation and a reflection are both provided in physical rather than coordinate form, they differ in terms of their geometric paradigm. The definition of translation is a GIp definition, as the description of a translation requires the accompanying diagram to make any sense. On the other hand, the definition of a reflection is a GIIp definition where full details of the properties of a reflection are provided.

7.5.1.2 Local trajectories from GI to GII

The definitions are followed by work in GIp. Although, as discussed previously, there is no unidirectional movement from GI to GII through the chapter, there are places where, over the course of a couple of blocks there is a development from GI to GII relating to a particular topic. I have termed such a development a local trajectory. For both translations and reflections a local trajectory occurs in that the GIp understanding of the transformation is then used in the context of the coordinate plane. Examples of the effect on coordinates are demonstrated (GIc) from which a general rule for the effect of that transformation on coordinates (GIIc) is given. For translations the move to GIc and then to GIIc takes place in a single instructional narrative block. This is shown in textbook extract 28 below.



Translation of points

A translation changes the position of a point by "sliding" the point to another position.

Example 1

We translate the point P(4; 2) to the point Q(5; −1) by moving it 1 unit to the right and 3 units down. This is a translation of (1; −3).

We translate the point $(x; y)$ to the point $(x + a; y + b)$ by a translation of $(a; b)$, where a is a horizontal move and b is a vertical move.
If $a > 0$ the horizontal translation is to the right.
If $a < 0$ the horizontal translation is to the left.
If $b > 0$ the vertical translation is upward.
If $b < 0$ the vertical translation is downward.

We can write the translation of $(x; y)$ by $(a; b)$ as $(x; y) \rightarrow (x + a; y + b)$

$(x; y) \rightarrow (x + a; y + b)$ is said as:
$(x; y)$ maps onto $(x + a; y + b)$
or $(x; y)$ goes to $(x + a; y + b)$
or $(x; y)$ transforms to $(x + a; y + b)$

We also say: The image of $P(x; y)$ after a translation of $(a; b)$ is $P'(x + a; y + b)$

The arrow in the drawing indicates the direction of the translation.

Textbook extract 28: Instructional narrative on translation (CM, p149)

This is followed by exercises that can be done in GIc (i.e. by plotting points on the coordinate plane and counting out the moves to do the translation) or GIIc (using the general rule). A similar process is repeated for reflections although some of the work done there brings additional complications, which will be discussed below.

7.5.1.3 Results of the movement between paradigms

One of the issues that emerges because of the continued movement between paradigms is a lack of clarity as to when a justification is needed. For example, in an instructional narrative it is stated:

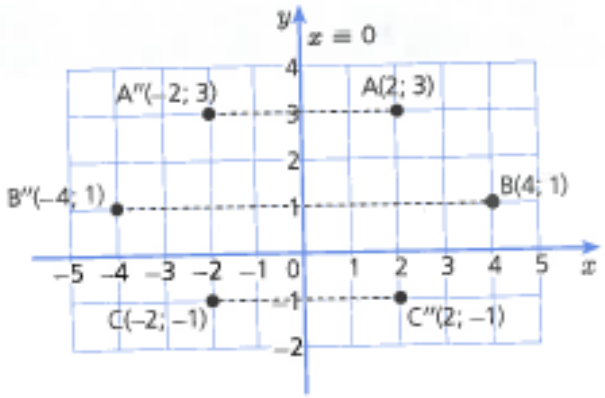
A translation changes the position of a region, but not its orientation, size or shape. The regions are congruent (they have not changed size or shape). (CM, p195)

This is stated with no justification. Within the GIp paradigm this is perhaps obvious – if you slide an object from one place to another then it does not change shape. But within GIc, GIIc or GIIp this would be something that needs justification. A similar statement that the shape and size of reflected regions remain constant follows the work on reflections.

A key issue that emerges from the movement between paradigms is the resultant circularity in argument it produces. Recall that the reflections were introduced through a GIIp definition in which it is clearly stated that the line of reflection must be the perpendicular bisector of the line joining the point and its image under the reflection. However when the exploration of reflections begins the learners implicitly need to ignore this definition and work from an understanding of reflections based in GIp. For example, in an exercise set there are five questions similar to the example extracted from it, shown in textbook extract 29 below, which ask learners to establish that the perpendicular distance from a point to the line of reflection and from the reflected point to the line of reflection are equal.

This exploration can only be sensible if one is relying on a GIp understanding of reflection (i.e. something like folding the piece of paper along the reflection line and making a pin prick where the point is). The GIIp understanding of reflection has the idea that the line of reflection is the perpendicular bisector of the line joining the point to its reflection as part of the definition. The experimentation in GIc which led to the GIIc definition requires that the idea of the perpendicular bisector at least be implicitly understood in order for the reflections to be performed.

2.



In the diagram A″, B″ and C″ are the reflections in the y-axis of the points A, B and C respectively.

a) Copy and complete the following table:

Point	Perpendicular distance from the y-axis ($x = 0$)	Sign of the x-coordinate	Sign of the y-coordinate
A			
A″			
B			
B″			
C			
C″			

b) Comment on the perpendicular distance from the y-axis ($x = 0$) of each pair of points A, A″, B, B″ and C, C″.

c) Comment on the sign of the x-coordinate in each pair of points.

d) Comment on the sign of the y-coordinate in each pair of points.

e) A″ is called the reflection of A about the y-axis ($x = 0$). A″ is also called the image of A obtained by reflection in the y-axis. If $P(x; y)$ is any point and P″ is the reflection of P about the y-axis comment on:
   (i) the perpendicular distance of P″ from the y-axis
   (ii) the sign of the x-coordinate of P″
   (iii) the sign of the y-coordinate of P″.

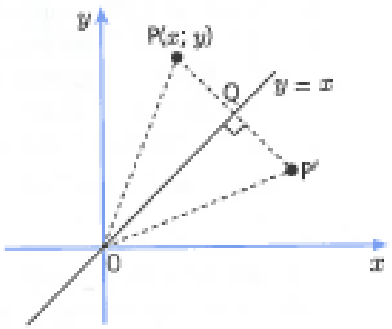
Textbook extract 29: Exercise on transformations (CM, p200)

Further, if we look at question 4 which appears later in the exercise set (shown in textbook extract 30 below), we see that learners are required to prove the fact that PQ = P'Q. This would need to be done using the fact that the coordinates of P' are (y; x).

4. a) P$(x; y)$ is reflected in $y = x$ to give P′. Give the coordinates of P′. (See figure alongside.)

b) Comment on the perpendicular distances of P and P′ to the line $y = x$. (See figure alongside.)

c) A **challenge**: Use congruency to show that your conclusion in **b)** is always true. (See figure alongside.)



Textbook extract 30: Question 4 on reflection (CM, p201)

The GIIc definition is thus used to prove the fact that the line of reflection is the perpendicular bisector of the line joining the point to its reflection i.e. to establish the GIIp definition which was the starting point.

It is of interest to note that, despite the attempt to develop some more formal GII work with transformations through the chapter, the end of chapter exercises are situated firmly in GIp.

7.5.1.4 Use of transformation geometry in other geometry chapters

In CM, chapter 14, which deals with the properties of quadrilaterals primarily using Euclidean geometry ideas, contains two instances of work using transformation geometry. This work is restricted to a worked example and an exercise. In the worked example the distance formula is used to show that the size of corresponding sides in a quadrilateral that has been reflected remain the same after reflection. In the exercise the learners are led through steps to show that a parallelogram that has been translated remains a parallelogram and a rhombus that has been reflected in the y-axis remains a rhombus. In each of these cases a quadrilateral with given vertices is used. Calculations are done to show equal lengths or equal gradients. Thus the work is situated in GIc. It is of interest to note that facts that are stated in the transformation geometry chapter (i.e. that translations and reflections preserve shape and size) are now not assumed, but need to be shown in the individual cases. Transformation geometry

is not used as an aid to exploring or showing properties of quadrilaterals in general.

7.5.2 Paradigms of geometry in transformation geometry chapter in OTM

A summary of the number of blocks in which each of the paradigms is provided below:

	GI	**GII**	**Total**
p (physical)	1	1	2
c (coordinates)	8	9	17
Total	9	10	19

Table 32: Paradigms of geometry in transformation chapter in OTM

This summary indicates that in OTM work was fairly evenly split between GI and GII and was done almost entirely with coordinates. As for CM, I needed to look at the starting point – both in terms of the introductory narrative and in terms of the notions of transformation learners are expected to start from. Then, because GIc and GIIc predominate in this chapter, it was important to investigate the nature of the relationship between these 2 paradigms established in this chapter. The fact that GIp and GIIp only occurred in 2 blocks meant that it was important to analyse how the physical was being used. Finally, as for CM, I looked through the other geometry chapters to see if transformation geometry was used in them and discuss how it manifests in these chapters.

7.5.2.1 Starting point

In OTM, although the chapter opener has a picture depicting a transformation in an everyday situation, the introductory narrative sets out a movement away from GIp :

> At primary school you made interesting tessellations and border patterns by translating, reflecting and rotating 2-dimensional shapes … In this Chapter,

> we take a different approach. Instead of using transformations to study other shapers, we study the transformations themselves and look at the properties of these movements. (OTM, p241)

Although the statement: "we study the transformations themselves and look at the properties of these transformations" indicates a move to GII, it is not clear whether this is GIIp or GIIc. The opening activity, which asks learners to use the words translate, reflect and rotate to describe the transformations in depictions of some patterns in traditional Southern African artwork, is the only block set in GIp. It is stated as being there for the learner to make sure they know the meaning of the words translate, reflect and rotate. Thus a GIp understanding is signalled as the starting understanding that learners need to have – but is also clearly signalled as a point from which the learners will move on.

7.5.2.2 The relationship between GIc and GIIc

The very first substantial activity starts in GIc and uses an exploration in GIc to establish a GIIc definition of a translation. The beginning of this activity is shown in textbook extract 31 below. In this activity no definition of a translation is given. It is assumed that learners know what a translation is and the single example illustrating the translation stands in place of a definition.

We begin by studying what is probably the easiest transformation to work with, the **translation**.®

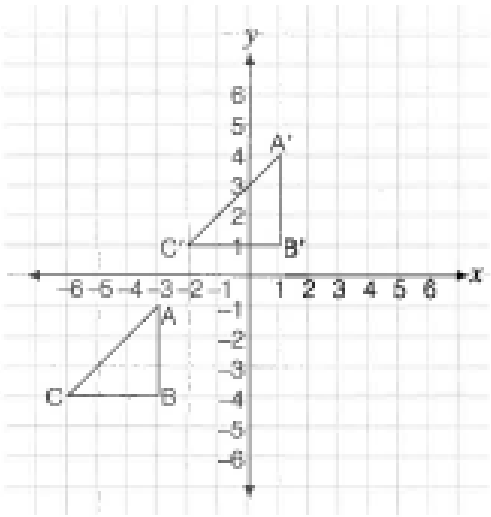
1. Do you think it is the easiest? Why?

As you work through this section take careful note of the mathematical terminology we use for transformations.

Refer to the picture alongside.

2. Write down the co-ordinates of the vertices of △ABC.

The original △ABC has been translated 4 units to the right and 5 units up. We call the **image**® △A'B'C'. The image is the shape that results after the transformation.



Textbook extract 31: Activity on translation (OTM, p242)

Learners are then asked to explore the questions shown in textbook extract 32.

3. If you know the co-ordinates of △ABC, how can you find the co-ordinates of △A'B'C' without using the picture? Write down the rule that will help you to do this. Remember to test your conjecture.
4. Draw different triangles on squared dotty paper. Translate the triangles to different positions on the co-ordinate grid. Test out your conjecture in question 3. Make sure that you move the triangles left and right **and** up and down.

Textbook extract 32: Questions from the activity on translation (OTM, p243)

The learners' explorations in this activity are used to develop a rule for translations which is summarised in OTM as follows:

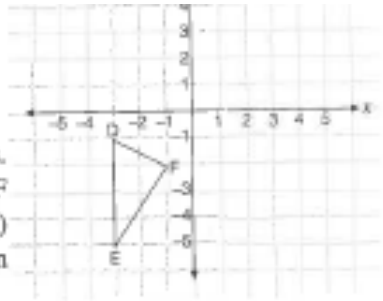
If I translate the point P$(x;\ y)$ a units horizontally and b units vertically, Where a and b are real numbers) then the image of P will be the point P'$(x + a;\ y + b)$. (OTM, p243)

This rule can be viewed as a definition of a translation in GIIc. This is followed immediately by exercises in which the learners can use either GIc or GIIc to perform translations on coordinates or to determine which translations have been performed.

A very similar sequence is followed for reflections. The key difference here is that in the introduction a picture of the reflected image is not given. Learners are asked to reflect a triangle in the x-axis, the y-axis and the line $y = x$ and draw the reflected image on the given coordinate grid. In each case they are encouraged to do the reflection mentally. Again no definition of a reflection is given and, in this case, learners are expected to be able to operate competently in GIc to produce the required transformation. In a similar vein to the work in translations, for each of the reflections, the exploration in GIc is used to produce a general rule for the reflection in terms of coordinates (e.g. The rule $(x; y) \rightarrow (-x; y)$ is a reflection of point $(x;\ y)$ in the y-axis.) This is a move into GIIc. Again, after the rules are established a series of exercises in which it is possible for the learner to choose to work in GIIc or GIc follows.

In all this work a strong emphasis is placed on making links between the graphical and symbolic form of the transformation and also with a verbal description of the transformation. A typical example of this is shown in textbook extract 33 below.

6. (a) Describe each of these transformations in words.
   (i) $(x;y) \rightarrow (-x;y)$
   (ii) $(x;y) \rightarrow (x;y + 5)$.
   (b) Draw $\triangle$DEF on co-ordinate axes. Draw the image $\triangle$D'E'F' of $\triangle$DEF under the transformation in (a)(i) followed by the transformation in (a)(ii).



The transformation in question 6(b) is a reflection followed by a translation.

Textbook extract 33: Verbal description of transformation (OTM, p247)

There is also a strong encouragement to perform the transformations mentally, rather than physically. For example, in every question in the activity where learners are required to reflect a triangle over various lines, the instruction "Try to do the transformation mentally!" is given.

Within one of the exercise sets there is an instructional narrative which explains the importance of making links between the "rules for transformation" (which have been given in coordinate form) and the physical or visualised movements:

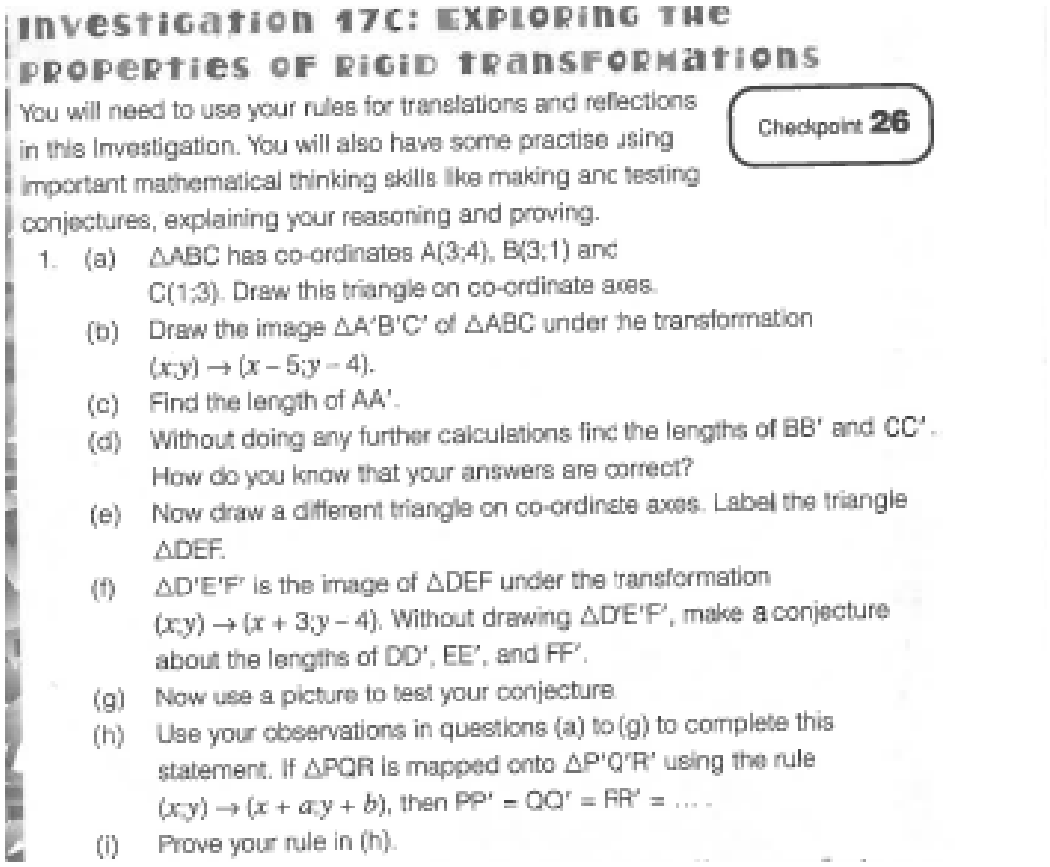
> In your group, talk about how you answered question 3 and 4. Did you fold the paper? Did you visualise the movement of the shapes? Did you rely solely on the rules? If we understand and can use the rules for transformations, we can answer the questions without physically moving the shapes **or** visualising the movements. But it always helps to be able to form a mental picture of the transformation – then we can check that our rules **do** work! (OTM, p247)

Thus for the major part of this chapter there is a focus on GIc and GIIc and making the link between the two. The underlying assumption appears to be that the previous experience learners have had with transformations in GIp is sufficient basis to work with the transformation in GIc and GIIc. Thus no definitions of the transformations are attempted. In addition a few examples in GIc are used to establish the "rules" (i.e. the generalised form of the transformation given in symbols) in GIIc and no further attempt is made to justify these rules.

7.5.2.3 Instances where the work is in GIp or GIIp

As discussed earlier the block in which work is done in GIp is the opening activity which asks learners to remind themselves of the words used to describe transformations by identifying them in Southern African artwork. Thus work in GIp is seen as belonging to the prior knowledge that is used as a starting point to move into GIc and GIIc.

The only block in which work in GIIp is required is the final activity of the chapter. An extract of which is shown in textbook extract 34 below.



Investigation 17C: Exploring the properties of rigid transformations

Checkpoint 26

You will need to use your rules for translations and reflections in this investigation. You will also have some practise using important mathematical thinking skills like making and testing conjectures, explaining your reasoning and proving.

1. (a) $\triangle ABC$ has co-ordinates A(3;4), B(3;1) and C(1;3). Draw this triangle on co-ordinate axes.
   (b) Draw the image $\triangle A'B'C'$ of $\triangle ABC$ under the transformation $(x;y) \rightarrow (x - 5; y - 4)$.
   (c) Find the length of AA'.
   (d) Without doing any further calculations find the lengths of BB' and CC'. How do you know that your answers are correct?
   (e) Now draw a different triangle on co-ordinate axes. Label the triangle $\triangle DEF$.
   (f) $\triangle D'E'F'$ is the image of $\triangle DEF$ under the transformation $(x;y) \rightarrow (x + 3; y - 4)$. Without drawing $\triangle D'E'F'$, make a conjecture about the lengths of DD', EE', and FF'.
   (g) Now use a picture to test your conjecture.
   (h) Use your observations in questions (a) to (g) to complete this statement. If $\triangle PQR$ is mapped onto $\triangle P'Q'R'$ using the rule $(x;y) \rightarrow (x + a; y + b)$, then PP' = QQ' = RR' = ...
   (i) Prove your rule in (h).

Textbook extract 34: Extract from the block contain GIIp work (OTM, p249)

In question 1 learners are asked to establish that if P is mapped to P' using the rule $(x; y) \rightarrow (x + a; y + b)$ then $PP' = \sqrt{a^2 + b^2}$. This is done through experimenting with a few examples to produce a conjecture which learners are then asked to prove. Similarly they are asked to experiment with reflections and note that if K is reflected over a line to K' then the line is the perpendicular bisector of KK'. Learners are asked to do this by experimenting with reflections in different lines of reflection. They are then asked to use the observations they have made to perform reflections. This work moves into the GIIp paradigm in that physical properties of the transformations are established. The second question, in particular, illustrates some difficulty in the movement between paradigms. This is similar to the issue noted in the analysis of CM. The original assumption in the chapter is that learners come with a GIp "definition" of a

reflection. This is used to work with reflections in GIc and that work "establishes" the generalised symbolic rule for reflections in GIIc. In order for any of this work to be done one has to work (implicitly at least) with an understanding that a reflection moves a point across the line of reflection to a point equidistant from the line in a direction perpendicular to the line. Formalising this understanding would result in a GIIp definition of a reflection and this, in turn, could be a way to establish the GIIc definition mathematically (which, as it stands, has been generalised from a few examples). This "messiness" (i.e. working with something that is only informally defined and ultimately circling back to define it formally and establish facts that have already been conjectured from the informal) is tricky to work with and yet an essential part of understanding mathematics as a discipline. OTM does not deal with it directly and instead avoids substantive work in GIIc and GIIp. OTM accepts the establishment the rules of GIIc by experiment and avoids any attempt at a more formal definition of the transformations that could allow work in GIIp.

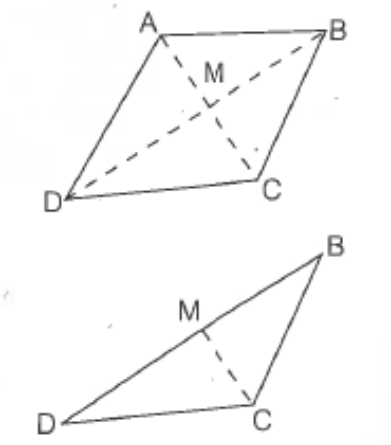
7.5.2.4 Use of transformation geometry in other geometry chapters

In OTM transformations are used in chapter entitled "Using inductive reasoning for study shape". Here transformations are shown as a way of justifying properties of geometric figures. See, for example, textbook extract 35.

LEARNER 1

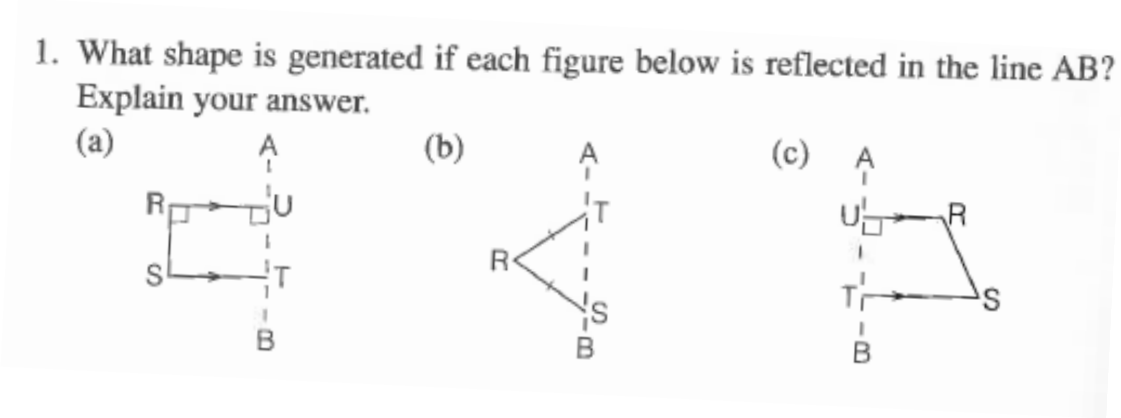
I began by drawing in the diagonals. I called the point where they meet M. Then I cut out the kite and folded it along DB, so that A was on C. Then I could see that AM = MC.

When I folded the kite along the line MC, B did not get as far as D, so BM and MD cannot be the same length.



Textbook extract 35: Transformation geometry used to justify properties (OTM, p210)

Transformations are also used as a means of investigation. See textbook extract 36.


1. What shape is generated if each figure below is reflected in the line AB? Explain your answer.

(a)

(b)

(c)

Textbook extract 36: Transformation geometry used for exploration (OTM, p212)

In this chapter the work using transformation geometry is set in the GIp paradigm. The idea that one should attempt to visualize the effect of the transformation (rather than performing the transformation physically) is encouraged.

It is worth noting that this work in GIp is not picked up again in the chapter on transformation geometry discussed previously. As shown the work in the transformation geometry chapter is set in GIc and GIIc and this is not presented as an aid in making deductions about geometric figures. Also there is no work on transformation geometry in the chapter "Using deductive reasoning to study shape". This positions transformation geometry as useful in the informal work but not as part of the rigorous deductive work.

