BOUNDS FOR THE RELIABILITY OF BINARY COHERENT SYSTEMS

Pumeza Yvonne Mteza

A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfillment of the requirements for the degree of Master of Science.

Johannesburg, 5 September 2013

Contents

DECLARATION

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

____
(Signature of candidate)

________________________day of_________________2013

ABSTRACT

This dissertation aims to conduct a comprehensive and up to date review of reliability bounds for the binary coherent systems. We assume that the components work independently of each other. The bounds are compared numerically by using simple structures.

DEDICATION

"Education is the most powerful weapon which you can use to change the world"- Nelson Mandela

I dedicate this dissertation to my two nieces, Zusiphe and Lilitha Mteza. I will die telling them that Education is the most powerful weapon they can use to conquer many obstacles that will be on their way as they grow up.

ACKNOWLEDGEMENTS

Had it not been for God on my side, I would have never reached this point in my academic path. At those times when I felt my strength was gone, His grace was sufficient for me, for His power is perfected in weakness.

I acknowledge the support I always get from my family. They stood by me through thick and thin. I will forever be grateful to God for seeing it fit that I should be raised by such a loving and caring family. They are the only people who always believe in me, and support me even if they don't understand some of my decisions. They have walked paths with me, where no one ever dared to walk. For this I say "THANK YOU" Antana (my mom), Libode (my dad) and Pozisa (my sister).

There can never be proper words to describe the gratitude I have towards my supervisor, Professor Frank Beichelt. His dedication , patience and support in supervising this dissertation goes beyond my expectations. The skills he taught me are life skills, that won't only benefit me in my career, but in my personal life as well.

Many thanks go to Professor Peter Fridjhon and Professor Jacqueline Galpin who both worked hard to make sure that I get funding for my studies. Their selfless support has never gone unnoticed.

I will forever be indebted to Nedbank and National Research Foundation for their financial support. Without these two organizations, furthering my studies would have been a mission impossible.

List of Figures

List of Tables

1 INTRODUCTION

1.1 General Introduction

Reliability theory evolved in world war II. Since then no other branch of engineering science, with the exception of computer technology and environmental engineering, has developed and advanced so fast as reliability engineering.This area has been developed to meet the needs of modern technology, particularly in electronics, communication, computer and transportation networks, control systems, aviation, space technology et al. The technological development in these areas has resulted in the creation of enormous variety of complex, sophisticated and automated devices. Most of these devices do not come cheap, and our reliance on them is continually increasing. Hence there is an increasing demand that these devices are of high reliability, particularly in areas where replacement costs are prohibitive (e. g. space and undersea).

Most devices are subject to random failures. Whether these failures simply increase costs and inconvenience or gravely threaten the public safety, the user demands assurance that the device he uses is dependable and efficient. In particular, high risk systems such as nuclear power plants and some chemical plants require operational safety of the highest order. A recent example of a catastrophic failure occurred in Japan's Fukushima Daiichi nuclear plant, caused by earthquakes. Although most experts believed that the woes would not produce major health or environmental consequences, they all concluded that the clean-up would cost Tokyo billions of Yen. But even producers of common goods such as electric irons or cell phones have to take into account reliability and quality aspects, otherwise competitors will win the economic battle. With so many of these products nowadays available in the market, if a brand is known to be unreliable, chances of it being picked up are slim. That is why in any engineering design, reliability is a key factor.

In all international standards, *reliability* of a device or a process is essentially defined as 'its suitability to adequately perform its defined purpose under specified operating conditions over a given time interval'. Unfortunately, this concept of reliability is not directly quantifiable. Hence, mathematically strictly defined concepts, so called *reliability criteria*, have to be introduced. The most important ones are:

- *Survival probability* of a system, i.e. the probability that the system does not fail within a given time interval.
- *Availability* of a system, i.e. the probability that the system is working at a given time point (or able to do its job when needed).
- *Mean lifetime* of a system, i.e. the expected time to its first failure.

Computing reliability criteria of a system based on reliability criteria of its components has been the key problem in reliability theory from the beginning. Up to now no computational techniques for exactly solving this problem have been developed, which can handle systems of more than 500 components. This is because this problem belongs from the computational point of view to the most difficult ones, namely to the NP-hard problems. As a consequence of this property, with increasing number n of components the computation time for determining the system reliability from the reliability of its components generally increases exponentially fast. Hence algorithms for the exact system reliability may require substantial computation even with the aid of a most up to date computer (in this case the amount of storage capacity required is often larger than the one which is available). Even when there is enough storage, the accumulation of round-off errors due to the excessive computation required can lead to inaccurate results.

To tackle the problem, special attention has been given to the development of computationally efficient approximative algorithms for the system reliability. One approximation method which has been found to be particularly suitable is the bounding method. This involves finding lower and upper bounds for system reliability. In this study we give a comprehensive survey on these bounds for binary coherent systems, and will carry out detailed numerical comparisons, which enable us to give recommendations under which assumptions on the component reliabilities which bounds are preferable to other ones.

1.2 Aims of Study

In this study we aim to cover the following:

1. Up to date survey on bounds for the reliability of binary coherent systems.
2. Comparison of these bounds under qualitative, quantitative and efficiency aspects.
3. Recommendations to the reliability engineer on which bounds to apply under what conditions.
4. Suggestions for further theoretical and numerical research.

The dissertation is organized as follows: Chapter 2 gives the necessary theoretical foundations of binary coherent systems. Chapter 3 reviews the most important reliability bounds found in the literature. Chapter 4 provides numerical comparisons of the reliability bounds, and finally chapter 5 gives the key references.

2 THEORETICAL FOUNDATIONS- BINARY COHERENT SYSTEMS

In this section we consider the structural relationship between a system and its components from the reliability point of view. We will consider the system at a fixed time point and that the present state of the system depends only on the present states of its components. We will consider only independent components because most of the times explicit numerical results can only be obtained under the independence assumption.

2.1 Basic concepts

Let a complex technical system S consist of n components $c_1, c_2, \ldots, c_n$. We consider systems and components that can only be in one of the two states: working or failed. Hence, we can assign to each component c_i and system S Boolean (binary) indicator variables for these states:

$$x_i = \begin{cases} 1 & \text{if component } c_i \text{ is working} \\ 0 & \text{if component } c_i \text{ has failed} \end{cases},$$

for $i = 1, 2, \ldots, n$ and

$$x_s = \begin{cases} 1 & \text{if system } S \text{ is working} \\ 0 & \text{if system } S \text{ has failed} \end{cases}.$$

We assume that the states of the components uniquely determine the state of the system. Hence the relationship between the state of the system and its components is given by the so- called structure function φ :

$$x_s = \varphi(\mathbf{x}) \qquad \mathbf{x} \in \mathbf{V}_n ,$$

where an n- tuple $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is called the state vector of the system and expresses, for a given realization of the system, the state that each component c_i of the system is in. $\mathbf{V}_n$ denotes the set of all those n- dimensional vectors the components of which are 0 or 1. Obviously, the cardinality of $\mathbf{V}_n$ is 2^n.

Boolean operators

Let x and y be any Boolean variables. Conjunction and disjunction of x and y and the negation of x are defined as follows:

i. Conjunction Boolean operator $\wedge$

$$x \wedge y = xy = \begin{cases} 1 & \text{if } x = y = 1 \\ 0 & \text{otherwise} \end{cases} .$$

ii. Disjunction Boolean operator $\vee$

$$x \vee y = x + \bar{x}y = x + y - xy = \begin{cases} 0 & \text{if } x = y = 0 \\ 1 & \text{otherwise} \end{cases} \tag{2.1}$$

iii. Negation

$$\bar{x} = 1 - x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases} .$$

We note that

$$x \wedge y = \min(x, y), \quad x \vee y = \max(x, y).$$

Generalization:

Let $x_1, x_2, \ldots, x_n$ be n Boolean variables. Then

$$\bigwedge_{i=1}^{n} x_i = x_1 \wedge x_2 \wedge \ldots \wedge x_n = \begin{cases} 1 & \text{if } x_1 = x_2 = \cdots = x_n = 1 \\ 0 & \text{otherwise} \end{cases} ,$$

$$\bigvee_{i=1}^{n} x_i = x_1 \vee x_2 \vee \ldots \vee x_n = \begin{cases} 0 & \text{if } x_1 = x_2 = \cdots = x_n = 0 \\ 1 & \text{otherwise} \end{cases} .$$

Equivalently,

$$\bigwedge_{i=1}^{n} x_i = x_1 x_2 \ldots x_n = \min(x_1, x_2, \ldots, x_n),$$

$$\bigvee_{i=1}^{n} x_i = 1 - \bar{x}_1 \, \bar{x}_2 \, \ldots \bar{x}_n = \max(x_1, x_2, \ldots, x_n).$$

Reliability block diagrams

The relationship between the reliability of the system and the reliabilities of its components is graphically represented by so-called reliability block diagrams. A reliability block diagram is a graph with an entrance node (entrance vertex) and an exit node (exit vertex), and other nodes in-between. The edges of the graph represent the components of the system. But, depending on the way the graphical representation is being done, there may be edges which do not symbolize components, but which are necessary to fully determine the structure of the system and which are thought to be always available (see Figure 2.3 below). Nodes represent stations, assumed to be completely reliable, and edges represent unreliable connections, that is, connections which are available on the average only for a given percentage of time. The system is operating if and only if there is at least one path from the entrance node to the exit node with all components, symbolized by the edges of this path, operating.

In a reliability block diagram, edges are generally directed, but some may be undirected. The directions of the edges need not be explicitly marked if they are obvious. For instance, in Figure 2.1 only c_3 is undirected, but the other edges have an obvious direction. Undirected edges allow for paths from the entrance node to the exit node which pass these edges in opposite directions.

Figure 2.1 below is a bridge structure. $c_1, c_2, \ldots, c_5$ are the edges and the dots are the nodes. c_3 is undirected. Entrance node is indicated by "EN" and exit node by "EX". This is the structure we will use in most of our examples.

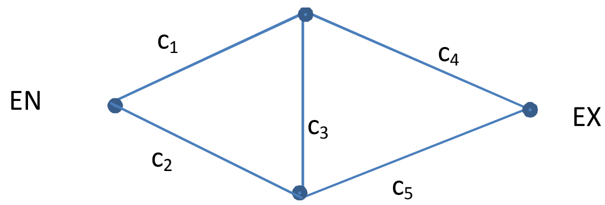

Figure 2.1: Bridge structure

Special systems

There are some special systems that are famous among reliability analysts. These are (1) series, (2) parallel, and (3) k-out-of-n systems.

(1) Series system

A series system functions if and only if all of its components function. Its structure function is

$$\varphi(\mathbf{x}) = \bigwedge_{i=1}^{n} x_i = \prod_{i=1}^{n} x_i = \min(x_1, x_2, \ldots, x_n).$$

Hence, the reliability block diagram of a series system (Figure 2.2) consists only of one path from the entrance node to the exit node.

Figure 2.2: Series system

Systems that do not require that all of its components must be working for the system to be working are called (structural) redundant. In redundant systems, a failure of a component may not lead to system failure. Thus, when a redundant system is working, some of its components are active, whereas other components may not or only partially included in keeping the system operating. Hence, at a given point in time, components of a redundant system can be classified as follows:

(a) Active or working components.

(b) Redundant components.

The redundant components may be in cold redundancy, that is if they are not subjected to any stress, and therefore cannot fail. Components that are exposed to a lower stress level than active components are said to be in warm redundancy. Components in warm redundancy may fail, but with lower probability than when being fully active. Components in hot redundancy fully participate in the work of the system and, hence, they are exposed to the same stress level as active components.

The two special systems below are examples of redundant systems.

(2) Parallel system

A parallel system functions if at least one of its components functions (Figure 2.3). Its structure function is

$$\varphi(\mathbf{x}) = \bigvee_{i=1}^{n} x_i = \coprod_{i=1}^{n} x_i = \max(x_1, x_2, \ldots, x_n),$$

where $\coprod_{i=1}^{n} x_i = 1 - \prod_{i=1}^{n} \bar{x}_i$.

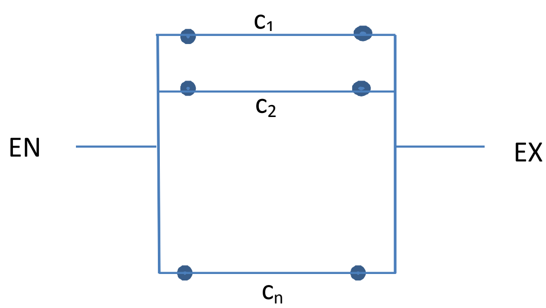

Figure 2.3: Parallel system

We assume that if a new parallel system starts operating, all of its components are active. The system fails as soon as the last of its components fails.

To include redundancy in a system, engineers frequently design *series- parallel -systems* (Figure 2.4), i.e, parallel systems connected in series or *parallel-series-systems* (Figure 2.5), i.e, series systems connected in parallel.

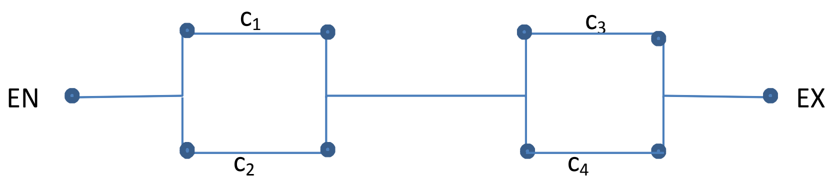

Figure 2.4: Series-parallel system

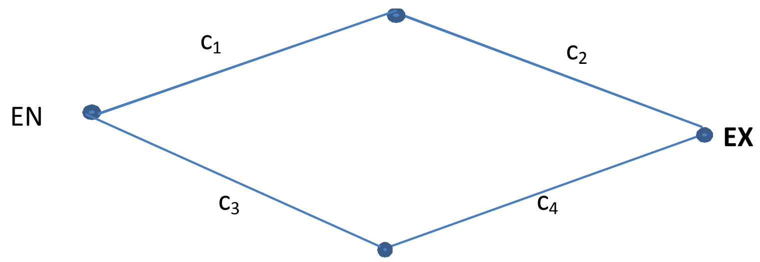

Figure 2.5: Parallel-series system

(3) *k-out-of-n* system

A *k-out-of-n* system functions if and only if at least k of the n components function. Its structure function is

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0 & \text{otherwise} \end{cases} .$$

Note that the series system is an *n-out-of-n* system and a parallel system is a 1-*out-of-n* system.

2.2 Coherent (Monotone) systems

Notation 1 $(0_i, \mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$
$(1_i, \mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$

This means that the state of component c_i is fixed.

Definition 1 *Component c_i is irrelevant if*

$$\varphi(1_i, \mathbf{x}) = \varphi(0_i, \mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{V}_n.$$

Otherwise c_i is relevant, $i = 1, 2, \ldots, n$.

This means that functioning or failure of an irrelevant component does not increase or decrease the reliability of the system.

Component c_5 in the reliability block diagram (Figure 2.6) below is an irrelevant component.

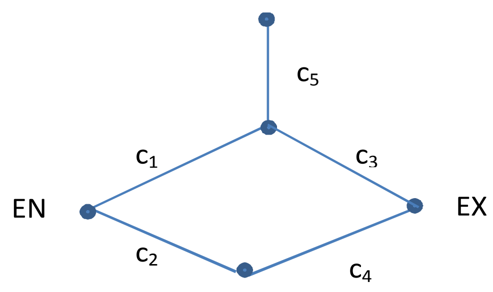

Figure 2.6: Example of an irrelevant component

Since irrelevant components can be deleted from a system without decreasing its reliability, we will generally assume that all components in our systems are relevant.

Definition 2 *A binary system with structure function* $\varphi(\mathbf{x}), \mathbf{x} \in \mathbf{V}_n$, *is called coherent or monotone if:*

(a) *every component is relevant, and*

(b) *The structure is non- decreasing, i.e.* $\varphi(1_i, \mathbf{x}) \leq \varphi(0_i, \mathbf{x})$, *for all* $\mathbf{x} \in \mathbf{V}_n$.

Roughly, a coherent system is the one whose reliability does not decrease when failed components are replaced by functioning ones and the failure of components does not increase the reliability of the system.

Let $\mathbf{1} = (1, 1, \ldots, 1)$ and $\mathbf{0} = (0, 0, \ldots, 0)$. From definition 2, the structure function of any coherent system S has the following two properties:

(1) $\varphi(\mathbf{1}) = 1, \quad \varphi(\mathbf{0}) = 0$

(2) For all $\mathbf{x} \in \mathbf{V}_n$,

$$\prod_{i=1}^{n} x_i \leq \varphi(\mathbf{x}) \leq \coprod_{i=1}^{n} x_i \tag{2.2}$$

To verify the two properties above, let us assume that $\varphi(\mathbf{0}) = 1$. Then since φ is non-decreasing, $\varphi(\mathbf{1}) = \varphi(\mathbf{0}) = 1$. But this implies that all components of S are irrelevant, contrary to (a). Again assume that $\varphi(\mathbf{1}) = 0$. Then since φ is non-decreasing, $\varphi(\mathbf{0}) = \varphi(\mathbf{1}) = 0$. But this implies that all components of S are irrelevant, contrary to (a). For the second property, the left-hand side of (2.2) is trivial if $\prod_{i=1}^{n} x_i = 0$. If $\prod_{i=1}^{n} x_i = 1$, then $\mathbf{x} = \mathbf{1}$ so that by (1) $\varphi(\mathbf{1}) = 1$. The right-hand side is trivial if $\coprod_{i=1}^{n} x_i = 1$. If $\coprod_{i=1}^{n} x_i = 0$, then $\mathbf{x} = \mathbf{0}$ so that by (1) $\varphi(\mathbf{0}) = 0$.

Property 2 means that every binary coherent system is structurally stronger (weaker) than a series (parallel) system with the same number of components. From here on we will assume that all systems are coherent.

If in a design of a system redundancy has to be included to meet the required reliability level, then the designer has two basic options (if not excluded by technological or economic reasons): either two systems each with structure function φ are connected in parallel or the individual components of the system are replaced by two components in parallel. What option is preferable from the reliability point of view? The answer to the question is summarized in the theorem below.

Theorem 1 *Let φ be a structure of a coherent system S with* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{V}_n$, $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbf{V}_n$. *Then*

$$\varphi(\mathbf{x} \vee \mathbf{y}) \geq \varphi(\mathbf{x}) \vee \varphi(\mathbf{y}),$$

where $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$.

Proof. $x_i \vee y_i = \max(x_i, y_i) \geq x_i$ for all $i = 1, 2, \ldots, n$, so that $\varphi(\mathbf{x} \vee \mathbf{y}) \geq \varphi(\mathbf{x})$ since φ is increasing. Similarly, $x_i \vee y_i = \max(x_i, y_i) \geq y_i$ for all $i = 1, 2, \ldots, n$, so that $\varphi(\mathbf{x} \vee \mathbf{y}) \geq \varphi(\mathbf{y})$. It follows that

$$\varphi(\mathbf{x} \vee \mathbf{y}) \geq \max(\varphi(\mathbf{x}), \varphi(\mathbf{y})) = \varphi(\mathbf{x}) \vee \varphi(\mathbf{y}).$$

■

Theorem 1 above simply shows that redundancy at component level is preferable to redundancy at system level.

Analogously to theorem 1 for the series connection (although this has nothing to do with redundancy) we have the following theorem

Theorem 2 *Let φ be a structure of a coherent system S with* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{V}_n$, $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbf{V}_n$. *Then*

$$\varphi(\mathbf{x} \wedge \mathbf{y}) \leq \varphi(\mathbf{x}) \wedge \varphi(\mathbf{y}),$$

where $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n)$.

Proof. $x_i \wedge y_i = \min(x_i, y_i) \leq x_i$ for all $i = 1, 2, \ldots, n$, so that $\varphi(\mathbf{x} \wedge \mathbf{y}) \leq \varphi(\mathbf{x})$ since φ is increasing. Similarly, $x_i \wedge y_i = \min(x_i, y_i) \leq y_i$ for all $i = 1, 2, \ldots, n$, so that $\varphi(\mathbf{x} \wedge \mathbf{y}) \leq \varphi(\mathbf{y})$. It follows that

$$\varphi(\mathbf{x} \wedge \mathbf{y}) \leq \min(\varphi(\mathbf{x}), \varphi(\mathbf{y})) = \varphi(\mathbf{x}) \wedge \varphi(\mathbf{y}).$$

■

Theorem 2 simply shows that connecting two systems, each with structure function φ, in series is from the reliability point of view more effective than replacing each component by a series connection of the two corresponding system components.

2.3 Pivotal decomposition of the structure φ

Lemma 1 *The following identity holds for any structure function φ of order n:*

$$\varphi(\mathbf{x}) = x_i\, \varphi(1_i, \mathbf{x}) + \bar{x}_i\, \varphi(0_i, \mathbf{x}) \qquad \textit{for all } x_i,\ i = 1, 2, \ldots, n.$$

Proof.
$x_i = 1 : \varphi(\mathbf{x}) = 1.\varphi(1_i, \mathbf{x}) + 0.\varphi(0_i, \mathbf{x}) = \varphi(1_i, \mathbf{x})$
$x_i = 0 : \varphi(\mathbf{x}) = 0.\varphi(1_i, \mathbf{x}) + 1.\varphi(0_i, \mathbf{x}) = \varphi(0_i, \mathbf{x})$ ■

Component c_i is called the pivotal component. Every component can be selected as a pivotal component. However one will select the pivotal component in such a way that the Boolean functions $\varphi(1_i, \mathbf{x})$ and $\varphi(0_i, \mathbf{x})$ become as simple as possible.

The pivotal decomposition can be applied to $\varphi(1_i, \mathbf{x})$ and $\varphi(0_i, \mathbf{x})$.So, for instance, decomposing again with pivotal component j yields

$$\begin{aligned} \varphi(\mathbf{x}) \;=\; & x_i x_j \varphi(1_i, 1_j, \mathbf{x}) + x_i \bar{x}_j \varphi(1_i, 0_j, \mathbf{x}) \\ & + \bar{x}_i x_j \varphi(0_i, 1_j, \mathbf{x}) + \bar{x}_i \bar{x}_j \varphi(0_i, 0_j, \mathbf{x}). \end{aligned}$$

By repeated applications such that all $x_i^{'}s$ in $\mathbf{x}$ are fixed by 0 or 1, gives the following representation of $\varphi(\mathbf{x})$:

$$\varphi(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbf{V}_n} \varphi(\mathbf{y}) \prod_{k=1}^{n} x_k^{y_k} \, (1 - x_k)^{1-y_k} ,$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\quad \mathbf{y} = (y_1, y_2, \ldots, y_n)$

Example 1 *From the bridge structure (Figure 2.1), making c_3 the pivotal component, we have:*

$$\begin{aligned} \varphi(1_3, \mathbf{x}) &= (x_1 \vee x_2)(x_4 \vee x_5) \\ &= (x_1 + x_2 - x_1 x_2)(x_4 + x_5 - x_4 x_5) \end{aligned}$$

and

$$\begin{aligned} \varphi(0_3, \mathbf{x}) &= x_1 x_4 \vee x_2 x_5 \\ &= x_1 x_4 + x_2 x_5 - x_1 x_2 x_4 x_5 \end{aligned}$$

so that

$$\begin{aligned} \varphi(\mathbf{x}) &= x_3 \, \varphi(1_3, \mathbf{x}) + \bar{x}_3 \, \varphi(0_3, \mathbf{x}) \\ &= x_1 x_4 + x_1 x_3 x_5 + x_2 x_3 x_4 - x_1 x_2 x_3 x_4 - x_1 x_2 x_3 x_5 \qquad (2.3) \\ &\quad + x_2 x_5 - x_1 x_2 x_4 x_5 - x_1 x_3 x_4 x_5 - x_2 x_3 x_4 x_5 + 2x_1 x_2 x_3 x_4 x_5 \end{aligned}$$

The representation in (2.3) is called the polynomial form of the structure function. Generally, the polynomial form of the structure function of a monotone system is given by:

$$\varphi(\mathbf{x}) = \sum_{i=1}^{n} a_i x_i + \sum_{\substack{i,j=1 \\ i<j}}^{n} a_{i\,j} x_i x_j + \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} a_{i\,j\,k} x_i x_j x_k + \cdots + a_{1\,2 \cdots n} x_1 x_2 \cdots x_n ,$$

where the coefficients a_i , a_{ij} , ... , $a_{1\,2 \cdots n}$ are integers.

2.4 Path and cut representation of coherent structures

For a state space $\mathbf{V}_n \ni \mathbf{x}$, define:

$$\mathbf{V}_n^{(1)} = \{\mathbf{x}\, | \varphi(\mathbf{x}) = 1\}\,, \quad \mathbf{V}_n^{(0)} = \{\mathbf{x}\, | \varphi(\mathbf{x}) = 0\}\,.$$

Obviously,

$$\mathbf{V}_n = \mathbf{V}_n^{(1)} \cup \mathbf{V}_n^{(0)}, \qquad \mathbf{V}_n^{(1)} \cap \mathbf{V}_n^{(0)} = \varnothing.$$

A vector $\mathbf{x} \in \mathbf{V}_n^{(1)}$ is called a path vector and a vector $\mathbf{x} \in \mathbf{V}_n^{(0)}$ is called a cut vector. For $\mathbf{x}$ a path vector, the set

$$P = \{j \mid x_j = 1\}\,,$$

is called a path set. For $\mathbf{x}$ a cut vector, the set

$$C = \{k \mid x_k = 0\}\,,$$

is called a cut set. Trivially $\mathbf{1} = (1, 1, \ldots, 1)$ is always a path vector and $\mathbf{0} = (0, 0, \ldots, 0)$ is always a cut vector.
Note that by definition the components with indices outside a path set have failed, whereas the components with indices outside a cut set are working.

Example 2 *A series system has only one path vector, i.e,* $\mathbf{1} = (1, 1, \ldots, 1)$ *and a parallel system has only one cut vector, i.e,* $\mathbf{0} = (0, 0, \ldots, 0)$.

A structure function can be generated from the path and cut sets. However, it is more efficient to only use the so-called minimal path and cut sets. To introduce these concepts, we firstly need to introduce a partial order in the state space $\mathbf{V}_n$. Let

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \text{ and } \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbf{V}_n.$$

Then we write $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$, $i = 1, 2, \ldots, n$. If $\mathbf{x} \leq \mathbf{y}$ and there exist at least one index j such that $x_j < y_j$, then we write $\mathbf{x} < \mathbf{y}$. In the latter case, vector $\mathbf{x}$ is said to be smaller than vector $\mathbf{y}$.

Definition 3 *A path vector* $\mathbf{x}$ *is said to be minimal, if for all state vectors* $\mathbf{y} \in \mathbf{V}_n$*, which satisfy* $\mathbf{y} < \mathbf{x}$, $\varphi(\mathbf{y}) = 0$*. The corresponding path set is called a minimal path set.*

A failure of only one component operating under a minimal path vector induces a system failure. Thus, " minimal" in connection with minimal path vectors refers to the number of components that are working. The components in a minimal path set may be viewed as connection in series. Let system S have the minimal path sets $P_1, P_2, \ldots, P_M$. Define $\rho_j(\mathbf{x})$ as the structure function of the minimal path set P_j. Then:

$$\rho_j(\mathbf{x}) = \prod_{i \in P_j} x_i .$$

$\rho_j(\mathbf{x})$ is called the jth minimal path series structure, $j = 1, 2, \ldots, M$. If $\rho_j(\mathbf{x}) = 1$, then $\mathbf{x}$ must be a path vector of S. Now if there exist at least one j such that $\rho_j(\mathbf{x}) = 1$ then the system is working. Thus the system can be represented as the parallel arrangement of the minimal paths, i.e,

$$\begin{aligned} \varphi(\mathbf{x}) &= \coprod_{j=1}^{M} \rho_j(\mathbf{x}) \\ &= \coprod_{j=1}^{M} \prod_{i \in P_j} x_i \\ &= \max_{1 \le j \le M} \min_{i \in P_j} x_i \end{aligned} \tag{2.4}$$

Definition 4 *A cut vector* $\mathbf{x}$ *is said to be minimal, if for all state vectors* $\mathbf{y} \in \mathbf{V}_n$*, which satisfy* $\mathbf{y} > \mathbf{x}$, $\varphi(\mathbf{y}) = 1$*.The corresponding cut set is called a a minimal cut set.*

A repair of only one component which has failed under a minimal cut vector will result in a system working. Thus "minimal" in connection with minimal cut vectors refers to the number of components not working. Now for a system to be working only one component in each minimal cut set is required to be working. Thus components of the minimal cut set can be viewed as the connection in parallel. Let $C_1, C_2, \ldots, C_N$ be the minimal cut sets of system S.Define $\kappa_k(\mathbf{x})$ as the structure function of the minimal cut set C_k. Then

$$\kappa_k(\mathbf{x}) = \coprod_{i \in C_k} x_i.$$

$\kappa_k(\mathbf{x})$ is called the kth minimal cut parallel structure, $k = 1, 2, \ldots, N$. All cut sets must have at least one working component for the system to function. Thus the system can be viewed as a series arrangement of minimal cut sets, i.e,

$$\begin{aligned} \varphi(\mathbf{x}) &= \prod_{k=1}^{N} \kappa_k(\mathbf{x}) \\ &= \prod_{k=1}^{N} \coprod_{i \in C_k} x_i \\ &= \min_{1 \le k \le N} \quad \max_{i \in C_k} x_i \end{aligned} \tag{2.5}$$

Example 3 *Minimal path and cut sets of a bridge structure (Figure 2.1)*

The minimal paths are:

$$P_1 = \{1,4\},\ P_2 = \{2,5\},\ P_3 = \{1,3,5\},\ P_4 = \{2,3,4\},$$

and the minimal path series structures are

$$\rho_1(\mathbf{x}) = x_1\, x_4\ , \quad \rho_2(\mathbf{x}) = x_2\, x_5\ , \quad \rho_3(\mathbf{x}) = x_1\, x_3\, x_5\ , \quad \rho_4(\mathbf{x}) = x_2\, x_3\, x_4\ ,$$

so that

$$
\begin{aligned}
\varphi(\mathbf{x}) &= \coprod_{j=1}^{4} \rho_j\,(\mathbf{x}) \\
&= 1-(1-x_1\,x_4\,)(1-x_2\,x_5\,)(1-x_1\,x_3\,x_5\,)(1-x_2\,x_3\,x_4\,) \\
&= x_1\,x_4\,+x_2\,x_5\,+x_1\,x_3\,x_5\,+x_2\,x_3\,x_4\,-x_1\,x_2\,x_3\,x_4\,-x_1\,x_2\,x_3\,x_5 \\
&\quad -x_1\,x_2\,x_4\,x_5\,-x_1\,x_3\,x_4\,x_5\,-x_2\,x_3\,x_4\,x_5\,+2x_1\,x_2\,x_3\,x_4\,x_5\,,
\end{aligned}
$$

which is the same as (2.3).

The minimal cuts are:

$$C_1=\{1,2\}, \qquad C_2=\{4,5\}, \qquad C_3=\{1,3,5\}, \qquad C_4=\{2,3,4\},$$

and the minimal cut parallel structures are

$$
\begin{aligned}
\kappa_1\,(\mathbf{x}) &= 1-\bar{x}_1\,\bar{x}_2\,, \quad \kappa_2\,(\mathbf{x})=1-\bar{x}_4\,\bar{x}_5 \\
\kappa_3\,(\mathbf{x}) &= 1-\bar{x}_1\,\bar{x}_3\,\bar{x}_5\,, \quad \kappa_4\,(\mathbf{x})=1-\bar{x}_2\,\bar{x}_3\,\bar{x}_4\,.
\end{aligned}
$$

2.5 Inclusion- Exclusion Form of $\varphi(\mathbf{x})$

We have just seen that using minimal path sets, a structure function φ is given as

$$\varphi(\mathbf{x})=\bigvee_{j=1}^{M}\rho_j\,(\mathbf{x}).$$

From this we can generate the linear form of the structure function by applying the principle of inclusion- exclusion. We start by writing the disjunction $\rho_1\,\vee\rho_2$ in terms of arithmetic operations, that is,

$$\rho_1\,\vee\rho_2\,=\rho_1\,+\rho_2\,-\rho_1\,\rho_2\,.$$

The disjunction $\rho_1\,\vee\rho_2\,\vee\rho_3$ is given as

$$\rho_1 \vee \rho_2 \vee \rho_3 = \rho_1 + \rho_2 + \rho_3 - \rho_1 \rho_2 - \rho_1 \rho_3 - \rho_2 \rho_3 + \rho_1 \rho_2 \rho_3 .$$

Generally, the inclusion- exclusion form of $\varphi(\mathbf{x})$ is

$$\varphi(\mathbf{x}) = \bigvee_{j=1}^{M} \rho_j = \sum_{j=1}^{M} (-1)^{j-1} T_j, \tag{2.6}$$

where $T_j = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_j \le M} \rho_{i_1} \rho_{i_2} \cdots \rho_{i_j}$.

