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The life and mathematics of Ismail Mohamed

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Abstract
Ismail Mohamed's major contributions, which were in collaboration with Hermann Heineken, was to provide a procedure for constructing groups with prescribed characteristics. In particular, they constructed examples of non-nilpotent groups in which every subgroup is subnormal and nilpotent. These have become known as the Heineken–Mohamed groups. This construction led to settling a few questions posed, in the 1940s, by Kurosh and Cernikov in their survey of various generalisations of nilpotency. He also studied properties of series of subgroups of a group G that are constructed from arbitrary subgroups of automorphisms of a group.

Keywords Heineken–Mohamed group · Nilpotency · Normaliser · Subnormal · Metabelian

MSC Codes 20 03; History of group theory · 01A70; Biographies, obituaries, personalia, bibliographies

1 Background

On the occasion of the centenary of the Royal Society of South Africa in 1977, the Council of the Society published a collection of essays in the form of a book, "A History of Scientific Endeavour in South Africa" Brown and (Ed) [2] on the history of science in South Africa. In the Preface Brown and (Ed) [2], the Editor Professor A.C. Brown remarks:

This paper is one of the two articles in which we focus on Black South Africans who managed to complete their PhDs, who pursued research, and published in mathematics, despite the formidable odds. For completeness sake, we include the background of both articles in the first section.

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> "The problem of what fields to select for inclusion to some extent resolved itself for, while all the authors whose essays appear here responded immediately and enthusiastically, no one could be found to write essays on the mathematical sciences, on inorganic and theoretical chemistry or several other branches of science. The Council was also of the opinion that the volume should to some extent reflect the activities of the Society, so that priority would be given to the biological sciences, including palaeontology".

Thus this volume excludes the story of the evolution of South African mathematical research. The South African Institute of Physics (SAIP) commissioned "a more recent history of South Africa physics, coinciding more-or-less with the lifespan of the SAIP, and picking up approximately at the point where the Brown-book left off" [15, p. 1]. The story of the evolution of the mathematical sciences research landscape in South Africa has yet to be written. There would be a wide range of topics to be covered which may fill more than one volume and such themes would also depend on the interests, expertise, and insights of mathematicians and historians with an interest in such a project.

In Kikianty and Nongxa [13], we looked at the South African mathematics research landscape for the period 1910 to 1961 which is the period which starts with the Union of the two British colonies and two Boer Republics to form the Union of South Africa in 1910 up to the establishment of the Republic of South Africa in 1961. This covers the period from 1948 when the apartheid regime came into power and legalised segregation in all spheres of South African life including separate universities for different racial groups and the unequal provision of resources for South Africans of different races to participate in and contribute to the scientific endeavour in the country. In particular the architect of apartheid, Hendrik Verwoerd (1901–1966) once infamously stated [24, p. 310]:

> "There is no place for the Bantu in the European community above certain forms of labour. What is the use of teaching the Bantu child mathematics when it cannot use it in practice? That is quite absurd. Education must train people in accordance with their opportunities in life, according to the sphere in which they live."

In the books that have been written about the evolution of scientific ideas and activities in South Africa, unsurprisingly the traditionally marginalised group in general, and Black South Africans and women, in particular, are conspicuously invisible. It is natural to explore what has been the involvement of these groups to the scientific endeavour in South Africa as well as their intellectual contributions.

The choice of this theme has been inspired by the book "Beyond Banneker: Black Mathematicians and the Paths to Excellence" by Erica N. Walker which states [25, p. 2]

> "Works focusing on the formative and educational experience of mathematicians are relatively rare. Within these works, Black mathematicians are often absent altogether or represented by one or two individuals."

Very little is known, even in South African mathematical circles, about the early Black South Africans who completed their PhDs during the dark days of apartheid. Furthermore, almost nothing has been written about their mathematical contributions.

The first two Black South Africans to be awarded doctorates in mathematics were the late Dr. Joseph Albert Mashite Mokoena (1919–1969) and the late Professor Ismail Jacob Mohamed (1930–2013). Mokoena received his PhD in 1958 from the University of the Witwatersrand under the supervision of James Hyslop (1908–1984). Mohamed received his PhD in 1960 from the University of London under the supervision of Kurt Hirsch (1906–1986).