7.5.3 Discussion of transformation geometry

One of the key issues that emerge from the analysis of both textbooks is the difficulty in avoiding circular arguments when moving between paradigms. Here the distinction between both GI and GII and the physical (p) and coordinate (c) versions of the paradigms was useful. Translating or reflecting points or shapes in GIc relies on an intuitive or implicit understanding of the GIIp definition of

translations or reflections. Both textbooks follow the curriculum assessment standard which asks learners to

> Investigate, generalise and apply the effect of the following transformations on the point (x; y)
>
> a) A translation p units horizontally and q units vertically
>
> b) A reflection in the x-axis, y-axis or the line y = x (MNCS, p34)

and use some investigation in GIc to develop a GIIc definition of translations and reflections. However, in further investigating properties of the translations and reflections both textbooks engage in circular arguments as they ask learners to establish facts that had to implicitly underlie the development of the GIIc definition in the first place.

In both textbooks the fact that one might prove the size and shape of objects do not change after translation or reflection is stated and not proved in the transformation geometry chapter.

Thus transformation geometry is not given a rigorous basis. In addition its key work seems to lie in GIc with a movement into GIIc, but with little further work in GIIc.

This has implications for how it can be used. We see this in the fact that in OTM it is a GIp version of transformation geometry that is used to explore some facts about quadrilaterals and in CM no substantive geometric work using transformation geometry is attempted.

A systematic look at various components of the theoretical framework helps provide some insight into what the difficulties behind this might be. I looked at the mathematics that could be recontextualised, the writing in mathematics education – particularly in South Africa on transformation geometry, the interviews with the textbook authors (PRF), the curriculum and interviews with the curriculum writers (ORF). From these I identified a key theoretical underpinning, the van Hiele theory that informed the work in transformation

geometry. The discussion in relation to this connected most strongly to work in GI. I then discuss the progression envisaged beyond GI that is identified and show how this illuminates a real struggle to make an intelligible transition to GII. This is discussed in sections 7.5.3.1 and 7.5.3.2 below.

7.5.3.1 Transformation geometry and the van Hiele levels

All interviewees were clear and explicit about the idea that notions from the van Hiele theory of geometric thinking had strongly influenced the curriculum and the textbooks. MC_1 , MRC, A_{CM}, MC_2 and A_{OTM} all mentioned Michael de Villiers (GC_F) as being particularly infuential in the thinking around the geometry curriculum and the textbook work. MRC and MC_1 also both mentioned Retha van Niekerk (GC_G) as influential in this regard, although more so at the level of the GET curriculum. When I interviewed these two geometry consultants they spoke extensively about a strong van Hiele underpinning to their work and ideas. I also consulted written work by GC_F for further elaboration of his ideas. The key role of transformation geometry as an aid to development of geometry in accordance with van Hiele theory is laid out in GC_F's 1993 paper "Transformations: A golden thread in school mathematics" (De Villiers, 1993) where GC_F argues that starting with tessellations at primary schools allows for the development of a visual basis for geometric content that will be worked on more formally at a later stage. This type of use is demonstrated in van Hiele's work in relation to properties of parallel lines (Fuys et al., 1988). De Villiers also argues that in the junior-secondary phase transformations can be used to deduce properties of figures informally. This use of transformations at an informal level is seen by both GC_G and GC_F as happening prior to the FET phase:

> GC_F: I mean the van Hieles their whole idea, I mean tessellations is in the primary curriculum. And the idea there is to develop visualization. Well, it's not only in the primary, it's also in the GET. I mean it's right up to grade 9 if I'm not mistaken. And the idea is that tessellations should be helpful, the visualization and the properties.

> GC_G: In the primary stages, in the primary school transformations as … you have to understand transformations as a tool, then when you move up you have to understand how the tool opens up the mathematics.

> GC_F: I mean as with the stuff that I'm talking about, the idea with tranformations is really (also taking from the van Hieles) is really that you should move from visualization to properties and then to formalize definitions

Thus the work they describe in relation to transformation geometry at primary and junior secondary school level is aimed at providing a strong basis in the first 2 levels of geometric thought (visualisation and informal deduction) of the van Hiele model. They imply that at the FET phase work in transformation geometry needs to be work that moves from level 2 (informal deduction) into level 3 (formal deduction). The way in which this movement is discussed by the various interviewees and elaborated in the literature is discussed further below. What is important to note at this stage is that the importance of transformation geometry in developing visualisation and informal deduction had an impact on the textbook writers even though they were writing at FET level. We can see this in the emphasis on visualisation in OTM and the extensive work in GIp in CM.

Although the authors of both textbooks spoke about being aware of the van Hiele underpinnings in the curriculum, neither spoke specifically about how one might move from work involving level 2 on the van Hiele model to work involving level 3 on the van Hiele model in relation to transformation geometry. Their emphasis in relation to van Hiele was that it gave them an awareness of the importance of not introducing formal proof work early and the importance of having experience of working in the visual and the concrete.

For example, A_{OTM}, after mentioning that work she had previously done had involved work with the van Hiele levels, responded to my question "So was there a kind of consicous van Hiele underpinning?" as follows:

> A_{OTM}: In some way, ja. And I think it was based on my experience of teaching and my knowledge of the literature and my masters was also on space and

> shape and van Hiele stuff so I think that all sort of contributed to my thinking that students in grade 10 were often not ready for the formal geometry.

This is reflected in both textbooks.

The struggle to see the role of transformation geometry beyond its usefulness in visualisation is evident in comments such as the following from the authors:

> A_{OTM}: I think I put in, I mean as far as I can remember, the curriculum stuff on the transformations is more using coordinates which I found quite limiting. I was much keener on the visual kind of stuff and the kind of stuff that I did in the chapter where you're using your reflections, your rotations to generate other shapes and then to draw conclusions about the properties of those shapes. So I saw the value of the transformations really as a tool to study shapes which I don't think the coordinate geometry gives you the flexibility to do.

> CM_1: (speaking about GC_G) And she is very much pro transformation geometry… I think she's done some interesting stuff. For Mindset[22] for example, I think she has been involved in some of the stuff where for instance they said that if you take any right angled triangle and rotate it about the right angle four times, then that's the proof of the fact that the diagonals of a rhombus bisect each other at right angles … But for me it wasn't something that should replace Euclidian proof.

Both of these comments suggest the authors are comfortable with the idea of transformations as a tool for exploration and visualisation. However A_{OTM}'s comment suggests that in the movement to working with transformations on the coordinate plane, as prescribed by the curriculum, she did not see transformation geometry as remaining a useful tool for studying shape. CM_1's comment suggests seeing less of a role for transformation geometry in a GIIp

[22] Mindset produces a series of television programmes related to various topics in the school curriculum. They used subject experts to help produce the material for these programmes.

paradigm. Thus although the van Hiele theory talks of a progression through all the stages of development of geometric thinking, the key idea that seems to have been appropriated from the theory in developing the transformation geometry work in the textbooks is the importance of work at level 1 and 2, incorporating work on visualisation and exploration. This difficulty with seeing a trajectory for transformation geometry at school level is discussed further below.

7.5.3.2 Progression envisaged for transformation geometry

The interviewees gave various reasons for the inclusion of transformation geometry in the curriculum and textbooks and the way they saw the work progressing. I have summarised these into three categories. The first relates to a link into higher mathematics, the second is in terms of its applications and the third is its ability to be used as a tool.

The link into higher mathematics

Many of the interviewees spoke about the work in transformation geometry as having links to higher mathematics.

> A_{CM}: Transformation geometry can relate very strongly to functions and the notation that one can use to indicate those transformations indeed is a function and a mapping notation. The arrow going from the x y to what should become subsequently after the transformation. And that in turn links up very strongly to another very important aspect of pure maths – that being mapping and function. So that's the usefulness of transformation geometry in the overall curriculum, the way in which it links to pure maths, to functions, also to linear algebra.

> MC_1: (with reference to a diagram of a mural in the grade 12 CM textbook): And I extracted this thing and then looked at actually groups and fields kind of stuff. Just to play around.

> A_{OTM}: I suppose eventually it's getting to when we then start studying the actual transformations and looking at the commutativity and all those kind

> of things then you sort of getting that, but that can get quite complicated quite fast. So, and it's not, I don't know if it can be done at school.

These three quotes speak of seeing a horizon for transformation geometry in a GIII paradigm where the work becomes increasingly abstract and detached from the physical referents. The discussion of transformations as functions speaks of a path from a GIIc version of transformations into a GIII paradigm. However all these interviewees were clear that this horizon was not the intended schoolwork. For many of the interviewees this was coupled with a lack of certainty as to what the purpose of the GIIc work at school was.

A_{OTM} is explicit about the curriculum not giving an indication of why transformation geometry is there and what the school horizon for it is.

> A_{OTM}: I think the next level is if you can study the reflections and the all the, I mean like as functions and get towards your matrices and those kind of things. So that's the way it could go but as I say I don't know whether that's … I doubt it's appropriate for school level and I don't think that the curriculum has an idea of gives a sense of where, of why it's there.

MC_2 concurs that the transformation geometry trajectory in the curriculum was not clear. He suggests that some of the difficulty in getting beyond the initial visualisation is because teachers themselves struggle to visualise the transformations.

> MC_2: There are a couple of things that I'm not sure we had our head around in terms of a developmental process. And we've tried to get our head around it here[23]. When do the transformations happen on a set of axes? Now in the FET there's (depending again on which book you look at) there's a whole lot of stuff that happens just spatially – so you've got your tessellations you know and all that kind of stuff probably in Grade 7, but a lot of that in

[23] The interview took place at the school MC_1 was teaching at the time of the interview. "Here" refers to at that school.

> schools have carried through to Grade 8 and 9 and again, spatial visualisation is not a South African strong point so what often happens is the teachers struggle to visualise how things rotate and turn and flip and so forth so they tend to neglect it.

Thus, while they do not use the language of geometric paradigms, a strong sense emerged from the interviewees that although they had a clear vision of transformation geometry's role in a GI paradigm and a GIII paradigm, they were less certain about what they were trying to achieve with it at FET level. The two possible suggestions that emerged were that of transformation geometry as having useful applications, and transformation geometry as a tool.

Transformation geometry as having useful applications

The authors of both textbooks mention the idea of applications. A_{OTM} describes having difficulty in doing that. This is reflected in the chapter in OTM where the few illustrations she uses at the start of the chapter are just used to get learners to describe the transformations they see. In the chapter the work in GIIc remains detached from these ideas. CM_1, on the other hand, feels more positive about having been able to integrate culture, design and other subject areas in the chapter on transformation geometry. However the analysis of the chapter from CM above indicates that this work took place largely in GIp disconnected from the development of work in GIIc that happens in the chapter.

> A_{OTM}: I did at the time try and explore some of the ethnomaths stuff that had been done, David, what's his name[24] and I looked I mean these were from, what's his name? The Mozambican Paulus Gerdes. I mean I looked at one of his books so, I did try and actually explore ways to try and bridge that, but I never got very far.

[24] As mentioned earlier she is referring to David Mogari who has done work on ethnomathematics in South Africa.

> MC_1: I think to some extent just showing that there is mathematics involved, that kind of thing that people actually use maths. Geometry in filmmaking and things like that. I set a number of wallpaper patterns and all those patterns wallpapers and designs are transformations. So it's linking up culture and design and odds and kind of, which was another brief that we had – to try and integrate subjects and subject areas and that kind of thing. So I thought it was a really nice opportunity to do some of that.

MC_2 further suggests the inclusion of transformation geometry in the FET curriculum was simply because it was there in the GET curriculum. He argues its presence in the GET curriculum was because of the experience of NGOs who argued it was useful in developing spatial abilities.

> MC_2 (asked about why transformation geometry was included in the curriculum): The transformation I think the sense was that it was nice and spatial; it had quite a lot of applications. This was just a sense I got because some of the NGOs had been doing work with transformations lower down because little bits of it were in the GET, so there was a sense that for follow through it needed to be in the FET as well. And so I think that was one of the issues, was to see the follow through of transformation into FET so teachers in GET would have a reason to teach it, you know, and where it went from there. I don't … I never got a sense that there was an underlying theoretical underpinning as to why a shift from Euclidean to transformation. We never …. I remember comments about it being, you know, you could easily apply it to advertising logos. But again I got the sense that was coming from some of the NGOs, that those were little things they were doing in FET and they thought it would be fun to continue with them into FET.

However the applications described remain in a GIp level. They do not link into a GII paradigm and this is reflected in the textbook analysis.

Transformation geometry as tool

When I asked GC_G about what her vision had been of where the work they described in the curriculum for the GET phase would go in the FET phase, she answered:

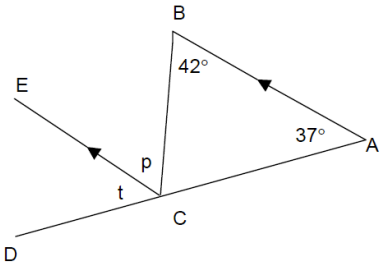
> GC_G: I saw that going that your … in the primary stages, in the primary school transformations as … you have to understand transformations as a tool, then when you move up you have to understand how the tool opens up the mathematics.

This idea is echoed by A_{OTM} who describes the influence of her work in Malati, the mathematics materials development organisation she had worked for.

> A_{OTM}: I think when I first saw it I was really excited. I thought again because at Malati we were doing a lot of the transformation stuff so I was sort of aware of the possibilities and I was excited because I thought it's sort of a way of students seeing they have different tools to study the same shapes.

The Malati materials A_{OTM} refers to are explicitly aimed at students in the GET phase. Within these materials Malati argues that transformations can provide alternative ways to study plane geometry. They argue that many traditional definitions in geometry can be redefined to make them amenable to work in transformation geometry. For example they argue that one can define parallel lines as follows: Two lines are parallel if there is a translation that maps one line onto another. Malati argues that transformations can be used to study the properties of plane figures and give an example of this as shown in figure 4 below.

Find the value of each of the angles p and t. Give reasons for your answers



Pupils could argue that, if one translates ∠ECD along line AD so that it lies on ∠BAC, then t = 37°.
∠ECB can be rotated 180° about the midpoint of CB so that it lies on ∠ABC. Hence p = 42°.

Figure 4: Transformation used to study the properties of plane figures (Malati, 1999, pp., p1)

De Villiers (1993, p15) provides an example of how one could use transformation geometry in a deductive system:

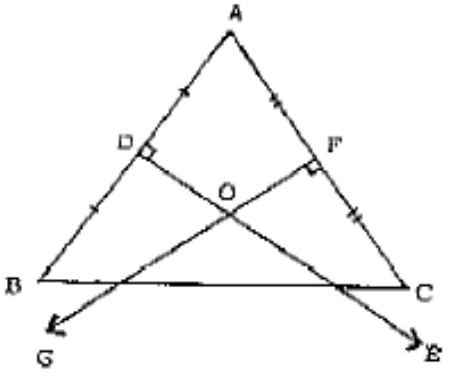


Fig 15

To prove:

The perpendicular bisectors of any ΔABC are concurrent.

Construction:

Draw DE and FG the respective perpendicular bisectors of AB and DC to intersect in O as shown.

Proof:

We now want to prove that O lies on the perpendicular bisector of BC. DE is, however, an axis of symmetry of AB, therefore there exists a reflection around DE so that A and B coincide. According to the definition of a reflection DE is therefore the locus of all the points equidistant from A and B. FG is similarly the locus of all the points equidistant from A and C, and therefore the point of intersection of DE and FG must be equidistant from B and C. In other words, O lies on the axis of symmetry of BC, or rather its perpendicular bisector. QED

Figure 5: de Villiers proof using transformation geometry (1993, p15)

Note that in order to do this proof he relies on definitions of reflections and makes deductions based on these. This requires a far more rigorous treatment of transformations than we saw in either textbook. It is also interesting to note that

neither the informal work proposed by Malati nor the more formal deductive work illustrated by de Villiers is linked to the coordinate plane.

It thus appears that within the work on transformation geometry the textbook authors interpreted the strong van Hiele underpinning in the curriculum to suggest the importance of exploration and visualisation that remained in the realms of GI. Both textbooks struggled to make a coherent move into GII. Although the authors talked of applications in relation to transformation and the ability to use it as a tool in plane geometry, in neither book was this realised. The move into GII that dominated was a move to GIIc. This was not developed further and did not link to the potential of transformation geometry that the authors had raised.

7.6 Coordinate geometry

As discussed earlier coordinate geometry is used as a tool to explore the properties of geometric shapes in both textbooks. Thus I have incorporated most of the analysis of coordinate geometry into section 7.7 that deals with the chapters on properties of geometric shapes in the textbooks. Here I simply discuss how the formulae and facts of coordinate geometry are established in the textbooks.

In CM, coordinate geometry is introduced in a chapter entitled "Linear functions and coordinate geometry" and the following formulae and facts are established:

The distance formula: This is established by using the Theorem of Pythagoras first in a case where specific coordinates are given and secondly in a case where coordinates of the form $(x_a; y_a)$ and $(x_b; y_b)$ are given. In both cases there is a strong reliance on the diagram in deriving the formula (e.g. the diagram has point $(x_b; y_b)$ above and to the right of point $(x_a; y_a)$ which makes it possible to see the vertical distance between the two is $(y_b - y_a)$ and the horizontal distance is $(x_b - x_a)$).

The midpoint formula: The derivation of the formula for the midpoint of a line segment is, in a similar way to the length of a line segment, derived through a logical argument from a diagram where points (x_a; y_a) and (x_b; y_b) are shown.

Slope of a line: The slope of line is defined as $\frac{change\ in\ dependent\ variable\ y}{change\ in\ independent\ variable\ x}$ which is shown to be $\frac{y_a - y_b}{x_a - x_b}$. This is based on a diagram, in a similar way to the distance formula and midpoint formula, but in this case (x_a; y_a) is to the left and above (x_b; y_b). No facts about the slopes of perpendicular or parallel lines are discussed.

In summary one can see that the derivation of the formulae are justified via a deductive argument, but are not fully rigorous in terms of accounting for all possibilities. They rely strongly on the diagram for the deduction. Thus they incorporate both elements of GI and GII.

Further work in this chapter provides learners with practice in using these formulae.

In OTM coordinate geometry is introduced in the chapter "Using inductive reasoning to study shape".

The distance formula: This is established in the same way in OTM as it was in CM.

The midpoint formula: This is stated without justification. After it is stated learners are asked to check whether it gives the correct answer for a horizontal line segment. They are then given two line segments with particular numerical coordinates and asked to find the midpoints using the formula and confirm they are correct by finding the distance from the endpoints to the midpoints.

Slope of a line: This is introduced in an earlier chapter on straight lines. It is given as $\frac{change\ in\ y}{change\ in\ x}$. The fact that parallel lines have equal gradients and that the product of the gradients of perpendicular lines equals -1 is established empirically through looking at a couple of examples.

Thus in OTM only the distance formula is justified via a deductive argument. Both the midpoint formula and facts of the gradients of lines are justified empirically. Thus this work in OTM lies more strongly in the GI paradigm.

7.7 The properties of geometric shapes[25]

In both CM and OTM there are two chapters that deal with the properties of geometric shapes. In CM these are *Chapter 3: Basic Geometry* and *Chapter 14: Quadrilaterals*. In OTM these are *Chapter 15: Using inductive reasoning to study shape* and *Chapter 16: Using deductive reasoning to study shape*. Although in both textbooks there are a few blocks within these chapters that use coordinate geometry or transformation geometry, the dominant form of work in these chapters is the analysis of the properties of geometric shapes either through experimentation with physical representations of the shapes or through logical deduction. Because of this, the description of GI and GII summarized in the table 2 on p54 aligns well with geometry in these chapters. However, as noted earlier, some problems emerged in trying to use these ideas to categorise textbook blocks according to the geometric paradigms they represent. The key issue in the chapters dealing with the properties of geometric shapes related to the difficulty in identifying the nature of deduction and validation and the nature of the objects. As discussed in section 7.4, there are points at which certain facts need to be used or objects are referred to where it is not clear whether those facts have been logically or empirically deduced and whether the object referred to is a drawing or an object governed by a mathematical definition. However it remained important to provide a picture of the nature of geometry that the textbooks appeared to be portraying. I have termed this the "apparent dominant paradigm". The notion of the apparent dominant paradigm indicates the paradigm in which it appears the learner is meant to understand the work

[25] The curriculum specifies that the shapes dealt with in this grade should be triangles and quadrilaterals. Throughout this chapter I have used the term "special quadrilaterals" where necessary to refer collectively to parallelograms, rectangles, squares, rhombi, kites and trapezia.

presented or do the work given. The use of the word "apparent" indicates that I am not making a claim that, in the case of blocks indicated as belonging to GII, the necessary definitions and axioms are in place to enable the work required in the paradigm.

The indicators I used to classify blocks into the paradigms drew on the combination of the Kuzniak's notion of the GWS and GI and GII as described in chapter 2. This is summarized in table 33 below.

	GI is apparent dominant paradigm	**GII is apparent dominant paradigm**
Nature of objects	Objects are physical. Drawings are used as objects and are what is measured, manipulated or drawn to come to conclusions about properties.	Objects are theoretical. Drawings are used to support reasoning.
Nature of tools	Drawing instruments (for accurate construction), measuring instruments, manipulation or computer software present (or implied).	No physical tools present or implied.
Nature of deduction and validation	Validation and deduction based on perception or use of instruments or through experimentation.	Logical deduction from a set of facts is required.

Table 33 Indicators used to classify blocks according to apparent dominant paradigm

I illustrate how I used this table in analysis through the following examples:

Example 1:

General discussion

Counter-examples

A counter example is a particular example that shows a conjecture to be false.

Example

Conjecture: All hexagons are similar to each other

Counter-example: A regular and an irregular hexagon are not similar.

∴ The conjecture is false.

In Exercise 3.2 question **4 c)**, the counter-example of a square and a rhombus with the corresponding sides equal shows that the statement is false.

Exercise 3.3 (AS 10.3.2 b)

- Work in pairs.
- In each case state whether the conjecture is true or false. If it is false give a counter-example to prove this.

1. All polygons have the same number of sides.
2. All pentagons are similar.
3. All pentagons are congruent.
4. All equilateral triangles are congruent.
5. All rectangles are squares.
6. All squares are rectangles.
7. All congruent hexagons are similar.
8. All similar hexagons are congruent.
9. All rectangles resulting from an original rectangle being translated many times are congruent.
10. All hexagons in a tessellation of hexagons are congruent.

Textbook extract 37: Example of apparent dominant paradigm GII (CM, p63)

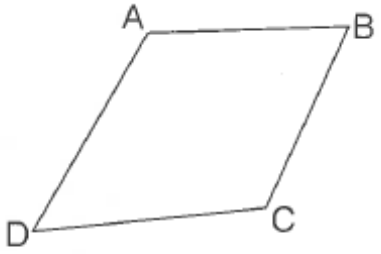
This block would be classified as having an apparent dominant paradigm of GII. This is because the objects (hexagons, polygons, pentagons etc.) are theoretical. No drawings are provided, and no physical tools are presented or implied. In order to show the claims are false a counter-example needs to be produced which requires evaluating examples against a definition and this fits in with a logical deductive system of reasoning. The fact that the definition of a square is only given later in the textbook does not alter the classification of the block. The way the instructional narrative and exercise set are set up give the impression that work in GII is required.

Example 2:

Investigation 15A: Different ways to study quadrilaterals

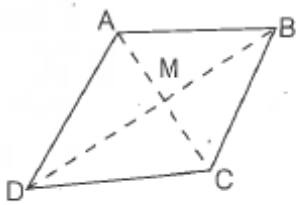
1. (a) Investigate the lengths of the diagonals of a kite. Make a conjecture, test it, and then explain your reasoning.

   (b) Now look at the responses of three Grade 10 learners to the task given in (a). Make sure that you can follow the reasoning in each solution. Try their methods yourself if you are not convinced.

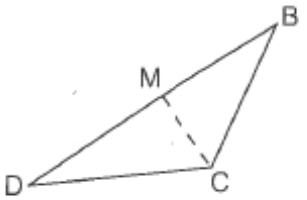


LEARNER 1

I began by drawing in the diagonals. I called the point where they meet M. Then I cut out the kite and folded it along DB, so that A was on C. Then I could see that AM = MC.

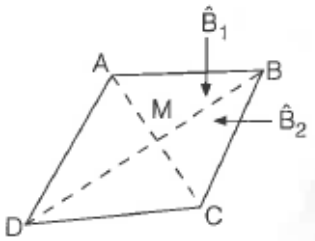


When I folded the kite along the line MC, B did not get as far as D, so BM and MD cannot be the same length.