The sum T_j contains exactly $\binom{M}{j}$ terms. Hence, the total number of terms in the inclusion- exclusion form is

$$\binom{M}{1} + \binom{M}{2} + \cdots + \binom{M}{M} = 2^M - 1.$$

Thus, the complexity of the inclusion- exclusion form increases exponentially fast with increasing number of minimal path sets of the system.

Example 4 *The minimal path sets of a bridge structure (Figure 2.1) are*

$$P_1 = \{1, 4\},\ P_2 = \{2, 5\},\ P_3 = \{1, 3, 5\},\ P_4 = \{2, 3, 4\}.$$

Hence, formula (2.6) yields

$$\begin{aligned} \varphi(\mathbf{x}) \;=\; & x_1 x_4 + x_2 x_5 + x_1 x_3 x_5 + x_2 x_3 x_4 - x_1 x_2 x_4 x_5 \\ & - x_1 x_3 x_4 x_5 - x_1 x_2 x_3 x_4 - x_1 x_2 x_3 x_5 - x_2 x_3 x_4 x_5 \\ & - x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 \\ & + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 - x_1 x_2 x_3 x_4 x_5 . \end{aligned}$$

After two pairs of terms in this representation of $\varphi(\mathbf{x})$ cancel each other, we get the linear form (2.3). Satyanarayana and Prabhakar [23] were the first to show that identifying those terms before determining the structure function via the inclusion- exclusion principle may substantially reduce the computational effort for obtaining φ.

2.6 Disjoint sum form for $\varphi(\mathbf{x})$

We have seen that we are able to derive the structure function of any system by using pivotal decomposition method or by the path/cut set representation. Both methods yield sums of Boolean products. The difference is that in the pivotal decomposition method, the Boolean products represent all the elementary events (obviously disjoint), while in the path/cut representation method the Boolean products represent non-disjoint events.

The complexity of the pivotal decomposition method is that it generates a large number of elementary events which can lead to computational difficulties as the number of components become very large (e.g. for the bridge structure, with only 5 components, $2^5 = 32$ elementary events have to be considered). The problem with the path/cut representation method is that it is numerically difficult to compute the probability of the sum of non-disjoint events.

To overcome these challenges, a great amount of work has been done in finding more efficient methods that will represent a structure function as a sum of non-elementary disjoint products, where the number of these products is not so large. This can be done by performing simple algebraic operations on the initial form obtained from the set of all minimal path/cut sets. This is explained below.

Given the disjunctive normal form

$$\varphi(\mathbf{x}) = \bigvee_{j=1}^{M} \rho_j \ ,$$

procedures are of interest which transform this representation into a disjoint sum form:

$$\varphi(\mathbf{x}) = \bigvee_{j=1}^{M} \delta_j, \qquad \delta_i\, \delta_j = 0, \quad i \neq j.$$

Using disjunction form, we have

$$\rho_1 \vee \rho_2 = \rho_1 + \bar{\rho}_1\, \rho_2 \,,$$

and

$$\begin{aligned} \rho_1 \vee \rho_2 \vee \rho_3 &= (\rho_1 + \bar{\rho}_1\, \rho_2\,) \vee \rho_3 \\ &= \rho_1 + \bar{\rho}_1\, \rho_2 + \bar{\rho}_1\, \bar{\rho}_2\, \rho_3 \,. \end{aligned}$$

By induction,

$$\varphi(\mathbf{x}) = \bigvee_{j=1}^{M} \rho_j = \sum_{j=1}^{M} \delta_j \,,$$

where the disjoint Boolean functions δ_j are defined by

$$\delta_1 = \rho_1 \,, \quad \delta_j = \bar{\rho}_1\, \bar{\rho}_2\, \ldots\, \bar{\rho}_{j-1}\, \rho_j \,, \quad j = 2, \ldots, M,$$

and contrary to the ρ_j, the δ_j are mutually disjoint products of some x_i and $\bar{x}_i$.

Analogously, disjoint products can be obtained by using minimal cut sets $C_1, C_2, \ldots, C_N$ as follows:
Let $\gamma_k = 1 - \kappa_k$, $k = 1, 2, \ldots, N$. Then

$$1 - \varphi(\mathbf{x}) = \bigvee_{k=1}^{N} \gamma_k = \sum_{k=1}^{N} \vartheta_k \,,$$

where the disjoint Boolean functions ϑ_k are defined by

$$\vartheta_1 = \gamma_1 \,, \quad \vartheta_k = \bar{\gamma}_1\, \bar{\gamma}_2\, \ldots \bar{\gamma}_{k-1}\, \gamma_k \,, \quad k = 2, \ldots, N.$$

For computational efficiency, starting from the conjunctive normal form, a disjoint sum form should be constructed in such a way that it has a low complexity, i.e. δ_j or ϑ_k should have as little as possible factors and the number of terms should be small as well. This problem has been a subject of intensive research for a few decades [see [3], [5], [19], [22], and [24]].

There are many algorithms that represent φ as a sum of disjoint simple products, but most of them are derived from the algorithm developed by Abraham [1]. There are also some algorithms that were derived before him [see [2] and [14]], but compared to these, his algorithm drastically reduces the amount of computation needed to obtain the disjoint sum. Hence in this study we will only present the Abraham algorithm.

The Abraham Algorithm
The aim of the algorithm is to generate sets PD_j of disjoint simple products B so that

$$\delta_1 = \rho_1 , \quad \delta_j = \bar{\rho}_1 \, \bar{\rho}_2 \, \dots \, \bar{\rho}_{j-1} \, \rho_j = \sum_{B \in PD_j} B, \qquad j = 2, \dots, M.$$

Thus each δ_j is represented as a disjoint sum form. At step j in the algorithm, each $B \in PD_j$ is made to be disjoint with $\rho_1 , \rho_2 , \dots , \rho_{j-1}$. Before we give the actual algorithm we need some notation and a theorem.

Table 1: Notation used for Abraham's Algorithm

Notation	**Description**
B_i	Boolean product of $x_i' s$ and $\bar{x}_i' s$ which implies some ρ_j (the B_i is derived from the ρ_j)
$PD_{0,j}, \dots, PD_{j-1,j} \quad j = 2, \dots, M$	Sets $\{B_i\}$ of disjoint products derived from ρ_j , each B_i being disjoint with $\rho_1 , \rho_2 , \dots , \rho_{j-1}$
PD_j	Sets that are successively obtained from $PD_{0,j}, \dots, PD_{j-1,j}$ with $PD_{0,j} = \{\rho_j\}$ and $PD_{j-1,j} = PD_j$
$C(\rho_i , \rho_j)$	$\{x_k \mid x_k \in P_i \text{ and } x_k \notin P_j, \; i \neq j\}$

Theorem 3 *Let ρ_j be a Boolean product with only uncomplemented variables and B_i be any product.*

a) *If there is at least one variable which exists (uncomplemented) in* ρ_j *and complimented in* B_i*, then* ρ_j *and* B_i *are disjoint.*

b) *If* ρ_j *and* B_i *are not disjoint, let* $\mathbf{x} = \{x_a, x_b, \ldots, x_m\}$ *be the set of variables which exists (uncomplemented) in* ρ_j *and which do not exist in* B_i*. Then*

(i) *If* $\mathbf{x} = \varnothing$*, then* $\rho_j \cup B_i = B_i$

(ii) *If* $\mathbf{x} \neq \varnothing$*, then* $\rho_j \cup B_i = \rho_j \cup \bar{x}_a B_i \cup x_a \bar{x}_b B_i \cup \ldots \cup x_a x_b \ldots \bar{x}_m B_i$*, and all the products in the right-hand side are mutually disjoint.*

Proof. a) and b) (i) are trivially true.
b) The equality obviously holds if ρ_j and B_i are both complemented, or if ρ_j is uncomplemented regardless of B_i. Now consider the case where ρ_j is complemented and B_i is uncomplemented (the left-hand side is uncomplemented for this case). This condition can happen only when one or more of the variables $x_a, x_b, \ldots, x_m$ are complemented. If x_a is complemented, then $\bar{x}_a B_i$ is uncomplemented and the right-hand side is uncomplemented. If x_a is uncomplemented, then if x_b is complemented, the right-hand side is again uncomplemented. Continuing in this fashion, it can be seen that the right-hand side has to be uncomplemented. It is also clear that all the products in the right-hand side are mutually disjoint. ■

Algorithm

1. Find all the minimal path sets, $P_1, P_2, \ldots, P_M$ of system S.
2. List all the minimal path series structures ρ_j, $j = 1, 2, \ldots, M$, in lexicographical order.
3. Define $PD_{0,1} = \{\rho_1\}$.For $2 \leq j \leq M$,
4. Define $PD_{0,j} = \{\rho_j\}$.
5. Compare $PD_{0,1}$ and $PD_{0,j}$ to form a set $PD_{1,j}$.

- if they are disjoint, then $PD_{1,j} = PD_{0,j}$
- if they are not disjoint, use (b) in theorem 3 to find all $B_i \in PD_{1,j}$ i.e find $C(\rho_1, \rho_j)$
  - if $C(\rho_1, \rho_j) = \varnothing$, then drop ρ_j from $PD_{0,j}$.
  -if $C(\rho_1, \rho_j) \neq \varnothing$, replace ρ_j in $PD_{0,j}$ by $\bar{x}_a \rho_j, x_a \bar{x}_b \rho_j, \ldots, x_a x_b \ldots \bar{x}_m \rho_j$ where $\{x_a, x_b, \ldots, x_m\} = C(\rho_1, \rho_j)$. The new set formed is $PD_{1,j}$ and ρ_1 is disjoint with all $B_i \in PD_{1,j}$

6. Repeat step 5 to form $PD_{s,j}$, $s = 2, 3, \ldots, j-1$, by comparing each $B_i \in PD_{s-1,j}$ with ρ_s.

7. Define $PD_{j-1,j} = PD_j$. You will notice that each $B_i \in PD_j$ is disjoint $\rho_1, \rho_2, \ldots, \rho_{j-1}$.

Now δ_j is defined as

$$\delta_j = \sum_{B_i \in PD_j} B_i.$$

Thus, each δ_j is represented as a disjoint sum form.

Example 5 *The minimal path sets of a bridge structure (Figure 2.1) are*

$$P_1 = \{1, 4\},\ P_2 = \{2, 5\},\ P_3 = \{1, 3, 5\},\ P_4 = \{2, 3, 4\},$$

and the minimal path series structures, listed lexicographically, are

$$\rho_1(\mathbf{x}) = x_1 x_4, \quad \rho_2(\mathbf{x}) = x_2 x_5, \quad \rho_3(\mathbf{x}) = x_1 x_3 x_5, \quad \rho_4(\mathbf{x}) = x_2 x_3 x_4.$$

$$PD_{0,1} = \{\rho_1\} = \{x_1 x_4\}.$$

$j = 2:$

$$\begin{aligned} PD_{0,2} &= \{B_1 = x_2 x_5\} \\ C(\rho_1, B_1) = \{x_1, x_4\} & \\ PD_{1,2} &= PD_2 = \{B_1 = \bar{x}_1 x_2 x_5,\ B_2 = x_1 x_2 \bar{x}_4 x_5\}. \end{aligned}$$

$j = 3:$

$$\begin{aligned}
PD_{0,3} &= \{B_1 = x_1\, x_3\, x_5\,\} \\
C(\rho_1\,, B_1) &= \{x_4\,\} \\
PD_{1,3} &= \{B_1 = x_1\, x_3\, \bar{x}_4\, x_5\,\} \\
C(\rho_2\,, B_1) &= \{x_2\,\} \\
PD_{2,3} &= PD_3 = \{B_1 = x_1\, \bar{x}_2\, x_3\, \bar{x}_4\, x_5\,\}.
\end{aligned}$$

$j = 4:$

$$\begin{aligned}
PD_{0,4} &= \{B_1 = x_2\, x_3\, x_4\,\} \\
C(\rho_1\,, B_1) &= \{x_1\,\} \\
PD_{1,4} &= \{B_1 = \bar{x}_1\, x_2\, x_3\, x_4\,\} \\
C(\rho_2\,, B_1) &= \{x_5\,\} \\
PD_{2,4} &= \{B_1 = \bar{x}_1\, x_2\, x_3\, x_4\, \bar{x}_5\,\} \\
& \quad \rho_3 \textit{ and } B_1 \textit{are disjoint, hence} \\
PD_{3,4} &= PD_4 = \{B_1 = \bar{x}_1\, x_2\, x_3\, x_4\, \bar{x}_5\,\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\delta_1 &= x_1\, x_4 \\
\delta_2 &= \bar{x}_1\, x_2\, x_5\, + x_1\, x_2\, \bar{x}_4\, x_5 \\
\delta_3 &= x_1\, \bar{x}_2\, x_3\, \bar{x}_4\, x_5 \\
\delta_4 &= \bar{x}_1\, x_2\, x_3\, x_4\, \bar{x}_5
\end{aligned}$$

And

$$\begin{aligned}
\varphi(\mathbf{x}) &= \sum_{j=1}^{4} \delta_j \\
&= x_1\, x_4\, + \bar{x}_1\, x_2\, x_5\, + x_1\, x_2\, \bar{x}_4\, x_5\, + x_1\, \bar{x}_2\, x_3\, \bar{x}_4\, x_5\, + \bar{x}_1\, x_2\, x_3\, x_4\, \bar{x}_5\,.
\end{aligned}$$

From example 5, it is easy to see how the terms of φ have been reduced to only 5 when using the disjoint sum form, unlike when using path/cut representation or the pivotal decomposition method (which both have 10 terms each (examples 1 and 3)).

Analogously, ϑ_k , $k = 1, 2, \ldots, N$, can be obtained by listing $\gamma_k = 1 - \kappa_k$ as the minimal cut series structures and follow the same way of obtaining δ_j.

2.7 Modules (subsystems) of systems

Most devices are comprised of many components. Then it becomes difficult to calculate the structure function of such systems. Reliability engineers often "package" some components into subsystems that can be removed and replaced as a whole. Each package is called a module of the system. The system structure is defined as the function of the structure of its modules, and the state of each module is a function of the components that comprise it. Formally the definition of a module is given as follows:

Definition 5 *Let A be a subset of components of system S with structure function φ. Then A is a modular set of S if:*

$$\varphi(\mathbf{x}) = \varphi(\mathbf{x}^A, \mathbf{x}^{A'}) = \psi[\chi_A(\mathbf{x}^A), \mathbf{x}^{A'}],$$

where (A, χ_A) is a coherent binary subsystem with structure function χ, and ψ is a structure function of a coherent binary system . When A is a modular set, then (A, χ_A) is called a module of system S.

Note that $\mathbf{x}^A$ denotes the vector with indicator variables x_i , where $x_i \in A$. Intuitively, a module (A, χ_A) is a coherent subsystem that acts as if it were just a component.

Using a pivotal decomposition of a structure function

$$\varphi(\mathbf{x}) = x_i\, \varphi(1_i, \mathbf{x}) + (1 - x_i)\varphi(0_i, \mathbf{x}),$$

and (A, χ_A) a module of S, we have

$$\varphi(\mathbf{x}) = \chi_A(\mathbf{x}^A)\psi(1, \mathbf{x}^{A'}) + [1 - \chi_A(\mathbf{x}^A)]\psi(0, \mathbf{x}^{A'}).$$

For $\chi_A(\mathbf{x}^A) = 1$, we have $\varphi(\mathbf{x}^A, \mathbf{x}^{A'}) = \psi(1, \mathbf{x}^{A'})$, and for $\chi_A(\mathbf{x}^A) = 0$, then $\varphi(\mathbf{x}^A, \mathbf{x}^{A'}) = \psi(0, \mathbf{x}^{A'})$. Since $\chi_A(\mathbf{1}^A) = 1$ and $\chi_A(\mathbf{0}^A) = 0$, it follows that: either

$$\varphi(\mathbf{x}^A, \mathbf{x}^{A'}) = \varphi(\mathbf{1}^A, \mathbf{x}^{A'}) \qquad \text{for every } \mathbf{x}^{A'} \tag{2.7}$$

or

$$\varphi(\mathbf{x}^A, \mathbf{x}^{A'}) = \varphi(\mathbf{0}^A, \mathbf{x}^{A'}) \qquad \text{for every } \mathbf{x}^{A'} \tag{2.8}$$

(2.7) and (2.8) corresponds to the intuitive notion that since the system is affected only by the functioning or failure of a modular structure, every performance state for the components of a module is comparable either to the state in which all modular components function or the state in which all modular components fail.

In practise, the process of designing a system automatically yields a modular decomposition of the system, that is, decomposition of the system into disjoint modules. Hence the following definition.

Definition 6 *A modular decomposition of a coherent system S is a set of disjoint modules $\{(A_1, \chi_{A_1}), \ldots, (A_r, \chi_{A_r})\}$ together with an organizing structure ψ such that*

(a) $S = \bigcup_{i=1}^{r} A_i$, *where* $A_i \cap A_j = \varnothing$ *for* $i \neq j$;

(b) $\varphi(\mathbf{x}) = \psi[\chi_{A_1}(\mathbf{x}^{A_1}), \chi_{A_2}(\mathbf{x}^{A_2}), \ldots, \chi_{A_r}(\mathbf{x}^{A_r})]$.

Generally systems require that all modules function properly in order for the system to operate satisfactory, hence, while it is not always the case, the modules often comprise a series structure.

In practise, given a modular decomposition of a coherent system, one can obtain a refinement of the decomposition, that is, one can decompose the modules into smaller modules.

2.8 Association of Random Variables

Definition 7 *A collection of random variables $x_1, x_2, \ldots, x_n$ is said to be associated if for every pair of non-decreasing binary functions f and g*

$$cov[f(\mathbf{x}),\ g(\mathbf{x})] \geq 0.$$

2.8.1 Properties of associated random variables

(P_1) - any subset of associated random variables are associated.

(P_2) - if two sets of associated random variables are independent of one another, then their union is a set of associated random variables.

(P_3) - a single random variable is associated.

(P_4) - coordinatewise non-decreasing functions of associated random variables are associated.

(P_5) -if $x_1, x_2, \ldots, x_n$ are binary associated random variables then $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ are also associated.

Theorem 4 *If $x_1, x_2, \ldots, x_n$ are associated binary random variables such that $E(x_i),\ E(x_i x_j) < \infty\ i, j = 1, 2, \ldots, n,\ i \neq j,$ then*

$$0 \leq E\left(\prod_{i=1}^{n} x_i\right) - \prod_{i=1}^{n} E(x_i) \leq \sum_{i<j} cov(x_i,\ x_j).$$

2.9 Reliability of Coherent Systems

Definition 8 *Independent components*
Two components c_i *and* c_j *are independent of each other if the state of component* c_i *is not influenced by the state of component* c_j *. Probabilistically this means that their indicator variables* x_i *and* x_j *are independent random variables, that is,*

$$\Pr[x_i = 1 \,|x_j = 1] = \Pr[x_i = 1 \,|x_j = 0] = \Pr[x_i = 1].$$

In this dissertation we will only consider systems with statistically independent components.

We will denote the reliability of component c_i , i.e, the probability that the component c_i is working as:

$$p_i = \Pr[x_i = 1].$$

for $i = 1, 2, \ldots, n$. The unreliability of component c_i is given as $q_i = 1 - p_i$.

Since $x_i^{'} s$ are $(0, 1)$- variables, the definition of reliability coincides with the expected value of x, i.e,

$$\begin{aligned} E(x_i) &= 1 \cdot \Pr[x_i = 1] + 0 \cdot \Pr[x_i = 0] \\ &= 1 \cdot p_i + 0 \cdot q_i \\ &= p_i \end{aligned}$$

The reliability of the system is given as:

$$R = \Pr[\varphi(\mathbf{x}) = 1].$$

Because components are independent, system reliability can be represented as a function of component reliabilities:

$$R = R(\mathbf{p}),$$

where $\mathbf{p} = (p_1, p_2, ..., p_n)$.

Again because the state of the system is 0 or 1, the system reliability is the expected value of the structure function:

$$\begin{aligned} E[\varphi(\mathbf{x})] &= 1 \cdot \Pr[\varphi(\mathbf{x}) = 1] + 0 \cdot \Pr[\varphi(\mathbf{x}) = 0] \\ &= \Pr[\varphi(\mathbf{x}) = 1] \\ &= R(\mathbf{p}) \end{aligned}$$

Example 6 *In this example we give the reliabilities of special systems with n components:*

(1) Series system

$$R(\mathbf{p}) = \prod_{i=1}^{n} p_i$$

(2) Parallel system

$$\begin{aligned} R(\mathbf{p}) &= \coprod_{i=1}^{n} p_i \\ &= 1 - \prod_{i=1}^{n} q_i \end{aligned}$$

(3) *k-out-of-n* system with identical component reliabilities, i.e, $p_1 = p_2 = \ldots = p_n = p$.

$$R(\mathbf{p}) = \sum_{i=k}^{n} \binom{n}{i} p^i q^{n-i}$$

The reliability of a series system is low for n large. On the other hand the reliability of a parallel system is high for n large, but it tends to be very expensive to set up a parallel system. Hence reliability engineers have turned their attention to *k-out-of-n* systems, which tend to have high reliabilities and are of comparatively low cost.

Using the pivotal decomposition with component c_i as the pivotal component, the reliability of a system is given as:

$$R(\mathbf{p}) \;=\; p_i\, R(1_i, \mathbf{p}) \;+\; q_i\, R(0_i, \mathbf{p}),$$

for $i = 1, 2, \ldots, n$.

Example 7 *From example 2.3 the structure function of a bridge structure (figure 2.1) with c_3 as the pivotal component is derived as*

$$\begin{aligned}\varphi(\mathbf{x}) \;&=\; x_1\, x_4 \;+ x_2\, x_5 \;+ x_1\, x_3\, x_5 \;+ x_2\, x_3\, x_4 \;- x_1\, x_2\, x_3\, x_4 \;- x_1\, x_2\, x_3\, x_5 \\ &\quad -x_1\, x_2\, x_4\, x_5 \;- x_1\, x_3\, x_4\, x_5 \;- x_2\, x_3\, x_4\, x_5 \;+ 2x_1\, x_2\, x_3\, x_4\, x_5\;.\end{aligned}$$

So the reliability of a bridge structure with independent components and c_3 as the pivotal component is

$$\begin{aligned}R(\mathbf{p}) \;&=\; p_1\, p_4 \;+ p_2\, p_5 \;+ p_1\, p_3\, p_5 \;+ p_2\, p_3\, p_4 \;- p_1\, p_2\, p_3\, p_4 \;- p_1\, p_2\, p_3\, p_5 \\ &\quad -p_1\, p_2\, p_4\, p_5 \;- p_1\, p_3\, p_4\, p_5 \;- p_2\, p_3\, p_4\, p_5 \;+ 2p_1\, p_2\, p_3\, p_4\, p_5\;,\end{aligned}$$

When $p_1 = p_2 = \cdots = p_5 = p$, we have

$$R(\mathbf{p}) \;= 2p^2 + 2p^3 - 5p^4 + 2p^5.$$

We can also give the reliability of a system in terms of its minimal path sets as:

$$R(\mathbf{p}) = E \coprod_{j=1}^{M} \prod_{i \in P_j} x_i\;,$$

and in terms of its minimal cut sets as:

$$R(\mathbf{p}) = E\prod_{k=1}^{N}\coprod_{i\in C_k} x_i.$$

Using the inclusion- exclusion method, the reliability of a system is

$$R(\mathbf{p}) = \sum_{j=1}^{M}(-1)^{j-1}E(T_j), \tag{2.9}$$

where $T_j = \sum_{1\leq i_1 \leq i_2 \leq\cdots\leq i_j \leq M} \rho_{i_1}\rho_{i_2} \cdots \rho_{i_j}$.

Using the disjoint sum form of $\varphi(\mathbf{x})$, the reliability of a system is

$$R(\mathbf{p}) = \sum_{k=1}^{M} E(\delta_k), \tag{2.10}$$

Using modular decomposition, the reliability of a system is

$$R(\mathbf{p}) = R_\psi[R_{\chi_{A_1}}(\mathbf{p}), \ldots, R_{\chi_{A_r}}(\mathbf{p})],$$

where $R_{\chi_{A_i}}(\mathbf{p})$ is the reliability function of module (A_i, χ_{A_i}), $i = 1, 2 \ldots, r$.

2.10 Birnbaum's Importance Measure

Some components in a system may play a more important role than others. Hence it is in every engineer's interest to know how each component contributes to the successful operation of the whole system. Birnbaum [7] was the first to quantify this importance, and the quantitative definition is given in definitions 9 and 10 below.

There are situations where only the structure function $\varphi(\mathbf{x})$ of the system is given, but no information is available about the component reliabilities. In such situations the relative importance of various components is called *structural importance.* In other instances, both the structure function φ and component reliabilities $\mathbf{p}$ are known. In such situations the importance of components is called *reliability importance.*

2.10.1 Birnbaum's Structural Importance

Definition 9 *Birnbaum's Structural importance*

- *The structural importance of component c_i for the functioning of system S is given as*

$$I_i^{(B)}(\varphi, 1) = 2^{-n} \sum_{\mathbf{x} \in \mathbf{V}_n} \bar{x}_i \, [\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})]. \tag{2.11}$$

- *The structural importance of component c_i for the failure of system S is given as*

$$I_i^{(B)}(\varphi, 0) = 2^{-n} \sum_{\mathbf{x} \in \mathbf{V}_n} x_i \, [\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})]. \tag{2.12}$$

- *The structural importance of component c_i for system S is given as*

$$\begin{aligned} I_i^{(B)}(\varphi) &= I_i^{(B)}(\varphi, 1) + I_i^{(B)}(\varphi, 0) \qquad (2.13) \\ &= \sum_{\mathbf{x} \in \mathbf{V}_n} [\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})]. \end{aligned}$$

Theorem 5 *For any coherent system S the following is true*

$$I_i^{(B)}(\varphi, 1) = I_i^{(B)}(\varphi, 0) = \frac{1}{2} I_i^{(B)}(\varphi).$$

Proof. Suppose that c_i is relevant for some state vector $\mathbf{x}$. Then component c_i is relevant for either the functioning or failure of the system. If c_i is relevant at $\mathbf{x}$ for the functioning of the system, then c_i is relevant at $(1_i, \mathbf{x})$ for failure. If c_i is relevant at $\mathbf{x}$ for for the failure of the system, then c_i is relevant at $(0_i, \mathbf{x})$ for functioning. Thus there is one-to-one correspondence between those state vectors at which c_i is relevant for functioning and those at which it is relevant for failure. It then follows from 2.11, 2.12 and 2.13 that

$$I_i^{(B)}(\varphi, 1) = I_i^{(B)}(\varphi, 0) = \frac{1}{2} I_i^{(B)}(\varphi).$$

■

Theorem 5 above states that there is no use in distinguishing between structural importance for functioning and for failure. This is not the case with reliability importance, and this will be revealed in the next section.

2.10.2 Birnbaum's Reliability Importance

Definition 10 *Birnbaum's Reliability Importance*

(i) *The reliability importance of component c_i for the functioning of system S is*

$$RI_i^{(B)}(\varphi, 1, \mathbf{p}) = \Pr[\varphi(\mathbf{x}) = 1 \mid x_i = 1] - \Pr[\varphi(\mathbf{x}) = 1].$$

(ii) *The reliability importance of component c_i for the failure of system S is*

$$RI_i^{(B)}(\varphi, 0, \mathbf{p}) = \Pr[\varphi(\mathbf{x}) = 0 \mid x_i = 0] - \Pr[\varphi(\mathbf{x}) = 0].$$

(iii) *The reliability importance of component c_i for system S is*

$$RI_i^{(B)}(\varphi, \mathbf{p}) = RI_i^{(B)}(\varphi, 1, \mathbf{p}) + RI_i^{(B)}(\varphi, 0, \mathbf{p}).$$

The higher the value of the reliability importance of a component, the more important to the system the component is thought to be.

Corollary 1 *For any structure function the following is true*

$$cov[x_i\ ,\ \varphi(\mathbf{x})] = p_i\, q_i\, E[\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})].$$

Proof. Define $\varphi(\mathbf{x})$ in terms of pivotal decomposition and use straightforward algebra. ■

Corollary 2 *The following identities are always true*

(i)

$$RI_i^{(B)}(\varphi, 1, \mathbf{p}) = q_i \; E[\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})] = q_i \; \frac{\partial R(\mathbf{p})}{\partial p_i}.$$

(ii)

$$RI_i^{(B)}(\varphi, 0, \mathbf{p}) = p_i \; E[\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})] = p_i \; \frac{\partial R(\mathbf{p})}{\partial p_i}.$$

(iii)

$$RI_i^{(B)}(\varphi, \mathbf{p}) = E[\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})] = \frac{\partial R(\mathbf{p})}{\partial p_i}.$$

Proof. Using corollary 1

(i)

$$\begin{aligned}
RI_i^{(B)}(\varphi, 1, \mathbf{p}) &= \Pr[\varphi(\mathbf{x}) = 1 \mid x_i \; = 1] - \Pr[\varphi(\mathbf{x}) = 1] \\
&= \frac{\Pr[\varphi(\mathbf{x}) = 1, \; x_i \; = 1]}{\Pr[x_i \; = 1]} - \Pr[\varphi(\mathbf{x}) = 1] \\
&= \frac{cov[x_i \;, \; \varphi(\mathbf{x})] + E(x_i \;) E(\varphi(\mathbf{x}))}{p_i} - R(\mathbf{p}) \\
&= \frac{p_i \; q_i \; E[\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})] + p_i \; R(\mathbf{p})}{p_i} - R(\mathbf{p}) \\
&= q_i \; E[\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})] \\
&= q_i \; \frac{\partial R(\mathbf{p})}{\partial p_i}
\end{aligned}$$

(ii)

$$
\begin{aligned}
RI_i^{(B)}(\varphi, 0, \mathbf{p}) &= \Pr[\varphi(\mathbf{x}) = 0 \mid x_i = 0] - \Pr[\varphi(\mathbf{x}) = 0] \\
&= \frac{\Pr[\varphi(\mathbf{x}) = 0,\ x_i = 0]}{\Pr[x_i = 0]} - \Pr[\varphi(\mathbf{x}) = 0] \\
&= \frac{cov[\bar{x}_i, (1 - \varphi(\mathbf{x}))] + E(\bar{x}_i)E(1 - \varphi(\mathbf{x}))}{q_i} - 1 + R(\mathbf{p}) \\
&= \frac{p_i\, q_i\, E[\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})] + q_i\, E(1 - \varphi(\mathbf{x}))}{q_i} - 1 + R(\mathbf{p}) \\
&= p_i\, E[\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})] \\
&= p_i\, \frac{\partial R(\mathbf{p})}{\partial p_i}
\end{aligned}
$$

(iii)

$$
\begin{aligned}
RI_i^{(B)}(\varphi, \mathbf{p}) &= RI_i^{(B)}(\varphi, 1, \mathbf{p}) + RI_i^{(B)}(\varphi, 0, \mathbf{p}) \\
&= q_i\, E[\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})] + p_i\, E[\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})] \\
&= E[\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})] \\
&= \frac{\partial R(\mathbf{p})}{\partial p_i}
\end{aligned}
$$

■

Example 8 *Special systems*

1. *Series system*

$$R(\mathbf{p}) = \prod_{k=1}^{n} p_k$$

$$
\begin{aligned}
RI_i^{(B)}(\varphi, \mathbf{p}) &= \frac{\partial R(\mathbf{p})}{\partial p_i} \\
&= \prod_{\substack{k=1 \\ k \neq i}}^{n} p_k
\end{aligned}
$$

i.e the component with the smallest reliability has the largest importance.