Both Mokoena and Mohamed became political activists in opposition to the apartheid regime and its policies. Both were honoured posthumously by the new South African democratically elected government. Although it is understandable that citations for these honours would not give details about their mathematical contributions, on reading these, one finds greater emphasis being placed on their political activism and their mathematical contributions given a cursory attention.

Our main objective is to address these omissions by giving an overview of the intellectual contributions of these two first Black PhDs in mathematics from South Africa. In this part of the two articles, we cover the life and the mathematics of Ismail Jacobus (Josef) Mohamed.[1]

2 Biography of Ismail Mohamed

Ismail Jacob Mohamed (1930–2013) was the second black South African to receive a doctoral degree in mathematics. Our main source for Mohamed's biography is the entry on the MacTutor History of Mathematics[2] O'Connor and Robertson [22].

2.1 Early life and education

Mohamed was born on 27 July 1930, in Barkly East, Eastern Cape, where his mother worked as a cook for white families. In the late 1940s, to earn money for school fees, he worked for white families. One of the white families, realising his mathematical talent, had him teach mathematics to their children. He worked as a painter and builder's mate in 1950 and finally earned enough to enrol at the University of the Witwatersrand (WITS) in 1951. He graduated BSc in mathematics and physics in 1953, and received a First Class Honours Degree in mathematics in 1954. Working as a teacher at William Hills High School since 1955, he also enrolled for MSc at WITS (part-time). Finally he was awarded a Shell Postgraduate Bursary [19, p. 711] to pursue his doctoral degree at the University of London in the UK O'Connor and Robertson [22].

2.2 Work with Heineken

Together with Hermann Heineken (b. 1936), he proved the following theorem: "For each prime p there is a metabelian p-group with trivial centre, such that every proper subgroup of is nilpotent and subnormal". This paper had a major impact on the development of group

[1] The other article covers the life and the mathematics of Joe Mokoena Kikianty and Nongxa [14].

[2] MacTutor is a free online resource containing biographies of more than 3000 mathematicians and over 2000 pages of essays and supporting materials. MacTutor was created and is maintained by Edmund Robertson and John O'Connor of the School of Mathematics and Statistics at the University of St Andrews, and is hosted by the University.

theory. Such a group is referred to as the Heineken–Mohamed type group. We discuss Mohamed's mathematical work in the final section. Mohamed published two papers based on his doctoral dissertation in 1963 and 1964, and four papers with Herman Heineken in the period of 1968–1973. After 1974, there are no further papers in mathematics by Mohamed.

2.3 Activism

The following is a quote written to Mohamed when he applied for a teaching position at the University of the Witwatersrand in 1974:

> "One can see that to appoint you in a permanent position of authority over white students and junior staff would be to wound the very heart of baaskap [white supremacy] and that there are limits to which we can go to offend a government."

Mohamed returned to South Africa in 1960, and he continued his fight against apartheid. His academic career was affected by his activism, for instance he lost his position as the Head of the Department of Mathematics at the University of the Western Cape after he delivered an address at the South African Students Organisation arguing against apartheid education, in response to the Soweto uprising in 1976.

Roseinnes Phahle[3] stated that "Ismail never had a problem of finding employment in universities in UK or USA" Phahle [23]. Following is an excerpt of the letter that Mohamed sent to Phahle in 1980, during his employment at the University of the Witwatersrand as an Associate Professor of Mathematics.

> I had an offer of a post at York University from D. [Solitar][4] but decided not to take this up because of various ties here. Gilbert Baumslag[5] visited here in December 1979/January 1980 and offered to arrange a position for me if ever I want to move.
>
> An AMS[6] member from Howard College, Washington DC, spoke to me about arranging a Maths position for me at Howard. The NAS[7] called the other day to say they can arrange a Maths position for me at Brown University, Providence R.I. This is one of the top Ivy League colleges. I am sorely tempted.

[3] Like Mohamed, Phahle studied at WITS and graduated with BSc degree in Pure and Applied Mathematics. Upon his return from exile, he taught mathematics at Vista University, launched the non-profit Dipalo School of Information and Communication Technology in Soweto, collaborated with researchers at the Women's Development Foundation and acted as a statistical advisor on a project located in the Office of the President.