LEARNER 2

I used congruency:
I drew in the diagonals DB and AC.
In an earlier activity I used congruency to prove $\Delta ABD \equiv \Delta CBD$
So $\hat{B}_1 = \hat{B}_2$
Then I focused on ΔABM and ΔCBM:



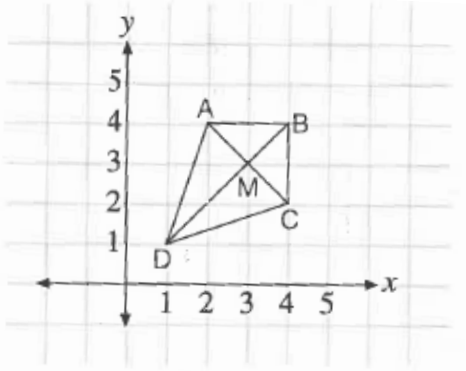
BM is a common side
AB = BC (sides of a kite)
$\hat{B}_1 = \hat{B}_2$ (angles of a kite, proved)
So $\Delta ABM \equiv \Delta CBM$ (SAS)
$\therefore$ AM = MC

It is not possible to prove that ΔADM and ΔABM are congruent, so $BM \neq MD$.

LEARNER 3

I drew a kite on a system of axes. Then I looked at the co-ordinates.

I could read that the co-ordinates of M are (3;3). I can see that AM goes across 1 block and down 1 block. MC also goes across 1 block and down 1 block, so they must be equal.



In the same way I can show that DM ≠ BM. From D to M, I go across 2 and up 2, but from M to B, I go up 1 and along 1. I tried putting a kite in a different position on the grid and it was the same.

(c) Discuss these explanations in your group.

(i) Was your response in (a) the same, or similar to any of the above? Explain.

(ii) Is one argument more convincing than another?

Textbook extract 38: Example of apparent dominant paradigm GI and GII (OTM, p210-211)

In this block, the response by learner 1 has an apparent dominant paradigm of GI. The drawing is the object. The drawing is cut out and folded showing the use of manipulation as a tool and the deduction is based on this manipulation.

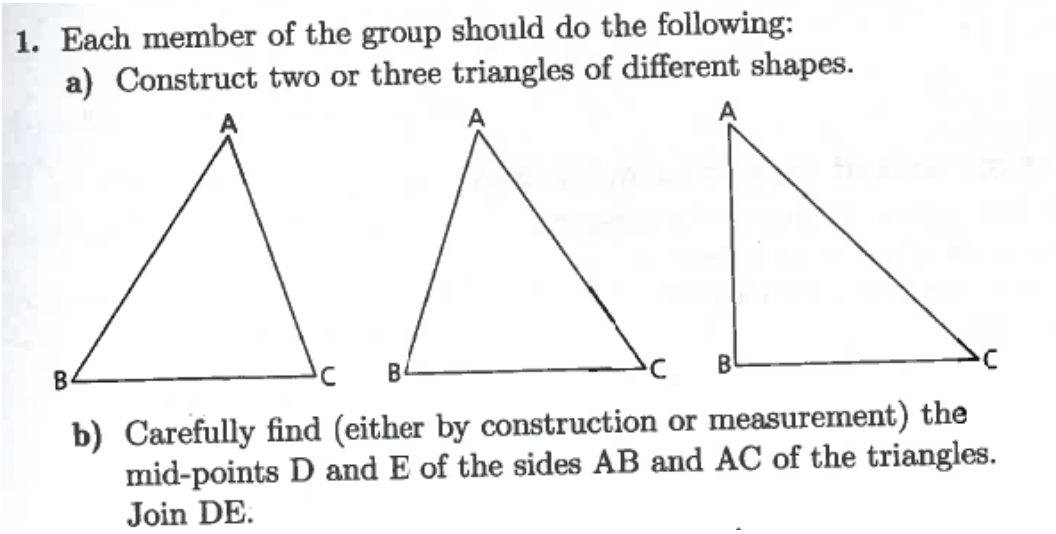
In the response by learner 2, although the imagined learner draws in diagonals DB and AC, this is a case of using the drawing to support reasoning. Although clearly a pencil would have been needed to draw in the diagonals, the diagonals did not need to be constructed accurately so no tools are present or implied. The validation is made through a series of logical deductions. For these reasons this would be classified as showing an apparent dominant paradigm of GII. Again the fact that no definition of a kite is given in the textbook and it is not clear how the cases for congruency have been established do not mitigate against it being classified as having an **apparent** dominant paradigm of GII.

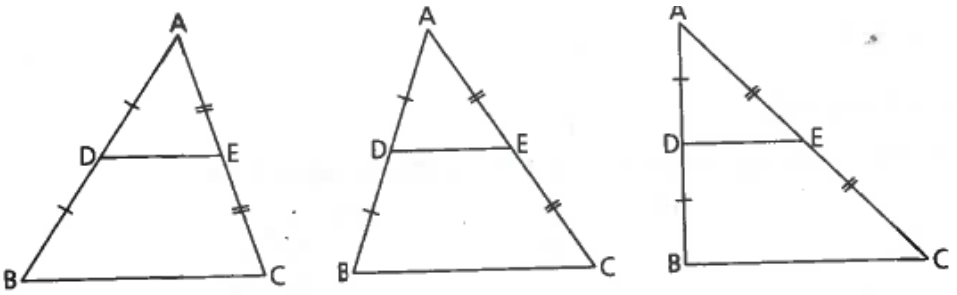
Learner 3's response uses coordinates. In section 7.5 above, we provided a classification of work using coordinates according to paradigms GIc, GIIc, GIp

and GIIp. Using those definitions this would be classified as GIc. In addition as, in this case, coordinate geometry is being used to make conclusions about properties of shapes, it is important that there is no contradiction between the classification of GIc (as described earlier) and our classification system here. In this example we see the object is a physical object, tied to specific coordinates, the grid acts as a tool with which the length of the sides of the object can be measured and the validation of the fact is based on this measurement. Thus the apparent dominant paradigm is GI.

Clearly in this activity there is no single dominant paradigm. In the 1st and 3rd parts GI is the apparent dominant paradigm and in the 2nd part GII is the apparent dominant paradigm. Thus, in a similar way to what was done for in the table showing the analysis of blocks in the transformation geometry chapter, I will use "GI, GII" to indicate that each paradigm is dominant for some portion of the block.

Example 3:





c) Measure the following pairs of angles in each triangle:
(i) $A\hat{D}E$ and $A\hat{B}C$ (ii) $A\hat{E}D$ and $A\hat{C}B$

d) Measure the lengths of the following line segments in each triangle: (i) DE (ii) BC

2. Put all your results together. Use them to make a conjecture about the lines DE and BC.

3. Use what you know about similar triangles to justify your conjecture.

Textbook extract 39: Example of apparent dominant paradigm GI and GII (CM, p71)

In question 1, learners are asked to construct triangles and use measurement to make a conjecture. This clearly indicates an apparent dominant paradigm of GI as the drawings are treated as objects and deductions are made using drawing and measuring tools.

In question 3, learners are asked to use similarity to justify their conjectures. This appears to be asking learners to prove a general conclusion for all triangles (theoretical objects) based on logical deduction. Thus the apparent dominant paradigm here is GII.

Again, similarly to what was done in the transformation geometry analysis, this will be indicated as GI -> GII, in order to indicate that there is a deliberate movement from GI to GII.

The curriculum analysis indicates that a movement from GI to GII is required in the FET band. Many authors have discussed the move from working with concrete objects to more abstract work as a difficult point of transition. Learners often struggle to understand when a move from one paradigm to the other has been made and what the rules of each respective paradigm are (Fischbein, 1993;

Kuzniak & Rauscher, 2011; Shaughnessy & Burger, 1985). As the move between an informal Euclidean geometry toward a more formal Euclidean geometry is likely to be where this features most prominent, I included in my analysis discussion of whether there was any teaching in the textbook directed at orienting learners towards the features of, and differences between, the two paradigms.

In particular I examined whether the following were present in the textbook chapters related to the properties of geometric shapes:

- Discussing what inductive reasoning is or what deductive reasoning is: This would be any statement that seeks to explain what either of these types of reasoning is or to clarify an aspect of this type of reasoning.
- Evaluating arguments: This would be when learners are asked to evaluate whether an argument is convincing or to explain which of two arguments is better.
- About elements of geometry in GI e.g. visualization, construction and measurement, investigation and conjecturing: This would involve direct discussion of these elements e.g. "Investigation is an important part of mathematics, it involves exploring different examples in order to try and uncover a pattern". Blocks where learners are just asked to investigate a particular issue would not be included here as what I am looking for is discussion **about** elements of the two paradigms.
- Discussion about elements of geometry in GII e.g. definition, classification systems (specifically a hierarchical or partitional classification system for quadrilaterals), proof, counterexample, axioms, theorems and axiomatic systems (any discussion of local axiomatic systems or "chains of reasoning" which show how definitions, axioms and theorems interlink is included here): In a similar way to the elements of GI discussed above, I am looking for

discussion about these issues and not simply mention of or use of them.

I thus included a column in the analysis table (in Appendix E) where I indicated if any of these were present in each block and then indicated whether this related to GI, GII or both.

7.7.1 Paradigms of geometry in chapters 3 and 14 of CM

The categorisation of the blocks that appear in the chapters on properties of shapes in CM is given in the table in Appendix E.

A summary of the number of blocks in which each of the paradigms appeared is provided below:

	GI	**GII**
Chapter 3	3	18
Chapter 14	10	23
Total	13	40

Table 34: Summary of blocks in CM in which each paradigm appeared

A summary of the number of blocks in which elements of GI or GII are discussed is provided below:

	GI
Total	0

GII	**GII**
Counterexample	1
Definition	5
Proof	3

Axiomatic system	5
Classification system	1
Total	15

Table 35: Summary of the blocks in CM in which elements of GI or GII are discussed

The summary indicates that in both chapter 3 and chapter 14, GII was the apparent dominant paradigm in the majority of cases. In chapter 3, GI was hardly present at all. In contrast in chapter 14, GI was the apparent dominant paradigm in about 1/3 of the blocks. This difference between the two chapters suggested it would be useful to discuss them separately.

The summary also indicates that no elements of GI are discussed whereas elements of GII are discussed. The notions of definition and axiomatic systems are the elements of GII that are discussed most.

7.7.1.1 Discussions of the paradigms in CM Chapter 3: Basic geometry

The chapter opener states clearly that part of this chapter is revision of work done in earlier grades and that new work involving counter-examples, definitions and the mid-point theorem will be introduced. This immediately points to an important issue to analyse. In the earlier grades geometry is intended to be studied informally. This means that the dominance of what appears to be a GII paradigm in the chapter represents a significant shift. This indicated that it would be important to look at:

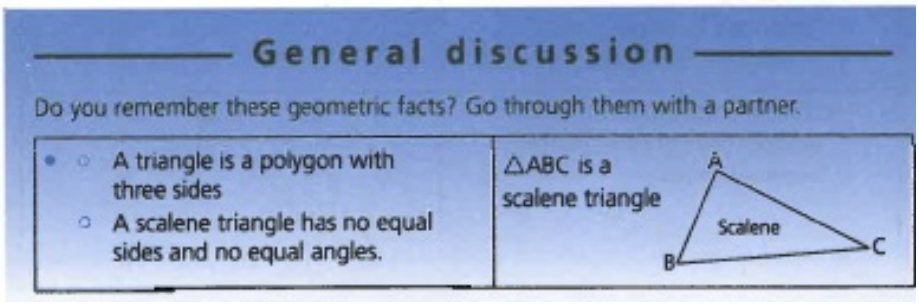
a) How the revision of work done in earlier grades is treated and whether it is structured in a way that helps the move into GII

b) The new work and the way in which GII manifests and is discussed within this

The revision of work done in previous grades

There are three summary narrative blocks (p58, p60 and p63) in the chapter in which a list of facts that learners should have previously seen is revised. These

all have GII as the apparent dominant paradigm. However the nature of the facts presented within each of these blocks differs and no clear distinction is made between a definition or axiom and a derived fact. I use the block that starts on p63 (textbook extract 40 below) to illustrate what I mean by this and discuss some of the issues that arise because of it.

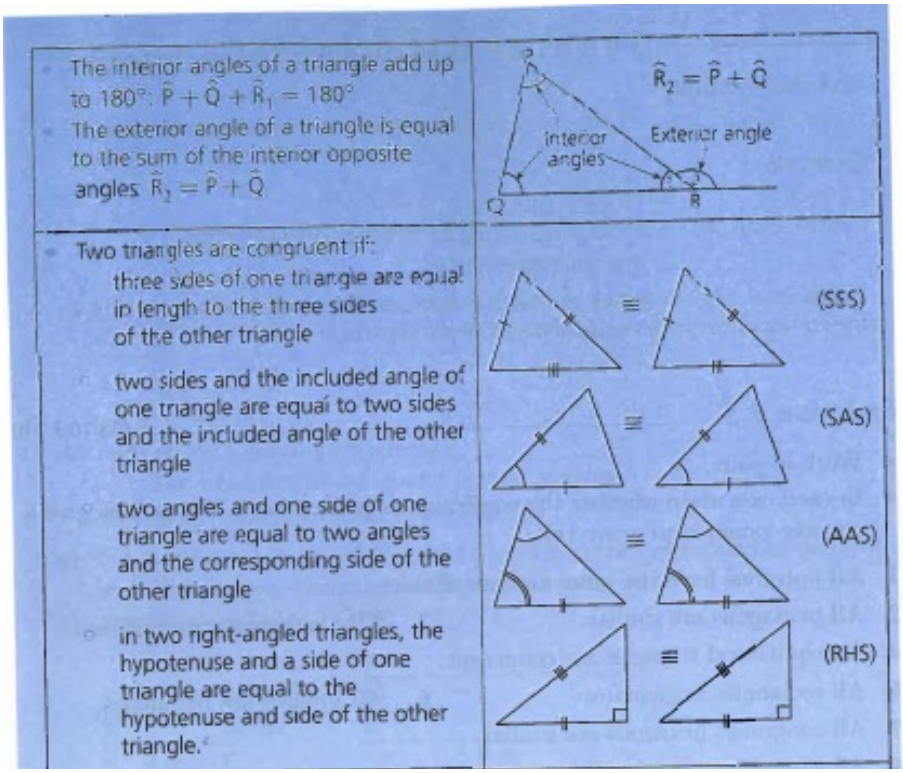


General discussion

Do you remember these geometric facts? Go through them with a partner.

- A triangle is a polygon with three sides
- A scalene triangle has no equal sides and no equal angles.

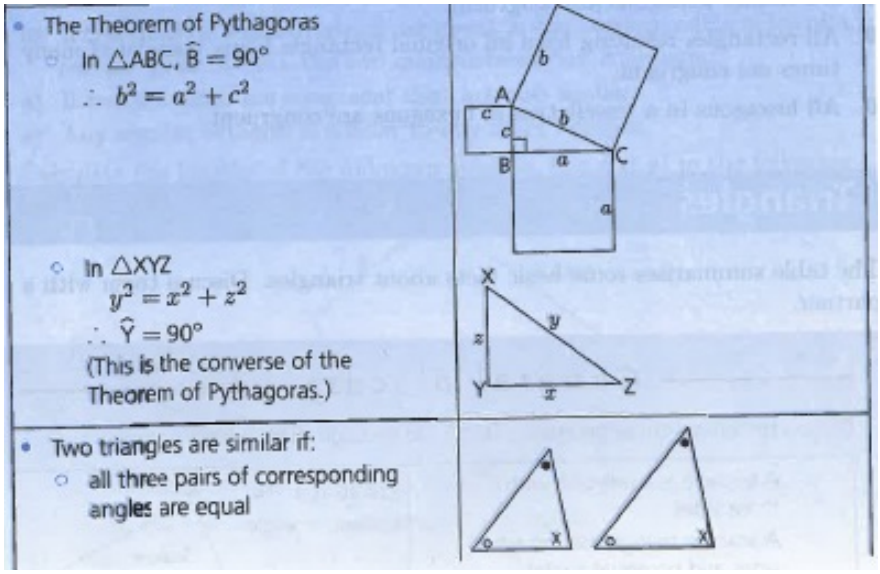
△ABC is a scalene triangle

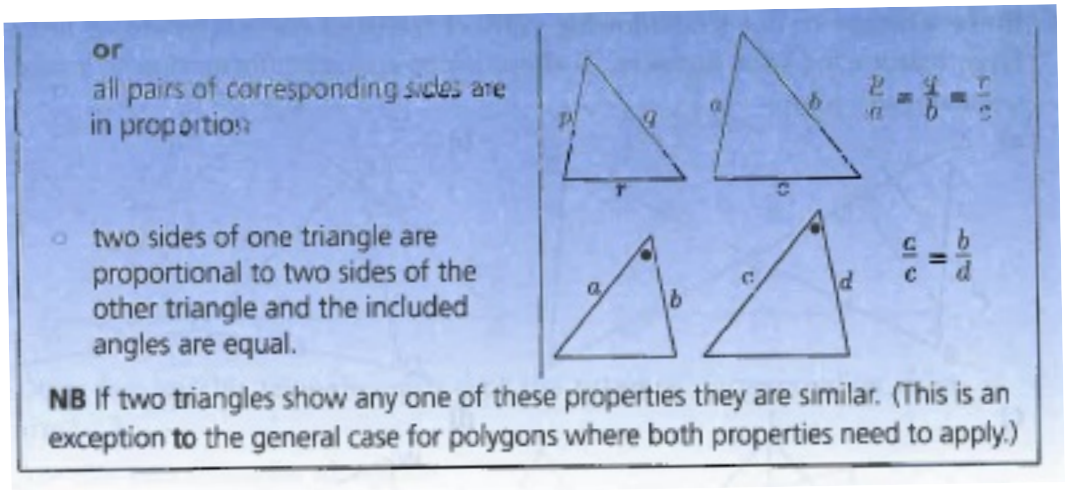


- The interior angles of a triangle add up to 180°: $\hat{P} + \hat{Q} + \hat{R}_1 = 180°$
- The exterior angle of a triangle is equal to the sum of the interior opposite angles: $\hat{R}_2 = \hat{P} + \hat{Q}$

$\hat{R}_2 = \hat{P} + \hat{Q}$

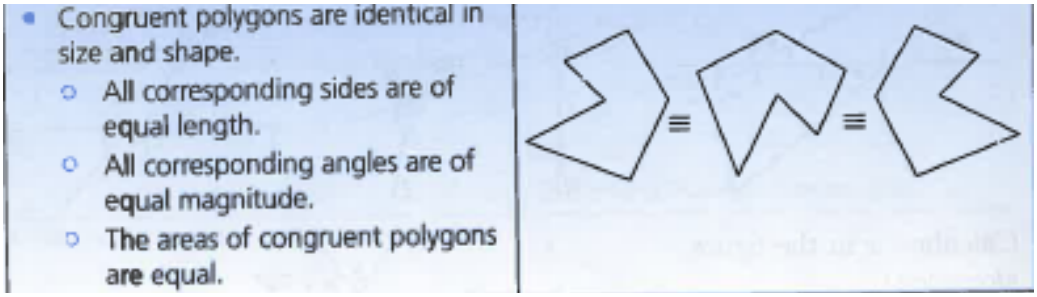
- Two triangles are congruent if:
  - three sides of one triangle are equal in length to the three sides of the other triangle (SSS)
  - two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle (SAS)
  - two angles and one side of one triangle are equal to two angles and the corresponding side of the other triangle (AAS)
  - in two right-angled triangles, the hypotenuse and a side of one triangle are equal to the hypotenuse and side of the other triangle. (RHS)





Textbook extract 40: Summary table (CM, p63-65)

We see here that the facts presented range from the definitions (the definition of a triangle and of a scalene triangle) to derived facts (the sum of the interior angles of a triangle, the relationship between the exterior angle of a triangle and the interior opposite angles, the theorem of Pythagoras and its converse.) This raises the issue of the status of the facts presented. The confusion inherent in this can be illustrated through looking at the way congruency is presented. In the previous summary narrative (on p61) congruency is presented as shown in textbook extract 41.



Textbook extract 41: Congruency in polygons (CM, p60)

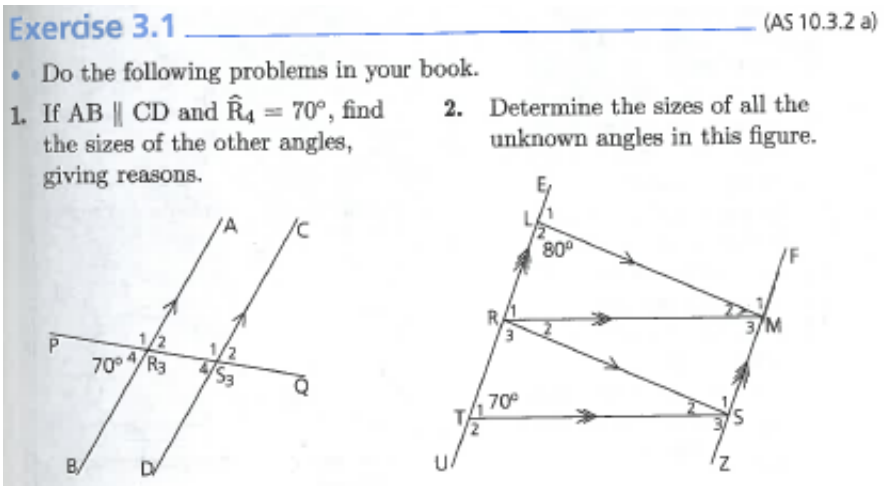
Here a definition of congruency is provided as being polygons that are "identical in size and shape". The status of the facts given in the bullet points is unclear. On one level they can simply be seen as consequences of two polygons being "identical in space and shape" or alternatively the first two bullet points combined could be seen as defining congruent polygons more formally and the third bullet point is a fact that can be derived from that definition.

In textbook extract 40 the statement about congruency in triangles is a set of derived facts. Any of the cases is sufficient to prove that congruency in triangles is derived from the definition of congruency in polygons and properties of triangles. However the layout of these facts is identical to that of the facts about congruency in polygons on p60. This gives the appearance that congruency in triangles is different to congruency in polygons. We see a similar issue in the statement about similarity in triangles: "If two triangles show any one of these properties they are similar. (This is an exception to the general case for polygons where both properties need to apply)" (CM, p65).

Immediately after each of these summary narratives is an exercise set in which the facts need to be applied in order to make deductions.

Thus we see that the recalled facts are presented with little regard for the mathematical structure and are given as tools with which to do the exercises. Although they have the appearance of presenting work in a GII paradigm they do not have a principle underlying GII structure.

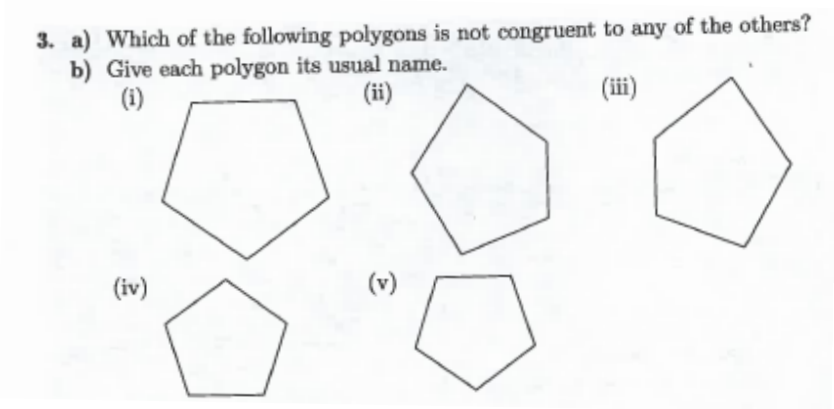
In the majority of exercises in this section learners are required to make deductions based on facts that have just been stated and to treat the diagrams simply as a support for their reasoning. Typical examples of this would be the questions shown in textbook extract 42 below which follow a summary narrative that provides facts about angles on parallel lines.



Textbook extract 42: Exercise on parallel lines (CM, p59)

Thus although there is some confusion in terms of the status of summarized facts, most of the work that learners are required to do supports an induction into the way of working required in the GII paradigm.

However there is one instance where this does not happen which is shown in textbook extract 43 below.



Textbook extract 43: An exercise where GI work is required (CM, p62)

This can clearly only be done by treating the drawings as objects and using measuring tools and thus requires work in GI.

In summary we see in the revision of old work in chapter 3 that although the focus is on the presentation of facts with the appearance of work in the GII paradigm, and the lack of differentiation between the status of the facts and some reliance on work in GI presents constraints on understanding the geometric work that is required.