2. *Parallel system*

$$R(\mathbf{p}) = 1 - \prod_{k=1}^{n}(1 - p_k)$$

$$\begin{aligned} RI_i^{(B)}(\varphi, \mathbf{p}) &= \frac{\partial R(\mathbf{p})}{\partial p_i} \\ &= \prod_{\substack{k=1 \\ k \neq i}}^{n} (1 - p_k) \end{aligned}$$

that is, the component with maximum reliability has the largest importance.

Definition 11 *Equivalence of Birnbaum's Structural and Reliability Importance*
Birnbaum's structural and reliability importance are equivalent in all senses if $p_1 = p_2 = \cdots = p_n = \frac{1}{2}$, *that is*

$$I_i^{(B)}(\varphi) = RI_i^{(B)}(\varphi, \mathbf{p}) \mid_{\mathbf{p}=\left(\frac{1}{2}, \cdots, \frac{1}{2}\right)} . \tag{2.14}$$

3 SYSTEM RELIABILITY BOUNDS

When computing the exact value of system reliability in quite large systems one is faced with a large amount of computations required and sometimes it is impossible to evaluate the exact value. When such a situation arises, there is a need for approximative methods. To be good approximations, these methods require a substantially reduced amount of computation time and should yield a result close to the true value as much as possible.

One approximative method that has been proven to be effective is the reliability bound method. This involves finding lower and upper bounds to system reliability. In this chapter we look at the bounds that are available in the literature.

3.1 Trivial bounds

Theorem 6 *Let φ be a structure function of a coherent system of n independent components with reliabilities $p_1, p_2, ..., p_n$. Then*

$$LB_{Triv} = \prod_{i=1}^{n} p_i \leq R(\mathbf{p}) \leq \coprod_{i=1}^{n} p_i = UB_{Triv}.$$

Proof. From (2.2) we have

$$\prod_{i=1}^{n} x_i \leq \varphi(\mathbf{x}) \leq \coprod_{i=1}^{n} x_i$$

Taking expectations we have

$$E\left(\prod_{i=1}^{n} x_i\right) \leq E[\varphi(\mathbf{x})] \leq E\left(\coprod_{i=1}^{n} x_i\right) \tag{3.1}$$

Because the components are independent, (3.1) becomes

$$\prod_{i=1}^{n} E(x_i) \leq R(\mathbf{p}) \leq \coprod_{i=1}^{n} E(x_i)$$

i.e

$$\prod_{i=1}^{n} p_i \;\leq R(\mathbf{p}) \leq \coprod_{i=1}^{n} p_i$$

■

Theorem 6 states that the reliability function of a coherent system is greater or equal to that of a series system and less or equal to that of a parallel system.

3.2 Esary- Proschan bounds

Esary and Proschan published a pioneering work showing that an improvement can be made to the trivial bounds by using minimal path and cut representation of the structure function. Before we state the bounds, we need the following four lemmas.

Lemma 2 *Let $x_1 , x_2 , \ldots , x_n$ be independent binary random variables. For any functions f_1 , f_2*

$$\begin{aligned} cov[f_1 (\mathbf{x}), f_2 (\mathbf{x})] &= p_i \, cov[f_1 (1_i, \mathbf{x}), f_2 (1_i, \mathbf{x})] + q_i \, cov[f_1 (0_i, \mathbf{x}), f_2 (0_i, \mathbf{x})] \\ &\quad + p_i \, q_i \, \cdot E[f_1 (1_i, \mathbf{x}) - f_1 (0_i, \mathbf{x})] \cdot E[f_2 (1_i, \mathbf{x}) - f_2 (0_i, \mathbf{x})], \end{aligned}$$

for $i = 1, 2, \ldots, n$.

Proof. Using pivotal decomposition, we have

$$E[f_1 (\mathbf{x}) f_2 (\mathbf{x})] = p_i \, E[f_1 (1_i, \mathbf{x}) f_2 (1_i, \mathbf{x})] + q_i \, E[f_1 (0_i, \mathbf{x}) f_2 (0_i, \mathbf{x})]$$

and

$$E[f(\mathbf{x})] = p_i \, E[f(1_i, \mathbf{x})] + q_i \, E[f (0_i, \mathbf{x})]$$

Now

$$
\begin{aligned}
cov[f_1(\mathbf{x}), f_2(\mathbf{x})] &= E[f_1(\mathbf{x}) f_2(\mathbf{x})] - E[f_1(\mathbf{x})] \cdot E[f_2(\mathbf{x})] \\
&= p_i\, E[f_1(1_i, \mathbf{x}) f_2(1_i, \mathbf{x})] + q_i\, E[f_1(0_i, \mathbf{x}) f_2(0_i, \mathbf{x})] \\
&\quad - (p_i\, E[f_1(1_i, \mathbf{x})] + q_i\, E[f_1(0_i, \mathbf{x})]) \\
&\quad (p_i\, E[f_2(1_i, \mathbf{x})] + q_i\, E[f_2(0_i, \mathbf{x})]) \\
\\
&= p_i\, E[f_1(1_i, \mathbf{x}) f_2(1_i, \mathbf{x})] + q_i\, E[f_1(0_i, \mathbf{x}) f_2(0_i, \mathbf{x})] \\
&\quad - p_i^2\, E[f_1(1_i, \mathbf{x})] \cdot E[f_2(1_i, \mathbf{x})] \\
&\quad - p_i\, q_i\, E[f_1(1_i, \mathbf{x})] \cdot E[f_2(0_i, \mathbf{x})] \\
&\quad - p_i\, q_i\, E[f_1(0_i, \mathbf{x})] \cdot E[f_2(1_i, \mathbf{x})] \\
&\quad - q_i^2\, E[f_1(0_i, \mathbf{x})] \cdot E[f_2(0_i, \mathbf{x})] \\
\\
&= p_i\, cov[f_1(1_i, \mathbf{x}), f_2(1_i, \mathbf{x})] + q_i\, cov[f_1(0_i, \mathbf{x}), f_2(0_i, \mathbf{x})] \\
&\quad + p_i\, q_i\, E[f_1(1_i, \mathbf{x})] \cdot E[f_2(1_i, \mathbf{x})] \\
&\quad + p_i\, q_i\, E[f_1(0_i, \mathbf{x})] \cdot E[f_2(0_i, \mathbf{x})] \\
&\quad - p_i\, q_i\, E[f_1(1_i, \mathbf{x})] \cdot E[f_2(0_i, \mathbf{x})] \\
&\quad - p_i\, q_i\, E[f_1(0_i, \mathbf{x})] \cdot E[f_2(1_i, \mathbf{x})] \\
\\
&= p_i\, cov[f_1(1_i, \mathbf{x}), f_2(1_i, \mathbf{x})] + q_i\, cov[f_1(0_i, \mathbf{x}), f_2(0_i, \mathbf{x})] \\
&\quad + p_i\, q_i\, E[f_1(1_i, \mathbf{x}) - f_1(0_i, \mathbf{x})] \cdot E[f_2(1_i, \mathbf{x}) - f_2(0_i, \mathbf{x})]
\end{aligned}
$$

∎

Lemma 3 *Let $x_1, x_2, \ldots, x_n$ be independent binary random variables. Let f_1, f_2 be increasing functions. Then*

$$cov[f_1(\mathbf{x}), f_2(\mathbf{x})] \geq 0$$

Proof. The proof is by induction on the order n of the functions. For $n = 1$, from Lemma 2

$$cov[f_1(x_1), f_2(x_1)] = pq \cdot E[f_1(1) - f_1(0)] \cdot E[f_2(1) - f_2(0)]$$

an expression which is clearly non-negative for increasing f_1 and f_2.
Assume that the covariance of any two increasing functions of order $n-1$ is non-negative. If f_1, f_2 are increasing and of order n, then the related functions $f_j(1_i, \mathbf{x}), f_j(0_i, \mathbf{x})$, $j = 1, 2$, are all increasing and of order $n-1$. Thus the first two terms of the representation of Lemma 2 are non-negative. Since $f_j(1_i, \mathbf{x}) \geq f_j(0_i, \mathbf{x})$, $j = 1, 2$, the third term of the same representation is also non-negative. ■

Lemma 4 *Let $x_1, x_2, \ldots, x_n$ be independent binary random variables. Define $f_j(\mathbf{x}) = \prod_{i \in A_j} x_i$, where A_j is a subset of $\{1, 2, \ldots, n\}$, $j = 1, 2, \ldots, M$. Then*

$$\Pr[f_1 = 0, \ldots, f_M = 0] \geq \Pr[f_1 = 0, \ldots, f_r = 0] \cdot \Pr[f_{r+1} = 0, \ldots, f_M = 0],$$

for $r = 1, 2, \ldots, M$.

Proof. Define

$$F_1 = \begin{cases} 0 & \text{if each } f_j = 0, \quad j = 1, 2, \ldots, r \\ 1 & \text{otherwise} \end{cases}$$

$$F_2 = \begin{cases} 0 & \text{if each } f_j = 0, \quad j = r+1, r+2, \ldots, M \\ 1 & \text{otherwise} \end{cases}$$

Thus $F_1 = 1 - \prod_{j=1}^{r}(1 - f_j)$, $F_2 = 1 - \prod_{j=r+1}^{M}(1 - f_j)$. Since each f_j is increasing, F_1 and F_2 are increasing. By Lemma 3

$$E(F_1 F_2) \geq E(F_1) \cdot E(F_2)$$

or equivalently

$$E(1-F_1)(1-F_2) \geq E(1-F_1) \cdot E(1-F_2)$$

or equivalently

$$\Pr[F_1 = 0,\ F_2 = 0] \geq \Pr[F_1 = 0] \cdot \Pr[F_2 = 0]$$

■

Lemma 5 *under the same hypothesis as in Lemma 4,*

$$\Pr[f_1 = 0, \ldots, f_M = 0] \geq \prod_{j=1}^{M} \Pr[f_j = 0].$$

Proof. Repeated application of Lemma 4 yields the results. ■

Now we are ready to state the Esary- Proschan bounds. They are summarized in the theorem below.

Theorem 7 *Let S be a coherent system of n independent components with reliabilities $p_1, p_2, ..., p_n$. Let $\mathbf{P} = \{P_1, P_2, \ldots, P_M\}$ and $\mathbf{C} = \{C_1, C_2, \ldots, C_N\}$ be the minimal path and cut sets respectively. Then*

$$LB_{EP} = \prod_{k=1}^{N} \coprod_{i \in C_k} p_i \leq R(\mathbf{p}) \leq \coprod_{j=1}^{M} \prod_{i \in P_j} p_i = UB_{EP}.$$

Proof. Let $\beta = 1 - \varphi$ and $\mu_k = 1 - \kappa_k$ for $k = 1, 2, \ldots, N$. Then

$$\begin{aligned}
\Pr[\beta &= 0] = \Pr[\varphi(\mathbf{x}) = 1] \\
&= \Pr[\kappa_1(\mathbf{x}) = 1, \kappa_2(\mathbf{x}) = 1, \ldots, \kappa_N(\mathbf{x}) = 1] \\
&= \Pr[\mu_1 = 0, \mu_2 = 0, \ldots, \mu_N = 0] \\
&\geq \prod_{k=1}^{N} \Pr[\mu_k = 0] \text{ from Lemma 5} \\
&= \prod_{k=1}^{N} \Pr[\kappa_k(\mathbf{x}) = 1] \\
&= \prod_{k=1}^{N} \coprod_{i \in C_k} p_i
\end{aligned}$$

which proves the lower bound.
Similarly

$$\begin{aligned}
\Pr[\varphi(\mathbf{x}) &= 0] = \Pr[\rho_1(\mathbf{x}) = 0, \rho_2(\mathbf{x}) = 0, \ldots, \rho_M(\mathbf{x}) = 0] \\
&\geq \prod_{j=1}^{M} \Pr[\rho_j(\mathbf{x}) = 0] \quad \text{from Lemma 5} \\
&= \prod_{j=1}^{M} \big(1 - \Pr[\rho_j(\mathbf{x}) = 1]\big) \\
&= \prod_{j=1}^{M} \left(1 - \prod_{i \in P_j} p_i\right)
\end{aligned}$$

Hence

$$\begin{aligned}
R(\mathbf{p}) &= \Pr[\varphi(\mathbf{x}) = 1] \\
&= 1 - \Pr[\varphi(\mathbf{x}) = 0] \\
&\leq 1 - \prod_{j=1}^{M} \left(1 - \prod_{i \in P_j} p_i\right) \\
&= \coprod_{j=1}^{M} \prod_{i \in P_j} p_i
\end{aligned}$$

Which proves the upper bound. ■

If the structure happens to have minimal path sets which are mutually disjoint, then $\rho_1, \rho_2, \ldots, \rho_M$ are independent random variables and $UB_{EP} = R(\mathbf{p})$. Similarly, if the structure has minimal cut sets which are disjoint, then $LB_{EP} = R(\mathbf{p})$.

Example 9 *In Example 7 the reliability of a bridge structure with* $p_1 = p_2 = \cdots = p_5 = p$ *is given as*

$$R(\mathbf{p}) = 2p^2 + 2p^3 - 5p^4 + 2p^5.$$

Esary- Proschan bounds are given as follows:

$$LB_{EP} = (1 - q^2)^2 \cdot (1 - q^3)^2,$$

and

$$UB_{EP} = 1 - (1 - p^2)^2 (1 - p^3)^2.$$

In Table 2 below we evaluate the bounds with low and high values of p.

Table 2: Comparison of Esary-Proschan bounds with reliability function for a bridge structure

p	LB_{EP}	$R(\mathbf{p})$	UB_{EP}
0.05	0.000193375	0.005219375	0.005242486
0.10	0.00265122	0.02152	0.02185922
0.15	0.011466193	0.049370625	0.050932534
0.85	0.949067466	0.950629374	0.988533807
0.90	0.97814078	0.97848	0.99734878
0.95	0.994757514	0.994780624	0.999806625

From Table 2 it is clear that when components have high reliabilities, Esary-Prschan's lower bound yields a good approximation to system reliability. When the components have low reliabilities, then it is the upper bound that yields a good approximation to system reliability.

3.3 Min- max bounds

Contrast to Esary and Proschan bounds, we can obtain upper bounds which depend on minimal cut sets and lower bounds which depend on minimal path sets. The bounds are summarized in the theorem below.

Theorem 8 *Let S be a coherent system of n independent components with reliabilities p_1 , p_2 , ..., p_n. Let $\mathbf{P}$ =$\{P_1, P_2, \ldots, P_M\}$ and $\mathbf{C} = \{C_1, C_2, \ldots, C_N\}$ be the minimal path and cut sets respectively. Then*

$$LB_{\min - \max} = \max_{1\leq j\leq M} \prod_{i\in P_j} p_i \ \leq R(\mathbf{p}) \leq \min_{1\leq k\leq N} \coprod_{i\in C_k} p_i \ = UB_{\min - \max}.$$

Proof. From (2.4) and (2.5) we have

$$\varphi(\mathbf{x}) = \max_{1\leq j\leq M} \quad \min_{i\in P_j} x_i \ = \min_{1\leq k\leq N} \quad \max_{i\in C_k} x_i \ .$$

Thus

$$\min_{i\in P_j} x_i \ \leq \varphi(\mathbf{x}) \leq \max_{i\in C_k} x_i \ ,$$

i.e.

$$\prod_{i\in P_j} x_i \ \leq \varphi(\mathbf{x}) \leq \coprod_{i\in C_k} x_i.$$

Taking expectations, we have

$$E\left[\prod_{i\in P_j} x_i\right] \leq E[\varphi(\mathbf{x})] \leq E\left[\coprod_{i\in C_k} x_i\right].$$

Because components are independent, this becomes

$$\prod_{i\in P_j} E(x_i\ ) \leq R(\mathbf{p}) \leq \coprod_{i\in C_k} E(x_i).$$

Hence

$$\max_{1\leq j\leq M} \prod_{i\in P_j} p_i \ \leq R(\mathbf{p}) \leq \min_{1\leq k\leq N} \coprod_{i\in C_k} p_i \ .$$

■

3.4 Inclusion-exclusion bounds

From (2.9) the reliability of a system is given as

$$R(\mathbf{p}) = \sum_{j=1}^{M} (-1)^{j-1} E(T_j),$$

where $T_j = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_j \leq M} \rho_{i_1} \rho_{i_2} \ldots \rho_{i_j}$. It is clear that

$$\begin{aligned} R(\mathbf{p}) &\leq E(T_1) \\ &\geq E(T_1) - E(T_2) \\ &\leq E(T_1) - E(T_2) + E(T_3) \\ &\geq E(T_1) - E(T_2) + E(T_3) - E(T_4) \\ &\quad \vdots \end{aligned}$$

that is, for odd k, $1 \leq k \leq M$:

$$R(\mathbf{p}) \leq \sum_{j=1}^{k} (-1)^{j-1} E(T_j),$$

and for even k, $2 \leq k \leq M$:

$$R(\mathbf{p}) \geq \sum_{j=1}^{k} (-1)^{j-1} E(T_j).$$

These bounds converge to $R(\mathbf{p})$, but not necessarily monotone. In practice it may be necessary to calculate only a few T_j's to obtain a close approximation. Hence

$$LB_{Inc-Ex} = E(T_1) - E(T_2) \leq R(\mathbf{p}) \leq E(T_1) = UB_{Inc-Ex},$$

gives a close approximation.

3.5 Spross Bounds

Motivated by an observation that most approximation methods are partially based on a rather complicated mathematics, Spross, under the supervision of Professor Beichelt [6], emphasizes the usefulness of a simple upper and lower bound on system reliability. The construction of these bounds requires only a few minimal path and cut sets of the system and the generation of disjoint sums (section 2.6). The bounds are derived as follows:
From (2.6) the structure function of a coherent system is given as

$$\varphi(\mathbf{x}) = \sum_{j=1}^{M} \delta_j,$$

and

$$\varphi(\mathbf{x}) = 1 - \sum_{k=1}^{N} \vartheta_k \, .$$

From (2.10) the reliability of a system is given as

$$R(\mathbf{p}) = \sum_{j=1}^{M} E[\delta_j],$$

and

$$1 - R(\mathbf{p}) = \sum_{k=1}^{N} E[\vartheta_k \,].$$

Without loss of generality, let $P_1, P_2, \ldots, P_g$ and $C_1, C_2, \ldots, C_h$ be the given, known minimal path sets and minimal cut sets, respectively, where $1 \leq g \leq M$ and $1 \leq h \leq N$.

Define

$$\underline{b}(g) = \sum_{j=1}^{g} E[\delta_j],$$

and

$$\bar{b}(h) = 1 - \sum_{k=1}^{h} E[\vartheta_k \,].$$

Then

$$LB_{BS} = \underline{b}(g) \leq R(\mathbf{p}) \leq \bar{b}(h) = UB_{BS}.$$

The calculation of $\underline{b}(g)$ and $\bar{b}(h)$ is stopped when, for a given measure of accuracy ϵ, $d(g,h) \leq \epsilon$, where $d(g,h) = \bar{b}(h) - \underline{b}(g)$. Otherwise g and h have to be enlarged.

Example 10 *This example is extracted from [6]*

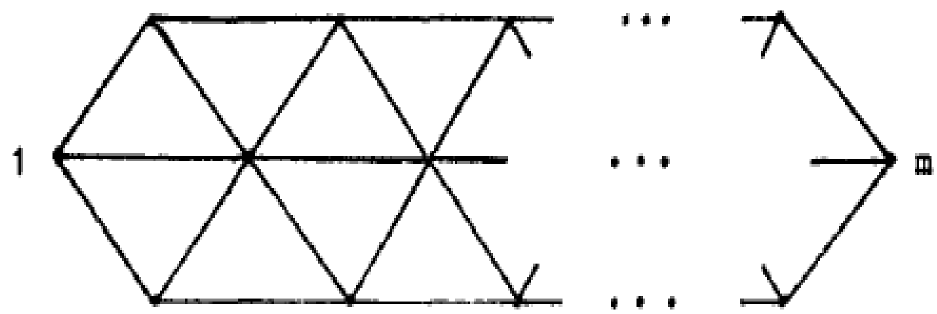

Figure 3.1: Example of a structure of variable size

Figure 3.1 above is a reliability block diagram with $m = 3k+1$ nodes and $n = 5 + 7(k-1)$ edges, $k = 1, 2 \ldots$. Table 3 below presents the number of minimal path and cut sets with respect to m. The table shows that the exact computation of $R(\mathbf{p})$ for $m \geq 19$ is not possible by applying the disjoint sum approach.

Table 3: Number of minimal path and cut sets

m	**# minimal paths**	**# minimal cuts**
4	3	4
7	21	18
10	151	56
13	1081	148
16	7739	356
19	55405	806

Table 4 below contains Spross lower and upper bounds for a structure with $m = 19$ nodes. To compute these bounds only minimal paths (cuts) with cardinalities ≤ 8 (≤ 5) have been used. From Table 4, $d(43, 15) = 0.0104$. Note that the number of minimal paths and cuts required to estimate the system reliability is much more less than the total number of paths/cuts.

Table 4: Lower and upper bounds

g	$\underline{b}(g)$	h	$\bar{b}(h)$
5	0.6453	2	0.9980
10	0.8069	4	0.9978
20	0.9040	8	0.9976
40	0.9765	12	0.9976
43	0.9871	15	0.9975

3.6 Fu- Koutras bounds

Motivated by the observation that Esary and Proschan's bounds cannot be effectively used for approximating system reliability from both sides, Fu and Koutras [13] have derived improved bounds to system reliability. Contrary to Esary and Proschan's bounds, Fu and Koutras' upper bound is derived from minimal cut sets and the lower bound is derived from minimal path sets.

Before we derive the bounds, we need the following definition:

Definition 12 *Let $C_1, C_2, \ldots, C_N$ be the minimal cut sets of system S.Define $L_1^* = \varnothing$, $L_k^* = \{i \mid C_i \cap C_k \neq \varnothing, 1 \leq i < k\}$, $k = 2, \ldots, N$. For each nonempty L_k^*, let L_k be an index set satisfying the following conditions:*

1. *$L_k \cap C_i \neq \varnothing \quad$ for every $i \in L_k^*$*
2. *$L_k \cap C_k = \varnothing$*

If $L_k^* = \varnothing$ we conventionally set $L_k = \varnothing$.

Fu and Koutras upper bound (UB_{FK}) is derived in the theorem below:

Theorem 9 *If $\kappa_k(\mathbf{x})$ is the kth minimal cut parallel structure generated by cut set C_k , $k = 1, 2, \ldots, N$, then*

$$R(\mathbf{p}) \leq \prod_{k=1}^{N} \left[1 - \left(\prod_{i \in L_k} p_i\right) \Pr[\kappa_k(\mathbf{x}) = 0]\right] = \prod_{k=1}^{N} \left[1 - \left(\prod_{i \in L_k} p_i\right)\left(\prod_{i \in C_k} q_i\right)\right].$$

Proof. The derivation is based on expressing the reliability $R(\mathbf{p})$ as the probability of intersection of events. Let S_k be the event that all components of cut set C_k have failed and $S_k^{'}$ the complimentary event, i.e, there exists at least one functioning component in cut set C_k. It is clear that $R = \Pr\left(\bigcap_{k=1}^{N} S_k^{'}\right)$.

Using the chain rule- probability we have

$$\begin{aligned} R &= \Pr\left(\bigcap_{k=1}^{N} S_k^{'}\right) \\ &= \Pr(S_1^{'}) \cdot \prod_{k=2}^{N} \Pr\left(S_k^{'} \mid \bigcap_{j=1}^{k-1} S_j^{'}\right) \qquad \text{[chain rule]} \\ &= [1 - \Pr(S_1)] \cdot \prod_{k=2}^{N} \left[1 - \Pr\left(S_k \mid \bigcap_{j=1}^{k-1} S_j^{'}\right)\right] \\ &= \prod_{k=1}^{N} (1 - \alpha_k), \end{aligned}$$

where $\alpha_1 = \Pr(S_1)$, $\alpha_k = \Pr\left(S_k \mid \bigcap_{j=1}^{k-1} S_j^{'}\right)$, $k = 2, 3, \ldots, N$.

Let D_k, $k = 1, 2, \ldots, N$, be the event that all the components contained in index set L_k are working (if L_k is empty, D_k is considered as an event with probability 1). Note $\Pr(D_k) = \prod_{i \in L_k} p_i$ and $\Pr(S_k) = \Pr(\kappa_k(\mathbf{x}) = 0)$.

If $L_k^* = \varnothing$ then S_k is independent of $\bigcap_{j=1}^{k-1} S_j^{'}$, therefore

$$\alpha_k = \Pr(S_k) = \Pr(S_k) \cdot \Pr(D_k),$$

hence

$$\begin{aligned} R(\mathbf{p}) &= \prod_{k=1}^{N} \left[1 - \left(\prod_{i \in L_k} p_i \right) (\Pr(\kappa_k (\mathbf{x}) = 0)) \right] \\ &= \prod_{k=1}^{N} \left[1 - \left(\prod_{i \in L_k} p_i \right) \left(\prod_{i \in C_k} q_i \right) \right], \end{aligned}$$

which proves the equality.
For $L_k^* \neq \varnothing$ it is obvious that

$$\begin{aligned} \alpha_k &\geq \Pr \left(S_k \mid D_k \cap \left(\bigcap_{j=1}^{k-1} S_j^{'} \right) \right) \cdot \Pr \left(D_k \mid \bigcap_{j=1}^{k-1} S_j^{'} \right) \\ &= \beta_k \, \eta_k \,, \end{aligned}$$

where $\beta_k = \Pr \left(S_k \mid D_k \cap \left(\bigcap_{j=1}^{k-1} S_j^{'} \right) \right)$ and $\eta_k = \Pr \left(D_k \mid \bigcap_{j=1}^{k-1} S_j^{'} \right)$. Condition 1 of Definition 12 implies that $D_k \cap S_j^{'} = D_k$ for all $i \in L_k^*$ and, therefore,

$$D_k \cap \left(\bigcap_{j=1}^{k-1} S_j^{'} \right) = D_k \cap \left(\bigcap_{\substack{j=1 \\ j \notin L_k^*}}^{k-1} S_j^{'} \right) = A_k \,. \tag{3.2}$$

By (3.2), definition 12 of L_k^* and condition 2 of definition 12, we conclude that the components of C_k are not used in event A_k. Therefore, S_k and A_k are independent and

$$\beta_k = \Pr \left(S_k \mid A_k \right) = \Pr \left(S_k \right).$$

Next, let

$$L_k^o = \{ j \mid C_j \cap L_k = \varnothing, \ 1 \leq j < k \}.$$

Then

$$\begin{aligned}
\Pr(A_k\ ) &= \Pr\left[D_k \cap \left(\bigcap_{j\in L_k^o} S_j^{'}\right)\right] \\
&= \Pr(D_k)\cdot \Pr\left[\bigcap_{j\in L_k^o} S_j^{'}\right] \\
&\geq \Pr(D_k)\cdot \Pr\left(\bigcap_{j=1}^{k-1} S_j^{'}\right).
\end{aligned}$$

Now

$$\begin{aligned}
\eta_k &= \frac{\Pr(A_k\ )}{\Pr\left(\bigcap_{j=1}^{k-1} S_j^{'}\right)} \\
&\geq \frac{\Pr(D_k)\cdot \Pr\left(\bigcap_{j=1}^{k-1} S_j^{'}\right)}{\Pr\left(\bigcap_{j=1}^{k-1} S_j^{'}\right)} \\
&= \Pr(D_k).
\end{aligned}$$

Hence

$$\begin{aligned}
\alpha_k &\geq \beta_k\, \eta_k \\
&= \Pr\left(S_k\right)\cdot \Pr\left(D_k \mid \bigcap_{j=1}^{k-1} S_j^{'}\right) \\
&\geq \Pr\left(S_k\right)\cdot \Pr(D_k).
\end{aligned}$$

So that

$$1-\alpha_k\ \leq 1-\Pr\left(S_k\right)\cdot \Pr(D_k).$$

Hence

$$\begin{aligned} R &= \prod_{k=1}^{N} (1 - \alpha_k\ ) \\ &\leq \prod_{k=1}^{N} [1 - \Pr(S_k) \cdot \Pr(D_k)] \\ &= \prod_{k=1}^{N} \left[1 - \left(\prod_{i \in L_k} p_i \right) \left(\prod_{i \in C_k} q_i \right) \right], \end{aligned}$$

which proves the upper bound. ■

Numerically, from Example 9 above and Example 11 below, we can see that Fu and Koutras' upper bound is very close to Esary and Proschan's lower bound. Contrary to Esary and Proschan's lower bound though, Fu and Koutras' upper bound depends on the numbering of minimal cut sets. It remains an open question which is the optimal arrangement of the minimal cut sets of a given coherent system yielding the lowest possible UB_{FK}. Another interesting point is that, even after having specified the arrangement to be used, the selection of index sets L_k may not be unique. If such a situation occurs, Fu and Koutras suggest that L_k should be chosen such that $\Pr(D_k) = \prod_{i \in L_k} p_i$ is maximized, hence bringing the product terms of LB_{EP} and UB_{FK} as close as possible. In the i.i.d case, the maximization of $\Pr(D_k)$ is equivalent to the minimization of the cardinality of L_k.

Next we derive Fu and Koutras' lower bound (LB_{FK}), but before we do that we need the following definition:

Definition 13 *Let $P_1, P_2, \ldots, P_M$ be the minimal path sets of system S.Define $K_1^* = \varnothing$, $K_j^* = \{i \mid P_i \cap P_j \neq \varnothing,\ 1 \leq i < j\}$, $j = 1, 2, \ldots, M$. For each nonempty K_j^*, let K_j be an index set satisfying the following conditions:*

1. $K_j \cap P_i \neq \varnothing$ *for every* $i \in K_j^*$

2. $K_j \cap P_j = \varnothing$

If $K_j^* = \varnothing$ we conventionally set $K_j = \varnothing$

LB_{FK} is derived in the theorem below:

Theorem 10 *If $\rho_j(\mathbf{x})$ is the jth minimal path series structure generated by path set P_j , $j = 1, 2, \ldots, M$, then*

$$R(\mathbf{p}) \geq 1 - \prod_{j=1}^{M} \left[1 - \left(\prod_{i \in K_j} q_i\right) \Pr[\rho_j(\mathbf{x}) = 1]\right] = 1 - \prod_{j=1}^{M} \left[1 - \left(\prod_{i \in K_j} q_i\right) \prod_{i \in P_j} p_i\right].$$

Proof. The derivation is based on expressing the reliability $R(\mathbf{p})$ as the probability of intersection of events. Let T_j be the event that all components of path set P_j are working and $T_j^{'}$ the complimentary event, i.e, there is at least one component that has failed in path set P_j. It is clear that $1 - R = \Pr\left(\bigcap_{j=1}^{M} T_j^{'}\right)$.

Using the chain rule- probability we have

$$\begin{aligned}
1 - R &= \Pr\left(\bigcap_{j=1}^{M} T_j^{'}\right) \\
&= \Pr(T_1^{'}) \cdot \prod_{j=2}^{M} \Pr\left(T_j^{'} \mid \bigcap_{i=1}^{j-1} T_i^{'}\right) && \text{[chain rule]} \\
&= [1 - \Pr(T_1)] \cdot \prod_{j=2}^{M} \left[1 - \Pr\left(T_j \mid \bigcap_{i=1}^{j-1} T_i^{'}\right)\right] \\
&= \prod_{j=1}^{M} (1 - \omega_j)
\end{aligned}$$

where $\omega_1 = \Pr(T_1)$, $\omega_j = \Pr\left(T_j \mid \bigcap_{i=1}^{j-1} T_i^{'}\right)$, $j = 2, 3, \ldots, M$.

Let B_j, $j = 1, 2, \ldots, M$, be the event that all the components contained in index set K_j are working (if K_j is empty, B_j is considered as an event with probability 1). Note $\Pr(B_j) = \prod_{i \in K_j} p_i$ and $\Pr(T_j) = \Pr(\rho_j(\mathbf{x}) = 1)$.

