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Radicals and Anti-radicals of near-rings and Matrix near-rings. [1]

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

_____

Anthony Mpho Matlala

This _____ day of _____ , at Johannesburg, South Africa.

Abstract

In this thesis the ideal structure of near-rings and the associated matrix near-rings is studied. Firstly, two aspects of near-rings are investigated, these are: the decomposition of a near-ring; and the nilpotency of the Jacobson s-radical, via the nil-rigid series. Secondly, the interplay of ideals of a near-ring and the ideals of the associated matrix near-ring is investigated.

The socle ideal (an anti-radical) of a near-ring R is characterized as an intersection of annihilators of R-groups of a special type, called type-$\mathcal{K}$. That is, the socle ideal is an **annihilator ideal**.

The R-groups of type-$\mathcal{K}$ are shown to separate the Jacobson 0-radical and the Jacobson s-radical, if there is at least one R-group of type-$\mathcal{K}$ contained in some type-0 R-group.

Alternating chains of R-groups of type-0 and type-$\mathcal{K}$ are defined and will be called **Ω-chains**. It is then proved that the maximal length of the Ω-chains give a lower bound on the nil-rigid length of the near-ring. Consequently, one can directly determine that the nil-rigid length of the matrix near-ring, $\mathbb{M}_n(R)$, is equal or bigger than the nil-rigid length of R from the structure

of any one of the finite faithful R-groups of R.

For an ideal I being a Jacobson type radical, socle ideal or s-socle ideal of a near-ring, open-questions on the relationship between I^+, I^* and the matrix near-ring ideals of the same type are answered.

A systematic theory is developed that connects each annihilator ideal I of R to its corresponding matrix near-ring ideal, $\mathcal{I}$. This is made possible by the use of Action 2 of the matrix near-ring on the $\mathbb{M}_n(R)$-groups. Specific conditions are given which ensure that $I^+ \subseteq \mathcal{I}$ and $\mathcal{I} \subseteq I^*$. Examples illustrate cases in which no such relationships exist. Necessary and sufficient conditions are given for $I^+ \subseteq \mathcal{I} \subseteq I^*$ to hold.

This thesis thus illustrates the relationship between the structure of a faithful R-group and the ideal structure of the associated matrix near-rings.

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Chapter 1

Introduction

In this thesis the ideal structure and specific decompositions of zero symmetric near-rings, and their associated matrix near-rings, are studied. For this, a natural point of departure is the radical theory of near-rings.

There is a large literature on the analogues of various ring radicals in the near-ring context. These results have been collected in, the now standard, text-books on near-rings, by Pilz [23] and Meldrum [14].

Radicals of near-rings differ remarkably from their analogous ring radicals. For instance, it is well-known, [1] and [12], that there exist finite distributively generated near-rings whose prime ideals are not maximal.

The focus of this thesis is the connection between Jacobson type radicals and an anti-radical called the socle ideal. Unlike in the ring case, for near-rings there are at least three distinct Jacobson type radicals which are two-sided ideals, viz. $J_\nu(R)$, $(\nu = 0, s, 2)$. Substantial work has been done on the Jacobson 0-radical, $J_0(R)$, and the Jacobson 2-radical, $J_2(R)$. Special attention is therefore given here to the Jacobson s-radical, $J_s(R)$, introduced by Hartney in [5]. The Jacobson s-radical contains the Jacobson 0-radical, and is itself contained in the Jacobson 2-radical, that is, $J_0(R) \subseteq J_s(R) \subseteq J_2(R)$. The s-radical is crucial to understanding the structure of near-rings and is fundamental in this thesis.

Another ideal which plays an integral part in elucidating the structure of near-rings, introduced by Hartney, in [5], is the socle ideal of a near-ring. The socle ideal, denoted $Soi(R)$, is referred to as an anti-radical as it annihilates one of the Jacobson type radicals, namely the Jacobson 0-radical. It is known that a zero-symmetric near-ring, R, with identity and satisfying the descending chain condition on left ideals, can be decomposed as

$$R = Soi(R) \oplus L,$$

where L is a left ideal of R containing $J_0(R)$, see Hartney [5].

Hartney, in [5], showed that there is a duality between the socle ideal and the Jacobson s-radical, given by

$$J_s(R) = (0) \text{ if and only if } Soi(R) = R.$$

The intricate relation between the socle ideal and the Jacobson s-radical is beautifully displayed by the nil-rigid series of a near-ring. The nil-rigid series, first defined by Scott in [25], is an ascending alternating series of Jacobson 0-radicals and socle ideals of quotient near-rings. The length of the nil-rigid series of a near-ring can be used as a measure of how far the Jacobson s-radical is from being nilpotent, see [6].

It is well known, [23] and [14], that Jacobson type radicals can be characterized as intersections of annihilators of monogenic R-groups of type-ν ($\nu = 0, s, 2$). It is proved, in this thesis, that the socle ideal can also be characterized as an intersection of annihilators of monogenic R-groups of yet another type, referred to here as type-$\mathcal{K}$. The ideal structure of a near-ring is thus seen to be dictated to by the structure of its faithful R-groups.

The structure of a near-ring, R, is further revealed by considering alternating chains, called Ω-chains, of monogenic R-subgroups of types 0 and $\mathcal{K}$ of any faithful R-group. The lengths of the Ω-chains are related to the length of the nil-rigid series and in some cases determine it.

The structure of near-rings can be further unravelled by considering the ideal structure of their matrix near-rings.

In near-rings, the lack of one distributive law results in complications when matrices with entries from a near-ring are considered. For only particular classes of near-rings do the matrices over near-rings form a near-ring, called a matrix near-ring, under the usual matrix addition and multiplica-

tion. Heatherly [10], and Ligh [13], studied such classes of near-rings and the associated matrix near-rings. In 1986, Meldrum and Van der Walt took a different approach to form matrix near-rings from an arbitrary near-ring. They regarded an $n \times n$ matrix, over a near-ring, R, as a function from the group R^n to itself. For a near-ring which is a ring, its matrix near-ring constructed this way coincide with its matrix ring. Here we shall use the definition of a matrix near-ring as given by Meldrum and Van der Walt, [18], denoted by $\mathbb{M}_n(R)$.

For an ideal I of a near-ring R, Meldrum and Van der Walt [18], introduced two ways to construct an ideal in $\mathbb{M}_n(R)$, namely I^+ and I^*, with $I^+ \subseteq I^*$. In addition, Van der Walt [27], showed that the inclusion can be strict. The gap between these two ideals, I^+ and I^*, can be wide, in that there may exist chains of ideals between them, see Meyer [21]. He refers to these ideals of $\mathbb{M}_n(R)$ as intermediate ideals. In a later paper, [17], Meldrum and Meyer furnished an example showing that the Jacobson 0-radical of a matrix near-ring might be an intermediate ideal.

A substantial amount of work has been done in studying this interplay of ideals of a near-ring R and corresponding ideals of $\mathbb{M}_n(R)$. In particular, it is known that

$$(J_2(R))^+ \subseteq J_2(\mathbb{M}_n(R)) = (J_2(R))^*,$$

proved by Van der Walt in [26]. The theory of s-primitivity in matrix near-rings, by Hartney and Mavhungu, [7, 9], showed that

$$J_s(\mathbb{M}_n(R)) \subseteq (J_s(R))^*.$$

That

$$(J_s(R))^+ \subseteq J_s(\mathbb{M}_n(R)),$$

is proved in this thesis, published in [8]. In his doctoral thesis, [19], Meyer proved that

$$J_0(\mathbb{M}_n(R)) \subseteq (J_0(R))^*.$$

The relationship between $(J_0(R))^+$ and $J_0(\mathbb{M}_n(R))$ has been an open question. Meldrum and Meyer, [17], conjectured that, for zero symmetric near-rings, $(J_0(R))^+ \subseteq J_0(\mathbb{M}_n(R))$. However, counter examples to this conjecture exist as is shown in this thesis.

For socle ideals, as with the Jacobson 0-radicals, it is shown, in this thesis, that there is in general no relationship between $(Soi(R))^+$ and $Soi(\mathbb{M}_n(R))$. On the other hand, we prove that

$$Soi(\mathbb{M}_n(R)) \subseteq (Soi(R))^*.$$

The s-socle of a near-ring R is the minimal ideal modulo which $J_s(R)$ is non-zero nilpotent. One would thus expect the s-socles to have relationships resembling those of the Jacobson type radicals and those of the socle ideals. Surprisingly, there is in general no relationship between the s-socle of $\mathbb{M}_n(R)$, denoted $\mathcal{A}$, and the ideal A^*, where A is the s-socle of R. Examples of near-rings are given where $A^* \subseteq \mathcal{A}$, and where A^* and $\mathcal{A}$ not comparable. Yet, for zero symmetric near-rings satisfying the descending chain condition on R-subgroups (DCCS), it is proved that

$$A^+ \subseteq \mathcal{A}\,.$$

Although, we are mainly interested in ideals which are intersections of annihilators of monogenic R-groups, monogenic R-groups are not the only R-groups needed to understand the ideal structure of matrix near-rings. Meldrum and Meyer, in [15], showed that, for a locally monogenic R-group, Δ, which is not monogenic, its corresponding $\mathbb{M}_n(R)$-group, Δ^n, is monogenic under Action 2. Non-monogenic R-groups, Δ, such that Δ^n is monogenic as an $\mathbb{M}_n(R)$-group, are classified into three classes, those of 0_n-form, s_n-form and $\mathcal{K}_n$-form. These R-groups play a key role in understanding the relationship between the ideals of R and those of $\mathbb{M}_n(R)$. For example, Meldrum and Meyer, [15], used them to show that it is possible to have

$$J_0(\mathbb{M}_n(R)) \neq J_2(\mathbb{M}_n(R)),$$

even though

$$J_0(R) = J_2(R).$$

In [9], Hartney and Mavhungu showed that, in this example by Meldrum and Meyer,

$$J_0(\mathbb{M}_n(R)) = J_s(\mathbb{M}_n(R)) \neq J_2(\mathbb{M}_n(R)),$$

while

$$J_0(R) = J_s(R) = J_2(R).$$

To further illustrate the effect of these non-monogenic R-groups on the ideals of $\mathbb{M}_n(R)$, an example of a near-ring R is given here, for which

$$J_0(\mathbb{M}_n(R)) \neq J_s(\mathbb{M}_n(R)) \neq J_2(\mathbb{M}_n(R)),$$

with

$$J_0(R) = J_s(R) = J_2(R).$$

The existence of non-monogenic R-groups of 0_n-form and $\mathcal{K}_n$-form affects the lengths of the Ω-chains, and hence the length of the nil-rigid series of R. In general, a lower bound for the length of the nil-rigid series of $\mathbb{M}_n(R)$ can be determined from the lengths of the Ω-chains. Deeper studies into group-representation theory should yield a better understanding of the ideal structure of near-rings.

This thesis is structured as follows.

In Chapter 2, definitions of basic concepts and preliminary results on near-rings and their associated matrix near-rings are collected.

Chapter 3 presents a characterization of the socle ideal as an intersection of annihilators of monogenic R-groups of type-$\mathcal{K}$ and gives a description of the nil-rigid series in terms of annihilators of monogenic R-groups which are members of chains of R-groups, called Ω-chains. A series of ideals, constructed from the Ω-chains, is defined and shown to contain the nil-rigid series. This new series of ideals is called the **Ω-ideal series**. Examples are given to illustrate how the maximal length of Ω-chains is related to the nil-rigid length.

In Chapter 4, the connection between annihilator ideals of a near-ring and those of the associated matrix near-rings is studied. By an annihilator ideal we are referring to any intersection of annihilating ideals of monogenic R-groups (or $\mathbb{M}_n(R)$-groups). The results are aided by the recognition of some non-monogenic R-groups whose corresponding matrix near-ring groups are

monogenic. This theory is immediately applied to the socle ideal, as an example of an annihilator ideal.

In Chapter 5, some open-questions on ν-primitive ideals of matrix near-rings are answered. In addition, questions on the relationship between the s-socle ideal of a near-ring and the s-socle ideal of the associated matrix near-ring are answered. Examples of near-rings are given illustrating cases where no such relationships exist.

Chapter 6 presents an application of a combination of the theory of Ω-chains, from Chapter 3, and the theory of annihilator ideals, from Chapter 4, to the nil-rigid series of matrix near-rings. As a consequence, one can give a lower bound on the nil-rigid length of the matrix near-ring from the Ω-chains. Examples of near-rings, with their nil-rigid lengths strictly less than those of their associated matrix near-rings, are given.

Throughout this thesis, R denotes a zero symmetric right near-ring with identity, and $\mathbb{M}_n(R)$ the Meldrum-Van der Walt matrix near-ring associated with R. Both R and $\mathbb{M}_n(R)$ are assumed to satisfy the DCCS, unless otherwise stated.

The term *R-group* is used instead of R-module. Likewise, the term *R-kernel* is used instead of R-ideal. An *ideal* will always mean a two-sided ideal. A faithful R-group of R will usually be denoted by Ω. A direct-sum of n copies of an R-group, Ω, will be denoted by Ω^n, where n is a natural number. The Greek letter α is reserved to denote the nil-rigid length. To

make a distiction between the nil-rigid length of R and $\mathbb{M}_n(R)$, we use α_R and $\alpha_{\mathbb{M}_n(R)}$, respectively.

Chapter 2

Preliminaries

In this chapter we collect definitions of basic concepts and preliminary results needed in this thesis. Concepts of a near-ring, an R-group and an ideal of a near-ring are defined in Section 2.1. In Section 2.2, ν-primitive ideals are defined and the relevant results on ν-radical theory are collected. Section 2.3 introduces anti-radicals: the socle ideal; the crux; and the critical ideal. Relevant known theory on how these ideals relate is recorded here. Meldrum-Van der Walt matrix near-rings are defined in Section 2.4. Basic facts on the ideal theory of matrix near-rings are collected here.

2.1 Basic concepts

In this section we collect basic definitions and introduce the notation we will use throughout this thesis.

Definition 2.1.1 *A* **near-ring** $(R, +, \cdot)$ *is a set with two binary operations, satisfying the following conditions:*
(i) $(R, +)$ *is a group, written additively (not necessarily abelian);*
(ii) $(R, \cdot)$ *is a semi-group;*
(iii) $(a + b) \cdot c = a \cdot c + b \cdot c$ *, for all* $a, b, c \in R$.

A near-ring, such as defined above, is called a **right** distributive near-ring (or right near-ring). If (iii) is replaced by the left distributive rule, the resulting near-ring is referred to as a **left** near-ring.

For right distributive near-rings

$$yx = (y + 0)x = yx + 0x,$$

for all $x, y \in R$. Hence,

$$0x = 0, \quad \text{for all} \quad x \in R.$$

However, it need not be the case that $x0 = 0,$ for all $x \in R.$ If the condition

$$x0 = 0, \quad \text{for all} \quad x \in R,$$

holds on the near-ring $R,$ then R is said to be a **zero-symmetric** near-ring.

A typical example of a zero-symmetric near-ring is a set of zero respecting mappings from a group, Ω, to itself, i.e.,

$$M_0(\Omega) = \{f : \Omega \to \Omega \mid f(0) = 0\},$$

under pointwise addition and composition of maps.

Throughout this thesis, $M_0(\Omega)$ denotes the near-ring of zero respecting mappings from a group, Ω, to itself, and R denotes a right near-ring.

Definition 2.1.2 *Let* $(\Omega, +)$ *be a group (not necessarily abelian) and let* R *be a right near-ring with identity. Let* μ *be a map* $\mu : R \times \Omega \to \Omega$ *and denote* $\mu(r, \omega) =: r\omega$*, for* $r \in R, \omega \in \Omega$*. Then* (Ω, μ) *is called an* R**-group** *if for* $r_1, r_2 \in R$, $\omega \in \Omega$,
(i) $(r_1 + r_2)\omega = r_1\omega + r_2\omega$,
(ii) $(r_1 r_2)\omega = r_1(r_2\omega)$,
(iii) $1\omega = \omega$.
We shall denote the R*-group* (Ω, μ) *simply by* Ω *if the meaning of* μ *is clear. A subgroup* $(\Delta, +)$ *of* $(\Omega, +)$ *is called an* R**-subgroup** *of* (Ω, μ) *if* $(\Delta, \mu|_{R\times\Delta})$ *is itself an* R*-group, in this case we will say that* Δ *is an* R*-subgroup of* Ω.
An R*-group* Δ *is said to be* **monogenic** *if there exists a* δ *in* Δ *such that* $\Delta = R\delta$.

Since R is a right near-ring, then an R-group is a near-ring analogue of a left-module of a ring. For each right near-ring R with identity, the group $(R, +)$ is also an R-group under left-multiplication.

Definition 2.1.3 *Let R be a right near-ring with identity, and let Ω be an R-group. An R-subgroup, Δ, of Ω, is an* R**-kernel** *of Ω if*
(i) $(\Delta, +)$ *is a normal subgroup of* $(\Omega, +)$,
(ii) $r(\omega + \delta) - r\omega \in \Delta$, *for all* $r \in R$, $\omega \in \Omega$, *and* $\delta \in \Delta$.

Definition 2.1.4 *Let R be a right near-ring with identity, Ω be an R-group, and let L be an R-kernel of Ω, and*

$$\Omega/L = \{\omega + L \mid \omega \in \Omega\}$$

denote the quotient group Ω/L. Then $(\Omega/L, +)$ is called the **factor** R**-group** *of Ω by L, where R acts on Ω/L by*

$$r(\omega + L) := r\omega + L,$$

for all $\omega \in \Omega$, $r \in R$.

Note that if the R-group, Ω, in Definition 2.1.4, is monogenic, then the quotient group, Ω/L, is a monogenic R-group.

Throughout this thesis, for any R-group Ω, the expression $\Omega = \bigoplus_{\lambda \in \Lambda} \Omega_\lambda$ will always mean that Ω is a near-ring direct sum of the *R-kernels* Ω_λ, $\lambda \in \Lambda$.

Definition 2.1.5 *Let R and R' be right near-rings with identity, and Ω and Δ be R-groups. Then*
(i) $\Psi : R \to R'$ *is a* **(near-ring) homomorphism** *if*

$$\Psi(r_1 + r_2) = \Psi(r_1) + \Psi(r_2) \text{ and } \Psi(r_1 r_2) = \Psi(r_1)\Psi(r_2),$$

for all $r_1, r_2 \in R$.

(ii) $\Psi : \Omega \to \Delta$ *is an* R**-homomorphism** *if*

$$\Psi(\omega + \omega') = \Psi(\omega) + \Psi(\omega') \quad \text{and} \quad \Psi(r\omega) = r\Psi(\omega),$$

for all $\omega, \omega' \in \Omega$, $r \in R$.

The *ideals* of the near-ring R may be considered as the kernels of (near-ring) homomorphisms of R.

Definition 2.1.6 *Let* R *be a right near-ring. A normal subgroup* $(I, +)$ *of* $(R, +)$ *is an* **ideal** *of* R *if*

(i) $IR \subseteq I$,

(ii) $r(r' + \alpha) - rr' \in I$, *for all* $r, r' \in R$, $\alpha \in I$.

If (ii) holds then I is called a **left ideal** of R. Likewise I is called a **right ideal** if (i) holds. The left ideals of R are precisely the kernels of R-homomorphisms of $(R, +)$.

Quotient near-rings are defined in the usual sense.

Definition 2.1.7 *Let* R *be a right near-ring and let* I *be an ideal of* R. *Then the (additive) factor group* R/I *is a near-ring under the binary operations:*

$$(r + I) + (s + I) = (r + s) + I, \quad r, s \in R$$

and

$$(r + I)(s + I) = (rs) + I, \quad r, s \in R.$$

This near-ring is called the **quotient near-ring** *of* R *by* I.

The annihilating ideal of an R-group Ω is defined to be

$$(0:\Omega) := \{\, x \in R \mid x\omega = 0,\ \forall\, \omega \in \Omega \,\}.$$

An R-group Ω is a **faithful** R-group if $(0:\Omega) = (0)$.

Proposition 2.1.8 *[23, Proposition 1.48] Let* R *be a zero-symmetric right near-ring. If* Ω *is a faithful* R*-group, then*

$$R \hookrightarrow M_0(\Omega).$$

That is R *has an embedding in* $M_0(\Omega)$.

Hence every zero-symmetric right near-ring can be considered as a subnear-ring of $M_0(\Omega)$, for some group $(\Omega, +)$.

If the near-ring R satisfies the descending chain condition for R-subgroups, we say R satisfies the DCCS. Similarly, if R satisfies the descending chain condition for left ideals, we say R satisfies the DCCL.

2.2 Jacobson type radicals

In this section we define the ν-radicals, $\nu = 0, \frac{1}{2}, s, 2,$ and recall the known radical theory needed in this thesis.

Definition 2.2.1 *Let R be a right near-ring with identity. A monogenic R-group, $\Omega = R\omega$, for some $\omega \in \Omega$, is of*

(i) **type-0** *if it has no non-trivial R-kernels,*

(ii) **type-s** *if it is of type-0 and for all $\omega' \in \Omega$ with $R\omega' \neq (0)$ we have that there exists a near-ring direct sum decomposition,*

$$R\omega' = \bigoplus_{i=1}^{k} \Delta_i,$$

where each Δ_i is an R-kernel of $R\omega'$ and an R-group of type-0,

(iii) **type-2** *if it has no non-trivial R-subgroups.*

It is clear from Definition 2.2.1 that *type-2* R-groups are of *type-s*, hence of *type-0*.

Lemma 2.2.2 *[5] Let R be a right near-ring and Ω an R-group. If $\Omega = \bigoplus_{\lambda\in\Lambda} \Delta_\lambda$, a direct sum of R-groups Δ_λ, $\lambda \in \Lambda$, all of type-0, then any non-trivial R-kernel, H, of Ω, is a direct summand of Ω with a direct sum of some of the Δ_λ's as a co-summand.*

Theorem 2.2.3 *[7, Theorem 2.1] Let R be a right near-ring satisfying the DCCL and let Ω be a faithful R-group. Any R-group of type-0 is a homomorphic image of some monogenic R-subgroup $R\omega$ of Ω.*

The following proposition due to Betsch (see Pilz, [23, Proposition 3.14]), gives that for an R-group, Ω, to be an (R/I)-group, where R/I is the quotient near-ring of R by I, it is necessary and sufficient that I annihilates Ω.

Proposition 2.2.4 *Let R be a right near-ring, I be an ideal of R, and Γ a group and $\nu \in \{0, s, 2\}$.*
(a) If Γ is an R-group with $I \subseteq (0 : \Gamma)$ and we define

$$(r + I)\gamma := r\gamma,$$

then Γ is an (R/I)-group under this operation. If Γ is an R-group of type-ν, then it is an (R/I)-group of type-ν. If Γ is a faithful R-group, then it is faithful as an (R/I)-group.
(b) If Γ is an (R/I)-group and we define

$$r\gamma := (r + I)\gamma,$$

then Γ is an R-group under this operation with $I \subseteq (0 : \Gamma)_R$. If Γ is an (R/I)-group of type-ν, then it is of type-ν as an R-group. If Γ is a faithful (R/I)-group, then $I = (0 : \Gamma)_R$.

We note that for any two R-groups, Δ_1 and Δ_2,

$$\Delta_1 \supseteq \Delta_2 \Rightarrow (0 : \Delta_1) \subseteq (0 : \Delta_2).$$

Corollary 2.2.5 *Let R be a zero-symmetric right near-ring, and let Δ_1 and Δ_2 be R-groups. If $\Delta_1 \supseteq \Delta_2$, then Δ_2 is an (R/I)-group, where $I = (0 : \Delta_1)$.*

Proposition 2.2.6 *Let R be a zero-symmetric right near-ring with identity, I be an ideal of R, and Δ any R-group. If $(0 : \Delta) \supseteq I$ then*

$$(0 : \Delta)_{R/I} = (0 : \Delta)/I,$$

where $(0:\Delta)$ *is the annihilating ideal of* Δ *in* R*, and* $(0:\Delta)_{R/I}$ *is the annihilating ideal of* Δ *in* R/I*.*

Proof The identification, $(r+I)\delta = r\delta$, of Proposition 2.2.4, yields the equality as follows,

$$\begin{aligned}(0:\Delta)_{R/I} &= \{\, x+I \in R/I \,|\, (x+I)\delta = 0,\, \forall\, \delta \in \Delta \,\} \\ &= \{\, x+I \in R/I \,|\, (x+I)\delta = x\delta = 0,\, \delta \in \Delta\} \\ &= \{\, x+I \in R/I \,|\, x \in (0:\Delta) \,\} \;=\; (0:\Delta)/I.\end{aligned}$$

□

Definition 2.2.7 *Let* R *be a right near-ring, and* Ω *be an* R*-group.*
(i) An ideal I *of* R *is a* ν-**primitive** *ideal if*

$$I = (0:\Omega) = \{x \in R \;\mid\; x\omega = 0, \forall\, \omega \in \Omega\,\},$$

where Ω *is of type-*ν*,* $\nu = 0, s, 2$*.*
(ii) For $\nu = 0, s, 2$*, the* **Jacobson-type radical***,* $J_\nu(R)$*, is the intersection of all* ν*-primitive ideals.*
(iii) The near-ring R *is* ν*-primitive if it possesses a faithful* R*-group of type-*ν*,* $\nu = 0, s, 2$*. That is, if the zero ideal of* R *is a* ν*-primitive ideal.*

The radicals $J_\nu(R)$, $\nu = 0, s, 2$, are related as follows

$$J_0(R) \;\subseteq\; J_s(R) \;\subseteq\; J_2(R).$$

Definition 2.2.8 *[14, page 87] Let R be a right near-ring. The left ideal L of R is a* **ν-modular** *left ideal, $\nu = 0, s, 2$, if the factor group R/L is an R-group of type-ν and R has a right identity modulo L, that is, an element e such that $xe - x \in L$ for all $x \in R$.*

Definition 2.2.9 *[14, page 93] Let R be a right near-ring.*

$$J_{1/2}(R) = \bigcap \{ L \mid L \text{ is a } 0\text{-modular left ideal of } R \}.$$

Whereas $J_\nu(R)$ is a two-sided ideal of R, for each $\nu = 0, s, 2$, $J_{1/2}(R)$ is, in general, only a left ideal of R.

Proposition 2.2.10 *[14, Theorem 5.27] Let R be a right near-ring. $J_0(R)$ is the largest two-sided ideal of R contained in $J_{1/2}(R)$.*

Proposition 2.2.11 *[4] If R is a right near-ring satisfying the DCCL , then $J_s(R)$ is the smallest ideal of R containing $J_{1/2}(R)$.*

Theorem 2.2.12 *[4] If R is a right near-ring satisfying the DCCL, then any ideal I of R, $I \neq (0)$, which contains $J_{1/2}(R)$ is an intersection of s-primitive ideals of R.*

From Propositions 2.2.10 and 2.2.11, it follows that

$$J_0(R) \subseteq J_{1/2}(R) \subseteq J_s(R) \subseteq J_2(R).$$

Proposition 2.2.13 *[24] If R is a right near-ring satisfying the DCCS , then $J_0(R)$ is a nilpotent ideal of R.*

For the definition of d.g. near-rings we refer the reader to [14], and for such near-rings we have the following theorem.

Theorem 2.2.14 *[11] Let R be a distributively generated right near-ring. Then*

(i) $J_0(R)$ contains all nilpotent ideals of R.

(ii) $J_{1/2}(R)$ contains all nilpotent left ideals of R.

(iii) $J_2(R)$ contains all nilpotent R-groups.

Lemma 2.2.15 *[11] Let R be a right near-ring satisfying the DCCS.*

(i) If I is an intersection of maximal left ideals, then

$$R/I = \bigoplus_{i=1}^{t} \Delta_i,$$

where each Δ_i is an R-group of type-0.

(ii) If $I = J_{1/2}(R)$, then any type-0 R-group is isomorphic to one of the Δ_i appearing in the direct decomposition

$$R/I = \bigoplus_{i=1}^{t} \Delta_i.$$

2.3 Anti-radicals

In this section, three anti-radicals are defined, and their theory relevant to this thesis is presented.

Lemma 2.3.1 *[2] (Left distribution over R-kernels)*
Let R be a right near-ring, and Ω be an R-group such that $\Omega = \bigoplus_{i\in\Lambda} \Omega_i$. Then

$$r(\omega_1 + \omega_2 + \cdots + \omega_t) = r\omega_1 + r\omega_2 + \cdots + r\omega_t$$

for all $r \in R$ and $\omega_i \in \Omega_i$.

Definition 2.3.2 *[5] Let R be a right near-ring, and let $\mathcal{F}$ be the collection of all ideals I of R such that*

$$I = \bigoplus_{i\in\Lambda} Re_i$$

where
(i) Re_i is of type-0 for each $i \in \Lambda$,
(ii) $e_i^2 = e_i$, $e_i e_j = 0$ for $i > j$ with some ordering on the index set Λ.
An ideal I of R is said to have an **$\mathcal{F}$-decomposition**, *or to be* **$\mathcal{F}$-decomposable**, *if $I \in \mathcal{F}$. If $\mathcal{F} \neq \emptyset$, then the unique maximal element of $\mathcal{F}$ is called the* **socle-ideal** *of R, denoted $Soi(R)$. If $\mathcal{F} = \emptyset$, then $Soi(R)$ is defined to be zero.*

Lemma 2.3.3 *[5] If R is a right near-ring satisfying the DCCL, then $Soi(R) \cap J_{1/2}(R) = (0)$. In particular, if $R = Soi(R)$, then $J_{1/2}(R) = (0)$.*

It follows, therefore, that

$$(Soi(R))(J_{1/2}(R)) = (Soi(R))(J_0(R)) = (0)$$

and so $Soi(R)$ is an anti-radical.

Theorem 2.3.4 *[5] If R a right near-ring satisfying the DCCL, then*

$$R = Soi(R) \oplus L$$

where $Soi(R) = \bigoplus_{i=1}^{t} Re_i$ *and* $L = \bigcap_{i=1}^{t} (0 : e_i)$.

Definition 2.3.5 *Let R be a right near-ring satisfying the DCCL. The decomposition $R = Soi(R) \oplus L$ is called the* **socle-decomposition of** *R.*

Theorem 2.3.6 *[5] Let R be a right near-ring satisfying the DCCS and I be an ideal of R, then $I \cap J_{1/2}(R) = (0)$ if, and only if, $I \subseteq Soi(R)$.*

Theorem 2.3.7 *[5] Let R be a near-ring with DCCL and I be an ideal of R, then $Soi(R/I) = R/I$ if, and only if, $J_{1/2}(R) \subset I$.*

Corollary 2.3.8 *[5] $J_s(R)$ is the unique smallest ideal amongst all ideals I of R such that $Soi(R/I) = R/I$.*

Thus $Soi(R) = R$ if, and only if, $J_s(R) = (0)$.

In [5], Hartney investigated the connection between $Soi(R)$ and the $Crux$ of R. The $Crux$ was first defined by Scott, see [25].

An ideal I of R is a **nil-ideal** if every element in I is nilpotent. The **nil-radical** of R, $nil(R)$, is the sum of all nil-ideals of R and is itself a nil-ideal.

Definition 2.3.9 *Let R be a right near-ring. An ideal I of R is said to be* **rigid** *if, whenever J is an ideal of R contained in I, then $(I/J) \cap$*

$nil(R/J) = (0)$. The sum of all rigid ideals of R is called the **Crux** *of R, denoted $Crux(R)$.*

Because of its strong nil "avoidance" property, $Crux(R)$ annihilates the nil-radical, $nil(R)$, and is thus an anti-radical.

Theorem 2.3.10 *[5] For each right near-ring, R,*

$$Soi(R) \subseteq Crux(R).$$

Concerning the nilpotence of $J_s(R)$, Hartney proved the following result.

Theorem 2.3.11 *[5] For the right near-ring, $R = Soi(R) \oplus L$, let $W := J_s(R) \cap Soi(R)$. Then $J_s(R/W)$ is nilpotent if, and only if,*

$$J_s(R) = W \oplus J_{1/2}(R).$$

In [6], Hartney used the *nil-rigid series* of a near-ring R to further explore the nilpotence of $J_s(R)$.

Definition 2.3.12 *[6, 25] For the right near-ring R satisfying the ascending chain condition for ideals, let $L_1 = nil(R)$ and C_1 be the ideal containing L_1 such that*

$$C_1/L_1 = Crux(R/L_1).$$

Let L_2 be the ideal of R containing C_1 such that

$$L_2/C_1 = nil(R/C_1).$$

If α *is a non-limit ordinal, define* L_α *to be the ideal of* R *containing* $C_{\alpha-1}$ *such that*

$$L_\alpha/C_{\alpha-1} = nil(R/C_{\alpha-1}),$$

and define C_α *to be the ideal of* R *containing* L_α *such that*

$$C_\alpha/L_\alpha = Crux(R/L_\alpha).$$

If α *is a limit ordinal, define*

$$C_\alpha = \bigcup_{\beta<\alpha} C_\beta,$$

and

$$L_\alpha = \bigcup_{\beta<\alpha} L_\beta.$$

The transfinite ascending sequence of ideals,

$$(0), L_1, C_1, L_2, C_2, \ldots$$

is called the **nil-rigid series** *of the near-ring* R*. The least* α *for which* $C_\alpha = R$ *is called the* **nil-rigid length** *or* **nil-rigid class number** *of* R*.*

The symbol α will be used exclusively to denote the nil-rigid length of R.

For a right near-ring R satisfying the DCCS, the following ideals coincide, see [6],

(a) $Soi(R) = Crux(R)$,

(b) $nil(R) = J_0(R)$.

Consequently, whenever R satisfies the DCCS, $Crux(R)$ and $nil(R)$ in the definition of the nil-rigid series may be replaced by $Soi(R)$ and $J_0(R)$, respectively.