The new work in chapter 3

In contrast to the set of recalled facts where no distinction is made between definitions and derived facts, the new work deals precisely with core features of GII directly viz. disproving facts by counter-example, definition and theorem.

Definition

The sequence of blocks dealing with definition starts with an instructional narrative reminding learners of the definition of an isosceles triangle as a triangle with at least two sides equal. It goes on to state you can use this definition to prove that the base angles of an isosceles triangle are equal. It then says that alternatively one could define an isosceles triangle as a triangle with at least two angles equal. The example and exercises that follow let learners derive properties of an isosceles triangle using each definition as a starting point. The final question in the exercise set asks learners to "Discuss two ways that you could define an equilateral triangle" (p70). The difficulty here is that on one hand an equilateral triangle only exists through its definition, and on the other hand the need to define the object we now call an equilateral triangle can only arise once we have had experience of that thing.

Thus the first mention of definition in geometry in CM contains deep mathematical ideas that include the nature of definition, equivalence of definitions and the need to decide on a definition to use as a starting point.

However none of these ideas are explicitly discussed which leaves these ideas undeveloped.

Disproving by counter-example

The work on disproving by counter-example is contained in the instructional narrative and exercise shown in textbook extract 2 on p86.

Interestingly, very similar ideas to those raised in the discussion of definition above, emerge from analysis of how this discussion of counter-example relates to CM's treatment of GII.

The first of these is that the notion that one disproves by counter-example is presented as just another fact. Here again we see how a fairly fundamental and deep point about the nature of deduction in mathematics is not discussed, but simply stated.

The second reinforces the ideas discussed above about the difficulty of mathematical objects in some sense pre-existing their definition. In particular no definitions of rectangles or squares are given in this chapter and the definition of an equilateral triangle is only discussed later in the chapter. Thus learners need to make deductions here based on an informal understanding of what these objects are. Although in some cases this may be possible, in order to correctly decide if all squares are rectangles, learners need recourse to the formal definitions of a square and a rectangle.

Theorem

The midpoint theorem and its converse are explored in an activity, stated formally (without proof) in an instructional narrative and then applied in an exercise. What is of interest here in terms of CM's treatment of GII is the manner in which the theorem is established. In the activity learners are first asked to draw triangles, join the midpoints of two sides and measure the angles, the length of the base of the triangle and the line joining the midpoint to establish a conjecture. They are then asked to use what they know about similar triangles to

justify the conjecture. They follow a similar process in exploring and justifying the converse of the midpoint theorem. Here we see work in GI to establish a conjecture and work in GII to prove it. Again no discussion is raised about these two different processes. In the work in GI, learners are asked to work in small groups and each member of the group is asked to construct 2 or 3 triangles and then do the constructions and measurements. The fact that experimentation with that many triangles is insufficient to establish the theorem or that the reason further justification is needed is not explicitly discussed.

Chapter 3 has no orienting narratives. None of the instructional narratives have GI as an apparent dominant paradigm. Thus we can see that there is no meta-level discussion about GI in this chapter. Similarly we see very little meta-level discussion about the elements of GII. Although there is mention of definition and counter-example, these are presented at the level of fact with no discussion of how they fit into the overall picture of geometric thinking and proof.

7.7.1.2 Discussions of the paradigms in CM Chapter 14: Quadrilaterals

The chapter opener for chapter 14 presents this chapter as covering a variety of tasks in relation to quadrilaterals. In particular it talks about investigating and making conjectures about the properties of quadrilaterals. This, together with the fact that we have seen that GI is the apparent dominant paradigm in about 1/3 of the blocks in chapter 14, meant that investigating the way in which GI is used in chapter 14 was important. As Fujita and Jones (2003b) have pointed out, the transition between informal and formal geometry is a significant step for learners and so, in a chapter that combines both GI and GII, it is important to look at how that link is made. In addition the chapter opener says that the learner will "learn how to develop proofs" and "investigate different ways of classifying and defining special quadrilaterals" (CM, p335). In addition, in contrast to chapter 3 where there was little focus on discussing aspects of the paradigms, in chapter 14 there are blocks discussing proof, axiomatic systems, classification systems and definition. This aligns with the signals of what is to be

learnt given in the chapter opener. Thus, in the discussion of the move from GI to GII, I discuss each of these aspects in turn.

How GI is used in chapter 14

There are three main types of blocks in which GI is the apparent dominant paradigm. The first of these is where there is an attempt to model a real-life scenario, the second where coordinate and transformation geometry are used and the third where an exploration of a concrete example is used to establish properties of a type of quadrilateral. I discuss these below.

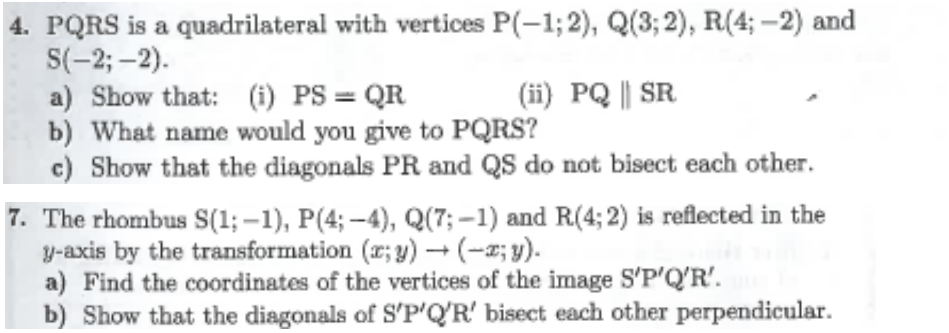
Modelling

This occurs in five blocks. Three of the five blocks are extra information narratives and as such are not central to the development of the work of the chapter. They illustrate how some of the facts about quadrilaterals can be applied to real-world scenarios. The fourth and fifth blocks are activities. In one learners are asked to explore how a carpenter might create a rectangle, and in the other mechanical linkages are explored. The learners create a physical model of a car jack and explore the action of the linkages. They are then asked to prove the conjecture they make on the basis of this physical exploration. In all these cases there is an attempt to illustrate the usefulness of the geometric facts that have been established to real world situations. Kuzniak argues that historically work in GI developed out of the need to solve practical problems (Kuzniak, 2008, 2011b, 2012), so this link between GI and modelling is a natural one.

Coordinate and transformation geometry

This work occurs in a section on its own entitled “Using coordinates”. Here the midpoint and distance formula are used to show lengths of line segments are equal, the formula for the gradient of a line is used to show that lines are parallel or perpendicular and the formulae that show the effect on the coordinates of a transformation are used to find transformed points. All of this work happens in the context of specific examples (i.e the end points of the lines have numeric

coordinates). The only link to previous work is that having found, using the distance formula and gradient, the opposite sides of a quadrilateral PQRS (where P, Q, R and S are given points), the learner is asked to name the quadrilateral or to use calculation to confirm known properties of that type of quadrilateral. Two typical examples are shown below.



4. PQRS is a quadrilateral with vertices P(−1; 2), Q(3; 2), R(4; −2) and S(−2; −2).
   a) Show that: (i) PS = QR (ii) PQ ∥ SR
   b) What name would you give to PQRS?
   c) Show that the diagonals PR and QS do not bisect each other.

7. The rhombus S(1; −1), P(4; −4), Q(7; −1) and R(4; 2) is reflected in the y-axis by the transformation $(x; y) \rightarrow (-x; y)$.
   a) Find the coordinates of the vertices of the image S′P′Q′R′.
   b) Show that the diagonals of S′P′Q′R′ bisect each other perpendicular.

Textbook extract 44: Examples of exercises using coordinates (CM, p345)

The essence of this work is calculation using the various coordinate geometry formulae and then application of the properties of quadrilaterals established empirically earlier in the chapter to name quadrilaterals. Of interest to note here is that although the curriculum stipulation is "Through investigations, produce conjectures and generalisations related to triangles, quadrilaterals and other polygons, and attempt to validate, justify, explain or prove them, using any logical method (Euclidean, co-ordinate and/or transformation)" (MNCS, p32), this has not been realized here. The strong location of this work using coordinate and transformation geometry in GI mitigates it being used to make more general claims.

Establishing the properties of quadrilaterals

GI is the dominant paradigm in the first two activities of this chapter. Textbook extract 1 on p40 shows the activity where learners are given instructions to investigate the properties of squares and rectangles by folding or measuring a drawing of a particular square and a particular rectangle. The activity that follows asks learners to repeat the process for a rhombus, parallelogram, kite

and trapezium. The learners are asked to record their results by ticking off properties in the following table shown in textbook extract 45 below.

No.	Property	Square	Rectangle	Rhombus	Parallelogram	Kite	Trapezium
1.	All sides are equal						
2.	At least two pairs of adjacent sides are equal						
3.	Both pairs of opposite sides are equal						
4.	At least one pair of opposite sides is parallel						
5.	Both pairs of opposite sides are parallel						
6.	Interior angles sum to 360°						
7.	All angles are right angles						
8.	At least one pair of opposite angles is equal						
9.	Both pairs of opposite angles are equal						
10.	Diagonals bisect each other						
11.	At least one diagonal is bisected at right angles						
12.	Both diagonals are bisected at right angles						
13.	Diagonals are equal in length						
14.	Diagonals bisect at least one pair of opposite interior angles						
15.	Diagonals bisect both pairs of opposite interior angles						
16.	All angles between diagonals and sides are 45°						
17.	Number of axes of symmetry						
18.	Order of rotational symmetry						

Textbook extract 45: List of properties of quadrilaterals (CM, p337)

Here we see that what is shown in the case of a particular example is generalized to be the properties of all of that type of special quadrilateral.

This list of properties plays an important role in the rest of the chapter. The properties are treated as established fact and the development of certain proofs and the definitions of the special quadrilaterals rely on it. As the blocks yet to be discussed in the chapter all have GII as the apparent dominant paradigm, the relationship between this list of properties established in GI and the way the development of GII play out is important. This is discussed in detail in below.

The relationship between GI and GII in chapter 14

Proof

After the initial activities in GI, the move to an apparent dominant paradigm of GII is immediate. The orienting narrative that begins the section that comes immediately after these activities states "In this section you will do calculations and proofs based on the properties of quadrilaterals." (CM, p339)

In most of the questions that follow learners are simply asked to explore relationships between properties such as the worked example shown in textbook extract 46 below. The proofs required in the questions and examples rely largely on the facts from chapter 3.

2. In quadrilateral ABCD, $\hat{A} = \hat{C}$ and $\hat{D} = \hat{B}$.
Prove, giving reasons, that AB ∥ DC and AD ∥ BC.



Solution

Let $x = \hat{A} = \hat{C}$; $y = \hat{B} = \hat{D}$



$\hat{A} + \hat{B} + \hat{C} + \hat{D} = 360°$ (Angles of a quad. add up to 360°)
$\therefore\ 2x + 2y = 360°$
$\therefore\ x + y = 180°$
i.e. $\hat{A} + \hat{D} = 180°$
$\therefore$ AB ∥ DC ($\hat{A}$ and $\hat{D}$ are co-interior angles)
and $\hat{A} + \hat{B} = 180°$
$\therefore$ AD ∥ BC ($\hat{A}$ and $\hat{B}$ are co-interior angles)

Textbook extract 46: Worked example (CM, p340)

The instructional narrative in this section provides the information about proof as shown in textbook extract 47.

General discussion

- Giving reasons at each step is a way of explaining how you get from one step to the next in your solution.
- It is important to work only from what is given when providing a proof or doing a calculation.
- In Example 2, you cannot assume that the quadrilateral is a parallelogram even though it looks like it is. You could claim that the properties of quadrilaterals show that when both pairs of opposite angles are equal, we have a square, rhombus, rectangle or a parallelogram. All of these figures have opposite sides parallel and so AB || DC and AD || BC.
- The proof given for Example 2 relies on basic known and accepted geometrical facts recorded in Chapter 3.

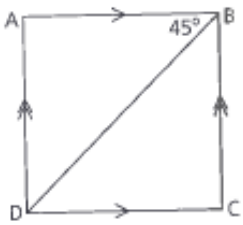
Textbook extract 47: Instructional narrative about proof (CM, p340)

On one level this instructional narrative signals a clear move into GII and sets out the importance of not making assumptions about a shape just because of how it looks and of making logical deductions from one step to the next. On the other hand it places the list of properties that has been generated in a GI paradigm as definitive.

The statement "You could claim that the properties of quadrilaterals show that when both pairs of opposite angles are equal, we have a square, rhombus, rectangle or a parallelogram" (CM, p340) is based on the assumption that the list of quadrilaterals in the table is complete. It makes the assumption that there is no other quadrilateral that has both pairs of opposite angles equal. And whilst this is clearly true it requires proof and that required proof is the proof shown in extract 46 above. Thus the recourse to the work in GI, when taken to its logical conclusion, results in a circular argument.

We see a further problem in the question shown in textbook extract 48 below.

5. Given that BD bisects $\hat{D}$, prove that ABCD is a square.



Textbook extract 48: A further proof (CM, p342)

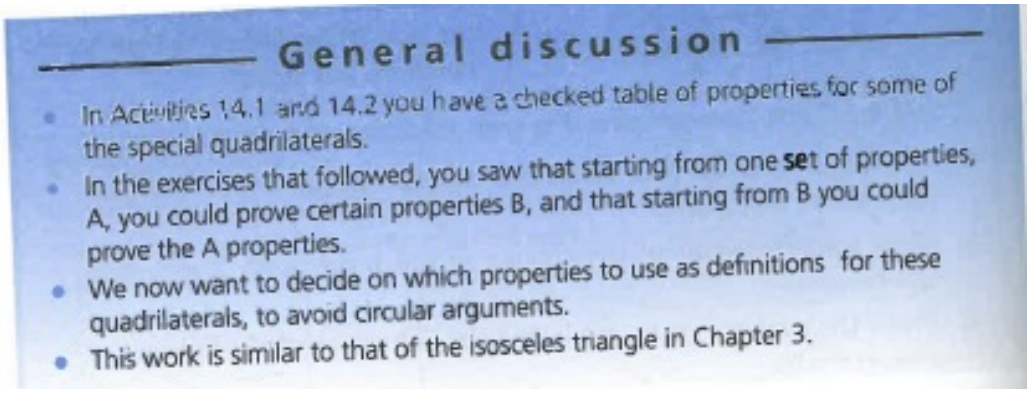
In order to do this, learners must make assumptions about what it means to prove something is a square. As the notion of square exists only in a GI paradigm at this point it is uncertain what it would take to produce such a proof.

Despite these two instances of confusion the majority of work that learners are engaged in requires them to work with making logical deductions showing that one property leads to another (e.g. if in a quadrilateral both pairs of opposite sides are parallel, then both pairs of opposite sides are equal).

The further instructional narratives about proof are more "technical" (e.g. you can add construction lines) and speak little about the nature of proof or the reasons one might want to prove something.

Definition

The instructional narrative on shown in textbook extract 49 signals a change in focus and from this point until the last few exercises at the end of the chapter the focus is on the process of defining the special quadrilaterals.



General discussion

- In Activities 14.1 and 14.2 you have a checked table of properties for some of the special quadrilaterals.
- In the exercises that followed, you saw that starting from one set of properties, A, you could prove certain properties B, and that starting from B you could prove the A properties.
- We now want to decide on which properties to use as definitions for these quadrilaterals, to avoid circular arguments.
- This work is similar to that of the isosceles triangle in Chapter 3.

Textbook extract 49: Instructional narrative starting section on defining (CM, p352)

As suggested in the instructional narrative, in the activities and exercises that follow learners are asked to explore which of the properties from the table of properties can be used to define each of the special quadrilaterals. They next investigate properties the quadrilaterals have in common and properties that

distinguish the quadrilaterals from each other. They then choose particular properties to define different types of quadrilaterals and see how the other properties can be derived from these.

Freudenthal (1973) and de Villiers (1998) refer to this type of defining as descriptive defining. De Villiers (1998, p250) outlines this practice as follows:

> With the descriptive (a posteriori) defining of a concept is meant here that the concept and its properties have already been known for some time and is defined only afterwards …. A posteriori defining is usually accomplished by selecting an appropriate subset of the total set of properties of the concept from which all the other properties can be deduced. This subset then serves as the definition and the other remaining properties are then logically derived from it as theorems.

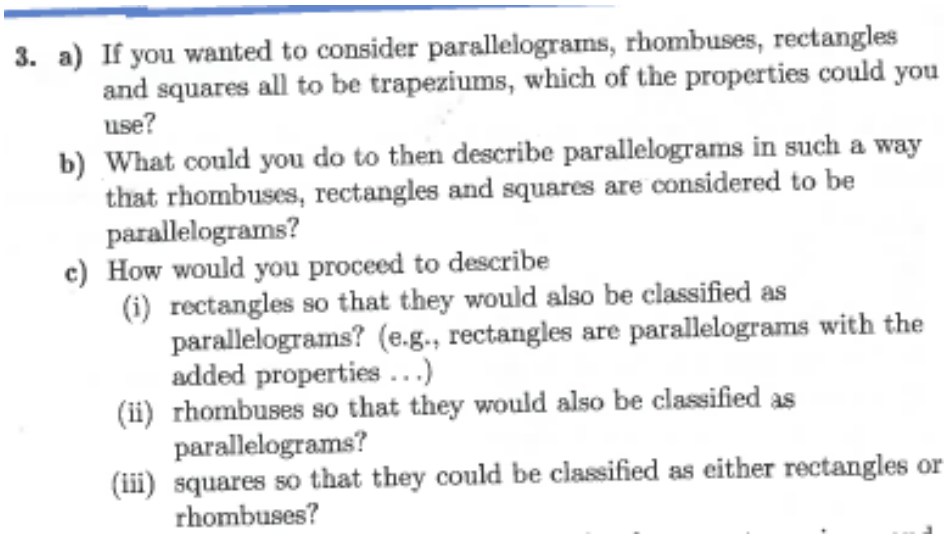
This type of defining is common in mathematics. CM attempts to recreate this process and give learners an experience of building an axiomatic system. However, recreating this process in a school mathematics textbook is a difficult task that offers layers of logical and pedagogical challenges that are illustrated in CM.

For example, in the first activity of chapter 14 learners are asked to cut out a rectangle which is 10cm by 7cm and by measuring or folding it explore the properties of the rectangle (see textbook extract 1 on p40). Here learners are working in GI and are dealing with the properties of a very particular physical shape. In this paradigm it would acceptable to state that one noticed that the diagonals of this shape do not meet at right angles or that adjacent sides differ in length. However the notion of a rectangle is an established one in GII and using the definition that exists in that paradigm neither of the two properties mentioned would be acceptable. CM gets around this possible conflict by providing a list of properties that learners need to tick if they are present for a particular quadrilateral, but "problematic" properties (such as the two mentioned above) are not included in the list. Thus the nature of the object explored in GI is implicitly controlled by its known nature in GII even though in

the textbook it does not as yet exist in GII as it has yet to be defined. This tension, which we can view as working with an object that exists in GI, but not in GII for the learners and yet exists in GII in mathematics, is a hard one to work with. This is the same issue, discussed on p259, that arose in dealing with defining isosceles and equilateral triangles in CM chapter 3.

Classification systems

In the activities and instructional narratives there is considerable discussion about how making different choices of defining properties could create either a partitional or hierarchical classification of the quadrilaterals. Learners are asked, for example, to discuss the questions shown in textbook extract 50.



3. a) If you wanted to consider parallelograms, rhombuses, rectangles and squares all to be trapeziums, which of the properties could you use?
   b) What could you do to then describe parallelograms in such a way that rhombuses, rectangles and squares are considered to be parallelograms?
   c) How would you proceed to describe
      (i) rectangles so that they would also be classified as parallelograms? (e.g., rectangles are parallelograms with the added properties ...)
      (ii) rhombuses so that they would also be classified as parallelograms?
      (iii) squares so that they could be classified as either rectangles or rhombuses?

Textbook extract 50: Exploring partitional and hierarchical definitions (CM, p354)

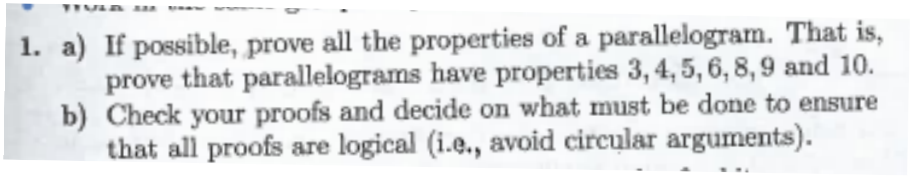
The curriculum asks that learners explore alternative definitions of quadrilaterals.

Here the notion is of alternative definitions that lead to alternative classification systems and clearly alternative axiomatic systems.

Axiomatic systems

Definition and axiomatic systems are inextricably linked. As we see in the instructional narrative above (textbook extract 49), CM talks about this as

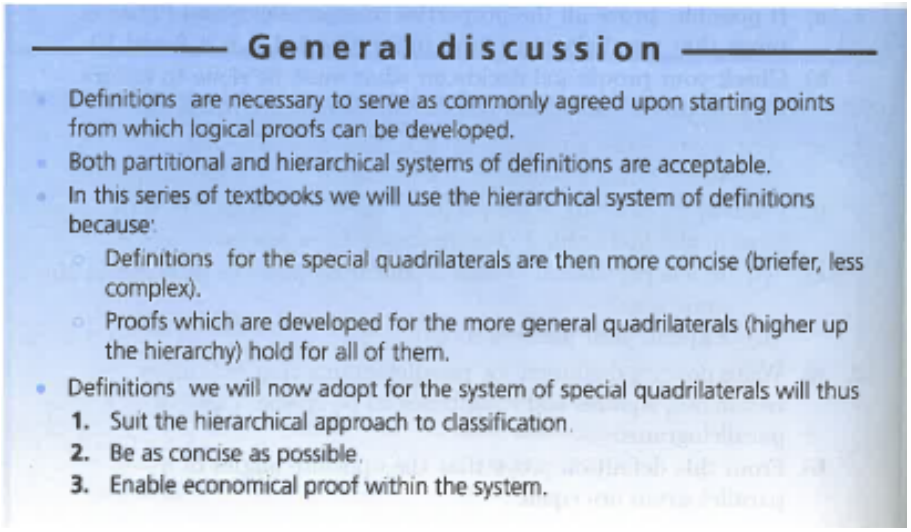
creating a system in order to prevent circular argument. The learners are asked to explore this in the activities. They are referred back to the list of properties (shown in textbook extract 45 on p264) and asked to explore what they need to do to prevent circular argument. This is exemplified in textbook extract 51below



1. a) If possible, prove all the properties of a parallelogram. That is, prove that parallelograms have properties 3, 4, 5, 6, 8, 9 and 10.
   b) Check your proofs and decide on what must be done to ensure that all proofs are logical (i.e., avoid circular arguments).

Textbook extract 51: Circular arguments (CM, p355)

After these explorations the following instructional narrative shown in textbook extract 52 below brings together this work.



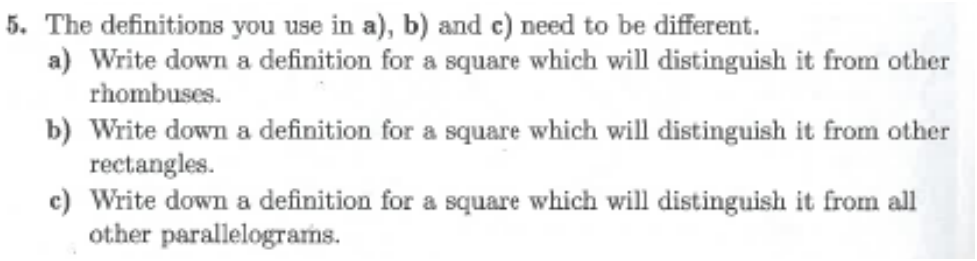
General discussion

- Definitions are necessary to serve as commonly agreed upon starting points from which logical proofs can be developed.
- Both partitional and hierarchical systems of definitions are acceptable.
- In this series of textbooks we will use the hierarchical system of definitions because:
  - Definitions for the special quadrilaterals are then more concise (briefer, less complex).
  - Proofs which are developed for the more general quadrilaterals (higher up the hierarchy) hold for all of them.
- Definitions we will now adopt for the system of special quadrilaterals will thus
  1. Suit the hierarchical approach to classification.
  2. Be as concise as possible.
  3. Enable economical proof within the system.

