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STUDENT-TEACHERS LEARNING MATHEMATICS FOR TEACHING: LEARNER THINKING AND SENSE MAKING IN ALGEBRA

by

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A thesis submitted in fulfilment of the requirements for the degree of

Doctor of Philosophy



University of the Witwatersrand

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JOHANNESBURG

SOUTH AFRICA

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SUPERVISOR:

Professor Jill Adler

DECLARATION

I declare that this thesis is my own work, except as indicated in the acknowledgements, the text and the references. It is being submitted in fulfilment of the requirements for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg, South Africa. It has not been submitted before for any degree or examination at any other institution.



_____ 08th May 2014

Signed Date

Patricia Phiri Nalube

ABSTRACT

This is a qualitative case study that draws on Bernstein's theory of *the pedagogic device* to analyse to what extent and how the discourse of engaging with learner mathematical thinking (LMT) is recognized and focused on in mathematics teacher education in Zambia, crucially in algebra that underlies progress towards further mathematics study. Four mathematics education teacher-educators and twenty of their final year student-teachers of a particular university in Zambia participated in this study. Moreover, a total of one hundred and five learners (45 grade 9s and 60 grade 12s learners) from a selection of four Government schools in a particular province of Zambia participated in the study.

My study specifically explored three critical questions, which are:

1. What do teacher-educators select and privilege with respect to the discourse of engaging with learner mathematical thinking?
2. What are student-teachers' recognitions and realizations of the discourse of engaging with learner mathematical thinking?
3. What is the relationship between teacher-educators' discourses of engaging with learner mathematical thinking and that of their student-teachers, and how might this be explained?

Semi-structured interviews were used to collect the data that enabled me to answer these research questions. These included three interview schedules that were used to:

1. carry out individual interviews with the teacher-educators;
2. interview the student-teachers in their respective two focus groups of 10 student-teachers in each; and
3. interview 8 pairs of student-teachers on 3 scenarios of common learner errors in algebra described in literature and the other 3 scenarios that were elicited from learners' own working.

The findings show that teacher-educators' privileged selections of what entails LMT is weakly classified and framed, hence implicit messages being relayed to student-teachers. Whether teacher-educators' focus is on developing in learners both relational and

instrumental understanding, or learner errors, or creating an environment where teacher can listen to learners, there is a range of mixed messages being relayed, hence messages within them are spread out. That is, the criteria for what counts as LMT are weak because they are spread out. Moreover, it is also evident that across all the three major categories, LMT is a practical accomplishment as principles that would guide discussions around it are not so clear. LMT is talked about when focus is on principles that guide discussions on topics/courses in the mathematics education curriculum. The indication here is that the privileging by the teacher-educators while in these three domains, some of the big discourses in the specialized fields of mathematics education research and mathematics education are filtering through but in a very weak way. As expected, student-teachers' realization of what entails LMT is also weakly classified and framed but they were able to recruit or take-up some of the messages from their teacher-educators with further elaboration in some cases. While there is attention to LMT in the courses and it is discussed as within the courses by both the teacher-educators and the student-teachers, neither seem to be informed directly by the literature. This suggests that the talk about error while it relates to the literature is not discursively organised. In terms of positioning of student-teachers/teachers/teacher-educators or learners or the curriculum, teacher-educators as well as student-teachers hold contradictory views in referring to what is there pertaining to LMT as absences or presences.

I have also shown that given scenarios on common learner errors in school algebra, student-teachers hold contradictory views on what teaching and learning entails despite constructivism being the theory espoused in their mathematics education courses. They equate teaching to learning; hence a transmission view as well as have the understanding that learners are not empty vessels but come with prior learning to the teaching and learning situation. This suggests that focus on LMT in terms of the nature of errors and strategies for carrying out error analysis is not a principled focus in these student-teachers' mathematics education courses. This is further confirmed in that student-teachers spoke in the everyday professional knowledge of teaching and learning. I have therefore argued that ways in which LMT ought to be explicitly structured and focused on in teacher education be sought, for example, to include use of discursive resources from research.

I have also made methodological and theoretical contributions to the specialized field of mathematics education research. In particular, methodologically, I have extended the use of the notion of *an evaluative event* as a unit of analysis on interviews unlike before when it was used on assessment tasks and instruction in teacher education. Theoretically, I have shown how my study has been an attempt towards developing the internal and external languages of description concerning the discourse of engaging with LMT, an issue of potential future development.

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"I have the strength to face all conditions by the power that Christ gives me" Philippians 4, verse 13.

I glorify God for having given me the strength to endure and walk this long and challenging journey towards the completion of this thesis to obtain a PhD.

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I would like to also thank colleagues in the community of mathematics education research (students and mathematics education lecturers) for the thought provoking feedback they provided to the presentations I made at different forums including local and international conferences, and PhD weekends provided for in the School of Education. Thank you to the participants of my study (mathematics education lectures and their student teachers, and learners in secondary school) and their authorizing agencies, who without their consenting and holding on up to completion of data collection, this study would not have been possible.

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I end by sincerely thanking my husband Sidney, the children, Mainga, Maimbo and Alinafe for enduring and walking with me on this long journey to obtaining a PhD. I hope this will inspire them to one day walk a similar journey. I will always cherish the love and support they gave me by coping with my long absence at home.

DEDICATIONS

To my loving husband, Sidney Muntanga Nalube, and my three children, Mainga, Maimbo, and Alinafe who without their support and love this thesis would not have been possible.

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LIST OF ABBREVIATIONS

CCK: Common content knowledge

ID: Instructional discourse

KCS: Knowledge of content and student

KCT: Knowledge of content and teaching

LMT: Learner mathematical thinking

MfT: Mathematics for teaching

MKT: Mathematical knowledge for teaching

MOE: Ministry of Education

OLP: Orientation towards learner and presences

OMA: Orientation towards mathematics and absences

OPRF: Official pedagogic recontextualising field

ORF: Official recontextualising field

OTA: Orientation towards teaching and absences

OTP: Orientation towards teacher and absences

PCK: Pedagogic content knowledge

PD: Pedagogic device

PRF: Pedagogic recontextualising field

RD: Regulative discourse

SCK: Specialised content knowledge

SMK: Subject Matter Knowledge

UPRF: Unofficial pedagogic recontextualisation field

IRE: Initiate, Response, and Evaluation

CHAPTER 1

1 INTRODUCTION TO THE STUDY

1.1 Introduction

In this study, I investigate and explain what and how mathematics teacher education in Zambia prepares student-teachers[1] to engage with learner mathematical thinking (LMT) in general, and algebra in particular. As I elaborate in forthcoming chapters, the overarching theory that informs my study is Bernstein's (1996, 2000) theory of the pedagogic device, particularly *evaluation criteria* of the what and how of the entailments pertaining to the discourse[2] of engaging with LMT. Following Adler & Davis (2006) and Davis, Adler & Parker (2007), the unit of analysis for my study is thus *an evaluative event*. I view both learning and knowledge as social. With respect to learning, and following Peressini, Borko, Romagnano, Knuth, & Willis (2004), I take a participationist or situative perspective that sees what student -teachers learn as (also) a function of how and where they learn it. Learning is becoming a participant in a community and so in its discourses.

The student–teachers in my study are becoming participants in the discourses of teaching, one of which is engaging with LMT. In Bernstein's (2000) terms, the theory of learning is talked about in terms of acquisition. What then do the student-teachers recognize and realize[3] as the discourse of engaging with LMT, hence their participation in the pedagogic discourse. However, Bernstein (op cit) does not only talk about recognition and realisation but also positionings by the discourse, hence my focus in the study. With respect to knowledge, and

[1] In literature, the reference to teachers, students, etc. gets confusing in teacher education in a way that one would not be sure about who is a learner, a teacher, or a student. In this study, students in school will be referred to as learners, and University students pursuing their degree programmes in mathematics education irrespective of whether they are in-service or pre-service will be referred to as student teachers. When it becomes apparent in the course of the study, a distinction might be made on whether the student teachers are in-service or pre-service.

[2] I am using Bernstein's (2000) notion of discourse from a social standpoint to mean ways of talking about something that bring some people together while excluding others. When I refer to the discourse of engaging with LMT, it is in ways Even and Tirosh (2002) have described it in terms of three categories as elaborated in Chapter 3, and focus in my study is on how teacher-educators and their student-teachers have come to reflect and substantiate them.

[3] These are some of key terms in Bernstein's (2000) work that are elaborated in Chapter 4.

following Bernstein (1996; 2000), what is made available to learn, to participate in, and here the focus is LMT, is a function of the pedagogic structuring – or what he calls the recontextualisation - of this knowledge.

While the social is my broad frame, my study foregrounds the errors, difficulties and misconceptions that learners display as they participate in a discourse of and about algebra, considering them critical to the discourse of and about engaging with LMT. It is important to acknowledge that most of the research on error is located in work on misconceptions and thus a constructivist framework where errors are viewed as part of learners' conceptions not aligned with conventional mathematics [for example, Smith, diSessa, & Roschelle (1993), Nesher (1987), Borasi (1987), Olivier (1989)]. More recently there are reinterpretations of this work from a social constructivist and sociocultural perspective [Peng & Luo (2009), Ryan & Williams (2007)], and a discursive perspective (Sfard, 2007, 2008). In Sfard's (op cit) terms, errors are located in social interaction and are an indication that learners are participating in a different discourse from expert discourse. My concern in the study is with student-teachers' learning and knowledge of errors, difficulties and misconceptions, and thus my study engages this relative wide region of research in mathematics education, albeit that this work is not theoretically unified as will be elaborated in Chapter 3 (Section 3.3).

Algebraic thinking or reasoning (Kendal & Stacey, 2004; Kieran, 2004; Kilpatrick, Swafford, & Findell, 2001) includes simplifying expressions and solving equations; constructing equations from given problems; and interpreting from context into algebraic language. If learners are unable to do these, this could be due to errors, difficulties or misconceptions they might have. Doerr (2004) would argue that these errors learners display could be due to the teaching of school algebra as a set of procedures, disconnected from meaning and purpose . The question to ask then is: How prepared are student-teachers to participate in a discourse of and about engaging with LMT in general, and algebra in particular?

I foreground errors, difficulties and misconceptions because they provide the most persuasive window into student-teachers' thinking: it is easy to see when things are going wrong and difficult to see when things are going right. Referring to, Freudenthal (1978:78), Sfard & Linchevski (1994, p. 195) state that "If learning is to be observed, the moments that count are its discontinuities, the jumps in the learning process". The significance of error in learning

and teaching is well documented. In the original works of Shulman (1986, 1987), engaging with learners' misconceptions is viewed as one of the components of pedagogical content knowledge (PCK). Moreover, Ball, Thames & Phelps (2008) interpret error as fundamental to being mathematical.

Thames, Ball & Bass (2008) have identified some of the mathematical tasks a teacher enacts during their teaching. These include explaining mathematical ideas, representing and recording mathematical ideas, interpreting and eliciting mathematical thinking, analysing errors, using mathematical language, generating and analysing alternative solution methods, and creating, analysing and modifying mathematics tasks. It is thus reasonable to assert that the discourse of and about engaging with LMT is paramount in that it is implicated in the other mathematical tasks of teaching identified.

Many researchers also argue that mathematical knowledge that is directly linked to the work teachers do and mathematics in the school curricula has implications for teaching and learning (Ball & Bass, 2000; Ball, Bass, & Hill, 2004; Ball, et al., 2008; Hill, Rowan, & Ball, 2005; Thames, et al., 2008). These authors further argue that teacher education should strive for the development of such knowledge. The assumption I then make for my study is that focusing on the discourse of and about engaging with LMT in teacher education is of much benefit to teaching and learning in general, and algebra in particular. If teachers are to have opportunities to identify and plan for LMT in advance of actual teaching, they will also be better prepared to engage with the discourse of and about LMT during their teaching, and so providing support to learners.

1.2 The Problem

On one hand, as a secondary school teacher, I expected my learners to provide correct responses whenever they were asked a mathematical question. If they gave incorrect answers, I believed such responses reflected a lack of thinking. Comments such as 'no', 'wrong', 'try again', 'what is wrong with your thinking?' were common in my lessons. I never imagined that such comments would be discouraging to the learners. My expectation was that after I had taught; my learners should demonstrate understanding by providing correct solutions to mathematical questions. These solutions were required to be mathematical rather than non mathematical in nature such that they corresponded to what was stipulated in the textbooks I

was using. Any working that was not in line with textbook demands was considered wrong. Errors, difficulties or misconceptions did not form part of the classroom discussion – I did not encourage such learners by asking them to explain how and why they thought about their solutions in the way they did; or by following through and understanding their patterns of thinking so that ways of assisting such learners could be sought.

On the other hand, as a mathematics teacher-educator, I experienced instances during the practicum where student-teachers' understanding of a particular concept was not well developed, and hence their communication flawed. This meant that their learners were likely to hold these misconceptions themselves. In this case, LMT was influenced by student-teachers' understanding of a particular topic in school mathematics. Indeed, in my discussions with student-teachers they did not realise that they were dealing with misconceptions. When it came to assessing learners' work, the misconceptions went unnoticed. From my experiences described above, I raise the following questions for teacher education: What do student-teachers think is the importance of participating in a discourse of and about engaging with LMT? How prepared are student-teachers to participate in a discourse of and about engaging with LMT?

Current education policy in Zambia has indicated that in order to attain high achievement levels in mathematics, teacher education should ensure that future teachers know the subject they are going to teach as well as have the skills for communicating the subject (MoE, 1996). This means that teachers should be knowledgeable about the mathematics they are going to teach and ways of teaching it. The notion of mathematical knowledge for teaching which emphasises the aspect of teachers 'knowing the mathematics as well as knowing how to teach it' (Ball & Bass, 2000) is not in any way different from what the policy stipulates. Teachers knowing the mathematics includes being aware of the rationale for the mathematical problem as well as bringing to the fore other ways of solving the problem (Ma, 1999). Therefore, for a teacher to participate in a discourse of and about engaging with LMT, they need a thorough comprehension of that particular topic; to understand the learners' various solutions to the problem; know how and why learners come up with them; know the relationship between the standard ways and the non standard ways; and know the conceptions underlying all the different ways (Ma, 1999). These are some of the mathematical demands a teacher is required to grapple with in their work of teaching. The question for teacher education in Zambia then

is; in what ways does it provide such learning or participation opportunities for student-teachers?

In his study on teachers' knowledge for teaching, Shulman (1986, 1987) found that there are three sources of the knowledge base for teaching namely, Subject Matter Knowledge (SMK), Pedagogical Content Knowledge (PCK) and Curriculum Knowledge. Using these constructs, Krauss, Neubrand, Blum & Baumert (2008) in the COACTIV[4] project explored whether teachers' PCK and SMK (they referred to this as 'content knowledge') were in any way related to the number of years of experience in their teaching. Their findings show that there is no positive correlation between teachers' knowledge base for teaching (i.e. PCK and Content Knowledge) and their years of teaching experience. They argue, with others such as Kotsopoulos & Lavigne (2008), that teachers tend to put into practice what they have learned during their training. As a result, both sets of authors argue that teacher education could be at the centre of the development of these knowledge sources, hence one of the reasons for situating my study in teacher education. Of course, this is not to suggest that everything that teachers come to do in schools is a function of their learning in teacher education.

Shulman's (1986) notion of the knowledge base for teaching has led to research in many disciplinary domains, and in particular, in science and mathematics education. In mathematics education, these notions have been reconceptualised as 'Mathematics for Teaching' (MfT), and identified as a form of mathematical knowledge, produced in, and used for, the practice of teaching (Adler & Davis, 2006). Developmental studies to advance this aspect have been on-going. For example, Thames et al. (2008); Ball et al. (2008); Hill et al. (2005); Ball et al. (2004); and Ball & Bass (2000); have engaged in empirical studies in elementary mathematics classrooms. From these studies, these authors have argued that teachers' mathematical knowledge that is linked to the work teachers do and the mathematics in school curricula, impacts positively on teaching and learning. Furthermore, the authors argue that the mathematics that is directly linked to the work teachers do and the mathematics in school curricula should be experienced in teacher education from a teaching perspective. The assumption here is that revisiting the mathematics in school curricula at teacher

[4] COACTIV refers to the project on secondary mathematics teachers' professional knowledge, Cognitively Activating Instruction, and the development of students' mathematical literacy.

education level offers a different mathematical experience for teachers. Included in such would be the discourse of and about engaging with LMT. The appropriate question to ask then is: What aspects of the discourse of and about engaging with LMT are dealt with in mathematics teacher education programmes in Zambia?

Moreover, using the notion of MfT, Kazima, Pillay & Adler (2008); Adler & Pillay (2007); Huillet (2007); Tatolo (2007); Adler & Davis (2006); focused on how MfT comes to be constituted in mathematics teacher education as well as in school mathematics teaching through the QUANTUM[5] project and research is on-going. It is hoped that this focus will enable a consideration of what, how and why the mathematical education of teachers aligns with the mathematical work demanded of them in their teaching. This body of research has focused on the one hand on school mathematics teaching (Adler & Pillay, 2007); and on the other hand on teacher education, for example, Adler & Davis (2006) or Davis, Adler & Parker (2007). Specifically, the project has focused on what and how mathematics is selected into formalised in-service programmes. Increasing and recent studies of MfT in pre-service or in-service (Beswick, Callingham, & Watson, 2012; Charalambous, Hill, & Ball, 2011; Rowland, Huckstep, & Thwaites, 2005) focused more broadly on the wider conception. However, very little is known about what and how teacher education prepares student-teachers to participate in a discourse of and about engaging with LMT[6]. My study will therefore make a contribution by addressing this gap.

Mathematics Education is a very broad field of study with many domains, but to advance my study, I focus on the discourse of and about learner *algebraic thinking*. The choice of algebra

[5] QUANTUM is a name given to a research and development project on quality mathematical education for teachers in South Africa. The issue at hand is that as the field of teacher education provides opportunities for teacher learning in South Africa, what mathematical practices are enabled and constrained – hence its focus on QUAlifications for Teachers Underqualified in Mathematics (QUANTUM). This project is being conducted under the research thrust *Mathematics for Teaching.* However, its tasks were completed in 2003 but it has continued to be a collaborative research project. The project is directed by Professor Adler and funded by the National Research Foundation (NRF). Therefore, this study gives a cross country perspective i.e. the Zambian perspective on MfT.

[6] In conversation, there have been attempts by Adler and Davis to elaborate the aspect of learner mathematical thinking but with very diverse effects. Therefore, it is important to follow up the aspect of learner mathematical thinking because Adler and Davis have dealt with it in in-service and not pre-service.

is based on the reason that research in Zambia has shown that teachers do not consider algebra to be a difficult topic to teach (Haambokoma, et al., 2002). However, the teachers do acknowledge that learners in secondary school experience problems with algebra. Also, teacher education in Zambia does not consider school algebra as one of the topics teachers experience problems to teach, therefore it is not explicitly engaged with when dealing with school mathematics within mathematics education courses. From the baseline survey conducted by Haambokoma et al. (op cit), neither the difficulties that learners experience in algebra nor how teachers might assist learners in overcoming the difficulties they face are known. Therefore, in this study, providing opportunity for student-teachers to engage in the discourse of and about LMT could illuminate some of the difficulties that learners experience in learning algebra and how they might be addressed by the teacher. More specifically, I raise the following questions: How do student-teachers talk about engaging with LMT in algebra? Is the discourse of and about engaging with LMT an intended aspect of the mathematics teacher education curriculum?

A great deal of research in mathematics education has focused on algebra and there is a substantial body of knowledge as to student difficulties, varying curricula across the world, and changes due to technological tools (Bell, 1995; Brown & Drouhard, 2004; Kendal & Stacey, 2004; Kieran, 2004; Kilpatrick, et al., 2001; MacGregor, 2004; Usiskin, 1988, 2004; Vermeulen, 2007). Despite this vast body of knowledge, learners continue experiencing difficulties in algebra; and the preparation and support of teachers in and for this particularly central part of their work remains a black box (Doerr, 2004). In line with this argument, Stacey proposes a "didactic transposition" from raw research results to learnable and organised material which could form the basis of a new teacher education (Doerr & Wood, 2004, p. 169).

Moreover, Doerr (2004) argues that researchers that have focused on teachers' knowledge of algebra (teachers' SMK and PCK) tend to frame their investigations in ways that position teachers and teaching in a deficit or deficient mode. Focus has been on what is inadequate in teachers' knowledge in relation to the activities of teaching algebra. Therefore, Doerr (2004, p. 273) in referring to Doerr and Lesh (2003) proposes a shift from a deficit type of research to focus on "what is it teachers do know and how they interpret the complex situations of

teaching". Doerr's (2004) proposal resonates with the focus of my study in that I am interested in exploring what it is student-teachers know and are able to do pertaining to the discourse of engaging with LMT. This will enable me describe and explain what and how the discourse is focused on in mathematics teacher education in Zambia; and in turn illuminate more generally, MfT in relation to the discourse of engaging with LMT.

Kieran (2004) argues that the teacher has a responsibility for bringing algebra representations to the fore and making their manipulation by learners a venue for epistemic growth. This implies that teachers' understanding of algebra is crucial as this influences how it is presented to the learners. However, research has shown that various approaches to algebra (through its 'language' and structure, through functions, or through problem solving) remain contested in curricula, as well as in school classroom teaching and teacher education (Adler & Pillay, 2007; Doerr, 2004; Huillet, 2007; Tatolo, 2007). What strategies then are suggested by student-teachers as they use the discourse of and about engaging with LMT in algebra?

Moreover, in relation to teacher education, the argument is that "the work of teacher educators has not itself been subject to the scrutiny of research and subsequent revision by the larger community of researchers and practitioners" (Doerr, 2004, p. 268). This suggests, in Bernstein (2000) terms, a lack of scrutiny from the specialised fields of mathematics education research and mathematics education. However, the work of Adler & Davis (2006) or Davis et al. (2007) which focuses on the constitution of mathematics in three formalised institutions of higher learning is an attempt to do so. Therefore, due to limited research on both teacher-educators' work, and teachers' understanding of learners' errors pertaining to algebraic thinking, my study will make a contribution to theory building to enable the description and explanation of what is recognised and focused on as the discourse of engaging with LMT in teacher education in Zambia.

1.3 Purpose of the study

This study sets out to explore and explain what and how teacher education in Zambia prepares University mathematics student-teachers to participate in a discourse of and about engaging with LMT in general, and algebra in particular. That is, what are the intentions of the mathematics teacher education curriculum in relation to the discourse? What

mathematical knowledge for teaching secondary school algebra do student-teachers know and are able to enact as they participate in the discourse of and about engaging with LMT? As student-teachers participate in a discourse of and about engaging with LMT, through the study, their thinking will be illuminated and so provide insights into their knowledge and understanding of the discourse. An underlying assumption of teacher education programmes, however they are organised, including when they are organised as higher level mathematics and pedagogy, is that teachers will be able to use mathematics to do their work of teaching. This suggests that however structured, working with the discourse of and about LMT is a presumed outcome of teacher education.

1.4 Research Questions

My study is a qualitative case study answering one main question. The main question is followed by three critical questions – the first and second pertaining to teacher-educators' and to student-teachers' involvement, respectively. The third one is about the relationship between teacher-educators' and student-teachers' involvement. Each critical question is also followed by sub questions which helped in answering the research question:

To what extent and how is the discourse of engaging with LMT recognized and focused on in mathematics teacher education in Zambia, and crucially in algebra that underlies progress for further mathematical studies?

1. What do teacher-educators select and privilege in relation to the discourse of engaging with LMT?
    - What do they say engaging with LMT is?
    - How do they say they "teach" for it in terms of what student-teachers should know and be able to do?
    - In talking about what it is and how they teach for it, how do they position the student-teachers/teachers, learners and the curriculum?
    - Where in their mathematics education courses do they say it is focused on?
2. What are student-teachers' recognitions and realizations of the discourse of engaging with LMT?

- What do they say engaging with LMT is?
- How do they say they "learn" for it in terms of what they should know and be able to do?
- In talking about what it is and how they learn for it, how do they position student-teachers/teachers/teacher-educators, learners and the curriculum?
- Where in their mathematics education courses do they say it is focused on?
- With specific focus on learner errors in school algebra in form of scenarios, what do they recognize as errors and how do they explain their sources and suggested remediation strategies?
- In recognizing the errors, explaining their sources and suggesting remediating strategies, what are their positionings and why are their discourses as they are?

3. What is the relationship between teacher-educators' discourses of engaging with LMT and that of their student-teachers, and how might this be explained?

- What messages from teacher-educators' discourses do student-teachers recruit or take-up?
- What in their discourses is different?

1.5 Outline of thesis

In Chapter 2, I provide the background to and context of the study. The motivation for this study is fourfold: policy on the teaching profession and achievement levels in school mathematics in Zambia, secondary mathematics teacher education programmes in Zambia, algebra in the Zambian school curriculum, and the genesis of the knowledge base for teaching, in particular PCK.

In Chapter 3, I review literature relevant to my study. I focus on three main aspects. Firstly, I focus on literature related to perspectives on teacher knowledge and learning. Included in this are Shulman's seminal contribution; mathematical knowledge in the context of teaching; elaborating MKT; MfT and its constitution in mathematics teacher education; a situative

perspective on teacher knowledge and learning of MfT; looking at the integration of content and pedagogy with a closer eye; and perspectives on the literature on MfT and the gap. Secondly, I focus on literature related to learner errors and teachers' engagement. Included in this are learner errors and PCK; distinguishing errors, errors and theory; errors from a constructivist/cognitive perspective; strategies for carrying out error analysis; and importance of carrying out error analysis by the teacher. Thirdly, I focus on literature related to algebraic thinking. Included in this are the definition of algebra; constitution of the school algebra curriculum; and learner errors and difficulties in algebra.

In Chapter 4, I discuss the theoretical gaze that informs my study. Focus is on Bernstein's (2000) theory of the pedagogic device, and more specifically evaluation criteria of what and how of the discourse of engaging with LMT. Therefore, I discuss distributive, recontextualising, and evaluative rules, and the respective fields they operate in. In relation to recontextualising rules, I also discuss the context of Zambian mathematics education teacher-educators operating in the PRF and the influence of the ORF. Pertaining to evaluative rules, focus is on three aspects. Firstly, I discuss classification and framing, and the regulation of recognition and realization rules at the level of the individual. Secondly, I elaborate the notion of positioning and how it informs my study. Thirdly, I describe form of knowledge of the discourse of engaging with LMT.

In Chapter 5, I discuss the research design and methodology for my study. I describe ways in which my study is a qualitative case study in one of the teacher education institutions in Zambia. Research participants are described to include teacher-educators and student-teachers involved in the teaching and learning of mathematics education courses, respectively. Methods that I used to collect data to enable me answer the research questions are also discussed. These are semi-structured interviews in threefold, that is, individual interviews with four teacher-educators, focus group interviews with two groups of student-teachers, and scenario-based interviews with eight pairs of student-teachers. I also describe how this data from interviews was analysed using the methodology of an evaluative event as a unit of analysis. How trustworthiness was ensured in terms of reliability and validity in the context of my study which is qualitative is also discussed. Ethical considerations in terms of my study and informed consent are also discussed. Delineating a study comes with its own limitations, and for my study, these are also discussed in Chapter 5.

In Chapter 6, I illuminate what teacher-educators say they select and privilege as the discourse of engaging with LMT. I bring Even and Tirosh's (2002) three broad categories of what entails the discourse to bear on teacher-educators' talk. These include: developing in learners both instrumental and relational understanding; focusing on learner errors and misconceptions; and creating an environment where teacher can listen to learners. The discussion is based on what it is they say LMT is, where it is focused on in the mathematics education courses, positioning of teachers/student-teachers, learners, and the curriculum, and the 'teaching' of it.

In Chapter 7, I discuss student-teachers' entailments pertaining to the discourse of engaging with LMT, and how these resonate with Even and Tirosh's (2002) broader categories. The broader categories include: developing in learners both instrumental and relational understanding, focus on learner errors and misconceptions, and creating an environment where teacher can listen to learners. In each category, discussions are on what student-teachers say LMT is; where in the courses LMT is focused on; how they say they 'learn' for it in terms of what they should know and be able to do; and positioning. Included in the discussions are the messages student-teachers recruit or take-up from teacher-educators' discourses.

In Chapter 8, I discuss student-teachers' discourses of LMT when engaged with a selection of three scenarios (Scenarios 1, 2 and 3) on common learner errors in school algebra, and as described in the specialized field of mathematics education research. A preamble of the nature of the algebraic problem is also provided for each scenario. Scenario 1 is discussed in detail while Scenarios 3 and 4 in summary. The detailed analysis of Scenarios 3 and 4 with evidence and extracts is provided for in the appendices. The main focus is on strategies of carrying out error analysis, and informed by Peng & Luo (2009) and Jacobs et al. (2010). These involve whether student-teachers do recognize or misrecognize the error, explanations for the sources of errors they provide, and suggested remediating strategies they propose, and how these resonate with mathematics education research. Included in the discussion is how these steps of carrying out error analysis might be explained in terms of positionings involving "voice" and "form of practice". Messages that student-teachers recruit or take-up from teacher-educators' discourses and their discourses in Chapter 7 also form part of the discussion.

In Chapter 9, I discuss student-teachers' discourses of error analysis and positionings on Scenarios 4, 5, and 6, and I am following the same structure as I did in Chapter 8 with Scenario 4 in detail and Scenarios 5 and 6 in summary. The detailed analysis of Scenarios 5 and 6 with evidence and extracts are also provided for in the appendices. Also included in the discussion are messages student-teachers recruit or take-up from teacher-educators' discourses or their initial discourses. The scenarios are also on common learner errors in school algebra described in the specialized field of mathematics education research but elicited from learners' own working, hence in learners' own handwriting. For each scenario, prior to discussion of student-teachers' discourses, a preamble of the nature of the algebraic problem is also provided.

In Chapter 10, I discuss lessons learned from my study by referring to the findings and discussing them in relation to the implications for policy, research and practice. Included in the discussion are the research questions and how my study has attempted to answer them. I also discuss how my study has made a theoretical and methodological contribution, and the challenges I experienced in the process.

CHAPTER 2

2 BACKGROUND TO AND CONTEXT OF THE STUDY

2.1 Introduction

The motivation for my study is fourfold: policy on the teaching profession and achievement levels in school mathematics in Zambia, secondary mathematics teacher education programmes in Zambia, algebra in the Zambian school curriculum, and the genesis of the knowledge base for teaching, in particular PCK. Reference to policy is made because it has emphasised mathematics for teaching as one of the ways in which access to mathematics could be enhanced among learners. Secondary mathematics teacher education programmes provide the context within which my study is situated; hence illustrating how mathematics secondary school teachers are trained in Zambia.

Since the object of focus for this study is student-teachers' thinking of and about learner mathematical thinking in general, and algebra in particular, it is necessary that algebra in the Zambian school curriculum is also discussed. It is important to note here that my goal was not to explore student-teachers' readiness of the whole curriculum but rather the more general goal of LMT in algebra. Shulman's (1986, 1987) notions provide the basis from which studies on what teachers need to know and be able to do in order for them to teach effectively have advanced. My study will make a contribution to such research, although at teacher education level.

2.2 Policy on the Teaching Profession and Achievement Levels in Mathematics

Similar to many researchers in mathematics education and other disciplines who have attached great importance to the teachers' knowledge base for teaching, the Ministry of Education (MoE) in Zambia has placed emphasis on the knowledge teachers are required to develop to enable them to teach any school subject. The policy states that

> *The essential competencies required in every teacher are mastery of the material that is to be taught, and skills in communicating that material to pupils. These deceptively simple formulations cover a great array of knowledge, understanding and skills that must become integral to every teacher* (MoE, 1996, p. 108).

Since the policy is inclusive of all subjects offered in schools, mathematics is also a concern. As a result, for teachers of mathematics to be able to teach mathematics successfully, that is, in ways which will enable their learners achieve their potentials in mathematics, they are required to have a broader and deeper understanding of the mathematics they are going to teach as well as ways of helping their learners achieve a similar understanding. I therefore argue that the expectations the policy has put forward in line with what is required of the teacher suggests the kind of work they are required to do in their work place. One of the teachers' tasks is therefore what and how of their recognition and positioning towards the discourse of and about engaging with LMT. The MoE (1996) also acknowledges that attaining the knowledge base required of teachers to teach is not as easy as it has been stated; there are multiple factors at play.

Irrespective of the kind of work that teachers of mathematics are required to do in their work place, MoE (1996) has been concerned with the low achievement levels in mathematics – hence a point of focus in the arenas of practice and research. On average, less than two-thirds of the candidates obtain a pass or better each year in the grade 12 School Certificate leaving examinations (MoE, 1996), and the trend has continued to date. The MoE has attributed these low achievement levels in mathematics to deficiencies at the school level such as facilities, teaching and learning resources, curriculum, and learners' expectations set by learners themselves or by other people. As a result, the MoE has realised an urgent need to find ways of improving the teaching and learning of school mathematics.

Suggested ways of addressing the problem include suitable interventions at school and teacher training levels (MoE, 1996). It is argued that if such interventions are not put in place it would result in subsequent impairment of the national potential for science and technological development. Since teacher education has been identified as one of the areas that need attention to address the low achievement levels in school mathematics, MoE further argues that professional competences of teachers identified (i.e. knowing the mathematics they are going to teach and the skill of communicating) rely heavily on initial teacher training as well as continued professional development of teachers. Hence my study and its location in the Zambian context. Specifically, how prepared are student-teachers in using the discourse of and about engaging with LMT in ways that would benefit their learners?

2.3 Secondary Mathematics Teacher Education Programmes in Zambia

There are two routes to preparation of teachers of mathematics for secondary school, namely, diploma or degree. Those who are trained at diploma level receive a secondary teachers' diploma from Colleges of Education and are required to teach grades 8 and 9. Those trained at degree level receive either a Bachelor of Science with Education (BScEd), a Bachelor of Arts with Education (BAEd) or a Bachelor of Education Secondary [BEd(Sec)] from the University and could teach all the grade levels in secondary school. The difference in the degree programmes is the number of pure mathematics courses student-teachers take from the School of Natural Sciences (there are courses which are core and required to be done by every student-teacher preparing to become a teacher of mathematics and other courses are optional). Otherwise, all student-teachers training to become mathematics teachers learn the same methodology courses and professional courses in the school of education.

The diploma programmes are offered for a period of three years although previously they were of two years duration. However, the degree programmes are offered for a period of three or four years, i.e. BScEd and BAEd programmes are of four years duration while BEd(Sec) runs for a period of three years and it is specifically offered to in-service teachers who intend to upgrade from a diploma to a degree level, although the in-service teachers could also upgrade through BScEd and BAEd programmes.

Students who enrol for a diploma programme in mathematics education are taught content and methodology courses by teacher-educators with relevant education degrees. At the University level, student-teachers pursuing BScEd, BAEd or Bed(Sec) programmes are taught content courses by mathematicians[7] while mathematics education is taught by mathematics teacher-educators who hold a masters degree or higher in the field of mathematics education.

At both diploma and degree levels, student-teachers are exposed to a wider range of topics in mathematics at levels higher than the levels they are going to teach in their work place [student-teachers following a diploma programme learn Advanced Level (A-Level) mathematics while those following the degree programme learn A-Level and higher

[7] These are holders of masters degrees or higher in mathematics whose first degree may or may not have included a component on education.

mathematics]. As shown in Table 1, methodology courses are pedagogical in nature and for the University I engaged with include: MSE 131 (Foundation mathematics for teachers), MSE 331 (Mathematics Education I), MSE 332 (Mathematics Education II) and MSE 431 (Mathematics Education III). MSE 131 acts as a bridge course to enable in-service teachers also manage first year pure mathematics offered in the school of natural sciences. This means that the course is meant to extend secondary school mathematical skills, concepts and processes and use them in the context of more advanced techniques. MSE 331 is designed to equip student-teachers with the skills and attitudes necessary to successfully teach secondary mathematics, and it is offered to student-teachers in the first semester at third year level. MSE 332 is aimed at strengthening student-teachers with the theoretical basis for teaching mathematics in secondary schools, and it is offered to student-teachers in the second semester of their third year. MSE 431 is meant to enable student-teachers consolidate the knowledge of psychological and pedagogical aspects of teaching mathematics in secondary schools, and it is offered to student-teachers at fourth year and in the first semester.

For each course, objectives and the content in terms of a list of topics to focus on are provided including a three-hour practical session slot on the timetable per week (called laboratory sessions). Practical sessions are usually activity-based and are aligned with the topic in focus during lectures. There is no prescription on what teacher-educators should focus on pertaining to a topic, hence suggesting that it depends on what each one knows and available resources. Each course outline also contains a list of recommended and prescribed references. In terms of assessment, for each course the examination as well as continuous assessment each weighs 50%. Continuous assessment activities for each course include two assignments, a test, peer/micro teaching, and practical activities.

Of interest to my study among the methodology topics offered is the aspect of engaging with school mathematics with a view of student-teachers being able to analyse its content. This is meant to ensure that student-teachers are groomed to be competent teachers who are able to teach effectively; and more emphasis has been on topics which teachers experience difficulties in teaching. Ironically, and as already noted, the baseline survey conducted by Haambokoma et al. (2002) shows that algebra was not considered as one of such topics contrary to the vast body of research on the difficulties the learners experience in algebraic thinking, and continuing poor learner performance in algebra.

Thus the question: how is the discourse of and about engaging with LMT structured in teacher education and how does this structure help student-teachers use the discourse in general; and algebra in particular?

Table 1: Objectives and content of the mathematics education courses

COURSE	OBJECTIVES	CONTENT
MSE 331 (Mathematics Education I)	On completion of the course, students should be able to: • justify the teaching of mathematics; • design realistic objectives for learning experience; • plan adequately for teaching experience; • use teaching/learning resources efficiently, and • use tests to improve the teaching/learning process.	1. Aims and objectives of teaching mathematics 2. Domains of learning and behavioural objectives 3. Sequencing instruction 4. Teaching methods 5. Use of basic teaching aids 6. Organising for teaching: Syllabuses, schemes of work, lesson plan and records of work 7. Assessment 8. Peer teaching 9. Micro teaching
MSE 332 (Mathematics Education II)	On completion of the course, students should be able to: • analyse the psychological basis of learning mathematics; • analyse mathematics skills, concepts and processes; • analyse classroom interaction; • exhibit gender consciousness; • analyse the content of school mathematics; • use Audio-Visual Aids efficiently and effectively; and • carry out small scale research projects.	1. Psychology of learning mathematics 2. Strategies for teaching mathematics skills, concepts and processes 3. Problem solving 4. Classroom interaction analysis 5. Gender issues in mathematics education 6. Use of Audio-Visual Aids 7. Use of textbooks 8. School mathematics 9. Research in Mathematics Education 10. Practical classroom teaching
MSE 431 (Mathematics Education III)	On completion of the course, students should be able to: • reflect on experiences and lessons learnt during Student Teaching Practice (STP); • analyse teaching strategies; • organise and manage effectively the mathematics classroom; • make use of aspects of the history of mathematics in teaching; • identify a personal philosophy of mathematics education; • plan adequately for teaching children with special needs; • use examinations and assessment to teaching and learning; • describe the process and history of curriculum development in mathematics education; • identify and justify the part played by mathematics clubs and projects in the mathematics education of children; • manage effectively and efficiently the mathematics department; and • identify avenues for the professional development of secondary mathematics teachers.	1. Reflection on experiences during STP 2. Analysis of teaching strategies 3. Classroom organisation and management 4. Teaching children with special needs 5. Aspects of the history of mathematics 6. Examinations and assessment 7. Philosophy of mathematics education 8. Curriculum development in mathematics 9. Professional development of mathematics teachers 10. Mathematics clubs and projects 11. Managing a department
MSE 131 (Foundation mathematics for teachers)	On completion of the course, students should be able to: • interpret and use mathematical symbols and terminology; • recognize the appropriate mathematical procedure for a given situation; • formulate problems into mathematical terms and select and apply appropriate techniques of solution	1. Rectangular Cartesian co-ordinates 2. Functions 3. The quadratic function 4. Simultaneous equations 5. The use f the expansion of $(a + b)^n$ for positive integral n 6. Circular measure 7. The six trigonometric functions of angles of any magnitude 8. Vectors in two dimensions 9. The idea of a derivative function 10. Integration as a reverse process of differentiation

2.4 Algebra in the Zambian School Curriculum

Before moving on to a discussion of teachers' knowledge base and PCK, it is important to clarify and describe the construction of algebra in the Zambian school curriculum. The curriculum for algebra is discussed in two phases namely, Grades 8 and 9 referred to as upper basic education or junior secondary school and Grades 10, 11 and 12 referred to as high school education or senior secondary school.

2.4.1 Algebra Curriculum at junior secondary school levels

The aims of basic education curriculum are to:

- equip the learners with science and technological tools which will enable them to live effectively and contribute to the social and economic development of Zambia;
- stimulate and encourage creativity and problem solving;
- develop to full potential the mathematical abilities of learners so that they can further the study of mathematics as a discipline as well as use it as a tool in relevant subject areas; and
- enable learners develop conceptual understanding of mathematics so that they can understand the environment around them (MoE, 2003, p. v).

The curriculum further identifies productive skills in mathematics and among them is the ability by learners to manipulate algebraic expressions. In Grade 8, the learners are introduced to algebra for the first time and they learn 'the basic processes of algebra'. These include the use of letters to represent numbers; index notation; directed numbers; use of brackets when combining the four algebraic operations; application of the four operations to algebraic expressions, use of the commutative, associative and distributive laws; substitution and evaluation in an algebraic expression. The learners are also introduced to 'equations and inequations[8]'. Aspects of focus include sequences of ordered pairs resulting in graphs of

[8] In the Zambian algebra curriculum, the term inequations is used to mean inequalities – the term widely used in many algebra curricula. However, you will notice that the term inequalities is used interchangeably with the term inequations from Grades 10 to 12 levels.

linear equations; solutions of simple linear equations and inequations in one variable; and solution sets of simple linear inequations. At the end of grade 8, a learner is expected to demonstrate ability in the outlined areas of algebra.

At grade 9 level, 'the basic processes of algebra' and 'equations and inequations' are approached at a level slightly higher than what is experienced in grade 8. In 'Basic processes of algebra' the focus is on simplification of algebraic expressions; Highest Common Factor (H.C.F) and Lowest Common Multiple (L.C.M) of algebraic expressions; construction of formulae; evaluation and easy manipulation of formulae; simple factors; and simple fractions. With reference to 'equations and inequations', focus is on graphical representation of a linear equation in one or two variables; solution set of simple linear inequalities; and solution of a system of simultaneous linear equations. Similarly, at the end of grade 9, the learners are expected to demonstrate ability in the outlined areas of algebra.

2.4.2 Algebra Curriculum at senior secondary school levels

The curriculum for Grades 10, 11 and 12 is structured in such a way that learners are expected to focus on mathematical concepts, principles and creative thinking processes. The aims of teaching mathematics at these grade levels are to:

- provide the learners with mathematical background necessary for terminating and further education;
- develop logical and abstract thinking in the learners;
- enable the learners develop mathematical language and skills as a means of communication and investigation;
- develop in the learners the ability to use mathematics as a tool in the environment;
- provide learners with mathematical skills to enable them perform adequately in other subject areas;
- develop in the learners a spirit of satisfaction and confidence in the understanding of mathematical concepts and mastery of mathematical skills; and

- stimulate and encourage creativity and a spirit of enquiry in the learners (MoE, 2002, p. vi).

In high school, at the end of Grade 12, learners are expected to develop the ability to explore mathematical situations and recognise patterns and structures in a variety of situational forms and justify generalisations. To attain this, the learners should have been introduced to 'basic processes of algebra' in grade 10 at levels slightly higher than the grade 9 level. Aspects covered include collecting and simplifying like terms; interpreting and using brackets; and adding and subtracting algebraic expressions. 'Factorisation' includes factorising algebraic expressions such as $ax \pm ay$, $ax \pm bx \pm kay \pm kby$, $a^2x^2 - b^2y^2$, $a^2 \pm 2ab \pm b^2$, $ax^2 \pm bx \pm c$. 'Formulae' includes constructing and using formulae; and changing the subject of the formulae. 'Equations and inequalities' include solving linear equations in one variable; solving simultaneous equations in two variables; solving linear inequations in one variable. In solving simultaneous equations, a variety of methods are encouraged except the matrix method. This exclusion could be as a result of learners having not yet been introduced to some skills with regard to matrices that would enable them to solve the simultaneous equations using the matrix method.

Learners in Grade 11 are introduced to relations and functions, graphs of polynomials and quadratic equations. In 'relations and functions', learners are expected to relate members of two or more sets according to a given rule; determine relationships that are functions; use function notation; and find inverse of a function. Areas of emphasis in the teaching and learning of relations and functions include familiarity with terminologies of domain, range and mapping; examples of function notation include $f(x) = 2x - 7$ or $f: x \rightarrow 2x - 7$; and examples of inverse function notion are $f^{-1}(x) = (x + 7)/2$ and $f^{-1}: x \rightarrow (x + 7)/2$. 'Graphs of polynomials' emphasize completing tables of values of functions; the equation of a line $y = mx + c$ and recognising what y, m, x and c stand for; and determining gradient and equation of a line given two points. 'Solving quadratic equations' includes use of methods such a factorisation, completing the square, and use of the formula.

At Grade 12 level, learners are introduced to 'graphical representation of inequalities' at levels higher than the preceding grades. The learners are expected to solve simple linear inequations; represent inequations in one variable on a number line; represent inequations in

two variables on the graph; form a linear inequation arising from a given situation; form a system of linear inequations arising from given situations (mathematical models); and apply the solution set of a system of linear inequations to solve linear programming problems. The emphasis in all these abilities stated is that in illustrating inequations on a Cartesian diagram, the unwanted region should be shaded. This is meant to allow for the easy visibility of the solution.

In this section, I have outlined what is covered in the curriculum in terms of algebra from Grades 8 to 12. However, focus of school algebra in Zambia seems to be more on algebraic syntax with emphasis on algebraic expressions and equations. Moreover, most of the common learners' errors in school algebra reported in the field of mathematics education research are around the syntax of algebra. Therefore, the emphasis is in what Kieran (2004) would mainly refer to either as transformational or generational algebraic activity, and are described in Chapter 3 as part of the constructs of the core of school algebra. This informs the scenarios for my study as will become evident in the methodology section (Chapter 5). Even though the school algebra curriculum is broader, the research on errors and the errors displayed by learners that formed part of the scenarios were mainly around the syntax of algebra. The issue then is what and how student-teachers' engagement in the discourse of and about LMT in general, and algebra in particular is at levels appropriate to the demands of the school algebra curriculum[9]? I was also interested in finding out what different issues related to LMT would need to be engaged with a different curriculum. I thus proceed to discuss the professional knowledge of teachers.

2.5 The Genesis of Teachers' Knowledge Base for Teaching, in Particular PCK

He who knows does, and he who teaches understands (Shulman, 1986)

Shulman's (1986, 1987) work focused on teachers' knowledge base for teaching, and this was necessitated by the way teacher competencies were being assessed in that pedagogical

[9] It is interesting to note how different the Zambian algebra school curriculum is from say the new South African curriculum and curriculum elsewhere. For example, in Zambia, the algebra curriculum at grades 10, 11 and 12 has not been impacted on in terms of functions as they emerge when technology such as the graphic calculators are used. The use of this technology illuminates functions and their transformations. In the South African algebra and functions curriculum, and curricular elsewhere, this is integrated.

aspects were in focus while content knowledge had disappeared. He alluded to the fact that in the late 1800's, 90-95% of teacher competence tests covered content knowledge and the remainder were pedagogical in nature. From the experiences reflected in the teachers' autobiographies, it became evident that for a teacher to competently teach any subject matter to learners, they needed to demonstrate knowledge of that subject matter as a prerequisite to teaching. With the eroding of the curriculum content in evaluation items designed to assess teachers' competencies in the 1980's, Shulman (op cit) wondered where the subject matter had gone and what had happened to it. As a result, Shulman (op cit) argued that knowing some teaching procedures was not enough, one needed to know what it is that they were going to teach.

The strong alignment to pedagogical practices as a basis for assessing teachers' competences was research driven in that research of that time focused on what was considered to be "good practice", i.e. patterns of teacher behaviour that would bring about improved academic performance among learners. Shulman (1986) argued that while it was good to implement practices which are informed by research, it was equally important to realise that research has its own constraints in terms of narrowing the scope, focusing the view, and formulating questions in an effort to simplify a complex reality. Subject matter was used for subdividing data sets rather than considering it in its own right. There was no research that focused on how subject matter was transformed from the knowledge of the teacher into the content of instruction, and on how particular formulations of that content related to what learners came to know or misconstrue (although cognitive research on learning grappled with the idea).

Therefore, Shulman and his colleagues considered the missing link between subject matter and various studies that focused on teaching as the "missing paradigm" problem – hence problematic for both policy and research. It was problematic in the sense that policy makers were inclined to pedagogical practices rather than subject matter in formulating policy because that was what research had focused on. Sole focus on pedagogy was also problematic for research in that the studies about teaching did not focus on how the teachers organised content knowledge in terms of what questions are asked, and what explanations are given during the teaching and learning process.

To substantiate his claim for the missing paradigm, Shulman (1986) made reference to the history of the academy, i.e. medieval university in the 1950's, in that content and pedagogy were not separated, implying that what is known (content) was not separate from how to teach it (pedagogy). This indicates that understanding of content was related to how well one was able to teach it. Therefore, understanding was viewed as a demonstration of "what is known" (content) as well as "how it is taught" (pedagogy) – hence a connection between knowing and teaching. Going by Shulman's (op cit) concern, what is it that teachers are able to teach if assessing their competencies is pedagogically inclined? How can one access that which they are teaching? I ask this because a competent teacher should be able to demonstrate that they understand what they are teaching in ways that would benefit their learners. I therefore agree with Shulman's (op cit) argument that teacher competencies cannot be assessed pedagogically devoid of subject matter – the two complement each other. What this means for my study is that as student-teachers engage in the discourse of and about LMT, their understanding of school algebra in terms of content and pedagogy would come to the fore.

To address the imbalance that was realised in assessing teacher competencies, Shulman and his colleagues (1986) embarked on a longitudinal study entitled "Knowledge Growth in Teaching" – how teachers' knowledge developed over time. They engaged with student-teachers that had finished their teaching programme three-quarter way and some were followed in their first year of full time teaching to track their "intellectual biography", and their study was qualitative in nature. The researchers assumed that most teachers were experts in the subject matter they taught; as a result, they were concerned with the transition from expert student to novice teacher – and asked how that knowledge was transformed into a form which their learners could understand, hence the crux of PCK. Therefore, Shulman's (op cit) focus of their study raises a question: How does knowing the subject matter and the teaching of it unfold in my study.

Krainer & Goffree (1999) as pointed out by Adler et al. (2005) identified four types of research related to teacher education. Firstly, *research in the perspective of teacher education* with a focus on teachers' mathematical beliefs, teachers' knowledge and aspects of teaching. Krainer and Goffree argue that although this kind of research is not directly linked to empirical studies in teacher education, the findings would inform the design of programmes

in teacher education. Secondly, *research in the context of teacher education* that focuses on teachers' professional development learning, establishing the gap between what is learned in pre-service training and the work they do in school, and tracking what changes in terms of teachers' beliefs and practices. In this instance, the understanding is that such research could inform teacher education but that teacher education itself is not the object of focus in research. Thirdly, *research on teacher education*, where studies are directly linked to teacher education with a focus on the interaction practices going on in teacher education. Fourthly, *research as teacher education* in that research is the focus as a way of enhancing teacher development. Examples of such studies include all forms of action research and reflective practice.

From the four categories of research in teacher education identified, I argue that Shulman's (1986) concept of PCK emerged from the study of teachers in the practice of teaching, and not from the study of teacher education. Shulman followed teachers' knowledge growth while they were in practice. The difference with my study is that I want to establish the preparedness of final year mathematics education student-teachers. Thus locating my study in the first category of research in teacher education with a focus on what mathematical knowledge student-teachers know and enact when they engage in the discourse of and about LMT in general, and algebra in particular. The findings would illuminate mathematics teacher education in Zambia in relation to the discourse. This study, therefore, would make an original contribution by focusing on what and how the discourse of and about engaging with LMT is recognized and focused on in the mathematics education courses, and in the context of school algebra.

2.6 Significance of the Study

Student-teachers' participation in the discourse of and about engaging with LMT has implications for both teaching and teacher education. Most research that has focused on teachers' mathematical knowledge for teaching has looked at the practice of teaching in both elementary and secondary mathematics classrooms (Adler & Pillay, 2007; Ball & Bass, 2000; Ball, et al., 2004; Ball, et al., 2008; Hill, et al., 2005; Huillet, 2007; Kazima, et al., 2008; Tatolo, 2007), and on how mathematics comes to be constituted in mathematics teacher education (Adler & Davis, 2006; Z. Davis, et al., 2007). While these varied research studies are useful, they do not help us in understanding and explaining the mathematical knowledge

for teaching school algebra that student-teachers know and are able to enact as they reach the end of their teacher education. Therefore, the findings of this study would contribute to reducing the knowledge gap in the body of mathematics education research that attends to this kind of knowledge in teacher education.

As argued, teacher education is a very crucial area that should not be overlooked since the student-teachers develop their initial teaching knowledge and skills during their training (Krauss, et al., 2008) and other authors have made recommendations for such focus in teacher education (Adler & Davis, 2006; Adler & Pillay, 2007; Ball & Bass, 2000; Thames, et al., 2008). Doing this research in teacher education in Zambia would illuminate the understanding of what and how student-teachers think in relation to the discourse of and about engaging with LMT in general, and algebra in particular. This would in turn inform teacher-educators as to preparing the student-teachers on how to deal with challenges that learners experience when learning school algebra in Zambia. Similarly, student-teachers' involvement in this research would provide insights into how they would plan and prepare for LMT before teaching occurs and during teaching.

Of course, significance extends beyond need and relevance to contribution to knowledge. Some indication of why my study is original has been discussed, however, originality and development of this from the knowledge base in the literature is also important and hence the focus of Chapter 3.

CHAPTER 3

3 REVIEW OF RELATED LITERATURE

3.1 Introduction

This study is informed by relevant literature relating to its three main aspects: perspectives on teachers' knowledge and learning and how teacher education and PCK are implicated; LMT and teachers' engagement with this; and school algebra. Perspectives on teachers' knowledge and learning provide insight into arguments that arise from different researchers who have engaged with teachers' knowledge base for teaching, in particular, mathematical knowledge, and what implications this research has for the teaching and learning of mathematics and teacher education. I pay particular attention to literature focused on PCK in teacher education, and whether they have worked with LMT. This is in light of my hypothesis that PCK should be a presumed aspect in teacher education.

Literature on LMT and teachers' engagement informed me of what LMT entails, including the nature of errors and strategies for dealing with these. The literature review on what LMT entails later became part of the framework for describing teacher-educators' and student-teachers' discourses. A review of literature on school algebra allowed me a chance to explore the question of what constitutes school algebra, and the role misconceptions, errors and difficulties learners experience in algebra play in the learning of algebra. This literature review also guided me in designing relevant tasks which student-teachers engaged with so as to establish their discourses of LMT in algebra.

3.2 Perspectives on teacher knowledge and learning

As alluded to in Chapter 2, Shulman's (1986, 1987) seminal work has had significant impact on how people think about teachers' knowledge. He was able to shift the debate from how often teachers carried out certain actions, to a very strong content focus. Shulman (op cit) might not have been the first one to develop this view of teacher knowledge. However, he provided a language to talk about elements of teacher knowledge. In Bernstein's (2000) terms, and as I discuss further in Chapter 4, Shulman's work could be described as strengthening the grammar and so the development of a more specialised language for teachers' professional knowledge base.

Shulman's work has been advanced and critiqued. In mathematics, in particular, is the development of what is referred to as mathematics knowledge for teaching (MKT) or more simply mathematics for teaching MfT (Adler & Davis, 2006; Ball, et al., 2008; B. Davis & Simmt, 2006; Z. Davis, et al., 2007; Even & Tirosh, 2002; Rowland, et al., 2005; Zazkis & Leikin, 2010). Adler et al. describes how LMT is part of teacher education; Ball et al. explored knowledge of content and student (KCS) among other constructs; Davis & Simmt explored categories of knowledge and those of knowing, and their co-existence; Even & Tirosh report on discursive resources for working with LMT; Rowland et al. developed a practice-based framework, called the knowledge quartet, used for exploring ways in which pre-service teachers' PCK and SMK come into play in the classroom; and Zazkis & Leikin examined secondary school teachers' perceptions of the usefulness of the mathematics they learn in college or university, referred to as Advanced Mathematical Knowledge (AMK), in the context of teaching.

While the work of Davis & Simmt (op cit), Rowland et al. (op cit) and Zazkis & Leikin (op cit) has focused on LMT at the peripheral, works done by Adler et al. (op cit), Ball et al. (op cit), and Even & Tirosh (op cit) have had particular focus on LMT. Therefore, I am focusing in on those studies that bring LMT in as part of PCK, hence insightful into my problem which is student-teachers' knowledge about LMT. Added to this, I review literature pertaining to what it means to learn to teach mathematics from a situative/participationist perspective (Peressini, et al., 2004). Student-teachers' participation in the sites of teacher learning, and in particular in the discourse of engaging with LMT, in Bernstein's (2000) terms, suggests their recognition and realization rules, and are described in Chapter 4. I also review literature on how the relationship between content and pedagogy has been critiqued.

3.2.1 Shulman's seminal contribution

While PCK as a construct has been the focus of many studies on teacher knowledge and learning, it is only one of three categories that Shulman identifies as comprising content knowledge for teaching, namely subject matter (SM) content knowledge and curricular knowledge. Shulman (1986) quoting Schwab (1978) defines content knowledge as including both substantive and syntactic structures. He further explains substantive structures as "the variety of ways in which the basic concepts and principles of the discipline are organised to incorporate its facts"; and the syntactic structures of a discipline as "the set of ways in which

truth or falsehood, validity or invalidity, are established" (Shulman, 1986, p. 9). Shulman elaborates on this by stating that syntax of a discipline could be linked to the grammar, which is a set of rules that regulate what is appropriate to say and what is not. In referring to the described distinction, Rowland and Turner (2008) critique Schwab's use of the term 'syntax'. In the first place, they explain that the distinction "comes close to that between content (substantive) and process (syntactic) knowledge, although syntax seems to entail greater epistemological awareness than process knowledge" (Rowland & Turner, 2008, p. 92). They thus argue that Schwab's use of the term 'syntax' is rather limiting since it "suggests formal structure only, whereas the heuristics of enquiry are at the heart of the intended meaning" (Rowland & Turner, 2008, p. 92).

Despite this circumstance, Shulman (1986, p. 9) argues that teachers are required to define "for students the accepted truths in a domain, explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and outside, and both in theory and in practice". The teacher should also realise that there are a variety of ways of organising the discipline. These various ways and alternative forms of organisation and the pedagogical grounds for selecting one under some circumstances and the other under different circumstances should be recognised by the teacher. Moreover, understanding the syntax of the discipline is crucial in that when competing claims are offered, the teacher would be able to follow up on how the controversy was adjudicated, and how a similar controversy would be adjudicated in present times. Shulman (op cit) further argues that a teachers' subject matter knowledge should at least equal those specialised in the discipline alone. Therefore, a teacher should understand *that* something is so, and *why* it is so. This suggests that substantive structure focuses on 'what' of the discipline while syntactic structure focuses on 'how' of the discipline.

Referring to PCK, Shulman (1986) argues that it addressed subject matter knowledge for teaching, and includes the most regularly taught topics in one's subject area; the most useful ways of representing those ideas; the most powerful analogies, illustrations, examples, explanations, and demonstrations – the ways of representing and formulating the subject that make it comprehensible to others. In light of this, it is important that teachers comprehend a variety ways of representations, some derived from research and others originating from the wisdom of practice. PCK also includes an understanding of what makes the learning of

specific topics easy or difficult – the conceptions and preconceptions that learners of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. In the event that these preconceptions turn out to be misconceptions, Shulman (op cit) states that teachers should have knowledge of strategies that would help in reorganizing the understanding of learners.

Shulman (1986) explains that a curriculum contains programmes that are designed for the teaching of particular subjects and topics at given levels of education. A variety of instructional materials are available in relation to the programmes, including sets of characteristics that serve as indications and contraindications for use of particular curriculum or programme materials in particular circumstances. He further describes a curriculum as a pool from which the teacher draws tools of teaching that present or exemplify particular content and remediate or evaluate learners' adequacy. Therefore, Shulman (op cit) asserts that teachers should look out for alternative curriculum materials for a given subject or topic; and be familiar with the curriculum of other subjects their learners are studying. He argues that if teachers are aware of the curriculum of that particular subject, the alternative curriculum of the subject, and the curriculum of the other subjects their learners are taking, connectivity is enhanced. This implies that teachers will be able to relate content of a given course or lesson to topics or issues being discussed simultaneously in other classes.

With a view of what it means to assess teacher competence, Shulman (1986) wonders what it would mean if teachers were to posses the varieties of content knowledge described above (i.e. SM, PCK and curriculum knowledge). He argues that

> … *the examination would measure deep knowledge of the content and structures of the subject matter, the subject and topic-specific pedagogical knowledge associated with the subject matter and the curricular knowledge of the subject* (Shulman, 1986, p. 10)

Shulman (op cit) further argues that the examination would illuminate the potentials of a professional, and not merely assess the subject matter. In addition, questions about the most likely misunderstandings, and the appropriate strategies most likely to be useful in overcoming the difficulties would be asked, and thus distinguish between a subject matter major and a subject matter teacher, and in a pedagogically relevant and important way. The questions to consider and reflect on as my study unfolds are whether the student-teachers are

able to demonstrate their understanding of the subject matter in school algebra in ways that are expected of a teacher when they engage with LMT. Are there opportunities for developing such a teacher in teacher education in Zambia?

3.2.2 Mathematical knowledge in the context of teaching

Building from Shulman's notions of teachers' knowledge base for teaching, the seminal work is Ball et al.'s (2008) but there have been synthesis of other studies, for example, the cognitively guided instruction (CGI) and Manor projects reviewed by Even & Tirosh (2002). I discuss Ball et al. and Even & Tirosh. Ball et al. (op cit) embarked on a longitudinal empirical study in a selected number of elementary mathematics classrooms in the United States so as to develop a practice-based theory of mathematical knowledge as it is entailed by and used in teaching. Their argument is that viewing subject matter as it is used in practice would bring out what teachers need to know and what they need to be sensitive to regarding content for them to teach well (Ball & Bass, 2000). Moreover, they argue that inquiries which begin with practice bring to the fore subject matter demands of teachers, work that cannot be seen when the initial focus is with lists of content to be taught derived from the school curriculum. This implies that the content demands emerge from analysing what constrains and what enables the teacher in the course of practice as they mediate learners' ideas, make choices about representations of content, and modify curriculum materials, to name but a few (Ball & Bass, 2000).

Ball et al.'s empirical work started with teaching practice to study its knowledge demands, and instructional tasks that provided opportunities to learn mathematical knowledge for teaching in the context of its use. As a result, records of practice such as videotapes of classroom teaching, copies of learners' work and of teachers notes, and curriculum materials from which the teachers were teaching, provided opportunities to engage in the kinds of mathematical thinking, reasoning, and communication used in teaching (Thames, et al., 2008). Records of practice I engaged with in my study on learner errors included scenarios drawn from specialised mathematics education research and learners' own working.

The notion of mathematical knowledge for teaching has been described as the mathematical knowledge needed to carry out the work of teaching mathematics, and so the tasks involved in teaching and the mathematical demands of these tasks (Ball, et al., 2008). In line with this

notion, it is also recognised that as teachers teach mathematics, the mathematical demands of teaching require that teachers need to hold and use mathematics in a specific way in order to teach mathematics successfully (Adler & Davis, 2006; Ball & Bass, 2000). Adler & Davis (op cit) argue that the way teachers of mathematics use mathematics in their teaching and so how they come to know it in practice differs from the way mathematicians hold and use mathematics; and that programmes that prepare and support mathematics teachers should provide opportunities for learning this mathematics. Following Adler & Davis (op cit) who state that both mathematics and teaching are implicated in mathematics teacher education curricula, my concern is whether the mathematical education of teachers can and does provide opportunities to learn these ways of knowing and using mathematics.

Moreover, Thames et al. (2008) state that practice-based approaches, although becoming common in methods courses and professional development programmes, are rarely used in mathematics courses for teachers. In most of the courses, mathematics is foregrounded and teaching is backgrounded (mainly used as a context to justify or apply the content being taught). Moreover, this orientation risks preparing teachers who are very good at doing mathematics in their courses but are unable to use their mathematical knowledge in actual teaching situations. Therefore, suggesting that

> "*... the mathematics taught in courses for teachers should be the mathematics required for the work of teaching, and that this mathematical knowledge for teaching should be integrated with and learned in the contexts of its use in practice* (Thames, et al., 2008, p. 6)

This means that the work of teaching is foregrounded, and mathematics is learned through its use in teaching practice.

Even & Tirosh (2002), in an overview of research that looked at teachers' understanding of learners' thinking, bring into focus two key projects that exemplify practice-based approaches and are relevant for my study since the concern is on teachers or teacher-educators and learners' thinking. They discuss what it might mean for teacher education to focus on learners' mathematical thinking and learning. Firstly, they report on one of the most effective projects in building teachers' knowledge entitled Cognitively Guided Instruction (CGI) by Fennema et al. (1996). In this study in-service elementary school teachers were presented with a model of children's thinking of word problems concerning basic addition, subtraction,

multiplication and division. The model distinguishes between several problem types and identifies the relative difficulty of each category. Through engagement with this model in workshops, teachers were able to: recognise differences among word problems, identify solution strategies that learners might use to solve different problems, and organise these strategies into hierarchical levels of thinking.

The results showed that there was great improvement by the participating teachers in terms of their beliefs and instruction. Their roles changed from demonstrating procedures to helping their learners build on their mathematical thinking by giving them support in a variety of problems and encouraging them to talk about their mathematical thinking – and consequently changes in learners' achievement. In my study, where student-teachers discuss scenarios designed around algebraic thinking, (see chapters 8 and 9) my interest is whether they recognize learner errors, and are able to explain possible reasons for the errors and suggest possible remediation strategies and their positions in them and explain why their discourses are as they are. I will then be able to reflect on what this means for the quality of instruction student-teachers propose.