Lemma 2.3.13 *[6] Let R be a right near-ring satisfying the DCCS and let α be its nil-rigid length. The s-radical, $J_s(R)$, is nilpotent if, and only if, $\alpha = 1$.*

Lemma 2.3.14 *[6] Let R be a right near-ring satisfying the DCCS and having nil-rigid length, $\alpha > 1$. Then there exists a unique ideal A of R such that A is minimal amongst all ideals I of R for which $J_s(R/I)$ is non-zero and nilpotent. Moreover,*

$$A \subseteq J_s(R) \cap C_{\alpha-1} \text{ and } J_s(R/A) = J_s(R)/A.$$

Definition 2.3.15 *The unique minimal ideal A in Lemma 2.3.14 will be called the* **s-socle** *of R.*

Theorem 2.3.16 *[6] Let R be a right near-ring satisfying the DCCS and having nil-rigid length, $\alpha > 2$. Then the s-socle of R is not contained in $Soi(R)$.*

Theorem 2.3.17 *[6] Let R be a right near-ring satisfying the DCCS and let α be the nil-rigid length of R. Then the s-socle, A, of R, is contained in C_1 if, and only if, $\alpha = 2$.*

Theorem 2.3.18 *[6] Let R be a right near-ring satisfying the DCCS and having nil-rigid length, $\alpha = 2$. If $Soi(R) \oplus J_0(R) = C_1$, then the s-socle, A of R is contained in the $Soi(R)$.*

Theorem 2.3.19 *[6] Let R be a right near-ring satisfying the DCCS and having nil-rigid length, $\alpha > 1$. Then the s-radical $J_s(R)$ can be decomposed*

as

$$J_s(R) = J_{1/2}(R) + A + B,$$

where A *is the s-socle of* R *and* $B = J_s(R) \cap L_{\alpha-1}$.

The last anti-radical to be considered is the **critical ideal**. This ideal was first defined by Laxton and Machin for distributively generated (d.g.) near-rings, in [12].

If Ω and Δ are R-groups and Δ is an R-homomorphic image of an R-subgroup of Ω, then we say Δ is a **subfactor** of Ω, denoted $\Delta \ll \Omega$.

For each faithful R-group Ω, the R-isomorphism classes of irreducible R-groups which appear as subfactors of monogenic R-subgroups of Ω can be split into two distinct classes $\mathcal{H}_1$ and $\mathcal{H}_2$. Here $\mathcal{H}_1$ contains only R-groups of type-0 and $\mathcal{H}_2$ contains the rest, Laxton and Machin in [12]. The monogenic R-subgroups of Ω now can be split into two classes $\mathfrak{H}_1$ and $\mathfrak{H}_2$ such that

(a) $\mathfrak{H}_1$ contains those monogenic R-subgroups of Ω which are direct sums of elements of $\mathcal{H}_1$.

(b) $\mathfrak{H}_2$ contains those monogenic R-subgroups of Ω with complete series and whose subfactors are elements of $\mathcal{H}_2$.

Definition 2.3.20 *Let* R *be a right d.g. near-ring with identity satisfying the DCCL, and* Ω *be a faithful* R*-group. The* **critical ideal** *of* R *is defined*

by

$$Crit(R) = \bigcap_{R\omega \in \mathfrak{H}_2} (0 : R\omega),$$

where $\mathfrak{H}_2$ *is the class of monogenic* R*-subgroups of* R *with complete series and whose subfactors are elements of* $\mathcal{H}_2$*.*

It was shown in [12] that, if R is a d.g. near-ring satisfying DCCL, then for each faithful R-group Ω,

$$R = L \oplus Crit(R),$$

where L is a left ideal whose irreducible subfactors are isomorphic to elements of $\mathcal{H}_2$. Here

$$Crit(R) = \bigoplus_{i=1}^{q} Re_i$$

where the Re_i are minimal left ideals isomorphic to elements of $\mathcal{H}_1$.

Theorem 2.3.21 *[12, Corollary 3] Let* R *be a right d.g. near-ring with identity satisfying the DCCL. The maximal class,* $\mathcal{H}_1$*, of* R*-groups of type-*0 *is the same for all faithful* R*-groups and consequently,* $\mathcal{H}_1$ *and* $Crit(R)$ *depend only on* R*.*

Thus, the critical ideal, $Crit(R)$, is unique up to isomorphism.

Hartney, in his doctoral thesis, extended the Laxton-Machin results to near-rings which are not necessarily d.g. near-rings and showed that if R satisfies the DCCS , then

$$Crit(R) \subseteq Soi(R).$$

Here monogenic R-subgroups of a faithful R-group Ω, are classified into three classes: $\mathcal{K}(\Omega)$, $\mathcal{G}(\Omega)$ and $\mathcal{B}(\Omega)$. For a near-ring R with DCCL, the following facts are established in Chapter 3:

1. The analogue of the Critical ideal, denoted by $C(\Omega)$, is equal to the Socle ideal.

2. If the near-ring R decomposes into $R = (Soi(R)) \oplus L$, then every element in $\mathcal{G}(\Omega)$ has an isomorphic copy in $Soi(R)$, and every element in $\mathcal{B}(\Omega)$ has an isomorphic copy in L.

2.4 Matrix near-rings

In this section, we introduce matrix near-rings, as defined by Meldrum and Van der Walt. Also, basic results on the ideals of matrix near-rings are collected here.

The set $M_n(R)$ of $n \times n$ matrices, with entries from a given near-ring R, does not usually form a near-ring under the usual matrix operations. In 1986, Meldrum and Van der Walt gave an alternative definition for a matrix near-ring, cf. [18]. They regarded an $n \times n$ matrix over a near-ring R as a function from R^n to R^n, where R^n is the direct sum of n copies of the group $(R, +)$.

Definition 2.4.1 *Let R be a right near-ring with an identity. The* **matrix near-ring**, $\mathbb{M}_n(R)$, *associated with a near-ring R, is the subnear-ring of*

$M_0(R^n)$ generated by the set

$$\{ f_{ij}^r \,|\, r \in R,\, 1 \leq i, j \leq n\}.$$

Here $f_{ij}^r : R^n \to R^n$ is given by

$$f_{ij}^r(\bar{\rho}) \;=\; \iota_i(r\pi_j(\bar{\rho})),$$

for $\bar{\rho} \in R^n$, where $\iota_i : R \to R^n$ and $\pi_i : R^n \to R$ are the i-th injection and projection functions, respectively.

Unlike earlier attempts to define a matrix near-ring using matrix ring operations, this definition gives a matrix near-ring which is always a near-ring. If the base near-ring is a ring, then the matrix near-ring defined as in Definition 2.4.1 is a matrix ring.

Lemma 2.4.2 *[18] Let R be a right near-ring with an identity, and f_{ij}^r be as defined in Definition 2.4.1. For any a, b, $a_i \in R$, $i = 1, 2, \ldots, n$,*
(i) $f_{ij}^a + f_{ij}^b \;=\; f_{ij}^{a+b}$,
(ii) $f_{ij}^a + f_{kl}^b \;=\; f_{kl}^b + f_{ij}^a$, *if* $i \neq k$,

(iii) $$f_{ij}^a f_{kl}^b \;=\; \begin{cases} f_{il}^{ab} & \text{if } j = k, \\ f_{il}^0 & \text{if } j \neq k, \end{cases}$$

(iv) $f_{ij}^a(f_{1k_1}^{a_1} + f_{2k_2}^{a_2} + \cdots + f_{nk_n}^{a_n}) \;=\; f_{ij}^a\, f_{jk_j}^{a_j} \;=\; f_{ik_j}^{aa_j}$,
(v) a is zero-symmetric in R if, and only if, f_{ij}^a is zero-symmetric in $\mathbb{M}_n(R)$,
(vi) a is distributive in R if, and only if, f_{ij}^a is distributive in $\mathbb{M}_n(R)$.

It follows that $\mathbb{M}_n(R)$ is a right distributive, zero-symmetric near-ring with identity, $1_{\mathbb{M}} = f^1_{11} + f^1_{22} + \cdots + f^1_{nn}$.

Definition 2.4.3 *Let R be right near-ring with identity. Any matrix U in $\mathbb{M}_n(R)$ can be represented as an expression, X, involving only f^r_{ij}'s, $r \in R$. The length, $l(X)$, of an expression X is the number of f^r_{ij}'s in it.*
The weight, $w(U)$, of a matrix U is the length of an expression of minimal length for U.

Known theory about of near-rings can therefore be applied to matrix near-rings. For instance, the Jacobson-type radicals of matrix near-rings can be defined in the usual way and are related, as in R, by

$$J_0(\mathbb{M}_n(R)) \subseteq J_{1/2}(\mathbb{M}_n(R)) \subseteq J_s(\mathbb{M}_n(R)) \subseteq J_2(\mathbb{M}_n(R)).$$

The relationship between ideals of a near-ring R and the ideals of its associated matrix near-ring $\mathbb{M}_n(R)$ form the major part of this thesis.

For an ideal I of R, there are two natural ways to construct an ideal in $\mathbb{M}_n(R)$ which relate naturally to I, as follows.

Definition 2.4.4 *[18] Let R be a right near-ring with an identity, and I be an ideal of R. Then*

$$I^+ := Id\,\{\, f^a_{ij} \,|\, a \in I, 1 \leq i, j \leq n \,\}$$

and

$$I^* := (I^n : R^n) = \{\, U \in \mathbb{M}_n(R) \,|\, U\bar{\rho} \in I^n, \forall\, \bar{\rho} \in R^n \,\}$$

are ideals of $\mathbb{M}_n(R)$.

It follows readily that $I^+ \subseteq I^*$. The ideals I^+ and I^* can be distinct, as shown in [27]. In fact, chains of ideals of $\mathbb{M}_n(R)$ may exist between I^+ and I^*, as shown by Meyer in [16, 21], and are called **intermediate ideals**.

Lemma 2.4.5 *[20] Let R be a right near-ring with an identity. If L is a left ideal of R and I is the largest two-sided ideal of R contained in L, then $L^* = I^*$, where $L^* := \{ W \in \mathbb{M}_n(R) \,|\, W\bar{\rho} \in L^n, \forall\, \bar{\rho} \in R^n \}$.*

It follows immediately that $(J_{1/2}(R))^* = (J_0(R))^*$.

Proposition 2.4.6 *[27] Let R be a right near-ring with an identity. For ideals I and J of R we have,*
(i) $I \subseteq J$ implies $I^+ \subseteq J^+$ and $I^* \subseteq J^*$,
(ii) if $I \nsubseteq J$, then $I^+ \nsubseteq J^*$,
(iii) $(I \cap J)^* = I^* \cap J^*$, in fact, this holds for arbitrary intersections,
(iv) $(I \cap J)^+ \subseteq I^+ \cap J^+$,
(v) $(I + J)^* \supseteq I^* + J^*$,
(vi) $(I + J)^+ = I^+ + J^+$.

Lemma 2.4.7 *[19, Corollary 1.54] Let R be a zero-symmetric right near-ring with identity satisfying the DCCS and I be any ideal of R. Then I^+ is nilpotent in $\mathbb{M}_n(R)$ if, and only if, I is nilpotent in R.*

Ideals of $\mathbb{M}_n(R)$ can be related back to ideals of R, in the following manner.

Definition 2.4.8 *Let $\mathcal{D}$ be an ideal of $\mathbb{M}_n(R)$, then*

$$\mathcal{D}_* := \{\, r \in R \,|\, r = \pi_j(U\bar{\rho}), \text{ for some } U \in \mathcal{D},\ \bar{\rho} \in R^n,\ \forall\, 1 \leq j \leq n\},$$

is a two-sided ideal of R.

Proposition 2.4.9 *[27, 18] Let R be a right near-ring with identity. For all ideals I of R and $\mathcal{J}$ of $\mathbb{M}_n(R)$, we have,*

(i) $\mathcal{J} \subseteq (\mathcal{J}_*)^*$,

(ii) $(I^*)_* = I$,

(iii) $((\mathcal{J}_*)^*)_* = \mathcal{J}_*$,

(iv) $(\mathcal{J}_*)^+ \subseteq \mathcal{J}$.

Proposition 2.4.10 *(Corollary 8, [27]) Let R be a right near-ring with an identity. The ideal I of R is nilpotent if, and only if, I^+ is nilpotent in $\mathbb{M}_n(R)$.*

Combining the results of [7, 9, 15, 26, 20] we have the following theorem.

Theorem 2.4.11 *Let R be a zero-symmetric right near-ring with identity, satisfying the DCCS. Then*

(i) $(J_2(R))^+ \subseteq J_2(\mathbb{M}_n(R)) = (J_2(R))^*$,

(ii) $J_s(\mathbb{M}_n(R)) \subseteq (J_s(R))^*$,

(iii) $J_0(\mathbb{M}_n(R)) \subseteq (J_0(R))^* = (J_{1/2}(R))^*$.

Note that in [26], part (i) was proved for a more general class of near-rings than we have considered in Theorem 2.4.11, whereas in [7], (ii) was proved for R satisfying the DCCL.

Chapter 3

Some Characterizations

In this chapter, firstly, we characterize the socle ideal of a near-ring as an intersection of annihilators of R-groups from a special class of monogenic R-groups. These monogenic R-groups are referred to as **R-groups of type-$\mathcal{K}$**. Secondly, we define alternating chains of R-groups called **Ω-chains**. We study two main questions.

(a) How does the existence of R-groups of type-$\mathcal{K}$ relate to the existence of a gap between the 0-radical $J_0(R)$ and the s-radical $J_s(R)$?

(b) Do the lengths of the Ω-chains determine the nil-rigid length of a near-ring?

The socle ideal is characterized in Section 3.1, and in Section 3.2 we introduce Ω-chains and a new series of ideals of R, called the **Ω-ideal series**. It is shown how the Ω-ideal series relates to the nil-rigid series.

In Section 3.3, the theory developed in the first two sections is illustrated with examples.

3.1 A characterization of $Soi(R)$

In this section, a new type of monogenic R-group is defined, and will be called **type-$\mathcal{K}$**. Monogenic R-subgroups of a faithful R-group, Ω, are classified into three classes, $\mathcal{K}(\Omega)$, $\mathcal{G}(\Omega)$ and $\mathcal{B}(\Omega)$. It is then proved that the socle ideal is an intersection of annihilators of elements of $\mathcal{K}(\Omega)$. It is further shown that every member of $\mathcal{G}(\Omega)$ is isomorphic to a direct summand of the socle ideal.

Throughout this section, R is assumed to be a zero symmetric right near-ring with identity satisfying the $DCCL$ and Ω is assumed to be a faithful R-group. We refer to the decomposition $R = Soi(R) \oplus L$, of Theorem 2.3.4, as the **socle-decomposition** of R. Here L is a left ideal such that $L \supseteq J_{1/2}(R)$.

For near-rings satisfying the DCCS, as noted in Theorem 2.3.6, the socle ideal is the largest ideal that annihilates $J_{1/2}(R)$. We extend this result to near-rings satisfying the DCCL.

In Theorem 3 of [11], it is proved that for a d.g. near-ring, R, satisfying the DCCS, we have the decomposition

$$R/J_{1/2}(R) = \bigoplus_{i=1}^{n} \Delta_i,$$

where each Δ_i is an R-group of type-0, for $i = 1, 2, \ldots, n$. The fact that R is distributively generated plays no role in the proof of this result, while the chain condition was simply needed to ensure that $J_{1/2}(R)$ could be written as an intersection of finitely many maximal left ideals. The proof of the following lemma is identical to the proof of Theorem 3 in [11].

Lemma 3.1.1 *Let L be a left ideal of the right near-ring, R, with identity. Suppose $L = \bigcap_{i=1}^{n} L_i$, an intersection of finitely many maximal left ideals L_i. Then the factor R-group, R/L, can be decomposed as*

$$R/L = \Delta_1 \oplus \Delta_2 \oplus \cdots \oplus \Delta_n,$$

where $\Delta_i = R\delta_i$, for some $\delta_i \in \Delta_i$, is of type-0, and L_i is the annihilating left ideal of δ_i, $i = 1, 2, \ldots, n$.

Proposition 3.1.2 *Let R be a zero symmetric right near-ring with identity. If Γ is a monogenic R-group with a near-ring direct sum decomposition,*

$$\Gamma = \Delta \oplus H,$$

then the R-subgroups, Δ and H, are monogenic.

Proof Let $\Gamma = R\gamma$. Since $\Gamma = \Delta \oplus H$ is a near-ring direct sum, each of the summands, Δ and H, is an R-kernel of Γ. Let the unique

representation of γ be

$$\gamma = \delta + h,$$

where $\delta \in \Delta$ and $h \in H$. Now, let $d \in \Delta$ and $h' \in H$, then $d + h' \in \Gamma$. Since Γ is monogenic, we have $d + h' = r\gamma$ for some r in R. By left distributivity over R-kernels, Lemma 2.3.1,

$$d + h' = r\gamma = r(\delta + h) = r\delta + rh,$$

hence $-r\delta + d = rh - h'$. Since Δ and H are R-subgroups of Γ, $-r\delta \in \Delta$ and $-h', rh \in H$. Thus, we have $-r\delta + d = rh - h' \in \Delta \cap H = (0)$, which gives $d = r\delta$ and $h' = rh$. Since d and h are arbitrary elements in Δ and H, respectively, we conclude that $R\delta = \Delta$ and $Rh = H$. □

3.1.1 Type-$\mathcal{K}$ R-groups and the Jacobson s-radical

In this subsection we show that a gap between the 0-radical and the s-radical can be attributed to the presence of an R-group of type-$\mathcal{K}$ contained in some R-group of type-0.

The symbol $\oplus_G$ denotes the group theoretic direct sum.

Definition 3.1.3 *Let R be a right near-ring with identity. A monogenic R-group, $\Delta = R\delta$, for some $\delta \in \Delta$, is of* **type-$\mathcal{K}$** *if it is not of type-0, and it has no type-0 R-kernels as its near-ring direct summands.*

Example 3.1.1 *Let* $\Gamma := \mathbb{Z}_2 \oplus_G \mathbb{Z}_2$, $\Delta_1 := \mathbb{Z}_2 \oplus_G \{0\}$ *and* $\Delta_2 := \{0\} \oplus_G \mathbb{Z}_2$. *Let* $R := \{ f \in M_0(\Gamma) \mid f(\Delta_i) \subseteq \Delta_i,\ i = 1, 2;\ \gamma - \gamma' \in \Delta_2 \Rightarrow f(\gamma) - f(\gamma') \in \Delta_2,\ \gamma, \gamma' \in \Gamma \}$, *then* R *is a near-ring with a multiplicative identity under point-wise addition and map composition. The group* Γ *is a monogenic* R*-group, in particular* $\Gamma = R(1, 1)$. *However,* Γ *is not of type-*0 *because* Δ_2 *is its only non-trivial* R*-kernel. Since* Δ_1 *is not an* R*-kernel of* Γ, *the type-*0 R*-group* Δ_2 *is not a near-ring direct summand of* Γ. *Hence* Γ *is an* R*-group of type-*$\mathcal{K}$ *as it has no non-trivial* R*-kernels of type-*0, *other than* Δ_2.

Example 3.1.2 *Let* $\Gamma := \mathbb{Z}_4 \oplus_G \mathbb{Z}_2 \oplus_G \mathbb{Z}_2$, $H_1 := \mathbb{Z}_4 \oplus_G \{0\} \oplus_G \{0\}$, $H_2 := \{0\} \oplus_G \mathbb{Z}_2 \oplus_G \mathbb{Z}_2$, $D_1 := \{0, 2\} \oplus_G \{0\} \oplus_G \{0\}$, $D_2 := \{0\} \oplus_G \mathbb{Z}_2 \oplus_G \{0\}$ *and* $D_3 := \{0\} \oplus_G \{0\} \oplus \mathbb{Z}_2$. *Consider*

$$\begin{aligned} R := \{ f \in M_0(\Gamma) \mid & f(H_i) \subseteq H_i,\ i = 1, 2;\ f(D_j) \subseteq D_j,\ j = 1, 2, 3; \\ & \gamma - \gamma' \in H_2 \Rightarrow f(\gamma) - f(\gamma') \in H_2,\ \gamma, \gamma' \in \Gamma; \\ & h_1 - h_2 \in D_2 \Rightarrow f(h_1) - f(h_2) \in D_2,\ h_1, h_2 \in H_2 \}. \end{aligned}$$

Then R *is a near-ring with a multiplicative identity under point-wise addition and map composition. We make the following observations.*

(a) The group Γ *is a monogenic* R*-group, in particular* $\Gamma = R(3, 1, 1)$.

(b) The subgroup H_2 *of* Γ *is the only non-trivial* R*-kernel of* Γ. *Thus,* Γ *is not of type-*0.

(c) The subgroup $D_2 = R(0, 1, 0)$ *is an* R*-group of type-*0 *and it is the only non-trivial* R*-kernel of* H_2. *Hence,* H_2 *is not of type-*0.

(d) The R*-kernel,* D_2, *of* H_2, *is not a near-ring direct summand of* H_2, *because* D_1 *is not an* R*-kernel of* H_2. *Since* H_2 *has no other non-trivial* R*-kernels, it is thus of type-*$\mathcal{K}$.

(e) The subgroup $H_1 = R(3,0,0)$ *is of type-*0*, but it is not an* R*-kernel of* Γ*. On the other hand,* H_2 *is an* R*-kernel of* Γ *but it is not of type-*0*. Therefore,* Γ *does not have a non-trivial near-ring direct sum decomposition and thus* Γ *is an* R*-group of type-*$\mathcal{K}$*.*

Example 3.1.3 *Let* R *be a right near-ring with identity and* Δ *be a monogenic* R*-group with* $\Delta \cong \mathbb{Z}_4$*, if such exists. Now if* Δ *has an* R*-kernel* $H \cong \{0,2\} \subseteq \mathbb{Z}_4$*, then* Δ *is not of type-*0*. In addition, since* $\Delta \cong \mathbb{Z}_4$ *and its only direct sum decomposition into* R*-kernels is itself,* $\mathbb{Z}_4$*, and since* Δ *is not of type-*0*, it follows that* Δ *is of type-*$\mathcal{K}$*.*

We now show that every type-0 R-group which is not of type-s is a monogenic R-group which has no R-kernels but contains an R-group of type-$\mathcal{K}$.

Lemma 3.1.4 *Let* R *be a zero-symmetric right near-ring with identity satisfying the DCCS, then each* R*-group of type-*0 *which is not of type-*s *contains an* R*-group of type-*$\mathcal{K}$*.*

Proof Let Γ be an R-group of type-0 which is not of type-s, then there exists a monogenic R-subgroup $\Delta \neq (0)$ of Γ, such that each near-ring direct sum decomposition of Δ has at least one direct summand which is not of type-0. Although Δ is monogenic, it can not be of type-0, as this would contradict the assumption that Γ is not of type-s. Thus, if Δ has no near-ring direct summands of type-0, it is of type-$\mathcal{K}$, and the proof is complete.

On the other hand, if Δ has at least one near-ring direct summand of type-0, then it can be written as a near-ring direct sum decomposition,

$$\Delta = \Delta_1 \oplus H_1,$$

where H_1 is a non-trivial type-0 R-subgroup of Δ. Since Δ is not of type-0, its R-subgroup $\Delta_1 \neq (0)$, and $\Delta \supsetneq \Delta_1$. By Proposition 3.1.2, Δ_1 is monogenic. Again, Δ_1 can not be of type-0 as this would contradict the assumption that Γ is not of type-s. If Δ_1 has no near-ring direct summand of type-0, it is of type-$\mathcal{K}$, which completes the proof for this case.

Otherwise, Δ_1 has a least one near-ring direct summand of type-0, and hence, can be decomposed into a near-ring direct sum,

$$\Delta_1 = \Delta_2 \oplus H_2,$$

where H_2 is a non-trivial type-0 R-subgroup of Δ_1. Note that $\Delta_1 \supsetneq \Delta_2 \neq (0)$, because Δ_1 is not of type-0. Also, it follows from Proposition 3.1.2 that Δ_2 is monogenic. But Δ_2 can not be of type-0, because Γ is not of type-s.

Continuing inductively, if Δ_i is not of type-$\mathcal{K}$, this procedure can be repeated, but it terminates by the DCCS at, say

$$\Delta_{n-1} = \Delta_n \oplus H_n,$$

where Δ_n is a non-zero type-$\mathcal{K}$ R-subgroup of Δ_{n-1}, and H_n is a non-trivial type-0 R-subgroup of Δ_{n-1}. $\square$

Theorem 3.1.5 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS. Then $J_0(R) \neq J_s(R)$ if, and only if, there exists an R-group Γ of type-0 with an R-subgroup Δ of type-$\mathcal{K}$.*

Proof $\Rightarrow$ Suppose $J_0(R) \neq J_s(R)$. Then there exists an R-group of type-0 which is not of type-s, say Γ. By Lemma 3.1.4, Γ contains an R-group of type-$\mathcal{K}$.
$\Leftarrow$ Conversely, let Γ be an R-group of type-0, and Δ be a type-$\mathcal{K}$ R-subgroup of Γ. By definition of type-$\mathcal{K}$, the R-group Δ is monogenic but not of type-0, and it has no type-0 R-kernel as a direct summand. Since $\Gamma \supseteq \Delta$, it follows that Γ is of type-0 but not of type-s. Therefore $J_0(R) \neq J_s(R)$. □

The next result extends Proposition 2.2.4 to R-groups of type-$\mathcal{K}$.

Proposition 3.1.6 *Let R be a zero-symmetric right near-ring with identity, and Γ be an R-group. Let I be an ideal of R with $I \subseteq (0:\Gamma)$, then Γ is an (R/I)-group of type-$\mathcal{K}$ if, and only if, Γ is an R-group of type-$\mathcal{K}$.*

Proof By Proposition 2.2.4, Γ is an (R/I)-group, and for each R-subgroup Δ of Γ, we have $I \subseteq (0:\Gamma) \subseteq (0:\Delta)$, and Δ is an (R/I)-subgroup of Γ. Conversely, by Proposition 2.2.4, if Δ is an (R/I)-subgroup of Γ, then Δ is an R-subgroup of Γ.

Now, Δ is of type-0 as an R-group if, and only if, it is of type-0 as an (R/I)-group, by Proposition 2.2.4. Thus, Γ is not of type-0 as an R-group

if, and only if, it is not of type-0 as an (R/I)-group.

Note that Δ is a (group) direct summand of Γ as an R-group if, and only if, Δ is a (group) direct summand of Γ as an (R/I)-group. A straight forward calculation, using the identification, $(r+I)\gamma := r\gamma$, made in Proposition 2.2.4 (a), shows that

$$(r+I)[\gamma+\delta]-(r+I)\gamma \;=\; r(\gamma+\delta)-r\gamma,$$

for all $\gamma \in \Gamma$, $r \in R$ and $\delta \in \Delta$. Thus, Δ is an R-kernel of Γ if, and only if, Δ is an (R/I)-kernel of Γ. Hence, as an R-group, Γ has no near-ring direct summand of type-0 if, and only if, as an (R/I)-group, it has no near-ring direct summand of type-0. □

3.1.2 A classification of monogenic R-groups

Lemma 3.1.7 *Let R be a right near-ring with identity which satisfies the DCCL. For any ideal P of R, $P \cap J_{1/2}(R) = (0)$ if, and only if, $P \subseteq Soi(R)$.*

Proof From Lemma 2.3.3, $P \subseteq \mathrm{Soi}(R)$ implies $P \cap J_{1/2}(R) = (0)$.

Conversely, suppose $P \cap J_{1/2}(R) = (0)$. We show that P has an $\mathcal{F}$-decomposition. The maximality of $\mathrm{Soi}(R)$ will then imply that $P \subseteq Soi(R)$. Putting $J_{1/2}(R) = J$ we have, by Lemma 3.1.1,

$$R/J = (L_1/J) \oplus \cdots \oplus (L_n/J),$$

L_i left ideals of R and L_i/J of type-0 for $i = 1, \ldots, n$. Reordering, if necessary, we have, by Lemma 2.2.2,

$$R/J = \big((P \oplus J)/J\big) \oplus (L_{m+1}/J) \oplus \cdots \oplus (L_n/J).$$

Since P is an ideal, $PL_j \subseteq P \cap L_j \subseteq P \cap J = (0),\ j = m+1, \ldots, n$. The decomposition of the coset $1 + J$ gives

$$1+J = (u+J)+(x_{m+1}+J)+\cdots+(x_n+J), \quad u \in P,\ x_j \in L_j,\ j = m+1, \ldots, n.$$

For any $a \in P$ we have, by left distribution modulo J,

$$a + J = (au + ax_{m+1} + \cdots + ax_n) + J.$$

Thus $a - au \in J \cap P = (0)$, because $ax_j = 0,\ j = m+1, \ldots, n$. Hence u is a right identity of P.

We can also decompose the coset $1 + J$ as

$$1 + J = y_1 + \cdots + y_n + J, \quad y_i \in L_i,\ i = 1, \ldots, n.$$

Thus for any $a \in P$ we have

$$a + J = (ay_1 + \cdots + ay_m + ay_{m+1} + \cdots + ay_n) + J.$$

Since $ay_j = 0,\ j = m+1, \ldots, n$ it follows that $-a + (ay_1 + \cdots ay_m) \in J \cap P = (0)$. Hence

$$P = P \cap L_1 \oplus P \cap L_2 \oplus \cdots \oplus P \cap L_m.$$

Now if $P \cap L_i \neq 0$, then $J + (P \cap L_i)$ is an R-kernel of L_i properly containing J. Since L_i/J is of type-0, this implies that $L_i = J + (P \cap L_i)$. In this

case we have

$$\begin{aligned} L_i/J = (J + P \cap L_i)/J &\cong (P \cap L_i)/(P \cap L_i \cap J) \\ &\cong P \cap L_i. \end{aligned}$$

Thus if $P \cap L_i \neq 0$, then it is of type-0. The decomposition of the right identity u of P namely,

$$u = u_1 + \cdots + u_m, \quad u_i \in P \cap L_i,\ i = 1, \ldots, m$$

gives an orthogonal idempotent set $\{u_i\}_{i=1}^m$ because left distribution over the $P \cap L_i$ is valid. Thus P has an $\mathcal{F}$-decomposition. □

Let Ω be a faithful R-group and let $\mathcal{S}$ be the collection of all monogenic R-subgroups of Ω. We define subclasses $\mathcal{B}(\Omega)$, $\mathcal{G}(\Omega)$ and $\mathcal{K}(\Omega)$ of $\mathcal{S}$ as follows.

Definition 3.1.8 *Let R be a right near-ring with identity, and Ω be a faithful R-group. Let $\mathcal{S}$ be the collection of all monogenic R-subgroups of Ω. Define*

(a) $\mathcal{K}(\Omega)$ to be the subclass of $\mathcal{S}$ of all R-groups of type-$\mathcal{K}$,

(b) $\mathcal{B}(\Omega)$ to be the subclass of $\mathcal{S}$ of all type-0 R-groups which are R-isomorphic to subfactors of at least one member of $\mathcal{K}(\Omega)$,

(c) $\mathcal{G}(\Omega)$ to be the subclass of $\mathcal{S}$ of all type-0 R-groups which are not in $\mathcal{B}(\Omega)$.

We note that the zero R-subgroup, (0), of Ω is of type-0 as it has no non-trivial R-kernels, hence it is not of type-$\mathcal{K}$. Thus the class $\mathcal{K}(\Omega)$ is distinct from each of the classes, $\mathcal{B}(\Omega)$ and $\mathcal{G}(\Omega)$, of type-0 R-subgroups of Ω.

When there is no ambiguity about the faithful R-group Ω, we will simply write $\mathcal{B}, \mathcal{G}$ and $\mathcal{K}$, for the classes $\mathcal{B}(\Omega), \mathcal{G}(\Omega)$ and $\mathcal{K}(\Omega)$, respectively.

Lemma 3.1.9 *[3] Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and let $R\omega$ be a monogenic R-subgroup of Ω. If $R\omega \notin \mathcal{K}(\Omega)$, then $R\omega = (\bigoplus_{i=1}^{m} \Delta_i) \oplus \Delta$, where each summand is an R-kernel of $R\omega$, and each Δ_i is of type-0 $(i = 1, 2, \ldots, m)$ and $\Delta \in \mathcal{K}(\Omega)$.*

Definition 3.1.10 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCL, and Ω a faithful R-group. We define the ideal $C(\Omega)$ and the left ideal $N(\Omega)$ as follows*

$$C(\Omega) = \bigcap_{K \in \mathcal{K}(\Omega)} (0\colon K), \qquad N(\Omega) = \bigcap_{\substack{\omega \in \Omega \\ R\omega \in \mathcal{G}(\Omega)}} (0\colon \omega).$$

Note. If $\mathcal{K} = \emptyset$, then $C(\Omega) = R$. Since $(0\colon \omega)$ is a maximal left ideal, it is clear that $J_{1/2}(R) \subseteq N(\Omega)$.

Theorem 3.1.11 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCL, and Ω a faithful R-group. Then*

$$C(\Omega) \cap N(\Omega) = (0) \quad \textit{and} \quad C(\Omega) \subseteq \mathrm{Soi}(R).$$

Proof As $J_{1/2}(R) \subseteq N(\Omega)$, it follows that, if $C(\Omega) \cap N(\Omega) = (0)$, then $C(\Omega) \cap J_{1/2}(R) = (0)$, and the result will follow from Lemma 3.1.7.

We now show that $C(\Omega) \cap N(\Omega) = (0)$. Let $x \in C(\Omega) \cap N(\Omega)$ and $\omega \in \Omega$. If $R\omega \in \mathcal{K}$, then $x\omega = 0$, by the definition of $C(\Omega)$. Otherwise,

$R\omega = (\bigoplus_{i=1}^{m} \Delta_i) \oplus \Delta,\ m \geq 1,\ \Delta_i$ of type-0, $\Delta \in \mathcal{K}$ by Lemma 3.1.9 and $x\Delta = (0)$. If $\Delta_i \in \mathcal{B}$, then $x\Delta_i = 0$, again by the definition of $C(\Omega)$.

