Pathwidth and component number of links.

by

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in fulfillment of the requirements
for the degree of Master of Science

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Declaration

I declare that this Dissertation is my own, unaided work. It is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

_____ (Signature of candidate)

_____ day of _____ 20 _____ in _____

Abstract

Knot theory is a branch of algebraic topology that is concerned with studying the interesting geometrical structures known as knots. The idea of a knot in the theory of knots is entirely different from everyday's idea of knots, that is, a knot has free ends. In knot theory a knot is defined as a knotted loop of string which does not have free ends unless we cut it using a pair of scissors.

The interesting aspect of knot theory is that it enables us to transfer techniques from graph theory, algebra, topology, group theory and combinatorics to study different classes of knots. In this dissertation we are only concerned with the relationship between knot theory and graph theory.

It is widely known in knot theory literature that a knot has a corresponding signed planar graph and that a signed planar graph also has a corresponding knot which depends on the signs of the edges of its signed planar graph. This provides a foundation of a solid relationship between knot theory and graph theory, and it allows for some of the notions in graph theory to be transferred to knot theory. In this dissertation we study the pathwidth and component number of links through their corresponding graphs.

Acknowledgements

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Chapter 1

Introduction

Graph theory was discovererd in 1736 by Leonhard Euler when he made an attempt to solve a prominent unsolved problem regarding the bridges in the city of Konigsberg, see [7]. The problem is known today as the Königsberg Bridge Problem, and it is concerned about the existence of a walk that can cross each of the seven bridges of Königsberg exactly once, and this concern led Euler to his famous notion of Eulerian graphs, see [23]. In 1840, August Ferdinand Möbius made an important discovery of two classes of graphs, that is, the class of complete graphs and the class of bipartite graphs, see [23]. In the same year Kazimierz Kuratowski gave a proof that the complete graphs and bipartite graphs are planar through the use of recreational problems, see [23]. In 1847, Gustav Kirchhoff advanced the notion of trees and used it to solve a system of linear equations relating to electric networks, see [7]. In 1852, Thomas Guthrie discovered another prominent problem in graph theory known as the Four Color Problem, see [23]. The Four Color Problem was solved by Kenneth Appel and Wolfgang Haken after a century and this milestone was marked as the inception of graph theory, see [23]. Graph theory is known today as a prodigious mathematical field with many diverse applications, see [7] and [23].

Knot theory as we know it today began in late 1860s when William Thomson also known as Lord Kelvin claimed that atoms behaved more like *knotted vortices in the ether*, see [8]. This discovery by Lord Kelvin influenced Peter Guthrie Tait to research more about knots and his quest led him to the establishment of the first tabulation scheme of knots, see [8]. There are many mathematicians who contributed to the development of knot theory and amongst them was Carl Gauss Friedrich who was known for his famous paper titled *A collection of knots*, see [20]. In 1847 Johann Benedict Listing a PhD student who was supervised by Gauss also made an important contribution to the theory of knots by stating in his monograph, *Vorstudien zur Topologie*, that the right trefoil knot and the left trefoil knot are not equivalent, see [20]. Note that this was the same year Gustav Kirchhoff established his graph theoretic notion that has applications to electric networks, see [7]. Kirchhoff's discovery was closely related to knot theory and this was shown after hundred years, see [20]. Knot theory today still remains a growing mathematical field with fascinating key applications to biology, chemistry and physics, see [1].

Overview of the dissertation

In Chapter 1, we give some basic definitions in graph theory and knot theory relevant to this work. In Chapter 2, we give some results on pathwidth of graphs. In Chapter 3, we give the pathwidth of pretzel links and we relate it to its bridge number if known. In Chapter 4, we count the number of components of pretzel links. The work in Chapter 2 and Chapter 3 have been submitted to Journal of Knot Theory and its ramifications as a single paper.

1.1 Preliminaries

In this chapter we will discuss some basic notions in graph theory and in knot theory. We will only define the notions which are relevant to this dissertation.

1.1.1 Graphs

In this section, we discuss basic concepts and notions in graph theory which are relevant to this dissertation. We refer the reader to [3] and [4] unless otherwise stated.

Definition 1.1.1. *A graph H is a pair $(V(H), E(H))$, where $V(H)$ is a set which consists of elements known as vertices and $E(H)$ is a set which consists of two-element subsets of $V(H)$ known as edges. The two sets $V(H)$ and $E(H)$ are called the vertex set and the edge set, respectively.*

In this dissertation we only consider finite graphs, thus $V(H)$ is a nonempty finite set of vertices and $E(H)$ is a finite set of edges consisting of unordered pairs of distinct vertices from $V(H)$. The vertex set is given by $V(H) = \{v_1, v_2, v_3, \cdots, v_k\}$ consisting of k vertices and the edge set is $E(H) = \{e_1, e_2, e_3, \cdots, e_n\}$ which consist of n two-element subsets of $V(H)$ and each edge e_i is of the form $\{v_a, v_b\}$ for all $i = 1, 2, \cdots, n$ and for all $a, b \in \{1, 2, \cdots, k\}$. There are various ways of denoting an edge, and in this dissertation we interchange between these two notations $\{v_i, v_j\}$ and v_iv_j.

Example 1.1.2. Let H be the graph shown in the diagram of Figure 1.1. Then the vertex set is given by $V(H) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$, and the edge set is given by $E(H) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}\}$.

Definition 1.1.3. *Let H be a graph. Then the number of vertices of H is denoted by $|H| = |V(H)|$, and it is called the **order** of a graph H and the number of edges is denoted by $\|H\| = |E(H)|$, and it is called the **size** of a graph H.*

Example 1.1.4. By applying Definition 1.1.3 to the graph H shown in the diagram of Figure 1.1 we see that the **order** of the graph H is given by $|H| = 9$ and the **size** of the graph H is given by $\|H\| = 12$.

Definition 1.1.5. *A pair of vertices $v_i, v_j \in V(H)$ are **adjacent** if $v_iv_j \in E(H)$, otherwise $v_i, v_j \in V(H)$ are **nonadjacent**.*

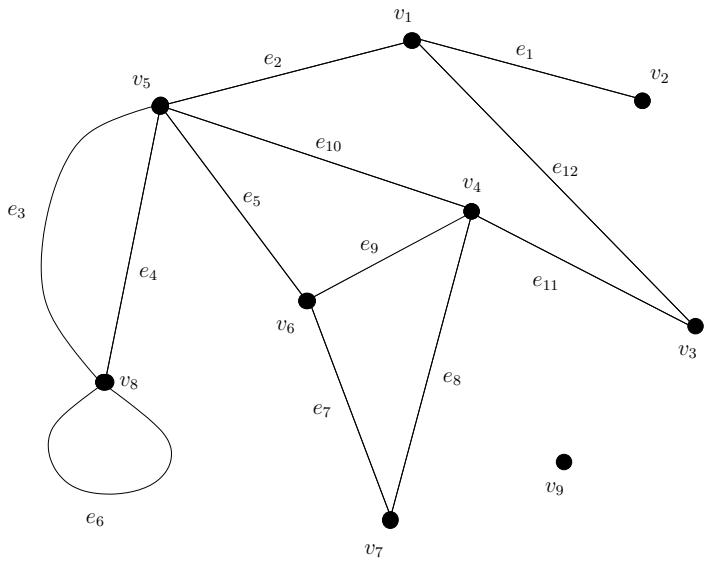

Figure 1.1: A general example of a graph G.

Definition 1.1.6. *If $v_iv_j \in E(H)$, then it is said that this edge is **incident** to vertices v_i and v_j.*

Remark 1.1.7. *Note that a **loop** is an edge which is **incident** to a vertex $v_i \in V(H)$ twice. In the diagram of Figure 1.1 the edge e_6 is the **loop**.*

Definition 1.1.8. *The **degree** of a vertex $v \in V(H)$ denoted by $deg(v)$ or $d(v)$ is the number of edges in $E(H)$ incident to v.*

Note that a vertex of degree zero is known as an **isolated vertex** and a vertex of degree one is known as an **end vertex**.

Remark 1.1.9. *Note that a loop contributes 2 to the degree of a vertex which is incident to.*

Example 1.1.10. Calculating the degree for some of the vertices of graph H shown in the diagram of Figure 1.1, we see that vertex v_1 has degree $d(v_1) = 3$, vertex v_9 is an **isolated vertex** since it has degree zero, and vertex v_2 is an **end vertex** since it has degree one. By Remark 1.1.9, the degree of vertex v_8, that is, $d(v_8) = 4$.

Remark 1.1.11. *Let H be a graph. Then $\delta(H)$ denotes the* ***minimum degree*** *of H and $\Delta(H)$ denotes the* ***maximum degree*** *of H.*

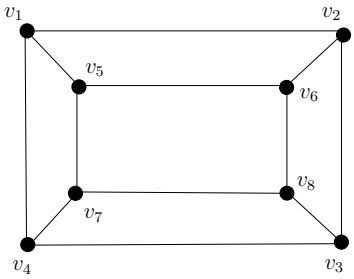

Figure 1.2: A 3-regular graph.

Definition 1.1.12. *Let H be a graph which has all vertices with the same degree, that is, $\delta(H) = \Delta(H)$. Then H is said to be a* ***regular graph****.*

Note that if a graph H has all vertices with degree r, then H is called an r-**regular** graph.

Example 1.1.13. In the diagram of Figure 1.1, we see that the graph H has $\Delta(H) = 5$ and $\delta(H) = 0$. The diagram in Figure 1.2 is an example of a 3-regular graph since each vertex has degree 3.

Definition 1.1.14. *Let H be a graph. Then a* ***walk*** *in a graph H is defined as a non-empty alternating sequence $v_0a_0v_1a_1 \cdots a_{q-1}v_q$ of vertices and edges in H such that $a_i = v_iv_{i+1}$ for all $i < q$.*

Example 1.1.15. In the diagram of Figure 1.3 let $a_0 = AC$, $a_1 = CE$, $a_2 = EB$, $a_3 = BF$, then $Aa_0Ca_1Ea_2Ba_3F$ is an example of a **walk**.

Remark 1.1.16. Let H be a graph with a **walk** $v_0a_0v_1a_1 \cdots a_{q-1}v_q$. If $v_0 = v_q$, then we say that a **walk** $v_0a_0v_1a_1 \cdots a_{q-1}v_q$ is **closed**. In the diagram of Figure 1.3 let

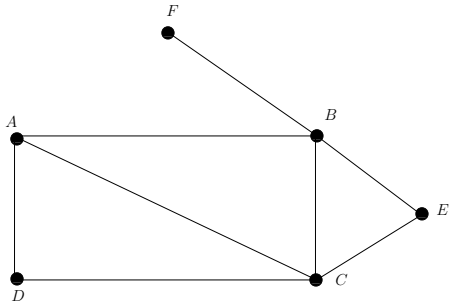

Figure 1.3: A graph that has a walk $Aa_0Ca_1Ea_2Ba_3F$.

$a_0 = AC$, $a_1 = CE$, $a_2 = EB$, $a_3 = BA$, then we see that the **walk** $Aa_0Ca_1Ea_2Ba_3A$ is **closed**. The **length** of a **walk** is defined as the number of edges in a **walk**. For instance the **walk** $Aa_0Ca_1Ea_2Ba_3F$ has length 4. Note that the vertices and edges may be repeated in a **walk**. A **trail** is a **walk** with no repeated edges. A **closed trail** is called a **circuit**.

Definition 1.1.17. *A **path** is a non-empty graph $\mathcal{P}$ which is formed by a vertex set $V(\mathcal{P}) = \{v_0, v_1, \cdots, v_{k-1}, v_k\}$ and an edge set $E(\mathcal{P}) = \{v_0v_1, v_1v_2, \cdots, v_{k-1}v_k\}$ where all the v_j's are not equal.*

Note that a **path** is a **walk** with no repeated vertices.

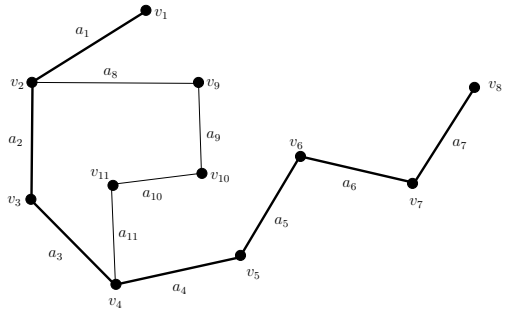

Figure 1.4: A path.

Remark 1.1.18. Let H be a graph shown in the diagram of Figure 1.4. Then we see that the **path** $\mathcal{P}$ with the vertex set $V(\mathcal{P}) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and the edge set $E(\mathcal{P}) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ is given by

$$v_1a_1v_2a_2v_3a_3v_4a_4v_5a_5v_6a_6v_7a_7v_8.$$

The notion of **length** also applies to **paths**, and so we have that the number of edges of a **path** determine its **length**. Therefore the **path** $\mathcal{P}$ shown in the diagram of Figure 1.4 is of **length** 7. We will denote by $\mathcal{P}^k$ a **path** of length k and by $\mathcal{P}_k$ a **path** with k vertices.

Definition 1.1.19. *A **path** that starts and ends at the same vertex is called a **cycle graph**. We denote a **cycle graph** on n vertices by C_n.*

In the diagram of Figure 1.4 we see that the **path** $v_2a_2v_3a_3v_4a_{11}v_{11}a_{10}v_{10}a_9v_9a_8v_2$ is an example of a **cycle**.

Definition 1.1.20. *A graph H which contains at least one **cycle graph** is called a **cyclic graph**. Otherwise H is called an **acyclic graph**.*

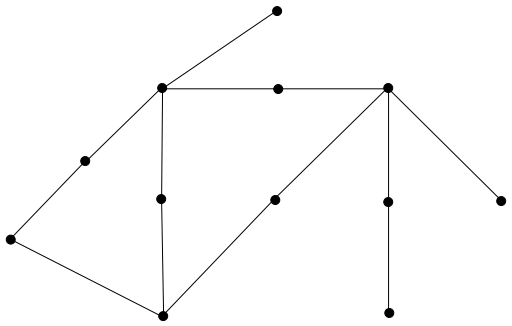

Figure 1.5: A connected graph.

Definition 1.1.21. *Let H be a graph. Then H is said to be **connected** if for any two vertices $v_i, v_j \in V(H)$ there is a path from v_i to v_j or vice versa. Otherwise we say that a graph H is **disconnected**.*

The diagram in Figure 1.5 shows an example of a **connected** graph.

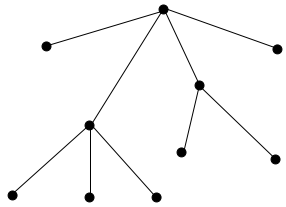

Figure 1.6: A tree.

Definition 1.1.22. *A graph H is said to be a **tree** if H is connected and acyclic.*

The diagram in Figure 1.6 shows an example of a **tree**.

Remark 1.1.23. *Note that a **forest** is a disjoint union of trees.*

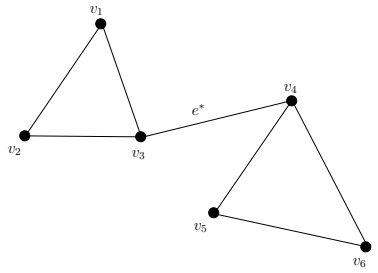

Figure 1.7: An example of a graph with a bridge or isthmus e^*.

Definition 1.1.24. *Let H be a graph. Then an edge $e \in E(H)$ is called a **bridge or isthmus** if its removal from the graph **disconnects** the graph H.*

Example 1.1.25. Let H be the graph shown in the diagram of Figure 1.7, we see that the edge e^* is an example of a **bridge or isthmus** because its removal from H results in H being disconnected.

Definition 1.1.26. *Let H be a graph. Then any two or more edges in $E(H)$ are said to be* ***parallel*** *if they are incident with the same pair of vertices which are in $V(H)$.*

Example 1.1.27. In the diagram of Figure 1.1 edges e_3 and e_4 are **parallel edges** since they are incident with vertices v_8 and v_5.

Definition 1.1.28. *Let H be a graph. Then H is said to be a* ***multigraph*** *if it has* ***parallel edges*** *between any pair of vertices $v_i, v_j \in V(H)$.*

From Definition 1.1.28 we deduce that a graph H is a **multigraph** if its vertex set $V(H)$ is not a multiset and its edge set $E(H)$ is a multiset of two-element subsets of $V(H)$. Multigraphs are also allowed to have **loops**. The graph H shown in the diagram of Figure 1.1 is an example of a **multigraph**, because the vertices v_8 and v_5 have **parallel edges** e_3 and e_4 between them. This shows that $E(H)$ is a multiset.

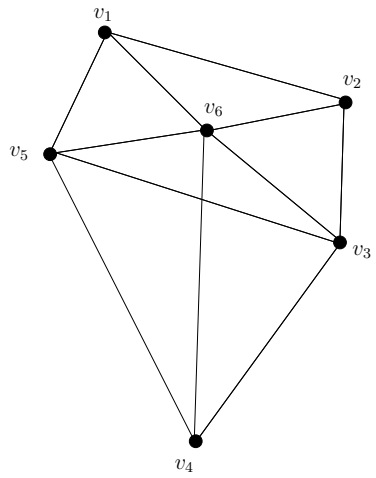

Figure 1.8: A simple graph.

Definition 1.1.29. *A graph H is called a* ***simple graph*** *if both its finite vertex set $V(H)$ and its finite edge set $E(H)$ are not multisets.*

The diagram in Figure 1.8 illustrates a general example of a **simple graph**. Unlike multigraphs, **simple graphs** are not allowed to have **loops** and **parallel** edges.

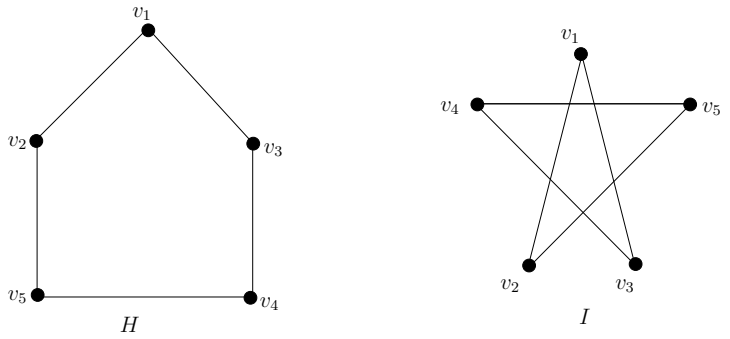

Figure 1.9: Two isomorphic graphs.

Definition 1.1.30. *Let H and I be graphs. Then H is **isomorphic** to I, that is, $H \cong I$ if there exists a one-to-one and onto mapping $\phi : V(H) \to V(I)$ such that any two vertices $v_i, v_j \in V(H)$ of H are adjacent $\iff$ $\phi(v_i), \phi(v_j) \in V(I)$ are adjacent in I.*

The diagram in Figure 1.9 shows an example of two isomorphic graphs, H and I.

1.1.1.1 Subgraphs and Supergraphs

In this section we introduce the reader to the notions of **subgraphs** and **supergraphs**. For further details we refer the reader to [3].

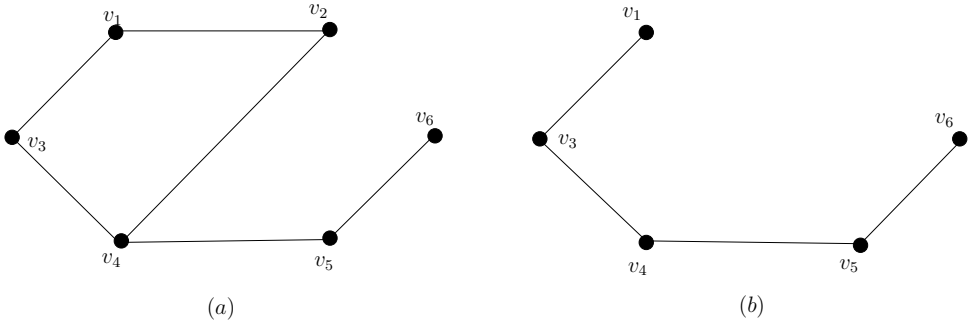

Figure 1.10: (a) A supergraph. (b) A subgraph.