Textbook extract 52: Instruction narrative on definition (CM, p 356)

The idea that there is more than one possible equivalent definition is mentioned briefly as well. This happens after an activity where learners have worked on using particular properties to define a kite. They are then told in an instructional narrative that "We could use properties 11 and 14 as the basis of a definition for the kite" (CM, p357). However the reason why the definitions they created in the activity and the one using properties 11 and 14 are equivalent is not discussed.

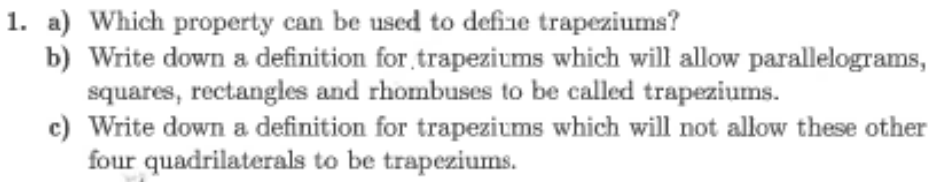
The idea of equivalent definitions is explored (and again not discussed) in an exercise shown in textbook extract 53



5. The definitions you use in **a)**, **b)** and **c)** need to be different.
   a) Write down a definition for a square which will distinguish it from other rhombuses.
   b) Write down a definition for a square which will distinguish it from other rectangles.
   c) Write down a definition for a square which will distinguish it from all other parallelograms.

Textbook extract 53: Exploring equivalent definitions (CM, p358)

However within the same exercise set there is a question that deals with alternative, non-equivalent definitions (textbook extract 54).



1. a) Which property can be used to define trapeziums?
   b) Write down a definition for trapeziums which will allow parallelograms, squares, rectangles and rhombuses to be called trapeziums.
   c) Write down a definition for trapeziums which will not allow these other four quadrilaterals to be trapeziums.

Textbook extract 54: Alternate non-equivalent definitions (CM, p358)

Thus we see here some of the difficulty of two aspects of this chapter: firstly the attempt to deal with objects in the two paradigms, and secondly the attempt to deal with very deep aspects of GII at the same time as exposing learners to a first encounter with it.

7.7.2 Paradigms of geometry in chapter 15 and 16 of OTM

The categorisation of the blocks which appear in the chapter on properties of chapters in OTM is provided in Appendix E.

A summary of the number of blocks in which each of the paradigms appears is presented below:

	GI	GII	?[26]
Chapter 15	18	5	0
Chapter 16	10	28	2
Total	28	33	2

Table 36: Summary of blocks in OTM in which each paradigm appeared

A summary of the number of blocks in which elements of GI or GII are discussed is presented below:

	GI
Inductive reasoning	4
Evaluate argument	2
Visualisation	1
Investigation	2
Construction	1
Total	10

GII	GII
Deductive reasoning	4
Evaluate argument	1
Proof	1
Axiomatic system	7
Definition	7
Axiom	2
Theorem	2
Counterexample	1
Total	25

Table 37: Summary of blocks in OTM in elements of GI or GII are discussed

The summary table shows that GI features prominently in chapter 15 and that GII is the dominant paradigm in chapter 16. This is unsurprising as chapter 15 is

[26] Two blocks dealt with definitions of invented objects. It was impossible to classify this work as falling into GI or GII.

entitled “Using inductive reasoning to study shape” and chapter 16 “Using deductive reasoning to study shape”. However, GI also features in a significant portion of the blocks in chapter 16. Table 28 also shows that discussion about elements of GI and GII feature prominently in the textbook. A closer analysis of how these elements are used indicates that they in fact are the core structuring device for the chapters. This is discussed below.

7.7.2.1 Discussion of the paradigms in OTM chapter 15: Inductive reasoning

The chapter opener states clearly that the chapter is about inductive reasoning and defines this as “when we draw conclusions from our observations”. This clearly sets the chapter within the GI paradigm. It also privileges “reasoning” rather than a specific content area (e.g. triangles or quadrilaterals). This focus on reasoning is reflected in the way in which the chapter is structured. Some elements of the geometric paradigms identified (evaluating argument, investigation and construction) form the basis of the structure of the chapter and so I discuss each in turn. In addition, despite the fact that reasoning is the focus of the chapter, the reasoning clearly has to be about something and so it was also important to look at the nature of the objects in the chapter.

Evaluating arguments

From table 44 in Appendix E we can see that after the chapter opener one of the key elements of the GI and GII paradigms that is brought into focus is the evaluation of arguments. The first activity asks learners to investigate the lengths of the diagonals of a kite and then uses three imaginary learners to present three arguments using transformations, congruency and coordinates. This is shown in textbook extract 38 on p248.

Learners are asked to compare their arguments to those given and to decide which is the most convincing. Two of the arguments (using transformation and coordinates) are clearly GI arguments: in the transformation argument manipulation of a cut-out kite is used and in the coordinate argument the grid is

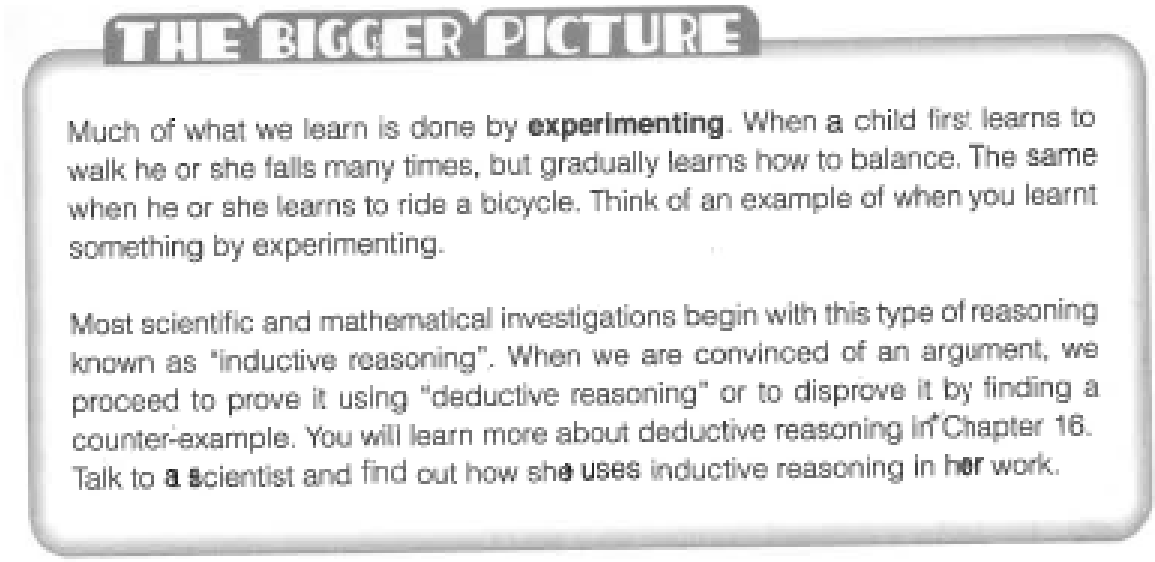
used to compare lengths. The congruency argument is given in the form of a typical GII argument.

Although learners are asked to describe which of the arguments is most convincing, at other points in the activity arguments are referred to as "explaining your reasoning" or "methods". In the instructional narrative the 3 imaginary learners are described as having "come up with the same conclusions about the lengths of diagonals of a kite, but each has used a different method to get to it" (OTM, p211).

Learners are asked to state whether they feel one argument is more convincing than another but in the activity that follows the importance of being able to work with all the "methods" is emphasized.

This activity is followed by an activity and an exercise where reflection is used to generate quadrilaterals and learners are again asked to "explain your reasoning" and to decide whether another learner's argument is convincing.

All of this is followed by the extra-information narrative shown in textbook extract 55 below:



THE BIGGER PICTURE

Much of what we learn is done by **experimenting**. When a child first learns to walk he or she falls many times, but gradually learns how to balance. The same when he or she learns to ride a bicycle. Think of an example of when you learnt something by experimenting.

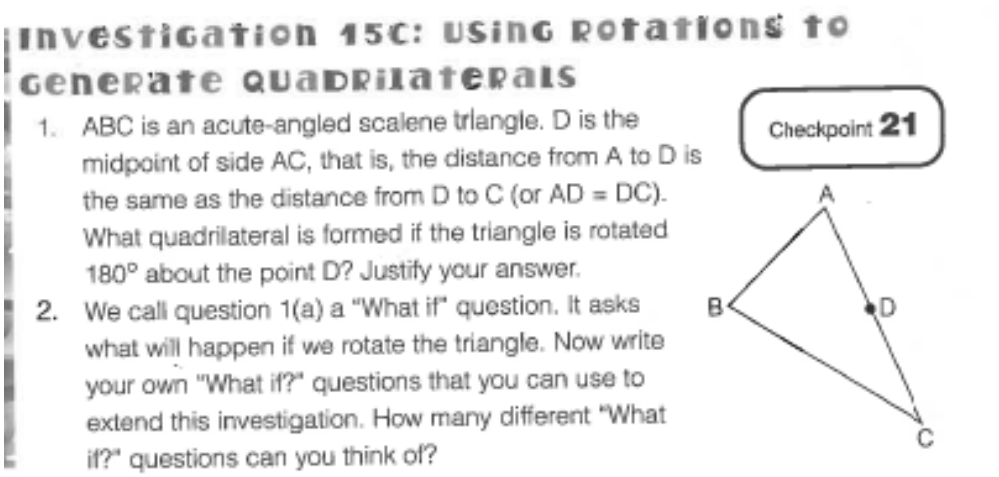
Most scientific and mathematical investigations begin with this type of reasoning known as "inductive reasoning". When we are convinced of an argument, we proceed to prove it using "deductive reasoning" or to disprove it by finding a counter-example. You will learn more about deductive reasoning in Chapter 16. Talk to a scientist and find out how she uses inductive reasoning in her work.

Textbook extract 55: Extra-information narrative (OTM, p213)

This formulation, together with the use of words in the activities, suggests that what learners have been busy with up until that point is not deductive reasoning. It also indicates clearly a split between what is dealt with in this chapter and deductive reasoning that will dealt with in the next chapter. What is not discussed here is what differentiates the arguments provided from "proof". The nature of proof is, however, discussed in chapter 16.

Investigation

The next discussion about the type of geometry learners are meant to be involved in raises the nature of investigation. In the activity shown in textbook extract 56 below the learners are explicitly pointed to the idea that the question they have just worked with is a "What if?" and are then asked to think of as many different "What if?" questions as they can. Although a number of the questions in the previous two activities were also in fact "What if" questions, this explicit focus on this type of question and asking learners to attempt to generate their own, makes a core feature of investigations central to the learners' work.



Investigation 15C: Using rotations to generate quadrilaterals

1. ABC is an acute-angled scalene triangle. D is the midpoint of side AC, that is, the distance from A to D is the same as the distance from D to C (or AD = DC). What quadrilateral is formed if the triangle is rotated 180° about the point D? Justify your answer.
2. We call question 1(a) a "What if" question. It asks what will happen if we rotate the triangle. Now write your own "What if?" questions that you can use to extend this investigation. How many different "What if?" questions can you think of?

Checkpoint 21

Textbook extract 56: Activity emphasising "What if" questions (OTM, p213)

Although the two activities and the exercise that follow this one focus on introducing the basics of coordinate geometry, the next activity returns to investigation by asking learners to investigate the midpoint theorem using

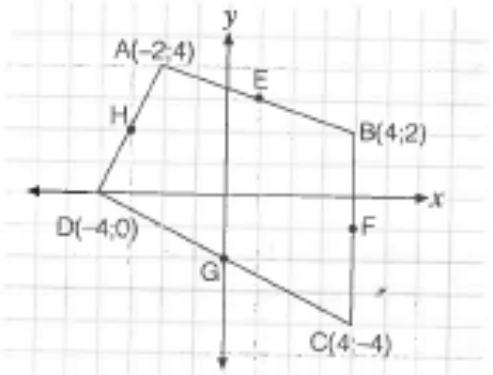
coordinates and the activity after that (shown in textbook extract 57 below) again raises the issue of investigation explicitly.

Through these kinds of activities the process of investigation itself becomes the focus of attention. The type of questions asked in the investigations also direct learners towards trying to see why a conjecture that results from an investigation may be true. We see this, for example, in question 3 in textbook extract 57 below and in the attention to arguments discussed above.

Investigation 15G: Using co-ordinates to explore quadrilaterals

Checkpoint **22**

1. (a) Copy this large irregular quadrilateral ABCD onto co-ordinate axes.
   (b) Calculate the co-ordinates of the points E, F, G and H, the midpoints of the sides as shown in the figure. Show your calculations.
   (c) What kind of quadrilateral is EFGH? Explain how you know.
   (d) Draw more irregular quadrilaterals on co-ordinate axes and test your conjecture in (c).



The fascination in the study of Geometry lies in the surprises that are awaiting our discovery. A true mathematician is always curious and wants to find out more. When meeting a surprising result as in question 1 above, the mathematician asks "What if?" questions, such as "What would happen if I joined the midpoints of the sides of a parallelogram, a rectangle, a rhombus, etc.? Surely if I make the quadrilateral ABCD more specialised, the parallelogram EFGH formed inside it will become more specialised!"

Here is your chance to be a mathematician.

2. Use co-ordinates to investigate the figures formed by joining the midpoints of the sides of a parallelogram, rhombus, kite, rectangle and square. Copy and complete the table as shown below.
(**Hint**: make sure you plot the figures so that you can pinpoint the midpoint easily.)

Type of quadrilateral	Parallelogram	Rhombus	Kite	Rectangle	Square
Shape formed by joining midpoints of its sides					

Show all your calculations and explain how you identified each shape.

3. Refer back to your investigation in question 2. Look carefully at the properties of the diagonals of the original quadrilateral. Can you use these properties to predict what type of quadrilateral you will form when you join the midpoints?
What property or properties must the diagonals of the original quadrilateral have so that the new quadrilateral is
(a) a rectangle
(b) a rhombus
(c) a square?

Textbook extract 57: Activity focused on investigation (OTM, p219)

Construction

Construction is introduced as being done with straight-edge and compass only and as being "one of the oldest methods used to study geometric figures" (OTM, p220). However no motivation for the restriction to those implements is given. Learners are asked to use construction to create the special quadrilaterals in various ways. For example:,

Now construct a rhombus:
(a) using two isosceles triangles
(b) using the diagonals of the rhombus.
In each case explain how you know that your figure is a rhombus

Textbook extract 58: Construction (OTM, p221)

The nature of the questions in this section directs learners' attention towards investigation and exploration of the relationship between different properties of the special quadrilaterals.

Nature of the objects in OTM chapter 15:

Throughout the chapter the special quadrilaterals are referred to without being defined. Learners are asked to investigate their properties, construct them or identify which of them is produced (either through transformation, on the coordinate plane or through a process of construction). However comments later in the textbook as well as comments in the accompanying teacher's guide make it clear that the author is assuming that the definitions of the special quadrilaterals are known. For example, the section of the teacher's guide that accompanies the activity where learners are asked to explore the properties of a parallelogram states "Remind learners that all they can assume about the parallelogram is that the opposite sides are parallel" (Bennie, 2005b, p181) and in the textbook in chapter 16 when the definition of a parallelogram is introduced for the first time in this book it is prefaced with the statement "We begin with a familiar definition" (OTM, p230).

7.7.2.2 Discussion of the paradigms in OTM Chapter 16: Deductive reasoning

It can be noted from table 44 in Appendix E that Chapter 16 is structured by a discussion of the key elements of GII even more strongly than chapter 15 was around key elements of GI.

The chronological flow of this structure in the chapter can be summarized from table 44 as follows:

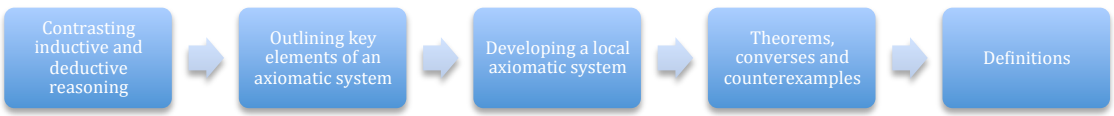


Figure 6: The chronological flow of the structure of OTM chapter 16

Each of these elements is discussed below.

Contrasting inductive and deductive reasoning

The chapter opener and first instructional narrative make clear distinctions between inductive and deductive reasoning. The chapter opener points to the fact that inductive reasoning was the mode of the previous chapter and that this chapter focuses on deductive reasoning. Deductive reasoning is described as "when you use existing results and a logical argument to draw conclusions" (OTM, p227). This is illustrated through using what is described as a known fact (that adjacent angles on a straight line are supplementary) to prove that vertically opposite angles are equal. Thus the start of this chapter indicates a clear change in the nature of work from that of the previous chapter and thus, implicitly, a clear contrast between the nature of reasoning required in GI and GII.

Outlining elements of an axiomatic system

Immediately after the discussion on the nature of deductive reasoning, an axiomatic system is described as where "theorems are deduced from axioms about undefined terms" (OTM, p228). The meaning of undefined term, axiom, theorem and corollary are provided. The link to previous work is provided through the statement "Many of the results you call 'properties' of straight lines, triangles, quadrilaterals and other polygons are really theorems or corollaries" (OTM, p228). Thus this instructional narrative explains what an axiomatic system is. The orienting narrative that introduces the following activity informs learners that they will "have a chance to work like a mathematician and to organize a part of your work into a deductive system"[27]. Here a link is made to the work of the previous chapter in the following way:

[27] OTM uses the term deductive system to describe what I have termed an axiomatic system

You will be working with parallelograms again. After working though Chapter 15, you should be very familiar with these shapes. In this chapter you will organize your knowledge into a localized logical deductive system. (OTM, p229)

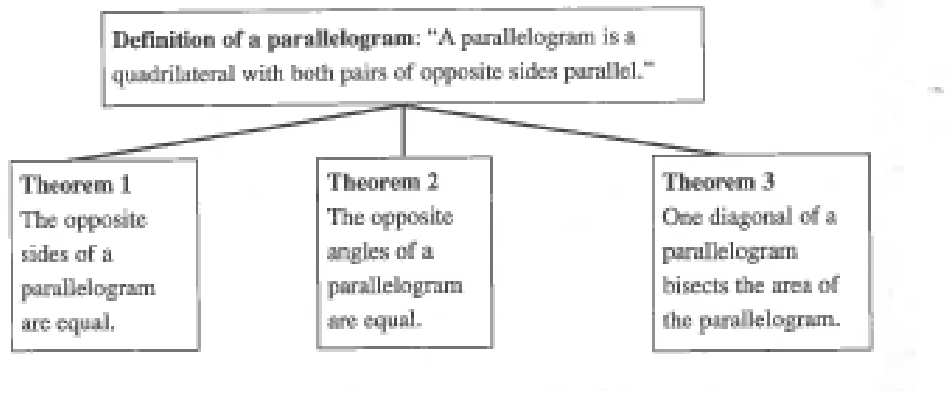
Thus the work in this chapter is not seen as producing new facts, but as organizing existing knowledge. Thus we see in OTM some signalling of GI and GII as two contrasting paradigms with different roles in geometric work.

Developing a local axiomatic system

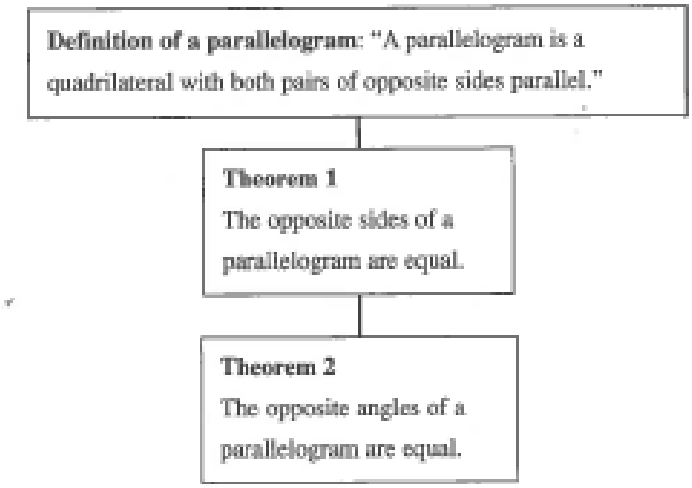
As indicated above, in this section learners are guided through a series of activities and instructional narratives to build a local axiomatic system. Here the idea, put forward above, that they are not generating new facts but organizing existing knowledge, appears through the sequence in which questions are asked. This sequence of questions is as follows:

> Learners are first asked to write down all the properties of the special quadrilateral that they know i.e. the properties are deemed to be known facts that are recalled.
> They are then provided with a definition of the special quadrilateral. This is similarly presented as a known fact.
> Next they are asked to construct the special quadrilateral using the definition and to check that it has all the properties listed. This task is not relevant to the fact that the sequence demonstrates that the sequencing of the questions indicates that what is being done is organizing existing knowledge. However I include it here for completeness as it is part of the sequence of questions. It is of interest to note that it represents a brief move to GI in the midst of a section that is dealing directly with the nature of reasoning in GII.
> They are then guided through using deductive reasoning to deduce the properties from the definition or from the definition and an already proved property.
> Finally they are shown (or are asked to create) a flow diagram that shows the chain of reasoning that has been used.

Textbook extracts 59 and 60 show two examples of the flow diagrams produced summarising a chain of reasoning used.



Textbook extract 59: Chain of reasoning 1 (OTM, p231)



Textbook extract 60: Chain of reasoning 2 (OTM, p232)

The learners are guided to work through chains of reasoning about the properties of parallelograms in two class activities and then on the properties of kites in an exercise set. They do not do proofs about the other special quadrilaterals in this chapter. Thus it is apparent that the focus here is on the practice of developing a local axiomatic system and not on confirming or establishing a list of properties of the special quadrilaterals.

Proof, theorems, converses and counterexamples

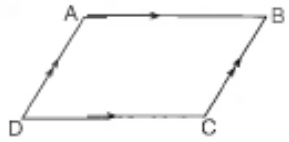
In addition to the broader level focus on system there is also specific focus on what proof is, stating theorems and converses formally and disproving by counterexample.

Proof is defined as ""a sequence of statements that we can use to explain something or to establish the truth of something" (OTM, p230). In the activity shown in textbook extract 61 below learners are provided with a proof of the fact that the opposite sides of a parallelogram are equal. Accompanying this worked example is a set of instructions that describe how one goes about constructing a proof.

Theorem 1: The opposite sides of a parallelogram are equal

Given: A parallelogram with opposite sides parallel.
To prove: The opposite sides are equal.

Identify what is given to you and what you must prove.

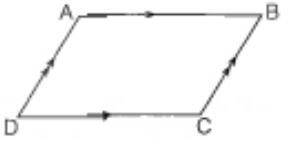


Draw and label a figure. Fill in the information we have about parallelogram ABCD.

Given: Parallelogram ABCD

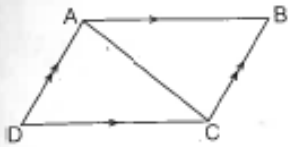
To prove: AB = CD and AD = BC.

Rewrite what is given to you as well as what you must prove in terms of the figure.



Now **plan** your proof on paper. Make sure you only use the definitions and axioms you already know. You must be able to explain the reasoning you use in your plan.

Construction: Join A and C.



You might realise that you need a construction in your proof. In this case we need to draw in one of the diagonals of parallelogram ABCD. When you are satisfied, write the proof.

Proof:

Can you follow the proof using congruency shown on the left?

In ΔABC and ΔDCA

AC is a common side

$B\hat{A}C = D\hat{C}A$ (alternate angles, $AB \parallel CD$)

$A\hat{C}B = C\hat{A}D$ (alternate angles, $AD \parallel BC$)

$\therefore \Delta ABC \equiv \Delta CDA$ (SAA)

$\therefore AB = CD$ and $AD = BC$.

Textbook extract 61: Proof (OTM, p230-231)

Thus learners are given direct instruction about how to create a proof.

A further activity looks specifically at writing the statements of theorems more formally. Learners are shown the "if…then" format of theorem statements and what the converse would be. Learners are asked to prove converses that are true and disprove converses that are false by producing a counter-example. The discussion is around the general idea of theorem statements and converses. However the tasks learners have to do all centre on proving or disproving statements about whether a quadrilateral having a particular property of a parallelogram proves that the quadrilateral is a parallelogram (e.g. the statement "If the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram").

Learners are asked to recall that a counter-example is "one example that shows a statement is not true" (OTM, p234). In a similar way to CM the fact that a single example is sufficient to disprove a statement is presented as fact and the reason this is so is not discussed.