If $\Delta_i \in \mathcal{G}$, then $\Delta_i = R\omega_i$, and since Δ_i is of type-0, any R-subgroup $R(r'\omega_i)$ is $R\omega_i$ or (0), for each $r' \in R$. Thus, if $R(r'\omega_i) = R\omega_i$, then $(0 : r'\omega_i) = (0 : \omega_i)$, so $xr'\omega_i = 0$. Otherwise, $R(r'\omega_i) = (0)$ so again $xr'\omega_i = 0$. Hence, if $x \in (0 : \omega_i)$, then $x \in (0 : r'\omega_i)$, and $x\Delta_i = (0)$.

Now, let δ_i be the component, in Δ_i, of the decomposition of ω as a sum $\omega = \sum_{j=1}^{m} \delta_j + \delta$, where $\delta_j \in \Delta_j$, $\delta \in \Delta$, $j = 1, \ldots, m$. Then $x\delta_i = 0$ as $x\Delta_i = (0)$, for all $i = 1, 2, ..., m$, and $x\delta = 0$. Thus $x\omega = 0$ and since Ω is faithful it follows that $x = 0$. □

Theorem 3.1.12 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCL, and Ω a faithful R-group. $C(\Omega)$ is a direct sum of elements from $\mathcal{G}$. More precisely, $C(\Omega)$ is a direct sum of R-groups of type-0 each of which has an isomorphic copy in $\mathcal{G}$.*

Proof Since R satisfies the $DCCL$, $N := N(\Omega)$ is an intersection of finitely many maximal left ideals. It follows from Lemma 3.1.1 that

$$R/N = \bigoplus_{R\omega_i \in \mathcal{G}} R\omega_i.$$

But $\big(C(\Omega) \oplus N\big)/N$ is an R-kernel of R/N, and hence it follows readily from Lemma 2.2.2 that it is a direct sum of elements from $\mathcal{G}$. The result now follows from the fact that $C(\Omega) \cap N = (0)$, Theorem 3.1.11, and the

isomorphisms

$$\begin{aligned}\big(C(\Omega)\oplus N\big)/N &\cong C(\Omega)/\big(C(\Omega)\cap N\big)\\ &\cong C(\Omega).\end{aligned}$$

□

Theorem 3.1.13 *Let R be a zero-symmetric right near-ring, with identity 1, satisfying the DCCL, and Ω a faithful R-group. Then*

$$C(\Omega) = Soi(R).$$

Proof Since $C(\Omega) \subseteq \mathrm{Soi}(R)$ by Theorem 3.1.11, we need only show that $Soi(R)K = (0)$, for every $K \in \mathcal{K}$. Let $Rk = K$ be in $\mathcal{K}$ and let $R = Soi(R) \oplus L$ be the socle-decomposition of R (cf. Definition 2.3.5.) Then

$$K = Rk = Soi(R)k + Lk. \tag{3.1}$$

If $Soi(R)k \not\subseteq Lk$, then $Re_ik \not\subseteq Lk$ for some type-0 summand Re_i in the $\mathcal{F}$-decomposition, $Soi(R) = \bigoplus_{i=1}^{n} Re_i$. But then the map $Re_i \to Re_ik$ given by $re_i \mapsto re_ik$ is an R-isomorphism of Re_i onto Re_ik, and thus Re_ik is of type-0 as Re_i is of type-0. Since $Re_ik \cap Lk$ is an R-kernel of Re_ik, the fact that Re_ik is of type-0 and $Re_ik \not\subseteq Lk$, gives $Re_ik \cap Lk = (0)$. Thus, we can form a near-ring direct sum, $Re_ik \oplus Lk$. On the other hand, if $Re_ik \subseteq Lk$, then $Re_ik = Re_i1e_ik \subseteq Re_iRe_ik \subseteq (Re_iL)k = (0)$, hence $Re_ik = (0)$.

Now, suppose $Re_ik \not\subseteq Lk$ for $i = 1, 2, \ldots, m$, and $Re_ik \subseteq Lk$ for $i = m+1, \ldots, n$, then $Re_ik \cap Lk = (0)$ and $K \supseteq Re_ik \oplus Lk$ if $i = 1, 2, \ldots, m$, and $Re_ik = (0)$, $i = m+1, \ldots, n$. If $l \in \{2, \ldots, m\}$ and $Re_lk \subseteq Re_1k \oplus Lk$, then discard Re_lk. Note that $m \geq 1$, since $Soi(R)k \not\subseteq Lk$. So, after relabelling $e_2, \ldots, e_m$, without loss of generality, $Re_jk \not\subseteq Re_1 \oplus Lk$, for $j \in \{2, \ldots, m_1\}$, where $m_1 \leq m$, and $Re_jk \subseteq Re_1 \oplus Lk$, $j = m_1 + 1, \ldots, n$. Since Re_2k is of type-0 and $Re_2k \cap (Re_1k \oplus Lk)$ is an R-kernel of Re_2k, it follows that $Re_2k \cap (Re_1k \oplus Lk) = (0)$, hence $Re_2k \oplus Re_1k \oplus Lk \subseteq K$.

Continuing this way we obtain $K = \bigoplus_{j=1}^{t} Re_jk \oplus Lk$, $m \geq t \geq 1$. But this contradicts the fact that K has no R-group of type-0 as a near-ring direct summand. Thus $Soi(R)K \subseteq Lk$ and by (3.1), $K \subseteq Lk$, so that $Soi(R)K \subseteq \big(\mathrm{Soi}(R)L\big)k$. But $Soi(R)L \subseteq Soi(R) \cap L = (0)$, hence $Soi(R)K = (0)$. Therefore $\mathrm{Soi}(R) \subseteq C(\Omega)$. □

We remark that the above result shows that $C(\Omega)$ depends only on the near-ring R and not on the faithful R-group Ω. Indeed, $C(\Omega)$ is precisely the Laxton-Machin critical ideal in the case of a d.g. near-ring R, cf. [12]. Also, we know that $C(\Omega) = Soi(R)$ is a direct sum of elements from $\mathcal{G}$, but it is not clear whether each element from $\mathcal{G}$ has an isomorphic copy as a summand in $\mathrm{Soi}(R)$.

Theorem 3.1.14 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCL, and Ω a faithful R-group. Let $\mathcal{G}$ and $\mathcal{B}$ be the classes of R-groups of type-0 as previously constructed in Definition 3.1.8, and $R = Soi(R) \oplus L$ be the socle-decomposition of R. Each element of $\mathcal{G}$ has*

an isomorphic copy as a summand of $Soi(R) = C(\Omega)$ *and each element of* $\mathcal{B}$ *is a subfactor of* L.

Proof Let L be written as a near-ring direct sum

$$L = \left(\bigoplus_{i=1}^{m} L_i\right) \oplus \left(\bigoplus_{j=1}^{r} L'_j\right) \oplus L'',$$

where $L_i \in \mathcal{G}$, $i = 1, \ldots, m$, $L'_j \in \mathcal{B}$, $j = 1, \ldots, r$, with $m, r \geq 0$, and $L'' \in \mathcal{K}$.

Put $M = Soi(R) \oplus (\bigoplus_{i=1}^{m} L_i)$. Then

$$MR = \left(\mathrm{Soi}(R) \oplus \bigoplus_{i=1}^{m} L_i\right)\left(\mathrm{Soi}(R) \oplus \left(\bigoplus_{i=1}^{m} L_i\right) \oplus \left(\bigoplus_{j=1}^{r} L'_j\right) \oplus L''\right). \tag{3.2}$$

If $m, r \geq 1$ and $L_i L'_j \neq (0)$, then there exists $x \in L'_j$ such that $L_i x \neq (0)$. It follows that the mapping $L_i \to L_i x$ given by $y \mapsto yx$, for $y \in L_i$, gives an R-isomorphism of L_i onto $L_i x$. Since $L_i x$ is an R-kernel of L'_j, which is of type-0, and $L_i x \neq (0)$, we have $L_i x = L'_j$. But $L'_j \ll K$ for some $K \in \mathcal{K}$, consequently $L_i \ll K$ and thus $L_i \in \mathcal{B}$. This contradicts the fact that $L_i \in \mathcal{G}$ and shows that $L_i L'_j = (0)$, for all $i = 1, \ldots, m$, and $j = 1, \ldots, r$.

Now suppose $m \geq 1$ and $L_i L'' \neq (0)$. Then there exists $K \in \mathcal{K}$ such that $L_i L'' K \neq (0)$ as otherwise $L_i L'' \subseteq Soi(R)$ giving $L_i L'' \subseteq Soi(R) \cap L'' = (0)$. Let $k \in K$ and $x \in L''$ be such that $L_i xk \neq (0)$. Then the mapping $L_i \to L_i xk$, defined by $l_i \mapsto l_i xk$, for $l_i \in L_i$, is an R-isomorphism of L_i onto $L_i xk$. Thus, $L_i xk$ is a type-0 R-kernel of K, but

then $L_ixk \oplus \big(K/(L_ixk)\big) \cong K$ gives K a near-ring direct summand of type-0, a contradiction. Hence $L_iL'' = (0)$.

Now consider (3.2) above. By right distribution and left distribution over a direct sum of R-kernels we have $MR \subseteq M$ so that M is a two-sided ideal. But M has an $\mathcal{F}$-decomposition so that $M \subseteq Soi(R)$ by the maximality of $Soi(R)$, i.e., $m = 0$. By Theorem 2.3.4, $J_{1/2}(R) \subseteq L$, and we may thus write the factor R-group R/J, $J := J_{1/2}(R)$, as

$$R/J = \big(\big(\mathrm{Soi}(R) + J\big)/J\big) \oplus \big(L/J\big).$$

Since every R-group Δ of type-0 has an R-isomorphic copy as a direct summand of R/J, Lemma 2.2.15 (ii), it follows that either Δ is R-isomorphic to a summand of $Soi(R) \cong \big(Soi(R) + J\big)/J$ or is a subfactor of L. By the above, since $m = 0$, if $\Delta \in \mathcal{G}$, then Δ is a summand of $Soi(R)$. $\square$

Consider a type-0 R-subgroup, $\Delta \in \mathcal{B}(\Omega)$, of a faithful R-group Ω. There exists an R-isomorphism $\phi : \Delta \to K/H$ for some $K \in \mathcal{K}(\Omega)$ and H an R-kernel of K. For any $s \in Soi(R)$ and $\delta \in \Delta$ such that $\phi(\delta) = k + H$, $k \in K$, we have

$$\phi(s\delta) \;=\; s\phi(\delta) \;=\; s(k + H) \;=\; sk + H \;=\; H.$$

Since ϕ is an R-isomorphism, $\phi(0) = H$, so that $\phi(0) = \phi(s\delta)$, and hence $s\delta = 0$. This yields the following proposition.

Proposition 3.1.15 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCL, and Ω a faithful R-group. For any $\Delta \in \mathcal{B}(\Omega)$,*

$$Soi(R) \;\subseteq\; (0 : \Delta).$$

3.2 A characterization of the nil-rigid series

In this section, alternating chains, called **Ω-chains**, of type-0 R-subgroups and type-$\mathcal{K}$ R-subgroups of a faithful R-group Ω are considered. In Subsection 3.2.2, an ascending series of ideals of R is defined in terms of the Ω-chains. This series of ideals is called the **Ω-ideal series**. In Subsection 3.2.3, it is proved that the Ω-ideal series contains the nil-rigid series.

3.2.1 The class $\mathcal{K}(\Omega)$ and the nil-rigid length

The following results indicate a connection between the class, $\mathcal{K}(\Omega)$ and α_R, the nil-rigid length of R.

Proposition 3.2.1 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and Ω a faithful R-group. If Ω is of type-s, then $\alpha_R = 1$.*

Proof Since Ω is faithful, $J_0(R) = J_s(R) = (0)$. By Corollary 2.3.8, $Soi(R) = R$. Hence the nil-rigid series of R is :

$$L_1 = (0) \quad \subsetneq \quad C_1 = Soi(R) = R.$$

So that $\alpha_R = 1$. □

Proposition 3.2.2 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and Ω a faithful R-group. If Ω is of type-0 but not of type-s, then $\alpha_R > 1$.*

Proof As Ω is faithful, $J_0(R) = (0)$. Since Ω is of type-0 but not of type-s, it has a non-zero R-subgroup Δ of type-$\mathcal{K}$, by Lemma 3.1.4. Now, by Theorem 3.1.13, $Soi(R) = C(\Omega)$, thus the first two members of the nil-rigid series of R are such that

$$L_1 = J_0(R) = (0) \subsetneq C_1 = Soi(R) \subseteq (0 : \Delta) \neq R.$$

Hence the nil-rigid series cannot end at C_1. Consequently $\alpha_R > 1$. □

Proposition 3.2.3 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and Ω a faithful R-group. If $\mathcal{K}(\Omega) = \emptyset$, then $\alpha_R = 1$.*

Proof If $\mathcal{K}(\Omega) = \emptyset$, then $Soi(R) = R$. This implies, by Corollary 2.3.8, that $J_s(R) = (0)$. Hence $J_0(R) = J_s(R)$, and $\alpha_R = 1$. □

3.2.2 The Ω-chains

In this subsection, Ω-chains are defined and some of their characteristics are noted.

The following lemma is an immediate consequence of Corollary 2.2.5.

Lemma 3.2.4 *Let R be a zero-symmetric right near-ring with identity, and let*

$$\Delta_1 \supsetneq \Delta_2 \supsetneq \Delta_3 \supsetneq \ldots \supsetneq \Delta_\beta \tag{3.3}$$

be a strictly descending chain of R-groups for some positive integer β. Then Δ_{i+1} is an (R/I_i)-group, where $I_i = (0 : \Delta_i)$, $i = 1, 2, \ldots, \beta - 1$. Moreover,

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_\beta,$$

is an ascending chain of ideals of R.

Definition 3.2.5 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and let Ω a faithful R-group. Let*

$$\Delta_1 \supsetneq \Delta_2 \supsetneq \ldots \supsetneq \Delta_{\beta-1} \supsetneq \Delta_\beta, \tag{3.4}$$

be a strictly descending chain of R-subgroups of Ω such that Δ_1 is maximal amongst type-0 R-subgroups of Ω, and Δ_β is an R-group of type-s.
The chain (3.4) is an ***Ω-chain*** *if for every pair (Δ_i, Δ_{i+1}), $1 \leq i < \beta$, of consecutive members of chain (3.4), we have that: if Δ_i is of type-0, then Δ_{i+1} is of type-$\mathcal{K}$ and it is maximal amongst type-$\mathcal{K}$ R-subgroups of Δ_i; and if Δ_i is of type-$\mathcal{K}$, then Δ_{i+1} is of type-0 and it is maximal amongst type-0 R-subgroups of Δ_i.*

We define the ***Ω-length*** *of the Ω-chain (3.4) to be $\frac{\beta-1}{2}$.*

We collect some facts about Ω-chains in the next Proposition.

Proposition 3.2.6 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and Ω be a faithful R-group.*
(a) Every type-0 R-subgroup of Ω which is maximal amongst type-0 R-subgroups of Ω is a member of some Ω-chain.
(b) If a type-0 R-subgroup of Ω is maximal amongst type-0 R-subgroups of Ω and is of type-s, then it forms a trivial Ω-chain on its own.
(c) If no type-0 R-subgroup of Ω contains an R-subgroup of type-$\mathcal{K}$, then the Ω-length of every Ω-chain is 0.
(d) The Ω-chains are alternating sequences of type-0 R-subgroups of Ω and type-$\mathcal{K}$ R-subgroups of Ω, starting with a type-0.
(e) If a type-$\mathcal{K}$ R-subgroup, Δ, of Ω, is not contained in any type-0 R-subgroup of Ω, then Δ is not a member of any Ω-chain.
(f) If two type-$\mathcal{K}$ R-subgroups of Ω, say Δ and Δ', are members of an Ω-chain where $\Delta \supsetneq \Delta'$, then there exists a type-0 R-group, say Δ'', with $\Delta \supsetneq \Delta'' \supsetneq \Delta'$.

We note, from Proposition 3.2.6 (d), that β in (3.4) is odd.

Examples of Ω-chains and their lengths are given in Section 3.3.

Theorem 3.2.7 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, Ω a faithful R-group, and let*

$$\Delta_1 \supsetneq \Delta_2 \supsetneq \cdots \supsetneq \Delta_\beta,$$

be an Ω-chain, and

$$L_1 \subsetneq C_1 \subsetneq L_2 \subsetneq C_2 \subsetneq \ldots \subsetneq L_{\alpha_R} \subsetneq C_{\alpha_R} = R,$$

be the nil-rigid series of R. Then, the nil-rigid length, α_R, is such that

$$\alpha_R \geq \frac{\beta+1}{2},$$

and for any $m = 1, 2, \ldots, (\beta-1)/2$,

(a) Δ_{2m} is an (R/L_m)-group of type-$\mathcal{K}$ and $(0 : \Delta_{2m}) \supseteq C_m$,

(b) Δ_{2m+1} is an (R/C_m)-group of type-0 and $(0 : \Delta_{2m+1}) \supseteq L_{m+1}$,

where L_m, C_m and L_{m+1} are ideals of R in the nil-rigid series of R.

Proof The proof follows by induction on m. For $m = 1$, note that $(0 : \Delta_1) \supseteq J_0(R) = L_1$, because Δ_1 is an R-group of type-0. Since $\Delta_1 \supsetneq \Delta_2$, we have $(0 : \Delta_2) \supseteq (0 : \Delta_1)$, and hence $(0 : \Delta_2) \supseteq L_1$. It follows by Proposition 2.2.4 that Δ_2 is an (R/L_1)-group, and since Δ_2 is of type-$\mathcal{K}$ as an R-group, it is an (R/L_1)-group of type-$\mathcal{K}$, by Proposition 3.1.6. Since Δ_2 is an (R/L_1)-group of type-$\mathcal{K}$ and $Soi(R/L_1)$ is the intersection of annihilators of all (R/L_1)-groups of type-$\mathcal{K}$, see Theorem 3.1.13, we have $(0 : \Delta_2)_{R/L_1} \supseteq Soi(R/L_1)$. By Proposition 2.2.6 and the definition of the nil-rigid series,

$$(0 : \Delta_2)/L_1 \supseteq Soi(R/L_1) = C_1/L_1,$$

thus $(0 : \Delta_2) \supseteq C_1$, giving $R \neq C_1$ and $\alpha_R \geq 2$.

Now consider Δ_3. Since $\Delta_3 \subsetneq \Delta_2$, then $(0 : \Delta_3) \supseteq (0 : \Delta_2)$, consequently $(0 : \Delta_3) \supseteq C_1$ and Δ_3 is an (R/C_1)-group by Proposition 2.2.4. In addition, since Δ_3 is of type-0 as an R-group, it is a type-0 (R/C_1)-group by Proposition 2.2.4. As the 0-radical $J_0(R/C_1)$ is the intersection of annihilators of all (R/C_1)-groups, $(0 : \Delta_3)_{R/C_1} \supseteq J_0(R/C_1)$. By the definition of the

nil-rigid series, $J_0(R/C_1) = L_2/C_1$, as the nil-rigid length of R is $\alpha_R \geq 2$. By Proposition 2.2.6,

$$(0 : \Delta_3)/C_1 \supseteq J_0(R/C_1) = L_2/C_1,$$

and hence $(0 : \Delta_3) \supseteq L_2$.

Assume the result holds for $m-1$, where $1 < m \leq (\beta - 1)/2$. Then, $\alpha_R \geq m$, the R-group Δ_{2m-2} is an (R/L_{m-1})-group of type-$\mathcal{K}$ and $(0 : \Delta_{2m-2}) \supseteq C_{m-1}$, also the R-group Δ_{2m-1} is an (R/C_{m-1})-group of type-0 and $(0 : \Delta_{2m-1}) \supseteq L_m$.

For the R-group Δ_{2m}, since $\Delta_{2m} \subsetneq \Delta_{2m-1}$, then $(0 : \Delta_{2m}) \supseteq (0 : \Delta_{2m-1})$. It follows from the fact that $(0 : \Delta_{2m}) \supseteq (0 : \Delta_{2m-1}) \supseteq L_m$ and Proposition 2.2.4 that Δ_{2m} is an (R/L_m)-group. But Δ_{2m} is an R-group of type-$\mathcal{K}$, and hence an (R/L_m)-group of type-$\mathcal{K}$, by Proposition 3.1.6. Thus $(0 : \Delta_{2m})_{R/L_m} \supseteq Soi(R/L_m)$ as the socle ideal, $Soi(R/L_m)$, is the intersection of all (R/L_m)-groups of type-$\mathcal{K}$. By Proposition 2.2.6 and the definition of the nil-rigid series,

$$(0 : \Delta_{2m})/L_m \supseteq Soi(R/L_m) = C_m/L_m,$$

thus $(0 : \Delta_{2m}) \supseteq C_m$, and $R \neq (0 : \Delta_{2m})$, $C_m \neq R$, giving $\alpha_R \geq m + 1$.

Now, since $\Delta_{2m+1} \subsetneq \Delta_{2m}$, we have $(0 : \Delta_{2m+1}) \supseteq (0 : \Delta_{2m})$. Again, $(0 : \Delta_{2m+1}) \supseteq (0 : \Delta_{2m}) \supseteq C_m$ and Proposition 2.2.4 implies that Δ_{2m+1} is an (R/C_m)-group. Since Δ_{2m+1} is an R-group of type-0, it follows from Proposition 2.2.4 that Δ_{2m+1} is of type-0 as an (R/C_m)-group. But then

$(0 : \Delta_{2m+1})_{R/C_m} \supseteq J_0(R/C_m)$ by definition of the 0-radical, $J_0(R/C_m)$. By the definition of the nil-rigid series, $J_0(R/C_m) = L_{m+1}/C_m$. Then by Proposition 2.2.6,

$$(0 : \Delta_{2m+1})/C_m \supseteq J_0(R/C_m) = L_{m+1}/C_m.$$

Hence $(0 : \Delta_{2m+1}) \supseteq L_{m+1}$. □

From Theorem 3.2.7, we observe that for each faithful R-group Ω, the lengths of the Ω-chains have an upperbound of $2\alpha_R - 1$. Hence we can find a maximum for the lengths of the Ω-chains.

Definition 3.2.8 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and let Ω a faithful R-group. An Ω-chain with the greatest Ω-length is called a $\mathcal{KG}$-**chain** of Ω, and the Ω-length of a $\mathcal{KG}$-chain is called the $\mathcal{KG}$-**length** of Ω.*

Note. If η is the $\mathcal{KG}$-length of Ω and α_R is the nil-rigid length of R, then by Theorem 3.2.7,

$$\alpha_R \geq \eta + 1.$$

Since the $\mathcal{KG}$-length is 0 if, and only if, $J_0(R) = J_s(R)$, the next proposition follows from Lemma 2.3.13.

Proposition 3.2.9 *Let R be a zero symmetric right near-ring with identity satisfying the DCCS, Ω a faithful R-group, and let α_R be the nil-rigid length of R. The $\mathcal{KG}$-length of Ω, $\eta = 0$ if, and only if, $\alpha_R = 1$.*

Theorem 3.2.10 *Let R be a zero symmetric right near-ring with identity satisfying the DCCS, and let Ω be a faithful R-group. If the nil-rigid length of R is $\alpha_R = 2$, then Ω has a $\mathcal{KG}$-length $\eta = 1$.*

Proof Suppose $\alpha_R = 2$, then by Theorem 3.2.7, $\alpha_R = 2 \geq \eta+1$. Assume that $\eta = 0$. Then every R-group of type-0 has no R-subgroup of type-$\mathcal{K}$. It follows from Lemma 3.1.4 that every R-subgroup of Ω which is maximal amongst type-0 R-subgroups of Ω is of type-s. That is, $J_0(R) = J_s(R)$, and by Lemma 2.3.13, $\alpha_R = 1$, contradicting the supposition that $\alpha_R = 2$. Therefore, $\eta = 1$. □

3.2.3 Ω-ideal series and the nil-rigid series

In this subsection, the Ω-ideal series is defined, and it is proved that the Ω-ideal series contains the nil-rigid series in the following sense.

Given two series of ideals

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{l-1} \subseteq I_{l(I)}, \tag{3.5}$$

and

$$J_1 \subseteq J_2 \subseteq \cdots \subseteq J_{l-1} \subseteq J_{l(J)}, \tag{3.6}$$

with their lengths such that $l(I) \geq l(J)$, we say the series (3.5) *contains* the series (3.6) if the corresponding ideals are such that

$$I_i \supseteq J_i, \quad \text{for each } i = 1, 2, \ldots, l(J).$$

Definition 3.2.11 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and Ω a faithful R-group. Let*

$$\Delta_{\lambda,1} \supsetneq \Delta_{\lambda,2} \supsetneq \ldots \supsetneq \Delta_{\lambda,(\beta(\lambda)-1)} \supsetneq \Delta_{\lambda,\beta(\lambda)},\ \lambda \in \Lambda, \tag{3.7}$$

*be all Ω-chains. Define the j-**th** Ω-**ideal**, T_j, of R to be the intersection of annihilators of the j-th members of each Ω-chain of length $\beta(\lambda) \geq j$, i.e.*

$$T_j := \bigcap_{\substack{\lambda\in\Lambda \\ \beta(\lambda)\geq j}} (0 : \Delta_{\lambda,j}), \quad j = 1, 2, \ldots, \tau(\Omega),$$

where $\tau(\Omega) = max\{\, \beta(\lambda) \mid \lambda \in \Lambda\}$. We also define

$$T_0 := (0) \quad \text{and} \quad T_{\tau(\Omega)+1} := R.$$

Note. In the above definition $\tau(\Omega)$ is always odd (as the Ω-chains are alternating chains starting and ending with type-0 R-groups), and $\frac{\tau(\Omega)-1}{2}$ is the $\mathcal{KG}$-length of Ω. That is $\tau(\Omega)$ is the length of the longest Ω-chains.

Proposition 3.2.12 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and let Ω be a faithful R-group. The Ω-ideals form an ascending series of ideals of R. That is,*

$$T_1 \subseteq T_2 \subseteq T_3 \subseteq \ldots \subseteq T_{\tau(\Omega)}.$$

We refer to the series

$$T_1 \subseteq T_2 \subseteq T_3 \subseteq \ldots \subseteq T_{\tau(\Omega)},$$

of the j-th Ω-ideals, $j = 1, 2, \ldots, \tau(\Omega)$, as the **$\Omega$-ideal series.**

Proposition 3.2.13 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, Ω a faithful R-group, and let T_1 the first Ω-ideal of R. Then,*

$$J_0(R) = T_1 = L_1,$$

where L_1 is as in the definition of the nil-rigid series.

Proof Let $\Delta_{\lambda,1}$, $\lambda \in \Lambda$, be as defined in (3.7), then $T_1 = \bigcap_{\lambda\in\Lambda}(0 : \Delta_{\lambda,1})$. Since each $\Delta_{\lambda,1}$ is of type-0, then $T_1 \supseteq J_0(R)$. Each $\Delta_{\lambda,1}$ is a type-0 R-subgroup of Ω which is maximal amongst type-0 R-subgroups of Ω, and by Proposition 3.2.6(a), every such maximal type-0 R-subgroup of Ω is a member of some Ω-chain. Thus T_1 annihilates all type-0 R-subgroups of Ω, because every type-0 R-subgroup of Ω is either maximal amongst type-0 R-subgroups of Ω or it is contained in a type-0 R-group which is maximal amongst type-0 R-subgroups of Ω. As a result $T_1 = J_0(R)$. □

Proposition 3.2.14 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, Ω a faithful R-group, with $\mathcal{KG}$-length $\eta \geq 1$, and let T_2 be the second Ω-ideal of R. Then*

$$T_2/L_1 \supseteq Soi\,(R/L_1) \quad \textit{and} \quad T_2 \supseteq C_1,$$

where L_1 and C_1 are as in the definition of the nil-rigid series of R.

Proof Let $\Delta_{\lambda,2}$, $\lambda \in \Lambda$, be as defined in (3.7), then by definition of Ω-ideals, $T_2 = \bigcap_{\substack{\lambda\in\Lambda \\ \beta(\lambda)\geq 2}}(0 : \Delta_{\lambda,2})$, where each $\Delta_{\lambda,2}$, $\beta(\lambda) \geq 2$, is a type-$\mathcal{K}$ R-subgroup of Ω. From Propositions 3.2.12 and 3.2.13, $T_2 \supseteq T_1 = L_1$.

Also, by Theorem 3.2.7, each $\Delta_{\lambda,2}$ is an (R/L_1)-group of type-$\mathcal{K}$. But, for each faithful (R/L_1)-group, the socle ideal, $Soi(R/L_1)$, is the intersection of the annihilators of all its (R/L_1)-subgroups of type-$\mathcal{K}$, see Theorem 3.1.13, hence $T_2/L_1 \supseteq Soi(R/L_1)$. By the definition of the nil-rigid series,

$$T_2/L_1 \supseteq Soi(R/L_1) = C_1/L_1,$$

thus $T_2 \supseteq C_1$. □

Proposition 3.2.15 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, Ω a faithful R-group with $\mathcal{KG}$-length $\eta \geq 1$, and let T_3 the third Ω-ideal of R. Then*

$$T_3/C_1 \supseteq J_0(R/C_1) \quad \textit{and} \quad T_3 \supseteq L_2,$$

where C_1 and L_2 are as in the definition of the nil-rigid series of R.

Proof Let $\Delta_{\lambda,3}$, $\lambda \in \Lambda$, be as defined in (3.7), then by definition of the Ω-ideal series, $T_3 = \bigcap_{\substack{\lambda\in\Lambda \\ \beta(\lambda)\geq 3}} (0 : \Delta_{\lambda,3})$, and $T_3 \supseteq T_2 \supseteq C_1$ by Propositions 3.2.12 and 3.2.13. Since each $\Delta_{\lambda,3}$, $\beta(\lambda) \geq 3$, is a type-0 R-group, it follows from Theorem 3.2.7 that each $\Delta_{\lambda,3}$ is an (R/C_1)-group of type-0. By definition of the 0-radical, we have

$$T_3/C_1 \supseteq J_0(R/C_1).$$

Since $L_2/C_1 = J_0(R/C_1)$, then $T_3 \supseteq L_2$. □

Theorem 3.2.16 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and Ω a faithful R-group with $\mathcal{KG}$-length $\eta \geq 2$. Let*

$$T_1 \subseteq T_2 \subseteq T_3 \subseteq \ldots \subseteq T_{\tau(\Omega)},$$

be the Ω*-ideal series. For* $m = 1, 2, \ldots, \frac{\tau(\Omega)-1}{2}$,

(a) $T_{2m}/L_m \supseteq Soi\,(R/L_m)$ *and* $T_{2m} \supseteq C_m$,

(b) $T_{2m+1}/C_m \supseteq J_0\,(R/C_m)$ *and* $T_{2m+1} \supseteq L_{m+1}$,

where L_m, C_m *and* L_{m+1} *are ideals of* R *in the nil-rigid series of* R.

Proof For $m = 1, 2$ the statement follows from Propositions 3.2.14 and 3.2.15, respectively. We proceed by induction on m. Assume that the statement holds for m, with $m < \frac{\tau(\Omega)-1}{2}$. That is,

$$T_{2m}/L_m \supseteq Soi\,(R/L_m) \quad \text{and} \quad T_{2m} \supseteq C_m.$$

Also,

$$T_{2m+1}/C_{m+1} \supseteq J_0\,(R/C_{m+1}) \quad \text{and} \quad T_{2m+1} \supseteq L_{m+1}.$$

Now for $m+1$. Let

$$\Delta_{\lambda,1} \supsetneq \Delta_{\lambda,2} \supsetneq \ldots \supsetneq \Delta_{\lambda,(\beta(\lambda)-1)} \supsetneq \Delta_{\lambda,\beta(\lambda)},\ \lambda \in \Lambda,$$

be all Ω-chains. By definition of the Ω-ideal series,

$$T_{2m+2} = \bigcap_{\substack{\lambda\in\Lambda \\ \beta(\lambda)\geq 2m+2}} (0 : \Delta_{\lambda,\,2m+2}).$$

By Proposition 3.2.12 and the induction hypothesis,

$$T_{2m+2} \supseteq T_{2m+1} \supseteq L_{m+1}.$$

Since each $\Delta_{\lambda,2m+2}$, $\beta(\lambda) \geq 2m+2$, is an R-group of type-$\mathcal{K}$, it follows from Theorem 3.2.7 that each $\Delta_{\lambda,2m+2}$ is an (R/L_{m+1})-group of type-$\mathcal{K}$. But for each faithful (R/L_{m+1})-group, the socle ideal, $Soi(R/L_{m+1})$, is the

intersection of the annihilators of all its (R/L_{m+1})-subgroups of type-$\mathcal{K}$, see Theorem 3.1.13, hence

$$T_{2m+2}/L_{m+1} \supseteq Soi(R/L_{m+1}).$$

By definition of the nil-rigid series,

$$T_{2m+2}/L_{m+1} \supseteq Soi(R/L_{m+1}) = C_{m+1}/L_{m+1},$$

thus $T_{2m+2} \supseteq C_{m+1}$.

Again, by definition of the Ω-ideal series,

$$T_{2m+3} = \bigcap_{\substack{\lambda\in\Lambda \\ \beta(\lambda)\geq 2m+3}} (0 : \Delta_{\lambda,\, 2m+3}).$$

By Proposition 3.2.12, we have

$$T_{2m+3} \supseteq T_{2m+2} \supseteq C_{m+1}.$$

Since each $\Delta_{\lambda,2m+3}$, $\beta(\lambda) \geq 2m+3$, is an R-group of type-0, then by Theorem 3.2.7, each $\Delta_{\lambda,2m+3}$ is an (R/C_{m+1})-group of type-0. But the 0-radical, $J_0(R/C_{m+1})$, is the intersection of annihilators of all (R/C_{m+1})-groups of type-0, hence

$$T_{2m+3}/C_{m+1} \supseteq J_0(R/C_{m+1}).$$

By definition of the nil-rigid series,

$$T_{2m+3}/C_{m+1} \supseteq J_0(R/C_{m+1}) = L_{m+2}/C_{m+1},$$

hence $T_{2m+3} \supseteq L_{m+2}$. □

The Ω-ideal series and the nil-rigid series are therefore related as follows.