Secondly, Even & Tirosh (2002) also report on the Manor project by Even (1999a). In this project focus is on the development of a professional group of secondary school mathematics teacher-leaders and in-service teacher-educators. The aim of the project was to deepen and expand the participants' understanding about learners' conceptions and ways of learning different topics in mathematics – hence reflecting and viewing mathematics learning from a learner's perspective, and from a constructivist standpoint. They also hoped that the participants would become familiar with research-based key features of learner thinking in different mathematical topics (i.e. cognitive development and aspects of mathematical thinking in algebra, analysis, geometry and probability). The authors initially purposed to challenge and expand the participants' understanding of learners' ways of making sense of the subject matter and the instruction. What was different with this project compared to the one previously discussed was that there was no model to use but instead they used research articles. The participants were required to read, make presentations and participate in the discussions, hence build on and interpret their experience-based knowledge using research-based knowledge. To support this, participants were also asked to replicate studies of their own choice.

The findings of the Manor project are that participants' initial understanding about learners' mathematical learning, which were intuitive, naive and implicit ideas, tended to develop into more formal, deliberate, and explicit knowledge. The main common thing about Even & Tirosh's (2002) reviews of the CGI and Manor projects that dealt directly with LMT is that in both projects, discursive resources are provided with which participants worked with. Teachers or teacher-educators did not only have their own experience to draw on but they were provided with discursive resources for working with LMT. In the case of the CGI project, a model is provided as a resource while in the case of the Manor project, research papers are provided, hence for both projects providing a trajectory of teacher learning. This suggests, in Davis et al. (2007) terms, that both the CGI and Manor projects function at the realm of the "intelligible" and not at the realm of the "sensible": teaching is not simply a practical accomplishment, but it has a knowledge base and so discursive resources that can be and are made available for teachers or teacher-educators to work with. The notions of "intelligible" and "sensible" will be further elaborated in Section 3.2.4. The question to think about as the study unfolds is: What discursive resources do teacher-educators say they make available to their student-teachers in relation to the discourse of engaging with LMT.

3.2.3 Elaborating mathematical knowledge for teaching

Unpacking or decompressing of mathematical ideas, which is an aspect of *specialised content knowledge* (SCK – discussed below) has been identified as an important element of the knowledge- in- action (mathematical practices) that mathematics teachers need to enact as they do their work (Ball & Bass, 2000; Ball, et al., 2004). Elements of unpacking are identified to include designing mathematically accurate explanations that are comprehensible and useful for learners; interpreting and making mathematical and pedagogical judgements about learners' questions, solutions, problems, and insights which are both predictable and unusual. The authors further argue that content that is in the curriculum should be part of the mathematical knowledge that teachers need. Ma (1999) also holds the same view and argues that a teacher's subject matter knowledge which develops in the context of teaching and learning is relevant to teaching and is likely to be used in teaching.

This understanding contradicts Krauss et al.'s (2008) findings as discussed in Chapter 1 (Section 1.2) in that they found no positive correlation between teachers' knowledge base for teaching and their years of teaching experience. What this suggests, and Ball et al. (2008)

also argue such, is that this is inevitable as teacher education is crucial. However, both Ball et al. and Ma (1999) argue about the importance of developing teachers' knowledge base for teaching through deliberate study of practice. For Ma, this requires experience in practice while Ball et al. argues for the importance of teacher education. This is interesting because Krauss et al. in their Coactive project are working at the secondary school level and Ball et al. and Ma are working at the elementary school level, and are arguing contrary positions pertaining to sites for developing teachers' knowledge base for teaching. Ma is saying teachers' knowledge base for teaching is learned in the context of teaching and Krauss et al. and Ball et al. are saying what is significant is what you have learned before entering the school.

Apart from being able to solve mathematical problems, teachers should also enable learners to access the content; interpret learners' questions and productions; generate contexts in which the content arises; explain and represent the content in multiple ways (Ma, 1999; Thames, et al., 2008). In addition, all these tasks that the teacher has to engage with require mathematical knowledge and reasoning. My study thus explores the kind of mathematical knowledge and reasoning entailed by student-teachers as they engage in a discourse of and about LMT.

Ball et al. (2008), building on Shulman (1986), and from their study of practice, categorise mathematical knowledge for teaching (MKT) as subject matter knowledge (SMK) comprising of *common content knowledge* (CCK), *specialised content knowledge* (SCK), and *Horizon content knowledge*. They further categorise pedagogical content knowledge (PCK) as comprising *knowledge of content and students* (KCS), *knowledge of content and teaching* (KCT) and *knowledge of content and curriculum* (Thames, et al., 2008).

Figure 1: Domains of Mathematical Knowledge for Teaching (MKT) by Ball et al. (2008)

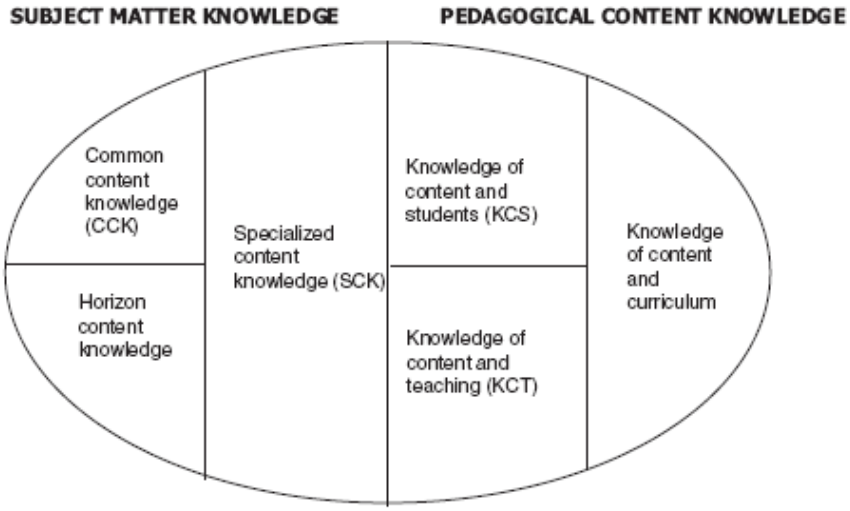


Figure 1 is an illustration of this proposed model of MKT whose knowledge domains are analytically distinct. CCK has been described as the ability by the teacher to solve mathematics problems in a way any other person would manage, for example, the mathematician or other users of mathematics. SCK describes the mathematical knowledge required for teachers to do their work, for example, looking for patterns in learners' errors, difficulties and misconceptions. SCK is also about unpacking or decompressing of mathematical ideas as earlier described. CCK is likely to resonate with what Shulman means by SMK (concerned with the substantive and syntax of the discipline of mathematics and is discussed in Section 3.2) while SCK is new in terms of its conceptualization but are both mathematical knowledge (Hill et al. 2008). For Hill et al. CCK and SCK do not entail any specific knowledge of learners or instruction.

To exemplify these sub-domains, Ball et al. (2008) states that manipulation of a mathematical problem and knowing that the manipulation is faulty is CCK in that everyone who has been enculturated in the discipline of mathematics is able to do. However, carrying out an analysis of why the error has been made and how it can be corrected by using a variety of representations requires a specialised way of knowing mathematics (SCK) – an indication that teachers work with mathematics in its decompressed or unpacked form while learners strive to develop fluency with compressed mathematical knowledge (Ball & Bass, 2000; Ball,

et al., 2008). Furthermore, the complexity of SCK in this instance is that if the teacher is aware of the errors or difficulties their learners experience, then they will require KCS. In this instance, for their work, Ball et al. assert that making distinctions between categories of knowledge was not easy in that it affected precision. They further define *horizon content knowledge* as awareness of how mathematical topics are related over the spam of mathematics included in the curriculum.

KCS focuses on knowledge about students and knowledge about mathematics, for example, learners' anticipated thoughts and likely confusion; or interesting, motivating, anticipated easiness or hardness of a task by learners. A specific example Hill et al. (2008) provide is that of teachers' knowledge and awareness that given two fractions to add, learners who are unable to conceptualize the multiplicative nature of fractions would possibly add numerators and denominators. This suggests that the teacher has to attend to both the specific content of adding fractions and the likely learner mistakes or misconceptions that might arise. Such knowledge would then guide the teacher to design appropriate instruction to address the error prior to it occurring or when it occurs. This is based on the argument that:

> "*Logically, teachers must be able to examine and interpret the mathematics behind students' errors prior to invoking knowledge of how students went astray*" (Hill, et al., 2008, p. 390).

To help clarify what KCS is, Hill et al. (2008) state that it is not about knowledge of best teaching strategies to build on learners' mathematical thinking including their errors. It is also not about knowledge of curriculum materials. This suggests that KCS resonates with the first two of the three stages for carrying out error analysis, which are identifying the possible errors learners make, and explaining reasons for the error (interpret and evaluate) as conceptualized by Peng & Luo (2009) and Jacobs, Lamb & Philipp (2010). In this case KCS is not about stage three which entails suggesting appropriate remediating strategies. The three stages of carrying out error analysis are discussed in detail in Section 3.3 of this chapter. Therefore, Hill et al. (2008) argue that there is a distinction between teachers' SMK and their KCS. This implies that one can have strong SMK without necessarily understanding how learners learn particular content, and vice versa.

In trying to conceptualize and measure KCS by writing, piloting, and analysing results from multiple – choice items, Hill et al. (2008) found that teachers were able to draw from their

KCS and/or mathematical reasoning. They were familiar with learners' mathematical thinking such as common learner errors as one element of knowledge for teaching. From the results, they claim that they could talk of how skilful, insightful, and full of wisdom their teachers were compared to the mathematician. However, they could not make claims on how these rich resources teachers had could translate into learner gains in their teaching of mathematics.

Similarly, Even & Tirosh (2002) report on a study done by Tirosh, Even, and Robinson (1998) and argue that there is a link between teachers' knowledge and understanding about learner learning, and their instructional practice. This implies a relation between teachers' awareness of learners' errors and the strategy for instruction. In light of this, Tirosh, Even and Robinson worked with four teachers (two novices and two experienced) on simplification of open algebraic expressions. Their findings are that the two novices who were not aware of learners' tendency to conjoin used a method of 'collecting like terms' that had implications for the quality of instructions given to learners. One teacher drew on the application of rules while the other drew on the 'fruit salad' approach. The other two experienced teachers who were aware of learners' tendency to conjoin planned comprehensive lessons meant to familiarise the learners with the notion of like and unlike terms prior to teaching simplification of algebraic expressions. One teacher focused on identifying like terms while the other teacher used multiple strategies that included substitution, order of operations, and going backwards.

These findings are important for my study in that as you will observe in the unfolding Chapters and more explicitly in Chapter 8, one of the strategies used by the novice teachers, which is the application of rules formed scenario 2, which I named *the conjoining problem*. One of the questions I ask my data pertaining to scenario 2 is: In what ways do student-teachers talk about the conjoining problem and how might this be explained in relation to their preparedness for the task of engaging with the discourse of LMT?

Moreover, Thames et al. (2008) refer to KCT as knowledge about teaching and content and focuses on knowing how to design instruction, for example, sequencing particular content for instruction and making instructional decisions related to which learners' contributions to ignore, pursue or reserve for future reference. They further stated that *knowledge of content*

and curriculum was provisionally placed while work was still in progress. Overall, this framework of MKT depicts the kind of mathematical problem solving that teachers engage with as they teach mathematics. In my study then, what kind of mathematical problem solving skills do student-teachers demonstrate as they engaged in a discourse of and about LMT in general and in the context of school algebra? To what extent would the kind of mathematical problem solving include aspects of CCK, SCK, Knowledge on the horizon, KCS, KCT and curriculum knowledge?

From the foregoing discussion, I have demonstrated how Ball et al. (2008) carefully mapped and measured their knowledge domains. In this respect, they argue that Shulman's notions of teachers' knowledge base for teaching lack clarity on the boundaries between PCK and other forms of teacher knowledge. This, they assert, has lead to researchers who have engaged with Shulman's constructs to overlook his initial invitation which was directed towards refinement. Ball et al. have observed that, instead, the researchers have used the constructs as though they were fully developed. In particular, PCK can hardly be distinguished from other forms of teacher knowledge. For example, PCK has been referred to what could be viewed as content knowledge and sometimes referred to something that could be considered, to a larger extent, as pedagogical skills. Ball et al. have further observed that the definitions of PCK as a construct that is an amalgam of content and teaching has been broadened to include any package of teacher knowledge and beliefs. Therefore, Ball et al. argue for precision about the concepts and methods involved, and hence their development.

Adler & Patahuddin (2012) also hold similar views on the lack of precision on how researchers have used Shulman's constructs of PCK and other forms of teacher knowledge. In referring PCK to something that could be viewed as content knowledge, Adler & Patahuddin draw on Ball et al.'s (2008) and Krauss et al.'s (2008) works. They point to that while Ball et al. consider working with different mathematical representations as SCK; Krauss et al. see such an aspect as focus on PCK. For Ball et al., you have to know different representations to mathematical problems irrespective of who the learners are; while for Krauss et al., you are only bothered by different representations because you are teaching and therefore you need to have different ways of explaining. In referring to the issue of being knowledgeable about multiple representations to mathematical problems, the argument is that:

> "*One will rarely be able to move flexibly across representations if one has poor understanding of a representation within itself and, vice versa, a representation within itself is likely to be poorly understood if one is not able to carry the meanings over to other representations*" (Arcavi, 1995, p. 155).

This suggests that one cannot work with multiple representations if they do not know what it means to get to the answer given any mathematical problem. Despite the debate around multiple representations, my study whose focus is on student-teachers' preparedness to participate in the discourse of engaging with LMT falls inside figure 1 in PCK. While there is controversy around what is PCK and what is not PCK, there is no controversy even coming from Shulman that engaging with the discourse of LMT is part of PCK, it is not SCK. Therefore, my study is about PCK. Following Even & Tirosh (2002), my conceptualisation of the notion of engaging with LMT is not only about focusing on learner errors but also about developing in learners both instrumental and relational understanding, and creating an environment where teacher can listen to learners. The detail of this is discussed in Section 3.3. However, QUANTUM project by focusing on MfT is concerned with the relationship between SMK and PCK, and I discuss what aspects inform my study.

3.2.4 MfT and its constitution in mathematics teacher education

Research in the QUANTUM project pursues the question of what and how mathematics is constituted across varying contexts, including teacher education and secondary school classrooms. In engaging with the notion of mathematics for teaching, it has been defined as a form of mathematical knowledge, produced in, and used for, the practice of teaching (Adler & Davis, 2006). Researchers in the QUANTUM project are equally concerned with the mathematics, in terms of how much and what kind do secondary school teachers need to know and know how to use in order to teach mathematics successfully in South Africa's diverse classroom contexts. Also of concern is how and in what ways do programmes that prepare and support mathematics teachers can/do provide opportunities for learning this mathematics. Their assumption is that the mathematics that teachers need to know and know how to use is specific to their work of teaching, hence view unpacking or decompressing of mathematical ideas as critical.

In their study of teacher education, Adler & Davis (2006) focused on mathematical practices revealed in formalised assessments across a range of mathematics teacher education courses

in South Africa. Their analysis showed that across a range of courses, especially in mathematics courses specifically designed for teachers, compression or abbreviation of mathematical ideas dominates formal evaluation (Adler & Davis, 2006, p. 291). Instances of unpacking or decompression of mathematical ideas as valued mathematical practice are limited. Moreover, in some more integrated courses, attempts to merge mathematical and teaching ideas in evaluation reveal an interesting spread of formal evaluative events such as the appearance of tasks where the demands of the tasks are not clear. Adler & Davis raise a question of what this means for research and practice in teacher education in South Africa. In my study, I am concerned with whether and how what teacher-educators say they select and privilege as the discourse of engaging with LMT provide opportunity for student-teachers' recognition of unpacking or decompressing of mathematical ideas as an important aspect of SCK.

In a related study, Davis et al. (2007) studied three courses in formalised in-service mathematics teacher education programmes in terms of their identification with images of the teacher and teaching. Their analysis reveals that all the courses appeal to images of teacher and teaching, with two of these more in a "sensible" and not "intelligible" manner (Z. Davis, et al., 2007, p. 54). For example, as a central resource for modelling the teaching and learning of mathematics, one of the courses considered the image first in elaborating mathematics itself. The authors' argument is that

> "*intelligibility matters for principled reproduction of both mathematics teaching and of school mathematics in mathematics teacher education, then it matters how, in teacher education practice, the relation between sensible experience and the intelligible is regulated*" (Davis et al., 2007, p. 33).

Therefore, learning at the realm of the sensible involves copying and imitating what is being modelled as valued practice, hence experiential and a "practical accomplishment" while intelligibility involves using theoretical or discursive resources. The authors' argument is that there must be a way of regulating the relation between the two.

The three models of teaching in teacher education observed by Davis et al. (2007) included: Firstly, look at me; I am going to show you how to teach. While I teach you, I am going to demonstrate what it means to teach well, that is, look at me and then you see the practice of teaching. The second one is in relation to reflection. Look at yourself, and then try out

activities in your classroom. The third one is in relation to records of practice, for example, look at other teachers teaching from video tapes. The authors also observed that in each case, there was dominant, though not exclusive, model at work, and alleged that the model in use impacts on what mathematics is offered. Each of the courses observed attended to learners' thinking but in very different ways with different effects. This shows that engaging with LMT is a focus in teacher education but is not being written about. Davis et al. (op cit) have attempted to do this but with diverse effects. In my study, I contribute to this knowledge gap with a focus on what and how of thinking about LMT and sense making in algebra. Is 'what' and 'how' the discourse of engaging with LMT focused on in mathematics teacher education at the realm of the sensible and/or intelligible?

Also of interest to my study is research that has focused on secondary school classrooms. These studies have identified and described the mathematical work of teaching that teachers grapple with as they teach specific mathematics topics (Adler & Pillay, 2007; Kazima & Adler, 2006; Kazima, et al., 2008). Kazima and Adler (2006) studied the teaching of probability in a Grade 8 multilingual classroom in South Africa. Instances of teaching to identify the mathematical problem solving that teachers encounter as they work with learners' ideas both expected and unexpected were used. From these instances, the authors show restructuring of tasks as an inevitable feature of teachers' work. Moreover, apart from scaling up or scaling down of tasks (Ball & Bass, 2000), this work can entail shifting the mathematical outcomes from those intended. They then proposed further work 'on tasks and the mathematical work teachers do and need to do as they set up and implement these in and across a range of classroom contexts' (Kazima & Adler, 2006, p. 57). In terms of my study then, as student-teachers engage with learners' thinking involving scenarios designed around common learner errors in school algebra, in what ways does their talk relate to restructuring/reorganising in terms of scaling up or scaling down of mathematical tasks involved.

Adler & Pillay (2007) investigated the mathematical work of a specific teacher (Nash) as he taught linear functions to his Grade 10 learners in South Africa. Evaluative events as well as appeals made by the teacher overtime were studied. The findings reveal that the mathematical work Nash engaged with was largely dependent on mathematics as it is constituted in curricula texts, as well as his own successful prior experience of teaching mathematics i.e.

wisdom of practice (Shulman, 1987), and so from and within his approach. The authors propose a need for external intervention such as researching how learners' misconceptions can be a source of knowledge construction. These findings are illuminating for my study in that I attempt to establish the extent to which student-teachers draw on the following resources as they participate in the discourse of LMT: mathematics as reflected in the school curriculum texts; their experience in mathematics teacher education; their experience as teachers; or their experience as learners. The focus on learners' misconceptions pertaining to school algebraic thinking for my study is in part responding to how such focus can be a resource for knowledge construction.

However, I am aware that using what student-teachers experienced as learners as a resource for knowledge construction would result in what Doerr (2004, p. 269) refers to as the "dilemma of experience". This is described as the challenge teacher-educators encounter in an effort "to simultaneously build on pre-service teachers' experience as pupils in schools and to break the mould of that experience" (op cit). Her argument, following Doerr & Lesh (2003), is that methods pre-service teachers observed their mathematics teachers use during lessons "does not necessarily yield any insight into *why* they acted as they did or what alternative courses of action they considered in particular situations" (op cit). Therefore, teacher-educators are faced with a responsibility to shift pre-service teachers' intuitive thinking of how school mathematics is learned to alternative methods which would enhance conceptual understanding.

In focusing on studies in secondary school classrooms pertaining to MfT, Kazima et al. (2008) pose a challenge for teacher education in relation to teacher preparedness and support given to teachers in order for them to teach mathematics well. They also argue that teacher education focuses on courses in the subjects that do not extend to include topic-specific tasks of teaching and how these tasks are associated to teaching mathematics. The questions that these authors raise include: How then do mathematics teacher education programmes include these mathematical foci that are specific to teaching? How do these programmes provide sufficient engagement with all the relevant topics in the school curriculum (Kazima, et al., 2008, p. 296)? I therefore argue that by exploring and explaining what and how LMT is dealt with in teacher education, and how student-teachers use the notion, some of what my study does is provide insights to both questions raised with respect to school algebra.

For Adler et al., knowledge is constituted in practice because they focus on the constitution of mathematics in teacher education. This has resonance with the situative perspective that problematizes where teachers learn "what" and "how".

3.2.5 A situative perspective on teacher knowledge and learning of MfT

The literature so far has focused on a practice-based theory of mathematical knowledge for teaching and emerging subdomains that provide refinements to Shulman's notions and consequently how work in the QUANTUM project has emerged and it is ongoing. I now present a discussion of a conceptual framework that applies a situative perspective on learning to the study of learning to teach mathematics (Peressini, et al., 2004). Although the authors have not explicitly referred their work to mathematics for teaching, I have done so because their study focuses on the tasks of teaching which teachers have to wrestle with in their work and how these develop for a teacher. From this perspective, Peressini et al. (op cit) argue that learning to teach mathematics occurs in many different situations such as in *mathematics and teacher preparation courses*, *pre-service field experiences*, and *schools of employment*. They suggest that teachers' participation in these varied sites over time results in refinement of their conceptions about their work of teaching. The refinements of teachers' conceptions about their work of teaching include the big ideas of mathematics, mathematics-specific pedagogy, and sense of self as a mathematics teacher.

The argument on teacher learning described above support both sides of the earlier debate on the significance of what is learned prior to practice as well as in practice with reference to knowledge base for teaching. This suggests that teacher learning depends on where you learn "what", hence what you learn in the context of teaching as well as what you have learnt before are significant. My study therefore explores the teacher education preparation courses with particular focus on what teacher-educators say they privilege as the discourse of engaging with LMT in the mathematics education courses. How then, does the experience in learning to teach mathematics in teacher education help student-teachers in participating in the discourse of and about engaging with LMT in general, and algebra in particular?

Peressini et al.'s (2004) study (which happens to be a project) that is guided by a situative perspective traced the learning trajectories of student-teachers from two reform-based teacher education programmes into their early teaching careers in the United States. The focus was

on prospective teachers' content knowledge in their mathematics courses as well as in their mathematics education courses. The mathematics content areas selected included functions, rate, and proof to enable them trace prospective teachers' beliefs and knowledge growth over time, and how these beliefs and knowledge about mathematics emerge in practice. Research tasks were constructed and the participants in the study were observed and interviewed in their roles as students as well as teachers.

To substantiate their argument, Peressini et al. (2004) provide two examples from their research to illustrate how this framework has helped them understand the process of learning to teach. The first example shows how Ms Audry Savant demonstrated different conceptions of proof on several occasions during her teacher training and her first year of teaching to enable her participate successfully in the multiple contexts of teacher education and initial teaching. To be specific, she drew on 'formal proofs that prove' in her teacher education, i.e. supporting her participation as a student in her content course and on the research tasks. She also drew on 'informal proofs that explain', in her teaching to support her successful participation as a teacher where the emphasis in her mathematics education courses and field placement was to enhance learner understanding through explaining and justifying. In Bernstein's (2000) terms, the question to ask would be: In what ways are student-teachers positioned as teachers or as student-teachers. It was interesting in my study to observe whether the way student-teachers engaged with LMT demonstrated their participation as teachers or as students.

The second example, from Peressini et al.'s (2004) study, shows how Mr Hanson's lessons during teaching practice were activity-based in that his learners explored important mathematical ideas either in groups or as a whole class. He also chose a variety of activities to explore the same mathematical concept and learners were allowed to explain their justifications and conclusions to their solutions. Mr Hanson's lessons were successful because his mentor, Ms Largent had created such a discourse in her class. Artigue, Assude, Grugeon, & Lenfant (2001) cited in Kieran (2007, p. 740) argues that "teachers begin to become sensitive to students' learning difficulties during their first year of fieldwork, but that the length of the training period is too short for the trainees to develop the means for analysing these difficulties or for generating remedial strategies".

However, the experience for Mr Hanson was different in his first year of teaching. His teaching orientation was characterised by asking many questions to elicit particular correct answers from learners, and more specifically, finding solutions to tasks using specific sets of procedure - I would classify this approach as traditional. Mr Hanson's change in approach was necessitated by the classroom environment he was found in as well as the socio-cultural context of the school. It did not support the ideal mathematics teacher he was striving to become (someone who understands the mathematics and is able to explain things in multiple ways) and as demonstrated in his teaching practice. Similarly, the MUSTER[10] project indicates that trainees are powerfully influenced by the norms of the schools, in which they work, as well as the working practices and the systems of reward and discouragement they find (Lewin, 2003).

The issue at play in Mr Hanson's experience resonates with two of the three categories of engaging with LMT identified by Even & Tirosh (2002) and further elaborated in Section 3.3. The first one is about developing in learners both relational and instrumental understanding in Skemp (1976) terms, or similarly, developing in learners both conceptual and procedural understanding in Kilpatrick et al. (2001) terms. The second one is about creating a classroom environment where teacher can listen to learners. With reference to the first one, during the training, Mr Hanson had trained as a teacher who would teach for conceptual understanding while during his first year of teaching, he had to adjust and teach for procedural understanding. There was a shift from teaching mathematics for "knowing both what to do and why" to "rules without reasons" (Skemp, 1976, p. 21). This in Skemp's terms suggests that in the former approach Mr Hanson strived for learner understanding while in the later approach, there was no understanding at all. Boaler's (1997) study shows that learners who were taught in ways that they understand the underlying concepts to mathematical problems performed better in their examinations despite not completing the curriculum than those who were taught "rules without reasons" and covered the whole curriculum. Ma (1999) in her

[10] MUSTER stands for Multi-Site Teacher Education Research project, which explored initial teacher education in five countries, namely Ghana, Lesotho, Malawi, South Africa, and Trinidad and Tobago with a view of looking at the characteristics of those selected for training, the curriculum process they experience, the outcomes of training, the reflections of newly trained teachers in schools, analysis of supply and demand for new teachers, and projections of the resources and cost implications of meeting national targets to universalise primary schooling (Lewin, Samuel, and Sayed, 2003).

study also described the Chinese teachers as having conceptual knowledge of elementary mathematics and their counterparts, the US teachers as having procedural knowledge. However, (Brodie, 2004) argues that it is time we started thinking about knowledge as going beyond the conventional distinction between conceptual and procedural knowledge as well as beyond dichotomies between knowledge and practice. Each is mutually constitutive. As my study unfolds, in what ways do teacher-educators and their student-teachers talk about LMT as developing in learners both instrumental and relational ways of knowing and understanding mathematics.

With reference to the second category of what entails engaging with LMT described, Mr Hanson's experience can be linked to what Even & Tirosh (2002) observed about classroom culture and how it influences mathematical activity. They demonstrated how a teacher not taking into consideration learners' understanding as an important aspect of mathematical communication in the classroom but what the teacher has in mind – hence the authority for determining the correctness of answers, can have an effect on learners' learning. Therefore, Even & Tirosh (op cit) argue that there is need to change the traditional roles and responsibilities of teachers and learners in the classroom discourse by making alterations. They also assert that many researchers and mathematics educators have attempted to support such a culture in mathematics classrooms by examining the change in behaviours that characterise classroom culture such as explaining, justifying, arguing and intellectual autonomy. As student-teachers in my study engage with the practice of teaching, how then do they make meaning of learners' understanding? What algebraic representations do student-teachers suggest for helping learners develop the required algebraic thinking? However, I am aware, and in this instance informed by Mr Hanson's experience, that the socio-cultural context of the classroom or school can have an effect on how one chooses to present concepts.

Based on the two examples discussed above depicting how learning to teach mathematics is situative, I argue that Ms Audry Savant's experience can be linked to what Shulman (1986) said about teachers being able to transform the knowledge they learn in teacher education in ways that bring about learner understanding. Mr Hanson's experience may be linked to what Ball et al. (2008) proposed in their study by stating that research was needed to see how their

knowledge domains developed are culturally influenced – in this case the culture of the classroom or school.

Having looked at the three perspectives of teacher learning, what has become evident is that Ball et al.'s (2008) work which is more pragmatic and practice-based is framed by a theory of mathematics on the one hand, and the wisdom of practice on the other. Adler et al.'s (2006, 2007) and Peressini et al.'s (2004) focus is more about that what is learned is a function of where it is learned and the practices within it. What Adler et al. (op cit) then start to do is point to that the pedagogic practices in the different sites of teacher education are not neutral. This informs the gaze that I bring to my study, in Bernstein's (2000) terms, in that the relay is not neutral, hence necessary to focus on the relay and what is relayed with reference to the discourse of engaging with LMT. Neither Peressini et al. (op cit) nor Ball et al. (op cit) talk about the relay, that is, what is actually going on inside teacher education, and that is my concern and I return to this in developing my theoretical gaze. I now focus on the relationship between content and pedagogy.

3.2.6 Looking at the integration of content and pedagogy with a closer eye

While it is well known that many proponents of PCK have focused on the integration between content and pedagogy as teachers and teacher-educators work in classrooms, careful examination of their relationship is equally important. In this regard, issues of what content and pedagogy entail; how they interact on/with/against each other; to provide meaning and experience should also be of focus. Segall's (2004) argument is that pedagogy should not be seen as external to content as many authors have indicated. For example, Adler & Davis (2006) say that there is always the one in relation to the other and depending on what the practice is, the one would be foregrounded while the other is backgrounded and vice versa.

Moreover, Segall (2004) argues that knowledge is already pedagogical in nature in that it is always by someone and for someone, always positioned and positioning. Furthermore, that what teachers "pedagogize" is already pedagogically pre-inscribed. To elaborate: Segall (op cit) states that texts brought into the classroom are of themselves pedagogical invitations for learning and should not be considered as finished works of content awaiting pedagogical transformation. In other words, content always has pedagogy in its text form and pedagogy always has content, and what the teacher does is to add to what already exists. Therefore, the

teacher and the text are considered pedagogical devices in that what a teacher says and does or what or how a text utters are both invitations to inquiry. Segall (op cit) asserts that it is thus important, in exploring a text, that aspects of 'what it says', 'how it says it', 'for what that saying does' i.e. how it invites readers to know, think and imagine, should be taken into consideration. This eventually influences the reader's production of meaning. In referring to Masterman (1985), Segall (2004, p. 497) states that:

> *Texts offer readers positions from which they are invited to see experience (and to see and experience) in particular ways.*

In explaining the relationship that exists between content and pedagogy, Segall (2004) states that teacher education should continue making content instructional as well as include ways of examining how content is already instructional and instructing. Thus prospective teachers should not only be taught how to manage ideas and theories in the classroom, as pointed out by Shulman (1987) but should also explore how the use of these ideas and theories in classrooms shapes those who attempt to engage them (Segall, 2004). With this view, Segall has questioned the rationale of having prospective teachers acquire content area knowledge in departments specialised in that discipline while receiving courses on pedagogy in the department of education. Neither should the argument be based on how well they perform in the subject courses, but on how pedagogy relates to content.

From this debate, it is clear that Segall (2004) is pointing to the notion that mathematics for teaching is different from the mathematics the specialised mathematician has to learn because the contexts in which this mathematics is going to be used are different. On the one hand teachers use the mathematics in the classroom for the purposes of enabling others to learn mathematics. Mathematicians/engineers on the other hand, use it in industry, or others use it in different work places, including the academy where purposes are to build the knowledge of mathematics itself. This suggests that mathematics requires different foci on how it is construed.

From the aforesaid, something important to be reflexive about for my study is on how student-teachers' responded to interview questions based on the scenarios designed around selected common learner errors in school algebra, and described and explained in Chapters 8 and 9 of this thesis. The reflection also includes the relationship between how distinct the

tasks in form of scenarios are. I raise the following questions to guide the reflection process: How do the tasks that student-teachers engage with in this study influence their responses? I ask this because the tasks in which ever form they are presented, communicate to student-teachers in particular ways and hence would influence their responses. Moreover, the questions I ask or the instructions I give to student-teachers in the process of engaging with tasks could also have influence on their responses. In short, the tasks and I as a researcher are pedagogical devices inviting student-teachers to engage in this study in particular ways.

In Bernstein's (2000) terms, this reminds us that the relay is never neutral - the thing we are doing in of itself is never a neutral carrier, it carries a whole range of things. For example, Adler & Patahuddin (2012) problematise the boundary between content and pedagogy. In using a selection of Hill et al.'s (2007) test items, meant to measure teachers' SCK, in the interview setting to read teachers' professional knowledge, Adler & Patahuddin (2012) found that the teachers responded differently. For example, a task that was meant to measure teachers' SMK in the Learning Mathematics for Teaching project resulted in establishing teachers' PCK when used in the interview context. The teachers justified their responses and ideas by reference to the learner and/or instruction – they did not provide mathematics as rationale in their responses. Therefore, Adler & Patahuddin (op cit) argue that teacher knowledge in the test is a function of the test items, and similarly, teacher knowledge in the interview is a function of the interview. This suggests that teacher knowledge in whatever context, including the test or interview, cannot be referred to as their fixed knowledge but that it is dependent on what the teachers do in relation to the test or the interview, hence leading to a situative perspective. For my study, 'what' and 'how' in terms of teacher knowledge about the discourse of engaging with LMT is dependent on the teacher-educators' and student-teachers' talk in relation to the context of varied interview settings they are involved in.

3.2.7 Perspectives on the literature on MfT and the gap

Where is mathematics for teaching?

Some researchers such as Ball et al. (2008) argue that you can only see mathematics for teaching in schools because it is for teachers' use while they are teaching. Adler & Davis (2006) have argued that actually teacher education has to be offering this kind of

mathematical knowledge. Therefore, what is constituted as MfT in teacher education is what those institutions and those sets of practices think MfT is. That is not to say teachers use that in the practice but it is what is being done there. Most of the work has actually focused on how to study teaching. To see what teachers actually use and so what is the profession with regards to teaching. My interest is what and how we prepare teachers for that. What many researchers offer are implications of their research for teacher education. I am starting by stating that whether teacher education does anything like that or not, it assumes it is learned - that is my argument. The issue here is not to see whether mathematics teacher education does or does not, but in what ways does it provide opportunities for student-teachers to engage with LMT. Thus, what does it do and what does it not do, hence the contribution of my study to the gap identified.

Can one learn mathematics for teaching in practice?

On the basis of the research reviewed in this Section 3.2, I would answer yes and no to the question above; yes depending on the practices the school supports; and no if the teacher has not experienced some aspects of MfT in teacher education, they might not put them into practice and therefore will not learn from it. Krauss et al. (2008) argues that what does matter is what is learned in teacher preparation, and prior to entering the field of practice by setting up conditions for learning it. This prior learning provides the conditions to learn in practice. Depending on what the conditions were, in Ma's (1999) terms they might have to learn in practice by deliberate study of practice. However, Artigue et al. (2001) argues that conditions for learning MfT might not be provided in teacher education due to time constraints despite student-teachers being aware of, for example, learner errors as experienced during their school experience. If learning is situated, then what one learns is a function of where they are and the practices that are supported.

How is the student-teachers' participation in this study going to impact on what they know and are able to do about LMT?

What I would like to see in this exploration is what student-teachers' ideas of what LMT are, therefore, testing the assumption that this is learned. What is it that student-teachers know and are aware of and about LMT through their experience of mathematics teacher education? I nevertheless hypothesize that as I give the student-teachers the tasks to do, they will become

aware of the issue of LMT and this would impact on how they talk about it. Therefore, I am interested in describing and explaining the 'what' and 'how' of what teacher-educators say they privilege and their student-teachers' thinking as far as LMT is concerned.

3.3 Learner errors and teachers' engagement[11]

LMT and teachers' engagement with this has been widely researched in the field of mathematics education. Some focus on the nature of errors (Borasi, 1987; Nesher, 1987; Olivier, 1989; Ryan & Williams, 2007; Smith, et al., 1993) and others focus on strategies for dealing with these (Jacobs, Lamb, & Philipp, 2010; Peng & Luo, 2009). Even & Tirosh (2002) focus more directly on what it means for a teacher to be aware of and knowledgeable about LMT, asserting that this impacts positively on the practice of teaching. They talk about the nature and strategies of learner thinking, and go further to conceptualise the kind of environment that is required if teachers are to access LMT.

Even & Tirosh (2002, p. 229) encourage a classroom culture where learners are to "make conjectures, explain their reasoning, validate their assertions, discuss and question their own thinking and the thinking of others, and argue about what is mathematically true". More specifically they argue that a teacher's awareness of learner thinking is multi-dimensional. As shown in Table 2, it extends from being aware of learner conceptions, to awareness of developing in the learner both instrumental and relational understanding, and to awareness of a classroom culture that supports such. They assert that while these dimensions are analytically distinct, the one always involves the other. As the study shows and is elaborated through the thesis, mathematics education teacher-educators' and student-teachers' discourses of LMT align with these three categories in one way or another.

Table 2: A framework of what entails LMT

Focus on learner errors
Developing in learners both instrumental and relational understanding
Creating a classroom environment where teacher can listen to learners

[11] First published in the Association for Mathematics Education in South Africa (AMESA) proceedings, 209 – 222, (2011).

3.3.1 Learner errors and PCK

That dealing with learner thinking is an important part of teachers' professional knowledge has long been recognised. Shulman's (1986, p. 9) seminal paper put this into the foreground as part of PCK and emphasized teachers having "an understanding of what makes the learning of specific topics easy or difficult; the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons." Shulman (op cit) states that in the event that these conceptions and preconceptions turn out to be misconceptions, teachers should be knowledgeable of the strategies that would help in reorganizing the understanding of learners.

This is in line with teacher education current practices, be it in-service or pre-service, in that they aim at providing opportunities for teachers to develop deeper understanding of the mathematics they are to teach as well as enhancing their understanding of learners' mathematical thinking (Even & Tirosh, 2002). Therefore, developing in a teacher the blend of mathematical and pedagogical knowledge, which includes knowledge of mathematics and of learner's conceptions, complemented by expertise in making connections between mathematics, models and tasks used in teaching (Ryan & Williams, 2007). In view of this Ryan & Williams (op cit) assert that subject matter growth in learners is inevitable if their teacher shares with them the question "why it is so?" while teacher's development of PCK is likely to develop if they address the question "How can I best introduce and justify this mathematics?"

Ryan & Williams (2007) then argue that teacher education is vital since it transforms SMK to PCK. For example, the development of enriched PCK among teachers or pre-service teachers is possible if they investigate their own mathematical errors, misconceptions and strategies in order to reorganise their own SMK and encourage teacher reflection. This example is based on the assumption that teachers have the potential to transmit errors and misconceptions to their learners and so necessary that they are exposed and dealt with (Ryan and Williams, 2007). Hence the crux of this study and its focus on what gets constructed as the discourse of engaging with LMT in the mathematics teacher education curriculum, and how do the student-teachers use the discourse in tasks designed around learner errors in school algebra. It is interesting that despite a great deal of research following Shulman's (1986) work little has focused on the actual engagement of this at the pre-service level.

3.3.2 Distinguishing errors, errors and theory

As stated in Section 1.1, most of the work on learner errors has been from a constructivist/cognitive perspective [for example, Smith, diSessa, & Roschelle (1993), Nesher (1987), Borasi (1987), Olivier (1989)]. More recent, focus on learner errors has been from socio-constructivist/sociocultural perspective [Peng & Luo (2009), Ryan & Williams (2007)]. Constructivist notions of error are in terms of cognition or learners' conceptions while sociocultural approach towards learner errors is situative as focus is on cognition as well as where the learning occurs. I am bringing in Bernstein's (2000) social theory to bear on this because I am concerned with the relay and the relayed pertaining to the discourse of engaging with LMT in teacher education with focus on what teacher-educators and their student-teachers say. In the field of mathematics education, research has distinguished between slips and errors. Errors are systematic. For constructivists, the sources of errors are misconceptions. I begin with a discussion on slips, and then discuss errors from a constructivist perspective.

Slips are wrong answers that are sporadically or carelessly made by both experts and novices; they are easily detected and are spontaneously corrected Olivier (1989, p. 3). Ryan and Williams (2007) talk of 'slips' or 'uncertain diagnosis' in that they are mistakes with no obvious developmental conceptual explanation. They argue that slips can be a result of misreading, misremembering facts, suffering from 'cognitive overload' or jumping to conclusions. I do not focus on slips in the rest of this study.

3.3.3 Errors from a constructivist/sociocultural perspective

Constructivists view errors as systematic wrong answers because they are applied regularly in the same circumstances and are symptoms of the underlying conceptual structures (Olivier, 1989, p. 3). This means that misconceptions form part of learner's conceptual structures which interact with new concepts, and influence new learning, mostly in a negative way: misconceptions generate errors (Olivier, 1989, p. 3). Moreover, from a sociocultural perspective, errors are described as superficial behavioural results of actions performed on a task and that correcting the error disregarding reason for the error committed is detrimental to a leaner's intellectual mathematical development (Ryan & Williams, 2007). In Ryan and Williams' (2007) terms systematic errors result from 'conceptual limitations' or 'identifiable misconceptions' and that diagnosis of such errors are linked to modelling, prototype,

overgeneralization, and process-object reification. The issue of cognition in terms of learner conceptions is implicated in how errors are described from a constructivist and sociocultural perspective.

Modelling error occurs when learners model a situation contrary to the rules of the mathematical game in the academic context of school. An error is diagnostic of prototypical thinking if it is as a result of a culturally 'typical example' of the concept. Error as a result of overgeneralization arises when generalisations that make sense to a set of cases are inappropriately extended (Ryan & Williams, 2007). The extension can either be forward or backwards. This can be as a result of what Lima and Tall (2008, p. 6) term "met-before" and "met-after" in that old experiences can affect new learning and similarly new learning can affect the remembering of previous knowledge, respectively (Lima & Tall, 2008). An example of met-before would be learners' experience in arithmetic affecting their conceptions of algebra, that is, in arithmetic learners have learned that $8 + 9 = 17$, and when it comes to algebra, they do the following $5x + 3 = 8x$ when asked to simplify. An example of met-after would be the learners' experiences in algebra such as $5x^4 \times 3x^2 = 15x^6$ can be misapplied in arithmetic in that if asked to simplify $5^4 \times 3^2$, they will give 15^6.

When learners fail to navigate between process and object in the learning of mathematics then the error is as a result of lack of completion of the process-object reification (Ryan & Williams, 2007, p. 25). This suggests learners' inability to have an operational as well as structural conception of the mathematical tasks (Sfard & Linchevski, 1994). For my study, overgeneralization and process-object conception emerged as the two possible dominant explanations for sources of learner errors in school algebra from the analysis of each of the six scenarios prior to analysing student-teachers' talk. The question for my study then is how these explanations for sources of learner errors manifest in student-teachers' engagement with tasks.

In Table 3, I exemplify errors from a constructivist/sociocultural perspective.

Table 3: Errors and possible social constructivist/sociocultural descriptions

Error	Possible constructivists description
Modelling error: $6 \div \frac{1}{2} = 3$ or "division makes smaller"	"Division makes smaller" is only appropriate for whole numbers where the numerator is bigger than the denominator. This could also serve as an overgeneralization.
Prototypical error: Failing to identify that a square or a rectangle with different orientations is a rectangle	As a result of a culturally 'typical example' of the concept in that a rectangle lies flat with its longest side horizontal.
Error as a result of overgeneralisation: 'Multiplication always makes bigger'	It is appropriate for whole numbers and it becomes an overgeneralization when applied to all rational numbers, thus including proper fractions or decimal fractions between 0 and 1.
Inadequate 'process' conception of the equal sign	A learner saying 548 is the answer in the problem $\square - 1452 = 2000$

It is argued that in order to understand misconceptions, you have to appeal to a theory of learning (Olivier, 1989; Sfard, 2008; Peng & Luo, 2009) and that not doing so renders the task of error analysis ineffective (Peng & Luo, 2009). Error analysis is regarded as an important aspect of teaching mathematics although viewed as challenging for teachers. Constructivists and Social constructivists view prior knowledge as the primary source for acquiring new knowledge, hence viewing error as a natural stage in knowledge construction (Hatano, 1996; Nesher, 1987; Peng & Luo, 2009; Smith, et al., 1993). For example, when confronted with new knowledge, learners' prior knowledge can constrain a range of possible operations and answers as well as their understanding of mathematical entities, such as how to formulate a given problem mathematically (Hatano, 1996, p. 209). This process results in learners creating misconceptions which from the constructivist perspective are viewed as part of learning. Prior knowledge, though constraining, plays a foundational role in developing understanding (Smith, et al., 1993).

This view is unlike earlier research where errors and misconceptions, from a behaviourist perspective, were rendered redundant in that learners' prior knowledge was not considered relevant to learning as cited in Olivier (1989). Errors and misconceptions were viewed as

though something has gone wrong in the computer's memory, that is – *if we don't like what is there, it can simply be erased or written over, by telling the pupil the correct view of the matter* (Strike, 1983). To be more elaborate, Gagne (1983 p. 15) states:

> *The effects of incorrect rules of computation, as exhibited in faulty performance, can most readily be overcome by deliberate teaching of correct rules ... This means that teachers would best ignore the incorrect performances and set about as directly as possible the rules for correct ones.*

The indication here is that the role prior knowledge plays in expert reasoning, in that it gets refined and reused (Smith et al., 1993) is overlooked. Moreover, Smith et al. (1993) would argue that earlier views of errors and misconceptions as interfering with learning (they must be replaced by expert concepts; and that they resist instruction) is limiting if viewed from the constructivist perspective. Here misconceptions are viewed as faulty extensions of productive prior knowledge in that conceptions that result in erroneous conclusions in one context can be quite useful in others. For example, "multiplication makes bigger" is useful when dealing with natural numbers but this might not always hold for some rational or real numbers. This suggests that "*... commonly reported misconceptions represent knowledge that is functional but has been extended beyond its productive range of application*" (Smith et al. 1993, p. 152).

Therefore, misconceptions could arise from learners' overgeneralization of mathematical knowledge from one domain to another, for example, overgeneralization of number and number properties seems to be the major cause of learner misconceptions (Olivier, 1989). Interestingly, misconceptions can result in correct answers, and hence can be very difficult to identify (Nesher, 1987). For example, the conception that "the more digits in the number, the bigger it is" will yield a correct answer in the case of 0,567 being bigger than 0,45 but not necessarily for 0, 567 and 0,67, and yet the mathematical grounds on which it is made are faulty.

Borasi (1987) argues that seeing errors as a sign that something has gone wrong in the learning process and therefore it needs remediation is quite limiting. He acknowledges that researchers who have worked with errors as a powerful tool to diagnose learning difficulties and consequently direct remediation have made a valuable contribution to mathematics education. They have brought to the fore an understanding that learners are different, that they experience difficulties in learning mathematics, and that the strategy of simply

explaining the same topic over again or giving learners additional practice exercises does not remove the errors. In Arcavi's (1995, p. 149) terms, the issue of giving learners additional practice exercises suggests "the harmful advocacy of the drill-and-practice rationale of learning mathematics where the concern is: Let's take care of the drill, meanings will emerge from practice". Doerr & Wood (2004, p. 175) argue that student-teachers, from their own experience as learners, come to the initial teacher training with the understanding that 'doing maths' involves application of discrete rules "best learned through repeated practice", for example, solving forty mathematical problems of similar focus. This in Doerr (2004) terms could be one of the "dilemma of experience" teacher-educators have to grapple with.

However, the issue of giving more practice exercises as a remediating strategy is contested in mathematics education literature. For example, Arcavi (1995, p. 149) agrees with Sfard that thinking at a structural level is not so straight forward, it requires "a great deal of doing, of practicing, of tolerating and living with partial understandings". He contends that if practicing is going to be foregrounded, then the question that needs addressing other than just focusing on practicing the solution process to a number of similar mathematical problems is: 'What kinds of practices, environments, and tasks and projects should be made available to learners if they are to develop this understanding?' Moreover, these practices should take into consideration learners' "engagement, perception of usefulness, and construction of meaning (even when partial), and their sense of ownership of the learning process" (Arcavi, 1995, p. 149).

Borasi (1987, p. 2) therefore proposes viewing errors as "*springboards for inquiry*", and this is elaborated in Section 3.3.5. Early research has also shown the relevance of focusing on error in that among the early mathematicians, failure to experience success on a particular goal resulted in unexpected and revolutionary outcomes, for instance the role error has played in the history of mathematics (Borasi, 1987). She argues that mathematicians in their search for mathematical knowledge make use of their errors in that wrong conjectures, unjustified guesses, and partial results are all important steps in the creation of new mathematical results, hence growth of the discipline. Therefore, the experience of error as motivational and a means of inquiry in mathematics should not only be felt by the mathematicians but also mathematics learners at their own levels of education. This means that errors play an important role in restructuring a learner's conceptual understanding.

The explanation that can be given to misconceptions that are persistent and resistant to change is that their experiential foundations are broad and deep rooted (Smith et al., 1993). Smith et al. (op cit) argue that misconceptions are not always resistant to change in that if interventions are designed appropriately, they can yield rapid and deep conceptual change within the short space of time. Others are resistant because they lack conceivable alternatives; and others because they are part of conceptual systems that contain useful elements whose breadth and utility are not immediately apparent. Thus understanding the strength of a particular conception depends on the characterization of the knowledge systems that embed that element.

Moreover, Smith et al. (1993) assert that viewing misconceptions as something needing replacement is not the norm in the constructivist perspective because learning processes are much more complex than what replacement suggests. They state that knowledge is reused in new contexts, and is refined into more productive forms, and in such a process, you see the reappearance of misconceptions that were thought to have been resolved. This has implications for instruction in that it should provide the experiential basis for complex and gradual processes of conceptual change. As discussed, some of the research is constructivist, and more recent research is social constructivist and sociocultural – and thus indicating the significance of error in mathematics teaching and learning, and that this is part of the learning process.

3.3.4 Strategies for carrying out error analysis

As identifying, interpreting, evaluating and remediating learner errors are important (Peng & Luo, 2009), this is viewed from a theory that is not behaviourist but from the theory that sees errors as socially and cognitively produced. Identifying entails "knowing the existence of mathematical error"; interpret entails "interpreting the underlying rationality of mathematical error"; evaluate entails "evaluating students' levels of performance according to the mathematical error"; and remediate entails "presenting teaching strategy to eliminate mathematical error" (Peng & Luo, 2009, p. 23). This part of the framework was used by the authors to establish mathematics teacher knowledge as used in error analysis. In the first phase of Peng & Luo's (op cit) study, the teachers analysed learners' errors and then the teachers' analysis was re-analysed by the researchers. They then mapped the re-analysis onto their framework.

Similarly, the strategy for carrying out error analysis is described by Jacobs et al. (2010, p. 169) using the construct "*professional noticing of children's mathematical thinking*" as an expertise involving three skills. The first skill is *attending to learners' strategies*. In this skill the focus is on the mathematical detail in learners' strategies as this provides a window into learners' understanding. Question to ask: *Please describe in detail what you think each child did in response to this problem?* For example, a mathematical activity that is modeled by $19 - 7 = \blacksquare$; and learners are asked to work out the problem in any way that makes sense to them. So the mathematical detail in this example would focus on how learners counted rather than the postures they took while carying out the activity.

The second skill is *interpreting children's understanding*. In this skill the focus is on teachers' interpretation of learners' understanding as reflected in their strategies. Attention is given to the extent to which the teachers' reasoning is consistent with the details of the learner's strategies and the research on learners' mathematical development. Question to ask: *Please explain what you learned about these children's understanding.* The third skill is *deciding how to respond on the basis of children's understanding*. In this skill the focus is on the reasoning that teachers use when deciding how to respond. That is, the extent to which teachers use what they have learned about the learners' understandings from specific situations, and whether their reasoning is consistent with the research on learners' mathematical development. Question to ask: *Pretend that you are the teacher in this classroom; what problem or questions might you pose next.* Moreover, Jacobs et al. (2010, p. 173) argue that these three skills "happen in the background, almost at the same time, as if constituting a single integrated teaching move, before a teacher responds".

I therefore argue that Jacobs et al. (2010) first skill resonates with Peng & Luo (2009) first step which is identify. The second skill with interpret/evaluate, and the third skill with remediate. In Table 4, I provide the main steps for carrying out error analysis so far described and how they inform the framework for carrying out error analysis for my study.

Table 4: A framework for carrying out error analysis[12]

Peng & Luo's (2009) strategy	**Jacobs et al.'s (2010) strategy**	**Strategy for my study**
Identify	Attending to learners' strategies	Identify
Interpret	Interpreting learners' understanding	Explain
Evaluate		
Remediate	Deciding how to respond on the basis of learners' understanding	Remediate

As already argued in Section 3.2.3, the skill of identifying and explaining (involving interpret/evaluate) in carrying out error analysis resonates with one of Ball et al. (2008) knowledge domain, which is KCS. For my study, Peng and Luo (2009) and Jacobs et al. (2010) strategies for carrying out error analysis formed part of the framework with which to interrogate teacher-educators' and student-teachers' discourses of LMT in general, and in particular scenarios designed around common learner errors in school algebra. Moreover, ways in which they referred to the nature of errors and misconceptions were also explored. As pointed out in Sections 3.2.5 and 3.2.7, this is despite the argument that although student-teachers become aware of learner errors during their school experience, there is not enough time in teacher education for them to develop the required skills to carry out error analysis (Artigue et al. 2001 cited in Kieran 2007). In the midst of this argument, as already indicated, since dealing with LMT is a presumed outcome in teacher education, my interest is to explore and explain opportunities that are made available to student-teachers pertaining to the discourse of engaging with LMT at pre-service level. This is irrespective of the argument about the length of time spent in teacher education to expect that student-teachers would develop the skill of carrying out error analysis.

[12] Strategies in the first two rows are identical while the ones in the third row are very different in that "remediation" is a much narrower strategy than "responding" which is quite a broader strategy.

3.3.5 Importance of carrying out error analysis by the teacher

The strategies for carrying out error analysis suggest the importance for the teacher to understand what learners' alternative strategies in their worked out solutions are as well as why they work out the mathematical problems in the way they do. This can be achieved by probing the causes of the mistake made, and that this mistake could provide valuable information about the specific difficulties and conceptions in learning of particular concepts (Borasi, 1987). For the teacher to address the errors made by learners, Borasi (op cit) suggests three strategies. The first one is, the teacher being able to hypothesize and verify the possible causes of the error, and establishing the relevant ones for each learner making such a mistake.

The second one is, diagnosing a particular error so that teachers become aware of the difficulties learners are likely to encounter as they study a particular topic. As a result, teachers may prepare the teaching of such a topic in ways that minimize the identified difficulties, hence preventing future occurrence of similar error with other learners. The third one, which resonates with viewing errors as "*springboards for inquiry"*, is involving learners themselves experience the process of error analysis. This is done by asking learners exploratory questions that can stimulate learners' erroneous strategies, including questioning whether there were cases or contexts where their strategies would be considered correct. Therefore, indicating that errors, other than being diagnostic and remediating, also illuminate strengths and weaknesses of available strategies, for example, $\frac{2x}{3x}+\frac{x}{3x}$ can easily be simplified by adding the numerators and dividing by $3x$ but applying the same strategy to $\frac{2x}{3x}+\frac{x}{5x}$ would be problematic.

I would therefore argue that for the teacher to be able to hypothesize and verify the possible causes of error, diagnosing a particular error to prevent future occurrences, and involving learners themselves experience the process of error analysis requires KCS. Moreover, the first and second strategies illuminate error being viewed as diagnostic and remediating, and only the teacher and researcher are involved. For Borasi (1987), the first and second strategies are limiting compared to the third one in that they do not exhaust the educational potential of the error in discussion. They deprive the learners themselves the opportunity of being involved in explaining and fixing up their own errors – an activity that could prove to be highly

motivating and challenging. In these two contexts (first and second strategies), error is also viewed as a deviation from what is accepted by the conventional mathematicians, and not thought of as a possible challenge to the standard results as discussed in the third strategy.

The role of the teacher is therefore central in helping learners construct new knowledge by being aware of previous knowledge. As the teacher verbalises the target knowledge or when ensuring that learners' understanding approximates expert understanding the learner must partially construct knowledge in the teaching and learning process (Hatano, 1996). This transmitted knowledge becomes useful to problem-solving only when it is reconstructed, that is interpreted, enriched and connected to prior knowledge of the learner. As elaborated by Borasi (1987) and Hatano (op cit), for my study, I am concerned with ways in which teacher-educators' and student-teachers' talk refer to these ways of knowing how to deal with learner errors.

From the review of literature on error, I would like to state that constructivism as a theory of learning has dominated research on error in recent years. As I have already indicated in sections 3.3.2 and 3.3.3, my use of error in this study is not as a result of slips but misconceptions. Therefore, I use error or misconceptions interchangeably with greater emphasis on error[13]. I have talked about errors and misconceptions in general and from a constructivist and sociocultural perspectives, which are not in conflict in terms of cognition or learner conceptions (see for example Sections 3.3.4 and 3.3.5). They also agree that errors are not slips and that they are not easily or simply corrected. They would differ perhaps on how they describe the sources of error and then perhaps too on what strategies could be for remediation. Moreover, while some studies from a constructivist perspective do not situate themselves, sociocultural studies, in general, do situate themselves. I have been saying all along that in teacher education, the relay – the inside of teacher education is very important including how learner errors are dealt with. Most of the work on errors assumes the relay, it does not problematise the relay, and hence my interest in situating my study in teacher education. In my study, I also go with Ryan & Williams (2007) categorization of error, which is sociocultural in its discourse. I now discuss algebraic thinking and the specificity of error.

[13] Other examples of how literature would refer to learner errors and misconceptions include 'learner conceptions' or 'alternate conceptions' or 'partial understandings'.

3.4 Literature related to school algebraic thinking

As indicated earlier, school algebra is also used to engage with the problem for my study. The choice of algebra is based on the vast body of knowledge relating to learners' difficulties, varying curricular across the world and technological tools. Irrespective of this, the problem of algebra among learners has persisted, and the preparation and support of teachers in and for this particular central part of their work is out of focus. Bernstein (2000) enables us to understand that knowledge production occurs across a number of fields. He distinguishes between the field of production and the field of recontextualization, and this is further elaborated in Chapter 4. Algebra in his terms is talked about in mathematics, the field of production of mathematics; and in the school, the field of recontextualization of mathematics for school purposes. I exemplify how algebra is defined in these two fields.

3.4.1 The definition of Algebra

Algebra has been in existence for a long time and it took many centuries to be developed; that is, as far back as the Babylonian times. Some psychological and pedagogical roots and some difficulties learners encounter are evident in the historical record. Writing from a mathematician's perspective - the field of production of mathematics, Derbyshire (2006), for example, defines algebra as a part of advanced mathematics that is not calculus. He argues that functions, and sequences and series belong to the aspect of mathematics called analysis. In higher levels of mathematics, algebra is considered to be a discipline in itself. Algebra developed from using of numbers without a referent to use of letter symbols to represent unknown numbers, to use of letter symbols when carrying out arithmetic (such as use of arithmetic laws of addition, subtraction and multiplication) and solving equations. When these letter symbols started detaching themselves from numbers, new mathematical objects such as groups, matrices, manifolds, etc. were discovered.

Mason, Graham, & Johnson-Wilder (2005), also by way of example, view algebra from the school algebra perspective – the field of recontextualization of mathematics for school purposes, and how it enhances understanding of further algebra. On one hand, they define algebra as an extension of arithmetic that organises counting, i.e. finding a structured way to get results through counting, structuring and organising. They further assert that algebra is generalised arithmetic, thus doing arithmetic with letters or algebra as the expression of the rules of arithmetic (associative, commutative, and distributive). On the other hand, they view

algebra as a specialised language for expressing generality. For example, $2n+1$ is a structure of odd numbers for every integer.

Mason et al. (2005) argue that algebraic thinking is more likely to develop in learners if they are given opportunity to work from generalisations to specifics and vice versa, that is, making sense of specifics in the generalisations and making sense of generalisations in the specifics. They identify four major roots of algebra, namely, expressing generality; multiple expressions; freedom and constraint; and experiencing structure, leading to generalised arithmetic. They argue that these four strands could inform every mathematics lesson. They further argue that algebra is not only limited to use of letters in place of numbers, solution of simultaneous linear equations and factoring of quadratic expressions, etc. but it is also implied in more sophisticated mathematical structures, which in turn enable one to solve more complex problems.

Although algebra has been viewed from two perspectives as exemplified above, one thing is coming through. They have viewed algebra as generalised arithmetic explicitly and implicitly. Algebra has been recognised as having its genesis in arithmetic, way through up to the time when letter symbols start detaching from numbers and arriving at higher levels of abstraction. For my study, the concern is school algebra, and more specifically student-teachers' thinking of learners' sense making of algebraic expressions, hence operating in the field of recontextualization of mathematics for school purposes.

3.4.2 Constitution of the school algebra curriculum

Algebra is part of the curriculum in secondary education internationally. The questions that drive the formation or discussion of a curriculum addresses the *why* and the *what* of the subject. Therefore, why learn algebra? And what should constitute school algebra? MacGregor (2004) argues that learning algebra is important in that it is a necessary general knowledge for an educated and democratic society; it is a gateway to further study of mathematics, certain higher education courses, and many fields of employment; it is a nation's technological future and economic progress; and for problem solving and training reasoning. However, MacGregor argues that justifications for these reasons are complex and not self-evident if algebra is viewed from outside.

In referring to Crawford (2001), MacGregor (2004, p. 314) states that competence in algebra is identified by the following abilities: to think in symbolic language, to understand algebra as generalised arithmetic as well as the study of mathematical structures; to understand equality and equations of algebra and to apply these within real world problem solving settings; to understand relationships of quantities through patterns, defining functions, and applying mathematical modelling; and to work with graphs of functions and relations on the XOY - plane. These competencies resonate with what Watson (2009) describes algebra to be at school level.

Vermeulen (2007) has also outlined requirements of an algebra curriculum, Vermeulen (op cit) outlines three main requirements which an algebra curriculum should attain. Firstly, it should enable learners to experience algebra as generalised arithmetic. Secondly, it should enable learners use algebra as a means of describing relationships between variables. Thirdly, it should enable learners use algebra as a tool to solve problems. In summary, a well designed algebra curriculum should aim at strengthening learners' arithmetic problem solving methods as well as providing opportunities for the growth of algebraic thinking (MacGregor, 2004). Although there are differences and similarities in selections and foci in terms of what constitutes school algebra curriculum, Kieran (2004) has developed a model of the core of school algebra by focusing on its main activities. She identifies three principal activities of school algebra, namely, *generational*, *transformational*, and *global/meta-level* activities. These relate to Kilpatrick et al.'s (2001) activities of school algebra, which are *representational*, *transformational*, and *generalizing and justifying*.

Kieran (2004, p. 22) describes *generational activities* of algebra to include "the forming of the expressions and equations that are the objects of algebra". For example, equations containing an unknown that represent quantitative problem situations; expressions of generality arising from geometric patterns or numerical sequences; and expressions of the rules governing numerical relationships. Therefore, the main aim of *generational activities* is to represent or interpret situations, properties, patterns, and relations using expressions or equations. The *transformational activities*, sometimes referred to as *rule-based activities*, include "collecting like terms, factoring, expanding, substituting, adding and multiplying polynomial expressions, exponentiation with polynomials, solving equations, simplifying expressions, working with equivalent expressions and equations" (Kieran, 2004, p. 24).

Lastly, *global/meta-level mathematical activities* in Kieran (2004) terms include the use of algebra as a tool in situations which are not exclusively algebra.

I therefore argue that Vermeulen's (2007) requirements of an algebra curriculum, although specific examples have not been given, fit in Kieran's (2004) models of the core of the algebra curriculum. For example, the first description could be linked to transformational activities, the second to generational activities, and the third one to global/meta-level activities. Moreover, while Brown & Drouhard (2004) recommend Kieran's model in that it prevents us from putting emphasis only on the transformational aspects of algebraic thinking, they argue that global/meta-level activities can be used in an interactive way so as to generate transformational and generational activities. They further argue that if transformational activities are developed from contexts full of meaning, it will support the development of new theory.

To show cause for their argument, Brown and Drouhard (2004) report on a study that was conducted in secondary classrooms in the UK, using examples of practice with 11 to 12 years old learners and their teacher Alf Cole who happened to be a co-researcher and a co-teacher. Learners were given contexts which encouraged them to seek reasons for why something worked without explicitly teaching them the transformational techniques. Alf suggested that the contexts provided to the learners could be explored further using algebra as a tool. To enable the learners experience the power of algebra for explaining and justifying, Alf demonstrated the use of algebraic thinking and manipulation. The results showed that the learners developed interest in using transformational activities within their mathematics. This was done in ways that justifying and proving became part of the classroom culture – hence the need to re-stress transformational algebraic activity within the mathematics curriculum.

To have a cross-country view of what constitutes algebra in the school curriculum and the variations that occur, Kendal & Stacey (2004) conducted a comparative study. They established a criterion for their evaluation which included differences in algebraic activities with focus on the degree of formalism, the nature of problems, the relation to the real world, and the use of technology. Samples of secondary school national state examination questions and a selection of primary and secondary school textbooks were also used in the comparison. On one hand, Kendal & Stacey (op cit) found that in settings where learners were of mixed

ability, generational activities were emphasised as they tended to give meaning to the building blocks of algebra. On the other hand, in streamed settings, higher ability classes emphasised on symbolic transformational aspects of algebra while lower ability classes had restricted goals for algebra such that in some circumstances, they never did any algebra at all – an indication that the decisions that are made about the structure of schooling impacts on the teaching of algebra. Therefore, Kendal & Stacey (op cit) argue that what one considers to be school algebra depends on cultural as well as other factors that vary widely across and even within communities.