[4] Donald Solitar (1932–2008) was a Canadian mathematician working in combinatorial group theory.

[5] Gilbert Baumslag (1933–2014) graduated from the University of the Witwatersrand (WITS) with a BSc Honours and DSc (unknown year) [22]. The Mathematics Genealogy Project recorded his second PhD under the supervision of Bernard Neumann from the University of Manchester. He worked on group theory, and the use of mathematical research for the reconstitution of learning systems, game theory, and security protocols [22]. The Baumslag–Solitar group was named after his contribution with co-author Donald Solitar (1932–2008), whom was referred to in this excerpt by Mohamed, published in 1962.

[6] American Mathematical Society

[7] National Academy of Sciences

> Phahle further stated that Mohamed declined job offers, because he wanted to be where the struggle was: in South Africa. In the letters he wrote to Phahle [23], Mohamed wrote of his conflict:As usual I am torn between mathematics and making a worthwhile contribution in resolving the problems at home[8];

and Mohamed always chose the latter. He was arrested along with other leaders of the United Democratic Front in 1985 and was released in 1986. Mohamed was elected as an African National Congress (ANC) Member of Parliament and served three terms as a member of parliament before retiring in 2009.

Mohamed was awarded posthumously the Order of Mapungubwe in Silver in 2014 and the reference to his mathematical contributions in his citation reads "Prof Mohamed with Professor Heineken solved an outstanding problem in Group Theory and stimulated new areas of mathematics research". He also received an honorary doctorate from the University of the Witwatersrand and the citation reads "It was during this period that his well-known and highly respected work with Hermann Heineken was published. One of their papers involved a major problem of group theory of more than twenty years standing. Other joint work followed. They constructed a collection of counter examples now internationally known and cited as the Heineken–Mohamed groups. As a result of their work, several important mathematical conjectures in group theory were resolved".

3 Mathematical contributions of Ismail Mohamed

3.1 Background

Ismail Mohamed was a group theorist whose contributions are on solvability and nilpotency of groups and related themes. In this paper we provide an overview of his mathematical contributions that should be accessible to any mathematician who has done a group theory course that introduces the fundamentals of solvable and nilpotent groups. His major contribution, which was in collaboration with Heineken, was to provide a procedure for constructing groups with prescribed characteristics. Specifically, Heineken and Mohamed constructed, for an arbitrary prime p, examples of non-nilpotent p-groups in which every proper subgroup is subnormal and nilpotent. These have become known as the *Heineken–Mohamed groups*. This construction led to settling a number of questions posed, in the 1940s, by Kurosh and Cernikov [16] in their survey of various generalisations of nilpotency. Mohamed also introduced and studied properties of generalised central series of subgroups of a group G that are constructed from arbitrary subgroups of the automorphism group of G.

We first give a general overview of some of the concepts studied in Mohamed's first two papers which were from his University of London PhD thesis. Let G be a group and A be any arbitrary subgroup of $\mathrm{Aut}(G)$, the group of automorphisms of G. For any $x \in G$ and any $\alpha \in A$, denote by $[x, \alpha]$ the element $x^{-1}\alpha(x)$ of G and note that if α is any inner automorphism of G then $\alpha(x) = g^{-1}xg$ for some $g \in G$ and so $[x, \alpha] = x^{-1}g^{-1}xg = [x, g]$, the *commutator of x and g*. The element $[x, \alpha]$, for an arbitrary $x \in G$ and any $\alpha \in A$, is called

[8] Here, 'home' means South Africa. This excerpt was taken from his letter dated 12 May 1987 which he wrote during his visit to the City University of New York.

the commutator of x and α. For any subgroup H of G, denote by $[H, A]$ the subgroup of G generated by $\{[x, \alpha] : x \in H \text{ and } \alpha \in A\}$.

A series

$$G = H_0 \geq H_1 \geq \cdots \geq H_k \geq \ldots \tag{1}$$

of subgroups of G will be called a *descending A-central series* if, for every $k \geq 1$, $[x, \alpha] \in H_k$, for all $\alpha \in A$, and for every $x \in H_{k-1}$. This is equivalent to the condition that $\alpha(xH_k) = xH_k$ for all $\alpha \in A$, and for every $x \in H_{k-1}$.