Corollary 3.2.17 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and Ω a faithful R-group. The series of Ω-ideals,*

$$T_1 \subseteq T_2 \subseteq T_3 \subseteq \ldots \subseteq T_{\tau(\Omega)} \subseteq T_{\tau(\Omega)+1} = R,$$

and the nil-rigid series of R,

$$L_1 \subsetneq C_1 \subsetneq L_2 \subsetneq C_2 \subsetneq \ldots \subsetneq L_{\alpha_R} \subsetneq C_{\alpha_R} = R,$$

are related by

$$T_{2m-1} \supseteq L_m \quad \text{and} \quad T_{2m} \supseteq C_m,$$

for $1 \leq m \leq \frac{\tau(\Omega)-1}{2}$.

The fact that the Ω-ideal series and the nil-rigid series are distinct is due to the existence of the quotient R-groups of the next proposition. This is illustrated in the Section 3.3.

Proposition 3.2.18 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and let Ω be a faithful R-group of $\mathcal{KG}$-length $\eta = \frac{\tau(\Omega)-1}{2}$. For any two consecutive members Δ_{2m} and Δ_{2m+1}, of a $\mathcal{KG}$-chain,*

$$\Delta_1 \supsetneq \Delta_2 \supsetneq \Delta_3 \supsetneq \ldots \supsetneq \Delta_{2\eta+1},$$

we have that, if Δ_{2m+1} is an R-kernel of Δ_{2m}, then $\Delta_{2m}/\Delta_{2m+1}$ is an (R/C_m)-group.

Proof Since Δ_{2m+1} is an R-kernel of Δ_{2m}, the quotient group, $\Delta_{2m}/\Delta_{2m+1}$, is an R-group. By Theorem 3.2.7, $(0 : \Delta_{2m}) \supseteq C_m$. Hence Δ_{2m} is an

(R/C_m)-group by Proposition 2.2.4. Since $(0 : \Delta_{2m}/\Delta_{2m+1}) \supseteq (0 : \Delta_{2m})$, we have

$$(0 : \Delta_{2m}/\Delta_{2m+1}) \supseteq (0 : \Delta_{2m}) \supseteq C_m,$$

thus $\Delta_{2m}/\Delta_{2m+1}$ is an (R/C_m)-group by Proposition 2.2.4. □

Lemma 3.2.19 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and let*

$$(0) \subseteq L_1 \subsetneq C_1 \subsetneq L_2 \subsetneq C_2 \subsetneq \cdots \subsetneq L_{\alpha_R} \subsetneq C_{\alpha_R} = R,$$

be the nil-rigid series of R. Then the nil-rigid length of R/C_i is

$$\alpha_R - i,$$

where $1 \leq i < \alpha_R$.

Proof By definition of the nil-rigid series, the quotient near-ring R/C_i has the nil-rigid series

$$(0) \subseteq L_1(i) \subsetneq C_1(i) \subsetneq L_2(i) \subsetneq C_2(i) \subsetneq \cdots \subsetneq L_{\alpha_R}(i) \subsetneq C_{\alpha_R - i}(i),$$

where $L_1(i)/C_i$ is an ideal of R/C_i such that

$$L_1(i)/C_i = J_0(R/C_i) = L_{i+1}/C_i,$$

so that $L_1(i) = L_{i+1}$. Also,

$$C_1(i)/L_1(i) = Soi(R/L_1(i)) = Soi(R/L_{i+1}) = C_{i+1}/L_{i+1},$$

hence $C_1(i) = C_{i+1}$. Similarly,

$$L_2(i)/C_1(i) = J_0(R/C_1(i)) = J_0(R/C_{i+1}) = L_{i+2}/C_{i+1},$$

which means $L_2(i) = L_{i+2}$, and

$$C_2(i)/L_2(i) = Soi(R/L_2(i)) = Soi(R/L_{i+2}) = C_{i+2}/L_{i+2},$$

so that $C_2(i) = C_{i+2}$. Therefore,

$$L_{\alpha_R - i}(i) = L_{\alpha_R} \quad \text{and} \quad C_{\alpha_R - i}(i) = C_{\alpha_R} = R,$$

because

$$L_{\alpha_R - i}(i)/C_{\alpha_R-(i+1)}(i) = J_0(R/C_{\alpha_R-(i+1)}(i)) = J_0(R/C_{\alpha_R - 1}) = L_{\alpha_R}/C_{\alpha_R - 1}$$

and

$$C_{\alpha_R - i}(i)/L_{\alpha_R - i}(i) = Soi(R/L_{\alpha_R - i}(i)) = Soi(R/L_{i+(\alpha_R - i)}) = C_{\alpha_R}/L_{\alpha_R},$$

and $C_{\alpha_R}/L_{\alpha_R} = R/L_{\alpha_R}$.

We conclude that the nil-rigid series of R/C_i is the nil-rigid series of R starting with L_{i+1} and ending with $C_{\alpha_R} = R$, giving $\alpha_{R/C_i} \leq \alpha_R - i$. But since $C_i \subseteq C_{\alpha_R - 1} \subsetneq C_{\alpha_R} = R$, we get $C_{\alpha_R - 1}/C_i \neq C_{\alpha_R}/C_i = R/C_i$. Hence, the nil-rigid length of R/C_i, $\alpha_{R/C_i} = \alpha_R - i$. □

Conjecture 3.1 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and Ω a faithful R-group. If the $\mathcal{KG}$-length of Ω is η, then the nil-rigid length of R,*

$$\alpha_R = \eta + 1.$$

3.3 Examples on Ω-chains

In this section, examples of near-rings and Ω-chains are given, these examples also illustrate the fact that the nil-rigid series and the Ω-ideal series may be distinct. Here, the symbol $\oplus_G$ will denote the group theoretic direct sum.

Example 3.3.1 *Let* $\Omega = \mathbb{Z}_{32}$*;* $H_1 = \{0, 2, 4, \ldots, 30\}$*;* $H_2 = \{0, 4, 8, \ldots, 28\}$*;* $H_3 = \{0, 8, 16, 24\}$*;* $H_4 = \{0, 16\}$.

Define the near-ring R *as follows,*

$R := \{\, f \in M_0(\Omega) \,|\, f(H_i) \subseteq H_i,\ i = 1, 2, 3, 4;$

$h - h' \in H_{2j} \Rightarrow f(h) - f(h') \in H_{2j},\ \forall\, h, h' \in H_{2j-1},\ j = 1, 2\,\}.$

Then R *is a near-ring with a multiplicative identity under point-wise addition and map composition. We make the following observations.*

(a) The group $\Omega = \mathbb{Z}_{32}$ *is a type-*0 *faithful* R*-group as it has no non-trivial* R*-kernels.*

(b) The subgroup H_1 *is a monogenic* R*-group with only one type-*0 R*-kernel,* H_2*. Since* H_2 *is not a direct summand of* H_1*, then* H_1 *is of type-*$\mathcal{K}$*.*

(c) The subgroup H_4 *is a type-*0 R*-kernel to the monogenic* R*-group* H_3 *and it is not a direct summand of* H_3*, hence* H_3 *is of type-*$\mathcal{K}$*.*

There is only one Ω*-chain, namely*

$$\Omega \supsetneq H_1 \supsetneq H_2 \supsetneq H_3 \supsetneq H_4,$$

which is a $\mathcal{KG}$*-chain of* $\mathcal{KG}$*-length* $\eta = 2$*, i.e.* $\tau(\Omega) = 5$*. The* Ω*-ideal series is now,*

$$(0) = T_0 = T_1 \subseteq T_2 \subseteq T_3 \subseteq T_4 \subseteq T_5 \subseteq T_6 = R,$$

where

$$T_1 = (0:\Omega) = (0),\ T_2 = (0:H_1),\ T_3 = (0:H_2),$$

$$T_4 = (0:H_3), \quad \textit{and} \quad T_5 = (0:H_4).$$

The nil-rigid series of R *is*

$$(0) = L_1 \subsetneq C_1 \subsetneq L_2 \subsetneq C_2 \subsetneq L_3 \subsetneq C_3 = R,$$

where

$$L_1 = (0:\Omega) = (0), \quad C_1/L_1 = Soi(R/L_1) = (0:H_1),$$

$$L_2/C_1 = J_0(R/C_1) = (0:H_2) \cap (0:H_1/H_2),$$

$$C_2/L_2 = Soi(R/L_2) = (0:H_3),$$

$$L_3/C_2 = J_0(R/C_2) = (0:H_4) \cap (0:H_3/H_4)$$

and

$$C_3/L_3 = Soi(R/L_3) = R/L_3.$$

Therefore $C_3 = R$ *and* $\alpha_R = 3 = \eta + 1$. *However, the nil-rigid series and the* Ω*-ideal series are distinct because* $T_3 \supsetneq L_2$ *and* $T_5 \supsetneq L_3$.

The next example shows that there exist type-$\mathcal{K}$ R-groups which are not members of any Ω-chains; that is, the socle ideal $Soi(R)$ is not a member of the nil-rigid series ($Soi(R) \neq C_1$). Again, we have that the Ω-ideal series is distinct from the nil-rigid series.

Example 3.3.2 *Let* $\Omega = \mathbb{Z}_8 \oplus_G \mathbb{Z}_4$, $\Delta_1 = \{0,4\} \oplus_G \mathbb{Z}_4$, $\Delta_2 = \{0,2,4,6\} \oplus_G \{0\}$, $\Delta_3 = \{0\} \oplus_G \mathbb{Z}_4$, $\Delta_4 = \{0,4\} \oplus_G \{0\}$,

$\Delta_5 = \{0\} \oplus_G \{0, 2\}$.

Define the near-ring R *as follows,*

$R := \{ f \in M_0(\Omega) \mid f(\Delta_j) \subseteq \Delta_j,\ j = 1, \ldots, 5;$

$\omega - \omega' \in \Delta_2 \Rightarrow f(\omega) - f(\omega') \in \Delta_2,\ \forall\, \omega, \omega' \in \Omega;$

$\delta - \delta' \in \Delta_4 \Rightarrow f(\delta) - f(\delta') \in \Delta_4,\ \forall\, \delta, \delta' \in \Delta_2;$

$\delta_1 - \delta_2 \in \Delta_5 \Rightarrow f(\delta_1) - f(\delta_2) \in \Delta_5,\ \forall\, \delta_1, \delta_2 \in \Delta_3 \}$.

Then R *is a near-ring with a multiplicative identity under point-wise addition and mapping composition. We make the following observations.*

(a) The group Ω *is a faithful monogenic* R*-group but it is not of type-*0 *because it has* Δ_2 *as an* R*-kernel.*

(b) The subgroup Δ_1 *is a type-*0 R*-group which is not of type-s as it contains a monogenic* R*-group* Δ_3 *which in turn has a type-*0 R*-subgroup* Δ_5*, but* Δ_5 *is not a direct summand of* Δ_3*. Thus* Δ_3 *is of type-*$\mathcal{K}$*.*

(c) The subgroup Δ_2 *is a monogenic* R*-group containing a type-*0 R*-subgroup* Δ_4*, which is not a direct summand of* Δ_2*. Therefore* Δ_2 *is of type-*$\mathcal{K}$*.*

(d) The near-ring has two other R*-groups of type-*2*,* Δ_3/Δ_5 *and* Δ_2/Δ_4*.*

The R*-group* Ω *has one non-trivial* Ω*-chain,*

$$\Delta_1 \supsetneq \Delta_3 \supsetneq \Delta_5,$$

and one trivial Ω*-chain consisting just of* Δ_4*. Hence* Δ_2 *is not a member of any* Ω*-chain. Also, the* $\mathcal{KG}$*-length is* $\eta = 1$*.*

The Ω*-ideal series is*

$$\{0\} = T_0 \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq T_4 = R,$$

where

$$T_1 = (0 : \Delta_4) \cap (0 : \Delta_1),$$

$$T_2 = (0 : \Delta_3), \quad \textit{and} \quad T_3 = (0 : \Delta_5).$$

Note that, since $(0 : \Delta_3/\Delta_5) \supseteq (0 : \Delta_3)$, *the quotient* R*-group* Δ_3/Δ_5 *is an* (R/C_1)*-group.*

The nil-rigid series of R *is*

$$\{0\} \subseteq L_1 \subsetneq C_1 \subsetneq L_2 \subsetneq C_2 = R,$$

where

$$L_1 = (0 : \Delta_4) \cap (0 : \Delta_1) \cap (0 : \Delta_2/\Delta_4) = J_0(R),$$

$$C_1/L_1 = Soi(R/L_1) = (0 : \Delta_3),$$

$$L_2/C_1 = J_0(R/C_1) = (0 : \Delta_5) \cap (0 : \Delta_3/\Delta_5).$$

Hence the nil-rigid length $\alpha_R = 2 = \eta + 1$,. *The two series are distinct as* $L_1 \neq T_1$ *and* $L_2 \neq T_3$.

Chapter 4

Annihilator Ideals in $\mathbb{M}_n(R)$

Ideals which are intersections of annihilators of monogenic R-groups (or $\mathbb{M}_n(R)$-groups) are herein referred to as **annihilator ideals**.

In Section 4.1, known theory on the connection between R-groups and $\mathbb{M}_n(R)$-groups is collected. Also, two ways in which $\mathbb{M}_n(R)$ acts on Ω^n, where Ω is a faithful R-group, are defined. It is well known that for some non-monogenic R-groups, direct sums of n copies of themselves are monogenic as $\mathbb{M}_n(R)$-groups. Three classes of non-monogenic R-groups are defined according to the type of their corresponding $\mathbb{M}_n(R)$-groups. That is, a non-monogenic R-group is said to be of ν_n**-form** if its corresponding monogenic $\mathbb{M}_n(R)$-group is of type-ν $(\nu = 0, s, \mathcal{K})$.

In Section 4.2, a theory is developed on how annihilator ideals of R and their corresponding ideals of $\mathbb{M}_n(R)$ are related. As an example of an annihilator ideal, the theory in Section 4.2 is applied to the socle ideal in Section 4.3. Independent of the theory of annihilator ideals in Section 4.2, it is proved in Section 4.3 that

$$Soi(\mathbb{M}_n(R)) \subseteq (Soi(R))^*.$$

Examples which illustrate relationships between socle ideals of R and $\mathbb{M}_n(R)$ are given in Section 4.4.

4.1 Modules in matrix near-rings

In order to relate annihilator ideals in R to their corresponding ideals in $\mathbb{M}_n(R)$, we recall known results on the relationships between R-groups and $\mathbb{M}_n(R)$-groups. Some of these results, by Van der Walt in [26], use a generalized notion of a monogenic R-group called a *connected* R-group. These results are given below in the terminology of Meldrum and Meyer, in [15], which has been used throughout this thesis.

Definition 4.1.1 *Let R be a zero-symmetric right near-ring with identity. Suppose Ω is an R-group. Then Ω is said to be* **locally monogenic** *if for any finite subset H of Ω there is an $\omega \in \Omega$ such that $H \subseteq R\omega$.*

Note that a monogenic R-group is locally monogenic. Van der Walt defined a natural action of $\mathbb{M}_n(R)$ on Ω^n as follows.

Lemma 4.1.2 *[26] Let R be a zero-symmetric right near-ring with identity. For Ω, a locally monogenic R-group, Ω^n is a locally monogenic $\mathbb{M}_n(R)$-group under the action*

$$U\langle\omega_1, \omega_2, \ldots, \omega_n\rangle := (U\langle r_1, r_2, \ldots, r_n\rangle)\omega$$

for $U \in \mathbb{M}_n(R)$, $\omega_i, \omega \in \Omega$, $r_i \in R$, and $r_i\omega = \omega_i$.

Theorem 4.1.3 *[26] Let R be a zero-symmetric right near-ring with identity. A locally monogenic R-group, Δ, has no non-trivial R-subgroups (resp. R-kernels) if, and only if, Δ^n has no non-trivial $\mathbb{M}_n(R)$-subgroups (resp. $\mathbb{M}_n(R)$-kernels).*

Theorem 4.1.4 *[26] Let R be a zero-symmetric right near-ring with identity, and let Δ be a locally monogenic R-group. Then any $\mathbb{M}_n(R)$-kernel (resp. $\mathbb{M}_n(R)$-subgroup) of Δ^n is of the form H^n, where H is an R-kernel (resp. R-subgroup) of Δ.*

It is a consequence of Theorem 4.1.3 that, any R-group, $R\delta$, is of type-ν ($\nu = 0, s, 2$) as an R-group if, and only if, $(R\delta)^n$ is of type-ν as an $\mathbb{M}_n(R)$-group. We extend this result to R-groups of type-$\mathcal{K}$.

Corollary 4.1.5 *Let R be a zero-symmetric right near-ring with identity, and let Δ be a monogenic R-group. The R-group Δ has a non-trivial R-kernel if, and only if, Δ^n has a non-trivial $\mathbb{M}_n(R)$-kernel. That is, Δ is not of type-0 if, and only if, Δ^n is not of type-0.*

Lemma 4.1.6 *Let* R *be a zero-symmetric right near-ring with identity, and let* Δ *be a monogenic* R*-group. The* R*-group* Δ *has no near-ring direct summands of type-*0 *if, and only if,* Δ^n *has no near-ring direct summands of type-*0 *as* $\mathbb{M}_n(R)$*-subgroups.*

Proof $\Rightarrow$ Suppose Δ^n has a near-ring direct summand of type-0, say

$$\Delta^n = D \oplus B,$$

where B is an $\mathbb{M}_n(R)$-kernel of type-0 and D is some $\mathbb{M}_n(R)$-kernel. By Theorem 4.1.4, $D = H^n$ and $B = S^n$, where both H and S are R-kernels of Δ. Since $H^n \oplus S^n \cong_{\mathbb{M}_n(R)} (H \oplus S)^n$, it follows that $\Delta = H \oplus S$. Since Δ is monogenic, by Proposition 3.1.2 both H and S are monogenic, and consequently S is of type-0, by Theorem 4.1.3. Thus Δ has a near-ring direct summand of type-0.

$\Leftarrow$ Conversely, suppose Δ has a near-ring direct summand of type-0, say

$$\Delta = H \oplus P,$$

where P is an R-kernel of type-0 and H is an R-kernel. Since Δ is monogenic, it follows from Proposition 3.1.2 that H is monogenic. Hence Δ^n, H^n and P^n are monogenic $\mathbb{M}_n(R)$-groups, by Lemma 4.1.2. It is easy to show that both H^n and P^n are $\mathbb{M}_n(R)$-kernels of Δ^n. By Theorem 4.1.3, P^n is a type-0 $\mathbb{M}_n(R)$-kernel of Δ^n. Since

$$\Delta^n = (H \oplus P)^n \cong_{\mathbb{M}_n(R)} H^n \oplus P^n,$$

it follows that Δ^n has a type-0 $\mathbb{M}_n(R)$-group as a direct summand. $\square$

The next theorem follows immediately from Corollary 4.1.5 and Lemma 4.1.6.

Theorem 4.1.7 *Let R be a zero-symmetric right near-ring with identity, and let Δ be a monogenic R-group. The R-group Δ is of type-$\mathcal{K}$ if, and only if, Δ^n is of type-$\mathcal{K}$ as an $\mathbb{M}_n(R)$-group.*

The next result is a generalization of Lemma 1.6 of [20], and the proof extends easily, using Lemma 4.1.2, to any group, $(\Gamma/\Delta)^n$, where Γ is a monogenic R-group and Δ an R-kernel of Γ. The action of $\mathbb{M}_n(R)$ on $(\Gamma/\Delta)^n$ is defined by

$$U\langle r_1\gamma + \Delta,\, r_2\gamma + \Delta,\, \ldots\,, r_n\gamma + \Delta\rangle \,:=\, \langle s_1\gamma + \Delta,\, s_2\gamma + \Delta,\, \ldots\,, s_n\gamma + \Delta\rangle,$$

where

$$U\langle r_1, r_2, \ldots, r_n\rangle \,=\, \langle s_1, s_2, \ldots, s_n\rangle,$$

for some $s_i \in R$, $1 \leq i \leq n$ and $U \in \mathbb{M}_n(R)$. This action is well-defined, see Appendix A.

Lemma 4.1.8 *Let R be a zero-symmetric right near-ring with identity, Γ a monogenic R-group, and let Δ be an R-kernel of Γ. Then*

$$(\Gamma/\Delta)^n \,\cong_{\mathbb{M}_n(R)}\, \Gamma^n/\Delta^n.$$

Theorem 4.1.9 *Let R be a zero-symmetric right near-ring with identity, Ω a faithful R-group, and let Γ and Δ be monogenic R-subgroups of Ω. Then, Δ is an R-homomorphic image of Γ if, and only if, Δ^n is an $\mathbb{M}_n(R)$-homomorphic image of Γ^n.*

Proof $\Rightarrow$ Suppose Δ is a homomorphic image of Γ. Then there exists an R-homomorphism $\phi : \Omega \to \Delta$ such that $\Gamma/ker(\phi) \cong_R \Delta$. Clearly, $\big(\Gamma/ker(\phi)\big)^n \cong_{\mathbb{M}_n(R)} \Delta^n$. But, by Lemma 4.1.8,

$$\big(\Gamma/ker(\phi)\big)^n \cong_{\mathbb{M}_n(R)} \Gamma^n/(ker(\phi))^n.$$

Thus

$$\Gamma^n/(ker(\phi))^n \cong_{\mathbb{M}_n(R)} \Delta^n.$$

$\Leftarrow$ Conversely, suppose $\Phi : \Omega^n \to \Delta^n$ is an $\mathbb{M}_n(R)$-homomorphism such that

$$\Delta^n \cong_{\mathbb{M}_n(R)} \Gamma^n/ker(\Phi).$$

By Theorem 4.1.4, $ker(\Phi) = H^n$, for H some R-kernel of Γ. Thus, $\Delta^n \cong_{\mathbb{M}_n(R)} \Gamma^n/H^n$. Again, by Lemma 4.1.8,

$$\Gamma^n/H^n \cong_{\mathbb{M}_n(R)} \big(\Gamma/H\big)^n.$$

It follows that

$$\big(\Gamma/H\big)^n \cong_{\mathbb{M}_n(R)} \Delta^n$$

and hence $\Gamma/H \cong_R \Delta$. □

From Theorems 4.1.3 and 4.1.9, it follows, under the conditions thereof, that for a faithful R-group, Ω, if $\mathbb{M}_n(R)$ has DCCS, then

$$\Delta \in \mathcal{B}(\Omega) \quad \textit{iff} \quad \Delta^n \in \mathcal{B}(\Omega^n). \tag{4.1}$$

Also, if $\Delta \in \mathcal{G}(\Omega)$ then Δ is of type-0, making Δ^n a type-0 $\mathbb{M}_n(R)$-group and $\Delta^n \notin \mathcal{B}(\Omega^n)$, by (4.1). Thus $\Delta^n \in \mathcal{G}(\Omega^n)$.

Conversely, $\Delta^n \in \mathcal{G}(\Omega^n)$ implies that Δ^n is an $\mathbb{M}_n(R)$-group of type-0 and $\Delta^n \notin \mathcal{B}(\Omega^n)$. Since Δ is monogenic, it is of type-0, by Theorem 4.1.3. So that $\Delta \notin \mathcal{B}(\Omega)$, by (4.1), hence $\Delta \in \mathcal{G}(\Omega)$. To summarize we have the following theorem.

Theorem 4.1.10 *Let R be a zero-symmetric near-ring with identity and Ω a faithful R-group. For any monogenic R-subgroup Δ of Ω,*
(a) $\Delta \in \mathcal{B}(\Omega)$ if, and only if, $\Delta^n \in \mathcal{B}(\Omega^n)$,
(b) $\Delta \in \mathcal{G}(\Omega)$ if, and only if, $\Delta^n \in \mathcal{G}(\Omega^n)$.

The action of the matrix near-ring $\mathbb{M}_n(R)$ is not limited to $\mathbb{M}_n(R)$-groups of the form Ω^n, where Ω is locally monogenic as in Lemma 4.1.2. There is another action of $\mathbb{M}_n(R)$ on Ω^n called *Action* 2, for which Ω need not be locally monogenic. Meldrum and Meyer used Action 2 to show that type-0 $\mathbb{M}_n(R)$-groups exist in several non-isomorphic ways, see [15]. This action is explained below.

Suppose

$$\Omega = \bigoplus_{\lambda=1}^{m} \Omega_\lambda$$

is a group theoretic direct sum of monogenic R-subgroups $\Omega_\lambda = R\omega_\lambda$, $\lambda = 1, 2, ..., m$. Then, as a group theoretic direct sum,

$$\Omega^n := \bigoplus_{i=1}^{n} \left(\bigoplus_{\lambda=1}^{m} \Omega_\lambda \right)$$

and there is a group isomorphism

$$\Psi : \Omega^n \to \bigoplus_{\lambda=1}^{m} \left(\bigoplus_{i=1}^{n} \Omega_\lambda \right) = \bigoplus_{\lambda=1}^{m} \Omega_\lambda^n$$

defined by

$$\Psi(\langle\langle r_{11}\omega_1, r_{12}\omega_2, \ldots, r_{1m}\omega_m\rangle, \cdots, \langle r_{n1}\omega_1, r_{n2}\omega_2, \ldots, r_{nm}\omega_m\rangle\rangle)$$

$$= \langle\langle r_{11}\omega_1, \ldots, r_{n1}\omega_1\rangle, \langle r_{12}\omega_2, \ldots, r_{n2}\omega_2\rangle, \cdots, \langle r_{1m}\omega_m, \ldots, r_{nm}\omega_m\rangle\rangle.$$

This can be rewritten as

$$\langle\langle r_{11}, \ldots, r_{n1}\rangle\omega_1, \langle r_{12}, \ldots, r_{n2}\rangle\omega_2, \cdots, \langle r_{1m}, \ldots, r_{nm}\rangle\omega_m\rangle.$$

Each $\langle r_{1j}, \ldots, r_{nj}\rangle$, $1 \leq j \leq m$, is an element in R^n on which a matrix acts naturally. Action 2 is now defined as follows.

Definition 4.1.11 *(**Action 2**) [15] Let R be a zero-symmetric right near-ring with identity, $U \in \mathbb{M}_n(R)$ and $\bar{\rho} = \langle\omega_1, \omega_2, \ldots, \omega_n\rangle$ be any element in Ω^n. Then*

$$U\bar{\rho} \quad = \quad U\langle\omega_1, \omega_2, \ldots, \omega_n\rangle \quad := \quad \Psi^{-1}U\Psi(\langle\omega_1, \omega_2, \ldots, \omega_n\rangle),$$

is a well-defined $\mathbb{M}_n(R)$ action on Ω^n.

Proposition 4.1.12 *Let R be a zero-symmetric right near-ring with identity. Let $\Gamma = \bigoplus_{i=1}^{m} R\delta_i$ be an R-group which is a group theoretic direct sum of monogenic R-groups. If $n \geq m$, then Γ^n is a monogenic $\mathbb{M}_n(R)$-group under Action 2.*

Proof Consider the following matrices, $V_{(r_1,r_2,..,r_n)} = f_{11}^{r_1} + f_{12}^{r_2} + \cdots + f_{1n}^{r_n}$ and $U_{(a_1,a_2,..,a_n)} = f_{11}^{a_1} + f_{21}^{a_2} + \cdots + f_{n1}^{a_n}$, for some $r_i, a_i \in R$, $1 \leq i \leq n$. Then

$$\Psi^{-1}U_{(a_1,a_2,..,a_n)}V_{(r_1,r_2,..,r_n)}\Psi\langle\langle\delta_1,0,\ldots,0\rangle,\langle 0,\delta_2,0,\ldots,0\rangle,\cdots$$

$$\cdots,\langle 0,\ldots,0,\delta_m\rangle,\langle 0,\ldots,0\rangle,\cdots,\langle 0,\ldots,0\rangle\rangle$$

$$= \Psi^{-1}U_{(a_1,a_2,..,a_n)}V_{(r_1,r_2,..,r_n)}\langle\langle\delta_1,0,\ldots,0\rangle,\langle 0,\delta_2,\ldots,0\rangle,\cdots,\langle 0,..,0,\delta_m,0,..,0\rangle\rangle$$

$$= \Psi^{-1}U_{(a_1,a_2,..,a_n)}\langle\langle r_1\delta_1,0,\ldots,0\rangle,\langle r_2\delta_2,0,\ldots,0\rangle,\cdots,\langle r_m\delta_m,0,\ldots,0\rangle\rangle$$

$$= \Psi^{-1}\langle\langle a_1r_1\delta_1,a_2r_1\delta_1,\ldots,a_nr_1\delta_1\rangle,\cdots,\langle a_1r_m\delta_m,a_2r_m\delta_m,\ldots,a_nr_m\delta_m\rangle\rangle$$

$$= \langle\langle a_1r_1\delta_1,a_1r_2\delta_2,\ldots,a_1r_m\delta_m\rangle,\cdots,\langle a_nr_1\delta_1,a_nr_2\delta_2,\ldots,a_nr_m\delta_m\rangle\rangle.$$

Thus, for an arbitrary element in Γ^n,

$$\langle\langle r_{11}\delta_1,r_{12}\delta_2,\ldots,r_{1m}\delta_m\rangle,\ \cdots,\ \langle r_{n1}\delta_1,r_{n2}\delta_2,\ldots,r_{nm}\delta_m\rangle\rangle$$

$$= \Psi^{-1}(\ f^1_{11}\,V_{(r_{11},r_{12},..,r_{1m})}+\cdots+f^1_{n1}\,V_{(r_{n1},r_{n2}..,r_{nm})})\Psi(\underline{\varepsilon}),$$

where

$$\underline{\varepsilon}:=\langle\langle\delta_1,0,\ldots,0\rangle,\langle 0,\delta_2,0,\ldots,0\rangle,\cdots,\langle 0,\ldots,0,\delta_m\rangle,\langle 0,\ldots,0\rangle,\cdots,\langle 0,\ldots,0\rangle\rangle.$$

Hence $\underline{\varepsilon}$ is an $\mathbb{M}_n(R)$-generator of Γ^n. □

We can now classify non-monogenic R-groups such as Γ in Proposition 4.1.12, according to the type-ν ($\nu = 0, s, \mathcal{K}$) of their corresponding monogenic $\mathbb{M}_n(R)$-group Γ^n.

Definition 4.1.13 *Let Γ be a non-monogenic R-group which is a group-theoretic direct sum of monogenic R-subgroups of Γ. Let $n \in \mathbb{N}$, $n \geq 1$.*
(a) If Γ^n is an $\mathbb{M}_n(R)$-group of type-0 but not of type-s, then Γ is said

to be an R-group of $\mathbf{0}_n$**-form**.
(b) If Γ^n *is an* $\mathbb{M}_n(R)$*-group of type-s but not of type-2, then* Γ *is said to be an R-group of* $\mathbf{s}_n$**-form**.
(c) If Γ^n *is an* $\mathbb{M}_n(R)$*-group of type-*$\mathcal{K}$*, then* Γ *is said to be an R-group of* $\mathcal{K}_n$**-form**.

4.2 Intersections of annihilators

Jacobson-type radicals and the socle ideals in R, and in $\mathbb{M}_n(R)$, are intersections of annihilators of monogenic R-groups, and monogenic $\mathbb{M}_n(R)$-groups, respectively. Consequently we are able to develop a general theory relating annihilator ideals of R and the corresponding ideals of $\mathbb{M}_n(R)$.

Lemma 4.2.1 *Let* R *be a zero-symmetric right near-ring with identity. For any* $r \in R$ *and monogenic* R*-group,* Δ*,*

$$r \in (0 : \Delta) \quad \text{if, and only if,} \quad f_{ij}^r \in (\bar{0} : \Delta^n).$$

Proof Let r be any non-zero element in R and $\bar{\rho} = \langle \delta_1, \delta_2, \ldots, \delta_n \rangle$ be any element in Δ^n where $\Delta = R\delta$. For each $\delta_i = r_i\delta$, with r_i some element of R, we have

$$f_{ij}^r \, \bar{\rho} = f_{ij}^r \, \langle r_1\delta, r_2\delta, \ldots, r_n\delta \rangle = f_{ij}^r \, \langle r_1, r_2, \ldots, r_n \rangle \, \delta$$

$$= \langle 0, \ldots, 0, rr_j, 0, \ldots, 0 \rangle \, \delta = \langle 0, \ldots, 0, r\delta_j, 0, \ldots, 0 \rangle$$

where $r_i\delta$ appears in the i-th place. Clearly,

$$r \in (0:\Delta) \quad \text{if, and only if,} \quad f_{ij}^r \in (\bar{0}:\Delta^n).$$

□

Theorem 4.2.2 *Let R be a zero-symmetric right near-ring with identity, and Ω a faithful R-group. Let $\{ R\delta_\lambda \mid \lambda \in \Lambda\}$ be a collection of monogenic R-subgroups of Ω, then*

$$\left(\bigcap_{\lambda\in\Lambda}(0:R\delta_\lambda)\right)^+ \subseteq \bigcap_{\lambda\in\Lambda}(\bar{0}:(R\delta_\lambda)^n).$$

Proof Let $Q := \bigcap_{\lambda\in\Lambda}(0:R\delta_\lambda)$ and U be any matrix in Q^+. We prove the result by induction on the weight of U.

Firstly, let $U = f_{ij}^a$, where $a \in Q$. Then, by Lemma 4.2.1,

$$f_{ij}^a \in \bigcap_{\lambda\in\Lambda}(\bar{0}:(R\delta_\lambda)^n).$$

Secondly, let $U = f_{ij}^{a_1}(f_{jk}^{a_2} + f_{jl}^{a_3})$, where $a_1, a_2, a_3 \in Q$, and let $\bar{\rho} = \langle \omega_1, \omega_2, \ldots, \omega_n\rangle \in (R\delta_\lambda)^n$. We now have

$$U\bar{\rho} = \langle 0, \ldots, 0, a_1(a_2\omega_k + a_3\omega_l), 0, \ldots, 0\rangle.$$

Clearly, $U\bar{\rho} = \bar{0}$ because $a_2, a_3 \in Q$ and $\bar{\rho} \in (R\delta_\lambda)^n$.

