IRREDUCIBLE OPERATORS ON RIESZ SPACES

Nicholas Charles Christopher Davies

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ABSTRACT

The purpose of this dissertation is to discuss some of the more important results which have come to light following the proof by de Pagter (1986) that every positive compact (ideal-) irreducible operator on a Banach lattice of dimension greater than one has a strictly positive spectral radius.

An extension of this result to positive σ-order continuous compact band irreducible operators is discussed, and consequences thereof proved.

The main results of this thesis include de Pagter's proof of the result mentioned in the first paragraph, and a proof of the Ando-Krieger Theorem due to Grobler which does not make use of any representation theory. The final result is a generalisation of the two well known theorems originally proved by Jentzsch and Perron in the one case, and by Frobenius in the other. Again both these generalisations are due to Grobler and further, the proofs are free of any representation.

DECLARATION

I declare that this dissertation is my own unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted for any other degree or examination in any other university.

NICHOLAS CHARLES CHRISTOPHER DAVIES

June 1987

TO MY PARENTS

PREFACE

A problem which was unsolved for many years, was whether every bounded linear operator on a Banach space had a non-trivial closed T-invariant subspace. In 1954 the two mathematicians N.Aronszajn and K.Smith proved that the above was true for compact operators. In 1966 A.Bernstein and A.Robinson extended this result to operators T having the property that $\rho(T)$ was compact, for some non-zero polynomial $\rho(x)$, and showed that such operators did have non-trivial closed T-invariant subspaces. It wasn't until 1973 that V.V. Lomonosov proved a generalisation of the Bernstein-Robinson result. He proved that if K is a bounded operator on a Banach space (where K is not a scalar multiple of the identity operator) such that there exists at least one non-zero operator commuting with K, then there exists a non-trivial closed subspace invariant under all operators which commute with K, and in particular for K itself.

It is the proof of Lomonosov's theorem which is greatly responsible for our main result, a theorem by de Pagter (theorem 2.11) which ensures that each positive non-zero irreducible compact operator on a Banach lattice has strictly positive spectral radius. Although de Pagter proved this result in 1986, it had been a long standing conjecture of H.H.Schaefer. The affirmation of this conjecture of Schaefer's saw a number of new results, closely related to this newly proved fact, emerge.

It is the aim of this dissertation to highlight and discuss these results and any interesting consequences that they may have.

Most of the results of this dissertation involve spectral theory to some degree. Consequently it is necessary to include in chapter one a fairly lengthy section devoted to the definitions and results of spectral theory. As is the case throughout the whole of chapter one, references for results are given wherever possible. Also introduced in this first chapter are the fundamental results of the complexification of real Riesz spaces, f-algebras and orthomorphisms, as well as general notation.

Chapter two sees the introduction of ideal irreducible operators. Most of the prerequisites for the proof of de Pagter's theorem, which is also proved in this chapter, are included. The idea of ideal irreducible operators is extended in chapter three to band irreducible operators. Once again all definitions and results required to prove Grobler's generalisation of de Pagter's theorem are introduced. A proof, again by Grobler, of the Ando-Krieger Theorem is also included in this chapter.

The final chapter, chapter four, deals with the generalisations by Grobler of Frobenius' theorem and the theorem of Jentzch and Perron. This chapter in particular relies heavily on the results of the spectral theory results given in chapter one.

Without the encouragement, enthusiasm, guidance and knowledge offered me by my supervisor, this dissertation would never have materialised. It is fitting then, that I extend my sincere thanks, in the first instance, to my supervisor and friend Dr. Conrad Labuschagne, for all the support he gave me and all that he taught me during our many lectures and discussions. I hope that this work is of a sufficiently high standard to pay tribute to his dedication to his students in general, and myself in particular.

My appreciation is also due to the University of the Witwatersrand and to the South African Council for Scientific and Industrial Re-

search, both of whom have assisted me financially throughout my post graduate studies.

I would also like to express my appreciation to Professor J.J. Grobler of the University of Potchefstroom, for making available to me his paper on the theorems of Frobenius and Jentzsch-Perron, [7], which, at the time of printing this dissertation, has yet to be published.

A word of thanks to Dr. Gunther Heimbeck, now of the Windhoek Academy, South West Africa, for the interest he took in my personal situation and progress, and the sincere advice he so readily offered me while he was at this university.

Finally, I extend my loving thanks to my parents for giving me the opportunity to pursue and reach this goal.

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CHAPTER 1

PRELIMINARY RESULTS AND NOTATION

It shall be assumed that the reader of this dissertation is familiar with the theory of Riesz spaces and has a basic knowledge thereof. For any background information the reader is referred to the standard works in this field, Luxemburg and Zaanen's 'Riesz Spaces I', [9], and Zaanen's 'Riesz Spaces II', [14].

§1 General Notation

Let E be a real Riesz space. The *positive cone of E* will, as usual, be denoted by E^+. Assume F is also a real Riesz space. A linear operator T from E into F is said to be *positive* if T maps the positive cone of E into the positive cone of F. Equivalently, T is positive if $Tu \geq 0$ in F for all $u \geq 0$ in E. If T is positive we write $T \geq 0$ and if $T \geq 0$ but $T \neq 0$, i.e. there exists some $u \in E$ such that $Tu \neq 0$, then we write $T > 0$. We say T is *strictly positive* if for every $u > 0$ in E, it follows that $Tu > 0$ in F, i.e. $Tu \geq 0$ in F but $Tu \neq 0$.

The vector space consisting of all linear operators $T:E \to F$ is denoted by $L(E,F)$. It is well known that if we define an ordering $\leq$ on $L(E,F)$ by $T \leq S$ if and only if $S-T \geq 0$, for T and S in $L(E,F)$, then $L(E,F)$ is a partially ordered vector space. A linear operator

T from E into F is said to be *order bounded* if for every bounded set U in E, the set $\{ |Tu| \,/\, u \in U \}$ is bounded in F. The set $L_b(E,F)$ of all order bounded linear operators from E into F is clearly a vector subspace of $L(E,F)$. In the particular case that $F = \mathbb{R}$ we shall denote the space $L(E,F)$ by E^*, and the members of E^* are termed *linear functionals* on E. If $E = F$ then $L(E,F)$ and $L_b(E,F)$ are replaced by $L(E)$ and $L_b(E)$ respectively. For any $T \in L(E,F)$, T' will denote the adjoint of T. If E is norm complete, we shall denote the Banach algebra of all norm bounded, and hence continuous linear operators on E by $\mathcal{L}(E)$. We shall denote by E' the subspace of E^* consisting of all norm bounded (i.e., continuous) linear functionals. If $T \in \mathcal{L}(E)$, then $T' \in \mathcal{L}(E')$.

From this point onwards we shall understand the word 'operator' to mean 'linear operator'.

If U is an arbitrary subset of the Riesz space E, the ideal generated in E by U will be written as $I(U)$, and the band generated in E by U will be denoted by $B(U)$ or $\{U\}$. In the case that $U = \{z\}$ is a singleton set, we may use the notation I_z instead of $I(\{z\})$. Note that

$$I_z = \{ g \in E \,/\, |g| \le n|z| \text{ for some } n \in \mathbb{N} \}$$

and that if $0 \le g \in B(U)$, then

$$g = \sup\{ \inf(g,nz) \,/\, n \in \mathbb{N} \}$$

([9], theorem 20.2).

An operator $T \in L(E,F)$ is said to be *σ-order continuous* if whenever $u_n \downarrow 0$ in E, then $Tu_n \downarrow 0$ in F. We say that T is *order continuous* if whenever $x_\alpha \downarrow 0$ in E, then $Tx_\alpha \downarrow 0$ in F, where $(x_\alpha)_\alpha$ is a net in E. The space of all σ-order continuous operators from E into F will be denoted by $L_c(E,F)$, while that of all order continuous operators from E into F will be denoted by $L_n(E,F)$. We clearly have that $L_n(E,F) \subseteq L_c(E,F) \subseteq L_b(E,F)$. For the case that $F = \mathbb{R}$, E'_n and E'_c will denote the spaces of the order continuous and σ-order continuous functionals on E respectively. Furthermore if $E = F$ we shall replace $L_c(E,F)$ and $L_n(E,F)$ by $L_c(E)$ and $L_n(E)$ respectively.

§2 Complexification of a real Riesz space

Since the main results of this work use spectral theory and since complex spaces play an important role in spectral theory, it is highly desirable that we consider a method of obtaining some type of complex Riesz space from a real Riesz space.

Given the real Riesz space L we make the Cartesian product $L \times L = \{ (f,g) \ / \ f \in E, \ g \in L \}$ into a complex vector space in the usual manner by defining that

$$(f_1,g_1) + (f_2,g_2) = (f_1+f_2,g_1+g_2),$$
$$(\alpha+i\beta)(f,g) = (\alpha f-\beta g,\beta f+\alpha g), \text{ for } \alpha,\beta \in \mathbb{R}.$$

The complex vector space so defined is denoted by $L+iL$ and is called the complexification of L. If (f,g) is represented by $f+ig$, then we define $\mathrm{Re}(f+ig) = f$, where $f,g \in L$, and if $E = L+iL$ then $\mathrm{Re}(E) = L$. By identifying $f \in L$ with the ordered pair $(f,0) \in L+iL$, the space L is embedded in the complex vector space $L+iL$ as a real linear subspace.

The above defined complexification is only of interest if for each element $h = f+ig \in L+iL$, it is possible to define the absolute value of h, $|h|$, such that this absolute value on $L+iL$ coincides with the original absolute value on L.

Theorem 1.1 ([14], theorem 91.1)
If L is an Archimedean and uniformly complete real Riesz space then

$$\sup\{\ f.\cos\theta + g.\sin\theta\ /\ 0 \le \theta \le 2\pi\ \}$$

exists in L for all f and g in L.

□

In view of this fact we make the following definition:

Definition 1.2
If $E = L+iL$ is the complexification of the Archimedean uniformly complete real Riesz space L, we define an absolute value on E by

$$|h| := \sup\{\ f.\cos\theta + g.\sin\theta\ /\ 0 \le \theta \le 2\pi\ \}$$

where $h = f+ig$ and $f,g \in L$.

Clearly $|h| \in L^+$. It is easily seen that the two absolute values agree on E.

Having introduced an absolute value on the complex Riesz space E, we say that the elements $f,g \in E$ are *disjoint*, written $f \perp g$, if $\inf(|f|,|g|) = 0$.

The following theorem collects together the main characteristics of the absolute value.

Theorem 1.3
For L an Archimedean uniformly complete (real) Riesz space, and with the absolute value as defined in definition 1.2 and assuming throughout that $f,g \in L$, we have

i) $|f| \le |h|$, $|g| \le |h|$ and $|h| \le |f|+|g|$, where $h = f+ig$.

ii) $|h| = 0$ if and only if $h = 0$,
$|\lambda h| = |\lambda||h|$,
$|h_1+h_2| \le |h_1|+|h_2|$ and

$||h_1|-|h_2|| \leq |h_1-h_2|$

where $h, h_1, h_2 \in L+iL$ and $\lambda \in \mathbb{C}$.

iii) (Dominated Decomposition Property)

If $h = f+ig$ and $f_1, f_2 \in L^+$ satisfy $|h| \leq f_1+f_2$, then there exists $h_1, h_2 \in L+iL$ such that $h = h_1+h_2$ with $|h_1| \leq f_1$ and $|h_2| \leq f_2$.

iv) If h_1 and h_2 are elements of L+iL satisfying $|h_1| \perp |h_2|$ then

$$\begin{aligned}|h_1-h_2| &= |h_1+h_2| \\ &= |h_1|+|h_2| \\ &= ||h_1|-|h_2|| \\ &= \sup(\ |h_1|, |h_2|\ ).\end{aligned}$$

v) $|f+ig| = |f|+|g|$.

vi) If $h_1 = f_1+ig_1$ and $h_2 = f_2+ig_2$ with $|f_1| \leq |f_2|$ and $|g_1| \leq |g_2|$, then $|h_1| \leq |h_2|$.

Proof

i),ii) [14], theorem 91.2.

iii) [14], theorem 91.3.

iv)-vi) [14], theorem 91.4.

□

For the remainder of this section on complexification we shall assume that L is Archimedean and uniformly complete.

Definition 1.4

i) The subset A of L+iL is an *ideal* in L+iL if A is a complex linear subspace of L+iL and if $|h| \leq |f|$ implies $h \in A$ whenever $f \in A$ and $h \in L+iL$.

ii) The set of all real elements in the ideal A will be denoted by A_r.

Theorem 1.5 ([14], theorem 91.6)
If A is an ideal in L+iL then, A_r is an ideal in L and $A = A_r+iA_r$. Conversely, if B is an ideal in L, then B+iB is an ideal in L+iL and $(B+iB)_r = B$.

□

Definition 1.6

i) The ideal A in L+iL is called a *band* in L+iL if the real part A_r is a band in L.

ii) For a non empty subset D of L+iL, the *disjoint complement* D^d, of D, is the band

$$D^d = \{ h \in L+iL \; / \; |h| \perp |f| \text{ for all } f \in D \}$$

in L+iL.

The following proposition is easily proved using the above definitions and results.

Proposition 1.7
Let L be a Riesz space and let E = L+iL. If $U \subseteq L$ is an arbitrary subset of L then the ideal generated by U in L is exactly the real part of the ideal generated in E by U, i.e. if $\mathcal{I}(U)$ denotes the ideal generated in E by U and I(U) the ideal in L generated by U then $\mathcal{I}(U) = I(U)+iI(U)$.
A similar result holds for bands.

□

Turning now to linear operators on complex Riesz spaces, if T is a linear operator from V into W where V and W are both real vector spaces then T can be extended uniquely to a linear operator from V+iV to W+iW by defining

$$T(f+ig) := Tf+iTg \quad \text{for all } f,g \in L.$$

The set of all linear operators $L(V,W)$ from V into W, can thus be viewed as a real linear subspace of the space of all complex linear operators $L(V+iV,W+iW)$ from $V+iV$ into $W+iW$. The elements of the space $L(V,W)$ have the characteristic that $T(V) \subseteq W$ and for this reason they are called *real linear operators* from $V+iV$ into $W+iW$.

If $T \in L(V+iV,W+iW)$ define the real linear operators T_1 and T_2 from $V+iV$ into $W+iW$ as follows; let $x \in E$, then $Tx = y+iz \in L+iL$. Define

$$T_1x := y, \text{ and } T_2x := z.$$

It then follows that for any $z = x+iy \in V+iV$

$$\begin{aligned} Tz = T(x+iy) &= Tx + iTy \\ &= T_1x+iT_2x + i(T_1y+T_2y) \\ &= T_1(x+iy) + i(T_2(x+iy)) \\ &= T_1z + iT_2z \end{aligned}$$

Hence $T = T_1+T_2$ From this argument it is clear that $T:V+iV \to W+iW$ is real if and only if $T = T_1$ and $T_2 = 0$.

Definition 1.8

If T is a linear operator from the complex Riesz space $V+iV$ into the complex Riesz space $W+iW$ then T is *positive*, written $T \geq 0$, if $Tu \geq 0$ for all $0 \leq u \in V+iV$.

It is easily seen that T is positive if and only if T is real and $T_1:E \to F$ is positive in the sense of real Riesz spaces.

Let E and F be real Riesz spaces such that E is Archimedean and uniformly complete and F is Dedekind complete (note that Dedekind σ-completeness implies uniform completeness, so that both E and F admit an absolute value). In view of the fact that the operator T from $E+iE$ into $F+iF$ can be written as $T = T_1+T_2$, it follows that $L_b(E+iE,F+iF)$ is the complexification of $L_b(E,F)$.

Theorem 1.9

Let E be an Archimedean uniformly complete Riesz space and let F be a Dedekind complete Riesz space.

i) If $T \in L_b(E,F)$ is real and $h \in E+iE$ then

$$|Th| \leq |T|(|h|)$$

where $|T| = \sup\{ \cos\theta.T_1+\sin\theta.T_2 \ / \ 0 \leq \theta \leq 2\pi \}$.

ii) If $T \in L_b(E+iE,F+iF)$ and $f \in E$ then

$$|Tf| \leq |T|(|f|).$$

iii) If $T \in L_b(E+iE,F+iF)$ then for $u \in E^+$

$$|T|(u) = \sup\{ |T(f+ig)| \ / \ |f+ig| \leq u \}$$

In particular, $|Th| \leq |T|(|h|)$ for all $h \in L+iE$.

Proof

i),ii) [14], lemma 92.5.

iii) [14], lemma 92.6.

□

For further results on complex operators the reader is referred to [14], §92.

In the case that E is a complex Riesz space many of the results that hold in the Riesz space Re(E) can be extended to the whole of E.

Throughout this work we shall, in the main, be working in (real and complex) Banach lattices. In order to consider the complexification of a real Banach lattice it is necessary for us to know that every real Banach lattice is Archimedean and uniformly complete. Since a normed Riesz space is immediately Archimedean, a Banach lattice in particular is Archimedean. It can also be proved that every Banach lattice is uniformly complete ([7], page 2). Such a proof is omitted here.

§3 Spectral Theory

Throughout this section E denotes a complex Banach lattice and we assume that $T \in \mathcal{L}(E)$ unless otherwise stated.

Definition 1.10

i) The *resolvent set* $\rho(T)$ is defined to be the set of all complex numbers λ such that $\lambda I-T$ is a bijection on E. (Note that $(\lambda I-T)^{-1}$ exists for each $\lambda \in \rho(T)$.)

ii) For each $\lambda \in \rho(T)$ *the resolvent of T at the point* λ is defined to be the operator $R(\lambda,T) := (\lambda I-T)^{-1}$.

iii) The set theoretic complement of the set $\rho(T)$ in the complex plane is called the *spectrum of T* and is denoted by $\sigma(T)$.

Definition 1.11

i) If there exists a non zero element $x \in E$ and a complex number λ satisfying $Tx = \lambda x$, then x is called an *eigenvector of T* and λ is called an *eigenvalue of T*.

ii) The set of all eigenvalues of T is called the *point spectrum* of T and is denoted by $\sigma_p(T)$. Clearly $\sigma_p(T) \subseteq \sigma(T)$.

iii) For any $\lambda \in \mathbb{C}$, the null-space of the operator $\lambda I-T$ is called the *eigenspace corresponding to* λ .(Note that if λ is an eigenvalue of T, the eigenspace corresponding to λ consists precisely of the eigenvectors of T pertaining to λ and the point zero.)

iv) If λ is an eigenvalue of T, then the dimension of the eigenspace corresponding to λ is called the *geometric multiplicity of* λ .