If $K_j^* = \varnothing$ then T_j is independent of $\bigcap_{i=1}^{j-1} T_i^{'}$, therefore

$$\omega_j = \Pr(T_j) = \Pr(T_j) \cdot \Pr(B_j).$$

Hence

$$\begin{aligned}
1-R(\mathbf{p}) &= \prod_{j=1}^{M}(1-\omega_j\ ) \\
&= \prod_{j=1}^{M}(1-\Pr(T_j)\cdot\Pr(B_j)) \\
&= \prod_{j=1}^{M}\left[1-\left(\prod_{i\in K_j} p_i\right)\left(\Pr(\rho_j\ (\mathbf{x})=1)\right)\right] \\
&= \prod_{j=1}^{M}\left[1-\left(\prod_{i\in K_j} p_i\right)\left(\prod_{i\in P_j} p_i\right)\right].
\end{aligned}$$

So that

$$R(\mathbf{p})=1-\prod_{j=1}^{M}\left[1-\left(\prod_{i\in K_j} p_i\right)\left(\prod_{i\in P_j} p_i\right)\right],$$

which proves the equality.
For $K_j^* \neq \varnothing$ it is obvious that

$$\begin{aligned}
\omega_j &\geq \Pr\left(T_j \mid B_j \cap \left(\bigcap_{i=1}^{j-1} T_i^{'}\right)\right)\cdot\Pr\left(B_j \mid \bigcap_{i=1}^{j-1} T_i^{'}\right) \\
&= \lambda_j\ \cdot\upsilon_j\ ,
\end{aligned}$$

where $\lambda_j\ =\Pr\left(T_j \mid B_j \cap \left(\bigcap_{i=1}^{j-1} T_i^{'}\right)\right)$ and $\upsilon_j\ =\Pr\left(B_j \mid \bigcap_{i=1}^{j-1} T_i^{'}\right)$. Condition 1 in Definition 13 implies that $B_j \cap T_i^{'} = B_j$ for all $i \in K_j^*$ and, therefore,

$$B_j \cap \left(\bigcap_{i=1}^{j-1} T_i^{'}\right) = B_j \cap \left(\bigcap_{\substack{i=1\\ i\notin K_j^*}}^{j-1} T_i^{'}\right) = G_j. \tag{3.3}$$

By (3.3), definition 13 of K_j^* and condition 2, we conclude that the components of P_j are not used in event G_j. Therefore, T_j and G_j are independent and

$$\lambda_j\ =\Pr\left(T_j \mid G_j\right)=\Pr\left(T_j\right).$$

Next, let

$$K_j^o = \{i \mid P_i \cap K_j = \varnothing,\ 1 \leq i < j\}.$$

Then

$$\begin{aligned}
\Pr(G_j) &= \Pr\left[B_j \cap \left(\bigcap_{i \in K_j^o} T_i^{'}\right)\right] \\
&= \Pr(B_j) \cdot \Pr\left[\bigcap_{i \in K_j^o} T_i^{'}\right] \\
&\geq \Pr(B_j) \cdot \Pr\left(\bigcap_{i=1}^{j-1} T_i^{'}\right).
\end{aligned}$$

Now

$$\begin{aligned}
\upsilon_j &= \frac{\Pr(G_j)}{\Pr\left(\bigcap_{i=1}^{j-1} T_i^{'}\right)} \\
&\geq \frac{\Pr(B_j) \cdot \Pr\left(\bigcap_{i=1}^{j-1} T_i^{'}\right)}{\Pr\left(\bigcap_{i=1}^{j-1} T_i^{'}\right)} \\
&= \Pr(B_j).
\end{aligned}$$

Hence

$$\begin{aligned}
\omega_j &\geq \lambda_j \cdot \upsilon_j \\
&= \Pr(T_j) \cdot \Pr\left(B_j \mid \bigcap_{i=1}^{j-1} T_i^{'}\right) \\
&\geq \Pr(T_j) \cdot \Pr(B_j).
\end{aligned}$$

So that

$$1 - \omega_j \leq 1 - \Pr(T_j) \cdot \Pr(B_j).$$

Hence

$$\begin{aligned} 1 - R &= \prod_{j=1}^{M} (1 - \omega_j) \\ &\leq \prod_{j=1}^{M} [1 - \Pr(T_j) \cdot \Pr(B_j)] \\ &= \prod_{j=1}^{M} \left[1 - \left(\prod_{i \in K_j} p_i \right) \left(\prod_{i \in P_j} p_i \right) \right]. \end{aligned}$$

So that

$$R(\mathbf{p}) = 1 - \prod_{j=1}^{M} \left[1 - \left(\prod_{i \in K_j} p_i \right) \left(\prod_{i \in P_j} p_i \right) \right],$$

which proves the lower bound. ■

Example 11 *Bridge structure*

$$L_1^* = L_2^* = \varnothing, \qquad L_3^* = \{1, 2\}, \qquad L_4^* = \{1, 2, 3\}$$

$$L_1 = L_2 = \varnothing, \qquad L_3 = \{2, 4\}, \qquad L_4 = \{1, 5\}$$

$$\begin{aligned} UB_{FK} &= \prod_{k=1}^{4} \left[1 - \left(\prod_{i \in L_k} p_i \right) \left(\prod_{i \in C_k} q_i \right) \right] \\ &= (1 - q^2)^2 (1 - p^2 q^3)^2 \end{aligned}$$

$$K_1^* = K_2^* = \varnothing, \qquad K_3^* = \{1, 2\}, \qquad K_4^* = \{1, 2, 3\}$$

$$K_1 = K_2 = \varnothing, \qquad K_3 = \{2, 4\}, \qquad K_4 = \{1, 5\}$$

$$\begin{aligned} LB_{FK} &= 1 - \prod_{j=1}^{4} \left[1 - \left(\prod_{i \in K_j} q_i \right) \prod_{i \in P_j} p_i \right] \\ &= 1 - (1 - p^2)^2 (1 - p^3 q^2)^2 \end{aligned}$$

Table 3 below gives the numerical values of LB_{FK}, $R(\mathbf{p})$ and UB_{FK} for low and high values of p.

Table 5: Fu-Koutras bounds

p	LB_{FK}	$R(\mathbf{p})$	UB_{FK}
0.05	5.2182×10^{-3}	0.005219375	9.4655×10^{-3}
0.10	2.1487×10^{-2}	0.02152	3.5576×10^{-2}
0.15	4.9148×10^{-2}	0.049370625	7.4893×10^{-2}
0.85	0.92511	0.950629374	0.95085
0.90	0.96442	0.97848	0.97851
0.95	0.99053	0.994780624	0.99478

3.7 Koutras-Papastavridis-Petakos Bounds

Koutras, Papastavridis and Petakos also derive bounds that are an improvement to Esary and Proschan's bounds.

The bounds are derived as follows: define L_k^* and K_j^* as in section 3.6 above. Let κ_k^* , $k = 1, 2, \ldots, N$, be the structure function of the system with minimal cut sets C_i, $i \in L_k^*$ and ρ_j^* , $j = 1, 2, \ldots, M$, a structure function of a system with minimal path sets P_i, $i \in K_j^*$. That is,

$$\kappa_k^* = \prod_{i \in L_k^*} \kappa_i \text{ and } \rho_j^* = \coprod_{i \in K_j^*} \rho_i \,.$$

For $L_k^* = \varnothing$ or $K_j^* = \varnothing$, we set $\kappa_k^* = 1$ or $\rho_j^* = 1$ respectively.

Before we derive Koutras, Papastavridis and Petakos upper bound (UB_{KPP}), we need the following definition:

Definition 14 *A family of non- decreasing binary functions $\{\mu_k = \mu_k(\mathbf{x})$, $k = 1, 2, \ldots, N\}$ will be called $\mu-$cut structures for the system S with minimal cut sets $C_1, C_2, \ldots, C_N$ if $\mu_1 = 1$ and the following conditions are satisfied for all $k = 2, 3, \ldots, N$:*

(i) *The components c_i, $i \in C_k$ are irrelevant for the structure defined by μ_k , i.e. the value of the function μ_k does not depend on the variables x_i , $i \in C_k$.*

(ii) *$\mu_k(\mathbf{x}) \leq \kappa_k^*(\mathbf{x})$ for every state vector $\mathbf{x} \in \mathbf{V}_n$.*

In words, condition (ii) of Definition 14 states that, if the structure defined by μ_k is up, then there is at least one working component in every C_i, $i \in L_k^*$. It is also worth mentioning that the set of all feasible $\mu-cut$ structures depends on the specific ordering of the minimal cut sets, since a rearrangement of them will manifestly alter the index sets L_k^*, $k = 2, 3, \ldots, N$.

Now we are ready to state UB_{KPP}. This is done in the theorem below.

Theorem 11 *If* $\{\mu_k\ ,\ k = 1, 2, \ldots, N\}$ *is any* $\mu - cut$ *structure for the system* S *with minimal cut sets* $C_1, C_2, \ldots, C_N$, *then*

$$\begin{aligned} R(\mathbf{p}) &\leq \prod_{k=1}^{N} [1 - \Pr(\mu_k\ = 1) \cdot \Pr(\kappa_k\ (\mathbf{x}) = 0)] \\ &= \prod_{k=1}^{N} \left[1 - \Pr(\mu_k\ = 1) \cdot \left(\prod_{i \in C_k} q_i \right) \right]. \end{aligned}$$

Proof. Let S_k be the event that all components of cut set C_k have failed and $S_k^{'}$ the complimentary event, i.e, there exists at least one functioning component in cut set C_k. It is clear that $R = \Pr\left(\bigcap_{k=1}^{N} S_k^{'} \right)$ .Also let A_k be the event $\{\mu_k\ = 1\}$.
Using the chain rule- probability we have

$$\begin{aligned} R &= \Pr\left(\bigcap_{k=1}^{N} S_k^{'} \right) \\ &= \Pr(S_1^{'}) \cdot \prod_{k=2}^{N} \Pr\left(S_k^{'} \mid \bigcap_{j=1}^{k-1} S_j^{'} \right) \\ &= [1 - \Pr(S_1)] \cdot \prod_{k=2}^{N} \left[1 - \Pr\left(S_k \mid \bigcap_{j=1}^{k-1} S_j^{'} \right) \right] \\ &= \prod_{k=1}^{N} (1 - \alpha_k\ ), \end{aligned}$$

where $\alpha_1\ = \Pr(S_1)$, $\quad \alpha_k\ = \Pr\left(S_k \mid \bigcap_{j=1}^{k-1} S_j^{'} \right)$, $k = 2, 3, \ldots, N$. Note that $\Pr(S_k) = \Pr(\kappa_k\ (\mathbf{x}) = 0)$.

If $L_k^* = \varnothing$ then S_k is independent of $\bigcap_{j=1}^{k-1} S_j^{'}$, therefore

$$\begin{aligned}\alpha_k &= \Pr(S_k) \\ &\geq \Pr(S_k) \cdot \Pr(A_k\ ).\end{aligned}$$

So that

$$1 - \alpha_k\ \leq 1 - \Pr(S_k) \cdot \Pr(A_k\ ).$$

Hence

$$\begin{aligned}R(\mathbf{p}) &= \prod_{k=1}^{N} (1 - \alpha_k\ ) \\ &\leq \prod_{k=1}^{N} [1 - \Pr(S_k) \cdot \Pr(A_k\ )] \\ &= \prod_{k=1}^{N} \left[1 - \Pr(\mu_k\ = 1) \cdot \left(\prod_{i \in C_k} q_i\right)\right],\end{aligned}$$

which proves the upper bound.

For $L_k^* \neq \varnothing$ it is obvious that

$$\alpha_k\ \geq \Pr\left(S_k \mid A_k \cap \left(\bigcap_{j=1}^{k-1} S_j^{'}\right)\right) \cdot \Pr\left(A_k \mid \bigcap_{j=1}^{k-1} S_j^{'}\right).$$

Condition (ii) of Definition 14 implies that

$$A_k \subseteq \bigcap_{j \in L_k^*} S_j^{'},$$

and therefore

$$A_k \cap \left(\bigcap_{j=1}^{k-1} S_j^{'}\right) = A_k \cap \left(\bigcap_{\substack{j=1 \\ j \notin L_k^*}}^{k-1} S_j^{'}\right) = E_k\ .$$

The event E_k is independent of S_k because the binary functions in $\bigcap_{j=1}^{k-1} S_j^{'}$ do not make use of the indicator variables associated with cut set C_k. Therefore

$$\Pr\left(S_k \mid A_k \cap \left(\bigcap_{j=1}^{k-1} S_j^{'}\right)\right) = \Pr\left(S_k \mid E_k\right) = \Pr(S_k).$$

On the other hand

$$\begin{aligned}
\Pr(E_k) &= \Pr\left[A_k \cap \left(\bigcap_{\substack{j=1 \\ j \notin L_k^*}}^{k-1} S_j^{'}\right)\right] \\
&= \Pr(A_k) \cdot \Pr\left(\bigcap_{\substack{j=1 \\ j \notin L_k^*}}^{k-1} S_j^{'}\right) \\
&\geq \Pr(A_k) \cdot \Pr\left(\bigcap_{j=1}^{k-1} S_j^{'}\right).
\end{aligned}$$

Now

$$\begin{aligned}
\Pr\left(A_k \mid \bigcap_{j=1}^{k-1} S_j^{'}\right) &= \frac{\Pr(E_k)}{\Pr\left(\bigcap_{j=1}^{k-1} S_j^{'}\right)} \\
&\geq \frac{\Pr(A_k) \cdot \Pr\left(\bigcap_{j=1}^{k-1} S_j^{'}\right)}{\Pr\left(\bigcap_{j=1}^{k-1} S_j^{'}\right)} \\
&= \Pr(A_k)
\end{aligned}$$

Hence

$$\begin{aligned}
\alpha_k &\geq \Pr\left(S_k \mid A_k \cap \left(\bigcap_{j=1}^{k-1} S_j^{'}\right)\right) \cdot \Pr\left(A_k \mid \bigcap_{j=1}^{k-1} S_j^{'}\right) \\
&= \Pr(S_k) \cdot \Pr\left(A_k \mid \bigcap_{j=1}^{k-1} S_j^{'}\right) \\
&\geq \Pr(A_k) \cdot \Pr(S_k).
\end{aligned}$$

So that

$$1 - \alpha_k \ \leq 1 - \Pr(A_k) \cdot \Pr(S_k).$$

Hence

$$\begin{aligned} R(\mathbf{p}) &= \prod_{k=1}^{N} (1 - \alpha_k \ ) \\ &\leq \prod_{k=1}^{N} [1 - \Pr(A_k) \cdot \Pr(S_k)] \\ &= \prod_{k=1}^{N} \left[1 - \Pr(\mu_k \ = 1) \cdot \left(\prod_{i \in C_k} q_i \right) \right], \end{aligned}$$

again proving the upper bound. ■

Again, before we derive Koutras, Papastavridis and Petakos lower bound (LB_{KPP}), we need the following definition:

Definition 15 *A family of non-decreasing binary functions* β_j $\{\beta_j = \beta_j(\mathbf{x})$, $j = 1, 2, \ldots, M\}$, *will be called* $\beta - path$ *structure for the system* S *with minimal path sets* $P_1, P_2, \ldots, P_M$ *if* $\beta_1 = 1$ *and the following conditions are satisfied for all* β_j, $j = 2, 3, \ldots, M$:

(i) *The components* c_i , $i \in P_j$ *are irrelevant for the structure defined by* β_j, *i.e the value of the function* β_j *does not depend on the variables* x_i , $i \in P_j$.

(ii) $\beta_j(\mathbf{x}) \geq \rho_j^* (\mathbf{x})$ *for all* $\mathbf{x} \in \mathbf{V}_n$.

In words, condition (ii) of Definition 15 states that if the structure function defined by β_j is down, then there is at least one component not working in every P_i, $i \in \kappa_j^*$. Note that the $\beta - path$ structure depends on the specific ordering of system's minimal path sets.

LB_{KPP} is stated in the theorem below.

Theorem 12 *If $\{\beta_j,\ j = 1, 2, \ldots, M\}$ is any $\beta -$ path structure for the system S with minimal path sets $P_1, P_2, \ldots, P_M$, then*

$$\begin{aligned} R(\mathbf{p}) &\geq 1 - \prod_{j=1}^{M} [1 - \Pr(\beta_j = 0) \cdot \Pr(\rho_j\ = 1)] \\ &= 1 - \prod_{j=1}^{M} \left[1 - \Pr(\beta_j = 0) \cdot \prod_{i \in P_j} p_i \right]. \end{aligned}$$

Proof. Let T_j be the event that all components of path set P_j are working and $T_j^{'}$ the complimentary event, i.e, $T_j^{'}$ is the event that there is at least one component that has failed in path set P_j. It is clear that $1 - R = \Pr\left(\bigcap_{j=1}^{M} T_j^{'}\right)$.
Also let B_j be an event $\{\beta_j = 0\}$.
Using the chain rule- probability we have

$$\begin{aligned} 1 - R &= \Pr\left(\bigcap_{j=1}^{M} T_j^{'}\right) \\ &= \Pr(T_1^{'}) \cdot \prod_{j=2}^{M} \Pr\left(T_j^{'} \mid \bigcap_{i=1}^{j-1} T_i^{'}\right) \qquad \text{[chain rule]} \\ &= [1 - \Pr(T_1)] \cdot \prod_{j=2}^{M} \left[1 - \Pr\left(T_j \mid \bigcap_{i=1}^{j-1} T_i^{'}\right)\right] \\ &= \prod_{j=1}^{M} (1 - \omega_j\ ), \end{aligned}$$

where $\omega_1 = \Pr(T_1)$, $\omega_j = \Pr\left(T_j \mid \bigcap_{i=1}^{j-1} T_i^{'}\right)$, $j = 2, 3, \ldots, M$. Note that $\Pr(T_j) = \Pr(\rho_j\ (\mathbf{x}) = 1)$.
If $K_j^{*} = \varnothing$ then T_j is independent of $\bigcap_{i=1}^{j-1} T_i^{'}$, therefore

$$\begin{aligned} \omega_j &= \Pr(T_j) \\ &\geq \Pr(T_j) \cdot \Pr(B_j). \end{aligned}$$

So that

$$1 - \omega_j\ \leq 1 - \Pr(T_j) \cdot \Pr(B_j).$$

Hence

$$\begin{aligned} 1 - R &= \prod_{j=1}^{M} (1 - \omega_j\ ) \\ &\le \prod_{j=1}^{M} \left[1 - \Pr(T_j) \cdot \Pr(B_j)\right]. \end{aligned}$$

So that

$$\begin{aligned} R(\mathbf{p}) &\geq 1 - \prod_{j=1}^{M} \left[1 - \Pr(T_j) \cdot \Pr(B_j)\right] \\ &= 1 - \prod_{j=1}^{M} \left[1 - \Pr(\beta_j = 0) \cdot \prod_{i \in P_j} p_i\ \right], \end{aligned}$$

which proves the lower bound.
For $K_j^* \neq \varnothing$ it is obvious that

$$\begin{aligned} \omega_j &\geq \Pr\left(T_j \mid B_j \cap \left(\bigcap_{i=1}^{j-1} T_i^{'}\right)\right) \cdot \Pr\left(B_j \mid \bigcap_{i=1}^{j-1} T_i^{'}\right) \\ &= \lambda_j\ \cdot \upsilon_j\ , \end{aligned}$$

where $\lambda_j\ = \Pr\left(T_j \mid B_j \cap \left(\bigcap_{i=1}^{j-1} T_i^{'}\right)\right)$ and $\upsilon_j\ = \Pr\left(B_j \mid \bigcap_{i=1}^{j-1} T_i^{'}\right)$.
Condition (ii) of definition 15 implies that

$$\bigcap_{i \in K_j^*} T_i^{'} \subseteq B_j,$$

and therefore

$$B_j \cap \left(\bigcap_{i=1}^{j-1} T_i^{'}\right) = B_j \cap \left(\bigcap_{\substack{i=1 \\ i \notin K_j^*}}^{j-1} T_i^{'}\right) = D_j.$$

The event D_j is independent of T_j because the binary functions in $\bigcap_{i=1}^{j-1} T_i^{'}$ do not make use of the indicator variables associated with path set P_j. Therefore

$$\Pr\left(T_j \mid B_j \cap \left(\bigcap_{i=1}^{j-1} T_i^{'}\right)\right) = \Pr\left(T_j \mid D_j\right) = \Pr(T_j).$$

On the other hand

$$
\begin{aligned}
\Pr(D_j) &= \Pr\left[B_j \cap \left(\bigcap_{\substack{i=1\\ i\notin K_j^*}}^{j-1} T_i^{'}\right)\right] \\
&= \Pr(B_j) \cdot \left(\bigcap_{\substack{i=1\\ i\notin K_j^*}}^{j-1} T_i^{'}\right) \\
&\geq \Pr(B_j) \cdot \left(\bigcap_{i=1}^{j-1} T_i^{'}\right).
\end{aligned}
$$

Now

$$
\begin{aligned}
\Pr\left(B_j \mid \bigcap_{i=1}^{j-1} T_i^{'}\right) &= \frac{\Pr(D_j)}{\Pr\left(\bigcap_{i=1}^{j-1} T_i^{'}\right)} \\
&\geq \frac{\Pr(B_j) \cdot \left(\bigcap_{i=1}^{j-1} T_i^{'}\right)}{\Pr\left(\bigcap_{i=1}^{j-1} T_i^{'}\right)} \\
&= \Pr(B_j).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\omega_j &\geq \Pr\left(T_j \mid B_j \cap \left(\bigcap_{i=1}^{j-1} T_i^{'}\right)\right) \cdot \Pr\left(B_j \mid \bigcap_{i=1}^{j-1} T_i^{'}\right) \\
&= \Pr(T_j) \cdot \Pr\left(B_j \mid \bigcap_{i=1}^{j-1} T_i^{'}\right) \\
&\geq \Pr(T_j) \cdot \Pr(B_j).
\end{aligned}
$$

So that

$$
1 - \omega_j \leq 1 - \Pr(T_j) \cdot \Pr(B_j).
$$

Hence

$$\begin{aligned}
1-R &= \prod_{j=1}^{M}(1-\omega_j\ ) \\
&\leq \prod_{j=1}^{M}\left[1-\Pr(T_j)\cdot\Pr(B_j)\right] \\
&= \prod_{j=1}^{M}\left[1-\Pr(\beta_j=0)\cdot\prod_{i\in P_j}p_i\right].
\end{aligned}$$

So that

$$R(\mathbf{p})\geq 1-\prod_{j=1}^{M}\left[1-\Pr(\beta_j=0)\cdot\prod_{i\in P_j}p_i\right],$$

again proving the lower bound. ■

3.8 Boutsikas- Koutras Bounds

Boutsikas and Koutras [11] generalize Esary-Proschan, Fu-Koutras and min-max bounds by considering an arbitrary partition of the family of minimal cut and path sets. Before we derive these bounds, we need the following definition:

Definition 16 *$\acute{C}$ and $\acute{P}$ will denote an arbitrary partition of* $\mathbf{C}$ *and* $\mathbf{P}$ *respectively, i.e.* $\acute{C}=\{\mathbb{C}_{\mathit{1}}\ ,\mathbb{C}_2,\ldots,\mathbb{C}_v\}$, $\acute{P}=\{\P_1,\P_2,\ldots,\P_\mu\}$, *where*

- $\mathbb{C}_s\subset\mathbf{C}$, $s=1,2,\ldots,v$, $\mathbb{C}_i\cap\mathbb{C}_j=\varnothing$ *for* $i\neq j$ *and* $\bigcup_{s=1}^{v}\mathbb{C}_s=\mathbf{C}$
- $\P_s\subset\mathbf{P}$, $s=1,2,\ldots,u$, $\P_i\cap\P_j=\varnothing$ *for* $i\neq j$ *and* $\bigcup_{s=1}^{u}\P_s=\mathbf{P}$.

3.8.1 Generalized Esary-Proschan Bounds

We recall that Esary and Proschan's lower bound is obtained by considering a series system with N independent subsystems, each subsystem being a parallel structure whose components are the ones contained in $C \in \mathbf{C}$. Their upper bound is obtained by considering a parallel system with M independent subsystems, each subsystem being a series structure whose components are the ones contained in $P \in \mathbf{P}$. Boutsikas and Koutras have explored the possibilities of combining several minimal cut (path) sets together, therefore reducing the number of series (parallel) subsystems and increasing subsystem's size. Their result is summarized in the theorem below.

Theorem 13 *The reliability $R(\mathbf{p})$ of any coherent system is bounded below ($LB_{EP}^{(G)}$) and above ($UB_{EP}^{(G)}$) as follows:*

$$\prod_{s=1}^{v}\left[\prod_{C\in\mathbb{C}_s}\left(1-\prod_{i\in C} q_i\right)\right] \leq R(\mathbf{p}) \leq 1-\prod_{s=1}^{u}\left[1-\prod_{P\in\P_s}\left(1-\prod_{i\in P} p_i\right)\right].$$

Proof. The random variables $\prod_{C\in\mathbb{C}_s}\left(1-\prod_{i\in C}\bar{x}_i\right)$, $s=1,2,\ldots,v$ are coordinatewise non-decreasing functions of the independent random variables x_i, $i=1,2,\ldots,n$. Therefore by making use of 5 we deduce

$$\begin{aligned} R(\mathbf{p}) &= E\left[\prod_{k=1}^{N}\coprod_{i\in C_k} x_i\right] \\ &= E\left[\prod_{s=1}^{v}\left(\prod_{C\in\mathbb{C}_s}\coprod_{i\in C} x_i\right)\right] \\ &\geq \prod_{s=1}^{v} E\left(\prod_{C\in\mathbb{C}_s}\coprod_{i\in C} x_i\right) \\ &= \prod_{s=1}^{v}\left[\prod_{C\in\mathbb{C}_s}\left(1-\prod_{i\in C} q_i\right)\right], \end{aligned}$$

which proves the lower bound.

The random variables $1-\coprod_{P\in\P_s}\prod_{i\in P} x_i$, $s=1,2,\ldots,u$ are coordinatewise non-decreasing functions of the independent random variables x_i, $i=1,2,\ldots,n$.

Therefore by making use of 5 we deduce

$$\begin{aligned}
R(\mathbf{p}) &= E\left[\coprod_{j=1}^{M}\prod_{i\in P_j} x_i\right] \\
&= 1 - E\left[\prod_{s=1}^{u}\left(1 - \coprod_{P\in\P_s}\prod_{i\in P} x_i\right)\right] \\
&\leq 1 - \prod_{s=1}^{u} E\left(1 - \coprod_{P\in\P_s}\prod_{i\in P} x_i\right) \\
&= 1 - \prod_{s=1}^{u}\left[1 - \prod_{P\in\P_s}\left(1 - \prod_{i\in P} p_i\right)\right]
\end{aligned}$$

which proves the upper bound. ■

3.8.2 Generalized Fu-Koutras Bounds

Definition 17 *For given partitions $\acute{C}$ and $\acute{P}$ we denote $\mathbb{C}_s^*$, $\P_s^*$ the next two collections of minimal cut and path sets.*

- $\mathbb{C}_s^* = \{D \in \bigcup_{l<s} \mathbb{C}_l : D \cap \left(\bigcup_{C\in\mathbb{C}_s} C\right) \neq \varnothing\}, \quad s = 2, 3, \ldots, v.$
- $\P_s^* = \{D \in \bigcup_{l<s} \P_l : D \cap \left(\bigcup_{P\in\P_s} P\right) \neq \varnothing\}, \quad s = 2, 3, \ldots, u.$

The generalized Fu-Koutras bounds are summarized in the theorem below.

Theorem 14 *The reliability $R(\mathbf{p})$ of any coherent system is bounded below by*

$$\begin{aligned}
LB_{FK}^{(G)} &= 1 - \left[\prod_{P\in\P_1}\left(1 - \prod_{i\in P} p_i\right)\right] \cdot \prod_{s=2}^{u}\left[1 - \left(\coprod_{P\in\P_s^*\cup\P_s}\prod_{i\in P} p_i - \coprod_{P\in\P_s^*}\prod_{i\in P} p_i\right)\right] \\
&= 1 - \left[1 - E\left(\coprod_{P\in\P_1}\prod_{i\in P} x_i\right)\right] \cdot \prod_{s=2}^{u}\left[1 - E\left(\left(\coprod_{P\in\P_s}\prod_{i\in P} x_i\right)\left(1 - \coprod_{P\in\P_s^*}\prod_{i\in P} x_i\right)\right)\right],
\end{aligned}$$

and bounded above by

$$
\begin{aligned}
UB_{FK}^{(G)} &= \left[\prod_{C\in\mathbb{C}_1}\coprod_{i\in C} p_i\right]\cdot\prod_{s=2}^{v}\left[1-\left(\prod_{C\in\mathbb{C}_s^*}\coprod_{i\in C} p_i - \prod_{C\in\mathbb{C}_s^*\cup\mathbb{C}_s}\coprod_{i\in C} p_i\right)\right] \\
&= E\left[\prod_{C\in\mathbb{C}_1}\coprod_{i\in C} x_i\right]\cdot\prod_{s=2}^{v}\left(1-E\left[\left(1-\prod_{C\in\mathbb{C}_s}\coprod_{i\in C} x_i\right)\left(\prod_{C\in\mathbb{C}_s^*}\coprod_{i\in C} x_i\right)\right]\right).
\end{aligned}
$$

Proof. To establish the upper bound $UB_{FK}^{(G)}$ we observe that for $s = 1, 2, \ldots, v$, we have

$$
\begin{aligned}
& E\left[\prod_{C\in\ \cup_{j\leq s-1}\mathbb{C}_j}\ \coprod_{i\in C} x_i\right] - E\left[\prod_{C\in\ \cup_{j\leq s}\mathbb{C}_j}\ \coprod_{i\in C} x_i\right] \\
= \ & E\left[\left(1-\prod_{C\in\mathbb{C}_s}\coprod_{i\in C} x_i\right)\cdot\prod_{j=1}^{s-1}\left(\prod_{C\in\mathbb{C}_j}\coprod_{i\in C} x_i\right)\right],
\end{aligned}
$$

or alternatively,

$$
\begin{aligned}
E\left[\prod_{C\in\ \cup_{j\leq s}\mathbb{C}_j}\ \coprod_{i\in C} x_i\right] &= E\left[\prod_{C\in\ \cup_{j\leq s-1}\mathbb{C}_j}\ \coprod_{i\in C} x_i\right] \\
&\quad -E\left[\left(\left(1-\prod_{C\in\mathbb{C}_s}\coprod_{i\in C} x_i\right)\cdot\prod_{C\in\mathbb{C}_j^*}\coprod_{i\in C} x_i\right)\left(\prod_{C\in\ \cup_{j\leq s-1}\mathbb{C}_j\setminus\mathbb{C}_s^*}\ \coprod_{i\in C} x_i\right)\right].
\end{aligned} \tag{3.4}
$$

The bracketed terms inside the mean value operator are coordinatewise non-decreasing functions of the independent random variables $x_i'\ s$, $i \in I \setminus \cup_{C\in\mathbb{C}_s} C$. Hence they are associated (by P2, P3 and P4). Therefore by making use of Lemma 5 we may write

$$
\begin{aligned}
E\left[\left(1-\prod_{C\in\mathbb{C}_s}\coprod_{i\in C} x_i\right)\cdot\prod_{j=1}^{s-1}\left(\prod_{C\in\mathbb{C}_j}\coprod_{i\in C} x_i\right)\right] &\geq E\left[\left(1-\prod_{C\in\mathbb{C}_s}\coprod_{i\in C} x_i\right)\cdot\prod_{C\in\mathbb{C}_j}\coprod_{i\in C} x_i\right] \\
&\quad \cdot E\left[\prod_{C\in\ \cup_{j\leq s-1}\mathbb{C}_j\setminus\mathbb{C}_s^*}\ \coprod_{i\in C} x_i\right]
\end{aligned} \tag{3.5}
$$

Also note that

$$E\left[\prod_{C\in\,\cup_{j\le s-1}\mathbb{C}_j\setminus\mathbb{C}_s^*}\ \coprod_{i\in C} x_i\right] \ge E\left[\prod_{C\in\,\cup_{j\le s-1}\mathbb{C}_j}\ \coprod_{i\in C} x_i\right] \tag{3.6}$$

Applying (3.6) into (3.5) we get

$$\begin{aligned} E\left[\left(1-\prod_{C\in\mathbb{C}_s}\coprod_{i\in C} x_i\right)\cdot\prod_{j=1}^{s-1}\left(\prod_{C\in\mathbb{C}_j}\coprod_{i\in C} x_i\right)\right] \;\ge\; & E\left[\left(1-\prod_{C\in\mathbb{C}_s}\coprod_{i\in C} x_i\right)\cdot\prod_{C\in\mathbb{C}_j}\coprod_{i\in C} x_i\right] \\ & \cdot E\left[\prod_{C\in\,\cup_{j\le s-1}\mathbb{C}_j}\ \coprod_{i\in C} x_i\right] \end{aligned} \tag{3.7}$$