The converses that are proved show the idea of defining properties. For example, if you have shown that a parallelogram has opposite angles equal as one of its properties and then you go on to show the converse is also true (i.e. if the opposite angles of a quadrilateral are equal the quadrilateral is a parallelogram) you have in fact shown that "opposite angles equal" is a defining property of a parallelogram. This link between these converses and definitions is not discussed here nor is this worked used during the discussion of alternative definitions that comes later.

Definitions

The discussion of definitions starts by defining definition as giving "the meaning of a word, in terms of other words you already know" (OTM, p234). This is followed by an activity where learners are asked to create a definition of an invented "creature":

A **definition** gives the meaning of a word, in terms of other words you already know.

1. Look at the creatures below.
   The "Cheeries"



Not "Cheeries"



(a) Which of these figures are "Cheeries"? Explain your choices.

(i) (ii) (iii) (iv)



(b) Write a definition of a "Cheerie".

(c) Compare your definitions of a "Cheerie". Does each definition describe a "Cheerie" and no other figure? Is there one correct definition of a "Cheerie"? Explain.

2. Now design you own "creature".
   (a) Give your creature a name.
   (b) Write a definition to describe your creature.
   (c) Give a set of examples of your creature. Give a set of non-examples of your creature.
   (d) Show your examples and non-examples to your partner. Ask him/her to write a definition.
   (e) Compare your definitions. Does your partner's definition describe your creature? Are the definitions the same?
3. What makes a good definition?

A good definition should be economical – it should give just enough information to ensure that it describes the word. It should also be useful. Mathematicians choose their definitions carefully from the available properties.

Textbook extract 62: Definition using an invented creature (OTM, p234-235)

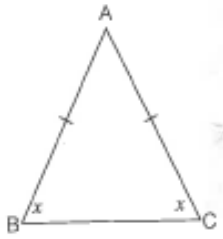
Thus learners' experience of the process of defining is not done in the context of a known shape and so sidesteps the issue of getting the learners to define something that already exists.

The remainder of the section on definitions discusses a good definition as economical and explores alternative equivalent definitions of a parallelogram.

This is first illustrated through the following example:

Investigation 16E: Definitions in Geometry

Here are two examples of definitions from Geometry:



EXAMPLE 1

The rough sketch alongside shows properties of an isosceles triangle. If you want to draw an **accurate** sketch of an isosceles triangle, do you **need** to use or are you **able** to use all of these properties?

1. Use your compass and a ruler and try it!

You will notice that if you draw a triangle with two equal sides **or** if you draw a triangle with two equal angles, you will get the shape that we think of as an isosceles triangle. So we could choose different definitions of an isosceles triangle:



Definition 1: An isosceles triangle is a triangle with two equal **sides**.



Definition 2: An isosceles triangle is a triangle with two equal **angles**.

Textbook extract 63: Definitions as economical (OTM, p 235-236)

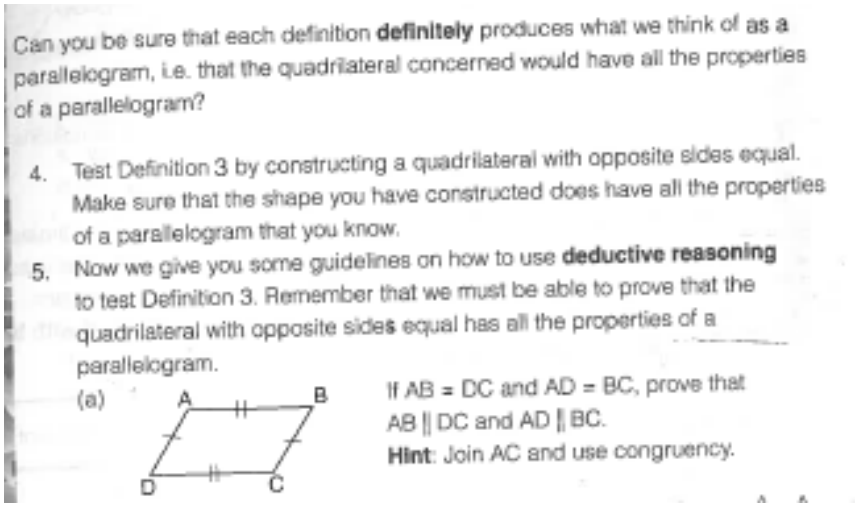
This has similarities to the difficulties of taking learners through a process of a posteriori defining as discussed in the analysis of defining in CM. In this example in OTM the author uses the phrase "the shape we think of as an isosceles triangle". The author thus signals a difference between the formally definition of an isosceles triangle and the idea of an isosceles triangle that exists prior to use defining it.

OTM then goes on to explore the definition of a parallelogram. In contrast to CM, OTM takes a known definition of a parallelogram as a starting point.

Earlier in this chapter we used this definition of a parallelogram: "A parallelogram is a quadrilateral with both pairs of opposite sides parallel." (Definition 1)
We showed that using this definition as a basis, it is possible to logically deduce all the other properties of a parallelogram. This is a good example of an economical definition. It would not be necessary to use this definition:
"A parallelogram is a quadrilateral with both pairs of opposite sides parallel **and** equal." (Definition 2)

Textbook extract 64 Definition of a parallelogram (OTM, p236)

Learners are then presented with four further possible definitions of a parallelogram and are asked to consider the following


Can you be sure that each definition **definitely** produces what we think of as a parallelogram, i.e. that the quadrilateral concerned would have all the properties of a parallelogram?

4. Test Definition 3 by constructing a quadrilateral with opposite sides equal. Make sure that the shape you have constructed does have all the properties of a parallelogram that you know.
5. Now we give you some guidelines on how to use **deductive reasoning** to test Definition 3. Remember that we must be able to prove that the quadrilateral with opposite sides equal has all the properties of a parallelogram.

(a)

If AB = DC and AD = BC, prove that AB ∥ DC and AD ∥ BC.
Hint: Join AC and use congruency.

Textbook extract 65: Defining a parallelogram (OTM, p237)

We see here that although OTM's explicit starting point is different to CM (OTM present a parallelogram as being defined as a quadrilateral with both pairs of opposite sides parallel whereas CM uses a GI notion of a parallelogram to produce a list of its properties), the use of the idea of "**all** the other properties" (my emphasis) indicates that the implicit starting point in OTM is in fact similar to CM. Parallelograms have many other properties other than those proved here (e.g. each diagonal bisects the area of parallelogram) and so this idea of proving "**all** the other properties" suggests that a list of properties exists, even if it is not given.

If one is working in GII and views a parallelogram as governed by its definition (quadrilateral with both pairs of opposite sides parallel[28]) then to show an alternative definition (let's call it definition 2) is equivalent to this definition, one would need to show

$$\text{definition 1} \Leftrightarrow \text{definition 2}$$

[28] For ease of reference I'll term this definition 1.

In the case of parallelograms each of the directions of the implication has, in fact, been shown earlier in the chapter (in the discussion of deductive systems and in the discussion of converses).

7.8 Discussion

In this section I link what emerged from the analysis of the textbooks with extracts from the interviews with the authors of the textbooks, those involved in creating the curriculum and relevant literature.

7.8.1 Influences on choice of paradigm

In all the interviews I conducted the difficulty learners and teachers in South Africa have traditionally experienced with geometry was discussed.

For example,

> Lynn: Was there a sense of dissatisfaction with what was in the old textbook in the geometry section?
>
> A_{CM}: Dissatisfaction in that so many kids couldn't actually grasp proof, couldn't carry the proofs out. So it was a very … It seemed to be a very difficult and a very demanding section to teach and I would say the number of teachers who had actual success in teaching this type of Euclidian geometry at this level particularly was very small, and as a consequence, kids just wrote Euclidian geometry off and teachers as well. They'd get them to learn theorems as you know I'm sure if you learn theorems off by heart when it came to the matric exam see what theorems there are, do your theorems first, leave blank pages for the riders. If you get a chance then try to do them before you have to hand your book in, your exam book in. And marking exam papers (as I used to do) – real blood money that was. You know if you were given the geometry paper to mark, pages and pages of emptiness, then the proof of a theorem that had been learnt off by heart. Change the letters and then the kid is in trouble. It was terrible.

As discussed in detail in chapter 4, there was considerable concern about whether teachers and learners would cope with the new curriculum and clearly a deep political and social concern about the consequences of a too difficult or overloaded mathematics curriculum. Euclidean geometry was seen as a particularly challenging part of the curriculum and, in particular, as a section of work that could stand in the way of providing equal access to mathematics for all learners:

> MRC: *(Speaking about the situation under apartheid education)* And de facto your black schools didn't teach higher grade and they didn't teach geometry. I mean you only have to ask anyone who has ever marked second paper they love the second paper from those kinds of schools because there is no geometry to mark. So our argument was that if you insisted on geometry being in then those kids would all end up in maths literacy.

As a result of discussions like this a special committee set up by the Minister of Education made Euclidean geometry optional after the MNCS was published and after the grade 10 textbooks had been written.

Although this decision was taken after the curriculum and grade 10 textbooks had been written, the sentiments expressed give an indication of the type of feedback from the field of reproduction that, in conjunction with the general regulative discourse, was influencing the ORF and PRF. The strong discourse in South Africa at the time about creating an education system that would reverse the legacy of apartheid and provide all children with equal access to quality education prioritised the notion of "maths for all". The experience of poor performance in geometry meant there was particular concern about the teaching and learning of geometry.

For the authors of the curriculum and the textbook the response to this concern was influenced by work from the field of mathematics education.

> A_{OTM}: But, generally my, I think my philosophy also, I think I think it's also based, I mean it's based on, I think, my, my experience of teaching Geometry, where a lot of students just felt, I think felt totally excluded because they

> couldn't make any sense of anything that had any sort of deductive reasoning, they couldn't work out where these things were coming from and why it was structured like it was. And I remember, in my teaching to try and explain it to the students, but I don't think I ever had any success. So, I think that, sort of my personal experience of that, teaching and also, I think, a lot from my work at Malati where we did a lot of van Hiele levels. So, I think, and also, I mean, from the research that we did at Malati that showed that students who are in grade 10, were certainly, or according to that theory, were not ready to be doing it formally were being expected to do in the curriculum.

We see here that A_{OTM} talks about the influence of van Hiele levels and the work done in the South African context by Malati. She talks, in particular, about how this work indicated that learners should be not rushed into having to tackle geometry at a formal level. However she also comments on the importance of formal geometry in terms of thinking skills and the structure of mathematics. The idea that one needed to start more informally because research and/or classroom experience suggests that learners are not ready for formal geometry was echoed by MC_1, MC_2, A_{CM}, GC_F and GC_G in their interviews. Thus a clear point that emerged from the interviews was a feeling that learners needed a strong basis in GI before they would be able to start work in GII.

Despite the importance given to GI it was largely viewed as an aid for entry to GII. GII was discussed as giving learners access to the structure and practices of mathematics. MC_1, MC_2, A_{OTM}, A_{CM} and GC_F reflected the importance of this in building towards formal geometric proof and developing an understanding of a local axiomatic system. We see this, for example, in comments like

> A_{OTM}: But then also I do think the, sort of traditional Euclidean geometry is very important in terms of the kind of thinking skills and to be able to understand the structure of Mathematics.

> A_{CM}: Nevertheless we thought (and I in particular feel it's very important) that that aspect not be lost because so much of modern mathematics

> depends upon axiomatics. And if one doesn't understand how axiomatics work, then a lot of modern mathematics gets lost.

Thus we see a strong sense of GI as a necessary pedagogical support and building block towards work in GII. This is reflected in the structure of chapter 14 of CM and in the placement of a chapter based largely in GI before a chapter based largely in GII in OTM.

In addition to GI's role as a pedagogical tool, there was also discussion about the importance of investigation and conjecturing:

> A_{CM}: So indeed I think of packages such as Geometer's Sketchpad and the like, and of course the investigation – that type of thing can be applied to any geometry, it doesn't have to be similarity or congruence or whatever, or circle geometry. So to set that approach going that one investigates, and that was also apparent in the Grade 10 where, in which the assessment standards were set up that investigation needed to play an important role in terms of results that one would arrive at and that one would then tentatively accept in terms of the investigations, call those results conjectures and then try to get some way of proving those conjectures in a more logical way. As we learn from studies that have been done around the learning of geometry that tends to be the most difficult part were the abstract proofs in the geometry.

The first point to note here is the mention of dynamic geometry software (DGS). The role of DGS in reinvigorating the study of geometry and opening opportunities for learners to investigate shapes is well documented in the literature (Jones, 2000; Laborde, 2000). In addition it was mentioned in this regard by A_{CM}, MC_2 and GC_F in their interviews. Clearly some of this work and the possibilities afforded by DGS had an impact on the MNCS and the textbook work despite the fact that ready access to computers and DGS is not common in South African schools.

The second point to note is the emphasis placed on investigation as a practice. Investigation and conjecturing was also mentioned by MC_2, GC_F and A_{OTM}. The

idea put forward in the comments by these interviewees are that investigation is not just a pedagogical support for work in GII, but an important part of mathematical work:

> GC_F: I think we owe an obligation to children to show them how mathematics is made. If we just show them the end product of mathematics we are short-changing them. I mean it's like teaching somebody how to cook by just showing them or dishing them up a meal … So also part of this idea is to show them mathematics in the making, that we can make mistakes, we can have false conjectures … So, you know, and that's a simple change to make in textbooks, have kids (not give the game away) let them actually investigate it. Now of course it makes it harder for kids in some respects because they don't have all … But I think they are gaining more from it, from what genuine mathematics is about.

Thus we see two views on the relative importance of the different paradigms in geometry: the first is the view of GI as a necessary pedagogical stepping stone into GII, and the second is a view of GI, particularly the investigating and conjecturing, as an important part of mathematical practice. Both these views are reflected in the work of both the textbooks, but they manifest in different ways. This is discussed further below.

The division of the chapters in OTM into one on inductive reasoning and the other on deductive reasoning puts mathematical practices at the centre of the geometrical work in OTM. In addition, as we saw in the analysis of the chapters, mathematical practices (e.g. evaluating arguments, investigation) helped structure the chapters. Throughout both chapters learners' attention is regularly drawn to aspects of the mathematical practices and there is explicit discussion of the mathematical practices. In particular, in GI, investigation is a focus. Learners are encouraged to ask "what if" questions. Learners are given guidance in the process of investigation (e.g. "draw a different triangle to test your conjecture"). The kind of investigations is largely focused on giving learners a genuine experience of investigating. They are not asking learners to explore or "discover" facts that are either already known to them or immediately obvious. See, for

example, the activity shown in textbook extract 57 on p278 in which learners investigate what quadrilaterals are formed by joining the midpoints of the sides of other quadrilaterals.

Although there is clear focus here on the practice of investigation, e.g. learners are specifically directed to think about what might happen if the originating quadrilateral was made more specialized, there are also strong parallels between this kind of work and the work of Godfrey and Siddons discussed by Fujita and Jones (Fujita & Jones, 2002a, 2003b). Fujita and Jones discuss geometric textbooks written by Godfrey and Siddons in the first half of the 20th century in which carefully designed experimental tasks were aimed at helping learners to develop what Godfrey termed the *geometrical eye*. This he defined as "the power of seeing geometrical properties detach themselves from a figure" (Godfrey, 1910 as quoted in (Fujita & Jones, 2003b, p47). Fujita and Jones argue that the design of the experimental tasks in Godfrey and Siddons's textbooks focus learners' attention on particular features of a drawing and thus help build their geometric intuition. They further state "Thus the place of these tasks in the teaching of geometry by Godfrey and Siddons is very important, not only for the sake of discovery, but also for the developing of the geometrical eye of students" (Fujita & Jones, 2003b, p56). The type of tasks provided in the chapter on inductive reasoning in OTM share similarities with the experimental tasks discussed by Fujita and Jones. However Fujita and Jones go on to emphasise the importance of linking these experimental tasks to deductive reasoning and they discuss how the tasks are designed so that they lead the learners towards requiring a proof and having the necessary geometric insight for the proof. The author of OTM expresses a similar idea:

> A_{OTM}: I mean, I think what I've tried to get through here is that we can explore and we can investigate but then we want to get to a stage where we want to, I mean, I think my idea about the Euclidean geometry stuff is that we are now going to try and explain why these things are like that, so that the idea of proof as explaining why things are like that.

However the structure of OTM militates against a strong link being established between the tasks in GI and those in GII. Firstly the separation of chapters into one on inductive reasoning and one on deductive reasoning means that there is not an immediate flow from an investigation into a proof. Secondly the focus in the chapter on deductive reasoning is on showing learners how to build a small axiomatic system and so the proofs that learners are engaged in are those that relate to the properties of parallelograms and a considerable portion of the work is on seeing how these proofs link to one another. Thus learners are not asked to consider how they might prove in GII some of the conjectures they made in GI (e.g. the investigation about the shape formed when joining the midpoints of the sides of quadrilaterals shown in textbook extract 57 on p278 is not followed up with a proof). The chapter opener for chapter 16 on deductive reasoning suggests that the chapter will formalize and explain some of the results that have been explored informally in chapter 15:

> Your investigation of different 2-dimensional figures in Chapter 15 may have convinced you of certain results. In this chapter, we show why these rules work. The path you have followed when studying Geometry is much like that used by all good mathematicians. They investigate and experiment until they are convinced of a result, that is they use inductive reasoning. Explaining "why" is the final step in a process. (OTM, p226)

However what we see is that the focus on practices means that the content explored in each chapter differs, i.e. to allow for studying how definitions and proofs fit into a axiomatic system the focus in the deductive reasoning chapter was narrowed to a few standard properties of parallelograms, whereas for proper investigation and exploration in the inductive reasoning chapter, the scope needed to be broader.

In CM in contrast, the chapters are not separated according to paradigms and the mathematical practices are not the predominant structuring mechanism for the chapters. The exception to this is the major focus on defining and axiomatic systems towards the end of chapter 14. Chapter 3, in particular, places very little emphasis on practices and instead presents geometric facts. As discussed in the

analysis above there is a degree to which GI and GII are conflated in much of this chapter. Much of the work in this chapter has the appearance of being in GII whilst relying on facts that have been established in GI. Thus we see the chapter prioritizing work in GII with little focus on the transition from the work in GI of previous grades. There is one place in the chapter where investigation in GI is specifically asked for: the exploration of the midpoint theorem shown in textbook extract 39 on p250. Here we see the movement from exploration in GI followed by proof in GII that was alluded to above. So we see, in contrast to OTM, here the focus on establishing the mid-point theorem allows a process of exploration, conjecturing and then proving. However the fact that the practices themselves are not prioritized means that they are not as rich or clear as they were in OTM. For example, learners are told exactly what to do here and no real process of investigation is carried out. In addition although learners can use a fact provided earlier (that two triangles are similar if two sides of one triangle are proportional to two sides of the other triangle and the included angles are equal) to justify the conjecture, how this fact relates to our understanding of what similarity means is not discussed.

In chapter 14, the key use made of GI is to use investigation to establish a list of the properties of the special quadrilaterals that is used as the basis for the discussion of defining, exploring hierarchical versus partitional classification of quadrilaterals and creating an axiom system. This approach echoes what GC_F had proposed to the curriculum committee as what should be done:

> GC_F: So what I suggested was that they try to focus on the process of defining rather than just presenting ready-made definitions and having students churn out theorems from those definitions. So, for example, I (and also with reference to the van Hiele theory) what I suggested was that children should for example first explore what a rectangle is without having a formal definition: A rectangle is something that looks like this. You can have it in Sketchpad, and you can have a figure like this. Let us investigate, you know, nothing more formal. Just like you don't learn what a table is by a formal definition, you know, this is a rectangle. And they then start investigating the properties of whatever the figure is, let's say it's a

> rectangle. But also by the way before they get to the properties let them investigate different variations of rectangles, including and particularly if you have Sketchpad, seeing that for example, if I have a rectangle I can move it until it becomes a square so that you actually get the classification at a visual intuitive level already in place. I know in the van Hiele theory it says it only arrives at van Hiele level 3, but I actually think it is important to address this. And I think particularly with Sketchpad because it is dynamic – you can actually move, you can show with a parallelogram that you can move it into a rectangle.
>
> And so this was one of my suggestions I think, for the curriculum, that in terms of the quadrilaterals, that they shouldn't start with a definition and just give students one definition, but they should give them visualization opportunities, introduce the objects intuitively, informally, then have them explore the properties of the figures and then only at the last stage start talking about well how do we define them. And then maybe create a little axiomatic system.

This approach then is clearly intended to link with the van Hiele approach and provide learners with a strong basis in visualization and analysis before moving on to informal deduction and later formal deduction. However, particularly in the context of a textbook, it is difficult to set up tasks that really foster investigation and visualization skills and at the same time provide the information required for building the definitions and axiomatic system. So we see in CM a strongly guided investigation (see textbook extract 19, p206) where learners are asked to tick off a pre-defined set of properties on a list (see textbook extract 45, p264). Below I discuss how the textbooks have dealt with defining in more detail. However here I note that we see tensions between wanting to engage learners in the mathematical practice of defining (a fairly deep and complex task at the heart of GII), whilst at the same time allowing this to develop out of an exploration in GI, and in all of this being guided by the fact that there are existing established definitions in the canon of mathematics that learners need to get to.

7.8.2 Defining

The quote above set out in detail the vision that GC_F presented to the curriculum committee. In addition his papers (de Villiers, 1994, 1998) discuss the ideas of giving learners access to the process of defining and of exploring hierarchical and partitional classification systems for the quadrilaterals. The authors of both textbooks alluded to being familiar with GC_F's work and aware of his influence on the curriculum. The curriculum itself states this work simply as follows:

Investigate alternative definitions of various polygons (including the isosceles, equilateral and right-angled triangle, the kite, parallelogram, rectangle, rhombus and square). (MNCS, p32)

This statement is vague and certainly does not describe the full process laid out by GC_F as discussed above. Investigating alternative definitions of polygons can imply investigating alternative equivalent definitions and potentially doing it formally in a GII system (i.e. showing definition 1 implies and is implied by definition 2), or one could understand it as exploring different possible ways of defining the quadrilaterals so that one would end up with a different classification system. We have seen in the analysis of defining in the textbooks, that OTM and CM reflect different understandings of this assessment standard.

One of the key ideas that emerge from analysis of defining in the textbooks rests in the fact that they are engaged in what de Villiers (1998) calls descriptive (a posteriori) defining. This refers to the fact that the concept and its properties are already known before it is defined. This contains an inherent tension. Although in mathematics the process of defining might start with an awareness of the existence of an object which is then defined, the object only actually exists mathematically once it is defined. The definitions of mathematical objects are carefully constructed (and adapted over time) to fit into a coherent mathematical system. These ideas are echoed by Nachlieli and Tabach, who using Vygotsky's work on scientific concepts state that "new objects make their way into

mathematical discourse via their explicit definitions" (Nachlieli & Tabach, 2012, p11), and Fischbein, who states "The properties of geometrical figures are imposed by, or derived from definitions in the realm of a certain axiomatic system. From this point of view, also, a geometrical figure has a conceptual nature. A square is not an image drawn on a sheet of paper. It is a shape controlled by its definition (though it may be inspired by a real object)" (Fischbein, 1993, p141). But at the same time many authors have highlighted that definitions are not necessarily the best entry point for learners when meeting the object (de Villiers, 1998; Freudenthal, 1973; Leikin & Winicki-Landman, 2000; van Dormolen & Zaslavsky, 2003; Vinner, 1991). In the case of geometric objects the inherent duality of the object allows us to introduce geometric objects to learners prior to defining them formally, but as we have seen in the textbooks this creates tensions. In CM a parallelogram, for example, is originally brought into existence through its physical representation. Thereafter a list of its properties is produced on the basis of exploration with that representation. This allows learners an entry point other than the formal definition. The list plays a crucial role in that properties such as "Two sides are long and two sides are short" are omitted from the list, so properties learners are likely to ascribe to the prototypical representation of a parallelogram or rectangle are not included in the list. Learners then look at ways they can use these properties to define the special quadrilaterals. Thus although learners are ostensibly being given the opportunity to define the special quadrilaterals, the existence of formal definitions for them in GII circumscribes the list of properties they are to produce for the object in GI. But almost paradoxically the list of properties produced in GI is seen as pre-existing the definition in that it is a selection from these properties that is used to produce the definitions.

Kuzniak and Rauscher (2005) comment on the same phenomenon and summarise it as follows: "So, the geometrical figure, totally determined by its definition, is confronted by a drawing which in turn, is the basis for the definition" (p3). They comment that this interplay between two paradigms of geometry

provides some of the explanation as to why learners experience difficulty in understanding geometry.