Definition 1.12

We define the real number r(T) by

$$r(T) := \sup\{ \, |\lambda| \; / \; \lambda \in \sigma(T) \, \}.$$

We call r(T) *the spectral radius of T*.

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We call r(T) *the spectral radius of T*.

It follows from the above definition that if $\lambda \in \mathbb{C}$, then $|\lambda| > r(T)$ implies $\lambda \in \rho(T)$.

Proposition 1.13 ([3], proposition 1.8)

i) If $T \in \mathcal{L}(E)$, then the limit of $\|T^n\|^{1/n}$ as $n \to \infty$ exists and equals $r(T)$.

ii) $r(T) = \inf\{ \|T^n\|^{1/n} \ / \ n \in \mathbb{N} \}$.

□

Definition 1.14

T is said to be *quasi-nilpotent* if $r(T) = 0$.

(Note that T is quasi-nilpotent if and only if $\sigma(T) = \{0\}$).

Definition 1.15

For $\lambda \in \rho(T)$, the series $\sum_{n=0}^{\infty} T^n/\lambda^{n+1}$ is called the *Neumann Series for* $R(\lambda,T)$.

Proposition 1.16

i) If $|\lambda| > r(T)$, then the Neumann Series for $R(\lambda,T)$ converges in norm to $R(\lambda,T)$ and we write

$$R(\lambda,T) = \sum_{n=0}^{\infty} T^n/\lambda^{n+1}$$

ii) If $|\lambda| > r(T)$, then

$$R(\lambda,T)x = \sum_{n=0}^{\infty} T^n x/\lambda^{n+1} \text{ for all } x \in E$$

(convergence being in norm).

iii) If $\lambda > r(T)$ then $TR(\lambda,T) = R(\lambda,T)T$.

iv) If T is a positive operator and $\lambda > r(T)$, then

$$R(\lambda,T) \geq 0 \text{ and } 0 \leq TR(\lambda,T) \leq \lambda R(\lambda,T).$$

□

Theorem 1.17 (Spectral Mapping Theorem) ([3], Theorem 1.20)
Let $T \in \mathcal{L}(E)$ and let f be a complex valued function which is analytic on some neighbourhood of $\sigma(T)$. Then $f(\sigma(T)) = \sigma(f(T))$.

□

Proposition 1.18 ([3], proposition 1.12)
If T' denotes the adjoint of $T \in \mathcal{L}(E)$ then
i) $\sigma(T') = \sigma(T)$ and
ii) $R(\lambda,T') = R(\lambda,T)'$ for $\lambda \in \rho(T)$,
i.e. $(\lambda I'-T')^{-1} = ((\lambda I-T)^{-1})'$.

□

Proposition 1.19 ([14], lemma 135.1 (ii))
If $0 < T \in \mathcal{L}(E)$ then $r(T) \in \sigma(T)$.

□

We now discuss briefly the case of an isolated pont of the spectrum. Recall that an isolated point of a set is a point which is not a cluster point.

Proposition 1.20 ([3], pages 29 - 31)
Suppose λ_0 is an isolated point of the spectrum $\sigma(T)$ where $T \in \mathcal{L}(E)$.

i) There exist sequences (A_n), (B_n) contained in $\mathcal{L}(E)$ such that
$R(\lambda,T) = \sum_{n=0}^{\infty} (\lambda-\lambda_0)^n A_n + \sum_{n=0}^{\infty} (\lambda-\lambda_0)^{-n} B_n$.
This series representation of $R(\lambda,T)$ is valid for all $\lambda \in \mathbb{C}$ satisfying $0<|\lambda-\lambda_0|<\delta$, where $0 < \delta$ is such that all of $\sigma(T)$ except λ_0 lies on or outside the circle $|\lambda-\lambda_0| = \delta$.

ii) The sequences (A_n) and (B_n) are such that
$B_{n+1} = (T-\lambda_0 I)B_n$, and therefore
$B_{n+1} = (T-\lambda_0 I)^n B_1$ for all $n \in \mathbb{N}$, and
$A_n = (T-\lambda_0 I)A_{n+1}$, and hence

$$(T-\lambda_0 I)A_0 = B_1 - I.$$

□

Definition 1.21

If $T \in \mathcal{L}(E)$ and λ_0 is an isolated point of $\sigma(T)$, then λ_0 is said to be *a pole of $R(\lambda,T)$ of order m* if $B_m \neq 0$ and $B_{m+1} = 0$.

Proposition 1.22 ([3], proposition 1.40)

If $T \in \mathcal{L}(E)$ and λ_0 is an isolated point of $\sigma(T)$, then λ_0 is a pole of $R(\lambda,T)$ of order m if and only if $(\lambda_0 I-T)^m B_1 = 0$ and $(\lambda_0 I-T)^{m-1} B_1 \neq 0$.

□

We now consider the case that T is a Riesz operator.

Definition 1.23

i) For X and Y complex Banach spaces, the linear mapping T from X into Y is said to be compact if for every bounded sequence (f_n) in X, the corresponding sequence (Tf_n) in Y contains a norm convergent sub-sequence.

ii) For $T \in \mathcal{L}(E)$ define the number $\kappa(T)$ by

$$\kappa(T) = \inf\{ \|T-C\| \ / \ C \in \mathcal{L}(E),\ C \text{ is compact } \}.$$

If $(\kappa(T^n))^{1/n} \to 0$ as $n \to \infty$, then T is called a *Riesz operator*.

Theorem 1.24 ([3], theorem 3.14)

Let T be a Riesz operator. Then

i) $\sigma(T)$ is countable and has no cluster point except possibly zero.

ii) $\sigma(T)\backslash\{0\} \subseteq \sigma_p(T)$ and each non zero element of $\sigma(T)$ is a pole of the resolvent of T.

Let λ be a non zero point in $\sigma(T)$. Denote by $m(\lambda)$ the order of the pole at λ.

iii) For all $n \in \mathbb{N}$, $N\{(\lambda I-T)^n\}$ is finite dimensional, (where $N\{S\}$ denotes the null space of the operator S) and $m(\lambda) = \min\{k \in \mathbb{N} \,/\, N\{(\lambda I-T)^{k+1}\}\} = N\{(\lambda I-T)^k\}$.

iv) For all $n \in \mathbb{N}$, $R\{(\lambda I-T)^n\}$ is finite dimensional, (where $R\{S\}$ denotes the range of the operator S) and $m(\lambda) = \min\{k \in \mathbb{N} \,/\, R\{(\lambda I-T)^{k+1}\}\} = R\{(\lambda I-T)^k\}$.

v) The B_n's of proposition 1.20 are operators of finite rank.

□

It is necessary to define the concept of the 'algebraic multiplicity' of an eigenvalue. In most cases this concept is defined in terms of 'the spectral projection', which we shall not make use of in this dissertation. For this reason we make the following equivalent definition.

Definition 1.25

If λ is a non zero point in $\sigma(T)$ and $m(\lambda)$ denotes the order of the pole of the resolvent at λ, then the *algebraic multiplicity of* λ, $d(\lambda)$, is defined as

$$d(\lambda) := \dim(\ N\{(\lambda I-T)^{m(\lambda)}\}\ )$$

For the equivalence of this definition and the one in terms of the spectral projection, the reader is referred to [3], theorem 3.14.

Further information on spectral theory may be found in [3] or on pages 601 to 613 of [14].

§4 f-Algebras

The latter part of §140, and the majority of §142 of [14] deals extensively with f-algebras and their properties. This section then will only briefly outline f-algebras and mention some of their properties.

The real Riesz space E is called a *Riesz algebra* if E has an associative multiplication defined on it satisfying the usual algebra properties, and with the additional condition that if $u \geq 0$ and $v \geq 0$ in E then the product $uv \geq 0$. If in addition to the above condition we also have that $\inf(u,v) = 0$ implies $\inf(uw,v) = 0 = \inf(wu,v)$ for all $0 \leq w \in E$, then E is called an *f-algebra*.

The multiplication on an f-algebra satisfies $|xy| = |x||y|$. Furthermore if E is Archimedean then the multiplication on the Riesz algebra E is commutative. In the case that E is an Archimedean uniformly complete f-algebra, we can extend the multiplication on E to the complexification E+iE as follows: if $x = a+ib$ and $y = c+id$, where a,b,c,d are in E, then $xy := (ac-bd)+i(ad+bc)$. With this multiplication on E+iE we have that if $x,y,z \in E+iE$ such that $\inf(|x|,|y|) = 0$, then $\inf(|zx|,|y|) = 0 = \inf(|xz|,|y|)$. Of course it still holds true that if $u,v \geq 0$ in E+iE then $uv \geq 0$ in E+iE. In [7] it is shown that if in addition to being Archimedean and uniformly complete, the f-algebra E is also semiprime, then for $x = a+ib \in E+iE$, $|x| = (a^2 + b^2)^{\frac{1}{2}}$. Note in particular that an Archimedean f-algebra possesing an order unit is semiprime ([14], theorem 142.5).

§5 Orthomorphisms

A real linear operator U on the real Riesz space E is said to be an *orthomorphism* if it is order bounded and $\inf(|x|,|y|) = 0$ implies $\inf(|Ux|,|y|) = 0$. The set of orthomorphisms on E will be denoted by Orth(E). If E is an Archimedean Riesz space then Orth(E) can be made into an Archimedean f-algebra with multiplicative unit by defining a multiplication on Orth(E) as follows: for $\pi_1,\pi_2 \in$ Orth(E) the product $\pi_1\pi_2$ is defined by

$$\pi_1\pi_2(h) := \pi_1(\pi_2 h) \text{ for all } h \in E.$$

If I denotes the identity operator on E then the ideal generated in Orth(E) by I is called *the centre of E* and is denoted Z(E). It follows that $\pi \in Z(E)$ implies that there exists $0 \leq \lambda \in \mathbb{R}$ such that $|\pi x| \leq \lambda|x|$ for all $x \in E$. If E is a Banach lattice then Orth(E) = Z(E) ([14], corollary 144.3).

It can also be shown that if E is uniformly complete then so too is Orth(E). Hence if E is a uniformly complete Archimedean Riesz space we may consider the complexification Orth(E)+iOrth(E) of Orth(E).

If E is an Archimedean f-algebra and $u \in E$ we define an operator π on E by

$$\pi x := ux \text{ for all } x \in E.$$

It then follows that $\pi \in$ Orth(E) ([14], theorem 141.1). Furthermore, if π is an orthomorphism on E, then there exists an element $u \in E$ such that

$$\pi f = uf \text{ for all } f \in E$$

([14], theorem 141.1).

If E is Archimedean and uniformly complete, then an order bounded linear operator U on E+iE is an element of Orth(E)+iOrth(E) if and only if $\inf(|z|,|w|) = 0$ implies that $\inf(|Uz|,|w|) = 0$ for

$z,w \in E+iE$. We say that an order bounded operator U is an *orthomorphism on E+iE* if $U \in \mathrm{Orth}(E)+i\mathrm{Orth}(E)$.

It follows from the above two paragraphs that if π is an orthomorphism on $E+iE$, where E is a uniformly complete Archimedean Riesz space with multiplicative unit, then there exists some $u \in E+iE$ such that

$$\pi x = ux \text{ for all } x \in E+iE.$$

In fact, the correspondence between the points of $E+iE$ and the orthomorphisms on $E+iE$ can be shown to be a Riesz homomorphism.

CHAPTER 2

IDEAL IRREDUCIBLE OPERATORS

The aim of this chapter is to show, using the proof of B. de Pagter ([10]), that the conjecture by Schaefer, that a positive compact ideal irreducible operator on a Banach lattice (of dimension greater than one) has a strictly positive spectral radius, ([11], §V.6), is true.

Definition 2.1

Let E be a (real or complex) Banach lattice and let $T \in \mathcal{L}(E)$. If U is a subspace of E such that $T(U) \subseteq U$ then we say that U is *T-invariant*.

Definition 2.2

If E is a Banach lattice and $T \in \mathcal{L}(E)$, then T is said to be *ideal-irreducible*, or simply irreducible, if the only closed T-invariant ideals in E are $\{0\}$ and E itself. (Note that $\{0\}$ is closed in E because E, being a normed space, is Hausdorff and every finite set is hence closed. ([4], VII 1.4)).

Proposition 2.3

Let $T \in \mathcal{L}(E)$, where E is a normed complex Riesz space, such that $T > 0$ and T is irreducible, and let $0 < x \in E$. Then there does not exist an $n \in \mathbb{N}$ such that $T^n x = 0$, i.e, $T^n x > 0$ for all $n \in \mathbb{N}$.

Proof

Suppose there does exist an $n \in \mathbb{N}$ such that $T^n x = 0$. Let m be the least such natural number and define $z := T^{m-1}x > 0$. Let I_z be the ideal generated in E by z. Let s be an arbitrary element of I_z, then there exists $\lambda \in \mathbb{R}^+$ such that $|s| \le \lambda z$. It then follows that

$$|Ts| \le T|s| \le \lambda Tz = \lambda T^m x = 0$$

which implies that $Ts = 0$. Since $0 \in I_z$, we conclude that $Ts \in I_z$. It follows that I_z is T invariant and hence the closure $\overline{I_z}$ is T-invariant. Since T is irreducible we must have that $\overline{I_z} = E$, since $z \ne 0$. However in the course of our proof we have shown that $T = 0$ on I_z and hence by continuity, on the closure $\overline{I_z}$. Thus $T = 0$ on E which contradicts the fact that $T > 0$. Thus $T^n x > 0$ for all $n \in \mathbb{N}$.

□

The following proposition highlights a number if interesting and useful facts related to positive ideal irreducible operators.

Proposition 2.4

Let E be a complex Banach lattice and let $0 \le T \in \mathcal{L}(E)$. Then the following statements are equivalent.

i) T is irreducible.

ii) For each $0 \ne x \in E^+$, the closed ideal generated in E by $\{ T^n x \ / \ n \in \mathbb{N} \}$ is exactly E.

iii) For all $0 < x \in E$ and for all $0 < f \in E'$ there exists an $m \in \mathbb{N}$ such that $f(T^m(x)) > 0$.

Proof

i) $\Rightarrow$ ii): Let J be the closed ideal generated in E by $\{ T^n x \ / \ n \in \mathbb{N} \}$. For each $y \in J$ there exists $\alpha_1,\ldots,\alpha_p$, all elements of $\mathbb{R}$, where p is some natural number depending on the particular y, and $n(1),n(2),\ldots,n(p)$, elements of $\mathbb{N}$ such that

$$|y| \le \sum_{k=1}^{p} \alpha_k T^{n(k)} x \ .$$

Then $|Ty| \leq T|y|$ which is an element of J, and hence $Ty \in J$. It follows that J is T-invariant and hence that $J = \{0\}$ or $J = E$. We show $J \neq \{0\}$. Since the zero operator leaves all ideals invariant, it must follow that $T > 0$. By proposition 2.3, $T^n x > 0$ for all $n \in \mathbb{N}$, and thus $J \neq \{0\}$. Hence J must then be precisely E as required.

ii) $\Rightarrow$ i): Let $\{0\} \neq I \subseteq E$ be a closed T-invariant ideal in E. Choose $0 \neq x \in I^+$. For all $n \in \mathbb{N}$ we have that $T^n x \in I$ and therefore $\{ T^n x \,/\, n \in \mathbb{N} \} \subseteq I$, which implies that the closed ideal generated in E by the collection $\{ T^n x \,/\, n \in \mathbb{N} \}$ is contained in I. By assumption however this ideal is exactly E and it follows that $I = E$, from which we infer that T is irreducible.

ii) $\Rightarrow$ iii): Assume that $0 < x_0 \in E$ and $0 < f \in E'$. Let $J := \{ x \in E \,/\, f(|x|) = 0 \}$. Then J is the null ideal of f. It is easily seen that J is closed in E. Since $f > 0$, there exists an x, $0 \leq x \in E$, such that $f(x) > 0$ and hence $J \neq E$. In view of the assumption, we cannot have $\{T^n x_0 \,/\, n \in \mathbb{N} \} \subseteq J$ otherwise $J = E$. Thus there exists an $n \in \mathbb{N}$ such that $T^n x_0 \notin J$, i.e., $f(|T^n x_0|) \neq 0$. Since $T^n x_0 \geq 0$ and $f \geq 0$, it follows that $f(T^n x_0) > 0$.

iii) $\Rightarrow$ ii): Suppose that iii) is true but that ii) fails to hold. Then there exists an x_0, $0 < x_0 \in E$, such that the closed ideal generated by $J := \{ T^n x_0 \,/\, n \in \mathbb{N} \}$ is a proper subset of E. According to the Hahn Banach theorem there exists a $g \in E'$ such that $g \neq 0$ but $g(x) = 0$ for all $x \in J^+$. Let $f := |g|$. Then for $x \in J$ we have

$$\begin{aligned} f(x) &= |g|(x) \\ &= \sup\{ |g(y)| \,/\, 0 \leq |y| \leq x \} \\ &= 0 \qquad \text{(since if } |y| \leq x, \text{ then } y \in J.) \end{aligned}$$

In particular for any $n \in \mathbb{N}$, $f(T^n x_0) = 0$ which contradicts iii).

□

Proposition 2.5

Let E be a complex Banach lattice and let $0 \le T \in \mathcal{L}(E)$ be irreducible. If $0 < x \in E$ and $|\lambda| > r(T)$, then $R(\lambda,T)x > 0$.

Proof

In view of proposition 1.16 we can write $R(\lambda,T)x$ as

$$R(\lambda,T)x = \sum_{k=0}^{\infty} T^k x/\lambda^{k+1}.$$

From proposition 2.3, $T^k x > 0$ for each $k \in \mathbb{N}$, and thus $R(\lambda,T)x > 0$.

□

Definition 2.6

Let E be a Banach lattice (real or complex) and let $x \in E$. We say that x is a *quasi-interior point* of E if the closure $\overline{I_x}$ of the ideal generated by x in E, is precisely E.

Proposition 2.7

Let $0 < T \in \mathcal{L}(E)$ where $E = L+iL$ is a Banach lattice. If T is irreducible, then $R(\lambda,T)x$ is a quasi-interior point of E for each $\lambda > r(T)$ and for each $x > 0$.

Proof

Firstly let $T \in \mathcal{L}(L)$ where L is a real Banach lattice, and let $\lambda > r(T)$ and $x > 0$. If $z \in I_{R(\lambda,T)x}$ then so too is Tz, for if $|z| \le \kappa R(\lambda,T)x$ for some $\kappa \in \mathbb{R}^+$, then

$$|Tz| \le T|z| \le \kappa TR(\lambda,T)x \le (\kappa\lambda)R(\lambda,T)x.$$

Hence $I_{R(\lambda,T)x}$, and consequently the closure $\overline{I_{R(\lambda,T)x}}$, is T-invariant.