Let I be a graph with a vertex set $V(I)$ and an edge set $E(I)$. We say that a graph J with a vertex set $V(J)$ and an edge set $E(J)$ is a **subgraph** of I if $V(J) \subseteq V(I)$ and $E(J) \subseteq E(I)$. Note that for each edge $a = v_i v_j \in E(J)$ we have that $v_i, v_j \in V(J)$. Basically, a **subgraph** J is constructed from I by removing vertices and/or edges from I in such a way that when a vertex $v_i \in V(I)$ is removed, all the edges incident with v_i are also removed. When a graph J is a **subgraph** of a graph I, we call I a **supergraph** of J. The diagram of Figure 1.10 shows an example of a **subgraph** and its **supergraph**.

1.1.1.2 Planar Graphs

In this section we introduce the reader to the notion of **planar graphs**. We conclude this section by introducing an important class of planar graphs called the **signed planar graphs**. For further details we refer the reader to [1], [3] and [4] unless otherwise stated.

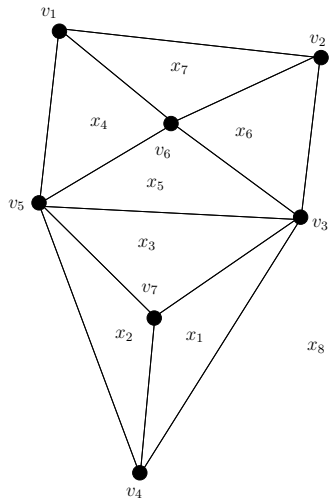

Figure 1.11: A planar graph.

Definition 1.1.31. *A graph H is said to be a **planar graph** if it can be depicted in*

the plane as a graph with no intersecting edges. Otherwise we say that a graph H is a ***nonplanar graph****.*

The diagram in Figure 1.11 is an example of a **planar graph**.

Definition 1.1.32. *Let H be a planar graph, then the depiction of H in the plane divides the plane into regions known as* ***faces****.*

The planar graph shown in the diagram of Figure 1.11 has the **faces** :

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \text{ and } x_8.$$

In the diagram of Figure 1.11 the faces $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7 are known as the **inner faces** and the face x_8 is known as the **outer face**.

Definition 1.1.33. *Let H be a planar graph. Then any pair of faces x_i and x_j are said to be adjacent if they share one or more edges.*

In the diagram of Figure 1.11 we see that the faces x_1 and x_2 are adjacent, x_1 and x_3 are adjacent, x_2 and x_3 are also adjacent.

Definition 1.1.34. *Let H be a planar graph with faces $x_1, x_2, x_3, \cdots, x_n$. Then a graph H^* is said to be a* ***dual graph*** *of H if H^* is formed by a pair $(V(H^*), E(H^*))$ where*

$$V(H^*) = \{x_1, x_2, x_3, \cdots, x_n, \}$$

and

$$E(H^*) = \{x_i x_j : x_i \text{ and } x_j \text{ are adjacent } faces\ 1 \leq i, j \leq n\}.$$

Example 1.1.35. When we apply Definition 1.1.34 to diagram (a) shown in Figure 1.12 we start by assigning vertices to each face including the **outer face** as shown in diagram (b) of Figure 1.12. This is followed by joining **adjacent faces** with an edge as shown in diagram (a) of Figure 1.13. This process results in the **dual graph** shown in diagram (b) of Figure 1.13.

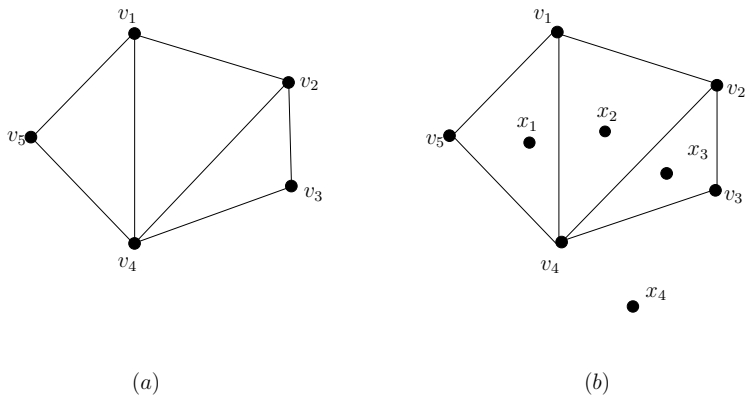

Figure 1.12: (a)A planar graph. (b)Assigning a vertex to each face.

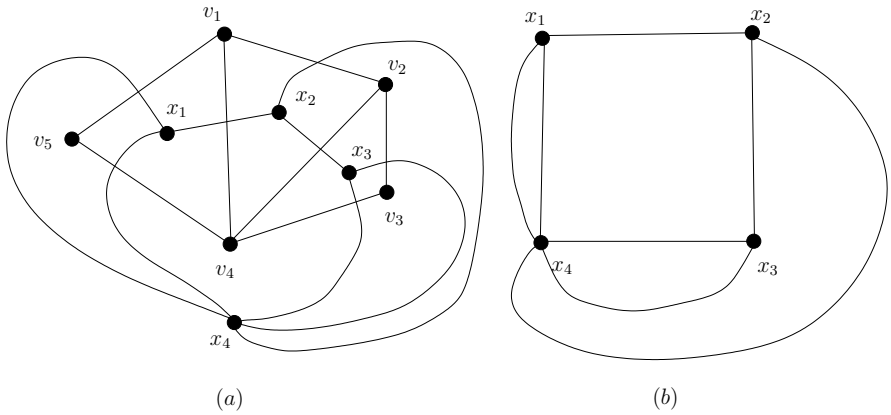

Figure 1.13: (a)Joining neighbouring vertices with an edge. (b)A dual graph.

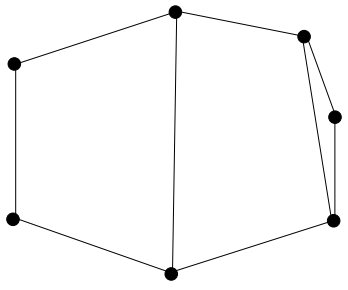

Figure 1.14: An outeplanar graph.

Definition 1.1.36. *Let H be a planar graph. If H has a depiction in the plane such that all its vertices form part of the outer face, then H is called an* ***outerplanar graph****.*

The diagram in Figure 1.14 is an example of an **outerplanar graph**.

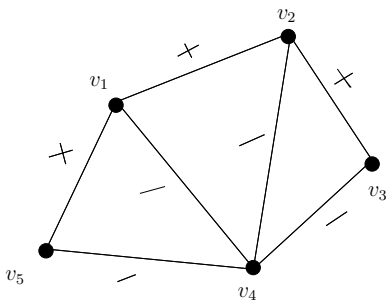

Figure 1.15: A signed graph.

Definition 1.1.37. *A* ***signed planar graph*** *is a graph whose underlying graph H has been assigned either a $+$ (positive) sign or a $-$ (negative) sign on its edges. The edges to which the $+$ or $-$ is assigned are called* ***positive*** *or* ***negative*** *edges.*

The diagram in Figure 1.15 is an example of a **signed planar graph**. A **signed planar graph** that has been assigned a + sign on all its edges will be considered to be the same as its underlying **unsigned planar graph**.

1.1.1.3 Graph operations

There are many graph operations which are known in the literature. In this section we discuss some graph operations which are relevant to this dissertation. We refer the reader to [3], [4] and [21] unless otherwise stated.

Let H be a graph. There are two basic graph operations, see Figure 1.16. These operations are as follows:

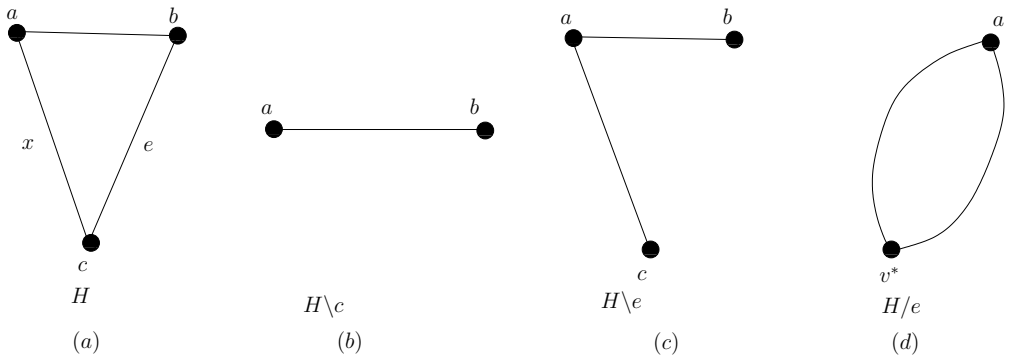

Figure 1.16: Edge deletion and edge contraction.

1. $H\backslash v_i$, which is the **vertex deletion** or the removal of the vertex v_i and all its incident edges from H;
2. $H\backslash e$, which is the **edge deletion** or the removal of the edge e from H and;
3. H/e, which is the **edge contraction** of the edge e, that is, e is removed and its incident vertices are merged, and this can lead to the occurrence of multiple edges and loops.

Example 1.1.38. Let H be the graph shown in diagram (a) of Figure 1.16. Then the graph $H\backslash c$ shown in diagram (b) of Figure 1.16 is constructed from H by removing the vertex c and its incident edges x and e. The graph $H\backslash e$ shown in diagram (c) of Figure 1.16 is constructed from H by removing the edge e from H. The graph H/e shown in diagram (d) of Figure 1.16 is also constructed from H by first removing the edge $e = bc$ and then merging the vertices incident with e, that is, $b = c$ and we obtain the new vertex v^*. Note that the vertex v^* is adjacent to the vertex a that had been adjacent to both b and c.

Definition 1.1.39. *Let H be a graph. If a graph F can be constructed from H by vertex deletion, and/or edge deletion, and/or edge contraction, then F is said to be a* ***minor*** *of H.*

In Figure 1.16, diagrams (b), (c) and (d) are **minors** of the graph H shown in Figure 1.16 diagram (a).

1.1.1.4 Tutte polynomial

We now turn to one of the most important notions in graph theory known as the **Tutte polynomial**. For this section we refer the reader to [6] and [24].

Definition 1.1.40. *A **Tutte polynomial** $T(H, x, y)$ of a graph $H = (V(H), E(H))$ is a two-variable polynomial which is defined as follows:*

$$T(H,x,y) = \begin{cases} 1, & \text{if } E(H) = \emptyset \\ xT(H\backslash i, x, y), & \text{if } i \in E(H) \text{ is a bridge} \\ yT(H/i, x, y), & \text{if } i \in E(H) \text{ is a loop} \\ T(H\backslash i, x, y) + T(H/i, x, y), & \text{if } i \in E(H) \text{ is a neither a bridge nor a loop} \end{cases}$$

Definition 1.1.40 is known throughout graph theory literature as the deletion-contraction algorithm and this will be demonstrated when we apply it to a given graph. Note that the variables x and y of a **Tutte polynomial** are independent.

Example 1.1.41. In this example $H = \{v_1v_2, v_1v_4, v_2v_4, v_3v_4, v_4v_5\}$ denotes a graph and each element is an edge incident with its vertices, for example $v_1v_2 \in H$ is the edge of the graph H incident with vertices v_1 and v_2. In this example we calculate the **Tutte polynomial** $T(H, x, y)$ of the graph H and we refer to Figure 1.17 for each step of our calculation.

$$\begin{aligned} T(H = \{v_1v_2, v_1v_4, v_2v_4, v_3v_4, v_4v_5\}, x, y) &= xT(H_\alpha = H\backslash v_3v_4, x, y). \\ T(H_\alpha = \{v_1v_2, v_1v_4, v_2v_4, v_4v_5\}, x, y) &= xT(H_\beta = H_\alpha\backslash v_4v_5, x, y). \\ T(H_\beta = \{v_1v_2, v_1v_4, v_2v_4\}, x, y) &= T(H_\delta = H_\beta\backslash v_1v_4, x, y) + T(H_\epsilon = H_\beta/v_1v_4, x, y). \\ T(H_\delta = \{v_1v_2, v_2v_4\}, x, y) &= xT(H_\lambda = H_\delta\backslash v_2v_4, x, y). \\ T(H_\epsilon = \{v_2v_4, v_4v_2\}, x, y) &= T(H_\psi = H_\epsilon\backslash v_4v_2, x, y) + T(H_\theta = H_\epsilon/v_4v_2, x, y). \\ T(H_\lambda = \{v_1v_2\}, x, y) &= xT(H_\lambda\backslash v_1v_2, x, y) \end{aligned}$$

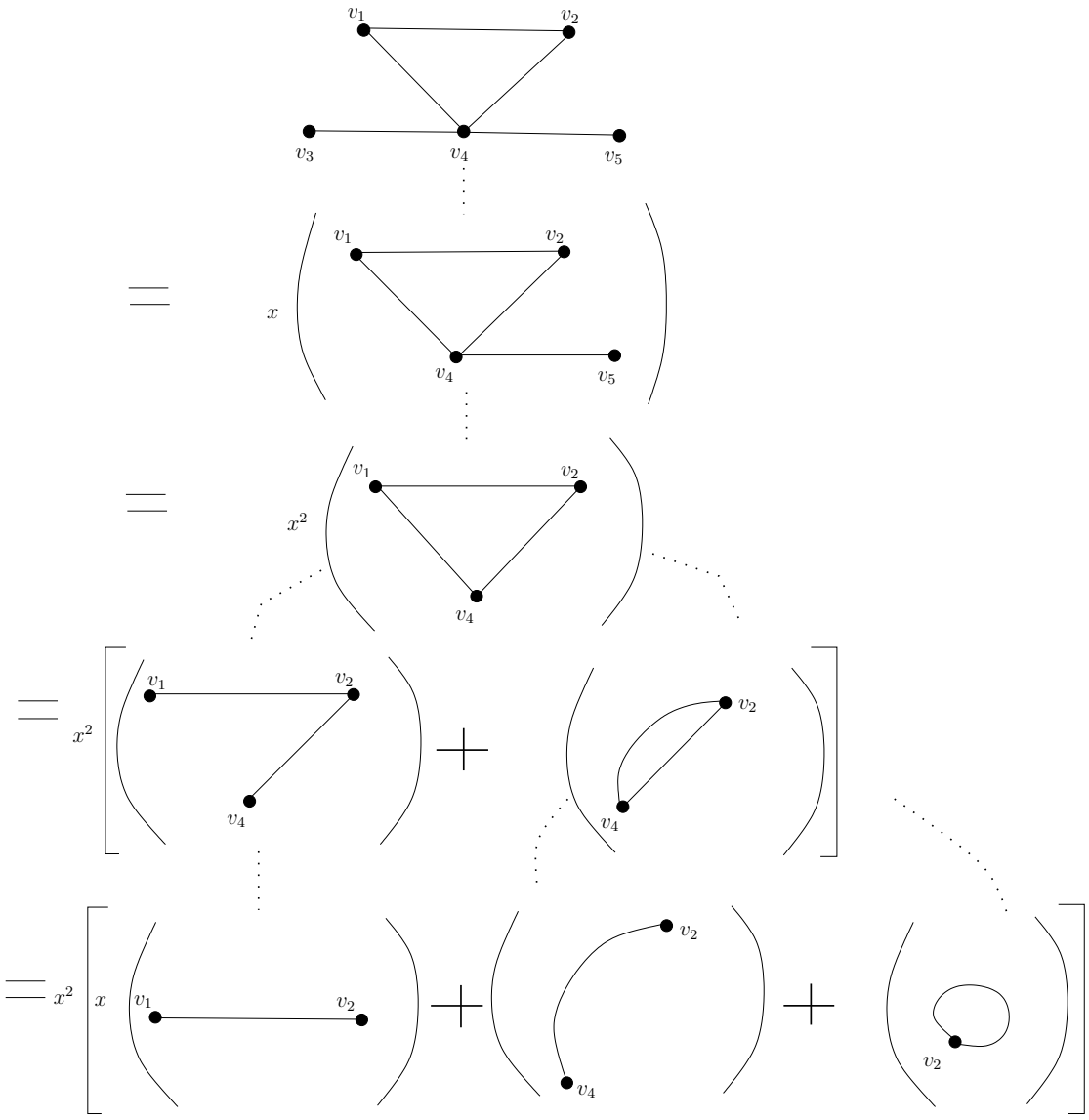

Figure 1.17: Applying the Tutte polynomial recursive algorithm on a graph G.

$$
\begin{aligned}
&= x(1) = x. \\
T(H_\psi = \{v_2v_4\}, x, y) &= xT(H_\psi \backslash v_2v_4, x, y) \\
&= x(1) = x. \\
T(H_\theta = \{v_2v_2\}, x, y) &= yT(H_\theta \backslash v_2v_2, x, y) \\
&= y(1) = y.
\end{aligned}
$$

By backward substitution we get that the **Tutte polynomial** of the graph $H = \{v_1v_2, v_1v_4, v_2v_4, v_3v_4, v_4v_5\}$ is given by $T(H, x, y) = x^4 + x^3 + x^2y$.

Note that in Example 1.1.41 the graphs H_α, H_β, H_δ, H_ϵ, H_λ, H_ψ and H_θ are all minors of $H = \{v_1v_2, v_1v_4, v_2v_4, v_3v_4, v_4v_5\}$. The diagrams in Figure 1.17 show all these minors of H.

1.1.2 Knots

The simplest (non-trivial) form of the theory of knots is concerned with the embedding of disjoint circles, it can be one or more disjoint circles, into three dimensional space. A **knot** might be defined to be such an embedding of a simple closed curve or a circle in three dimensional space. We can visualize it as a **knotted loop** of a string that cannot be untangled unless we cut it. An informal definition of a **knot** would be such a **knotted loop** of string with a crossing point being a single point. The concept of a **mathematical knot** is different from everyday's idea of a **knot** where a knot is a piece of string with free ends. In this section we refer the reader to [1] and [14].

In this dissertation we will only consider **mathematical knots** and for simplicity we will just call them **knots**. We denote a **knot** by K. The following definitions are some basic concepts in knot theory:

Definition 1.1.42. *A **knot** K is a closed curve in space with no self-intersection anywhere.*

In the theory of knots a closed curve in space that has no knot in it is regarded as the **unknot** or **trivial knot**, see Figure 1.18 diagram (b).

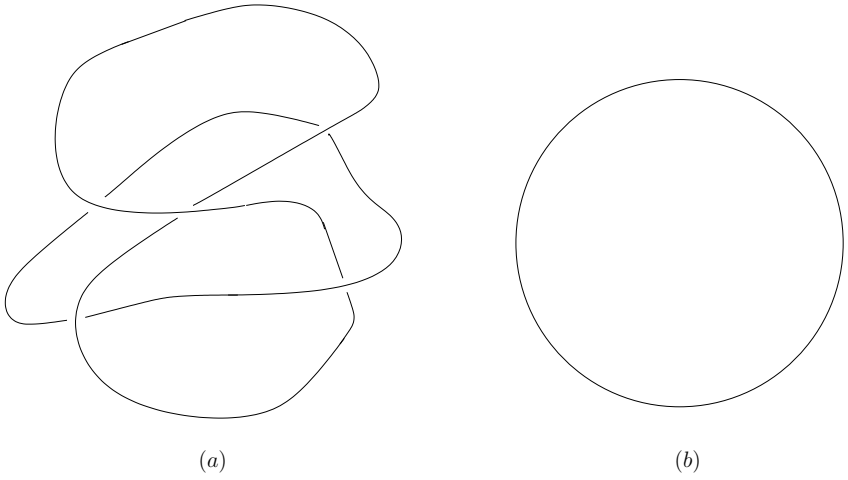

Figure 1.18: (a) A knot projection. (b) The unknot.