OTM sidesteps some of these problems by originally locating the practice of defining in a context with invented objects and by starting with an assumption that learners have been given a definition of a parallelogram. However the same paradoxical situation occurs. This is exemplified in the following statement:

> Earlier in this chapter we used this definition of a parallelogram: "A parallelogram is a quadrilateral with both pairs of opposite sides parallel." (Definition 1). We showed that using this definition as a basis, it is possible to logically deduce all the other properties of a parallelogram. (OTM, p236)

Here we see that although a GII definition for a parallelogram is provided, the pre-existence of a thing called a parallelogram with a specific list of properties is assumed. Note also that this pre-existing thing called a parallelogram is implicitly controlled by the GII definition in that if one can deduce "all the other properties of a parallelogram" from the given definition these properties cannot include the idea of a pair of equal long sides and a pair of equal shorter sides, for example.

Thus we see that the decision to involve learners in the practice of defining means the textbooks involve themselves in some of the deeper aspects of mathematics and result in a paradoxical situation.

8 Conclusion

8.1 Introduction

This study has shown that incorporating multiple goals into the school curriculum places demands that are difficult to manage and can produce tensions. The imperative to include socially relevant contexts did not fit easily with development of the mathematical ideas in the geometry sections of the textbooks and, where an attempt was made to bring them together, either the mathematical work was backgrounded or the social context trivialized. The pedagogical decision to work via inductive reasoning into deductive reasoning created particular tensions in the geometry produced in the textbooks. This working between the GI and GII paradigms was managed differently in each of the textbooks but nevertheless produced mathematical difficulties in both.

In this final chapter I reflect on the study as a whole. I begin by revisiting Bernstein's pedagogic device and and show how this tool has helped provide a lens for looking at my data and I also discuss how my study has contributed to the development of this theoretical tool. I elaborate on the tensions (discussed above) that emerged from the empirical analysis and draw on the theoretical framework to present a deeper understanding of them. In particular I show how Kuzniak's notion of the Geometric Working Space has provided a useful way of understanding the geometry that is produced in the textbooks. I conclude by offering some implications for both practice and research from my study.

8.2 The pedagogical device and GWS

The study has contributed to understanding the pedagogic device as incorporating more than a unidirectional recontextualisation of scholarly mathematics into school mathematics. The model I proposed (reproduced in figure 7 below) indicates that, in the creation of school mathematics, agents of the PRF and ORF recontextualise scholarly mathematics, mathematics education research and aspects of the general regulative discourse. Bernstein himself

alluded to this and pointed out that the recontextualising principle would not only recontexualise the scholarly discipline in question, but also what he calls "the how; that is the theory of instruction" (Bernstein, 1996, pp., p49). The empirical results of my study bear out the model proposed and thus enhance our understanding of the pedagogic device. My study shows that elements from different areas in the field of production (mathematics, mathematics education and the general regulative discourse) and from the field of reproduction interact in the field of recontextualisation, mutually influence each other and are each recontextualised into the pedagogical discourse of the curriculum and textbooks.

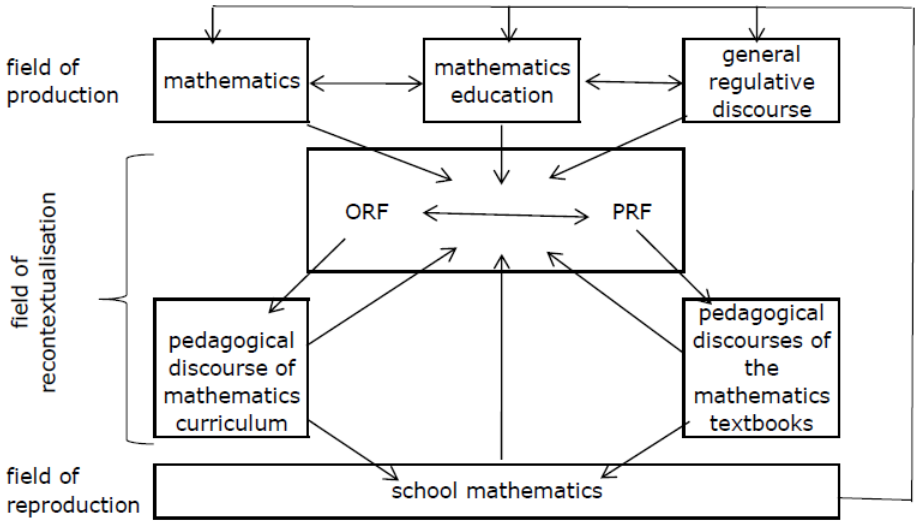


Figure 7 My model of the pedagogic device

For example, my study has shown that elements that were part of the general regulative discourse in post-apartheid South African were incorporated into the MNCS and the two textbooks. However this incorporation happened through a process of recontextualisation. As indicated in my model of the pedagogic device (figure 7) this recontexualisation takes place in the field of recontextualisation where influences from mathematics, mathematics education and the field of reproduction together with interaction between the ORF and PRF combine to produce the pedagogical discourse of the curriculum and textbooks. This was seen clearly in the study where, for example, the need to redress apartheid

inequalities combined with elements from the field of critical mathematics education and was recontextualised in the curriculum into a call for the incorporation of socially relevant contexts. In the textbooks it was apparent that the authors struggled to realise this in this face of the demands of the mathematical content. Thus not only was an aspect of the GRD recontextualised in the production of pedagogical discourse, but also this aspect of the GRD had an effect on the recontextualisation of another component of the field of production (i.e. the theory of critical mathematics education). In a similar vein we saw the van Hiele theory of the stages of thinking in geometry from the field of mathematics education being recontextualised into the MNCS and again in the textbooks. However this recontextualisation was influenced by and influenced other elements in the process of recontexualisation. So, for example, we saw the authors of the textbooks bringing their experience from the field of reproduction to bear on their understanding of how the van Hiele theory should be implemented in the textbooks. At the same time we saw that the van Hiele theory had an impact on the way in which the authors thought about their own classroom experience in relation to what they produced in the textbooks. Thus we saw the mutual influence of an element of the field of production (mathematics education) and experience from the field of reproduction resulting in a recontextualisation of both in the production of the pedagogic discourse of the textbooks.

In the synthesis of such a multiplicity of influences it is unsurprising that each aspect that is recontextualised often undergoes significant changes and that tensions and sometimes contradictions arise in trying to make them work together.

In pulling together the findings discussed in the previous chapters, two points of particular interest emerged. The first was the way in which, in the process of recontextualisation, different elements from the field of production and the field of reproduction were conflated, and in some cases confused. And secondly, although the agents of the ORF and PRF I interviewed largely did not speak of conflict between the ideas contained in the curriculum and those

recontextualised in the textbook, there were a number of instances in which what appeared realisable in the ORF could not be translated into the PRF. I will discuss each of these two points in turn.

8.2.1 Ideas from the ORF that were not realised in the PRF

The analysis of the MNCS shows that orientation 1 was a strong focus in the products of the ORF. Interestingly it was less present in the interview discussions I conducted with agents of the ORF. Thus we see that although ideas about access to mathematics and using mathematics to question and challenge social and economic inequalities were clearly present in the ORF, they were not foregrounded by the people I interviewed. This is perhaps unsurprising because within the MNCS the analysis showed that orientation 1 was strongest in the chapters that were not mathematics-specific and the interviews I conducted were with members of the curriculum committee who dealt specifically with mathematics. Thus orientation 1 remained largely at the level of rhetoric in the MNCS. Attempts to translate it into the mathematical heart of the MNCS and into mathematical activity in the textbooks were largely unsuccessful. The general regulative discourse present in South Africa at the time combined with notions of ethnomathematics and critical mathematics education had some impact on the ORF, but were never sufficiently integrated with the mathematical demands of the MNCS to have anything other than a superficial impact on the pedagogical discourse of the mathematics curriculum and the mathematics textbooks. In my study it appeared that the gaze of the mathematics curriculum committee members and authors that I interviewed was on the mathematics and thus the social and political were trivialised. Brantlinger (2011) discusses the same problem of bringing what he terms the "critical and mathematical components" (p409) together, but shows how, in his case, a focus on real-world concerns lead to a compromising of the mathematics.

8.2.2 Conflation of ideas

A strong imperative emerging from the GRD was the importance of redressing the effects of apartheid and building a democratic South Africa. Alongside this

were the scholarly writings in mathematics education advocating for reform. A motivating factor in the reform was providing access to mathematics for all. In the South African scenario experience in the field of reproduction in relation to geometry had earmarked this area of mathematics as a particular barrier to mathematics learning for large sections of learners. This, in the opinions of the agents of the ORF and PRF that I interviewed, necessitated a change in the approach to geometry in the MNCS. As discussed in chapter 4, the curriculum emphasises the importance of ensuring redress of apartheid inequalities and of winning all learners to mathematics. We saw that in the MNCS the section on "Contexts: inclusivity, human rights and indigenous knowledge" most directly addressed this issue. Although much of this section focused on using culturally sensitive contexts there was explicit mention of winning learners to Mathematics by "not formalising in the abstract prematurely but first taking care to develop understanding and process skills" (MNCS, p62). This gave the necessity for changing school geometry a moral and political slant and, in particular, suggested that a gradual move to abstraction was important for allowing all learners access to mathematics.

This sentiment aligned well with experience from the field of reproduction. For example, we saw the author of OTM, in talking about what motivated her approach to the geometry chapters in OTM talked about the difficulties she believed learners experienced in learning geometry presented in the traditional theorem-proof format and suggested that the van Hiele theory suggested an alternative that might be more accessible. Thus the idea of the concrete before the abstract and inductive before deductive reasoning is supported by her experience in the field of reproduction together with the van Hiele theory from the field of production.

The author of CM spoke similarly about a combination of classroom experience and evidence from education research suggesting a movement away from the products of mathematics to the processes of mathematics. This is seen in statements from the MNCS such as the following:

> Competence in mathematical process skills such as investigating, generalizing and proving is more important than the acquisition of content knowledge for its own sake. (MNCS, p9)

The MNCS emphasizes the importance of not presenting mathematics as a fixed body of facts but allowing learners to see it as developing and as being contestable over time. It thus re-emphasizes the need to involve learners in the process of investigating and creating mathematics.

In the MNCS Euclidean geometry was particularly identified as a place in which learners would get a sense of how mathematics is created and formalized. This included allowing for experimentation and conjecture, involving learners in the process of defining and ultimately in the creation of a local axiomatic system. At the level of giving learners experience of Euclidean geometry and proof, the idea was to move away from presenting learners with a quantity of theorems and proofs to learn and instead to take small sections of work, explore the shapes empirically and then work in a way similar to how mathematicians are perceived to work in order to create a local axiomatic system.

Summarising these ideas fairly crudely indicates that from the field of production there is

- a moral imperative from the GRD to open access to mathematics to all learners that is recontextualised into a suggestion that delaying the move to abstraction might be necessary for this.
- a view of the process by which mathematics is constructed is recontextualised as experimentation followed by deduction.
- the van Hiele theory of levels of thinking in geometry suggests a careful approach to the teaching of geometry that allows learners to move through the levels. This theory is recontextualised to as the importance of including more experiential work prior to deduction.

And these were combined with feedback from the field of reproduction that the "definition – theorem – proof" presentation of mathematics that was the dominant mode in previous versions of the curriculum and textbooks was not accessible to learners and that a process that allowed learners to be more active in constructing the mathematics was preferable.

At the intersection of all of these ideas is the notion of moving from inductive to deductive reasoning or from the concrete to the abstract. The fact that this is reinforced from multiple standpoints makes it a particularly strong notion. This is certainly evident in what was produced in the MNCS and textbooks. However the analysis of the textbooks indicates that these ideas create tensions in the geometry presented in the textbooks. I argue that in the conflation of disparate influences some of the nuances and understanding necessary to work with the inherent difficulties has been lost.

The key distortion that happened in the conflation process is the characterization of the practices of mathematicians. For example, the movement between the concrete and the abstract is very different and serves different purposes for an experienced geometer or for a learner in school. Although there is no doubt that inductive reasoning, visualization and experimentation play an important role in the work of an experienced geometer, the experienced geometer remains aware and in control of his/her passage between the different paradigms of geometry. As Kuzniak explains:

> When specialists are trying to solve geometric problems they go back and forth between the paradigms and they use figures in various ways, sometimes as a source of knowledge and, at least for a while, as a source of validation of some properties. However, they always know the exact status of their hypotheses and the confidence they can give to each one of their conclusions. (Kuzniak, 2011b, p4)

The problem with attempting to mimic this process within an educational setting is that novice geometers are not aware of the exact status of their hypotheses and it is difficult to make them so. As we saw in looking at material from the PRF,

the textbooks struggled with this notion in different ways. In OTM the strategy was to separate the two into chapters on inductive thinking and deductive thinking. Although this signposted a distinction between the two modes of thinking and perhaps made sense from a pedagogical point of view, it did not induct learners into the process of moving between the two as an experienced geometer might. In CM there was a constant movement between the GI and GII paradigms with little explicit attention to the status of results. In CM, in particular, "facts" discovered in GI become, in part, accepted and used as the building blocks to develop definitions in GII. The geometric working space (GWS) developed in both textbooks is thus unclear.

Interestingly there are similarities between what my research project has demonstrated and the findings from Kuzniak's work in France. He shows that currently in France at this level of schooling, the reference GWS "can be characterized as relevant to parcelled Geometry II. Numerous demonstrative islets are introduced to show well the link between geometry and space intuition" (Kuzniak, 2011a, p10). He argues that this type of GWS and the "constant emphasis on the transition to GII based on GI" (p10) suggests that a mixed Geometry, i.e. a GI – GII mixture, is possible. He shows that this translates into an appropriate GWS that is unstable and swings in a confusing way between paradigms. The nature of the Geometry waivers according to the level of understanding of the learners instead of being based on a principled epistemology. In the French scenario, the use of dynamic geometry software (DGS) is cited as having a particular impact on the current situation. However, in the South African scenario where very few schools and learners use DGS, we have seen in the development of the MNCS a reference GWS similar to that in France, and in the two textbooks studied an appropriate GWS that is also largely unstable, oscillating between the two paradigms.

One of the key problems this study highlights is that, in an attempt to open up access to mathematics, learners have in fact been denied access to some of the crucial aspects of mathematics. This dilemma is at the heart of some of the issues surrounding recent reforms in mathematics education. Sfard (2000, p180)

identifies similar problems in her discussion of the reform movement, particularly in relation to work in the USA:

> Similarly, the request for rigorous definitions which may count as "truly mathematical" cannot sound convincing without it being related to the idea of mathematical proof; and the mathematical rules of proving, in their turn, cannot be understood without the agreement that the ultimate criterion of a proper argumentation is the logical bond between propositions, and not relations between these propositions and physical reality. None of these meta-rules can be arbitrarily removed or changed without affecting the congruity and cohesiveness of the discourse. On a closer look, therefore, it turns out that because of the keen wish to respect students' need to understand, one may end up compromising the very feature of the mathematical discourse which is the basic condition of its comprehensibility: its inner coherence.

In designing this research project I originally chose geometry because I felt the nature of geometric objects as both concrete and abstract would lever up particular challenges in the development of teaching materials and thus prove an interesting topic to analyse. Although this has certainly proved to be true, the discussion from Sfard above suggests that what might have been illuminated quite starkly through my exploration of geometry are also challenges present in other areas of mathematics. We have seen in geometry that the process of helping learners make the move from GI to GII is not a simple one. But in studying the MNCS it was apparent that movement from the concrete to the abstract, embedding mathematics in real-life situations and the placing of an active learner capable of constructing mathematics (and possibly contesting it) was part of the overall philosophy of school mathematics in South Africa. Although, as we have seen in the case of geometry, these ideas are appealing and appear to emerge with support from findings in the field of production and from experience in the field of reproduction, their realisation in the ORF and PRF are far from straightforward. Sierpinska (1995a, p10), quotes Ryle (1969) as claiming that "if there were a smooth transition between practical and theoretical reasoning, most of the well known paradoxes, such as Dichotomy or

Achilles, would not even exist", which reinforces the notion that despite the inherent appeal there is significant work to be done in effecting the transition.

8.3 Some implication for further research and for practice

Although my research cannot purport to provide easy answers as to how to effect the transition from concrete to abstract or practical to theoretical in mathematics education, I would like to venture suggestions, some which have implications at the classroom level and others that identify areas for further research.

The first clear implication is in terms of the kind of mathematical knowledge required in order to realise a mathematical learning programme at school level. In order to teach mathematics in a way that allows learners access to what Sfard (2000, p179) terms its "inner consistency and overall coherence", teachers need to have knowledge that extends beyond mathematical content to knowledge about mathematics and its process of creation and organisation.

Secondly, the learner-centred philosophy has too often been taken as a suggestion that learners should reinvent mathematics for themselves and that teachers should refrain from showing learners how to do it. The analysis in this research report shows that at the FET phase (grades 10 – 12) learners need to grapple with ideas that take them into the heart of mathematical thinking and into mathematics as a discipline. Along with Sfard (2000) I assert that it is naïve to think that learners will be able to achieve this without the active guidance of a more knowledgeable other.

Finally there is a need for teachers to be deeply aware of the difficulties in making the transition from the concrete to the abstract. Alongside this is the necessity for research that looks at the way mathematics and this transition is presented in school curricula, textbooks and classrooms and for understanding the efficacy of different learning programmes. Houdement and Kuzniak's (2003)

notions of the geometric paradigms and associated notion of the Geometric Work Space has proved very useful in this study. My experience in using these same ideas with teachers suggest these theoretical tools might also be useful in helping teachers think about their classroom work. More recent work by Kuzniak and his colleagues suggest that the notion of a Mathematical Work Space might be a useful tool in exploring other area of mathematics teaching and learning. The utility of GWS in providing the kind of insights it has in my study suggest to me that further development of this theoretical work to develop a good understanding of the kinds of Mathematical Work Spaces, at the level of the reference, appropriate and personal, will be very useful in aiding a research programme understanding the nature of movement from the concrete to the abstract in mathematics education as well as providing a tool to help teachers think about these ideas in relation to their classroom practice.

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Appendix A: Interview outlines

Interview with members of curriculum committee

Overall sense of the process of the committee

composition of the committee
how committee went about their work
brief given to committee
sources and resources used

Process in particular around space and shape aspects of curriculum

who
sources and resources
Thrust of new space and shape/geometry
what are main changes from the old curriculum and what motivated these changes
did committee have an overall approach to what the main aim of geometry in school is? If so, what was this
GET – FET transition
how closely did you work with the GET curriculum and the GET curriculum committee in developing the FET curriculum.
do you think there is a clear trajectory in the development of transformation geom. and Euclidean geom. from GET through FET? Describe it.
What in particular is the progression in transformation geom. in FET from the GET band?

Format of the curriculum

prescribed?

useful for material developers and teachers? what other support did you envisage material developers and teachers needing/using in interpreting curriculum.

Clear van Hiele underpinning to Euclidean geom. section – how would material developers and teachers access this?

Proof

proof – what was envisaged?
what discussions? about proof in transformation geom., coord geom., Euclidean geom.
the rest of the curriculum (i.e. outside of geom.)

Interview with textbook authors

Did you have an overall philosophy that guided you in the development of

the Euclidean geometry sections?
transformation geometry sections?
coordinate geometry sections>
overall to space and shape measurements

How easy did you find it to interpret the curriculum?

Comparing the old edition of the textbook and the new edition of the textbook

what would you say are the main differences
generally
in terms of geom..
how much did you use the old in creating the new?

In relation to space and shape (as a whole) what do you think the strengths are that the new edition has that the old edition didn't have as strongly – and vice versa

Same qu as above but in relation to Euclidean geom..

Teachers use of the book

how would you want teachers to use the book?

what criteria did you use for deciding what goes into the teacher guide?

how much support do you think the teacher guide gives to teachers for implementing geom..? What else/more do you think they might need?

Content specifics

For the chapters

main aim of the chapter?

moves between informal/experimental work and formal/proof work

what is the purpose of this?

there is some tension for me between the desire to give students an experience of geom. that is grounded in some practical experience but at the same time provide a notion of a mathematical system. How do you feel you dealt with that tension? What are some of the issues it raised for you as a textbook writer?

Transformation geometry and coordinate geometry

proofs?

guiding principle?

Interview with geometry consultants

Their involvement in the curriculum process

how were they brought in

what role did they play

Thrust of new space and shape/geometry

what are main recommendations you made in terms of space and shape/geometry in the new curriculum?

Why did you make these recommendations?

To what extent do you think your recommendations were realised in the new curriculum

Appendix B: Tables detailing orientation 1 in blocks of the textbooks

In Appendices B – E I provide the categorisation of the blocks of the textbooks according to the various orientations. The tables in Appendix B list all blocks in the textbooks. For the sake of space and ease of reading in Appendix C, D and E I have present only those blocks relevant to the orientation being analysed.

Orientation 1 in CM

ch	pg	type	title	aspect	description
3	57	chapter opener	Basic Geometry	1	photo Ndebele woman and hut
3	58	summary narrative	Intersecting and parallel lines		
3	59	exercise set	exercise 3.1		
3	60	summary narrative	polygons		
3	61	exercise set	exercise 3.2		
3	63	instructional narrative	counter-examples		
3	63	exercise set	exercise 3.3		
3	63	orienting narrative			
3	63	summary narrative	triangles		
3	65	exercise set	exercise 3.4		
3	68	instructional narrative	definitions for the isosceles triangle		
3	68	worked example			
3	69	activity	activity 3.5		

3	70	instructional narrative			
3	71	activity	activity 3.6		
3	72	instructional narrative	midpoint theorem and converse		
3	72	exercise set	exercise 3.7		
3	73	exercise set	exercise 3.8 Check your skills		
3	75	activity	activity 3.9 Apply your skills, Exploration of a fallacy		
3	76	puzzle	problem solving		
3	76	project	project	1&3	Research Euclid and Pythagoras's biography and contribution to mathematics
4	77	chapter opener	Linear functions and analytical geometry		
4	78	instructional narrative			
4	78	worked example			
4	79	activity	activity 4.1		
4	80	instructional narrative	definition		
4	80	worked example			
4	81	instructional narrative			
4	81	exercise set	exercise 4.1		
4	83	activity	activity 4.3		
4	83	instructional			

		narrative			
4	83	worked example			
4	84	exercise set	exercise 4.4	2	Seating placement to make HIV/AIDS ribbon logo
4	85	activity	activity 4.5		
4	87	instructional narrative			
4	87	worked example			
4	87	exercise set	exercise 4.6		
4	88	instructional narrative			
4	88	worked example			
4	89	instructional narrative	did you know?		
4	89	exercise set	exercise 4.7		
4	90	instructional narrative	general discussion		
4	90	activity	activity 4.8		
4	91	instructional narrative	general discussion		
4	92	worked example			
4	93	exercise set	exercise 4.9		
4	94	instructional narrative	general discussion		
4	94	activity	activity 4.9		
4	95	exercise set	exercise 4.11		
4	96	exercise set	check your skills exercise 4.12		
4	97	activity	apply your skills activity 4.13		

4	97	puzzle	Problem solving: A linear plot?		
6	133	chapter opener	Length, area and volume		
6	134	instructional narrative			
6	134	activity	activity 6.1		
6	135	extra information narrative	did you know?		
6	136	instructional narrative	general discussion		
6	136	exercise set	exercise 6.2		
6	137	exercise set	exercise 6.3		
6	137	exercise set	exercise 6.4		
6	138	instructional narrative	general discussion		
6	139	exercise set	exercise 6.5		
6	140	extra information narrative	did you know?		
6	140	extra information narrative	did you know?		
6	140	instructional narrative	general discussion		
6	141	exercise set	exercise 6.6		
6	141	activity	activity 6.7		
6	142	instructional narrative	general discussion		
6	143	exercise set	exercise 6.8		
6	147	exercise set	exercise 6.9		
6	148	instructional narrative	general discussion		

6	148	exercise set	exercise 6.10		
6	149	exercise set	exercise 6.11 check your skills		
6	150	activity	apply your skills	1	photograph of three different size potjies
6	150	puzzle	problem solving		
9	189	chapter opener	Transformational geometry	1	photograph of decorated calabash
9	190	instructional narrative	general discussion		
9	191	activity	activity 9.1		
9	193	instructional narrative	general discussion		
9	194	exercise set	exercise 9.2		
9	195	instructional narrative	general discussion		
9	195	exercise set	exercise 9.3		
9	197	instructional narrative	general discussion		
9	197	activity	activity 9.4		
9	199	exercise set	exercise 9.5		
9	202	instructional narrative	general discussion		
9	203	exercise set	exercise 9.6 more reflections on points		
9	204	instructional narrative	general discussion		
9	204	exercise set	exercise 9.7		
9	206	exercise set	exercose 9.8		
9	207	instructional narrative	general discussion		