Similarly, a series

$$\{i_G\} = G_0 \leq G_1 \leq \cdots \leq G_k \ldots \tag{2}$$

of subgroups of G will be called an *ascending A-central series* if, for every $k \geq 1$, $[x, \alpha] \in G_{k-1}$, for all $\alpha \in A$, and for every $x \in G_k$. This is equivalent to the condition that $\alpha(xG_{k-1}) = xG_{k-1}$ for all $\alpha \in A$, and for every $x \in G_k$. Hence, if (1) is a normal series of G and A is the subgroup of inner automorphisms of G, then (1) is a descending central series of G. Similarly, if (2) is a normal series of G and A is the subgroup of inner automorphisms of G, then (2) is an ascending central series of G.

For the subgroup A of the automorphism group $\mathrm{Aut}(G)$ of G, one can define a descending series $(\Gamma_k^A(G) : k = 0, 1, 2, \ldots)$ of subgroups of G, recursively as follows:

$$G = \Gamma_0^A(G) \geq \cdots \geq \Gamma_k^A(G) = [\Gamma_{k-1}^A(G), A] \geq \ldots \tag{3}$$

Analogously for the subgroup A of $\mathrm{Aut}(G)$ one can define recursively an ascending series of subgroups $Z_k^A(G)$ of G for $k = 0, 1, \ldots$ as follows:

$$Z_0^A(G) = \{i_G\} \leq \cdots \leq Z_k^A(G) = \langle \{g \in G : [g, \alpha] \in Z_{k-1}^A(G) \text{ for all } \alpha \in A\} \rangle \leq \ldots \tag{4}$$

By definition, both (3) and (4) are A-central series of G.

We show by induction on k that, if (1) is a descending lower A-central series of A, then $\Gamma_k^A(G) \leq H_k$ for all k, the conclusion being trivially true if $k = 0$. Assume that $\Gamma_{k-1}^A(G) \leq H_{k-1}$ and consider a generator $[x, \alpha]$ of $\Gamma_k^A(G)$, where $x \in \Gamma_{k-1}^A(G)$ and $\alpha \in A$. Since $\alpha(xH_k) = xH_k$, then $\alpha(x) = xh$ for some $h \in H_k$, and therefore $[x, \alpha] = x^{-1}\alpha(x) = h \in H_k$. Thus, every generator of $\Gamma_k^A(G)$ is in H_k which implies that $\Gamma_k^A(G) \leq H_k$. Since, for all $k \geq 0$, we have $\Gamma_k^A(G) \leq H_k$ and, for all $x \in \Gamma_{k-1}^A(G)$ and every $\alpha \in A$, we have $[x, \alpha] \in \Gamma_k^A(G)$; and the series (3) is called a *lower A-central series of G*.

We now show that for each $k \geq 0$, $\Gamma_{k+1}^A(G)$ is a normal subgroup of $\Gamma_k^A(G)$. Let x be any element of $\Gamma_k^A(G)$ which means that $[x, \alpha]$ be an arbitrary generator of $\Gamma_{k+1}^A(G)$. For any g element of $\Gamma_k^A(G)$, we want to show that $g^{-1}[x, \alpha]g$ belongs to $\Gamma_{k+1}^A(G)$. Since $\alpha(x\Gamma_{k+1}^A(G)) = x\Gamma_{k+1}^A(G)$ for all x in $\Gamma_k^A(G)$, then both $[xg, \alpha]$ and $[g, \alpha]$ belong to $\Gamma_{k+1}^A(G)$. It implies that

$$[xg, \alpha][g, \alpha]^{-1} = g^{-1}x^{-1}\alpha(x)\alpha(g)\alpha(g)^{-1}g = g^{-1}[x, \alpha]g$$

also belongs to $\Gamma^A_{k+1}(G)$. Thus, every conjugate of a generator of $\Gamma^A_{k+1}(G)$ by an element of $\Gamma^A_k(G)$ belongs to $\Gamma^A_{k+1}(G)$ which means that $\Gamma^A_{k+1}(G)$ must be a normal subgroup of $\Gamma^A_k(G)$. In particular $\Gamma^A_1(G)$ is a normal subgroup of G.