As you will see in Chapter 5 and the scenarios designed around learner algebraic thinking, the type of description I assign in terms of activities for school algebra for five of the scenarios is that they are transformational and only one is generational. I focus my scenarios more on transformational activity because most research on learner errors in the field of mathematics education relate to transformational activity. This is not to suggest that the scenarios are exhaustive. Moreover, as already argued in Chapter 2, the Zambian school algebra curriculum mostly focuses on the transformational activity. In Zambia, more emphasis on the transformational activity is meant for all learners irrespective of their levels of learning abilities, and hence contradicting Kendal & Stacey's (2004) findings. I now discuss problems experienced by learners in the learning of algebra, as documented in the field of mathematics education research.

3.4.3 Learner errors and difficulties in algebra

The field of research in algebra is wide. Usiskin (2004, p. 150) argues that "algebra needs to turn on students rather than turn them off". To this effect, literature has suggested ways in which teachers could assist learners make sense of school algebra and eventually develop conceptual understanding. These strategies include introducing algebra in the early years of schooling, use of representations, use of technological tools, and focus on relationships between quantities before number is introduced (Kendal & Stacey, 2004; Kieran, 2004; Kilpatrick, et al., 2001; Usiskin, 2004; Watson, 2009). These are not dominant parts of the school algebra curriculum in Zambia, and they are not in focus here. Although with regard to representations, some algebraic problems are evident in the textbooks in form of word problems which require translation into symbolic expressions before manipulation.

Of interest for my study are the errors and difficulties learners experience in the learning of school algebra. Many researchers have attributed the difficulties that learners experience in learning school algebra to the teaching approaches used in that they are more traditional rather than activity based (Kilpatrick, et al., 2001; MacGregor, 2004; Usiskin, 2004; Watson, 2009). They argue that learners are required to memorise facts, rules and techniques and apply them in different algebraic tasks in a routine manner without developing underpinning understanding of why they had to carry out the manipulations in the way they did. They further argue that many textbooks have focused on rules to be followed in manipulating symbolic expressions and equations rather than on the concepts that support those rules or give meaning to the expressions or equations being manipulated. This suggests, in Skemp's (1976) terms, teaching for instrumental and not relational understanding. For example, Usiskin (2004, p. 150) states:

> *Instead of being taught as a living language with a logical structure and many connections between its topics and other subjects, algebra is taught as a dead language with a myriad of rules that seem to come from nowhere, and within applications that are viewed as puzzles, like chess problems.*

The problem has been exacerbated by the reason that even where teachers have tried to use reform based approaches such as engaged learners in discussion on a complex task; the traditional approach in manipulating the tasks has remained the focus. This includes the orientation of the tasks the learners have to engage with. As a result, learners tend not to develop conceptual understanding of the algebraic concepts involved.

Because of the way algebra is approached in the classroom, i.e. rule-based instructional approaches, learners are faced with many difficulties. These difficulties result from, in a broader sense, dominance in manipulation over reasoning; applying learnt procedures; inability to read and use algebraic symbolism in a meaningful way; failure to distinguish data from conclusion; and tendency to write arithmetical at the expense of algebraic expressions (Bell, 1995; Watson, 2009). Watson (op cit) argues that over-reliance on rules result in learners misapplying them, or misremembering them, or might not think of the appropriate situation in which to apply them. Such rules could include, guess-and-check used when solving equations; and Bracket of Division, Multiplication, Addition and Subtraction (BODMAS) used when simplifying expressions with different operations.

In referring to the tendency by learners to misapply arithmetical meanings to algebraic expressions, Watson (2009) argues that reasons for this include: seeing algebra as calculations rather than relations; not understanding the inverses and operations in solving the unknown values; not seeing that some situations need to be transformed into algebraic form before solving; not treating letters and numbers as symbols in a structure in the event that they are used together [2(3 + b) has a different structure from 6 + 2b]; and seeing the equal sign as 'calculate' in an attempt to express an answer and not 'is equal to' or 'is equivalent to'. Seeking closure to obtain one term is common in this situation, since learners might not realize that for example in the notation $a + b = ab$, ab mean 'multiply'. A desire for an answer is also evident when learners 'solve' an expression as if it is an equation (Ryan & Williams, 2007; Wagner & Parker, 1993). Watson (2009) argues that this is an indication of how problematic the aspect of interpreting is while Nickson (2000) argues that learning arithmetic is not necessarily the same as learning algebra and therefore needs careful attention.

In Kieran's (1989, 1992) terms, Watson (2009, p. 15) describes 'structure' in algebra as: "(1) surface structure of expression: arrangement of symbols and signs; (2) systemic: operations within an expression and their actions, order, use of brackets, and so on; (3) structure of an equation: equality of expressions and equivalence". Watson (op cit) also argues for flexibility in how mathematical expressions are acted upon in that they should be seen as an answer to a particular question, an object in itself (for example, $3x + 4$), and also an algorithm or process for calculating a particular number. In Gray and Tall (1994) terms, Watson (op cit) refers to the dual meaning of an algebraic expression as *proceptual thinking* while in Sfard & Linchevski's (1994) terms, it is referred to as the *process-object duality*.

Research has also shown learners' possible interpretations of letters in their early learning of algebra (Hart et al., 1981). This was in the Chelsea diagnostic test instrument involving 2 900, 12 – 16 year old learners. The possible learner interpretations of letters include 6 aspects as described by Hart et al. (1981, p. 104): (1) *letter evaluated* "where the letter is assigned a numerical value from the outset"; (2) *letter not used* where the letter is ignored or its existence is acknowledged but without giving it a meaning; (3) *Letter used as an object* where "the letter is regarded as a shorthand for an object or as an object in its own right"; (4) *Letter used as a specific unknown* where a letter is regarded as a specific but unknown

number, and can be operated upon directly; (5) *Letter used as a generalized number* where "the letter is seen as representing, or at least as being able to take, several values rather than just one"; (6) *Letter used as a variable* where "the letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between such sets of values". In light of these possible learner actions on letters, Watson (2009, p. 4) argues that "teachers have to understand that students may use any one of these approaches and students need to learn when these are appropriate or inappropriate".

In seeing the prevalence of similar errors in studies twenty years apart (see Hart et al. (1981 and Ryan and Williams 2007), the argument is that it "is evidence that these are due to students' normal sense-making of algebra, given their previous experiences with arithmetic and the inherent non obviousness of algebraic notation" (Watson, 2009, p. 19). For my study, of the six possible learner interpretations of letters, the first four descriptions were used in setting up and then analysing the six scenarios, and in describing student-teachers' discourses of these scenarios. Moreover, in what ways did student-teachers' explanations of the sources of learner errors in the scenarios resonate with over-reliance on rules, or the tendency by learners to misapply arithmetical meanings to algebraic expressions, or inadequate process-object duality?

As a result learners' tendencies with learning algebra that I engaged with in the scenarios included: (1) equating an open algebraic expression to zero when asked to simplify (Ryan & Williams, 2007; Wagner & Parker, 1993); (2) conjoin or 'finish' open algebraic expressions when asked to simplify (Even & Tirosh, 2002); (3) dividing by x when asked to solve the quadratic equation $2x^2 = 6x$ (Extracted from Learning Mathematics for Teaching project test items meant to measure teachers' SMK by Hill et al., 2004); (4) distributing the power over the sum of terms in an expression, for example, $(x + y)^2 = x^2 + y^2$ (Vermeulen, 2007); (5) multiplying numbers only when asked to 'Multiply $n + 5$ by 4' (Hart et al., 1981); and (6) equating each factor to 4 when asked to solve for x in the quadratic equation $(x - 1)(x + 2) = 4$ (Bell, 1995).

Since my study is more concerned with school algebra, I work with the view that algebra is generalized arithmetic as well as a specialized language for expressing generality. I also work with an understanding that activities of school algebra, as modelled by Kieran (2004), include

transformational, generational, and global/meta-level, and for my study the algebraic activity is predominantly transformational. The understanding that learners experience difficulties with learning school algebra is not a new phenomenon in mathematics education research. Types of common errors learners make are well documented and some reasons for these errors provided. A selection of these common errors was then used in designing scenarios which I engaged with in my study. Some suggestions on how these errors in algebra could be addressed have also been outlined. My argument with these suggested strategies is that they could also generate errors of their own nature.

3.5 Conclusion to Chapter 3

A number of issues have risen from reviewing the relevant literature for my study. These include the analytic resources and questions to consider and reflect on as the study unfolds. Concepts drawn include teacher education as intelligible and/or sensible (a practical accomplishment) in Davis et al. (2007) terms. These concepts enable the explanation of whether the discourse of engaging with LMT dealt with in mathematics teacher education is in the realm of the sensible and/or intelligible. Categorization of what entails the discourse of engaging with LMT following Even & Tirosh (2002) has also been illuminated to include developing in learners both instrumental and relational understanding, focusing on learner errors, and creating a classroom environment where teacher can listen to learners. This enables the establishment of ways in which teacher-educators' and student-teachers' talk resonate with these three categories.

In dealing with learner errors, a three-stage process for carrying out error analysis, in Peng and Luo's (2009) terms and Jacobs et al.'s (2010) include identify, explain, and remediate. Analytic resources for explaining sources of learner error in Ryan & Williams' (2007) terms include modelling error, prototypical error, error as a result of overgeneralization, and error as a result of inadequate process-object conception [proceptual thinking in Watson's (2009) terms or process-object duality in Sfard and Linchevski's (1994) terms]. Error as a result of overgeneralization is re-described, in Lima & Tall (2008) terms, as an issue of *met-before* or *met-after*. Other concepts for explaining sources of learner error in Hart et al. (1981) terms relate to learners' interpretation of letters, and for my study these include *letter evaluated, letter not used, letter used as an object,* and *letter used as a specific unknown*. Having pulled the conceptual map from the literature review, one has to gaze on this through an

understanding of how the relay produces some of these, and I now turn to Bernstein (2000). He provides the theoretical gaze that informs this study, hence the theories together with the literature reviewed make my theoretical field (Dowling & Brown, 2010).

CHAPTER 4

4 A THEORETICAL PERSPECTIVE AND CONCEPTUAL FRAMEWORK

4.1 Introduction

My study is informed by Bernstein's (1982, 1996, 2000) theory of the pedagogic device (PD). Given my research questions, the relevant literature reviewed, and that I want to figure out what student-teachers know in relation to the discourse of engaging with LMT, I have to establish how they deal with this knowledge. Therefore, Bernstein's (op cit) theory of the PD provides me with a special language to describe how educational knowledge is produced and reproduced in context, hence necessary to use in exploring and explaining the problem of engaging with LMT at teacher education level. Bernstein (op cit) argues that learning is about acquiring recognition and realization rules based on classification and framing values that are guided by contextual rules. In working towards establishing student-teachers' preparedness for the tasks of teaching, focus is mainly on *what is* entailed in the discourse of engaging with LMT, *how* it is 'taught' and 'learned', and positionings, in particular, its use when dealing with school algebraic thinking.

From the aforesaid, I am in a way being responsive to Bernstein's (2000, p. 27) invitation to not only focus on '*the carried*' (what is relayed) but also on '*the carrier*' (or relay). Brown and Dowling (1999) argue that a conceptual frame for research (its language of description), develops through the interaction between the theoretical and empirical fields of the study. As will become evident, my conceptual framework has just this interactive base. I thus need to briefly state what constitutes the empirical for the study, as the theoretical elaborations following are a function of my empirical foci. Details of the methodology follow in Chapter 5.

Teacher-educators' discourses were elicited from in-depth interviews with four relevant mathematics teacher-educators, where they spoke about what and how they taught for LMT, and how they thought it was learned. These interviews were followed by focus group interviews with student-teachers, who similarly were asked to speak about what, how and

where they learned LMT in their formal programme. Following this, student-teachers, in pairs were then interviewed on a range of algebraic tasks related to common learner errors from the literature and generated from tests that were administered to learners in selected local schools. In the discussion below, teacher-educators' discourses thus refers to their interviews. Student-teachers' discourses refer to the two distinct contexts in which this was elicited – interviews on LMT per se, and interviews where they engaged with learner errors.

As I progressed through the study, engaging iteratively with the empirical and further reading of literature reviewed in Chapter 3, the resonance between threads in the teacher-educators' discourses of LMT and some of the literature became apparent, and so more explicit in the conceptual framing of the study. Specifically, to describe what engaging with LMT entails, I explored how teacher-educators' and their student-teachers' talk resonated with Even and Tirosh's (2002) categories and described in Chapter 3. Moreover, student-teachers' talk was mapped onto a framework drawn from Peng and Luo (2009), and Jacobs et al. (2010) to explore how engaging with LMT was used when focus was on scenarios designed around common algebraic learner errors. As for positionings, Morgan, Tsatsaroni and Lerman's (2002) elaboration of Bernstein's (2000) notions of classification and framing were used. These discourses and positionings then made it possible to establish and explain how student-teachers' discourses of LMT relate to their teacher-educators' discourses. As I elaborate my theoretical and conceptual framing below, the interaction between the theoretical and empirical will become apparent.

Finally, by way of introduction to this chapter, it is important to note that while Bernstein's (1982, 1996, 2000) work is concerned with understanding the production or reproduction of social inequalities in terms of who gets what between the working class and the middle class learners through schooling, my work does not have social class as its focus. I want to understand the what and how of student-teachers' acquisition of LMT. I am concerned with the opportunities for LMT the mathematics education teacher-educators make available for their student-teachers, and then whether and how what student-teachers acquire is similar or different from what is made available. In explaining this, Bernstein's pedagogic device (the PD) is pertinent. It can be used to analyse "the processes by which discipline-specific or domain-specific expert knowledge is converted or pedagogised to constitute school knowledge (classroom curricula, teacher-student talk, online learning)" (Singh, 2002, p. 572).

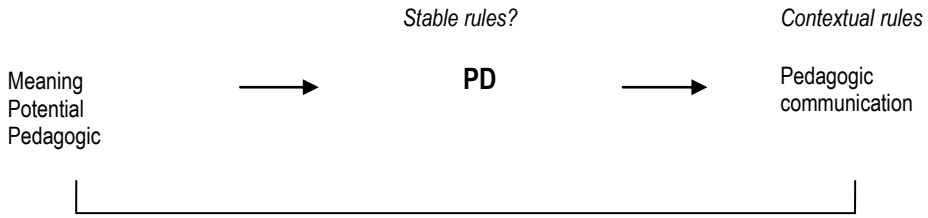
It is important to note that this process applies to any institutional knowledge such as a University, and not just the school. In line with this understanding the question I ask for my study is: What is constituted as the discourse of engaging with LMT in the mathematics education curriculum in formalized institutions of higher learning? In the sections that follow, I describe the PD and its intrinsic rules which regulate what counts as pedagogic communication, and how this provides a lens for my study.

4.2 The pedagogic device

Bernstein (2000) initially makes an assumption that the transformation of knowledge into pedagogic communication is always governed by general principles. This is irrespective of the knowledge being intellectual, practical, expressive, official or local. He describes pedagogic communication as "a carrier, a relay for ideological messages and for external power relations", and not "an apparently neutral carrier or relay of skills of various kinds" (Bernstein, 2000, p. 25). This means that the relay is never neutral, it carries with it pedagogic messages depending on what the practices are. The general principles could take the nature of either official or local knowledge. Therefore, he asserts that it is important for studies to focus on both the carried/relayed (pedagogic messages and their institutional and ideological base) and what constitutes the carrier or relay (social grammar) which makes the message possible. As already indicated in the introduction to this chapter, my focusing on the *what*, *how*, and *why* of the discourse of engaging with LMT suggests that I am taking into consideration both the relay and the relayed. Moreover, I have argued in Chapter 3 that the relay – the inside of teacher education is very important including how the discourse of engaging with LMT is dealt with.

Bernstein (2000) defines what the PD is by describing its characteristics in terms of the flow of knowledge as shown in figure 2.

Figure 2: The pedagogic device (PD) as elaborated by Bernstein



> "*This device has internal rules which regulate the pedagogic communication which the device makes possible. Such pedagogic communication acts selectively on the meaning potential. By meaning potential we simply mean the potential discourse that is available to be pedagogised. The pedagogic device regulates fundamentally the communication it makes possible, and in this way it acts selectively on the meaning potential. The device continuously regulates the ideal universe of potential pedagogic meanings in such a way as to restrict or enhance their realisations*" (Bernstein, 2000, p. 27).

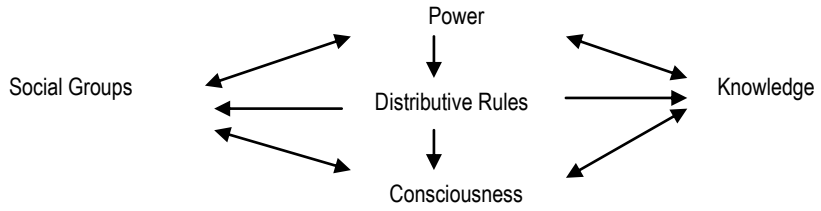
Bernstein (op cit) further argues that the intrinsic rules of the PD are relatively stable and they relate to each other, and are not ideologically free. The rules are an indication of the dominant groups' emphases of what entails meaning potential. So their stability is informed by what the dominant groups in society are concerned with, hence bias is inevitable. Moreover, contextual rules regulate forms of realization of the PD, and these forms can restrict or enhance the potential discourse available to be pedagogised. Therefore, rules of the PD play a vital role in terms of the realization of various forms of consciousness pertaining to what is distributed and its constraints. So my concern is to understand what is regulated as the discourse of engaging with LMT by establishing what is communicated and how it is communicated. As for how biased these rules could turn out to be is out of focus for my study since my concern is not on *who* gets what pertaining to the discourse.

However, the intrinsic rules in question, referred to by Bernstein (2000) in a metaphoric sense as intrinsic grammar, that are responsible for what pedagogic messages are relayed include distributive rules, recontextualising rules, and evaluative rules. Bernstein (2000, p. 28) argues that "these rules are hierarchically related, in the sense that recontextualising rules are derived from the distributive rules, and evaluative rules are derived from the recontextualising rules". He further argues that this interrelationship between the rules is necessary, and that power relations also exist between them. Bernstein (op cit) also states that these rules are

responsible for the creation of fields, which are fields of production, recontextualisation, and reproduction, respectively. Just as the rules are in hierarchical relation, so are the fields they create in that recontextualisation of knowledge is possible if such knowledge is produced, and reproduction is possible if there is recontextualization. I now discuss the intrinsic grammar of the PD which regulates the pedagogic communication it makes possible and the fields they create in turn.

4.2.1 Distributive rules and the field

Figure 3: Distributive rules and their regulation as illustrated by Bernstein (2000, p. 31)



The rules of the PD regulate what are privileged texts of school knowledge, and they are inter-related. Nevertheless, they have different functions to play. According to Bernstein (2000, p. 28) and as illustrated in figure 3, the distributive rules "regulate the relationships between power, social groups, forms of consciousness and practice". They "specialise forms of knowledge, forms of consciousness and forms of practice to social groups". They "distribute forms of consciousness through distributing different forms of knowledge". So distributive rules "regulate the power relationships between social groups by distributing different forms of knowledge, and thus constituting different orientations to meaning or pedagogic identities" (Singh, 2002, p. 573). Moreover, Bernstein (op cit) argues that meanings that would bring the two groups together are those that have no direct relation to a specific material base.

In terms of fields (social space of conflict and competition), the distributive rules create sociologically a specialized field of the production of knowledge, with specialized rules of access, and specialized power controls (Bernstein, 2000, p. 31). The state is usually in control of this field of production of knowledge. Since my study is not concerned with understanding

about who gets what in terms of different forms of knowledge distributed to different social groups, the distributive rules are not dominantly in focus. My concern is to understand what and how of student-teachers' discourse of engaging with LMT. However, it is necessary to describe the distributive rules since they do not stand in isolation from the other two rules. They are implicated in a way pertaining to the recontextualising rules as production of knowledge is crucial.

4.2.2 Recontextualising rules and the field

Bernstein (2000) says that recontextualising rules regulate the formation of a specific pedagogic discourse. Moreover, he states that since the pedagogic discourse is constructed by a recontextualising principle which "selectively appropriates, relocates, refocuses and relates other discourses to constitute its own order" (Bernstein 2000, p. 32), it cannot be identified with any of the discourses it has recontextualized. Therefore, the pedagogic discourse as a principle is responsible for the creation of specialised pedagogic subjects through its contexts and contents. Bernstein further states that imaginary subjects are as a result of pedagogic discourse principle at play. For example, outside pedagogy a discourse could be called carpentry but inside pedagogy, it could be called woodwork which is an imaginary subject. The transformation of real discourses from its field of production to imaginary subjects is as a result of the pedagogic discourse which is a recontextualising principle. In a sense the recontextualized discourse does not look like the original discourse because it has appropriated itself a space to give itself a unique name. For my study, the recontextualization is twofold, from the field of production of knowledge which happens to be the mathematics education research field or mathematics education field to what teacher-educators chose to privilege as the discourse of engaging with LMT. The other recontextualization is from what teacher-educators chose to privilege as the discourse of engaging with LMT to what student-teachers realize, hence my concern to establish the relationship between teacher-educators' discourses and that of their student-teachers.

Just as distributive rules create fields of the production of knowledge, Bernstein (1996, 2000) states that recontextualising rules also create recontextualising fields and agents with recontextualising functions. It is the recontextualising functions that become the means for creating a specific pedagogic discourse. He then argues that the crucial role of creating a fundamental autonomy of education is dependent on the recontextualising field. Therefore, he

distinguishes between two types of fields: (1) an official recontextualising field (ORF) "created and dominated by the state, and its selected agents and ministries"; and (2) a pedagogic recontextualising field (PRF) "consists of pedagogies in schools and colleges; and departments of education, specialized journals, private research foundations" (Bernstein, 2000, p. 33).

Although I have talked briefly about the ORF (the Zambian school algebra curriculum) in Chapter 2, my main focus is on the PRF which we do know that already it is a recontextualising both from mathematics and from the official discourses. For my study, teacher-educators' talk is in the PRF and not the voice of mathematics. The recontextualising field is where production of curriculum for schools and mathematics education courses is made possible. Hence the recontextualising rule as a principle frames my study. I set up the study to engage with teacher-educators so that they revealed how the PRF is working and what is in it. In this way, teacher-educators' privileging of what is entailed in the discourse of engaging with LMT and how they make it available to their student-teachers becomes evident.

4.2.2.1 The context of Zambian mathematics education teacher-educators operating in the PRF and the influence of the ORF

The mathematics teacher-educators are responsible for designing the mathematics education courses, and making choices about what to focus on and what to ignore. This is in line with the Zambian education policy on Universities, which states that:

> "*academically, each university is responsible for determining its own programmes of instruction at all levels, determining and regulating the requirements for admission, regulating and conducting examinations, conferring degrees and other awards, and promoting, coordinating and controlling the direction of research*" (MoE, 1996, p. 98).

In Bernstein's (1996) terms, since Universities are located in the PRF, the mathematics education teacher-educators operate in this field and act as recontextualizers of the mathematics education courses. As noted, this recontextualization occurs in two ways, firstly they recontexualize the mathematics education courses from its field of production, which is the mathematics education community (involved with research and writing books pertaining

to mathematics education) into mathematics teacher education. Because of this movement, the mathematics education courses they design no longer resemble the actual or unmediated discourse. There is movement from unmediated discourse to an imaginary or mediated discourse (Bernstein, 1996), meaning that the mathematics education courses so designed are an imaginary discourse. Transformation takes place because 'every time a discourse moves from one position to another, there is a space in which ideology can play' (Bernstein, 1996, p. 47). Moreover, Bernstein suggests that in higher education such as the University it is possible that the mathematics teacher-educators can also be the producers of knowledge in that they would be involved in research and writing books. Therefore, they would influence and contribute to knowledge growth in the mathematics education community. Indeed, policy on Universities in Zambia points to Universities being responsible for "coordinating and controlling the direction of research" (MoE, 1996, p. 98).

Secondly, the mathematics education teacher-educators are also responsible for the recontextualising of the mathematics education courses into their classrooms during lectures since they are the ones who also offer the same courses they design. This includes making choices on what gets assessed as continuous assessment and examinations. In Bernstein's (1996, 2000) terms, in this case the mathematics education teacher-educators have control over selection of what is included or excluded from the mathematics education courses offered, hence strongly framed.

It is evident that the Universities in Zambia enjoy the autonomy granted to them by the state and its selected agents and ministries operating in the ORF, especially academically as I have illuminated above. Bernstein (1996, p. 48) states that "if the PRF can have an effect on pedagogic discourse independently of the ORF, then there is both some autonomy *and* struggle over pedagogic discourse and its practices". All the state and its selected agents and ministries do is to provide a general policy framework within which the Universities should operate. This includes two overarching principles, namely:

(1) the Universities must be responsive to the real needs of Zambia; and

(2) they must, on merit win the respect and proper recognition of the university world (MoE, 1996, p. 98).

Before proceeding to discuss the evaluative rules and the field it creates, it is important to further discuss what the pedagogic discourse entails. This is because evaluative rules are concerned with the transformation of the pedagogic discourse into a pedagogic practice. Moreover, I am interested in understanding the transmission and acquisition of the discourse of engaging with LMT. Having said this, it becomes necessary to also discuss the principles of classification and framing and their regulation of recognition and realization rules, respectively.

Other than the role played by the pedagogic discourse as a recontextualising principle in the creation of specialized pedagogic subjects through its contexts and content, Berrnstein (2000) states that it also consists of two discourses. These are the instructional discourse (ID) and the regulative discourse (RD), and illustrated as: Pedagogic Discourse $= \frac{ID}{RD}$. The former is embedded in the latter, hence the argument that it results in one text and one discourse with the RD being dominant. The ID is described as a "discourse which creates specialized skills and their relationship to each other", and the RD as "the moral discourse which creates order, relations, and identity" (Bernstein, 2000, p. 32). Moreover, Bernstein explains that the ID is regulated by rules of discursive order, which are selection, sequence, pacing, and criteria of the knowledge while the RD is regulated by rules of social order in terms of expectations about conduct, character and manner. He further says that the RD is dominant because its components activate rules of the internal order of the instructional discourse. This means, for example, that rules of order of physics in school (selection, relation, sequence and pace) are a function of the regulative discourse.

To elaborate on this, Morais (2002) describes pedagogic discourse as a set of rules that regulate the transmission/acquisition of knowledge in terms of the contents and competencies, and their transmission and evaluation. It is about "the *what* that is transmitted, *how* it is transmitted, and also which student realizations are considered legitimate" (Morais, 2002, p. 560). This suggests that ultimately the recontextualisation is twofold: (1) "the *what* of the pedagogic discourse, what discourse is to become subject and content of pedagogic practice" and (2) "the *how* that is *the theory of instruction*" (Bernstein, 2000, p. 34). Selection of the theory of instruction is both instrumental and "also belongs to the regulative discourse,

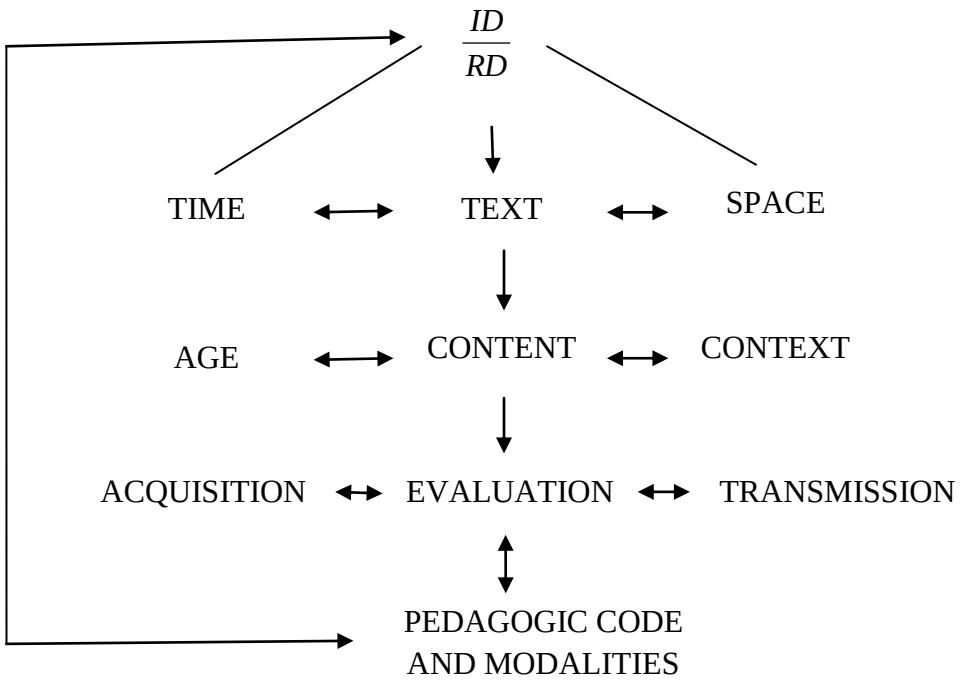
and contains within itself a model of the learner and of the teacher and of the relation" (Bernstein, 2000, p. 35).

Since recontextualisation is a social fact as indicated, it will occur as we move across contexts. The questions to ask for my study are: What is selected as the discourse of engaging with LMT, and how does this selection relate to other selections in the mathematics education courses? In what ways has the general preparation or the preparation in other areas enabled student-teachers to recontextualise and look at the discourse of engaging with LMT in terms of school algebra? How is the image of learner, teacher and relation portrayed in view of the recontextualisation of the discourse of engaging with LMT? Therefore, what are evaluation criteria for the transmission/acquisition of the discourse of engaging with LMT?

4.2.3 Evaluative rules and the field

As already stated, the evaluative rules regulate the transformation of pedagogic discourse to pedagogic practice, and this transformation is in stages from the most abstract level to the classroom level as shown in figure 4. The pedagogic practice is described as "forms of communication where classificatory principles – whether strong or weak – form consciousness in the process of their acquisition". It is "the form of control which regulates and legitimises communication in pedagogic relations: the nature of the talk and the kinds of spaces constructed" (Bernstein, 2000, p. 12). This description suggests that the concepts of classification and framing are more appropriate in analysing the different forms of legitimate communication realized in any pedagogic practice. The reasons for this are that classification as a principle – strong or weak – carries with it power to establish relations between contents, categories or discourses. Framing also as a principle carries the boundary relations of power and socializes individuals into these relationships. I shall further describe the concepts of classification and framing and what they create at the level of the acquirer later in the discussion.

Figure 4: Transformation of pedagogic discourse to pedagogic practice by Bernstein (2000, p. 36)



The first level is the specialization of time, text and space, and at the centre of the specialization is the pedagogic discourse in that it brings them into a special relationship with each other. This suggests that "pedagogic discourse specialises meaning to time and space, and that the specialization of time, text and space marks us cognitively, socially and culturally" (Bernstein 2000, p. 35). The punctuation or dislocation of time results in imaginary and arbitrary age stages. Bernstein (op cit) states that at the second level, which is more obvious and real, time transforms into age, text into a specific content, and space into a specific context. At level three, age is transformed into acquisition, content into evaluation, and context into transmission. Acquisition, evaluation and transmission are the "social relations of pedagogic practice and the crucial features of the communication" (Bernstein 2000, p. 36).

Therefore, Bernstein's (2000) argument, and as can be seen in figure 4, is that continuous evaluation is key to pedagogic practice, hence the role of pedagogic practice is to transmit criteria. He further argues that "evaluation condenses the meaning of the whole device" (Bernstein 2000, p. 36). Evaluation criteria are then described as "rules that regulate the extent to which legitimate text is made explicit to acquirers" (Morais, 2002, p. 560). In

Bernstein's terms, these rules could include classification and framing, which at the level or the acquirer translate into recognition and realization rules. He then concludes by stating that the whole purpose of the device is to produce a symbolic ruler for consciousness. This suggests that evaluative rules are responsible for the creation of the field of reproduction of knowledge where already recontextualised discourses are transformed again for the purposes of transmission and acquisition. For my study, what are evaluation criteria for the transmission and acquisition of the discourse of engaging with LMT? What do student-teachers think it is and how do they say they 'learn' for it? How is what they say different or similar from what their teacher-educators say?

4.2.3.1 Classification and framing, and the recognition and realization rules

Classification refers to the strength in insulation between boundaries of contents or categories or discourses, and that power relations yield principles of strong and weak classification (Bernstein, 1982, 1996, 2000). In explaining this, Bernstein states that if contents are well insulated from each other, then the contents have a closed relation to each other – hence strong classification and a space in which a category develops its own unique identity with its own internal rules and special voice. Identity in Bernsteinian terms means a sense of membership in a particular class and this could be strong or weak. If there is a reduction in the insulation between contents, then the contents have an open relation to each other – hence weak classification. This results into less specialised discourses, less specialised identities and less specialised voices (Bernstein, 2000). Bernstein (op cit) says that classification whether weak or strong always carry power relations and is therefore viewed as a distinguishing feature of the division of labour of educational knowledge. My concern in the study is to establish whether the discourse of and about engaging with LMT is weakly or strongly classified and what it means for its identity in the mathematics education curriculum. It will therefore be interesting to see how these identities are constrained and enabled by the student-teachers as they talk about the discourse of and about engaging with LMT.

Framing refers to the degree of control, for instance, teacher and taught or social worker and client possess over the selection of communication, sequencing, pacing, criteria, and social base of the knowledge transmitted and received in the pedagogical relationship (Bernstein, 1982, 1996, 2000). In a similar way to classification, framing may also be strong or weak.

Strong framing means the transmitter has explicit control over selection, sequencing, pacing, criteria and social base while weak framing means the acquirer would have more apparent control over communication and social base. Bernstein (op cit) says that framing, strong or weak can vary depending on the elements of the practice, for example, weak framing over pacing and strong framing over the other aspects of the discourse.

Bernstein (2000) also alludes to that strong framing results in visible pedagogic practice in that rules of the ID and RD are explicit, hence evaluation criteria are made explicit to the acquirer. Moreover, weak framing results in invisible pedagogic practice where rules of ID and RD are implicit, and largely unknown to the acquirer, hence evaluation criteria are implicit to the acquirer. The argument is that strong framing at the level of school is crucial if learners are to acquire the recognition and realization rules of the school context (Morais, 2002). However, Davis (2010) would argue to the contrary in that it is about 'the grounds' (iconic, empirical, propositional, or procedural) on which the mathematical objects were built that could not have been well articulated that would result in learners not gaining access to recognition and realization rules. For my study, the concern is about how implicit or explicit are evaluation criteria pertaining to the discourse of engaging with LMT.

As earlier stated, the principles of classification and framing regulate recognition and realisation rules, respectively. This implies that classification is concerned with *what* meanings provide the limits of the discourse and framing is concerned with *how* meanings are to be put together to produce a legitimate text (Bernstein, 1996, 2000). A text is defined as "anything which attracts evaluation, and this can be no more than a slight movement", and "evaluation condenses into itself the pedagogic code (elaborate or restricted) and its classification and framing procedures, and the relationships of power and control that have produced these procedures" (Bernstein, 2000, p. 18). In relation to this, Bernstein (op cit) asserts that being aware of the power relations in which individuals are involved, and their positions in them, does not guarantee speaking the expected legitimate text. This suggests that having recognition rules to enable one distinguish the speciality of the context does not always translate into producing legitimate communication. The production of legitimate communication requires acquiring of both recognition and realization rules. For my study, student-teachers' acquiring of both recognition and realisation rules pertaining to the discourse of engaging with LMT is about their participation.

Moreover, in using the principles of classification and framing, Bernstein realised two types of curriculum namely, the collection code and the integrated code. The collection type refers to the strengthening of classification and framing while the integrated type refers to the weakening of classification and framing (Bernstein, 1982, 1996, 2000). Therefore, the recognition and realization rules in the collection code are different from those in the integrated code. The question I ask then is: What is focused on as the discourse of and about engaging with LMT and what does it mean for the pedagogic practice, including student-teachers' recognition and realisation rules of the type of curriculum, and their positionings? I am interested in establishing how student-teachers recognize and realize the discourse of engaging with LMT by the way they communicate. I will then be able to explain what they communicate and how they communicate about the discourse of engaging with LMT. More specifically, if student-teachers are given scenarios on learner errors, what do they recognize as the error, and what is their realization in terms of the explanations for why the error occurred and suggested remediating strategies? Here I am concerned with the "what' and "how", and since the "how" regulates the "what", when I look at the relationship between the "what" and the "how", I should be able to establish student-teachers' discourses pertaining to LMT.

4.2.3.2 Elaboration of the notion of positionings and my study

In my study, I also use a model developed by Morgan et al. (2002) to explore and explain what and how of teacher-educators' and their student-teachers' talk about transmission and acquisition of the discourse of engaging with LMT. I will first describe the model and then explain how it contributes to my theoretical field and the extensions I have added.

The purpose of the model developed by Morgan et al. (2002) was to enable the understanding of teachers' assessment practices and positionings using Bernstein's framework. More specifically, "to identify and explain the positions available to teachers within assessment practices, and to address the question of how the official discourse of assessment, including the explicit criteria provided for teacher-assessors, may be transformed within the school … by drawing on the various resources available to them" (Morgan et al., 2002, p. 447). This was made possible by revisiting Morgan's (1998) earlier work on mathematics coursework assessment in UK schools that identified a number of positions teachers adopt as they read and assess learners' texts. Moreover, her results showed that "teachers drew on resources

from different, sometimes contradictory, discourses, and that the various ways they were positioned within these discourses could lead to different evaluations of the same learner text" (Morgan et al. 2002, pg 446). Examples of positions identified were: Examiner, using externally determined criteria; examiner, setting and using their own criteria; and teacher-advocate, looking for opportunities to give credit to students. One or more of these positions were adopted by the teachers in their work of evaluating learners' work and justifying their evaluations.

Among the theoretical resources and the set of concepts Morgan et al. (2002) developed to revisit the teacher positions identified, was to re-describe the PRF to constitute the official pedagogic recontextualising field (OPRF) and the unofficial pedagogic recontextualising field (UPRF). The former would include the official discourses of school mathematics as informed by the ORF, and the latter would include specialized (mathematics) education discourses or specialized languages of (mathematics) education research. As can be observed, these are the pedagogies operating in the PRF. The purpose of the re-description of the PRF was to ascertain teachers' positions within it in the process of examining the discourse in practice. This includes "their positions in relation to the official discourses, the resources on which they draw, and their assessment practices (i.e. the criteria and their orientation to the task)" – (Morgan et al. 2002, p. 452).

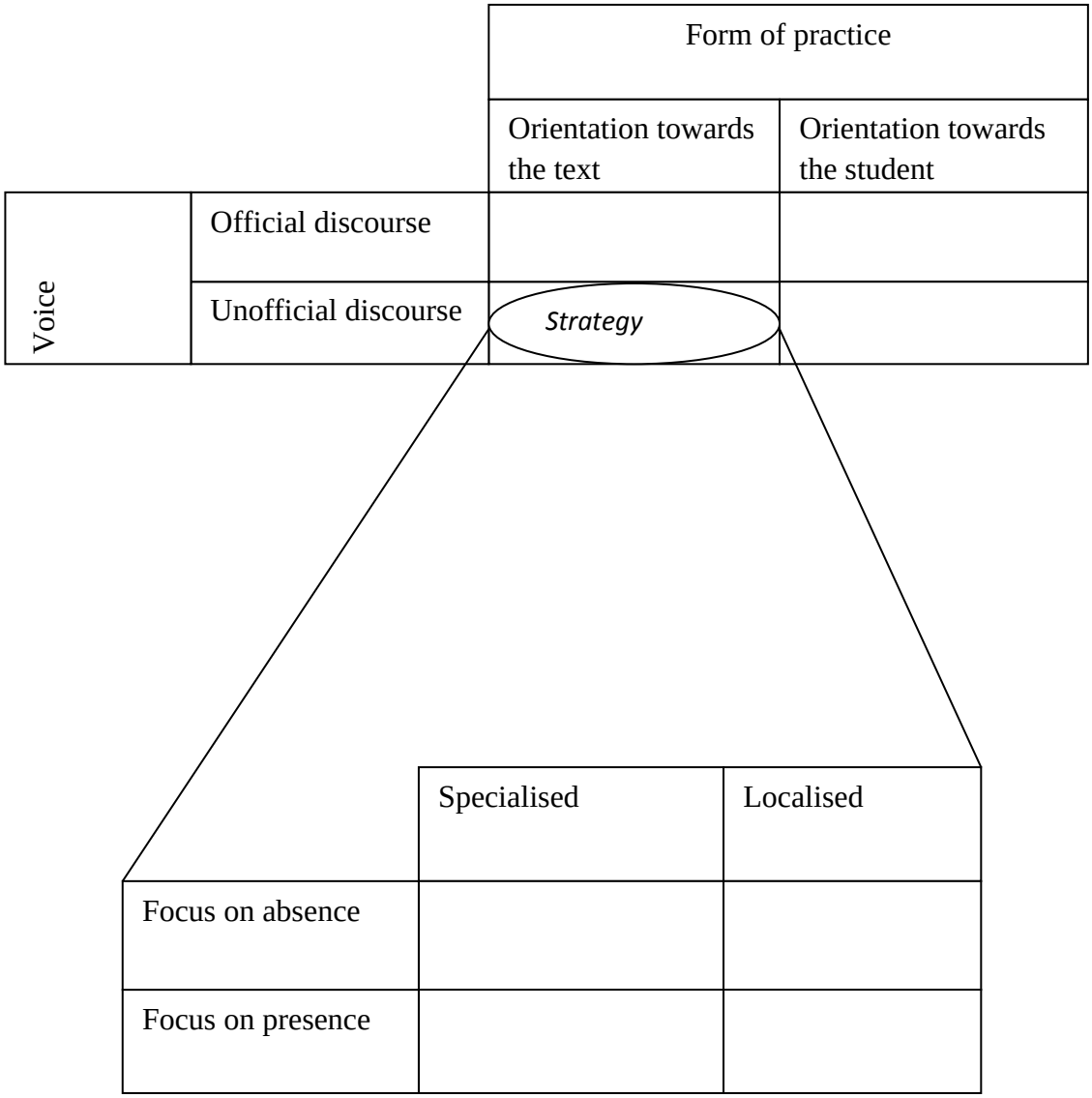
The assumption they make about teachers is that "the strategies they develop in doing their tasks, and in justifying their practices, depend on how they interpret their own schoolwork activity" (Morgan et al. 2002, p. 452). Their strategies could be drawn from either vertical or horizontal discourses. The latter could be associated with forms of knowledge described as 'everyday' or 'common sense', for example, drawing from teachers' everyday professional experience of teaching and learning. The former could be associated with forms of knowledge that are specialized and elaborate, and underpin performance. I shall return to horizontal and vertical discourses later in Section 4.2.4 when I discuss the form of knowledge of the discourse of engaging with LMT.

To now build the model that would enable the understanding of teachers' assessment practices, the theoretical consideration was that:

> "*... the official discourse of evaluation shapes teachers' assessment activity, constructing teachers' positions and forms of practice, and that structurally there are two (pre-)dominant subject positions for teachers: speaking the voice of the official (legitimate) discourse of evaluation, or speaking the voice of other discourses*" (Morgan et al. 2002, p. 452).

Therefore, the guiding question towards their attempt to build the model was whether teachers accept or reject the official discourse. This question was based on the understanding that teachers' resistance to the discourse of evaluation was inevitable, and this could easily be established during interviews. As can be seen in figure 5, the discourse of evaluation is in the form of a four-dimensional model consisting of two matrices, the second one subsumed in the first. The structural positions pertaining to the 'voice' constitution are presented in the first two rows of the main matrix. Here, teachers are positioned as either speaking the voice of the official discourse of evaluation or the voice(s) of other discourses. The other dimension is the positions pertaining to 'form of practice'. This is presented in the columns and the focus is on "defining teachers' orientations and strategies" (Morgan et al. 2002, p. 453). Focus here is also on the two oppositional 'forms of practice', which are: "orientation towards the text produced by the student, as in performance models; and orientation towards the student, as in the competence model" (Morgan et al. 2002, p. 453). Their argument is that these positions are possible when there is 'consistency' in the way the discursive resources are drawn; otherwise it might be possible to identifying all four positions described in the model.

Figure 5: Positions and practices in discourses of assessment by Morgan et al. (2002, p. 454)



The second matrix represents 'strategies' following Dowling (1998), and is described by Brown (1999) as 'focus directing teachers' actions' or 'generality of teachers' commentary' (in Morgan et al., 2002). The rows refer to teachers' focus and whether this is on what is present or absent, which results in two possibilities. The first possibility is that 'consistent' uses of resources results in the teacher focusing on what is absent in the learners' text, hence the orientation being defined by the performance model. The second possibility is that if the

teacher is focused on presences in terms of the quality of learners' thinking exhibited, then the orientation is defined by competence models. The columns refer to the level of generality of teachers' comments, which could either be localised or specialised judgements. If teachers see their activity as a specialised one drawn from vertical discourse then reference is towards pedagogic principles and theories as justification of evaluation of learners' text. Similarly, if teachers see their activity as an everyday one, drawn from horizontal discourses, then they will refer to common sense notions as a way of justifying their evaluation of a learners' text.

Therefore, Morgan et al.'s (2002) argument that criteria used by teachers and their degree of explicitness would be affected by their structural position in the model, more especially 'form of practice', which includes orientation and strategies. This suggests that criteria of the official discourse could be explicitly adopted or implicitly reinforced. Alternatively, criteria of the official discourse could be explicitly rejected and alternative criteria adopted whose interpretation (or re-interpretation) would be according to everyday resources.

As indicated in the elaboration of Morgan et al. (2002) model, focus was on the discourse of evaluation and criteria at play pertaining to teachers' assessment practices in the presence of official guidelines in the context of school mathematics. My study is theoretically similar as I am concerned with evaluation criteria pertaining to the discourse of engaging with LMT. As teacher-educators and student-teachers talk about what the discourse of engaging with LMT is and how they 'teach' or 'learn' for it, respectively, their positions are crucial. Ascertaining their positions is important in the process of exploring and explaining the discourse of engaging with LMT in pedagogic practice as it would enable me say how the image of the teacher, learner, curriculum, or relation are implicated in their descriptions. This would in turn make the descriptions of evaluation criteria for the discourse more richly described. Moreover, the discourse of engaging with LMT is about discourses of student-teachers' learning what and how to teach as it is located in teacher education. Therefore, it also becomes necessary to consider the opportunities their teacher-educators make available. However, the resources used to describe the discourse could be drawn from the OPRF or the UPRF, hence it is necessary for me to understand both teacher-educators and student-teachers' positions, and thus elaborate the model.

Having said this, the two assumptions I make about teacher-educators and their student-teachers are that:

(1) What teacher-educators say engaging with LMT is and how they make it accessible to their student-teachers depends on the knowledge resources they have access to, and this in turn shapes the messages they make available to their student-teachers.

(2) What student-teachers say engaging with LMT is and how they come to know and understand the discourse, including its use in the context of school algebra also depends on the knowledge resources they have available about the 'best' practices of teaching and learning, which are partially as a result of the messages they recruit or take-up from their teacher-educators.

I now discuss two contexts in my study in which the notion of positioning becomes useful. The first context is where both teacher-educators and their student-teachers have to talk about what they think engaging with LMT is and how they 'teach' or 'learn' for it, respectively. Here the structural positions, namely, 'form of practice' in terms of 'strategy', specifically 'focus directing teacher-educators'/student-teachers' actions (which include 'focus on absences' or 'focus on presences') was used. Therefore, teacher-educators'/student-teachers' positioning pertaining to what they said LMT is and how they 'teach' or 'learn' for it respectively, was looked at in terms of absences and presences. In the case of teacher-educators, the positioning was looked at in relation to teachers/student-teachers, or learners, or the curriculum (school curriculum or teacher education curriculum). Similarly, in the case of student-teachers, the positioning was looked at in relation to teachers/student-teachers/teacher-educators, or learners, or the curriculum (school curriculum or teacher education curriculum).

It is important to mention here that 'focus on absences' for my study in this first context I am describing does not refer to 'consistent' use of resources provided (assessment guidelines) as in the Morgan et al. (2002) model. What 'focus on absences' for my study means is that: when the talk points to what is thought to be not paid attention to in terms of teaching or learning with reference to teachers/student-teachers, or teacher-educators, or learners, or the curriculum. On the contrary, 'focus on presences' in the model refers to when the teachers are concerned with qualities of learners' thinking exhibited. Following a similar line of thought,

the meaning of ‘focus on presences’ in my study is recognized when the talk is in solidarity with what teachers/student-teachers, or teacher-educators, or learners, or the curriculum are able to do or address. However, in this context and the one to be discussed next, I did not take into consideration how student-teachers’ or teacher-educators’ positionings were pedagogically oriented in terms of performance or competence models. This is because doing so would have limited my re-descriptions.

The second context is when there is specific focus on school algebra. Here student-teachers engaged with scenarios on common learner errors drawn from the UPRF, specifically, from the specialized languages of mathematics education research. From the Morgan et al. (2002) model, positions pertaining to ‘voice’ and ‘form of practice’ (orientations and strategies) is useful for my study, of course, also with some extensions because of the context. For Morgan et al. (op cit), teachers are positioned either speaking voice of official discourse of evaluation or voice(s) of other discourses. In my study, student-teachers are positioned either speaking the voice of the official discourse of school algebra or voice(s) of unofficial discourse. Voice(s) of unofficial discourse was further categorized into two, that is, either everyday professional knowledge of teaching and learning discourses, or discursive/theoretical discourses by drawing on terms or concepts developed in the mathematics education literature. The level of generality of their commentaries is also referred to as localised or specialised, respectively.

As for ‘form of practice’, what is useful for my study from the model are ‘orientations’ and ‘strategy’ pertaining to focus on absences or presences. The orientations in my study are divided into four categories, unlike in the model where they are only two, namely, orientation towards the text (mathematics) or orientation towards the learner, hence focus is on absences or presences, respectively. In my study the orientations extend to include orientation towards teaching or orientation towards teacher, hence focus on absences or presences, respectively. I would like to state here that there is some resonance with the discourse of evaluation since student-teachers in my study were concerned with learner thinking of their mathematical texts. This is based on the understanding that errors are part of learners’ learning as explained in Chapter 3 of this thesis.

Table 5: Synopsis of the applicability of the device in my study

Rules	Field	Concept	How I am using the concepts in my study
Distributive Rules	Production of knowledge		Necessary to talk about how different forms of knowledge are distributed to different social groups. For without creation of knowledge there would be no recontextualising process.
Recontextualising Rules	Field of recontextualising	Official recontextualising field (ORF)	Positionings in relation to voice of official discourse of school algebra (school algebra curriculum)
		Pedagogic recontextualising field (PRF)	What and how of the discourse of engaging with LMT teacher-educators privilege their student-teachers, and the positionings in terms of form of practice (absences and presences). That is, what they recontextualise and how they make it accessible to their student-teachers. The recontextualising could be from the field of mathematics education or mathematics for use in another context such as mathematics education courses offered to student-teachers.
Evaluative rules	Reproduction of knowledge	Classification and framing (selection and evaluation criteria); and Recognition and realization rules	There is also a recontextualising process from what teacher-educators priviledge as the discourse of engaging with LMT to what their student-teachers think it is and how they make sense of it in terms of appropriate practices for working with the discourse. Whether what is privileged as the discourse of engaging with LMT is in closed or open relation with other contents of the mathematics education curriculum. Whether evaluation criteria are implicit or explicit to the acquirers (student-teachers) pertaining to the discourse of engaging with LMT and what this means for their access to recognition and realisation rules from the way they communicate and the positionings (in terms of voice and form of practice) when asked to talk about the transmission and acquisition of the discourse including explaining their knowledge and understanding of scenarios on selected common learner errors in school algebra.

In re-describing Tables 5, I provide a summary of what I have done so far in the chapter. In the ORF, I am interested in establishing whether given scenarios on common errors in algebra, student-teachers speak with voice of the official discourse of school algebra as outlined in the curriculum. To set the context in which the teacher-educators are operating, I focused on what the state has outlined in terms of giving guidelines on the operations of Universities, and how this informs teacher-educators operating in the PRF. Therefore, in the PRF, I established that the teacher-educators recontextualise what is to constitute the mathematics education curriculum, and this includes the discourse of engaging with LMT. The teacher-educators enjoy the autonomy given to them by the state, which is operating in the ORF. At this stage I could not use classification principle to establish whether the discourse of engaging with LMT was in closed or open relation with other contents since the mathematics education curriculum is just a list of topics to be focused on, as indicated in Section 2.3.

In the same PRF the second recontextualisation is a transformation of what has been selected as mathematics education curriculum as it becomes active in pedagogic process in the mathematics education classrooms. Since I did not enter the mathematics education classrooms to observe what the teacher-educators privilege as the discourse of engaging with LMT, I relied on what they said. Thus, using classification principle, I could establish whether focus on the discourse is in closed or open relation with other contents in the curriculum, but only from what was said. The assumption I would make here, following Bernstein (1996, 2000) is that the discourse in question could be in open relation with other contents in the curriculum given the nature of courses in higher education.

Since I am concerned with evaluation criteria pertaining to the discourse of engaging with LMT, focus on the 'what' and 'how' of the discourse also includes positionings. To help with further elaboration of evaluation criteria, again in the PRF, I also focus on mathematics education research to establish how it resonates with the descriptions teacher-educators or student-teachers provide for the 'what' and 'how' of the discourse in general, and when specific focus is given on selected common learners' errors in school algebra. From literature (see Chapter 3), categories of what entails engaging with the discourse of LMT, following Even & Tirosh (2002) include, developing in learners both instrumental and relational understanding, focusing on learners' errors, and creating an

environment where teacher can listen to learners. Still drawing from literature (see Chapter 3), strategies for carrying out error analysis, following Peng and Luo (2009) and Jacobs et al. (2010) include, identifying the error (identifying strategy for the error), explaining the sources of the error (interpreting and evaluating), and thinking of remediation strategies.

Analytic resources for explaining the sources of error, also drawn from literature, include modelling, prototype, overgeneralization, process-object duality, letter evaluated, letter not used, letter used as an object, and letter used as a specific unknown. In Davis et al. terms (2007), intelligibility and/or sensibility (a practical accomplishment) enable the explanation of how the discourse of engaging with LMT is dealt with. Bringing in these resources is necessary because, as already mentioned, the recontextualisation of the mathematics education curriculum is from specialized mathematics education research, specialised mathematics education, mathematics and other discourses. These sites have influence on what comes to be the mathematics education curriculum.

Following from Morgan et al. (2002), I would argue that since specialised mathematics education research, and specialised mathematics education are in the UPRF as opposed to school mathematics which is in the OPRF and regulated by the ORF, it is important to interrogate these forms of knowledge in the UPRF or OPRF and so I move on to discussing the form of knowledge of the discourse of engaging with LMT.

4.2.4 Form of knowledge of the discourse of engaging with LMT

I have discussed how selections into what should be considered mathematics education curriculum, and so the discourse of engaging with LMT is as a result of a recontextualising principle. Therefore, it is necessary to understand the form of knowledge of the discourse of engaging with LMT using Bernstein's (2000) description of the internal principles of discourses, in terms of their construction and social base. Understanding the forms of knowledge of discourses is part of Bernstein's (op cit) later work since his earlier work, as already discussed, was concerned with different principles of pedagogic transmission/acquisition, and the contexts that generate them and the implications for change. More specifically, he was concerned with the pedagogic communication the device makes possible with focus on who gets access to what in terms of forms of knowledge made available within the context of education.

Bernstein (2000) distinguishes two forms of knowledge, which are *Vertical discourse* and *Horizontal discourse* where the former is the written form concerned with symbolic mastery; and the latter is the oral form concerned with practical mastery. These forms are referred to by Bourdieu as "the function to which they give rise, one form creating symbolic, and the other practical mastery" (Bernstein, 2000, p. 155). In the educational field, he describes *Vertical discourse* as "school(ed) knowledge" and *Horizontal discourse* "as everyday common sense knowledge or 'official' and 'local' knowledge" (Bernstein, 2000, p. 156). These two forms of knowledge have different structural features and are organised differently although at school level, segments of *Horizontal discourse* could be fused into the *Vertical discourse* to enhance the access of school knowledge. Such fusion could provide learners with possible ways of solving problems in their everyday world if they arise. This means "*Vertical discourses* are reduced to a set of strategies to become resources for allegedly improving the effectiveness of the *repertoires* made available in *Horizontal discourse*" (Bernstein, 2000, p. 169).

Horizontal discourse as a 'form of knowledge' is referred to as 'common' because it is easily accessible to everyone, it is applicable to all, and has a common history pertaining to people's common life experiences (Bernstein, 2000). Moreover, its features are that "it is likely to be oral, local, context dependent and specific, tacit, multi-layered and contradictory across but not within contexts" (Bernstein, 2000, p. 157). Most of them all, it is "segmentally organised" in terms of "the sites of realization of this discourse" (Bernstein, 2000, p. 157). These segments are differentiated with some being more important than others. Circulation of knowledge within *Horizontal discourse* is made possible if forms of recontextualising are implicit and that the organising principles are not that systematic. The distributive rules within *Horizontal discourse* regulate "the circulation of knowledge, behaviour and expectations according to status/position" (Bernstein 2000, p. 157).

Horizontal discourse is further described as consisting of "a set of strategies which are local, segmentally organised, context specific and dependant, for maximising encounters with persons and habitats" (Bernstein 2000, p.157). To describe how this set of strategies plays out, Bernstein introduces two terms, which are *repertoire* and *reservoir*. *Repertoire* entails "the set of strategies any one individual possesses and their analogic potential for contextual transfer" while *reservoir* entails "the total set of strategies possessed by all

members of this community" (Bernstein, 2000, p. 158). Based on these descriptions, he explains that members of a particular community could have a common focus but each bringing with them different *repertoires* to bear on it. The differences between *repertoires* are a function of experiences members of a community encounter in terms of contexts, activities and their associated issues.

Bernstein (2000) says that the regulation on the relation between *reservoir* and *repertoire* depends on the structuring of social relationships. That is, if members of a community are isolated or excluded from each other, then the social base in which the *reservoir* or *repertoire* should develop is weakened. This results in a reduction in effectiveness because circulation or exchange of strategies is restricted. Moreover, restriction to circulation or exchange of strategies specialises, classify and privatise knowledge. If there is a huge reduction in isolation or exclusion between members of a community, then the social base pertaining to "circulation of strategies, of procedures and their exchange" is enhanced (Bernstein, 2000, p. 158). This suggests an expansion of *repertoire* and *reservoir* and the effectiveness that is brought about because of the circulation and exchange of strategies.

Therefore, Bernstein (2000, p. 158) argues that "structuring of the social relationships generate the forms of discourses but the discourse in turn is structuring a form of consciousness, its contextual mode of orientation and realisation, and motivates forms of social solidarity". This suggests that acquisition is segmentally structured, and pedagogic practice vary with segment and across social groups or class. Bernstein (op cit) also says that common place for segment pedagogy is in the family, among peers or local community, and transmission of pedagogy is "by modelling, by showing, or by explicit modes". Moreover, in a segmental pedagogy acquisition of competence is through repeating; pedagogy is exhaustive in the context of its enactment; and acquiring common competence is important and competition is inevitable among peers. For example, pedagogies for acquiring competences in counting change are distinct from those required for acquiring competences in addressing different individuals. As already stated, an individual operating in a *Horizontal discourse* need not necessarily have one strategy to bear on a particular context. This is because of the implicitness of forms of recontextualising, and unsystematic organising principles.

The oppositional 'form of knowledge' of a *Horizontal discourse* is a *Vertical discourse*. Circulation of knowledge within a *Vertical discourse* takes the form of strong distributive rules regulating access, transmission and evaluation (Bernstein 2000). He therefore states that circulation is made possible if forms of recontextualising affecting distribution in terms of time, space and actors are made explicit. That is recontextualisation and evaluation ought to be explicit, and influenced by strong distributive rules. A *Vertical discourse* is described in two ways. The one description is that it is about *Hierarchical Knowledge Structures* consisting of "coherent, explicit and systematically principled structure, hierarchically organised as in the sciences" (Bernstein 2000, p. 157). That is, *Hierarchical Knowledge Structures* are produced by integrating codes at the level of meanings, and their users appear "to be motivated towards greater and greater integrating propositions, operating at more and more abstract levels" (Bernstein 2000, p. 161). A sign of development is recognised if there is progress in terms of theory in that it has become more general and integrating than the previous one, that is, the oppositions within this knowledge structure are between theories.

The other description is that a *Vertical discourse* is about *Horizontal Knowledge Structures* which take "the form of a series of specialised languages with specialised modes of interrogation and specialized criteria for the production and circulation of texts as in the social sciences and humanities" (Bernstein, 2000, p. 157). He states that *Horizontal Knowledge Structures* are produced by collection or serial codes concerned with the 'integration *of* language' and the 'accumulation of *languages*'. Bernstein's (op cit) argument is that these languages are contextual in that they are specific to a particular *Horizontal Knowledge Structure* and therefore not translatable to others because assumptions made for each are usually different and in most cases opposing. Moreover, criteria for legitimate text for each language are different in terms of what is considered as evidence and legitimate questions or a legitimate problem. Bernstein goes on to say that just as each language is specialised and excluding, so are its users. A sign of development is then recognized if a new language is introduced, hence "a fresh perspective, a new set of questions, a new set of connections, and an apparently new problematic, and most importantly a new set of speakers" (Bernstein, 2000, p. 162). In contrast to *Hierarchical knowledge structures*, the oppositions within this knowledge structure are between languages.

Bernstein (2000) further makes a distinction within *Horizontal Knowledge Structures* between languages which have theoretical concepts or discursive resources that could be used to interrogate the empirical field to generate descriptions or relations by applying rigorous restrictions, and those languages which do not have such powers. The former description is referred to as having strong grammars while the latter is referred to as having weak grammars. Examples of strong grammars include Mathematics, Logic, Economics, Linguistics, and parts of Psychology, although Mathematics and Logic might not necessarily address an empirical phenomenon but have "a set of discrete languages for particular problems" and possess "the strongest grammars" (Bernstein, 2000, p. 163). Moreover, examples of weak grammars include Sociology, Social Anthropology, and cultural studies.

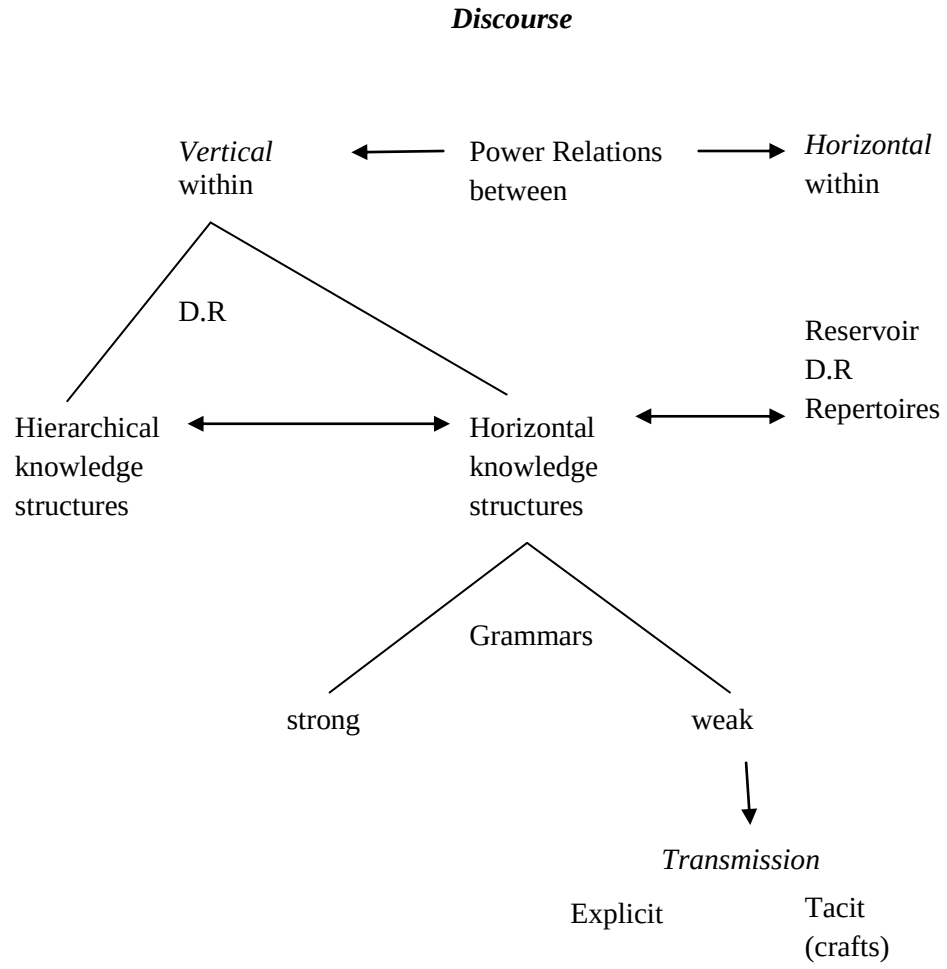
Acquisition in the *Hierarchical Knowledge Structure*, according to Bernstein (2000) is by correct usage of the theories, for example, those developed in Physics since as a strong grammar the theories clearly outline what it is. There will be no worry of what one is speaking or writing in terms of language since there is explicitness in what Physics entails. What is required here is mastery of procedures of investigation and instruments of observation to enhance understanding of the theory. A shift from one theory to another is an indication that there is "an extension of its explanatory/descriptive powers" rather than "a break in the language" (Bernstein, 2000, p. 163). Moreover, weak grammars generate problems of acquisition in that acquirers might not necessarily recognise what they are speaking or writing.

Since *Horizontal Knowledge Structure* consist of a range of languages, for any one transmission to occur, selections and what is privileged in terms of recontextualisation is inevitable (Bernstein, 2000). This requires taking into consideration the social base of the recontextualising principle in terms of "whose perspective is it? How is it generated and legitimated" (Bernstein, 2000, p. 164)? Bernstein (2000, p. 164) says that "the dominant perspective within any transmission could depend on power relations among the teachers, or of pressure from groups of acquirers, or indirect or direct pressures from the market or state". That is, the *Horizontal Knowledge Structure* to be acquired is constructed by a perspective which becomes a principle of the recontextualisation. This at the level of the acquirer translates into possessing the recognition and realization rules of what could be considered 'truth' and this would influence how the acquirer reads, evaluates and creates

texts, hence acquisition of a ‘gaze’. Bernstein’s argument is that if acquisition of the specialised language is to be made possible, oral transmission and experience as a result of a social interactional relationship with those who poses the ‘gaze’ is necessary.