The above two cases provide a basic step to an inductive proof based on the fact that the ideal Q^+ is generated by

$$\left\{ f_{ij}^a \;\middle|\; a \in Q = \bigcap_{\lambda\in\Lambda}(0:R\delta_\lambda),\ 1 \le i,j \le n \right\}.$$

Assume that the theorem holds for any matrix U in Q^+ of weight less than n, where n is a positive integer.

Suppose $U = V + W$ where V and W are matrices in Q^+ each of a weight less than n. Then, for any $R\delta_\lambda,\ \lambda \in \Lambda$, and $\bar{\delta} = \langle \delta_1, \delta_2, \ldots, \delta_n \rangle \in (R\delta_\lambda)^n$, we have

$$U\bar{\delta} = V\bar{\delta} + W\bar{\delta} = \bar{0} + \bar{0}.$$

Thus, $U \in Q^+$.

Suppose $U = VW$ where V and W are matrices in Q^+ each of a weight less than n. Then, for any $R\delta_\lambda,\ \lambda \in \Lambda$, and $\bar{\delta} = \langle \delta_1, \delta_2, \ldots, \delta_n \rangle \in (R\delta_\lambda)^n$, we have

$$U\bar{\delta} = V\left(W(\bar{\delta})\right) = V(\bar{0}) = \bar{0}.$$

Hence the result follows. □

Let Ω a faithful R-group, and let $E_\nu(R)$ denote the class of all type-ν R-subgroups of Ω, $\nu = 0, s, 2, \mathcal{K}$, and $\mathbb{E}_\nu(\mathbb{M}_n(R))$ denote the class of all type-ν $\mathbb{M}_n(R)$-subgroups of Ω^n, $\nu = 0, s, 2, \mathcal{K}$.

Theorem 4.2.3 *Let R be a zero-symmetric right near-ring with identity, and Ω a faithful locally monogenic R-group. Then*

$$\bigcap_{\Gamma \in \mathbb{E}_\nu(\mathbb{M}_n(R))} (\bar{0} : \Gamma) \subseteq \bigcap_{\Delta \in E_\nu(R)} (\bar{0} : \Delta^n).$$

Moreover, if Ω has no R-subgroups of ν_n-form, then the two ideals are equal.

Proof By Theorems 4.1.3 and 4.1.7,

$$\Delta \in E_\nu(R) \;\Rightarrow\; \Delta^n \in \mathbb{E}_\nu(\mathbb{M}_n(R)),$$

giving

$$\mathbb{E}_\nu(\mathbb{M}_n(R)) \;\supseteq\; \{\; \Delta^n \mid \Delta \in E_\nu(R) \;\}.$$

Hence

$$\bigcap_{\Gamma \in \mathbb{E}_\nu(\mathbb{M}_n(R))} (\bar{0} : \Gamma) \;\subseteq\; \bigcap_{\Delta \in E_\nu(R)} (\bar{0} : \Delta^n),$$

which proves the result. □

If a faithful locally monogenic R-group Ω of a zero-symmetric near-ring, R, with identity, has no R-subgroups of ν_n-form, then every type-ν $\mathbb{M}_n(R)$-subgroup, Γ, of Ω^n, is of the form $\Gamma = \Delta^n$, for some R-group Δ, see Theorem 4.1.3. Hence

$$\mathbb{E}_\nu(\mathbb{M}_n(R)) \;=\; \{\; \Delta^n \mid \Delta \in E_\nu(R) \,\}$$

and

$$\bigcap_{\Gamma \in \mathbb{E}_\nu(\mathbb{M}_n(R))} (\bar{0} : \Gamma) \;=\; \bigcap_{\Delta \in E_\nu(R)} (\bar{0} : \Delta^n).$$

This and Theorem 4.2.2 yield the next corollary.

Corollary 4.2.4 *Let R be a zero-symmetric right near-ring with identity, and Ω a faithful locally monogenic R-group. If Ω has no R-group of ν_n-form, $\nu = 0, s, \mathcal{K}$, then*

$$\left(\bigcap_{\Delta \in E_\nu(R)} (0 : \Delta) \right)^+ \subseteq \bigcap_{\Gamma \in \mathbb{E}_\nu(\mathbb{M}_n(R))} (\bar{0} : \Gamma).$$

Lemma 4.2.5 *Let R be a zero-symmetric right near-ring with identity, and Γ an R-group. For each $\delta \in \Gamma$ we have*

$$(0 : R\delta)^* \subseteq (\bar{0} : (R\delta)^n).$$

Proof Let $\bar{\omega} \in (R\delta)^n$, say $\bar{\omega} = \bar{\rho}\delta$ where $\bar{\rho} = \langle r_1, r_2, r_3, \ldots, r_n \rangle \in R^n$. For $W \in \mathbb{M}_n(R)$ we write

$$W\bar{\rho} = \langle r_1^{'}, r_2^{'}, r_3^{'}, \ldots, r_n^{'} \rangle \in R^n.$$

Suppose $W \in (0 : R\delta)^*$, then

$$W\bar{\rho} = \langle r_1^{'}, r_2^{'}, r_3^{'}, \ldots, r_n^{'} \rangle \in (0 : R\delta)^n.$$

Equivalently,

$$r_i' \in (0 : R\delta) \quad \text{for each} \quad 1 \leq i \leq n.$$

Now, in terms of $\bar{\omega}$ we have

$$W\bar{\omega} = W\langle r_1\delta, r_2\delta, r_3\delta, \ldots, r_n\delta \rangle = \langle r_1', r_2', r_3', \ldots, r_n' \rangle\delta,$$

where each $r_i' \in (0 : R\delta)$, giving $W\bar{\omega} = \bar{0}$. Since $\bar{\omega} \in (R\delta)^n$ was arbitrary $W \in (0 : (R\delta)^n)$. □

Corollary 4.2.6 *Let R be a zero-symmetric right near-ring with identity. For any monogenic R-group, $R\delta$, such that $|R\delta| = 2$,*

$$(0 : R\delta)^* = (0 : (R\delta)^n).$$

Proof By Lemma 4.2.5 we need only prove that

$$(0 : (R\delta)^n) \subseteq (0 : R\delta)^*.$$

Let $W \in (0 : (R\delta)^n)$. Assume that $W \notin (0 : R\delta)^*$. Then there exists some $\langle b_1, b_2, b_3, \ldots, b_n\rangle$ in R^n such that $W\langle b_1, b_2, b_3, \ldots, b_n\rangle \notin (0 : R\delta)^n$. That is, if

$$W\langle b_1, b_2, b_3, \ldots, b_n\rangle = \langle b'_1, b'_2, b'_3, \ldots, b'_n\rangle$$

then there exists $j \in \{1, 2, .., n\}$ such that $b'_j \notin (0 : R\delta)$, but since $|R\delta| = 2$, we have $(0 : R\delta) = (0 : \delta)$. This gives

$$W\langle b_1, b_2, b_3, \ldots, b_n\rangle\delta = \langle b'_1\delta, b'_2\delta, b'_3\delta, \ldots, b'_n\delta\rangle \neq \bar{0},$$

which contradicts $W \in (0 : (R\delta)^n)$. □

Note that in Lemma 4.2.5, if $|R\delta| \neq 2$ and $R\delta$ is not faithful, then $(0 : \delta)$ may strictly contain $(0 : R\delta)$, in which case, $(0 : R\delta)^* \subsetneq (0 : (R\delta)^n)$.

Lemma 4.2.7 *Let R be a zero-symmetric right near-ring with identity, and Γ be an R-group. For each $\delta \in \Gamma$,*

$$(\bar{0} : (R\delta)^n) \subseteq (0 : \delta)^*.$$

Proof Let $U \in (\bar{0} : (R\delta)^n)$ and $\bar{\rho}$ be an element of R^n, then $\bar{\omega} := \bar{\rho}\,\delta \in (R\delta)^n$. Hence $\bar{0} = U\bar{\omega} = U\bar{\rho}\,\delta$, giving $U\bar{\rho} \in (0 : \delta)^n$. Thus $U \in (0 : \delta)^*$. □

We now extend the earlier results to R-groups which are direct sums of R-groups of order 2.

Consider the R-group, $\Delta = \bigoplus_{i=1}^{m} R\delta_i$, where each $R\delta_i$ is an R-kernel of Δ, and $|R\delta_i| = 2$. By left distributivity over R-kernels (cf. Lemma 2.3.1)

$$(0 : \Delta) = \bigcap_{i=1}^{m} (0 : R\delta_i).$$

Again, by Theorem 2.4.6 (iii),

$$\left(\bigcap_{i=1}^{m} (0 : R\delta_i) \right)^* = \bigcap_{i=1}^{m} (0 : R\delta_i)^* .$$

Hence

$$(0 : \Delta)^* = \bigcap_{i=1}^{m} (0 : R\delta_i)^*. \tag{4.2}$$

On the other hand,

$$\Delta^n = \left(\bigoplus_{i=1}^{m} R\delta_i \right)^n \cong_{\mathbb{M}_n(R)} \bigoplus_{i=1}^{m} \left(R\delta_i\right)^n,$$

and by left distributivity over $\mathbb{M}_n(R)$-kernels,

$$(\bar{0} : \Delta^n) = \bigcap_{i=1}^{m} (\bar{0} : (R\delta_i)^n).$$

By Corollary 4.2.6,

$$(\bar{0} : (R\delta_i)^n) = (0 : R\delta_i)^*,$$

for each $i = 1, 2, \ldots, m$. Thus

$$(\bar{0} : \Delta^n) = \bigcap_{i=1}^{m} (0 : R\delta_i)^*. \tag{4.3}$$

The next theorem follows from (4.2) and (4.3).

Theorem 4.2.8 *Let R be a zero-symmetric right near-ring with identity, and let Δ be an R-group such that $\Delta = \bigoplus_{i=1}^{m} R\delta_i$. If each R-subgroup $R\delta_j$ is an R-kernel of Δ and a group of order 2, then*

$$(\bar{0} : \Delta^n) = (0 : \Delta)^*.$$

Theorem 4.2.9 *Let R be a zero-symmetric right near-ring with identity, and let Δ be an R-group such that $\Delta = \bigoplus_{i=1}^{m} R\delta_i$. If each R-subgroup $R\delta_j$ is an R-kernel of Δ and a group of order 2, then*

$$(0 : \Delta)^* = \left(\bigcap_{i=1}^{m}(0 : \delta_i)\right)^*.$$

Proof Since $(\bar{0} : \Delta^n) = (0 : \Delta)^*$, by Theorem 4.2.8, we need only show that

$$(\bar{0} : \Delta^n) = \left(\bigcap_{i=1}^{m}(0 : \delta_i)\right)^*.$$

We note that $(0 : R\delta_i) = (0 : \delta_i)$ as $|R\delta_i| = 2$, $i = 1, 2, \ldots, m$. Hence

$$(0 : R\delta_i)^* = (0 : \delta_i)^*.$$

By Corollary 4.2.6,

$$(\bar{0} : (R\delta_i)^n) = (0 : R\delta_i)^*,$$

for each $i = 1, 2, \ldots, m$. By left distributivity over $\mathbb{M}_n(R)$-kernels,

$$(\bar{0} : \Delta^n) = \bigcap_{i=1}^{m}(\bar{0} : (R\delta_i)^n),$$

as $\Delta^n \cong_{\mathbb{M}_n(R)} \bigoplus_{i=1}^{m} \left(R\delta_i\right)^n$. Hence

$$(\bar{0} : \Delta^n) = \bigcap_{i=1}^{m}(0 : \delta_i)^*,$$

and the result follows by Proposition 2.4.6 (iii). □

4.3 The ideals $Soi(R)^+$ and $Soi(R)^*$

By Theorem 3.1.13 the socle ideal is an annihilator ideal. The containment of $Soi(\mathbb{M}_n(R))$ in $(Soi(R))^*$ is established here, and we apply the theory developed in the previous section to study the relationship between $(Soi(R))^+$ and $Soi(\mathbb{M}_n(R))$.

Since $(Soi(R))^+ = \left(\bigcap_{\Delta \in \mathcal{K}(\Omega)} (\bar{0} : \Delta)\right)^+$, the following theorem is a consequence of Theorem 4.2.2.

Theorem 4.3.1 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCL, and let Ω be a faithful locally monogenic R-group. Then*

$$(Soi(R))^+ \subseteq \bigcap_{\Delta \in \mathcal{K}(\Omega)} (0 : \Delta^n).$$

According to Theorem 3.1.13, the socle ideal is characterizable as

$$Soi(\mathbb{M}_n(R)) = \bigcap_{\Gamma \in \mathcal{K}(\Omega^n)} (\bar{0} : \Gamma).$$

The next theorem is a consequence of Theorem 4.2.3.

Theorem 4.3.2 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCL, and let Ω be a faithful locally monogenic R-group. Then*

$$Soi(\mathbb{M}_n(R)) \subseteq \bigcap_{\Delta \in \mathcal{K}(\Omega)} (\bar{0} : \Delta^n).$$

The next theorem in a direct consequence of Theorem 4.1.7 and Theorem 4.3.2 with $\nu = \mathcal{K}$, along with Theorem 4.3.1.

Theorem 4.3.3 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCL, and let Ω be a faithful locally monogenic R-group. If Ω has no R-groups of $\mathcal{K}$-form, then*

$$Soi(\mathbb{M}_n(R)) = \bigcap_{\Delta \in \mathcal{K}(\Omega)} (0 : \Delta^n).$$

Moreover, $(Soi(R))^+ \subseteq Soi(\mathbb{M}_n(R))$.

In general, there is no relationship between $(Soi(R))^+$ and $Soi(\mathbb{M}_n(R))$. This is illustrated by Example 4.4.3, where an example is given in which $(Soi(R))^+ \not\subset Soi(\mathbb{M}_n(R))$.

The next theorem is a consequence of Proposition 2.4.6 (iii) and Lemma 4.2.5.

Theorem 4.3.4 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCL, and let Ω be a faithful locally monogenic R-group. Then*

$$(Soi(R))^* \subseteq \bigcap_{\Delta \in \mathcal{K}(\Omega)} (0 : \Delta^n).$$

We now apply the theory of $\mathcal{F}$-decompositions of a near-ring, R, as in Section 3.1, to study the relationship between $(Soi(R))^*$ and $\text{Soi}\left(\mathbb{M}_n(R)\right)$ for near-rings satisfying the $DCCL$. Recall that, $R = Soi(R) \oplus L$, if R satisfies the $DCCL$.

Theorem 4.3.5 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCL. Let $R = Soi(R) \oplus L$ be the socle-decomposition of R, and Ω be a faithful locally monogenic R-group. If $\mathbb{M}_n(R)$ satisfies the DCCL, then*

$$(Soi(R))^* = (\bar{0} : L^n),$$

the annihilating ideal of L^n.

Proof Since $R = Soi(R) \oplus L$, each r in R is of the form $r = s + l$ where $s \in Soi(R)$ and $l \in L$. Clearly $R^n = (Soi(R))^n \bigoplus L^n$ and any $\bar{\rho} := \langle r_1, r_2, \ldots, r_n \rangle$ in R^n is of the form $\bar{\rho} = \langle s_1, \ldots, s_n \rangle + \langle l_1, \ldots, l_n \rangle$ where $s_i \in Soi(R)$ and $l_i \in L$, $1 \leq i \leq n$.

Since $\big(Soi(R)\big)^n$ and L^n are $\mathbb{M}_n(R)$-kernels of R^n, for any $U \in \mathbb{M}_n(R)$,

$$U\bar{\rho} = U\left[\langle s_1, \ldots, s_n \rangle + \langle l_1, \ldots, l_n \rangle\right] = U\langle s_1, \ldots, s_n \rangle + U\langle l_1, \ldots, l_n \rangle,$$

where $U\langle s_1, \ldots, s_n \rangle =: \langle s'_1, \ldots, s'_n \rangle \in (Soi(R))^n$ and $U\langle l_1, \ldots, l_n \rangle =: \langle l'_1, \ldots, l'_n \rangle \in L^n$, also we may write

$$U\bar{\rho} = \langle s'_1, \ldots, s'_n \rangle + \langle l'_1, \ldots, l'_n \rangle =: \langle s''_1, \ldots, s''_n \rangle.$$

If $U \in (Soi(R))^* = \left\{ V \in \mathbb{M}_n(R) \mid V\,\bar{\rho} \in \big(\mathrm{Soi}(R)\big)^n, \ \forall\, \bar{\rho} \in R^n \right\}$, then $U\bar{\rho} \in (Soi(R))^n$, giving

$$l'_i = s''_i - s'_i \in Soi(R) \cap L = (0), \quad \forall\, 1 \leq i \leq n.$$

Thus $U\langle l_1, \ldots, l_n \rangle = \langle l'_1, \ldots, l'_n \rangle = \langle \bar{0} \rangle$. That is, $(Soi(R))^* \subseteq (\bar{0} : L^n)$.

Conversely, if $U \in (0 : L^n)$, then $U\langle l_1, \ldots, l_n\rangle = \langle \bar{0}\rangle$ for any $\langle l_1, \ldots, l_n\rangle$ in L^n. This implies that $U\bar{\rho} = U\langle s_1, \ldots, s_n\rangle \in (Soi(R))^n$ for all $\bar{\rho} \in R^n$. Therefore $U \in (Soi(R))^*$ so that $(0 : L^n) \subseteq (Soi(R))^*$. □

Theorem 4.3.6 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCL. Let $R = Soi(R) \oplus L$ be the socle-decomposition of R, and Ω be a faithful locally monogenic R-group. If $\mathbb{M}_n(R)$ satisfies the DCCL, then*

$$Soi(\mathbb{M}_n(R)) \subseteq (Soi(R))^*.$$

Proof From Theorem 4.3.5, we need only show that $Soi\big(\mathbb{M}_n(R)\big)L^n = (\bar{0})$. By Theorem 3.1.14, since $R = Soi(R) \oplus L$, the left ideal L may be written as a direct sum

$$L = \left(\bigoplus_{j=1}^{r} L'_j\right) \oplus L'',$$

where $L'_j \in \mathcal{B}(\Omega)$ and $L'' \in \mathcal{K}(\Omega)$. Now $R^n = (Soi(R))^n \bigoplus L^n$ and thus

$$L^n = \left(\bigoplus_{j=1}^{r} (L'_j)^n\right) \oplus (L'')^n.$$

By Theorems 4.1.10 and 4.1.7, each $(L'_j)^n \in \mathcal{B}(\Omega^n)$ and $(L'')^n \in \mathcal{K}(\Omega^n)$. Since $Soi(\mathbb{M}_n(R))$ annihilates all $\mathbb{M}_n(R)$-groups of type-$\mathcal{K}$ and all $\mathbb{M}_n(R)$-groups in $\mathcal{B}(\Omega^n)$, from Proposition 3.1.15, it follows that

$$Soi\big(\mathbb{M}_n(R)\big)L^n = (\bar{0}).$$

□

The inclusion in Theorem 4.3.6 above can be strict as shown in the Example 4.4.3. That is, there are cases when $Soi\left(\mathbb{M}_n(R)\right) \subsetneq (Soi(R))^*$.

Corollary 4.3.7 *Let R be a zero-symmetric right near-ring with identity satisfying the DCCL. Let $R = Soi(R) \oplus L$ be the socle-decomposition of R, and Ω be a faithful locally monogenic R-group. If Ω has no R-subgroups of $\mathcal{K}$-form, then*

$$Soi(\mathbb{M}_n(R)) = (Soi(R))^* = \bigcap_{\Delta \in \mathcal{K}(\Omega)} (0 : \Delta^n).$$

Proof By Theorem 4.3.6, $Soi(\mathbb{M}_n(R)) \subseteq (Soi(R))^*$, and by Theorem 4.3.4, $(Soi(R))^* \subseteq \bigcap_{\Delta \in \mathcal{K}(\Omega)} (0 : \Delta^n)$. But, $Soi\,(\mathbb{M}_n(R)) = \bigcap_{\Delta \in \mathcal{K}(\Omega)} (\bar{0} : \Delta^n)$ by Theorem 4.3.3. Hence the result follows. □

4.4 Examples on Socle Ideals

In this section, examples are given to illustrate the relationships between $(Soi(R))^+$ and $Soi(\mathbb{M}_n(R))$, and between $(Soi(R))^*$ and $Soi(\mathbb{M}_n(R))$, in the context of the theory of Section 4.3. We give examples of near-rings where the above ideals are not comparable and examples for which they are comparable and the inclusion is strict. The symbol $\oplus_G$ is used to denote the group theoretic direct sum. The subgroup of a group, G, generated by $\bar{x} \in G$ is denoted by $grp\{\bar{x}\}$.

The following example illustrates that the inclusion in Theorem 4.3.1 can be strict. That is, $(Soi(R))^+ \subsetneq \bigcap_{\Delta \in \mathcal{K}(\Omega)} (\bar{0} : \Delta^n)$.

Example 4.4.1 *Let* $\Omega := \mathbb{Z}_2 \oplus_G \mathbb{Z}_2 \oplus_G \mathbb{Z}_2$,
$\Omega_1 := \{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\}$,
$\Omega_2 := \{(0,0,0),(0,1,0),(0,0,1),(0,1,1)\}$,
$\Omega_3 := grp\{(0,1,0)\}$ *and* $\Omega_4 := grp\{(0,0,1)\}$.

Define a near-ring R *by*
$R := \{ \ f \in M_0(\Omega) \ \mid \ f(\Omega_i) \subseteq \Omega_i, \ \ i = 1,2,3,4;$
$\forall \ \omega, \omega' \in \Omega_2, \ \ \omega - \omega' \in \Omega_j \Rightarrow f(\omega) - f(\omega') \in \Omega_j, \ \ where \ \ j = 3,4;$
$\forall \ \omega_1, \omega_2 \in \Omega_1, \ \ \omega_1 - \omega_2 \in \Omega_3 \Rightarrow f(\omega_1) - f(\omega_2) \in \Omega_3 \ \}.$

Then R *is a near-ring with a multiplicative identity under point-wise addition and map composition. We observe the following facts about* R*:*
1. The group Ω *is a faithful* R*-group of type-*0*. Thus,* $J_0(R) = (0)$.
2. The subgroups Ω_1 *and* Ω_3 *are monogenic* R*-groups. The* R*-subgroup* Ω_3 *is an* R*-kernel of both* Ω_1 *and* Ω_2*. The* R*-subgroup* Ω_4 *is an* R*-kernel of* Ω_2.
3. The R*-group* Ω_1 *is monogenic but has no* R*-group of type-*0 *as a direct summand, therefore it is of type-*$\mathcal{K}$*. As a result the faithful* R*-group* Ω *is not of type-s, and hence* $J_s(R) \neq (0)$.
4. The classes of monogenic R*-subgroups of* Ω *are;* $\mathcal{K}(\Omega) = \{\Omega_1\}$, $\mathcal{B}(\Omega) = \{\Omega_3, \Omega_1/\Omega_3\}$, $\mathcal{G}(\Omega) = \{(0), \Omega, \Omega_2, \Omega_4\}$, $(\Omega_4 \cong \Omega_2/\Omega_3)$.
5. The socle ideal $Soi(R) = (0 : \Omega_1)$, *and*
$J_s(R) = (0 : \Omega_4) \cap (0 : \Omega_3) \cap (0 : \Omega_1/\Omega_3) = J_2(R)$.

Consider the matrix near-ring $\mathbb{M}_2(R)$*. We show that there exists a matrix* U *in* $(\bar{0} : \Omega_1^2)$ *which is not in* $(Soi(R))^+$. *Let* $U = f_{11}^c(f_{12}^a + f_{11}^b)$ *where*

a, b, and c are maps in R defined by

a	b	c
$(1,1,1) \mapsto (1,0,1)$,	$(1,1,1) \mapsto (1,1,1)$,	$(1,1,1) \mapsto (1,1,1)$,
$(1,0,1) \mapsto (1,0,1)$,	$(1,0,1) \mapsto (1,1,1)$,	$(1,0,1) \mapsto (1,0,1)$,
$(0,1,1) \mapsto (0,0,1)$,	$(0,1,1) \mapsto (0,0,1)$,	$(0,1,1) \mapsto (0,1,1)$,
$(0,0,1) \mapsto (0,0,1)$,	$(0,0,1) \mapsto \bar{0}$,	$(0,0,1) \mapsto (0,0,1)$,
$(0,1,0) \mapsto \bar{0}$,	$(0,1,0) \mapsto (0,1,0)$,	$(0,1,0) \mapsto \bar{0}$,
$(1,0,0) \mapsto (0,1,0)$,	$(1,0,0) \mapsto (0,1,0)$,	$(1,0,0) \mapsto (0,1,0)$,
$(1,1,0) \mapsto (0,1,0)$,	$(1,1,0) \mapsto (0,1,0)$,	$(1,1,0) \mapsto (0,1,0)$,
$\bar{0} \mapsto \bar{0}$.	$\bar{0} \mapsto \bar{0}$.	$\bar{0} \mapsto \bar{0}$.

We note that $a, b, c \notin (0 : \Omega_1)$, where $\Omega_1 := \{\bar{0}, (0,1,0), (1,0,0), (1,1,0)\}$, so that f_{ij}^a, f_{ij}^b and f_{ij}^c are not in $(0 : \Omega_1)^+ = (Soi(R))^+$, $i, j \in \{1, 2\}$.

Consider the elements r' and r of R, defined below, and note the resulting near-ring element $c(ar' + br)$, where a, b and c are as previously defined.

r'	r	$c(ar'+br)$
$(1,1,1)\mapsto(1,1,1)$,	$(1,1,1)\mapsto(1,0,1)$,	$(1,1,1)\mapsto\bar{0}$,
$(1,0,1)\mapsto(1,0,1)$,	$(1,0,1)\mapsto(1,1,1)$,	$(1,0,1)\mapsto\bar{0}$,
$(0,1,1)\mapsto(0,0,1)$,	$(0,1,1)\mapsto(0,1,1)$,	$(0,1,1)\mapsto\bar{0}$,
$(0,0,1)\mapsto(0,0,1)$,	$(0,0,1)\mapsto(0,0,1)$,	$(0,0,1)\mapsto(0,0,1)$,
$(0,1,0)\mapsto\bar{0}$,	$(0,1,0)\mapsto(0,1,0)$,	$(0,1,0)\mapsto\bar{0}$,
$(1,0,0)\mapsto(1,0,0)$,	$(1,0,0)\mapsto(0,1,0)$,	$(1,0,0)\mapsto(0,1,0)$,
$(1,1,0)\mapsto(1,1,0)$,	$(1,1,0)\mapsto\bar{0}$,	$(1,1,0)\mapsto(0,1,0)$,
$\bar{0}\mapsto\bar{0}$.	$\bar{0}\mapsto\bar{0}$.	$\bar{0}\mapsto\bar{0}$.

Then $\langle r,r'\rangle \in R^2$ *and we have* $U\langle r,r'\rangle = \langle c(ar'+br),0\rangle$. *Observe that* $c(ar'+br) \notin (0:\Omega_1) = Soi(R)$. *Thus* $U \notin (Soi(R))^*$. *Since* $(Soi(R))^+ \subseteq (Soi(R))^*$, *we conclude that* $U \notin (Soi(R))^+$.

The above example also illustrates that $(Soi(R))^+$ can be strictly contained in $Soi(\mathbb{M}_n(R))$, as noted below.

Example 4.4.2 *The near-ring* R *given in Example 4.4.1 has no* R*-groups of* $\mathcal{K}_2$*-form, therefore* $\mathcal{K}(\Omega^2) = \{\Omega_1^2\}$ *and* $Soi(\mathbb{M}_2(R)) = (0:\Omega_1^2)$. *Thus,* $(Soi(R))^+$ *is strictly contained in* $Soi(\mathbb{M}_2(R))$.

The next example gives a near-ring, R, for which $(Soi(R))^+ \not\subset Soi\left(\mathbb{M}_n(R)\right)$. Indeed, the inequality or lack of containment between the two ideals is due to the presence of R-groups of $\mathcal{K}_2$-form. This example further illustrates that $Soi(\mathbb{M}_n(R))$ may be strictly contained in $(Soi(R))^*$.

Example 4.4.3 *Let* $\Omega := \mathbb{Z}_2 \oplus_G \mathbb{Z}_4 \oplus_G \mathbb{Z}_2$;
$\Omega_1 := grp\{(1,0,0),(0,2,0),(0,0,1)\}$; $\Omega_2 := grp\{(0,1,0)\}$;
$\Omega_3 := grp\{(0,2,0),(0,0,1)\}$; $\Omega_4 := grp\{(1,0,0),(0,2,0)\}$;
$\Delta_1 := grp\{(1,0,0)\}$; $\Delta_2 := grp\{(1,2,0)\}$; $\Delta_3 := grp\{(0,2,0)\}$;
$\Delta_4 := grp\{(0,0,1)\}$; $\Delta_5 := grp\{(0,2,1)\}$.

Define a near-ring R *as follows,*
$R := \{\, f \in M_0(\Omega) \,|\, f(\Omega_i) \subseteq \Omega_i,\ i = 1,2,3,4;\ f(\Delta_j) \subseteq \Delta_j,$
$j = 1,2,..,5;\ \forall\, \omega, \omega' \in \Omega_l, (l = 2,3)\, \omega - \omega' \in \Delta_3 \Rightarrow f(\omega) - f(\omega') \in \Delta_3 \,\}$.

Then R *is a near-ring with a multiplicative identity under point-wise addition and map composition. Each of the subgroups* Ω_i $(i = 1,2,3,4)$ *and* Δ_j $(j = 1,2,3,4,5)$, *is an* R*-subgroup of* Ω. *All these* R*-groups are monogenic, except for* Ω_3 *and* Ω_4. *In addition,* Δ_3 *is an* R*-kernel of both* Ω_2 *and* Ω_3. *Hence,* Ω_3 *is of* $\mathcal{K}_2$*-form, and* Ω_4 *is of* 0_2*-form.*

The classes of monogenic R*-subgroups of* R *with respect to* Ω *are*

$$\mathcal{K}(\Omega) = \{\Omega_2\},\ \mathcal{B}(\Omega) = \{\Delta_3\},$$

$$\mathcal{G}(\Omega) = \{\, (0), \Omega, \Omega_1, \Delta_j \ \ (j = 1,2,4,5) \,\}.$$

The socle ideal of R *is therefore* $Soi(R) = (0 : \Omega_2)$.

We now consider Ω^2 *as a monogenic faithful* $\mathbb{M}_2(R)$*-group (because* Ω *is monogenic as an* R*-group). We also note that* Ω_3^2 *and* Ω_4^2 *are monogenic* $\mathbb{M}_2(R)$*-groups.*

The classes of monogenic $\mathbb{M}_2(R)$*-subgroups of* Ω^2 *are*

$$\mathcal{K}(\Omega^2) = \{\Omega_2^2, \Omega_3^2\},\ \mathcal{B}(\Omega^2) = \{\Delta_3^2, \Delta_4^2, \Delta_5^2\},$$

$$\mathcal{G}(\Omega^2) = \{(\bar{0}),\ \Omega^2,\ \Omega_1^2,\ \Delta_1^2,\ \Delta_2^2,\ \Omega_4^2\}.$$

Thus, the socle ideal is $Soi\left(\mathbb{M}_2(R)\right) = (\bar{0} : \Omega_2^2) \cap (\bar{0} : \Omega_3^2)$.

Consider the maps w *and* u *in* R *defined by*

w	u
$(0,3,0) \mapsto (0,3,0)$,	$(0,3,0) \mapsto (0,2,0)$,
$(0,1,0) \mapsto (0,1,0)$,	$(0,1,0) \mapsto (0,2,0)$,
$(0,2,0) \mapsto \bar{0}$,	$(0,2,0) \mapsto \bar{0}$,
$(0,0,1) \mapsto (0,0,1)$,	$(0,0,1) \mapsto (0,0,1)$,
$(0,2,1) \mapsto (0,2,1)$,	$(0,2,1) \mapsto \bar{0}$,
$(1,0,0) \mapsto \bar{0}$,	$(1,0,0) \mapsto (1,0,0)$,
$(1,2,0) \mapsto (1,2,0)$,	$(1,2,0) \mapsto (1,2,0)$,
$\bar{0} \mapsto \bar{0}$.	$\bar{0} \mapsto \bar{0}$,
$\bar{x} \mapsto \bar{x}, \forall\ \ \bar{x} \notin (\Omega_2 \cup \Omega_3 \cup \Omega_4)$,	$\bar{x} \mapsto \bar{x}, \forall\ \ \bar{x} \notin (\Omega_2 \cup \Omega_3 \cup \Omega_4)$.

We note that

$$(wu)\bar{y} = (w \circ u)\bar{y} = \bar{0},$$

for all $\bar{y} \in \Omega_2$. *Meanwhile,*

$$wu(0,0,1) = (0,0,1) \neq \bar{0}.$$

Thus, $wu \notin (0 : \Omega_3)$. *Hence, the matrix* $U := f_{12}^{wu}$ *is an element of* $(Soi(R))^+$ *but not of* $Soi\left(\mathbb{M}_2(R)\right)$.

Let us also note that the matrix

$$U = f_{12}^{wu} \in (Soi(R))^*,$$

but not in $Soi\,(\mathbb{M}_2(R))$. □

Chapter 5

ν-Primitivity in $\mathbb{M}_n(R)$

In this chapter, the connection between ν-primitive ideals ($\nu = 0, s, 2$) of R and $\mathbb{M}_n(R)$ is considered. A similar connection is investigated for the s-socle ideal, which is the minimal ideal modulo which the Jacobson s-radical is nilpotent.

5.1 ν-Radicals of $\mathbb{M}_n(R)$

Here we apply the theory of annihilator ideals of Section 4.2 to the ν-primitive ideals. Let $E_\nu(R)$ be the class of all type-ν R-subgroups of a faithful locally monogenic R-group, and $\mathbb{E}_\nu(\mathbb{M}_n(R))$ be the class of type-ν $\mathbb{M}_n(R)$-subgroups of a faithful locally monogenic $\mathbb{M}_n(R)$-group, $(\nu = 0, s, 2)$.