From the definition of the subgroups $Z^A_k(G)$ it is obvious that $\alpha(xZ^A_{k-1}(G)) = xZ^A_{k-1}(G)$ for all $x \in Z^A_k(G)$. We prove by induction on k that for each $k \geq 0$, the term G_k in (2) is contained in $Z^A_k(G)$ the conclusion being true for $k = 0$. Assume that $G_{k-1} \leq Z^A_{k-1}(G)$ and let $g \in G_k$ and note that for all $\alpha \in A$, we have that $[g, \alpha] \in G_{k-1}$. By induction hypothesis $G_{k-1} \leq Z^A_{k-1}(G)$ and therefore $[g, \alpha] \in Z^A_{k-1}(G)$ which means that $g \in Z^A_k(G)$. Since, for all $k \geq 0$, we have $G_k \leq Z^A_k(G)$ and $\alpha(xZ^A_k(G)) = xZ^A_k(G)$, for all x in $Z^A_{k+1}(G)$ and all $\alpha \in A$, the series (4) is called an *upper A-central series of G*. It is easy to show that if A is the group of inner automorphisms of G the series (3) is simply the *lower central series of G* and the series (4) will be the upper central series of G. In this case, $\Gamma^A_k(G)$ will simply be denoted by $\Gamma_k(G)$ and $Z^A_k(G)$ by $Z_k(G)$.

For an arbitrary subgroup A of the group of automorphisms of G, the lower and upper A-central series of G were defined in Mohamed's papers Mohamed [19] and Mohamed [20].

3.2 Nilpotency class of the stability group of a finite subgroup chain

Mohamed published two papers from his PhD dissertation, both dealing with the properties of the terms, as well as corresponding quotients, of a lower A-central series of a group G where A is a subgroup of $\mathrm{Aut}(G)$.

In Kaloujnine [12], Kaloujnine considered a descending sequence

$$G = H_0 \geq H_1 \geq \cdots \geq H_m = \{i_G\}$$

and $A = \{\alpha \in \mathrm{Aut}(G) : \alpha(xH_k) = xH_k \text{ for all } x \in H_{k-1} \text{ and } k = 1, 2, \ldots, m\}$, called the *stability subgroup* of the sequence. Since $\Gamma^A_m(G) \leq H_m = \{i_G\}$, G is called *A-nilpotent of A-class n* where n is the smallest integer such that $\Gamma^A_n(G) = \{i_G\}$. He proved that if G is A-nilpotent of A-class n, then A is always solvable of derived length at most $n - 1$. He showed further that if each $\Gamma^A_k(G)$ is a normal subgroup of G, then A is nilpotent of class not exceeding $n - 1$. Kaloujnine conjectured that each $\Gamma^A_k(G)$ is normal in G. In Hall [4] and Hall [5], Hall proved that, if G is A-nilpotent of A-class n, then A is always nilpotent of class not exceeding $n(n-1)/2$ and gave an example of a group G and a subgroup A of $\mathrm{Aut}(G)$, such that G is A-nilpotent of A-class 3 and A is nilpotent of class 3. It implies that in this example, $\Gamma^A_2(G)$ is not normal in G, otherwise from Kaloujnine's theorem, A would be nilpotent of class 2.

In Mohamed [19], Mohamed investigated necessary and sufficient conditions for $\Gamma^A_k(G)$ to be a normal subgroup of G for each $k \geq 2$ and also studied the nilpotency of the factors $\Gamma^A_k(G)/\Gamma^A_{k+1}(G)$ in the case where G is A-nilpotent of A-class n. He proved the following theorem.