A “deeper resemblance” is recognised between *Horizontal Knowledge Structure* weak grammar and *Horizontal discourse* that is characterised as a specialised practice responsible for satisfying the material requirements of its segments (Bernstein, 2000, p. 165). Common features between these two forms of knowledge include aspects that they are horizontally organised, serial, segmented, and their contents are volatile. Volatility in the case of a horizontal discourse “refers to the referents of this discourse”. Moreover, volatility in the case of *Horizontal Knowledge Structure* with a weak grammar modality “refers to additions and omissions of the specialised language of a particular *Horizontal Knowledge Structure*” (Bernstein 2000, p. 165). Acquisition is also implicit and contextual with reference to the segmentation of a *Horizontal discourse* and specialised languages of a *Horizontal Knowledge Structure* weak grammar.

Figure 6: A map of discourses and knowledge structures by Bernstein (2000, p. 168)



D.R = Distributive Rules

Figure 6 provides a synopsis of Bernstein's (2000) descriptions of forms of knowledge with reference to discourses and knowledge structures as already described. Having provided Bernstein's (2000) descriptions of forms of knowledge, I can now be in a position to describe the form of knowledge of the discourse of engaging with LMT. I would like to state here that the discourse of engaging with LMT is a *Vertical discourse* of the form *Horizontal Knowledge Structure* with modality of a weak grammar, hence resembling a *Horizontal discourse*. I say so because mathematics education courses of which the discourse of engaging with LMT is part of is a *Vertical discourse* of the form *Horizontal Knowledge Structure* because the production and circulation of texts is

dependent on specialised languages. This includes modes of interrogation and criteria. Moreover, the fields of specialised mathematics education research or mathematics education from which the recontextualisation is possible for the constitution of mathematics education courses has seen the development of new languages. Some of these specialised languages in these fields could have a strong or weak grammar.

As for strong grammar, the fields have seen the introduction of concepts which have been used to interrogate the empirical field, hence the expansion of their conceptualisation. For example, as illuminated in Chapters 2 and 3 of this thesis Shulman's (1986, 1987) seminal work on knowledge base for teaching has seen the development of concepts such as PCK and SMK. There is also an invitation for researchers to engage with these concepts so that they could be further elaborated as in of themselves were not conclusive. For example, Ball et al. (2008) in their effort to respond to Shulman's invitation developed knowledge domains to illuminate their conception of MKT. This has seen SMK to include CCK, SCK, and knowledge on the horizon while PCK includes KCT, KCS, and curricular knowledge. This development of a language of description is contextual in that the concern is about knowledge base for teaching, hence MKT.

Since the discourse of engaging with LMT is part of PCK, it follows that it is a *Horizontal Knowledge Structure* with modality of weak grammar whose transmission and acquisition could either be explicit or implicit. Drawing from the literature reviewed in Chapter 3 of this thesis, the specialized language that entails the discourse of engaging with LMT includes three categories, namely, developing in learners both instrumental and relational understanding which in a grounded way has been described as mathematical reasoning; focusing on learners' errors; and creating an environment where teacher can listen to learners. With specific focus on learners' errors, the specialised language is on the nature of error and the strategies for carrying out error analysis.

In my study, as already stated, one of my concerns is what teacher-educators select and privilege in terms of recontextualising the discourse of engaging with LMT for transmission to occur, that is, from the perspective of teacher-educators. Therefore, what each teacher-educator recontextualises from the fields of specialised mathematics education research, specialised mathematics education, and other discourses could vary. In Chapter 6 of this thesis, I am going to show how these selections and privileging resonate with categories of the discourse of engaging with LMT established in the

literature, and their positions in them. Since teacher-educators' responsibility is to teach their student-teachers what and how to teach school mathematics, choices of what they select and privilege as the discourse of engaging with LMT depends on their knowledge and understanding of what teaching and learning school mathematics entails. This is based on the assumption that each teacher-educator's experience is unique, hence implicit transmission. Therefore, if for each teacher-educator, aspects of what entails the categories of LMT are diverse, the expectation is that the messages their student-teachers receive will also be diverse although contextual with reference to what is in focus.

It will be interesting, in Chapter 7 of this thesis, to see what and how of student-teachers' recontextualisation, which at the level of the acquirer is about their recognition and realisation rules of the discourse of engaging with LMT, hence their participation. Student-teachers' recognition and realisation rules could have an influence on their reading, evaluating, and creating texts. Of interest also is how this resonates with their teacher-educators' selections and privileging of the discourse of engaging with LMT. This is as a result of a social interactional relationship with their teacher-educators that happen to possess the gaze to enable acquisition of what entails the specialised language of the discourse of engaging with LMT. Moreover, other aspects such as student-teachers' own experiences that are contextual could also be factored into the social interactional relationship. Recontextualising is inevitable as shown by Ensor (2001) when she followed the pre-service teachers into their classrooms to establish how what they had learned in the mathematics education courses was recontextualised into their classrooms. My interest in the study is to establish the nature of this recontextualising in relation to the discourse of engaging with LMT.

Exploring and explaining student-teachers' recontextualising (that is their recognition and realisation rules) is also extended to when the focus is on a selection of common algebraic errors reported in the field of mathematics education research. In Chapters 8 and 9 of this thesis, I establish possible *reservoirs* of how the sources of errors for each scenario (3 scenarios for each Chapter) could be explained from the specialised field of mathematics education research. For each scenario, I also establish possible student-teachers' shared *repertoires* in terms of the task of carrying out error analysis, and their positions in them. Focus here is on *what* and *how* of student-teachers' recognition of the errors. This suggests student-teachers' descriptions of the errors in terms of their recognition and

realization rules of what entails the discourse of engaging with LMT. In doing so, I also establish the relationship between the possible *reservoirs* and student-teachers' shared *repertoires* with reference to descriptions of the errors in the scenarios. Conclusions are then made about the nature of a social interactional relationship between teacher-educators and their student-teachers pertaining to the discourse of engaging with LMT.

4.3 Conclusion to Chapter 4

Table 6 is a summary of the theoretical field for my study which includes the theories and the analytic resources from the literature.

Table 6: Synopsis of the theoretical field for my study

Concepts from the theory	Classification Framing (Selection, Evaluation criteria) Recontextualisation		Recognition Rules Realisation Rules Text		Reservoirs Repertoires
Analytic resources drawn from the literature	Even & Tirosh's (2002) categories of the discourse of engaging with LMT: • Mathematical reasoning (Developing in learners both instrumental and relational understanding) • Focus on learners' errors • Creating an environment where teacher can listen to learners	Peng & Luo (2009) and Jacobs et al. (2010) strategies of carrying out error analysis: • Identify (Recognise) the error • Explain (interpret/ evaluate) the error • Suggest remediating strategies	Resources for explaining sources of errors: • Overgeneralization (met-before/met-after) • Process-object duality • Prototype • Modelling • Letter evaluated • Letter not used • Letter used as an object • Letter used as a specific unknown	Morgan et al. (2002) notions of positioning: • Absences • Presences • Voice [official discourse or unofficial discourse (Everyday or discursive / theoretical] • Form of practice in terms of orientations and strategy (OTA, OLP, OMA, OTP)	Adler and Davis (2006) notion of evaluative events: • Teacher-educators' and student-teachers' discourses of engaging with LMT: an event is a piece of text that describes what LMT is and how it is justified. • Student-teachers' discourses of scenarios on learner errors described in literature/elicited from learners' own working: each scenario viewed as a new event based on what student-teachers say on learner errors exhibited. • 'Text' as a smaller unit of analysis Davis et al. (2007) descriptions of intelligible and/or sensible (a practical accomplishment)

Overall, as already stated my concern in this study is to explore and explain the evaluation criteria with reference to the *what* and *how* of the discourse of engaging with LMT. Critical to this is the selection and privileging of the discourse. Therefore, I elaborate the construct 'evaluative events' as a methodology in relation to the context of my study, and this is discussed in Chapter 5. As shown in Table 6, an attempt towards developing the language of description for my study includes concepts from Bernstein's (2000) theory of the pedagogic device. These include classification, framing (selection, and evaluation criteria), recontextualisation, and at the level of an individual recognition and realisation rules. The other resources for analysis drawn from the literature to enable explanation of student-teachers preparedness for the discourse of engaging with LMT refer to Even & Tirosh's (2002) three categories of what entails the discourse. These include developing in learners both instrumental and relational thinking, focus on learners' errors, and creating an environment where teacher can listen to learners.

With a specific focus on learners' errors, a further set of analytic resources refer to strategies for carrying out error analysis as outlined by Peng & Luo (2009) and Jacobs et al. (2010). These include identify (recognition) the error, explain (interpret/evaluate) the sources of error, and suggest remediating strategies. Resources for explaining the sources of error include overgeneralization (met-before/met-after), process-object duality, prototype, and modelling (Ryan & Williams, 2007). In terms of interpretation of letter, the resources include *letter evaluated*, *letter not used*, *letter used as an object*, *letter used as a specific unknown* (Hart et al. 1981). From these possible explanations for the sources of learner errors, possible *reservoirs* and shared *repertoires* are described. Morgan et al.'s (2002) work provides resources for looking at teacher-educators and their student-teachers' positioning. Concepts drawn from their work include absences, presences, voice, and form of practice. Adler & Davis (2006) also provide me with the notion of *an evaluative event*, a methodology I use to describe the smaller unit of analysis for my study which is a text. Moreover, Davis et al. (2007) provide resources such as intelligible and/or sensible to enable the explanation of how the discourse of engaging with LMT is dealt with.

In the chapter that follows I describe the design of my study which includes among other things data collection instruments, participants, and how data was analysed using the resources outlined in Table 6, and drawn from the theory and the literature reviewed.

CHAPTER 5

5 RESEARCH DESIGN AND METHODOLOGY

5.1 Introduction

Having surveyed the field and developed a theoretical and conceptual framing, I now focus in on the design and methodology of the study. This section discusses the research paradigm and approaches, the participants; methods of data collection, data analysis, trustworthiness, research ethics, and limitations of the study. My concern in the study is to establish student-teachers' preparedness pertaining to the discourse of engaging with LMT. Specific focus is on what teacher-educators select and privilege; and student-teachers' realisations with respect to the discourse of engaging with LMT. Moreover, establishing the relationship between what teacher-educators say they select and privilege as the discourse of engaging with LMT and student-teachers' realisations will provide an understanding of messages student-teachers recruit or take-up, hence addressing the 'why' question. Sources of data are interviews which were in threefold with teacher-educators and student-teachers all framed by a Bernsteinian gaze. How these interviews were analysed is also described, hence illuminating the methodology for my study as an *evaluative event*.

5.2 Research Paradigm and Approach

This study is qualitative focusing on student-teachers' knowledge and understanding of the discourse of and about engaging with LMT in general, and algebra in particular. Thus, what is it that they know and are able to do in relation to the discourse of and about engaging with LMT in general, and algebra in particular? I need to establish how the discourse is structured in terms of classification and framing and how student-teachers use the discourse. To get at the structure of the discourse, teacher-educators' interviews are analysed with focus on what they think the discourse of engaging with LMT entails and the 'teaching' of it in terms of: what student-teachers are supposed to know and be able to do; their positioning; and where in the mathematics education courses/topics LMT is focused on. This suggests teacher-educators' selections and privileging pertaining to the discourse, hence the recontextualisation from the specialised fields of mathematics education research, mathematics education, and other discourses as they operate in the PRF. Getting at how the discourse is realized by student-teachers involved substantial amounts of narrative data obtained from interviews and focus group discussions. The data obtained from this study was

therefore analysed qualitatively and included the coding of the data and production of a verbal synthesis (Gay, 1996, p. 11).

The qualitative approach, therefore, provides a careful and logical in-depth analysis of student-teachers' discourses as they participated in the discourse of and about engaging with LMT in general, and algebra in particular. Student-teachers' discourses were compared with their teacher-educators' to establish the relation in terms of messages they recruit or take-up and what is different, and how this might be explained. The advantages of using qualitative studies are that they are carefully planned just like quantitative studies are carefully planned. However, qualitative studies provide room for change such as asking questions different from what was originally planned, thereby the researcher following the direction of the study while in quantitative research; the original plans cannot easily be disrupted (Best & Kahn, 2003). Moreover, Best and Kahn (op cit) state that qualitative studies are more open and responsive to its participants. In my study, as teacher-educators and their student-teachers participated in the discourse of and about engaging with LMT, I prepared lead questions with probes beforehand which resulted in other probing questions to arise in the process of interviews. These further probes depended on the responses participants gave, hence participant-driven to a larger extent although within the confines of the intensions of the study.

The approach which this study is following is a case study in one teacher education institution in Zambia whose context I have provided in Section 2.3. The context is in terms of how mathematics teacher education is organized, who teaches what courses, and what available information there is on each of the courses. I am interested in seeing what teacher-educators select and privilege as the discourse of engaging with LMT, and their student-teachers' recognition and realization rules pertaining to the discourse in general, and algebra in particular, hence their participation. Participation for my study means using the discourse by recognizing where you are and produce the appropriate talk – talk the right talk and ask the right questions. That is, what and how of student-teachers talk about the discourse of engaging with LMT when there is no specific task given, and then how do they participate in the discourse when something specific, such as scenarios on common learner errors in algebra, is given. This suggests that my study is interventionist in some way. In trying to establish what entails the discourse of engaging with LMT, a detailed analysis of teacher-educators' discourses and that of their student-teachers was carried out, as earlier indicated, to describe and explain the what, how, and why for my study. Bogdan & Biklen (1982, p. 58)

as pointed out by Wellington (2000) provide a suitable operational definition of a case study. They define a case study as "a detailed examination of one setting, or one single subject, or one single depository of documents, or one particular event". A unit in my study includes mathematics teacher-educators in a university together with their student-teachers participating in one particular event 'the discourse of engaging with LMT'. Therefore, my case for the study is one Zambian teacher education institution.

Having explained the design for this proposed study as a qualitative case study in its nature, I now discuss the research participants and how they were selected.

5.3 Research Participants

The main participants for this study included four mathematics education teacher-educators, twenty mathematics education student-teachers.

5.3.1 Teacher-educators

Four mathematics education teacher-educators from among all the teacher educators in a University participated in the study. The choice of four was based on the reason that they are the ones who teach mathematics education courses at that particular University. Their role in the study, as already indicated in section 5.1, was to provide information which would enable me explain the social structuring of the discourse of engaging with LMT in terms of selections and privileging they make available. This enables the assumption and explanation of the relation between what teacher-educators select and privilege pertaining to the discourse of engaging with LMT and what their student-teachers recognise and realise pertaining to the messages they recruit or take-up. This suggests the provision of an overall understanding of the situation under study.

5.3.2 Student-teachers

Twenty student-teachers, in their final year (4^{th} year) of teacher training, were also selected from among all student-teachers in the University to engage in this study. Five of them are in-service while the other fifteen are pre-service. As I earlier explained in 'the background and context of the study' section (Section 2.3), these two sets of student-teachers take the same courses in mathematics and mathematics education at the University. While the number twenty is arbitrary, it is considered sufficient bearing in mind that the study is qualitative and dealing with extensive amounts of narrative data. I limit the number of student-teachers participating to twenty as ideal for the type of study. Furthermore, the choice of twenty

student-teachers was meant to provide me with a deeper understanding of what my study purposes to explore and explain. Purposive sampling was, therefore, used as a procedure to select the twenty student-teachers. This implies building up a set of participants that is satisfactory to the specific needs of a researcher (Cohen, Manion, & Morrison, 2000).

The twenty student-teachers that participated in my study were those that were nearing the end of their mathematics teacher education programme, and from a situative perspective according to Peressini et al. (2004), about to be deployed to another site of teacher-learning, the work place which happens to be the school. My assumption for this selection is that it would enable me develop a description of what it means in terms of student-teachers' preparedness in participating in a discourse of and about engaging with LMT. Moreover, since LMT in algebra is not explicit content in the mathematics education courses in that some of the specific aspects around school algebra are not dealt with in the programme, choosing a group nearing the end of their final year of training would be of more benefit to my study.

5.4 Methods of Data Collection

For me to be able to collect valuable data, which would enable the provision of answers to the research questions for my study as outlined in Section 1.4, interviews and focus group discussions were used.

5.4.1 Interviews

These included three interview schedules that were used to:

(1) carry out individual interviews with teacher-educators;

(2) interview the student-teachers in their respective two focus groups of ten student-teachers in each; and

(3) interview eight pairs of student-teachers on three scenarios of common algebraic errors described in the specialized field of mathematics education research and the other three scenarios that were elicited from learners' own working.

The interviews were semi-structured. This involves the use of an interview guide where topics and issues to be covered are specified and outlined before hand; and the interviewer

decides the sequencing and wording of questions in the course of the interview (Best & Kahn, 2003). This includes deeper probing of the interview questions as the interaction progresses.

5.4.1.1 Individual interviews with teacher-educators

As stated, the semi-structured interviews were conducted at different stages during the process of data collection for this study. Mathematics education teacher-educators were interviewed individually to find out what is focused on and how they deal with the discourse of and about engaging with LMT in their courses, in particular its classification and framing. I explained the purpose of my study in relation to the discourse of and about engaging with LMT. The questions that I asked were designed in such a way that teacher-educators talked in terms of selection, sequencing, pacing, and criteria. This included their rationales for these and their experiences, hence their interpretation of MKT. The responses from these questions enabled me say something on whether and how the discourse is weakly or strongly classified and framed. See Appendix A for the interview guide, which included main questions and their probes; further probes arose from the responses that teacher-educators provided in most of the cases.

5.4.1.2 Focus group interviews with student-teachers to establish their positionings towards LMT

Student-teachers were interviewed in two phases to establish their realization of LMT. The first interviews were conducted in focus groups to establish what they say is focused on and how, pertaining to the discourse. Therefore, similar questions as those asked as I interviewed their teacher-educators were used. Focus was on how they talk about LMT; whether they see engaging with LMT feasible; and whether they associate themselves with LMT or they disassociate themselves from it. Focus groups are small structured groups with selected participants, and normally led by a moderator with an aim of exploring specific topics as well as individuals' views and experiences through group interactions (Litosseliti, 2003). Moreover, those participating in the discussions share and respond to comments, ideas and perceptions in a comfortable and enjoyable manner. From the aforesaid description of focus groups, as a researcher I moderated the discussions to ensure that the talk was in focus pertaining to the purpose of the focus group interviews for my study. See Appendix B for the interview guide that was used, which included main questions with probes. Again further probes arose from the responses student-teachers provided during the interview process.

5.4.1.3 Scenario-based interviews with student-teachers

The second interviews for student-teachers were conducted in pairs and these were based on the six scenarios on selected common learner errors in school algebra reported in the field of specialised mathematics education research (operating in the UPRF). The assumption I made on the choice of selected common learner errors was that even if they are not taught they are more likely to be known if they are the common ones rather than the obscure ones. This suggests that the errors were authentic as demonstrated in Section 3.4.3, although not exhaustive in terms of the whole school algebra curriculum. Moreover, as already argued in Chapter 2, the algebraic activity for the school algebra curriculum in Zambia operating in the OPRF is dominantly transformational. It became necessary that the algebraic activity for the choice of scenarios also be dominantly transformational. The first three scenarios, discussed in Chapter 8, were elicited from the literature while the other three scenarios, discussed in Chapter 9, were drawn from local learners' own working on a test.

Following Kieran (2004) and as described in Chapter 3 (Section 3.4.2), the algebraic activity in these scenarios is that scenarios 1, 2, 3, 4, and 6 are transformational while scenario 5 is generational. Scenario 1, which I have called 'the expression-equation problem' requires learners to simplify the open algebraic expression which is '$2x+5+3x-7$'; but the problem arose when they equated the expression to zero and solved for *x*. Scenario 2 called 'the conjoining problem' also requires learners to simplify the expression '$3m+2+2m$'; but the problem that arose was conjoining $5m+2$ to get $7m$. Scenario 3, which I have named 'the quadratic-linear equation problem' requires learners to solve the quadratic equation '$2x^2=6x$'; but the problem that arose was dividing both sides of the equation by *x*. Scenario 4, which I have called 'the expression-expansion by index problem' requires learners to expand the binomial '$(x+3)^2$'; but the problem that arose was distributing the power 2 to each term and get x^2+9. Scenario 5, which I have called 'the expression-expansion by number problem' requires learners to come up with a symbolic representation of 'Multiply *n* + 5 by 4' and then multiply out; but the problem that arose was multiplying 4 by 5 only to get *n* + 20 and a further 20*n*. Lastly, Scenario 6, which I have named 'the quadratic equation-factors problem', requires learners to solve for *x* in the quadratic equation $(x-1)(x+2)=4$; but the problem that arose was equating each factor to 4. See Appendix C for the interview guide and the scenarios that were used. In the same manner, probes to the key questions of

the interview schedule depended on student-teachers' responses during the process of interviewing.

The local learners' own working on the test enabled the elicitation of common learner errors in algebra reported in the field of specialised mathematics education research (UPRF). Selections of these common learners' errors, in their own handwriting, were used as scenarios and engaged with pairs of student-teachers in interviews. The three scenarios included what I have called the expression-expansion by index problem (Scenario 4), the expression-expansion by number problem (Scenario 5), and the quadratic equation-factors problem (Scenario 6). These are described in detail in the analysis chapter (see Chapter 9). Using learners' own handwriting in the scenarios was meant to enable an understanding among student-teachers that algebraic errors reported in the specialised mathematics education research field could also arise in the Zambian context. Moreover, I reflect on how such a context would also influence the way student-teachers talk about error analysis, hence their shared *repertoires* (described in Section 4.2.4). The analysis of each of these scenarios in terms of how literature would explain possible sources of the errors formed what I also described in Section 4.2.4 as *reservoirs*. Therefore, for each scenario, possible *reservoirs* and student-teachers' shared *repertoires* shape part of the analysis as described in section 5. 5; and shown in Chapters 8 and 9 of this thesis.

It is important to mention here that I worked with eight pairs of student-teachers, not 10 pairs because 4 student-teachers withdrew from the study as it progressed and I respected their decision. This was in line with the ethics for carrying out research which I explicitly made clear to them and are discussed in Section 5.7. For example, I informed the student-teachers that if they feel uncomfortable to continue as participants in the study, they could withdraw at any stage. The focus in the paired interviews was on what and how of student-teachers' explanations pertaining to the errors presented in each of the scenarios. Therefore, *what* do the student-teachers recognise as errors, and *how* do they explain and justify the recognised errors in terms of the sources of errors and suggested remediating strategies to enable learners develop the required algebraic thinking, hence focus on learner thinking and sense making in algebra. Student-teachers were given ample time to familiarize themselves with the scenarios prior to being interviewed. The interviews were conducted in two phases. Phase one involved scenarios 1 to 3 while phase two involved scenarios 4 to 6.

As I already explained in Chapter 1 (Section 1.2) of this thesis, as student-teachers participate in the discourse of and about engaging with LMT in their mathematics education programmes, some of the specific issues around algebra are not dealt with. Thus the question: As student-teachers engage with specific learner errors in algebra, what do they do with these? What has been the general preparation or the preparation in other areas that has enabled the student-teachers to recontextualise and look at LMT in terms of algebra? Thus, I am trying to see what is there in general pertaining to the discourse of engaging with LMT and then what is there when something specific such as scenarios on common learner errors in algebra are dealt with.

5.5 Data Analysis

Since this study is qualitative in design and that the main instrument for collecting data is interviews, as indicated in Sections 5.2 and 5.4, respectively, the main data sources are transcripts focusing on what teacher-educators say they select and privilege pertaining to the discourse of engaging with LMT; and their student-teachers' recognition and realization of this discourse. What then do student-teachers actually say as they participate in a discourse of and about engaging with LMT in general, and algebra in particular? To establish student-teachers' thinking of the discourse, the general focus is on their understanding of what the discourse entails, and a specific focus is on algebraic scenarios on common learner errors through what they communicate. Moreover, of importance is establishing the relationship between what student-teachers say and what their teacher-educators privilege as the discourse of engaging with LMT. This suggests that what teacher-educators say they select and privilege pertaining to the discourse provides the context within which the discourse occurs within the PRF, hence their legitimate criteria. Depending on how implicit or explicit the criteria are, it will influence the acquisition of what entails the discourse of engaging with LMT, hence my interest in exploring and explaining the evaluation criteria pertaining to the transmission and acquisition of the discourse of engaging with LMT.

Therefore, a unit of analysis for my study is an *evaluative event*. I recruit from Adler & Davis (2006) and Davis et al. (2007) the notion of an *evaluative event* as a unit of analysis in teacher education research but use it differently in that I use it for interviews. Their empirical data were assessment tasks used in selected formalized teacher education sites (Adler & Davis, 2006) and instruction in teacher education classes (Davis et al. 2007). An event was related to object and criteria where criteria are central. The notion of an evaluative event is

based on the assumption that pedagogic discourse unfolds over time and that central to any pedagogic practice, is the transmission of criteria, be these implicit or explicit transmits criteria. In short, pedagogy is highly evaluative.

From the aforesaid, for Adler & Davis (2006), the constitution of mathematics in any practice is reflected through what and how criteria come to work, hence through evaluation. For example, in Davis et al. (2007) in focusing on the models of teaching in teacher education and discussed in Section 3.2.4, activities in the classroom are broken down by seeing when an object is introduced and how it is legitimated. They called this an evaluative event because criteria are transmitted as to what the object is and what it is not, and why. Similarly, an evaluative event was used as a unit of analysis for assessment tasks used in selected formalized institutions of teacher education because an assessment item in itself has a beginning and an end in talking about it, and implied evaluative criteria. The focus was on establishing what was privileged in terms of the reasoning that was required between mathematics (M or m) and teaching (T or t) and whether there was some "unpacking" (U^+ or U^-) of the knowledge in the tasks. This was based on their assumption that both mathematics and teaching are implicated in the formulation of the tasks. For example, if mathematics was foregrounded with no aspects of unpacking, and teaching backgrounded but with some aspects of unpacking, the assessment task was coded MU^-tU^+. A new piece of teaching then signals a new event or a new assessment task is an event. In both cases Adler & Davis (2006) and Davis et al. (2007) were concerned with the evaluative criteria in terms of appeals and what was legitimated in the practice.

In my study, firstly I have the broad interviews with teacher-educators and student-teachers as conversations, and then I have scenarios. Adler & Davis's (2006) evaluative event in using assessment tasks is similar to what I do with student-teachers' participation in the scenarios. I view each scenario as a new event because when scenarios are discussed with student-teachers, they have to legitimate what they say in terms of what the learners have done. I am then interested in establishing criteria the student-teachers give for why they think the learners make the errors as exemplified in the scenarios. This suggests that each analysis of student-teachers' interviews on scenarios is a piece of text that attracts evaluation. This is based on my assumption that stages of carrying out error analysis, and informed by Peng & Luo (2009) and Jacobs et al. (2010), which include identify, explain, and remediate are implicated in student-teachers' talk on the scenarios.

The broad interview with teacher-educators and student-teachers cannot be described as events in the way Adler & Davis (2006) and Davis et al. (2007) talk about an event. As already indicated, since my data is completely different from Adler & Davis (2006) and Davis et al. (2007) because it is not pedagogy but conversations, my assumption is that even in conversations aspects of pedagogy are discussed. That is, what entails the discourse of engaging with LMT is discussed, and that here too judgements will be made about what it is and what it is not, and why; and criteria transmitted. What then I have called events in this instance is recognized very differently. Since teacher-educators and student-teachers do legitimate the descriptions of what entails the discourse of engaging with LMT, I consider a piece of text that describes what engaging with LMT is and how it is justified, as an event. The recognition is in terms of how their talk resonates with any one of Even & Tirosh (2002) broader categories of what entails the discourse of engaging with LMT and positionings. These categories include developing in learners both instrumental and relational understanding, focusing on learner errors, and creating an environment where teacher can listen to learners. Therefore, in both cases (firstly, broad interviews with teacher-educators and student-teachers and then scenario-based interviews with student-teachers), I am looking for how judgements are made about what the discourse of engaging with LMT is, and so the criteria.

5.5.1 Analysis of broad interviews with teacher-educators and student-teachers

In Table 7 below, I exemplify the coding of the broad interviews with teacher-educators' (in their individual interviews) and their student-teachers' (in their focus group discussions). I highlight the talk that was coded in the first instance, in terms of its resonances with Even & Tirosh's (2002) categories of the discourse of engaging with LMT. Developing in learners both instrumental and relational understanding is one such category. In the first column in Table 7, Kenneth, a teacher-educator says that engaging with LMT is about developing in learners reasoning that is a particular mathematical way of reasoning. This reasoning includes arguing (A), evidencing (E), and systematic thinking (S), I coded this as MR-R-AES (Mathematical reasoning that is relational). When Kenneth contrasted this description of reasoning with showing learners solutions (SS) and finding answers (FA) to mathematical problems, I coded this latter talk as MR-I-SS and MR-I-FA (Mathematical reasoning that is instrumental), respectively. As argued in Chapter 4, and this illustrative data suggests, there

could be multiple texts pertaining to this category depending on teacher-educators' different descriptions of MR-R and/or MR-I.

In their talk, I also focused on the 'how' of LMT in terms of the 'teaching' of it, and what they say about teachers/student-teachers (T or ST), learners (L), and the curriculum (C) (in school or teacher education) in the process. This is in terms of Morgan et al.'s (2002) notion of positioning pertaining to absences and presences, this entails for example, teacher-educators' positioning of teachers/student-teachers, or learners or the curriculum (in school or teacher education) as they talk about what LMT is and how they 'teach' for it. Similarly, for student-teachers' talk focus was on the 'how' of LMT in terms of 'learning' for it and what they say in the process about teachers/teacher-educators, learners, and the curriculum (in school or teacher education). Therefore, focus was on student-teachers' positioning of teachers/student-teachers/teacher-educators, or learners or the curriculum in terms of absences and presences as they talk about what LMT is and how they 'learn for it. For example, I coded as absences (A) when teacher-educators' or student-teachers' talk focused on what it is that teachers or learners or the curriculum were doing wrong in terms of what they think LMT is and how they 'teach' or 'learn' for it. Similarly, I coded as presences (P) when teacher-educators' or student-teachers' talk focused on competencies of teachers or learners or the curriculum pertaining to what they say LMT is and how they 'teach' or 'learn' for it.

Also exemplified in Table 7 is the coding of 'where' the teacher-educators and student-teachers said in their mathematics education courses or topics the discourse of engaging with LMT is focused on. I was interested in establishing whether what they say LMT is and its 'teaching' or 'learning' has a direct focus in the mathematics education curriculum or is embedded in other topics explicitly outlined. Analysis of this enabled the description of LMT in terms of Bernstein's (1982, 1996, 2000) notion of classification, that is, whether it is in closed or open relation with other contents of the mathematics education curriculum; thus determining whether the discourse of engaging with LMT is a strongly or weakly classified discourse and what this means for its identity and voice.

The analysis in the first phase enables the descriptions of teacher-educators' and student-teachers' discourses of LMT in terms of what it is and how it is 'taught' or 'learned', respectively, and positioning (absences or presences) in them. I also describe some teacher-educators' or student-teachers' talk in relation to the questions that were raised from review

of relevant literature in Chapter 3. For example, one of the questions is concerned with possibility of transforming SMK to PCK. Following up on this, Sam (one of the teacher-educators) points to how he emphasizes to his student-teachers the importance of teaching for understanding (TU) and not showing their learners how much mathematics they know. It is also possible to establish how teacher-educators' discourses relate to their student-teachers' discourses. This is in terms of what messages student-teachers recruit or take-up from what teacher-educators select and privilege as the discourse of engaging with LMT and what is different, hence how this might be explained.

Table 7: Examples of how broad interviews on LMT were analysed drawn from Kenneth's and Sam's talk

Excerpt	Where	What	How			
			Positionings			'Teaching'
		LMT	T or ST	C	L	
Extract 6.2 from Kenneth's talk on LMT I: OK. Do you follow any sequence in the way aspects of learner mathematical thinking are introduced in your courses? K: No definite sequence except that my first entry point in that perhaps it doesn't address directly learner mathematical thinking, but I devote a lot of time at the beginning of those coming new to mathematics teacher education, I do have/dwell a lot of time on the aims for which we teach mathematics. And we make big discussion on the fact that teaching mathematics is not the same thing as showing pupils how to find answers. And sometimes this may look like a small issue, sometimes people from colleges are said to be better teachers and yet what they mean is that we are able to show pupils how to find answers. So I do dwell a lot that teaching maths is not the same as showing someone how to solve. It is with the expectation that you achieve a certain way of mathematical reasoning. And to me I would take it that what is meant by learner mathematical thinking is that we expect to achieve a certain way in which a mathematician must reason, a mathematical way of reasoning. And you won't achieve that by simply showing a way of solving a problem, but there must be a way of arguing, some way of showing evidence that this is equal to that, therefore this, hence that starts thinking process which is a mathematical way of reasoning. And sometimes you can show children how to find answers without cultivating this type of thinking. And it's this type of thinking that makes mathematics compulsory – that people must be systematic in thinking in whatever they endeavour. Even if it's a job that does not involve mathematics but your type of thinking must be	Aims for which mathematics is taught	**MR-I-FA** **MR-I-FA** **MR-I-SS** **MR** **MR-I-SS** **MR-R-AE** **MR-I-FA** **MR-R-S**			

systematic. And we hope that Maths is better placed to achieve that. I: Mmm K: So in terms of learner thinking, um, that's my entry point. Beyond that the order may not be consistent.		**MR-R-S**				
Extract 6.30 from Sam's talk on LMT						
So at a certain stage I'd said, 'When you are introducing this topic to children at Grade 10 level it's important to write the steps how you are going to arrive at the answer.' They are not interested in that because to them they 'I want an answer.' I said, 'No, I want you to tell me what you want students, pupils to be doing from one stage to the other. Write it down for me so that I can see whether you are helping the children or you are solving the problem. 'Ah but Sir, you know how, it's so easy. From there you do this, there are a few other things that have been done here in there.' I said, 'Well, if you tell me that you have thought about it here, how sure are you that the children have also understood it? How sure are you that the pupils were following your thinking? I'm not interested in an answer now, I'm interested in how you would have to… Because most of the times you'll find that pupils will actually make silly errors that will actually approach the children, the question from a different angle that you did not anticipate.' . . . If you are teaching mathematics, I'm sorry, nobody is going to understand. But if you try to teach them mathematics, for example, giving them the steps that you'll follow when you are solving these equations, when they are in Grade 12 even if they don't write those steps you are sure that they should be able to understand it because you've covered it at that particular level. But many times you say me I'm a graduate from the university I know how to solve this question. This is what you do, you children, from there you move to this, do this, do this. Well you have solved the problem, but they never understood the steps.' *(laughs)*		MRR – TU with focus on necessary steps and not just solving the problem	There is a sense of absences in student teachers in terms of focusing on explaining the necessary steps required in helping a learner understand since they are interested most in the answer. A			Teacher educator encourages student teachers to write all the necessary steps of arriving at the answer to help learners understand.

The rubric in Table 8 is a summary of the three categories of what entails LMT and indicators of how each might be recognised as an evaluative event. This rubric was used to carry out the analysis of teacher-educators' and student-teachers' discourses of LMT as exemplified in Table 7 in relation to developing in learners both relational and instrumental understanding. Although these categories are analytically distinct, they could feed into each other empirically. If none of the teacher-educators' or student-teachers' talk resonated with any of the three categories, I provided a category that I named 'other'. However, none of their talk fell in this category according to my analysis. In Appendix D, I provide detailed examples of the analysis of teacher-educators' interviews on LMT in relation to how the rubric was used. Student-teachers' initial interviews were also analysed using the same process.

Table 8: Categories of LMT and their indicators

Nature of teacher - educators'/student – teachers' discourses of LMT				
Excerpt	**Where**	**What** [refers to the principle of classification, hence providing the limits of any discourse (Bernstein, 1996)] (What teacher-educators or student-teachers say LMT is)	**How** [refers to the principle of framing, hence providing the form of the realization of that discourse (Bernstein, 1996, 2000)] (How teacher-educators or student-teachers say they 'teach' or 'learn' for LMT, respectively and their positionings)	
Key excerpts extracted from the data as evidence on the teacher-educators'/student-teachers' discourses on what knowledge and awareness is required pertaining to LMT.	This refers to a content topic area in the mathematics education courses around which the discussion on LMT takes place. -Where is it in focus? This will tell me how classified or framed it is, i.e. weak or strong; -if it is embedded in other aspects or it is on its own; -is it inside a topic in the course or is it a topic on its own in a course.	Following Even and Tirosh (2002), LMT includes a discussion on three aspects namely: (1)Learner errors and misconceptions: Learners tend to build their knowledge of mathematical concepts and ideas in ways that are not aligned with conventional mathematical knowledge, hence generating errors in the process. Therefore, it is necessary for a teacher to attend to and understand LMT so that appropriate ways can be designed to enhance knowledge construction. **Code to be used: LEM** (**L**earner **E**rrors and **M**isconceptions, **LEM**) – I will recognize this when teacher-educators/student-teachers talk about learners who make errors or have misconceptions as LMT. That is LMT is about errors and misconceptions, that is, in terms of nature of errors – View error as a natural stage in knowledge construction in that they provide opportunity for learning, are normal and necessary for the process of learning; and that errors are persistent and systematic (Borasi, 1987; Nesher, 1987; Olivier, 1989; Ryan and Williams 2007; Smith, diSessa &Roschelle, 1993). I will also recognize **LEM** when teacher-educators/student-teachers talk in terms of strategies of carrying out error analysis as conceptualized by Peng & Luo (2009) and Jacobs et al. (2010), namely: **Identify (I):** Knowing the existence of mathematical error. **Explain (E) [Interpret/evaluate]:** Interpreting the underlying rationality of mathematical error. Evaluating learners' levels of performance according to mathematical error. **Remediate (R):** Presenting teaching strategy to eliminate mathematical error. **Code to be used: LEM - IER**	I will recognize this when teacher-educators/student-teachers talk about ways of becoming knowledgeable and aware of LEM. That is, how teacher-educators/student-teachers think LEM is learned. For example, Kenneth talks of how he emphasizes to his student-teachers the importance of giving specific rather than general strategies when focus is on strategies for remediating errors learners make in selected school mathematics questions.	I will recognize teacher-educators' and student-teachers' orientations towards student-teachers/teacher-educators/teachers (T)/learner (L)/the curriculum in school or teacher education (C) in terms of presences or absences from how they talk about the 'what' and the 'how' of the discourse of engaging with LMT. I will recognize presences when teacher-educators' or student-teachers' talk resonates with what they think are potentials in relation to working with LMT. Similarly, I will recognize absences when teacher-educators' or student-teachers' talk resonates with what they think is inadequate focus in relation to LMT. **Codes to be used: A (for**
		(2)Enhance the development of both instrumental and relational or procedural and conceptual	I will recognize this	

		understanding of mathematical activities in the learners (Quoting Skemp, 1976 and Kilpatrick et al. 2001 respectively). **Code to be used: MR** (**M**athematical **R**easoning) – I will recognize this when teacher educators/student teachers talk about learner reasoning as LMT. That is LMT is about ways of reasoning such as relational (**R**) and/or instrumental (**I**). I will recognize Instrumental reasoning when the talk resonates with mathematical rules without reasons where focus is only on finding the answer. I will recognize relational reasoning when the talk extends to include explanations for the steps taken to finding the answer, hence the centrality of meaning. So **MRR and/or MRI**	when teacher-educators/student-teachers talk about ways of becoming knowledgeable and aware of MR. That is, how the teacher educators/student teachers think MR is learned. For example, Kenneth talks of how he emphasizes to his student-teachers the importance of putting more time and effort in the planning of a lesson.	**absences) and/or P (for presences)**
		(3)Access what learners are thinking by encouraging an autonomous classroom culture where learners are free to make conjectures, explain their reasoning, validate their assertions, discuss and question their own thinking and the thinking of others, and argue about what is mathematically true. **Code to be used: Env** (**Env**ironment) – I will recognize this when teacher-educators/student-teachers talk about strategies for eliciting MR and LEM.	I will recognize this when teacher-educators/student-teachers talk about ways of becoming knowledgeable and aware of Env. That is how teacher-educators/student-teachers think Env is learned. For example, Titus talks about how he encourages his student-teachers ask their learners questions from the onset in the process of teaching in that learner thinking so realized could be engaged in a discussion to arrive at common understanding.	
		Other: To be used as a code when teacher educators/student teachers talk about LMT as something else other than MR, LEM and Env. Specific codes might then develop.		

5.5.2 Analysis of scenario-based interviews with student-teachers

The second phase of data analysis focused on the scenarios, each of which is realised as an evaluative event for my study is when student-teachers (in their paired interviews on scenarios of errors described in literature and those elicited from learners' own working) talk. I analysed the talk in relation to Peng & Luo (2009) and Jacobs et al. (2010) steps of carrying out error analysis. For each scenario as illustrated in Table 9, I focused on (1) whether each pair recognised or misrecognised the error (2), their explanations for the sources of error recognised, and (3) their suggested remediating strategies to enable the development of the required algebraic thinking in learners. The analysis thus includes both the 'what' of the recognition of error in the scenarios, and the 'how' of the explanation of the sources and suggested remediation strategies. The main explanations for the possible sources of errors that emerged from the analysis are that they are as a result of teaching emphasis or teaching sequence (ordering or overgeneralization) or a problem of interpretation. If the source of error was as a result of teaching sequence, and in particular overgeneralization, then in Lima & Tall (2008) terms, it was further categorised as an issue of either met-before or met-after as discussed in Section 3.3.3. It is unusual to discuss results here but this has been done for the purposes of further categorization.

In each of the three steps of carrying out error analysis, using Morgan et al. (2002) terms and described in Section 4.2.3.2, positioning in terms of 'voice' and 'form of practice' were analysed. In terms of voice, I focused on whether student-teachers spoke with voice of official discourse of school algebra (VOD) as stipulated in the algebra curriculum in Zambia (the ORF and outlined in Chapter 2) or voice of unofficial discourse (VUD) referring to either everyday professional experience of teaching and learning (E) or discursive/theoretical (D/T) tools developed in the specialised mathematics education research field. In relation to form of practice, the analysis focused on orientations and strategy. Four forms of practices emerged from the analysis, which are orientation towards teaching and absences (OTA), orientation towards learner and presences (OLP), orientation towards mathematics and absences (OMA), and orientation towards teacher and presences (OTP). The indicators for recognition of voice and forms of practice are described in Table 10 below.

Table 9: Example of how scenario-based interviews were analysed

An Evaluative Event: Scenario 6 on errors elicited from learners' own working (Interview 1)							
Scenario 6	**Excerpt**	**What do student teachers recognize as the error**	**How do they recognize the error**				
			Explain (interpret/evaluate)				**Remediation (How do they act on the error)**
				T	**C/M**	**L**	
	I: So, we can move on to Task six, they were asked to solve for *x* in this equation, *x minus one times x plus two is equal to four*. So the learners said this implies *x plus two is equal to four* or *x minus one is equal to four*, giving *x is equal to four minus two* or *x is equal to four plus one*, giving us *x is equal to two* or *x is equal to five*. So, how do you interpret learners' understanding in this task? ST1: The learners' understanding in this task is a problem of, as a teacher was explaining on the part of how to solve quadratic equations, expressions equal to zero and when it is equal to any other number. So, on the part where it is equal to zero, that is where you can do it the way they are doing it. Now the problem is that these pupils misinterpreted just because all the examples, may be the teacher tackles were all equal to zero in that form and we are saying either this is equal to that or that is equal to that. So, the pupils take it that in as long as this is in brackets and the other is in brackets, whatever it is equal to, then you must say either this is equal to that or that is equal to that. Yes. I: So, is it right? ST1: It is wrong.	Learners are solving the non-standard quadratic equation as if it were standard. (In recognizing the error, they speak voice of official discourse of school algebra – VOD)	Error stems from teaching emphasis: -Teacher could just have focused on quadratic equations which are already in standard form, hence the misinterpretation on the part of the learners. (VUD – E)	**Oriented towards teaching and absences (OTA)**	**OMA**		

	I: Why? ST1: It is wrong and it is very, very wrong. I: Why? ST1: Because on the part of quadratic expressions when are we allowed to do that? It is when on the other side it is equal to zero. That is when we can say these in brackets would be equal to that or that. Now if it is not equal to zero, then we must first of all expand what is in brackets and then go back to quadratic equations, how we solve them. Are they factorizable, later on by taking the like terms together and grouping all of them together making sure that one side of the equation is zero. In that way, we can solve it so that in this case like I try to pass through it to be x minus two multiplied by x plus three equals zero, which means either x minus two is equal to zero or x plus three is equal to zero so that x is two or x is negative three. I: Oh, so you found x as being two or negative three? ST1: Yes. I: So we have an x is equal to two there. ST1: Of course there is an x equals two, we cannot take this method to be strictly correct, because if we try to change anything on another one that would be similar in the same way. I don't think I would emphasize to say it would be same value that we get. Somehow somewhere there will be a change.						

	I: What do you mean? ST1: What I mean is, this two of course there is a chance that a wrong answer could be found to be the exact answer that the teacher wanted but the method you have used is a wrong method, anyway. I: So, how would you interpret x being equal to two or x being equal to five? ST1: [chuckles] I: And then you solve it in the way you are saying it should have been solved that you get x is equal to… ST1: …two and x equals to negative three. ST2: There are times when you use a wrong method and it gives you the correct answer, but the fact is pupils should be able to use the correct ways of solving. Like he said, when it is equal to zero, this one was to be correct. Now in this case, it is not equal to zero. It is a number four. I: Why should it always be equal to zero? ST1: No, when I am saying if it was equal to zero… I: Yes, I am saying why should it equal to zero? ST2: When we are doing quadratic equations and then while there are being as in factorized, while a point is emphasized, when we were introduced to quadratic equations I remember very well, that at a point where you have factorized and the other side	The getting x = 2 is coincident and cannot be considered to be correct because of the wrong method used.					

	is zero, then you can take them to say either this part is equal to zero or the other part will be zero. And now that is the background of where we say a variable times another variable equal zero is the same…is either one variable is zero or the other one is zero, or vice versa. But, one of them could be zero, the other one, or both of them could zero. That is why we take either that to be zero or the other one to be zero because there will be a situation where may be the other one is not zero and may be two, so that whenever we say two times zero it will still give us zero. I: Okay. ST2: Yes. I: So, how would you help the learners develop the required algebraic thinking in this context? ST1: It is the logical way. I think our pupils must be taught how to handle such first of all. We should tell them to say number one, when it is of this form, make sure that on one side it is zero and then what is on the other side is completely factorized then you can apply this method. So, the logical way of how to do it, you give them you show your pupils, you emphasize it and in that way the pupils will be able to learn and see how they would apply it. Now, if you don't emphasize, you just give examples that are almost similar all of them, they are all equal to zero, they wouldn't tell…it is not always that all the pupils will ask: what about if it was equal to four on this side? They would not ask that, and then imagine now you bring a question and put a four on the other side, instead of a zero. And then pupils		Error stems from teaching emphasis: -Teacher could just have focused on quadratic equations which are already in standard form, hence the misinterpretation on the part of the learners.				Emphasize key concepts or issues during teaching: -Emphasize procedure in terms of expressing the quadratic equation in standard form and then factorizing.(VOD; OTP)

	will misinterpret like that. I: Okay. ST1: Yes.						

I: Interviewer, ST1 – Student-teacher 1, ST2 – Student-teacher 2

In Table 10, I present a glossary of statements and their descriptions which enabled me identify possible sources of errors and positioning in terms of 'voice' and 'form of practice' in reference to what student-teachers said when they engaged with the six scenarios. This rubric was used for analysing each of the eight pairs' utterances and the findings were then synthesized. Of importance was establishing the relationship between what teacher-educators privilege as the discourse of engaging with LMT, and student-teachers' realizations in the initial interviews (focus groups), and how these two foci together relates to student-teachers' engagement with scenarios in pairs. As the analysis chapters unfold in Chapters 6, 7, 8 and 9 of this thesis, it will become clearer on how the codes in both Sections 5.5.1 and 5.5.2 helped in carrying out my own analysis.

Table 10: A glossary of statements and their descriptors

Statements	Description
Error stems from teaching emphasis	This is referred to when student-teachers locate the source of the error in teaching in terms of the teacher having not emphasized key concepts or terms or issues.
Error stems from teaching sequence	This is referred to when student-teachers locate the source of error in two aspects: (1) In the sequencing of topics or concepts during teaching, for example, the teacher could have overlooked the teaching of simplification of expressions and went straight to teaching solving equations with the anticipation that simplification of expressions will be learned in the process. (2) In the learners' knowledge of related concepts that could have interfered with the learning of the concept in question, that is, the issue of overgeneralization – the case of met-before or met-after.
Error stems from an interpretation problem	This is referred to when student-teachers locate source of error in the definitional meaning of an algebraic expression, or direct translation of the mathematical statement.
Orientation towards mathematics and absences (OMA)	This is referred to when student-teachers criticize learners for what they have done wrong in terms of the school mathematics.
Orientation towards learner and presences (OLP)	This is referred to when student-teachers' talk is in solidarity with what the learner has done. The issue of overgeneralization is included here because the learner is seen to have some competences.
Orientation towards teaching and absences (OTA)	This is referred to when the teacher is criticized for having not emphasized key concepts or issues in their teaching.
Orientation towards teacher and presences (OTP)	This is referred to when student-teachers talk in solidarity with teachers in terms of what they ought to be thinking of incorporating in their teaching. That is teachers' reasoning when deciding how to respond.
Speaking with the voice of the official discourse (VOD)	This is referred to when student-teachers talk of ways of working with school algebra in ways that are stipulated in the school curriculum.
Speaking with the voice of the unofficial discourse (VUD)	This is referred to when student-teachers talk is in two ways: (1)Experiential, that is everyday professional experience of what it means to teach (at the level of the sensible) (2)Discursive or theoretical by drawing on terms or concepts developed in the field of specialized mathematics education literature (at the level of the intelligible)

In Table 11, I provide a summary of the three main research questions for my study, and then the various data and analysis methods that were used to answer these.

Table 11: A summary of research questions, data sources and analytic frame

Main research questions	Data sources	Analytic frame
1.What do teacher-educators select and privilege with respect to the discourse of engaging with LMT?	Individual interviews with four teacher-educators.	Analytic resources drawn from literature which describe LMT to include: developing in learners both instrumental and relational understanding; focus on learner errors; and creating an environment where teacher can listen to learners. Evaluation criteria, classification and framing, recontextualization 'sensible' and/or 'intelligible' – a practical accomplishment, Teacher-educators' positioning of student-teachers/teachers/teacher-educators, or learners, or the curriculum in terms of absences and presences.
2.What are student-teachers' recognitions and realizations of the discourse of engaging with LMT?	Two focus group interviews of ten student-teachers in each discussing what they think LMT entails.	Analytic resources drawn from literature which describe LMT to include: developing in learners both instrumental and relational understanding; focus on learner errors; and creating an environment where teacher can listen to learners. Evaluation criteria, classification and framing, 'sensible' and/or 'intelligible' – a practical accomplishment, recognition and realization rules Student-teachers' positioning of student-teachers/teachers/teacher-educators, or learners, or the curriculum in terms of absences and presences.
	Eight paired interviews on six scenarios of common learner errors in school algebra of which three are described in literature and the other three elicited from learners' own working.	Analytic resources from literature on strategy for carrying out error analysis involving identify, explain and remediate. Reservoirs, Repertoires Modelling error, prototypical error, error as a result of overgeneralization – met-before/met-after, process-object conception. Learners' interpretation of letters: Letter evaluated, letter not used, letter used as an object, and letter used as a specific unknown Positionings in terms of 'voice' and 'form of practice'
3.What is the relationship between teacher-educators' discourses of engaging with LMT and that of their student-teachers, and how might this be explained?	Teacher-educators' discourses established in research question 1 and student-teachers' discourses established in research question 2	Relate teacher-educators' discourses with student-teachers' discourses to establish what student-teachers recruit or take-up from teacher-educators' privileged selections of LMT.

5.6 Trustworthiness: Reliability and Validity

Reliability is the extent to which a test, a method or a tool yields consistent results across a range of settings, and if used by a range of researchers (Wellington, 2000). Wellington argues that this is linked to the aspect of 'replicability' which implies the extent to which a piece of research can be copied or replicated in order to yield the same results in a different context with different researchers. Validity refers to the degree to which a method, a test or a research tool actually measures what it is supposed to measure (J. Bell, 1999; Wellington, 2000). The two terms reliability and validity are contentious when it comes to qualitative studies. Many researchers argue that the two terms are never certain when a researcher is studying naturalistic behaviour or unique phenomena because human behaviour is never static (Merriam, 1998; Wellington, 2000). Therefore, in qualitative studies, what counts as reliability is the extent to which the already discovered categories are recognisable to others (Adler, 1996). This implies an observer looking at the same data according to the researcher's categories so as to establish whether the categories are recognisable in the data (Silverman, 1993).

For my study, data was reviewed by experts to establish whether they were able to recognise the categories I developed. These experts included my supervisor and colleagues in the research community. I also checked with them if I was applying my categories consistently by carrying out the analysis in the same way throughout; and systematically by assigning the correct label to the right category. I provided these experts with a selection of transcripts of data and a system of categorization and the indicators I used to describe my data. I also provided a synopsis of how I have used the indicators to organize my data. The analysis was also kept 'close to the data' by providing readers with quotations from teacher-educators' or student-teachers' own working or words so that they can check my conclusions or draw their own conclusions - hence letting data speak for themselves (Carr, 1996).

Similarly, validity in qualitative studies is referred to be the relationship between interpretation and evidence (Adler, 1996). This implies that the interpretations the researcher makes about the data should be supported by evidence. Two types of validity are evident in qualitative studies, namely, descriptive and interpretive validity; and theoretical and explanatory validity (Maxwell, 1992). Descriptive validity requires accuracy in reporting facts while interpretive validity requires accuracy in the interpretation of the facts found (Maxwell, 1992). Both descriptive and interpretive validity can be achieved through careful

transcriptions and interpretations. For my study, to ensure careful transcriptions, the interviews and focus group discussions were audio taped. Therefore, every word spoken by the teacher-educators or student-teachers was transcribed without any alterations. Careful interpretations were ensured by categories that are recognisable and 'remain close to the data'. Theoretical and explanatory validity is achieved by establishing a systematic link between theoretical concepts and the theoretical framework of the study (Maxwell, 1992). For my study, theoretical validity was achieved by using theoretical concepts and terms, which are coherent to the theoretical framework of the study, and these are detailed in Chapter 4

5.7 Research Ethics

Wellington (2000) asserts that an 'ethic' is a moral principle or a code of conduct which guides what people do. Ethics are moral principles which guide conduct and are held by individuals, a group, or a profession. Just like many authors have placed importance on ethical considerations in any research, Wellington (op cit) argues that these ethics override all other aspects of research. For example, research could be unethical in its design, questions, methods, analysis, presentation, and findings. Based on this, I applied for ethics clearance for scrutiny by the Ethics Committee of the Witwatersrand School of Education. Upon recommending that the study was ethical, authority was granted for the study to be conducted as was proposed. Most of the research ethics hinge on the aspect of 'informed consent' in that:

> *Participants in a research study have the right to be informed about the aims, purpose and likely publication of findings involved in the research and of potential consequences for participants, and to give their informed consent before participating in research* (BERA[14], 1992 in Wellington, 2000, p. 56).

Written consent was sought from the relevant authorities for me to enter the research sites and conduct the study. These authorities include the Dean, School of Education, and consequently the Head, Department of Mathematics and Science Education in a University; as well as the Permanent Secretary in the Ministry of Education in Zambia and the Head Teachers of secondary schools I involved in the study. The aim, purpose and benefits of doing the research in the University and in the secondary schools were explained in the

[14] BERA stands for British Educational Research Association who have since published a set of ethical guidelines on research.

request. Written consent was also sought from the twenty student-teachers, four teacher-educators, and secondary school learners with a cross range of attainment in school mathematics (with the help of their mathematics teachers). Similarly, the aim, purpose, and the processes they would be involved in, as well as how their participation in the study would be of benefit to them was explained. Participants were informed that if they decide not to continue their participation in the study, they were free to withdraw. As already explained in Section 5.4.1.3, four student-teachers withdrew from participating in the study.

To ensure confidentiality and anonymity, I informed the participants that their identity would be protected in that their names and that of the institution would remain anonymous in reporting the findings in form of thesis, journal publications, and conference presentations, by using pseudonyms or numbers. Moreover, the participants were told that raw data which may contain their names will be stored in a locked cabinet, and will be destroyed when no longer required. They were also informed that only researchers directly involved in my study, such as my project supervisor and members of my supervisory committee, will see the data, for example notes from interviews and transcripts of the audiotapes. I also explained to the participants that their participation in my study would not affect or change their status in the University and the school as student-teachers or learners, respectively. Teacher-educators were also informed that their participation in the study would not in any way affect their positions as lecturers in the University. Moreover, student-teachers and learners were also informed that their participation in the study would not in any way affect their marks. See appendix E for examples of the actual consent letters that the authorising agents and the participants signed, including the ethics clearance letter from the ethics committee of the Witwatersrand School of Education.

5.8 Limitations of the Study

The purpose of my study is to explore and explain what student-teachers know and are able to do when they participate in the discourse of and about engaging with LMT in general, and algebra in particular. Thus, how prepared are the student-teachers to participate in the discourse. There is wide agreement that what participants say might not necessarily reflect what they actually do in practice. I did not observe teacher-educators and their student-teachers interacting with focus on the discourse of engaging with LMT in their mathematics education courses. During the design of my study and informed by their curriculum, which is a list of topics to be covered as described in Section 2.3, I realized that it would be difficult to

identify specific topics to observe where LMT is focused on. The notion of engaging with LMT is also weakly classified as the results have shown in Chapters 6 and 7 of this thesis. Data on discourses of LMT in teacher education is thus limited to what was said.

Moreover, I did not follow the student-teachers into schools and observe their teaching as this would have had implications on the delineation of the study since observing them in school is a different context with different effects. It is possible that the scenarios the student-teachers engaged with might not necessarily reflect the exact aspects of LMT to be experienced in their classrooms because the learners they will interact with might be different from the ones involved in this study. The reason could be that the learners have different potentials and are bound to make errors depending on the culture of the classroom. For example, the way algebra is introduced in the classroom can result in varied errors and difficulties by learners.

As already explained in Chapter 2 and Section 5.3, both the pre-service and in-service teachers participated in the study. The study was not designed to effect any comparison - the intention was to have a holistic understanding of what, how and with what effects a focus on LMT is occurring in the programme. This was irrespective of whether they were in-service or pre-service teachers with different experiences but that they attended the same mathematics education courses.

I worked with four teacher-educators and twenty mathematics student-teachers and from a University, small numbers in both situations, and obviously not representative of all the teacher-educators and mathematics student-teachers in the University or in the whole country for that matter. The findings of this study cannot be generalised to the entire population of University teacher-educators or mathematics student-teachers. Instead, the findings offer in-depth insights on the structuring of the discourse of engaging with LMT through what teacher-educators select and privilege as the discourse. Moreover, the findings offer further in-depth insights on student-teachers' knowledge and understanding of LMT in general, and algebra in particular. The insights raise issues for policy, research and practice. In line with this understanding, Gay (1996) argues that in qualitative studies generalisations are minimal and sometimes non-existent because the choice of participants is usually purposive and small in size. Gay (op cit) further argues that what could be on offer are the insights which are a result of lengthy and intensive engagement with the participants.

Having explained how the data was collected and the methodology used to analyse the data, I now present detailed analysis of the data that enabled the answering of the research questions. The analysis is presented in four consecutive chapters, which are Chapters 6, 7, 8 and 9. In Chapter 6, I illuminate what teacher-educators say they select and privilege as the discourse of engaging with LMT. Chapters 7, 8 and 9 illuminate student-teachers' realizations of the discourse. The relationship between what teacher-educators say they select and privilege as the discourse of engaging with LMT and student-teachers' realizations is also established to illuminate the messages student-teachers recruit or take-up and how these might be explained.

CHAPTER 6

6 TEACHER-EDUCATORS' TALK OF THE DISCOURSE OF ENGAGING WITH LMT

6.1 Introduction

This chapter focuses on the analysis of teacher-educators' discourses of engaging with LMT, that is, on what teacher-educators say they select and privilege as the discourse, hence exploring the first main research question for my study. There are three dominant foci in the teacher-educator's talk namely, developing in learners both relational and instrumental understanding, errors and misconceptions, and creating an environment where teacher can listen to learners, and these are read in terms of Even & Tirosh's (2002) as discussed in Chapters 4 and 5. The analysis is thus structured by Even and Tirosh's (2002) broader categories of what the discourse entails. Specifically, I explore the following questions: What is included in what the teacher-educators say LMT is? How do they position the teacher or student-teachers, learners in school, and the curriculum either in teacher education or in school with respect to the discourse?

As summarized in Table 12 below, all four teacher-educators (Kenneth, Sam, Titus and David)[15] talked about learner error and misconceptions; three referred to either *instrumental* or *relational* ways to mathematical reasoning or to both; and two made reference to creating a classroom environment where teachers can listen to learners. In the sub sections that follow, I discuss, for each teacher-educator, these different ways. I also discuss how these different ways compare across each teacher-educator, including their positionings.

[15] These are pseudonyms for the four teacher-educators that participated in my study.

Table 12: Distribution of what teacher-educators say they enact as LMT

Teacher - educators	Focus on LMT		
	Mathematical Reasoning (MR)	Learner errors and misconceptions (LEM)	Environment (Env)
Kenneth	√	√	
Sam	√	√	
Titus	√	√	√
David		√	√

The three themes under which teacher-educators' discourses are discussed are evidenced by excerpts extracted from their interviews. The numbering of each event that referred to any of the three categories of LMT and how this was grounded, from Kenneth's to David's talk, was continuous and ranged from 6.1 to 6.81. The '6' is because the data is being drawn on to evidence the discussion in chapter 6.

6.2 Kenneth's talk of the discourse of engaging with LMT

In his talk, Kenneth pointed to two broader aspects of the discourse: mathematical reasoning and learner errors and misconceptions. I discuss each of these in turn together with how learners in school, student-teachers or teachers and the curriculum are positioned in this discourse. He did not refer to creating an environment where teachers can listen to learners.

6.2.1 LMT is about mathematical reasoning (MR)

- *What is LMT*?

For Kenneth, LMT is about particular kinds of reasoning. He talked about mathematical reasoning over and again in extracts and I use extract 6.2 as a typical example.

> "*... what is meant by learner mathematical thinking is that we expect to achieve a certain way in which a mathematician must reason,* ***a mathematical way of reasoning.*** *And you* ***won't achieve that by simply showing a way of solving a problem****, but there must be* ***a way of arguing, some way of showing evidence*** *that this is equal to that, therefore this, hence that starts thinking process which is a mathematical way of reasoning. And* ***sometimes you can show children how to find answers without cultivating this type of thinking****. And it's this type of thinking that makes mathematics compulsory – that people must be* ***systematic in thinking*** *in whatever they endeavour*". (Extract 6.2)

Kenneth pointed to the purpose of mathematics in the school curriculum in that it is meant to develop in learners a sense of thinking that is a mathematical way of reasoning (MR), and he says this over and again. For him, developing reasoning means developing in learners ways of arguing (A), evidencing (E) and systematic thinking (S), that is MR-R-AES. He thus foregrounds relational understanding (R) in Skemp's (1976) terms, and contrasts this with "*showing pupils how to find answers*" (MR-I-FA) or "*showing someone how to solve*" (MR-I-SS) and this relates to instrumental understanding (I) which in his view is inadequate. LMT for Kenneth is thus associated with mathematical reasoning, and a relational view of understanding and this is contrasted with an instrumental view of understanding.

- *Where does mathematical reasoning get learned in the courses?*

  "*my first entry point in that perhaps it doesn't address directly learner mathematical thinking, but I devote a lot of time at the beginning of those coming new to mathematics teacher education, I do have/dwell a lot of time on the aims for which we teach mathematics*". (Extract 6.2)

Kenneth deals with the issue of mathematical reasoning when focusing on "Aims and objectives of teaching mathematics". As discussed in Section 2.3, this happens to be the first topic in Mathematics Education I (MSE 331), an introductory course to mathematics education courses offered to student-teachers. In Bernstein's (1996, 2000) terms, this suggests that LMT with specific focus on mathematical reasoning is weakly classified. It is not given specific focus.

- *Positionings of teachers/student-teachers, learners and the curriculum with respect to mathematical reasoning*

In talking about mathematical reasoning, there is a strong sense of absences (A) in each of teachers or student-teachers, the teacher education curriculum, and learners in school.

> "*I am* ***not sure that I see it in the, when we see students in the school teaching,*** *that they have that at the back of their mind. … that we expect a certain kind of reasoning from children who have done school mathematics. So I think they have a duty to inculcate that sense of mathematical reasoning in the, in children. So even in their teaching, in their assessment, in whatever other activities, to amalgamate them in a way that the objectives, the aims for which we were teaching mathematics in the end are achieved.* ***It's a duty that I do not think many maths teachers are conscious of****".* (Extract 6.1)

Kenneth posits that student-teachers or teachers do not recognize their role of instilling a sense of mathematical reasoning in their learners. This is an indication that Kenneth locates the problem in absences in the teacher resulting in a sense of absences in learners. Learners are not seen as coming with some knowledge to the teaching and learning situation because the teacher will have to develop in them the reasoning he values. He attributes, though with hedging, lack of consciousness of this aspect among teachers to absences in mathematics education courses in terms of not doing enough in terms of practice of teaching such as peer teaching and school teaching practice as indicated in an extension of extract 6.1 below:

> "*… but **it still goes back to maybe more exposure to peer teaching** so that you can question, 'Did you have this in mind when you were presenting that lesson?' **And maybe more teaching practice** as I said, **so that you can gauge what you have discussed in the lecture room with what they are able to do in a class.** 'Look, we have discussed these issues, do you think it came through in your teaching today?' And 'No' this and that. Then you'll have an opportunity next time to see the student again. **But we don't have that opportunity."*** (Extract 6.1)

He suggests that if student-teachers were given more time to put into practice the theory they have learned about developing in learners mathematical reasoning then the teacher-educators would be able to ensure that this skill manifests. Moreover, the issue of more time for practice of teaching being critical is an indication that Kenneth is aware that the integration of theory and practice does not occur in an instance but that it happens over time with more practice of teaching. The practice of teaching such as school experience or peer teaching happen to be the focus in mathematics education courses, namely, Mathematics Education II (MSE 332) and Mathematics Education I (MSE 331), respectively.