Using (3.4) and (3.7) we verify that

$$\begin{aligned} E\left[\prod_{C\in\,\cup_{j\le s}\mathbb{C}_j}\ \coprod_{i\in C} x_i\right] &\le E\left[\prod_{C\in\,\cup_{j\le s-1}\mathbb{C}_j}\ \coprod_{i\in C} x_i\right]\left(1-E\left[\left(1-\prod_{C\in\mathbb{C}_s}\ \coprod_{i\in C} x_i\right)\cdot\prod_{C\in\mathbb{C}_s^*}\ \coprod_{i\in C} x_i\right]\right) \\ &= E\left[\prod_{C\in\,\cup_{j\le s-1}\mathbb{C}_j}\ \coprod_{i\in C} x_i\right] \\ &\quad \left(1-\left(E\left[\prod_{C\in\mathbb{C}_s^*}\ \coprod_{i\in C} x_i\right]-E\left[\prod_{C\in\mathbb{C}_s\cup\mathbb{C}_s^*}\ \coprod_{i\in C} x_i\right]\right)\right). \end{aligned}$$

The upper bound $UB_{FK}^{(G)}$ is easily obtained by induction. ■

3.8.3 Generalized Min-Max Bounds

Theorem 15 *The reliability $R(\mathbf{p})$ of any coherent system is bounded below ($LB_{\min-\max}^{(G)}$) and above ($UB_{\min-\max}^{(G)}$) as follows:*

$$LB_{\min-\max}^{(G)} = \min_{1\le s\le v}\prod_{C\in\mathbb{C}_s}\ \coprod_{i\in C} p_i \;\le R(\mathbf{p}) \le \max_{1\le s\le u}\prod_{P\in\P_s}\ \left(1-\prod_{i\in P} p_i\right) = UB_{\min-\max}^{(G)}.$$

Proof. The random variables $\prod_{C\in\mathbb{C}_s}\left(1-\prod_{i\in C}\bar{x}_i\right)$, $s=1,2,\ldots,v$ are coordinatewise non-decreasing functions of the independent random variables x_i ,

$i = 1, 2, \ldots, n$. Therefore by making use of 5 we deduce

$$\begin{aligned}
R(\mathbf{p}) &= E\left[\prod_{k=1}^{N} \coprod_{i \in C_k} x_i\right] \\
&= E\left[\prod_{s=1}^{v}\left(\prod_{C \in \mathbb{C}_s} \coprod_{i \in C} x_i\right)\right] \\
&\geq \prod_{s=1}^{v} E\left(\prod_{C \in \mathbb{C}_s} \coprod_{i \in C} x_i\right) \\
&= \prod_{s=1}^{v}\left(\prod_{C \in \mathbb{C}_s} \coprod_{i \in C} p_i\right) \\
&= \min_{1 \leq s \leq v}\left\{\prod_{C \in \mathbb{C}_s} \coprod_{i \in C} p_i\right\},
\end{aligned}$$

which proves the lower bound.
Similarly,

$$\begin{aligned}
1 - R(\mathbf{p}) &= 1 - E\left[\coprod_{j=1}^{M} \prod_{i \in P_j} x_i\right] \\
&= E\left[\prod_{s=1}^{u}\left(1 - \coprod_{P \in \P_s} \prod_{i \in P} x_i\right)\right] \\
&\geq \prod_{s=1}^{u} E\left(1 - \coprod_{P \in \P_s} \prod_{i \in P} x_i\right) \\
&= \prod_{s=1}^{u}\left[1 - \prod_{P \in \P_s}\left(1 - \prod_{i \in P} p_i\right)\right]
\end{aligned}$$

So

$$\begin{aligned}
R(\mathbf{p}) &\leq 1 - \prod_{s=1}^{u}\left[1 - \prod_{P \in \P_s}\left(1 - \prod_{i \in P} p_i\right)\right] \\
&= \coprod_{s=1}^{u}\left(\prod_{P \in \P_s}\left(1 - \prod_{i \in P} p_i\right)\right) \\
&= \max_{1 \leq s \leq u}\left\{\prod_{P \in \P_s}\left(1 - \prod_{i \in P} p_i\right)\right\},
\end{aligned}$$

which proves the upper bound. ■

4 NUMERICAL COMPARISON OF RELIABILITY BOUNDS

In this chapter we compare numerically the bounds derived in chapter 3. Firstly we will use the extended bridge structure (Figure 4.1) below to compare the bounds. Secondly we will use Figure 4.4 to establish whether it is better to take the system as a whole or to break it into modules and then evaluate modules individually. And finally we will investigate the effect a component with a lowest/highest Birnbaum importance has on the bounds, using Figure 4.1. For the first and second sections we will assume that $p_1 = p_2 = \ldots = p_n = p$. We will compare low and high values of p. For the third section we will only vary the reliability function of the component with the highest/lowest Birnbaum importance, and the reliability functions of the other components will be held constant, i.e. all equal to p.

4.1 Evaluation of Bounds

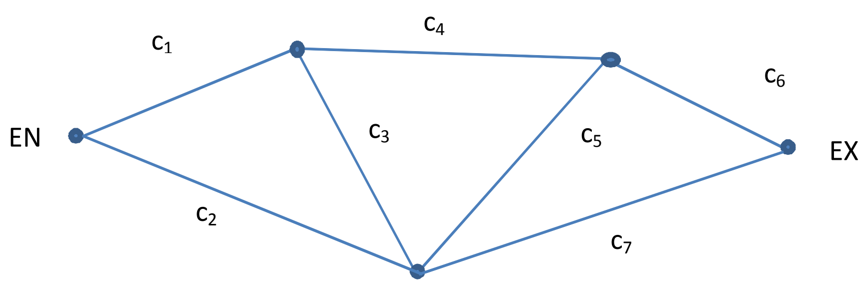

Figure 4.1: Extended bridge structure

Figure 4.1 above is an undirected graph of an extended bridge structure. The structure has 7 components, $c_1, c_2, \ldots, c_7$. Path is from the entrance node (EN) to the exit node (EX).

The minimal path sets for this structure are:

$$\begin{aligned} P_1 &= \{2,7\}, \quad P_2 = \{1,3,7\}, \quad P_3 = \{1,4,6\}, \quad P_4 = \{2,5,6\}, \\ P_5 &= \{1,3,5,6\}, \quad P_6 = \{1,4,5,7\}, \quad P_7 = \{2,3,4,6\}, \end{aligned}$$

and minimal cut sets are:

$$\begin{aligned} C_1 &= \{1,2\}, \quad C_2 = \{6,7\}, \quad C_3 = \{4,5,7\}, \\ C_4 &= \{1,3,5,7\}, \quad C_5 = \{2,3,5,6\}. \end{aligned}$$

Note that the minimal path and cut sets are arranged in lexicographical order.

The structure function φ is

$$\begin{aligned} \varphi(\mathbf{x}) \;=\;& x_2\, x_7 \;+ x_1\, \bar{x}_2\, x_3\, x_7 \;+ x_1\, \bar{x}_2\, \bar{x}_3\, x_4\, x_6 \;+ x_1\, x_2\, x_4\, x_6\, \bar{x}_7 \;+ x_1\, \bar{x}_2\, x_3\, x_4\, x_6\, \bar{x}_7 \\ & + \bar{x}_1\, x_2\, x_5\, x_6\, \bar{x}_7 \;+ x_1\, x_2\, \bar{x}_4\, x_5\, x_6\, \bar{x}_7 \;+ x_1\, \bar{x}_2\, x_3\, \bar{x}_4\, x_5\, x_6\, \bar{x}_7 \\ & + x_1\, \bar{x}_2\, \bar{x}_3\, x_4\, x_5\, \bar{x}_6\, x_7 \;+ \bar{x}_1\, x_2\, x_3\, x_4\, \bar{x}_5\, x_6\, \bar{x}_7\;, \end{aligned}$$

and the reliability function $R(\mathbf{p})$ is

$$R(\mathbf{p}) = p^2 + p^3q + 2p^3q^2 + p^4q + 2p^4q^2 + 3p^4q^3.$$

4.1.1 Trivial bounds

$$LB_{Triv} = p^7$$

$$UB_{Triv} = 1 - q^7$$

From Tables 10, 11 and 13 (page 94) and Figures 4.2 and 4.3 below it is clear that the lower bound is too wide and the distance is wider when the p values are in the interval (0.5, 0.8). The upper bound is a bad estimator for almost all values of p. All in all, these bounds are too weak, and should be avoided.

4.1.2 Esary- Proschan bounds

$$
\begin{aligned}
LB_{EP} &= \prod_{k=1}^{5}(1-q^{|C_k|}) \\
&= (1-q^2)^2(1-q^3)(1-q^4)^2
\end{aligned}
$$

$$
\begin{aligned}
UB_{EP} &= 1-\prod_{j=1}^{7}\left(1-p^{|P_j|}\right) \\
&= 1-(1-p^2)(1-p^3)^3(1-p^4)^3
\end{aligned}
$$

As was mentioned by Esary and Proschan, the lower bound yields good approximation to system reliability when component reliabilities are high, whereas the upper bound yields good approximations when component reliabilities are low (see Tables 10, 11 and 13 and Figures 4.2 and 4.3).

4.1.3 Min- max bounds

$$
\begin{aligned}
LB_{\min-\max} &= \max_{1\le j\le 7}\{p^{|P_j|}\} \\
&= \max\{p^2, p^3, p^4\}
\end{aligned}
$$

$$
\begin{aligned}
UB_{\min-\max} &= \min_{1\le k\le 5}\{1-q^{|C_k|}\} \\
&= \min\{1-q^2, 1-q^3, 1-q^4\}
\end{aligned}
$$

From Table 11 it is clear that $LB_{\min-\max}$ is better than LB_{EP} when p is very small. Similarly, Table 13 shows that UB_{EP} is a better upper bound than $UB_{\min-\max}$ when p is large. For independent components it is advisable to take $\max\{LB_{EP}, LB_{\min-\max}\}$ as a lower bound and $\min\{UB_{EP}, UB_{\min-\max}\}$ as an upper bound.

4.1.4 Inclusion-exclusion bounds

$$\begin{aligned} E(T_1) &= \sum_{i=1}^{7} E(\rho_i) \\ &= p^2 + 3p^3 + 3p^4 \\ &= UB_{Inc-Ex} \end{aligned}$$

$$\begin{aligned} E(T_2) &= \sum_{1\leq i<j\leq 7} E(\rho_i \, \rho_j) \\ &= 3p^4 + 11p^5 + 6p^6 + p^7 \end{aligned}$$

$$\begin{aligned} LB_{Inc-Ex} &= E(T_1) - E(T_2) \\ &= p^2 + 3p^3 - 11p^5 - 6p^6 - p^7 \end{aligned}$$

LB_{Inc-Ex} gives a good approximation when p is small. LB_{Inc-Ex} and UB_{Inc-Ex} are very bad approximators when p is large and should be avoided as much as possible (see Tables 10, 11 and 13).

4.1.5 Spross bounds

Abraham Algorithm

$$\rho_1 = x_2\,x_7\,, \quad \rho_2 = x_1\,x_3\,x_7\,, \quad \rho_3 = x_1\,x_4\,x_6\,, \quad \rho_4 = x_2\,x_5\,x_6\,,$$
$$\rho_5 = x_1\,x_3\,x_5\,x_6\,, \quad \rho_6 = x_1\,x_4\,x_5\,x_7\,, \quad \rho_7 = x_2\,x_3\,x_4\,x_6$$

$$PD_{0,1} = \{\rho_1\} = \{x_2\,x_7\}$$

$j = 2$:

$$\begin{aligned} PD_{0,2} &= \{\rho_2\} = \{x_1\,x_3\,x_7\} \\ C(\rho_1\,,\rho_2) &= \{x_2\} \\ PD_{1,2} &= \{B_1 = x_1\,\bar{x}_2\,x_3\,x_7\} = PD_2 \end{aligned}$$

$j = 3$:

$$\begin{aligned} PD_{0,3} &= \{\rho_3\} = \{\, x_1\,x_4\,x_6\,\} \\ C(\rho_1\,,\rho_3) &= \{x_2\,,x_7\} \\ PD_{1,3} &= \{\, B_1 = x_1\,\bar{x}_2\,x_4\,x_6\,,\ B_2 = x_1\,x_2\,x_4\,x_6\,\bar{x}_7\,\} \\ C(\rho_2\,,B_1) &= \{x_3\,,x_7\},\quad \rho_2 \text{ and } B_2 \text{ are disjoint, hence} \\ \{B_1 &= x_1\,\bar{x}_2\,\bar{x}_3\,x_4\,x_6\,, B_2 = x_1\,x_2\,x_4\,x_6\,\bar{x}_7\,, B_3 = x_1\,\bar{x}_2\,x_3\,x_4\,x_6\,\bar{x}_7\,\} \\ &= PD_{2,3} \\ &= PD_3 \end{aligned}$$

$j = 4:$

$$\begin{aligned}
PD_{0,4} &= \{\rho_4\} = \{\ x_2\, x_5\, x_6\,\} \\
C(\rho_1\,, \rho_4\,) &= \{x_7\,\} \\
PD_{1,4} &= \{B_1 =\ x_2\, x_5\, x_6\, \bar{x}_7\,\} \\
&\quad\ \rho_2 \text{ and } B_1 \text{ are disjoint, hence} \\
PD_{2,4} &= \{B_1 =\ x_2\, x_5\, x_6\, \bar{x}_7\,\} \\
C(\rho_3\,, B_1) &= \{x_1\,, x_4\,\} \\
PD_{3,4} &= \{B_1 =\ \bar{x}_1\, x_2\, x_5\, x_6\, \bar{x}_7\,,\ B_2 = x_1\, x_2\, \bar{x}_4\, x_5\, x_6\, \bar{x}_7\,\} \\
&= PD_4
\end{aligned}$$

$j = 5:$

$$\begin{aligned}
PD_{0,5} &= \{\rho_5\} = \{\ x_1\, x_3\, x_5\, x_6\,\} \\
C(\rho_1\,, \rho_5\,) &= \{x_2\,, x_7\,\} \\
PD_{1,5} &= \{B_1 =\ x_1\, \bar{x}_2\, x_3\, x_5\, x_6\,,\ B_2 = x_1\, x_2\, x_3\, x_5\, x_6\, \bar{x}_7\,\} \\
C(\rho_2\,, B_1) &= \{x_7\,\}, \quad \rho_2 \ \text{ and } B_2 \text{ are disjoint, hence} \\
PD_{2,5} &= \{B_1 =\ x_1\, \bar{x}_2\, x_3\, x_5\, x_6\, \bar{x}_7\,,\quad B_2 = x_1\, x_2\, x_3\, x_5\, x_6\, \bar{x}_7\,\} \\
C(\rho_3\,, B_1) &= \{x_4\,\}, \quad C(\rho_3\,, B_2) = \{x_4\,\} \\
PD_{3,5} &= \{B_1 = x_1\, \bar{x}_2\, x_3\, \bar{x}_4\, x_5\, x_6\, \bar{x}_7\,,\ B_2 = x_1\, x_2\, x_3\, \bar{x}_4\, x_5\, x_6\, \bar{x}_7\,\} \\
C(\rho_4\,, B_2) &= \varnothing, \quad \rho_4 \ \text{ and } B_1 \text{ are disjoint, hence} \\
PD_{4,5} &= \{B_1 = x_1\, \bar{x}_2\, x_3\, \bar{x}_4\, x_5\, x_6\, \bar{x}_7\,\} \\
&= PD_5
\end{aligned}$$

$j = 6:$

$$
\begin{aligned}
PD_{0,6} &= \{\rho_6\} = \{\, x_1\, x_4\, x_5\, x_7 \,\} \\
C(\rho_1\,, \rho_6\,) &= \{x_2\} \\
PD_{1,6} &= \{B_1 \;= x_1\, \bar{x}_2\, x_4\, x_5\, x_7 \,\} \\
C(\rho_2\,, B_1) &= \{x_3\} \\
PD_{2,6} &= \{B_1 \;= x_1\, \bar{x}_2\, \bar{x}_3\, x_4\, x_5\, x_7 \,\} \\
C(\rho_3\,, B_1) &= \{x_6\} \\
PD_{3,6} &= \{B_1 \;= x_1\, \bar{x}_2\, \bar{x}_3\, x_4\, x_5\, \bar{x}_6\, x_7 \,\} \\
& \quad \rho_4 \text{ and } B_1 \text{ are disjoint, hence} \\
PD_{4,6} &= \{B_1 \;= x_1\, \bar{x}_2\, \bar{x}_3\, x_4\, x_5\, \bar{x}_6\, x_7 \,\} \\
& \quad \rho_5 \text{ and } B_1 \text{ are disjoint, hence} \\
PD_{5,6} &= \{B_1 = x_1\, \bar{x}_2\, \bar{x}_3\, x_4\, x_5\, \bar{x}_6\, x_7 \,\} \\
&= PD_6
\end{aligned}
$$

$j = 7:$

$$\begin{aligned}
PD_{0,7} &= \{\rho_7\} = \{\, x_2\, x_3\, x_4\, x_6\, \} \\
C(\rho_1\,, \rho_7\,) &= \{x_7\} \\
PD_{1,7} &= \{B_1 \ = x_2\, x_3\, x_4\, x_6\, \bar{x}_7\,\} \\
& \quad \rho_2 \text{ and } B_1 \text{ are disjoint, hence} \\
PD_{2,7} &= \{B_1 \ = x_2\, x_3\, x_4\, x_6\, \bar{x}_7\,\} \\
C(\rho_3\,, B_1) &= \{x_1\} \\
PD_{3,7} &= \{B_1 \ = \bar{x}_1\, x_2\, x_3\, x_4\, x_6\, \bar{x}_7\,\} \\
C(\rho_4\,, B_1) &= \{x_5\} \\
PD_{4,7} &= \{B_1 \ = \bar{x}_1\, x_2\, x_3\, x_4\, \bar{x}_5\, x_6\, \bar{x}_7\,\} \\
& \quad \rho_5 \text{ and } B_1 \text{ are disjoint, hence} \\
PD_{5,7} &= \{B_1 \ = \bar{x}_1\, x_2\, x_3\, x_4\, \bar{x}_5\, x_6\, \bar{x}_7\,\} \\
& \quad \rho_6 \text{ and } B_1 \text{ are disjoint, hence} \\
PD_{6,7} &= \{B_1 \ = \bar{x}_1\, x_2\, x_3\, x_4\, \bar{x}_5\, x_6\, \bar{x}_7\,\} \\
&= PD_7
\end{aligned}$$

Therefore

$$\begin{aligned}
\delta_1 &= x_2\, x_7 \\
\delta_2 &= x_1\, \bar{x}_2\, x_3\, x_7 \\
\delta_3 &= x_1\, \bar{x}_2\, \bar{x}_3\, x_4\, x_6 \ + x_1\, x_2\, x_4\, x_6\, \bar{x}_7 \ + x_1\, \bar{x}_2\, x_3\, x_4\, x_6\, \bar{x}_7 \\
\delta_4 &= \bar{x}_1\, x_2\, x_5\, x_6\, \bar{x}_7 \ + x_1\, x_2\, \bar{x}_4\, x_5\, x_6\, \bar{x}_7 \\
\delta_5 &= x_1\, \bar{x}_2\, x_3\, \bar{x}_4\, x_5\, x_6\, \bar{x}_7 \\
\delta_6 &= x_1\, \bar{x}_2\, \bar{x}_3\, x_4\, x_5\, \bar{x}_6\, x_7 \\
\delta_7 &= \bar{x}_1\, x_2\, x_3\, x_4\, \bar{x}_5\, x_6\, \bar{x}_7
\end{aligned}$$

Therefore, a disjoint sum form for φ in terms of simple products is

$$\begin{aligned}\varphi(\mathbf{x}) &= \sum_{j=1}^{7} \delta_j \\ &= \sum_{j=1}^{7} \sum_{B_i \in PD_j} B_i \\ &= x_2\, x_7 \,+ x_1\, \bar{x}_2\, x_3\, x_7 \,+ x_1\, \bar{x}_2\, \bar{x}_3\, x_4\, x_6 \,+ x_1\, x_2\, x_4\, x_6\, \bar{x}_7 \,+ x_1\, \bar{x}_2\, x_3\, x_4\, x_6\, \bar{x}_7 \\ &\quad +\bar{x}_1\, x_2\, x_5\, x_6\, \bar{x}_7 \,+ x_1\, x_2\, \bar{x}_4\, x_5\, x_6\, \bar{x}_7 \,+ x_1\, \bar{x}_2\, x_3\, \bar{x}_4\, x_5\, x_6\, \bar{x}_7 \\ &\quad +x_1\, \bar{x}_2\, \bar{x}_3\, x_4\, x_5\, \bar{x}_6\, x_7 \,+ \bar{x}_1\, x_2\, x_3\, x_4\, \bar{x}_5\, x_6\, \bar{x}_7\end{aligned}$$

We again use the Abraham algorithm to find ϑ_k , $k = 1, 2, \ldots, 5$.

Algorithm:

$$\begin{aligned}\gamma_1 &= \bar{x}_1\, \bar{x}_2\,, \qquad \gamma_2 = \bar{x}_6\, \bar{x}_7\,, \qquad \gamma_3 = \bar{x}_4\, \bar{x}_5\, \bar{x}_7\,, \\ \gamma_4 &= \bar{x}_1\, \bar{x}_3\, \bar{x}_5\, \bar{x}_7\,, \qquad \gamma_5 = \bar{x}_2\, \bar{x}_3\, \bar{x}_5\, \bar{x}_6\end{aligned}$$

$$PD_{0,1} = \{\gamma_1\} = \{\bar{x}_1\, \bar{x}_2\}$$

$k = 2$:

$$\begin{aligned}PD_{0,2} &= \{\gamma_2\} = \{\bar{x}_6\, \bar{x}_7\} \\ C(\gamma_1\,, \gamma_2) &= \{\bar{x}_1\,, \bar{x}_2\} \\ PD_{1,2} &= \{B_1 = x_1\, \bar{x}_6\, \bar{x}_7\,, \quad B_2 = \bar{x}_1\, x_2\, \bar{x}_6\, \bar{x}_7\} \\ &= PD_2\end{aligned}$$

$k = 3:$

$$\begin{aligned}
PD_{0,3} &= \{\gamma_3\} = \{\bar{x}_4\,\bar{x}_5\,\bar{x}_7\} \\
C(\gamma_1,\gamma_3) &= \{\bar{x}_1,\bar{x}_2\} \\
PD_{1,3} &= \{B_1 = x_1\,\bar{x}_4\,\bar{x}_5\,\bar{x}_7, \quad B_2 = \bar{x}_1\,x_2\,\bar{x}_4\,\bar{x}_5\,\bar{x}_7\} \\
C(\gamma_2,B_1) &= \{\bar{x}_6\}, \qquad C(\gamma_2,B_2) = \{\bar{x}_6\} \\
PD_{2,3} &= \{B_1 = x_1\,\bar{x}_4\,\bar{x}_5\,x_6\,\bar{x}_7, \quad B_2 = \bar{x}_1\,x_2\,\bar{x}_4\,\bar{x}_5\,x_6\,\bar{x}_7\} \\
&= PD_3
\end{aligned}$$

$k = 4:$

$$\begin{aligned}
PD_{0,4} &= \{\gamma_4\} = \{\bar{x}_1\,\bar{x}_3\,\bar{x}_5\,\bar{x}_7\} \\
C(\gamma_1,\gamma_4) &= \{\bar{x}_2\} \\
PD_{1,4} &= \{B_1 = \bar{x}_1\,x_2\,\bar{x}_3\,\bar{x}_5\,\bar{x}_7\} \\
C(\gamma_2,B_1) &= \{\bar{x}_6\} \\
PD_{2,4} &= \{B_1 = \bar{x}_1\,x_2\,\bar{x}_3\,\bar{x}_5\,x_6\,\bar{x}_7\} \\
C(\gamma_3,B_1) &= \{\bar{x}_4\} \\
PD_{3,4} &= \{B_1 = \bar{x}_1\,x_2\,\bar{x}_3\,x_4\,\bar{x}_5\,x_6\,\bar{x}_7\} \\
&= PD_4
\end{aligned}$$

$k = 5:$

$$\begin{aligned}
PD_{0,5} &= \{\gamma_5\} = \{\bar{x}_2\,\bar{x}_3\,\bar{x}_5\,\bar{x}_6\} \\
C(\gamma_1\,,\gamma_5) &= \{\bar{x}_1\} \\
PD_{1,5} &= \{B_1 = x_1\,\bar{x}_2\,\bar{x}_3\,\bar{x}_5\,\bar{x}_6\} \\
C(\gamma_2\,,B_1) &= \{\bar{x}_7\} \\
PD_{2,5} &= \{B_1 = x_1\,\bar{x}_2\,\bar{x}_3\,\bar{x}_5\,\bar{x}_6\,x_7\} \\
& \quad \gamma_3 \text{ and } B_1 \text{ are disjoint, hence} \\
PD_{3,5} &= \{B_1 = x_1\,\bar{x}_2\,\bar{x}_3\,\bar{x}_5\,\bar{x}_6\,x_7\} \\
& \quad \gamma_4 \text{ and } B_1 \text{ are disjoint, hence} \\
PD_{4,5} &= \{B_1 = x_1\,\bar{x}_2\,\bar{x}_3\,\bar{x}_5\,\bar{x}_6\,x_7\} \\
&= PD_5
\end{aligned}$$

Therefore

$$\begin{aligned}
\vartheta_1 &= \bar{x}_1\,\bar{x}_2 \\
\vartheta_2 &= x_1\,\bar{x}_6\,\bar{x}_7 + \bar{x}_1\,x_2\,\bar{x}_6\,\bar{x}_7 \\
\vartheta_3 &= x_1\,\bar{x}_4\,\bar{x}_5\,x_6\,\bar{x}_7 + \bar{x}_1\,x_2\,\bar{x}_4\,\bar{x}_5\,x_6\,\bar{x}_7 \\
\vartheta_4 &= \bar{x}_1\,x_2\,\bar{x}_3\,x_4\,\bar{x}_5\,x_6\,\bar{x}_7 \\
\vartheta_5 &= x_1\,\bar{x}_2\,\bar{x}_3\,\bar{x}_5\,\bar{x}_6\,x_7
\end{aligned}$$

Spross lower bound is given as

$$\begin{aligned}
g &= 1: \quad \underline{\mathrm{b}}(g) = p^2 \\
g &= 2: \quad \underline{\mathrm{b}}(g) = p^2 + p^3 q \\
g &= 3: \quad \underline{\mathrm{b}}(g) = p^2 + p^3 q + p^3 q^2 + p^4 q + p^4 q^2 \\
g &= 4: \quad \underline{\mathrm{b}}(g) = p^2 + p^3 q + 2p^3 q^2 + p^4 q + 2p^4 q^2
\end{aligned}$$

and their upper bound is given as

$$\begin{aligned} h &= 1: \quad \bar{b}(h) = 1 - q^2 \\ h &= 2: \quad \bar{b}(h) = 1 - [q^2 + pq^2 + pq^3] \\ h &= 3: \quad \bar{b}(h) = 1 - [q^2 + pq^2 + pq^3 + p^2q^3 + p^2q^4] \end{aligned}$$

The numerical values of the bounds are given in Tables 6 and 7 below

Table 6: Spross lower bounds

p	$\underline{b}(1)$	$\underline{b}(2)$	$\underline{b}(3)$	$\underline{b}(4)$
0.05	0.0025	2.6188×10^{-3}	2.7431×10^{-3}	2.8616×10^{-3}
0.10	0.01	0.0109	1.1881×10^{-2}	1.2772×10^{-2}
0.15	0.0225	2.5369×10^{-2}	2.8603×10^{-2}	3.1407×10^{-2}
0.85	0.7225	0.81462	0.91848	0.94405
0.90	0.81	0.8829	0.96236	0.97621
0.95	0.9025	0.94537	0.99027	0.99445

Table 7: Spross upper bounds

p	$\bar{b}(1)$	$\bar{b}(2)$	$\bar{b}(3)$
0.05	0.0975	9.5063×10^{-3}	5.3265×10^{-3}
0.10	0.19	0.0361	2.2249×10^{-2}
0.15	0.2775	7.7006×10^{-2}	5.1443×10^{-2}
0.85	0.9775	0.95551	0.9527
0.90	0.99	0.9801	0.97921
0.95	0.9975	0.99501	0.99489

Tables 6-9 below give the distance $d(g,h) = \bar{b}(h) - \underline{b}(g)$

Table 8: numerical values of d(1,h)

p	$\bar{b}(1) - \underline{b}(1)$	$\bar{b}(2) - \underline{b}(1)$	$\bar{b}(3) - \underline{b}(1)$
0.05	0.095	$7.006\,3 \times 10^{-3}$	$2.826\,5 \times 10^{-3}$
0.10	0.18	0.026 1	$1.224\,9 \times 10^{-2}$
0.15	0.255	$5.450\,6 \times 10^{-2}$	$2.894\,3 \times 10^{-2}$
0.85	0.255	0.233 01	0.230 2
0.90	0.18	0.170 1	0.169 21
0.95	0.095	0.092 51	0.092 39

Table 9: numerical values of d(2,h)

p	$\bar{b}(1) - \underline{b}(2)$	$\bar{b}(2) - \underline{b}(2)$	$\bar{b}(3) - \underline{b}(2)$
0.05	$9.488\,1 \times 10^{-2}$	$6.887\,5 \times 10^{-3}$	$2.707\,7 \times 10^{-3}$
0.10	0.179 1	0.025 2	$1.134\,9 \times 10^{-2}$
0.15	0.252 13	$5.163\,7 \times 10^{-2}$	$2.607\,4 \times 10^{-2}$
0.85	0.162 88	0.140 89	0.138 08
0.90	0.107 1	0.097 2	0.096 31
0.95	0.052 13	0.049 64	0.049 52

Table 10: numerical values of d(3,h)

p	$\bar{b}(1) - \underline{b}(3)$	$\bar{b}(2) - \underline{b}(3)$	$\bar{b}(3) - \underline{b}(3)$
0.05	$9.475\,7 \times 10^{-2}$	$6.763\,2 \times 10^{-3}$	$2.583\,4 \times 10^{-3}$
0.10	0.178 12	$2.421\,9 \times 10^{-2}$	$1.036\,8 \times 10^{-2}$
0.15	0.248 90	$4.840\,3 \times 10^{-2}$	0.022 84
0.85	0.059 02	0.037 03	0.034 22
0.90	0.027 64	0.017 74	0.016 85
0.95	0.007 23	0.004 74	0.004 62

Table 11: numerical values of d(4,h)

p	$\bar{b}(1) - \underline{b}(4)$	$\bar{b}(2) - \underline{b}(4)$	$\bar{b}(3) - \underline{b}(4)$
0.05	9.4638×10^{-2}	6.6447×10^{-3}	2.4649×10^{-3}
0.10	0.177 23	2.3328×10^{-2}	9.477×10^{-3}
0.15	0.246 09	4.5599×10^{-2}	2.0036×10^{-2}
0.85	0.033 45	0.011 46	0.008 65
0.90	0.013 79	0.003 89	0.003
0.95	0.003 05	0.000 56	0.000 44

Taking $\epsilon = 0.01$, we realize that $d(4,3) = \bar{b}(3) - \underline{b}(4) < \epsilon$, for all the p values we have used. Hence we stop the calculation and take $\underline{b}(4)$ as Spross lower bound and $\bar{b}(3)$ as Spross upper bound.

Spross lower bound (LB_{BS}) is a very good approximation for all values of p. Their upper bound (UB_{BS}) is also a good approximation, but is weaker than the generalized Fu-Koutras upper bound (UB_{FK}). The interval (LB_{BS}, UB_{FK}) will result in very sharp bounds (see tables 10, 11 and 13 and Figures 4.2 and 4.3). The downside of the Spross bounds is that there are many computations involved, but with so much advancement in technology in our days, this problem is easily solved.