Thus $\overline{I_{R(\lambda,T)x}} = \{0\}$ or $\overline{I_{R(\lambda,T)x}} = L$. The fact that $R(\lambda,T)x \ne 0$ (proposition 2.5) then shows that $\overline{I_{R(\lambda,T)x}} = L$, i.e., $R(\lambda,T)$ is a quasi-interior point in L.

Suppose now that $0 < x \in E = L+iL$ and that $0 < T \in \mathcal{L}(E)$. Then $0 < T \in \mathcal{L}(L)$. Also because $0 < x \in E$ it follows from what was said above that $\overline{I_{R(\lambda,T)x}} = L$ (where the ideal generated and the norm closure are taken with respect to L). Thus the norm closure in $E = L+iL$ of the ideal $I_{R(\lambda,T)x}$ generated by $R(\lambda,T)x$ in E is exactly E, i.e.,

$$\overline{I_{R(\lambda,T)x} + iI_{R(\lambda,T)x}} = \overline{I_{R(\lambda,T)x}} + i\overline{I_{R(\lambda,T)x}} = L+iL = E.$$

□

Proposition 2.8

Let E be a complex Banach lattice. Suppose that $T \in \mathcal{L}(E)$ and that Tu is a quasi-interior point of E for all $0 < u \in E$. Then T is irreducible.

Proof

Let J be a non-zero closed T-invariant ideal in E. If $0 < x \in J$ then $Tx \in J$ and hence $\overline{I_{Tx}} \subseteq J$, i.e., $E \subseteq J$. Thus $E = J$ and we conclude that T is irreducible.

□

Lemma 2.9

Let L be a real Banach lattice with quasi-interior point. If $0 \leq u \leq |f|$ in L, then there exists a sequence (π_n) in $Z(L)$ such that $\pi_n f \to u$ in norm and $|\pi_n| \leq I$ for all $n \in \mathbb{N}$ (where I denotes the identity operator on L).

Proof

Suppose $0 \neq w$ is a quasi-interior point of L. We may assume that $w \geq 0$ ([9], page 135). Further, without loss of generality we may assume that $|f| \leq w$, (for if not, consider $w' := \sup(w,|f|)$). So $0 \leq u \leq |f| \leq w$.

It is well known that I_w is an Archimedean f-algebra with multiplicative unit w. Since it is true that $|f|+n^{-1}w \geq n^{-1}w$, for all $n \in \mathbb{N}$, it follows from [14], theorem 146.3, that $(|f|+n^{-1}w)^{-1}$ exists. Noting this, define

$$p_n := uf((|f|+n^{-1}w)^{-1})^2 \text{ for all } n \in \mathbb{N}.$$

Because I_w is an f-algebra, $p_n \in I_w$ for all $n \in \mathbb{N}$. Furthermore,

$$\begin{aligned}
|p_n| &= |uf(|f|+n^{-1}w)^{-2}| \\
&= |u||f||(|f|+n^{-1}w)^{-2}| \\
&\leq |f||f||(|f|+n^{-1}w)^{-2}| \\
&= |f|^2(|f|+n^{-1}w)^{-2} \\
&\leq (|f|+n^{-1}w)^2(|f|+n^{-1}w)^{-2} \\
&= w,
\end{aligned}$$

and using the fact that every Archimedean f-algebra is commutative ([14], theorem 140.10),

$$\begin{aligned}
p_nf &= uf(|f|+n^{-1}w)^{-2}f \\
&= uf^2(|f|+n^{-1}w)^{-2} \\
&\leq u(|f|+n^{-1}w)^2(|f|+n^{-1}w)^{-2} \\
&= uw \\
&= u
\end{aligned}$$

Therefore $p_nf \leq u$ for each $n \in \mathbb{N}$.

Hence
$$\begin{aligned}
0 \leq u-p_nf &= u-uf(|f|+n^{-1}w)^{-2}f \\
&= u-uf^2(|f|+n^{-1}w)^{-2} \\
&= u-u|f|^2(|f|+n^{-1}w)^{-2} \qquad (f^2 = |f|^2) \\
&= u[w-|f|^2(|f|+n^{-1}w)^{-2}] \\
&= u[(|f|+n^{-1}w)^2-|f|^2](|f|+n^{-1}w)^{-2} \\
&= [2n^{-1}u|f|+n^{-2}u](|f|+n^{-1}w)^{-2} \\
&= 2n^{-1}(u|f|+(2n)^{-1}u)(|f|+n^{-1}w)^{-2} \\
&\leq 2n^{-1}(|f|^2+(2n)^{-1}|f|)(|f|+n^{-1}w)^{-2} \\
&= 2n^{-1}[(|f|+(4n)^{-1}w)^2-(4n)^{-2}w][|f|+n^{-1}w]^{-2}.
\end{aligned}$$

But $|f|+(4n)^{-1}w \leq |f|+n^{-1}w$.

Thus
$$\begin{aligned}
0 \leq u-p_nf &\leq 2n^{-1}[(|f|+n^{-1}w)^2-(4n)^{-2}w](|f|+n^{-1}w)^{-2} \\
&\leq 2n^{-1}[(|f|+n^{-1}w)^2+(4n)^{-2}w](|f|+n^{-1}w)^{-2} \\
&= 2n^{-1}[w+4^{-2}(n^{-1}w)^2(|f|+n^{-1}w)^{-2}]
\end{aligned}$$

$$\le 2n^{-1}[w+4^{-2}(|f|+n^{-1}w)^2\ (|f|+n^{-1}w)^{-2}]$$
$$= 2n^{-1}[w+4^{-2}w]$$
$$= (17/8)\ w/n.$$

Hence $u-p_nf \to 0$ w-uniformly in I_w.
We now define the orthomorphism π_n on I_w by

$$\pi_n x := p_n x \text{ for each } x \in I_w.$$

Note that this is an orthomorphism by [14], theorem 141.1. By the same theorem it follows that $|\pi_n|$ then pertains to the element $|p_n|$, and the identity orthomorphism on I_w pertains to w.

Now if $x \ge 0$ in I_w then for any $n \in \mathbb{N}$

$$|\pi_n|x = |p_n|x \le wx = x = Ix,$$

i.e., $|\pi_n| \le I$ for each $n \in \mathbb{N}$. Therefore $\pi_n \in Z(L)$ for each $n \in \mathbb{N}$ ([14], page 659) and hence π_n is norm bounded on I_w, ([14], theorem 144.3), i.e., π_n is continuous on I_w for all $n \in \mathbb{N}$. Using the fact that $\overline{I_w} = L$, we can extend $\pi_n : I_w \to I_w$ by continuity to a unique continuous mapping $\tilde{\pi}_n$ from L into L. (See [12], I.6.4 theorem 2).

Since $|\pi_n| \le I$ for all $n \in \mathbb{N}$, it is easily seen that for each $n \in \mathbb{N}$ $|\tilde{\pi}_n| \le I$, from which it follows that this extension is an orthomorphism on L for each $n \in \mathbb{N}$.

Because $\pi_n f \to u$ w-uniformly on I_w and $f \in I_w$ $(|f| \le w)$, and because π_n is just the restriction of $\tilde{\pi}_n$, we have that $\tilde{\pi}_n f \to u$ w-uniformly, and since relative uniform convergence implies convergence in norm we have that $\tilde{\pi}_n f$ converges in norm to u as $n \to \infty$.

$\square$

Proposition 2.10

If E is a complex Banach lattice such that the dimension of E is greater than one and if $0 < T \in \mathcal{L}(E)$ is a positive compact quasi-nilpotent linear operator, then there exists a non-trivial closed T-invariant ideal in E.

Proof

Suppose $E = L+iL$, then because T is positive, $T \in \mathcal{L}(L)$. Suppose now that the result is true for a real Banach lattice, then there exists $I \subseteq L$ such that I is a non trivial closed T-invariant ideal. It follows that $J := I+iI$ is a closed non-trivial T-invariant ideal in E. Thus it remains to prove the theorem for the case of real Banach lattices.

Assume that L is a real Banach lattice. Since $T > 0$ there exists u, $0 < u \in L$, such that $Tu > 0$. Define $S_n := \sum_{k=0}^{n} 2^{-k} (T^k u)/(\|T\|^k)$, which is an element of L. Then for all $m < n$,

$$\begin{aligned}\|S_n - S_m\| &\leq \sum_{k=m+1}^{n} 2^{-k}\|u\| \\ &\leq \|u\|\sum_{k=m+1}^{\infty} 2^{-k} \\ &\downarrow 0 \text{ as } m \to \infty\end{aligned}$$

Thus (S_n) is a Cauchy sequence in L which is a Banach lattice and hence there exists $w \in L$ such that $S_n \to w$ in norm. However $S_n\uparrow$ and thus $S_n \uparrow w$, i.e., $w = \sum_{k=0}^{\infty} 2^{-k}(T^k u)/(\|T\|^k)$. Note that $w \geq 2^{-1}\, Tu/\|T\| > 0$.

The principal ideal I_w generated by w in E is T-invariant, for if $0 \leq v \in I_w$, then $0 \leq \kappa w$ for some $\kappa \in \mathbb{R}^+$, i.e.,

$$0 \leq v \leq \kappa\sum_{k=0}^{\infty} 2^{-k} (T^k u)/(\|T\|^k)$$

and hence

$$\begin{aligned}0 \leq Tv &\leq \kappa\sum_{k=0}^{\infty} T^{k+1}u/(2^k\|T\|^k) \\ &= 2\kappa\|T\|\sum_{k=0}^{\infty} T^{k+1}u/(2^{k+1}\|T\|^{k+1}) \\ &\leq 2\kappa\|T\|\sum_{k=0}^{\infty} T^k u/(2^k\|T\|^k) \\ &= 2\kappa\|T\|w.\end{aligned}$$

Hence $Tv \in I_w$. It has thus been shown that I_w is T-invariant and consequently so too is the closure $\overline{I_w}$. If $\overline{I_w} \neq L$ then $\overline{I_w}$ is a non-trivial closed T-invariant ideal, and the proof is complete. Alternatively $\overline{I_w} = L$ and from here on we may assume that L has a quasi-interior point.

The following notational definitions are now required;

$$\mathscr{C}^+(T) := \{\ 0 \le R \in L_b(L)\ /\ RT = TR\ \},$$

$$\mathscr{J}^+ := \{\ 0 \le S \in L_b(L)\ /\ 0 \le S \le R \text{ for some } R \text{ in } \mathscr{C}^+(T)\ \},$$

$$\mathscr{J} := \{\ S_1 - S_2\ /\ S_1, S_2 \in \mathscr{J}^+\ \}.$$

Firstly, one can easily verify that $\mathscr{J}$ is a linear subspace of $L_b(L)$ and secondly we claim that if $S \in \mathscr{J}$ and $\pi \in Z(L)$ then $\pi S \in \mathscr{J}$. To see this let $S = S_1 - S_2$, $0 \le S_1, S_2 \in L_b(L)$ and $R_1, R_2 \in \mathscr{C}^+(T)$ such that $0 \le S_i \le R_i$, $i = 1,2$. Then

$$\begin{aligned}\pi S &= (\pi^+ - \pi^-)(S_1 - S_2)\\ &= \pi^+ S_1 - \pi^+ S_2 - \pi^- S_1 + \pi^- S_2\\ &= (\pi^+ S_1 + \pi^- S_2) - (\pi^+ S_2 + \pi^- S_1).\end{aligned}$$

Now $S_1 \ge 0$ and $\pi^+ \ge 0$, therefore $\pi^+ S_1 \ge 0$. Similarly $\pi^- S_2 \ge 0$, and hence $\pi^+ S_1 + \pi^- S_2 \ge 0$. Also

$$\begin{aligned}\pi^+ S_1 + \pi^- S_2 &\le |\pi| S_1 + |\pi| S_2\\ &\le \lambda S_1 + \lambda S_2\\ &\le \lambda(R_1 + R_2) \text{ for some } \lambda \in \mathbb{R}^+.\end{aligned}$$

For this specific λ,

$$\begin{aligned}\lambda(R_1 + R_2)T &= \lambda(R_1 T + R_2 T)\\ &= \lambda(TR_1 + TR_2) \qquad (R_1, R_2 \in \mathscr{C}^+(T))\\ &= \lambda T(R_1 + R_2)\\ &= T(\lambda(R_1 + R_2)).\end{aligned}$$

Thus $\lambda(R_1 + R_2) \in \mathscr{C}^+(T)$.

So $0 \le \pi^+ S_1 + \pi^- S_2 \le \lambda(R_1 + R_2)$ and $\lambda(R_1 + R_2) \in \mathscr{C}^+(T)$, from which it follows that $\pi^+ S_1 + \pi^- S_2 \in \mathscr{J}^+$. Similarly $\pi^+ S_2 + \pi^- S_1 \in \mathscr{J}^+$ and therefore $\pi S \in \mathscr{J}$.

Thirdly it is easily checked, using the fact that $T \ge 0$, that $S \in \mathscr{J}$ implies that $TS \in \mathscr{J}$.

For all $f \in L$, we define $\mathscr{J}[f] := \{\ Sf\ /\ S \in \mathscr{J}\ \}$. It is easy to that for each $f \in L$, $\mathscr{J}[f]$ is a T-invariant subspace of L. Our next claim is that $\overline{\mathcal{J}[f]}$ is an ideal in L. We have already observed that $\mathcal{J}[f]$ is a vector subspace and consequently so too is $\overline{\mathcal{J}[f]}$. Suppose now that $0 \le u \le |Sf|$ in L, where $S \in \mathcal{J}$. By lemma 2.9 there exists $(\pi_n) \subseteq \mathrm{Orth}(L)$ (with $|\pi_n| \le I$ for all $n \in \mathbb{N}$) such that $\pi_n(Sf) \to u$ in norm. Now $\pi_n S \in \mathcal{J}$ for all $n \in \mathbb{N}$ and therefore $\pi_n(Sf) \in \mathcal{J}[f]$ for all $n \in \mathbb{N}$ which implies that $u \in \overline{\mathcal{J}[f]}$.

Now let $0 \le v \in I_{\mathcal{J}[f]}$, (the ideal generated in L by $\mathcal{J}[f]$). Then there exist $f_1,\dots,f_n \in \mathcal{J}[f]$ such that

$$0 \le v \le \sum_{k=1}^{n} \alpha_k |f_k|,$$

where $\alpha_k \in \mathbb{R}^+$ for $k = 1,\dots,n$.
In other words there exist $S_1,\dots S_n \in \mathcal{J}$ such that

$$0 \le v \le \sum_{k=1}^{n} |S_k f|.$$

Now by the Dominated Decomposition Property ([14], theorem 91.3) there exist $v_1,\dots v_n$ in L satisfying $0 \le v_i \le |S_i f|$ for $i = 1\dots,n$, and such that $v = \sum_{k=1}^{n} v_i$.

By what we have proved above, $v_i \in \overline{\mathcal{J}[f]}$ for $i = 1,\dots,n$, and thus $v \in \overline{\mathcal{J}[f]}$. It has thus been shown that $\mathcal{J}[f] \subseteq I_{\mathcal{J}[f]} \subseteq \overline{\mathcal{J}[f]}$, and it must then follow that $\overline{I_{\mathcal{J}[f]}} = \overline{\mathcal{J}[f]}$. However $I_{\mathcal{J}[f]}$ being an ideal implies that its norm closure is an ideal, ([11], II 5.2 corollary 1). Hence $\overline{\mathcal{J}[f]}$ is a closed ideal in L which is T-invariant.

If we can now show that there exists a non-zero $f \in L$ such that $\overline{\mathcal{J}[f]} \ne L$, then we shall be through. To this end let us suppose that $\overline{\mathcal{J}[f]} = L$ for all $f \in L\backslash\{0\}$. Let $0 \le u \in L$ such that $Tu > 0$; let $d := \frac{1}{2}\|Tu\| > 0$, $\delta := \frac{1}{2}\|u\|$, and let $\kappa := \min\{\ \delta, d/\|T\|\ \}$.

Define $\mathcal{A} := \{\ z \in L\ /\ \|z-u\| < \kappa\ \}$. It follows that $\mathcal{A}$ has the following properties;

$0 \notin \overline{\mathcal{A}}$: for otherwise there exists $(x_n) \subseteq \mathcal{A}$ such that $x_n \to 0$ in norm. Thus there exists an $N \in \mathbb{N}$ such that $n \ge N$ implies that $\|x_n\| < \frac{1}{2}\|u\|$. Choose such an n, then

$$\|u\| = \|u-x_n+x_n\|$$

$$
\begin{aligned}
&\leq \|u-x_n\|+\|x_n\| \\
&< \kappa+\|x_n\| \\
&\leq \tfrac{1}{2}\|u\|+\tfrac{1}{2}\|u\| \\
&= \|u\| \qquad \text{which is a contradiction.}
\end{aligned}
$$

$0 \notin \overline{T(\mathscr{A})}$: for otherwise there exists $(y_n) \subseteq T(\mathscr{A})$ such that $y_n \to 0$ in norm, i.e., there exists a sequence $(x_n) \subseteq \mathscr{A}$ such that $Tx_n \to 0$ in norm. Consequently there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies that $\|Tx_n\| < \frac{1}{2}\|Tu\| = d$. Choose such an n, then

$$
\begin{aligned}
\|Tu\| &= \|Tu-Tx_n+Tx_n\| \\
&\leq \|Tu-Tx_n\|+\|Tx_n\| \\
&\leq \|T\|\|u-x_n\|+\|Tx_n\| \\
&\leq \|\text{[illegible]}+\|Tx_n\| \\
&< \|\text{[illegible]}(\text{[illegible]}/\|T\|)+\|Tx_n\| \\
&\leq d+\|Tx_n\| \\
&\leq \tfrac{1}{2}\|Tu\|+\tfrac{1}{2}\|Tu\| \\
&= \|Tu\|, \qquad \text{a contradiction.}
\end{aligned}
$$

$\mathscr{A}$ is norm bounded: for if $z \in \mathscr{A}$ then

$$
\begin{aligned}
\|z\| &= \|z-u+u\| \\
&\leq \|z-u\|+\|u\| \\
&< \kappa+\|u\|.
\end{aligned}
$$

If $f \in \overline{T(\mathscr{A})}$ then $f \neq 0$ and therefore $\overline{\mathscr{J}[f]} = L$ by assumption. Fix $f \in \overline{T(\mathscr{A})}$. Since $u \in \overline{\mathscr{J}[f]} = L$, $V \cap \mathscr{J}[f] \neq \phi$ for all open sets V containing u. In particular, $\mathscr{A} \cap \mathscr{J}[f] \neq \phi$. Thus there exists $S_f \in \mathscr{J}$ such that $S_f f \in \mathscr{A}$. Now S_f is norm bounded and hence continuous, so let $\mathscr{A}_f$ be an open neighbourhood of f such that $S_f(\mathscr{A}_f) \subseteq \mathscr{A}$.