9	207	worked example			
9	209	activity	activity 9.9		
9	212	activity	activity 9.10 check your skills		
9	213	activity	activity 9.11 apply your skills		
9	213	puzzle	problem solving	1	analyse transformation used on calabash
14	335	chapter opener	Quadrilaterals		
14	336	orienting narrative			
14	336	activity	Activity 14.1		
14	338	orienting narrative			
14	338	activity	activity 14.2		
14	339	orienting narrative			
14	339	worked example			
14	340	instructional narrative	general discussion		
14	340	exercise set	exercise 14.3		
14	342	instructional narrative	general discussion		
14	342	worked example			
14	344	exercise set	exercise 14.4		
14	346	extra information narrative	did you know?		
14	346	activity	activity 14.5		
14	347	instructional narrative	general discussion		

14	347	extra information narrative	did you know?	1	Discusses how shape of rectangular and circular houses in rural areas can be marked out.
14	348	activity	activity 14.6		
14	349	instructional narrative	general discussion		
14	349	exercise set	exercise 14.7		
14	350	extra information narrative	did you know?		
14	351	orienting narrative			
14	351	activity	activity 14.8		
14	352	extra information narrative	did you know?		
14	352	instructional narrative	general discussion		
14	353	orienting narrative			
14	353	activity	activity 14.9		
14	354	instructional narrative	general discussion		
14	355	orienting narrative			
14	355	activity	activity 14.10		
14	356	instructional narrative	general discussion		
14	357	activity	activity 14.11		
14	357	instructional narrative	general discussion		

14	358	exercise set	exercise 14.12		
14	359	instructional narrative	general discussion		
14	359	exercise set	exercise 14.13		
14	360	exercise set	exercise 14.14		
14	363	exercise set	exercise 14.15 check your skills		
14	365	activity	activity 14.16 apply your skills		
14	365	puzzle	problem solving		

Table 38: Orientation 1 in CM

Orientation 1 in OTM

ch	pg	type	title	aspect	description
14	198	chapter opener	Formulae for 3-dimensional shapes		
14	199	instructional narrative			
14	199	activity	Investigation 14A: A packaging problem		
14	199	instructional narrative			
14	200	activity	Investigation 14A: A packaging problem ctd		
14	200	orienting narrative			
14	201	extra information narrative	The bigger picture		

14	201	activity	Activity 14B: Building formulae for 3-dimensional shapes		
14	202	learning reflection	Am I on track?		
14	202	instructional narrative			
14	203	exercise set	Exercise 14.1		
14	203	activity	Investigation 14C: Changes in the dimensions of 3-dimensional shapes		
14	203	learning reflection	Am I on track?		
14	204	activity	Investigation 14D: Using transformations to generate 3-dimensional objects		
14	205	activity	Investigation 14E: 3-dimensional objects and graphs		
14	206	project	Conic sections	3	Brief discussion of conic sections in Greek mathematics
14	208	learning reflection	tracking my progress		
15	209	chapter opener	Using inductive reasoning to study shape	1&3	Brief discussion on Egyptian and Babylonians use of measurement and experimentation
15	210	activity	Investigation 15A: Different ways to study quadrilaterals		

15	210	instructional narrative			
15	211	activity	Investigation 15A: Different ways to study quadrilaterals ctd		
15	211	learning reflection	Am I on track?		
15	212	orienting narrative			
15	212	activity	investigation 15B: Using reflections to generate quadrilaterals		
15	212	learning reflection	Am I on track?		
15	212	exercise set	Exercise 15.1		
15	213	orienting narrative	The bigger picture		
15	213	activity	Investigation 15C: Using rotations to generate quadrilaterals		
15	214	activity	Investigation 15D: Using co-ordinates and gradient to investigate 2-dimensional shapes		
15	214	instructional narrative	Investigation 15E: Using coordinates and length to investigate 2-dimensional shapes		
15	215	activity	Investigation 15E: Using coordinates and length to investigate 2-dimensional shapes		
15	216	extra information	The bigger picture	3	Information on

		narrative			Descartes and Fermat and Cartesian coordinate system
15	216	exercise set	exercise 15.2		
15	218	learning reflection	Am I on track?		
15	218	orienting narrative			
15	218	instructional narrative			
15	218	activity	Investigation 15F: Investigating midpoint properties		
15	218	orienting narrative			
15	218	learning reflection	Am I on track?		
15	219	activity	Investigation 15F: Investigating midpoint properties ctd		
15	219	activity	Investigation 15G: Using coordinates to explore quadrilaterals		
15	219	instructional narrative			
15	220	orienting narrative			
15	220	activity	Investigation 15G: Using coordinates to explore quadrilaterals ctd		
15	220	extra information narrative			

15	220	orienting narrative			
15	220	activity	investigation 15H: Using geometric constructions to explore quadrilaterals		
15	222	extra information narrative	The bigger picture		
15	222	exercise set	exercise 15.3 (mixed)		
15	223	learning reflection	tracking my progress		
15	224	project	architecture in africa	1	Project on architecture in Africa
16	226	chapter opener	using deductive reasoning to study shape		
16	227	instructional narrative	Deductive versus inductive reasoning		
16	229	learning reflection	The bigger picture		
16	229	orienting narrative		3	discussion of Greeks and deductive reasoning
16	230	instructional narrative			
16	230	activity	investigation 16A: Developing a logical deductive system		
16	230	instructional narrative			
16	230	activity	investigation 16A: Developing a logical deductive system ctd		

16	230	orienting narrative			
16	230	instructional narrative	theorem 1: the opposite sides of a parallelogram are equal		
16	230	activity	theorem 1: the opposite sides of a parallelogram are equal		
16	231	activity			
16	231	instructional narrative			
16	232	activity			
16	232	instructional narrative			
16	232	activity	investigation 16B: Linking proofs about parallelograms		
16	233	orienting narrative			
16	233	exercise set	exercise 16.1		
16	233	learning reflection	Am I on track?		
16	233	instructional narrative			
16	233	activity	investigation 16C: Theorems and converses		
16	233	instructional narrative			
16	234	activity	investigation 16C:		

			Theorems and converses ctd		
16	234	learning reflection	Am I on track?		
16	234	instructional narrative			
16	234	activity	Investigation 16D: Definitions		
16	235	instructional narrative			
16	235	activity	Investigation 16E: Definitions in geometry		
16	236	instructional narrative			
16	236	activity	Investigation 16E: Definitions in geometry ctd		
16	236	instructional narrative			
16	236	activity	Investigation 16E: Definitions in geometry ctd		
16	236	instructional narrative			
16	237	activity	Investigation 16E: Definitions in geometry ctd		
16	237	orienting narrative			
16	237	instructional narrative			
16	237	activity	Investigation 16E:		

			Definitions in geometry ctd		
16	237	instructional narrative			
16	237	activity	Investigation 16E: Definitions in geometry ctd		
16	238	learning reflection	Am I on track?		
16	238	instructional narrative			
16	238	exercise set	exercise 16.2		
16	239	extra information narrative	The bigger picture	3	Euclid's axioms and the development of other geometries
16	240	learning reflection	tracking my progress		
17	241	chapter opener	transformations	1	Photo of use of printed packaging as wallpaper, reference to Greek mathematics
17	242	activity		1	Pictures of patterns provided to help learners talk about various transformations
17	242	orienting narrative			
17	242	activity	Investigation 17A: Translating 2-dimensional shapes		
17	242	instructional narrative			

17	243	activity	Investigation 17A: Translating 2-dimensional shapes ctd		
17	243	instructional narrative			
17	243	activity	Investigation 17A: Translating 2-dimensional shapes ctd		
17	243	instructional narrative			
17	243	activity	Investigation 17A: Translating 2-dimensional shapes ctd		
17	244	exercise set	exercise 17.1		
17	244	learning reflection	Am I on track?		
17	245	orienting narrative			
17	245	activity	investigation 17B: Reflecting 2-dimensional shapes		
17	246	exercise set	exercise 17.2		
17	246	learning reflection	Am I on track?		
17	247	orienting narrative			
17	247	exercise set	exercise 17.2 ctd		
17	247	orienting narrative			
17	248	exercise set	exercise 17.3 (mixed)		
17	249	orienting narrative			

17	249	activity	Investigation 17C: Exploring the properties of rigid transformations		
17	250	learning reflection	tracking my progress		

Appendix C: Tables detailing orientation 2 in blocks of the textbooks

In the category of context column, S = social issues, P = physical world, F = financial, E = everyday

Orientation 2 in CM

ch	start page	type	title	description	modelling (M), interpretation (I) or none (no)	need to work with maths	category of context
3	65	exercise set	exercise 3.4	find height of tower	no	yes	P
4	80	Worked example		Length of river	no	yes	P
4	81	Instructional narrative		length of river	I	no	P
4	83	activity	activity 4.3	centre of classroom	no	yes	E
4	84	exercise set	exercise 4.4	HIV logo in chairs	no	no	S, E
4	85	activity	activity 4.5	match graph to context described	no	yes	P, E
4	87	worked example		rate of readings in wpm	no	yes	E
4	89	exercise set	exercise 4.7	create linear eq from situation	M	yes	E

4	90	activity	activity 4.8	perim and area of rectangles out of string	no	yes	P
4	94	instructional narrative	general discussion	discussion about modelling with linear eq	M	n/a	n/a
4	94	activity	activity 4.9	depreciation with straight line	I	yes	F
4	95	exercise set	exercise 4.11	various modelling with linear eq	M	yes	E
4	97	activity	apply your skills activity 4.13	linear eq as model for winning time - interpret	I	yes	E
6	134	instructional narrative		explains Russian nested dolls	n/a	n/a	E
6	134	activity	activity 6.1	dimensions of the dolls and their ratio	M	yes	E
6	135	extra info narrative	did you know?	about Russian dolls	n/a	n/a	E
6	136	instructional narrative	general discussion	definition of ratio of heights	no	yes	E
6	136	exercise set	exercise 6.2	more Russian dolls - approx	no	yes	E

				area as rect			
6	137	exercise set	exercise 6.3	more Russian dolls - approx area as rect	no	yes	E
6	139	exercise set	exercise 6.5	enlarging a photo	no	yes	E
6	141	activity	activity 6.7	Find and explore scale factors	no	yes	E
6	143	exercise set	exercise 6.8	Russian doll	no	yes	E
6	147	exercise set	exercise 6.9	box in room	no	yes	E
6	148	exercise set	exercise 6.10	cool-drink can	no	yes	E
6	149	exercise set	exercise 6.11 check your skills	Russian dolls	no	yes	E
6	150	activity	apply your skills	potjies and other objs	M	yes	E
9	191	activity	activity 9.1	cards and transformation geom	no	yes	E
9	195	exercise set	exercise 9.3	cards and transformation geom	no	yes	E
9	197	activity	activity 9.4	cards and transformation geom	no	yes	E
9	204	exercise set	exercise 9.7	cards and transformatio	no	yes	E

				n geom			
9	212	activity	activity 9.10 check your skills	gourd patterns	no	yes	S
9	213	activity	activity 9.11 apply your skills	design a pack of cards	no	yes	E
9	213	puzzle	problem solving	calabash	no	yes	S
14	335	Chapter opener	quadrilaterals	Says will apply geometry to model	M	n/a	n/a
14	346	activity	activity 14.5	cutting out rectangles from wood	M	yes	P
14	347	extra information narrative	did you know?	rural house design - making a rectangle	M	yes	P
14	350	extra information narrative	did you know?	network of roads	M	yes	P
14	351	activity	activity 14.8	car jack	M, I	yes	P
14	352	extra information narrative	did you know?	steam engines and linkages	no	no	P

Table 39: Orientation 2 in CM

Orientation 2 in OTM

ch	start page	type	title	description	modelling (M), interpretation (I) or none (no)	need to work with maths	category of context
14	199	activity	Investigation 14A: A packaging problem	pizza box – figure out dimension to hold pizza and make net	no	yes	P
14	199	instructional narrative		calculate surface area of pizza box	no	yes	P
14	200	activity	Investigation 14A: A packaging problem ctd	surface area	no	yes	P
14	201	extra information narrative	The bigger picture	packaging	no	no	P
14	206	project	Conic sections	mentions conic sections exist in world	no	no	P
15	209	chapter opener	Using inductive reasoning to study shape	egyptians and babylonians used maths to solve everyday probs	no	no	P

15	216	extra information narrative	The bigger picture	cartesian coord used in maps, classroom as illustration for 3-D space	no	no	P
15	222	extra information narrative	The bigger picture	vectors for velocity	no	no	P
15	224	project	architecture in africa	archi in africa	no	yes	P, S
17	242	activity		patterns used to discuss transformation words	no	yes	S

Table 40: Orientation 2 in OTM

Appendix D: Geometric paradigms in transformation geometry chapters

Geometry paradigms in transformation geometry chapters in CM

chapter	page	block type	Title	paradigm
9	189	chapter opener	Transformation geometry	n/a
9	190	instructional narrative	general discussion	GIp, GIIp
9	191	Activity	activity 9.1	GIp
9	193	instructional narrative	general discussion	GIp, GIc, GIIc
9	194	exercise set	exercise 9.2	GIc
9	195	instructional narrative	general discussion	GIp, GIc
9	195	exercise set	exercise 9.3	GIc
9	197	instructional narrative	general discussion	GIIc
9	197	Activity	activity 9.4	GIc
9	199	exercise set	exercise 9.5	GIc, GIIp, GIp, GIIc
9	202	instructional narrative	general discussion	GIIp, GIIc
9	203	exercise set	exercise 9.6 more reflections on points	GIc or GIIc
9	204	instructional narrative	general discussion	GIc, GIp
9	204	exercise set	exercise 9.7	GIp, GIc,

				GIIc
9	206	exercise set	exercise 9.8	GIc
9	207	instructional narrative	general discussion	GIIp, GIIc
9	207	worked example		GIIc and GIIp
9	209	Activity	activity 9.9	GIc -> GII, GIc
9	212	Activity	activity 9.10 check your skills	GIp
9	213	Activity	activity 9.11 apply your skills	GIp
9	213	Puzzle	problem solving	GIp

Table 41: Geometric paradigms in transformation geometry chapters in CM

Geometric paradigms in transformation geometry chapters in OTM

chapter	page	block type	Title	paradigm
17	241	chapter opener	Transformations	n/a
17	242	activity		GIp
17	242	orienting narrative		n/a
17	242	activity	Investigation 17A: Translating 2- dimensional shapes	GIc
17	242	instructional narrative		GIc
17	243	activity	Investigation 17A: Translating 2- dimensional shapes ctd	GIc

17	243	instructional narrative		n/a
17	243	activity	Investigation 17A: Translating 2- dimensional shapes ctd	GIIc
17	243	instructional narrative		GIIc
17	243	activity	Investigaton 17A: Translating 2-dimensional shapes ctd	GIIc
17	244	exercise set	exercise 17.1	GIc and GIIc
17	244	learning reflection	Am I on track?	n/a
17	245	orienting narrative		n/a
17	245	activity	investigation 17B: Reflecting 2-dimensional shapes	GIc to GIIc
17	246	exercise set	exercise 17.2	GIc or GIIc
17	246	learning reflection	Am I on track?	n/a
17	247	orienting narrative		n/a
17	247	exercise set	exercise 17.2 ctd	GIIc
17	247	orienting narrative		n/a
17	248	exercise set	exercise 17.3 (mixed)	GIIc or GIc
17	249	orienting narrative		n/a
17	249	activity	Investigation 17C: Exploring the properties of rigid transformations	GIc, GIIc, GIIp
17	250	learning reflection	tracking my progress	n/a

Table 42: Geometric paradigms in transformation geometry chapters of OTM

Appendix E: Geometric paradigms in the chapters on properties of shapes

Geometric paradigms in the chapters on properties of shapes in CM

ch	page	type	title	apparent dominant paradigm	discussion about elements of GI or GII
3	57	chapter opener	Basic Geometry	n/a	
3	58	summary narrative	Intersecting and parallel lines	GII	
3	59	exercise set	exercise 3.1	GII	
3	60	summary narrative	polygons	GII	
3	61	exercise set	exercise 3.2	GI, GII	
3	63	instructional narrative	counter-examples	GII	Counterexamples
3	63	exercise set	exercise 3.3	GII	
3	63	orienting narrative		n/a	
3	63	summary narrative	triangles	GII	
3	65	exercise set	exercise 3.4	GII	
3	68	instructional narrative	definitions for the isosceles triangle	GII	Definition
3	68	worked example		GII	

3	69	activity	activity 3.5	GII	
3	70	instructional narrative		GII	
3	71	activity	activity 3.6	GI -> GII	
3	72	instructional narrative	midpoint theorem and converse	GII	
3	72	exercise set	exercise 3.7	GII	
3	73	exercise set	exercise 3.8 Check your skills	GII	
3	75	activity	activity 3.9 Apply your skills, Exploration of a fallacy	GII	
3	76	puzzle	problem solving	GI or GII	
3	76	project	project	n/a	
14	335	chapter opener	Quadrilaterals	n/a	
14	336	orienting narrative		n/a	
14	336	activity	Activity 14.1	GI	
14	338	orienting narrative		n/a	
14	338	activity	activity 14.2	GI	
14	339	orienting narrative		n/a	
14	339	worked example		GII	
14	340	instructional narrative	general discussion	GII	Proof
14	340	exercise set	exercise 14.3	GII	
14	342	instructional	general	GII	

		narrative	discussion		
14	342	worked example		GI	
14	344	exercise set	exercise 14.4	GI	
14	346	extra information narrative	did you know?	GI	
14	346	activity	activity 14.5	GI	
14	347	instructional narrative	general discussion	GII	Proof
14	347	extra information narrative	did you know?	GI	
14	348	activity	activity 14.6	GII	
14	349	instructional narrative	general discussion	GII	Proof
14	349	exercise set	exercise 14.7	GII	
14	350	extra information narrative	did you know?	GI	
14	351	orienting narrative		n/a	
14	351	activity	activity 14.8	GI	
14	352	extra information narrative	did you know?	n/a	
14	352	instructional narrative	general discussion	GII	axiomatic systems
14	353	orienting narrative		n/a	
14	353	activity	activity 14.9	GII	

14	354	instructional narrative	general discussion	GII	classification systems
14	355	orienting narrative		n/a	
14	355	activity	activity 14.10	GII	axiomatic systems, definitions
14	356	instructional narrative	general discussion	GII	axiomatic systems, definitions
14	357	activity	activity 14.11	GII	axiomatic systems, definitions
14	357	instructional narrative	general discussion	GII	axiomatic systems, definitions
14	358	exercise set	exercise 14.12	GII	
14	359	instructional narrative	general discussion	GII	
14	359	exercise set	exercise 14.13	GII	
14	360	exercise set	exercise 14.14	GII	
14	363	exercise set	exercise 14.15 check your skills	GI, GII	
14	365	activity	activity 14.16 apply your skills	GII	
14	365	puzzle	problem solving	GII	

Table 43: Geometric paradigms in chapters on properties of shapes in CM

Geometric paradigms in the chapters on properties of shapes in OTM

ch	page	block type	title	apparent dominant paradigm	information about GI/GII
15	209	chapter opener	Using inductive reasoning to study shape	n/a	inductive reasoning
15	210	Activity	Investigation 15A: Different ways to study quadrilaterals	GI, GII	arguments evaluated
15	211	instructional narrative		n/a	
15	211	Activity	Investigation 15A: Different ways to study quadrilaterals ctd	GI, GII	
15	211	learning reflection	Am I on track?	n/a	
15	212	orienting narrative		n/a	visualisation
15	212	Activity	investigation 15B: Using reflections to generate quadrilaterals	GI	
15	212	learning reflection	Am I on track?	n/a	
15	212	exercise set	Exercise 15.1	GI	arguments evaluated
15	213	orienting narrative	The bigger picture	n/a	inductive reasoning, deductive

					reasoning
15	213	Activity	Investigation 15C: Using rotations to generate quadrilaterals	GI	investigation
15	214	Activity	Investigation 15D: Using co-ordinates and gradient to investigate 2-dimensional shapes	GI	
15	214	instructional narrative	Investigation 15E: Using coordinates and length to investigate 2-dimensional shapes	GI	
15	215	Activity	Investigation 15E: Using coordinates and length to investigate 2-dimensional shapes	GI-> GIIc	
15	216	extra information narrative	The bigger picture	n/a	
15	216	exercise set	exercise 15.2	GI, GII	
15	218	learning reflection	Am I on track?	n/a	
15	218	orienting narrative		GI	
15	218	instructional narrative		GIIc	

15	218	Activity	Investigation 15F: Investigating midpoint properties	GI	
15	218	orienting narrative		n/a	
15	218	learning reflection	Am I on track?	n/a	
15	219	Activity	Investigation 15F: Investigating midpoint properties ctd	GI	
15	219	Activity	Investigation 15G: Using coordinates to explore quadrilaterals	GI	
15	219	Instructional narrative		GI	Investigation
15	220	orienting narrative		n/a	
15	220	Activity	Investigation 15G: Using coordinates to explore quadrilaterals ctd	GI	
15	220	extra information narrative		n/a	construction
15	220	orienting narrative		n/a	
15	220	Activity	investigation 15H: Using geometric	GI	

			constructions to explore quadrilaterals		
15	222	extra information narrative	The bigger picture	GI	
15	222	exercise set	exercise 15.3 (mixed)	GI	
15	223	learning reflection	tracking my progress	n/a	
15	224	Project	architecture in Africa	GI	
16	226	chapter opener	using deductive reasoning to study shape	n/a	inductive reasoning, deductive reasoning
16	227	instructional narrative	Deductive versus inductive reasoning	GI, GII	inductive reasoning, deductive reasoning
16	229	learning reflection	The bigger picture	GII	deductive reasoning discussed, axioms, theorems
16	229	orienting narrative		n/a	axiomatic system
16	230	instructional narrative		GII	axiomatic system, proof
16	230	Activity	investigation 16A: Developing a logical	GI	

			deductive system		
16	230	instructional narrative		GII	
16	230	Activity	investigation 16A: Developing a logical deductive system ctd	GI	
16	230	orienting narrative		n/a	
16	230	instructional narrative	theorem 1: the opposite sides of a parallelogram are equal	GII	
16	230	Activity	theorem 1: the opposite sides of a parallelogram are equal	GII	
16	231	Activity		GII	
16	231	instructional narrative		GII	axiomatic system
16	232	Activity		GII	
16	232	instructional narrative		GII	axiomatic system
16	232	Activity	investigation 16B: Linking proofs about parallelograms	GII	axiomatic system
16	233	orienting narrative		n/a	
16	233	exercise set	exercise 16.1	GI, GII	
16	233	learning	Am I on track?	n/a	

		reflection			
16	233	instructional narrative		GII	Theorems
16	233	Activity	investigation 16C: Theorems and converses	GII	
16	233	instructional narrative		GII	counterexample
16	234	Activity	investigation 16C: Theorems and converses ctd	GI, GII	
16	234	learning reflection	Am I on track?	n/a	
16	234	instructional narrative		?	Definitions
16	234	Activity	Investigation 16D: Definitions	?	Definitions
16	235	instructional narrative		GII	Definitions
16	235	Activity	Investigation 16E: Definitions in geometry	GI	
16	236	instructional narrative	Investigation 16E: Definitions in geometry ctd	GII	definitions
16	236	Activity	Investigation 16E: Definitions in geometry ctd	GII	
16	236	instructional narrative	Investigation 16E: Definitions in geometry ctd	GII	definitions

16	236	Activity	Investigation 16E: Definitions in geometry ctd	GI	
16	236	instructional narrative	Investigation 16E: Definitions in geometry ctd	GII	definitions
16	237	Activity	Investigation 16E: Definitions in geometry ctd	GI	
16	237	orienting narrative	Investigation 16E: Definitions in geometry ctd	n/a	
16	237	instructional narrative	Investigation 16E: Definitions in geometry ctd	GII	axiomatic system
16	237	Activity	Investigation 16E: Definitions in geometry ctd	GII	
16	237	instructional narrative	Investigation 16E: Definitions in geometry ctd	GII	
16	237	Activity	Investigation 16E: Definitions in geometry ctd	GI, GII	
16	238	learning reflection	Am I on track?		
16	238	instructional narrative		GII	definitions
16	238	exercise set	exercise 16.2	GI and GII	
16	239	extra	The bigger picture	GII	axioms,

		information narrative			axiomatic systems
16	240	learning reflection	tracking my progress	n/a	

Table 44: Geometric paradigms in chapters on properties of shapes OTM