Definition 1.1.43. *The diagram of a knot K in two-dimensional space is called a* ***knot projection****.*

In Figure 1.18 diagram (a) is an example of a **knot projection**.

Definition 1.1.44. *A* ***crossing point*** *denoted by c, in a knot projection is the part where the knot projection crosses itself.*

At each crossing point one segment of the knot passes under the other segment. We will call the segment that passes under the **underpass** and we will call the the segment that passes over the **overpass**.

Definition 1.1.45. *Given a knot K, an* ***orientation*** *is established on K by choosing a direction to move around K. We then say that the knot K is* ***oriented****.*

An orientation on a knot K is commonly shown as the directed arrows along the knot projection in a given direction. An example of an oriented knot is shown in the diagram of Figure 1.19.

Definition 1.1.46. *Two or more disjoint knots which are tangled up together form a* ***link****.*

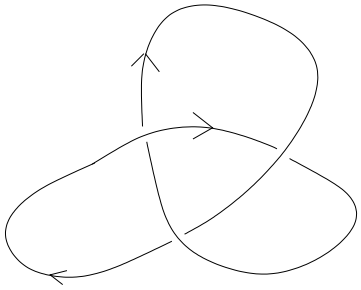

Figure 1.19: The oriented knot.

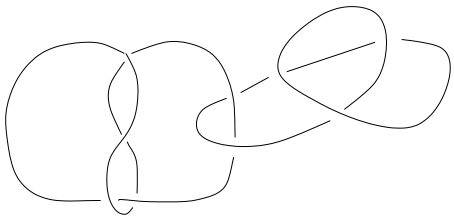

Figure 1.20: A link.

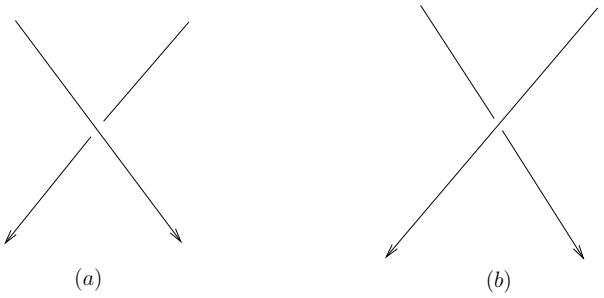

Figure 1.21: (a) Negative crossing and (b) Positive crossing.

The diagram of Figure 1.20 is an illustration of a **link**.

Remark 1.1.47. A crossing point c is negative if the overpass is rotated clockwise an angle $< \pi$ to align it with the underpass (such that the arrow heads coincide, see Figure 1.21 diagram (a)) and it is positive if the overpass is rotated anticlockwise an angle $< \pi$ to align it with the underpass (such that the arrow heads coincide, see Figure 1.21 diagram (b)).

1.1.2.1 Composition of knots

If we have two knot projections, we can form a new knot projection by removing a small arc from each knot projection and then joining the four endpoints by two new arcs. This new knot projection will be called the **composition** of the two knot projections. Let K_1 and K_2 be two disjoint knot projections, then their composition is denoted by $K_1 \# K_2$. An illustration of the operation of composition of two knot projections A and B shown in Figure 1.22 is given in Figure 1.23. This way of defining composition assumes that the two knot projections do not overlap and that the two small arcs that we remove are on the outside of each knot projection to avoid any crossings between the two knot projections. This definition further assumes that the two new arcs are chosen in such a way that they do not cross either the original knot projections or each other.

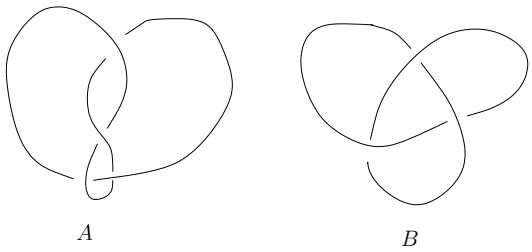

Figure 1.22: Two disjoint knots A and B.

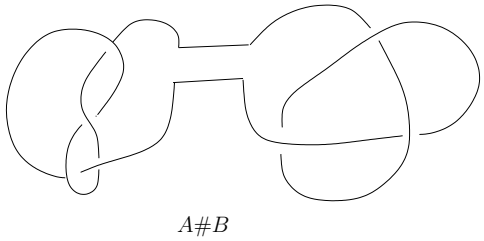

Figure 1.23: $A\#B$ composition of knots A and B.

Definition 1.1.48. *A **knot projection** is said to be **composite** if it can be expressed as the **composition** of two nontrivial knots. If a knot cannot be expressed as the **composition** of any two nontrivial knots, we call it a **prime knot**.*

In the diagram of Figure 1.23 the knot projection $A\#B$ is an example of a **composite** knot projection.

Remark 1.1.49. The knots which formulate a **composite knot** are called the **factor knots**, for example the knot projections A and B shown in Figure 1.22 are examples of **factor knots**.

1.1.3 How to extract planar graphs from knots and vice versa.

In this section we look at how to extract a planar graph from a given knot and the reverse process. There are several methods for doing this, but in this section we only consider one method called the **checkerboard shading**. For further details we refer the reader to [1] and [14].

1.1.3.1 Checkerboard Method

For this method we assume that a given knot is projected onto a sphere other than a plane. Thus the knot divides the sphere into a finite number of regions, say, $S_1, S_2, \cdots, S_n$. Starting with the outer unbounded region, the **checkerboard method** says that we must shade a given knot projection in an alternating manner,

see Figure 1.24. Note that there are only two possible **checkerboard shadings** for each knot. The diagram in Figure 1.24 demonstrates the two possible checkerboard shading of a trefoil knot.

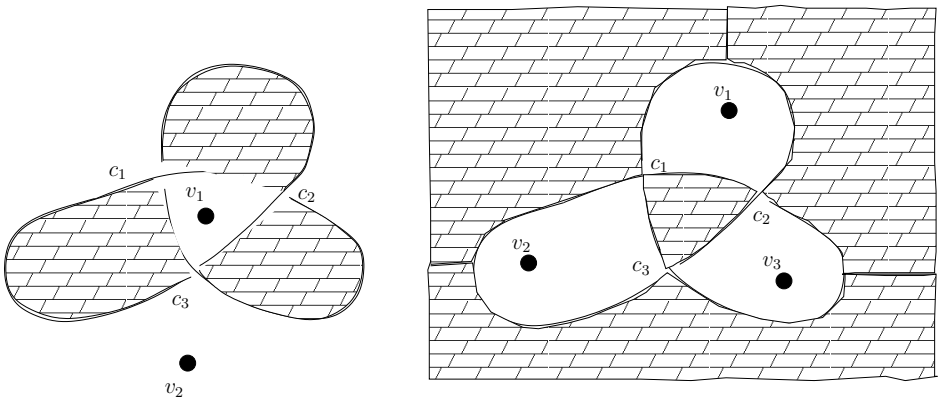

Figure 1.24: Two possible checkerboard shadings.

To extract a graph from a knot we start by assigning a vertex v_k to each unshaded region, see Figure 1.24. Let's say that the set of unshaded regions is $S_{k_1}, S_{k_2}, \cdots, S_{k_m}$. Then if two unshaded regions S_{k_i} and S_{k_j} meet at a crossing point c_p, then we join vertices v_i and v_j corresponding to regions S_{k_i} and S_{k_j} respectively by an edge e_k. The graph produced in this manner is a **signed planar graph** where the signs on the edges are induced by the type of a crossing point, as shown in the diagram of Figure 1.25.

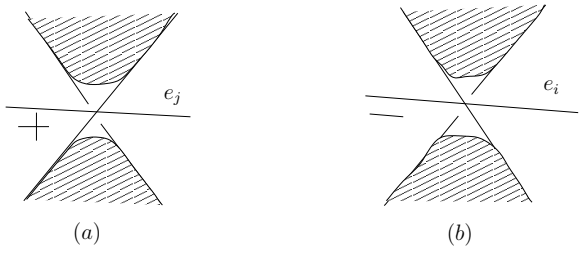

Figure 1.25: A sign that an edge will take on depending on the type of crossing.

Following these steps for the knots shown in the diagram of Figure 1.24 and Figure 1.25 one should obtain graphs similar to Figure 1.26 respectively. Note that when we carry out the inverse checkerboard shading, that is, starting by shading the exterior region we obtain a dual graph of the planar graph we obtained by not shading the exterior region. This can be verified by checking the graphs in Figure 1.26 are duals of each other.

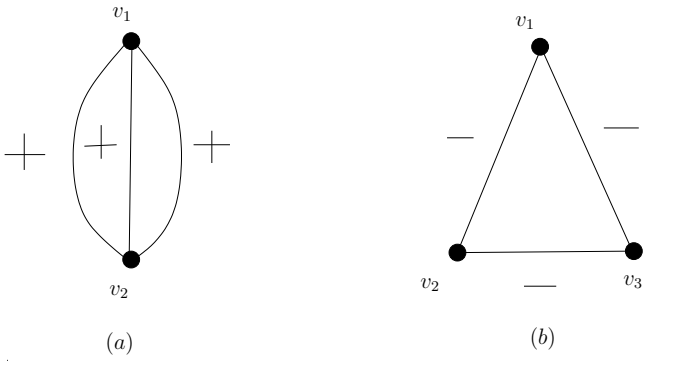

Figure 1.26: A graph and its dual graph.

Now, given a planar graph one may ask *is it possible to extract a knot projection corresponding to this planar graph?* The answer is yes! and it is done in the following way:

Given the planar graph shown in diagram (a) of Figure 1.27 we mark each edge with a cross as shown in diagram (b) of Figure 1.27 and then we join the adjacent crosses in the manner shown in diagram (c) of Figure 1.27.

After we are done joining the adjacent crosses we shade each region that has a vertex as shown in diagram (a) of Figure 1.28. Following the convention we have introduced earlier regarding the negative and positive crossings we arrive at the knot projection shown in diagram (b) of Figure 1.28.

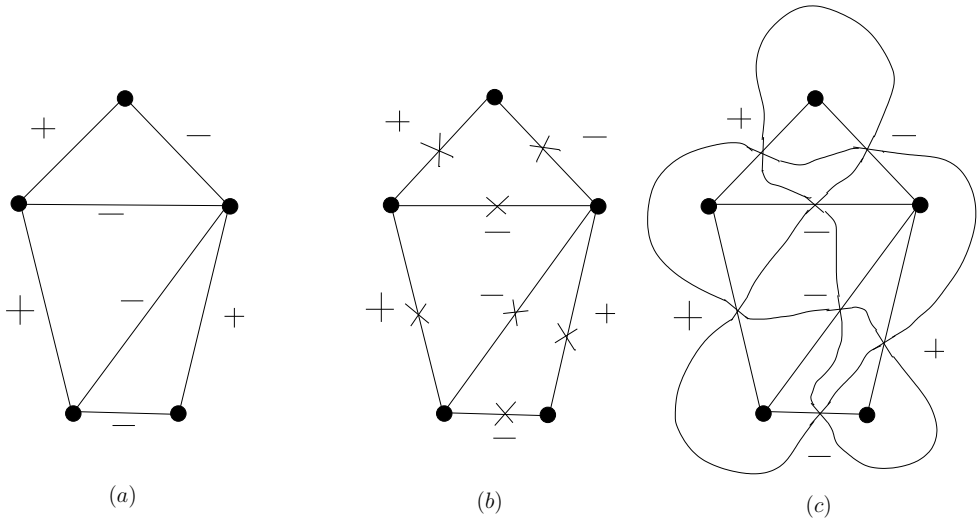

Figure 1.27: Extracting knot from a given signed planar graph.

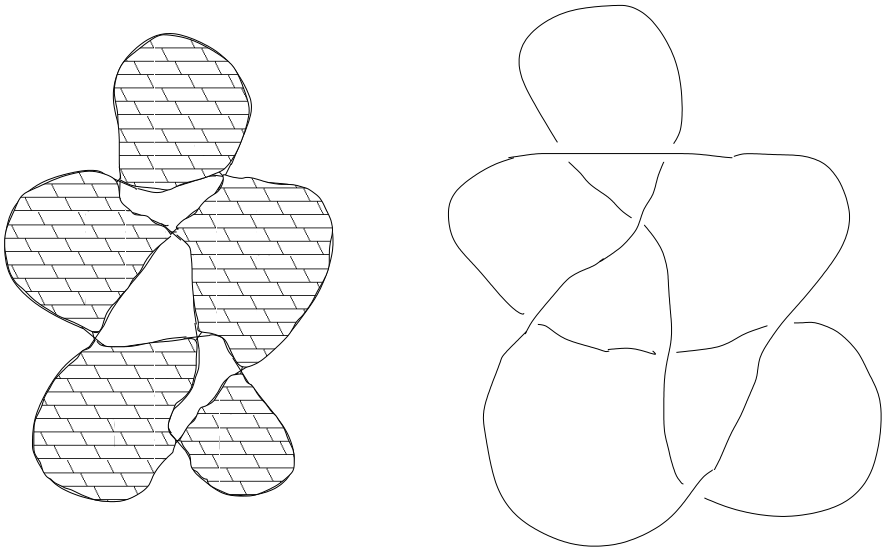

Figure 1.28: Extracting knot from a given signed planar graph.

Chapter 2

Characterisation of pathwidth three graphs

2.1 Introduction

Robert and Seymour [21] introduced the concept of **pathwidth**. In [13] Kinnersley proved a very useful result that we make use of, that is, the vertex separation number of a graph equals its pathwidth. The characterization of graphs with pathwidth at most two was studied by Barat, Hajnal, Lin and Yang, see [2]. Unfortunately, their definition of pathwidth does not correspond with the one that will be given in this dissertation.

In this chapter we begin by giving a definition of general pathwidth of a graph. We then state and prove a theorem on general pathwidth of a graph. Finally, we describe two classes of graphs and show that these classes of graphs have pathwidth three.

2.2 Pathwidth of a graph

We begin this section by introducing the notion of m-separation then the important definition of **pathwidth** of graphs. For further details, we refer the reader to [19].

Definition 2.2.1. *Let H be a graph, $E_1(H)$, $E_2(H)$ be edge sets of H and $V(E_1(H))$, $V(E_2(H))$ be the sets of vertices incident with edges in $E_1(H)$ and $E_2(H)$ respectively. A pair $(E_1(H), E_2(H))$ is an m-**separation** if $|V(E_1(H)) \cap V(E_2(H))| \leq m$.*

Definition 2.2.2. *A graph H with k edges is said to have pathwidth m if and only if m is the minimum value such that there exists an ordering $a_1, a_2, a_3, \cdots, a_k$ where $(\{a_1, a_2, a_3, \cdots, a_i\}, \{a_{i+1}, a_{i+2}, \cdots, a_k\})$ is an m-**separation** for all $1 \leq i \leq k-1$. We denote by $pw(H)$ the pathwidth of a graph H.*

We state the following lemma without proof, and for further details we refer the reader to [19].

Lemma 2.2.3. *Let the graph H be a planar graph without any isthmuses. Then $pw(H) = 2$ if and only if the graph H is an outerplanar graph.*

We state and prove the following lemma which is of interest in proving other lemmas.

Lemma 2.2.4. *Let C_n be a cycle graph on n vertices. Then the pathwidth of C_n is two, that is, $pw(C_n) = 2$.*

Proof. It is clear that if we order the vertices of C_n, $v_1, v_2, v_3, \cdots, v_n$, such that v_i is adjacent to v_{i+1} and v_1 is adjacent to v_n. Then any subgraph of C_n with vertex set $v_1, v_2, \cdots, v_i$, is isomorphic to a path P_i for some $i \in \{1, 2, \cdots, n\}$ and $C_n \backslash P_i \cong P_j$ where $j = n + 1 - i$. Clearly, $|V(P_i \cap P_j)| = 2$. Assume that $pw(C_n) = 1$. Then there is an ordering $a_1, \cdots, a_k$ of edges of C_n, such that for all $1 \leq i \leq k-1$ the pair $(\{a_1, \cdots, a_i\}, \{a_{i+1}, \cdots, a_k\})$ is a 1-separation. But if $\{a_1, \cdots, a_i\}$ is a path then $C_n \backslash \{a_1, \cdots, a_i\}$ is also a path, hence the two paths $\{a_1, \cdots, a_i\}$ and $C_n \backslash \{a_1, \cdots, a_i\}$ intersect at two vertices. If $\{a_1, \cdots, a_i\}$ is a forest then $C_n \backslash \{a_1, \cdots, a_i\}$ is also a forest hence intersect at more than one vertices, which is a contradiction. □

We are in a good position to state and prove the following lemma.

Lemma 2.2.5. *Let H be a graph. If $pw(H) = 1$ then H is a tree.*

Proof. Assume H is not a tree and has pathwidth 1. Then H has a cycle C_n for some $n \in \mathbb{Z}$ as a subgraph. In any ordering of H, when we take the edge set E_1 and E_2 such that E_1 contains part of $E(C_n)$ and E_2 contains the remaining part of $E(C_n)$ not in E_1. Then by application of Lemma 2.2.4, we get vertex separation $|V(E_1) \cap V(E_2)| = 2$. Hence pathwidth of $H \geq 2$ which is a contradiction. □

We illustrate the notion of pathwidth of a graph through the following example.

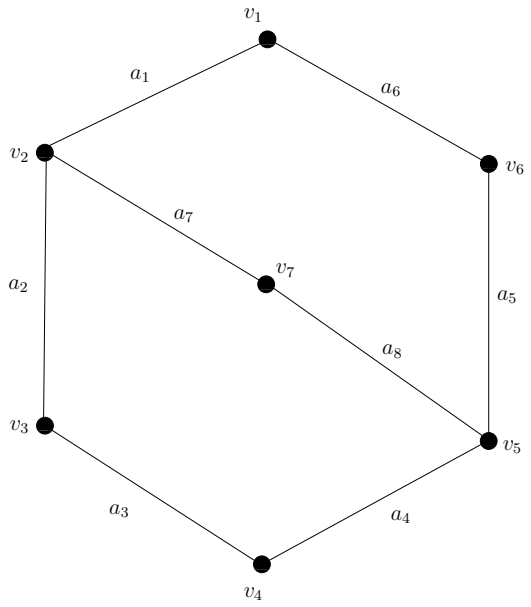

Figure 2.1: A planar graph H.