We then have that $\{ \mathscr{A}_f \,/\, f \in \overline{T(\mathscr{A})} \}$ is an open cover of $\overline{T(\mathscr{A})}$ which is a compact subspace of L. Consequently there exist $f(1),\ldots,f(n) \in \overline{T(\mathscr{A})}$ such that $\overline{T(\mathscr{A})} \subseteq \cup\mathscr{A}_{f(i)}$, where the union is taken over $i = 1,2,\ldots,n$.

Let $\mathcal{A}_i := \mathcal{A}_{f(i)}$, and $S_i := S_{f(i)}$, for $i = 1,2,\ldots,n$.

Since $Tu \in T(\mathcal{A}) \subseteq \overline{T(\mathcal{A})}$, there exists $j(1) \in \{1,\ldots,n\}$ satisfying $Tu \in \mathcal{A}_{j(1)}$, and therefore $S_{j(1)}(Tu) \in \mathcal{A}$. This then implies that $T(S_{j(1)}(Tu)) \in T(\mathcal{A}) \subseteq \overline{T(\mathcal{A})}$.

Repeat the above argument with Tu replaced by $TS_{j(1)}(Tu)$. Repeating this argument, we obtain a sequence $\{ j(m) \,/\, m \in \mathbb{N} \}$ contained in $\{1,\ldots,n\}$ such that $g_m := S_{j(m)}TS_{j(m-1)}\ldots TS_{j(1)}Tu \in \mathcal{A}$ for $m = 1,2,\ldots$.

Now each $S_j \in \mathcal{J}[f]$ so that $S_j = S^1{}_j - S^2{}_j$ where $0 \le S^1{}_j \le R^1{}_j$ for some $R^1{}_j \in \mathcal{E}^+(T)$, and $0 \le S^2{}_j \le R^2{}_j$ for some $R^2{}_j \in \mathcal{E}^+(T)$.

Define $C := \max\{ \|R^i_j\| \,/\, j = 1,\ldots,n \text{ and } i = 1,2 \}$.
Then

$$|g_m| = |S_{j(m)}T\ldots S_{j(1)}Tu|$$
$$\le (R^1{}_{j(m)}+R^2{}_{j(m)})T\ldots(R^1{}_{j(1)}+R^2{}_{j(1)})Tu.$$

Since T commutes with all the $R^i_{j(k)}$'s, where $i = 1,2$ and $k = 1,\ldots m$,

$$|g_m| \le (R^1{}_{j(m)}+R^2{}_{j(m)})\ldots(R^1{}_{j(1)}+R^2{}_{j(1)})T^m u.$$

Thus
$$\|g_m\| \le \|R^1{}_{j(m)}+R^2{}_{j(m)}\|\ldots\|R^1{}_{j(1)}+R^2{}_{j(1)}\|\,\|T^m\|\,\|u\|$$
$$\le (\|R^1{}_{j(m)}\|+\|R^2{}_{j(m)}\|)\ldots(\|R^1{}_{j(1)}\|+\|R^2{}_{j(1)}\|)\,\|T^m\|\,\|u\|$$
$$\le (2C)^m\|T^m\|\,\|u\| \text{ for all } m \in \mathbb{N}.$$

This inequality then implies that

$$\|g_m\|^{1/m} \le 2C\|u\|^{1/m}\|T^m\|^{1/m} \text{ for all } m \in \mathbb{N}.$$

In view of the hypothesis, $\|T^m\|^{1/m} \to 0$, and it then follows that $\|g_m\|^{1/m} \to 0$, and immediately we have that $g_m \to 0$ in norm.

Now $g_m \in \mathcal{A}$ for all $m \in \mathbb{N}$, and therefore $0 \in \overline{\mathcal{A}}$, which is a contradiction since we proved earlier that this was not possible. It has thus been proved that there does exist a non-zero $f \in L$ such that $\overline{\mathcal{J}[f]} \ne L$.

□

Theorem 2.11 (de Pagter's Theorem)

If T is a positive compact irreducible operator in $\mathcal{L}(E)$ where E is a complex Banach lattice of dimension greater than one, then $r(T) > 0$.

Proof

Suppose $r(T) = 0$, then T is quasi-nilpotent and hence by proposition 2.10 there exists a non-trivial closed T-invariant ideal in E, contradicting the fact that T is irreducible. Thus $r(T) > 0$.

□

Note that we have to exclude the possibility that the dimension of E is one in theorem 2.11 for in this case the zero operator satisfies the conditions of the hypothesis but $r(T) = 0$, where T is the zero operator. If we wish to include the case that the dimension of E is one then we must insist that T is not the zero operator, and the conclusion of 2.11 is still valid.

We now make use of proposition 2.11 to prove an Ando-Krieger type result for (ideal-) irreducible operators.

Proposition 2.12

Let E be a complex Banach lattice with $\dim(E) > 1$, and let $T \in \mathcal{L}(E)$ be a positive irreducible operator on E. Suppose that $S,Q \in \mathcal{L}(E)$ are such that $0 < S \leq Q$, and suppose further that Q is compact. If $S \leq T^n$ for some natural number n, then $r(T) > 0$.

Proof

Consider first the case that S itself is compact. Fix $\lambda > r(T)$. The resolvent $R(\lambda,T)$ then has the property that $R(\lambda,T)u$ is a quasi-interior point of E for all $0 < u \in E$, (proposition 2.7). Define $A := R(\lambda,T)S$; then $A \geq 0$. In fact we have that $A \neq 0$. To see this, there exists an $x \in E^+$ such that $Sx > 0$, hence by proposition 2.5 $R(\lambda,T)Sx > 0$, i.e., $Ax \neq 0$.

It is also true that A is compact. This follows from the fact that $\lambda > r(T)$ implies that $\lambda \in \rho(T)$, which in turn implies that $R(\lambda,T) \in \mathcal{L}(E)$. It then follows that because S is compact, so too is A.

Define $N_s := \{ f \in E \,/\, S|f| = 0 \}$. Evidently N_s is an A-invariant ideal. Further, N_s is closed in E for if $x \in \overline{N_s}$ then there exists a sequence $(x_n) \subseteq N_s$ such that $x_n \to x$ in norm. Thus

$$\begin{aligned}|S(|x|)| &= S|x| \\ &\leq S(|x-x_n|+|x_n|) \\ &= S|x-x_n|+S|x_n| \\ &= S|x-x_n|,\end{aligned}$$

which implies that $\|S(|x|)\| \leq \|S|x-x_n|\| \leq \|S\|\|x-x_n\|$, and this last term can be made arbitrarily small. Thus $\|S(|x|)\| = 0$. It follows immediately that $S|x| = 0$, from which we infer that $x \in N_s$, and that N_s is closed in E.

Because N_s is closed we may consider the quotient space E/N_s, and because N_s is A invariant the operator A_0 induced by A on E/N_s, which is given by

$$A_0[f] := [Af] \text{ for all } [f] \in E/N_s,$$

is well defined. Since A is compact so too is A_0 ([3], theorem 2.14).

We now claim that $r(A_0) \leq r(A)$. In order to show this, let $\mu \in \mathbb{C}$ such that $|\mu| > r(A)$. Then for $x \in N_s$

$$\begin{aligned}|R(\mu,A)x| &\leq |R(\mu,A)||x| \\ &= R(\mu,A)|x| \\ &= \textstyle\sum_{k=0}^{\infty} A^k|x|/\mu^{k+1} \\ &= \textstyle\sum_{k=0}^{\infty} (R(\lambda,T)S)^k|x|/\mu^{k+1} \\ &= |x|/\mu,\end{aligned}$$

since for $k > 0$, $(R(\lambda,T)S)^k|x| = 0$ because $S|x| = 0$.

It thus follows that $R(\mu,A)x \in N_s$, since N_s is an ideal; thus if $|\mu| > r(A)$ then N_s is $R(\mu,A)$-invariant.

Now let $[x] \in E/N_s$ be such that $(\mu I-A_0)[x] = 0$, (where it is understood that 0 represents the zero element of the space E/N), then $[\mu x-Ax] = 0$ and so $\mu x-Ax \in N_s$. But then $R(\mu,A)(\mu I-A)x \in N_s$ which means that $(\mu I-A)^{-1}(\mu I-A)x \in N_s$, i.e., $x \in N_s$. From this we deduce that $[x] = 0$. It is clear that $(\mu I-A_0)[0] = 0$. Thus we have that if $|\mu| > r(A)$ then $(\mu I-A_0)[x] = 0$ if and only if $[x] = 0$. Furthermore $|\mu| > r(A)$ implies that $\mu \in \rho(A)$ which in turn implies that the range of the operator $\mu I-A$ is the whole of L, from which it follows immediately that the range of the operator $\mu I-A_0$ is the whole of the quotient space E/N_s. We have thus shown that if $|\mu| > r(A)$ then $\mu \in \rho(A_0)$. Using the definition of the spectral radius we deduce that $r(A_0) \leq r(A)$.

We next show that A_0 is an irreducible operator and to this end we make use of proposition 2.8. Let $0 \leq u \in E$ such that $[u] > 0$ in E/N_s. Clearly $u \notin N_s$. If $Su = 0$ then $S|u| = 0$ which means that $u \in N_S$ which is a contradiction. Thus $Su > 0$. Using proposition 2.7, we have that $Au = R(\lambda,T)Su$ is a quasi-interior point of L. It then follows that $[Au] = A_0[u]$ is a quasi-interior point of E/N_s, ([11], II corollary 1 of proposition 6.2). Proposition 2.8 then implies that A_0 is irreducible, and theorem 2.11 then yields that $r(A_0) > 0$. From what was shown above, namely that $r(A_0) \leq r(A)$, we have that $r(A) > 0$.

Now by hypothesis $0 < S \leq T^n$ for some $n \in \mathbb{N}$.
Hence $0 \leq A = R(\lambda,T)S$

$\leq R(\lambda,T)T^n$ (since $\lambda > r(T)$ implies that $R(\lambda,T) \geq 0$)

which means that $r(A) \leq r(R(\lambda,T)T^n)$. It is then evident that $r(R(\lambda,T)T^n) > 0$. Thus by the spectral mapping theorem, $r(T) > 0$. This completes the proof if S is compact.

Suppose now we only have that $0 < S \leq Q$, where Q is compact; again let $\lambda > r(T)$. Now $(R(\lambda,T)S)^3 > 0$, and to see this suppose that $(R(\lambda,T)S)^3 = 0$. It then follows that $(R(\lambda,T)S)^2 z = 0$ for all $z \in I_{R(\lambda,T)Su}$. However $I_{R(\lambda,T)Su}$ is dense in E because $R(\lambda T)Su$ is a quasi-interior point (propositions 2.5 and 2.7), and therefore $(R(\lambda,T)S)^2 = 0$ on E ([12], I.6.4, theorem 1). Repeating the same argument again we can show that $R(\lambda,T)S = 0$ on E, but this contradicts the fact that $A = R(\lambda,T)S \neq 0$ on E. Hence $(R(\lambda,T)S)^3 > 0$.

So we have $0 \leq R(\lambda,T)S \leq R(\lambda,T)Q$, where the latter composite operator is compact. It then follows from a well known theorem of Aliprantis and Burkinshaw ([14], theorem 124.5) that $(R(\lambda,T)S)^3$ is compact. We once again make use of proposition 2.8 to prove that $R(\lambda,T)T^n$ is irreducible . Let $u > 0$ in E, then $T^n u > 0$ (by proposition 2.3). By proposition 2.7, $R(\lambda,T)T^n u$ is a quasi-interior point, from which we conclude that $R(\lambda,T)T^n$ is irreducible.

We now have exactly the same situation as the first part of the proof with T and S replaced by $R(\lambda,T)T^n$ and $(R(\lambda,T)S)^3$ respectively. We thus deduce that $r(R(\lambda,T)T^n) > 0$. Invoking the spectral mapping theorem yields $r(T) > 0$.

□

CHAPTER 3

BAND IRREDUCIBLE OPERATORS

The main result of chapter two was theorem 2.11 which we proved using a method of de Pagter. The goal of this chapter then is twofold; firstly we shall obtain the Ando-Krieger Theorem as a result of theorem 2.11 and secondly we shall show that 2.11 also holds for σ-order continuous band irreducible operators. The results in this chapter are, in the main, taken from the two papers of Grobler, [5] and [6].

In definition 2.2 we introduced the concept of an ideal irreducible operator. We now generalise this concept to include not only ideals, but bands as well.

Definition 3.1

Let E be a Banach lattice and let $T \in \mathcal{L}(E)$. We say that T is *band irreducible* if the only T-invariant bands in E are $\{0\}$ and E itself.

If A is a band in the (complex) Riesz space E such that $E = A \oplus A^d$, the *projection* P_A of E onto A is defined by $P_A(x) := a_1$, where $x = a_1+a_2$ with $a_1 \in A$ and $a_2 \in A^d$. The Riesz space E is said to have *the projection property* if $E = B \oplus B^d$ for all bands B in E.

Proposition 3.2

Let E be a complex Riesz space with the projection property. If $0 < T \in \mathcal{L}(E)$ then T is band irreducible if and only if $P_{(B^d)}\, T\, P_B > 0$ for every non-trivial band B in E.

Proof

Assume T is band irreducible. Let B be a band in E such that $\{0\} \neq B \subset E$, (where $\subset$ denotes proper inclusion). Then since T is band irreducible, B is not T-invariant. Consequently there exists $z \in B^+$ such that $Tz \notin B$. This means that $P_{(B^d)} \neq 0$, and thus $P_{(B^d)}\, T(P_B z) \neq 0$ (because $P_B z = z$). However, since $T > 0$ and every projection is positive, it follows that $P_{(B^d)}\, T\, P_B \geq 0$, and we may hence conclude that $P_{(B^d)}\, T\, P_B > 0$.

Conversely suppose that $P_{(B^d)}\, T\, P_B > 0$ for each band B in E. Assume now that B is a band in E which is T-invariant. Suppose further that $B \neq \{0\}$ and $B \neq E$. Because $P_{(B^d)}\, T\, P_B > 0$, there exists $0 \neq x \in E^+$ such that $P_{(B^d)}\, T\, P_B(x) > 0$. Let $y := P_B x$; then $y \in B$ and $P_{(B^d)}(Ty) > 0$. It follows that $Ty \notin B$ and thus B is not T-invariant, which is a contradiction to the assumption. So either $B = \{0\}$ or $B = E$, i.e., the only T-invariant bands in E are $\{0\}$ and E, which implies that T is band irreducible.

□

We shall now prove results analogous to those proved in propositions 2.3, 2.5, 2.7 and 2.8, for the case that T is a band irreducible σ-order continuous operator.

Proposition 3.3

Let E be a complex Banach lattice and let $0 < T \in \mathcal{L}(E)$ be a band irreducible σ-order continuous operator. Then $T^n x > 0$ for all $n \in \mathbb{N}$, whenever $0 < x \in E$.

Proof

Suppose there exists an $n_0 \in \mathbb{N}$ such that $T^{n_0}x = 0$, where x is some non-zero positive element in E. Let n be the least such natural number. If $z := T^{n-1}x$, then $z > 0$. Denote by $B(z)$ the band in E generated by z. Then $B(z) = \{z\}+i\{z\}$, where $\{z\}$ denotes that band in $Re(E)$ generated by z.

If $x \in \{z\}$, then $|x| \in \{z\}$ and thus

$$|x| = \sup\{ \inf(|x|,nz) \ / \ n \in \mathbb{N} \}.$$

Since T is σ-order continuous it follows that

$$T|x| = \sup\{ T(\inf(|x|,nz)) \ / \ n \in \mathbb{N} \}.$$

However

$$0 \leq T(\inf(|x|,nz)) \leq nTz = 0 \text{ for all } n \in \mathbb{N},$$

and thus $T|x| = 0$. It follows that $Tx = 0$ for all $x \in \{z\}$, and consequently $Ts = 0$ for all $s \in \{z\}+i\{z\} = B(z)$. Clearly $B(z)$ is then T-invariant and therefore $B(z) = E$ since $z \neq 0$. However this then implies that T is the zero operator on E, contradicting the assumption that $T > 0$.

Thus there does not exist a natural number n such that $T^n x = 0$. Since $T^n \geq 0$ for all $n \in \mathbb{N}$ and since $x > 0$, we conclude that $T^n x > 0$ for all natural numbers n, whenever $0 < x \in E$.

□

Corollary 3.4

If E is a complex Banach lattice and $0 < T \in \mathcal{L}(E)$ is a band irreducible σ-order continuous operator, then $R(\lambda,T)x > 0$ for all $\lambda \in \mathbb{C}$, $|\lambda| > r(T)$, and for all $0 < x \in E$.

Proof

Analogous to that of proposition 2.5.

□

Proposition 3.5

Let E be a complex Banach lattice and let $0 < T \in \mathcal{L}(E)$. If $B(Tu) = E$ (i.e., the band generated by Tu in E equals E) for all $0 < u \in E$, then T is band irreducible.

Proof

Suppose B is a T-invariant band in E such that $B \neq \{0\}$. Thus there exists $0 < x \in B$ such that $Tx \in B$, and hence $B(Tx) \subseteq B$. This then implies that $B = E$ which in turn implies that the only T-invariant bands in E are $\{0\}$ and E. Thus T is irreducible.

□

Proposition 3.6

Let E be a complex Banach lattice with $0 < T \in \mathcal{L}(E)$ a σ-order continuous band irreducible operator. If $\lambda \in \mathbb{C}$ and $|\lambda| > r(T)$, then if $0 < x \in E$, the band generated in E by $R(\lambda,T)x$ is the whole of E.