More exposure to practice of teaching such as peer teaching and school teaching practice also suggests ways of how student-teachers would be knowledgeable and aware of instilling in learners a sense of mathematical reasoning as discussed above. Moreover, in following up on student-teachers' teaching, the teacher-educator would ask questions. The other strategy suggested by Kenneth of how he 'teaches' for developing in learners a sense of mathematical reasoning is that he emphasizes the issue of lesson planning in that the student-teachers should be aware of:

> *"…the importance and the effort and time that need to be devoted to planning the lessons to begin with, so that **they have thought through what they intend to achieve in that lesson**, that **they have thought through what key questions they need to raise***

> ***in class***, *questions which will, whose answers will be a reflection of learner understanding. ...* ***So we encourage that, to invest sufficient effort in planning. We encourage them to reflect seriously on what activities they have lined up for themselves, what activities they have lined up for the children***. *What will these activities achieve?"* (Extract 6.3)

Lesson planning is one of the topics in mathematics education courses, Mathematics Education I (MSE 331). In emphasizing the issue of lesson planning, what is key is "… *the effort and time* …" that should be put into planning in terms of "… *what they intend to achieve in that lesson*…" and "…*what key questions they need to raise in class* …" The key questions which "*will bring out the thinking process in the pupils. Questions which will demonstrate understanding of the concept*" (Extract 6.3). For Kenneth, what should be in the teacher's mind is developing in learners the thinking which is a mathematical way of reasoning.

From the two strategies described above, it suggests that LMT is a skill that is learned in the practice of teaching, be it peer teaching or teaching practice, and in planning lessons; and it is seen through observing student-teachers teach. At the same time, there is a sense of absence (A) of these strategies in the student-teachers or teachers as Kenneth observed during teaching practice.

> "...***we were reflecting on teaching practice and I was saying to the students how little we saw this element of having thought through key questions to ask.*** *And equally how you can assess a successful lesson from the quality of questions children are asking. You say, 'OK, this child has really followed. For him or her to have asked that question tells you a measure of understanding.'* ***… once you are in the field teaching I don't think they invest as much effort as they would like to to thinking through their lesson, elaborate planning lessons.*** *But really essentially to have thought through, 'What do I intend to achieve by this example? How different will be my next example? What point do I want emphasized in this lesson/example?' So this thinking process we hope if it was adhered to faithfully and addressed some of the questions I think that you ask would not arise.*" (Extract 6.3)

This suggests that the quality of key questions asked by student-teachers during teaching practice did not reflect that they were thoughtful about their lessons. Kenneth emphasizes that the student-teachers did not use the questions learners asked during the lessons to determine whether they succeeded in developing in learners mathematical reasoning in relational ways or not. The importance of teacher questions in the process of teaching and learning has been

emphasized in mathematics education research in that they assume different roles such as probing, generating discussion or targeting concepts among others (Boaler & Brodie (2004).

At the same time, and in contrast to extract 6.3 above, there is a sense of presences (P) in learners in terms of understanding learner learning from the quality of answers they provide to teachers' key questions, the quality of questions they ask, and the activities lined up for them. A sense of presences in learners is an indication that learners come with some knowledge to the learning situation which the teacher has to mould in the teaching and learning situation. Here I see evidence of Kenneth's contradictory positioning of learners.

6.2.1.1 Summary and conclusion of the analysis

For Kenneth, engaging in a discourse of LMT is about developing in learners a sense of mathematical reasoning. Skemp's (1976) distinction between relational and instrumental understanding is evident in Kenneth's discussion. Developing in learners relational understanding, for Kenneth, involves arguing, evidencing and thinking that is systematic while developing in learners instrumental understanding involves finding answers or solving solutions to mathematical problems. The way he makes this distinction is that it is accompanied by positioning of a teacher or a student-teacher or a learner who does not know (A). He doubts whether teachers are conscious of the role of mathematics in the school curriculum in that it is meant for them to develop in learners a sense of mathematical reasoning.

In terms of how they 'teach' for developing in learners a sense of mathematical reasoning, two strategies are identified: teaching practice, and lesson planning where aspects of theory to structure it with are discussed. The expectation is that student-teachers should therefore demonstrate during practice of teaching that time and effort was spent in planning their lessons from the key questions they ask "*questions which will bring out the thinking process in the pupils, questions which will demonstrate understanding of the concept*" (Extract 6.3). As noted, aspects of learner mathematical reasoning become evident only when the lesson plan is put into practice of teaching, an indication that a lesson plan, in this instance, is a discursive resource constructed in practice. Learner understanding becomes evident as learners carry out activities planned for them by the teacher, or respond to key questions asked by the teacher, or when learners ask questions. This posits a contradictory positioning in terms of seeing a learner who knows (P).

The interpretation here is that developing in learners a sense of mathematical reasoning, be it in relational or instrumental ways, is a skill that is accomplished practically in that it lies in the practice of teaching such as peer teaching or teaching practice and it is supported by lesson planning. This is a skill that needs to be learned by student-teachers and so be taught. What is interesting about Kenneth's discussion is an assumption of an automatic transfer from being taught to being able to do it in practice. Yet the practicing of this skill during the practice of teaching is not evident anywhere. Indeed, that engaging with LMT is accomplished practically is a common thread across all four teacher-educators, as will become evident through this chapter.

This analysis also suggests that LMT and in this case mathematical reasoning, following Bernstein (2000), is a weakly classified discourse. Kenneth described how he deals with it when focusing on specific topics in the mathematics education courses, which are: 'aims and objectives of teaching mathematics'; 'lesson planning'; 'peer teaching'; and 'teaching practice'. This means that mathematical reasoning has no identity or voice of its own since it is not a topic of focus on its own, but it is embedded in other topics in the courses, a phenomenon that is apparent across all four teacher-educator interviews.

6.2.2 LMT is about anticipating learner difficulties and suggesting remediating strategies

- *What is LMT?*

For Kenneth, learner errors and misconceptions are part of the discourse of engaging with LMT. He said that focusing on learner errors and misconceptions involves identifying learner error and suggesting likely strategies for remediation (LEM-IR). He says this is achieved by giving his student-teachers school mathematical questions and asks them to:

> "*Look at this question. What knowledge, skills, and concepts will a child need in order to do this? What strategies would you adopt in order to enable students to do this?* ***What difficulties do you anticipate from children? And arising from that what strategies will you arm yourself with in view of the difficulties you will anticipate?***" (Extract 6.4; see extract 6.8 for more emphasis)

In this case, identifying learner errors is talked about in terms of anticipating learner difficulties. Following Peng & Luo (2009) and Jacobs et al. (2010) strategies of carrying out error analysis, Kenneth's focus on error analysis does not include discussion of sources of errors, of why the errors are occurring. Kenneth does not raise the sources of the errors as

something that is problematic and therefore needs to be thought about. Of course, overlooking the sources of errors could also affect the quality of remediating strategies suggested. In working with school questions student-teachers are given opportunity to think through the required knowledge in terms of skills and concepts to solve the mathematical problem, and the likely strategies for teaching for it. Only then can they think of identifying the errors learners would make and the likely strategies for remediating the errors. This suggests that student-teachers not knowing and understanding the mathematical demands of the school questions can result in difficulties in thinking of learner errors and misconceptions and maybe that is why it is emphasized.

- *Where are learner errors and misconceptions (LEM-IR) learned in the courses?*

Error analysis which for Kenneth is about identifying the error and suggesting remediating strategies (LEM-IR) is focused on in an effort to familiarize student-teachers with the school mathematics curriculum. Kenneth states that they "*... make effort to make them understand the school curriculum that they will be implementing, the mathematics syllabuses at school. In fact a component of our curriculum is to look at school questions*" (Extract 6.4). This is "*… dealt with in the context of what we have got on our course outline, school mathematics, from the context of specific topics that they will meet at school*" (Extract 6.8). Therefore, Kenneth focuses on learner errors and misconceptions (LEM-IR) by looking at "school mathematics", a topic in the course Mathematics Education II (MSE 332) with specific emphasis on school questions. Similar to mathematical reasoning, learner errors and misconceptions with specific focus on identifying the error and suggesting possible remediating strategies (LEM-IR) is also weakly classified in that it is not given specific focus. It is talked about when focus is on a selection of school mathematics questions.

- *Positionings of student-teachers, learners, and the curriculum in mathematics education*

In a situation where student-teachers experience difficulties in working out a mathematical problem also suggests that absences (A) realized in them are likely to be the anticipated absences (A) in learners as indicated by Kenneth:

> "*… ironically,* ***those problems the students will make themselves as trainee teachers which actually you anticipate from children****. So sometimes when you notice the*

difficulties because they are making the problems, they are making the errors themselves." (Extract 6.7)

Furthermore, for Kenneth, a sense of absences (A) in learners in terms of working with mathematics questions is a result of absences (A) in teachers:

> "*Even when you are examining Grade 12 examination, **you could see how poorly they attempted**. **It reflects on how poorly the teaching was**. So we pick up such topics and make an effort to prepare our students for the tackling of these topics.*" (Extract 6.4)

It is thus possible to infer that Kenneth locates the source of errors and misconceptions in teaching. When learners are making errors, teachers should reflect on their teaching to establish what could have gone wrong and work with that. A contrary positioning of student-teachers is also observed in that Kenneth sees presences (P) in student-teachers' suggested remediating strategies during lecture sessions in that they provide reasonable answers and a sense of absences (A) during the practice of teaching in that he does not see in them the aspect of working with errors and misconceptions. This is indicated in the excerpt below:

> "*Yes, **in class they do attempt to come up with reasonable suggestions**. …But when I saw them **in the field on teaching practice it didn't come through that they had done sufficient homework.** But they'll give you good suggestions in a classroom context when you are in the university discussing in the classroom. But what I saw in the field when there was school teaching practice, sometimes I was wondering, 'Are these the students I trained?' (laughs) … It didn't quite come through. So I'm reflecting where did I go wrong?*" (Extract 6.5)

The issue of reflecting on ones teaching is deep rooted in Kenneth in that even for his own practice if student-teachers fail to provide reasonable responses, then there could be something wrong with the way he has been putting his ideas across during lecturing. Moreover, when he is focusing on errors and misconceptions during lectures, he expects the manifestation of this skill during practice of teaching such as school teaching practice or peer teaching; and unfortunately it is missing. As discussed in Chapter 3 and following Peressini et al. (2004), this points to the issue of putting into practice through peer teaching or teaching practice what has been learned theoretically, and teaching practice is considered as one of the sites of teacher learning.

- *Strategies of how Kenneth "teaches" for learner errors and misconceptions (LEM-IR)*

Similar to the two strategies identified earlier on how Kenneth "teaches" for developing in learners a sense of mathematical reasoning, he suggests the practice of teaching and the issue of lesson planning to include anticipated learner errors and suggested remediating strategies. Added to these two ways is the aspect of challenging student-teachers with more school mathematics. I discuss what is entailed in each of the two strategies suggested.

The first strategy is about the practice of teaching such as peer teaching and school teaching practice. As already pointed out in Extract 6.5, Kenneth expects to see the manifestation of what student-teachers learn about identifying errors and suggested remediation strategies in the practice of teaching. The practice is that "*For now we do it theoretically then ask them to practice it*" (Extract 6.5). It is during the practice of teaching that the student-teachers should demonstrate understanding of what is being taught during lectures. This means that identifying and remediating (LEM-IR) for Kenneth is still a skill that is learned in practice but there is no mention of how he would deal with it in practice other than making observations on whether what student-teachers learned theoretically was put into practice. This shows that for Kenneth there is a direct transfer of knowledge from theory to practice. However, he contemplates introducing student-teachers to peer teaching from the onset so that theory is developed out of practice. Kenneth's option is illustrated in the excerpt below:

> "*And I've been discussing with my colleagues in the Mathematics Education section. We haven't agreed on it but I've suggested that my preference would be to* ***have peer teaching right from the word go***. *Even when they have not learnt anything on making a lesson plan. …What I think,* ***build all the theory from the context of whatever mistakes they may be making in their teaching. Make that as a platform for the theory that you want to do.*** *If you want to say, 'A lesson is planned this way', it should be from the context of what you have seen them doing. 'Where did you think this, it doesn't seem that the teacher was clear on the objective of the lesson.' Then you go into your discussions of lesson objectives. 'What do we need in a lesson?'… But if the theory and the practice can go hand in glove, maybe some of these can be nice.*" (Extract 6.5, see also extract 6.7 for emphasis)

For the strategy of introducing peer teaching from the onset, what it would mean for focusing on errors and misconceptions for Kenneth is that he would observe the mistakes the student-teachers are making with working with errors. He would then discuss with them the importance of including in the planning anticipated learner difficulties and strategies of how they would remediate the errors. I would argue that maybe his preference of theory

developing out of practice could address the issue of transfer earlier discussed. Moreover, his preference, as discussed in Section 3.2.4, could be one of the ways of attending to the "dilemma of experience" and is described as one of the challenges teacher-educators are faced with when working with pre-service teachers' experiences (Doerr, 2004, p. 269). From Kenneth's preference, there is a sense of presences (P) in student-teachers in terms of how he thinks learning how to teach should be done. This means that he sees student-teachers as coming with some understanding of what teaching entails and then the teacher-educator builds up from what they know.

Directly linked to the strategy of practice of teaching is the strategy of planning for teaching to include specific ways of remediating errors. He suggests that this could be made possible by, for example, working with learners on the identified difficulty with focus on specific strategies to solutions to mathematical problems that could address the error.

> *"And* ***in your illustrations lead the children to this common difficulty that they face and encourage a system of working which will minimize that error****. Simplification of algebraic fractions where you have maybe two terms on the numerator in one fraction, you have two terms on the numerator in one fraction and two terms as the numerator on one fraction. And you are required to subtract. A common difficulty is how to deal with the subtraction and the brackets so that the signs are not messed up. So I was citing that as an example as this is one difficulty you can anticipate from children.* ***Maybe make sure that the first stage where brackets are maintained is always there before they proceeded to simplify because it is a difficulty that you anticipate.*** *So you should* ***build in your planning some mechanism of how can I alleviate this difficulty?****"* (Extract 6.4)

This confirms Kenneth's earlier point that the source of error is located in teaching: the nature of the suggested remediating strategy in the example he gives is teacher oriented. If the teacher does not emphasize the issue of maintaining the brackets in solving the mathematical problem, then the learners will always experience the identified error. However, in Hill et al. (2008) terms, and as discussed in Section 3.2.3, Kenneth brings to student- teachers' attention the importance of KCS and how this enables the incorporation in the planning of lessons anticipated learner errors. Moreover, as argued by Even and Tirosh (2002), teacher awareness of learner errors has implications for the quality of instruction. Kenneth is also aware that errors are persistent in Smith et al.'s (1993) terms and as a result they cannot be completely eradicated by suggesting that the teacher should "*encourage a system of working which will minimize that error*". In anticipating learner difficulty, there is a sense of absences (A) in

learners in terms of subtracting algebraic fractions since they have a tendency of confusing the signs.

One of the suggested remediating strategies for errors is to include in the planning for teaching specific rather than general techniques to the task. For the identified error and suggested remediating strategy discussed above, Kenneth distinguishes between a strategy that is specific to the error and the one that is general and states that:

> *"First maintain your brackets until you have cleared the denominators. Then open the brackets. Then simplify.' So you are looking at* ***a strategy that is specific to that task****. But to say I would give them more homework, of course it's useful, but it's not just useful for that topic,* ***it's useful generally****."* (Extract 6.6)

Preference for Kenneth is on the specific strategy because it addresses the error directly while the general strategy does not. This implies that for the general strategy, more practice is required after all when learning mathematical concepts. Following Borasi (1987), Arcavi (1995) and Doerr & Wood (2004), and as discussed in Section 3.3.3, Kenneth is aware that general strategies such as giving learners more practice exercises would not necessarily remediate the errors. However, the explanation Kenneth gives as an example of a specific strategy to the error suggests what Borasi (op cit) and Arcavi (op cit) would describe as explaining the same process over and again. What is interesting about the general strategy is that Kenneth is contradicting himself by proposing challenging his student-teachers with more school mathematics as a way of making them experience more of what it means to work with learner errors as indicated in the excerpt below:

> *"...* ***give them more challenges on specific questions that they would meet at secondary school level*** *and to keep challenging them that what skills, what concepts do children need, what difficulties do you think pupils are likely to have ..."* (Extract 6.7)

There is also a contradiction here in Kenneth's talk in that in identifying learner errors and likely remediating strategies he sees absences (A) in learners and yet at some point he sees presences (P) in learners in terms of their conceptions and that they are just limited. The teachers' role is to work with what learners have to expand their limited framework. This is because he says misconceptions mean that "*...the concepts of pupils are never wrong but they're just operating in a limited framework and our job being expanding that framework*"

(Extract 6.9). This constructivist view is in resonance with how he views misconceptions "*through the avenue of the psychology of learning mathematics*" (Extract 6.9). 'Psychology of learning mathematics' is a topic in one of the mathematics education courses, Mathematics Education II (MSE 332), which focuses on theories of learning, more especially constructivism. The emphasis is on:

> *"… paying sufficient attention to what the implications are for teaching and what we know about how children learn. From what we know about* ***the nature of mathematics*** *it's a hierarchical nature and so on,* ***how it translates to children's learning*** *and the children's thought processes, and how we can minimize on the misconceptions (if one can call them misconceptions)."* (Extract 6.9)

6.2.2.1 Summary and conclusion of the analysis

What is interesting about Kenneth's strategy of error analysis is that the focus is only on identifying the error and suggesting a remediating strategy (LEM-IR). As argued, Kenneth's general view of the source of error is that it is an indication of inadequacies in teaching. The specific remediating strategies for an error attests to this general view of the source of error and his concern when he fails to see student-teachers model during school experience or peer teaching what they have been taught during lectures. The way Kenneth focuses on learner errors is accompanied by positioning of a teacher or a learner who does not know (A). He says learners' poor performance in school mathematics is a sign of poor teaching or that student-teachers make similar errors learners would make in manipulating school mathematics. A contradictory positioning of student-teachers is observed in that during lectures, Kenneth sees student-teachers who know (P) in terms of suggesting remediating strategies that are specific to the errors. Kenneth also suggests developing theory out of practice as his first preference, an indication of seeing student-teachers as coming with some understanding about identifying error and suggesting remediating strategies, and so a view of the student-teachers as not simply 'empty vessels' (P).

Some indication of the internal contradictions to Kenneth's expressed transmission view of knowledge and learning is also observed. He contradicts himself by putting up a constructivist view of errors and misconceptions in that they are persistent and can never be completely eradicated. Moreover, the view of learners is that they are never wrong but that their thinking is limited conceptually and it is the role of the teacher to expand their conceptual understanding. This is an indication that learners come with some knowledge to the teaching and learning situation from a constructivist perspective, hence seeing presences

in learners (P). What then does this mean for student-teachers in terms of receiving different messages about LMT?

As previously, working with learner errors and misconceptions for Kenneth is a skill that is accomplished practically. The skill lies in the practice of teaching such as peer teaching or school teaching practice and is supported by principles for generating a lesson plan, working with school mathematical problems, and the psychology of learning mathematics where LMT is discussed. This further reinforces Kenneth's transmission view of teaching and learning in that if student-teachers have been taught, they should be able to perform. This analysis of Kenneth's discussion of error as important in LMT confirms that LMT with respect to errors and misconceptions is also not assigned topic status in the mathematics education courses. LMT remains weakly classified with no identity and voice of its own. As Kenneth describes, he addresses issues of learner errors when dealing with selected topics in the mathematics education courses such as 'school mathematics', 'psychology of learning mathematics', and 'reflections on peer teaching and school teaching practice'.

In Table 13, I provide a synopsis of Kenneth's selection and privileging of the discourse of engaging with LMT in terms of mathematical reasoning, and learner errors and misconceptions.

Table 13: Synopsis of Kenneth's talk of LMT with focus on mathematical reasoning, and learner errors

"What" is LMT?		**"How" of LMT, that is positionings in terms of presences and absences and the "teaching" of it in terms of what student-teachers are supposed to know and be able to do.**				**"Where" in the courses/topics LMT is focused on**
	Elaboration of the "what"	**Positioning of teachers/student teachers**	**Positioning of learners**	**Positioning of the curriculum (in school or TE)**	**"teaching" of LMT**	
Mathematical reasoning is relational and not instrumental	Relational thinking includes arguing, evidencing, and systematic thinking (MR-R-AES). Instrumental thinking includes showing learners solutions (MR-I-SS) or finding answers to mathematical problems (MR-I-FA).	Absences – teachers and student-teachers do not develop mathematical reasoning in learners. A	Absences in learners as a result of absences in teachers. A Presences in learners in answers they give, questions they ask, and activities lined up for them during planning. P	Absences in TE curriculum - no time to integrate theory and practice in peer teaching and school teaching. A	Through peer teaching and school teaching practice Through lesson planning with focus on intensions of the lesson, what key questions to ask, and activities lined up for teacher and learners.	Aims and objectives of teaching mathematics (MSE 331) Peer teaching (MSE 331) School teaching practice (MSE 332) Lesson planning (MSE 331)
Errors are difficulties that need to be identified/ anticipated and remediating strategies sought (LEM-IR)	Locates source of error in teaching.	Absences: student-teachers do not understand the demands of school questions. A Presences: in student-teachers during lecture sessions; Absences in student-teachers during school teaching practice. PA Presences in student-teachers if theory could develop out of practice in working with errors. P	Absences: working with mathematical problems as a result of absences in teaching. AA Absences: subtracting algebraic fractions. A Presences: learners' conceptions not wrong but limited. P		Through practice of teaching and in lesson planning Through more study of school mathematics Through arguments on nature of mathematics, how learners learn mathematics	School mathematics (MSE 332) Peer teaching (MSE 331) School teaching practice (MSE 332) Lesson planning (MSE 331) Reflecting on school teaching practice (MSE 431) Psychology of learning mathematics (MSE 332)
LMT is mathematical reasoning that is relational and working with learner error	*Source of error in teaching*	*Discourse of absences dominate with some discourse of presences in student-teachers*	*Some discourse of presences, but dominantly absences in learners*	*Discourse of absence in the curriculum of TE*	*LMT is practical accomplishment learned through* • *School mathematics* • *theories of learning and mathematics* • *practice of teaching*	*Distributed across courses – weakly classified*

6.3 Sam's talk of the discourse of engaging with LMT

Like Kenneth, Sam pointed to two broader aspects of the discourse, namely, mathematical reasoning and learner errors and misconceptions. Of the two aspects, mathematical reasoning was Sam's most dominant discourse.

6.3.1 LMT is about teaching for understanding (TU)

- *What is LMT?*

Similar to Kenneth, critical to Sam's talk about LMT with particular focus on mathematical reasoning is developing in learners relational rather than instrumental understanding. Sam says:

> *"When a teacher stands in the classroom you can actually observe whether this* ***teacher is teaching pupils mathematics or he's teaching mathematics****. Teaching mathematics means he is solving the problems for the children, whether they're understanding or not is not much of his concern. A teacher who is teaching pupils mathematics will be concerned if he can, if he can observe that pupils are not following what he is doing*." (Extract 6.10)

Sam makes a distinction between teaching for relational understanding and teaching for instrumental understanding and talks about it in terms of "*teaching learners mathematics as opposed to teaching mathematics*". Teaching mathematics to learners means the teacher has to think about what learners understand and teaching mathematics means the teacher teaching mathematics for its own sake irrespective of learner understanding. Sam further explains that teaching for understanding (TU) entails "*teaching for real grasping of the concepts that we are trying to put across to the students*" (Extract 6.12), going "… *deeper than the answer*..." (Extract 6.18), "… *giving them the steps that you will follow* ..." (Extract 6.30), and "…*not giving them a restricted way of answering a question*" (Extract 6.30). He also points to that what merits one to be a good teacher is "*when you can impart knowledge to the students and the students understand*" (Extract 6.26). The four aspects of teaching for understanding identified by Sam point to developing in learners conceptual understanding (MR-R-TU). In the emergent themes that follow Sam elaborates what teaching for understanding (TU) entails by saying what it is as well as what it is not, and in how he positions the teacher/student-teacher, learners and the curriculum.

- *Positionings of teachers/student-teachers, learners and the curriculum with respect to TU*

6.3.1.1 LMT is about TU and not about curriculum completion or examination purposes

In talking about LMT with particular focus on TU, the teacher or student-teachers, the curriculum in teacher education, and learners in school are positioned in particular ways. There is a sense of absences (A) in teachers or student-teachers in terms of TU in that they are more attuned to teaching for examination purposes as shown in the extract below:

> "*I'm sorry to say most of the teachers were teaching mathematics and not pupils mathematics and that was a big concern for my part and I felt I wished there was a way of addressing this but you see. … Now most of our teachers were interested, because of the examination, to say,* ***'Here's an example which is likely to come in the exam and this is how you solve it' whether the pupils were following those steps or not, he was not really taking care of****, and I had a bit of concern about that to say, 'Let's not teach for examination.' Unfortunately our system is examination-oriented and therefore most of the teachers I can tell you were teaching for the examination because that's what counts mostly. Now I must say unfortunately even up to now,* ***even the teachers I'm training, they have that concern of: The pupils I'm going to teach should pass, whether they will understand maths or not is immaterial.***" (Extract 6.10; see also extract 6.37 for emphasis)

Sam attributes teachers' failure to teach for understanding to the education system that is examination-oriented. Teaching for instrumental understanding could be linked to Sam's concern of teachers teaching for examination purposes where teachers show learners how to solve mathematical problems so that they can pass examinations without being concerned with their understanding. This implies that the teacher would focus on the procedures which learners need to apply for them to get the mathematics right in the examination without paying attention to the underlying meaning of those procedures. This type of focus on teaching, Sam argues, is so deep rooted even in his student-teachers that he is left with a challenge of looking for "*a way of trying to ensure that my thoughts actually are effected in terms of helping these teachers understand that they have to teach for the learning of the concepts in mathematics*" (Extract 6.12). In this way, teaching would really mean that '*we should be educating for knowledge and not for the examination*' (Extract 6.11). If the teacher is concerned with learner understanding, the question he would be asking is "*What is it that I*

can do to help them to do it correctly and properly?" (Extract 6.18). Unfortunately even for student-teachers, as Sam observed, they are more oriented towards passing examinations in that "*Once they have passed the exams they can deal with vectors, they can deal with functions – that's it, but how do you teach the vectors and the functions to the children in the classroom for understanding purpose?*" (Extract 6.13). This suggests that Sam is pointing to the issue of CCK as elaborated by Ball et al. (2008) that a teacher knowing mathematics in ways anyone who has learned mathematics would do, for example a mathematician, does not translate into teaching for understanding. There is need for SCK, hence Sam's concern for student-teachers' views of learning mathematics in the University.

In this aspect, Sam shares his head of department's experience that learning mathematics in a university is different from teaching school mathematics:

> "*My Head of Department at one time said, 'It's only after I had graduated from here and gone into schools that I realized what we are learning about vectors at university. Oh, is this what it really means?' Why? Because he had the chance to be in a position to put it across to the children to understand. He understood it from the university point of view and he passed the exam, but when it came to putting it across to the children, to understand the concepts, he realized that actually there was something missing. He was actually learning the subjects for the sake of passing an exam. But when it came to him now to be able to put the concepts across to the children, ah, it was a different situation altogether*." (Extract 6.14)

This means that for the head of department, the underlying meaning of concepts about vectors made sense when he prepared to teach for it in school in ways that would make the learners understand. In Ball and Bass's (2000) terms, this suggests that he did not have to teach vectors to learners in a compressed form but he had to decompress or unpack the mathematical ideas for the purposes of learner understanding.

Other than teaching for examination purposes, a sense of absences (A) is also realized in teachers in terms of teaching for understanding because they are also mostly concerned with "*If I've covered the syllabus I'm satisfied with everything that is happening here*" (Extract 6.37). Sam says this is expressed by teachers when they are asked to share their experiences of teaching school mathematics. Because of this, Sam feels that "*we may be missing the point if we are not emphasizing on does the teacher teach for understanding?*" (Extract 6.37).

However, in sharing his experiences of teaching school mathematics for understanding, Sam worries about the time constraints of covering the syllabus in that:

> "*… there were times when I felt **'I'm not covering the syllabus within the time required' because I go slightly deeper to help these students to understand** the mathematics for the sake of mathematics itself, whereas the pupils, the students were actually mainly interested in, 'Can we cover the syllabus so that we can write the examination'."* (Extract 6.15)

From this experience, Sam is pointing to the importance for the teacher to be aware of embracing both curriculum coverage and teaching for understanding. As argued in Section 3.2.5, this contradicts Boaler's (1997) study which showed that key in school mathematics teaching and learning is developing in learners relational understanding rather than curriculum coverage where the concern is mastery of rules.

The absences realized in teachers or student-teachers in terms of teaching for understanding results in absences in learners (A), hence has implications for learners furthering their studies in mathematics related careers as Sam said:

> "*We lose so many students from the natural sciences here 1st year – 1 000 so many. Next year semester or, yes next semester you may have 300 – from 1 000 – because they have all failed. Now it's my wish that the students that are enrolled actually excel. The teachers that we train, when they go into the schools they should ensure that the children they teach actually learn the correct mathematics (I don't know whether there is such a word) the proper mathematics that you would like them to teach. Teach for understanding instead of 'I've taught, I've done my job' whether she/he has done it or not it doesn't matter.*" (Extract 6.27)

This interpretation of high failure rate in first year courses being blamed on secondary school teachers having not taught their learners for understanding but for purposes of passing examinations is not new. Sam is shifting the blame of high dropout rate in mathematics and sciences at University to secondary school teachers.

The absences realized in teachers or student-teachers are also seen as a result of absences in mathematics teacher education curriculum (A). Sam explains that:

> "*But as you know, **lecturers at the university I'm sorry to say, they teach the subject**. Whether the students go further in understanding whatever information we are giving them or not is not very much emphasized. A lecturer will come up with the basic notes of what he expects the students to grasp from that course and leave it to the students to research – which is good, to go and read more. But then it's that part of going to*

read more which worries me because not all the students realize that the more you read, the more you understand." (Extract 6.12)

For Sam, teaching for understanding in teacher education is dependent on enabling the student-teachers to research and read more. Sam's concern is that he is not sure whether the student-teachers have this realization.

Moreover, Sam contends that:

> "*So here in the university I think it's* ***left up to the individual lecturer to see to what extent he should go into the aspect of teaching mathematics for the children to understand the concepts*** *and the way we want them to be taught rather than teaching mathematics for the child to be able to pass an examination*" (Extract 6.14; see also Extract 6.15 for emphasis).

Here is another indication that the teacher education curriculum has no clear guidelines and structures of how the aspect of teaching for understanding rather than teaching for examination purposes should be dealt with by individual lecturers. This is further evidence that teaching for understanding is not a specialized discourse in this mathematics education programme, hence in Bernstein's (1996, 2000) terms, it is weakly classified.

The other aspect identified pertaining to mathematics teacher education curriculum is that a sense of absences (A) is realized in terms of having less focus on specific school mathematics topics student-teachers are going to teach. Mostly the focus is on the methodology which is rather general. This is reflected in extract 6.16:

> "*I'm not yet satisfied with the aspect of the mathematics that they are going to teach. We are mainly looking into the methodology (could I put it that way?)* ***we are looking at the strategies in some cases which to me seem to be general for every teacher, not necessarily a mathematics teacher.*** *How I wished there were components of the mathematics at grade, at senior secondary school the way we are doing the MSE 131. If there was a component of MSE 131 in the 2nd, 3rd and 4th year which would be focusing at specific topics in the syllabus. Right now we are teaching, we are teaching mathematics across the syllabus, not necessarily focusing on certain topics.*" (Extract 6.16)

This absence in teacher education curriculum has resulted in absences in student-teachers in terms of being uncomfortable in teaching specific school topics such as "earth geometry".

"*It's only when you are interacting with the students that you realize* ***there are certain topics which they are not comfortable with.*** *For example, earth geometry, even when you ask them to peer teach nobody will pick on geometry, they are so scared of it and so you start wondering what is it that, if a teacher now is uncomfortable at this level what more when he is in the school and there's nobody to tell him to do it, it will be ignored completely.*" (Extract 6.16)

This sense of absences in the mathematics teacher education curriculum is exacerbated by the assumption that mathematics teacher-educators make about their student-teachers.

"*…because they are graduates and therefore we think they know mathematics. They have been trained to teach and therefore we think that they are able to teach mathematics to a new learner who doesn't know mathematics as much as they do.* *Now that's where the problem comes in you see because whereas I, it's a question of I'm thinking that they'll be able to do that. But are they going to do it really when they are in the classroom? But to what extent will they be reflecting and saying, 'Have I taught for understanding?' Or they will be saying, 'Well I've covered all the topics in the syllabus, that's fine with me.' It's more of that which I thought we should be doing rather than the administration, the methods of teaching that we are teaching them which are actually more general.*" (Extract 6.19)

Clearly, for Sam, knowing mathematics for its own sake does not translate into knowing it for the purposes of teaching.

6.3.1.2 LMT is about TU and not about showing one's ability that you can solve mathematical problems

Tied to the assumption teacher-educators make about student-teachers knowing school mathematics and that they should be able to teach for understanding, Sam shares his experience of working with his student-teachers on solving quadratic equations:

"*... at a certain point I realized 'I hope I'm not teaching these student-teachers how to solve quadratic equations' because for them to get to university level they should have passed them at a lower level. ... And therefore would it be wise for me to give them tough quadratic equations – for what purpose?* ***So we went into discussing how to teach quadratic equations to pupils, how to teach that myself in the class, go through the steps that children should follow in solving quadratic equations. Then***

> ***I'd give them work to do. And I realized that they were all rushing to the answer****. I said, 'No, this is not the issue we are dealing with here now. You know, when a child moves from that step to this one, what thinking is there? Can you tell me that the child has understood by your telling them that you move from there to there?'* ***And you could see that in as far as the teachers were concerned, they were solving the problem rather than looking at how to make it easier for the pupils to understand the movements in the solution of a quadratic equation.*** *So I was saying to them, 'I'm not really interested in you giving me a correct answer to this question I have given you …*" (Extract 6.29; see also extract 6.30 for emphasis)

Sam's experience suggests that there is a sense of presences (P) in student-teachers in terms of them knowing how to solve quadratic equations and a sense of absences (A) in student-teachers in terms of them teaching for understanding. This is because they tended to focus on getting to the answer without necessarily focusing on how they would explain all the necessary steps to ensure learner understanding of the concepts involved. This suggests that the student-teacher's ways of solving the quadratic equations to get to the answer would only develop in their learners instrumental understanding. What Sam was trying to do is make his student-teachers engage with the underlying meanings behind each step they took for the purposes of enhancing relational understanding in their learners. Again in Ball and Bass's (2000) terms it is about decompressing the compressed form of solving quadratic equations for the purposes of learner understanding.

During peer teaching, Sam encountered similar experiences with student-teachers when he worked with them on quadratic equations pertaining to teaching for understanding. They tended to solve the mathematical problems for the purposes of finding the answer and not teaching for understanding:

> "*Sometimes you would find that they are distracted, they forget that they are teaching, they will end up solving the problem there. And when you tell them, 'We wanted you to teach, not to solve the problem,' some (at least 3) came to me and said,* ***'Sir this part of our work is very very useful because you know it really exposes you to certain things you took for granted which you don't know, which you would normally overlook in the classroom situation.*** *So I think this part of the work seems to be addressing what I'm going to do when I get into schools.'* ***Maybe my answer there as well, whereas that may be good, you may be solving the problem from your point of view as a teacher, but not from the point of view of a child who should be able to understand.*** *You are grappling with standing in front of the class. What you*

should be grappling with is: Are the children understanding what I'm teaching them?" (Extract 6.32; see also extracts 6.48 and 6.49 for emphasis)

This suggests that there is a sense of absences (A) in student-teachers in terms of teaching for understanding in that they get distracted during peer teaching and end up solving the mathematical problems. They do not explain in ways that would develop in learners understanding. However, through peer teaching, there is realization among some student-teachers (P) that the taken for granted work of teaching for conceptual understanding is not so straight forward and is not the same as just demonstrating that they are able to solve the mathematical problems. In Sam's terms, it is not about teaching mathematics but teaching learners mathematics. The importance of practice of teaching including peer teaching and school teaching practice (STP), according to Sam, is that it highlights this issue. Through peer teaching, Sam was able to differentiate between in-service and pre-service teachers in that "*They have classroom experience very well and you could actually see the difference when you are taking them through peer teaching that: This one I think he has already taught; this one is a novice, he actually has to learn how to teach*" (Extract 6.16). Moreover, through peer teaching, he was able to "*see the difference between teaching for understanding and teaching mathematics*" (Extract 6.14).

However, there is a sense of absences (A) in the mathematics teacher education curriculum in terms of reviewing student-teachers' experiences on school teaching practice and how this aligns with what was taught in the mathematics education courses.

> "*… our system is such that when the teacher has left the university even if the STP (school teaching practice)* ***they don't come back to review that and say, 'To what extent did we succeed in whatever we were doing here?'*** *What I mean is a 4th year student he goes for peer teaching, for STP, and after that they go into the schools. So I have no feedback from that particular moment.* ***So really I have still the problem of how can we be able to correct whatever we may have identified in the schools for this teacher or this course should we have a certain aspect which we really feel is missing from our teachers?*** *That part is actually missing on the… I have, I am still struggling with how to find a way of satisfying myself that whatever we do has it produced the results that we feel we need? … it's my hope that there should be a mechanism of me being satisfied that what I saw at the beginning, what I started with, has it changed to a point where I'm satisfied that this teacher now can go into teaching in a school*?" (Extract 6.21)

This shows how crucial STP is for Sam in terms of assessing his student-teachers and himself on whether the teachers he is sending to schools are able to model what his understanding of what teaching mathematics in school means. Sam's expectations of his student-teachers are that:

> *"…'Has this teacher grasped the skills of teaching mathematics when they go in the school?" … are they able to transmit the mathematics that we want children to learn across to the pupils? ... To what extent am I satisfied that these people are now capable of going to teach mathematics the way I view it myself?*" (Extract 6.20)

This shows that for Sam, teaching for understanding is a skill, it is modeled by the teacher-educator during lectures, and the student-teacher is expected to show understanding of this skill by demonstrating during discussions, peer teaching or school teaching practice. In doing so, learner understanding is foregrounded.

6.3.1.3 LMT is about TU and involves teachers' originality and being able to produce multiple routes to an answer

A sense of absences (A) is also realized in student-teachers in that they lacked originality: Sam observed this when he asked his student-teachers to research on the Pythagoras theorem:

> "*Unfortunately that was lacking again since when you ask somebody to present something what comes into their mind is how did the others do it? (laughs) So* ***it's more like you are teaching the way you were taught.*** *And I was trying to put across to them the fact that you know the way you are taught may not have been all that pleasant, you see, and you're not to imitate what somebody did, you are likely to make mistakes.* ***Can you be original? Can you come up with a different way of putting it across so that others can benefit,*** *… some of them maybe come up with a diagram showing the areas on the hypotenuse, the areas on the other side as a way of convincing the learner that that is the, how this man thought about it in those early days and* ***unfortunately we seem to be just going along the same line of thinking rather than the possibilities of looking at it from a different angle.***" (Extract 6.49)

Sam further says that over time through probing student-teacher can think of alternative strategies of presenting concepts to learners in original ways.

> "*It's not so easy but what I realized was* ***when you probe with the students, when you push the students against the wall, they are able to think of different ways of doing the same thing.*** *They may not succeed within it (laughs) within the time you have, you see, but my view is this may actually be in their mind for quite a longer time and be able to be applied at a later way, later stage.*" (Extract 6.49)

Sam is thus aware of the demands that go with the task, but this is meant to make the student-teachers be aware that different routes to an answer could be sought in explaining mathematical concepts to learners. However, as discussed in Chapter 3, the issue of multiple representations is contested in mathematics education research in terms of identifying it with SMK or PCK. Nevertheless, Arcavi (1995) argues that knowing the underlying concepts to get from the problem to the answer is crucial for relating one route to another. The issue of student-teachers teaching the way they were taught in school could also be linked to what Doerr (2004) calls the "dilemma of experience" where teacher-educators are expected to grapple with student-teachers' school experiences in the process of becoming mathematics teachers.

Alternatively, there is a sense of presences (P) in teachers in terms of being aware of the possible misconceptions irrespective of the learners' different routes to an answer. Sam points to that for the teacher:

> "*When you are marking a question you are not giving them a restricted way of answering a question. They are human beings, they think of it from a different angle*. *Now for you as a teacher* ***it's not possible to cover all those angles, but you must be aware of the possible misconceptions that may arise in the process,*** *the way you've demonstrated it to me here you have shown me that you can solve this equation here and, you're solving the same problem. This one solves it this way; this one solves it that way.*" (Extract 6.30)

While Sam encourages his student-teachers to explore different routes to an answer, he suggests that this awareness is also meant to help them work with learner multiple solutions to an answer in terms of identifying the misconceptions. This is because it might not be possible for them to work with all the alternate ways in the classroom. There is also a sense of presences (P) here in learners in that they could have different approaches to a mathematical problem, and some of these approaches could signal some misconceptions in their thinking. In sharing his student-teachers' query on whether you give full marks when a

learner uses an incorrect method that leads to the correct answer, Sam advised that "… *you must as a teacher be able to analyze the results of your students and see to what extent they've understood the mathematics, not the answer at the end of the day …*" (Extract 6.18). The emphasis here is on the teacher being able to explore the meanings that learners make of their routes to the correct answer, an indication that focusing on the correct answer does not translate into conceptual understanding.

6.3.1.4 Summary and conclusion of the analysis

For Sam, mathematical reasoning is about teaching for understanding (MR-R-TU) which in Skemp's (1976) terms is about teaching with a view of developing in learners relational understanding. He further explains that this means going deeper than the answer, focusing on the necessary steps that need to be followed, and encouraging learners to solve mathematical problems in multiple ways. Sam strongly argues that there is a sense of absences (A) in teachers or student-teachers in terms of teaching for understanding because they are more attuned to teaching for curriculum completion and examination purposes. The ultimate goal is that their learners should score highly in high stake examinations so their teaching is tailored towards achieving this goal. Moreover, during peer teaching a sense of presences (P) is realized in student-teachers in terms of them demonstrating their ability to solve a mathematical problem resulting in a sense of absences (A) in terms of explaining the underlying meanings to the steps. Student-teachers tend to teach in a compressed way rather than a decompressed way. A sense of absences (A) is also realized in student-teachers in terms of originality: they tend to use strategies their teachers in school exposed them to. Over time through probing, a sense of presences (P) in student-teachers is realized in that they are able to think of alternate ways of presenting concepts. Sam explains how important this is for the teacher in that it brings to the realization a sense of presences (P) in terms of being aware of possible learner misconceptions irrespective of their different routes to the answer. What is critical is that relating one route to another requires knowing the underlying concepts to get from the problem to the answer.

However, absences (A) realized in teachers or student-teachers are as a result of absences (A) in the mathematics teacher education curriculum. Sam says teaching for understanding is left to individual student-teachers to research and read more, and there are no clear guidelines and structures. This shows that teaching for understanding is not an explicit discourse in the mathematics education courses offered. Sam explains how he deals with teaching for

understanding during peer teaching and in the first year foundation mathematics course for in-service teachers (MSE 131) where originality and multiple routes to answer is encouraged. Despite this focus, there is a sense of absences (A) in the mathematics teacher education curriculum in terms of focusing on specific school topics student-teachers are going to teach. Moreover, Sam is concerned that a sense of absences (A) realized in the mathematics education courses in that teaching practice is left to the end of the programme makes it impossible for him to review how theory is put into practice.

In sum, for Sam, mathematical reasoning with particular focus on teaching for understanding (MR-R-TU) is a skill that is taught to the student-teachers and demonstrated during the practice of teaching, hence also accomplished practically. The skill lies in the practice of teaching such as peer teaching and school teaching experience and is supported by principles that guide the focus on a foundation mathematics course and teacher educators' views during lecture discussions. During practice of teaching, the teacher-educator could observe: who are teaching for curriculum completion and examination purposes, who are teaching in a compressed or decompressed form, and who are being original and demonstrating multiple routes to an answer.

6.3.2 LMT is about anticipating learner difficulties and suggesting remediating strategies

What is LMT?

Similar to Kenneth, Sam states that engaging with LMT is also about identifying learners' likely difficulties (I) and suggesting possible remediating strategies (R). This is looked at in terms of "*… if you ask children to solve this problem what are the likely difficulties, what is it that you can do to address these difficulties from the pupils?*" (Extract 6.19). He tells his student-teachers "*that they just have to work extra hard just to be able to identify flaws when they are teaching*" (Extract 6.18). Moreover, during tutorials in the MSE 131 course, Sam gives his student-teachers some mathematics problems to solve and guides them through by saying "*I'm not interested in your answer, I want you to focus on possible errors pupils may make and then we can discuss how you as a teacher can go round those difficulties that pupils may have*" (Extract 6.23). Sam also draws his student-teachers to realize that they are modeling a true classroom situation in school and says "*You are teaching in the class and you will realize that there are these misconceptions, there are these errors which children make. What is it that you can actually do to address them*?" (Extract 6.25).

The importance of focusing on learner errors and misconceptions in the courses is given attention by Sam "*so that we can actually try to produce mathematics teachers who can teach children mathematics*" (Extract 6.27). This means that by focusing on learner errors and misconceptions with particular focus on identifying and remediating (LEM-IR), Sam sees one of the ways of developing in teachers the aspect of teaching for relational understanding. Moreover, Sam says "*that's what actually we want them to go and do when they are in the school*" (Extract 6.19).

What is interesting with the way Sam focuses on LMT with particular focus on errors and misconception is that similar to Kenneth, the issue of explaining which includes interpreting and evaluating in carrying out error analysis, in Peng and Luo's (2009) terms is not included in the discussion. There is no distinction between the observed error and explanations as to how the error occurred to why the error occurred. From the constructivist perspective, it is only after such comprehensive analysis of error that decisions about remediation can be made (Borasi, 1987; Jacobs et al., 2010; Peng & Luo, 2009). Ryan & Williams (2007) and as discussed in Section 3.3.3., talk about how not taking into consideration reasons for the error can have an effect on a learners' intellectual mathematical development. In Sam's talk, there is no indication that reasons for the error are in teaching as Kenneth did. This could be because of where it is located in that it is talked about during tutorials in the MSE 131 course; focus on learner errors is not given specific attention. However, Sam says he does make his student-teachers aware that carrying out error analysis, although not exhaustive, is a task that occurs simultaneously during the process of teaching.

In talking about errors and misconceptions with particular focus on identifying and remediating (IR), the teacher or student-teachers, the curriculum, and learners in school are positioned in particular ways. I discuss these ways in terms of absences (A) and presences (P) in four emergent themes, namely: The essence of focusing on errors during teaching; student-teachers making similar errors their learners would make in class; errors can only be addressed in the next class or cohort of learners; and student-teachers to relate their awareness and knowledge of errors in a particular school topic to other school topics.

- *Positionings of teachers/student-teachers, learners and the curriculum with respect to LEM-IR*

6.3.2.1 LMT is about IR: The essence of focusing on errors during teaching

There is a sense of absences (A) in in-service teachers in terms of addressing learners' possible difficulties irrespective of going deeper in their teaching. This is illuminated by what Sam observed when the in-service student-teachers shared their experiences of working with school mathematics in their classrooms. Sam said that:

> *"When I tried some of these examples I am describing to you here, I realized that a few of the teachers would actually express their surprise that sometimes when you are, especially to the in-service teachers, when you are teaching in the classroom some of us we really go deeper to look at these issues in the context that we are discussing them here now, and as a result* ***we have not addressed the possible difficulties that pupils may have when they are teaching*** *these in the, in the schools."* (Extract 6.31)

The in-service teachers express awareness that they hardly worked with learner errors. This could suggest that they were not aware of the importance of working with learner errors in the classroom. Key to them was that they needed to explain mathematics to their learners in ways that would develop in them conceptual understanding. In Jacobs et al.'s (2010, p. 173) terms, and as discussed in Section 3.3.4., there was no realization among these student-teachers that the process of analyzing errors in the classroom "*happen in the background, almost simultaneously, as if constituting a single, integrated teaching move*", before a teacher responds. Neither was there realization that in responding, the teacher can use errors as "*springboards for inquiry*" to illuminate strengths and weaknesses of available strategies (Borasi, 1987, p. 2), as also discussed in Sections 3.3.3. and 3.3.5.

Moreover a sense of absences realized in student-teachers in terms of working with learner errors in the process of teaching mathematics in school is attributed to time constraints due to large classes, lack of resources and the urgency to complete the curriculum and produce good results. This is emphasized in the extract below:

> "*... the pressures that the teachers we are training are going to have when they go in the school system.* ***The pressure of handling 100 Grade 10 pupils even if the pupils, some pupils got a question wrong, to what extent does the teacher have the time to***

> ***go back and say this is what, this is where you are going wrong***. *We try here to bring it to their attention that that is going to arise in the school system when you are teaching. Don't put a blind eye to them.* ***Now the 4th year students, especially the in-service teachers, will tell you about the pressures that they have there. The Head of Department will be saying, 'You must complete the syllabus'. The Head will be saying, 'We want good results.*** *... But the poor teacher may have that feeling that, 'I really need to go back to that' but he may not have the time,* ***he may not have the resources to deal with that part of the, that part of the work***" (Extract 6.22; see also extract 6.18).

A sense of absences (A) is further realized in student-teachers in terms of working with learner errors during school teaching practice as they would respond by saying *"... well, we didn't do anything because there was no time, which is true, but at least they would have realized that they should have done something about that*" (Extract 6.35).

Sam understands that the context in which the student-teachers will work upon completion of their programme also matters. He tries to make the student-teachers realize the ideal situation which could turn out to be different when they are in school. It is likely that when teachers are in schools they are overtaken by the pressures they find. As discussed in Section 3.2.5., Peressini et al. (2004) points to how the school culture can influence a teacher in school in the sense that what he could have been holding as good practices from teacher education gets disoriented. In this case, key to the teacher would be completing the syllabus with an aim of producing good results. As earlier argued, this could have implications for learners' learning in terms of instrumental rather than relational understanding. But Sam does not say what strategies his student-teachers would have to work with learner errors in the midst of the pressures they would find in school.

However, there is a sense of presences (P) in in-service teachers in terms of identifying possible learner misconceptions from their experiences as teachers but a sense of absences (A) in terms of them thinking about possible remediating strategies. Sam says he benefits from having in-service teachers among his student-teachers because:

> *When I'm looking for examples of situations where there will be such misconceptions* ***I've benefited more from the in-service teachers because I tried to ask them to come out and help us to identify these things*** *because they've gone through them themselves and as they have alluded to say, 'Well we never thought along those lines,*

now can you please come out more and give us examples of this nature." (Extract 6.44)

More interestingly is that a sense of presences realized in in-service teachers in terms of the experiences they bring to the discussions could result in presences (P) in pre-service teachers in terms of not only theoretical but also practical experiences as indicated:

"*But when you have in-service students who talk about what they're going through, it gives them a slightly different view because what I may have presented the reaction could have been different from when an in-service teacher who also had that particular experience. So* ***these pre-service teachers seem to be actually looking at it from both the practical side of it and the theoretical*** *which seems to be coming out from our end here*." (Extract 6.46)

When discussing possible remediating strategies, a sense of presences (P) is realized in student-teachers in terms of suggesting varied remediating strategies which together as a class they work with to come up with even more powerful strategies.

"***If there are five students, there may be five different ways****. And I'd be interested to listen from them so that I can actually tell them to benefit from each other's view and hopefully they can come up with their own view which may be strengthening what they had in mind or modifying it to an extent where they could approach it in a different way. Now that different way would take slightly longer to deal with and if one is not careful you may really fail to cover all the work you are supposed to deal with because of that part.*" (Extract 6.41)

Working with learner errors in teacher education is also time constraining; it takes time to negotiate possible remediating strategies. Though constraining, it is also Sam's hope that by working with learner errors in teacher education, eventually a sense of presences (P) in some student-teachers might be realized. They would relate their awareness to the classroom situation since what they might think is so obvious to them might not be necessarily so for their learners.

"*Well hopefully* ***a few of them will relate to these when they are in the school situation, meaning if we were not addressing these issues now there's no way they would actually think of them when they're in the classroom situation*** *because* ***unless they're reading an article in a journal*** *where these things are highlighted, they would not find a way of addressing them, or be aware of possible misconceptions. They may be surprised when they find that children are not*

understanding what they think is obvious. So by addressing these here, by mentioning to them that there are possible misconceptions, it's hoped that it will actually be like a reminder to them when they are in the school situation that this can happen." (Extract 6.47)

Interestingly, while Sam indicates that student-teachers could become aware of learner errors by reading about them in journal articles, he does not appear to include such in his courses. This suggests that Sam's working with learner errors is not in the realm of the "intelligible" according to Davis et al. (2007). Learner errors are discussed at the level of experience. However, similar to Krauss et al. (2008) and Kotsopoulos and Lavigne (2008) argument, Sam points to the importance in teacher education of making student-teachers be aware of working with learner errors during teaching as not doing so would result in them overlooking this crucial task.

Absences realized in student-teachers or teachers in terms of working with learner errors in the classroom are also attributed to absences (A) in the teacher education curriculum. Sam states that "*over the content aspect of it, it seems to be defeating that particular purpose; examples are from the MSE 131course that I talked about"* (Extract 6.23). Moreover,

> "*... **if that was done properly here, say for example you identified certain topics where there are these errors and then you use them more to teach the teachers how to address them, hopefully they should be able to apply these when they are in schools** because if they are aware that there are these errors, **chances are they will address them**. But if they discover these errors when they are teaching, they will end up discovering that there's an error here, what can I do about it? There's nothing much that can be done.*" (Extract 6.24)

Focusing deeply on learner errors and misconceptions is also left to individual lecturers as argued by Sam:

> "*As I said, the current courses maybe they need the restructuring, readjusting in order to have this component we are discussing addressed here rather than giving the idea of 'When you meet these try to think of what is it that you can do to address these issues?' **The current course outlines we are having do not necessarily go deeper into that particular area. It's actually left up to each individual lecturer to notice that there are these problems that may arise at school level and help the teachers to address them when they meet them in the school environment.** And if the lecturer does not, if the lecturer goes according to course line analyzing teaching strategies*

> *for example both here organizing and managing effectively the mathematics classroom you are talking about administration,* ***not more of the mathematics that you are going to teach****. Plan adequately for teaching children with special educational needs. These are the topics that we have on our course outline to teach. … As I said, our course outline does not really align itself in that particular area and I think it's not correct. But then what can one do?*" (Extract 6.26)

From extracts 6.23, 6.24 and 6.26, it is evident that there is no specific topic in the mathematics education curriculum that has an in-depth focus on learner errors and misconceptions. Sam suggests that this could be successful if "… *say for example you identified certain topics where there are these errors and then you use them more to teach the teachers how to address them ...*" (Extract 6.24). This as earlier argued would make the student-teachers be aware of the likely errors pertaining to those particular topics focused on, and "*chances are they will address them*" (Extract 6.24) in school. Furthermore, Sam says that when the issue is left unattended to in mathematics teacher education, the student-teachers might not make sense of learner errors in their teaching when in school.

The issue of errors and misconceptions, according to Sam, is not given specialized attention. It is dependent on the knowledge an individual lecturer has pertaining to errors and misconceptions that are likely to be experienced in school. Sam decides to bring in this issue when dealing with MSE 131, a foundation first year course specifically designed for in-service teachers to help prepare them for first year university mathematics. This means that the pre-service teachers do not benefit from such focus. However, they do benefit when Sam is focusing on "teaching strategies" and peer teaching as will later be discussed. I infer that Sam's argument is supported by Ball et al. (2008) in that working with student-teachers in teacher education on the mathematics that is directly linked to the work they are going to do in school has benefits for learner learning. Moreover, Sam's argument resonates with Stacey's proposal cited in Doerr & Wood (2004) of the "didactic transposition" forming the basis of a new teacher education since there is overwhelming research on learners' errors and their development.

A sense of absences (A) is also realized in the mathematics education courses in terms of providing opportunity to discuss misconceptions as experienced by student-teachers from their school teaching practice (STP) since it is left to the end of the programme:

"*… but the real-life situation would be more useful to the teacher and hence if we had the school teaching practice before graduation, sorry,* ***if they had the school teaching practice in the first semester so that when you come to the second semester this discussion on the STP experiences would be more useful,*** *maybe during that time we could have more coming from the teachers' experiences on what they went through, then* ***one could ask the question, 'How did you deal with the issue of misconceptions with your pupils in the classroom? Those errors that they were making, what did you do to help the pupils***?" (Extract 6.35)

Moreover, pertaining to STP, there is a sense of absences (A) in the mathematics education curriculum in terms of giving it more time so that adequate assessment is done on how student-teachers are incorporating learner errors in their teaching. Sam indicates that:

"*I would have loved the* ***STP to be longer than it is now. Say the whole term the new teacher is teaching in the school, the assessment would be more realistic*** *and may be more helpful to the trainee teacher than it is now when it's for the sake of getting a certificate. All they are looking for is 'Was it satisfactory or unsatisfactory?' And the chances are everybody will be satisfactory because the time is so short to look at all the aspects.* ***For example, if you are going to observe a teacher, to what extent can you incorporate fully the aspect of or is the teacher focusing on the possibilities of errors that pupils make in the classroom?*** *The trainee teacher would ultimately be interested in, 'I want to impress this person in my teaching so that at the end of it all he will say I'm a good teacher.' Now you are a good teacher, but have you produced learners who understand? What's the assessment within the 40 minutes that you observe the teacher there? It may not be adequate enough to get you down to this level we are talking about. And those are my concerns which I don't know yet how to address them.*" (Extract 6.38)

The absences realized in the mathematics education courses pertaining to school teaching practice in that it is left to the end of the programme and also that it is allocated insufficient time as indicated in extracts 6.35 and 6.38 are crucial for Sam. His argument is that it is only through reviewing student-teachers' experiences on school teaching practice that one can have access to their stories on how they dealt with learner errors and misconceptions in their classrooms. These stories would be realistic only if student-teachers are observed over a long period of time, unlike the current situation where they are observed only once. For Sam school teaching practice is a place where observations can be made on whether theory learned on how to work with errors in school is being put into practice as a way of enhancing

relational understanding in learners. This means that for Sam to be convinced that the student-teachers have learned the skill of working with learner errors, he should observe them demonstrate this skill in school.

6.3.2.2 LMT is about IR: Student-teachers making similar errors their learners would make in class

There is a sense of absences (A) in student-teachers in terms of making similar errors learners in school would make. Sam observed the following when working with his student-teachers on some mathematics questions from MSE 131 course:

> "*... really the example I gave you when we were learning quadratic equations is one actually which brought up this part more seriously in my mind* ***because the misconception I saw in the teacher who was, in the trainee teacher who was solving the problem in front of the rest of the class, was a real situation to me and I was saying, 'If this happens, what more when this teacher is in the school? If he himself as a teacher can have this misconception about this particular topic, what more if they are going to teach it now because there'll be nobody to guide them and to say, 'Here you have gone in a different direction.'*** *There'll be nobody, uh uh. So it's more of trying to ensure that these teachers are aware of the fact that there will be misconceptions. They are aware that it's their responsibility to address these misconceptions.* (Extract 6.39)

Because of the misconceptions student-teachers were experiencing, Sam had to shift attention from just focusing on how they would teach the concepts to learners in school to include their understanding of the mathematical concepts.

> "*So we were more of learning the mathematics and how to teach it at that particular point. ...* ***Now let's look at it from both ends. The angle that you should be familiar, you should actually understand this concept better than when you are in the school level, but moreover now that you are going to be the teacher, how are you going to address it? How are you going to put it across to the pupils?*** *The fact that you've understood it slightly better now, it doesn't mean you will teach it better because now you are actually going to learn here, you are learning how to teach*." (Extract 6.40)

From Sam's experience, there is a sense of absences (A) in student-teachers in terms of appreciating the process of identifying possible learner errors and probable remediating strategies (LEM – IR) when dealing with a mathematical problem since to them it only involves finding the correct answer.

"*And there are times when I felt the students were saying, 'Ah, he's wasting our time, I thought he was asking us to solve this question.' (both laugh) When I tell them, 'I'm not interested in your answer' sometimes they are taken aback because learning is about giving a correct answer in mathematics unfortunately.*" (Extract 6.23)

Sam ensures that his student-teachers become aware that the focus is on the process of getting to the answer and then anticipating possible learner errors and remediating strategies. The importance of being aware of this process is also emphasized as it can be incorporated in their teaching be it during the planning or teaching processes.

"*So when you ask them to say, 'No, I'm not interested in this, I'm interested in how to get to the answer, and* ***I want you to focus on the possible areas where pupils make mistakes and prepare you to address that because definitely when you get in the school system you'll give pupils this question. They will, some of them will make these errors. So what are you going to do?*** *Alternatively, if you are aware of possible errors you can deal with them in your teaching and address those issues to the pupils and emphasize those areas that you have identified from here are common mistakes pupils make.*" (Extract 6.23)

The problem with such a situation where a sense of absences realized in teachers in terms of not knowing the mathematics is that it might result in absences in learners in terms of them reproducing similar errors. This is because "*Pupils very much copy what you have given them and if you write wrong things, they'll copy wrong things and they will produce wrong things. And then that's your fault as a teacher because you are not very careful*" (Extract 6.33).

Although Sam is locating the source of learner errors in teaching he has some words of encouragement for his student-teachers. He tells them that everyone, including himself is prone to error, and all they can do as teachers is create time where they can reflect back on their practices especially their lesson preparation as elaborated in the extract below:

"*Actually what I actually tell them is 'Even me as a lecturer when I stand in front of them when I'm teaching, I'm likely to make some errors. OK? So if I can make errors, please don't think that you're an angel, you're gonna be perfect.* *.... But when you are in a classroom, you are actually the boss, the authoritarian sort of person. Nobody is going to question you on whatever's going to happen there. So be careful and try by all means to reflect back. .... Do you stand in front of the board throughout the lesson or you can – you can stand back and see whatever you've written.' They would write the wrong things, wrong mathematical statements on the*

board and when they walk backwards in the classroom then they look at what they've given, they discover that they wrote wrong things…. And then you try to bring it to attention that these errors can happen if you are not very careful with the preparation of your work." (Extract 6.33)

However, a sense of presences (P) is realized in student-teachers in terms of being aware of possible learner errors by behaving like learners during peer teaching and providing wrong answers.

"*... it's* ***during the peer teaching you would actually tell the students****, the pupils* ***sometimes to behave like pupils and sometimes give wrong answers. … Ja, they give the wrong answers and I want to see how the teacher's going to address that****. So when you have naughty pupils in the class it's more interesting because it brings out more of the teacher's reaction to misconceptions*." (Extract 6.34)

The strategy which Sam uses to establish how his student-teachers would work with learner errors or indeed their own errors in class is always received with mixed feelings:

"*Some would get annoyed, some would realize, realize they have made a mistake and therefore how would you correct such a mistake. Some would say, 'But Sir,* ***if I tell children that I've made a mistake, it's very bad on my part.****' I said, 'Yes, that's why you should prepare well, correctly and present your work correctly in the classroom situation.' So that's one area and I agree with Kenneth that it should actually be given more time because it brings out these misconceptions which actually can be done by the teachers themselves. And then how do you address these? Maybe during this period when they are actually student-teachers they would realize more about the possible errors which can come from the pupils in the class*." (Extract 6.34)

The absences realized in student-teachers in that they make similar errors learners in school would make is worrying for Sam. Sam's concern resonates with Ryan & Williams (2007) argument and discussed in Section 3.3.1. His concern is that if student-teachers' understanding of school mathematics is faulty; the expectation is that the teaching will be faulty. This would result in learners learning wrong concepts especially if there is no one to bring to the teachers' attention that the mathematics that has been presented is faulty. Bringing to student-teachers' attention their own errors is meant to make them be aware that errors are inevitable and therefore need to be attended to during teaching. Therefore, what Sam has tried to do in such a situation is bring to student-teachers' attention the importance of being knowledgeable about both the SMK and PCK of the mathematical ideas. As

explained in extract 6.23, Sam sees absences (A) in his student-teachers since they do not recognize the importance of carrying out error analysis, which is about identifying and remediating the error (IR). They are more interested in solving mathematical problems with the ultimate goal of getting the answers contrary to what Sam thinks teaching for understanding entails.

However, in extract 6.33, following Smith et al. (1993), Sam reiterates the general nature of errors in that they are natural and are part of the learning process, hence everyone is prone to making errors. Therefore, Sam makes his student-teachers aware that what is key is that during planning and teaching processes, they should be able to reflect back and assess whether the mathematical ideas are correct and are presented in the rightful way for the enhancement of learners' learning. During peer teaching, as explained in extract 6.34, Sam sees presences (P) in his student-teachers in that they are able to emulate possible errors learners are likely to make in school. His interest in such a strategy is to assess whether and how the student-teachers would address the errors. Nevertheless, student-teachers are aware that making mathematical errors during teaching reduces learner confidence in their teachers.

6.3.2.3 LMT is about IR: Errors can only be addressed in the next class or cohort of learners

Sam has a number of views on errors and how to work with them that indicate the complexities of dealing with these. Sam talks of how working with learner errors involves recognition of the errors and addressing them in the next class or cohort of learners due to time pressures in school. He also says working with learner errors is a learning process in the entire teacher's career.