Example 2.2.6. Let $H = (V(H), E(H))$ be the graph shown in the diagram of Figure 2.1 with $E(H) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$ and $V(H) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. Just to clarify the concept, if $E_1(H) = \{a_1, a_2, a_3, a_4\}$ and $E_2(H) = \{a_5, a_6, a_7, a_8\}$ then $V(E_1(H)) = \{v_1, v_2, v_3, v_4, v_5\}$ and $V(E_2(H)) = \{v_5, v_6, v_1, v_2, v_7\}$. To compute the **pathwidth** of the planar graph H, we maintain the order $a_2, a_3, a_4, a_5, a_6, a_1, a_8, a_7$ of its edges.

$$
\begin{aligned}
|V(\{a_2\}) \cap V(\{a_3, a_4, a_5, a_6, a_1, a_8, a_7\})| &= |\{v_2, v_3\} \cap \{v_3, v_4, v_5, v_6, v_1, v_2, v_7\}| \\
&= |\{v_2, v_3\}| = 2. \\
|V(\{a_2, a_3\}) \cap V(\{a_4, a_5, a_6, a_1, a_8, a_7\})| &= |\{v_2, v_3, v_4\} \cap \{v_4, v_5, v_6, v_1, v_2, v_7\}| \\
&= |\{v_2, v_4\}| = 2. \\
|V(\{a_2, a_3, a_4\}) \cap V(\{a_5, a_6, a_1, a_8, a_7\})| &= |\{v_2, v_3, v_4, v_5\} \cap \{v_5, v_6, v_1, v_2, v_7\}| \\
&= |\{v_2, v_5\}| = 2. \\
|V(\{a_2, a_3, a_4, a_5\}) \cap V(\{a_6, a_1, a_8, a_7\})| &= |\{v_2, v_3, v_4, v_5, v_6\} \cap \{v_1, v_6, v_2, v_5, v_7\}| \\
&= |\{v_2, v_5, v_6\}| = 3. \\
|V(\{a_2, a_3, a_4, a_5, a_6\}) \cap V(\{a_1, a_8, a_7\})| &= |\{v_2, v_3, v_4, v_5, v_6, v_1\} \cap \{v_1, v_2, v_5, v_7\}| \\
&= |\{v_1, v_2, v_5\}| = 3. \\
|V(\{a_2, a_3, a_4, a_5, a_6, a_1\}) \cap V(\{a_8, a_7\})| &= |\{v_2, v_3, v_4, v_5, v_6, v_1\} \cap \{v_5, v_7, v_2\}| \\
&= |\{v_2, v_5\}| = 2. \\
|V(\{a_2, a_3, a_4, a_5, a_6, a_1, a_8\}) \cap V(\{a_7\})| &= |\{v_2, v_3, v_4, v_5, v_6, v_1, v_7\} \cap \{v_2, v_7\}| \\
&= |\{v_2, v_5\}| = 2.
\end{aligned}
$$

This order of edges gives us a 3-separation. But the planar graph H, is neither an outerplanar graph nor a tree, hence by Lemma 2.2.3 and Lemma 2.2.5 its pathwidth is not equal to 2 and it is certainly not equal to 1. Therefore the planar graph H has pathwidth 3.

An n-chain is a graph which is isomorphic to a path $\mathcal{P}_{n+1}$ thus a 2-chain is isomorphic to $\mathcal{P}_3$.

Definition 2.2.7. *Let H be a graph, we shall call a graph H' obtained by replacing every subgraph of H isomorphic to an i-chain, for some $i > 2$ by a 2-chain, a* ***source graph*** *of H.*

To clarify the concept of a source graph, the diagrams in Figure 2.2 and Figure 2.3 are examples of a planar graph and its source graph, respectively.

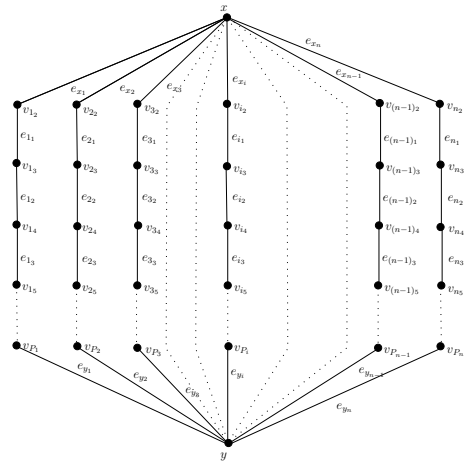

Figure 2.2: A planar graph H.

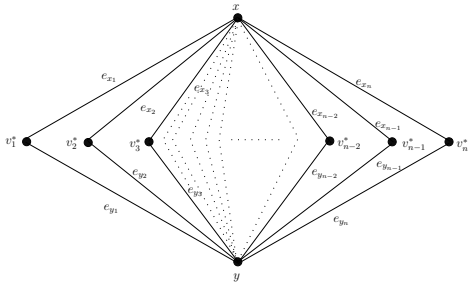

Figure 2.3: H' a source graph of H.

We are in a position to state and prove the main theorem of this chapter on general pathwidth.

Theorem 2.2.8. *Let H be a planar graph and let H' be its source graph. Then the pathwidth of H is equal to the pathwidth of H' i.e. $pw(H) = pw(H')$.*

Proof. Without loss of generality, let the graph H' be a source graph of the graph H as shown in Figure 2.4. Let H^* be the parts of H and H' which are identical as shown in Figure 2.4. Let the edge set of H^* be $E(H^*)$ such that $E(H^*) = E_1(H^*) \cup E_2(H^*)$.

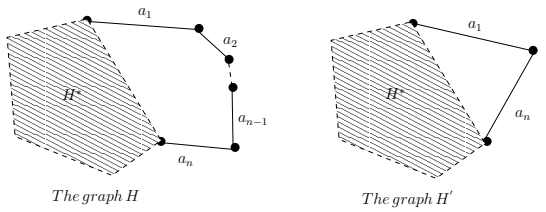

Figure 2.4: The planar graph H and its source graph $H^{'}$

Let $E_1(H^*), a_1, a_n, E_2(H^*)$ be the ordering of edges of H' which gives an n-separation. Then by definition of n-separation, $E_1(H') = \{E_1(H^*), a_1\}$ and $E_2(H') = \{a_n, E_2(H^*)\}$implies $|V(E_1(H')) \cap V(E_2(H'))| \leq n$. Suppose that $a_1 = (v_1, v_p)$, $a_n = (v_p, v_n)$ and $|V(E_1(H')) \cap V(E_2(H'))| = j$ where $j \leq n$, then it is clear that

$$|V(E_1(H^*)) \cap V(E_2(H^*))| = j - 1 \tag{2.1}$$

since $V(a_1) \cap V(a_n) = v_p$ hence $|V(a_1) \cap V(a_n)| = 1$. Now we construct the graph H from H' by adding a chain of $n-2$-edges, $a_2, \cdots, a_{n-1}$ between a_1 and a_n. Then the ordering of edges of H, $E_1(H^*), a_1, a_2, \cdots, a_{n-1}, a_n, E_2(H^*)$ then for all i it is clear that $|(\{a_1, a_2, \cdots, a_i\} \cap \{a_{i+1}, \cdots, a_{n-1}, a_n\})| = 1$. Hence by applying Equation 2.1 and letting $E_1(H) = \{E_1(H^*), a_1, , a_2, \cdots, a_i\}\}$ and $E_2(H) = \{a_{i+1}, \cdots, a_{n-1}, a_n, E_2(H^*)\}$ we get that $|V(E_1(H)) \cap V(E_2(H))| = j$. Hence it follows that we get an n-separation like for the graph H'. Thus $pw(H) = pw(H')$. □

We now define three classes of graphs and apply Theorem 2.2.8 to find the pathwidth of these classes of graphs. In the following Proposition 2.2.9, we let H_1 be the graph shown in the diagram of Figure 2.2.

Proposition 2.2.9. *The pathwidth of the planar graph H_1 is 3.*

Proof. For the planar graph H_1, its source graph $H_1^{'}$ is shown in the diagram of Figure 2.3. By Theorem 2.2.8 it is sufficient to show that the pathwidth of the source graph H_1' is 3. To find the pathwidth of the source graph H_1', we follow the order

$$e_{x_1}, e_{y_1}, e_{y_2}, e_{x_2}, e_{x_3}, e_{y_3}, \cdots, e_{x_{n-2}}, e_{y_{n-2}}, e_{y_{n-1}}, e_{x_{n-1}}, e_{x_n}, e_{y_n}$$

of its edges. Thus,

$$\begin{aligned}
& |V(\{e_{x_1}\}) \cap V(\{e_{y_1}, e_{y_2}, e_{x_2}, \cdots, e_{x_n}, e_{y_n}\})| \\
= \; & |\{x, v_1^*\} \cap \{v_1^*, y, v_2^*, x, \cdots, v_n^*\}| \\
= \; & |\{v_1^*, x\}| = 2. \\
& |V(\{e_{x_1}, e_{y_1}\}) \cap V(\{e_{y_2}, e_{x_2}, \cdots, e_{x_n}, e_{y_n}\})| \\
= \; & |\{x, v_1^*, y\} \cap \{y, v_2^*, x, \cdots, v_n^*\}| \\
= \; & |\{x, y\}| = 2. \\
& |V(\{e_{x_1}, e_{y_1}, e_{y_2}\}) \cap V(\{e_{x_2}, \cdots, e_{x_{n-1}}, e_{x_n}, e_{y_n}\})| \\
= \; & |\{x, v_1^*, y, v_2^*\} \cap \{v_2^*, x, \cdots, v_{n-1}^*, v_n^*, y\}| \\
= \; & |\{x, y, v_2^*\}| = 3. \\
& |V(\{e_{x_1}, e_{y_1}, e_{y_2}, e_{x_2}\}) \cap V(\{e_{x_3}, \cdots, e_{x_{n-1}}, e_{x_n}, e_{y_n}\})| \\
= \; & |\{x, y, v_1^*, v_2^*\} \cap \{x, v_3^*, \cdots, v_{n-1}^*, v_n^*, y\}| \\
= \; & |\{x, y\}| = 2. \\
& |V(\{e_{x_1}, e_{y_1}, e_{y_2}, e_{x_2}, e_{x_3}\}) \cap V(\{e_{y_3}, \cdots, e_{x_{n-1}}, e_{x_n}, e_{y_n}\})| \\
= \; & |\{x, y, v_1^*, v_2^*, v_3^*\} \cap \{v_3^*, y, \cdots, x, v_{n-1}^*, v_n^*\}|
\end{aligned}$$

$$
\begin{aligned}
&= |\{x, y, v_3^*\}| = 3. \\
&\quad \vdots \quad \vdots \quad \vdots \\
&\quad |V(\{e_{x_1}, e_{y_1}, e_{y_2}, \cdots, e_{x_{n-2}}, e_{y_{n-2}}, e_{y_{n-1}}\}) \cap V(\{e_{x_{n-1}}, e_{x_n}, e_{y_n}\})| \\
&= |\{x, y, v_1^*, v_2^*, \cdots, v_{n-2}^*, v_{n-1}^*\} \cap \{v_{n-1}^*, x, v_n^*, y\}| \\
&= |\{x, y, v_{n-1}^*\}| = 3. \\
&\quad |V(\{e_{x_1}, e_{y_1}, e_{y_2}, \cdots, e_{y_{n-2}}, e_{y_{n-1}}, e_{x_{n-1}}\}) \cap V(\{e_{x_n}, e_{y_n}\})| \\
&= |\{x, y, v_1^*, v_2^*, \cdots, v_{n-2}^*, v_{n-1}^*\} \cap \{x, v_n^*, y\}| \\
&= |\{x, y\}| = 2. \\
&\quad |V(\{e_{x_1}, e_{y_1}, e_{y_2}, \cdots, e_{y_{n-1}}, e_{x_{n-1}}, e_{x_n}\}) \cap V(\{e_{y_n}\})| \\
&= |\{x, y, v_1^*, v_2^*, \cdots, v_{n-2}^*, v_{n-1}^*, v_n^*\} \cap \{v_n^*, y\}| \\
&= |\{y, v_n^*\}| = 2.
\end{aligned}
$$

This order of edges gives us a 3-separation. But $H_1^{'}$ is neither an outerplanar graph nor a tree, thus by Lemma 2.2.3 and Lemma 2.2.5 $H_1^{'}$ has pathwidth greater than 2. Thus the minimal pathwidth of H_1 is 3. By Theorem 2.2.8, we get $pw(H_1') = pw(H_1) = 3$. □

In the following Proposition 2.2.10, we let H_2 be the graph shown in the diagram of Figure 2.5.

Proposition 2.2.10. *The pathwidth of the planar graph H_2 is 3.*

Proof. Consider the planar graph H_2 and its source graph $H_2^{'}$ as shown in Figure 2.5. By applying Theorem 2.2.8, we only need to compute the pathwidth of the source graph $H_2^{'}$. To compute the pathwidth of $H_2^{'}$, we follow the order

$$a_1, a_{1_1}, a_{1_m}, a_2, a_3, a_{2_1}, a_{2_m}, \cdots, a_h, a_{h+1}, a_{i_1}, a_{i_m}, a_{h+2}, a_{h+3}, \cdots, a_{l-1}, a_j, a_l, a_{n_1}, a_{n_m}$$

of its edges. But $H_2^{'}$ is neither an outerplanar graph nor a tree, thus by Lemme 2.2.3 and Lemma 2.2.5 $H^{'}$ has pathwidth greater than 2. Thus,

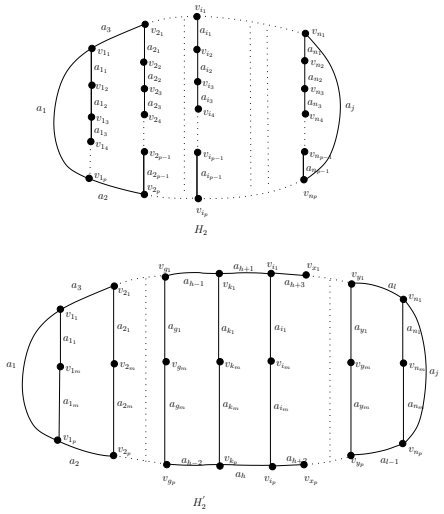

Figure 2.5: The graph H_2 and its source graph $H_2^{'}$.

$$
\begin{aligned}
& |V(\{a_1\}) \cap V(\{a_{1_1}, a_{1_m}, \cdots, a_j, a_l, a_{n_1}, a_{n_m}\})| \\
= \;& |\{v_{1_1}, v_{1_p}\} \cap \{v_{1_1}, v_{1_m}, v_{1_p}, \cdots, v_{y_1}, v_{n_m}\}| \\
= \;& |\{v_{1_1}, v_{1_p}\}| = 2. \\
& |V(\{a_1, a_{1_1}\}) \cap V(\{a_{1_m}, a_2, a_3, \cdots, a_l, a_{n_1}, a_{n_m}\})| \\
= \;& |\{v_{1_1}, v_{1_p}, v_{1_m}\} \cap \{v_{1_m}, v_{1_p}, v_{2_p}, v_{1_1}, \cdots, v_{y_1}, v_{n_m}\}| \\
= \;& |\{v_{1_1}, v_{1_p}, v_{1_m}\}| = 3. \\
& |V(\{a_1, a_{1_1}, a_{1_m}\}) \cap V(\{a_2, a_3, a_{2_1}, a_{2_m}, \cdots, a_l, a_{n_1}, a_{n_m}\})| \\
= \;& |\{v_{1_1}, v_{1_p}, v_{1_m}\} \cap \{v_{1_p}, v_{2_p}, v_{1_1}, \cdots, v_{y_1}, v_{n_m}\}| \\
= \;& |\{v_{1_1}, v_{1_p}\}| = 2. \\
& |V(\{a_1, a_{1_1}, a_{1_m}, a_2\}) \cap V(\{a_3, a_{2_1}, a_{2_m}, \cdots, a_l, a_{n_1}, a_{n_m}\})| \\
= \;& |\{v_{1_1}, v_{1_p}, v_{1_m}, v_{2_p}\} \cap \{v_{1_1}, v_{2_1}, v_{2_m}, v_{2_p}, \cdots, v_{y_1}, v_{n_m}\}| \\
= \;& |\{v_{1_1}, v_{2_p}\}| = 2. \\
& |V(\{a_1, a_{1_1}, a_{1_m}, a_2, a_3\}) \cap V(\{a_{2_1}, a_{2_m}, \cdots, a_l, a_{n_1}, a_{n_m}\})|
\end{aligned}
$$

$$
\begin{aligned}
&= |\{v_{1_1}, v_{1_p}, v_{1_m}, v_{2_p}, v_{2_1}\} \cap \{v_{2_1}, v_{2_m}, v_{2_p}, \cdots, v_{y_1}, v_{n_m}\}| \\
&= |\{v_{2_1}, v_{2_p}\}| = 2. \\
&\quad |V(\{a_1, a_{1_1}, a_{1_m}, a_2, a_3, a_{2_1}\}) \cap V(\{a_{2_m}, \cdots, a_j, a_l, a_{n_1}, a_{n_m}\})| \\
&= |\{v_{1_1}, v_{1_p}, v_{1_m}, v_{2_p}, v_{2_1}, v_{2_m}\} \cap \{v_{2_m}, v_{2_p}, \cdots, v_{y_1}, v_{n_m}\}| \\
&= |\{v_{2_p}, v_{2_m}\}| = 2. \\
&\quad \vdots \quad \vdots \quad \vdots \\
&\quad |V(\{a_1, a_{1_1}, \cdots, a_{k_1}, a_{k_m}, a_h, \cdots, a_{y_1}, a_{y_m}, a_{l-1}, a_j\}) \cap V(\{a_l, a_{n_1}, a_{n_m}\})| \\
&= |\{v_{1_1}, v_{1_p}, v_{1_m}, \cdots, v_{y_1}, v_{y_m}, v_{y_p}, v_{n_p}, v_{n_1}\} \cap \{v_{y_1}, v_{n_1}, v_{n_m}, v_{n_p}\}| \\
&= |\{v_{y_1}, v_{n_1}, v_{n_p}\}| = 3. \\
&\quad |V(\{a_1, a_{1_1}, \cdots, a_{k_1}, a_{k_m}, a_h, \cdots, a_{y_1}, a_{y_m}, a_{l-1}, a_j, a_l\}) \cap V(\{a_{n_1}, a_{n_m}\})| \\
&= |\{v_{1_1}, v_{1_p}, v_{1_m}, \cdots, v_{y_1}, v_{y_m}, v_{y_p}, v_{n_p}, v_{n_1}\} \cap \{v_{n_1}, v_{n_m}, v_{n_p}\}| \\
&= |\{v_{n_p}, v_{n_1}\}| = 2. \\
&\quad |V(\{a_1, a_{1_1}, \cdots, a_{k_1}, a_{k_m}, a_h, \cdots, a_{y_1}, a_{y_m}, a_{l-1}, a_j, a_l, a_{n_1}\}) \cap V(\{a_{n_m}\})| \\
&= |\{v_{1_1}, v_{1_p}, v_{1_m}, \cdots, v_{y_1}, v_{y_m}, v_{y_p}, v_{n_p}, v_{n_1}, v_{n_m}\} \cap \{v_{n_m}, v_{n_p}\}| \\
&= |\{v_{n_m}, v_{n_p}\}| = 2.
\end{aligned}
$$

This order of edges gives us a 3-separation and thus the source graph $H_2^{'}$ has pathwidth 3. By Theorem 2.2.8, we get $pw(H_2') = pw(H_2) = 3$. □

In the following Proposition 2.2.11, we let H_3 be the graph shown in the diagram of Figure 2.6.