Proof

From corollary 3.4, $R(\lambda,T)x > 0$ and is hence an element of $Re(E)$. Let z be a non-zero positive element in the band $\{R(\lambda,T)x\}$ generated in $Re(E)$ by $R(\lambda,T)x$. Then

$$z = \sup\{ \inf(z,nR(\lambda,T)x) \ / \ n \in \mathbb{N} \}.$$

By the σ-order continuity of T it follows that

$$Tz = \sup\{ T(\inf(z,nR(\lambda,T)x)) \ / \ n \in \mathbb{N} \}.$$

However, since $0 \leq \inf(z,nR(\lambda,T)x) \leq nR(\lambda,T)x$ for all $n \in \mathbb{N}$, it follows that

$$0 \leq T(\inf(z,nR(\lambda,T)x)) \leq nTR(\lambda,T)x \leq n\lambda R(\lambda,T)x \text{ for all } n \in \mathbb{N}.$$

This then implies that $T(\inf(z,nR(\lambda,T)x))$ is an element of the ideal in $Re(E)$ generated by $R(\lambda,T)x$, for all $n \in \mathbb{N}$. Thus $Tz \in \{R(\lambda,T)x\}$, showing that $\{R(\lambda,T)x\}$ is a T-invariant band in $Re(E)$. It follows immediately that the band $J := \{R(\lambda,T)x\}+i\{R(\lambda,T)x\}$ is a T-invariant band in E. Since $R(\lambda,T)x \neq 0$, we have that $J = E$.

□

Before stating the next theorem, which is of fundamental importance in obtaining a proof for the Ando-Krieger Theorem via 2.11, recall that $(E')_n$ denotes the set of all order continuous linear functionals on E.

Proposition 3.7

Let E be a Dedekind complete complex Banach lattice such that $(E')_n \neq \{0\}$. Suppose T is a positive order continuous band irreducible operator on E. Then E and $(E')_n$ both have weak order units.

Proof

Fix $\lambda > r(T)$, then $0 \leq TR(\lambda,T) \leq \lambda R(\lambda,T)$. Since $(E')_n \neq \{0\}$, there exists a non-zero positive $x \in E$. We shall show that $e := R(\lambda,T)x$ is a weak order unit in E.

From what has been shown above, $Te \leq \lambda e$. It follows that if I_e denotes the ideal in E generated by e, then $T(I_e) \subseteq I_e$, for if $v \in I_e$, then $|v| \leq \lambda_1 e$ for some $\lambda_1 \in \mathbb{R}^+$. Hence

$$|Tv| \leq T|v| \leq \lambda_1 Te \leq \lambda_1 \lambda e$$

which implies that $Tv \in I_e$. It is true however, that a stronger result holds, namely that $T(\{I_e\}) \subseteq \{I_e\}$, where $\{I_e\}$ denotes the band generated in E by the ideal I_e. To see this let $0 < v \in \{I_e\}$, then

$$v = \sup\{ \inf(v,ne) \ / \ n \in \mathbb{N} \}.$$

Since T is order continuous, it is also σ-order continuous and so

$$Tv = \sup\{ T(\inf(v,ne)) \ / \ n \in \mathbb{N} \}.$$

Since $0 \leq \inf(v,ne) \leq ne$, $0 \leq T(\inf(v,ne)) \leq nTe \leq n\lambda e$, for all $n \in \mathbb{N}$, and so for each natural number n, $T(\inf(v,ne)) \in I_e$, and hence $Tv \in \{I_e\}$.

In view of the fact that T is band irreducible, $\{I_e\} = \{0\}$ or $\{I_e\} = E$. The former equality cannot hold, for then $e = R(\lambda,T)x = 0$, which implies that $x = 0$, since for $\lambda \in \rho(T)$, $R(\lambda,T)$ is injective. This then contradicts our original choice of

x as being non-zero. Thus $\{I_e\} = E$, which proves our claim that $e = R(\lambda,T)x$ is a weak order unit of E.

To prove the existence of a weak order unit in $(E')_n$, fix $\lambda > r(T)$. Then we have that $0 \leq T'R(\lambda,T') \leq \lambda R(\lambda,T')$. It follows that for all non-zero positive linear functionals $x_0' \in (E')_n$,

$$T'R(\lambda,T')x_0' \leq \lambda R(\lambda,T')x_0'.$$

Since $(E')_n \neq \{0\}$ there exists a non zero order continuous linear functional x' on E such that $x' > 0$.

Defining $e' := R(\lambda,T')x'$ gives $T'e' < \lambda e'$. It will now be shown that $T'((E')_n) \subseteq (E')_n$. To this end let $f \in (E')_n$ and assume that $x_\alpha \downarrow 0$, where $(x_\alpha)_\alpha \subseteq E$. Because T is order continuous, $Tx_\alpha \downarrow 0$, and then since f is order continuous, $f(Tx_\alpha)\downarrow 0$. This then shows that $T'f$ is order continuous, i.e., $T'f \in (E')_n$, from which we can deduce that $T'((E')_n) \subseteq (E')_n$. It follows from this that $(T')^k((E')_n) \subseteq (E')_n$ for all $k \in \mathbb{N}$, and hence that $\sum_{k=0}^{m} T'^k x'/\lambda^{k+1} \in (E')_n$ for all $m \in \mathbb{N}$. Using the fact that $\sum_{k=0}^{m} T'^k x'/\lambda^{k+1} \uparrow$, and the fact that this sequence of partial sums converges in norm to $R(\lambda,T')x'$, we have that $\sum_{k=0}^{n} T'^k x'/\lambda^{k+1} \uparrow R(\lambda,T')x'$. In view of the fact that $(E')_n$ is a band it follows that $R(\lambda,T')x' \in (E')_n$, i.e., $e' \in (E')_n$.

Consider now the null ideal, $N_{e'} := \{ x \in E \ / \ e'(|x|) = 0 \}$ of e'. Since $e' \in (E')_n$, $N_{e'}$ is a band in E ([13], theorem 90.4). Furthermore, $T(N_{e'}) \subseteq N_{e'}$, for if $x \in N_{e'}$ then

$$0 \leq e'(|Tx|) \leq e'(T|x|) = T'e'(|x|) = T'(0) = 0.$$

Thus $Tx \in N_{e'}$, and hence $N_{e'}$ is T-invariant. Consequently $N_{e'} = \{0\}$ or $N_{e'} = E$ since T is band irreducible. If $N_{e'} = E$ then e' is the zero operator on E. However making use of the injective property of $R(\lambda,T')$, we can show that $e' \neq 0$. It follows then that $N_{e'} = \{0\}$ and thus $C_{e'} := N_{e'}{}^d = E$.

x as being non-zero. Thus $\{I_e\} = E$, which proves our claim that $e = R(\lambda,T)x$ is a weak order unit of E.

To prove the existence of a weak order unit in $(E')_n$, fix $\lambda > r(T)$. Then we have that $0 \leq T'R(\lambda,T') \leq \lambda R(\lambda,T')$. It follows that for all non-zero positive linear functionals $x_0' \in (E')_n$,

$$T'R(\lambda,T')x_0' \leq \lambda R(\lambda,T')x_0'.$$

Since $(E')_n \neq \{0\}$ there exists a non zero order continuous linear functional x' on E such that $x' > 0$.

Defining $e' := R(\lambda,T')x'$ gives $T'e' < \lambda e'$. It will now be shown that $T'((E')_n) \subseteq (E')_n$. To this end let $f \in (E')_n$ and assume that $x_\alpha \downarrow 0$, where $(x_\alpha)_\alpha \subseteq E$. Because T is order continuous, $Tx_\alpha \downarrow 0$, and then since f is order continuous, $f(Tx_\alpha)\downarrow 0$. This then shows that $T'f$ is order continuous, i.e., $T'f \in (E')_n$, from which we can deduce that $T'((E')_n) \subseteq (E')_n$. It follows from this that $(T')^k((E')_n) \subseteq (E')_n$ for all $k \in \mathbb{N}$, and hence that $\sum_{k=0}^{m} T'^k x'/\lambda^{k+1} \in (E')_n$ for all $m \in \mathbb{N}$. Using the fact that $\sum_{k=0}^{m} T'^k x'/\lambda^{k+1} \uparrow$, and the fact that this sequence of partial sums converges in norm to $R(\lambda,T')x'$, we have that $\sum_{k=0}^{n} T'^k x'/\lambda^{k+1} \uparrow R(\lambda,T')x'$. In view of the fact that $(E')_n$ is a band it follows that $R(\lambda,T')x' \in (E')_n$, i.e., $e' \in (E')_n$.

Consider now the null ideal, $N_{e'} := \{ x \in E \,/\, e'(|x|) = 0 \}$ of e'. Since $e' \in (E')_n$, $N_{e'}$ is a band in E ([13], theorem 90.4). Furthermore, $T(N_{e'}) \subseteq N_{e'}$, for if $x \in N_{e'}$ then

$$0 \leq e'(|Tx|) \leq e'(T|x|) = T'e'(|x|) = T'(0) = 0.$$

Thus $Tx \in N_{e'}$, and hence $N_{e'}$ is T-invariant. Consequently $N_{e'} = \{0\}$ or $N_{e'} = E$ since T is band irreducible. If $N_{e'} = E$ then e' is the zero operator on E. However making use of the injective property of $R(\lambda,T')$, we can show that $e' \neq 0$. It follows then that $N_{e'} = \{0\}$ and thus $C_{e'} := N_{e'}^d = E$.

The Riesz space $(E')_n$ has the projection property since it is a band in E', which is a Dedekind complete Banach lattice, and therefore has the projection property. If $B_{e'}$ denotes the band generated in $(E')_n$ by e', then $(E')_n = B_{e'} \oplus B_{e'}{}^d$. If $\phi \in (B_{e'}{}^d)^+$ then $\phi \perp e'$, which implies that $C_\phi \perp C_{e'}$ ([14], theorem 90.6). But $C_\phi = \{0\}$ because $C_{e'} = E$. Thus $N_\phi = E$ implying that $\phi = 0$ on E. It thus follows that $B_{e'}{}^d = \{0\}$ and then $(E')_n = B_{e'}$, which completes the proof.

□

Remark: It was shown in the first part of the above proof that for a fixed $\lambda > r(T)$ and a fixed $x > 0$, $e := R(\lambda,T)x$ is a weak order unit of E. However for the same λ and x, there does exist a weak order unit not necessarily equal to e, namely $TR(\lambda,T)x$. For the proof of this result the reader is referred to [14], theorem 136.2.

We now recall briefly the concept of a finite rank operator. An element $T \in \mathcal{L}(X)$, where X is a normed vector space, is said to be a finite rank operator if its range is finite dimensional. If $y \in X$ and $f \in X'$, define the bounded linear operator $f \otimes y$ on X by

$$(f \otimes y)(x) := f(x)y.$$

It can be shown that every finite rank operator can be written as a finite linear combination of the type of operators just mentioned ([3], theorem 2.4). One final result on finite rank operators is that every such operator is compact ([3], theorem 2.10).

Definition 3.8

If E is a Dedekind complete Banach lattice we denote by $(E' \otimes E)^{dd}$ the band generated in $L_b(E)$ by the finite rank operators $E' \otimes E$.

Proposition 3.9

Let E be a Dedekind complete Banach lattice such that $(E')_n \neq \{0\}$. If $0 < T \in \mathcal{L}(E)$ is an order continuous band ireducible operator on E, then $e' \otimes e$ is a weak order unit of $((E')_n \otimes E)^{dd}$, where e and e' are weak order units of E and $(E')_n$ respectively.

Proof

Our proof will make use of the following inequality,

$$\inf((f \otimes u),(e' \otimes e)) \geq (\inf(f,e')) \otimes (\inf(u,e))$$

for all $u \in E^+$ and for all $f \in ((E')_n)^+$. To prove this inequality, let $x \in E^+$ and let $f \in ((E')_n)^+$. There are two possibilities. Either $f(x) \geq e'(x)$ or $f(x) \leq e'(x)$.

In the first case,

$$\inf(f(x)u,e'(x)e) \geq \inf(e'(x)u,e'(x)e) = e'(x)(\inf(u,e)).$$

In the second case,

$$\inf(f(x)u,e'(x)e) \geq f(x)(\inf(u,e)).$$

These two inequalities together imply that

$$\inf(f(x)u,e'(x)e) \geq (\inf(e'(x),f(x))(\inf(u,e))).$$

Since $\inf(e',f) \leq e'$ and $\inf(e',f) \leq f$,

$$(\inf(e',f))x \leq \inf(e'(x),f(x)),$$

and so

$$\inf(f(x)u,e'(x)e) \geq (\inf(e',f))x(\inf(u,e)).$$

Since x was an arbitrary positive element it follows that

$$\inf(f \otimes u,e' \otimes e) \geq (\inf(e',f)) \otimes (\inf(u,e)).$$

This proves the inequality which we shall use in the proof which we now tackle.

Since e is a weak order unit of E, the sequence (u_n) defined by $u_n := \inf(u,ne)$ for all $n \in \mathbb{N}$, is such that $u_n \uparrow u$, for all $u \in E^+$.

Similarly the sequence (f_n), defined by $f_n := \inf(f, ne')$ for all $n \in \mathbb{N}$, is such that $f_n \uparrow f$, for all $f \in ((E')_n)^+$.

It follows that $f_n \otimes u_n \uparrow f \otimes u$. However, for all $n \in \mathbb{N}$,

$$(\inf(f, \sqrt{n}e')) \otimes (\inf(u, \sqrt{n}e)) \leq \inf((f \otimes u), n(e' \otimes e))$$

Thus

$$(\inf(f, \sqrt{n}e')) \otimes (\inf(u, \sqrt{n}e)) \uparrow f \otimes u.$$

Because $\inf(f \otimes u, n(e' \otimes e)) \leq n(e' \otimes e)$, it follows that $\inf(f \otimes u, n(e' \otimes e)) \in I_{e' \otimes e}$, the ideal generated in $L_b(E)$ by $e' \otimes e$.

Also, $\inf(f \otimes u, n(e' \otimes e)) \leq f \otimes u$, and from what was shown above we conclude that $\inf(f \otimes u\, n(e' \otimes e)) \uparrow f \otimes u$. Consequently $f \otimes u \in (e' \otimes u)^{dd}$, for all $u \in E^+$ and for all $f \in ((E')_n)^+$. Thus $(E')_n \otimes E \subseteq (e' \otimes e)^{dd}$, which implies that $((E')_n \otimes E)^{dd} \subseteq (e' \otimes e)^{dd}$. However since it is also true that

$$e' \otimes e \in (E')_n \otimes E,$$

and thus that

$$(e' \otimes e)^{dd} \subseteq ((E')_n \otimes E)^{dd},$$

it follows that $((E')_n \otimes E)^{dd} = (e' \otimes e)^{dd}$ as required.

□

Lemma 3.10

Let E be a Dedekind complete Banach lattice and let $0 < T \in ((E')_n \otimes E)^{dd}$ be band irreducible. If e' and e are weak order units of $(E')_n$ and E respectively, then $S := \inf(e' \otimes e, T)$ is band irreducible.

Proof

Note firstly that $(E')_n \neq \{0\}$, for otherwise $(E')_n \otimes E = \{0\}$ contradicting the existence of a non zero operator in $(E')_n \otimes E$. Let $\{0\} \neq B \subset E$ be a band in E. Then for any $n \in \mathbb{N}$, $T \geq (1/n)T$ and thus

$$\inf(e' \otimes e, T) \geq \inf(e' \otimes e, (1/n)T) = (1/n)(\inf(n(e' \otimes e), T)).$$

Hence

$$P_{(B^d)}(\inf(e'\otimes e,T))P_B \geq (1/n)P_{(B^d)}(\inf(n(e'\otimes e),T))P_B,$$

and so

$$nP_{(B^d)}(\inf(e'\otimes e,T))P_B \geq P_{(B^d)}(\inf(n(e'\otimes e),T))P_B.$$

Because $T \in ((E')_n \otimes E)^{dd}$ and since $e'\otimes e$ is a weak order unit of $((E')_n\otimes E)^{dd}$, we have that $(\inf(n(e'\otimes e),T)) \uparrow T$. By the order continuity of band projections it follows that

$$\begin{aligned} nP_{(B^d)}(\inf(e'\otimes e,T))P_B &\geq P_{(B^d)}(\inf(n(e'\otimes e),T))P_B \\ &\uparrow P_{(B^d)}\, T\, P_B > 0 \text{ (by 3.2)}. \end{aligned}$$

Hence there must exist an $n_0 \in \mathbb{N}$ such that

$$n_0 P_{(B^d)}(\inf(e'\otimes e,T))P_B > 0$$

and therefore

$$P_{(B^d)}(\inf(e'\otimes e,T))P_B > 0$$

From proposition 3.2 it follows that $S := \inf(e'\otimes e,T)$ is band irreducible.

□

Lemma 3.11

Let E be a Banach lattice and let $0 < T \in \mathcal{L}(E)$ be a σ-order continuous band irreducible operator on E. Then every non-zero closed T-invariant ideal in E contains a weak order unit of E.

Proof

Let I be a non-zero closed T-invariant ideal in E. Fix $\lambda > r(T)$ and let $0 < u \in I$. Set $w := R(\lambda,T)u$, then $w = \sum_{k=0}^{\infty} T^k u/\lambda^{k+1}$ and $w \geq 0$.

Since I is T-invariant, $T^k u \in I$ for all $k \in \mathbb{N}$, and thus $T^k u/\lambda^{k+1} \in I$ for all $k \in \mathbb{N}$. It immediately follows then that $\sum_{k=0}^{n} T^k u/\lambda^{k+1} \in I$ for all $n \in \mathbb{N}$. Because $\sum_{k=0}^{n} T^k u/\lambda^{k+1} \to w$ in norm, and since I is closed, $w \in I$. Also

$$0 \leq Tw = TR(\lambda,T)u$$

$$\leq \lambda R(\lambda,T)u$$
$$= \lambda w$$

and hence $Tw \in I_w$, the ideal in E generated by w. Consequently I_w is T-invariant, for if $z \in I_w$ then

$$|z| \leq \kappa w \quad \text{for some } \kappa \in \mathbb{R}^+$$

and therefore

$$|Tz| \leq T|z| \leq \kappa Tw \in I_w.$$

Thus $Tz \in I_w$.

If B(w) denotes the band in E generated by w then if $0 \leq v \in B(w)$, $v_n \uparrow v$, where $v_n := \inf(v, nw)$ for all $n \in \mathbb{N}$. Clearly the sequence (v_n) is contained in the ideal I_w. The σ-order continuity of T implies that $Tv_n \uparrow Tv$, but since I_w is T-invariant it follows that $Tv_n \in I_w$ for all $n \in \mathbb{N}$, which in turn implies that $Tv \in B(w)$. Thus B(w) is T-invariant and it follows that either $B(w) = \{0\}$ or $B(w) = E$. In view of the fact that $w \neq 0$ (corollary 3.4), $B(w) = E$ and furthermore, $w \in I$ as required.

□

Lemma 3.12

Let E be a Banach lattice and let $0 < T \in \mathcal{L}(E)$ be a band irreducible operator on E. Suppose there exists a σ-order continuous compact operator S on E such that $T \leq S$. Then if M is the closed ideal generated by T(E) (the range of T) in E, then the operator T restricted to M is irreducible.