Theorem 1 *Let G be a group and A any subgroup of $Aut(G)$, if $\Gamma^A_k(G)$ is a normal subgroup of G, then $\Gamma^A_{k+1}(G)$ will be a normal subgroup of G if and only if $[\Gamma^A_k(G), \Gamma^A_1(G)] \leq \Gamma^A_{k+1}(G)$.*

Since we have already observed that $\Gamma_1^A(G)$ is a normal subgroup of G, then $\Gamma_2^A(G)$ is normal in G if and only if $[\Gamma_1^A(G), \Gamma_1^A(G)]$ is a subgroup of $\Gamma_2^A(G)$, etc. Thus $\Gamma_k^A(G)$ is a normal subgroup of G for all k if and only if $[\Gamma_k^A(G), \Gamma_1^A(G)]$ is a subgroup of $\Gamma_{k+1}^A(G)$ for every $k \geq 1$. From the theorem, he was able to deduce a number of observations about the properties of the subgroups $\Gamma_k^A(G)$ in the event that G is A-nilpotent of A-class n. Namely that $\Gamma_{n-1}^A(G)$ will be an abelian group; that $\Gamma_k^A(G)$ is a normal subgroup of G for $k = 1, 2, \ldots, n-1$, if and only if $\Gamma_1^A(G)$ is nilpotent; and that $\Gamma_{k+1}^A(G)$ is a normal subgroup of $\Gamma_{k-1}^A(G)$ for every $k = 2, 3, \ldots, n-2$, if and only if $\Gamma_1^A(G)$ is solvable.

Mohamed gave a counterexample to Kaloujnine's conjecture referred to above. He constructed, for every $n \geq 3$, a group G and a subgroup A of $\mathrm{Aut}(G)$ where G is A-nilpotent A-class n, with $\Gamma_k^A(G)$ not normal in G for every $k \geq 2$. The group he constructed is a homomorphic image of a direct product of two copies of a dihedral group of order $2n$.

3.3 The quotients $\Gamma_k^A(G)/\Gamma_{k+1}^A(G)$

For elements x and y of a group G, and for $n \geq 1$, define the sequence $e_n[x, y]$ recursively as follows:

$$e_1[x, y] = [x, y] = x^{-1}y^{-1}xy, \ldots, \quad \text{and} \quad e_{n+1}[x, y] = [e_n[x, y], y], \ldots.$$

An element g of G is called a *left Engel element* if, for all $x \in G$, there exists an integer $n = n(g, x)$, depending on x, such that $e_n[x, g] = i_G$. If this n does not depend on x (i.e. for all x in G, we have $e_n[x, g] = i_G$), then g is called an *n-Engel element of G*. If every element of G is an Engel element, then G is called an *Engel group*; and if for a fixed n, every element of G is an n-Engel element, then G is called an *n-Engel group*. For example, a 1-Engel element of G belongs to $Z(G)$ and vice versa; and G is a 1-Engel group if and only if G is an abelian group.

In his investigation of the quotients $\Gamma_k^A(G)/\Gamma_{k+1}^A(G)$ for an A-nilpotent group G of A-class n, Mohamed showed that, for $1 \leq k \leq n-1$, the set of generators $[g, \alpha]\Gamma_{n-k+1}^A(G)$ of the group $\Gamma_{n-k}^A(G)/\Gamma_{n-k+1}^A(G)$, with $g \in \Gamma_{n-k-1}^A(G)$ and $\alpha \in A$, are k-Engel elements of the group. He also showed that every element of $\Gamma_1^A(G)$ is a left Engel element of G.

In connection with the upper A-central series of G, the subgroups $Z_k^A(G)$ are also not necessarily normal in G and Mohamed gave necessary and sufficient conditions for each $Z_k^A(G)$ to be normal in G for all $k \geq 1$.

Theorem 2 *Let G be any group and A be a subgroup of $Aut(G)$. If $Z_{k-1}(G)$ is a normal subgroup of G, then $Z_k(G)$ is normal in G if and only if $[Z_k(G), \Gamma_1^A(G)]$ is a subgroup of $Z_{k-1}(G)$.*

As is well-known, if $A = \mathrm{Inn}(G)$, the inner automorphism group of G, then $Z_n^A(G) = G$ for some positive integer n if and only if $\Gamma_n^G(G) = \{i_G\}$. Mohamed constructed a group G and a subgroup A of $\mathrm{Aut}(G)$ with $Z_2^A(G) = G$ but $\Gamma_n^G(G) \neq \{i_G\}$ for all n.