Proposition 2.2.11. *The pathwidth of the planar graph H_3 is 3.*

Proof. Consider the planar graph H_3 and its source graph $H_3^{'}$ as shown in Figure 2.6. By applying Theorem 2.2.8, we only need to compute the pathwidth of the source graph $H_3^{'}$. To compute the pathwidth of $H_3^{'}$, we follow the order

$$a_1, a_{x_1}, a_{1_m}, a_2, a_{x_2}, a_{2_m}, a_3, \cdots, a_{k+1}, a_{x_i}, a_{i_m}, a_{k+2}, \cdots, a_{x_y}, a_{y_m}, a_n, a_{x_n}, a_{n_m}, a_{n+1}$$

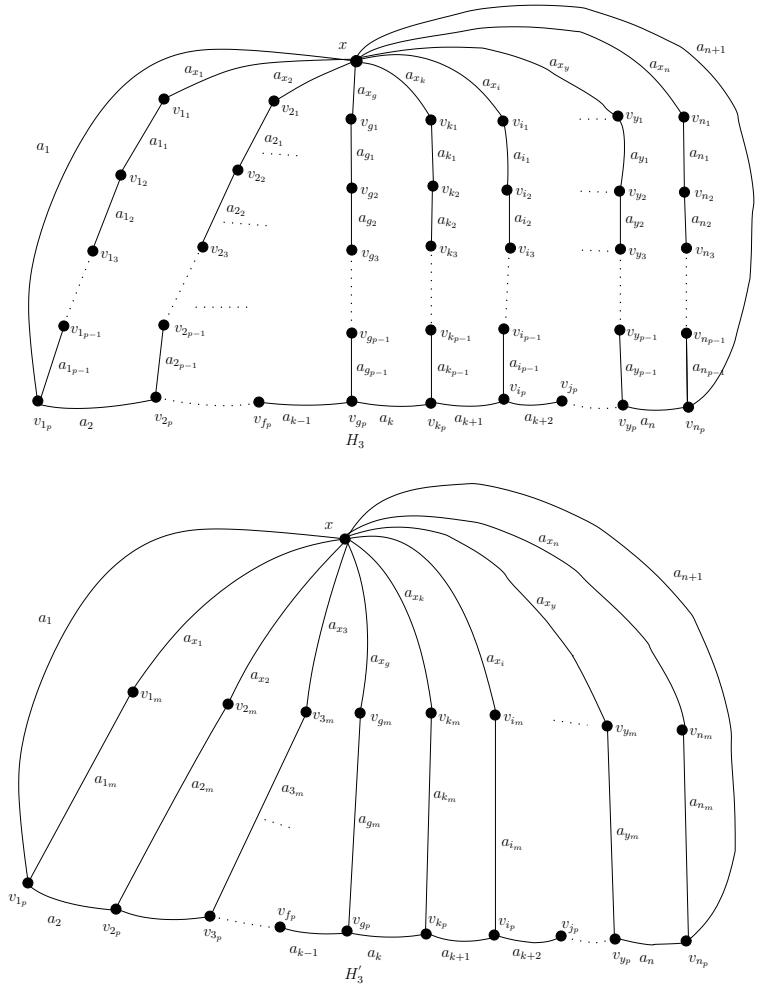

Figure 2.6: The graph H_3 and its source graph H_3'.

of its edges. But $H_3^{'}$ is neither an outerplanar nor a tree thus by applying Lemma 2.2.3 and Lemma 2.2.5 $H_3^{'}$ has pathwidth greater than 2. Thus,

$$
\begin{aligned}
& |V(\{a_1\}) \cap V(\{a_{x_1}, a_{1_m}, a_2, \cdots, a_{n_m}, a_{n+1}\})| \\
= \; & |\{x, v_{1_p}\} \cap \{x, v_{1_m}, v_{1_p}, v_{2_p}, \cdots, v_{n_p}\}| \\
= \; & |\{x, v_{1_p}\}| = 2. \\
& |V(\{a_1, a_{x_1}\}) \cap V(\{a_{1_m}, a_2, \cdots, a_{n_m}, a_{n+1}\})| \\
= \; & |\{x, v_{1_p}, v_{1_m}\} \cap \{v_{1_m}, v_{1_p}, v_{2_p}, \cdots, x, v_{n_p}\}| \\
= \; & |\{x, v_{1_p}, v_{1_m}\}| = 3. \\
& |V(\{a_1, a_{x_1}, a_{1_m}\}) \cap V(\{a_2, \cdots, a_{k-1}, a_{x_g}, a_{g_m}, \cdots, a_{n_m}, a_{n+1}\})| \\
= \; & |\{x, v_{1_p}, v_{1_m}\} \cap \{v_{1_p}, v_{2_p}, x, v_{2_m}, \cdots, v_{g_m}, v_{g_p}, \cdots, v_{n_p}\}| \\
= \; & |\{x, v_{1_p}\}| = 2. \\
& |V(\{a_1, a_{x_1}, a_{1_m}, a_2\}) \cap V(\{a_{x_2}, a_{2_m}, \cdots, a_{x_g}, a_{g_m}, \cdots, a_{n+1}\})| \\
= \; & |\{x, v_{1_p}, v_{1_m}, v_{2_p}\} \cap \{x, v_{2_m}, v_{2_p}, \cdots, v_{g_m}, v_{g_p}, \cdots, v_{n_p}\}| \\
= \; & |\{x, v_{2_p}\}| = 2. \\
& |V(\{a_1, a_{x_1}, a_{1_m}, a_2, a_{x_2}\}) \cap V(\{a_{2_m}, \cdots, a_{x_g}, a_{g_m}, \cdots, a_{n+1}\})| \\
= \; & |\{x, v_{1_p}, v_{1_m}, v_{2_p}, v_{2_m}\} \cap \{v_{2_m}, v_{2_p}, \cdots, x, v_{g_m}, v_{g_p}, \cdots, v_{n_p}\}| \\
= \; & |\{x, v_{2_p}, v_{2_m}\}| = 3. \\
& |V(\{a_1, a_{x_1}, a_{1_m}, a_2, a_{x_2}, a_{2_m}\}) \cap V(\{a_3, a_{x_3}, a_{3_m}, \cdots, a_{k-1}, \cdots, a_{n+1}\})| \\
= \; & |\{x, v_{1_p}, v_{1_m}, v_{2_p}, v_{2_m}\} \cap \{v_{2_p}, v_{3_p}, x, v_{3_m} \cdots, v_{n_p}\}| \\
= \; & |\{x, v_{2_p}\}| = 2. \\
& |V(\{a_1, a_{x_1}, a_{1_m}, a_2, a_{x_2}, a_{2_m}, a_3\}) \cap V(\{a_{x_3}, a_{3_m}, \cdots, a_{k-1}, \cdots, a_{n+1}\})| \\
= \; & |\{x, v_{1_p}, v_{1_m}, v_{2_p}, v_{2_m}, v_{3_p}\} \cap \{x, v_{3_p}, v_{3_m} \cdots, v_{n_p}\}| \\
= \; & |\{x, v_{3_p}\}| = 2. \\
& |V(\{a_1, a_{x_1}, a_{1_m}, a_2, a_{x_2}, a_{2_m}, a_3, a_{x_3}\}) \cap V(\{a_{3_m}, \cdots, a_{k-1}, \cdots, a_{n+1}\})|
\end{aligned}
$$

$$
\begin{aligned}
&= |\{x, v_{1_p}, v_{1_m}, v_{2_p}, v_{2_m}, v_{3_p}, v_{3_m}\} \cap \{v_{3_p}, v_{3_m} \cdots, x, v_{n_p}\}| \\
&= |\{x, v_{3_p}, v_{3_m}\}| = 3. \\
&\quad \vdots \quad \vdots \quad \vdots \\
&\quad |V(\{a_1, a_{x_1}, a_{1_m}, \cdots, a_k, a_{x_k}, \cdots, a_n\}) \cap V(\{a_{x_n}, a_{n_m}, a_{n+1}\})| \\
&= |\{x, v_{1_p}, v_{1_m}, \cdots, v_{g_p}, v_{k_p}, v_{k_m}, \cdots, v_{y_p}, v_{n_p}\} \cap \{x, v_{n_m}, v_{n_p}\}| \\
&= |\{x, v_{n_p}\}| = 2. \\
&\quad |V(\{a_1, a_{x_1}, a_{1_m}, \cdots, a_n, a_{x_n}\}) \cap V(\{a_{n_m}, a_{n+1}\})| \\
&= |\{x, v_{1_p}, v_{1_m}, \cdots, v_{y_p}, v_{n_p}, v_{n_m}\} \cap \{x, v_{n_m}, v_{n_p}\}| \\
&= |\{x, v_{n_m}, v_{n_p}\}| = 3. \\
&\quad |V(\{a_1, a_{x_1}, a_{1_m}, \cdots, a_n, a_{x_n}, a_{n_m}\}) \cap V(\{a_{n+1}\})| \\
&= |\{x, v_{1_p}, v_{1_m}, \cdots, v_{n_m}, v_{n_p}\} \cap \{x, v_{n_p}\}| \\
&= |\{x, v_{n_p}\}| = 2.
\end{aligned}
$$

This order of edges gives us a 3-separation and thus the source graph $H_3^{'}$ has pathwidth 3. The pathwidth of the planar graph H_3 is 3. □

Chapter 3

Pretzel links, their pathwidth and bridge number

3.1 Introduction

Pretzel links are a class of links which are of increasing interest in Knot theory research. In [11] Jin and Zhang studied the zeros of the Jones polynomials for families of pretzel links. In [12] Kim and Lee studied some invariants of pretzel links, and the Conway polynomials of pretzel links are included in their study of these invariants. Furthermore, Zentner [25] studied the representation spaces of pretzel knots.
In this chapter we study pretzel links and graphs extracted from pretzel links. We begin by describing the underlying notion of twists in a link (or knot). Then we state and prove the main results of this chapter. We conclude the chapter by relating the bridge number and pathwidth of certain pretzel links.

3.2 Pretzel links

In this section we define and study pretzel links. From this section to the end of this dissertation we shall only talk of a link, where a knot is a link with one component,

For further details we refer the reader to [1] and [15].

Definition 3.2.1. *A **twist** in a link is a part of a link which consists of strings which are twisted together once or more than once.*

A **twist** is said to be **positive** if and only if at each crossing point of the twist the slope of the overpass is **positive**; otherwise it is said to be **negative**. The diagram (a) in Figure 3.1 is a positive twist and diagram (b) is a negative twist, see [1]. Twists are either **odd** or **even** depending on the number of crossing points of a given twist, see [1]. The diagrams (a) and (b) in Figure 3.1 are examples of odd twists since both of them have five crossing points. We shall refer to diagram (a) in Figure 3.1 as a 5-twist and diagram (b) in Figure 3.1 as a -5-twist.

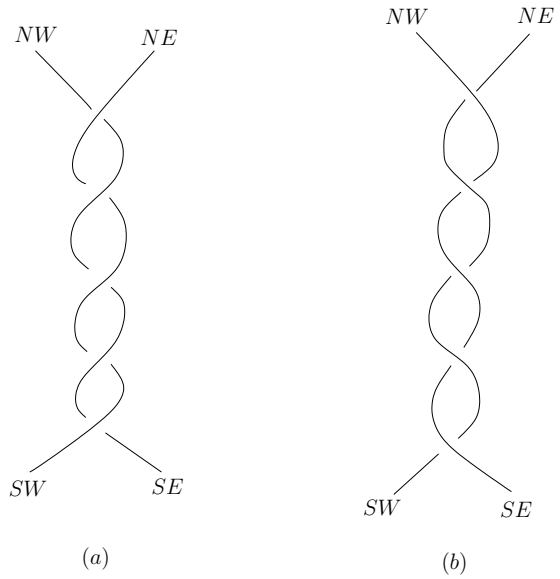

Figure 3.1: (a) positive twist.(b) negative twist.

Definition 3.2.2. *A **pretzel link** $(P_1, P_2, P_3, \cdots, P_n)$ is*

(a) made up of three or more twist, denoted by the P_is, connected together and

(b) the twists are connected in such a way that each northeast (NE) string connects to the adjacent northwest (NW) string, and each southeast (SE) string connects to the adjacent southwest (SW) string.

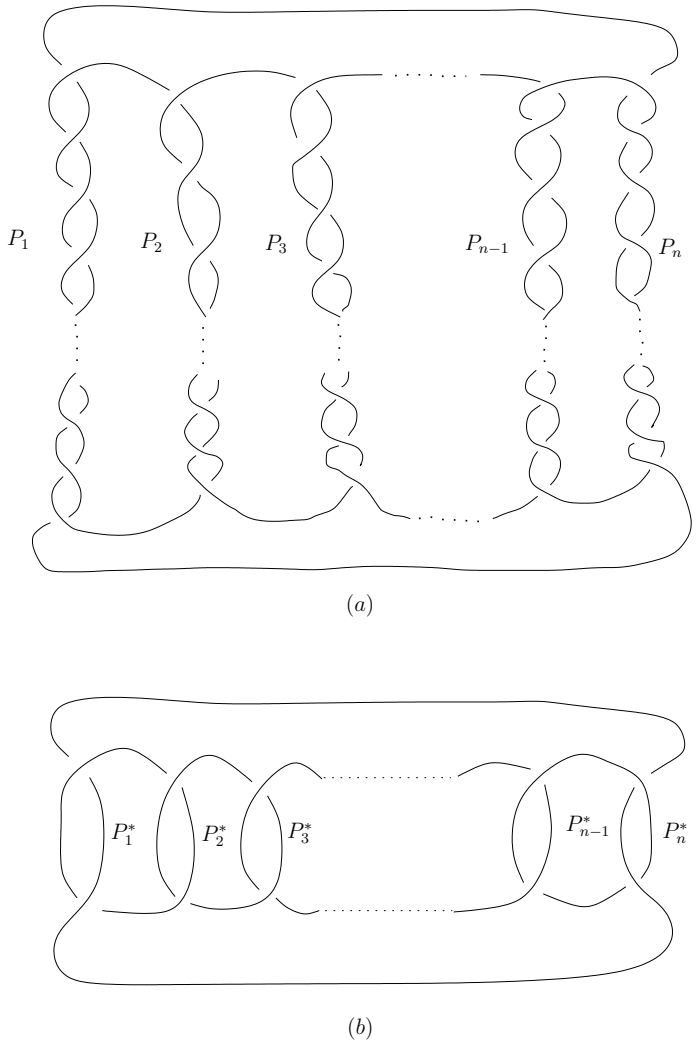

Figure 3.2: A $(P_1, P_2, P_3, \cdots, P_n)$ pretzel link and its source link.

A pretzel link is named according to the number of crossing points in each twist, see [15]. The diagram (a) in Figure 3.2 shows an example of a $(P_1, P_2, P_3, \cdots, P_n)$ pretzel link.

Recall that a source graph of a given planar graph H is obtained by replacing every subgraph of H isomorphic to an i-chain, for some $i > 2$, by a 2-chain. We define a **source link** as follows.

Definition 3.2.3. *Let $L(H)$ be a link corresponding to a planar graph H and $L(H')$ be a link corresponding to the source graph of H, H'. Then we say $L(H')$ is a* ***source link*** *for $L(H)$.*

To clarify the concept, diagrams (a) and (b) in Figure 3.2 are examples of a link and its **source link**, respectively.

We demonstrate how to extract a signed and an unsigned planar graph from pretzel links.

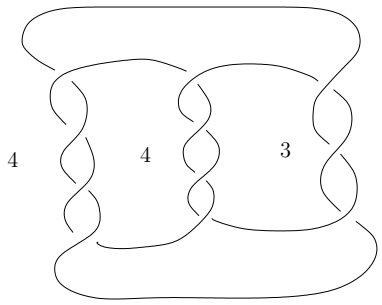

Figure 3.3: A $(4, 4, 3)$ pretzel link.

Without loss of generality, consider the $(4, 4, 3)$ pretzel link shown in Figure 3.3. By applying the checkerboard method described in Chapter 1, Section 1.1.3.1 to the $(4, 4, 3)$ pretzel link shown in Figure 3.3, we obtain the signed planar graph shown in Figure 3.4.

The diagrams in Figure 3.5 are examples of a general (p, q, r) pretzel links. In particular, diagram (b) is the (p, q, r) pretzel link with a slight modification, that is, one of $|p|, |q|, |r|$ equal to 1 which is of interest, and will be discussed later in this dissertation.

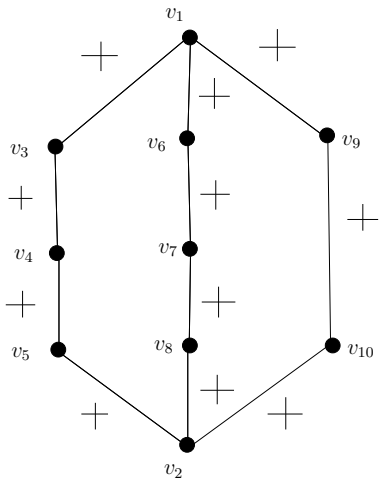

Figure 3.4: A signed graph extracted from a $(4, 4, 3)$ pretzel link.

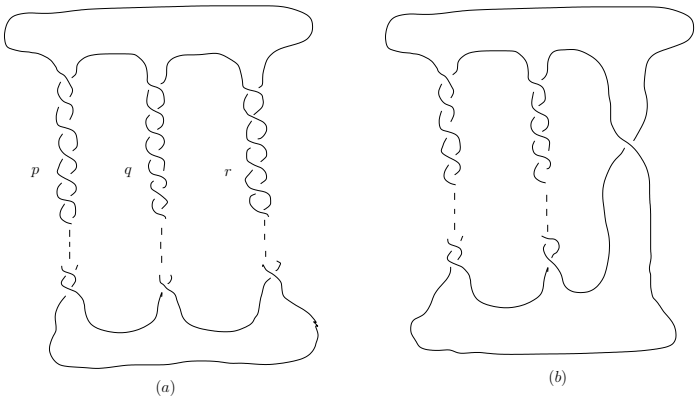

Figure 3.5: (a) The (p, q, r) pretzel link. (b) The (p, q, r) pretzel link with $|r|$ equal to 1.

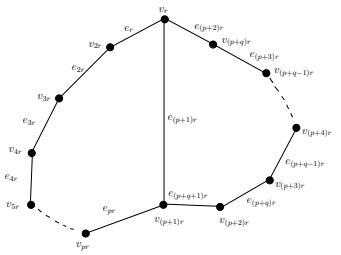

Figure 3.6: The graph H.

The diagram of Figure 3.6 is an example of a planar graph that is extracted from the (p, q, r) pretzel link using the checkerboard method described in Chapter 1, Section 1.1.3.1 with $r = 1$.

3.3 Pretzel links and pathwidth

In this section we state and prove some theorems on the pathwidth of pretzel links. Let $L(H)$ denote a link with an associated planar graph H. It is known that $L(H)$ has different link projections. So a link $L(H)$ has different planar graphs associated to it. A quick example is a planar graph H and its dual graph H^*, they represent the same link. A link $L(H)$ associated to the planar graph H, is said to have **pathwidth** m if H has **pathwidth** m.

Definition 3.3.1. *The* ***pathwidth of a link*** L, *denoted by* $pw(L)$, *is the least pathwidth of all the link projections of the link* L.

Proposition 3.3.2. *Let* K *be a pretzel link* (p, q, r) *with one of* $|p|, |q|, |r|$ *equal to* 1. *Then* K *has pathwidth* 2.