Proof

Denote by T_M the restriction of T to the ideal M. Let I be a closed non-zero T_M-invariant ideal in M. Then I is a closed non-zero T-invariant ideal in E. Since T satisfies the conditions of lemma 3.11, I contains a weak order unit w of E. Let $0 \leq x \in E$, and let $x_n := \inf(x, nw)$ for each $n \in \mathbb{N}$. Then $0 \leq x_n \leq nw$ and consequently, $x_n \in I$ for all $n \in \mathbb{N}$. Also $x_n \uparrow x$. It follows from the σ-order continuity of S that $Sx_n \uparrow Sx$. Because S is compact we have in addition that $Sx_n \to Sx$ in norm. This is easily seen to imply that $Tx_n \to Tx$ in norm. Now I is T-invariant and the sequence (x_n) is contained in I and hence $Tx_n \in I$ for each $n \in \mathbb{N}$. Also I is closed in E and hence $Tx \in I$. Since x was an arbitrary positive element of E it follows that $T(E) \subseteq I$, and hence $M \subseteq I$. By assumption we have the reverse inclusion so that we may conclude that $I = M$.

It has thus been shown that T_M is an ideal irreducible operator.

□

Corollary 3.13

Let E be a complex Banach lattice and let $0 < T \in \mathcal{L}(E)$ be a σ-order continuous operator on E which is band irreducible and compact. Then $r(T) > 0$.

Proof

Once again let M be the closed ideal generated in E by the image T(E) of T. Denote by T_M the restriction of T to M. By lemma 3.12, $T_M : M \to M$ is irreducible, and then by theorem 2.11, $r(T_M) > 0$. It then follows that $r(T) > 0$ since $r(T) \geq r(T_M)$.

□

We are now in a position to prove our first main result, namely the Ando-Krieger Theorem.

Theorem 3.14 (Ando-Krieger)

Let E be a Dedekind complete Banach lattice. If $0 < T \in ((E')_n \otimes E)^{dd}$ is a band irreducible operator, then $r(T) > 0$.

Proof

By proposition 3.7, E and $(E')_n$ have weak order units, e and e' respectively. Then $S := \inf(e'\otimes e, T)$ is a band irreducible by lemma 3.10. Since $0 < S \leq T$, and therefore $r(S) \leq r(T)$, we may assume that $T \leq e'\otimes e \in (E')_n \otimes E$. Note that $e'\otimes e$ is a σ-order continuous operator, and that since it is of finite rank, it is also compact. It then follows from [14], theorem 124.5 that T^3 is compact.

Let M be the closed ideal generated in E by $T(E)$. Then if T_M denotes the restriction of T to M, T_M is irreducible by lemma 3.12, and is compact since T^3 is compact. Let $\lambda > r(T_M)$. We then have that $[R(\lambda,T_M)T_M]^3 = [R(\lambda,T_M)]^3 T_M{}^3$ (Proposition 1.16 iii). Also $[R(\lambda,T_M)]^3 T_M{}^3$ is compact since $T_M{}^3$ is compact and the operator $R(\lambda,T_M)^3$ is bounded.

We now claim that $R(\lambda,T_M)^3 T_M{}^3$ is ideal irreducible, and to prove this we shall make use of proposition 2.8 of chapter two. For convenience we shall abbreviate $R(\lambda,T_M)$ as R until the end of this proof.

Let $0 < u \in M$; from proposition 2.3, $T^3 u > 0$ and thus $RT_M{}^3(u) > 0$ by proposition 2.5. Again by the same proposition we have that $R^2 T_M{}^3 > 0$. Thus $R^3 T_M{}^3(u)$ is a quasi-interior point of M by proposition 2.7. It then follows from proposition 2.8 that $R^3 T_M{}^3$ is (ideal-) irreducible.

From theorem 2.12, with $T = S = Q = [R(\lambda,T_M)T_M]^3$, and with $n = 1$, we obtain that $r(R(\lambda,T_M)^3 T_M{}^3) > 0$. Invoking the spectral mapping theorem yields $r(T_M) > 0$. Since $r(T) \geq r(T_M)$ we clearly have $r(T) > 0$ as required.

□

In proposition 2.12 of chapter two we proved that if T was a positive irreducible operator on E such that T^n majorised another operator $S > 0$, for some $n \in \mathbb{N}$, where S was in turn majorised by a compact operator, then $r(T) > 0$. It is our aim now to prove an analogous result for band irreducible operators. However in order to do this we require the following lemma.

Lemma 3.15

Let E be a complex Banach lattice and let $0 < T \in \mathcal{L}(E)$ be a positive σ-order continuous operator on E. If $\lambda > r(T)$ then the resolvent operator, $R(\lambda,T)$ is also σ-order continuous.

Proof

Define the sequence (S_n) by $S_n := \sum_{k=0}^{n} T^k/\lambda^{k+1}$ and let $S := R(\lambda,T)$. Then because all the terms of the sum are positive we have that $0 \le S_n\uparrow$, and since $S_n \to S$ in norm we have that $S_n \uparrow S$.

Suppose that $u_m \downarrow 0$ in E. We must show that $\inf\{ Su_m \ / \ m \in \mathbb{N} \} = 0$. To this end assume that $0 \le v \le Su_m$ for all $m \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Since $u_m \le u_1$ for every natural number m, and since $S-S_n \ge 0$, we have that $(S-S_n)u_m \le (S-S_n)u_1$. This then implies that

$$\begin{aligned} S_n u_m &\ge Su_m - Su_1 + S_n u_1 \\ &\ge v - Su_1 + S_n u_1 \quad \text{for all } m \in \mathbb{N}. \end{aligned}$$

Note that since S_n is σ-order continuous, $\inf\{ S_n u_m \ / \ m \in \mathbb{N} \}$ does exist, and in fact equals zero. Thus from what was shown above,

$$\inf \{ S_n u_m \ / \ m \in \mathbb{N} \} \ge v - Su_1 + S_n u_1$$

i.e., $0 \le v \le (S-S_n)u_1$

Since $n \in \mathbb{N}$ was arbitrary it has thus been shown that

$$\begin{aligned} 0 \le v &\le \inf(S-S_n)u_1 \\ &= 0 \qquad \text{because } S_n \uparrow S. \end{aligned}$$

We thus have that $v = 0$ from which we conclude that $\inf Su_m = 0$. It follows immediately that $S = R(\lambda,T)$ is σ-order continuous.

□

Theorem 3.16

Let E be a complex Banach lattice and let $0 < T \in \mathcal{L}(E)$ be a positive σ-order continuous band irreducible operator on E. Suppose S and Q are operators such that Q is compact and $0 < S \leq Q$. Suppose further that $S \leq T^n$ for some $n \in \mathbb{N}$. Then $r(T) > 0$.

Proof

We consider firstly the case that S is compact. Fix $\lambda > r(T)$ and define $A := R(\lambda,T)S$. Then $A > 0$, and to see this we have by hypothesis that $S > 0$ and since $R(\lambda,T) \geq 0$, it follows that the composition A is greater than or equal to zero. However because $S > 0$, there exists $0 < u \in E$ such that $Su > 0$, and it then follows from corollary 3.4 that $Au = R(\lambda,T)Su > 0$.

It is easy to see that A is a compact operator because S is compact and $R(\lambda,T)$ is bounded. Finally, A is σ-order continuous for if $u_m \downarrow 0$, then

$$\begin{aligned} 0 &\leq \inf |R(\lambda,T)Su_m| \\ &= \inf R(\lambda,T)Su_m \\ &\leq \inf R(\lambda,T)T^n u_m \end{aligned}$$

Now $0 \leq T^n$ is σ-order continuous (because T is σ-order continuous), and hence $T^n u_m \downarrow 0$. Since $R(\lambda,T)$ is σ-order continuous (proposition 3.15), it follows that $\inf\{ R(\lambda,T)T^n u_m \ / \ m \in \mathbb{N} \} = 0$.

It is an immediate consequence then that $\inf R(\lambda,T)Su_m = 0$, i.e., A is σ-order continuous.

Thus far we have that A is a non-zero positive σ-order continuous operator on E. Let M be the closed ideal generated in E by the image A(E) in E, and denote the restriction of A to M by A_M It follows that because A is compact so too is A_M ([3], theorem 2.12).

Let N := { $k \in M$ / $S|k| = 0$ }. Then N is exactly the null ideal of S in M. We claim firstly that N is A_M-invariant. To see this suppose that $k \in N$, then

$$\begin{aligned} |A_M(k)| &= |Ak| \\ &= |R(\lambda,T)Sk| \\ &\le R(\lambda,T)S|k| \\ &= 0. \end{aligned}$$

Hence $|A_M(k)| = 0$ and thus $S|A_M(k)| = S(0) = 0$. It then follows that $A_M(k) \in N$.

Our second claim is that N is closed, and the proof follows exactly as in the corresponding proof that N_s is closed in proposition 2.12 of chapter 2 .

Let A_0 be the operator induced on the complex quotient space A/M by the operator A_M. Since $A > 0$ it is easy to see that $A_0 > 0$. Moreover, A_0 is also compact ([3], theorem 2.14), and because $\|A_0{}^n\| \le \|A_M{}^n\|$ for all $n \in \mathbb{N}$, $r(A_0) \le r(A_M)$.

We now show that A_0 is irreducible on M/N. Let $j: M \to M/N$ be the quotient map, i.e., $j(f) = [f]$. Let J be a closed non-zero A_0-invariant ideal in M/N. We must show that $J = M/N$.

Since M is a Banach lattice and the quotient map is positive, it follows that j is continuous ([11], II theorem 5.3). Consequently $j^{-1}(J)$ is closed in M and furthermore it is A_M-invariant, for if $f \in j^{-1}(J)$ then

$$\begin{aligned} j(A_M(f)) &= [A_M] \\ &= A_0[f] \\ &= A_0(j(f)) \\ &\in A_0(J) \\ &\subseteq J, \qquad \text{since J is } A_0\text{-invariant.} \end{aligned}$$

It follows that $A_M(f) \in j^{-1}(J)$, and that $j^{-1}(J)$ is therefore A_M-invariant. Thus $j^{-1}(J)$ is a closed A_M-invariant ideal in M. Also since J is non-zero, there exists $u \in j^{-1}(J)$ such that $Su > 0$, for if not then $|x| \in j^{-1}(J)$ for all $x \in j^{-1}(J)$. This then means that $S|x| > 0$ does not hold. We must therefore have that $S|x| = 0$ because S is positive. Thus $x \in N$ whenever $x \in j^{-1}(J)$, i.e., $j^{-1}(J) \subseteq N$. Consequently $J \subseteq j(N) = \{[0]\}$ which contradicts the fact that J is non-zero.

Defining $w := A_M(u) = Au$ (and remembering that $u \in j^{-1}(J) \subseteq M$) we can show exactly as we did in lemma 3.11 that w is a weak order unit of E.

Assume now that $0 \leq x \in E$, and define $x_n := \inf(x, nw)$ for all $n \in \mathbb{N}$. It is clear that the sequence (x_n) is contained in $j^{-1}(J)$, and that $x_n \uparrow x$. The σ-order continuity of A then implies that $Ax_n \uparrow Ax$, and the compactness of A gives that $Ax_n \to Ax$.

We observed earlier that $(x_n) \subseteq j^{-1}(J) \subseteq M$, and for this reason $Ax_n = A_M(x_n) \in j^{-1}(J)$ for all $n \in \mathbb{N}$, since $j^{-1}(J)$ is A_M-invariant. This fact together with the facts that Ax is the norm limit of the sequence (Ax_n), and that $j^{-1}(J)$ is closed implies that $Ax \in j^{-1}(J)$. We thus conclude that $A(E) \subseteq j^{-1}(J)$, which is an ideal, and by definition of M (namely the closed ideal in E generated by $A(E)$) we have that $M \subseteq j^{-1}(J)$. Hence $M = j^{-1}(J)$ from which it follows immediately that $J = M/N$.

From the above argument we have shown that A_0 is irreducible and thus applying theorem 2.11, we obtain $r(A_0) > 0$. Noting the following inequalities,

$$0 < r(A_0) \leq r(A_M) \leq r(A) \leq r(R(\lambda,T)T^n),$$

where the last inequality follows from the fact that $0 < S \leq T^n$, which implies that $A \leq R(\lambda,T)T^n$, we have that $r(R(\lambda,T)T^n) > 0$. The spectral mapping theorem once again yields $r(T) > 0$. This proves the result for the case that S is compact.

Suppose now that $0 < S \leq Q$ with Q compact. Again fix $\lambda > r(T)$ and define $U := [R(\lambda,T)S]^3$. Then $U \leq [R(\lambda,T)T^n]^3$ in view of the fact that $S \leq T^n$. Exactly as in the corresponding part of the proof of proposition 2.12, we can show that $U > 0$. Further, $0 \leq R(\lambda,T)S \leq R(\lambda,T)Q$. Note that the last operator is compact because Q is compact and $R(\lambda,T)$ is bounded. We again employ the well known result of Aliprantis and Burkinshaw to obtain that U is a compact operator ([14], theorem 124.5). From proposition 3.15 we have that $R(\lambda,T)$ is σ-order continuous, as is T and hence the composition $R(\lambda,T)T^n$ is also a σ-order continuous operator.

We claim further, that $R(\lambda,T)T^n$ is band irreducible. Our proof of this will make use of proposition 3.5. Let $0 < u \in E$, then $T^n u > 0$ by proposition 3.3, and it then follows from proposition 3.6 that $B(R(\lambda,T)T^n u) = E$. Proposition 3.5 then yields that $R(\lambda,T)T^n$ is band irreducible.

Thus by the first part of the proof, with T replaced by $R(\lambda,T)T^n$ and S replaced by U, we have that $r(R(\lambda,T)T^n) > 0$. An application of the spectral mapping theorem yields $r(T) > 0$, as required.

□

The importance of theorem 3.16 is demonstrated by the following result.

Corollary 3.17

Let E be a Dedekind complete (complex) Banach lattice and let $\mathscr{C}$ denote the band in $L_b(E)$ generated by the positive compact operators. If $0 < T \in \mathscr{L}(E)$ is a σ-order continuous band irreducible operator on E satisfying $T^n \in \mathscr{C}$ for some $n \in \mathbb{N}$, then $r(T) > 0$.

Proof

Let I_c denote the ideal in $L_b(E)$ generated by the positive compact operators on E. Then $T^n = \sup\{ S \,/\, S \in I_c,\ 0 \leq S \leq T^n \}$ since $0 < T^n \in \mathscr{C} = \{I_c\}$. In view of the fact that $T^n > 0$, there must exist $S_0 \in I_c$ such that $0 < S_0 \leq T^n$. Now $S_0 \in I_c$ implies that there exist positive compact operators $Q_1, Q_2, \ldots, Q_n$ and positive real numbers $\alpha_1, \alpha_2, \ldots \alpha_n$ such that $S_0 \leq \sum_{k=1}^{n} \alpha_k Q_k$. Define $Q := \sum_{k=1}^{n} \alpha_k Q_k$, which is a compact operator ([3], theorem 2.8). Then $0 < S_0 \leq Q$ and $S_0 \leq T^n$ and it follows immediately from theorem 3.16 that $r(T) > 0$.

□

Remark: Theorem 3.16 improves a result on Harris operators of Casselles' paper, [2].

CHAPTER 4

THE THEOREMS OF JENTZSCH-PERRON AND FROBENIUS

The work done is this final chapter is largely based on Grobler's paper, [7]. The aim of that paper was, and consequently the aim of this chapter is, to prove extensions of the above two mentioned theorems. Prior to Grobler's paper, both theorems required that the Banach lattice be Dedekind complete, and that the operator which occurs is a specific type of operator, namely an abstract kernel operator. We shall show that these two theorems hold in *any* Banach lattice, and the only requirement on the operator is that it is σ-order continuous.

In order to gain further information from some of the theorems proved in this chapter, we shall assume from time to time that the Banach lattice is Dedekind σ-complete.

Let's briefly recall the definition of a Riesz operator. If X is a normed vector space and $T \in \mathcal{L}(X)$ we define the real number $\kappa(T)$ by

$$\kappa(T) := \inf\{ \|T-C\| \ / \ C \in \mathcal{L}(X) \text{ and } C \text{ is compact } \}.$$

If $[\ \kappa(T^n)\ ]^{1/n} \to 0$, then T is said to be a Riesz operator. It is easy to see that if T^k is a compact operator for some $k \in \mathbb{N}$, then T is a Riesz operator.

Throughout the remainder of this chapter we shall assume without further notice that all Riesz spaces are Archimedean. Furthermore we shall take E to be a complex Banach lattice unless otherwise stated.

The following three results are of a somewhat technical nature, but they shall prove invaluable in proving the main results of this chapter.

Theorem 4.1 (Krein-Rutman)
If $0 < T \in \mathcal{L}(E)$ is a Riesz operator with spectral radius $r(T) > 0$, then there exists a non-zero positive $u \in E$ such that $Tu = r(T)u$, i.e., $r(T)$ is an eigenvalue of T.

Proof
Assume firstly that $r(T) = 1$, then $1 \in \sigma(T)$, ([14], theorem 135.1) For every $\lambda > 1$, we have that 1 is a pole of $R(\lambda,T)$ ([3], theorem 3.14). If k is the order of the pole at 1, then

$$R(\lambda,T) = B_k/(\lambda-1)^k + \ldots + B_1/(\lambda-1) + A_0 + A_1(\lambda-1) + \ldots$$

from which it follows that

$$(\lambda-1)^k R(\lambda,T) = B_k + B_{k-1}(\lambda-1) + \ldots + B_1(\lambda-1)^{k-1} + A_0(\lambda-1)^k + \ldots$$

Taking the limit as λ approaches 1 from the right yields

$$B_k = \lim\,(\lambda-1)^k R(\lambda,T) \quad (\text{as } \lambda \to 1^+).$$

This limit is not equal to zero for otherwise the order of the pole at 1 is no greater than $k-1$, which is a contradiction.

Furthermore for $\lambda > r(T) = 1$, the operator $R(\lambda,T)$ is positive and so $(\lambda-1)^k R(\lambda,T) \geq 0$. It then follows that

$$\lim\,(\lambda-1)^k R(\lambda,T) > 0 \quad (\text{as } \lambda \to 1^+)$$

It is thus shown that $B_k > 0$. Let $0 < x \in E$ such that $u := B_k x > 0$, then

$$\begin{aligned}
Tu &= TB_k x \\
&= T(\lim(\lambda-1)^k R(\lambda,T)x) \\
&= \lim(\lambda-1)^k TR(\lambda,T)x \\
&= \lim(\lambda-1)^k [\lambda R(\lambda,T)x - x] \\
&\qquad\qquad (\text{ since } (T-\lambda I)R(\lambda,T)x = -x\ ) \\
&= \lim\,[\lambda(\lambda-1)^k R(\lambda,T)x] - \lim\,[(\lambda-1)^k x] \\
&= [\lim \lambda]\,[\lim\,((\lambda-1)^k R(\lambda,T)x)]
\end{aligned}$$

$= B_k x$
$= u$
$= r(T)u,$

(where it is understood that we are always considering the limit as λ approaches 1 from the right).