In Mohamed [20], Mohamed studied the nilpotency class of a stability group of a finite chain of subgroups of a group G. We recall that if

$$G = H_0 \geq H_1 \geq \cdots \geq H_n = \{i_G\}$$

is a finite chain of subgroups of G, the *stability group* of the series is, say $A = \{\alpha \in \mathrm{Aut}(G) : \alpha(xH_i) = xH_i \text{ for all } x \in H_{i-1} \text{ for every } i \geq 1\}$. Any subgroup of the stability group of the series is said to stabilise the sequence. Since $\Gamma_k^A(G) \leq H_k$ for every k, then G is A-nilpotent of A-class at most n. Hall [4, 5] proved that A is actually nilpotent of class at most $n(n-1)/2$. Mohamed proved the following theorem which encompasses Hall's and Kaloujnine's theorems as special cases.

Theorem 3 *Suppose G is A-nilpotent of A-class n. Then*

1. $[\Gamma_{n-i}^A(G), \Gamma_k(A)] \leq \Gamma_{n-i+j+1}^A(G)$, *where* $0 \leq j \leq i-1 \leq n-1$, *with* $k = \binom{i-1}{2} + j$, $\Gamma_k(A)$ *the k-th term of the lower central series of A.*
2. *If each* $\Gamma_k^A(G)$ *is normal in G, then* $[\Gamma_i^A(G), \Gamma_j(A)] \leq \Gamma_{i+j+1}^A(G)$ *for all* $0 \leq i, j \leq n$.

If one puts $i = n$ and $j = n-1$ in (a), then $k = n(n-1)/2$ and $[\Gamma_0^A(G), \Gamma_{n(n-1)/2}(A)] \leq \Gamma_n^A(G) = \{i_G\}$ which implies that $\Gamma_{n(n-1)/2}(A) = \{i_G\}$ and therefore A is nilpotent of class at most $n(n-1)/2$ which gives Hall's upper bound for the nilpotency class of the subgroup A. If one puts $i = 0$ and $j = n-1$ in (b), then $[\Gamma_0^A(G), \Gamma_{n-1}(A)] \leq \Gamma_n^A(G) = \{i_G\}$ which implies $\Gamma_{n-1}(A) = \{i_G\}$ which gives Kaloujnine's upper bound for the nilpotency class of A under the assumption that all the terms of the lower A-central series are normal in G.

3.4 Generalisations of nilpotency

In Baer [1], Baer provides a list of equivalent conditions for finite nilpotent groups, and he also points out that if the group is not assumed to be finite, then in general these conditions are no longer equivalent. Recall that a subgroup H of a group G is said to be *subnormal* if there exists a finite chain $H = H_0 < H_1 < \cdots < H_n = G$ such that H_k is a normal subgroup of H_{k+1} for $k = 0, 1, \ldots, n-1$. If n is the length of the shortest such chain, then H is said to be of *defect n*. If G is an arbitrary group such that for every proper subgroup H of G, we have that $H < N_G(H)$, then G is said to *satisfy the normaliser condition*. The class of all groups G such that every subgroup of G is subnormal is called the class of N_1 *-groups*. It is obvious that every N_1-group satisfies the normaliser condition. A finite group is nilpotent if and only if it satisfies the normaliser condition and all finite p-groups satisfy the normaliser condition.

3.5 Heineken–Mohamed groups

If p is prime, a p-group G is said to be a *Heineken–Mohamed group* if it is non-nilpotent, and every proper subgroup is nilpotent and subnormal in G. Thus Heineken–Mohamed groups satisfy the normaliser condition. In their seminal paper Heineken and Mohamed [9] gave the first examples of such groups. They proved the following result.

Theorem 4 *For each prime p there is a metabelian*[9] *p-group G with trivial centre,*[10] *such that every proper subgroup of G is nilpotent and subnormal.*

They showed that if G satisfies the theorem above, then

1. G is countable;
2. G/G' is isomorphic to the Prufer p-group[11]
3. for every $H \leq G$, we have $G'H = G$ if and only if $H = G$;
4. if G is hypercentral, then G will be nilpotent;
5. if G contains no element of finite order, then G is nilpotent.[12]

In their paper, Heineken and Mohamed noted that the existence of such a group settled a number of important questions that were open at the time, namely:

1. a group satisfying the normaliser condition need not be hypercentral;
2. a group with all its proper subgroups nilpotent and subnormal need not be nilpotent;
3. the direct product of two groups satisfying the normaliser condition need not satisfy the normaliser condition;
4. if all subgroups of a group are subnormal, there need not be a bound on the defects of these subgroups.