> "*But **unfortunately you may discover these misconceptions for this particular class and when you've taught this topic that's it, you can't come back to it until next year when you are teaching the other class.** Now it's my wish that when you are teaching the other class, make use of your discoveries in this class which has gone to rectify those problems. **But as you can see, it's not easy. People forget about some of these things in due time and you may not necessarily go back to them and look at this.***" (Extract 6.23)

> "*… You'll discover that the teachers will actually identify the difficulties and the misconceptions that may arise in the process of teaching this, but **there's little room for going back with the same class on those difficulties and find ways and means of resolving that.** As I gave an earlier example. It's more of be aware of these difficulties*

so that when you identify them in your class you ***try to address them more in the next class ...***"

"… ***So there may not be a serious correct way of doing it apart from when you are a teacher you are learning throughout.*** *You may even be learning the subject yourself by discovering that what you thought was easy is not as easy as you thought it was because pupils are going to make what you may call silly mistakes, they are not silly. They are actually mistakes arising from the* ***misconceptions which you may not have addressed at the time you were presenting that new topic to the children.*** *So* ***as a teacher you have a very, very big task of observing these and hoping that you can address them in the next cohort of students.*** *Rarely would a teacher sort out in that particular class. ... Time.* ***Maybe I could say it's time and the pressures that the teachers will have.*** *...*" (Extract 6.25; see also extract 6.29 for emphasis)

According to Sam, the problem that arises when errors can only be addressed in the next class or cohort of learners is that:

"*It's more of a sensitive area or critical area (if I could use that word) because* ***a lot of our children go with those errors, because with the misconceptions unaddressed and may only realize them when they are maybe teachers because that's when they'll realize 'I didn't understand that.'*** *Like what my Head of Department said, 'It's interesting to have somebody who can actually come up and say, 'Oh, is this what it meant?'*" (Extract 6.25)

As indicated in extracts 6.23 and 6.25, there is a sense of absences (A) in teachers using the experiences they have had in a particular class to address similar learner errors experienced by another class. Sam is pointing to that learner errors should be observed in one cohort or class of learners and later be addressed in the next cohort or class where similar errors reoccur. This understanding resonates with Borasi's (1987) diagnostic property of working with learner errors and is discussed in Section 3.3.5. This suggests that thinking about ways of remediating the observed learner errors is not something that requires immediate intervention from teachers since they need time to have thought through ways of doing so. Therefore, learners who would benefit from such observed errors are those in the next cohort or class when focus is on the similar mathematical concepts. Sam is contradicting his earlier assertion that he ensures that his student-teachers become aware that carrying out error analysis is part of the process of teaching and therefore should be immediately embraced.

Nevertheless, teachers are constructed as having absences (A) because they do not seem to have such realization.

Sam also contends that not having such realization could be due to that working with learner errors is a learning process in the teacher's entire career since there is no prescribed strategy of how such a process could best be carried out. He also says that learner errors are a sign that the teacher did not address them during teaching due to time constraints. Absences realized in student-teachers could result in absences (A) in most learners in that they finish school with some errors and misconceptions not remediated and only discover when they are maybe teachers that they never understood particular concepts. As a result learners complete school with unresolved errors. This suggests that despite Sam's understanding that errors are part of the process of learning; there is no recognition, in Smith et al. (1993) terms, that errors are persistent.

6.3.2.4 LMT is about IR: Student-teachers to relate their awareness and knowledge of errors in a particular school topic to other school topics

"*I think it's, it's* ***not possible to have a satisfactory coverage of all the areas on our syllabus in terms of the misconceptions, in terms of the errors that children can actually make****. It's not possible to cover all the topics on this syllabus. But when you are preparing them it's important to actually make them realize that there are these possible errors which they also went through. They may not have been assisted by their teacher on these, but it doesn't mean that you now who is aware of this should not help your children.* ***It's actually giving them an idea of the possible errors, common errors which children make in the schools when, which you should be aware of when you are teaching across the board, not necessarily in numbers, in the algebra, in the commercial arithmetic – not necessarily that sort of area, but to give them certain examples which they can use to spread to the other topics on the syllabus.*** *It's more of saying be aware of certain errors that may arise*." (Extract 6.24; see also extract 6.23, 6.42 and 6.43 for emphasis)

*"… in the courses that I'm teaching, for example, where they are just dealing with the teaching strategies I'll do, purposely bring in a topic or a mathematical problem which I will say to the teachers, 'How could you teach this topic using these strategies?' And hopefully within that discussion with the students we should be able to identify flaws that may arise in the classroom situation…****When we were looking at the different teaching strategies I would actually try to use specific examples from the mathematics content that we are dealing with at the moment and let the teacher go in front during a tutorial and try to use that particular strategy I've talked about***

and then within one hour hopefully we should be able to identify the difficulties and make suggestions of how to go round that. *That's why I was saying using some examples of that nature and emphasizing to the student-teachers that* ***'Please transfer this to algebra, to geometry' would be the only alternative of trying to help the teachers address this particular aspect.***" (Extract 6.27)

As indicated in extracts 6.24 and 6.27, Sam's argument is that it is impossible to enable a comprehensive discussion of possible learner errors in all school topics. This contradicts his earlier discussion in Section 6.3.2.1 of the need to cover more school topics. Here, Sam says what is possible to do is encourage his student-teachers to relate their awareness and knowledge of errors in particular school topics to other school topics. This suggests that focus on specific learner errors should not limit student-teachers' skill of carrying out error analysis which for Sam involves identifying and remediating (IR). Instead it should make them have a broader view that learner errors are inevitable and could arise in other school topics, hence necessary to think of ways of helping their learners in such circumstances. Moreover, as indicated in extract 6.27, Sam focuses on error analysis (IR) when dealing with "strategies for teaching"; one of the topics in one of the mathematics education courses MSE 332: Mathematics Education II. In focusing on particular teaching methods, Sam asks his student-teachers to explain how they would teach a particular school mathematics topic or problem. In the process of such deliberations, issues of error analysis (IR) form part of the discussions. Therefore, a sense of presences (P) is realized in student-teachers in terms of Sam's expectation that they are capable of relating what they have learned before pertaining to error analysis (IR) to new areas of focus.

6.3.2.5 Summary and conclusion of the analysis

Similar to Kenneth's ways of what and how he deals with learners' errors in the mathematics education courses, Sam says that learners' errors are difficulties that should be identified or anticipated and possible remediating strategies sought (LEM-IR). The issue of explaining how the error occurred and why as informed by researchers such as Peng & Luo (2009) and Jacobs et al. (2010) is not in focus. Despite this omission, Sam makes his student-teachers be aware that carrying out error analysis is one of the tasks a teacher enacts in the process of teaching to enhance teaching for understanding. However, Sam contradicts himself by stating that errors can only be addressed in the next class or cohort of learners, although this is in resonance with Borasi's (1987) diagnostic nature of errors.

The way Sam deals with learner errors and misconceptions is accompanied by positioning of a teacher/student-teacher, a learner, and the curriculum largely in terms of absences (A). He points to how school teaching practice is left to the end of the mathematics education programme and allocated less time. His argument is that it is through observing student-teachers teach overtime that one can assess how learner errors and misconceptions are being dealt with. He also sees student-teachers making similar errors learners in school would make, which he suggests would result in absences in learners since the teaching would be faulty. Moreover, bringing to student-teachers' awareness of their errors is meant for them to understand that errors are inevitable and they should be addressed in teaching. Sam's observation is also that student-teachers do not see the importance of carrying out error analysis since they are more focused on finding the answer, which contradicts his notion of teaching for understanding. A contradictory positioning of student-teachers is observed during lecture sessions in that Sam sees presences (P) in student-teachers working with learner errors but absences (A) during classroom teaching. Moreover, presences (P) are realized in student-teachers in that they are able to relate what they know about learner errors and misconceptions to other topics of focus. This again contradicts Sam's discussion of the need to cover more specific school topics. However, a sense of absences realized in student-teachers/teachers is attributed to focus on learner errors and misconceptions not having a prescribed focus, and time constraints teachers experience during teaching. This suggests that the persistent nature of errors is not taken into consideration.

As previously, for Sam focusing on learner errors and misconceptions, which involves identifying and remediating (LEM-IR), is a skill that is accomplished practically, hence taught. It is situated in the practice of teaching such as school teaching practice and peer teaching; and supported by principles that guide the focus on a foundation mathematics course, lesson planning, and strategies for teaching. Also, similar to Kenneth, focusing on learner errors for Sam is a weakly classified discourse given where it is focused on in the mathematics education courses or topics. Sam points to how having a specific focus on error analysis in mathematics education courses could result in an in-depth focus although he does not explain how that in-depth focus would look like.

Table 14 provides a brief overview of Sam's selection and privileging of the discourse of engaging with LMT in terms of teaching for understanding, and anticipating learner difficulties and suggesting possible remediating strategies.

Table 14: Synopsis of Sam's talk of LMT with focus on TU and learner errors

"What" is LMT?		**"How" of LMT that is positionings in terms of presences and absences and the "teaching" of it in terms of what student-teachers are supposed to know and be able to do.**				**"Where" in the courses/topics LMT is focused on**
	Elaboration of the "what"	**Positioning of teachers/student-teachers**	**Positioning of learners**	**Positioning of the curriculum (in school or TE)**	**"teaching" of LMT**	
TU is relational rather than instrumental i.e. teaching learners mathematics as opposed to teaching mathematic. (MR-R-TU)	Teaching for relational understanding includes going deeper than the answer, focusing on the steps to be followed and multiple ways of answering problems. Teaching for Instrumental understanding includes showing learners how to solve problems	Absences: teaching for curriculum completion and examination purposes. A Presences: finding solutions Absences: explaining the underlying meanings to the steps taken PA Absences: not original in concept presentation and of different routes to the answer A	Presences: have different routes to mathematical problems and some could signal misconceptions. P	Absences: teaching for conceptual understanding is left to individual students to research and read more; and no clear guidelines and structures on TU. A Absences: no focus on specific school topics. A Absences: no opportunity to reflect on the integration of theory and practice as teaching practice is left to the end of teacher education programme. A	Through peer teaching and school teaching practice. Through MSE 131 with emphasis that TU is not about covering the syllabus and examination purposes, or showing learners that they can solve mathematics. Through encouraging originality and multiple routes to an answer.	MSE 131, a foundation mathematics course Peer teaching (MSE 331) School teaching practice (MSE 332)
Errors are difficulties that need to be identified/ anticipated and remediating strategies sought (LEM-IR)	Locate source of error in teaching	Absences: not addressing learners' errors due to large classes, lack of resources, curriculum completion, and producing good results. A Presences: in student-teachers during lecture sessions. Absences in student-teachers during classroom teaching. PA Absences: make similar errors learners in school would make. A	Absences in working with mathematical problems as a result of absences in teaching. AA	Absences: no focus on specific school topics. A Absences: school teaching practice is left towards the end of final year programme; and is given insufficient time. A	Through making student-teachers aware that the focus is on the process and anticipating learner errors and remediating strategies. Through informing that errors are inevitable; everyone including himself is prone to errors. Through assessing student-teachers' remediating strategies during peer teaching.	MSE 131, a foundation mathematics course Peer teaching (MSE 331) School teaching practice (MSE 332) Lesson planning (MSE 331) Strategies for teaching (MSE 332)
LMT is teaching for relational understanding and working with learner error	*Locate source of error in teaching*	*Discourse of absences dominate with some discourse of presences in student-teachers*	*Discourse of presence as well as absence in learners*	*Discourse of absences in the curriculum of TE*	*LMT is practical accomplishment learned through:* *- practice of teaching* *- nature of errors* *- focus on error analysis* *- multiple strategies of solving* *- a foundation mathematics course*	*Distributed across courses – weakly classified*

6.4 Titus' talk of the discourse of engaging with LMT

In his talk, Titus pointed to the three broader aspects of the discourse, namely, mathematical reasoning, learner errors and misconceptions, and creating an environment where teachers can listen to learners. I discuss each of these in turn together with how learners in school, student-teachers or teachers and the curriculum are positioned in this discourse.

6.4.1 LMT is about finding multiple routes to an answer (MRA)

- *What is LMT?*

As already discussed, Sam also talks about multiple routes but in terms of encouraging his student-teachers to be original. For Titus, LMT is about finding multiple routes to an answer through problem solving:

> "*I think problem solving is a good example. You know when students,* ***in problem solving you normally want to, not so much looking for the answer, but the route to the answer.*** *And you'll find that there are different ways in which students think. My own study now is focused on what we call 'mental calculation'. 'Mental calculation' means students work out an answer, sometimes not necessarily, they may not be, they may not even have written it down, but what I've been trying to do is when a student works out an answer mentally, ask them to show you how they worked it out. … You will find out that* ***there are different ways to get to a correct answer****. And that's the thing that I try to do.* ***So problem solving is a very good example of where students will bring their own ways of thinking.***" (Extract 6.53)

Moreover, Titus says that in focusing on problem solving:

> "*I ask them questions like for example in problem solving what we try to do is the students, I understand that* ***before anyone can teach problem solving or can teach in a problem solving way, they should themselves have solved problems.*** *If you solve problems then you begin to know that there are different ways in which an answer can be arrived at.*" (Extract 6.59; see also extract 6.64)

In extracts 6.53 and 6.59, Titus is pointing to that finding multiple routes to an answer in solving a mathematical problem involves problem solving techniques. The process requires thinking, in Skemp (1976) or Kilpatrick et al. (2001) terms, which is relational or conceptual, respectively. As Arcavi (1995) states, you cannot judge whether these different routes are appropriate if you do not know what it means to get from the problem to the answer. Titus talks about the aspect of knowing multiple routes to an answer as SCK in Ball et al. (2008) terms by referring to the importance for teachers to develop these ways of thinking about

solving mathematical problems in the absence of teaching. However, he points to that these ways of thinking are important if teachers are to understand and guide the varied ways their learners would present solutions to mathematical problems. Titus contrasts relational ways of solving mathematical problems to instrumental ways where the focus is on "*looking for the answer*" (Extract 6.64). This resonates with Even & Tirosh (2002) argument that the two ways should complement each other. "Problem solving" is a topic in Mathematics Education II (MSE 332), a mathematics education course. LMT with focus on multiple routes to an answer is talked about when dealing with problem solving. This is an indication that the discourse has no specific focus of its own.

- *Positioning of student-teachers/teachers, learners and the curriculum with respect to finding multiple routes to an answer*

Other than the teacher getting to know that there are different routes to an answer through experiencing problem solving, Titus further says that he:

> *"… **prepares teachers to understand that even their own students will have different ways of approaching certain things**. So I usually simply tell them, or tell them or demonstrate to them that it's **important to understand the route that a child takes to get to an answer than necessarily the answer.** You know, when we mark Grade 12 papers, exam papers we always give marks for the thinking before the final answer. … "I try to tell them, not to tell them but actually **I encourage them to be open-minded**. **But, you know, at school level there is a curriculum to be gone through and then a syllabus to be completed. While you can allow students to do certain things in a certain way, you should also be thinking about how to get them to do the things correctly.** So you can allow them to think about something, but eventually you have to show them that: If this is how you think, this is a more appropriate way of doing what you were thinking."* (Extract 6.59)

A sense of presences (P) realized in student-teachers in terms of understanding that mathematical problems can have different routes to an answer could result in them seeing presences (P) in their own learners. Student-teachers "*will tolerate more their own students' different ways to, to (chuckles) different ways of getting to an answer*" (Extract 6.64). This suggests that some routes that learners may use to get to the answer may have errors in them but that is how the learners have come to understand the process. Titus, just like Kenneth and Sam, does not see errors as a problem but a natural stage in knowledge construction (Nesher, 1987; Smith et al., 1993; Peng & Luo, 2009). Teachers need to understand learners' routes to

an answer so that they can redirect their understanding in ways that are acceptable mathematically in line with curriculum demands. Therefore, student-teachers and learners in school are positioned in terms of presences (P). They are both seen as coming with competences to the teaching and learning situation. Moreover, the curriculum is also positioned in terms of presences (P) as it provides competences which should guide the teaching and learning of mathematics.

However, with reference to learner's multiple routes to an answer, a sense of absences (A) is realized in teachers as stated in the following example:

> **"***For example I was dealing with a Grade 2 class. If you said 29 – 18, for example, students, some students will say '29 is very close to 30 and 18 is close to 20. Why don't I just say 30 – 20 and then I take away, I add 2 and take away 1?' (laughs)*
>
> *I: OK.*
>
> *T: If you say 30 – 20 you get 10*
>
> *I: 10*
>
> *T: Add 2*
>
> *I: 12*
>
> *T: Take away 1*
>
> *I: 11*
>
> *T: 29 – 18 is 11 (chuckles)*
>
> *I: OK (laughs)*
>
> *T: So (laughs) so this is how children sometimes work. Now* ***the problem is when children get into school we want to teach them directly the procedures which have been refined already. What we are supposed to do is to take what they already know and eventually show them that actually what they are doing can be done more efficiently using the new procedures that have been developed already.*** *29 – 18 you write 29 on top and then – 18 below*." (Extract 6.53)

A sense of absences in teachers is because they do not work with learners' already existing strategies before they are introduced to conventional ones. The teachers do not see presences in their learners which they could work with to eventually lead to mathematically acceptable ways.

- *How Titus "teaches" for mathematical reasoning involving finding multiple routes to an answer*

Two strategies of how Titus "teaches" for mathematical reasoning involving finding routes to an answer are identified. These include providing opportunity to his student-teachers to experience problem solving so that they can realize the multiple routes to a mathematical problem and in turn see presences in learners' routes they come with to the teaching and learning situation, as already discussed. The second strategy is that:

> "*But students don't usually work things like that. They find things in their own way and then if you don't understand what they are doing, if you don't try to* ***ask them to explain to you what they are thinking****, then you'll go away and say, 'It's wrong. She can't even explain the answer.' It's just (laughs). So anyway, that's one of the things I've been doing*." (Extract 6.53)

Titus makes his student-teachers be aware that for them to access presences in their learners in terms of varied routes to working out mathematical problems, they should ask for learner explanations for their reasoning.

6.4.1.1 Summary and conclusion of the analysis

For Titus, LMT is about relational understanding involving working with multiple routes to an answer as experienced through the process of problem solving. Mathematical reasoning with specific focus on multiple routes to an answer is also a weakly classified discourse since it is discussed when focus is on problem solving, a topic in one of the mathematics education courses. Titus contrasts the relational way of understanding with instrumental ways whose focus is on the answer to a mathematical problem. The way Titus focuses on LMT is accompanied by a contradictory positioning of student-teachers in terms of absences (A) and presences (P). Key for Titus is that student-teachers should be knowledgeable and aware of multiple routes to an answer before they think of how this could enable their teaching (P). Presences realized in student-teachers would enable them see presences in their learners

during teaching in that they would take into consideration that learners come with multiple routes to a mathematical problem which the teacher has to work with in relation to what is provided for in the school curriculum. On the contrary, student-teachers are also positioned as having absences (A) since Titus contends that teachers do not seem to realize the important role learners' multiple routes to an answer play in the teaching and learning process. The way Titus deals with LMT in relation to multiple routes to an answer and the positionings suggests that LMT is a skill that is accomplished practically, hence taught. It is seen in the teaching and learning process and supported by principles that guide discussions on problem solving.

6.4.2 LMT is about learner errors and misconceptions

- *What is LMT?*

Two aspects are identified pertaining to LMT with focus on learner errors and misconceptions. These are prior knowledge as a resource for teaching, and anticipating difficulty and suggesting remediating strategy.

6.4.2.1 LEM is about using prior knowledge as a resource for teaching (PKRT)

For Titus, LMT is about getting student-teachers to think of using learners' prior knowledge as a resource for teaching and learning as indicated in the extract below:

> "*What I tell them usually is that the most important thing is not for you to learn the content of the course, my aim is to get you to start thinking like teachers do. If you start thinking like a teacher, then you can do almost anything.*
>
> *I: So what, what does it mean to 'think like a teacher'?*
>
> *To think like a teacher means* ***to understand what children already know and to think about, 'How do I get them to understand this? How do I develop their knowledge in this area?' Ja. Whatever you are going to teach, what do they already know about this? How can I connect them to this?*** *So this is what I, that's what I tell them that they should develop, they should learn to think like a teacher, not to think like somebody else who's not a teacher. (chuckles).*" (Extract 6.50)

In another instance, Titus said:

> "*So you cannot, as a teacher you cannot go ahead and talk about something when there's nothing, there's no basis for building on. You need to have students understand something before you can move onto something else*." (Extract 6.61)

The emphasis for Titus is "*to get you to start thinking like teachers do*" (Extract 6.50) and he explains what he means by this pertaining to the role of prior knowledge in teaching and learning. He states that it is about the teacher thinking about ways of connecting what he intends to teach to what the learners already know. He argues that if student-teachers become aware of the role of prior knowledge in their teaching then they will be "*thinking so much about how children learn, what they understand really*" (Extract 6.50) when teaching in school. This suggests that errors could be part of learner understanding upon which new knowledge is built through reconstructing, in Hatano's (1996) terms, hence important for the teacher to establish the nature of learner understanding. This is because the teacher could use such understanding as a resource for directing further teaching and learning.

- *Positioning of student-teachers/teachers and learners with respect to using learners' prior knowledge as a resource for teaching*

  "*... my understanding of how to use mathematics, children's mathematics thinking is not something that I used to think about very much as a teacher, it's something that I think about very much now. In fact I have been influencing, ... and I talked about how they can use children's thinking in their teaching of mathematics ... It was very well taken and* ***they were very surprised that in their own teaching they have not, they do not usually think that children know something already and that it's possible to get to what they know already and use it in shaping your lessons and so on.*** *They didn't do that*." (Extract 6.50)

  "*But I've done a lot of observation of teaching and I always see these students start something but they move on. For example, here's an example.* ***Somebody is teaching a class and then uses a term which they assume that students know, but students don't know it. They will just use a term and not define it.***

  .
  .
  .

  T: I do remember ***someone … who did not understand what a determinant is****. Not a determinant, but ja, a determinant of a 2 by 2 matrix. He thought that a determinant of a 2 by 2 matrix is the number you get, you know, when you work out the determinant itself, then you divide that number by, by the sum of the numbers inside. So somehow he would always write a determinant as something, as a fraction. He, what he was thinking was that since determinants are calculated when you want to get an inverse so the final determinant is always the determinant itself divided by some number. Now students were taught that.* ***Now I thought that was conceptually wrong.*** *So* ***a person like that can teach and people will still be confused at the end.***

I: Oh, the teacher is the one who had the problem?

T: ***The teacher had a problem so even if he taught well students would still be, have difficulty.***" (Extract 6.51)

In both extracts 6.50 and 6.51, there is a sense of absences (A) in teachers for not working with learners' prior knowledge in their lessons because they rarely think about learners' prior knowledge as a resource for teaching and learning. This is compounded by not defining new terminology and having knowledge gaps in teachers' mathematical conceptions as Titus observed during school teaching practice.

Using his experience as a secondary school teacher and the seminars he provides to teachers as a mathematics teacher-educator, Titus realizes that shaping instruction based on learners' prior knowledge is a taken for granted resource that teachers do not think of (See extract 6.50). This suggests that if the teacher is aware that learners' prior understanding of some concepts is erroneous, he could use this understanding as an entry point for further teaching and learning. Therefore, teachers need to start seeing presences in learners, and then work with these to mould and further advance required mathematical understanding. In Smith et al.'s (1993) terms, this suggests that teachers will not view learner errors as a problem but as part of the process of learning.

Moreover, using new words without defining them and teacher having misconstrued mathematical concepts, as indicated in extract 6.51, creates a sense of absences (A) in learners' mathematical conceptualization. The teacher could be skillful in pedagogy but if what they are communicating is faulty, as illustrated with finding the determinant of a 2 by 2 matrix, then the teacher's effort is futile because it is not helping the learners learn and advance the required mathematical concepts. This confirms Shulman's (1986) argument that knowledge base for teaching is the amalgam of pedagogy and content. The elaboration of this is "mathematical knowledge for teaching" which stresses the importance for the teacher to know the mathematics and how to teach it (Ball et al., 2008). It cannot be an either or situation but it has to be both. However, I can infer that using prior knowledge as a resource for teaching is a skill which is learned during lectures in that the teacher has to define terminology and know his mathematics for them to engage with learner understanding appropriately. Whether this has been achieved by the student-teacher is dependent on the teacher-educator's observations during teaching practice.

However, Titus had the following to say about some student-teachers or learners in school despite absences observed:

> **"***What I've done here is* ***when I'm talking about something there is always somebody in the classroom who understands something. Sometimes I ask that person to talk about what they know.*** *And usually students connect with some, with one of their own ways of understanding than with the lecturer. But I have never taught something that students are completely… Apart from maybe at secondary school level. But even at secondary school level there was always somebody who could explain something among the students. …* ***I will use that as an example, say, 'Over here, that's a good example.' So everyone will say, 'Ah, so that's what you mean?' (laughs) So it gets, it becomes clarified,*** *I think so, that way*." (Extract 6.61)

A sense of presences (P) is realized in student-teachers in terms of them having their own understanding of what the teacher-educator is focusing on. Similarly, in school there is a sense of presences (P) among learners. The teacher-educator or teacher's role is to ensure that what is meant in whatever is being focused on is clarified on the basis of what some student-teachers or learners already know.

- *How Titus "teaches" for using learners' prior knowledge as a resource for teaching*

For Titus, using prior knowledge as a resource for teaching is realized through sequencing subject matter appropriately as experienced in his teaching and makes his student-teachers be aware of this aspect as indicated below:

> "*Yes I think certainly from my own,* ***from the time when I began teaching I've always been very careful with the way I sequence the work, sequencing the subject matter.*** *When subject matter is sequenced appropriately (I'm not saying correctly) there's no such thing as sequencing correctly but* ***appropriately with regard to what children already know*** *there will not be any gaps in knowledge because you'll be able to pick them from the level where they are and bring them up to a certain level of understanding. So personally I also try to tell my students here and I try to demonstrate*." (Extract 6.51)

Moreover, Titus said that:

"***You could sequence something correctly but not make use of their understanding, the understanding they already have.*** *So even if you sequence it correctly and finish your teaching you have to lecture to them, they may not still have understood anything. And* ***it is also possible that during the lecture, during the lesson you may be asking questions, you ask students questions and then when the answer comes****, like I saw in the primary school, you clap for this one. So they clap. When they clap they don't know why they're clapping, they know it's correct but they don't know … why is it correct? So you can sequence things correctly and teach your lesson and yet* ***some students might not understand fully what has happened because those who answered questions answered them and you never probed the answers, you never tried to justify, to ask them to justify their thinking.*** *So that leaves room for (chuckles) for some incorrect thinking, ja."* (Extract 6.51)

In extract 6.51, Titus is pointing to that sequencing subject matter appropriately includes taking into consideration learners' prior knowledge and learners' explanations for the answers they provide to teacher questions through teacher probing and providing learners opportunity to justify their thinking. This is said in terms of absences in teaching, that is, what the teacher might overlook. The assumption here is that if teachers engage with learners' prior knowledge, which could have some errors, in their teaching then these could be addressed and therefore enhance learner learning. "Sequencing of instructions" is a topic in the mathematics education course MSE 331, and from what Titus says LMT is and how it is focused on in his teaching, I can infer that it is weakly classified. It has no specific structure as it gets discussed when focusing on "sequencing of instructions".

6.4.2.2 LEM is about anticipating learner difficulties and suggesting remediating strategies

Similar to Kenneth and Sam, for Titus LMT is also about identifying likely learner difficulties and suggesting ways of remediating these difficulties.

> *"when we ask questions we say 'What would, this is, you saw this question and after solving it* ***where do you think students in grade, in school, at school would have problems here? How would you try to help them overcome these problems?*** *Why do you…?' Things like that. So that helps them to, to see the problem that students might have, the mistakes and so on. (laughs).*" (Extract 6.64)

Interestingly for Titus, his student-teachers would say "*No, children always have difficulty doing this and this. At times they are not talking about children, they are talking about themselves*" (Extract 6.58). The way errors are focused on points to that in solving school

mathematical problems, student-teachers have the opportunity to identify likely learner errors and think of ways of how they would remediate the errors (LEM-IR). Similar to Sam's observation, the likely learner errors sometimes end up being the errors the student-teachers themselves make. It is difficult to say whether Titus engages with the sources of errors because it is not explicit in his talk. However, following Peng & Luo (2009) and Jacobs et al. (2010), I infer that because of the sequence in which errors are analyzed where they are first identified, explained and then remediated, similar to Kenneth and Sam, Titus overlooked explanations for the sources of errors. This is because you cannot think of ways of remediating the errors before understanding their sources. Anticipating learner difficulties and likely remediating strategies is focused on when dealing with problem solving as Titus states when asked to sight an activity where this was dealt with in the mathematics education courses "*An activity. I can only think about problem solving again*" (Extract 6.64) with focus on school mathematics.

- *Positioning of student-teachers/teachers, learners, and the curriculum with respect to anticipating learner difficulties and suggesting remediating strategies*

Arising from the errors the student-teachers would make which would eventually be identified as anticipated learner difficulties, Titus talked about student-teachers' experiences of solving vector problems and his thoughts about it in general.

> "*T: Um, I think a typical lecture session is when we were talking about vectors.* ***At one point we were talking about vectors and the teachers failed to, the teachers themselves got the vector questions wrong.***
>
> *I: You asked them to solve?*
>
> *T: Yes, I gave them, we gave them Grade 12 questions (chuckles) and they got them wrong. …* ***And then they said 'Our students also have a problem here.'*** *But then the problem it seems was with the teachers because the teachers got them wrong. It means if they are the ones who are teaching students they were just getting them wrong, then you understand why.*" (Extract 6.58)
>
> **"***Well I think that if student-teachers do the lab activities and they continue thinking about how students would do them in the school,* ***they would develop tolerance for wrong answers because they themselves go wrong from time to time****. They will develop tolerance for wrong answers and they also,* ***hopefully, develop the ability to ask students, the pupils why their answers are wrong.*** *So that's usually my aim. My*

aim is that as you get things wrong yourself then you'll learn that, 'Hey, if I can get this wrong, what about students there (chuckles) at that level?' And then you, you begin to understand that: Things can actually go wrong here. I have to find a way of making things easier for the pupils." (Extract 6.64)

From the experiences illustrated in Extract 6.58 and 6.64, it is evident that there is a sense of absences (A) in student-teachers in terms of solving vector problems. These absences could translate into absences (A) in the learners they teach since what they anticipate as absences in learners are actually absences in themselves. The advantage of student-teachers experiencing absences in solving school mathematics is that it could translate into them seeing presences (P) in their learners and tolerate their errors. This, in Borasi (1987) and Smith et al. (1993) terms, would make student-teachers view errors as part of the learning process, and hence think about ways that would enable learners advance the required mathematical understanding. I can infer here that anticipating learner difficulties and suggesting likely remediating strategies is a practical experience as it is focused on from student-teachers' own experience.

Moreover, Titus states how privileged he is to have in-service teachers in his class and the classroom experiences they bring to the discussions pertaining to what learners find difficult as indicated in the following extract:

"*Yes, **the good thing that will happen to our student-teachers who are former teachers and they bring in classroom situations** and say 'What if a student does this and this? How would you deal with it?' So we do get situations, yes, where students, student-teachers do bring up questions from the classroom directly. … Ja, **what learners found difficulty, and some of the questions the learners would ask. And 'How would you deal with this?'** But it's not like, **we don't get as many from those who are direct entrance from secondary school because most of them have never taught a class anyway. They will only teach when they go to teaching, student teaching, student is what, school experience.** … We do get, that's the beauty of having in-service people. **There are times when I actually tell them that, 'OK, this is what the books say. What have you observed in your own life as a school teacher?' And then they say, 'No, you know, these things don't work because this doesn't work.** When you do this and this people will do this.' So you actually learn what actually works in a school and what doesn't work. … Ja, the in-service people. So they tell you things that happened in the school and you compare it with what some people have written.*" (Extract 6.57)

In further elaborating experiences in-service teachers bring to the discussion on learner difficulties, Titus provided the following example:

> "*Now with regard to the classroom situation, I learnt about students who sometimes when they are subtracting they, they may get something wrong and the teacher will say, 'What do you do here?' For example you are subtracting some numbers and subtracting involves what we usually call 'borrowing'. ... (chuckles) which is not borrowing really but a (what's that word, I've forgotten that word now?). So sometimes the students will take 1 from somewhere which actually becomes a 10 and when they add it to something and they subtract they forget that they had actually got something from there, then they… So things like that happen.* ***So teachers were saying 'What do you do when this happens?' And I'll just say, 'Well try to find out from them. This is an error they've committed, but how, the best thing is how did this error come about?' Until the student understands how it came about they will not understand anything.***" (Extract 6.57)

From Extract 6.57, on one hand, in-service teachers are constructed as having presences (P) as they would ask questions on how they would respond to learner errors and some of the questions learners would ask as experienced during their teaching. Furthermore, the in-service teachers are able to identify the contradictions that arise between theory and practice. On the other hand, pre-service teachers are constructed as having absences (A) in asking similar questions in-service teachers would ask as they would not have had the teaching experience. What is interesting with Titus's response to in-service teachers' questions pertaining to working with learner errors is that, following other researchers such as Borasi (1987) and discussed in Section 3.3.5., we can only anticipate unless we ask the learners why they made such errors can the actual sources of the errors made be established. Therefore, learners are constructed as having presences (P) in terms of explaining their thinking processes pertaining to the errors they make. Similarly, in these explained circumstances LMT is a practical experience as only in-service teachers who have had teaching experience are likely to bring up classroom examples on learner difficulties and the type of questions learners would ask.

In talking about LMT as anticipating learner difficulties and suggesting remediating strategies, the curriculum in the colleges of education is positioned as follows:

> ***"…*** *but I've discovered that student-teachers who have a diploma in mathematics may not understand certain things even though when you look at* ***the syllabus that they followed at college the syllabus is very rich and shows that everything is covered. But the teaching there maybe is not… You know you can have a certain number of topics which are very detailed, but it's the level at which you pitch your teaching that determines how much they understand.*** *I suspect that sometimes they don't even do certain things, they just say, 'Oh no these you'll learn on your own.' But teachers have difficulty doing that*." (Extract 6.58)

As shown in extract 6.58, a sense of absences (A) realized in in-service teachers in terms of understanding some mathematical concepts could be due to absences (A) in teaching experienced at colleges of education where teaching could be pitched at a low level. This is with the expectation that student-teachers will learn particular topics on their own, an indication of a sense of presences (P) in them.

- *How Titus "teaches" for anticipating learner difficulties and suggesting remediating strategies*

Two strategies of how Titus teaches for anticipating learner difficulties and likely strategies are identified. Firstly, from the way student-teachers are positioned, it is clear that one of the strategies Titus teaches for anticipating learner difficulties and likely strategies is by focusing on their knowledge of school mathematical problems and the difficulties they experience in solving them. Titus had the following to say on how he worked with student-teachers experiencing difficulties with vectors:

> "*T: We went back to what is a vector? We talked about vector, the rectangular rule or whatever. Vector addition and subtraction and then certain laws that we use and the parallel, when 2 vectors are parallel and when they are equal. Those simple things are the things that they didn't understand. So*
>
> *I: So what was the importance of doing that?*
>
> *T: It was to give them not only theoretical understanding but also a correct conceptual understanding of what is, what was involved, because they were not teaching it correctly. They didn't understand it so students were not understanding them either*." (Extract 6.58)

This shows that the student-teachers are constructed as having absences (A) in that they lacked basic concepts on vectors, and Titus had to explain so as to develop in them both theoretical and conceptual understanding. In Skemp's (1976) terms developing in student-

teachers theoretical and conceptual understanding relates to developing in student-teachers relational understanding of the mathematical concepts. This understanding in teachers, it was hoped, would eventually result in presences (P) in learners.

Secondly, Titus also teaches for anticipated difficulties and likely remediating strategies from the point of view of learners learning in school. This is said in similar ways in extracts 6.58, 6.60 and 6.64 but extract 6.64 is provided as a typical example.

> "*the laboratory activities usually are intended to give the teachers themselves a feel of how it is like to be a student in a classroom. Some of the questions that we give them are not university level material but they are student, they come from the classroom. But most of the, most of the, let me say most of the lab activities that we do are not done for the benefit of the student him or herself. …* ***We are always assuming that if you were a student yourself how would you do this? If you had to get stuck here how would you explain it to them that this is where the problem is?*** *So that's the thing. So the lab activities are really a way of working like students.* ***We are actually saying they are intended to allow the student-teachers to experience learning as students would in the classroom, ja.*** *… That's the main aim of the lab activity. It's not mainly to do university work as such, but it's intended to let you experience how students would learn it somewhere else*." (Extract 6.64)

There is an indication here that Titus does not only work with his student-teachers absences in ways that would enable them develop relational understanding but he also brings into discussion experiences of learners learning mathematics in school. The approach, as suggested by Titus, would benefit both the teacher and the learners because the teacher would be thinking of presences (P) in the learners, for example, "… *What do they think about vectors*? *What do they already know*? …" (Extract 6.58). Moreover, Titus points to that "*... if we were training teachers to just give them knowledge and so on without relating what they are learning to the classroom, they would not be able to teach*" (Extract 6.60). A practical session where this relation is emphasized is a three-hour session per week allocated in each mathematics education course where student-teachers do practical work. The practical activities are aligned with the topic in focus during lectures at that particular time.

In relating to learners learning mathematics in school, Titus encourages his student-teachers to always probe learner answers:

> "*I convey to them in that I myself listen to what they do. And I always keep telling them that* ***it's important to understand an error that a student has committed, to***

probe it and understand why it's there than to move ahead and deal with only the correct things. Because even errors do teach something*, ja. … let's try and find out what does he mean by this.' So they understand that instead of people laughing, they should be given the chance to, students should be given the chance to explain what they think…. Whether the answer is correct or wrong, it needs to be probed further*." (Extract 6.63)

Moreover, Titus asks his student-teachers to explain their understanding of questions asked:

"*Ja, what I usually tell them (and this is, I've been saying this several times) is that if I give you a question, before you answer the question – I mean these essay questions for example –* ***before you answer the question I want you to tell me what you think the question means. I always like them saying, 'This question is asking us to do this and this and this.' (chuckles)… So that I understand what they think before they start answering.*** *I don't like students who, given a question, then they go straight into the answer. I don't entertain that. …* ***They always try to first of all unpack the question****, then… So that's the main thing that I encourage them there.* (Extract 6.63)

The aspect of teacher probing learner answers and asking learners to explain what the task demands of them so that the teacher is aware of learner thoughts about the questions asked suggest that learners or student-teachers are constructed as having presences (P). This further suggests that learner responses whether they have errors or not have something to offer in terms of the meanings they make of mathematical concepts or any learning situation they are involved in. For Titus, probing learner answers and asking learners their understanding of the demands of the tasks given to them is the way to go in establishing learners' thinking. Moreover, errors are also viewed, in Peng and Luo (2009) and Smith et al. (1993) terms, as part of the learning process. However, following strategies of error analysis suggested by Jacobs et al., (2010), Peng & Luo, (2009) and Borasi (1987), I would argue that probing learners' responses foregrounds the teacher focusing on the mathematical detail in learners' strategies so as to access their understanding. What is backgrounded is the teacher's interpretation of learners' understanding as reflected in their strategies, and how to remediate this understanding.

6.4.2.3 Summary and conclusion of the analysis

Titus' talk of LMT in relation to learner errors and misconceptions is in two ways: the role of prior knowledge as a resource for teaching and learning (PKRT); and anticipating learner difficulties and suggesting remediating strategies (LEM-IR). The role of prior knowledge as a resource for teaching and learning is talked about in terms of making connections between what the teachers intend to teach and what the learners already know. This suggests that errors could be part of learners' prior knowledge upon which new knowledge is built through reconstruction. The way Titus focuses on using prior knowledge as a resource for teaching and learning is largely accompanied by positioning of student-teachers and learners in terms of absences (A). Moreover, Titus observed during school teaching practice that the student-teachers had knowledge gaps in their mathematical conceptions, and they were also unable to define terminology. Absences (A) realized in student-teachers would also result in absences (A) in learners' mathematical conceptualizations. This suggests the importance for the teacher to know both the mathematics and how to teach it.

A contradictory positioning of student-teachers or learners is also observed in that presences (P) are realized in terms of having their own understanding of what is in focus, which the teacher-educator or teacher, respectively, ought to work with. Since seeing whether student-teachers/teachers take into consideration learners' prior knowledge as a resource for teaching and learning is observed during school teaching practice, LMT as a skill is accomplished practically, hence be taught. LMT is therefore supported by principles that guide discussions on "sequencing of instructions", a topic in one of the mathematics education courses. Moreover, LMT with focus on prior knowledge as a resource for teaching and learning is a weakly classified discourse given where it is focused on.

In referring to learner errors and misconceptions in terms of anticipating learner difficulties and suggesting possible remediating strategies, as previously with Kenneth and Sam, Titus does not take into consideration reasons for the error. Not taking into consideration reasons for the errors, as argued, would affect the quality of remediating strategies suggested. The way Titus says he works with learner errors and misconceptions is accompanied by positioning of student-teachers, learners, and the curriculum in terms of absences (A). Titus talks of how he observed student-teachers fail to solve school mathematical problems when

the focus was on vectors. The absences realized in in-service teachers in terms of mathematical concepts could be due to teaching in colleges of education that is pitched at low level. A contradictory positioning of student-teachers is observed in that it is possible that absences observed in student-teachers in solving school mathematics could translate into them seeing presences in their learners in terms of tolerating their errors. Student-teachers would then tend to view errors as part of the learning process.

Moreover, presences (P) in in-service teachers while absences (A) in pre-service teachers are observed in terms of the experiences they bring to discussions on learner errors and misconceptions. The experiences the in-service teachers bring are based on their experiences of teaching, which the pre-service teachers do not have. Learners are also positioned as having presences (P) in terms of the opportunity they would be given to explain their thinking processes to school mathematical problems. Therefore, for Titus LMT with focus on learner errors and misconceptions is a skill that is accomplished practically as it is focused on from student-teachers experiences with school mathematics, and hence taught. Similar to focus on multiple routes to answers, LMT is also supported by principles that guide discussions on "problem solving", one of the topics in the mathematics education courses, hence weakly classified.

6.4.3 LMT is about taking into consideration discussion as part of teaching and learning

- *What is LMT?*

For Titus, LMT is about taking into consideration discussion as part of teaching and learning.

> "*You see,* ***if you asked a question in class*** *you say, 'What is this and this?'* ***and then the child gives you an answer and then you say, 'Very good. Sit down.' The rest of the group in the class will wonder why that answer was correct.*** *... They may not know even why it was correct,* ***but if a child is asked to explain and encourage the others to question that person,*** *those were the things I was dealing with*" (Extract 6.50)

A similar approach is also suggested when learners provide a wrong answer:

> "*T: When a student, when they are teaching students if they ask a question and a student gives a wrong answer, 'wrong' (in inverted commas)*
>
> *I: Why in inverted commas?*

> *T: Because that's their understanding. That's their own understanding of what is happening, the teacher should try to ask the student to explain things. In fact what* ***I've been advising students to do is you don't necessarily need to say, 'That is wrong.*** *That answer is wrong.'* ***What you do is you ask the student to explain and then allow others to question that student. That student will actually discover on his or her own that 'my earlier answer was wrong' and revise it without you necessarily saying it's wrong.***" (Extract 6.52)

Titus further explains the important role discussion plays in working with learner thinking during the teaching and learning of mathematics.

> "***Discussion is one of the key things that to me, they call it 'argumentation' in the literature. 'Argumentation' meaning you argue for a particular point. You make a claim and support it.*** *Eventually you'll find that the claim you made was wrong and then you'll revise that claim within the classroom. So a teacher doesn't need to tell you that that is a wrong claim. But as you go on you begin to see that 'Oh, my colleagues actually think differently and they've taught me this' so you revise it. That's what I think teaching should be about really –* ***mathematics teaching. … Ja, it should be about making claims and revising them in the light of what others are saying***." (Extract 6.54)

However, although Titus points to the importance of using discussion as a strategy of teaching, he is aware that in and of itself is inadequate by stating that:

> "*…* ***in a given lesson there should be exposition, there should be some discussion,*** *there should be what? The more… You know in the past we used to talk about a lesson which is maybe a discussion lesson only,* ***but it can never be an actual, an actual discussion lesson only****. The more you put in some expositions, some what, some what, the better the lesson, because there are times when you have to explain something and then times when students ask questions and times when you do this and this. So that's what I try to ask them to do. Not to say lecture method. It means you are going to lecture throughout. No, but there should be a combination of (chuckles) ja*." (Extract 6.56)

The description of LMT in extracts 6.50, 6.52, 6.54 and 6.56 foreground the role of discussion in the teaching and learning of mathematics where learners' claims, whether right or wrong as it is their understanding, are interrogated leading to reformulation of ideas. Making reference to the specialized field of mathematics education research, Titus calls this type of focus the process of "argumentation". Titus' description, following Even & Tirosh

(2002), suggests creating an environment where teacher can listen to learners. Thus when the space for discussion is opened up, the teacher can access what learners are thinking with the help of peers or the teacher asking for explanations of the claims made. However, Titus takes into consideration that discussion alone as a process of teaching and learning is not adequate as there are times when exposition by the teacher is required where the teacher has to explain some mathematical concepts.

In talking about considering discussion as part of teaching, student-teachers or teachers, learners, and the curriculum are positioned in particular ways as follows:

- *Positioning of student-teachers/teachers, and learners with respect to considering discussion as part of teaching*

In talking about the aspect of making discussion as part of teaching, Titus said the following about teachers.

> **"***One thing I discovered is that* ***the idea of discussion is usually not part of teaching****. That discussion was not part of teaching. And whereas a child can answer a question when you ask a question, a direct question, and the child answers,* ***they were not asking the child to explain why the child thought the answer was correct***" (Extract 6.50).

> **"***Actually I think that* ***the teacher who teaches just starts talking about things does not only lack competency but also confidence*** *because he will just be saying what they know and they will not admit any questions from students, because if questions from students come and they are different from what they were thinking, they might say, 'No, that's not part of the lesson' and so on*" (Extract 6.56).

Extracts 6.50 and 6.56 suggest that the teachers are constructed as having absences (A) in terms of making discussion a part of teaching and learning where learners are asked to explain their responses and allow learners to question each other. A sense of absences is also realized in teachers in that by them not making discussion part of teaching and learning could be a sign that they are not competent and confident. This is because their teaching would only be dominated by teacher talk and learners would be considered as listeners who are denied opportunity to ask questions so as not to divert teacher's ways of how he has conceptualized the mathematical ideas.

In terms of seeing presences (P) in student-teachers' in terms of dealing with discussion as part of teaching, Titus had the following to say:

> "*I could think about what we usually do during tutorials and examinations. ... during our examinations at this university, in our department* ***we usually don't ask questions that students can answer by taking an answer straight from the book onto the paper.*** *... So we normally ask, for example, I give a statement. One of the things we normally say is, 'Explain your understanding of this.' ... So they'll explain it. Now when you ask them to explain their understanding of it then you'll be able to look at the subsequent questions and try to understand why they're answering the question the way they are doing because their own understanding of the question may be a little bit different. ... So* ***sometimes we give marks for people who understand the question differently and are consistent in looking at subsequent questions with that understanding. So (laughs) so that's an alternative way of thinking. So students are thinking differently than you thought. Now you can ask a question and not know that there are other things that students think about which are not part of your present ways of thinking****. So that's one of the things*"
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> .
>
> *T: So during tutorials I also do that. When we are discussing during tutorials I ask students sometimes a question and then I encourage them by saying, 'The thing that you thought about first, whatever it is it may sound ridiculous to you, tell us about it because then it represents what you are thinking.' Many students will want to wait until they know the correct way of thinking. Now we don't want that. I would rather students tell me what they think about this. So* ***I always encourage people who say, 'I'm just thinking' I say, 'Yes, yes, go ahead. You're just thinking like everybody else.' So 'I'm just thinking.' And then they tell me what they are thinking****.*
>
> *I: So how do you think that helps them*
>
> *T: It develops*
>
> *I: in becoming mathematics teachers?*
>
> *T:* ***It develops their own confidence.*** *They begin to learn that what I think is not necessarily wrong. What I think if I share it and I give reasons for it, other people understand something.*
>
> .
> .
> .

> *T:* ***I always hope that when they get to schools they will allow students to also think for themselves.*** *… Ja, that's what I always think, that if student-teachers learn to think on their own they will not impose their thinking on students. They will allow students to think*" (Extract 6.54).

In talking about the issue of making discussion as part of teaching, presences (P) realized in student-teachers are in terms of how the aspect is addressed during tutorials and examinations. During examinations, there is a sense of presences (P) in student-teachers in terms of alternate ways of thinking about a given statement, which are sometimes not aligned with teacher-educator's thinking. Titus says that if student-teachers' thinking is consistent in the subsequent questions asked, consideration is sometimes given. This is an indication that learner thinking is given attention. Similarly, during tutorials a sense of presences (P) is realized in that asking student-teachers to explain their understanding in ways they think appropriate helps develop confidence in themselves since they are able to explain their reasoning. This means that Titus gives his student-teachers opportunity to voice out their initial thoughts pertaining to the focus of the discussion, hence listen to learners. Moreover, the anticipation is that a sense of presences (P) realized in student-teachers as independent thinkers could translate into them developing in their learners such thinking. The way Titus works with his student-teachers, following the theory of constructivism, is that they do not come to the learning situation empty. They have some knowledge which can be accessed through encouraging them to explain their understanding.

- *How Titus "teaches" for making discussion part of teaching and learning*

Two strategies for how Titus teaches for creating an environment where discussion is part of teaching and learning are identified. Firstly, he encourages his student-teachers to probe learners' answers.

> "*So in handling alternate conceptions* ***I always ask student-teachers to always probe****. You could say, for example,* ***you ask a question and then somebody answers in a very strange fashion. You could say something like, 'I've never thought of that as an answer, tell me, why do you think that's an answer?'*** *Or you could say, 'That's a different way of thinking' or maybe, I don't know (laughs) 'That's something that I've never thought about before. Tell me, why do you think this is the thing?'* ***And then the child will explain.*** *So that's how I ask them to handle what we might call misconceptions. … Ja. I call them 'alternative conceptions' so that teachers should*

always want to listen to even a wrong answer because ***to probe a wrong answer means eventually you understand why that person thinks it's correct and after the other students have chipped in, there's what we call 'negotiation of meaning'*** *(chuckles).*" (Extract 6.52)

Secondly, he encourages his student-teachers to ask questions from the onset of the topic area of focus and giving opportunity to learners to demonstrate their understanding.

"*I should say I think experience is what has taught me to do certain things differently. … So when I focus on certain areas I always ask questions. Usually I ask questions about,* ***there are certain topics which I start by asking a question****: 'Has any one of you heard about such a thing?' 'Yes, yes we have.' 'What do you think it, what do you think it means? How…?', so that's how I sometimes get into a lesson.* ***So students will be talking about what they think that thing means. … And then the others will say, 'No, but it doesn't actually mean that.' And so on. Then we talk about … So that's, that's what it is. I, when I focus on something… And let me just say my way of teaching always tries to allow students to, to think, but I'm doing that more and more now than at the beginning.*** *At the beginning I really didn't know what is very important here. As you teach for some time you begin to pick up ideas about this is more important than this*." (Extract 6.55)

"*What I do is* ***I actually try to train students to allow their own students to talk.*** *… But this was a discussion. We were having a discussion and we were sharing ideas. So that's how I want my students to teach.*" (Extract 6.56).

As shown in extracts 6.52, 6.55 and 6.56, the indication is that through probing by asking the "why" question to learner responses or from asking questions from the onset of the topic, the teacher would eventually understand learners' thinking processes and come to a common understanding through classroom discussion. For Titus, what is dominant in the way he teaches for making discussion part of teaching and learning, as already argued in the previous section, is that he sees presences (P) in his student-teachers and works with that to attain shared meanings through discussion. Moreover, "where" Titus focuses on using discussion as part of teaching and learning is in his entire teaching since he says it is his way of teaching. By modeling to his student-teachers on how considering discussion as part of teaching and learning should be focused on, Titus's expectation is that they will be able to copy or imitate his strategies in their teaching. This strategy is similar to one of the strategies described by Davis et al. (2007) and discussed in Section 3.2.4., and hence an indication that LMT is a practical experience.

6.4.3.1 Summary and conclusion of the analysis

As discussed, for Titus LMT is also about creating an environment where discussion is considered as part of teaching and learning as one of the ways a teacher can listen to learners. The process involves a teacher asking questions and probing learners' answers including encouraging learners to question their peers' explanations, hence linking the description to the theory of argumentation. The argument is that such a process encourages learners to realize on their own the need to refine their understanding without the teacher's reinforcement. The way Titus focuses on LMT is accompanied by positioning of teachers and learners in terms of absences (A). His observation is that teachers hardly consider discussion as part of teaching and learning in order to access learner thinking, hence a sign that they are not competent and confident since their teaching is dominated by teacher talk and learners' role is to listen.

A contradictory positioning from that of teachers and learners is observed in student-teachers and learners. Titus talks of how during examinations and tutorials ensures that his student-teachers explain their understanding of questions asked. He asserts that doing so helps in building confidence in his student-teachers in that their initial thoughts on what is being discussed are engaged with to attain shared meanings through discussions. Titus argues that such focus would translate into student-teachers seeing presences in their learners. Focusing on making discussion as part of teaching and learning for Titus is a skill that is accomplished practically, hence taught. He talks of how he models to his student-teachers this aspect in his teaching with the expectation that they would emulate. The practice is supported by principles that guide what discussion entails. Moreover, LMT with focus on making discussion part of teaching and learning is weakly classified because it is not given specific focus in the mathematics education courses as it is Titus' way of teaching.

Table 15 provides a brief overview of Titus' selection and privileging of the discourse of engaging with LMT in terms of multiple routes to an answer, learner errors, and making discussion part of teaching and learning.

Table 15: Synopsis of Titus' talk of LMT with focus on multiple routes to an answer, learner errors, and making discussion part of teaching and learning

"What" is LMT?		**"How" of LMT, that is, positionings in terms of presences (P) and absences (A) and the "teaching" of it in terms of what student-teachers are supposed to know and be able to do**				**"Where" in the courses/topics LMT is focused on**
	Elaboration of the "what"	**Positioning of teachers/student-teachers**	**Positioning of learners**	**Positioning of the curriculum (in school or TE)**	**"teaching" for LMT**	
Focus on multiple routes to an answer is relational rather than instrumental whose focus is on the answer. (MRR-MRA)	Finding routes to an answer involves teaching in a problem solving way and requires the teacher having gone through the process of problem solving.	Presences: teachers experiencing process of problem solving enable understanding of learners' routes to an answer. P Absences: not working with learner strategies to mathematical problems before guiding them to conventional ways. A	Presences: have different routes to mathematical problems and some could signal misconceptions. P	Presences: in the school curriculum in competencies to guide teaching	Through providing opportunity to student-teachers to experience problem solving. Through teacher asking learner explanations for their routes to an answer.	Problem solving (MSE 332)
It is about using prior knowledge as a resource for teaching. (LEM-PKRT)		Absences: in teachers using prior knowledge as a resource for teaching. A Absences: in defining terminology and mathematical knowledge gap. A Presences: have own understanding for teacher-educator to clarify. P	Absences in learners due to absences in teachers. A Presences: have own understanding for teacher to clarify. P		Through sequencing subject matter appropriately using learners' prior knowledge and explanations for answers they provide to teacher questions.	School teaching practice (MSE 332) Sequencing instructions (MSE 331)
Errors are difficulties that need to be identified/ anticipated and remediating strategies sought. (LEM-IR)	Locate source of error in teaching	Absences: make similar errors learners in school would make. A Presences: in-service teachers bring classroom experience to the discussion. P Absences: pre-service teachers do not bring classroom experience to the discussion. A Presences: in student-teachers in colleges of education to learn particular concepts on their own. P	Absences: in working with mathematical problems as a result of absences in teaching. A Presences: teachers would tolerate learners' errors. P Presences: in learners in that they can explain their thinking if asked. P	Absences: in curriculum in colleges of education – pitch the teaching of some topics at a low level	Through focus on student-teachers' knowledge of school mathematics and difficulties they experience. Through focus on learners learning in school which include teacher to always probe learner answers, and ask learners to explain their understanding of the questions asked.	Laboratory activities; a 3 hour session per week set aside for student-teachers to do practical work. Problem solving (MSE 332)

Creating an environment where discussion is considered as part of teaching. Env-DPT	Teacher should ask learners to explain their responses (E), and allow learners to question each other (Q). Discussion is about "argumentation".	Absences: discussion not part of process of teaching and teacher talk dominates resulting in lack of competence and confidence. A Presences: alternate ways of thinking which are sometimes not aligned with teacher-educators, hence confidence building and independent thinking. P	Absences: in learners as teacher talk dominates. Presences: in learners as independent thinkers. P		Through encourages student-teachers to probe learners' answers. Through encouraging student-teachers to ask questions from the onset so that they can access learner thinking and engage with such thinking in a discussion to arrive at common understanding. In both strategies learners are positioned as having presences.	Tutorials and examinations
LMT is relational thinking involving multiple routes to an answer; and working with learner errors involving PKRT and IR; and environment where discussion is part of teaching	*Locate source of error in teaching*	*Discourse of absences as well as presences in teachers/student-teachers*	*Discourse of presences dominate with some discourse of absences*	*Discourse of absence in TE curriculum*	*LMT is practical accomplishment learned through:* • *Experience with problem solving* • *Asking for learner explanations to their thinking (probe learner answers)* • *Sequencing subject matter appropriately* • *Difficulties student-teachers experience in solving school mathematics*	*Distributed across courses – weakly classified*

6.5 David's talk of the discourse of engaging with LMT

David's talk of the discourse of engaging with LMT resonates with Even & Tirosh (2002) two broad categories, which are: creating an environment where teachers can listen to learners, and learner errors and misconceptions. He said nothing that referred to developing in learners both instrumental and relational understanding. I am going to discuss each of the two ways David talked about LMT in turn together with how learners in school, student-teachers or teachers and the curriculum are positioned in this discourse. Tied to the positionings is how David said he "teaches" for LMT in terms of what the student-teachers should know and be able to do.

6.5.1 LMT is about focusing on both content and context; and assessing learner understanding of what has been taught

- *What is LMT?*

For David, LMT is about teacher focusing on the context rather than only content; and assessing learner understanding of what has been taught.

> "*… **rather than just focusing on the content, they need to take into account many other aspects like taking the context**, they need to look at the learners they have, **the background of the learners, what else they know and how best to present the work** and to understand the learners, or their audience in a, in a much more professional way in the sense that if they look at the age focus **they need to go beyond just focusing on the, on the subject** and just saying, 'Oh I know or I understand vectors and therefore I can go and teach.' But they need to go beyond that, **understand their learners, look at the context and see how best they can prepare. Look at the resources they have and see how best they can, they can, they can do that.** And I feel this career, that's what our work should be, is here, to see if we can give them all these tools, resources and cultivate in them the right attitude in terms of how they should be approaching the learners, not just to, ja, teach them, focus on the subject and just teach regardless of whatever they, ja, they find in the school*." (Extract 6.65)

> "*… **how they want to assess, or how they want to understand how the learner, whether the learners have understood what they are teaching or not.** How they want to do that, how should they do it? They may have talked about whatever topic earlier, they may have talked about the numbers or sets, but I mean how would they get convinced that yes I think they are, the learners are with them, they understand sets and the what assessment procedures and what kind of attitudes should they adopt and whether the… Ja, **the learners are not just reproducing what they have taught them, but they can use the mathematics that they're learning and apply it elsewhere. And then explain it and understand it.** So how do they go about checking*

or convincing themselves that yes, I think this has happened. And I think you know problems most times we try to bring that awareness, that consciousness or that preparation that you know that that's one of the reasons why, ***that's what should make a difference, put them apart from someone who hasn't gone through the teacher preparation programme.*** *They should do better, do things differently from others who just, maybe just teach and give ...*" (Extract 6.66)

In extract 6.65, David's argument is that while knowing the mathematics the teacher is going to teach is important, it is not adequate if the context of the learners is overlooked. For example, the teachers have to take into consideration the background knowledge of their learners and understand them, and then using the resources they have, see how they can best present their work. David's argument is similar to what proponents of mathematical knowledge for teaching such as Ball et al. (2008) and Adler & Davis (2006) have argued that knowing mathematics per se does not translate into teaching it but that teachers encounter mathematical demands for teaching. The way they come to hold and use mathematics in order to teach it successfully is different from how the mathematician, for example, would hold and use mathematics. One of the demands then for teaching mathematics identified by David is that the teacher has to understand the context of his learners, and how best to present the mathematics in light of this knowledge and the resources readily available.

The other demand for teaching identified by David in extract 6.66 is the need for the teacher to assess learner understanding of what has been taught. The indicators of learner understanding are that "… *the learners are not just reproducing what they have taught them, but they can use the mathematics that they're learning and apply it elsewhere. And then explain it and understand it*" (Extract 6.66). The challenge for the teachers is to provide the learners with such tasks where they can demonstrate understanding of mathematical concepts taught by applying them in different contexts other than the contexts that were focused on during teaching. These demands for teaching, as David has argued, distinguishes between a teacher of mathematics and someone who is not. The way David talks of focusing on both content and context; and assessing learners' understanding resonates with Even and Tirosh's (2002) aspect of creating an environment where teacher can listen to learners.

- *Positioning of student-teachers/teachers with respect to focusing on both content and context; and assessing learners' understanding of what has been taught*

In talking about LMT as focusing on both content and context, and assessing understanding of what has been taught as discussed above, David said that he ensures that student-teachers

are aware of such principles. If the teachers do not show awareness, it is as a result of other factors. This was said in both extracts 6.66 and 6.67 but I only provide extract 6.67 as a typical example.

> "*D: And **I mean those principles they have.** And we try to cultivate that and try to give them that, but of course **we are also aware that the environment under which sometimes they work, because really I mean you can't pretend that to have that attitude and to do what is expected of you in those circumstances, that is a bit more work.** … But, ja, **many easily slip into where you'll just teach them as one group** and you'll treat them as a bunch and it's like they take the easier option"*
>
> *I: What do you think is the reason for opting for the easier option?*
>
> *D: I think it's, really they are not, **there's something to do with some motivation and, and sometimes in some situations it's about maybe the right numbers of learners and it's like the environment is such that the teacher finds himself or herself in a situation where it's just too many learners and poorly motivated and meanwhile the demands are so many placed on her.** And then they, they take the easier, the easier option which is really unfortunate. It doesn't really imply they are not prepared. Actually they are but they take the easier option and they slip back now to just being a teacher like an untrained teacher who is just picked from the streets and they do what they, they do what they want to do*." (Extract 6.67)

This suggests that student-teachers are constructed as having presences (P) in terms of what is expected of them as teachers of mathematics, and absences (A) are due to de-motivated teachers with large classes of poorly motivated learners. David argues that the absences realized in the teachers leaves them with no option but to consider their learners as a group and not individuals, and hence disregarding their capabilities. Therefore, this makes it difficult to distinguish between a trained teacher and the untrained one. This suggests that for David, LMT is a practical experience where student-teachers are expected to put into practice the principles learned about what is required of a teacher of mathematics.

- *How David "teaches" for focusing on both content and context; and assessing learners' understanding of what has been taught*

Strategies of how David "teaches" for LMT are in form of two dispositions, namely, developing in student-teachers the right attitude for teaching; and reaching common agreement between teacher's and learners' principles. These two strategies are described in extracts 6.67 and 6.68, respectively.

"… *Though I would really say I mean the attitude is really part and parcel of teacher preparation and I think* ***teachers ought to have the right attitude because teaching is not just teaching but it also involves having the right attitude.*** *You must have the right attitude to be a good teacher.*

I: And what does it involve to have the right attitude? (laughs)

D: (chuckles) Ja, the right attitude actually really involves really feeling for the learner. I think if you have the right attitude for teaching I think ***you would take the learners as individuals and you would see yourself more as someone who is there for the learners, to help them, facilitate their learning and create an environment where the learners are able to maximize on their, on their learning.*** *…* ***You really need to do more work to prepare****, if you are going to do, to really create an environment where learners can maximize their learning, you reach out to them, you look at them as individuals and it demands a lot from a teacher*." (Extract 6.67)

"… *where you, you kind of feel you kind of meet somewhere with your own thinking, your own ideas, with your own principles. And* ***with the learners as you present where you seem to, to meet somewhere where you, yes, I think this is good. It's like you begin to have a common general view, a common, a common idea or a common thinking on an issue I think over there.*** *There are times when I feel yes, I think this is good, I've done my, I've done my part. I think this is… Because there are times when you, you kind of move apart and you don't seem to,* ***not that we should be agreeing on every aspect, but there are some, some aspects that you'll see that this is what happens in the schools and you feel it shouldn't be happening this way.*** *Things should happen this way but sometimes you'll find the in-service teacher seems to suggest that no, you see there's no time, you need to produce results, this is what you need to do and they seem to stick to, to those viewpoints. And if you do succeed and you create a situation in other words where they can see the point and you meet somewhere there, those I think are the, the points where I feel, 'Oh yes, this is good.'* **"** (Extract 6.68)

In extract 6.67, David points to three attributes of what it means for a teacher to have the right attitude for teaching. These include considering learners as individuals, teacher as a facilitator of learner learning, and teacher creating an environment where learners could maximize their learning. David argues that the three attributes put demands on teachers in terms of preparation and how they would reach out to learners and consider them as individuals. Moreover, in extract 6.68, David also models the importance of teacher and learner reaching a common agreement in the teaching and learning situation. This is exemplified by how he deals with issues raised by, for example, in-service teachers that there is no time to consider individual learners' learning since you need to produce results, and therefore work towards ways of achieving this goal. The significant moment for David is when he sees shifts in the

thinking of his student-teachers almost aligning with what is considered ideal. David is convinced that what the in-service teachers are arguing for is not good enough if our focus is on learner learning. This suggests that, in Smith et al. (1993) terms, in the teaching and learning of mathematics, learners come with their own understanding of mathematical concepts. It is therefore important for the teacher to engage with learner understanding with a view of drawing learners to what is accepted as conventional mathematics. For David, a sense of presences (P) in teachers in terms of having the right attitude for teaching; and reaching common agreement between teachers' and learners' principles is crucial and he works towards developing this in his student-teachers.

6.5.1.1 Summary and conclusion of the analysis

For David, LMT is about two aspects, which are focusing on both content and context; and assessing learners' understanding of what has been taught. Focusing on both content and context involves teachers knowing the mathematics they are going to teach as well as understanding learners' background, and figuring out how best they can teach using available resources. Assessing learners' understanding of what has been taught involves the teacher giving learners tasks in which they can apply their knowledge in different context other than the context focused on during teaching. Both foci of LMT provide an environment where teacher can listen to learners. The way David focuses on LMT is accompanied by positioning of student-teachers in terms of presences (P). David says that his student-teachers know and are aware of what is expected of them as teachers.