This proves the result for the case that T has spectral radius equal to 1. If $r(T) > 0$, define an operator T^* by

$$T^* := T/r(T),$$

then T^* is a positive non-zero Riesz operator having $r(T^*) = 1$. From what we have shown above there exists $u \in E\backslash\{0\}$ such that $T^*u = u$ which obviously implies that $Tu = r(T)u$.

□

Lemma 4.2

Let (S_n) be a sequence of σ-order continuous positive operators on E.

i) If $S_n \uparrow S$, ie., $S_n x \uparrow Sx$ for all $x \in E^+$, then S is σ-order continuous.

ii) If $S_n \to S$ in norm and if $(E')_c$ separates the points of E, then S is also σ-order continuous.

Proof

i) Let $u_m \downarrow 0$. We must show $\inf Su_m = 0$. To this end let

$$0 \le v \le Su_m \quad \text{for all } m \in \mathbb{N}.$$

Fix $n \in \mathbb{N}$. Then

$$0 \le v \le (S-S_n)u_m + S_n u_m \quad \text{for all } m \in \mathbb{N}.$$

Since $(S-S_n) \ge 0$ and $u_m \le u_1$ for all $m \in \mathbb{N}$, it follows that

$$0 \le v \le (S-S_n)u_1 + S_n u_m \quad \text{for all } m \in \mathbb{N}.$$

In view of the fact that S_n is σ-order continuous, $\inf S_n u_m = 0$. Thus

$$0 \le v \le (S-S_n)u_1.$$

Consequently $v = 0$, because $S-S_n \downarrow 0$ implies that $(S-S_n)u_1 \downarrow 0$.

We have thus shown that $\inf Su_m = 0$, i.e., S is σ-order continuous.

ii) Suppose that $S_n \to S$ in norm. Let $u_m \downarrow 0$ and let $0 \le v \le Su_m$ for all $m \in \mathbb{N}$. It will be shown that for each positive $f \in (E')_c$, $f(v) = 0$. It will then follow from the assumption that $(E')_c$ separates the points of E, that $v = 0$ and the proof will then be complete.

Let $f \in ((E')_c)^+$ be fixed. Then for all $m \in \mathbb{N}$ we have

$$
\begin{aligned}
0 \le f(v) &\le f(Su_m) \\
&= S'(fu_m) \quad (S' \text{ denotes the adjoint of } S) \\
&= ([S'-S_n']f)u_m + f(S_n u_m) \\
&\le |(S'-S_n')f|u_1 + f(S_n u_m) \quad \text{for any } n \in \mathbb{N}.
\end{aligned}
$$

Thus for every fixed $n \in \mathbb{N}$,

$$
\begin{aligned}
0 \le f(v) &\le |(S'-S_n')f|u_1 + \inf\{ f(S_n u_m) \ / \ m \in \mathbb{N} \} \\
&= |(S'-S_n')f|u_1
\end{aligned}
$$

since S_n and f are σ-order continuous, and consequently $\|f(v)\| \le \| |(S'-S_n')f|u_1 \|$.

Because $\|(S'-S_n')f\| \ge (\| |(S'-S_n')f|u \|)/\|u\|$ for all $u \ne 0$,

$$\|(S'-S_n')f\| \|u_1\| \ge \| |(S'-S_n')f|u_1 \|,$$

and hence $0 \le \|f(v)\| \le \|(S'-S_n')f\| \|u_1\| \to 0$ as $n \to \infty$.
It follows immediately that $\|f(v)\| = 0$, from which we deduce that $f(v) = 0$.

□

Theorem 4.3
Let $0 < T \in \mathcal{L}(E)$ be a σ-order continuous band irreducible Riesz operator on E with $r(T) > 0$. If $(E')_c$ separates the points of E, or else if we assume that T^k is compact for some $k \in \mathbb{N}$, then there exists a strictly positive σ-order continuous linear functional ϕ on E, satisfying $T'\phi = r(T)\phi$

Proof

Assume firstly that $r(T) = 1$. Since T is a Riesz operator, so too is T' ([3], theorem 3.22). It follows in the same manner as in theorem 4.1 that we can find an element $x' \in E'$ such that $x' > 0$ and $\phi := \lim(\lambda-1)^p R(\lambda,T')(x') > 0$ (as $\lambda \to 1^+$), where p is the order of the pole at 1, and such that $T'\phi = \phi$.

It shall now be shown that ϕ is σ-order continuous in each of the two cases mentioned in the statement.

Firstly let us consider the case that T^k is compact for some $k \in \mathbb{N}$.

$$\begin{aligned} \text{Then } (T^k)'\phi &= (T')^k\phi \\ &= (T')^{k-1}T'\phi \\ &= (T')^{k-1}\phi \quad \text{because } T'\phi = \phi. \end{aligned}$$

Continuing in this way yields $(T^k)'\phi = \phi$.

Suppose now that $u_m \downarrow 0$ in E. For any $n \in \mathbb{N}$

$$\begin{aligned} \phi(u_n) &= (T^k)'\phi(u_n) \\ &= \phi(T^k u_n) \end{aligned}$$

In view of the fact that T^k is compact, $(T^k u_n)$ has a norm convergent subsequence, and because $(T^k u_n)$ is monotone, $(T^k u_n)$ is norm convergent. Let $q \in E$ denote the norm limit of $(T^k u_n)$. The σ-order continuity of T^k, which follows from that of T, implies that $T^k u_n \downarrow 0$. Therefore $q = 0$, and so we conclude that $T^k u_n \to 0$ in norm (as $n \to \infty$). Considering $\phi(u_n)$ we have

$$\begin{aligned} \|\phi(u_n)\| &= \|\phi(T^k u_n)\| \\ &\le \|\phi\|\|T^k u_n\| \to 0 \quad (\text{as } n \to \infty), \end{aligned}$$

i.e., $\phi(u_n)$ converges in norm to zero. However since ϕ is positive, $\phi(u_n)\downarrow$, and consequently $\phi(u_n)\downarrow 0$. This implies that ϕ is σ-order continuous.

Consider now the case that $(E')_c$ separates the points of E. Let $\lambda > r(T) = 1$, then $R(\lambda,T) = \sum_{k=0}^{\infty} T^k/\lambda^{k+1}$. Define the sequence (S_n) by $S_n = \sum_{k=0}^{n} T^k/\lambda^{k+1}$. As before, $S_n \uparrow R(\lambda,T)$. It follows from the σ-order continuity of T, and hence of $(1/\lambda^{k+1})T^k$ for all natural k, that S_n is σ-order continuous for all $n \in \mathbb{N}$. Employing part i) of the previous lemma yields that $R(\lambda,T)$ is σ-order continuous, while part ii) ensures that $B_p = \lim (\lambda-1)^p R(\lambda,T)$ is σ-order continuous (where p is the order of the pole of the resolvent operator at 1, and the limit is as $\lambda \to 1^+$).

In view of proposition 1.24, B_p is of finite rank and is hence compact. Also

$$\begin{aligned}\phi &= \lim (\lambda-1)^p R(\lambda,T')x' \\ &= \lim (\lambda-1)^p (R(\lambda,T))'\, x' \qquad \text{([3], proposition 1.12)} \\ &= [\lim (\lambda-1)^p R(\lambda,T)]'\, x' \\ &= (B_p)'\, x'\end{aligned}$$

(where again the limit is as λ approaches 1 from the right).

Let $u_n \downarrow 0$, then $\phi(u_n) = (B_p)'\, x'(u_n) = x'(B_p u_n)$

Exactly as in the preceding case of the theorem, $B_p u_n$ converges in norm to zero. Hence

$$\|x'(B_p u_n)\| \le \|x'\| \|B_p u_n\| \to 0 \qquad (\text{as } n \to \infty)$$

i.e., $x'(B_p u_n)$ converges in norm to zero. It is also true that $x' > 0$ and $B_p > 0$ from which it follows that $x'(B_p u_n)\downarrow$. These last two facts then imply that $x'(B_p u_n)\downarrow 0$, i.e., $(B_p)'\, x'(u_n)\downarrow 0$. This then gives that $\phi(u_n)\downarrow 0$.

It still remains to show that ϕ is strictly positive, i.e., $u > 0$ implies that $\phi(u) > 0$. To this end let $u > 0$ be given and let $\lambda > 1$ be fixed. Then

$$\begin{aligned}\phi(R(\lambda,T)u) &= (R(\lambda,T))'\phi(u) \\ &= R(\lambda,T')\phi(u) \\ &= \sum_{k=0}^{\infty} (1/\lambda^{k+1})(T'^k \phi(u)) \qquad (\text{since } r(T) = r(T'))\end{aligned}$$

$$= \sum_{k=0}^{\infty} (1/\lambda^{k+1})((T^k)'\phi(u))$$
$$= \sum_{k=0}^{\infty} (1/\lambda^{k+1})\phi(T^k u)$$
$$= \sum_{k=0}^{\infty} (1/\lambda^{k+1})\phi(u)$$
$$= \phi(u)/(\lambda-1) \quad \text{(geometric progression)}.$$

Thus $\phi(u) = (\lambda-1)\ \phi(R(\lambda,T)u)$.

However because $0 \le TR(\lambda,T) \le \lambda R(\lambda,T)$, T maps the ideal generated by $R(\lambda,T)u$ into itself. Since T is σ-order continuous, T then maps the band generated by $R(\lambda,T)u$ into itself. By hypothesis, T is band irreducible and it therefore follows that $B(R(\lambda,T)u) = \{0\}$ or $B(R(\lambda,T)u) = E$. By corollary 3.4 though, $R(\lambda,T)u \neq 0$ and so $B(R(\lambda,T)u) = E$.

It has already been shown that ϕ is positive and non-zero, and hence there exists $v \in E^+$ such that $\phi(v) > 0$. In view of the fact that $E = B(R(\lambda,T)u)$,

$$v = \sup\ \{\ \inf(\ v,nR(\lambda,T)u\ )\ /\ n \in \mathbb{N}\ \}$$

and therefore

$$\phi(v) = \sup\ \{\ \phi(\inf(\ v,nR(\lambda,T)u\ ))\ /\ n \in \mathbb{N}\ \}$$

since ϕ is σ-order continuous.

Since it is true that

$$0 \le \phi(\inf(v,nR(\lambda,T)u)) \le n\phi(R(\lambda,T)u) \text{ for all } n \in \mathbb{N},$$

we cannot have that $\phi(R(\lambda,T)u) = 0$ for then $\phi(v) = 0$ which is a contradiction. Thus $\phi(R(\lambda,T)u) > 0$, and from what was shown above, namely that $\phi(u) = (\lambda-1)\phi(R(\lambda,T)u)$, it follows that $\phi(u) > 0$. It has thus been shown that ϕ is strictly positive.

To prove the result for the general case in which $r(T) > 0$, define $T^* := T/r(T)$ and proceed in the same way as the second part of theorem 4.1.

□

We are now in a position to prove the first major result of this chapter.

Theorem 4.4 (Generalized Jentzsch Theorem)

Let $0 \le T \in \mathcal{L}(E)$ be a σ-order continuous band irreducible operator such that T^k is compact for some $k \in \mathbb{N}$. The following statements then hold:

i) $r(T) > 0$.

ii) The eigenspace of T corresponding to $r(T)$ contains a weak order unit of E.

iii) $r(T)$ is an eigenvalue of T of algebraic multiplicity 1.

Proof

i) Follows from corollary 3.16.

ii) Since T^k is compact, T is a Riesz operator, and from theorem 4.1 there exists $0 \le u \in E$ such that $Tu = r(T)u$. Let $B(u)$ be the band generated in E by u. If $0 \le v \in B(u)$, then $v \in (B(u))_r = \{I_u\}$, where $\{I_u\}$ is the band generated in $Re(E)$ by u. Thus

$$v = \sup \{ \inf(v,nu) \ / \ n \in \mathbb{N} \}$$

and hence

$$Tv = \sup \{ T(\inf(v,nu)) \ / \ n \in \mathbb{N} \}$$

However for all $n \in \mathbb{N}$, $0 \le T(\inf(v,nu)) \le nTu = nr(T)u$ and so $T(\inf(v,nu)) \in I_u \subseteq \{I_u\}$ for all $n \in \mathbb{N}$. It follows that $Tv \in \{I_u\} \subseteq B(u)$. Since v was an arbitrary positive element of $B(u)$, $T(B(u)) \subseteq B(u)$, and since T is band irreducible, $B(u) = E$. (Note that $u \neq 0$.)

iii) Let $N_{r(T)} := \{ z \in E \ / \ Tz = r(T)z \}$, the eigenspace of T corresponding to the eigenvalue $r(T)$. Let $x \in N_{r(T)}$. Then $r(T)x = Tx$, and hence $|r(T)x| = |Tx|$, which implies that $r(T)|x| \le T|x|$.

It follows that if $\phi \in (E')_c$ is a strictly positive linear functional on E satisfying $T'\phi = r(T)\phi$, (note that the existence of such a functional is guaranteed by theorem 4.3), then

$$\begin{aligned} 0 &\le \phi(T|x|-r(T)|x|) \\ &= T'\phi|x|-r(T)\phi|x| \\ &= r(T)\phi|x|-r(T)\phi|x| \\ &= 0, \end{aligned}$$

and because ϕ is strictly positive, $T|x|-r(T)x = 0$, i.e., $|x|$ is an element of the eigenspace corresponding to $r(T)$. The eigenspace $N_{r(T)}$ is clearly a subspace of E, and in view of the fact that $x \in N_{r(T)}$ implies that $|x| \in N_{r(T)}$, it follows that $N_{r(T)}$ is a Riesz subspace of E. Further, using the same method as we did for B(u) in ii), we can show that $B(|x|) = E$ for all $x \neq 0$ in the eigenspace corresponding to $r(T)$.

Thus far we have shown that $N_{r(T)}$ is a Riesz subspace of E in which every non-zero positive element is a weak order unit of E. Hence if $f \in Re(N_{r(T)})$ and $f = f^+-f^-$ where $f^+,f^- \neq 0$, then since $f^+ \perp f^-$, $B(f^+) \perp B(f^-)$, which implies that $E \perp E$. This is clearly not possible so we must conclude that for $f \in Re(N_{r(T)})$, either $f^+ = 0$, or $f^- = 0$, i.e., either $f \geq 0$ or $f \leq 0$. The real part of the eigenspace $N_{r(T)}$ is therefore a linearly ordered Riesz space and is hence one dimensional ([11], II proposition 3.4).

Suppose now that $x \in \{ z \in E \ / \ (r(T)I-T)^{n+1}z = 0 \}$, where $n \in \mathbb{N}$ is fixed. Then if $w := (r(T)I-T)^n x$, it follows that

$$w \in \{ z \in E \ / \ (r(T)I-T)z = 0 \} = N_{r(T)} = (N_{r(T)})_r + i(N_{r(T)})_r$$

where $(N_{r(T)})_r$ is one dimensional.

Consequently there exists $\alpha \in \mathbb{C}$ such that $w = \alpha u$, where u is some element of $N_{r(T)}$. Moreover,

$$\begin{aligned} \phi(w) &= \phi((r(T)I-T)^n)x \\ &= ((r(T)I-T)^n)'\phi(x) \\ &= (r(T)I'-T')^n\phi(x) \\ &= (r(T)I'-T')^{n-1}(r(T)I'-T')\phi(x) \\ &= (r(T)I'-T')^{n-1}(r(T)\phi(x)-T'\phi(x)) \end{aligned}$$

$= 0$

since $r(T)\phi = T'\phi$, i.e., $\phi(\alpha u) = 0$, which implies that $\alpha\phi(u) = 0$ and hence $\phi(u) = 0$. Since ϕ is strictly positive, $u = 0$, and consequently $w = 0$.

We have thus shown that

$$(r(T)I-T)^{n+1}x = 0 \text{ implies } (r(T)I-T)^n x = 0 \text{ for all } n \in \mathbb{N}.$$

It follows immediately that the order of the pole of $R(\lambda,T)$ at $r(T)$ is one (theorem 1.24 ii)). Thus the algebraic multiplicity of $r(T)$, which is the dimension of the space $\{ z \in E \ / \ (r(T)I-T)z = 0 \}$ which we have shown earlier to be one, is one as required.

□

Before attempting the second of our two main theorems, we require a number of preliminary results.

Lemma 4.5

Let E be a complex Banach lattice with weak order unit w (i.e., $\{w\} = E$). Let $U \in Orth(I_w)+iOrth(I_w)$. Then if E is Dedekind σ-complete, or of w is a quasi-interior point of E, U can be uniquely extended to an orthomorphism on E.

Proof

Since $U \in Orth(E)+iOrth(E)$, we can write U as $U = U_1+U_2$, where U_1, $U_2 \in Orth(I_w)$. Furthermore, it is possible to write U_1 as $U_1 = U_1^+-U_1^-$, and U_2 as $U_2 = U_2^+-U_2^-$, where U_i^+ and U_i^- are elements of $(Orth(I_w))^+$, for $i = 1,2$. It is thus sufficient to prove this lemma for the case that U is positive.

If $v \in E^+$, then $v = \sup v_n$, where $v_n := \inf(v,nw)$ for each $n \in \mathbb{N}$. (Note that $v_n \in I_w$ for all $n \in \mathbb{N}$). In view of the fact that $U \in Z(I_w)$, ([11], II theorem 5.3 and [14], corollary 144.3), there exists $\lambda \in \mathbb{R}^+$ such that

$$Uv_n \le \lambda v_n \le \lambda v \quad \text{for all } n \in \mathbb{N}.$$

In the case that E is Dedekind σ-complete, sup (Uv_n) exists and we hence define $Uv := \sup (Uv_n)$. The operator U then satisfies $Uv \le \lambda v$ for all $v \in E^+$, and consequently belongs to Z(E), i.e., U is an orthomorphism on E. Since two orthomorphisms agreeing on an order dense ideal are equal on the whole space, ([14], corollary 140.6), it follows that U is the unique extension to E of $U \in Z(I_w)$.