4.1.6 Fu- Koutras bounds

$$\begin{aligned} K_1^* &= \varnothing, \quad K_2^* = \{1\}, \quad K_3^* = \{2\}, \quad K_4^* = \{1,3\}, \\ K_5^* &= \{2,3,4\}, \quad K_6^* = \{1,2,3,4,5\}, \quad K_7^* = \{1,2,3,4,5,6\} \end{aligned}$$

$$\begin{aligned} K_1 &= \varnothing, \quad K_2 = \{2\}, \quad K_3 = \{2,3,7\}, \quad K_4 = \{1,3,4,7\}, \\ K_5 &= \{2,4,7\}, \quad K_6 = \{2,3,6\}, \quad K_7 = \{1,5,7\} \end{aligned}$$

$$\begin{aligned} LB_{FK} &= 1 - \prod_{j=1}^{7} \left[1 - p^{|P_j|} q^{|K_j|}\right] \\ &= 1 - (1 - p^3 q)(1 - p^3 q^3)(1 - p^3 q^4)(1 - p^4 q^3)^3 \end{aligned}$$

$$L_1^* = L_2^* = \varnothing, \qquad L_3^* = \{2\}, \qquad L_4^* = \{1, 2, 3\}, \qquad L_5^* = \{1, 2, 3, 4\}$$

$$\begin{aligned} L_1 &= \varnothing, \qquad L_2 = \{1, 2\}, \qquad L_3 = \{1, 2, 6\}, \\ L_4 &= \{2, 4, 6\}, \qquad L_5 = \{1, 4, 7\} \end{aligned}$$

$$\begin{aligned} UB_{FK} &= \prod_{k=1}^{5} (1 - p^{|L_k|} q^{|C_k|}) \\ &= (1 - p^2 q^2)(1 - p^3 q^3)(1 - p^3 q^4)^2 \end{aligned}$$

As was claimed by Fu and Koutras, UB_{FK} is very close to LB_{EP} when p is very large. Thus (LB_{EP}, UB_{FK}) will provide a satisfactory interval estimation of the exact reliability value. Their claim could not be verified for LB_{FK} and UB_{EP} where they claim that these two bounds are very close when p is very small (see Tables 10, 12 and 14 and Figures 4.2 and 4.3).

4.1.7 Koutras-Papastavridis-Petakos Bounds

$$\begin{aligned} K_1^* &= \varnothing, \qquad K_2^* = \{1\}, \qquad K_3^* = \{2\}, \qquad K_4^* = \{1, 3\}, \\ K_5^* &= \{2, 3, 4\}, \qquad K_6^* = \{1, 2, 3, 4, 5\}, \qquad K_7^* = \{1, 2, 3, 4, 5, 6\} \end{aligned}$$

$$\rho_1^* = 1$$

$$\begin{aligned}\rho_2^* &= \coprod_{j\in K_2^*}\prod_{i\in P_j} x_i\\ &= x_2\,x_7\end{aligned}$$

$$\begin{aligned}\rho_3^* &= \coprod_{j\in K_3^*}\prod_{i\in P_j} x_i\\ &= x_1\,x_3\,x_7\end{aligned}$$

$$\begin{aligned}\rho_4^* &= \coprod_{j\in K_4^*}\prod_{i\in P_j} x_i\\ &= 1-(1-x_2\,x_7\,)(1-x_1\,x_4\,x_6\,)\end{aligned}$$

$$\begin{aligned}\rho_5^* &= \coprod_{j\in K_5^*}\prod_{i\in P_j} x_i\\ &= 1-(1-x_1\,x_3\,x_7\,)(1-x_1\,x_4\,x_6\,)(1-x_2\,x_5\,x_6\,)\end{aligned}$$

$$\begin{aligned}\rho_6^* &= \coprod_{j\in K_6^*}\prod_{i\in P_j} x_i\\ &= 1-(1-x_2\,x_7\,)(1-x_1\,x_3\,x_7\,)(1-x_1\,x_4\,x_6\,)(1-x_2\,x_5\,x_6\,)\\ &\quad (1-x_1\,x_3\,x_5\,x_6\,)\end{aligned}$$

$$\begin{aligned}\rho_7^* &= \coprod_{j\in K_7^*}\prod_{i\in P_j} x_i\\ &\quad 1-(1-x_2\,x_7\,)(1-x_1\,x_3\,x_7\,)(1-x_1\,x_4\,x_6\,)(1-x_2\,x_5\,x_6\,)\\ &\quad (1-x_1\,x_3\,x_5\,x_6\,)(1-x_1\,x_4\,x_5\,x_7\,)\end{aligned}$$

$$\begin{aligned}\beta_1 &= 1, \qquad \beta_2 = x_2\,, \qquad \beta_3 = x_3\,x_7\,, \qquad \beta_4 = 1-\bar{x}_1\,\bar{x}_7\,,\\ \beta_5 &= 1-\bar{x}_2\,\bar{x}_4\,\bar{x}_7\,, \qquad \beta_6 = 1-\bar{x}_2\,\bar{x}_3\,\bar{x}_6\,, \qquad \beta_7 = 1-\bar{x}_1\,\bar{x}_5\,\bar{x}_7\end{aligned}$$

$$\begin{aligned}LB_{KPP} &= 1-\prod_{j=1}^{7}[1-p^{|P_j|}\cdot\Pr(\beta_j = 0)]\\ &= 1-(1-p^3q)(1-p^3q^2)^2(1-p^4q^3)^3\end{aligned}$$

$$\begin{aligned}L_1^* &= L_2^* = \varnothing, \qquad L_3^* = \{2\},\\ L_4^* &= \{1,2,3\}, \qquad L_5^* = \{1,2,3,4\}\end{aligned}$$

$$\kappa_1^* = \kappa_2^* = 1$$

$$\begin{aligned}\kappa_3^* &= \prod_{k\in L_3^*}\coprod_{i\in C_k} x_i\\ &= 1-\bar{x}_6\,\bar{x}_7\end{aligned}$$

$$\begin{aligned}\kappa_4^* &= \prod_{k\in L_4^*}\coprod_{i\in C_k} x_i\\ &= (1-\bar{x}_1\,\bar{x}_2\,)(1-\bar{x}_6\,\bar{x}_7\,)(1-\bar{x}_4\,\bar{x}_5\,\bar{x}_6\,)\end{aligned}$$

$$\begin{aligned}\kappa_5^* &= \prod_{k\in L_5^*}\coprod_{i\in C_k} x_i\\ &= (1-\bar{x}_1\,\bar{x}_2\,)(1-\bar{x}_6\,\bar{x}_7\,)(1-\bar{x}_4\,\bar{x}_5\,\bar{x}_6\,)(1-\bar{x}_1\,\bar{x}_3\,\bar{x}_5\,\bar{x}_7\,)\end{aligned}$$

$$\mu_1 = \mu_2 = 1, \qquad \mu_3 = x_1\, x_2\, x_3\, x_7\,, \qquad \mu_4 = x_2\, x_4\, x_6\,, \qquad \mu_5 = x_1\, x_4\, x_7$$

$$\begin{aligned} UB_{KPP} &= \prod_{k=1}^{5} [q^{|C_k|} \cdot \Pr(\mu_k = 1)] \\ &= (1-q^2)^2(1-p^4q^3)(1-p^3q^4)^2 \end{aligned}$$

The lower bound is not a good approximator especially for $p > 0.2$. The upper bound performs better but is still weaker than UB_{EP} and UB_{BS}. For large values of p, the interval (LB_{FK}, UB_{KPP}) gives a good estimation of the exact value of $R(\mathbf{p})$ (see Tables 10, 12, 13 and 14 and Figures 4.2 and 4.3)

4.1.8 Generalized Esary-Proschan Bounds

$$\begin{aligned} \mathbb{C}_1 &= \{\{1,2\},\ \{6,7\}\}, \qquad\qquad \mathbb{C}_2 = \{\{4,5,7\}\}, \\ \mathbb{C}_3 &= \{\{1,3,5,7\},\ \{2,3,5,6\}\} \end{aligned}$$

$$\begin{aligned} LB_{EP}^{(G)} &= \prod_{s=1}^{3} \left[\prod_{C\in\mathbb{C}_s} \left(1 - \prod_{i\in C} q_i \right) \right] \\ &= (1-q^2)^2(1-q^3)(1-q^4)^2 \end{aligned}$$

$$\begin{aligned} ¶_1 &= \{\{2,7\}\}, \qquad ¶_2 = \{\{1,3,7\},\ \{1,4,6\},\ \ \{2,5,6\}\}, \\ ¶_3 &= \{\{1,3,5,6\},\ \{1,4,5,7\},\ \{2,3,4,6\}\} \end{aligned}$$

$$
\begin{aligned}
UB_{EP}^{(G)} &= 1 - \prod_{s=1}^{3}\left[1 - \prod_{P\in\P_s}\left(1 - \prod_{i\in P} p_i\right)\right] \\
&= 1 - p^2[1-(1-p^3)^3][1-(1-p^4)^3]
\end{aligned}
$$

$LB_{EP}^{(G)}$ is exactly LB_{EP}. The upper bound is a very bad estimator and should be avoided. These bounds result in an increasing computational complexity.

4.1.9 Generalized Fu- Koutras bounds

$$\P_1^* = \varnothing, \quad \P_2^* = \{\{2,7\}\}, \quad \P_3^* = \{\{2,7\},\ \{1,3,7\},\ \{1,4,6\},\ \{2,5,6\}\}$$

$$\mathbb{C}_1^* = \varnothing, \qquad \mathbb{C}_2^* = \{\{6,7\}\}, \qquad \mathbb{C}_3^* = \{\{1,2\},\ \{6,7\},\ \{4,5,7\}\}$$

$$
\begin{aligned}
LB_{FK}^{(G)} &= 1 - \left[\prod_{P\in\P_1}\left(1 - \prod_{i\in P} p_i\right)\right]\cdot\prod_{s=2}^{3}\left[1 - \left(\coprod_{P\in\P_s^*\cup\P_s}\prod_{i\in P} p_i - \coprod_{P\in\P_s^*}\prod_{i\in P} p_i\right)\right] \\
&= 1 - [1-p^2][p^2 + (1-p^2)(1-p^3)^3][1-(1-p^2)(1-p^3)^3(1-(1-p^4)^3)]
\end{aligned}
$$

$$
\begin{aligned}
UB_{FK}^{(G)} &= \left[\prod_{C\in\mathbb{C}_1}\coprod_{i\in C} p_i\right]\cdot\prod_{s=2}^{3}\left[1 - \left(\prod_{C\in\mathbb{C}_s^*}\coprod_{i\in C} p_i - \prod_{C\in\mathbb{C}_s^*\cup\mathbb{C}_s}\coprod_{i\in C} p_i\right)\right] \\
&= (1-q^2)^2[1-q^3(1-q^2)][1-(1-q^2)^2(1-q^3)(1-(1-q^4)^2)]
\end{aligned}
$$

$LB_{FK}^{(G)}$ is not so good an estimator, and other bounds are preferred in this case. $UB_{FK}^{(G)}$ is a very good estimator. This is more evident for $p > 0.4$ (see Tables 10, 12 and 14 and Figures 4.2 and 4.3)

4.1.10 Generalized min-max bounds

$$\begin{aligned} LB^{(G)}_{\min-\max} &= \max_{1\le s\le 3}\left\{1-\prod_{P\in\P_s}\left(1-\prod_{i\in P}p_i\right)\right\} \\ &= \max\{1-(1-p^2),\ 1-(1-p^3)^3,\ 1-(1-p^4)^3\} \end{aligned}$$

$$\begin{aligned} UB^{(G)}_{\min-\max} &= \min_{1\le s\le 3}\left\{\prod_{C\in\mathbb{C}_s}\left(1-\prod_{i\in C}q_i\right)\right\} \\ &= \min\{(1-q^2)^2,\ (1-q^3),\ (1-q^4)^2\} \end{aligned}$$

Both $LB^{(G)}_{\min-\max}$ and $UB^{(G)}_{\min-\max}$ are good estimators, especially for large p values.

Tables 10-14 below give the numerical values for $R(\mathbf{p})$, lower and upper bounds for $p \in \{0.05, 0.10, 0.15, 0.85, 0.90, 0.95\}$.

Table 12: R(p)

p	$R(\mathbf{p})$
0.05	2.8777×10^{-3}
0.10	1.2991×10^{-2}
0.15	0.03234
0.85	0.94933
0.90	0.97818
0.95	0.99476

Table 13: Lower bounds

p	LB_{Triv}	LB_{EP}	$LB_{\min-\max}$	LB_{Inc-Ex}	LB_{BS}
0.05	7.8125×10^{-10}	4.6651×10^{-5}	0.0025	$\mathbf{2.8715 \times 10^{-3}}$	2.8616×10^{-3}
0.10	0.0000001	1.157×10^{-3}	0.01	$\mathbf{1.2884 \times 10^{-2}}$	1.2772×10^{-2}
0.15	0.000001708	6.7892×10^{-3}	**0.0225**	3.1720×10^{-2}	3.1407×10^{-2}
0.85	0.320577088	0.95132	0.7225	-4.8994	**0.94405**
0.90	0.4782969	**0.97892**	0.81	-7.1653	0.97621
0.95	0.698337296	**0.99487**	0.9025	-10.146	0.99445

Table 14: Lower bounds cont...

p	LB_{FK}	LB_{KPP}	$LB_{EP}^{(G)}$	$LB_{FK}^{(G)}$	$LB_{\min-\max}^{(G)}$
0.05	3.4377×10^{-4}	3.6041×10^{-4}	4.6651×10^{-5}	2.8917×10^{-3}	0.0025
0.10	2.5016×10^{-3}	2.736×10^{-3}	1.157×10^{-3}	1.3230×10^{-2}	0.01
0.15	7.6147×10^{-3}	8.6509×10^{-3}	6.7892×10^{-3}	3.3564×10^{-2}	**0.0225**
0.85	9.9061×10^{-2}	0.12169	0.95132	0.79799	0.94254
0.90	7.5466×10^{-2}	8.8165×10^{-2}	**0.97892**	0.84594	0.98010
0.95	4.3269×10^{-2}	4.7259×10^{-2}	**0.99487**	0.912	0.99710

Table 15: Upper bounds

p	UB_{Triv}	UB_{EP}	$UB_{\min-\max}$	UB_{Inc-Ex}	UB_{BS}
0.05	0.301662704	2.8927×10^{-3}	0.0975	2.8938×10^{-3}	5.3265×10^{-3}
0.10	0.5217031	1.3263×10^{-2}	0.19	0.0133	2.2249×10^{-2}
0.15	0.679422912	3.3833×10^{-2}	0.2775	3.4144×10^{-2}	5.1443×10^{-2}
0.85	0.999998292	0.99826	0.9775	4.1309	0.9527
0.90	0.9999999	0.99985	0.99	4.9653	0.97921
0.95	0.999999999	1.00000	0.9975	5.9181	0.99489

Table 16: upper bounds cont.....

p	UB_{FK}	UB_{KPP}	$UB_{EP}^{(G)}$	$UB_{FK}^{(G)}$	$UB_{\min-\max}^{(G)}$
0.05	0.99743	9.5043×10^{-3}	1.0000	8.7002×10^{-3}	3.4408×10^{-2}
0.10	0.98988	0.03605	1.00000	3.0832×10^{-2}	0.11827
0.15	0.97825	7.6711×10^{-2}	1.00000	6.2418×10^{-2}	**0.22848**
0.85	0.98109	0.95323	0.39338	**0.95144**	0.99899
0.90	0.99103	0.97931	0.23841	**0.97894**	0.9998
0.95	0.99763	0.99489	0.10586	**0.99487**	0.99999

Figures 4.2 and 4.3 give a graphical comparison of the bounds with $R(\mathbf{p})$. The description of colours is given in Table 15 below.

Table 17: Description of colours

Colour	Bounds
Black	$R(\mathbf{p})$
Light green	Trivial
Light blue green	Esary-Proschan
Green	Min-max
Brown	Inclusion-Exclusion
Light red	Spross
Purple	Fu-Koutras
Navy	Koutras-Papastavridis-Petakos
Yellow	Generalized Esary-Proschan
Light blue	Generalized Fu-Koutras
Magenta	Generalized Min-max

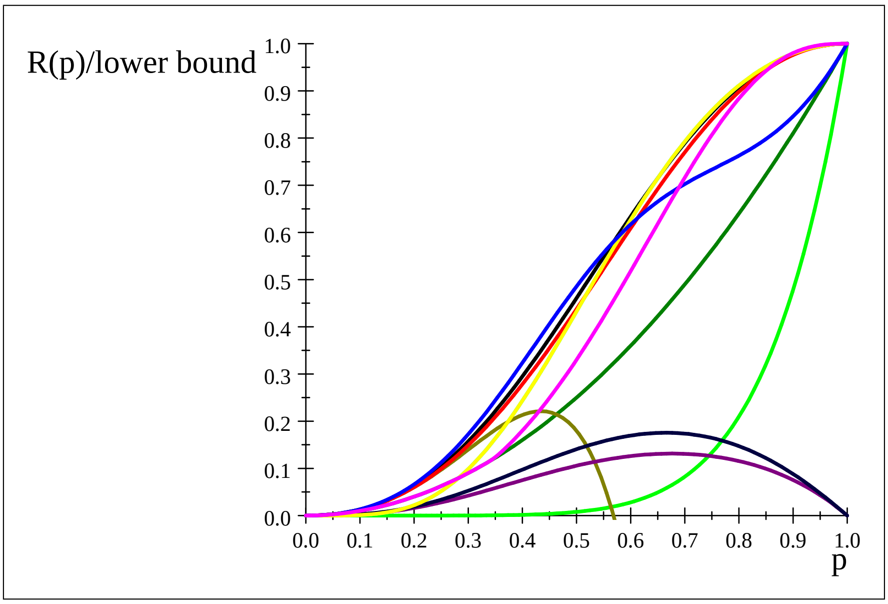

Figure 4.2: Comparison of lower bounds with R(p)

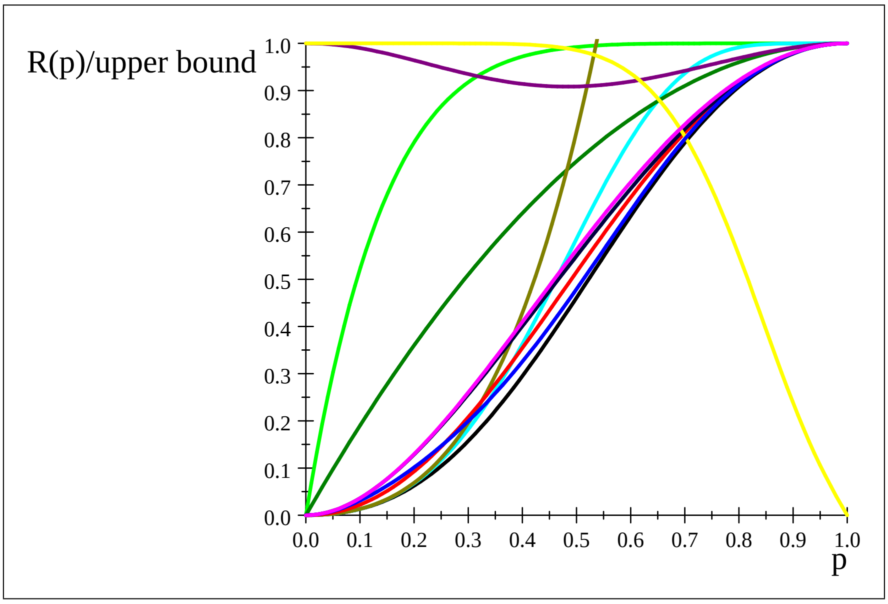

Figure 4.3: Comparison of upper bounds with R(p)

4.2 Structure with modules

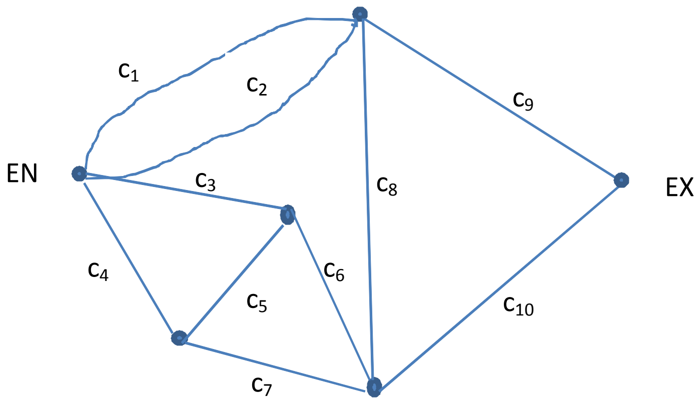

Figure 4.4: Structure with modules

Figure 4.4 above is an undirected graph of a structure with modules. The structure has 10 components, c_1 , c_2 , ..., c_{10} , with 5 modules.The first module, A_1, has two components c_1 and c_2, connected in parallel. The second module, A_2, is a bridge structure with five components, c_3 , c_4 , c_5 , c_6 , c_7 . The last three modules, A_3, A_4 and A_5, each has one component, c_8 , c_9 and c_{10} , respectively. We will use this Figure to evaluate which way is better: to take the system as a whole and find the system reliability bounds, or to decompose the system into modules, find reliability of the modules and then find the system reliability bounds in terms of the reliability of the modules. We will not evaluate all bounds, we will only take three, i.e. Esary-Proschan, min-max and Fu- Koutras bounds. We will assume that all components have the same reliability function p.

4.2.1 Using the system as a whole

The minimal path sets are:

$$\begin{aligned} P_1 &= \{1,9\}, \quad P_2 = \{2,9\}, \quad P_3 = \{1,8,10\}, \\ P_4 &= \{2,8,10\}, \quad P_5 = \{3,6,10\}, \quad P_6 = \{4,7,10\}, \\ P_7 &= \{3,5,7,10\}, \quad P_8 = \{3,6,8,9\}, \\ P_9 &= \{4,5,6,10\}, \quad P_{10} = \{4,7,8,9\} \end{aligned}$$

and minimal cut sets are:

$$\begin{aligned} C_1 &= \{9,10\}, \quad C_2 = \{1,2,3,4\}, \\ C_3 &= \{1,2,6,7\}, \quad C_4 = \{1,2,8,10\}, \\ C_5 &= \{3,4,8,9\}, \quad C_6 = \{6,7,8,9\}, \\ C_7 &= \{1,2,3,5,7\}, \quad C_8 = \{1,2,4,5,6\} \end{aligned}$$

The reliability of the system in terms of minimal path sets is

$$\begin{aligned} R(\mathbf{p}) &= E \coprod_{j=1}^{10} \prod_{i \in P_j} x_i \\ &= 2p^2 + 3p^3 + p^4 - 18p^5 + 7p^6 + 26p^7 - 33p^8 + 15p^9 - 3p^{10} \end{aligned}$$

Esary-Proschan bounds

$$\begin{aligned} LB_{EP} &= \prod_{k=1}^{8}(1-q^{|C_k|}) \\ &= (1-q^2)(1-q^4)^5(1-q^5)^2 \end{aligned}$$

$$\begin{aligned} UB_{EP} &= \coprod_{j=1}^{10} p^{|P_j|} \\ &= 1-(1-p^2)^2(1-p^3)^4(1-p^4)^4 \end{aligned}$$

Min-Max bounds

$$\begin{aligned} LB_{\min-\max} &= \max_{1\le j\le 10}\{p^{|P_j|}\} \\ &= \max\{p^2, p^3, p^4\} \end{aligned}$$

$$\begin{aligned} UB_{\min-\max} &= \min_{1\le k\le 8}\{1-q^{|C_k|}\} \\ &= \min\{1-q^2, 1-q^4, 1-q^5\} \end{aligned}$$

Fu-Koutras bounds

$$\begin{aligned} K_1^* &= \varnothing, \qquad K_2^* = K_3^* = \{1\}, \qquad K_4^* = \{2,3\}, \qquad K_5^* = \{3,4\}, \\ K_6^* &= \{3,4,5\}, \qquad K_7^* = \{3,4,5,6\}, \qquad K_8^* = \{1,2,3,4,5,7\} \\ K_9^* &= \{3,4,5,6,7,8\}, \qquad K_{10}^* = \{1,2,3,4,6,7,8,9\} \end{aligned}$$

$$
\begin{aligned}
K_1 &= \varnothing, \qquad K_2 = \{1\}, \quad K_3 = \{9\}, \qquad K_4 = \{1,9\}, \\
K_5 &= \{8\}, \quad K_6 = \{3,8\}, \quad K_7 = \{4,6,8\}, \quad K_8 = \{1,2,10\}, \\
K_9 &= \{3,7,8\}, \quad K_{10} = \{1,2,3,10\}
\end{aligned}
$$

$$
\begin{aligned}
LB_{FK} &= 1 - \prod_{j=1}^{10}(1 - p^{|P_j|}q^{|K_j|}) \\
&= 1 - (1-p^2)(1-p^2q)(1-p^3q)^2(1-p^3q^2)^2(1-p^4q^3)^3(1-p^4q^4)
\end{aligned}
$$

$$
\begin{aligned}
L_1^* &= L_2^* = \varnothing, \qquad L_3^* = \{2\}, \quad L_4^* = \{1,2,3\}, \quad L_5^* = \{1,2,4\}, \\
L_6^* &= \{1,3,4,5\}, \quad L_7^* = \{2,3,4,5,6\}, \quad L_8^* = \{2,3,4,5,6,7\}
\end{aligned}
$$

$$
\begin{aligned}
L_1 &= L_2 = \varnothing, \qquad L_3 = \{3\}, \quad L_4 = \{3,6,9\}, \quad L_5 = \{1,10\}, \\
L_6 &= \{1,3,10\}, \quad L_7 = \{4,6,8\}, \quad L_8 = \{3,7,8\}
\end{aligned}
$$

$$
\begin{aligned}
UB_{FK} &= \prod_{k=1}^{8}(1 - p^{|L_k|}q^{|C_k|}) \\
&= (1-q^2)(1-q^4)(1-pq^4)(1-p^2q^4)(1-p^3q^4)^2(1-p^3q^5)^2
\end{aligned}
$$

The numerical values of the bounds and $R(\mathbf{p})$ are summarized in Tables 16 and 17 below.

Table 18: Lower bounds and R(p)

p	LB_{EP}	$LB_{\min-\max}$	LB_{FK}	$R(\mathbf{p})$
0.05	1.0957×10^{-6}	0.0025	5.3509×10^{-3}	5.3758×10^{-3}
0.10	1.5327×10^{-4}	0.01	2.2539×10^{-2}	2.2929×10^{-2}
0.15	2.1428×10^{-3}	0.0225	5.2466×10^{-2}	5.4381×10^{-2}
0.85	0.97488	0.7225	0.80276	0.68918
0.90	0.98949	0.81	0.8524	0.54994
0.95	0.99747	0.9025	0.9151	0.33064

Table 19: Upper bounds

p	UB_{EP}	$UB_{\min-\max}$	UB_{FK}
0.05	5.516×10^{-3}	0.0975	1.7307×10^{-2}
0.10	2.4205×10^{-2}	0.19	6.0502×10^{-2}
0.15	5.9235×10^{-2}	0.2775	0.12004
0.85	0.99991	0.9775	0.97553
0.90	1.00000	0.99	0.98957
0.95	1.00000	0.9975	0.99747

The diagrams (Figures 4.5 and 4.6) below compare graphically the three lower and upper bounds above with the system reliability. The colours are explained in Table 18 below.

Table 20: Explanation of colours for bounds in figures 4.5 and 4.6

Colour	**Bound**
Black	$R(\mathbf{p})$
Yellow	Esary-Proschan
Purple	Min-max
Magenta	Fu-Koutras

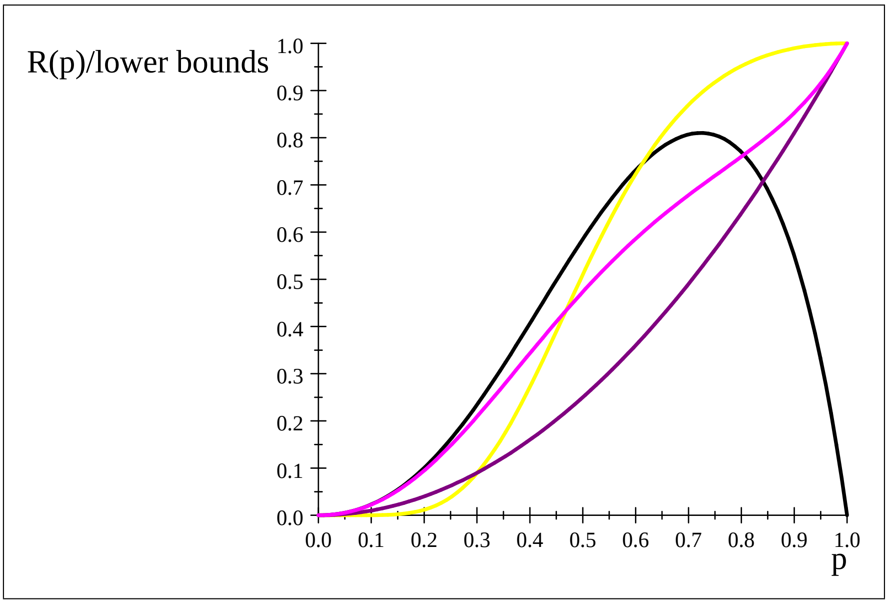

Figure 4.5: Comparison of lower bounds with R(p)

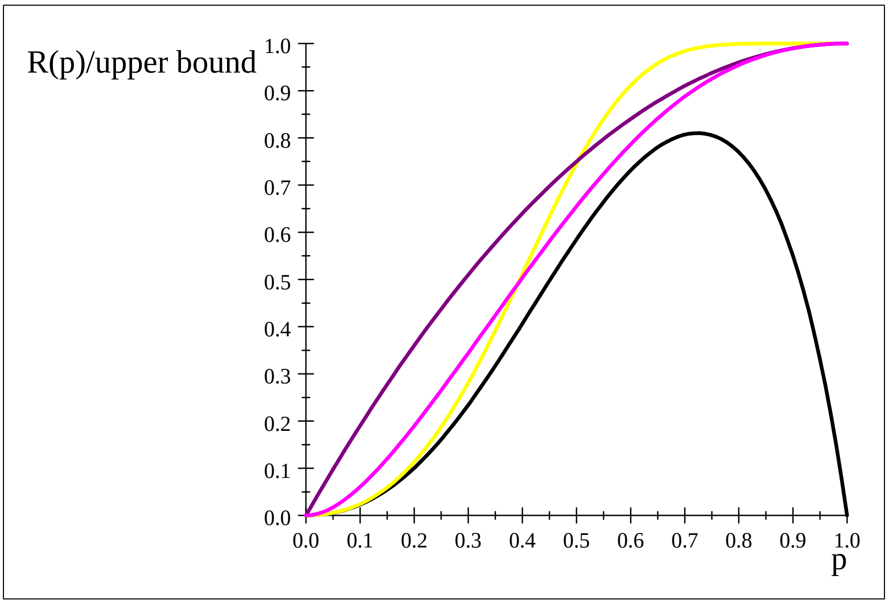

Figure 4.6: Comparison of upper bounds with R(p)

4.2.2 Modular decomposition

We can also decompose the system in Figure 4.4 into 5 modules, $A_1, A_2, \ldots, A_5$, where

$$\begin{aligned} A_1 &= \{c_1\,, c_2\,\}, \qquad A_2 = \{c_3\,, c_4\,, c_5\,, c_6\,, c_7\,\}, \\ A_3 &= \{c_8\,\}, \qquad A_4 = \{c_9\,\}, \qquad A_5 = \{c_{10}\,\} \end{aligned}$$

The structure function $\varphi_{A_1}(\mathbf{x})$ of A_1 is

$$\varphi_{A_1}(\mathbf{x}) = 1 - \bar{x}_1\,\bar{x}_2$$

Hence the reliability p_1 of A_1 is

$$p_1 = 2p - p^2$$

A_2 is a bridge structure so its structure function $\varphi_{A_2}(\mathbf{x})$ is

$$\varphi_{A_2}(\mathbf{x}) = x_3\,x_6 + \bar{x}_3\,x_4\,x_7 + x_3\,x_4\,\bar{x}_6\,x_7 + x_3\,\bar{x}_4\,x_5\,\bar{x}_6\,x_7 + \bar{x}_3\,x_4\,x_5\,x_6\,\bar{x}_7$$

and the reliability p_2 of A_2 is

$$p_2 = 2p^2 + 2p^3 - 5p^4 + 2p^5$$

A_3, A_4 and A_5 have each reliability function p_i , $i = 3, 4, 5$, equal to p.