Proof. Without, loss of generality, we consider K, the (p, q, r) pretzel link, shown in diagram (b) of Figure 3.5 and its corresponding planar graph shown in Figure 3.6, we denote it by H. We only need to show the case $r = 1$ and the other two cases are similar. For the case $r = 1$, to compute the pathwidth of the planar graph H we maintain the order

$$e_r, e_{2r}, e_{3r}, e_{4r}, \cdots, e_{pr}, e_{(p+1)r}, e_{(p+2)r}, e_{(p+3)r}, \cdots, e_{(p+q-1)r}, e_{(p+q)r}, e_{(p+q+1)r}$$

of its edges. Hence,

$$\begin{aligned} & |V(\{e_r\}) \cap V(\{e_{2r}, e_{3r}, \cdots, e_{(p+q+1)r}\})| \\ = \; & |\{v_r, v_{2r}\} \cap \{v_r, v_{2r}, \cdots, v_{(p+q)r}\}| \end{aligned}$$

$$
\begin{aligned}
&= |\{v_r, v_{2r}\}| = 2.\\
&\quad |V(\{e_r, e_{2r}\}) \cap V(\{e_{3r}, e_{4r}, \cdots, e_{(p+q+1)r}\})|\\
&= |\{v_r, v_{2r}, v_{3r}\} \cap \{v_r, v_{3r}, \cdots, v_{(p+q)r}\}|\\
&= |\{v_r, v_{3r}\}| = 2.\\
&\quad |V(\{e_r, e_{2r}, e_{3r}\}) \cap V(\{e_{4r}, e_{5r}, \cdots, e_{(p+q+1)r}\})|\\
&= |\{v_r, v_{2r}, v_{3r}, v_{4r}\} \cap \{v_r, v_{4r}, v_{5r}, \cdots, v_{(p+q)r}\}|\\
&= |\{v_r, v_{4r}\}| = 2.\\
&\quad |V(\{e_r, e_{2r}, e_{3r}, e_{4r}\}) \cap V(\{e_{5r}, e_{6r}, \cdots, e_{(p+q+1)r}\})|\\
&= |\{v_r, v_{2r}, v_{3r}, v_{4r}, v_{5r}\} \cap \{v_r, v_{5r}, v_{6r}, \cdots, v_{(p+q)r}\}|\\
&= |\{v_r, v_{5r}\}| = 2.\\
&\quad \vdots \quad \vdots \quad \vdots\\
&\quad |V(\{e_r, e_{2r}, e_{3r}, \cdots, e_{pr}\}) \cap V(\{e_{(p+1)r}, e_{(p+2)r}, \cdots, e_{(p+q+1)r}\})|\\
&= |\{v_r, v_{2r}, v_{3r}, v_{4r}, v_{5r}, \cdots, v_{(p+1)r}\} \cap \{v_r, v_{(p+1)r}, \cdots, v_{(p+q)r}\}|\\
&= |\{v_r, v_{(p+1)r}\}| = 2.\\
&\quad |V(\{e_r, e_{2r}, \cdots, e_{pr}, e_{(p+1)r}\}) \cap V(\{e_{(p+2)r}, e_{(p+3)r}, \cdots, e_{(p+q+1)r}\})|\\
&= |\{v_r, v_{2r}, \cdots, v_{(p+1)r}\} \cap \{v_r, v_{(p+1)r}, \cdots, v_{(p+q)r}\}|\\
&= |\{v_r, v_{(p+1)r}\}| = 2.\\
&\quad |V(\{e_r, e_{2r}, e_{3r}, \cdots, e_{(p+1)r}, e_{(p+2)r}\}) \cap V(\{e_{(p+3)r}, \cdots, e_{(p+q+1)r}\})|\\
&= |\{v_r, v_{2r}, \cdots, v_{(p+1)r}, v_{(p+q)r}\} \cap \{v_{(p+q)r}, v_{(p+q-1)r}, \cdots, v_{(p+1)r}\}|\\
&= |\{v_{(p+1)r}, v_{(p+q)r}\}| = 2.\\
&\quad \vdots \quad \vdots \quad \vdots\\
&\quad |V(\{e_r, e_{2r}, e_{3r}, \cdots, e_{(p+q-1)r}\}) \cap V(\{e_{(p+q)r}, e_{(p+q+1)r}\})|\\
&= |\{v_r, v_{2r}, v_{3r}, v_{4r}, v_{5r}, \cdots, v_{(p+q)r}, v_{(p+1)r}, v_{(p+3)r}\} \cap \{v_{(p+3)r}, v_{(p+2)r}, v_{(p+1)r}\}|\\
&= |\{v_{(p+1)r}, v_{(p+3)r}\}| = 2.\\
&\quad |V(\{e_r, e_{2r}, e_{3r}, \cdots, e_{(p+q)r}\}) \cap V(\{e_{(p+q+1)r}\})|
\end{aligned}
$$

$$= |\{v_r, v_{2r}, v_{3r}, v_{4r}, v_{5r}, \cdots, v_{(p+1)r}, v_{(p+2)r}, v_{(p+3)r}\} \cap \{v_{(p+1)r}, v_{(p+2)r}\}|$$
$$= |\{v_{(p+1)r}, v_{(p+2)r}\}| = 2.$$

This order of edges gives us a 2-separation. Therefore, $pw(H) = 2$. But, it is clear that H is not a tree, therefore the pathwidth of H is greater than 1 by applying Lemma 2.2.5. Hence the pathwidth of H is equal to 2. By Definition 3.3.1, it follows that K, has pathwidth 2. □

This result can also be verified by applying Lemma 2.2.3, since H in Proposition 3.3.2 is an outerplanar graph.

For the following Lemma 3.3.3, let H be the planar graph shown in the diagram of Figure 3.7.

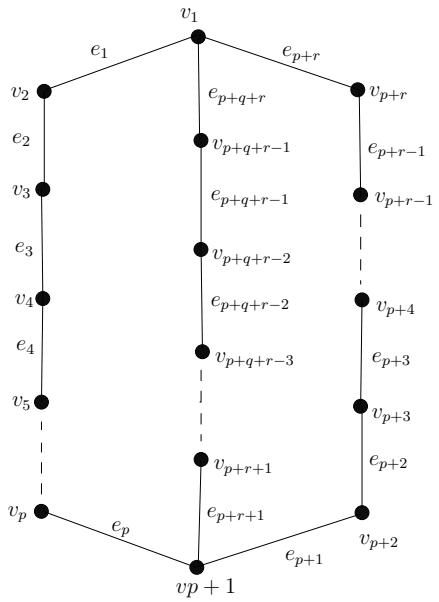

Figure 3.7: A planar graph extracted from a (p, q, r) pretzel link.

Lemma 3.3.3. *Let K be a pretzel link (p, q, r) with all of p, q, r not equal to 0 or ± 1. Then K has the corresponding planar graph H.*

Proof. Without loss of generality, consider the (p, q, r) pretzel link shown in Figure 3.5 diagram (a). Applying the checkerboard method described in Chapter 1, Section 1.1.3.1, to the (p, q, r) pretzel link results in the planar graph H. □

We are in a position to state and prove the main results of this section.

Proposition 3.3.4. *Let $K = (p, q, r)$ be a pretzel link. Then K has pathwidth 3 if and only if all of $|p|, |q|, |r|$ are strictly greater than 1.*

Proof. $\Rightarrow$ Assume that a (p, q, r) pretzel link has pathwidth 3. If one of $|p|, |q|, |r|$ is equal to 1 then a (p, q, r) pretzel link has pathwidth 2, by Proposition 3.3.2. This contradicts the fact that (p, q, r) pretzel link has pathwidth 3 and its corresponding planar graph has pathwidth 3. Hence, $|p|, |q|, |r|$ are strictly greater than 1.

$\Leftarrow$ Assume that $|p|, |q|, |r|$ are strictly greater than 1. Since $|p|, |q|, |r|$ are strictly greater than 1, by Lemma 3.3.3 the planar graph corresponding to a (p, q, r) pretzel link is as shown in the diagram of Figure 3.7, and to compute its pathwidth we follow the order

$$e_1, e_2, e_3, \cdots, e_p, e_{p+1}, e_{p+2}, \cdots, e_{p+q+r-2}, e_{p+q+r-1}, e_{p+q+r}$$

of its edges. Thus,

$$\begin{aligned}
& |V(\{e_1\}) \cap V(\{e_2, e_3, \cdots, e_{p+q+r}\})| \\
= \; & |\{v_1, v_2\} \cap \{v_1, v_2, \cdots, v_{p+q+r-1}\}| \\
= \; & |\{v_1, v_2\}| = 2. \\
& |V(\{e_1, e_2\}) \cap V(\{e_3, e_4, \cdots, e_{p+q+r}\})| \\
= \; & |\{v_1, v_2, v_3\} \cap \{v_1, v_3, \cdots, v_{p+q+r-1}\}| \\
= \; & |\{v_1, v_3\}| = 2. \\
& |V(\{e_1, e_2, e_3\}) \cap V(\{e_4, e_5, \cdots, e_{p+q+r}\})| \\
= \; & |\{v_1, v_2, v_3, v_4\} \cap \{v_1, v_4, v_5, \cdots, v_{p+q+r-1}\}| \\
= \; & |\{v_1, v_4\}| = 2.
\end{aligned}$$

$$
\begin{aligned}
& |V(\{e_1, e_2, e_3, e_4\}) \cap V(\{e_5, e_6, \cdots, e_{p+q+r}\})| \\
= \; & |\{v_1, v_2, v_3, v_4, v_5\} \cap \{v_1, v_5, v_6, \cdots, v_{p+q+r-1}\}| \\
= \; & |\{v_1, v_5\}| = 2. \\
& |V(\{e_1, e_2, e_3, e_4, e_5\}) \cap V(\{e_6, e_7, \cdots, e_{p+q+r}\})| \\
= \; & |\{v_1, v_2, v_3, v_4, v_5, v_6\} \cap \{v_1, v_6, v_7, \cdots, v_{p+q+r-1}\}| \\
= \; & |\{v_1, v_6\}| = 2. \\
& \vdots \quad \vdots \quad \vdots \\
& |V(\{e_1, e_2, e_3, \cdots, e_p\}) \cap V(\{e_{p+1}, e_{p+2}, \cdots, e_{p+q+r}\})| \\
= \; & |\{v_1, v_2, v_3, v_4, v_5, \cdots, v_{p+1}\} \cap \{v_1, v_{p+1}, \cdots, v_{p+q+r-1}\}| \\
= \; & |\{v_1, v_{p+1}\}| = 2. \\
& |V(\{e_1, e_2, \cdots, e_p, e_{p+1}\}) \cap V(\{e_{p+2}, e_{p+3}, \cdots, e_{p+q+r}\})| \\
= \; & |\{v_1, v_2, \cdots, v_{p+2}\} \cap \{v_1, \cdots, v_{p+1}, \cdots, v_{p+q+r-1}\}| \\
= \; & |\{v_1, v_{p+1}, v_{p+2}\}| = 3. \\
& |V(\{e_1, e_2, e_3, \cdots, e_{p+1}, e_{p+2}\}) \cap V(\{e_{p+3}, \cdots, e_{p+q+r}\})| \\
= \; & |\{v_1, v_2, \cdots, v_{p+1}, v_{p+2}, v_{p+3}\} \cap \{v_1, \cdots, v_{p+1}, \cdots, v_{p+q+r-1}\}| \\
= \; & |\{v_1, v_{p+1}, v_{p+3}\}| = 3. \\
& |V(\{e_1, e_2, e_3, \cdots, e_{p+2}, e_{p+3}\}) \cap V(\{e_{p+4}, \cdots, e_{p+r+1,\cdots,e_{p+q+r}}\})| \\
= \; & |\{v_1, v_2, \cdots, v_{p+3}, v_{p+4}\} \cap \{v_1, v_{p+1}, v_{p+4}, \cdots, v_{p+q+r-1}\}| \\
= \; & |\{v_1, v_{p+1}, v_{p+4}\}| = 3. \\
& \vdots \quad \vdots \quad \vdots \\
& |V(\{e_1, e_2, \cdots, e_{p+r-1}\}) \cap V(\{e_{p+r}, e_{p+r+1}, \cdots, e_{p+q+r}\})| \\
= \; & |\{v_1, v_2, v_3, \cdots, v_{p+r-1}, v_{p+r}\} \cap \{v_1, v_{p+1}, \cdots, v_{p+r}, v_{p+r+1}, \cdots, v_{p+q+r-1}\}| \\
= \; & |\{v_1, v_{p+r}, v_{p+1}\}| = 3. \\
& |V(\{e_1, e_2, \cdots, e_{p+r}\}) \cap V(\{e_{p+r+1}, \cdots, e_{p+q+r-1}, e_{p+q+r}\})| \\
= \; & |\{v_1, v_2, v_3, \cdots, v_{p+r}\} \cap \{v_1, v_{p+1}, v_{p+r+1}, \cdots, v_{p+q+r-1}\}|
\end{aligned}
$$

$$
\begin{aligned}
&= |\{v_1, v_{p+1}\}| = 2. \\
&\quad |V(\{e_1, e_2, \cdots, e_{p+r+1}\}) \cap V(\{e_{p+r+2}, \cdots, e_{p+q+r-1}, e_{p+q+r}\})| \\
&= |\{v_1, v_2, v_3, \cdots, v_{p+r+1}\} \cap \{v_1, v_{p+r+1}, v_{p+r+2}, \cdots, v_{p+q+r-1}\}| \\
&= |\{v_1, v_{p+r+1}\}| = 2. \\
&\quad \vdots \quad \vdots \quad \vdots \\
&\quad |V(\{e_1, e_2, \cdots, e_{p+q+r-1}\}) \cap V(\{e_{p+q+r}\})| \\
&= |\{v_1, v_2, v_3, \cdots, v_{p+q+r-2, v_{p+q+r-1}}\} \cap \{v_1, v_{p+q+r-1}\}| \\
&= |\{v_1, v_{p+q+r-1}\}| = 2.
\end{aligned}
$$

This order of edges gives us a 3-separation, hence $pw(H) = 3$. But H is neither an outerplanar graph nor a tree, so by Lemma 2.2.3 and Lemma 2.2.5 respectively, the pathwidth of H is greater than 2. By Definition 3.3.1, K, has pathwidth 3. □

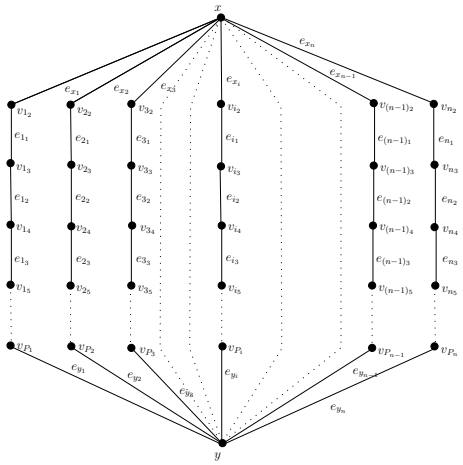

Figure 3.8: H, a planar graph extracted from a $(P_1, P_2, P_3, \cdots, P_n)$ pretzel link.

Theorem 3.3.5. *Let $K = (P_1, P_2, P_3, \cdots, P_n)$ be a pretzel link with all the P_is not equal to 0 or ± 1. Then K has pathwidth 3.*

Proof. Without loss of generality, let $K = (P_1, P_2, P_3, \cdots, P_n)$, then its corresponding

planar graph is the planar graph H shown in Figure 3.8. By Proposition 2.2.9, graph H has pathwidth 3. Hence by Definition 3.3.1, K, has pathwidth 3. □

3.4 Pathwidth vs Bridge Number of general links

In this section we introduce the reader to the famous notion of **bridge number** of a link and we relate it to the pathwidth of the link. For further details we refer the reader to [1].

3.4.1 Bridge Number

Let L be a link projection, then a **bridge** in a link projection, L, occurs in more than one section of L where we have one segment of L as an overpass in more than one crossing point of L, see diagram (a) of Figure 3.9.

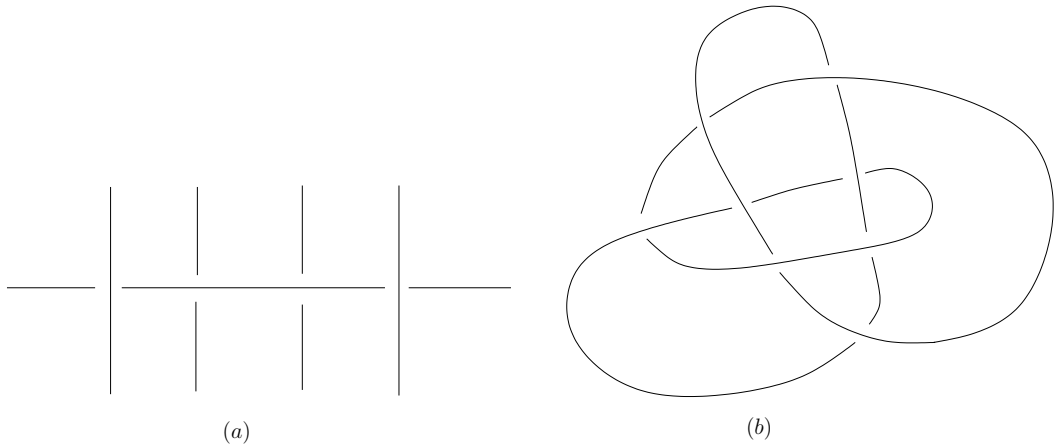

Figure 3.9: (a) Occurrence of a bridge.(b) A knot that has 5 bridges

Then the **bridge number** of a given link, L, is defined as follows:

Definition 3.4.1. *Let L be a link projection. Then the **bridge number**, $br(L)$, of a link projection L is given by the minimum number of **bridges** occurring in L.*

The diagram (b) in Figure 3.9 is an example of a link projection with 5 bridges.

3.4.2 Famous Knots with Bridge number 3 in relation to their pathwidth

In this section we consider some of the well known links that have bridge number 3. Our purpose for doing so, is to try and compare their pathwidth with their bridge number. For further details of the links considered in this section we refer the reader to [16].

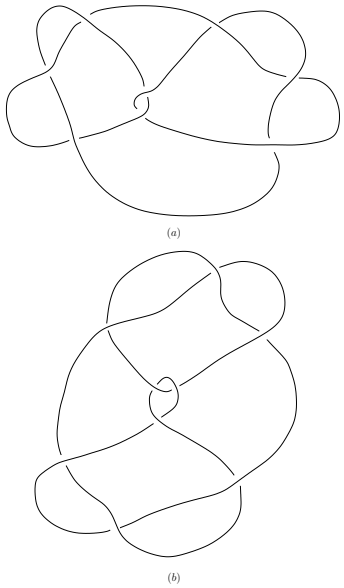

Figure 3.10: (a) 8_5 knot. (b) A 9_{16} knot.

Amongst other links with bridge number 3 we have the 8_5 knot and the 9_{16} knot, as shown in Figure 3.10. We compute the pathwidth of both the 8_5 knot and the 9_{16} knot. So we use the checkerboard method described in Chapter 1, Section 1.1.3.1 to extract the planar graph from each knot. We will compute the pathwidth of each knot separately.

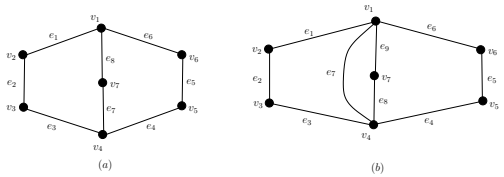

Figure 3.11: (a) A planar graph extracted from the 8_5 knot. (b) A planar graph extracted from the 9_{16} knot.

The diagrams of Figure 3.11 are illustrations of the planar graphs extracted from the 8_5 knot and the 9_{16} knot, respectively, using the checkerboard method described in Chapter 1, Section 1.1.3.1. We now compute the pathwidth of the planar graphs shown in Figure 3.11 as follows:

Example 3.4.2. In computing the pathwidth of the planar graph shown in the diagram of Figure 3.11 diagram (a), we maintain the order

$$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$$

of its edges.

$$\begin{aligned}
|V(\{e_1\}) \cap V(\{e_2, e_3, e_4, e_5, e_6, e_7, e_8\})| &= |\{v_1, v_2\} \cap \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}| \\
&= |\{v_1, v_2\}| = 2. \\
|V(\{e_1, e_2\}) \cap V(\{e_3, e_4, e_5, e_6, e_7, e_8\})| &= |\{v_1, v_2, v_3\} \cap \{v_1, v_3, v_4, v_5, v_6, v_7\}| \\
&= |\{v_1, v_3\}| = 2. \\
|V(\{e_1, e_2, e_3\}) \cap V(\{e_4, e_5, e_6, e_7\})| &= |\{v_1, v_2, v_3, v_4\} \cap \{v_1, v_4, v_5, v_6, v_7\}| \\
&= |\{v_1, v_4\}| = 2. \\
|V(\{e_1, e_2, e_3, e_4\}) \cap V(\{e_5, e_6, e_7, e_8\})| &= |\{v_1, v_2, v_3, v_4, v_5\} \cap \{v_1, v_4, v_5, v_6, v_7\}| \\
&= |\{v_1, v_4, v_5\}| = 3. \\
|V(\{e_1, e_2, e_3, e_4, e_5\}) \cap V(\{e_6, e_7, e_8\})| &= |\{v_1, v_2, v_3, v_4, v_5, v_6\} \cap \{v_1, v_4, v_6, v_7\}| \\
&= |\{v_1, v_4, v_6\}| = 3.
\end{aligned}$$

$$
\begin{aligned}
|V(\{e_1,e_2,e_3,e_4,e_5,e_6\}) \cap V(\{e_7,e_8\})| &= |\{v_1,v_2,v_3,v_4,v_5,v_6\} \cap \{v_1,v_4,v_7\}| \\
&= |\{v_1,v_4\}| = 2. \\
|V(\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\}) \cap V(\{e_8\})| &= |\{v_1,v_2,v_3,v_4,v_5,v_6,v_7\} \cap \{v_1,v_7\}| \\
&= |\{v_1,v_7\}| = 2.
\end{aligned}
$$

This order of edges gives us a 3-separation and thus the planar graph shown in diagram (a) of Figure 3.11 has pathwidth 3.