In the case that w is a quasi-interior point of E, we can extend $U \in Z(I_w)$ to an orthomorphism on E by means of continuity. This is possible since every positive element of E is the norm limit of a sequence of positive elements of I_w. The argument used is similar to that used in the case that E is Dedekind σ-complete.

□

The following lemma is a purely theoretical result concerned with the complexification of a Riesz space, but it is decisive in helping to prove the next theorem which in turn is the key result to a general Frobenius theorem which will be proved shortly.

Lemma 4.6

Let L be an Archimedean and uniformly complete Riesz space and let L+iL be its complexification. Furthermore, let $f := g+ih$, where $g,h \in L$. Suppose that $|f| = |g|$, then $h = 0$.

Proof

If $f' := |g|+i|h|$ then

$$|f| = |f'|, \qquad ([14], \text{theorem } 91.4)$$

$$= \sup\{\, |g|.\cos\theta+|h|.\sin\theta \;/\; 0 \le \theta \le \pi/2 \,\}$$

([14], theorem 91.4).

Since $|f| = |g|$ we have that

$$|g| \ge |g|.\cos\theta + |h|.\sin\theta \quad \text{for all } \theta,\ 0 \le \theta \le \pi/2.$$

which implies that

$$|h|.\sin\theta \le |g|.(1-\cos\theta) \text{ for all } \theta,\ 0 \le \theta \le \pi/2$$

whence

$$|h| \le |g|.(1-\cos\theta)/\sin\theta \text{ for all } \theta,\ 0 < \theta \le \pi/2.$$

It is easily proved that $(1-\cos\theta)/\sin\theta \to 0$ as $\theta \to 0^+$, and consequently $|h| \le 0$. It is then immediate that $h = 0$.

□

Lemma 4.7

Let E be a Banach lattice separated by $(E')_c$, and let $0 < T \in \mathcal{L}(E)$ be a σ-order continuous band irreducible Riesz operator on E with $r(T) = 1$. If λ, $|\lambda| = 1$, is an eigenvalue of T and if u is an eigenvector pertaining to λ, then there exists an injective orthomorphism $U:I_{|u|} \to I_{|u|}$ such that $\lambda T = U^{-1}TU$.

If either E is σ-Dedekind complete or if $|u|$ is a quasi-interior point of E, then U can be extended to E as an injective orthomorphism and $\lambda T = U^{-1}TU$ on E.

Proof

The fact that u is an eigenvector pertaining to λ implies that $Tu = \lambda u$, and in exactly the same way as was done in theorem 4.4, it is possible to show that $|u|$ is an eigenvector of T pertaining to the eigenvalue 1, and that it is a weak order unit in E.

If $0 \le x \in I_{|u|}$, then $x \le n|u|$ for some $n \in \mathbb{N}$, and hence $Tx \le nT|u| = n|u|$. It follows that $I_{|u|}$ is T-invariant.

Also as mentioned before $I_{|u|}$ is an Archimedean f-algebra with multiplicative unit $|u|$.

Define $U:I_{|u|} \to I_{|u|}$ by $Uz := uz$.
From what was said in chapter one, we have that $U \in \mathrm{Orth}(I_{|u|})$, and

$$|Ux| = |ux| = |u||x| = |x| \text{ for all } x \in I_{|u|}.$$

Suppose $u := a+ib$, and $u' := a-ib$. Then since $|u'| = (a^2+b^2)^{\frac{1}{2}} = |u|$, $u' \in I_{|u|}$, and again bearing in mind the preliminary results discussed in chapter one, the mapping $U':I_{|u|} \to I_{|u|}$ defined by $U'z := u'z$ is an orthomorphism on $I_{|u|}$. Furthermore,

$$\begin{aligned} U'Uz &= U'(uz) \\ &= u'(uz) \\ &= (a-ib)(a+ib)z \\ &= |u|^2 z \\ &= z \end{aligned}$$

and from this we deduce that U is injective. We can similarly show that $UU'z = z$ which implies that U is surjective, and consequently $U' = U^{-1}$

It now remains to show that $\lambda T = U^{-1}TU$ on $I_{|u|}$. To this end let $V := \lambda^{-1}U^{-1}TU$. If we can show that $V = T$ we are through.

Although $I_{|u|}$ need not be Dedekind complete, it is still true that $\sup\{ |Vf| \,/\, |f| \le v \}$ exists for all $v \ge 0$ in $I_{|u|}$ and equals Tv. To see this let v be as above. Then as f runs through all elements satisfying $|f| \le v$, then so too does Uf. Indeed, if $|f| \le v$ then

$$|Uf| = |uf| = |u||f| = |f| \le v.$$

Conversely, if $|Uf| \le v$ then letting $g := Uf$ we have that $f = U^{-1}g$, and thus

$$|f| = |U^{-1}g| = |u'||f| = |u||g| = |g| = |Uf| \le v.$$

Hence

$$\begin{aligned} \sup\{ |U^{-1}TUf| \,/\, |f| \le v \} &= \sup\{ |U^{-1}Tg| \,/\, |g| \le v \} \\ &= \sup\{ |Tg| \,/\, |g| \le v \} \\ &\qquad (|U^{-1}Tg| = |u'||Tg| = |Tg|) \\ &= Tv. \end{aligned}$$

(This last equality holds since Tv is clearly an upperbound, and v satisfies $|v| \le v$)

As a consequence of this equality,

$$Tv \geq |U^{-1}TUf| \quad \text{for all } f \text{ satisfying } |f| \leq v.$$

In particular,

$$Tv \geq |U^{-1}TUv|,$$

and since $|\lambda| = 1$,

$$T(|\lambda|v) \geq |U^{-1}TUv|$$

which implies that

$$Tv \geq |\lambda^{-1}U^{-1}TUv|,$$

i.e., $Tv \geq |Vv|$ for all v, $0 \leq v \in I_{|u|}$.

Consider the following:

$$\begin{aligned} V|u| &= \lambda^{-1}U^{-1}TU|u| \\ &= \lambda^{-1}U^{-1}Tu|u| \\ &= \lambda^{-1}U^{-1}Tu \\ &= \lambda^{-1}U^{-1}\lambda u \quad \text{(u is an eigenvector pertaining to } \lambda) \\ &= U^{-1}u \\ &= u'u \\ &= |u|^2 \\ &= |u|. \end{aligned}$$

Thus if $V = V_1+iV_2$ where V_1,V_2 are real, then $|u| = V|u| = V_1|u|+iV_2|u|$ and $V_2|u|$ must then be zero and $V_1|u| = |u|$. Hence for any $v \geq 0$ in $I_{|u|}$, since V_1v is the real part of Vv, $V_1v \leq |Vv| \leq Tv$ which implies that $T-V_1 \geq 0$ on $I_{|u|}$ and

$$\begin{aligned} (T-V_1)|u| &= T|u|-V_1|u| \\ &= |u|-|u| \\ &= 0. \end{aligned}$$

It then follows that $T = V_1$ on $I_{|u|}$. So $V_1v = Tv \geq |Vv|$ for all $0 \leq v \in I_{|u|}$. However we have already seen that the reverse inequality also holds on $I_{|u|}$, and consequently $V_1v = |Vv|$ for all $v \geq 0$ in $I_{|u|}$. Lemma 4.6 then implies that $V_2v = 0$ for all $v \geq 0$ in $I_{|u|}$.

In view of the above facts it has been shown that $V = V_1 = T$ on $I_{|u|}$ as was required.

It remains to show that these results also hold on E in the two cases mentioned.

Since U and U' are both orthomorphisms on $I_{|u|}$, it is possible to extend U and U' uniquely to orthomorphisms $\bar{U}$ and $\bar{U}'$ respectively on E (lemma 4.5). Using the fact that an orthomorphism is σ-order continuous, ([14], theorem 139.4) and appealing to theorem 20.2 of [9], it is an easy excercise to show that U is bijective on E by proving that $U'U = I$, and $UU' = I$ on E.

To see that the equality $\lambda T = U^{-1}TU$ is valid on E, consider first the case that E is σ-Dedekind complete. As shown before for a positive element v of E, $Uv = \sup v_n$ where $v_n = \inf(v,nw)$ for each natural n. Bearing this in mind and the facts that U,U' and T are all σ-order continuous, and that $\lambda T = U^{-1}TU$ on I_w, we have that

$$\begin{aligned} U^{-1}TU(v) &= U^{-1}TU(\sup v_n) \\ &= U^{-1}T\ (\sup\ (Uv_n)) \\ &= U^{-1}\ (\sup\ (TUv_n)) \\ &= \sup\ (U^{-1}TUv_n) \\ &= \sup\ (\lambda Tv_n) \\ &= \lambda T\ (\sup v_n) \\ &= \lambda Tv. \end{aligned}$$

It is then clear that $U^{-1}TU = \lambda T$ on E as required.

In the case that $|u|$ is a quasi-interior point of E, the claim is easily proved via continuity.

□

And now to the second of our two main results.

Theorem 4.8 (Generalized Frobenius Theorem)

Let $0 < T \in \mathcal{L}(E)$ be a σ-order continuous band irreducible operator on E such that T^k is compact for some $k \in \mathbb{N}$. Let $\lambda_1,\ldots,\lambda_m$ be the different eigenvalues of T satisfying $|\lambda_j| = r(T)$ for $j = 1,\ldots,m$. Then

i) λ_j is a root of the equation $\lambda^m - r(T)^m = 0$ for $j = 1,\ldots,m$,

ii) the eigenvalues $\lambda_1,\ldots\lambda_m$ are all of algebraic multiplicity one,

iii) if either E is Dedekind σ-complete or if T satisfies the additional condition that $0 < T \leq S$, for some compact and σ-order continuous S, then the spectrum $\sigma(T)$ of T is invariant under a rotation of $2\pi/m$, multiplicities included.

Proof

We may assume without loss of generality that $r(T) = 1$.

i) Let α and β be eigenvalues of T satisfying $|\alpha| = |\beta| = 1$, and let u and v be eigenvectors pertaining to α and β respectively. Then $Tu = \alpha u$ and $Tv = \beta v$. As shown before (in theorem 4.4) $|u|$ and $|v|$ are eigenvectors of T pertaining to the eigenvalue 1. It follows from what was discussed in the proof of Jentzsch's theorem, that the dimension of $\{ x \in E \,/\, Tx = x \}$ is 1, and hence $|v|$ is a scalar multiple of $|u|$. We may assume that $|v| = |u|$, for if $|v| = \kappa|u|$, then because $T(\kappa u) = \kappa u$, we may replace the above u by κu.

By theorem 4.7, there exist orthomorphisms U and V on the ideal $I_{|u|} = I_{|v|}$ such that

$$\alpha T = U^{-1}TU \text{ and } \beta T = V^{-1}TV.$$

Hence $T = \alpha^{-1}U^{-1}TU$ and thus $\beta\alpha^{-1}U^{-1}TU = V^{-1}TV$, whence $T = \alpha\beta^{-1}UV^{-1}TVU^{-1}$ on $I_{|u|}$

Let w be any non-zero eigenvector of T pertaining to 1 (i.e., $Tw = w$), then as we showed above with u and v, $I_{|w|} = I_{|v|}$ and so $w \in I_{|v|}$. Consider then the following.

$$T(UV^{-1}w) = \alpha\beta^{-1}(VU^{-1})^{-1}T(VU^{-1})UV^{-1}w$$

(Note that since $w \in I_{|v|}$ it does make sense to consider $V^{-1}w$).

Thus $T(UV^{-1}w) = \alpha\beta^{-1}(VU^{-1})^{-1}Tw$
$= \alpha\beta^{-1}(UV^{-1})w,$
which shows that $UV^{-1}w$ is an eigenvector of T pertaining to $\alpha\beta^{-1}$. Thus $\alpha\beta^{-1}$ is an eigenvalue of T satisfying $|\alpha\beta^{-1}| = |\alpha||\beta^{-1}| = 1$. Consequently $\{ \lambda_1,\ldots,\lambda_m \}$ is a multiplicative group, and the theorem of Lagrange then implies that $\lambda_j^m = 1$, for $j = 1,\ldots,m$. This proves i).

In view of this result we may hence assume that $\lambda_j = e^{2\pi ij/m}$.

ii) From the operator equality, $T-\alpha\beta^{-1}I = \alpha\beta^{-1}(VU^{-1})^{-1}(T-I)(VU^{-1})$ which holds on $I_{|u|}$, we conclude that the dimension of the null-space of $T-\alpha\beta^{-1}I$ contained in $I_{|u|}$ is equal to the dimension of the null-space of T-I, (consider the bijection from the null-space of $(T-\alpha\beta^{-1}I)$ to that of (T-I) which maps the element x to the element $VU^{-1}x$), which has already been shown to be one. However, the null space of $(T-\alpha\beta^{-1}I)$ is entirely contained in $I_{|u|}$. (In order to see this, it has already been proved above that if w is an eigenvector pertaining to the eigenvalue λ, $|\lambda| = 1$, then $w \in I_{|u|}$ and the result follows.) Therefore the dimension of the null-space of $(T-\alpha\beta^{-1}I)$ is one. Replacing β by 1, and replacing α successively by $\lambda_1,\ldots,\lambda_m$ yields that the null-space of $(T-\lambda_jI)$ has dimension one, for each $j = 1,\ldots,m$.

The inequality $|(\lambda-\lambda_j)^m R(\lambda,T)x| \le |\lambda-\lambda_j|^m R(|\lambda|,T)|x|$ which holds for all λ, $|\lambda| > 1$, shows that the order of the pole λ_j of $R(\lambda,T)$ is less than or equal to the order of the pole 1 of $R(\lambda,T)$. Hence each λ_j is a pole of order one of the resolvent operator.

This conclusion together with the result proved just before it implies that the algebraic multiplicity of each λ_j is 1 for $j = 1,\ldots,m$, ([3], theorem 3.14). This completes the proof of ii).

iii) Suppose E is Dedekind σ-complete; and define $\alpha := e^{2\pi i/m}$. By theorem 4.7 there exists an orthomorphism U on E such that $\alpha T = U^{-1}TU$. Thus

$$(T-\alpha\lambda I)^k = \alpha^k U(T-\lambda I)^k U^{-1} \text{ for all } k \in \mathbb{N} \text{ and for all } \lambda.$$

It follows that $\lambda \in \sigma(T)$ if and only if $\alpha\lambda \in \sigma(T)$ and $\alpha\lambda$ has the same multiplicities (both algebraic and geometric) as λ, which proves iii) for this case.

Finally if $0 < T \le S$, where S is compact and σ-order continuous, then by the well known theorem of Aliprantis and Burkinshaw ([14], theorem 124.4) T^3 is compact. Let F denote the closed ideal generated by T(E), the image of E under T, in E. Let $\lambda \in \sigma(T)$; the binomial expansion of $(T-\lambda I)^k$ yields

$$(T-\lambda I)^k = T^k + \lambda k T^{k-1} + \lambda^2 \binom{k}{2} T^{k-2} + \ldots + \lambda^k$$

and hence the null-space of $(T-\lambda I)^k \subseteq T(E) \subseteq F$, for if $x \in E$ and $(T-\lambda I)^k x = 0$ then

$$T^k x + \lambda k T^{k-1} x + \ldots + \lambda^k x = 0$$

which implies that

$$T(\lambda^{-k} T^{k-1} x + \lambda^{1-k} T^{k-2} x + \ldots + \lambda^{-1} kx) = x$$

and thus $x \in T(E) \subseteq F$. We then infer that
$\{ x \in E \ / \ (T-\lambda I)^k x = 0 \} \subseteq T(E)$.

Thus in our study of $\sigma(T)$ we may restict T to F. From lemma 3.12 it follows that T restricted to F is ideal irreducible and so if u is an eigenvector pertaining to α, $T|u| = |u|$ (see the proof of theorem 4.4 iii)). This equality implies that $|u|$ is a quasi-interior point of F, for if $0 \le x \in I_{|u|}$ then there exists an $n \in \mathbb{N}$ such that $0 \le x \le n|u|$. Thus $0 \le Tx \le nT|u| = n|u|$ which implies that $Tx \in I_{|u|}$, and so $I_{|u|}$ is T-invariant, and hence $\overline{I_{|u|}} = F$. (Note that $I_{|u|} \ne \{0\}$ because $|u| \ne 0$).

Now on F the operator T satisfies all the conditions of theorem 4.7 (note that an ideal irreducible operator in this case is also band irreducible, since in a normed Riesz space every band is closed).

Thus there exists an orthomorphism U of F such that $\alpha T = U^{-1}TU$. As above this implies that for all $\lambda \in \sigma(T)$, $\alpha\lambda \in \sigma(T)$ and $\alpha\lambda$ has the same multiplicities as λ.

□

Corollary 4.9

Let $0 < T \in \mathcal{L}(E)$ be a σ-order continuous band irreducible operator on E such that T^k is compact for some $k \in \mathbb{N}$. If T has the additional property that Tv is a weak order unit of E for every $0 < v \in E$, then the only eigenvalue λ of T satisfying $|\lambda| = r(T)$ is the number $r(T)$ itself.

Proof

By Jentzch's theorem, $r(T) > 0$. Assume that there exists $\lambda \in \sigma(T)$ such that $\lambda \neq r(T)$ and $|\lambda| = r(T)$. Let u be an eigenvector pertaining to λ. As shown in earlier proofs, $|u|$ is then an eigenvector pertaining to the eigenvalue $|\lambda| = r(T)$. Since $\lambda \neq r(T)$, u and $|u|$ are linearly independent. To see this linear independence, suppose that $u = \kappa|u|$. Then $Tu = \kappa T|u|$ which yields $\lambda u = \kappa r(T)|u|$, i.e., $\lambda u = r(T)u$. Obviously then, $\lambda = r(T)$ which is a contradiction.

By the generalised Frobenius Theorem we see that there exists an $m \in \mathbb{N}$ such that $\lambda^m = (r(T))^m$. It is then clear that T^m satisfies all the conditions of theorem 4.4 and $r(T^m) = (r(T))^m$

The eigenspace pertaining to $(r(T))^m$ then has dimension one, contradicting the fact that both u and $|u|$ belong to this eigenspace and are linearly independent. Thus any eigenvalue λ satisfying $|\lambda| = r(T)$ can only be $r(T)$.

□

Thus there exists an orthomorphism U of F such that $\alpha T = U^{-1}TU$. As above this implies that for all $\lambda \in \sigma(T)$, $\alpha\lambda \in \sigma(T)$ and $\alpha\lambda$ has the same multiplicities as λ.

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The eigenspace pertaining to $(r(T))^m$ then has dimension one, contradicting the fact that both u and $|u|$ belong to this eigenspace and are linearly independent. Thus any eigenvalue λ satisfying $|\lambda| = r(T)$ can only be $r(T)$.

□

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AND

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