Denote by $\mathcal{P}(\nu)$, the ideal

$$\bigcap_{R\delta \in E_\nu(R)} (\bar{0} : (R\delta)^n).$$

Collecting the results from Section 4.2, we obtain that the ideal $\mathcal{P}(\nu)$ relates to ν-primitive ideals of $\mathbb{M}_n(R)$ as follows.

Theorem 5.1.1 *Let R be a zero symmetric right near-ring with identity. Then, for $\nu = 0, s, 2$,*
(a) $(J_\nu(R))^* \subseteq \mathcal{P}(\nu)$,
(b) $J_\nu(\mathbb{M}_n(R)) \subseteq \mathcal{P}(\nu)$,
(c) $(J_\nu(R))^+ \subseteq \mathcal{P}(\nu)$.

It is well known that if $\nu = 2$ the inclusions in parts (a) and (b) of the above theorem are in fact equality, see Theorem 2.4.11 (i). Since $I^+ \subseteq I^*$, for any ideal I of R, it now follows that

$$(J_2(R))^+ \subseteq (J_2(R))^* = J_2(\mathbb{M}_n(R)) = \mathcal{P}(2).$$

For $\nu = 0, s$, the inclusions (a) and (b) may be strict. Also,

$$J_\nu(\mathbb{M}_n(R)) \subseteq (J_\nu(R))^*,$$

for $\nu = 0, s$, and thus

$$J_\nu(\mathbb{M}_n(R)) \subseteq (J_\nu(R))^* \subseteq \mathcal{P}(\nu), \quad \text{for } \nu = 0, s. \tag{5.1}$$

If a faithful locally monogenic R-group, Ω, of R has no R-subgroups of ν_n-form, then

$$\mathbb{E}_\nu(\mathbb{M}_n(R)) = \{ \Delta^n \mid \Delta \in E_\nu(R) \}$$

for $\nu = 0, s,$ and hence, by Theorem 4.2.3,

$$J_\nu(\mathbb{M}_n(R)) = \mathcal{P}(\nu).$$

This together with (5.1) yield the next proposition.

Proposition 5.1.2 *Let R be a zero symmetric right near-ring with identity, and Ω a faithful locally monogenic R-group. For $\nu = 0, s$, if Ω has no R-subgroups of ν_n-form then*

$$J_\nu(\mathbb{M}_n(R)) = (J_\nu(R))^* = \mathcal{P}(\nu).$$

The ideal $\mathcal{P}(\nu)$ turns out to be helpful when constructing examples of near-rings with either a strict inclusion between two annihilator ideals of $\mathbb{M}_n(R)$, or no inclusion at all. Meldrum and Meyer conjectured that $(J_0(R))^+ \subseteq J_0(\mathbb{M}_n(R))$ in [17]. However, counter examples to this conjecture exist, see Example 5.2.1.

The next theorem, a direct consequence of Corollary 4.2.4, shows that there is, subject to some conditions, a relationship between $(J_0(R))^+$ and $J_0(\mathbb{M}_n(R))$.

Theorem 5.1.3 *Let R be a zero symmetric right near-ring with identity and Ω a faithful locally monogenic R-group, and let $\nu = 0, s$. If Ω has no R-subgroups of ν_n-form, then*

$$(J_\nu(R))^+ \subseteq J_\nu(\mathbb{M}_n(R)).$$

We will now proceed to prove that $(J_s(R))^+ \subseteq J_s(\mathbb{M}_n(R))$ when $\mathbb{M}_n(R)$ has DCCL.

Let R be a zero-symmetric near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCL. Then, by Theorems 2.2.3 and 4.1.4, any $\mathbb{M}_n(R)$-group, Γ, of type-s is a homomorphic image of some monogenic $\mathbb{M}_n(R)$-subgroup of R^n, say K^n. Let $\Psi : K^n \to \Gamma$ be the $\mathbb{M}_n(R)$-homomorphism and $L^n := ker(\Psi)$. Then $K^n/L^n \cong_{\mathbb{M}_n(R)} \Gamma$. By Lemma 4.1.8,

$$(K/L)^n \cong K^n/L^n. \tag{5.2}$$

Lemma 5.1.4 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCL, and let K^n/L^n be an $\mathbb{M}_n(R)$-group of type-s. Then $(0 : K/L)$ is an intersection of s-primitive ideals of R.*

Proof. For $k \in K$, let $M := (\mathbb{M}_n(R))(\langle k, 0, ..., 0\rangle + L^n)$, a monogenic $\mathbb{M}_n(R)$-subgroup of K^n/L^n. Now K^n/L^n is an $\mathbb{M}_n(R)$-group of type-s, hence $M = \bigoplus_{i=1}^{l} M_i$, where each M_i is of type-0 and is an $\mathbb{M}_n(R)$-kernel of M. Then

$$\langle k, 0, ..., 0\rangle + L^n = (\langle k_{11}, ..., k_{n1}\rangle + L^n) + ... + (\langle k_{1l}, ..., k_{nl}\rangle + L^n),$$

where each $\langle k_{1i}, ..., k_{ni}\rangle + L^n$ is an $\mathbb{M}_n(R)$-generator of M_i. Applying the matrix f_{11}^1 to the above gives

$$\langle k, 0, ..., 0\rangle + L^n = (\langle k_{11}, 0, ..., 0\rangle + L^n) + ... + (\langle k_{1l}, 0, ..., 0\rangle + L^n),$$

where $\langle k_{1j}, 0, \ldots, 0\rangle + L^n \in M_j$, which in fact generates M_j. It follows readily that

$$(Rk + L)/L = (Rk_{11} + L)/L \oplus \cdots \oplus (Rk_{1l} + L)/L,$$

where, by Theorem 4.1.3, each $(Rk_{1j}+L)/L$ is of type-0 and an R-kernel of $(Rk+L)/L$. Thus

$$(0:k+L)=\bigcap_{j=1}^{l}(0:k_{1j}+L)$$

is the intersection of maximal left ideals. Consequently,

$$(0:K/L)=\bigcap_{k\in K}(0:k+L),$$

an intersection of s-primitive ideals, by Theorem 2.2.12. □

Theorem 5.1.5 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies DCCL. Then*

$$(J_s(R))^+\subseteq J_s(\mathbb{M}_n(R)).$$

Proof. Let Γ be an $\mathbb{M}_n(R)$-group of type-s. Then

$$\Gamma\ \cong_{\mathbb{M}_n(R)}\ K^n/L^n,$$

for some $\mathbb{M}_n(R)$-subgroups K^n, and L^n of R^n. By Lemma 5.1.4,

$$(0:K/L)\supseteq J_s(R).$$

Thus, for each $f_{ij}^{\sigma}\in(J_s(R))^+$, $\sigma\in J_s(R)$, and $\langle k_1,..,k_n\rangle\in K^n$, we have

$$f_{ij}^{\sigma}(\langle k_1,..,k_n\rangle+L^n)\ =\ \langle 0,..,0,\sigma k_j,0,..,0\rangle+L^n\ =\ \langle 0,0,\ldots,0\rangle+L^n.$$

Hence

$$f_{ij}^{\sigma}\in(0:K^n/L^n),\quad\text{and therefore}\quad f_{ij}^{\sigma}\in(0:\Gamma).$$

Since Γ is arbitrary, we have $f_{ij}^{\sigma}\in J_s(\mathbb{M}_n(R))$. □

The inclusion in Theorem 5.1.5 may be strict as shown in Example 1 of [8].

5.2 Examples on ν-radicals

In this section we give an example of a near-ring R where

1. $J_0(\mathbb{M}_n(R)) \neq J_s(\mathbb{M}_n(R)) \neq J_2(\mathbb{M}_n(R))$ while $J_0(R) = J_s(R) = J_2(R)$, and

2. $(J_0(R))^+ \not\subseteq J_0(\mathbb{M}_n(R))$ and $J_s(\mathbb{M}_n(R)) \subsetneq (J_s(R))^*$.

The symbol $\oplus_G$ is used to denote a group theoretic direct sum.

Example 5.2.1 *Consider the group* $\Omega := \mathbb{Z}_2 \oplus_G \mathbb{Z}_4 \oplus_G \mathbb{Z}_2 \oplus_G \mathbb{Z}_2$, *and the following subgroups:* $S := \mathbb{Z}_2 \oplus_G \mathbb{Z}_4 \oplus_G \{0\} \oplus_G \{0\}$;
$\Gamma := \{0\} \oplus_G \{0,2\} \oplus_G \mathbb{Z}_2 \oplus_G \{0\}$; $H_1 := \{\bar{0}, (1,0,0,0), (0,2,0,0), (1,2,0,0)\}$;
$H_2 := \{\bar{0}, (0,1,0,0), (0,2,0,0), (0,3,0,0)\}$; $T_1 := \{\bar{0}, (1,0,0,0)\}$;
$H_3 := \{\bar{0}, (0,2,0,0), (1,1,0,0), (1,3,0,0)\}$; $T_2 := \{\bar{0}, (0,2,0,0)\}$;
$T_3 := \{\bar{0}, (1,2,0,0)\}$; $T_4 := \{\bar{0}, (0,0,1,0)\}$; $T_5 := \{\bar{0}, (0,2,1,0)\}$.

Define a subnear-ring R *of* $M_0(\Omega)$ *as*
$R := \{ f \in M_0(\Omega) \mid f(S) \subseteq S;\ f(\Gamma) \subseteq \Gamma;\ f(H_i) \subseteq H_i, \forall\, i = 1,2,3;$
$f(T_j) \subseteq T_j, \forall j = 1,2,..,5;\ \omega - \omega' \in \Gamma \Rightarrow f(\omega) - f(\omega') \in \Gamma, \forall\, \omega, \omega' \in \Omega;$
$h - h' \in T_2 \Rightarrow f(h) - f(h') \in T_2, \forall\, h, h' \in H_l,\ (l = 1,2,3)\ \}$.

Then R *is a near-ring with a multiplicative identity under point-wise addition and mapping composition. Observe that:*
1. $\Omega = R(1,1,1,1)$ *is not of type-*0 *as* Γ *is a non-trivial* R*-kernel of* Ω. *Therefore* $J_0(R) \neq (0)$.
2. H_2 *and* H_3 *are* R*-groups of type-*$\mathcal{K}$.

3. $S = T_1 \oplus_G H_2$ *is an* R*-group of* 0_2*-form. That is,* S^2 *is an* $\mathbb{M}_2(R)$*-group of type-*0 *but not of type-*s*.*

4. $\Gamma = T_2 \oplus_G T_4$ *is of* s_2*-form. That is,* Γ^2 *is an* $\mathbb{M}_2(R)$*-group of type-*s *but not of type-*2*.*

*5. The type-*0 R*-subgroups,* T_j $(j = 1, 2, .., 5)$*, of* Ω *are of type-*2*. The quotient* R*-groups,* H_l/T_2 $(l = 1, 2, 3)$*, are of type-*0 *and of order* 2 *as groups, thus of type-*2*. Since every type-*0 R*-group has an isomorphic copy in* Ω*, these are all the* R*-groups of type-*0*, up to isomorphism. Therefore* $J_0(R) = J_s(R) = J_2(R)$*.*

(a) *We show that* $(J_0(R))^+ \not\subset J_0(\mathbb{M}_2(R))$*.*

Define a map z *in* R *by*

$$z(x) = \begin{cases} 0 & \text{if } \ x \in (H_1 \cup \Gamma) \\ (0,2,0,0) & \text{if } \ x \in (H_l \setminus T_2),\ l = 2,3 \\ x & \text{if } \ x \in \Omega \setminus (S \cup \Gamma)\ . \end{cases}$$

Note that z *annihilates all type-*0 R*-groups, thus* $z \in J_0(R)$*. Hence, for each* i, j*,* $f_{ij}^z \in (J_0(R))^+$*. For* $s_1 = (1, 3, 0, 0) \in S$*,*

$$f_{12}^z\langle s_1, s_1\rangle \ = \ \langle zs_1, 0\rangle \ = \ \langle (0,2,0,0), 0\rangle \ \neq \ \langle 0, 0\rangle.$$

Thus $f_{12}^z \notin (0 : S^2)$*. Since* S^2 *is a type-*0 $\mathbb{M}_2(R)$*-group, it follows that* $f_{12}^z \notin J_0(\mathbb{M}_2(R))$*.*

(b) *Here we show that* $J_0(\mathbb{M}_2(R)) \neq J_s(\mathbb{M}_2(R))$*.*

Consider the element z *of* R *as defined above, and let* $V := f_{11}^z + f_{12}^z$*. For any* $\langle r_1, r_2\rangle \in R^2$*,*

$$V\langle r_1, r_2\rangle \ = \ \langle zr_1 + zr_2, 0\rangle.$$

For $\langle s_1, 0\rangle \in S^2$, *where* $s_1 = (1,3,0,0)$, *we have*

$$V\langle s_1, 0\rangle = \langle zs_1, 0\rangle \;=\; \langle (0,2,0,0), 0\rangle \;\neq \langle 0, 0\rangle.$$

So $V \notin J_0(\mathbb{M}_2(R))$.

The type-s $\mathbb{M}_2(R)$*-groups are*

$$\Gamma^2,\; T_l^2,\; (H_l/T_2)^2 \;\cong_{\mathbb{M}_2(R)}\; H_l^2/T_2^2,\; l = 1,2,3.$$

By definition

$$V \in ((0:\Gamma) \cap (0:H_1))^+,$$

and hence

$$V \;\in\; (0:\Gamma)^+ \;\cap\; (0:H_1)^+ \;\subseteq\; (0:\Gamma^2) \;\cap\; (0:H_1^2)$$

by Proposition 2.4.6, and Lemma 4.2.1, respectively. A simple calculation shows that $V(H_l^2) \subseteq T_2^2$ *and hence*

$$V(H_l^2/T_2^2) \;\subseteq\; \bar{0} + T_2^2$$

for each $l = 1,2,3$. *Therefore* $V \in J_s(\mathbb{M}_2(R))$.

(c) *We now prove that* $J_s(\mathbb{M}_2(R)) \neq J_2(\mathbb{M}_2(R))$.
Define an R*-subgroup of* $(R,+)$ *as* $K := \{0, k_1, k_2, k_3\}$ *where*

$$k_i(x) = \begin{cases} b_i & \text{if } \; x = (0,0,0,1) \\ 0 & \text{if } \; x \neq (0,0,0,1) \end{cases}$$

and $b_1 = (0,2,0,0)$, $b_2 = (0,0,1,0)$ *and* $b_3 = (0,2,1,0)$.

Note that K *is of* s_2*-form as it is* R*-isomorphic to* $\Gamma = T_2 \oplus T_4$. *That is,* K^2 *is an* $\mathbb{M}_2(R)$*-group of type-s.*

Define elements t *and* s *of* R *by* $t(b_3) = b_3$ *and* $t(x) = 0$ *otherwise, and* $s(b_1) = b_1$, $s(b_2) = b_2$ *and* $s(x) = 0$, *otherwise.*

The matrix $B := f_{11}^t(f_{11}^s + f_{12}^s)$ *on* R^2 *yields*

$$B\langle r_1, r_2\rangle = \langle t(sr_1 + sr_2), 0\rangle.$$

For $\langle r_1, r_2\rangle = \langle k_1, k_2\rangle$, *observe that*

$$B\langle k_1, k_2\rangle = \langle t(sk_1 + sk_2), 0\rangle$$

and

$$t(sk_1 + sk_2)(0,0,0,1) = t(sb_1 + sb_2) = t(b_1 + b_2) = t(b_3) = b_3 \neq 0.$$

Thus,

$$B \notin (0 : K^2).$$

That is, there exists an $\mathbb{M}_2(R)$*-group of type-s which* B *does not annihilate. Therefore*

$$B \notin J_s(\mathbb{M}_2(R)).$$

Simple calculations show that, if

$$\langle b, b'\rangle \in \left(\bigcup_{i=1}^{5} T_i^2\right),$$

then $B\langle b, b'\rangle = \langle 0, 0\rangle$, *and if*

$$\langle b, b'\rangle \in \left(\bigcup_{l=1}^{3} H_l^2/T_2^2\right),$$

then $B\langle b, b'\rangle \in T_2^2$. *That is,* B *annihilates every* $\mathbb{M}_2(R)$*-group of type-*2*, therefore* $B \in J_2(\mathbb{M}_2(R))$.

(d) *In addition, since*

$$(J_s(R))^* = (J_2(R))^* = J_2(\mathbb{M}_2(R)),$$

and

$$J_s(\mathbb{M}_2(R)) \neq J_2(\mathbb{M}_2(R)),$$

we conclude that

$$J_s(\mathbb{M}_2(R)) \subsetneq (J_s(R))^*.$$

5.3 The s-socles

Recall that the s-socle of R is the minimal ideal, A, of R modulo which $J_s(R)$ is non-zero nilpotent, see Section 2.3. Note that, by definition the s-socle, A, is contained in $J_s(R)$, hence $A^+ \subseteq (J_s(R))^+$, by Proposition 2.4.6. For the remainder of this thesis the s-socle of R will be denoted by A, while the s-socle of $\mathbb{M}_n(R)$ will be denoted $\mathcal{A}$. For a subset J of R we denote

$$J^{(m)} = \{\, r_1 r_2 \cdots r_m \,|\, r_i \in J,\, i = 1, 2, \ldots, m \,\}.$$

The next proposition is a direct consequence of Lemma 2.4.7 and the definition of the s-socle of R.

Proposition 5.3.1 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS. Then $(J_s(R/A))^+$ is nilpotent if, and only if, $J_s(R/A)$ is nilpotent.*

Theorem 5.3.2 *[26, Lemma 4.2] Let R be a zero-symmetric right near-ring with identity satisfying the DCCS, and let I be an ideal of R. Then*

$$\mathbb{M}_n(R/I) \cong \mathbb{M}_n(R)/I^*.$$

Let I and J be non-trivial ideals of a zero-symmetric right near-ring with identity, R, such that $J \supseteq I$. Let $x, x_{ij}, z \in J$, $i, j = 1, 2, \ldots, n$, $r \in R$ and consider the map

$$\Phi : J^+ \to (J/I)^+$$

defined recursively by

$$\sum_{i,j=1}^{n} f_{ij}^{x_{ij}} \mapsto \sum_{i,j=1}^{n} f_{ij}^{(x_{ij}+I)},$$

and

$$f_{p\mu}^{r}\left(\sum_{i,j=1}^{n} f_{ij}^{x_{ij}}\right) \mapsto f_{p\mu}^{(r+I)}\left(\sum_{i,j=1}^{n} f_{ij}^{(x_{ij}+I)}\right).$$

This map, Φ, is the restriction to J^+ of the map used in the proof of Lemma 4.2 in [26]. Thus Φ is well-defined and is a near-ring homomorphism as shown in [26]. It follows easily that Φ is surjective and that $ker(\Phi) = J^+ \cap I^*$, so that $ker(\Phi) \supseteq I^+$. That $ker(\Phi) \subseteq I^+$ is established in Appendix B. The following lemma is a direct consequence of the above.

Lemma 5.3.3 *Let R be a zero-symmetric right near-ring with identity, and let I and J be non-trivial ideals of R such that $J \supseteq I$. Then*

$$J^+/I^+ \cong (J/I)^+.$$

Recall that $J_s(R/A)$ is non-zero and nilpotent, by Lemma 2.3.14. Also, $J_s(R/A) = J_s(R)/A$. It now follows from Proposition 5.3.1 that $(J_s(R/A))^+$ is nilpotent. By Lemma 5.3.3, $(J_s(R)/A)^+ \cong (J_s(R))^+/A^+$, which yields the following result.

Theorem 5.3.4 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS. Then $(J_s(R))^+$ is nilpotent modulo A^+.*

By (i) and (iv) of Proposition 2.4.9, we have the following theorem.

Proposition 5.3.5 *[27] Let R be a zero-symmetric right near-ring with identity. If $\mathcal{D}$ is an ideal of $\mathbb{M}_n(R)$, then*

$$(\mathcal{D}_*)^+ \subseteq \mathcal{D} \subseteq (\mathcal{D}_*)^*.$$

This enables us to prove the following lemma.

Lemma 5.3.6 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS. If $\mathcal{D}$ is an ideal of $\mathbb{M}_n(R)$ contained in $(J_s(R))^+$ such that $[(J_s(R))^+]/\mathcal{D}$ is nilpotent, then $A^+ \subseteq \mathcal{D}$.*

Proof For any $\sigma \in J_s(R)$, we have $f_{ij}^{\sigma} \in (J_s(R))^+$. Since

$$[(J_s(R))^+]^{(m)} \subseteq \mathcal{D},$$

for some positive integer m, we have $(f_{ij}^{\sigma})^m \in \mathcal{D}$. For each $\langle r_1, ..., r_n \rangle \in R^n$, we have

$$\pi_i(f_{ii}^{\sigma})^m \langle r_1, ..., r_n \rangle = \pi_i \langle 0, ..., \sigma^m r_i, ..., 0 \rangle = \sigma^m r_i \in \mathcal{D}_*.$$

In particular,

$$\pi_i(f_{ii}^{\sigma})^m \langle 1, ..., 1 \rangle = \pi_i \langle 0, ..., \sigma^m, ..., 0 \rangle = \sigma^m \in \mathcal{D}_*.$$

Thus

$$J_s(R)/\mathcal{D}_* \quad \text{is a nil ideal of} \quad R/\mathcal{D}_*.$$

Since R satisfies the DCCS, $J_s(R)/\mathcal{D}_*$ is nilpotent. By the minimality of A we have $A \subseteq \mathcal{D}_*$. Therefore, by Propositions 2.4.6 (i) and 5.3.5,

$$A^+ \subseteq (\mathcal{D}_*)^+ \subseteq \mathcal{D}.$$

□

Hence by Theorem 5.3.4 and Lemma 5.3.6 the next theorem follows.

Theorem 5.3.7 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS. Then A^+ is the unique minimal ideal modulo which $(J_s(R))^+$ is non-zero nilpotent.*

Since $(J_s(R))^+ \subseteq J_s(\mathbb{M}_n(R))$, by Theorem 5.1.5, and by definition, $J_s(\mathbb{M}_n(R))/\mathcal{A}$ is nilpotent, it follows that $(J_s(R))^+ + \mathcal{A}$ is nilpotent modulo $\mathcal{A}$. This implies that $(J_s(R))^+$ is nilpotent modulo $A^+ \cap \mathcal{A}$. By the minimality of A^+ we have the following theorem.

Theorem 5.3.8 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS. Then*

$$A^+ \subseteq \mathcal{A}.$$

Clearly, $A^+ \subseteq A^*$, but it is not clear what the relationship between A^* and $\mathcal{A}$ is. However, if $\mathcal{A} = (0)$, then $A^* = (0)$. To show this we need the following lemma.

Lemma 5.3.9 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS, and let Δ be a monogenic R-group. If Δ is an R-group of type-0 but not of type-s, then the $\mathbb{M}_n(R)$-group Δ^n is also of type-0 but not of type-s.*

Proof Suppose Δ^n is an $\mathbb{M}_n(R)$-group of type-s, and let $(R\delta)^n$ be any monogenic $\mathbb{M}_n(R)$-subgroup of Δ^n. Then, by definition of type-s, Definition 2.2.1 (ii), along with Theorem 4.1.4, $(R\delta)^n$ is a direct-sum of the form

$$(R\delta)^n = \bigoplus_{i=1}^{k} H_i^n,$$

where each H_i^n is an $\mathbb{M}_n(R)$-kernel of type-0, for $i = 1, 2, ...k$. It follows from Lemma 4.1.6 that $R\delta = \bigoplus_{i=1}^{k} H_i$, where each H_i is an R-kernel of type-0. That is, Δ is of type-s. □

Theorem 5.3.10 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS.*

$$\textit{If} \quad \mathcal{A} = (0), \quad \textit{then} \quad A^* = (0).$$

Proof Since $\mathcal{A} = (0)$, it follows that $J_s(\mathbb{M}_n(R)) = J_0(\mathbb{M}_n(R))$, by definition of the s-socle and Lemma 2.3.13. Assume that $A \neq (0)$. Then $J_0(R) \neq J_s(R)$, by definition of the s-socle and Lemma 2.3.13, and hence there exists an R-group Δ of type-0 which is not of type-s. It follows from Lemma 5.3.9 that Δ^n is an $\mathbb{M}_n(R)$-group of type-0 which is not of type-s. This contradicts the fact that $J_s(\mathbb{M}_n(R)) = J_0(\mathbb{M}_n(R))$. Therefore $J_0(R) = J_s(R)$, hence $A = (0)$. It now follows that $A^* = (0)$. □

Thus, if $A^* \neq (0)$, then $\mathcal{A} \neq (0)$. In fact, examples of near-rings exist where $\mathcal{A}$ properly contains A^*, see Example 5.4.1.

One would expect the connection between the s-socle ideals A and $\mathcal{A}$ to imitate that of their corresponding s-radicals, namely,

$$A^+ \subseteq \mathcal{A} \subseteq A^*.$$

However, it turns out to be more complicated. Examples show that there is, in general, no relationship between A^* and $\mathcal{A}$. Only in restricted cases does one find a relationship between these two ideals, one such case is given below in Theorem 5.3.12. To this end we note, below, Corollary 2 to Theorem 4.4 of [6] as a lemma.

Lemma 5.3.11 *[6] Let R be a zero-symmetric right near-ring with identity satisfying the DCCS. If $\alpha_R = 2$ and $A \subseteq Soi(R)$, then*

$$J_s(R) \cap Soi(R) = A.$$

Theorem 5.3.12 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS. If $\alpha_R = \alpha_{\mathbb{M}_n(R)} = 2$, $A \subseteq Soi(R)$ and*

$\mathcal{A} \subseteq Soi(\mathbb{M}_n(R))$, *then*

$$\mathcal{A} \subseteq A^*.$$

Proof By Lemma 5.3.11,

$$A = J_s(R) \cap Soi(R) \quad \text{and} \quad \mathcal{A} = J_s(\mathbb{M}_n(R)) \cap Soi(\mathbb{M}_n(R)).$$

By Proposition 2.4.6 (iii), we have

$$A^* = [J_s(R) \cap Soi(R)]^* = (J_s(R))^* \cap (Soi(R))^*.$$

Also, by Theorem 2.4.11 (ii),

$$\mathcal{A} \subseteq J_s(\mathbb{M}_n(R)) \subseteq (J_s(R))^*,$$

and by Theorem 4.3.6,

$$\mathcal{A} \subseteq Soi(\mathbb{M}_n(R)) \subseteq (Soi(R))^*.$$

Therefore, $\mathcal{A} \subseteq A^*$. □

5.4 Examples on s-socles

In this section, examples of near-rings are given to illustrate the relationship between the s-socle ideal of $\mathbb{M}_n(R)$ and the ideals A^+ and A^*, where A is the s-socle of R. The symbol $\oplus_G$ is used to denote a group theoretic direct sum.

The following example illustrates that A^* can be strictly contained in $\mathcal{A}$.

Example 5.4.1 Let $G := \mathbb{Z}_2 \oplus_G \mathbb{Z}_4 \oplus_G \mathbb{Z}_2 \oplus_G \mathbb{Z}_2$, $S'' := \mathbb{Z}_2 \oplus_G \mathbb{Z}_4 \oplus_G \mathbb{Z}_2 \oplus_G \{0\}$, and $S' := \{0\} \oplus_G \{0\} \oplus_G \mathbb{Z}_2 \oplus_G \mathbb{Z}_2$. Let $S_1 := \{(0,0,0,0), (0,2,1,1)\}$,
$S_2 := \{(0,0,0,0), (0,2,0,1)\}$, $S_3 := \{(0,0,0,0), (0,2,1,0)\}$,
$S_4 := \{(0,0,0,0), (0,0,1,1)\}$, $S_5 := \{(0,0,0,0), (0,0,0,1)\}$,
$S_6 := \{(0,0,0,0), (0,0,1,0)\}$, $S_7 := \{(0,0,0,0), (0,2,0,0)\}$.

Define a near-ring R by
$R := \{ f \in M_0(G) \mid f(S'') \subseteq S'', f(S_i) \subseteq S_i, \forall i \in \{1,2,\ldots,7\};$
$\forall s, s' \in S', s - s' \in S_5 \Rightarrow f(s) - f(s') \in S_5;$
$\forall g, g' \in G, g - g' \in S'' \Rightarrow f(g) - f(g') \in S'' \ \}.$

We note that

(a) Each S_i is an R-subgroup of G, for each $i \in \{1,2,\ldots,7\}$.

(b) G is a monogenic R-group generated by $(1,3,1,1)$, and S'' is an R-kernel of G. Thus G is not of type-0, so that $J_0(R) \neq (0)$.

(c) S'' is an R-group of type-s but not of type-2, because it has R-subgroups S_3, S_6 and S_7 none of which is an R-kernel. Thus

$$J_0(R) \neq J_2(R).$$

(d) $S' = S_6 \oplus_G S_5$ is a non-monogenic R-subgroup of G because all its proper subgroups are R-groups.

(e) S_5 is an R-kernel of S'.

(f) All the type-0 R-subgroups of G are also of type-s, hence

$$J_s(R) = J_0(R).$$

Therefore, the s-socle A of R is zero. Hence, $A^* = (0)$.

We now look at the R-subgroups of $(R,+)$, but first we define the following maps in R. Let $\bar{x} := (1,2,1,1)$, and $\bar{0} := (0,0,0,0)$. Define k_0 to be the zero-map;

$$k_1(\bar{x}) := (0,2,1,1);\ k_2(\bar{x}) := (0,2,0,1);\ k_3(\bar{x}) := (0,2,1,0);$$

$$k_4(\bar{x}) := (0,0,1,1);\ k_5(\bar{x}) := (0,0,0,1);\ k_6(\bar{x}) := (0,0,1,0);$$

$$k_7(\bar{x}) := (0,2,0,0); \quad \text{such that} \quad k_j(\bar{y}) = \bar{0}, \quad \text{if} \quad \bar{y} \neq \bar{x},$$

for each $j \in \{1,2,\ldots,7\}$.

Define $K := \{k_0, k_1, k_2, \ldots, k_7\}$, $H := \{k_0, k_4, k_5, k_6\}$, and $T := \{k_0, k_5\}$, and observe that

(1) K and H are non-monogenic R-groups, and can be written as group-theoretic direct-sums as follows ;

$$K = Rk_6 \oplus_G Rk_5 \oplus_G Rk_7, \quad \text{and} \quad H = Rk_6 \oplus_G Rk_5.$$

(2) The maps k_j, for $j = 1,2,\ldots,7$, are defined in such a way that we have the following R-isomorphisms between R-subgroups of $(R,+)$ and the R-subgroups of G

$$Rk_6 \cong_R S_6; \quad T = Rk_5 \cong_R S_5; \quad Rk_7 \cong_R S_7 \quad \text{and} \quad S' \cong_R H.$$

(3) $T = Rk_5$ is an R-kernel of H.

Consider K^2 as an $\mathbb{M}_2(R)$-group under Action 2. Let $\langle k, k' \rangle \in K^2$, where

$$k = a_1k_6 + a_2k_5 + a_3k_7 \quad \text{and} \quad k' = b_1k_6 + b_2k_5 + b_3k_7.$$

We may rewrite k and k' as

$$k = \langle a_1k_6, a_2k_5, a_3k_7\rangle \text{ and } k' = \langle b_1k_6, b_2k_5, b_3k_7\rangle.$$

There is an isomorphism $\Psi : K^2 \to K^2$ defined by

$$\Psi(\langle k, k'\rangle) = \langle\langle a_1k_4, b_1k_4\rangle, \langle a_2k_5, b_2k_5\rangle, \langle a_3k_7, b_3k_7\rangle\rangle,$$

which is an element of $(Rk_6)^2 \oplus_G (Rk_5)^2 \oplus_G (Rk_7)^2$.

A matrix $U \in \mathbb{M}_2(R)$ acts on K^2 by Action 2 as follows

$$U\langle k, k'\rangle = \Psi^{-1}U\Psi\langle k, k'\rangle = \Psi^{-1}\langle(U\langle a_1, b_1\rangle)k_4, U(\langle a_2, b_2\rangle)k_5, U(\langle a_3, b_3\rangle)k_7\rangle.$$

Taking $U := f^a_{11} + f^b_{21}$ for $a, b \in R$, we obtain

$$\begin{aligned} U\langle k_6 + k_5 + k_7, 0\rangle &= \Psi^{-1}\langle U\langle 1, 0\rangle k_6, U\langle 1, 0\rangle k_5, U\langle 1, 0\rangle k_7\rangle \\ &= \Psi^{-1}\langle\langle ak_6, bk_6\rangle, \langle ak_5, bk_5\rangle, \langle ak_7, bk_7\rangle\rangle \\ &= \langle\langle ak_6, ak_5, ak_7\rangle, \langle bk_6, bk_5, bk_7\rangle\rangle. \end{aligned}$$

Since a and b can be arbitrarily chosen in R, all the 64 elements of K^2 can be obtained by using $\langle k_6 + k_5 + k_7, 0\rangle$ as an $\mathbb{M}_2(R)$-generator. So then

$$K^2 = \mathbb{M}_2(R)\langle k_6 + k_5 + k_7, 0\rangle.$$

In a similar manner, it can be shown that the subgroup H^2 of K^2 is a monogenic $\mathbb{M}_2(R)$-group under Action 2, such that

$$H^2 = \mathbb{M}_2(R)\langle k_6 + k_5, 0\rangle.$$

We note that K^2 has no $\mathbb{M}_2(R)$-kernels, hence it is an $\mathbb{M}_2(R)$-group of type-0. Since H^2 is a monogenic $\mathbb{M}_2(R)$-subgroup of K^2 which is not

a direct sum of $\mathbb{M}_2(R)$-kernels, it follows that K^2 is not of type-s. In particular, H^2 is of type-$\mathcal{K}$ as it has only one $\mathbb{M}_2(R)$-kernel T^2. Hence $J_s(\mathbb{M}_2(R)) \neq J_0(\mathbb{M}_2(R))$, therefore the s-socle $\mathcal{A}$ of $\mathbb{M}_2(R)$ is non-zero. From this it follows that $A^* = (0) \subsetneq \mathcal{A}$.