Example 3.4.3. We now compute the pathwidth of the planar graph shown in the diagram of Figure 3.11 diagram (b), we maintain the order

$$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$$

of its edges.

$$
\begin{aligned}
|V(\{e_1\}) \cap V(\{e_2,e_3,e_4,e_5,e_6,e_7,e_8\})| &= |\{v_1,v_2\} \cap \{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}| \\
&= |\{v_1,v_2\}| = 2. \\
|V(\{e_1,e_2\}) \cap V(\{e_3,e_4,e_5,e_6,e_7,e_8\})| &= |\{v_1,v_2,v_3\} \cap \{v_1,v_3,v_4,v_5,v_6,v_7\}| \\
&= |\{v_1,v_3\}| = 2. \\
|V(\{e_1,e_2,e_3\}) \cap V(\{e_4,e_5,e_6,e_7\})| &= |\{v_1,v_2,v_3,v_4\} \cap \{v_1,v_4,v_5,v_6,v_7\}| \\
&= |\{v_1,v_4\}| = 2. \\
|V(\{e_1,e_2,e_3,e_4\}) \cap V(\{e_5,e_6,e_7,e_8\})| &= |\{v_1,v_2,v_3,v_4,v_5\} \cap \{v_1,v_4,v_5,v_6,v_7\}| \\
&= |\{v_1,v_4,v_5\}| = 3. \\
|V(\{e_1,e_2,e_3,e_4,e_5\}) \cap V(\{e_6,e_7,e_8\})| &= |\{v_1,v_2,v_3,v_4,v_5,v_6\} \cap \{v_1,v_4,v_6,v_7\}| \\
&= |\{v_1,v_4,v_6\}| = 3. \\
|V(\{e_1,e_2,e_3,e_4,e_5,e_6\}) \cap V(\{e_7,e_8\})| &= |\{v_1,v_2,v_3,v_4,v_5,v_6\} \cap \{v_1,v_4,v_7\}| \\
&= |\{v_1,v_4\}| = 2. \\
|V(\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\}) \cap V(\{e_8\})| &= |\{v_1,v_2,v_3,v_4,v_5,v_6,v_7\} \cap \{v_1,v_7\}| \\
&= |\{v_1,v_7\}| = 2.
\end{aligned}
$$

This order of edges gives us a 3-separation and thus the planar graph shown in diagram (b) of Figure 3.11 has pathwidth 3.

Since we know that the 8_5 knot and the 9_{16} knot have bridge number 3, that is, $br(8_5) = br(9_{16}) = 3$ and path-with 3, this leads us to the question *if a link has pathwidth* 3 *does that mean it has bridge number* 3*?* We also see that the pretzel link has pathwidth 3 and again it leads us to this question *does it have bridge number* 3*?*. The Table 3.1 is a table which compares some known bridge number 3 links with their pathwidth, For further details we refer the reader to [16].

Table 3.1: Bridge number vs. pathwidth.

Link (or Knot)	Bridge Number	pathwidth
8_5	3	3
8_{10}	3	3
8_{15}	3	3
8_{16}	3	3
8_{17}	3	3
8_{18}	3	3
8_{19}	3	3
8_{20}	3	3
8_{21}	3	3
9_{16}	3	3
9_{22}	3	3
9_{24}	3	3
9_{25}	3	3
9_{28}	3	3

3.5 Pathwidth vs Bridge number of certain pretzel links

Another famous bridge 3 link is the $(3,3,3)$ pretzel link. The diagrams in Figure 3.12 are the $(3,3,3)$ pretzel link and its corresponding planar graph.

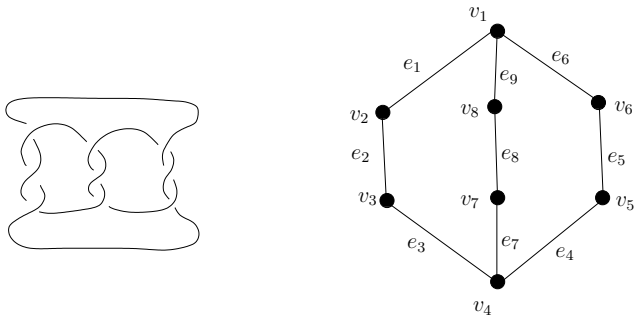

Figure 3.12: (a) The $(3,3,3)$ pretzel link. (b) The planar graph extracted from the $(3,3,3)$ pretzel link.

Example 3.5.1. The computation of the pathwidth of the planar graph shown in Figure 3.12 diagram (b) is done by maintaining the order

$$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$$

of its edges. Hence,

$$\begin{aligned}
|V(\{e_1\}) \cap V(\{e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\})| &= |\{v_1, v_2\} \cap \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}| \\
&= |\{v_1, v_2\}| = 2. \\
|V(\{e_1, e_2\}) \cap V(\{e_3, e_4, e_5, e_6, e_7, e_8, e_9\})| &= |\{v_1, v_2, v_3\} \cap \{v_1, v_3, v_4, v_5, v_6, v_7, v_8\}| \\
&= |\{v_1, v_3\}| = 2. \\
|V(\{e_1, e_2, e_3\}) \cap V(\{e_4, e_5, e_6, e_7, e_8, e_9\})| &= |\{v_1, v_2, v_3, v_4\} \cap \{v_1, v_4, v_5, v_6, v_7, v_8\}|
\end{aligned}$$

$$
\begin{aligned}
& & = & \ |\{v_1, v_4\}| = 2. \\
|V(\{e_1, e_2, e_3, e_4\}) \cap V(\{e_5, e_6, e_7, e_8, e_9\})| & = & |\{v_1, v_2, v_3, v_4, v_5\} \cap \{v_1, v_4, v_5, v_6, v_7, v_8\}| \\
& = & |\{v_1, v_4, v_5\}| = 3. \\
|V(\{e_1, e_2, e_3, e_4, e_5\}) \cap V(\{e_6, e_7, e_8, e_9\})| & = & |\{v_1, v_2, v_3, v_4, v_5, v_6\} \cap \{v_1, v_4, v_6, v_7, v_8\}| \\
& = & |\{v_1, v_4, v_6\}| = 3. \\
|V(\{e_1, e_2, e_3, e_4, e_5, e_6\}) \cap V(\{e_7, e_8, e_9\})| & = & |\{v_1, v_2, v_3, v_4, v_5, v_6\} \cap \{v_1, v_4, v_7, v_8\}| \\
& = & |\{v_1, v_4\}| = 2. \\
|V(\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}) \cap V(\{e_8, e_9\})| & = & |\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \cap \{v_1, v_7, v_8\}| \\
& = & |\{v_1, v_7\}| = 2. \\
|V(\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}) \cap V(\{e_9\})| & = & |\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \cap \{v_1, v_8\}| \\
& = & |\{v_1, v_8\}| = 2.
\end{aligned}
$$

This order of edges gives us a 3-separation and thus the planar graph in Figure 3.12 has pathwidth 3, and this also implies that the $(3, 3, 3)$ pretzel link has pathwidth 3.

We state the following theorem concerning the bridge number of a pretzel link without proof, For further details we refer the reader to [25]. This theorem is crucial in getting the main result of this chapter.

Theorem 3.5.2. *Let $K = (p, q, r)$ be a pretzel link with all of $|p|, |q|, |r|$ strictly greater than* 1. *If all of $|p|, |q|, |r|$ are pairwise coprime, then a pretzel link $K = (p, q, r)$ has bridge number* 3.

We are now in a position to state the following main result of this chapter.

Theorem 3.5.3. *Let $K = (p, q, r)$ be a pretzel link with all of $|p|, |q|, |r|$ strictly greater than* 1. *If all of $|p|, |q|, |r|$ are pairwise coprime, then $pw(K) = br(K)$.*

Proof. By Proposition 3.3.4 a pretzel link K has pathwidth 3 and by Theorem 3.5.2 K has bridge number 3. Therefore $pw(K) = br(K)$. □

Chapter 4

Component number of Pretzel links

4.1 Introduction

The **component number** of a link is one of the invariants of link which is of interest in Knot theory research. There are many methods for determining the **component number** of links. Determining the **component number** of links has also been the topic of major interest in Knot theory research. In [5] Endo studied the component number of suspended trees. In [10] Jin, Dong and Tay have studied the component number of links corresponding to lattices. Jiang and Jin [9] have studied the component number of links corresponding to Sierpinski graphs.

In this chapter we consider two methods for computing the **component number** of a link. In particular, we describe the Tutte polynomial and the deletion-contraction methods for computing the **component number** of a given link. We will conclude this chapter by applying the deletion-contraction method to Pretzel links. In this dissertation we define the **component number** of links as follows:

Definition 4.1.1. *Let $\mathcal{L}$ be a link. Then the* ***component number*** *of $\mathcal{L}$ denoted by $\zeta(\mathcal{L})$ is the number of components which form $\mathcal{L}$.*

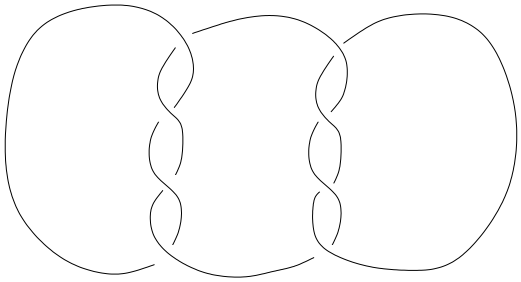

Figure 4.1: A 3-component link.

If the **component number** of $\mathcal{L}$ is **n** we say that $\mathcal{L}$ is an **n**-component link. The diagram of Figure 4.1 is an example of a 3-component link.

4.2 Tutte polynomial method

In this section we use the Tutte polynomial described in Chapter 1, Definition 1.1.40, to compute the component number of a link corresponding to an unsigned planar graph. We only compute the Tutte polynomial based on an **unsigned planar graph** corresponding to a link, since disregarding intentionally its signs will not change the number of components of a corresponding link, see [5].

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be an **unsigned planar graph** and let $\mathcal{L}_{\mathcal{H}}$ denote the **link projection** corresponding to $\mathcal{H}$. Through the Tutte polynomial described in Chapter 1, Definition 1.1.40, of $\mathcal{H}$ we can compute the **component number** of the **link projection**, $\mathcal{L}_{\mathcal{H}}$, corresponding to $\mathcal{H}$. This is done by evaluating the Tutte polynomial of an **unsigned planar graph**, $\mathcal{H}$, corresponding to a **link projection**, $\mathcal{L}_{\mathcal{H}}$, at $x = -1$ and $y = -1$. Thus we get that

$$T(\mathcal{H}, -1, -1) = (-1)^{|E(\mathcal{H})|}(-2)^{\zeta(\mathcal{L}_{\mathcal{H}})-1}$$

where $E(\mathcal{H})$ is the edge set of the **unsigned planar graph**, $\mathcal{H}$, and $\zeta(\mathcal{L}_{\mathcal{H}})$ is the **component number** of the **link projection**, $\mathcal{L}_{\mathcal{H}}$, corresponding to $\mathcal{H}$, see [22]. We demonstrate the Tutte polynomial method in the following example:

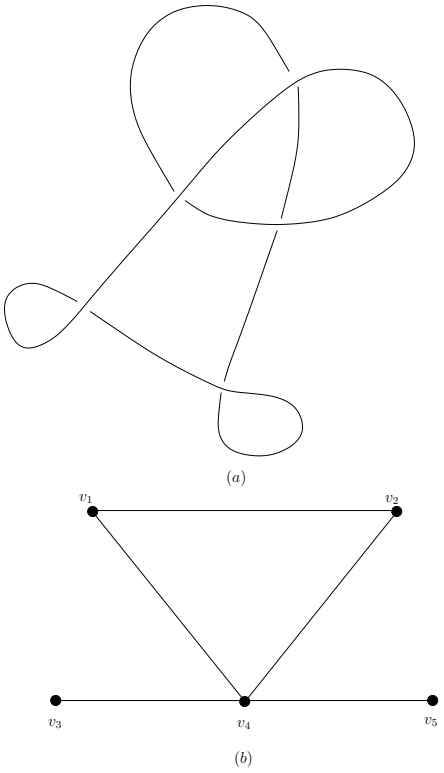

Figure 4.2: (a) The link $\mathcal{L}_{\mathcal{H}}$. (b) The unsigned planar graph $\mathcal{H}$.

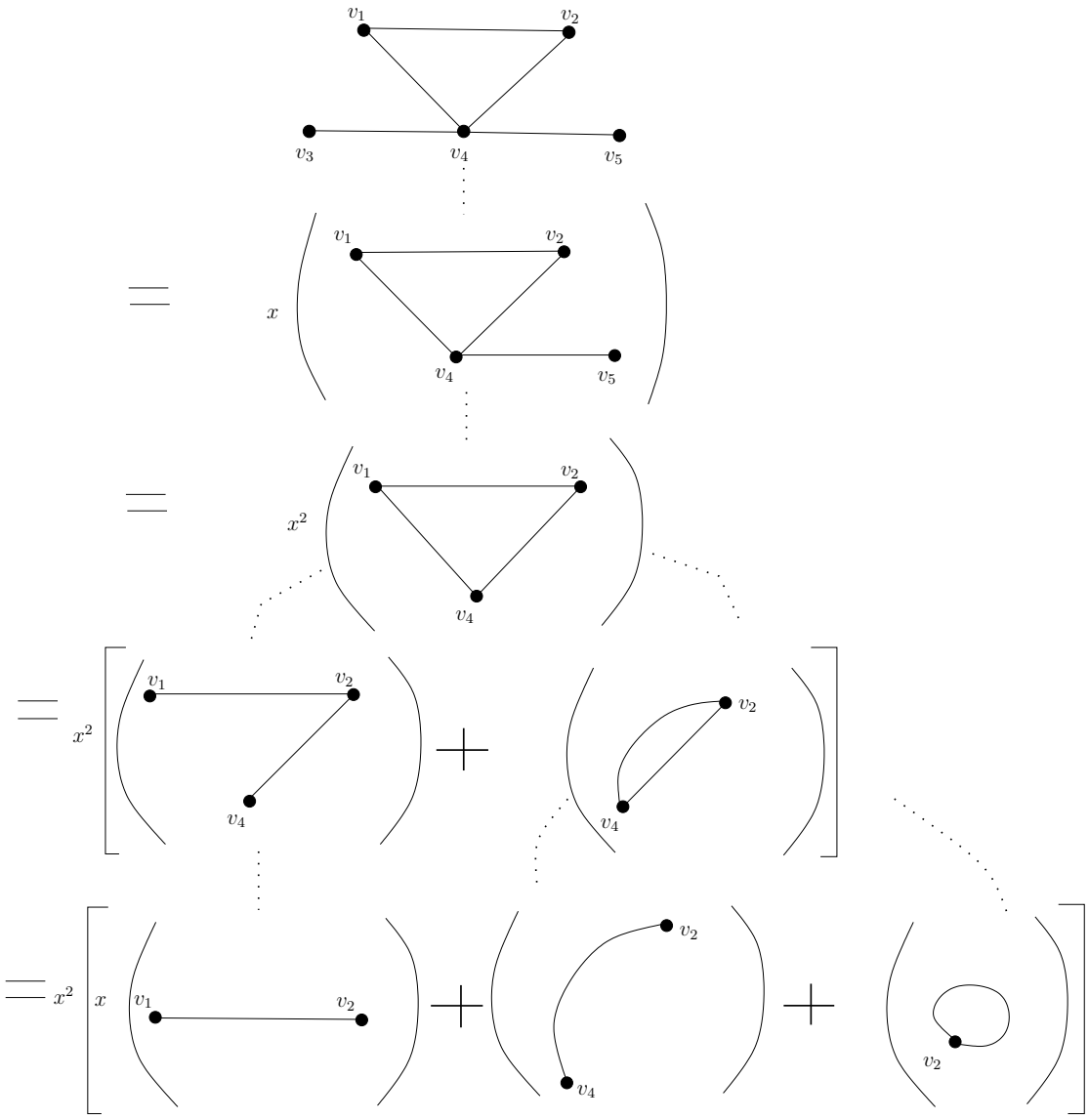

Figure 4.3: Tutte polynomial recursive algorithm on a graph H.

Example 4.2.1. Let $\mathcal{H} = \{v_1v_2, v_1v_4, v_2v_4, v_3v_4, v_4v_5\}$ be the unsigned planar graph shown in diagram (b) of Figure 4.2. The corresponding link of $\mathcal{H}$, $\mathcal{L}_{\mathcal{H}}$, is the diagram (a) of Figure 4.2. We calculate the Tutte polynomial $T(\mathcal{H}, x, y)$ of the unsigned planar graph $\mathcal{H}$ corresponding to the link $\mathcal{L}_{\mathcal{H}}$. The computation of the Tutte polynomial is demonstrated in Figure 4.3, where each diagram represents the Tutte polynomial of that graph. Let $\mathcal{H} = \{v_1v_2, v_1v_4, v_2v_4, v_3v_4, v_4v_5\}$, $\mathcal{H}_\alpha = \mathcal{H}\backslash v_3v_4 = \{v_1v_2, v_1v_4, v_2v_4, v_4v_5\}$, $\mathcal{H}_\beta = \mathcal{H}_\alpha\backslash v_4v_5 = \{v_1v_2, v_1v_4, v_2v_4\}$, $\mathcal{H}_\delta = \mathcal{H}_\beta\backslash v_1v_4 = \{v_1v_2, v_2v_4\}$, $\mathcal{H}_\epsilon = \mathcal{H}_\beta/v_1v_4 = \{v_2v_4, v_4v_2\}$, $\mathcal{H}_\lambda = \mathcal{H}_\delta\backslash v_2v_4 = \{v_1v_2\}$, $\mathcal{H}_\psi = \mathcal{H}_\epsilon\backslash v_4v_2 = \{v_2v_4\}$ and $\mathcal{H}_\theta = \mathcal{H}_\epsilon/v_4v_2 = \{v_2v_2\}$. Hence we can write the Tutte polynomial,

$$\begin{aligned}
T(\mathcal{H}, x, y) &= xT(\mathcal{H}_\alpha, x, y) \\
&= x^2T(\mathcal{H}_\beta, x, y) \\
&= x^2[T(\mathcal{H}_\delta, x, y) + T(\mathcal{H}_\epsilon, x, y)] \\
&= x^3T(\mathcal{H}_\lambda, x, y) + x^2T(\mathcal{H}_\psi, x, y) + x^2T(\mathcal{H}_\theta, x, y) \\
&= x^3(x) + x^2(x) + x^2y \\
&= x^4 + x^3 + x^2y.
\end{aligned}$$

Thus,

$$T(\mathcal{H}, -1, -1) = (-1)^4 + (-1)^3 + (-1)^2(-1) = 1 - 1 + 1(-1) = -1 = (-1)^5(-2)^{1-1}.$$

Therefore $|E(\mathcal{H})| = 5$ and $\zeta(\mathcal{L}_{\mathcal{H}}) = 1$ implying that the number of components of the link projection, $\mathcal{L}_{\mathcal{H}}$, is 1.