A contradictory positioning of student-teachers/teachers is also observed in terms of absences (A). David talks of how student-teachers/teachers tend not to focus on both the context of learners; and assess learners' understanding since they are de-motivated and have large classes of poorly motivated learners. Absences realized in teachers result in them not taking into consideration potentials of individual learners as they tend to treat them as a group, hence exhibiting practices similar to those of untrained teachers. Two dispositions are observed on how David "teaches" for LMT: developing the right attitude for teaching; and reaching consensus between teachers' and learners' principles. Developing the right attitude for teaching involves teacher: considering learners as individuals; as a facilitator of learner learning; and creating an environment where learners could maximize their learning. A sense of presences (P) in terms of the two dispositions ought to be realized in student-teachers since

David talks of how he ensures that they become aware of their importance. Therefore, for David LMT is a practical experience in that he expects his student-teachers to put into practice principles learned about what it means to be a teacher of mathematics.

6.5.2 LMT is about learner errors and misconceptions

Two aspects are identified pertaining to focusing on learner errors and misconceptions, namely, that errors are not conscious or deliberate but genuine; and that it is about carrying out error analysis where the error has to be recognized, explained and remediated. I discuss how David talks about these two aspects in turn. My decision to discuss these two separately although they are interlinked is that the first one is more inclined to a discussion around the nature of errors while the second one is more on the practical side focusing on strategies one would have to use when discussing a school mathematical activity.

6.5.2.1 Errors are not conscious or deliberate (learners make errors genuinely)

In focusing on learner errors and misconception, one of the things David emphasized is that learners' errors are genuine and therefore not made consciously or deliberately as stated in extract 6.69

> *"… The first one is that they should always understand or appreciate that* ***when learners have made a mistake or they've errored, it's not conscious, it's not deliberate.*** *They should always have it in their, in their head or mind that when they see something wrong – a wrong answer, a wrong interpretation – it's never deliberate. It's the pupils never or the learners never make mistakes deliberately. It's not something that they do consciously to make an error or two. So this is really what the learners,* ***when they make a mistake or they make an error it's really genuine and they should never take it to mean just one of those things that learners do.***" (Extract 6.69)

David's argument about learner errors and misconceptions, following Olivier, (1989); Hatano, (1996); Nesher, (1987); Peng and Luo (2009) and Smith et al. (1993), is an indication that they are viewed as a natural stage in knowledge construction as they form part of what the learner is thinking. This means, in Olivier's (op cit) terms, learners' conceptual structures which could have some misconceptions will interact with new concepts and influence new learning, and mostly in a negative way.

- *Positioning of student-teachers/teachers and learners with respect to learner errors being genuine*

In talking about learners' errors being genuine and not made consciously or deliberately, David talked about the experiences the student-teachers bring to the learning situation. This is contained in extracts 6.70, 6.74 and 6.79 but I provide extract 6.70 as a typical example with some illustrations from extracts 6.74 and 6.79.

> "*… **the input from the in-service teachers, sometimes it's valuable because they actually give examples that they've come across** because we just say, no, there are some learners who do this and they come up with this" . But **when you analyze the solutions and the methods, the working, the workings of the learners then you'll see most times that either it's the way they were taught in the past or by them or it's maybe the way they have interpreted the, the mathematics structure or syntax as the way it appears in there.** So the in-service teachers have those aspects that are very, very valuable and I find, ja, because **for the ones coming straight from school it does sound a bit more, a bit theoretical in the sense that you can bring out examples of areas where learners originally have errors and have misconceptions or whatever, but it really sounds as if it's a fairy tale.** They'll be amused, Oh yes, but for the in-service teachers it's something they've experienced and they bring out that. And when they discuss I think they follow. **But of course they'll always tell you. 'Oh no, that's not feasible you know. There's no time.'** All we do is they just mark the final answer and move on. Well **there's no time to start studying, look at where this could have come from.** You know there are many other things you know to, we need to chase money elsewhere', so… **But otherwise we, ja, they do have the skills and I'm sure these would do a good job.**" (Extract 6.70)

In-service teachers are constructed as having presences (P) since they are able to enrich the discussions by providing examples experienced during their teaching while absences (A) are realized in pre-service teachers since they have had no teaching experience. This means that the pre-service teachers see the aspect of analyzing learner methods used to solutions of mathematical problems in detecting the sources of errors made more theoretical rather than practical. This is because the experiences they have are "… *what they went through themselves as pupils, as learners, but really that's a different background*" (Extract 6.74). This suggests that working with learner errors is a practical experience in that you need to have the experience of teaching to get better at it especially that the sources of errors are located in teaching. Although student-teachers would wonder how this is so by saying:

> "… *'**No but the teachers they are the teachers, so how can errors come from**?' But the moment you actually cite or give examples, then they said, 'Oh yes, that's it',*

because the number of examples that we go through, ***we go through a number of examples where they, almost every topic seems to have the errors that they, they have seen or they have committed themselves while they were learners.*** *And then they can see and they say, 'Ah, OK, so this really happened.' …*" (Extract 6.79)

There is a sense of absences (A) in student-teachers in terms of accepting that errors are as a result of teaching. A sense of presences (P) is realized when David provides examples which they end up recognizing. A sense of absences (A) realized in in-service teachers in demonstrating the skill of working with learner errors are also as a result of the context, since they see the process to be time consuming, and therefore resort to focusing on the final answer rather than evaluating the source of error. One of the reasons, I infer, could be that they are lowly paid and therefore they need to look for extra jobs elsewhere to supplement their meagre salaries. Therefore, looking for reasons for learner errors is not an option for in-service teachers although they would have developed the skill and are aware of the importance of doing so.

Moreover, David states the importance of focusing on learner errors and spoke in similar ways in extracts 6.71 and 6.73 but I provide extract 6.71 as a typical example.

"*... because I think* ***that's one of, one of the lasting single window of opportunity or for them to get into how the learners think or the way they interpret that part or piece of mathematics because it's like when they look at the thinking well that reveals the thinking of the learner****, … And so you, it's like you'd give them a window of opportunity to see how the learners are thinking or this particular learner is thinking and we'll see how the concept is developed or not developed in the learner. And we'll see how you can reach out to the learner and see how you can address those practical areas. So* ***I think it's key in that well, in whatever they do in presenting the mathematics if they, if they don't spend time to reveal to the learners or to pay attention to how learners think or know or how they interpret those parts, then they may actually be missing the point.*** *I mean it's really like they say, 'Oh they've learnt vectors or geometry so we go, we move on' without them paying attention to what areas or what was the learner thinking.*

As people have said, whatever we, whatever we do, how well we plan our lessons, usually what we plan to teach is not what learners actually learn in that learners have their own way. You can't package something and then they get it. They always have their ways of learning. So in one particular lesson they will then pick up a few different things. So they want a time you can be sure and say, 'Yes I think, so this is the…' ***It's when you pay attention to these errors, these misconceptions and then for those you bring out that window of opportunity*** *where you say, 'Ah I think here*

we may have spent a lot of time but really there are these gaps that we saw and we need to attend to'." (Extract 6.71)

Teachers are constructed as having presences (P) if they focused on learner errors as this provides a "*window of opportunity*" for them to access learner thinking. Learner thinking would be explored in terms of their interpretation of the mathematics and see how the concepts are developed or not developed in them. Depending on the outcome, the teacher will seek ways of remediating the errors identified. Moreover, presences (P) are also realized in learners in that their take-up from a lesson does not completely resemble what the teacher has planned and taught; there could be errors and misconceptions which need to be attended to. This is in light of constructivist's theory that learners construct their own understanding of what the teacher has planned and taught. What has been constructed could have errors, hence providing a window of opportunity to establish the sources, and find ways of addressing them. The finding is also supported by Ball et al. (2008) that errors are persuasive and hence important to focus on them.

- *How David "teaches" for learner errors being genuine*

Three strategies of how David "teaches" for learner errors being genuine and not deliberately or consciously made are identified. Key among them and closely linked to each other is that learner errors are as a result of teaching – present or past, and the nature of mathematics. He spoke in similar ways in extracts 6.69, 6.70, 6.78 but I provide extract 6.69 as a typical example, and provide some illustrations from extract 6.78.

Errors are as a result of teaching – present or past; and the nature of mathematics

"*... **in fact most times they should actually know that the errors, the mistakes that learners make are usually the result of our teaching.** It may not be their teaching, but because of the teaching – somebody's teaching – **if it's not them but problem with the previous teacher or the previous grade level** or where they came from it's like those are usually where the source of the errors come from. So learners never make them deliberately and that mostly those errors come about as a result of our teaching and the way we do it. ... the errors and misconceptions that the, the, the learners will have are usually yes, out of the teaching and also **sometimes it's out of how mathematics is, the nature of mathematics, the way it is structured, its language and the like.** There are some aspects that confuse learners in either misrepresentation, in the structure, in the how it's presented, the symbols and the like. So it's some of the errors or misconceptions are arising from the nature of the subject itself. ... always when there's an error they should always try to search their, their*

teaching, search the way they present the work and also look into the way the mathematics was structured or presented." (Extract 6.69)

Below is an illustration of the two strategies described above.

"*... And normally we go through a number of examples that we, we look at. Normally they are at the lower levels,* ***they tend to introduce a lot of what they call a short, shortcuts.*** *Do one thing, do this. If you, if you do one thing to the top, make sure you do the same thing to the bottom. Multiplication makes, makes something bigger, makes a number bigger. But those are not really, they are things people say in particular instances whatever, but to say those are the practices of teachers and to say that and for learners to take it that yes if you multiply, something must get bigger.* ***But they won't tell the whole story then****. You know, if you multiply by a fraction it's not bigger. But multiplication is seen as a, yes, and… So you find there are all these short so-called shortcuts which now – and those are the kind of errors which come up later on, not really because learners, in fact they're all coming from the teachers' practices. ... So I think the bulk of the difficulties, the bulk of actually I said is as a result of teachers' practices, ja, I think that's what it is. ... pay attention to what the learners are doing in the classroom situation because then they will be saying, 'Where is this coming from? Where, how, was it in my teaching?' If it's not their teaching then maybe in the teaching of their predecessors*." (Extract 6.78)

In extract 6.69 and 6.78, the indication is that absences (A) in learners are as a result of absences (A) in teaching and the nature of mathematics. David makes his student-teachers know and be aware that if learners make errors, to establish the sources, the teacher should reflect on his teaching or previous teaching of other teachers who might have taught the learners. Due to the nature of mathematics, the learners could also misrepresent the structure due to the way it is presented by the teacher. Watson (2009) talks of the importance of seeing structure in the learning of school algebra, and that not being able to do so contributes to the difficulties learners experience in algebra, and this is discussed in Section 3.4.3. Moreover, the nature of mathematics could also include symbols and the language. Overall, David's thoughts are that teachers' practices owe an explanation to learners' errors. For example, one of the practices could be the tendency by teachers to use "shortcuts" in explaining solutions to mathematical problems, which do not necessarily provide a comprehensive picture of what is actually happening. He then contends that errors arising from such "shortcuts" end up resurfacing later in learners' mathematical learning.

This finding seems to agree with other researchers such as Nickson (2000) who argues that the way mathematics is presented to learners could offer explanations for errors learners

make. However, other researchers such as Borasi (1987), Nesher (1987); Sfard (2007); and Ball et al. (2008) would agree with the issue that learners' errors are genuine but disagree with the explanations for the sources of errors, and argue that irrespective of the type of teaching learners are exposed to; they will still be prone to making errors. This is because errors are fundamental to being mathematical and therefore a process of learning. Suffice to say that David's student-teachers are likely not to realize that, overgeneralization (Ryan & Williams, 2007; Lima & Tall 2008) and dual nature of mathematical conceptions (Sfard & Linchevski, 1994) could offer alternative explanations for learner errors.

The second strategy of how David teaches for learner errors being genuine is that:

> "*But we bring in, sometimes, you know,* ***mathematics is more like … because it's methods. So we introduce these methods,*** *they're supposed to, mathematics is supposed to help us handle things, quantify them, represent them in such a way that we, we can easily manipulate things and we get to what we want to do. So some short forms and* ***some different representations and they are contextual****. So but sometimes it may work in one context and then be different in another context. So I think they need to be aware of all these aspects that… And that's what makes them teachers as it were, trained teachers, or teachers who have gone through the teacher preparation*." (Extract 6.79)

As indicated in extract 6.79, David ensures that his student-teachers realize that school mathematics is about different methods and representations that are contextual. This suggests that a method or representation that might work for a particular mathematical problem might not necessarily work for a similar one because the contexts are different. Teachers need to take into consideration the context of the mathematical problems they are involved with as it matters for the choices of methods and representations they decide to use. I infer that this could be a way in which teachers can ensure that learners avoid making errors as a result of their teaching.

The third strategy of how David "teaches" for learner errors being genuine is that he tells student-teachers that it is important to sustain the skill of analyzing learner errors, which for David the sources are as a result of teaching and the nature of mathematics:

> "*… we tend to focus more on the principles and that you see this is very key and cardinal to their teaching and that* ***they must ensure that it's part of their department, because they'll be part of a department in the school.*** *It should be part of a department of, the departmental activity every time they meet as a department in the school, that's one aspect that they must attend to. They must, the head of*

department must always ***ensure that each teacher in the department shares the, the difficulties that learners are facing. And then they discuss that, then they try to find out, it may be a problem that is actually happening to the whole school because then you can say, 'Oh that's oh.' Then it's like a. But if it's a particular class then they can have suggestions, then you have the whole school improving in mathematics improving in their particular school****. And not, and I'm sure most times it's like, insist that in as much if it becomes just what I do in my class and not the same thing is done in another class it might die out and people may, but you need support from others.* ***Sometimes it may be so difficult to diagnose pupils' errors or misconceptions alone in the class as it were but once it's thrown to the colleagues in the department then we might find solutions coming out*** *and they will bring, they will lead you to other issues like, 'Oh,* ***that is coming like that because at a particular time in a particular grade this is what happened.****' So they would tend to identify the source of the error or the problem then you can easily deal with it there*." (Extract 6.72)

As shown in extract 6.72, to sustain the skill of analyzing errors, David points to how it should be dealt with as an activity in class and consequently at departmental level. The skill should be introduced as one of the practices in the department. If this is done, David suggests that it could lead to improved performance in mathematics among the learners. This means teachers in the department would benefit from each other's input on the errors experienced in their classrooms. Only when and after the sources are established would they then seek ways of remediating the errors. What is interesting about making the skill of analyzing errors an activity in the department of mathematics is that they would still be looking for sources for the errors learners make in teachers' practices of teaching. Therefore, critical for David is the idea that learners make errors genuinely as they are not consciously or deliberately made, and that the sources are in the teaching and how the mathematics is represented. The student-teachers should then realize that mathematical methods and representations are contextual.

- *Where are errors focused on in the courses?*

David stated that he discusses errors when focusing on "assessment", one of the topics in the mathematics education courses.

> "*I usually deal with that when I, I look at, I deal with different ways of assessing learners or using assessment to, to check learners learning. And in particular* ***when we look at the diagnostic forms of assessment and how you analyze the errors****, how you diagnose the problem areas for learners and like…*" (Extract 6.70)

Among the assessment techniques David focuses on include clinical interviews.

"*So you would actually go in to the other assessment techniques that we give them that there are so many ways you can assess what the learner actually knows. It's not just through written, written work or whatever, so* ***we need to sit down with them and do a clinical interview and ask them about, what they know about a particular topic*** *until you, you...*" (Extract 6.74)

This suggests that LMT is a less specialized discourse, hence weakly classified, since errors are focused on when dealing with "assessment" where the discussion is around how errors could be used to assess learners learning; and different assessment techniques that are available such as written work and clinical interviews. "Assessment" is a topic in the fourth year mathematics education course called Mathematics Education III (MSE 431).

6.5.2.2 Learner errors and misconceptions is about analyzing learner errors

David described how he deals with learner errors and misconceptions by focusing on school mathematics in particular ways. In extract 6.75 he describes the strategy and in extract 6.80 he further elaborates using a metaphor.

"*I've given them a situation of where they would work,* ***work out a series of problems and then suggest areas where learners make errors, or sometimes would present a case of a particular solution and putting some, some errors there and then ask them to comment on why the learner made certain errors*** *or miscon, why, ja, the learner came up with that,* ***and then ask them to see what, what they would do to, to help such learners*** *and the like. I think there has been something more like that, more in activities and the like,* ***during the lab, the lab sessions and tutorials rather than in a formal, formal lecture***." (Extract 6.75)

David further uses metaphor of a running tap to describe the process of working with learner errors. This is in extract 6.80, and in similar ways in extract 6.81 but I provide extract 6.80 as a typical example.

"*Yeah, but most times like for me I would really, usually* ***what I focus on is them showing understanding and awareness of you know the sources of the difficulties like in as much as, ja, I expect them to indicate where the errors or misconceptions are and how they can deal with them but they also have to try to go beyond that and search for areas of what the source is,*** *I mean how do these sources arise? How… What could be the circumstances so that instead of just marking what is there* ***they should try to see where the tap is so they close the tap and then move rather than the tap is running then they just keep on mopping the water because the source of the problem is still there*** *but they need to address, look at it, or is it in the*

circumstances, the background of the learner, or is it… ***They need to give me that broad picture of understanding of the situation***." (Extract 6.80)

As indicated in extracts 6.75 and 6.80, it is clear that David's expectation of his student-teachers when dealing with activities on learner errors is that they should provide "*that broad picture of understanding of the situation*" (Extract 6.80). The holistic view includes recognizing the errors, establishing and explaining the sources of the errors, and seeking ways of remediating the error. By using a running tap as a metaphor, David argues that recognizing errors and seeking ways of remediating them without exploring their sources does not make the errors be fully addressed. Unlike for Kenneth, Sam and Titus, critical for David when dealing with learner errors is to establish the sources. This finding is in resonance with other researchers such as Peng & Luo (2009) and Jacobs et al. (2010) who have categorically indicated the strategies of carrying out error analysis. Although these authors talk differently, their talk points to three steps in a hierarchical order, namely, identify, explain, and remediate. I would like to also say here that the discourse of engaging with LMT with specific focus on analysis of learner errors, in Bernstein's (2000) terms, is a less specialized discourse as it is focused on in practical sessions and tutorials rather than in formal lectures as indicated in extract 6.75.

- *Positioning of student-teachers/teachers with respect to analyzing learner errors*

In focusing on the process of analyzing errors, David had this to say about remediating strategies his student-teachers would suggest such as re-teaching the topic.

> **"…** *they tended to pick on easier ways of addressing, already addressing* ***the issue in a superficial way****. I mean that, what I mean is they would just say, 'Well OK,* ***if someone came up with this, what we all do is we re-teach the topic.'*** ***But really I mean that doesn't really solve the problem because re-teaching may actually mean here you're doing the same thing or the same error may still come up*** *and whatever. But, I mean, the argument is that 'no, it means that this particular learner did not follow or understand, so I repeat the teaching.' But I, my emphasis was and has always been that no, but the thing is that you are not really addressing the problem. It could be but really just, just one aspect. But most likely he's just at, I mean the approach used if it did not, you know, resonate with the learners, so you needed to bring, if you are going to be teachers then you have to do it in a different way. But also you need to pay attention to really why this particular error, is it really just that they missed the whole concept or there's something that is causing them to…? How are they thinking and how, how can you reach out to them?*

So re-teaching may mean a repeat of how you did that, and they've brought up this error, miscon, and then you go back to do it again and it's really like saying these guys, maybe they were, they didn't pay attention that time so now you expect them to pay. Hey, there is that chance, but most likely there are more fundamental issues to be, to be attended to. In other words this requires time in order to study the solution. Talk to the learner and find out exactly what their difficulties are." (Extract 6.76)

As indicated in extract 6.76, a sense of absences (A) in student-teachers is realized in terms of providing remediating strategies specific to the problem in that they tend to propose general ones such as re-teaching the topic in the same way because maybe learners did not understand. David argues, and in resonance with Borasi (1987), that such a strategy is likely to perpetuate the same errors, hence necessary to opt for a different approach. The approach could be sought only after establishing the reasons for the errors by studying solutions and talking to learners to establish what their difficulties are. Similarly, Borasi (1987) and as discussed in Section 3.3.5, suggests that the teacher could hypothesize sources of learners' errors based on his knowledge and understanding of errors, and then interview the learners to confirm which one is applicable.

However, David said the following pertaining to the importance of analyzing learner errors:

"*In a way I think for me it was really solidifying and really for them,* ***since we are preparing them to be competent teachers, it's really solidifying and really bringing it, bringing it really to the fore for them that this is where the heart of teaching is.*** *So they need really to, to focus, to focus on that, apart from all the other aspects of teacher preparation that we give them, that really is what is cardinal in terms of teaching and learning in a classroom situation. …* ***they appreciate and when you give them a task later, ja, they will do it, they will do it well,*** *so. And I'm sure you would say that they, it's one of the things that they, they appreciate and I hope it's one of the tools that they have with them now and if they haven't had it is good that they should have that with them too to be able to help the learners*." (Extract 6.76)

In extract 6.76, David is pointing to that focusing on analyzing learner errors is critical for teachers as it is at the heart of teaching and learning. Thus, by giving his student-teachers tasks for them to carry out the analysis of errors, a sense of presences (P) is realized in them as they tend to see the importance of doing so. It is David's hope that his student-teachers carry forth this skill of error analysis learned for the enhancement of learner learning.

- *How David "teaches" for analyzing learner errors*

Two strategies are identified of how David "teaches" for analyzing learner errors. These are described in extracts 6.75 and 6.77, respectively:

> "***But most of the difficulties that learners tend to have, from my experience and from what you see at the examinations council*** *it is more with the handling of directed numbers and the handling of I think it's algebra and its variable, values like, so the two seem to more like they arise in other areas of mathematics, the handling of directed numbers and the like. So normally I would give them situations where those things arise, either whether they're doing, they're solving quadratic equations and the like, but it's usually I write this on the handling of the directed numbers and the… and that. So it's like the problem may not, so the misconception, the errors may not have a reason out of their own circumstances there but could have come, it seemed to come from somewhere else where the learners don't seem to have a good grasp of the, of integers, or directed numbers as it were*." (Extract 6.75)

> "*But of course I think* ***in terms of focusing or strengthening ja, there may be need to ah, maybe to give it more emphasis in a way and perhaps to keep coming back to it each time you talk about many other or another topic area and the like, so that the others and because that attitude changes*** *what is in, sometimes the issue before we have these but people don't, attitudes tend to, to take time to change and that's why sometimes I feel the in-service, the* ***in-service teachers come really with pre-set attitudes*** *and really change their, because sometimes they just do it because I'm around and the like, but* ***they seem still to, to hold onto their, to their experiences and the like. And they will always tell you about the real situation, I mean in the classroom there is no time, there are large classes there.*** *So it's really like there because sometimes you'd say yes, OK, that's what is real, that is what is in the classroom but what can you do if you have seventy learners in your class? That's a challenge. What do you do? You have this information and how would you do it? And how would you? Not to resign and sit back and say, 'No they are…' But they are the ones who tend to be a bit, ja, I'm afraid to say, rigid and set and having some particular attitude which I feel, ja, needs to be changed. And you, and you see* ***here we try to emphasize that even when they're expected to go on this school experience we expect to see these aspects.*** *And they'll say, 'No, but you see we have large classes.' And I would say that but if you really want to do well in your (laughs) I expect to see these aspects. So, ja, I think in terms of ja maybe* ***it will mean refocusing, yes, and maybe strengthening it and maybe giving it more, more attention each time you deal with any other aspects of teaching the course.***" (Extract 6.77)

In extract 6.75, David points to that he focuses on difficulties that learners experience in school mathematics and informed by learner performance in examinations and from his

experience. These difficulties include largely working with directed numbers, and algebra with focus on the concept of a variable, which are applicable to other topics in school mathematics. This means that the focus is on learner errors in some basic concepts which manifest themselves in other areas of mathematics where they are applicable. Thus, David provides scenarios embedded with the identified learner errors. By stating that the errors identified are usually not learners' own making, I infer that David is suggesting that they stem from teaching. This is not surprising since in talking about learner errors being genuine, he repeatedly indicated that the bulk of them stem from teachers' practices.

In extract 6.77, David suggests that to develop in student-teachers the right attitude for teaching, analyzing learner errors needs to be emphasized when dealing with every topic in the mathematics education courses. The aim should be to ensure that the student-teachers, especially the in-service teachers see value in analyzing learner errors and stop using large class size and inadequate time as excuses but instead rise to the challenges. David also ensures that his student-teachers demonstrate the skill of analyzing learner errors in their classrooms during teaching practice. There is an indication that LMT with focus on analyzing learner errors, in Bernstein's (1996, 2000) terms, is a less specialized discourse, hence weakly classified since it could be focused on when dealing with any topic in the mathematics education courses. The problem with such focus is that analyzing learner errors would have no specialized language as it would be infiltrated by other messages from the core of the topic in the course being discussed. Moreover, analyzing learner errors is a practical experience as it is a skill that is taught during practical sessions and tutorials and student-teachers should copy or imitate this skill during their teaching. This finding is supported by other researchers such as Jacobs et al. (2010) that in the classroom, the steps of analyzing errors happen at the same time as if constituting a single act.

6.5.2.3 Summary and conclusion of the analysis

Focusing on learner errors and misconceptions for David is in two ways although interlinked. One way relates to the nature of errors in that they are genuine and not deliberately or consciously made. This view of errors is accompanied by a contradictory positioning of student-teachers in terms of presences and absences. David talks of the examples in-service teachers bring from their teaching experience to discussions on learner errors (P), which pre-service teachers are not able to because they do not have the teaching experience to draw from (A). This suggests that working with learner errors is a practical experience because the

experience of teaching is crucial since as Kenneth, David locates the source of error in teaching – present or past where the structure of the mathematics in focus could not have been well presented. Added to the inadequate presentation of structure are the symbols and the language used, and the use of shortcuts in explaining solutions to mathematical problems which could be the sources of learner errors. David talks of how crucial the context of the mathematical problem is in determining the methods and representations to be used. He also talks about the importance of sustaining the skill of analyzing learner errors at classroom and departmental level for the enhancement of learners' learning. Seeing errors as being genuine since they are not consciously or deliberately made is weakly classified because of where it is focused on. David says he talks about LMT with focus on learner errors when talking about "assessment", one of the topics in the mathematics education courses.

The other way of how David focuses on learner errors and misconceptions relates to strategies of carrying out error analysis, which involves identify, explain, and remediate. David says he gives his student-teachers scenarios on school mathematics embedded with learner errors. The choice of learner errors is informed by David's knowledge of learner performance in examinations and from his own experience. The way David deals with LMT with focus on error analysis is accompanied by a contradictory positioning of student-teachers in terms of absences and presences. Student-teachers tend to suggest re-teaching as one of the remediating strategies (A), which for David is too simplistic. With continued focus on school mathematics in form of scenarios on learner errors, student-teachers tend to realize the importance of carrying out error analysis for the enhancement of learner learning (P). LMT with focus on error analysis is also weakly classified given where it is focused on. David says he deals with error analysis during practical sessions and tutorials. He suggests that if the skill of error analysis could be focused on in every topic of mathematics education courses, and put into practice during school teaching practice, it would develop in student-teachers the right attitude for teaching. This suggests that LMT with focus on error analysis is a skill accomplished practically, hence taught. It is taught during practical sessions and tutorials using school mathematics in form of scenarios on learner errors as discursive resources.

Table 16 provides a brief overview of David's selection and privileging of the discourse of engaging with LMT. In terms of creating an environment where teacher can listen to learners,

focus is on both content and context; and assessing learners' understanding. Learner errors are focused on in the sense that they are genuine and not deliberately or consciously made; and the process of carrying out error analysis.

Table 16: Synopsis of David's talk of LMT with focus on both content and context, and assessing learners' understanding; and learner errors

"What" is LMT?		**"How" of LMT, that is, positionings in terms of presences (P) and absences (A) and the "teaching" of it in terms of what student-teachers are supposed to know and be able to do.**			**"Where" in the courses/topics LMT is focused on**
	Elaboration of the "what"	**Positioning of teachers/student-teachers**	**Positioning of learners**	**"teaching" for LMT**	
Focus on both content and context; and assessing learner understanding of what has been taught involves teacher listening to learners. (Env-CC or Env-AUT)	Focus on background knowledge of learners and teach based on the resources readily available. Ensure learners apply what has been taught in different contexts.	Presences: in teachers knowing and being aware of the principles outlined. P Absences: in teachers a result of de-motivation due to large classes of de-motivated learners. A		Through developing in student-teachers the right attitude for teaching: consider learners as individuals and teacher as a facilitator of learning; and ensure learners maximize their learning. Through reaching common agreement between teacher's and learners' principles. In both dispositions, there is a sense of presences in the learners.	Assessment (MSE 431)
Learner errors are genuine since they are not deliberately or consciously made. (LEM – LECD)	Locate source of error in teaching and nature of mathematics	Presences: in-service teachers bring classroom experiences to discussions. P Absences: in pre-service teachers as what they experienced as learners is a different context. A Absences: in student-teachers due to context such as time constraints. A Presences: in student-teachers as focus on learner errors provides a window of opportunity to access learner thinking. P	Presences: in constructing their own understanding of concepts. P	Through making student-teachers know and be aware that learners errors are as a result of teaching and the nature of mathematics. Through making teachers realize that school mathematics is about different methods and representations that are contextual. Through encouraging student-teachers to introduce error analysis as practice in the department of mathematics in schools.	Assessment (MSE 431)
Errors should be identified and explained before remediation. (LEM-IER)		Absences: in student-teachers providing remediating strategies specific to the problem other than the general ones. A Presences: in student-teachers appreciating the process of error analysis and hopefully for the learning of their learners as well. P		Through working with scenarios embedded with learner errors identified from their performance on examinations and from David's own experience. Suggests making error analysis part of discussion in every topic of mathematics education courses. Through school teaching practice	Laboratory sessions (practical sessions) and tutorials

LMT is about teacher listening to learners through focus on content and context, and assessing learner understanding; learner errors involving their genuine and analysis.	*Locate source of error in teaching and nature of mathematics*	*Discourses of absences as well as presences in teachers/student-teachers*	*Discourse of presence in learners*	*LMT is practical accomplishment learned through:* • *Developing the right attitude for teaching* • *Realization that errors are a result of teaching and nature of mathematics.* • *Making error analysis practice in school and mathematics education courses.* • *Working with scenarios on learner errors* • *Practice of teaching*	*LMT supported by principles that guide discussions on assessment – weakly classified*

6.6 A synthesis of teacher-educators' talk of the discourse of engaging with LMT

In this section, I discuss, across the four teacher-educators, discourses of what and how of LMT in relation to developing in learners both instrumental and relational understanding, learner errors, and creating an environment where teacher can listen to learners.

6.6.1 Discourses of what and how of LMT in relation to developing in learners both instrumental and relational understanding[16]

Teacher-educators talked of LMT as developing in learners relational or instrumental understanding, and foreground relational understanding. They talked of LMT in three diverse ways. LMT as relational understanding is being able to work with mathematics as processes which involve arguing, evidencing, systematic thinking, and conceptual thinking; recognizing that there are different strategies or methods of getting to the same answer; and not just giving answers. This suggests that teacher-educators speak through different discourses, an indication of diverse messages being relayed.

LMT as relational understanding is learned during the practice of teaching such as peer teaching or school teaching practice through effort and time dedicated to planning of a lesson; by not focusing on curriculum coverage for examination purposes; and by being aware of different routes to an answer. How they position student-teachers is largely in terms of absences in these listed aspects, which could also position learners in terms of absences in developing relational understanding. Absences in student-teachers are attributed to absences in the teacher education curriculum in terms of less time given to practice of teaching and it being focused on towards the end of the programme. Moreover, there is no in-depth focus on specific school mathematics. Learners are positioned as having presences in terms of different strategies of getting to the answer which teachers do not seem to work with. Therefore, how LMT is learned is through theory and practice, an indication that how it is relayed is a practical experience.

Where LMT is taught is distributed across topics in the mathematics education courses which involve theory and practice. Theoretical resources include: Aims and objectives of teaching

[16] This section was presented as a poster presentation entitled 'Teacher educators' perspectives on working with learner mathematical thinking: A Zambian study' at the 36th Conference of the International Group of the Psychology of Mathematics Education (PME 36): "Opportunities to learn in Mathematics Education". 1, 258, 18 – 22 July 2012, Taipei – Taiwan.

mathematics; Lesson planning; a foundation mathematics course (MSE 131); and Problem solving. Practice includes: Peer teaching; and School teaching practice. If what is learned is distributed over the curriculum, it is a number of things and there is not a unified message but mixed, as indicated. Therefore, in Bernstein (1982, 1996, 2000) terms, it is weakly classified, which means the relay is implicit. If LMT is learned in the practice of teaching as indicated, then how it is relayed is a practical activity. Since LMT suggests that messages are implicit and a practical activity, in Davis et al. (2007, p. 42) terms, it is a "practical accomplishment". It is more at the level of the "sensible" where student-teachers can copy or imitate what they have been taught about LMT and less at the realm of the "intelligible" where student-teachers could be given opportunity to engage with relational and/or instrumental ways of knowing. The anticipation here is that, to some degree, student-teachers would recruit or take-up a whole range of meanings. What then do the student-teachers recruit or take-up?

6.6.2 Discourses of what and how of LMT in relation to learner errors

Teacher-educators talked of LMT in relation to learner errors in two diverse ways, which are: a way of carrying out error analysis; and the general nature of errors. LMT as a way of carrying out error analysis involve recognizing learner errors which are difficulties that need to be identified/anticipated and likely remediating strategies sought. Sources of errors are said to be due to inadequate teaching in terms of presentation of structure of the mathematics in focus. During the process of teaching and learning, factors that prevent errors from being addressed include time constraints and teachers making similar errors to those of their learners. LMT as focus on the general nature of errors involves seeing errors as part of learners' process of learning in terms of using prior knowledge as a resource for teaching; and seeing errors as genuine since they are not deliberately or consciously made by learners. What LMT with focus on learner errors is also suggests that teacher-educators speak in different ways, an indication of diverse messages being relayed.

LMT as a way of carrying out error analysis; and the general nature of errors is also learned during the practice of teaching such as peer teaching or school teaching practice through incorporating anticipated learner errors in the planning of lessons and reflect back to ensure correct methods and representations of mathematics, sequencing subject matter appropriately, and probing learner answers. Positioning of student-teachers in these listed aspects is also largely in terms of absences, which could also translate into absences in learners in terms of developing the required school mathematical knowledge. Absences in student-teachers are

also attributed to absences in the teacher education curriculum which lacks topic specific in-depth focus on error analysis; and school teaching practice is programmed towards the end of training and is given less time making it difficult to engage discussions on errors. The suggestion of theory developing out of practice would position student-teachers as having presences while learners are also positioned as having presences in terms of having limited and not wrong mathematical conceptions. As also indicated, how LMT is learned and the positionings is through theory and practice, an indication that how it is relayed is a practical experience.

Where LMT is taught with specific focus on learner errors is distributed across topics in the mathematics education course, which are in terms of theory and practice. Theory includes School mathematics with specific focus on selected questions; Lesson planning; Psychology of learning mathematics; MSE 131, a content course with focus on advanced school mathematics; Methods of teaching; Sequencing instructions; Problem solving; and Assessment. Practice includes Peer teaching; School teaching practice; Practical sessions; and Tutorials. This suggests that LMT with focus on learner errors is also weakly classified, hence the relay is implicit. LMT with specific focus on learner errors is also learned in the practice of teaching and supported by principles that support discussions on the topics in the mathematics education courses as indicated, then how it is relayed is a practical activity.

Since LMT with specific focus on learner errors suggest that messages are implicit and a practical activity, in Davis et al. (2007) terms, it is a "practical accomplishment". It is more at the realm of the "sensible" where student-teachers can copy or imitate what they have been taught about learner errors and less at the realm of the "intelligible" where student-teachers can be given opportunity to read about what the specialized field of mathematics education research has informed us about learner errors. For example, the CGI model or MANOR project could be used as discursive resources, and are discussed in Section 3.2.2. Moreover, reading about learner errors would make them realize that errors can assume names depending on the sources such as "modeling error", "prototypical error", "error as a result of overgeneralization" and "error as a result of lack of completion of the process-object reification" (Ryan & Williams, 2007, p. 25), and these are discussed in Section 3.3.3. As previously, the anticipation here is that, to some degree, student-teachers would recruit or take-up a whole range of meanings. Therefore, what do the student-teachers recruit or take-up?

6.6.3 Discourses of what and how of LMT in relation to creating an environment where teacher can listen to learners

LMT as creating an environment where teacher can listen to learners is being able to work with mathematical practices in three ways involving learners' explanations of their responses and questioning each other, consideration of learners' background knowledge, and application of knowledge in different contexts. The indication here is that teacher-educators speak through different discourses, hence diverse messages being relayed. How LMT is learned is through encouraging student-teachers assume dispositions involving probing learners' answers, teacher asking questions from the onset, seeing learners as individuals and teacher as facilitator of learning.

Learners are positioned as having presences in the dispositions outlined above, which the teacher should listen to and work with. Student-teachers are positioned largely in terms of absences in the dispositions listed above mainly because they are de-motivated since they have to deal with large classes of learners who are also de-motivated. However, student-teachers are also positioned as having presences in terms of alternate ways of thinking which are sometimes not aligned with teacher-educators', which could translate into learners who are also independent thinkers. As also indicated, how LMT is learned and the positionings is through theory and practice, an indication that how it is relayed is a practical experience. LMT is a practical experience because the expectation is that student-teachers will put into practice principles learned about what it means to be a teacher of mathematics.

Where LMT is taught is distributed across a topic "Assessment", in one of the mathematics education courses; and other activities in the programme such as tutorials and examinations. Assessment provides the theoretical aspect of the LMT while tutorials and examinations provide the practical aspect. If what is learned is distributed over the curriculum, it is a number of things, and there is not a unified message but mixed, as indicated. Therefore, LMT with focus on creating an environment where teacher can listen to learners is weakly classified, which means the relay is implicit. Moreover, Since LMT with specific focus on creating an environment where teacher can listen to learners suggests that messages are implicit and a practical activity, it is also a "practical accomplishment".

6.6.4 Summary and conclusion of the synthesis

As can be observed, there is a whole set of mixed messages being relayed here, whether focus is on learner errors, developing in learners both relational and instrumental understanding, and creating an environment where teacher can listen to learners. While these three major categories of what entails the discourse of engaging with LMT are referred to by the teacher-educators, the messages within them are spread out, hence weakly classified and weakly framed. That is, the criteria for what counts here around LMT are weak because they are spread out. Moreover, it is also evident that across all the three major categories, LMT is a practical accomplishment as principles that would guide discussions around it are not so clear. LMT is talked about when focus is on principles that guide discussions on topics/courses in the mathematics education curriculum. It is interesting then to see what it is the student-teachers recruit or take-up. What do they realize because the privileging by the teacher-educators while in these three domains, some of the big discourses in the specialized fields of mathematics education research and mathematics education are filtered through but in a very weak way.

What is evidenced here is more of a reflection of what Doerr & Wood (2004) referring to Stacey, are saying is that LMT is not evident in teacher-education. Despite overwhelming research on learner thinking, it is not informing teacher-education in an explicit way. Artigue et al. (2001) in Kieran (2007) says there is no time for this while Ball et al. (2008) says some of them learn it in practice of teaching but not all of them and so it should be focused on in pre-service. What I have shown here is that learner thinking is not done explicitly but informally. Learner thinking is noise rather than a set of clear messages. Learner thinking is also a general thing, which suggests that student-teachers will just stay with what they know. It is at the level of everyday professional experience of teaching and learning, hence their sense making. Therefore, whatever ideas they have about LMT cannot be challenged.

Table 17 provides the synopsis of teacher-educators' synthesis of the discourse of engaging with LMT.

Table 17: Synopsis of teacher-educators' discourses of engaging with LMT

What is LMT?	How in terms of "teaching" of it and positionings	Where in the courses/topics LMT is focused on
LMT as relational understanding is about working with mathematics as processes involving: • Arguing, evidencing, systematic thinking, and conceptual thinking • Different strategies of getting to answer • Not just giving answers	During the practice of teaching - peer teaching or school teaching practice: • Through effort and time dedicated to planning of a lesson • Being aware of different routes to an answer • Not about curriculum coverage for examination purposes Positioning of student-teachers: • Absences largely due to absences in the teacher education curriculum in that: o Too little time is given to school teaching practice o School teaching practice is too late - towards end of programme. o No in-depth focus on specific school mathematics topics. Positioning of learners: • Presences in different strategies of getting to an answer which teachers do not utilize. LMT learned through theory and practice, hence how it is relayed is a practical experience.	Theoretical resources on: • Aims and objectives of teaching mathematics • Lesson planning • A foundation mathematics course MSE 131 • Problem solving Practice includes: • Peer teaching • School teaching practice LMT distributed over the curriculum - no unified message but mixed, hence weakly classified suggesting implicit relay. Therefore, messages are implicit and a practical activity, hence a practical accomplishment.
LMT is about working with errors as part of the process of teaching and learning and involves: • Prior knowledge • Seeing errors as genuine • Recognition of error • Suggesting remediating strategies Not attending to learner errors is due to: • Time constraints • Teachers not recognizing occurrence of errors Sources of error which are due to • inadequate teaching	During the practice of teaching - peer teaching or school teaching practice: • Incorporate anticipated learner errors in planning of lessons • Reflect back to ensure correct methods and representations of mathematics • Sequence subject matter appropriately • Probe learner answers Positioning of student-teachers: • Largely absences, due to absences in the teacher education curriculum in that: o There is no in-depth topic specific focus on error analysis o Programming of STP is towards end of training and is allocated less time to allow for discussions on learner errors. • Presences would occur if as suggested theory developed out of practice Positioning of learners: • Presences in terms of limited and not wrong mathematical conceptions As previously, LMT learned through theory and practice, hence how it is relayed is a practical experience.	Theoretical resources on: • School mathematics with specific focus on selected questions • Lesson planning • Psychology of learning mathematics • MSE 131, a foundation mathematics course • Strategies for teaching • Sequencing instructions • Problem solving • Assessment Practice includes: • Peer teaching • School teaching practice • Practical sessions • Tutorials. As previously, LMT is distributed over the curriculum - no unified message but mixed, hence weakly classified suggesting implicit relay. Therefore, messages are implicit and a practical activity, hence a practical accomplishment.
LMT as creating an environment where teacher can listen to learners is being able to work with mathematical practices	Through encouraging student-teachers to assume dispositions involving: • probing learners' answers; • teacher asking questions from the onset;	Theoretical resources on: • Assessment Practice includes:

in three ways involving: • learners' explanations of their responses and questioning each other; • consideration of learners' background knowledge; and • application of knowledge in different contexts.	• seeing learners as individuals; and • teacher as facilitator of learning. Positioning of student-teachers: • Largely absences in the outlined dispositions because of de-motivation due to large classes whose learners' are also de-motivated. • Presences are due to alternate ways of thinking not aligned with teacher-educators', and if such thinking is encouraged among their learners, it could also develop in them independent thinking. Positioning of learners: • Largely presences as evident in the outlined dispositions Again, as previously, how LMT is relayed is a practical experience because the expectation is that student-teachers will put into practice principles learned about what it means to be a teacher of mathematics.	• Tutorials • Examinations Again, as previously, LMT is distributed over the curriculum - no unified message but mixed, hence weakly classified suggesting implicit relay. Therefore, messages are implicit and a practical activity, hence a practical accomplishment.
Teacher educators' discourses resonates with mathematics education literature on LMT and its emphasis on relational understanding/conceptual knowledge, and error as part of learning as construction of knowledge, and so with a classroom pedagogy that elicits learner thinking. *There is an absence of discussion of sources of error, outside of these being a problem of teaching*	*How LMT is taught is a practical accomplishment through practice of teaching and student-teachers assuming particular dispositions – the relayed is a practical experience as principles that would guide discussions around LMT are not so clear since it is talked about when focus is on principles that guide discussions on a selection of topics/courses in the mathematics education curriculum.*	*Where LMT is taught is distributed across topics/course(s) – the relay is weakly classified as LMT has no specific focus.*

CHAPTER 7

7 STUDENT-TEACHERS' DISCOURSES OF ENGAGING WITH LMT

7.1 Introduction

In this Chapter, I present a discussion of the first level of analysis of student-teachers' talk of the discourse of engaging with LMT from the two focus group discussions I had with them prior to working with them on the scenarios presented and discussed in Chapters 8 and 9. The purpose of the focus group discussions, as already indicated in Chapter 5, was to establish student-teachers' realizations of the discourse of engaging with LMT. The questions were designed to elicit information on: What the student-teachers said constitutes the discourse of engaging with LMT, and how strong this is? How student-teachers said they went about working with the discourse of engaging with LMT in their mathematics education courses? This means that the focus group interviews focused on what the student-teachers foreground pertaining to the discourse of engaging with LMT; how they said this could be learned; how they positioned learners, student-teachers/teachers, teacher-educators, and the curriculum; and then some explanations as to why. The main purpose for this chapter is to describe student-teachers' discourses and then relate them to the teacher-educators' discourses. This provides insights into messages student-teachers recruit or take-up from teacher-educators' privileged selections.

As observed, in interviewing teacher-educators and student-teachers, I followed a similar structure in probing their thoughts about the discourse of engaging with LMT. Following from teacher-educators' talk; three dominant foci in resonance with Even & Tirosh's (2002), also emerged from student-teachers' talk. These are: developing in learners both instrumental and relational understanding, focus on errors and misconceptions, and creating an environment where teacher can listen to learners. Thus they are discussed within this frame and evidenced by excerpts extracted from the two focus group interviews. A lot of issues were raised in the two focus groups but I only focused on issues that were critical to the intentions of my study. As a result, student-teachers' critical discourses on LMT from both focus groups were organized under fifteen main headings following the interview questions and probes. The critical extracts were numbered from 1 to 50 across the two focus groups. Under each heading, I had one or more extracts depending on shifts in student-teachers' talk,

and these were drawn from either focus group 1 or focus group 2 or from both. For example, extracts 7.5 and 7.10 mean that the "7" in both extracts is representing the Chapter, which is 7 in this case; where extracts 5 and 10 are drawn on to evidence the discussion or argument.

7.2 LMT is about developing in learners both relational and instrumental understanding

The student-teachers talked about two aspect of developing in learners both relational and instrumental understanding, namely, working with learners' routes to answers and teaching for understanding.

7.2.1 LMT is about working with learners' routes to answers

- *What is LMT?*

For student-teachers, LMT is about working with learners' routes to answers. This is described in extract 7.10.

> ***"I think those solutions should not just be thrown away. What I mean is that it is from there where you can understand where that child has reached in the level of thinking and it is from there where you can build on and explain thoroughly to those children.*** *So that will try to give you the level at which that child is thinking, and then you can use that to explain, unlike a situation where you say that this solution is not what we want. You can use that to build on whatever knowledge the child has so that is, that is the one which shows you how thoroughly that child has understood the topic that you are trying to deliver."* (Extract 7.10; Focus group 1)

As indicated in extract 7.10, focus is on learners' routes to answers illuminating their level of understanding of the mathematical concepts which teachers can eventually engage with. This suggests, in Skemp's (1976) terms, teachers to think in relational ways in terms of what it means to work from the problem to the answer for them to work with learners' routes to answers. This is one way teachers can understand learners' routes to answers and guide their thinking by building on what they already know and explaining it thoroughly. Moreover, learners having varied mathematical routes to answers suggest that their thinking is also in relational ways subject to verification by their teachers. Therefore, LMT is about thinking in relational ways in terms of working with learners' mathematical routes to answers. Teacher-educators spoke in similar ways in terms of working with different routes to answers during teaching although student-teachers provide ways of working with learners' routes to answers.

- *How student-teachers say they "learn" for working with learners' routes to answers*

In a situation where teachers are not familiar with learners' routes to answers, six strategies of how teachers could work with learners' routes to answers are identified in extracts 7.12, 7.13, and 7.15. I use extracts 7.12 and 7.15 to exemplify as the issue raised in extract 7.13 is addressed in extract 7.12, and extract 7.15 is a different issue altogether.

> ***"****Maybe in that case* ***you can try it to problems similar to what he has solved, and if that method works, you can encourage the pupil to continue with his endeavors****. There is no way you can discard what the pupil has discovered and you know that so far it is working. So the best you can do is …*
> *[Inaudible] because* ***there must be a standard way of certain things to follow****, because even as the examiners, as they mark, they have given certain solution set as in how they are marking each step that the pupil was answering. So of course the answers could be correct but maybe the standard way to use,* ***you cannot say just because they have said use the quadratic equation and you have another way of doing it you do not apply the quadratic equation, you may fail*** *because that is what happens even here sometimes, you know the solutions, you have answered it, it is the correct one quite alright but because it is not the standard which the lecturer recommends, he knows there are certain weaknesses with that method you have used that is why he teaches you that particular method. But we are not restricting the pupils not to follow their way of reasoning but you can help them understand even the solution you have and marry it with the ones they have.*
> *And in examinations in most cases, if a question has got several ways of solving it they mostly specify, for example, they would say: Use the quadratic equation to solve find the solution to this? And there are some questions which they just ask and it is up to the pupil to find what way to use to solve that problem.* ***Now if they specify to say use this method and then a pupil uses the other one, it is ok to mark that person wrong but if they have not specified, it is better just to mark****.*
> ***I:*** *Yes, you want to counter react?*
> *I wanted to [inaudible]. So* ***if you reject a solution in that way you will not motivate the pupil in any way****, so* ***the best way you can do is follow-up his method, ask those that are in your system, this is what my pupil is doing, how do you go about it?*** *Is it a good way of doing things? How can this help our ways of solving such kinds of problems? In that way you are trying to motivate the pupil,* ***in that way you are trying to encourage discoveries in the learning of mathematics****.*
> *I did not refuse that, I said all the pupils,* ***you also have to marry that, the standard way which you know is the standard way with what their reasoning is because sometimes there are certain reasoning that seems to be ok at that particular time and then if you try to use it somewhere else, it may fail …****. Explore it quite alright but at the same time help the pupil understand the standard way of answering it in marrying what he knows and the standard way, or else you may mislead the pupil,*

one day he may try to use it on somewhere where it does not work and because he is used to that one, and it will fail. That is what I am saying.
Sometimes you may follow a certain method in a certain book but maybe a child used a different book from what you as a teacher is using. So what we just need is to follow it up and see if actually the working to that solution is ok *because different books have different methods of coming up with answers.*
… *maybe the best way might be ask for the book that that pupil used so that maybe you can go through it..”* (Extract 7.12; Focus group 1)

“*... I really appreciate one method, strategy of teaching which I have been introduced to and that is* ***investigations arising from the unusual in class****. What that really means is that investigations are not just about giving projects to pupils, they also arise from the short answers, short problems that arise in the classroom.* ***If a problem arises in the classroom and then as a teacher you feel you do not know much about the method that has been used by the pupil, the best would be to ask fellow pupils to try and see how they can understand that problem. As long as it is giving the correct answer, and it is establishing the truth in mathematics, in other ways, it is connecting ideas****. … But this thing of restricting them to the standard, me I am not in support of it because not everyone will be conversant with the standard, they can always come up with other explorations … .”* (Extract 7.15; Focus group 1)

In extract 7.12, five strategies are identified on how the teacher could work with learners' routes to answers: (1) the teacher should try the method the learner has used on similar mathematical problems and if it works learners should be encouraged to use it; (2) in a situation where the method to be used is specified, such as in examinations, the teacher should encourage the learners to follow the method that is specified; otherwise they could use any method. In the event that the learners do not follow the specified method to use, they should be marked wrong. However, there is some disagreement among student-teachers pertaining to marking a correct method wrong despite the circumstances as it de-motivates the learners. (3) The suggestion instead is that if teachers are not sure about the method used by their learners they should consult colleagues in the department. If the method turns out to be helpful in solving mathematical problems it will motivate the learners since the teacher is "*trying to encourage discoveries in the learning of mathematics*" (Extract 7.12). In agreeing with the idea of incorporating learners' methods of getting to the answer, student-teachers suggest that (4) it is also useful for the teacher to explore learners' methods used in light of the standard ways and see how things work out to avoid recommending a method which would mislead learners. Moreover, (5) student-teachers point to that if learners have used a

method from a different book which the teacher is not familiar with, the teacher should make a follow-up and check if the working in the solution conforms by asking for the book that the learners used to come up with the method.

In extract 7.15, the sixth strategy is that if the teacher is not sure of the method that the learners have used "*investigations arising from the unusual in class*" (Extract 7.15) could be helpful. This strategy is a situation where the teacher asks fellow learners to make inquiries and see how they could understand the problem at hand without necessarily restricting them to the standard way but as long as it is in line with what is considered as truth in mathematics in terms of connecting ideas. In Rowland's (2005) terms this strategy suggests "*contingency*" which means the teacher acting in a moment from the unexpected in the teaching and learning situation.

In the sections that follow, I discuss how student-teacher or teachers, and learners are positioned in relation to what it means to work with learners' routes to answers and how they "learn" for it in terms of what student-teachers should know and be able to do.

- *Positioning of student-teachers/teachers and learners in relation to working with learners' routes to answers*

Student-teachers shared their experiences of working with mathematical routes to answers and what this informed them of in terms of working with learners' routes to answers. This is exemplified in extract 7.16.

> ***"****I think there was one problem that, we were given one sequence at some point and* ***many of us we really failed how to solve it****.*
> ***I:*** *What were you required to find?*
> *We were required to find the next two numbers.*
> ***I:*** *Can you remember the sequence?*
> *The sequence, it was like 2, 4, 8, 16, 31, … and then they said can you tell us what are the next two numbers? Now in looking at that one, you find that from the sequence kind of mathematics way of trying to do it and you have done a lot of sequences and you want to apply your mathematics, your higher mathematics, trying to understand that you realize that actually there must have been a mistake and trying to change the 31 to 32, again you try, no I think there was something wrong here. By the end of the day we realize that actually, after submitting our solutions, and then* ***when we discussed it in class we realized that ok there was some other method which could***

have been done of which most of us said that was not more mathematical because it was like something away from the mathematics we do but actually it involved a lot of thinking*, ya.*
I: *So what was the correct approach to this problem?*
Well, the correct approach was, involved trying to group numbers trying to see their difference until you come to a point where there is uniformity in certain numbers, and now looking at the similarities or something, ya.
I: *So what were the next two numbers?*
Others: It was find the next number, and it was 57.
I: *57, ok.*
I: *So what was the importance of doing that?*
I think ***it was to help you change not to look at only the standard way of doing it*** *but also to explore that you can tackle certain questions not just using the standard method but you can think more than that and go a step further and seek other ways that can exist other than the standard way which you know.*
I do think ***it is important to do that because you are trying to help or you are trying to instill in a learner creativity****. The learner does not need to rely on standard ways of doing things always, he has to think of ways of how correctly he or she can do or work out that problem and still get the solution."* (Extract 7.16; Focus group 1)

As indicated in extract 7.16, there is a sense of absences (A) in student-teachers in that the methods of solutions to some mathematical problems they dealt with were not within their realm of understanding of how they have come to know mathematics. This experience taught them that there could be other methods of solving mathematical problems which one might not be familiar with. Student-teachers therefore argue that for teachers to have this understanding instills in learners creativity as teachers will encourage their learners to explore other useful ways of solving mathematical problems, hence seeing presences (P) in learners.

In talking about working with learners' routes to answers, student-teachers' views of learners is that they are not empty vessels.

> ***"****And in this same course, we are taught that* ***pupils are not empty vessels waiting to be filled****. So the teacher should not only expect them to reproduce what he taught them. If they produce what he has never seen and yet it is making sense, he should just accept.* (Extract 7.15; Focus group 1)

This view of learners as coming with competencies to the teaching and learning situation in terms of mathematical routes to answers suggest a sense of presences (P) in them in that they do not only reproduce methods taught by their teachers.

7.2.2 LMT is about teaching for understanding (TU)

- *What is LMT?*

Student-teachers talked about LMT as particular kinds of teaching for understanding and what this means for learner learning. This is contained in extracts 7.10, 7.23 and 7.25.

> ***"****Yes,* ***the only way we would try to teach our topics thoroughly and base our teaching on understanding because the person we have in mind here is the pupil****, we have the pupil in mind. So the best way would be to help our pupils to understand our topics as much as possible.* ***The topics that we are teaching let us try to relate them to life experiences****. When we are talking about inequalities, it is not just an issue of involving numbers in there, try as much as possible to bring things that they will be able to see. Suppose you are talking of how many oranges, say 20 oranges greater than how many tomatoes, say 11 tomatoes. There, there is a relation, then you can bring in the numbers, so it is a matter of moving from life time experiences in our preparation of these topics to the pupils."* (Extract 7.10; Focus group 1)

> ***"****… I mean* ***the issue of focus on learning****, it starts there because if you were to dish out anything that you want to dish out to the learners it would really not help giving them the whole syllabus or maybe the whole topic but really it would help them to* ***target those misconceptions because really that would be a starting point to learn other things.*** *You will then lay a very good foundation, so the issue of focus on learning starts there otherwise you will only end up dishing out information to learners. In other ways you end up* ***'teaching mathematics and not teaching learners'****.*
>
> ***When you say I want to teach mathematics you are really are, you are really burnt on teaching particular, particular concepts in mathematics and the skills but when you are saying I want to teach the learner the focus there is on learning****, how much do they learn, would they be able to get the concepts that you give them, the idea is not teaching all that you have. If there are three topics, the idea is to take time and make sure that they understand each and every one of those three topics, and not really plough through them and say I have taught these.*
>
> *Yes, maybe to elaborate further, the difference between teaching mathematics and focusing on the learning of the learners, if you, if you focus on just teaching, whatever you have,* ***you go to class and teach but then on the other side the learners might not have learnt during your teaching because teaching and learning really are different****, you can speak to them, and say I have taught, when they have not learnt anything. So teaching and learning in that sense are different."* (Extract 7.23; Focus group 1)

> ***"...*** *when we were learning in our lectures,* ***the lecturer used to emphasize that whatever step you are making, it has to be justified, it has to be justified****, so those*

mistakes can be seen as you are moving because a child would ask from this step what happened? So ***as a teacher, you need to know what you are doing and justify what you are doing****. … you need to know that from that step, a child should be able to follow, so from that step we come to this, what happened is this.*

The teacher should have a solid background of mathematics, even when he introduces a topic, he should not introduce it from nowhere, there should be some pre-requisites *like in the issue of equating vectors, if the teacher has got knowledge of equal vectors and has explained what equal vectors are, then even when he comes to equate vectors, the pupils will not have problems because they will understand that they are equating because the vectors are equal. So* ***it is the pre-requisites that the teacher should teach first before he teaches the rest of the topic.****"* (Extract 7.25; Focus group 2)

In general, teaching for understanding suggests relational ways of thinking in Skemp's (1976) terms since the focus is mainly on understanding that is conceptual. For example, in extract 7.10, student-teachers point to how teaching for understanding could be enhanced by teachers relating the school mathematical topics to everyday life experiences of their learners so that they understand the underlying meanings. However, the example they provide in Hart (1981) terms suggests one of the learners' interpretations of letters in their early learning of algebra, that is, 'letter used as an object' and the problem that it obscures the meaning of a letter. This is discussed in Section 3.4.3.

In extract 7.23, student-teachers suggest that the teachers' focus should be on learner learning with specific focus on learners' misconceptions and not necessarily concerned with covering the whole curriculum or topic. The argument is that such a focus is critical as it would lay a strong foundation for learners to learn other aspects of mathematics. Moreover, similar to teacher-educators' discourses, the teacher not focusing on learner learning for student-teachers suggests "*teaching mathematics and not teaching learners*" (Extract 7.23). *Teaching mathematics* means teaching particular skills and concepts in mathematics without necessarily being concerned with whether the learners have understood. This in Skemp's (1976) terms could be associated with teaching for instrumental understanding. *Teaching learners* foregrounds learners' understanding of the skills and concepts being taught.

Moreover, as indicated in extract 7.25, *teaching learners* would further be enhanced if the teachers focus on knowing and justifying each step taken when solving mathematical

problems to illuminate learners' errors. For a teacher to know and justify each step taken, they should have a strong background of mathematics by knowing the pre-requisites of a particular topic of focus. In extract 7.23, student-teachers further distinguish between teaching and learning in that teaching a particular topic does not necessarily mean learning on the part of the learner has occurred. This suggests that, in Hatano's (1996) terms, learning is a process and for some learners it does not happen during the process of teaching.

- *How student-teachers say they "learn" for TU*

Three strategies are identified on how student-teachers said they "learn" for teaching for understanding. These include the issues of emphasizing and clarifying concepts, verbalization, and using shortcuts. They are talked about in extracts 7.35, 7.36 and 7.38, respectively.

> ***"...*** *I remember very well when our lecturer was emphasizing the point that it is very good to emphasize and expand on even on some matters which you think they are very elementary, they are very simple, but* ***it is good to emphasize on each and every point and in that way it means that you are even polishing up all those, maybe misconceptions which learners might have.*** *And it does not just mean that maybe when you are solving a problem, you just go straight, ah no it goes to the other side, it is negative two, then you divide by what, it will be this, this, this. But it will be better if time allows,* ***make sure that you clarify points, you explain to them why things are happening like that****. I know that time is not very much enough, but if you are able to explain all concepts in every problem that you are looking at, I think it will be possible that you can even clear out all the problems that learners have, or what they might be facing.*
>
> ***I:*** *Maybe some people did not pick up your example, can you elaborate on the example you gave on how you can explain all the possible steps into finding a solution to a problem?*
>
> *Ok, for example, let us say maybe, this child, the example that I had given in the first place, this child does not know how to identify which one is your* a*, which one is your* b*, which one is your* c *in the quadratic equation. So you start, you just have to take your time because those learners, they do not know, you assume they are meeting that for the first time and they are not at your level whereby you are able to grasp concepts very fast. So you start slowly but sure, but you have to be very patient,* $ax^2 + bx + c = 0$*, you explain to them that* a *stands for the coefficient* x^2*,* b *stands for the coefficient of* x*,* c *is the constant, then zero, it can, it is not always that there is supposed to be a zero here, there can be any other number but this is just the standard form. Then you go to the next step. Maybe I can give an example after explaining that quadratic equation, the general form of it. Then if you clearly tell them to say the*

coefficient of x^2 is a, then the coefficient of x is b, the one, the other value which is not a coefficient is the constant which we call c, and then it should be equated to zero. If there is a number there, you have to make sure that you add the additive inverse of that number to both sides of the equation so that you remain with a zero because in the general equation, we have a zero. So in that way, ***whether the equation will be turned upside down, it will be rotated, it will be put in whatever direction, as long as you have already told them that the coefficient of x^2 is a, I think it will be very possible that they will not get misconceptions****."* (Extract 7.35; Focus group 2)

"*Just on that, because as a teacher, I may think that I have exhausted all the explaining, putting emphasis where I may think that children may follow but so* ***the other solution is to allow verbalization, meaning that you allow children, you would say John or Mary, what do you think of this way, explain****, just to say that because they may understand it in a different way, so as they are explaining then you can know that, ah, this person has not understood this thing, so in that way you will help, you even help the child and explain that this means this and not in the way you are taking it."* (Extract 7.36; Focus group 2)

"*The other thing is,* ***avoid shortcuts when you are teaching****, where you just ignore some stages, then definitely pupils will develop problems including future problems."* (Extract 7.38; Focus group 2)

In extracts 7.35, 7.36 and 7.38, student-teachers point to ways in which they could put into practice aspects of teaching for understanding. In extract 7.35, student-teachers point to the importance of the teacher to emphasize and expand on issues that may seem elementary but are important in explaining the steps being taken in solving a mathematical problem as doing so might address learners' errors. Moreover, it is about explaining and clarifying each step that is taken in the manipulation of mathematical problems despite time constraints. The benefits for the learners are that it does not matter in which form a mathematical problem could be phrased, they will be able to know what is required of them to do because they understand the underlying concepts.

Other than just the teachers thinking that they have explained and clarified each step taken in solving mathematical problems, student-teachers in extract 7.36 point to how important it is for them to assess learner understanding. This could be done by allowing learners to verbalize and from the explanations they provide, the teacher could then guide them accordingly. The

importance of explaining and clarifying rests on how the teachers could avoid making shortcuts in their teaching of mathematics. As student-teachers have indicated in extract 7.38, making shortcuts by ignoring some steps in the teaching of mathematics creates in learners difficulties which might also manifest in further learning. Teachers' use of shortcuts is also identified as one of the explanations for reasons for learner errors in Section 7.3.

In referring to what teaching for understanding entails and how the student-teachers said they learn for it in terms of what they are supposed to know and be able to do, student-teachers/teachers/teacher-educators and learners are positioned in particular ways. I discuss these ways in the section that follows.

- *Positioning of student-teachers/teachers/teacher-educators and learners in relation to TU*

Student-teachers shared the experience of explaining and clarifying each step taken in manipulating school mathematical problems in extract 7.24.

> ***"****... I remember in one of the labs, we were given a question on vectors, then we were given on position vectors, maybe you still remember, given, in part (a) we were given that, given that vector something is equals to that, then we were asked to find another vector, then we had to equate those vectors at some point, then after equating, we had to get the one with vector h equated to the other vector h, then the other one with vector q, we equated to vector q, then we solved, then there was a question to say under what condition, why do we equate those things,* ***but our lecturer left us in suspense, up to now I do not know why we equate those vectors, so I do not know where we can put ourselves to say, even when we look at those problems at secondary school****, we are finding solutions here or we are just inviting more problems to ourselves. I do not know maybe you still remember that…[interrupted]*
> ***I:*** *So what would you have hoped to see?*
> *What I wanted to see was,* ***since we did not know, I was expecting our lecturer to help us*** *to say, the reason why we do this is a, b, c, d. But if I am to go back to secondary school and teach, if a pupil asked me 'under what condition do you equate those vectors?' I will simply say this one I just found it in mathematics also, because I was not told. I do not know; we need to be helped in some of those things."* (Extract 7.24; Focus group 2)

As indicated in extract 7.24, a sense of absences (A) is realized in student-teachers in terms of explaining and clarifying why some steps are taken in solving school mathematics. Student-teachers argue that absences experienced are due to absences in teacher-educators for not explaining and clarifying to them. Moreover, student-teachers point to how absences realized would in turn result into absences in learners in knowing why some steps are taken in solving mathematical problems because the teacher is not in a position to explain.