When the system in Figure 4.4 is decomposed into modules as explained above, it is reduced into a bridge structure with components $A_1, A_2, \ldots, A_5$. The structure function of the system is thus given as

$$\begin{aligned} \varphi(\mathbf{x}) &= \mathbf{x}^{A_1}\mathbf{x}^{A_4} + \mathbf{x}^{A_2}\mathbf{x}^{A_5} + \mathbf{x}^{A_1}\mathbf{x}^{A_3}\mathbf{x}^{A_5} + \mathbf{x}^{A_2}\mathbf{x}^{A_3}\mathbf{x}^{A_4} - \mathbf{x}^{A_1}\mathbf{x}^{A_2}\mathbf{x}^{A_4}\mathbf{x}^{A_5} \\ &\quad - \mathbf{x}^{A_1}\mathbf{x}^{A_3}\mathbf{x}^{A_4}\mathbf{x}^{A_5} - \mathbf{x}^{A_1}\mathbf{x}^{A_2}\mathbf{x}^{A_3}\mathbf{x}^{A_4} - \mathbf{x}^{A_1}\mathbf{x}^{A_2}\mathbf{x}^{A_3}\mathbf{x}^{A_5} \\ &\quad - \mathbf{x}^{A_2}\mathbf{x}^{A_3}\mathbf{x}^{A_4}\mathbf{x}^{A_5} + 2\mathbf{x}^{A_1}\mathbf{x}^{A_2}\mathbf{x}^{A_3}\mathbf{x}^{A_4}\mathbf{x}^{A_5} \end{aligned}$$

Hence the reliability function $R(\mathbf{p})$ of the system is given as

$$R(\mathbf{p}) = p^2 \left(2 + 3p + p^2 - 16p^3 - 3p^4 + 47p^5 - 53p^6 + 24p^7 - 4p^8\right)$$

The minimal path sets are

$$\begin{aligned} P_1 &= \{1,4\}, \qquad P_2 = \{2,5\}, \\ P_3 &= \{1,3,5\}, \qquad P_4 = \{2,3,4\} \end{aligned}$$

and minimal cut sets are

$$\begin{aligned} C_1 &= \{1,2\}, \qquad C_2 = \{4,5\}, \\ C_3 &= \{1,3,5\}, \qquad C_4 = \{2,3,4\} \end{aligned}$$

Esary-Proschan Bounds

$$\begin{aligned} LB_{EP} &= \prod_{k=1}^{4} \coprod_{i \in C_k} p_i \\ &= p^4 \left(p^2 - 2p + 2\right) \left(2p^7 - 13p^6 + 32p^5 - 35p^4 + 12p^3 + 5p^2 - 4\right)^2 \end{aligned}$$

$$\begin{aligned} UB_{EP} &= \coprod_{j=1}^{4} \prod_{i \in P_j} p_i \\ &= 1 - [1 - p(2p - p^2)][1 - p(2p^2 + 2p^3 - 5p^4 + 2p^5)] \\ &\quad [1 - p^2(2p - p^2)][1 - p^2(2p^2 + 2p^3 - 5p^4 + 2p^5)] \end{aligned}$$

Min-Max Bounds

$$\begin{aligned} LB_{\min-\max} &= \max_{1\leq j\leq 4}\prod_{i\in P_j} p_i \\ &= \max\left\{\begin{matrix} p(2p-p^2), & p(2p^2+2p^3-5p^4+2p^5), \\ p^2(2p-p^2), & p^2(2p^2+2p^3-5p^4+2p^5) \end{matrix}\right\} \end{aligned}$$

$$\begin{aligned} UB_{\min-\max} &= \min_{1\leq k\leq 4}\coprod_{i\in C_k} p_i \\ &= \min\left\{\begin{matrix} 1-(1-2p+p^2)(1-2p^2-2p^3+5p^4-2p^5),\ 1-q^2, \\ 1-q^2(1-2p+p^2),\ 1-q^2(1-2p^2-2p^3+5p^4-2p^5) \end{matrix}\right\} \end{aligned}$$

Fu-Koutras Bounds

$$K_1^* = K_2^* = \varnothing, \qquad K_3^* = \{1,\ 2\}, \qquad K_4^* = \{1,\ 2, 3\}$$

$$K_1 = K_2 = \varnothing, \qquad K_3 = \{2,\ 4\}, \qquad K_4 = \{1,\ 5\}$$

$$\begin{aligned} LB_{FK} &= 1-\prod_{j=1}^{4}\left[1-\left(\prod_{i\in P_j} p_i\right)\prod_{i\in K_j} q_i\right] \\ &= 1-[1-p(2p-p^2)][1-p(2p^2+2p^3-5p^4+2p^5)] \\ &\quad [1-p^2q(2p-p^2)(1-2p^2-2p^3+5p^4-2p^5)] \\ &\quad [1-p^2q(1-2p+p^2)(2p^2+2p^3-5p^4+2p^5)] \end{aligned}$$

$$L_1^* = L_2^* = \varnothing, \qquad L_3^* = \{1,2\}, \qquad L_4^* = \{1,2,3\}$$

$$L_1 = L_2 = \varnothing, \qquad L_3 = \{2,4\}, \qquad L_4 = \{1,5\}$$

$$
\begin{aligned}
UB_{FK} &= \prod_{k=1}^{4}\left[1-\left(\prod_{i\in L_k} p_i\right)\left(\prod_{i\in C_k} q_i\right)\right] \\
&= [1-q^2][1-(1-2p+p^2)(1-2p^2-2p^3+5p^4-2p^5)] \\
&\quad [1-pq^2(1-2p+p^2)(2p^2+2p^3-5p^4+2p^5)] \\
&\quad [1-pq^2(2p-p^2)(1-2p^2-2p^3+5p^4-2p^5)]
\end{aligned}
$$

The numerical values of the bounds and $R(\mathbf{p})$ are summarized in Tables 19 and 20 below.

Table 21: Lower bounds and R(p)

p	LB_{EP}	$LB_{\min-\max}$	LB_{FK}	$R(\mathbf{p})$
0.05	1.8894×10^{-4}	4.875×10^{-3}	5.3750×10^{-3}	5.3762×10^{-3}
0.10	2.8115×10^{-3}	0.019	2.2902×10^{-2}	2.2941×10^{-2}
0.15	1.3009×10^{-2}	4.1625×10^{-2}	5.4167×10^{-2}	5.4450×10^{-2}
0.85	0.97484	0.83088	0.96778	0.97508
0.90	0.98947	0.891	0.98702	0.98951
0.95	0.99747	0.94763	0.99712	0.99747

Table 22: Upper bounds

p	UB_{EP}	$UB_{\min-\max}$	UB_{FK}
0.05	5.3902×10^{-3}	0.0975	9.9198×10^{-3}
0.10	2.3181×10^{-2}	0.19	3.8764×10^{-2}
0.15	5.5712×10^{-2}	0.2775	8.4094×10^{-2}
0.85	0.99701	0.9775	0.97511
0.90	0.99947	0.99	0.98951
0.95	0.99997	0.9975	0.99747

The diagrams (Figures 4.7 and 4.8) below compare graphically the three lower and upper bounds above with the system reliability. The colours are explained in Table 21 below.

Table 23: Explanation of colours for bounds in figures 4.7 and 4.8

Colour	**Bound**
Black	$R(\mathbf{p})$
Yellow	Esary-Proschan
Purple	Min-max
Magenta	Fu-Koutras

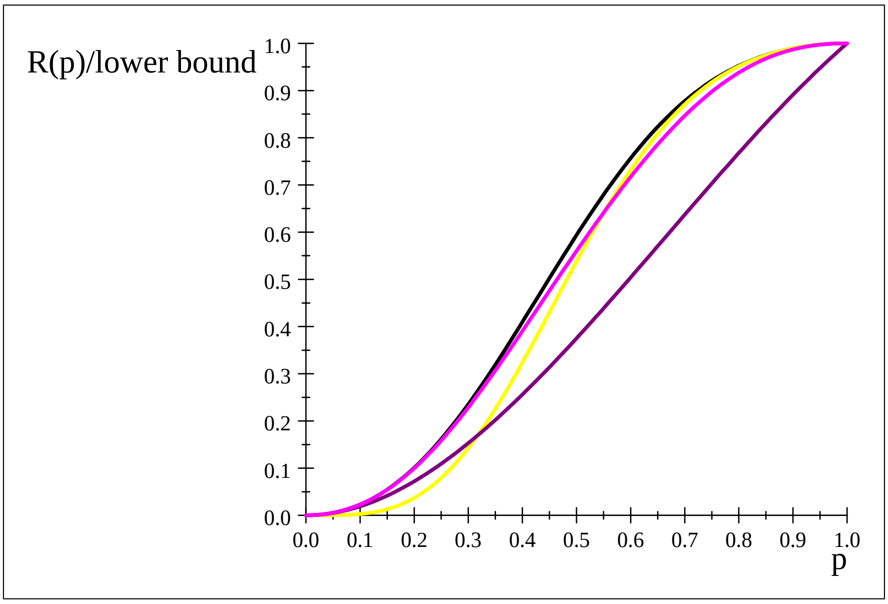

Figure 4.7: Comparison of lower bounds with R(p)

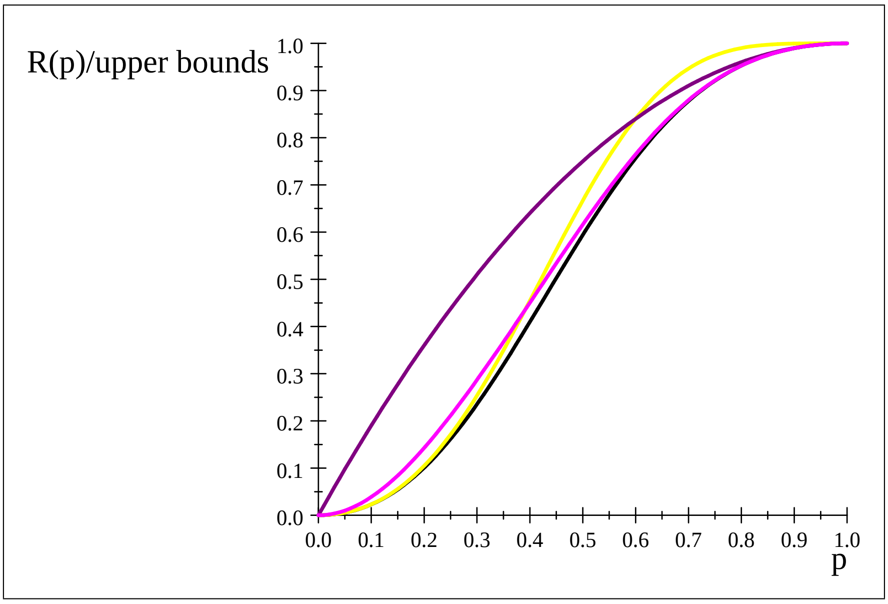

Figure 4.8: Comparison of upper bounds with R(p)

It is evident from Figures 4.5, 4.6, 4.7 and 4.8 that both the lower and upper bounds perform better when the system is first decomposed into modules.

4.3 Birnbaum's Reliability Importance

In this section we look at the reliability importances of the components in Figure 4.1. We will take the components with the highest and lowest reliability importance, and investigate which of these components improve the performance of the bounds. We will only consider Birnbaum's reliability importance, as discussed in section 2.10.

The reliability importances of the seven components are given below. We have used $p_1 = p_2 = \cdots = p_7 = \frac{1}{2}$.

$$
\begin{aligned}
RI_1^{(B)}(\varphi,\mathbf{p}) &= \frac{\partial R(\mathbf{p})}{\partial p_1} \\
&= q_2\, p_3\, p_7 \ + q_2\, q_3\, p_4\, p_6 \ + p_2\, p_4\, p_6\, q_7 \ + q_2\, p_3\, p_4\, p_6\, q_7 \\
&\quad -p_2\, p_5\, p_6\, q_7 \ + p_2\, q_4\, p_5\, p_6\, q_7 \ + q_2\, p_3\, q_4\, p_5\, p_6\, q_7 \\
&\quad +q_2\, q_3\, p_4\, p_5\, q_6\, p_7 \ - p_2\, p_3\, p_4\, q_5\, p_6\, q_7 \\
&= \frac{1}{8} + \frac{1}{16} + \frac{2}{32} + \frac{1}{64} \\
&= 0.265\,63
\end{aligned}
$$

$$
\begin{aligned}
RI_2^{(B)}(\varphi,\mathbf{p}) &= \frac{\partial R(\mathbf{p})}{\partial p_2} \\
&= p_7 \ - p_1\, p_3\, p_7 \ - p_1\, q_3\, p_4\, p_6 \ + p_1\, p_4\, p_6\, q_7 \ - p_1\, p_3\, p_4\, p_6\, q_7 \\
&\quad +q_1\, p_5\, p_6\, q_7 \ + p_1\, q_4\, p_5\, p_6\, q_7 \ - p_1\, p_3\, q_4\, p_5\, p_6\, q_7 \\
&\quad -p_1\, q_3\, p_4\, p_5\, q_6\, p_7 \ + q_1\, p_3\, p_4\, q_5\, p_6\, q_7 \\
&= \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{1}{64} \\
&= 0.421\,88
\end{aligned}
$$

$$
\begin{aligned}
RI_3^{(B)}(\varphi,\mathbf{p}) &= \frac{\partial R(\mathbf{p})}{\partial p_3} \\
&= p_1\, q_2\, p_7 \ - p_1\, q_2\, p_4\, p_6 \ + p_1\, q_2\, p_4\, p_6\, q_7 \ + p_1\, q_2\, q_4\, p_5\, p_6\, q_7 \\
&\quad -p_1\, q_2\, p_4\, p_5\, q_6\, p_7 \ + q_1\, p_2\, p_4\, q_5\, p_6\, q_7 \\
&= \frac{1}{8} - \frac{1}{16} + \frac{1}{32} + \frac{1}{64} \\
&= 0.109\,38
\end{aligned}
$$

$$
\begin{aligned}
RI_4^{(B)}(\varphi,\mathbf{p}) &= \frac{\partial R(\mathbf{p})}{\partial p_4} \\
&= p_1\, q_2\, q_3\, p_6 \ + p_1\, p_2\, p_6\, q_7 \ + p_1\, q_2\, p_3\, p_6\, q_7 \ - p_1\, p_2\, p_5\, p_6\, q_7 \\
&\quad -p_1\, q_2\, p_3\, p_5\, p_6\, q_7 \ + p_1\, q_2\, q_3\, p_5\, q_6\, p_7 \ + q_1\, p_2\, p_3\, q_5\, p_6\, q_7 \\
&= \frac{2}{16} + \frac{1}{64} \\
&= 0.140\,63
\end{aligned}
$$

$$
\begin{aligned}
RI_5^{(B)}(\varphi,\mathbf{p}) &= \frac{\partial R(\mathbf{p})}{\partial p_5} \\
&= q_1\, p_2\, p_6\, q_7 \; + p_1\, p_2\, q_4\, p_6\, q_7 \; + p_1\, q_2\, p_3\, q_4\, p_6\, q_7 \\
&\quad + p_1\, q_2\, q_3\, p_4\, q_6\, p_7 \; - q_1\, p_2\, p_3\, p_4\, p_6\, q_7 \\
&= \frac{1}{16} + \frac{1}{32} + \frac{1}{64} \\
&= 0.109\,38
\end{aligned}
$$

$$
\begin{aligned}
RI_6^{(B)}(\varphi,\mathbf{p}) &= \frac{\partial R(\mathbf{p})}{\partial p_6} \\
&= p_1\, q_2\, q_3\, p_4 \; + p_1\, p_2\, p_4\, q_7 \; + p_1\, q_2\, p_3\, p_4\, q_7 \; + q_1\, p_2\, p_5\, q_7 \\
&\quad + p_1\, p_2\, q_4\, p_5\, q_7 \; + p_1\, q_2\, p_3\, q_4\, p_5\, q_7 \; - p_1\, q_2\, q_3\, p_4\, p_5\, q_7 \\
&\quad + q_1\, p_2\, p_3\, p_4\, q_5\, q_7 \\
&= \frac{3}{16} + \frac{2}{32} + \frac{1}{64} \\
&= 0.265\,63
\end{aligned}
$$

$$
\begin{aligned}
RI_7^{(B)}(\varphi,\mathbf{p}) &= \frac{\partial R(\mathbf{p})}{\partial p_7} \\
&= p_2 \; + p_1\, q_2\, p_3 \; - p_1\, p_2\, p_4\, p_6 \; - p_1\, q_2\, p_3\, p_4\, p_6 \; - q_1\, p_2\, p_5\, p_6 \\
&\quad - p_1\, p_2\, p_4\, p_5\, p_6 \; - p_1\, q_2\, p_3\, q_4\, p_5\, p_6 \; + p_1\, q_2\, q_3\, p_4\, p_5\, q_6 \\
&\quad - q_1\, p_2\, p_3\, p_4\, q_5\, p_6 \\
&= \frac{1}{2} + \frac{1}{8} - \frac{2}{16} - \frac{2}{32} - \frac{1}{64} \\
&= 0.421\,88
\end{aligned}
$$

Components c_2 and c_7 have the highest reliability importance, both with reliability importance equal 0.421 88. We will investigate the effect on the bounds when we increase/decrease the reliability p_2 of component c_2 while the reliabilities of other components are held constant.

The reliability function $R(\mathbf{p})$, when component c_2 has different reliability, is given as

$$R(\mathbf{p}) = pp_2 + p^2 p_2 - p^3 p_2 - 7p^4 p_2 + 9p^5 p_2 - 3p^6 p_2 + 2p^3 + 2p^4 - 5p^5 + 2p^6$$

Components c_3 and c_5 have the lowest reliability importance, both with reliability importance equal 0.10938. We will investigate the effect on the bounds when we increase/decrease the reliability p_3 of component c_3 while the reliabilities of other components are held constant.

The reliability function $R(\mathbf{p})$, when component c_3 has different reliability, is given as

$$R(\mathbf{p}) = p^2 + p^2 p_3 q + p^3 q^2 + p^3 p_3 q^2 + 2p^3 p_3 q^3 + p^3 q q_3 + p^4 q + p^4 q^2 + p^4 q^2 q_3$$

4.3.1 Trivial bounds

When component c_2 has a different reliability function, the lower and upper bounds are given as

$$\begin{aligned} LB_{Triv} &= \prod_{i=1}^{7} p_i \\ &= p_2\, p^6 \end{aligned}$$

$$\begin{aligned} UB_{Triv} &= \coprod_{i=1}^{7} p_i \\ &= 1 - q_2\, q^6 \end{aligned}$$

Figure 4.9 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

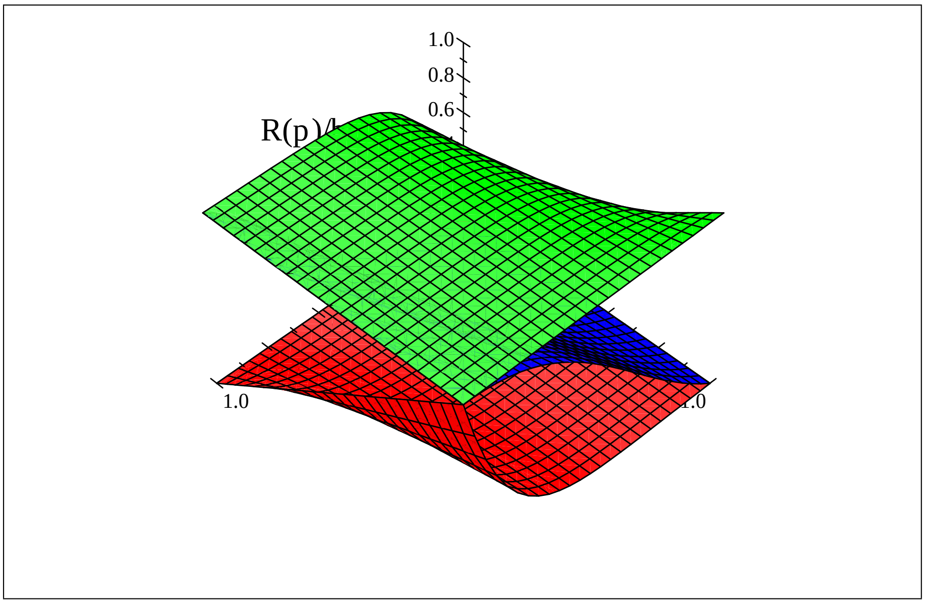

Figure 4.9: Comparison of R(p) with trivial bounds for p_2

When component c_3 has a different reliability function, the lower and upper bounds are given as

$$
\begin{aligned}
LB_{Triv} &= \prod_{i=1}^{7} p_i \\
&= p_3\, p^6
\end{aligned}
$$

$$
\begin{aligned}
UB_{Triv} &= \coprod_{i=1}^{7} p_i \\
&= 1 - q_3\, q^6
\end{aligned}
$$

Figure 4.10 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

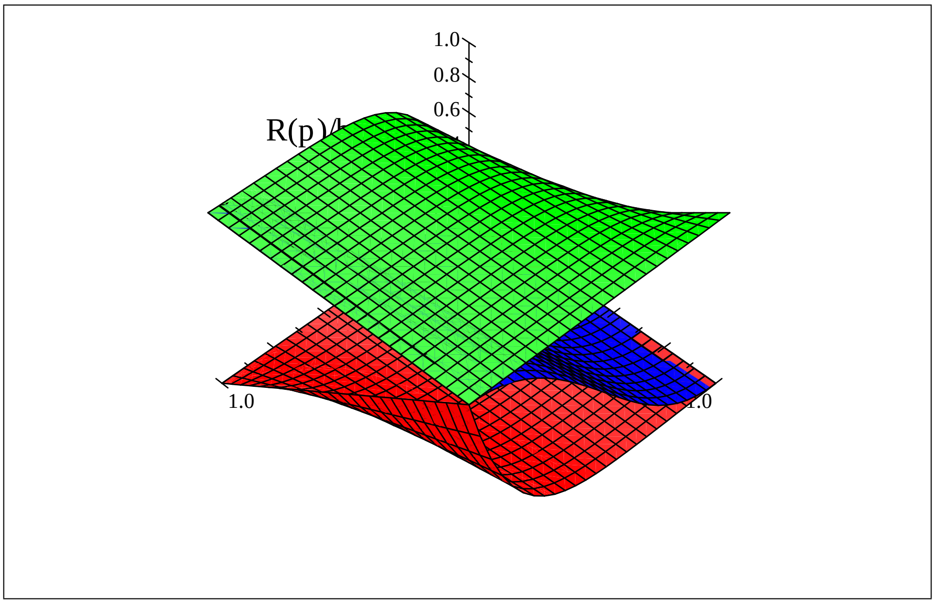

Figure 4.10: Comparison of R(p) with trivial bounds for p_3

For the trivial bounds it does not matter whether you invest on the component with the highest or lowest Birmbaum importance, the bounds will behave the same (see Figures 4.9 and 4.10)

4.3.2 Esary-Proschan bounds

When component c_2 has a different reliability function, the lower and upper bounds are given as

$$\begin{aligned} LB_{EP} &= \prod_{k=1}^{5} \coprod_{i \in C_k} p_i \\ &= (1-q_2 q)(1-q^2)(1-q^3)(1-q_2 q^3)(1-q^4) \end{aligned}$$

$$\begin{aligned} UB_{EP} &= \coprod_{j=1}^{7} \prod_{i \in P_j} p_i \\ &= 1-(1-p_2\, p)(1-p^3)^2(1-p_2\, p^2)(1-p^4)^2(1-p_2\, p^3) \end{aligned}$$

Figure 4.11 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

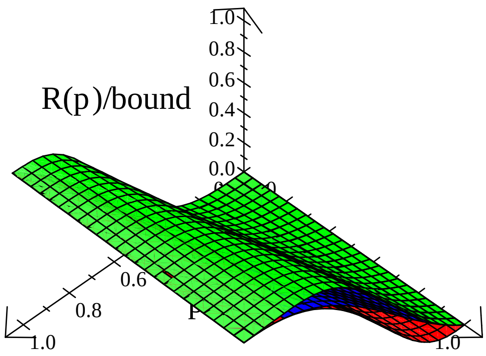

Figure 4.11: Comparison of Esary-Proschan bounds with R(p) for p_2

When component c_3 has a different reliability function, the lower and upper bounds are given as

$$\begin{aligned} LB_{EP} &= \prod_{k=1}^{5} \coprod_{i \in C_k} p_i \\ &= (1-q^2)^2(1-q^3)(1-q^3 q_3)^2 \end{aligned}$$

$$\begin{aligned} UB_{EP} &= \coprod_{j=1}^{7} \prod_{i \in P_j} p_i \\ &= 1-(1-p^2)(1-p^2 p_3)(1-p^3)^2(1-p^3 p_3)^2(1-p^4) \end{aligned}$$

Figure 4.12 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

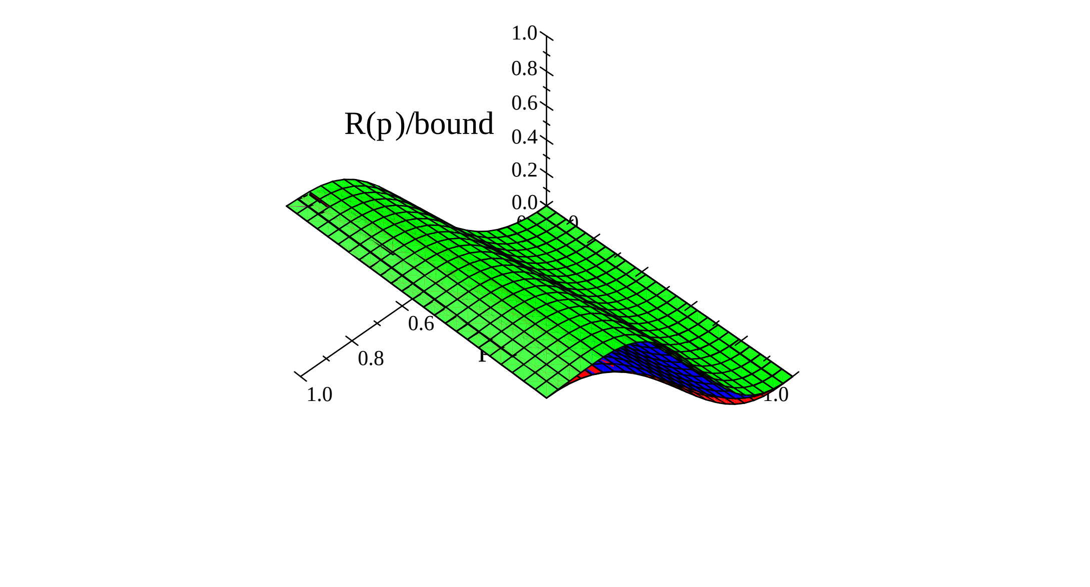

Figure 4.12: Comparison of Esary-Proschan bounds with R(p) for p_3

For the Esary-Proschan bounds, the component with the lowest Birnbaum importance improves the lower bound better than the component with the highest Birnbaum importance. While the component with the highest Birnbaum importance improves the upper bound better than the component with the lowest Birnbaum importance, especially when p is close to 0.5 for all values of p_2 /p_3 .

4.3.3 Min-max bounds

$$\begin{aligned} LB_{\min-\max} &= \max_{1\leq j\leq 7} \prod_{i\in P_j} p_i \\ &= \max\{p_2\, p,\ p_2\, p^2,\ p^3,\ p_2\, p^3,\ p^4\} \end{aligned}$$

$$\begin{aligned} UB_{\min-\max} &= \min_{1\leq k\leq 5} \coprod_{i\in C_k} p_i \\ &= \min\{(1-q_2\, q),\ (1-q^2),\ (1-q^3),\ (1-q_2\, q^3),\ (1-q^4)\} \end{aligned}$$

Figure 4.13 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

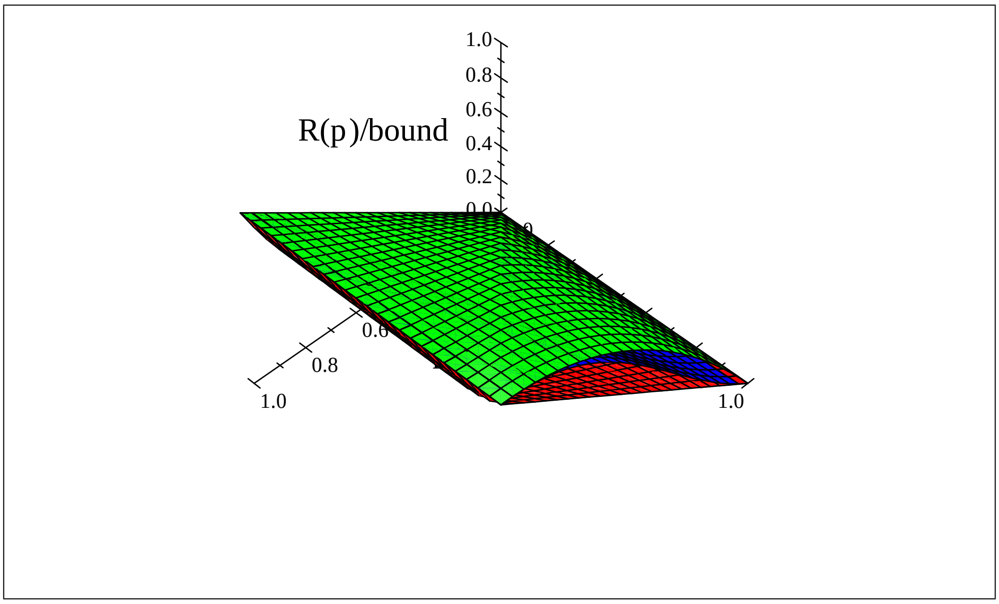

Figure 4.13: Comparison of min-max bounds with R(p) for p_2

When component c_3 has a different reliability function, the lower and upper bounds are given as

$$
\begin{aligned}
LB_{\min-\max} &= \max_{1\le j\le 7} \prod_{i\in P_j} p_i \\
&= \max\{p^2, p^2 p_3\ , p^3, p^3 p_3\ , p^4\}
\end{aligned}
$$

$$
\begin{aligned}
UB_{\min-\max} &= \min_{1\le k\le 5} \coprod_{i\in C_k} p_i \\
&= \min\{1-q^2, 1-q^3, 1-q^3 q_3\ \}
\end{aligned}
$$

Figure 4.14 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

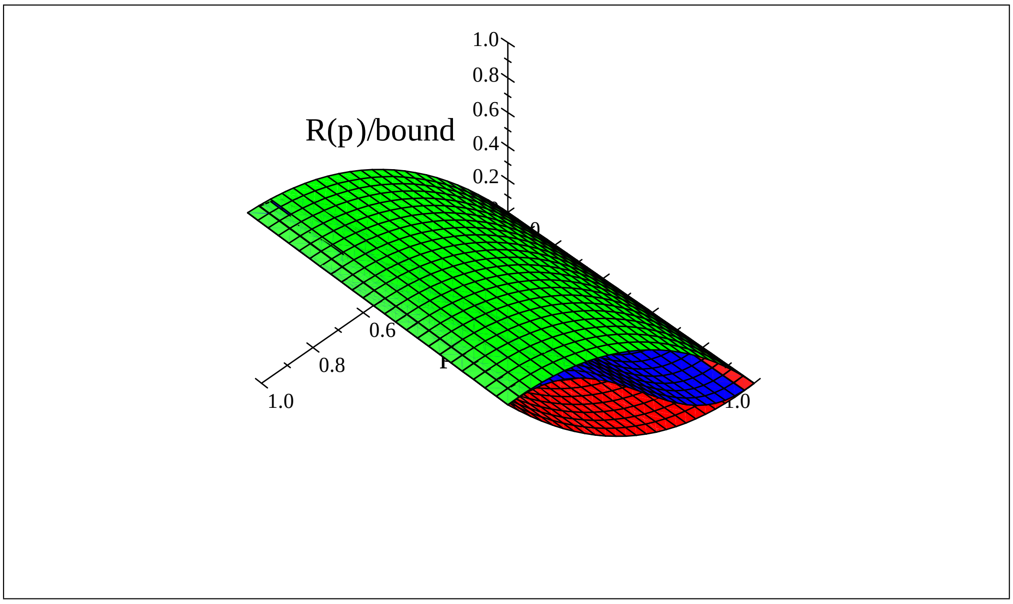

Figure 4.14: Comparison of min-max bounds with R(p) for p_3

For the min-max bounds, the component with the highest Birnbaum importance improves both bounds better than the component with the lowest Birnbaum importance.