From Example 4.2.1 we see that the link projection $\mathcal{L}_{\mathcal{H}}$ has one component. A link with one component is called a knot, see [1]. Therefore $\mathcal{L}_{\mathcal{H}}$ in Example 4.2.1 is a knot.

4.3 Deletion-Contraction method

In this section we describe the deletion-contraction method which uses the **deletion** and **contraction** operations discussed in Chapter 1, Section 1.1.1.3. For a graph $\mathcal{H}$ it is known that an evaluation

$$T(\mathcal{H}, -1, -1) = (-1)^{|E(\mathcal{H})|}(-2)^{\zeta(\mathcal{L}_{\mathcal{H}})-1}$$

where $\zeta(\mathcal{L}_{\mathcal{H}})$ is the **component number** of the link, $\mathcal{L}_{\mathcal{H}}$, corresponding to the graph $\mathcal{H}$. In this section we define a number $\Upsilon(\mathcal{L}_{\mathcal{H}}) = (-1)^{|E(\mathcal{H})|}(-2)^{\zeta(\mathcal{L}_{\mathcal{H}})-1}$ for an unoriented link projection, $\mathcal{L}_{\mathcal{H}}$, where $\zeta(\mathcal{L}_{\mathcal{H}})$ is the component number of $\mathcal{L}_{\mathcal{H}}$ and $E(\mathcal{H})$ is the edge set of $\mathcal{H}$. By [17], $\Upsilon(\mathcal{L}_{\mathcal{H}})$ has the following properties :

Proposition 4.3.1. *If $\mathcal{H}$ is an unsigned planar graph then :*

(a) $\Upsilon(\mathcal{L}_{\mathcal{H}}) = (-2)^{k-1}$ *if $\mathcal{H}$ is an* ***empty graph*** *with k vertices. This implies that* $\Upsilon(\mathcal{L}_{\mathcal{H}}) = 1$ *if $\mathcal{H}$ is a graph with only one vertex.*

(b) $\Upsilon(\mathcal{L}_{\mathcal{H}}) = -1$ *if $\mathcal{H}$ is a bridge or a loop.*

(c) $\Upsilon(\mathcal{L}_{\mathcal{H}}) = -\Upsilon(\mathcal{L}_{\mathcal{H}}\backslash t)$ *if t is a loop in $\mathcal{H}$.*

(d) $\Upsilon(\mathcal{L}_{\mathcal{H}}) = -\Upsilon(\mathcal{L}_{\mathcal{H}}/t)$ *if t is a bridge in $\mathcal{H}$.*

(e) $\Upsilon(\mathcal{L}_{\mathcal{H}}) = \Upsilon(\mathcal{L}_{\mathcal{H}}\backslash t) + \Upsilon(\mathcal{L}_{\mathcal{H}}/t)$ *if t is neither a bridge nor a loop.*

We state without proof the following theorem which describes the deletion-contraction method for computing the component number of a link projection corresponding to an unsigned planar graph. These operations are applied to a given unsigned planar graph, and do not change the number of components of the corresponding link, see [17]. Note that in the deletion-contraction method the number of isolated vertices is equal to the number of components, see [5]. In the following theorem **series pair** refers to two edges sharing one vertex that has degree 2 and **parallel pair** refers to parallel edges, see [17].

Theorem 4.3.2. *If $\mathcal{H}$ is an unsigned planar graph then :*

(a) $\Upsilon(\mathcal{L}_{\mathcal{H}}) = \Upsilon(\mathcal{L}_{\mathcal{H}}/t/w)$ *if* t *and* w *are a series pair in* $\mathcal{H}$.

(b) $\Upsilon(\mathcal{L}_{\mathcal{H}}) = \Upsilon(\mathcal{L}_{\mathcal{H}}\backslash t\backslash w)$ *if* t *and* w *are a parallel pair in* $\mathcal{H}$.

We demonstrate the application of Theorem 4.3.2 through the following example.

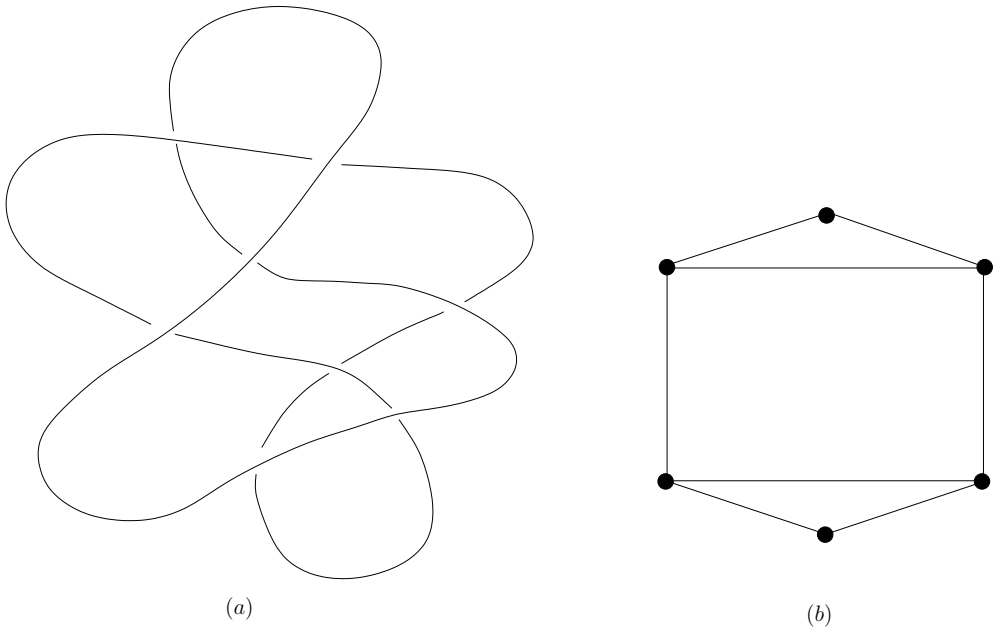

Figure 4.4: (a) The link $\mathcal{L}_{\mathcal{H}}$. (b) The unsigned planar graph $\mathcal{H}$.

Example 4.3.3. The diagrams in Figure 4.4(a) and Figure 4.4(b) show the link $\mathcal{L}_{\mathcal{H}}$ and its corresponding unsigned planar graph $\mathcal{H}$. The method is based on the unsigned planar graph $\mathcal{H}$ as shown in the diagram of Figure 4.5. We start by contracting the edges a_1 and a_2 as they are a **series pair**, and this results in the diagram on the right-hand side of Figure 4.5(A). When we contract another series pair found in the diagram on the right-hand side of Figure 4.5(A), that is, edges a_3 and a_4 we get the diagram in Figure 4.5(B). In the diagram of Figure 4.5(B) e_1 and e_2 are loops so we delete them, a_5 and a_6 are a parallel pair, we delete these. This results in the diagram in Figure 4.5(C) with two isolated vertices and this implies that the link $\mathcal{L}_{\mathcal{H}}$ shown in the diagram of Figure 4.4(a) has two components.

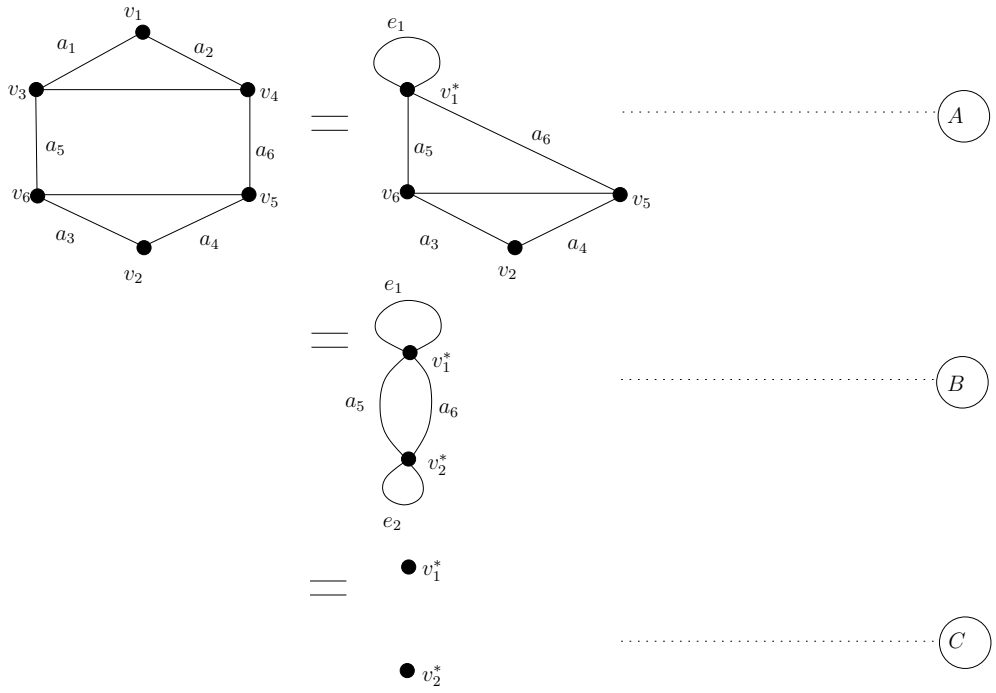

Figure 4.5: Deletion-Contraction method.

4.4 Computing the component number of Pretzel links

In this section we describe how to compute the component number of a pretzel link $(P_1, P_2, \cdots, P_n)$. The component number of a general pretzel link is known, see [11]. We start by stating and proving the following lemma:

Lemma 4.4.1. *Let H be a chain of n edges.*

(a) If n is even, then contracting series pairs will result in a 2-chain before coming to a single vertex.

(b) If n is odd, then contracting series pairs will result in a 1-chain.

Proof. Without loss of generality, assume we have a chain with an even number of edges as shown in diagram (1) of Figure 4.6(a). Applying the deletion-contraction

method dicussed in Section 4.3, Theorem 4.3.2, on each series pair indicated by braces in diagram (1) of Figure 4.6(a), it is clear that we get a 2-chain shown in diagram (2) of Figure 4.6(a) and this happens when we choose not to contract the remaining series pair. Similarly, for an even number of edges shown in diagram (3) of Figure 4.6(b) we get a 1-chain shown in diagram (4) of Figure 4.6(b) when we repeatedly contract series pairs. □

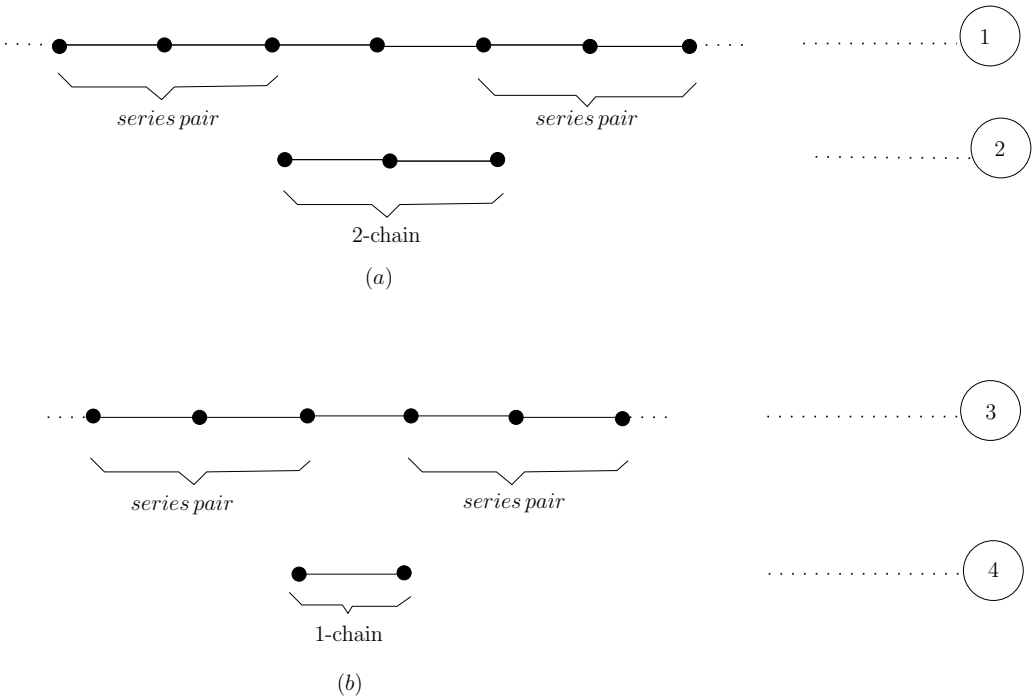

Figure 4.6: Chain of n edges.

We are now in a good position to state and prove the following propositions that give the number of components of a pretzel link $(P_1, P_2, \cdots, P_n)$ which were stated and not proved in [11].

Proposition 4.4.2. *Consider the pretzel link or pretzel knot* $(P_1, P_2, \cdots, P_n)$.

(1) When all the P_i*s are odd, then* $(P_1, P_2, \cdots, P_n)$ *is a* ***pretzel knot*** *if and only if*

n is odd.

(2) When all the P_is are odd, then $(P_1, P_2, \cdots, P_n)$ is a ***pretzel link*** *with* ***two components*** *if and only if n is even.*

Proof. (1) Let us assume that we have a pretzel link $(P_1, P_2, \cdots, P_n)$ as shown in Chapter 3, Figure 3.2, with all the P_is odd. Then the unsigned planar graph corresponding to $(P_1, P_2, \cdots, P_n)$ is as shown in Chapter 3, Figure 3.8, and will have chains with odd number of edges. Without loss of generality, assume we have a chain with an odd number of edges, by Lemma 4.4.1 it is clear that contracting series pairs will result in a single edge. Thus if we repeat this process on each chain of the graph we have a bunch of n parallel edges. Therefore by applying the deletion-contraction method discussed in Section 4.3, Theorem 4.3.2 (b), we delete parallel pairs, repeatedly. Since n is odd there will be one edge remaining. When we continue applying the process by contracting the edge we remain with a single vertex. This implies that $(P_1, P_2, \cdots, P_n)$ has one component and therefore it is a pretzel knot.
(2) Similarly, with all the P_is odd, we remain with a bunch of n parallel edges and since n is even, deleting pairs of parallel edges will result in two isolated vertices. Thus this implies that $(P_1, P_2, \cdots, P_n)$ has two components and therefore it is a pretzel link. □

Proposition 4.4.3. *Consider the pretzel link or pretzel knot $(P_1, P_2, \cdots, P_n)$. If there is some even P_i, then the component number of $(P_1, P_2, \cdots, P_n)$ is equal to the number of even P_is.*

Proof. Let us again assume that we have a pretzel link $(P_1, P_2, \cdots, P_n)$ as shown in Chapter 3, Figure 3.2, with j number of odd P_is and k number of even P_ls such that $j + k = n$. Then the unsigned planar graph corresponding to $(P_1, P_2, \cdots, P_n)$ will be as shown in Chapter 3, Figure 3.8, and will have j number of odd chains and k number of even chains. Then by applying Lemma 4.4.1 the j odd chains will result in a bunch of j, 1-chains and the k, even chains will result in a bunch of k 2-chains

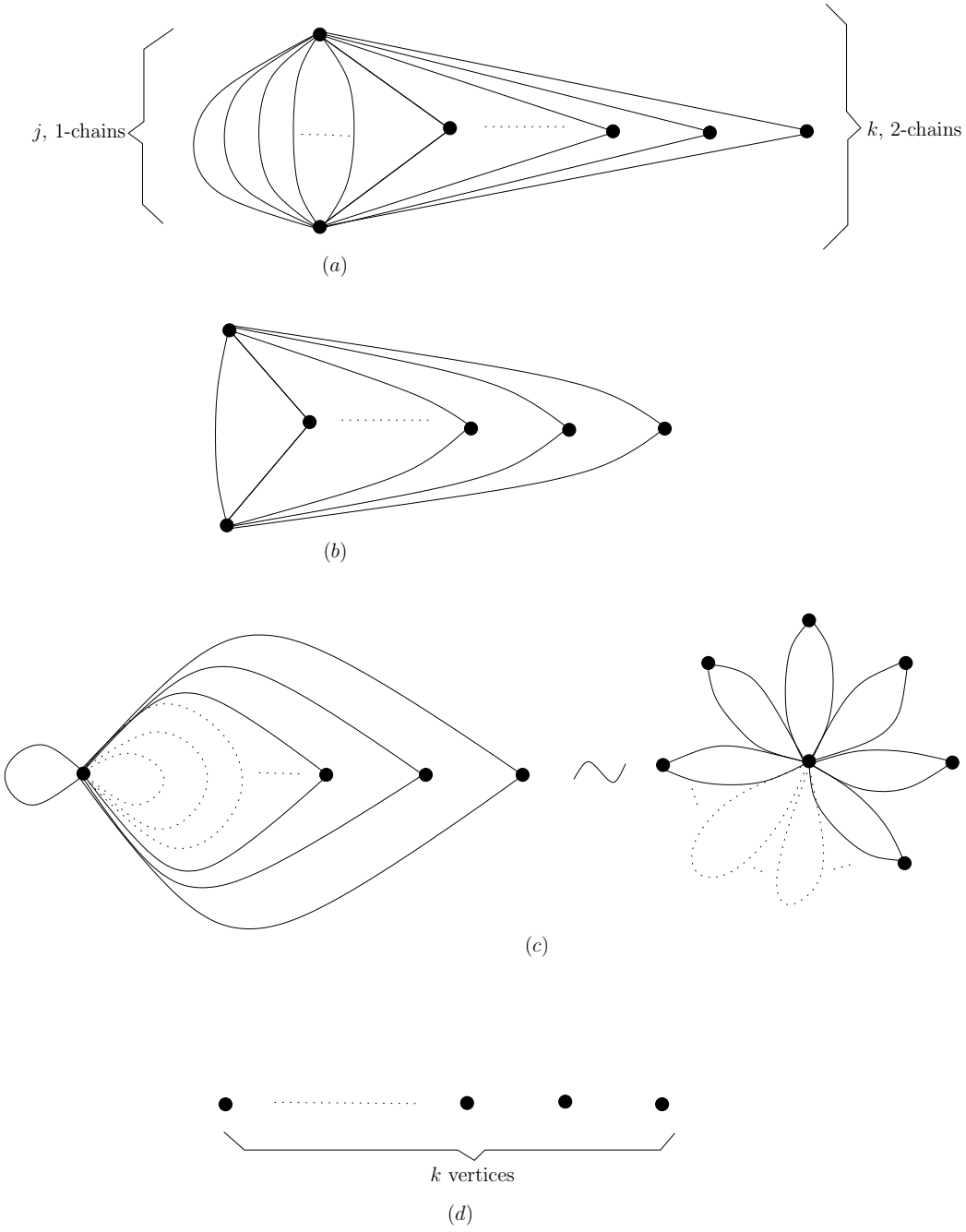

Figure 4.7: j, 1-chains and k, 2-chains.

as shown in diagram (a) of Figure 4.7 without loss of generality. By applying the deletion-contraction method discussed in Section 4.3, Theorem 4.3.2 (b), we delete pairs of parallel edges. For the j odd chains, it will result in the graph shown in diagram (b) of Figure 4.7. Continuing with the process we contract one series pair which will result in the graph in diagram (c) Figure 4.7 with $k-1$ pairs of parallel edges. Again, by Theorem 4.3.2 (b) we delete all the $k-1$ parallel edges and we also delete the loop. We therefore remain with k vertices as shown in diagram (d) of Figure 4.7 and this implies that the component number of $(P_1, P_2, \cdots, P_n)$ is k which is equal to the number of even P_ls. □

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