The next example illustrates that, in general, the ideals A^* and $\mathcal{A}$ are not comparable. For the near-ring in this example we observe that $A^* \not\subseteq \mathcal{A}$.

Example 5.4.2 Let $\Omega := \mathbb{Z}_4 \oplus_G \mathbb{Z}_2 \oplus_G \mathbb{Z}_4$;
$\Omega_1 := \mathbb{Z}_4 \oplus_G \mathbb{Z}_2 \oplus_G \{0\}$;
$\Omega_2 := \{0\} \oplus_G \mathbb{Z}_2 \oplus_G \mathbb{Z}_4$;
$\Delta_1 := \{\bar{0}, (2,0,0), (0,1,0), (2,1,0)\}$;
$\Delta_2 := \{\bar{0}, (1,0,0), (2,0,0), (3,0,0)\}$;
$\Delta_3 := \{\bar{0}, (0,1,0), (0,0,2), (0,1,2)\}$;
$T_1 := \{\bar{0}, (0,1,0)\}$; $T_2 := \{\bar{0}, (2,0,0)\}$;
$T_3 := \{\bar{0}, (2,1,0)\}$; $T_4 := \{\bar{0}, (0,0,2)\}$;
$T_5 := \{\bar{0}, (0,1,2)\}$.

Define a near-ring, R, by
$R := \{ f \in M_0(\Omega) \mid f(\Omega_i) \subseteq \Omega_i,\ i = 1,2;$
$f(\Delta_j) \subseteq \Delta_j,\ j = 1,2,3;\ f(T_l) \subseteq T_l,\ l = 1,2,\ldots,5;$
$\forall\, \delta, \delta' \in \Delta_j\, (j = 1,2),\ \delta - \delta' \in T_2 \Rightarrow f(\delta) - f(\delta') \in T_2 \}$.

1. Observations on R-subgroups of Ω.

(a) It follows immediately from definition of R that the group Ω is a faithful R-group, and Ω is of type-0 because it has no non-trivial R-kernels. But,

Ω contains a monogenic R-group, Δ_2, which is of type-$\mathcal{K}$ as Δ_2 has no R-group of type-0 as a direct-summand and is itself not of type-0 because it has an R-kernel, T_2. This implies Ω is not of type-s, and hence,

$$J_0(R) = (0) \neq J_s(R),$$

and therefore the s-socle $A \neq (0)$.

(b) The subgroup Ω_1 is of type-0 because it has no R-kernels. However, Ω_1 contains Δ_2, therefore Ω_1 is not of type-s just as Ω is not of type-s. The group Ω_2 is a type-s R-group as it has no R-kernels and all its monogenic R-subgroups are of type-2.

(c) The monogenic R-group Δ_2 is in $\mathcal{K}(\Omega)$, because it has no type-0 R-group as a direct-summand. The type-0 R-subgroups of Ω are :

$$\Omega_2,\ T_j,\ j = 1, 2, 3, 4, 5,$$

and all other R-groups of type-0 are isomorphic to

$$\Delta_i/T_2, \quad i = 1, 2.$$

(d) The subgroups Δ_1 and Δ_3 are both non-monogenic R-subgroups of Ω. The R-group T_2 is an R-kernel of Δ_1, while Δ_3 has no R-kernels. In fact, Δ_1 is of $\mathcal{K}_2$-form, whereas Δ_3 is of s_2-form.

(e) The classes of monogenic R-subgroups of Ω are:

$$\mathcal{K}(\Omega) := \{\Delta_2\};\ \ \mathcal{B}(\Omega) := \{T_2\};$$

$$\mathcal{G}(\Omega) := \{(0),\ \Omega,\ \Omega_1,\ \Omega_2,\ T_1,\ T_3,\ T_4,\ T_5\}.$$

The Socle ideal of R is therefore given by $Soi(R) = (0 : \Delta_2)$.

(f) The nil-rigid series of R is

$$(0) = L_1 \subset C_1 \subset L_2 \subset C_2 = R$$

where

$$L_1 = J_0(R) = (0),\ C_1 = Soi(R) = (0 : \Delta_2),$$

$$L_2/C_1 = J_0\left(R/C_1\right) \subseteq (0 : T_2),$$

and

$$C_2/L_2 = Soi\left(R/L_2\right) = R/L_2,$$

because $\mathcal{K}\left(R/L_2\right) = \emptyset$. Therefore the nil-rigid length of R, is $\alpha_R = 2$, and hence the s-socle

$$A \subseteq C_1 = Soi(R).$$

It follows from Lemma 5.3.11 that

$$A = J_s(R) \cap Soi(R),$$

and hence

$$A = \left[\bigcap_{j=1}^{5}(0 : T_j)\right] \cap \left[\bigcap_{i=1}^{2}(0 : \Delta_i/T_2)\right] \cap (0 : \Delta_2) \cap (0 : \Omega_2).$$

Since $\Delta_1 = T_1 \oplus_G T_2$, in fact $\Delta_1 = T_1 \cup T_2 \cup T_3$, the s-socle is

$$A = \left[\bigcap_{j=1}^{5}(0 : T_j)\right] \cap (0 : \Delta_2) \cap (0 : \Omega_2).$$

2. Classifications of $\mathbb{M}_2(R)$-subgroups of Ω^2.

(a) Let us first note that Ω^2 is a faithful monogenic $\mathbb{M}_2(R)$-group. Also, Δ_2^2 is a monogenic $\mathbb{M}_2(R)$-group containing a type-0 $\mathbb{M}_2(R)$-group T_2^2. But, T_2^2 is not a direct-summand of Δ_2^2. Hence, Ω^2 is not of type-s, it is however of type-0 as it has no $\mathbb{M}_2(R)$-kernels.

(b) Each $\mathbb{M}_2(R)$-group H^2, where H is a monogenic R-subgroup of Ω, has the same properties that H had as an R-group. As for the non-monogenic R-subgroups Δ_1 and Δ_3, their corresponding $\mathbb{M}_2(R)$-groups, Δ_1^2 and Δ_3^2 are monogenic under Action 2.

(c) Since Δ_1^2 contains a type-0 $\mathbb{M}_2(R)$-group, T_2^2, which is not its direct-summand, $\Delta_1^2 \in \mathcal{K}(\Omega^2)$. On the other hand, Δ_3^2 has no $\mathbb{M}_2(R)$-kernel and each of its proper $\mathbb{M}_2(R)$-subgroups are of type-2. Hence, Δ_3^2 is of type-s.

(d) The classes of monogenic $\mathbb{M}_2(R)$-subgroups of Ω_2^2 are

$$\mathcal{K}(\Omega^2) := \{(0), \Delta_2^2, \Delta_1^2\}; \mathcal{B}(\Omega^2) := \{T_2^2\};$$

$$\mathcal{G}(\Omega^2) := \{\Omega^2, \Omega_1^2, \Omega_2^2, \Delta_3^2, T_1^2, T_3^2, T_4^2, T_5^2\}$$

The socle ideal of $\mathbb{M}_2(R)$ is therefore given by

$$Soi(\mathbb{M}_2(R)) = (0 : \Delta_2^2) \cap (0 : \Delta_1^2).$$

(e) The nil-rigid series of $\mathbb{M}_2(R)$ is

$$(0) = \mathcal{L}_1 \subset \mathcal{C}_1 \subset \mathcal{L}_2 \subset \mathcal{C}_2 = \mathbb{M}_2(R)$$

where

$$\mathcal{L}_1 = J_0(\mathbb{M}_2(R)) = (0),$$

$$\mathcal{C}_1 = Soi(\mathbb{M}_2(R)) = (0 : \Delta_2^2) \cap (0 : \Delta_1^2),$$

$$\mathcal{L}_2/\mathcal{C}_1 = J_0\left(\mathbb{M}_2(R)/\mathcal{C}_1\right) \subseteq \bigcap_{j=1}^{3} \left(0 : T_j^2\right),$$

$$\mathcal{C}_2/\mathcal{L}_2 = Soi\left(\mathbb{M}_2(R)/\mathcal{L}_2\right) = \mathbb{M}_2(R)/\mathcal{L}_2$$

because $\mathcal{K}\left(\mathbb{M}_2(R)/\mathcal{L}_2\right) = \emptyset$.

The nil-rigid length of $\mathbb{M}_2(R)$ is $\alpha_{\mathbb{M}_2(R)} = 2$, hence we have that the s-socle of $\mathbb{M}_2(R)$, is

$$\mathcal{A} \subseteq Soi(\mathbb{M}_2(R)).$$

By Lemma 5.3.11

$$\mathcal{A} = Soi(\mathbb{M}_2(R)) \bigcap J_s(\mathbb{M}_2(R)),$$

so the s-socle $\mathcal{A}$ is the intersection

$$(0:\Omega_2^2) \cap (0:\Delta_1^2) \cap (0:\Delta_2^2) \cap \left(\bigcap_{j=1}^{3}(0:(T_j)^2)\right) \cap \left(\bigcap_{i=1}^{2}(0:(\Delta_i/T_2)^2)\right).$$

3. There is a matrix W in A^* which is not in $\mathcal{A}$.

(a) Define $K := \{k_0, k_1, k_2, k_3\}$, where k_0 is the zero-map,

$$k_1((3,1,1)) = (0,1,0),\ k_2((3,1,1)) = (2,0,0),\ k_3((3,1,1)) = (2,1,0),$$

and if $\bar{x} \neq (3,1,1)$,

$$k_1(\bar{x}) = k_2(\bar{x}) = k_3(\bar{x}) = \begin{cases} \bar{0} & \text{if} \quad \bar{x} \in \Omega_1 \\ \bar{x} & \text{if} \quad \bar{x} \in \Omega \setminus \Omega_1 \end{cases}.$$

Then K is a non-monogenic R-group and it is R-isomorphic to Δ_1. That is, K is of $\mathcal{K}_2$-form. So that

$$K^2 = \mathbb{M}_2(R)\langle k_1, k_2\rangle$$

is an $\mathbb{M}_2(R)$-group of type-$\mathcal{K}$ under Action 2. Hence $\mathcal{A} \subset (0:K^2)$.

(b) Define a matrix

$$W = f_{11}^s(f_{11}^t + f_{12}^t)$$

where

$$t((0,1,0)) = (0,1,0),\ t((2,0,0)) = (2,0,0),$$

and $\bar{0}$ otherwise,

$$s((2,1,0)) = (2,1,0),$$

and $\bar{0}$ otherwise. For any $\langle r, r' \rangle$ in R^2 we have

$$W\langle r, r' \rangle = \langle s(tr + tr'), 0 \rangle.$$

(i) We now show that $W \notin \mathcal{A} \subset (0 : K^2)$.

Note that

$$s(tr + tr')(0,1,0) \in s(t(T_1) + t(T_1)) \subseteq s(T_1) = (0), \tag{5.3}$$

$$s(tr + tr')(2,0,0) \in s(t(T_2) + t(T_2)) \subseteq s(T_2) = (0), \tag{5.4}$$

$$s(tr + tr')(2,1,0) \in s(t(T_3) + t(T_3)) = s((0)) = (0). \tag{5.5}$$

For the generator $\langle k_1, k_2 \rangle$ of K^2, we have

$$W\langle k_1, k_2 \rangle = \langle s(tk_1 + tk_2), 0 \rangle.$$

But

$$s(tk_1 + tk_2)(3,1,1) = s(t((0,1,0)) + t((2,0,0)))$$

$$= s((0,1,0) + (2,0,0)) = s((2,1,0)) \neq (0).$$

Hence, $W \notin (0 : K^2)$ and therefore $W \notin \mathcal{A}$.

(ii) We prove that $W \in A^*$.

In order to prove that $W \in A^*$, we need only show

$$W\langle r, r' \rangle \in A^2 \text{ for all } r, r' \in R.$$

Since

$$A = \left[\bigcap_{j=1}^{5}(0 : T_j)\right] \cap (0 : \Delta_2) \cap (0 : \Omega_2),$$

it suffices to show that

$$s(tr + tr')(x) = 0, \tag{5.6}$$

for any

$$x \in \left[\bigcup_{j=1}^{5} T_j\right] \cup \Delta_2 \cup \Omega_2.$$

If $x \in T_1 \cup T_2 \cup T_3$, then by (5.3), (5.4), and (5.5), it follows that (5.6) holds.

If $x \in T_4 \cup T_5$, then

$$s(tr + tr')x \in s(t(T_i) + t(T_i)) = s((0)) = (0),$$

$i = 4, 5$.

If $x \in \Delta_2$, then

$$s(tr + tr')x \in s(t(\Delta_2) + t(\Delta_2)) \subseteq s(T_2) = (0).$$

If $x \in \Omega_2$, then

$$s(tr + tr')x \in s(t(\Omega_2) + t(\Omega_2)) \subseteq s(T_1) = (0).$$

Therefore $W \in A^*$.

Chapter 6

Nil-rigid series of $\mathbb{M}_n(R)$

This chapter focuses on applying the theory of Chapter 3 to matrix near-rings. It is shown how R-groups of ν_n-form affect the Ω-chains and their lengths. Sufficient conditions under which the nil-rigid series of $\mathbb{M}_n(R)$ is longer than that of R are stated. The positions of the s-socle ideals; A^+, A^* and $\mathcal{A}$, with respect to the (Ω^n)-ideal series, are located. The $()^+$-series and $()^*$-series are defined in a natural way. Sufficient conditions under which the $()^+$-series, $()^*$-series and the nil-rigid series of $\mathbb{M}_n(R)$ are related to the (Ω^n)-ideal series are stated. Examples of near-rings, with their nil-rigid lengths strictly less than that of their associated matrix near-rings are given in Section 6.3.

Throughout this chapter R denotes a zero-symmetric right near-ring with identity satisfying the DCCS. The nil-rigid length of R is denoted by α_R.

6.1 The nil-rigid length of $\mathbb{M}_n(R)$

In this section, it is shown that, under certain conditions, knowing all Ω-chains (up to isomorphism) and where the R-groups of ν_n-form appear in the Ω-chains, provides sufficient information to determine the nil-rigid length of $\mathbb{M}_n(R)$. We refer the reader to Section 3.2 for definitions relating to Ω-chains.

The following theorem is a consequence of Theorem 4.1.3, Lemma 4.1.7 and the definition of R-groups of ν_n-form.

Theorem 6.1.1 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS. Let Ω be a faithful locally monogenic R-group, and*

$$\Delta_1 \supsetneq \Delta_2 \supsetneq \cdots \supsetneq \Delta_\beta \tag{6.1}$$

be an Ω-chain. If Ω has no R-subgroups of ν_n-form, $\nu = 0, \mathcal{K}$, then

$$\Delta_1^n \supsetneq \Delta_2^n \supsetneq \cdots \supsetneq \Delta_\beta^n$$

is an (Ω^n)-chain.

We now generalize this notion.

Theorem 6.1.2 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS. Let Ω be a faithful locally monogenic*

R-group, and

$$\Delta_1 \supsetneq \Delta_2 \supsetneq \cdots \supsetneq \Delta_\beta \tag{6.2}$$

be an Ω*-chain. An* (Ω^n)*-chain corresponding to (6.2) is formed as follows.*
(a) Let Γ_1 *be an R-subgroup of* Ω *which is maximal amongst the type-*0 *and* 0_n*-form R-subgroups of* Ω *containing* Δ_1*. Then* Γ_1^n *is a maximal type-*0 $\mathbb{M}_n(R)$*-subgroup of* Ω^n *containing* Δ_1^n *and has*

$$\Gamma_1^n \supsetneq \Delta_2^n.$$

(b) For each odd i*,* $1 \leq i \leq \beta - 1$*, let* Γ_{i+1} *be an R-subgroup of* Ω *which is maximal amongst type-*$\mathcal{K}$ *and* $\mathcal{K}_n$*-form R-subgroups of* Γ_i *containing* Δ_{i+1}*. Then* Γ_{i+1}^n *is a maximal type-*$\mathcal{K}$ $\mathbb{M}_n(R)$*-subgroup of* Γ_i^n *containing* Δ_{i+1}^n *and has*

$$\Gamma_{i+1}^n \supsetneq \Delta_{i+2}^n,$$

if $i+2 \leq \beta$*.*
(c) For each even i*,* $1 \leq i \leq \beta$*, let* Γ_{i+1} *be an R-subgroup of* Ω *which is maximal amongst type-*0 *and* 0_n*-form R-subgroups of* Γ_i *containing* Δ_{i+1}*. Then* Γ_{i+1}^n *is a maximal type-*0 $\mathbb{M}_n(R)$*-subgroup of* Γ_i^n *containing* Δ_{i+1}^n *and has*

$$\Gamma_{i+1}^n \supsetneq \Delta_{i+2}^n$$

if $i+2 \leq \beta$*.*

Note. If Γ_β^n is not of type-s then the above procedure can be continued, making the (Ω^n)-chain in $\mathbb{M}_n(R)$ strictly longer than the chain (6.2).

Using the above procedure we observe the following.

Theorem 6.1.3 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS. Let Ω be a faithful locally monogenic R-group, and*

$$\Delta_1 \supsetneq \Delta_2 \supsetneq \cdots \supsetneq \Delta_\beta \tag{6.3}$$

an Ω-chain, where Δ_1 is not contained in any 0_n-form R-subgroup of Ω. Also, let H and K be R-subgroups of Ω such that $H \supsetneq K$. Assume that one of the following cases occurs.

(a) There exists an odd i such that H is maximal amongst $\mathcal{K}_n$-form R-subgroups of Ω with

$$\Delta_i \supsetneq H \supsetneq \Delta_{i+1},$$

and there exists K maximal amongst type-0 and 0_n-form R-subgroups of Ω with

$$H \supsetneq K \supsetneq \Delta_{i+1}.$$

(b) There exists an even i such that H is maximal amongst 0_n-form R-subgroups of Ω with

$$\Delta_i \supsetneq H \supsetneq \Delta_{i+1},$$

and there exists K maximal amongst type-$\mathcal{K}$ and $\mathcal{K}_n$-form R-subgroups of Ω with

$$H \supsetneq K \supsetneq \Delta_{i+1}.$$

Then there is an (Ω^n)-chain corresponding to the Ω-chain (6.3) of length at least $\beta + 2$.

If the conditions of Theorem 6.1.3 hold and the Ω-chain (6.3) has at least

one pair of R-groups, H and K, as described in the above theorem, we say such an Ω-chain has a **refinement**.

Theorem 6.1.3 can now be rephrased, under the conditions stated in the theorem, as follows.

If the Ω-chain,

$$\Delta_1 \supsetneq \Delta_2 \supsetneq \cdots \supsetneq \Delta_{\beta_\Omega}, \tag{6.4}$$

has a refinement, then the length, β_{Ω^n}, of its corresponding (Ω^n)-chain satisfies

$$\beta_{\Omega^n} \geq \beta_\Omega + 2.$$

By definition, the Ω-length of (6.4) is $\frac{\beta_\Omega - 1}{2}$. Thus the (Ω^n)-length of the corresponding (Ω^n)-chain is

$$\frac{\beta_{\Omega^n} - 1}{2} \geq \frac{(\beta_\Omega + 2) - 1}{2} = \left(\frac{\beta_\Omega - 1}{2}\right) + 1. \tag{6.5}$$

Denote the $\mathcal{KG}$-length of Ω and Ω^n by η_R and $\eta_{\mathbb{M}_n(R)}$, respectively. The next corollary follows readily.

Corollary 6.1.4 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS, and let Ω be a faithful locally monogenic R-group. If a $\mathcal{KG}$-chain of Ω has a refinement, then*

$$\eta_{\mathbb{M}_n(R)} \geq \eta_R + 1.$$

Note that a refinement of an Ω-chain does not occur if Ω has no non-monogenic R-subgroups of ν_n-form, $\nu = 0, \mathcal{K}$. That is, under this condition, every Ω-chain of Ω-length l corresponds to an (Ω^n)-chain of (Ω^n)-length l.

Theorem 6.1.5 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS, and let Ω be a faithful locally monogenic R-group. If Ω has no R-subgroups of ν_n-form, $\nu = 0, \mathcal{K}$, then*

$$\eta_{\mathbb{M}_n(R)} = \eta_R.$$

6.2 Series of ideals of $\mathbb{M}_n(R)$

In this section, the positions of the s-socle ideals; A^+, A^* and $\mathcal{A}$, with respect to the (Ω^n)-ideal series, are located. The $()^+$-series, $()^*$-series are defined in a natural way. Sufficient conditions are given which allow the $()^+$-series, $()^*$-series and the nil-rigid series of $\mathbb{M}_n(R)$ to be easily related to the (Ω^n)-ideal series. Throughout this section, A denotes the s-socle of R, and $\mathcal{A}$ denotes the s-socle of $\mathbb{M}_n(R)$.

The next proposition relates the s-socle ideal of R to the Ω-chains of R. Recall: If $\Delta_1 \supsetneq \Delta_2 \supsetneq \cdots \supsetneq \Delta_\beta$ is a $\mathcal{KG}$-chain then the $\mathcal{KG}$-length, $\eta_R = \frac{\beta-1}{2}$. That is, $\beta = 2\eta_R + 1$. Thus, if $\eta_R > 0$ then $\beta > 1$.

Proposition 6.2.1 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS, and let Ω be a faithful locally monogenic R-group. Let*

$$\Delta_1 \supsetneq \Delta_2 \supsetneq \cdots \supsetneq \Delta_\beta, \tag{6.6}$$

be a $\mathcal{KG}$-chain. If the $\mathcal{KG}$-length, $\eta_R > 0$ and the nil-rigid length of R,

$\alpha = \frac{\beta+1}{2}$, *then*

$$(0 : \Delta_{\beta-1}) \supseteq A.$$

Moreover, $T_{\beta-1} \supseteq A$, *where* T_j, $j = 1, 2, \ldots, \beta$, *are* Ω*-ideals as defined in Definition 3.2.11.*

Proof Consider the Ω-ideal series of R, in Definition 3.2.11 and Proposition 3.2.12,

$$T_1 \subseteq T_2 \subseteq \ldots \subseteq T_{\beta-1} \subseteq T_\beta.$$

Since (6.6) is a $\mathcal{KG}$-chain,

$$(0 : \Delta_{\beta-1}) \supseteq T_{\beta-1}.$$

By Corollary 3.2.17,

$$T_{\beta-1} \supseteq C_{\alpha-1},$$

where $\alpha - 1 = [\beta - 1]/2$. Since

$$C_{\alpha-1} \supseteq A,$$

by Lemma 2.3.14, the result follows. □

The (Ω^n)-ideal series of $\mathbb{M}_n(R)$ is defined in the same way as the Ω-ideal series of R was defined in Definition 3.2.11 and Proposition 3.2.12. To avoid confusion, we denote the (Ω^n)-ideal series of $\mathbb{M}_n(R)$ by

$$\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \cdots \subseteq \mathcal{T}_{\gamma-1} \subseteq \mathcal{T}_\gamma.$$

Theorem 6.2.2 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS, and let Ω be a faithful locally monogenic R-group. If the $\mathcal{KG}$-length of R is $\eta_R > 0$, the nil-rigid length $\alpha = \frac{\beta+1}{2}$, and Ω has no R-subgroups of ν_n-form, $\nu = 0, \mathcal{K}$, then*

$$A^* \subseteq \mathcal{T}_{2\eta_R}.$$

Proof Since Ω has no R-subgroups of ν_n-form, $\nu = 0, \mathcal{K}$, by Theorem 6.1.1 each (Ω^n)-chain is of the form

$$\Delta^n_{l,1} \supsetneq \Delta^n_{l,2} \supsetneq \cdots \supsetneq \Delta^n_{l,\beta(l)}$$

corresponding to

$$\Delta_{l,1} \supsetneq \Delta_{l,2} \supsetneq \cdots \supsetneq \Delta_{l,\beta(l)},$$

where $l \in \Lambda$ indexes the Ω-chains, as in Definition 3.2.11. Thus the j-th Ω-ideal T_j has

$$T^*_j = \left[\bigcap_{\substack{l\in\Lambda \\ \beta(l)\geq j}} (0 : \Delta_{l,\,j}) \right]^* = \bigcap_{\substack{l\in\Lambda \\ \beta(l)\geq j}} (0 : \Delta_{l,\,j})^*.$$

Since

$$(0 : \Delta_{l,\,j})^* \subseteq (0 : \Delta^n_{l,\,j}),$$

by Lemma 4.2.5, it follows that $T^*_{\beta-1} \subseteq \mathcal{T}_{\beta-1}$. But $A \subseteq T_{\beta-1}$, by Proposition 6.2.1, and hence $A^* \subseteq T^*_{\beta-1}$, by Proposition 2.4.6(i). Therefore $A^* \subseteq \mathcal{T}_{\beta-1}$. This proves the theorem since $\eta_R = \frac{\beta-1}{2}$. □

Corollary 6.2.3 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS. Let Ω be a faithful locally monogenic*

R-group, and the $\mathcal{KG}$-length of R be $\eta_R > 0$ with the nil-rigid length $\alpha = \frac{\beta+1}{2}$. If Ω has no R-subgroups of ν_n-form, $\nu = 0, \mathcal{K}$, then

$$A^+ \subseteq \mathcal{T}_{2\eta_R}.$$

Denote the nil-rigid series of $\mathbb{M}_n(R)$ by

$$\mathcal{L}_1 \subsetneq \mathcal{C}_1 \subsetneq \mathcal{L}_2 \subsetneq \mathcal{C}_2 \subsetneq \cdots \subsetneq \mathcal{L}_{\alpha_{\mathbb{M}_n(R)}} \subsetneq \mathcal{C}_{\alpha_{\mathbb{M}_n(R)}},$$

and the nil-rigid length of $\mathbb{M}_n(R)$ by $\alpha_{\mathbb{M}_n(R)}$ to avoid ambiguity.

The following lemma and theorems follow as direct application of Theorems 3.2.7, 3.2.16 and 3.2.17, in terms of matrix near-rings.

Lemma 6.2.4 *Let $\mathbb{M}_n(R)$ be a zero-symmetric right near-ring with identity satisfying the DCCS, and having the nil-rigid length, $\alpha_{\mathbb{M}_n(R)} \gneq 3$. Suppose that Ω^n is a faithful $\mathbb{M}_n(R)$-group with $\mathcal{KG}$-length η. Let*

$$\Gamma_{i,1} \supsetneq \Gamma_{i,2} \supsetneq \ldots \supsetneq \Gamma_{i,\eta-1} \supsetneq \Gamma_{i,\eta},$$

be a $\mathcal{KG}$-chain of Ω^n.

(a) If j is even then $\Gamma_{i,j}$ is a type-$\mathcal{K}$ $\left(\mathbb{M}_n(R)/\mathcal{L}_{j/2}\right)$-group.

(b) If j is odd then $\Gamma_{i,j}$ is a type-0 $\left(\mathbb{M}_n(R)/\mathcal{C}_{(j-1)/2}\right)$-group.

Theorem 6.2.5 *Let $\mathbb{M}_n(R)$ be a zero-symmetric right near-ring with identity satisfying the DCCS, and Ω^n be a faithful $\mathbb{M}_n(R)$-group.*

(a) If j is even then

$$\mathcal{T}_j/\mathcal{L}_{(j/2)} \supseteq Soi\left(\mathbb{M}_n(R)/\mathcal{L}_{(j/2)}\right).$$

(b) If j is odd then

$$\mathcal{T}_j/\mathcal{C}_{(j-1)/2} \supseteq J_0\left(\mathbb{M}_n(R)/\mathcal{C}_{(j-1)/2}\right).$$

Here $\mathcal{T}_j$ is the j-th (Ω^n)-ideal of $\mathbb{M}_n(R)$, and $\mathcal{L}_{(j/2)}$ and $\mathcal{C}_{(j-1)/2}$ are ideals in the nil-rigid series of $\mathbb{M}_n(R)$.

Theorem 6.2.6 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS, and let Ω be a faithful locally monogenic R-group. If Ω has no R-subgroups of ν_n-form, $\nu = 0, \mathcal{K}$, then the (Ω^n)-ideal series of $\mathbb{M}_n(R)$,*

$$\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \cdots \subseteq \mathcal{T}_{\gamma-1} \subseteq \mathcal{T}_\gamma,$$

and the nil-rigid series of $\mathbb{M}_n(R)$,

$$\mathcal{L}_1 \subsetneq \mathcal{C}_1 \subsetneq \cdots \subsetneq \mathcal{L}_{\alpha_{\mathbb{M}_n(R)}-1} \subsetneq \mathcal{C}_{\alpha_{\mathbb{M}_n(R)}-1} \subsetneq \mathcal{L}_{\alpha_{\mathbb{M}_n(R)}} \subsetneq \mathcal{C}_{\alpha_{\mathbb{M}_n(R)}},$$

are related by $\mathcal{T}_{2i-1} \supseteq \mathcal{L}_i$ and $\mathcal{T}_{2i} \supseteq \mathcal{C}_i$, where $1 \leq i \leq \alpha_{\mathbb{M}_n(R)}$.

If a faithful R-group, Ω, has no R-subgroups of ν_n-form, $\nu = 0, \mathcal{K}$, it follows from Theorem 6.2.6 that

$$\mathcal{C}_{\alpha-1} \subseteq \mathcal{T}_{2(\alpha-1)}.$$

By definition of the s-socle,

$$\mathcal{A} \subseteq \mathcal{C}_{\alpha-1},$$

see Lemma 2.3.14, hence the next theorem.

Theorem 6.2.7 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS, and let Ω be a faithful locally monogenic R-group. Let the $\mathcal{KG}$-length of R be $\eta_R > 0$. If Ω has no R-subgroups of ν_n-form, $\nu = 0, \mathcal{K}$, then*

$$\mathcal{A} \subseteq \mathcal{T}_{2(\alpha-1)}.$$

Definition 6.2.8 *For the nil-rigid series of R;*

$$(0) \subseteq L_1 \subsetneq C_1 \subsetneq \cdots \subsetneq L_{\alpha-1} \subsetneq C_{\alpha-1} \subsetneq L_\alpha \subsetneq C_\alpha = R,$$

(a) the series of ideals of $\mathbb{M}_n(R)$,

$$(0) \subseteq L_1^+ \subsetneq C_1^+ \subsetneq \cdots \subsetneq L_{\alpha-1}^+ \subsetneq C_{\alpha-1}^+ \subsetneq L_\alpha^+ \subsetneq C_\alpha^+ = \mathbb{M}_n(R),$$

*will be called the $(\,)^+$-**series**, and*

(b) the series of ideals of $\mathbb{M}_n(R)$,

$$(0) \subseteq L_1^* \subsetneq C_1^* \subsetneq \cdots \subsetneq L_{\alpha-1}^* \subsetneq C_{\alpha-1}^* \subsetneq L_\alpha^* \subsetneq C_\alpha^* = \mathbb{M}_n(R),$$

*will be called the $(\,)^*$-**series**.*

We observe that, just as the annihilator ideal

$$\mathcal{P}(\nu) := \bigcap_{R\delta \leq \Omega} (\bar{0} : (R\delta)^n)$$

contains

$$(J_\nu(R))^+,\ J_\nu(\mathbb{M}_n(R)) \ \text{ and } \ (J_\nu(R))^*,$$

for $\nu = 0, s, 2$, the (Ω^n)-ideal series contains the $()^+$-series, the nil-rigid series, and the $()^*$-series, when R has no R-subgroups of ν_n-form. This fact is recorded in the next theorem.

Theorem 6.2.9 *Let R be a zero-symmetric right near-ring with identity such that $\mathbb{M}_n(R)$ satisfies the DCCS. Let Ω be a faithful locally monogenic R-group, and*

$$(0) \subseteq L_1 \subsetneq C_1 \subsetneq \cdots \subsetneq L_{\alpha-1} \subsetneq C_{\alpha-1} \subsetneq L_\alpha \subsetneq C_\alpha = R,$$

be the nil-rigid series of R. If Ω has no R-subgroups of ν_n-form, $\nu = 0, \mathcal{K}$, then the (Ω^n)-ideal series of $\mathbb{M}_n(R)$,

$$\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \cdots \subseteq \mathcal{T}_{\beta-1} \subseteq \mathcal{T}_\beta,$$

contains the $()^+$-series, the $()^*$-series and the nil-rigid series of $\mathbb{M}_n(R)$, in the sense of Theorem 6.2.6.

6.3 Examples on the nil-rigid length of $\mathbb{M}_n(R)$

In this section, examples of near-rings, with their nil-rigid lengths strictly less than that of their associated matrix near-rings are given. The symbol $\oplus_G$ is used denote group theoretic direct sum.

Throughout this section, let

$$\Omega := \mathbb{Z}_{32} \oplus_G \mathbb{Z}_2.$$

Consider the following subgroups of the group $\mathbb{Z}_{32}$;

$$H_1 = \{0, 2, 4, \ldots, 28, 30\},\ H_2 = \{0, 4, 8, 12, 16, 20, 24, 28\},$$

$$H_3 = \{0, 8, 16, 24\} \quad \text{and} \quad H_4 = \{0, 16\}.$$

Define the following subgroups of Ω;

$$\Omega_1 := H_1 \oplus_G \mathbb{Z}_2, \quad \Omega_2 := H_2 \oplus_G \mathbb{Z}_2,$$

$$\Omega_3 := H_3 \oplus_G \mathbb{Z}_2, \quad \Delta_1 := H_4 \oplus_G \mathbb{Z}_2,$$

$$\Delta_2 := H_3 \oplus_G \{0\} = \{(0,0), (8,0), (16,0), (24,0)\},$$

$$\Delta_3 := \{(0,0), (8,1), (16,0), (24,1)\},$$

$$S_1 := H_4 \oplus_G \{0\} = \{(0,0), (16,0)\},$$

$$S_2 := \{0\} \oplus_G \mathbb{Z}_2 = \{(0,0), (0,1)\},$$

$$S_3 := \{(0,0), (16,1)\}.$$

We now use Ω to give an example which illustrates the fact that α_R may be strictly less than $\alpha_{\mathbb{M}_n(R)}$. Let the near-ring in this example be R'. We will observe that $J_0(R') = (0)$ so that the socle ideal, $Soi(R') \neq (0)$, is a member of the nil-rigid series. Here, the faithful R'-group Ω is a member of a non-trivial Ω-chain.