In Heineken and Mohamed [10] Heineken and Mohamed showed, for every prime p, that there exist $2^{\aleph_0}$ non-isomorphic groups satisfying their theorem.

The Heineken–Mohamed construction is intricate. They consider the group algebra $R = F_p[C(p^\infty)]$ where F_p is a field with p elements and $C(p^\infty)$ is the group generated by $\{u_i : u_1^p = \text{identity}; u_{i+1}^p = u_i : i = 1, 2, \ldots\}$. They show that the set of all ideals of R is completely ordered and give an explicit description of each ideal of R. They proceed to construct a group F generated by $\{a, x_i : a^p = 1; e_2[a, b] = 1 \text{ for all } b \in F; x_1^p = 1 = (x_1 a)^p; x_{i+1}^p = x_i a : i = 1, 2, 3, \ldots\}$. They consider a subgroup A generated by the conjugacy class of a and observe that A is an elementary abelian normal subgroup F and that F induces an endomorphism ring of A which is a quotient of the group algebra R. Within F they construct a group G generated by $\{x_i a : i = 1, 2, 3, \ldots\}$ and proceed to show that every subgroup of G is subnormal and nilpotent; that $Z(G) = \{i_G\}$ and that G itself is not nilpotent.

4 Concluding remarks

We have not come across any published articles that report further research that has been undertaken on, for example, the properties of the lower/upper A-central series introduced in Mohamed's papers. In Hall and Hartley [6], Hall and Hartley study the stability groups

[9] A group is metabelian if it has an abelian normal group N such that G/N is also abelian. It implies that the group is solvable of derived length at most 2.

[10] In contrast to the fact that if G is a finite p-group then $Z(G)$ is non-trivial.

[11] In their paper Heineken and Mohamed assumed that $G \neq G$. In 1990, Möhres [21] proved that all N_1 -groups are solvable and therefore the assumption $G \neq G$ is superfluous.

[12] Heineken and Mohamed assumed that G is solvable but as noted in the previous footnote, all N_1-groups are solvable and therefore the assumption $G \neq G$ is superfluous.

of series with an infinite number of terms. Mohamed's inequality which gives, as a special case, the Hall upper bound for the nilpotency class of the stability subgroup of a finite subgroup chain does not give an improvement on the Hall upper bound. Further work in this direction can be found in Hurley [11].

The Heineken–Mohamed paper has attracted considerable attention over the years, for example Hartley [7], Hartley [8], Meldrum [17], and Menegazzo [18]. In Menegazzo [18], Menegazzo exploits the ideas in Heineken and Mohamed [9] to give a general construction of Heineken–Mohamed groups and uses it to construct Heineken–Mohamed groups of arbitrary derived length and those whose commutator subgroups have infinite exponent.[13] In his survey of N_1 groups in Casolo [3], Casolo devotes a chapter giving an overview of further developments on Heineken–Mohamed groups.

Mohamed's last paper appeared in 1974 and the following quote by Phahle throws some light on the lack of research output thereafter.

> "For me, knowing Ismail personally, his most outstanding characteristic was his two passions: pursuit of research in Mathematics and a striving throughout his life to be present and active at the coalface of the struggle for the national liberation of South Africa. Throughout his life, the two presented him with a conflict of interest: does he stay in Europe or United States where he could comfortably occupy a university teaching post that will enable him to pursue mathematics research; or does he forgo such an opportunity to indulge his passion for mathematics by remaining in South Africa in order to play a part in the struggle. Every time, the latter option trumped the former" Phahle [23].

In 1976, Ismail Mohamed was detained by the apartheid regime for his political activities and released after three months without being charged. He was subsequently fired from his job at the University of the Western Cape for these activities. He joined the ANC underground around 1980 and in 1985 was charged with treason. In 1994 he entered parliament as an ANC Member of Parliament after the first democratic elections in South Africa's history.±

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[13] In the original Heineken–Mohamed construction, G' is an elementary abelian p-group.

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