4.3.4 Fu-Koutras bounds

When component c_2 has a different reliability function, the lower and upper bounds are given as

$$\begin{aligned} LB_{FK} &= 1 - \prod_{j=1}^{7} \left[1 - \left(\prod_{i \in K_j} q_i \right) \prod_{i \in P_j} p_i \right] \\ &= 1 - (1 - p_2\, p)(1 - p^3 q_2\,)(1 - p^3 q_2\, q^2)(1 - p^2 p_2\, q^4)(1 - p^3 p_2\, q^3)(1 - p^4 q_2\, q^2)^2 \end{aligned}$$

$$\begin{aligned} UB_{FK} &= \prod_{k=1}^{5} \left[1 - \left(\prod_{i \in L_k} p_i \right) \left(\prod_{i \in C_k} q_i \right) \right] \\ &= (1 - q_2\, q)(1 - p_2\, pq^2)(1 - p_2\, p^2 q^3)(1 - p_2\, p^2 q^4)(1 - p^3 q_2\, q^3) \end{aligned}$$

Figure 4.15 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

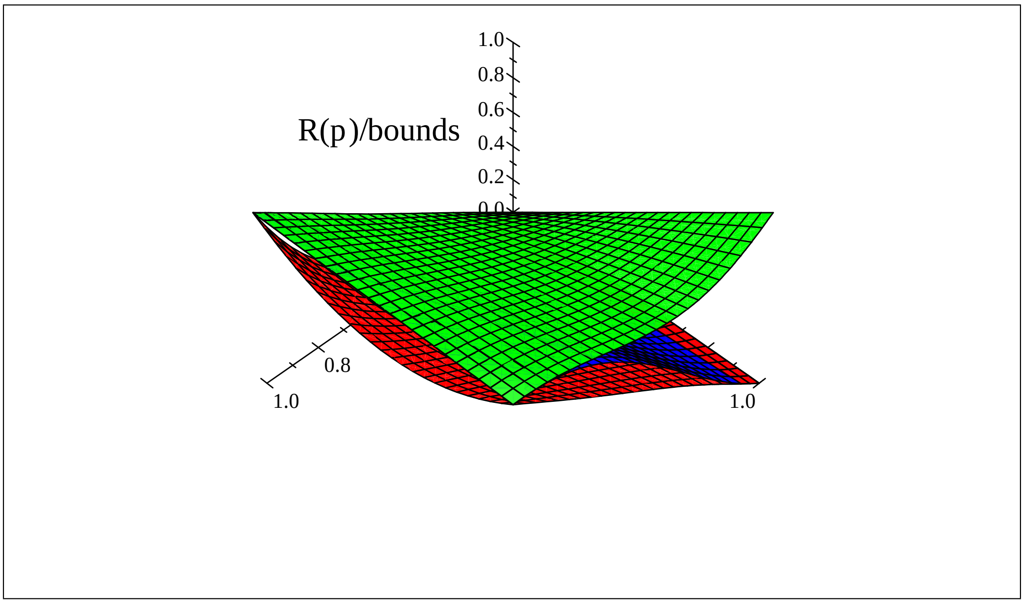

Figure 4.15: Comparison of Fu-Koutras bounds with R(p) for p_2

When component c_3 has a different reliability function, the lower and upper bounds are given as

$$\begin{aligned} LB_{FK} &= 1 - \prod_{j=1}^{7}\left[1 - \left(\prod_{i \in K_j} q_i\right)\prod_{i \in P_j} p_i\right] \\ &= 1-(1-p^2)(1-p^2qq_3)(1-p^3q^2q_3)(1-p^3p_3q^3)^2(1-p^3q^3q_3)(1-p^4q^2q_3) \end{aligned}$$

$$\begin{aligned} UB_{FK} &= \prod_{k=1}^{5}\left[1 - \left(\prod_{i \in L_k} p_i\right)\left(\prod_{i \in C_k} q_i\right)\right] \\ &= (1-q^2)(1-p^2q^2)(1-p^3q^3)(1-p^3q^3q_3)^2 \end{aligned}$$

Figure 4.16 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

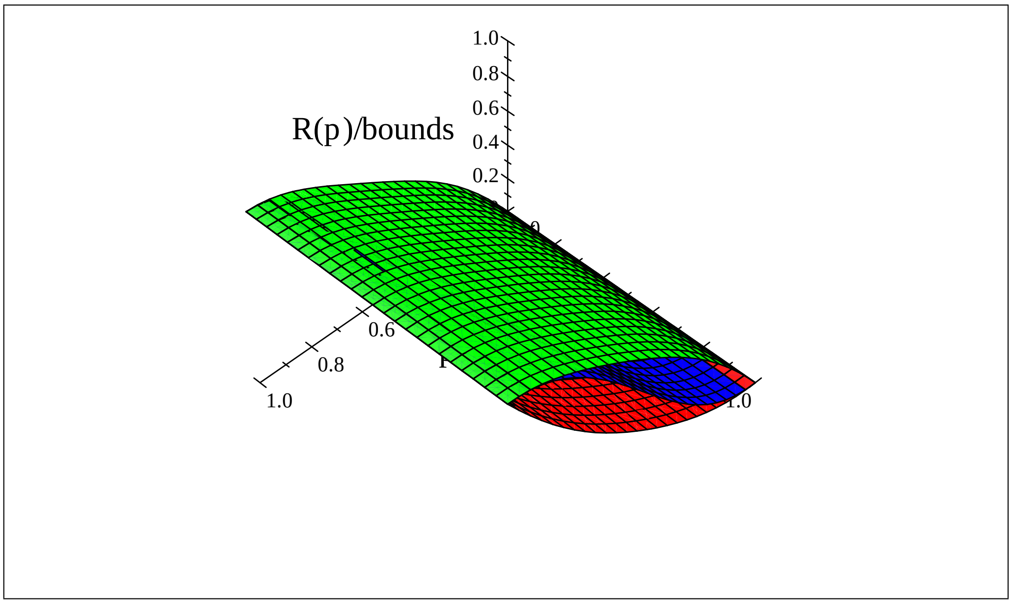

Figure 4.16: Comparison of Fu-Koutras bounds with R(p) for p_3

For the Fu-Koutras bounds, the component with the lowest Birnbaum importance improves both bounds better than the component with the lowest Birnbaum importance.

4.3.5 Koutras-Papastavridis-Petakos bounds

When component c_2 has a different reliability function, the lower and upper bounds are given as

$$\begin{aligned} LB_{KPP} &= 1 - \prod_{j=1}^{7} \left[1 - \Pr(\beta_j = 0) \cdot \prod_{i \in P_j} p_i \right] \\ &= 1 - (1 - p^3 q_2)(1 - p^3 q^2)(1 - p^2 p_2 q^2)(1 - p^4 q^2 q_2)^2 (1 - p^3 p_2 q^3) \end{aligned}$$

$$\begin{aligned} UB_{KPP} &= \prod_{k=1}^{5} \left[1 - \Pr(\mu_k = 1) \cdot \prod_{i \in C_k} q_i \right] \\ &= (1 - q q_2)(1 - q^2)(1 - p^3 p_2 q^3)(1 - p^2 p_2 q^4)(1 - p^3 q^3 q_2) \end{aligned}$$

Figure 4.17 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

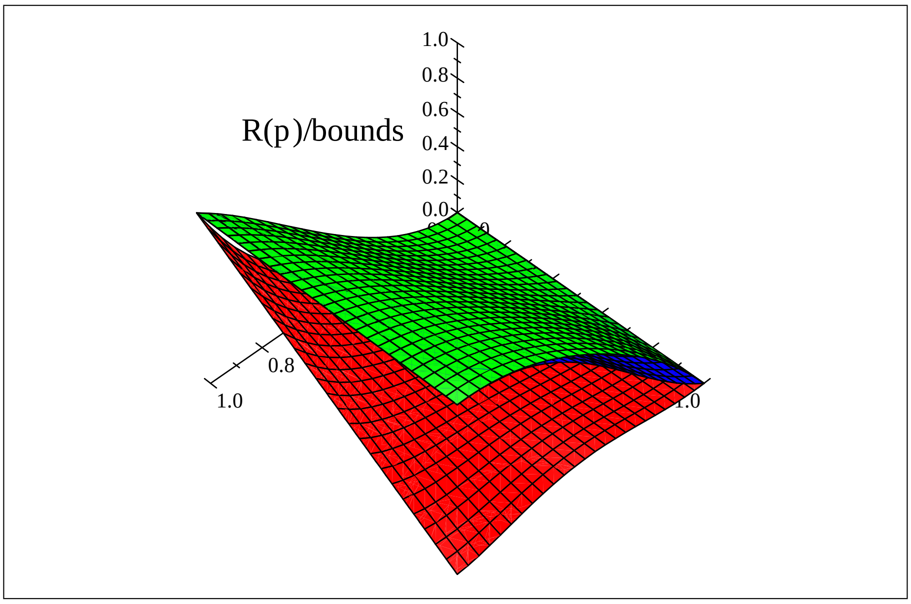

Figure 4.17: Comparison of KPP bounds with R(p) for p_2

When component c_3 has a different reliability function, the lower and upper bounds are given as

$$\begin{aligned} LB_{KPP} &= 1 - \prod_{j=1}^{7}\left[1 - \Pr(\beta_j = 0)\cdot \prod_{i\in P_j} p_i\right] \\ &= 1-(1-p^2p_3\,q)(1-p^3qq_3\,)(1-p^3q^2)(1-p^3p_3\,q^3)^2(1-p^4q^2q_3\,) \end{aligned}$$

$$\begin{aligned} UB_{KPP} &= \prod_{k=1}^{5}\left[1 - \Pr(\mu_k = 1)\cdot \prod_{i\in C_k} q_i\right] \\ &= (1-q^2)^2(1-p^3p_3\,q^3)(1-p^3q^3q_3\,)^2 \end{aligned}$$

Figure 4.18 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

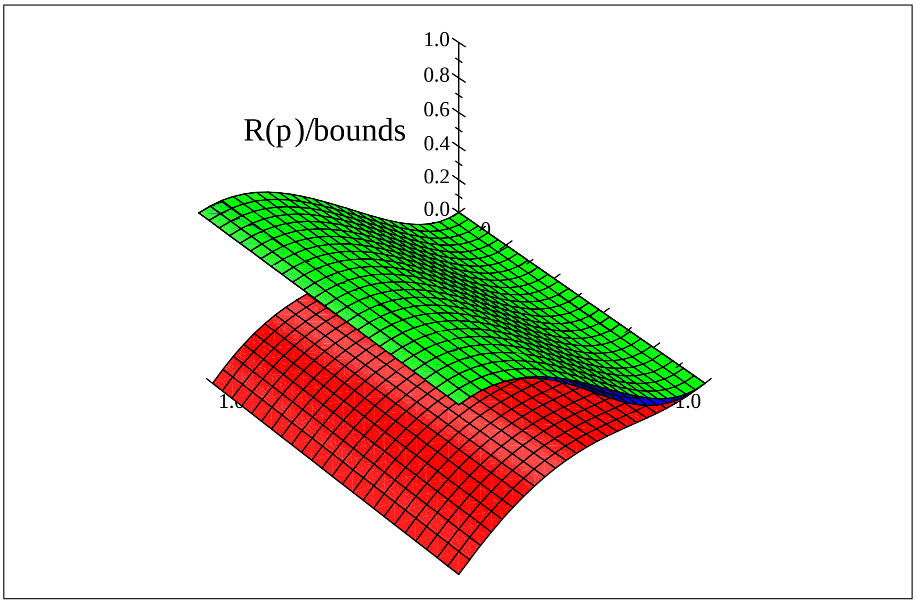

Figure 4.18: Comparison of KPP bounds with R(p) for p_3

For the Koutras-Papastavridis-Petakos bounds, the component with the highest Birnbaum importance improves the lower bound better than the component with lowest Birnbaum importance, especially for high values of p and low values of p_2 /p_3 .The component with the lowest Birnbaum importance improves the upper bound better (see figures 4.17 and 4.18).

4.3.6 Generalized Esary-Proschan bounds

When component c_2 has a different reliability function, the lower and upper bounds are given as

$$
\begin{aligned}
LB_{EP}^{(G)} &= \prod_{s=1}^{3}\left[\prod_{C\in\mathbb{C}_s}\left(1-\prod_{i\in C} q_i\right)\right] \\
&= (1-q_2q)(1-q^2)(1-q^3)(1-q_2q^3)(1-q^4)
\end{aligned}
$$

$$
\begin{aligned}
UB_{EP}^{(G)} &= 1 - \prod_{s=1}^{3}\left[1 - \prod_{P \in \P_s}\left(1 - \prod_{i \in P} p_i\right)\right] \\
&= 1 - [1-(1-pp_2\,)][1-(1-p^3)^2(1-p^2p_2\,)][1-(1-p^4)(1-p^3p_2\,)]
\end{aligned}
$$

Figure 4.19 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

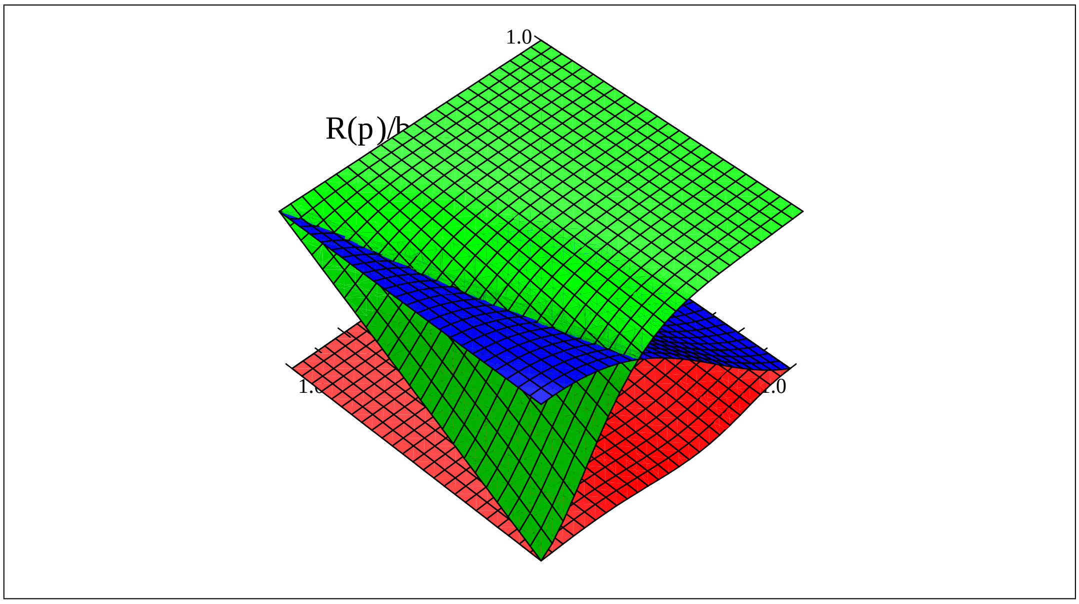

Figure 4.19: Comparison of generalized EP bounds with R(p) for p_2

When component c_3 has a different reliability function, the lower and upper bounds are given as

$$
\begin{aligned}
LB_{EP}^{(G)} &= \prod_{s=1}^{3}\left[\prod_{C \in \mathbb{C}_s}\left(1 - \prod_{i \in C} q_i\right)\right] \\
&= (1-q^2)^2(1-q^3)(1-q^3q_3\,)^2
\end{aligned}
$$

$$
\begin{aligned}
UB_{EP}^{(G)} &= 1 - \prod_{s=1}^{3} \left[1 - \prod_{P \in ¶_s} \left(1 - \prod_{i \in P} p_i \right) \right] \\
&= 1 - p^2[1 - (1 - p^2 p_3)(1 - p^3)^2][1 - (1 - p^3 p_3)^2 (1 - p^4)]
\end{aligned}
$$

Figure 4.20 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

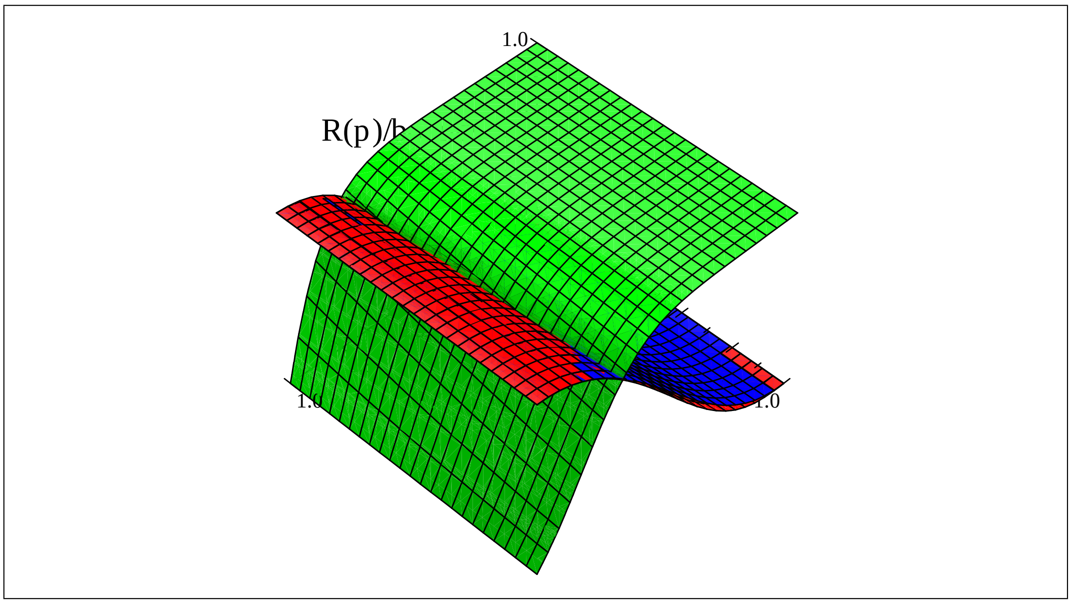

Figure 4.20: Comparison of generalized EP bounds with R(p) for p_3

For the generalized Esary-Proschan bounds the component with the lowest Birnbaum importance gives a better lower bound. The upper bound is not a good approximation for both situations (see Figures 4.19 and 4.20).

4.3.7 Generalized Fu-Koutras bounds

When component c_2 has a different reliability function, the lower and upper bounds are given as

$$
\begin{aligned}
LB_{FK}^{(G)} &= 1-\left[\prod_{P\in\P_1}\left(1-\prod_{i\in P}p_i\right)\right]\cdot\prod_{s=2}^{3}\left[1-\left(\coprod_{P\in\P_s^*\cup\P_s}\prod_{i\in P}p_i-\coprod_{P\in\P_s^*}\prod_{i\in P}p_i\right)\right]\\
&= 1-(1-p_2\,p)[p_2\,p+(1-p_2\,p)(1-p_2\,p^2)(1-p^3)^2][1-(1-p_2\,p)\\
&\quad(1-p_2\,p^2)(1-p^3)^2+(1-p_2\,p)(1-p_2\,p^2)(1-p^3)^2(1-p_2\,p^3)(1-p^4)^2]
\end{aligned}
$$

$$
\begin{aligned}
UB_{FK}^{(G)} &= \left[\prod_{C\in\mathbb{C}_1}\coprod_{i\in C}p_i\right]\cdot\prod_{s=2}^{3}\left[1-\left(\prod_{C\in\mathbb{C}_s^*}\coprod_{i\in C}p_i-\prod_{C\in\mathbb{C}_s^*\cup\mathbb{C}_s}\coprod_{i\in C}p_i\right)\right]\\
&= (1-q_2\,q)(1-q^2)[q^2+(1-q^2)(1-q^3)][1-(1-q_2\,q)(1-q^2)\\
&\quad(1-q^3)+(1-q_2\,q)(1-q^2)(1-q^3)(1-q_2\,q^3)(1-q^4)]
\end{aligned}
$$

Figure 4.21 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

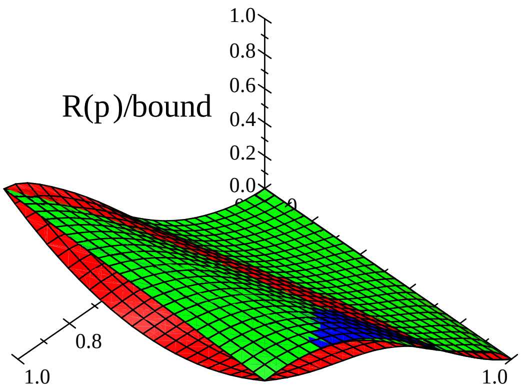

Figure 4.21: Comparison of generalized FK bounds with R(p) for p_2

When component c_3 has a different reliability function, the lower and upper bounds are given as

$$\begin{aligned}
LB_{FK}^{(G)} &= 1 - \left[\prod_{P \in \P_1} \left(1 - \prod_{i \in P} p_i \right) \right] \cdot \prod_{s=2}^{3} \left[1 - \left(\coprod_{P \in \P_s^* \cup \P_s} \prod_{i \in P} p_i - \coprod_{P \in \P_s^*} \prod_{i \in P} p_i \right) \right] \\
&= 1 - (1-p^2)[p^2 + (1-p^2)(1-p^2p_3\ )(1-p^3)^2][1-(1-p^2)(1-p^2p_3\ ) \\
&\quad (1-p^3)^2 + (1-p^2)(1-p^2p_3\ )(1-p^3)^2(1-p^3p_3\ )^2(1-p^4)]
\end{aligned}$$

$$\begin{aligned}
UB_{FK}^{(G)} &= \left[\prod_{C \in \mathbb{C}_1} \coprod_{i \in C} p_i \right] \cdot \prod_{s=2}^{3} \left[1 - \left(\prod_{C \in \mathbb{C}_s^*} \coprod_{i \in C} p_i - \prod_{C \in \mathbb{C}_s^* \cup \mathbb{C}_s} \coprod_{i \in C} p_i \right) \right] \\
&= (1-q^2)[q^3 + (1-q^2)(1-q^3)][1-(1-q^3q_3\ ) + (1-q^2)^2(1-q^3)(1-q^3q_3\ )^2]
\end{aligned}$$

Figure 4.22 below gives a graphical presentation of these bounds with the reliability function. Red is for the lower bound, blue for the reliability function and green for the upper bound.

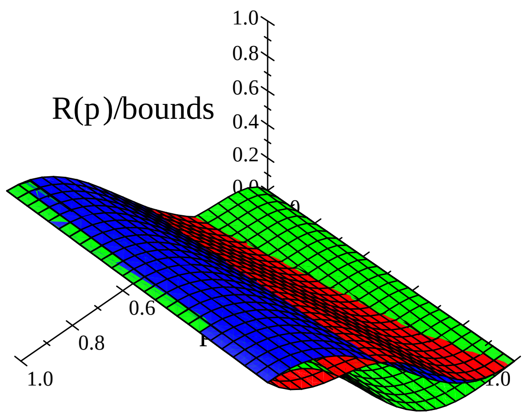

Figure 4.22: Comparison of generalized FK bounds with R(p) for p_3

Generalized Fu-Koutras bounds are not good approximations, for both cases, and should be avoided at all cost.

5 SUMMARY AND CONCLUSION

This dissertation deals with the key problem of reliability theory: Calculation of the reliability of a binary coherent system given both the reliabilities of its components (subsystems) and their structural mutual relationship. In a binary or Boolean coherent system, the system and its components can only assume two states: available or unavailable (operating or not, up or down), and the failure of a component can only lead to a system failure, or, if the component is redundant, the system may continue to operate. In this dissertation, under *system* (*component*) *reliability* we understand the *system* (*component*) *availability*, i.e. the probability that the system (component) is available at a given time point t or in the stationary regime. Since t is fixed, we need not explicitly consider time-dependent reliabilities. The theory of determining the reliability of a system from its component reliabilities has already been developed about four decades ago. But, never mind our simple two-state assumption, calculating the system reliability from its component reliabilities belongs, from the computational point of view, to the class of the most difficult problems, namely to the NP-hard problems. In particular, the NP-hard property implies that with increasing number of components, assuming a non-trivial structural relationship between them, the calculation time of the system reliability increases exponentially fast. This is also true under the assumption, usually made in this dissertation, that the components operate independently of each other. Algorithms for exactly calculating the system reliability, which work in polynomial time, only exist for simply structured systems as series- and parallel systems and their combinations. Hence, in the past much effort has been invested in methods and algorithms for efficiently obtaining information on the system reliability by making use of asymptotic and approximative techniques. This paper gives an overview on the most important approaches for constructing lower and upper bounds on the reliability of a system based on its minimal paths and minimal cuts. These bounds are qualitatively evaluated and quantitatively compared by numerically analyzing example systems. Unfortunately, determining all minimal paths and cuts of a binary coherent system is an NP-hard problem as well. But, determining the exact reliability of a system based on its structure function is, loosely speaking, "doubly NP-hard", since, on the one hand, finding its minimal paths- and cuts is NP-hard and, on the other hand, determining its structure function for given minimal paths and cuts is NP-hard as well. Since the calculation of reliability bounds with the minimal paths and cuts given can be done in polynomial time, the lower and upper reliability bounds discussed in this dissertation allow to derive information on the system reliability for more complex systems than the ones, which are tractable with exact methods. Moreover, some bounds may yield excellent results even if only a comparatively small percentage of the existing minimal paths and cuts are used.

From the point of view of the reliability engineer, the exact calculation of the system reliability is an academic exercise anyway, since the reliabilities of the components are in practice not known with absolute accuracy, so that the accumulation of round-off errors due to the required excessive computation can only lead to approximative values for the system reliability. In addition, there is always some dependency in the failure behavior between the components during the time periods they are operating, which is not exactly or not at all quantifiable. This has a significant influence on the accuracy of the predicted system reliability.

The research, done in this dissertation under the assumptions stated, allows the following conclusions:
1. General theoretical comparisons between the bounds, which are valid for all possible component reliabilities, are not possible. In particular, no bounds are uniformly sharper than other ones for all possible component reliabilities. Significant differences between the bounds mostly exist in the areas in which the component reliabilities are high and/or small.
2. As a corollary from 1, combinations of different bounds make sense: For instance:
a) Use the maximum of the lower Esary-Proschan bound and the lower min-max bound and the minimum of the upper Esary-Proschan bound and the upper min-max bound.

b) For low component reliabilities, use the Fu-Koutras lower bounds and the upper Esary-Proschan bounds. For high component reliabilities, use the lower Esary-Proschan bounds and the upper Fu-Koutras bounds.
3. Of special interest are the Spross bounds. Only a small percentage of all minimal paths and minimal cuts needs to be known to obtain sharp bounds. This property becomes the more important, the more complex the system is. Hence, the fact that listing all minimal paths and cuts is NP-hard, plays no significant role when applying the Spross bounds. However, determining the Spross bounds requires the development of a more sophisticated software than necessary for the other bounds. In addition, This software must determine the minimal paths and cuts in the order of their importance. (Generally, most important are the minimal path and cut sets with the smallest cardinality.)
4. All bounds can be most efficiently improved if the reliability of the component with the largest structural Birnbaum-importance is increased.
5. Modular decomposition of complex systems (if possible) has two advantages

a) Reduction of computation time.

b) The bounds obtained after modular decomposition are sharper than the ones obtained without modular decomposition.

Suggestions for further research are:
1. Analyzing more thoroughly reliability bounds for different component reliabilities.
2. Constructing reliability bounds for a system after its pivotal decomposition.
3. Applying the splitting approach along a separating node set used in network reliability analysis to reliability block diagrams for the construction of reliability bounds. (In network reliability analysis, this approach yields for planar graphs polynomial algorithms.)

6 References

[1] Abraham, J. A. (1979) An Improved Algorithm for Network Reliability, IEEE TRANSACTIONS ON RELIABILITY, vol. R- 28, no. 1, pp. 58- 61.

[2] Argawal, K.K. and Misra, K.B. (1975) A Fast Algorithm for Reliability Evaluation, IEEE TRANSACTIONS ON RELIABILITY, vol. R- 24, no. 1, pp. 83- 85.

[3] Balan, A. O. and Traldi, L. (2003) Preprocessing minpaths for sum of disjoint products, IEEE TRANSACTIONS ON RELIABILITY, vol. 52, no. 3, pp. 289- 295.

[4] Barlow, R.E. and Proschan, F. (1975) Statistical theory of Reliability and life testing, Holt, Rinehart and Winston, Inc.

[5] Beichelt, F. and Spross, L. (1987) An improved Abraham method for generating disjoint sums, IEEE TRANSACTIONS ON RELIABILITY, vol. R- 36, no. 1, pp. 70- 74.

[6] Beichelt, F. and Spross, L. (1989) Bounds on the Reliability of Binary Coherent Systems, IEEE TRANSACTIONS ON RELIABILITY, vol. 38, no. 4, pp. 425- 427.

[7] Birnbaum, Z. W. (1968) On the Importance of Different Components in a Multicomponent System, University of Washington Technical Report, no. 54, pp. 1- 24.

[8] Birnbaum, Z.W. and Esary, J.D. (1965) Modules of Coherent Binary Systems, Journal of the Society for Industrial and Applied Mathematics, vol. 13, no. 2, pp. 444-462.

[9] Birnbaum, Z.W., Esary, J.D. and Saunders, S.C. (1961) Multicomponent Systems and Structures and Their Reliability, Technometrics, vol. 3, no. 1, pp. 55-77.

[10] Bodin, D.L. (1970) Approximations to System Reliability Using a Modular Decomposition, Technometrics, vol. 12, no. 2, pp. 335- 344.

[11] Boutsikas, M.V. and Koutras, M.V. (2000) Generalized reliability bounds for coherent structures, Journal of Applied Probability, vol. 37, pp. 778-794.

[12] Esary, J.D. and Proschan, F. (1963) Coherent Structures of Non-Identical Components, Technometrics, vol. 5, no. 2, pp. 191-209.

[13] Fu, J.C. and Koutras, M.V. (1995) Reliability bounds for coherent structures with independent Components, Statistics and Probability Letters, vol. 22, pp. 137-148.

[14] Fratta, L. and Montanari, U.G. (1973) A Boolean Algebra Method for Computing the Terminal Reliability in a Communication Network, IEEE TRANSACTIONS ON CIRCUIT THEORY, vol. CT- 20, no. 3, pp. 203-211.

[15] Koutras, M.V., Papastavridis S.G. and Petakos, K.I. (1996) Bounds for coherent reliability structures, Statistics and Probability Letters, vol. 26, pp. 285-292.

[16] Kovalenko, N.I., Kuznetsov, N.Y. and Pegg, P.A. (1997) Mathematical Theory of Reliability of Time Dependent Systems with Practical Applications, John Wiley and Sons Ltd.

[17] Locks, M. O. (1980) Recursive Disjoint Products, Inclusion- Exclusion and Min- Cut Approximations, IEEE TRANSACTIONS ON RELIABILITY, vol. R- 29, no. 5, pp. 368- 371.

[18] Locks, M. O. (1982) Recursive Disjoint Products, IEEE TRANSACTIONS ON RELIABILITY, vol. R- 31, no. 1, pp. 33- 35.

[19] Locks, M. O. (1987) A minimizing algorithm for sum of disjoint products, IEEE TRANSACTIONS ON RELIABILITY, vol. R- 36, no. 4, pp. 445- 453.

[20] Modarres, M., Kaminskiy, M. and Krivtsov, V. (2010) Reliability Engineering and Risk Analysis: A Practical guide, Taylor & Francis Group.

[21] Nachlas, A. (2005) Reliability Engineering: Probabilistic Models and Maintenance Methods, Taylor & Francis Group.

[22] Rai, S., Veeraraghavan, M. and Trivedi, K.S. (1995) A survey of efficient reliability computation using disjoint products approach, Networks, vol. 25, pp. 147- 163.

[23] Satyanarayana, A. and Prabhakar, A. (1978) New Topological Formula and Rapid Algorithm for Reliability Analysis of Complex Networks, IEEE TRANSACTIONS ON RELIABILITY, vol. R-27, no. 2, pp. 82-100.

[24] Soh, S. and Rai, S. (1990) Experimental results on preprocessing of path/cut terms in sum of disjoint product technique, IEEE TRANSACTIONS ON RELIABILITY, vol. 39, pp. 42- 45.