Example 6.3.1 Define the near-ring R' by,
$R' := \{ f \in M_0(\Omega) \mid f(\Omega_i) \subseteq \Omega_i,\ i = 1, 2, 3;$
$f(\Delta_j) \subseteq \Delta_j,\ j = 1, 2, 3;\ f(S_l) \subseteq S_l,\ l = 1, 2;$
$\forall\, \omega, \omega' \in \Omega_1,\ \omega - \omega' \in \Omega_2 \Rightarrow f(\omega) - f(\omega') \in \Omega_2\,;$
$\forall\, \omega_1, \omega_2 \in \Omega_2,\ \omega_1 - \omega_2 \in \Omega_3 \Rightarrow f(\omega_1) - f(\omega_2) \in \Omega_3\,;$
$\forall\, \delta_1, \delta_2 \in \Delta_1,\ \delta_1 - \delta_2 \in S_1 \Rightarrow f(\delta_1) - f(\delta_2) \in S_1\,\}.$

Then R' is a near-ring with multiplicative identity under point-wise addition and map composition.

We observe the following.
(a) The group Ω is a faithful R'-group of type-0.
(b) The subgroup Ω_2 is an R'-kernel of Ω_1, and Ω_3 is an R'-kernel of Ω_2, and Ω_1/Ω_2 and Ω_2/Ω_3 are R'-groups of type-2. In addition, both Ω_1 and Ω_2 are of type-$\mathcal{K}$.
(c) The subgroup $\Omega_3 = \Delta_2 \oplus_G S_2$ is the only non-monogenic R'-subgroup of Ω. In fact, Ω_3 is an R'-group of 0_2-form.
(d) The subgroup Δ_1 is an R'-group of $\mathcal{K}_2$-form but has S_1 as an R'-kernel. Thus Δ_1/S_1 is an R'-group of type-2.
(e) The R'-groups of type-0 are Δ_2, Δ_3, S_1 and S_2.

There is only one non-trivial Ω-chain, up to isomorphism,

$$\Omega \supsetneq \Omega_1 \supsetneq \Delta_3.$$

Thus the $\mathcal{KG}$-length of Ω is $\eta_{R'} = 1$. Also, we observe that the nil-rigid length of R' is $\alpha_{R'} = 2$.

Now consider the faithful $\mathbb{M}_2(R')$-group, Ω^2. Note that Ω_3^2 is an $\mathbb{M}_2(R')$-group of type-0. The chain of $\mathbb{M}_2(R')$-groups corresponding to $\Omega \supsetneq \Omega_1 \supsetneq \Delta_3$ is

$$\Omega^2 \supsetneq \Omega_1^2 \supsetneq \Delta_3^2. \tag{6.7}$$

Here, Ω_1^2 is maximal type-$\mathcal{K}$ $\mathbb{M}_2(R')$-subgroup of Ω^2, but (6.7) is not an (Ω^2)-chain because Δ_3^2 is not maximal amongst type-0 $\mathbb{M}_2(R')$-subgroups of

Ω_1^2. Therefore, Ω^2 has an (Ω^2)-chain,

$$\Omega^2 \supsetneq \Omega_1^2 \supsetneq \Omega_3^2 \supsetneq \Delta_1^2 \supsetneq S_2^2,$$

which is the only (Ω^2)-chain, up to isomorphism, hence a $\mathcal{KG}$-chain. We conclude, by Theorem 3.2.7, that the nil-rigid length of $\mathbb{M}_2(R')$ is $\alpha_{\mathbb{M}_2(R')} \geq 3$, giving $\alpha_{\mathbb{M}_n(R')} \gneq \alpha_{R'}$.

The next example involves a refinement of a $\mathcal{KG}$-chain. The near-ring, R, in this example is constructed such that $J_0(R) \neq (0)$ and $Soi(R) = (0)$. Also, Ω is not a member of any Ω-chain.

Example 6.3.2 Define the near-ring R by,
$R := \{ f \in M_0(\Omega) \mid f(\Omega_i) \subseteq \Omega_i,\ i = 1, 2, 3;$
$f(\Delta_j) \subseteq \Delta_j,\ j = 1, 2, 3;\ f(S_l) \subseteq S_l,\ l = 1, 2, 3;$
$\forall\, \omega, \omega' \in \Omega,\ \omega - \omega' \in \Omega_1 \Rightarrow f(\omega) - f(\omega') \in \Omega_1\,;$
$\forall\, \omega_1, \omega_2 \in \Omega_2,\ \omega_1 - \omega_2 \in \Omega_3 \Rightarrow f(\omega_1) - f(\omega_2) \in \Omega_3\,;$
$\forall\, \delta_1, \delta_2 \in \Delta_1,\ \delta_1 - \delta_2 \in S_1 \Rightarrow f(\delta_1) - f(\delta_2) \in S_1\,\}.$

Then R is a near-ring with multiplicative identity under point-wise addition and map composition.

Observe the following.
(a) The group Ω is a faithful R-group of type-$\mathcal{K}$, because it has the type-0 R-subgroup Ω_1 as an R-kernel. Therefore $J_0(R) \neq (0)$.
(b) The R-group Ω_3 is an R-kernel of Ω_2, and S_1 is an R-kernel of Δ_1. Therefore, Ω_2 and Δ_1 are of type-$\mathcal{K}$.

(c) The R-group $\Omega_3 = \Delta_2 \oplus_G S_2$ is the only non-monogenic R-subgroup of Ω. In fact, Ω_3 is an R-group of 0_2-form.
(d) Since all subgroups of Δ_1 are R-groups, Δ_1 is non-monogenic and of $\mathcal{K}_2$-form.
(e) The type-0 R-subgroups of Ω are Ω_1, Δ_2, Δ_3, S_1 and S_2.

There is only one non-trivial Ω-chain, which is thus a $\mathcal{KG}$-chain, that is

$$\Omega_1 \supsetneq \Omega_2 \supsetneq \Delta_3. \tag{6.8}$$

The $\mathcal{KG}$-length of R is $\eta_R = 1$. Hence, the nil-rigid length of R is $\alpha_R = 2$.

Consider Ω^2 as a faithful $\mathbb{M}_2(R)$-group. Note that Ω_3 and Δ_1 give a refinement of (6.8), where Ω_3 is of 0_2-form and Δ_1 is of $\mathcal{K}_2$-form.

An (Ω^2)-chain corresponding to $\Omega_1 \supsetneq \Omega_2 \supsetneq \Delta_3$ is

$$\Omega_1^2 \supsetneq \Omega_2^2 \supsetneq \Omega_3^2 \supsetneq \Delta_1^2 \supsetneq S_2^2,$$

which is the only (Ω^2)-chain, up to isomorphism, and hence a $\mathcal{KG}$-chain of Ω^2. Therefore, $\eta_{\mathbb{M}_2(R)} = 2$, and by Theorem 3.2.7, the nil-rigid length of $\mathbb{M}_2(R)$ is $\alpha_{\mathbb{M}_2(R)} \geq 3$. Thus $\alpha_{\mathbb{M}_n(R)} \gneq \alpha_R$.

Chapter 7

Conclusion

7.1 A brief summary of results

A special type of monogenic R-groups, called R-groups of type-$\mathcal{K}$, was introduced. The socle ideal was then characterized as an intersection of annihilators of R-groups of type-$\mathcal{K}$.

It has been shown that, for near-rings satisfying the DCCL, if

$$R = Soi(R) \oplus L,$$

then $(Soi(R))^*$ is the annihilating ideal of L^n. That is,

$$(Soi(R))^* = (0 : L^n).$$

It was also proved that the 0-radical, $J_0(R)$, and the s-radical, $J_s(R)$, are distinct if, and only if, there is an R-group of type-$\mathcal{K}$ contained in a type-0 R-group.

Alternating chains of type-0 and type-$\mathcal{K}$ R-subgroups, called Ω-chains, of a faithful R-group Ω, satisfying the DCCS, were defined. The maximal length of the Ω-chains is closely related to the nil-rigid length.

Non-monogenic R-groups, whose corresponding $\mathbb{M}_n(R)$-groups are monogenic, are classified into three ν_n-forms, $\nu = 0, s, \mathcal{K}$. It is shown that a monogenic R-group is of type-$\mathcal{K}$ if, and only if, its corresponding $\mathbb{M}_n(R)$-group is of type-$\mathcal{K}$.

The cognizance of R-groups of ν_n-form, and where they fit into Ω-chains, enables one to give a lower bound on the nil-rigid length of $\mathbb{M}_n(R)$, without having to first calculate each ideal in the series.

For a zero-symmetric near-ring with identity, and Ω a faithful R-group, let

$$I_\nu = \bigcap_{\Delta_i \in E_\nu(R)} (0 : \Delta_i), \quad \mathcal{P}(\nu) = \bigcap_{\Delta_i \in E_\nu(R)} (0 : \Delta_i^n),$$

where $E_\nu(R)$ is the class of all type-ν R-subgroups of Ω, $\nu = 0, s, 2, \mathcal{K}$, and

$$\mathcal{I}_\nu = \bigcap_{\Gamma_j \in \mathbb{E}_\nu(\mathbb{M}_n(R))} (0 : \Gamma_j),$$

where $\mathbb{E}_\nu(\mathbb{M}_n(R))$ is the class of all type-ν, $\nu = 0, s, 2, \mathcal{K}$, $\mathbb{M}_n(R)$-subgroups of Ω^2. Then the following relationships were shown,

$$I_\nu^+ \subseteq \mathcal{P}(\nu), \quad I_\nu^* \subseteq \mathcal{P}(\nu), \quad \mathcal{I}_\nu \subseteq \mathcal{P}(\nu).$$

These ideals, which are intersections of annihilators of monogenic R-groups or $\mathbb{M}_n(R)$-groups, are referred to as annihilator ideals. Using this concept, the following relationships were proved,

$$Soi(\mathbb{M}_n(R)) \subseteq (Soi(R))^*; \quad (J_s(R))^+ \subseteq J_s(\mathbb{M}_n(R))$$

where R satisfies the DCCL.

Although the s-socle is not an annihilator ideal, it was shown that

$$A^+ \subseteq \mathcal{A},$$

where A and $\mathcal{A}$ are the s-socle ideals of R and $\mathbb{M}_n(R)$, respectively.

A counter-example to a conjecture by Meldrum and Meyer in [17] is given. That is, a near-ring R with

$$(J_0(R))^+ \not\subset J_0(\mathbb{M}_n(R)),$$

is constructed.

Several examples are presented illustrating the lack of relationships,

$$(Soi(R))^+ \not\subset Soi(\mathbb{M}_n(R)); \quad A^* \not\subset \mathcal{A}, \text{ and } \mathcal{A} \not\subset A^*,$$

where A and $\mathcal{A}$ are s-socles of R and $\mathbb{M}_n(R)$, respectively.

We conclude that the only annihilator ideal which strictly satisfies the relationship

$$I_\nu^+ \subseteq \mathcal{I}_\nu \subseteq I_\nu^*$$

is the Jacobson s-radical. The Jacobson 2-radical fails simply because $J_2(\mathbb{M}_n(R)) = (J_2(R))^*$. That is, a strict inclusion never occurs.

Other examples illustrate,
(a) a typical R-group of type-$\mathcal{K}$, and a typical R-group of type-0 but not of type-s,
(b) how the $\mathcal{KG}$-length relates to the nil-rigid length,
(c) that the nil-rigid length of a near-ring can be strictly less than the nil-rigid length of its associated matrix near-rings.

We concluded that, besides the lack of one distributive law, or the lack of commutativity on the additive group $(R, +)$, the discrepancies between ideals of a near-ring and the corresponding ideals of the associated matrix near-rings are ascribable to the existence of R-groups of ν_n-form $(\nu = 0, s, \mathcal{K})$.

7.2 Some Open Questions

Several questions of interest still remain unanswered.

1. What are the necessary and sufficient conditions for an annihilator ideal I_ν, $\nu = 0, s, 2, \mathcal{K}$, of R to be such that

$$\mathcal{I}_\nu = I_\nu^*, \tag{7.1}$$

where $\mathcal{I}_\nu$ is the ideal of $\mathbb{M}_n(R)$ corresponding to I_ν?

The condition that every R-subgroup of a faithful R-group must be monogenic turns out to be too strong. In particular, the Jacobson 2-radical satisfies (7.1) regardless of whether every R-group of R is monogenic or not.

2. What are the relationships between any two of the quasi-radicals,

$$(J_{1/2}(R))^+, \quad J_{1/2}(\mathbb{M}_n(R)) \text{ and } (J_{1/2}(R))^* ?$$

There is a close connection between the left ideal $J_{1/2}(R)$ and the s-radical, $J_s(R)$. For example, it is well known that,

(a) for a near-ring R satisfying the DCCL, $J_{1/2}(R)$ is a two-sided ideal if, and only if, $J_0(R) = J_s(R)$, cf. [4, Theorem 2.12];

(b) For a near-ring satisfying the DCCS and having nil-rigid length $\alpha > 1$,

$$J_s(R) = J_{1/2}(R) + A + B,$$

where A is the s-socle of R, $B = J_s(R) \cap L_{\alpha-1}$, and $L_{\alpha-1}$ is the member of the nil-rigid series of R as in Theorem 2.3.19.

3. When is the s-socle an annihilator ideal? This can be rephrased as: when is the s-socle, A, equal to the intersection $J_s(R) \cap C_{\alpha-1}$, where the nil-rigid length of R is $\alpha > 2$?

4. Is there an example of a near-ring R with

$$J_0(\mathbb{M}_n(R)) \neq J_{1/2}(\mathbb{M}_n(R)) \neq J_s(\mathbb{M}_n(R)) \neq J_2(\mathbb{M}_n(R))$$

while

$$J_0(R) = J_{1/2}(R) = J_s(R) = J_2(R)?$$

5. What conditions are needed on the near-ring, R, for $\alpha_R = \eta_R + 1$?

6. Can the theory presented here be extended to other radicals?

7. By considering a group near-ring, $R[G]$, as a subnear-ring of a matrix near-ring $\mathbb{M}_n(R)$, if $|G| = n$, see Meyer [22], can the theory developed here be transfered to the context of group near-rings?

Appendix A

Action of $\mathbb{M}_n(R)$ on factor-groups

Consider the group, $(\Gamma/\Delta)^n$, where $\Gamma = R\gamma$ is a monogenic R-group and Δ an R-kernel of Γ. Let

$$\langle r_1\gamma + \Delta,\, r_2\gamma + \Delta,\, \ldots\, , r_n\gamma + \Delta\rangle \in (\Gamma/\Delta)^n,$$

and $U \in \mathbb{M}_n(R)$.

Define the following action of $\mathbb{M}_n(R)$ on $(\Gamma/\Delta)^n$ as

$$U\langle r_1\gamma{+}\Delta,\, r_2\gamma{+}\Delta,\, \ldots\, , r_n\gamma{+}\Delta\rangle := \langle s_1\gamma{+}\Delta,\, s_2\gamma{+}\Delta,\, \ldots\, , s_n\gamma{+}\Delta\rangle, \quad \text{(A.1)}$$

where

$$U\langle r_1, r_2, \ldots, r_n\rangle = \langle s_1, s_2, \ldots, s_n\rangle,$$

for some $s_i \in R,\ 1 \leq i \leq n$.

To show that action (A.1) is well-defined, suppose

$$\langle r_1\gamma + \Delta,\, r_2\gamma + \Delta,\, \ldots\, , r_n\gamma + \Delta\rangle \,=\, \langle a_1\gamma + \Delta,\, a_2\gamma + \Delta,\, \ldots\, , a_n\gamma + \Delta\rangle,$$

for $r_i, a_i \in R,\ 1 \leq i \leq n$. It follows that $r_i\gamma + \Delta \,=\, a_i\gamma + \Delta,\ \ 1 \leq i \leq n$. But, for any $r \in R$, we have $rr_i\gamma + \Delta \,=\, ra_i\gamma + \Delta$, hence

$$rr_i\gamma \,-\, ra_i\gamma \,=\, (rr_i \,-\, ra_i)\gamma \,\in\, \Delta,\, 1 \leq i \leq n. \tag{A.2}$$

We now prove, by induction on the weight of U, that the action (A.1) is well-defined.

Let $U \,=\, f^r_{ij}$, then

$$U\langle r_1, r_2, \ldots, r_n\rangle \,=\, \langle 0, \ldots, 0, rr_j, 0, \ldots, 0\rangle$$

and

$$U\langle a_1, a_2, \ldots, a_n\rangle \,=\, \langle 0, \ldots, 0, ra_j, 0, \ldots, 0\rangle,$$

where both rr_j and ra_j appear in i-th places. Thus

$$U\langle r_1\gamma + \Delta,\, r_2\gamma + \Delta,\, \ldots\, , r_n\gamma + \Delta\rangle \,=\, \langle \Delta, \ldots, \Delta, rr_j\gamma + \Delta, \Delta, \ldots, \Delta\rangle$$

and

$$U\langle a_1\gamma + \Delta,\, a_2\gamma + \Delta,\, \ldots\, , a_n\gamma + \Delta\rangle \,=\, \langle \Delta, \ldots, \Delta, ra_j\gamma + \Delta, \Delta, \ldots, \Delta\rangle$$

where both $rr_j\gamma + \Delta$ and $ra_j\gamma + \Delta$ appear in i-th places.

Hence, by (A.2),

$$U\langle r_1\gamma + \Delta,\, r_2\gamma + \Delta,\, \ldots\, , r_n\gamma + \Delta\rangle \,-\, U\langle a_1\gamma + \Delta,\, a_2\gamma + \Delta,\, \ldots\, , a_n\gamma + \Delta\rangle$$

$$= \langle \Delta, \ldots, \Delta, (rr_j - ra_j)\gamma + \Delta, \Delta, \ldots, \Delta \rangle = \langle \Delta, \Delta, \ldots, \Delta \rangle.$$

Let $U = f_{ij}^{b_1} + f_{il}^{b_2}$, then

$$U\langle r_1, r_2, \ldots, r_n \rangle = \langle 0, \ldots, 0, b_1 r_j + b_2 r_l, 0, \ldots, 0 \rangle$$

and

$$U\langle a_1, a_2, \ldots, a_n \rangle = \langle 0, \ldots, 0, b_1 a_j + b_2 a_l, 0, \ldots, 0 \rangle,$$

where both $b_1 r_j + b_2 r_l$ and $b_1 a_j + b_2 a_l$ appear in i-th places.

Hence

$$U\langle r_1\gamma+\Delta, r_2\gamma+\Delta, \ldots, r_n\gamma+\Delta \rangle = \langle \Delta, \ldots, \Delta, (b_1 r_j + b_2 r_l)\gamma+\Delta, \Delta, \ldots, \Delta \rangle$$

and

$$U\langle a_1\gamma+\Delta, a_2\gamma+\Delta, \ldots, a_n\gamma+\Delta \rangle = \langle \Delta, \ldots, \Delta, (b_1 a_j + b_2 a_l)\gamma+\Delta, \Delta, \ldots, \Delta \rangle,$$

where both $(b_1 r_j + b_2 r_l)\gamma + \Delta$ and $(b_1 a_j + b_2 a_l)\gamma + \Delta$ appear in i-th places. It now follows by (A.2) and the first step of induction that

$$U\langle r_1\gamma + \Delta, r_2\gamma + \Delta, \ldots, r_n\gamma + \Delta \rangle - U\langle a_1\gamma + \Delta, a_2\gamma + \Delta, \ldots, a_n\gamma + \Delta \rangle$$

$$= \langle \Delta, \ldots, \Delta, \big((b_1 r_j + b_2 r_l) - (b_1 a_j + b_2 a_l)\big)\gamma + \Delta, \Delta, \ldots, \Delta \rangle$$

$$= \langle \Delta, \ldots, \Delta, \big(b_1 r_j + b_2 r_l - b_2 a_l - b_1 a_j\big)\gamma + \Delta, \Delta, \ldots, \Delta \rangle = \langle \Delta, \Delta, \ldots, \Delta \rangle.$$

Assume that the action (A.1) of $\mathbb{M}_n(R)$ on $(\Gamma/\Delta)^n$ is well-defined for any matrix of weight less than m, where m is a positive integer.

Let V and W be matrices each of weight less than m, and write
$V\langle r_1, r_2, \ldots, r_n\rangle = \langle r'_1, r'_2, \ldots, r'_n\rangle$, $V\langle a_1, a_2, \ldots, a_n\rangle = \langle a'_1, a'_2, \ldots, a'_n\rangle$,
$W\langle r'_1, r'_2, \ldots, r'_n\rangle = \langle r''_1, r''_2, \ldots, r''_n\rangle$ and $W\langle a'_1, a'_2, \ldots, a'_n\rangle = \langle a''_1, a''_2, \ldots, a''_n\rangle$.

Suppose $U = V + W$. It follows from by the induction hypothesis that

$$U\langle r_1\gamma + \Delta,\, r_2\gamma + \Delta,\, \ldots\, , r_n\gamma + \Delta\rangle - U\langle a_1\gamma + \Delta,\, a_2\gamma + \Delta,\, \ldots\, , a_n\gamma + \Delta\rangle$$

$$= \big(\langle r'_1\gamma + \Delta,\, r'_2\gamma + \Delta,\, \ldots\, , r'_n\gamma + \Delta\rangle + \langle r''_1\gamma + \Delta,\, r''_2\gamma + \Delta,\, \ldots\, , r''_n\gamma + \Delta\rangle\big)$$

$$- \big(\langle a'_1\gamma + \Delta,\, a'_2\gamma + \Delta,\, \ldots\, , a'_n\gamma + \Delta\rangle + \langle a''_1\gamma + \Delta,\, a''_2\gamma + \Delta,\, \ldots\, , a''_n\gamma + \Delta\rangle\big)$$

$$= \langle (r'_1 + r''_1 - a''_1 - a'_1)\gamma + \Delta, (r'_2 + r''_2 - a''_2 - a'_2)\gamma + \Delta, \ldots, (r'_n + r''_n - a''_n - a'_n)\gamma + \Delta\rangle$$

$$= \langle (r'_1 - a'_1)\gamma + \Delta, (r'_2 - a'_2)\gamma + \Delta, \ldots, (r'_n - a'_n)\gamma + \Delta\rangle = \langle \Delta, \Delta, \ldots, \Delta\rangle.$$

Now suppose $U = WV$. It follows again by the induction hypothesis that

$$U\langle r_1\gamma + \Delta,\, r_2\gamma + \Delta,\, \ldots\, , r_n\gamma + \Delta\rangle - U\langle a_1\gamma + \Delta,\, a_2\gamma + \Delta,\, \ldots\, , a_n\gamma + \Delta\rangle$$

$$= W\langle r'_1\gamma + \Delta,\, r'_2\gamma + \Delta,\, \ldots\, , r'_n\gamma + \Delta\rangle - W\langle a'_1\gamma + \Delta,\, a'_2\gamma + \Delta,\, \ldots\, , a'_n\gamma + \Delta\rangle$$

$$= \langle r''_1\gamma + \Delta,\, r''_2\gamma + \Delta,\, \ldots\, , r''_n\gamma + \Delta\rangle - \langle a''_1\gamma + \Delta,\, a''_2\gamma + \Delta,\, \ldots\, , a''_n\gamma + \Delta\rangle$$

$$= \langle (r''_1 - a''_1)\gamma + \Delta, (r''_2 - a''_2)\gamma + \Delta, \ldots, (r''_n - a''_n)\gamma + \Delta\rangle = \langle \Delta, \Delta, \ldots, \Delta\rangle.$$

Therefore action (A.1) is well-defined. □

Appendix B

Structure of ideals of the form I^+

Let I and J be non-trivial ideals of a zero-symmetric right near-ring, R, with identity, such that $J \supseteq I$. The zero-element of $(J/I)^+ \subsetneq \mathbb{M}_n(R/I)$ is denoted by $\underline{0}$.

Let $x, x_{ij}, z \in J,\ i,j = 1,2,\ldots,n,\ \ r \in R,$ and consider the map

$$\Phi : J^+ \to (J/I)^+$$

defined recursively by

$$\sum_{i,j=1}^{n} f_{ij}^{x_{ij}} \mapsto \sum_{i,j=1}^{n} f_{ij}^{(x_{ij}+I)},$$

and

$$f_{p\mu}^{r}\left(\sum_{i,j=1}^{n} f_{ij}^{x_{ij}}\right) \mapsto f_{p\mu}^{(r+I)}\left(\sum_{i,j=1}^{n} f_{ij}^{(x_{ij}+I)}\right).$$

Note. By right distributivity,

$$\left(\sum_{p,q=1}^{n} f_{pq}^{r_{pq}}\right)\left(\sum_{i,j=1}^{n} f_{ij}^{r_{ij}}\right) = \sum_{p,q=1}^{n}\left(f_{pq}^{r_{pq}}\sum_{i,j=1}^{n} f_{ij}^{r_{ij}}\right). \tag{B.1}$$

We now give some properties of the generating set,

$$\{\, f_{ij}^{a} \,|\, a \in I,\, 1 \le i,j \le n \,\},$$

of I^+, in line with Section E in Chapter 1 of [19].

(a) For $a \in I$ and $r \in R$ with $a + r \in I$, we note that

$$f_{ij}^{a} + f_{ij}^{r} \in I^+,$$

however, if $p \neq i$ or $\mu \neq j$, then

$$f_{p\mu}^{a} + f_{ij}^{r} \in I^+,$$

only if $r \in I$.

(b) For $a, b \in I$ and $r \in R$ with $a + r - b \in I$, we note that

$$f_{ij}^{a} + f_{ij}^{r} - f_{ij}^{b} \in I^+,$$

but if $p \neq i \neq l$ or $\mu \neq j \neq q$, then

$$f_{p\mu}^{a} + f_{ij}^{r} - f_{lq}^{b} \in I^+,$$

only if $r \in I$.

(c) For $a \in I$ and $r, r' \in R$ with $r(r'+a) - rr' \in I$, we note that

$$f_{ij}^{r(r'+a)} + f_{ij}^{rr'} \in I^+,$$

whereas, if $p \neq i$ or $\mu \neq j$, then

$$f_{p\mu}^{r(r'+a)} + f_{ij}^{rr'} \in I^+,$$

only if $r \in I$ or $r' \in I$.

(d) For any $\bar{\rho} = \langle r_1, r_2, \ldots, r_n \rangle$, $r, r' \in R$ and $z \in J$, let $U := f_{p\mu}^{r} \left(f_{ij}^{r'} + f_{lq}^{z} \right) - f_{p\mu}^{r} f_{ij}^{r'}$, then

$$U\bar{\rho} = \begin{cases} \langle 0, 0, \ldots, 0 \rangle, & \text{if } \mu \neq i \text{ and } \mu \neq l \\ \langle 0, \ldots, 0, rr'r_j - rr'r_j, 0, \ldots, 0 \rangle, & \text{if } \mu = i \text{ and } \mu \neq l \\ \langle 0, \ldots, 0, rzr_q, 0, \ldots, 0 \rangle, & \text{if } \mu \neq i \text{ and } \mu = l \\ \langle 0, \ldots, r(r'r_j + zr_q) - r(r'r_j), 0, \ldots, 0 \rangle, & \text{if } \mu = i = l \text{ and } j \neq q \\ \langle 0, \ldots, r(r'+z)r_j - rr'r_j, 0, \ldots, 0 \rangle, & \text{if } \mu = i = l \text{ and } j = q. \end{cases}$$

Now U can be written in terms of the generators, $\{ f_{ij}^{a} \mid a \in R,\ 1 \leq i, j \leq n \}$, of $\mathbb{M}_n(R)$ as

$$U = \begin{cases} f_{ij}^{0}, & \text{if } \mu \neq i \text{ and } \mu \neq l \\ f_{ij}^{0}, & \text{if } \mu = i \text{ and } \mu \neq l \\ f_{pq}^{rz}, & \text{if } \mu \neq i \text{ and } \mu = l \\ f_{p\mu}^{r}\left(f_{\mu j}^{r'} + f_{\mu q}^{z} \right) - f_{p\mu}^{r} f_{\mu j}^{r'}, & \text{if } \mu = i = l \text{ and } j \neq q \\ f_{pq}^{r(r'+z)-rr'}, & \text{if } \mu = i = l \text{ and } j = q. \end{cases}$$

Note that, in the fourth case, i.e., $U = f_{p\mu}^{r}\left(f_{\mu j}^{r'} + f_{\mu q}^{z} \right) - f_{p\mu}^{r} f_{\mu j}^{r'}$, we have

$$U\bar{\rho} = \langle 0, \ldots, r(r'r_j + zr_q) - r(r'r_j), 0, \ldots, 0 \rangle.$$

If $U \in J^+ \cap I^*$ then $r(r'r_j + zr_q) - rr' \in I$, for all $r_j, r_q \in R$. In particular, for $r_j = 1_R = r_q$, we have

$$r(r'1_R + z1_R) - r(r'1_R) = r(r' + z) - rr' \in I.$$

Hence, $U = f^r_{p\mu}\big(f^{r'}_{\mu j} + f^z_{\mu q}\big) - f^r_{p\mu} f^{r'}_{\mu j} \in I^+$.

Thus, in general, if $U := f^r_{p\mu}\left(f^{r'}_{ij} + f^z_{lq}\right) - f^r_{p\mu} f^{r'}_{ij} \in J^+ \cap I^*$, then $U \in I^+$.

As in Section E of Chapter 1 of [19], the ideal,

$$J^+ = Id\{\, f^\sigma_{ij} \,|\, \sigma \in J, 1 \le i, j \le n\},$$

is constructed by the following recursive rules:

(i) $f^a_{ij} \in J^+$, for all $a \in J$;

(ii) If $f^a_{ij}, f^b_{lq} \in J^+$, then $f^a_{ij} + f^b_{lq} \in J^+$;

(iii) If $f^a_{ij} \in J^+$ and $f^r_{p\mu} \in \mathbb{M}_n(R)$, then $f^a_{ij} f^r_{p\mu} \in J^+$;

(iv) If $f^a_{ij} \in J^+$ and $f^r_{p\mu} \in \mathbb{M}_n(R)$, then $f^a_{ij} + f^r_{p\mu} - f^a_{ij} \in J^+$;

(v) If $f^a_{ij} \in J^+$ and $f^r_{p\mu}, f^{r'}_{lq} \in \mathbb{M}_n(R)$, then

$$f^r_{p\mu}\left(f^{r'}_{ij} + f^a_{lq}\right) - f^r_{p\mu} f^{r'}_{ij} \in J^+;$$

(vi) Nothing else is in J^+.

Proposition B.0.1 *Let R be a zero-symmetric right near-ring with identity, and with non-trivial ideals I and J such that $J \supseteq I$. Let Φ be as defined above, then $ker(\Phi) \subseteq I^+$.*

Proof Let U be an element of $ker(\Phi) = J^+ \cap I^*$. That is, U is an

matrix in $\mathbb{M}_n(R)$ such that $\Phi(U) = \underline{0}$. We prove the result by induction on the weight of U.

Let $U = f_{ij}^x$, then

$$\Phi(U) = f_{ij}^{(x+I)} = f_{ij}^{(0+I)} = \underline{0},$$

which implies $x \in I$ and hence $U \in I^+$.

Let $U = f_{ij}^x + f_{lq}^z$, then

$$\Phi(U) = f_{ij}^{(x+I)} + f_{lq}^{(z+I)} = \underline{0}.$$

If $i = l$ and $j = q$, then $\Phi(U) = f_{ij}^{(x+z)+I} = \underline{0}$, hence $x + z \in I$, and $U \in I^+$. If $i \neq l$ or $j \neq q$, then $\Phi(U) = f_{ij}^{(x+I)} + f_{lq}^{(z+I)} = f_{ij}^{(0+I)} + f_{lq}^{(0+I)}$. Thus, $x \in I$ and $z \in I$, hence $U \in I^+$.

Let $U = f_{p\mu}^r \left(f_{ij}^{r'} + f_{lq}^z\right) - f_{p\mu}^r f_{ij}^{r'}$, then

$$\Phi(U) = \begin{cases} f_{ij}^{(0+I)}, & \text{if } \mu \neq i \text{ and } \mu \neq l \\ f_{ij}^{(0+I)}, & \text{if } \mu = i \text{ and } \mu \neq l \\ f_{pq}^{(rz+I)}, & \text{if } \mu \neq i \text{ and } \mu = l \\ f_{p\mu}^{(r+I)}\left(f_{\mu j}^{(r'+I)} + f_{\mu q}^{(z+I)}\right) - f_{p\mu}^{(r+I)} f_{\mu j}^{(r'+I)}, & \text{if } \mu = i = l \text{ and } j \neq q \\ f_{pq}^{(r(x+z)+I)}, & \text{if } \mu = i = l \text{ and } j = q. \end{cases}$$

In the fourth case of $\Phi(U)$, we have

$$f_{p\mu}^{(r+I)}\left(f_{\mu j}^{(r'+I)} + f_{\mu q}^{(z+I)}\right) - f_{p\mu}^{(r+I)} f_{\mu j}^{(r'+I)} = \underline{0},$$

only if

$$(r+I)\big((r'+I)+(z+I)\big) - (r+I)(r'+I) = \big(r(r'+z) - rr'\big) + I = 0 + I = \underline{0}.$$

That is, $r(r'+z)-rr' \in I$, in which case, $U \in I^+$. Thus, for all values of the indices, p, μ, i, j, l, and q, we have that

$$U = f^r_{p\mu}\left(f^{r'}_{ij} + f^z_{lq}\right) - f^r_{p\mu}f^{r'}_{ij} \in ker(\Phi) \text{ implies } U \in I^+.$$

Assume that, for any matrix, $U \in \mathbb{M}_n(R)$, of weight less than m, where m is a positive integer, if $U \in ker(\Phi)$, then $U \in I^+$.

Let U and V be matrices in $ker(\Phi)$, each of weight less than m. By the induction hypothesis, $U \in I^+$ and $V \in I^+$. It follows by the way in which I^+ is generated that $U+V \in I^+$ and $UV \in I^+$. □

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