7.2.3 Summary and conclusion of the analysis

For student-teachers, LMT is about relational ways of thinking involving working with different routes to answers and teaching for understanding, hence similar to teacher-educators' discourses established in Chapter 6. Working with different routes to answers for student-teachers involves encouraging use of learners' methods of solving mathematical problems if authentic; encouraging learners to use specified method if asked to do so; teacher consulting colleagues to establish if learners' methods are genuine; compare learners' methods with the standard way; consult resources used by learners; and turn unexpected incidents in class into investigations. The way student-teachers say they focus on LMT in relation to different strategies of getting to answer is accompanied by positioning themselves in terms of absences. They talked of how they did not know that there are different strategies of getting to answers which they are not familiar with. However, being aware that there are different strategies to getting to answer would result in seeing presences in learners and engage with their strategies.

Teaching for understanding for student-teachers includes relating the mathematics to learners' everyday life experiences for the purposes of meaning making; and that it is concerned with learners' conceptual understanding. Therefore, a teacher with strong mathematical background is crucial as they have to know and justify each step taken when solving mathematical problems; and identify pre-requisites to the mathematical topic in focus. A teacher has to also assess learners' understanding through verbalization; and avoid using shortcuts in the teaching of mathematics since what is important is to explain and clarify concepts. Student-teachers are also aware that teaching is not always equal to learning. Student-teachers are largely positioned in terms of absences as they do not seem to realize the importance of explaining and clarifying steps taken in solving school mathematics. The reasons for not realizing the importance of explaining and clarifying are due to teacher-

educators not modeling the skill. Moreover, absences realized in student-teachers could result in learners not knowing and understanding concepts because the teacher is not in a position to explain and clarify concepts. However, from student-teachers' talk, it was not easy to establish specific topics in the mathematics education courses where LMT is focused on.

7.3 LMT is about analyzing learner errors and misconceptions involving identifying and remediating (LEM-IR)

- *What is LMT?*

For the student-teachers, LMT with particular focus on errors and misconceptions is about carrying out error analysis. This is described in two ways. The one way is about recognizing learner errors and likely remediating strategies (LEM-IR) in light of working out solutions to school mathematical problems as talked about repeatedly in extracts 7.3, 7.11, 7.18 and 7.19. I provide extracts 7.3 and 7.11 as typical examples.

> *"In our courses* ***we have always been given the activities … to find out from us how we would help our learners if they find difficulties*** *in certain problems that we are given. … you are given a question from the secondary school level. Ok, supposed you are given on graphs and then they would ask us to say what would be your answer to this question? Then at the end of it, they would say what do you think are some of the problems that pupils would face in coming up with such graphs? So, that is where I think we have been helped because we have tried to look at both angles of how we can help them get the correct way of doing it, the idea there is the process of getting to the answer."* (Extract 7.3; Focus group 1)

> *"I talked about the activities that we have been doing here that relate to the mathematics that is there in our secondary schools. So in our solving those problems as teachers, we ought to also have in mind some of the difficulties at each stage because* ***we understand fully well that mathematics has got connected ideas, it is a subject based on logic****. …* ***It is from the procedures that we have and say at this stage it would really, because these are, we are people that have been pupils at one time, then we say at the time I was a pupil, I think at this stage there would be a problem here. So it is a matter of having the pupils' solution here and then marrying it with your own procedure here, say ok here, how connected is this to that idea, where did you get this idea to come to the next one?"*** (Extract 7.11; Focus group 1)

The other way is about evaluating learners' answers. This is talked about repeatedly in extracts 7.6 and 7.9, and I provide extract 7.6 as a typical example.

> *"I think the only way we can engage with our learners … **is by evaluating their answers** … let us not have this tendency of discarding the learners' answers; immediately the answer is given to some problem, we are not out rightly supposed to say this is wrong, ok, but all we are supposed to say is **ask the learner how he arrived at that answer,** what motivated the learner to arrive at that answer, look at the procedure and then try as much as possible to reason with that learner **and direct them in the correct procedures**. … Let us try to rectify what they have already done. That is the only way we can motivate them even to fully participate."* (Extract 7.6; Focus group 1)

As indicated in extracts 7.3 and 7.11, analyzing learner errors involves anticipating learner difficulties arising from student-teachers' procedures in solving school mathematics. This is informed by student-teachers' understanding of the nature of mathematics in terms of its connectedness and logic; and from the experience of having been learners before. To remediate the learners' anticipated difficulties, student-teachers suggest that they have to probe learners' procedures in light of teachers' procedures. Moreover, as shown in extract 7.6, student-teachers' description of evaluating learners' answers suggests that it involves the teacher probing learners' procedures and directing them towards the right procedures. There is an aspect of establishing whether the answer is right or wrong, and in the event that it is wrong, fixing it. Common in both ways of carrying out error analysis described in extracts 7.3 and 7.11; and 7.6 is that the focus is on recognizing the error and finding ways of remediating. There is no explanation of sources of error. This finding contradicts the strategies of carrying out error analysis so far established such as that by Peng & Luo (2009) and Jacobs et al. (2010) where explanation for the error is an important step before one can think of ways of remediating. However, for teacher-educators, while recognition of error and suggested remediating strategies are in focus, they locate sources of error in inadequate teaching. It will later become evident that even for student-teachers sources of error also stem from teaching.

- *Where in the courses analyzing learner errors is focused on?*

From the description of carrying out analysis of learner errors provided in extracts 7.3 and 7.11, it is clear that "school mathematics" is one of the topics in the mathematics education

courses (in particular Mathematics Education II, with course code MSE 332) where this is focused on. As stated in extract 7.19, student-teachers also discuss learner errors when focusing on peer teaching during which they also experience working with school mathematics.

> *"…, but maybe where I can say I experienced some school mathematics was during our peer teaching, we were allocated some topics which we were supposed to present in class, so it was like maybe, you have a component of quadratic functions, to teach about quadratic functions where you even identify the kinds of problems that pupils can face."* (Extract 7.19; Focus group 2)

This suggests that in their mathematics education courses, student-teachers are given school mathematics topics to teach their peers. It is during the teaching that anticipated learner difficulties pertaining to the topic in focus are discussed. In relation to teacher-educators' discourses, for student-teachers, LMT with focus on error analysis which involve recognition of the error and suggesting remediating strategies is also a skill that is accomplished practically, hence taught. The skill is situated in the practice of teaching such as peer teaching and supported by principles that guide discussions on school mathematics. LMT is also weakly classified because of where it is focused on in the mathematics education courses.

- *How student-teachers say they "learn" for analyzing learner errors*

In talking about how the student-teachers "learn" for analyzing learner errors in terms of what they should know and be able to do, four strategies are mentioned and I discuss each of them in turn.

(1) Plan for anticipated leaner errors

Student-teachers talked about how important it is for them to be aware of learner errors beforehand as it would help them in the planning of their lessons. This was said in similar ways in extracts 7.4, 7.22, 7.33 and 7.37. I provide extracts 7.22 and 7.37 as typical examples.

"I believe it is very very necessary and important to deal with those misconceptions because ***it is the only way that we will know the way ahead in terms of how well we will handle our pupils in their abilities****. ...* ***really gives the direction to the planning of our work.***

Ok, another aspect is that, you know ***when you are trying to plan for a learner you have to sought of go back and try to think the way they would actually think****. That would be one sure way of knowing exactly how to give out the information, looking at all those mistakes they might make, that is the only way you can know how to control them and* ***in that way you can be sure that your planning is actually very effective and learner-centred****.*

I: *What is your yardstick for gauging that they will make this mistake? How would you know that they will be capable of making such a mistake?*

Well like I said earlier on, ***you more like try to think the way they think****, try to reduce your mind from that huge academic point of view and take it down to the learners' point of view.*

Maybe just to add on, aa, ***we have been pupils also before and we suffered consequences then of having misunderstood some concepts****, so I think basically even before we learnt many things in the programme, we understand some things that we misunderstood before when we were learners then. I think it is very easy for you to imagine how pupils can understand some concept and go in a different direction all together.*

Ya, to also amplify his point there are other situations where you as a teacher maybe ***you have taught that topic before to another class and then you are dealing with another class on the same topic, you will be able to see the kind of mistakes these others had done*** *and how the misconceptions were with the pupils. By the end of the day, ... with your experience you will be able to come up with a proper planning having known how misconceptions are being made by the pupils in their mistakes."* (Extract 7.22; Focus group 1)

"*I think when doing the plan, you should question yourself and* ***find out the problems that you expect from the children and then turn those problems into maybe questions*** *so that as you go in class, you have a view, a clear view of what you expect from the children so that you address them with at least a clear emphasis."* (Extract 7.37; Focus group 2)

As indicated in extracts 7.22 and 7.37, student-teachers emphasize the importance of being aware of learners' errors in that it would help them plan ahead in terms of handling learners' abilities. Being aware of learner errors beforehand in Hill et al. (2008) terms suggests KCS.

Four strategies of how teachers would anticipate likely learner errors in their planning are identified. Firstly, is that the teachers should adjust their thinking to the way learners would think to enhance planning that is effective and learner-centred. Secondly, is that the teachers should draw on their experiences of having been learners before in terms of the problems they faced. Thirdly, is that teachers should identify errors their learners are likely to make in their teaching of a particular topic so that they can use this experience in their teaching of the same topic to another class. This means that one would incorporate such errors in planning for the benefit of the other learners rather than the ones that made the errors, hence seeing errors as diagnostic in Borasi's (1987) terms and this is discussed in Section 3.3.5. Fourthly, is that teachers should design questions that are aligned with the anticipated errors so that in case such errors arise the teacher is able to clarify. This in Borasi's (1987) terms suggests viewing errors as *springboards for inquiry*, and is also discussed in Section 3.3.5.

In relation to teacher-educators' discourses, student-teachers are also aware of the importance of incorporating in their lesson plans anticipated learner errors. The difference is that student-teachers have provided strategies of how anticipated learner errors could be recognized and planned for prior to teaching. As before, there is resonance but student-teachers fill out meanings.

(2) Sources of errors stem from teaching

By making reference to classroom activities, student-teachers said that they became aware that sources of errors are in teaching. This is talked about in four ways in extracts 7.5, 7.7, 7.9, 7.22 and 7.26 that errors are as a result of teachers: making shortcuts, not relating new learning to what learners already know, not motivating learner learning by relating the mathematical concepts to the history of mathematics, and not building solid foundation in learners by ensuring that they understand topics which are the basis for topics to come so as to enhance further learning. I discuss these in turn.

(i) Teachers making shortcuts

This was said in similar ways in both focus groups but I only provide the extract from focus group 2 to exemplify the point.

"I think ***teachers also contribute to misconceptions that pupils face because of the way they teach. The way they solve themselves, that is how they teach to the children, shortcuts*** *and those methods which themselves are able to understand, that is how they do it on the board or in the classroom. That creates misconceptions and problems to the children. The teacher is supposed to* ***teach in a way so that children should understand each and every stage of solving a problem****, not where you take shortcuts; crossing the bridge, it is the teacher who does that, not that the pupils have created, the teacher on the board will say, these two will cross the bridge and becomes a minus. ... Even me, I remember I was taught, taught the same way, it crosses the bridge, and I was crossing the bridge without knowing why it has become a minus. So as teachers, we should* ***teach in stages and explain each stage of the problem so that pupils are able to understand what they are doing.****"* (Extract 7.5; Focus group 2)

The main argument in extract 7.5 is that learners make errors because of the way teachers tend to teach using shortcuts which they themselves understand the underlying meanings. Therefore, it is important for the teacher to explain each step taken for the purposes of learner understanding. In Ball and Bass (2000) terms, teaching mathematics using shortcuts could be linked to teaching in a compressed form rather than in a decompressed form.

(ii) Teachers not relating new learning to what learners already know

"... most of the teachers fail to do that because if you look at, for example, the equation that was given, the equation was $x+2=5$*; the* x *here tends to confuse pupils and yet they did this in grade one. Instead of the teacher explaining that actually this* x *represents a box so that they are able to reconnect and relate to what they had actually learnt, teachers actually do not actually do that,* ***they will simply say ok,*** $x+2=5$ ***so*** $x=3$***, but how did they arrive at three, and where did actually this equation come from, it came from the box where we actually fail to relate. So that leads to most of the pupils fail to understand the mathematics.****"* (Extract 7.5; Focus group 2)

"... I think ***it may also help us in revising certain pre-requisite topics relating to one of the topics you intend to teach****. For example, when you are talking about quadratic equations, there are a number of mistakes that pupils make especially when you are focusing on a method like, a strategy like factorization, factorization, usually pupils find it very difficult to express the sum of a quadratic equation as a sum of its factors especially when it has a negative sign there. So for you to teach quadratic equations perfectly well, then you have to ensure that you revise how to factorize algebraic*

expressions very well so that you connect that with what they are going to encounter when you start teaching that topic." (Extract 7.22; Focus group 2)

In both extracts 7.5 and 7.22, the issue at hand is that it is important for teachers to relate the mathematical concepts they are introducing to their learners to what they are already familiar with for the purposes of connectivity. Student-teachers argue that doing so would prevent learners from making errors since they will be aware of how the new topic they are being introduced to connect to earlier topics learned. Therefore, being aware of learners' errors would help the teacher in determining which pre-requisite topics to revise with learners before teaching a particular topic.

(iii) Teachers not motivating learner learning by relating the mathematical concepts to the history of mathematics

"*Recently we learnt about some aspects of the history of mathematics. We are taught that* ***if we include when we are teaching some aspects of the history of mathematics we shall be able to motivate our children****. They will be able to get more interested when they learn how maths was developed and the people who were involved in the discoveries of what they are learning in mathematics today. …*
… Say if you are teaching number system, if you can talk about how numbers were developed by the Babylonians and the Hindus. For example, a number zero, it is said that it took a long time to include that number in the numerical system. So if you can explain to the children how the system was developed by the Babylonians, the Hindus, even the Greeks, definitely they will be very much interested in learning that subject." (Extract 7.5; Focus group 2)

The issue in extract 7.5 is about arousing interest in learners to learn mathematics because the teacher makes it interesting by providing the historical background of how the mathematics in focus was developed.

(iv) Teachers not ensuring that learners understand topics which are the basis for topics to come so as to enhance further learning.

"Yes, and I think ***when a teacher is teaching a topic which he knows is the basis for some other topics up there, he should make sure that his pupils understand*** *and he should take his time because if he just rushes through the topic, it means that the one who will come to build on it will just do his things in vain."* (Extract 7.7; Focus group 1)

"... If you are really so much concerned about the volume of work, then you will not achieve anything. ... ***the idea is to lay a very good foundation for them so that even if they were to do other topics, they would do the other topics with vigor and the interest that is needed.****"* (Extract 7.9; Focus group 1)

"... I was presented with, with a certain class, that was just my own arrangement handling grade elevens, so what I discovered was they were having problems with just the number line, just that number line, adding numbers on the number line, well when it came to addition it was alright but subtraction was a problem especially when it went to the negative direction, anything negative was giving them some problems, suppose you say -2 + (-3), that was really a problem to grade elevens that I handled at that point. So what I discovered was the whole issue was understanding the number line and not just the concept of subtraction as picked out as subtraction like that. So ***if I was to work on just subtraction there, yes I was assured of having addressed that misconception but I was not assured of other misconception arising, because there then I was not assured of having tackled the whole part of the number line. So the idea is to tackle the whole topic****, that scenario.*

In a way, I am prepared for the mistakes that I am yet to see my pupils make. And how I have responded to the activities is, ***I have been able to derive really the idea behind those mistakes, as the issue I referred to about basic concepts, missing basic concepts.*** *So I have been able to put ideas together and say this mistake is coming as a result of not understanding these concepts."* (Extract 7.26; Focus group 1)

In general, the argument in extracts 7.7, 7.9 and 7.26 is that teachers need to build in learners a strong mathematical background upon which further learning could be made possible. Student-teachers point to how motivating and interesting doing this would be for the learners as they would have opportunity to learn some mathematical concepts on their own. Student-teachers also point to how directly addressing errors is a short term remedy, rather focus on the basic concepts which could not have been well taught so that other unforeseen errors that might arise are addressed. Moreover, student-teachers explain how they have been helped by focusing on learner errors in that they are now aware of what is expected of them in their work of teaching. Especially that largely learner errors are as a result of learners not understanding basic mathematical concepts which the teacher should address to help them overcome the errors.

From student-teachers' discussions on that sources of errors stem from teaching, it is clear that they draw from teacher-educators' discourses an understanding of teaching for

conceptual understanding, and the importance of prior knowledge in teaching. However, student-teachers also include the importance of making reference to the history of mathematics and ensuring that learners have strong mathematical basics to build on as a way of enhancing further learning.

(3) Teachers should know the subject matter and be consistent

The importance of the student-teachers knowing the subject matter and being consistent in directing learner learning is also emphasized in the mathematics education courses. This was said in similar ways by both focus groups, but I use extracts from focus group 1 to exemplify.

> ***"**... **a teacher should be a willing teacher himself first, then he has to understand the subject matter so that as he gives out the concepts, explanations about the subject, everything is as much clear as possible**. Then the willing part, if he is not willing, I mean he will just be doing it out of frustration or just for the sake of teaching because I am a teacher let me just teach. It should not be like that but **he should first and foremost be a self motivated person himself, then he will be able to direct his motivation to motivate others as well**.*
> ***I:** What do you mean when you say he should understand the subject matter? What is involved in that?*
> *Understanding the subject matter, for example, let me say he has been talking about equations, you are teaching on equations; maybe they are fractional equations, whatever sought of equations. Now you have to understand exactly what you mean by equations, how you work them out, its addition, subtraction, multiplication; everything that concerns equations that you are teaching, you have to know exactly how you can explain that to the learners so that they get the understanding."* (Extract 7.7; Focus group 1)

> ***"**... one other point they have emphasized in our learning of teaching courses was "**the ability to say what we mean and mean what we say**"; **consistency in anything that we do**, and that is one way of learning mathematics according to the Cocroft report. ... And I think in as much as we learn all these things, the problems we encounter everywhere, we should be able to establish something, **and having the full knowledge of it is very important** so that we do not mislead anybody ..."* (Extract 7.21; Focus group 1)

In extract 7.7, student-teachers point to how important it is for teachers to know the subject matter as that is one way they can be self motivated and in turn motivate their learners. Knowing the subject matter for student-teachers implies understanding the meaning of the mathematical concepts in focus in terms of the steps involved in its manipulation, and how

these steps can be explained in detail to enhance learner understanding. In extract 7.21, student-teachers also point to the issue of the teacher being consistent to enhance learning, and for them consistency means "*the ability to say what we mean and mean what we say*" by having full knowledge of the mathematics in focus. This means that teachers' knowledge of the subject matter is directly linked to them being consistent.

Student-teachers' descriptions of SMK resonate with Ball et al.'s (2008) knowledge domains which are CCK and SCK; and these are discussed in Section 3.2.3. In relation to their teacher-educators, student-teachers' talk of SMK relate to relational understanding which is about working with mathematics as processes involving conceptual thinking, different strategies of getting to answer, and not just giving answers.

(4) *Teachers' recognition of learners' errors could be enhanced by them getting involved in marking of examinations and class work, and doing more of school mathematics*

> ***"****I am sure when we go out there we shall try by all means to put those things into practice, and one of the thing is* ***through practice****, and, like when you look at* ***if you are a maker of grade twelve or grade nine exams, that gives you an insight of most of the misconceptions which pupils have in these topics****, so when you do a lot of that, you may know now how to come and help the children when you come to meet them in those areas."* (Extract 7.34; Focus group 1)

> ***"****Sometimes there are situations like when you look at matrices, when pupils are to find for instance the determinant of maybe a 2 x 2 matrix where we talk about the major diagonal minus the minor diagonal. In most cases, pupils make mistakes in understanding which one is the major diagonal and which one is the minor diagonal. So it is from there that maybe when you have your time you* ***explain thoroughly on such kind of mistakes which arise because we have seen that, especially when we are marking the grade 12 examinations****."* (Extract 7.9; Focus group 1)

> ***"When you give work, definitely you mark work done by each child in that class****. So from what they have done in their books, you will be able to tell the misconceptions that the students have. So maybe in the next lesson, you can point out those which you observed from their books."* (Extract 7.44; Focus group 2)

> ***"*** *... What I was just simply trying to say is that at least* ***we should have been doing a little more of what is closer to what the learners are going to be dealing with****, like, for instance in some of the labs that we have been doing, we have been given, what is this, some, some things that students actually do, some, for example questions from*

grade twelve so that we can actually see the problems. Yes we are, ***we have very good aptitude for mathematics, so we understand those things but we needed more of those things so that we can actually see the various mistakes and misconceptions that learners bring up****. So it is like we are a bit removed from what we are actually going to do."* (Extract 7.28; Focus group 1)

In extract 7.34 and 7.9 student-teachers point to that one of the ways of how they would put into practice the aspect of focusing on learner errors is by becoming markers of examinations (examiners). Through marking examinations teachers would recognize most of the errors learners make in school mathematics. This would in turn enhance teachers' ways of addressing the errors if experienced by their learners during teaching. Marking of learners' class work, as indicated by student-teachers in extract 7.44, could also serve similar purposes as the teacher would be able to address the errors identified from marking the learners' books in the next meeting. Another way, as shown in extract 7.28, student-teachers say they would experience variety in terms of the errors learners make would be by doing more school mathematics during the practical sessions. They say focusing more on the mathematics learners do in school is more practical as it is directly linked to what they will be expected to deal with in school to enhance learning. This argument resonates with Ball et al.'s (2008) argument and is discussed in Section 3.2.2.

In talking about what LMT is in relation to analyzing learner errors student-teachers/teachers/teacher-educators, learners and the curriculum are positioned in particular ways in terms of absences (A) and presences (P).

- *Positioning of student-teachers/teachers/teacher-educators in relation to analyzing learners' errors*

In talking of analyzing learners' errors, the following was said about student-teachers working with school mathematics in extracts 7.17, 7.20, 7.30 and 7.48 but I provide extracts 7.20 and 7.30 to exemplify:

"*It was not, that is why there was also a provision where you, the one solving, would write the problems you faced when solving that same problem, then you relate that same problem to what the pupil would face because you were once a pupil. So* ***the problems you face are what the pupils might face also****.*

It was not smooth as some problems were requiring you to connect some concepts which might have been learnt maybe some time back*. So for you to connect them, sometimes it was quite difficult. In fact in some problems, we were even making just some silly mistakes. So as teachers, I think, we must be aware of such things.*

I: *So how do you think you would have avoided such mistakes?*

I ***think the problem of failing to solve such problems at school level could be that the mathematics we do here is far much advanced*** *compared to that. So what could be done maybe is for the teacher of mathematics to be at least be going through the school mathematics because when you go through it you can get used to solving the same problems.*

Another problem I remember facing is, we were given a certain function and we were told to find its inverse. So we did according to what we were taught by our teachers at our schools and there was a point where we were interchanging y *with* x*.* ***Now the lecturer asked us why we did so. We failed to answer because our teachers did not tell us why we should do it that way. Up to now I personally do not know why we are doing it that way.***

I: *Was it not resolved?*

It was not*.*

Others: The lecturer did not tell us*.*

I: *Ok, so how did this experience inform you as a student-teacher and a mathematics teacher to be?*

This informed us that we should keep on reading and keep practicing mathematics." (Extract 7.20; Focus group 1)

"*... **there are some topics which are really difficult to teach at secondary school, which we have not tackled here**, like a topic like transformation. There is a part when you reach the shear, and the stretch, those parts; people find it very difficult to deliver on those two parts. And then there are some topics like earth geometry, normally when I go back to my school they always say no, we did not do this at college. So, in most cases they just leave it and pupils do not benefit from them, and then they just find it in the exams.* ***So, if the focus was changed, at least for us to do more of what we are going to teach when we leave this place****."* (Extract 7.30; Focus group 1)

In both extracts 7.20 and 7.30, student-teachers or teachers are constructed as having absences (A) in that they experienced problems in solving school mathematics and found the teaching of school mathematics such as transformation and earth geometry difficult. The reasons for the absences in student-teachers when solving school mathematics could be

because of difficulties in remembering the concepts they learned the time they were learners. In some instances it is because of absences in their teachers then in school or their teacher-educators for not explaining to them reasons for some mathematical manipulations, for example, the interchanging of *y* with *x* when solving for the inverse of a function. Moreover, the absences in teachers of finding the teaching of some school mathematics such as transformation and earth geometry difficult could be as a result of absences in the teacher education curriculum in that the topics have not been given specific focus. As a result, the absences realized in student-teachers could also be considered as anticipated absences in learners. Student-teachers experiences discussed informed the student-teachers of the importance of familiarizing themselves with school mathematics by practicing and reading.

Moreover, student-teachers shared how difficult it was for them to think of the necessary basic concepts and how to adjust their thinking to that of learners in anticipating and remediating learner errors. The issue of basic concepts is talked about in extract 7.26; and the issue of adjusting their thinking to that of learners is talked about in extracts 7.27, 7.28, 7.29, 7.32 and 7.40 but I use extracts 7.27 and 7.28 to exemplify.

> *"I think* ***difficulties were there especially when it came to basic concepts, you would really, you would really wonder as to why certain misconceptions arise because of just the disregard of the basic concepts****, so what really was there was some tasks, some activities that were presented to us could have such problems that could be worked out by those learners from which we would identify the difficulties which they would face, and then* ***you would find out that those difficulties can really arise from just the basic concepts which are not emphasized****, so in order to work on those misconceptions or difficulties you have to go all the way to the basic concepts and not really just emphasizing on them, because basically just targeting them like that, of course it would help at some point just there it would help but the problem is not really addressed, the problem is the basic concepts. It could be they are really not doing it well at some point but because of some basic concepts which was not well taught, rather which was not well learnt. So* ***there was that difficult of bringing them into the picture unlike just addressing that misconception, because the idea is how well do you bring them into a picture of learning the whole****."* (Extract 7.26; Focus group 1)

> *"I think my friend referred to bringing,* ***bringing your mind down to their level of thinking, that really also is a problem*** *in terms of, because you know these things right,* ***you take it that you know it, and we are students that have been, that have been studying mathematics that is beyond the one that you teach****, so at some point*

you feel well, that is my concern, I have sometimes have that frustration in terms of really, is this a concept that can give someone problems [laugh]. So adjusting to that level has quite been a problem, adjusting to the level of explaining something if there is a misconception.

I: *What do you think is the cause of that problem?*

The cause is ***not being patient with the learner****, not being patient with the learner, you can have the skill yes but if you are not patient with the learner you get frustrated. The idea is to try and* ***reason with the learner and be at the pace at which they are learning certain things.****"* (Extract 7.27; Focus group 1)

"*I wanted to say well* ***going beyond is not a problem, the problem is reducing yourself to that point of a pupil****,* ***not that going beyond because it helps you understand better any concept given to you.*** *Why we learn all these things is for us to have a wider understanding, even the complex variables, there were times when we would go through some mathematics quizzes as grade 12s at some point and then I first heard of that word complex number from grade twelve but other guys had no opportunity. The question was phrased like: What was the root of negative one, and we were all stuck, we would say it is 1, no it is -1, it is what but fortunately somebody had an idea, and then after exploring it a lot, we realized that actually the answer was i, and we never knew what was the i."* (Extract 7.28; Focus group 1)

In extract 7.26, student-teachers point to how amazing it is for them to think that basic concepts are an issue pertaining to working with learner errors in that they could not have been well taught or learned. However, student-teachers are constructed as having absences (A) in that they do not realize that addressing the errors requires them to work with basic concepts in their totality unlike directly addressing a particular error. As already mentioned under how they "learn" for analyzing learner errors, working with basic concepts in its totality is a long term remedy as it addresses other errors that might arise other than only addressing the one that is being focused on. As indicated in extracts 7.27 and 7.28 a sense of absences (A) is also realized in student-teachers in terms of adjusting their thinking to that of their learners in working with learner errors. Absences in student-teachers to adjust their thinking to that of learners are due to lack of patience with the learner despite sometimes having the skill; and the teacher education curriculum in that the mathematics they do is far detached from the mathematics they are going to teach in school. Exercising patience with the learner means "*reason with the learner and be at the pace at which they are learning certain things*" (Extract 7.27). While some student-teachers are for the idea that doing mathematics higher than school mathematics in teacher education is problematic, others think

that the idea is good "*because it helps you understand better any concept given to you*" (Extract 7.28). However, in extract 7.31, student-teachers argue that the absences others see as a result of doing mathematics that is pitched higher than the level of school mathematics could be addressed by teachers undertaking In-Service Training (INSET) programmes to better and improve their teaching skills.

> ***"*** *I think it is not just an idea of learning topics in, at such a level, at University level, topics that are in schools. I think it is now of learning beyond what is there in schools because then even as we are learning in mathematics education,* ***there is what is known as in-service training, so even as we become teachers, there is still that room for us to advance in our teaching career****. What in-service training there is referring to is the activities that we will undertake to better our teaching skills and methods, because even as you are there, you still have to attend seminars, you still have to attend workshops, you still have to attend other programmes that will work towards your improvement."* (Extract 7.31; Focus group 1)

As already mentioned in how they "learn" for analyzing learner errors, student-teachers talked about the experiences they bring to the situation and how to deal with large classes. This is exemplified in a portion of extract 7.22 and in extract 7.44.

> ***"****Exactly, as a teacher I believe you really have to have a keen eye on the preparation that you are doing in terms of* ***if you are preparing some work, you also have to note down, rather to jot down those difficulties that you feel having been a student or pupil at some point, that you feel they can have problems in****, and obviously working on those will really help you address their problems."* (Extract 7.22; Focus group 1)

> ***"****I think it is a yes and no situation. ... How big is the class to manage? If it is a manageable class, I think it is possible you can engage with each and every learner in class, and if it is a very big class, I think it is not.*
>
> *But me, I think it is better to engage in the mistakes somebody makes, you see, because like* ***a mistake which is made by another person and if it is explained to the entire class then everyone can benefit from the kind of explanation you are giving****. So even a wrong answer, if it is very well exploited, it can be also a source of information for the whole class* ***than just only considering that you are concerned with ok, maybe because of the number of pupils****.* ***It is just a matter of being what, being aware of what is your intention when you are teaching****. If you only rely on saying that no, in my case teaching is assisting pupils to be getting correct answers, then in that way you may have the time to be disregarding wrong answers. But if you*

are aware that ok, these people are making mistakes just because maybe of the background they have. So if you want to uplift them, it is better you also exploit whatever mistakes that can be made by a learner in class." (Extract 7.44; Focus group 2)

As shown in extract 7.22, a sense of presences (P) is realized in student-teachers in terms of bringing in experiences of learner errors from when they were learners in school into their planning. Their argument is that thinking of learner errors during planning would help in addressing learners' difficulties as they are thought of before hand. As indicated in extract 7.44, a sense of presences (P) is also realized in student-teachers on how to deal with learner errors irrespective of large classes since the entire class would benefit from teacher explanations of an error made by a learner. This view is strong in student-teachers and that, in Borasi's (1987) terms, learner errors if well exploited could be springboards for inquiry. The issue at hand is what the intensions of the teacher are in terms of focusing on correct answers or learners' mistakes. However, a contradictory way of working with learner errors is evident in student-teachers' talk suggesting that the teaching is focused on correct answers. They are also of the view that working with learner errors is individualized and therefore not possible if the class size is big.

- *Positioning of learners in relation to analyzing learners' errors*

Student-teachers also talked of how learners' interest in mathematics could have an effect on working with learner errors. This is said in similar ways in extracts 7.39, 7.41, 7.42 and 7.45; and I use extract 7.42 and 7.45 to exemplify. Moreover, the role motivation plays in the issue of interest is discussed in extract 7.43.

> ***"****... learning is about time, you learn from day one to whatever day, then the exam comes. That is how it goes, and you find* ***a pupil who would not learn within a space of time, it means they need to be taught something else, we have got different gifts****, that is why some of us are mathematics teachers, it is because we loved mathematics while we were pupils, we saw that it was good, we found it interesting, there are those that found it hard, and they could not learn mathematics within the time period we were given to learn, so, ya, they have gone for RS [Religious Studies] and other things. So let us go and teach patiently, but when we find someone who does not learn within the space of time, we can always encourage them to do something else."* (Extract 7.42; Focus group 1)

***"I think a pupil should show some willingness to learn in the first place**. Whereby at least, it is not just about the teacher following, … . If given work, he is supposed to, like when you look at classroom organization, **the pupil should, if given work, should be able to do the work in the given time frame so that it makes the teacher's work easy and very cooperative** and at the end of the day, because it is one thing but it involves two parties, that is a pupil and a teacher. So for us to achieve the learning on the part of the pupil, the two parties have to be fully involved."* (Extract 7.45; Focus group 2)

***"**I do not know whether most of us have attended motivational talk. In life, **the best thing that you have to tell yourself is you can do it**. So even when you are talking to the pupils, tell them that they can do it, that is the only motivation, if you go on the negative, if you just tell yourself that you cannot do it, you can never do it, everything is in the mind, the mind is a very powerful machine, you tell it, it cannot, it cannot. Because there are situations where I have told myself that Real Analysis is hard, and those are the times when I got C, the times I told myself I can do it, those are the times when I really got good grades. So the same applies to everyone, in any motivational talk, you will always be told you can do it, it is about faith, it goes even biblical, even religious, everything, it is, you can do it.*

Positive conversion actually is the driving force.

Now, even when you are motivating your pupil, do not hide what it would be like.

***Others:** [Laugh]*

*If you tell them there are challenges, **yes they can do it but you even highlight the challenges**."* (Extract 7.43; Focus group 1)

Extracts 7.42 and 7.45 point to that absences (A) in learners could be due to lack of interest in learning mathematics despite the teacher being patient with them taking into consideration the time frame within which learning should occur and then examination time. This means that both the teacher and learners have roles to play in the learning situation. More specifically, a sense of presences (P) in learners could be realized if they showed willingness to learn so as to ease teachers' work, for example, by completing their work within the time frame given. However, as indicated in extract 7.43, absences (A) in learners could be addressed by self conviction that they can do mathematics irrespective of the challenges, and it is the teachers' role to instill this realization in them.

- *Positioning of the curriculum in relation to analyzing learners' errors*

Student-teachers talked of what they thought is missing in their mathematics education courses which would better their working with learner errors.

> ***"****The other thing, I think is* ***the need maybe to have a kind of what, a textbook where these writers could just write maybe that common mistakes*** *in maybe this one, in this one, just like that [interrupted]*
> ***Others:*** *[Laugh]*
> ***Common mistakes, to prepare maybe the new teachers****, you see, like if you look at English, they have like common mistakes in English, isn't it?*
> ***Others:*** *Yes*
> *They have got those books but* ***in mathematics, we have got nothing****, up to a certain level we are just moving like on our own, you see, you are having, maybe* ***even you teacher, you have not even cleared problems, then you go and convey the same mistake to pupils****, and so on, it is like a chain of mistakes, we make mistakes, you see, you are not conversant with whatever you are explaining, the learner asks you a question and you just brush out, oh, ya, that question so, so, so, so, you move on. So,* ***it is better we are prepared seriously because the new generation which is coming is supposed to be given what, proper information more especially with this globalization theory****, and so on, whatever is coming. You may end up remaining like with the same mistakes and these others will be moving on and on with perfection, and us will be unprepared to face questions or challenges from our learners."* (Extract 7.25; Focus group 2)

> ***"****I think as it has been said before, I think* ***we have been exposed to little components of things we are going to teach when we go into the school****.* ***If it was possible, maybe the exposure of what we are supposed to meet can be widened, so that we can have ideas on how we can deal with a number of topics and concepts unlike just limiting to maybe one or two concepts which we have dealt with so far****.*
> ***I:*** *Don't you think the experience you have in those few will enable you to know what strategies to use for a different topic?*
> *The strategy that I can take in problems faced in vectors is totally different to the strategies that I should take in transformation. It is just that because we are tackling two different topics and probably two different angles of misconceptions."* (Extract 7.47; Focus group 2)

> ***"****... And then one other thing, for instance, nowadays,* ***there is earth geometry, is not it, most of us did not encounter it in our school days****, and this was probably the time for us to get a feel of what it is all about, rather than for us, because we will not even be able to identify the mistakes properly because it is something that is so detached from what we did."* (Extract 7.46; Focus group 1)

"To strategize and say ok, during quadratic equations, I faced this problem, then this problem, so when I go now in the real case whereby I now will be in charge of a given class, it will just be a walk over because you know all the strategies you are supposed to use.

Others: *That is true.*

Just an emphasis on the same point. ***If we had time to go and come back, we would have an opportunity as teachers in mathematics to explore the errors and all the misconceptions*** *.... So that point should be looked at very seriously because if I come as a mathematics teacher and these from other schools, we will come and combine the ideas, so how did you handle this? Or how did you overcome this? How did you help the child in this? So you tend to bring the ideas together and then be able to move in a direction to help the learner.*

I: *So why, it is like there is a big outcry on teaching practice, why do you think teaching practice is very important?*

It is very important madam because ***we are supposed to meet these challenges so that when we come back here, we will be able to polish up****. Now without these challenges, some of us will go there for the first time and then maybe we will be full time, or maybe we will be meeting these challenges after we go into the field. So it will be very difficult for me to polish these challenges up compared to what I would have done if I went there before I graduated."* (Extract 7.50; Focus group 2)

In extract 7.25, student-teachers point to a sense of absences (A) in the teacher education curriculum due to lack of a textbook on common learner errors in school mathematics that could be used in the training of new teachers. Moreover, a sense of absences (A) in student-teachers as a result of errors not attended to in teacher education could result in absences in the learners they are going to teach. In extracts 7.46 and 7.47, similar to teacher-educators' discourses in terms of positioning, student-teachers point to a sense of absences (A) in the teacher education curriculum in terms of focusing on learner errors specific to particular school topics in detail since it is limited to few concepts which their teacher-educators select at random. The student-teachers argue that it is important for them to be exposed to in-depth focus of working with learner errors pertaining to particular school mathematics topics because errors and their likely remediating strategies is topic dependant. Moreover, a sense of absences (A) in student-teachers is exacerbated in that it might be challenging for them to work with learner errors in their teaching, especially when they are meeting the topic for the first time.

In extract 7.50, also similar to teacher-educators' discourses in terms of positioning, student-teachers point to a sense of absences (A) in the mathematics teacher education courses in

terms of providing opportunity for student-teachers to share and clarify with each other the challenges experienced pertaining to learner errors during school teaching practice. The reason being that school teaching practice is left towards the end of the programme and thereafter student-teachers are deployed in schools as teachers. The importance they attach to school teaching practice suggests that analyzing learner errors is a practical experience as the skill is taught theoretically and then learned in the practice of teaching.

7.3.1 Summary and conclusion of the analysis

For student-teachers, LMT is about focusing on learner errors in terms of analyzing them. Similar to teacher-educators' discourses, analyzing learner errors for student-teachers involves recognition of the errors, and suggesting possible remediating strategies. Sources of errors stem from teaching, and while teacher-educators talk about inadequate teaching due to misrepresentation of mathematical structures, student-teachers talk about teachers: making shortcuts; not relating mathematical concepts to learners' prior knowledge, and the history of mathematics; and not ensuring that basic concepts are grounded in learners. LMT for student-teachers and similar to the teacher-educators' discourses also involves planning for anticipated learner errors and the importance of the teacher to know the subject matter. As a way of recognizing most errors learners make in school mathematics, student-teachers suggest getting involved in marking examinations and class work, and doing more of school mathematics.

What LMT entails for student-teachers is accompanied by positioning of student-teachers, learners and the curriculum largely in terms of absences. In student-teachers, absences are as a result of: not remembering the concepts they learned in school; teachers in school or teacher-educators for not addressing school mathematics in relational ways where reasons for some mathematical manipulations are explained; and not realizing that basic concepts and adjusting their thinking to that of their learners are crucial in addressing learner errors. Moreover, absences in student-teachers are due to absences in the teacher education curriculum; and as outlined in teacher-educators' discourses in that there is no in-depth school mathematics topic specific focus on error analysis; and that school teaching practice is left towards the end of the programme. Student-teachers also pointed to that there is no textbook on common learner errors in school mathematics to be used in training of new

teachers; and that the university mathematics they learn is far detached from the mathematics they are expected to teach in school. However, contradictory positioning of student-teachers is realized in that they see presences in student-teachers for bringing their own experiences of learner errors into planning. They also have ways of dealing with learner errors irrespective of large classes, although they again contradict themselves by saying class size matters as focus on learner errors is individualized. Absences realized in learners are due to lack of interest in learning mathematics, hence requiring self conviction that they can learn mathematics irrespective of the challenges.

From student-teachers' discussions on error analysis, LMT is a practical accomplishment and weakly classified. It is focused on during the practice of teaching such as peer teaching and school teaching practice and supported by principles that guide discussions on school mathematics.

7.4 LMT is about teacher engaging with learners' thought processes

- *What is LMT?*

Student-teachers said that LMT is about the teacher facilitating learner learning based on what the learner knows, that learners are not 'empty vessels'.

> *"I think **it is important for the teacher to know the learner's way of thinking** because then you would reason with that learner and **you could know how to direct the learner**. In other ways, you would help the learner **because the teacher's role there is to facilitate the learning process**, not that he will be able to push in knowledge, or rather pump in knowledge in the brain of the learner, no, but the idea is to facilitate learning. **So you would engage with the learner in other words, as the teacher, you would be at the level of the learner** in terms of how can I help this learner if he reaches this stage and then he is stuck, how well can I help him.*
>
> ***Ya, as earlier on stated, the learner is not an empty vessel that has to be filled.** As you stated, the learners are also thinking on their own and their thinking is what the teacher has to direct, or put in the right direction."* (Extract 7.1; Focus group 1)

As indicated in extract 7.1, learners are viewed to come with some competencies to the learning situation which the teacher engages with through reasoning with them and re-directing their thoughts in situations where they seem to lose focus. Moreover, the argument is that key for teachers to achieve their role as facilitators is to know and understand learners'

thought processes (Env). In Even & Tirosh's (2002) terms, in this way student-teachers' discussion relates to creating an environment where a teacher can listen to learners (Env). Moreover, the issue of teacher facilitation of learning was raised by teacher-educators as one of the dispositions student-teachers have to assume in listening to learners. However, there was no mention, in student-teachers' talk, of the topics in their mathematics education courses where engaging with learners' thought processes is dealt with.

- *How student-teachers say they "learn" for engaging with learners' thought processes – learners are different*

Student-teachers said that for teachers to engage with learners' thought processes they have to be aware that learners have different learning abilities needing particular attention.

> ***"**I think **as teachers we are always prepared though sometimes situations which we find in schools make us not to think about this**. For instance, you might go into a school where you have a class of 40 to 60 pupils, now for you to come up with the methods which will tackle the learning abilities of those children will be very difficult in the sense that you will not have all that time to prepare for those children but we **are always aware of the fact that children have those learning varying abilities**.* (Focus group 1)
>
> *One, as you go to handle a class, you need to prepare yourself adequately, and then **as you go to class, you should be able to realize that a class is made of individual pupils. So to say that those individual pupils have different abilities, so as you handle those pupils, what should come in mind is that those who are lagging in** whatever you are presenting should at least be given remedial work to assist them. Let us say you give an exercise, there are some who will finish that exercise much earlier than the others, but you should have something for them so that you do not disturb those who are lagging behind.* (Focus group 2)
>
> ***… we need to be ready to receive wrong answers from pupils and then assist them in any way they can** so that they learn, we need not give up as you teach this person, you ask them whether they have understood and then, maybe results show that they know nothing, so we need not to be frustrated, our work is to make them learn. … **so we have to reduce our levels and maybe speak the language they speak, be available for them.*** (Focus group 2)
>
> ***One way I would say I am prepared is in the way I have been privileged to know about the questioning skills.** … setting questions as in trying to get from them and knowing their direction of the thinking. How will I do that? It is by asking questions like: why are you thinking like that? Or why that answer? What if we did this? Well it*

is, it is a matter of putting myself in their position and trying to reason with them, that way I would engage with them. (Focus group 1)" [Extract 7.2]

As shown in extract 7.2, in general, student-teachers are aware that learners have different learning abilities and how constraining and enabling this situation is for the teacher. For example, they point to how planning for strategies of working with varied abilities is rather problematic because of large class sizes and time constraints. The strategies of working with varied learners' abilities involve three aspects. Firstly, student-teachers point to how necessary it is for the teacher to provide remedial work to those who are lagging behind, and how important it is to also include in the planning extended work for faster learners. Secondly, they point to how important it is for a teacher to exercise patience with learners' learning by working towards helping them especially when they do not demonstrate understanding of the concepts. One such strategy that would help is teacher reducing his thinking to that of learners so that they can understand each other. Thirdly, they talk of how important it would be for teachers to have good questioning skills so that through them one can illuminate learner understanding. The issue of using questions to access what the learners are thinking is also talked about in similar ways in extract 7.8. Seeing learners as individuals is one of the dispositions teacher-educators pointed to as a way of teacher listening to learners, although they did not outline what this entails in the way student-teachers have done it. This is interesting – the student-teachers give meaning to this disposition.

In talking about teacher engaging with learners' thought processes, with the understanding that learners have different learning abilities, student-teachers and learners are positioned in particular ways, and these are discussed in the section that follows:

- *Positioning of student-teachers and learners with respect to teacher engaging with learners' thought processes*

In agreeing about their awareness of teacher engaging with learners' thought processes, student-teachers said:

"I also agree with Mathias about the time factor being one of the hindrances. … you look at the secondary education, you can quite alright complete that programme but if you fail the exam, even if you know the stuff, they will say you have failed, you are a

failure. So if you had to follow to what you are saying, you may be doing grade 10 or 11 work when you are supposed, when the pupil is supposed to be sitting for the grade 12 examinations. (Extract 7.2; Focus group 1)

"Well, there is this issue of time factor as far as lessons are concerned, … most of the things we have learnt, well they have been good, … but there is this issue of preparing a lesson plan, detailed lesson plan, which is very good, but again if you look at the time that we have for lessons in class, … ***you find that you just go half way your lesson plan and time is up, ok, because you are focusing on learning, and if you try to make sure that you follow that, you find that maybe you will always be behind time, and maybe you might not cover as you are expected to cover for the term, as in per syllabus. …*** *Maybe that is one of the causes why most teachers just go and lecture in classrooms…. What we learn here, you go to class, you find something different, time, time, time always, time always is working against the teacher, I do not know. Yes."* (Extract 7.14; Focus group 1)

As shown in extracts 7.2 and 7.14, there is a sense of presences (P) in student-teachers in terms of them knowing and being aware that their role as teachers is to facilitate learner learning based on their knowledge and understanding of learners. There is an aspect of reasoning with learners' thought process which manifest in different learning abilities. Absences (A) are as a result of large class sizes and time constraints to complete the curriculum in light of learner learning. This means that if they do not manifest their role as facilitators in their teaching practices it is because they cannot rise to the challenge of dealing with large classes within the given time constraints. Moreover, student-teachers argue that assuming the role of a facilitator affects the pacing of learners' learning especially that the curriculum is specific on what work should have been covered in a particular grade level in the process of preparing learners for grade 12 examinations. This contradicts teacher-educators' teaching that LMT is not about curriculum coverage for examination purposes.

However, student-teachers suggest the following on how teachers could rise to the challenge of time without losing focus of learner learning:

"… At the same time you should just look at, fine, you have prepared the lesson plan, how much can you cover within that lesson plan, then as much as you can cover, cover as much as you can … you can give them assignments after you have done all the explanation and everything, give them assignments so that they can do it at home, and then see how the mistakes they make now, and then you deal with them in the classroom, maybe." (Extract 7.14; Focus group 1)

The indication in extract 7.14 is that one could cover as much as they can in line with what has been planned, and then the remaining work could be given to learners in form of home work or assignments, which require follow-ups on the mistakes they make.

In making reference to teachers being skillful in asking questions to elicit learner understanding, student-teachers had this to say about learners:

> *"You are fighting so hard to involve them in the learning and see how much they have understood but only a few pupils will keep on responding to the question, same pupils and the others, it is like they have not been in class or they cannot do anything about what you did. So that is why maybe that point of arranging for extra classes after the usual periods that you have in a day because definitely those 40 or 80 minutes may not be enough, you will not attend to each pupil, you will only attend to only a few who I probably will label them fast learners."* (Extract 7.2; Focus group 1)

The argument in extract 7.2 is that no matter how skillful the teacher can be in asking questions to elicit learner understanding he might not be able to reach out to all the learners due to time constraints. This is because usually in the classroom the same learners who happen to be faster learners would be the ones always responding to teachers' questions. Therefore, knowing what other learners who are not responding to teachers' questions are thinking becomes difficult for the teachers. This suggests that there is a sense of presences (P) in faster learners in terms of responding to questions that arise in the lesson and absences (A) in slow learners.

7.4.1 Summary and conclusion of the analysis

For student-teachers, LMT is about creating an environment where teacher can listen to learners through engaging with their thought processes. Focusing on engaging with learners' thought processes for student-teachers involves their awareness that learners have different learning abilities needing particular attention which require the teacher to plan for in three main aspects: remedial and extended work for slow and faster learners, respectively; exercise patience with learners' rate of learning; and have good questioning techniques to enable the access of learner thinking.

What student-teachers say LMT is and how they "learn" for it is accompanied by positioning of themselves in terms of presences (P) in that they are aware of their role of facilitating learning. A contradictory positioning of student-teachers in terms of absences (A) would be

realized if they see attending to learners' different abilities not feasible due to large classes and time constraints which would prevent curriculum completion in readiness for examinations. However, teacher-educators would argue that LMT is not about curriculum completion for examination purposes. In terms of responding to questions asked by the teacher, faster learners are positioned as having presences (P) while slow learners are positioned as having absences (A).

Moreover, what student-teachers say LMT is, how they learn for it, and the positioning are largely in terms of the dispositions teacher-educators encourage them to assume for them to listen to learners. The dispositions involve teacher as facilitator of learning; seeing learners as individuals, and it resonates with the understanding that learners have different learning abilities; and probing learners' answers and teacher asking questions from the onset which could also relate to teacher having good questioning techniques. Student-teachers have also elaborated what the role of facilitation and strategies for working with learners' different learning abilities demand of the teacher, while the teacher-educators have not done so. Similarities in positionings between teacher-educators' talk and that of student-teachers relate to absences in listening to learners due to large classes, although student-teachers also include time constraints.

Table 18 provides a brief overview of student-teachers' realizations of the discourse of engaging with LMT in terms of working with different strategies to an answer, and teaching for understanding; analyzing learner errors; and engaging with learners' thought processes.

Table 18: Synopsis of student-teachers' discourses of LMT with focus on engaging with learners' thought processes, analyzing learner errors, and working with different strategies to an answer and teaching for understanding

"What" is LMT?		**"How" of LMT, that is positionings in terms of presences and absences and the "learning" of it in terms of what student-teachers are supposed to know and be able to do.**				**"Where" in the courses/topics LMT is focused on**
	Elaboration of the "what"	**Positioning of teachers/student-teachers/teacher-educators**	**Positioning of learners**	**Positioning of the curriculum (in school or TE)**	**"learning" of LMT**	
Working with learners' mathematical routes to answers is relational. MRR-RA		Absences: in student-teachers not being familiar with some methods of solutions to school mathematical problems. A Presences: in teachers understanding multiple routes to answers would result in learners being creative. P	Presences: in learners having multiple routes to answers. P		Strategies include teacher to: • Try the method learners have used on similar mathematical problems. • Encourage learners to use specified method to a mathematical problem. • Consult colleagues in the department if not sure of the methods of solving used by learners. • Explore learners' methods used in light of the standard ways. • Consult textbook used by learner in coming up with the method used. • Ask learners to make inquiries based on the usefulness of the methods they have come up with.	
Teaching for understanding is relational MRR-TU	TU involves relating the mathematics to everyday experiences of learners; focusing on learners' misconceptions as foundational in further learning; teaching learners mathematics and not teaching mathematics; and teachers knowing and justifying each step taken.	Absences: in student-teachers explaining why some steps are taken in solving school mathematics. A Absences: in teacher-educators explaining to student-teachers why some steps are taken. A			Strategies include teacher to: • Explain and clarify each step taken in the manipulation of mathematical problems • Allow learners to verbalize so that you assess their understanding. • Avoid using shortcuts in teaching.	
Errors need to be recognized and remediating strategies sought. LEM-IR	Locate source of error in teaching.	Absences: in student-teachers - not understanding the demands of solving school mathematics A; failing to teach some school mathematics A; thinking of necessary basic concepts A; adjusting their thinking to that of	Absences: in learners due to lack of interest in learning mathematics. A Presences: in learners if confident that they can learn	Absences: in TE curriculum – no book on misconceptions in school mathematics A; no focus on specific school mathematics topic A; and teaching practice left to end of	Through that: • they should plan for anticipated learner errors. • sources of errors stem from teaching. • teachers should know their subject matter and be consistent. • teachers' recognition of errors could be enhanced by them getting involved in	School mathematics (MSE 332) Peer teaching (MSE 331) Laboratory

		learners A. Presences: in student-teachers bringing experiences of learner errors; and dealing with large classes. P	mathematics amidst challenges. P	programme A.	marking examinations and practicing more school mathematics.	sessions (practical sessions)
Engaging with learners' thought processes through teacher facilitation. Env-facilitating learning		Presences: student-teachers aware of their role to facilitate learning. P Absences: in student-teachers are due to context such as large classes and time constraints. A	Presences: in faster learners responding to teacher questions. P Absences: in slow learners, hence difficult to assess their thinking. A		Through awareness that learners have different learning abilities. Strategies include: • working with varied learner abilities in light of large class size and time constraints. • provide different work for slow and fast learners • patience with learners' learning • good questioning skills	
LMT is relational thinking involving routes to answer and teaching for understanding; working with learner errors; and working with learners' thought processes through facilitation.	*Locate source of error in teaching*	*Discourse of absences dominate with some discourse of presences in student-teachers/teachers/teacher-educators.*	*Discourse of presences as well as absences in learners*	*Discourses of absences in the curriculum of TE*	*LMT is practical accomplishment learned through:* • *establishing how authentic learners' strategies to solving mathematical problems are; and emphasize use of a method if specified;* • *explaining and clarifying; verbalization; and no use of shortcuts;* • *focus on error analysis;* • *knowledge of subject matter and consistency;* • *planning for anticipated learner errors;* • *marking examinations;* • *school mathematics;* • *practice of teaching;* • *awareness and ways of working with learners' varied learning abilities.*	*Distributed across courses – weakly classified*

7.5 Messages student-teachers recruit or take-up from teacher-educators' discourses

From analysis of student-teachers' discourses on LMT, I have shown that despite LMT being an implicit discourse, hence weakly classified and framed, they recruit or take-up some messages from teacher-educators' discourses. Although in some cases student-teachers further elaborated or give a complete different description of LMT depending on their experiences. The observation I make is despite focus being on developing in learners both instrumental and relational understanding, learner errors, and creating an environment where teacher can listen to learners. Since the messages within these categories are further spread out and in some cases cutting across categories, criteria for what counts as LMT are also weak. While there is attention to LMT in the courses and it is discussed as within the courses by both the teacher-educators and the student-teachers, neither seem to be informed directly by the literature. This suggests that the talk about error while it relates to the literature is not discursively organised. It will be interesting to see what messages student-teachers recruit or take-up from teacher-educators' and their discourses when they engage with scenarios on learner errors in school algebra, and these are discussed in Chapters 8 and 9.

CHAPTER 8

8 STUDENT-TEACHERS' DISCOURSES OF ERRORS DESCRIBED IN LITERATURE (SCENARIOS 1, 2 and 3)

8.1 Introduction

In this chapter, I present the analysis of student-teachers' discourse of engaging with LMT from the discussions I had with them in pairs as they worked with scenarios on learner errors in school algebra, and described in the specialized field of mathematics education research. The scenarios were designed to provoke discussion of learners' mathematical thinking. These scenarios included *the expression-equation problem*, *the conjoining problem*; and *the quadratic-linear equation problem*. As described in Section 5.4.1.3, the algebraic activity in the three scenarios under discussion is transformational. The purpose of the paired discussions, as already indicated in chapter 5, was to elicit student-teachers discourses of the errors in the scenarios in terms of their recognition, explanations of the sources of errors (why the errors were being made), and suggested ways of how they would help their learners overcome such errors (teachers' reasoning in deciding how to respond). The analysis of the paired interviews thus focused on how the errors in the scenarios could be described in terms of literature; *what* student-teachers recognized as errors; and *how* they recognized the errors in terms of their descriptions of the sources and suggested remediating strategies.

The possible explanations of errors in terms of literature provided are from the constructivist/sociocultural perspective. There are two possible dominant explanations of sources of learner errors emerging from the analysis of the scenarios, namely, overgeneralization and operational thinking. These are large categories that apply to a number of different circumstances. Therefore, when I discuss each of these, it is in terms of how overgeneralization or operational thinking is at work in each of the scenarios. As discussed in the theoretical chapter (chapter 4) this is in an effort to construct the possible *reservoir* for each scenario, and this is drawn from the specialized field of mathematics education research. I then explored student-teachers *repertoires* that they used as they engaged with the scenarios, including how these repertoires related to the reservoir.

In analyzing the *what* and *how* of student-teachers' realizations of learner errors, their positionings of teachers, curriculum/mathematics and learners were also taken into

consideration. Therefore, a similar frame as used in chapters 6 and 7 in terms of the *"what*" and "*how*" of learners' mathematical thinking was used. I was looking for the criteria in use, and whether these related to "is this a teaching problem", "is it a learning problem", "is it a mathematical problem", or "is it a curriculum problem". The analysis shows that student-teachers discourses are predominantly oriented towards an everyday professional experience of teaching and learning in that if teaching is right, then learning will occur, hence a transmission[17] view of teaching and learning.

8.2 SCENARIO 1: The algebraic expression-linear equation problem

The scenario read as follows:

> A Grade 8 teacher noticed that when learners were asked to simplify the expression
>
> $2x + 5 + 3x - 7$, they did the following:
>
> $2x + 5 + 3x - 7 = 0$
>
> $5x - 2 = 0$
>
> $5x = 2$
>
> $x = \frac{2}{5}$
>
> Source: Adapted from Wagner and Parker, 1993

In trying to explain why learners have difficulties distinguishing between an open expression and an equation in their early learning of algebra, Wagner and Parker (1993, p. 127) posed one of the following questions:

[17] The use of 'transmission' in this context and other occurrences in Chapters 8 and 9 refers to equating teaching to learning in that when you teach then learning is certain and not in the non-pejorative sense Bernstein (2000) uses the term together with acquisition. In Davis et al. (2007) terms, such a view of teaching and learning is at the realm of the sensible, hence a practical accomplishment.

Have any of your students ever done this?

Simplify: $2x+5+3x-7$

Solution: $2x+5+3x-7=0$

$$5x-2=0$$

$$5x=2$$

$$x=\frac{2}{5}$$

As can be observed, there is not much difference between the above layout and what is in the scenario. The procedure supposedly carried out by learners in simplifying $2x+5+3x-7$ remained the same. What was adapted is the question in that it was rephrased to read: 'A Grade 8 teacher noticed that when learners were asked to simplify the expression $2x+5+3x-7$, they did the following: ...', hence the scenario. Therefore, student-teachers' interviews were based on Scenario 1, which I have named '*the expression-equation problem*'.

8.2.1 A Constructivist/ Sociocultural interpretation of Scenario 1 and sources of error

- *What is the possible reservoir from the mathematics education literature?*

In scenario 1, the mathematical question in focus is an open algebraic expression. This, in Tirosh et al. (1998) terms, could be simplified using the method of 'collecting like terms' to obtain a simpler open algebraic expression 5*x*-2. However, there is misrecognition of the open algebraic expression as it is equated to zero, and a value of *x* found. This means that the open algebraic expression has been transformed into a linear equation. From a constructivist/sociocultural perspective, Scenario 1 can be explained as a result of a tendency by learners to overgeneralize (Ryan & Williams, 2007) or in terms of operational thinking (Sfard & Linchevski, 1994), and these are discussed in Section 3.3.3. In terms of overgeneralization, there are at least two ways in which this can happen. The one explanation is that there is a relationship between open algebraic expressions and linear equations, where for learners *x* is viewed as a "letter as specific unknown" (Hart et al., 1981, pg. 108). They see letter *x* in the open algebraic expression '2*x* + 5 + 3*x* -7' as a specific unknown that needs to be evaluated, hence the equating of the expression to zero. This suggests overgeneralization of using operations for solving linear equations to simplifying open

algebraic expressions. Lima and Tall (2008) describe such a situation as met-after, which is overgeneralization of new learning (i.e. solving linear equations) over remembering previous learning (i.e. simplifying open algebraic expressions).

A second explanation for the overgeneralization in this situation could also be because of early closure, not in the same way that you cannot leave an operation but that you cannot leave the open algebraic expression without finding what *x* is. In terms of this issue, Wagner & Parker (1993, p. 126) point to that learners see an equal sign as an operation sign "write-the-answer" sign as experienced in arithmetic, and this is overgeneralized over simplification of an open algebraic expression such as $2x+5+3x-7$. This in Lima and Tall (2008) terms suggests an issue of met-before.

Sources of the error can also be explained in terms of process-object duality (Sfard & Linchevski, 1994). This is as a result of learners' inadequate 'process' and 'structural' conception of an open algebraic expression. Operationally, learners see $2x + 5 + 3x - 7$ as requiring the process of "adding 2 times *x* plus 5 plus 3 times *x* minus 7 as 'equal-to' zero", therefore, finding the value of *x* which will make the equation true. They also do not see structure in 2x + 5 + 3x – 7 which is difference of 'sum of the product of 2 and *x*, and 5, and product of 3 and *x*' and '7', hence unable to reify an open algebraic expression.

These could be possible ways of explaining sources of the error in Scenario 1, hence the *reservoir* drawn from the specialized field of mathematics education research. Therefore, what *repertoires* of the expression-equation problem do the student-teachers bring to this *reservoir* and how do these relate to their initial expressed views on errors and misconceptions and to teacher-educators' discourses.

8.2.2 Student-teachers' recognition/misrecognition of the error

Across all 8 paired interviews on this scenario, in Jacobs et al. (2010) terms, student-teachers attended to the learners' strategy in two distinct ways. Seven (pairs 1, 2, 3, 4, 5, 6, and 7) out of the 8 pairs recognized the problem in the strategy. Pair 8 did not. Typically, explanations included:

> *"... the only problem that arose there was equating the expression now to 0. Because it's like now they have a problem now in comparing the expression and the equation.*

They do not understand the difference between the expression and the equation" (Pair 2, turn 7)[18].

"*Because in this case, we are not solving for x, we are simply finding the simple expression to this complex sort of expression. We are not solving for x*" (Pair 6, turn 13). [Extract 8.1]

As indicated in extract 8.1, student-teachers' main repertoire in recognizing the error in the scenario is that learners did not distinguish an equation from an expression, hence collapsing an expression into an equation. All the seven pairs pointed to the problem of confusing an expression with an equation in some form or another. They identified learner error in this scenario in terms of their knowledge of what it means to simplify an expression and solve an equation in that simplifying an expression means writing it in its simplest form and solving an equation means finding the value of *x*.

In recognizing the error in the scenario, in Morgan et al. terms (2002), student-teachers spoke with the voice of the official discourse of school algebra in terms of what simplifying an open expression and solving an equation mean. In terms of form of practice, student-teachers' orientation is dominantly towards mathematics (in the context of school algebra) and absences. Focus is on what the learners have done wrong with simplifying an open expression is that they have treated it like an equation to be solved, hence absences. Student-teachers also make explicit the criteria of the official discourse of school algebra in terms of distinguishing between simplifying an open expression and solving an equation.

Pair 8 is an interesting outlier in that they did not recognize the error. They said:

"I think in this scenario there is, this learner was doing fine, he was able to collect the like terms. He or she was able to add the negative numbers and the positive number. He was also able to divide. So at least there was some thinking" (Pair 8, turn 2).

"OK, the learner didn't show all the working, but the answer is correct. So maybe to start with he was supposed to collect the like terms. You said 2x + 3x (to show the teacher that you know what you are doing). 2x + 3x + 5 – 7 = 0. Then you have 5x - 2 = 0. Then if you have time you say 5x – 2 you add the additive inverse of -2 to both sides. 5x - 2 + 2 = 0 + 2. Then the negative 2 and the 2 will cancel, then your 5x = 2. Then now you divide by 5. Now he didn't even divide." (Pair 8, turn 12). [Extract 8.2]

[18] Evidence provided by an example from one pair in each case, selected for typicality across the seven interviews.

As indicated in extract 8.2, pair 8 pointed to that there was nothing wrong with what the learners did in that they were able to add and subtract like terms except that they did not show all the necessary steps in arriving at the answer. The necessary steps being referred to are the grouping of like terms, and the additive and multiplicative inverses. In a sense, the pair is just focused on the solution to an equation and not the demands of the task in terms of what actions they needed to apply on the open expression. This suggests that the context of the algebraic task was out of focus. As explained in Chapter 5, these student-teachers were given ample time to engage with the scenarios prior to the interview being conducted. Since the focus of attention was on the solution learners wrote, in a sense these student-teachers display a similar error, and so their own misrecognition on misconceptions. What they did is not helpful in terms of giving me access to student-teachers' discourses of learner misconceptions. This might not be a function of cognition, it is not that maybe they do not know the difference between an expression and an equation; they just do not focus on it.

8.2.3 Student-teachers' explanation of the sources of error

Three major categories of the sources of error recognised in the scenario emerged from the analysis. These included teaching emphasis, teaching sequence, and a problem of interpretation, and teaching sequence was seen to be a dominant discourse. I discuss each of these in turn.

- Error stems from teaching emphasis

In locating the source of the error in teaching emphasis, three pairs (Pairs 1, 4 and 7) talked of how the problem was as a result of teachers' lack of emphasis of key terms or concepts in school algebra during teaching.

> *"... I think it gets back to the teacher of not, of not emphasizing certain terms in algebra. We have terms in algebra like simplify, terms like expression, and terms like equation, terms like solve for x or solve the equation. So when pupils have no clear picture of such terms then they'll end up confusing, 'What am I supposed to do with the given algebraic expression' Because a pupil, I think for a pupil in this case maybe he thought maybe the teacher asked to solve but you cannot solve an expression which is not an equation. You can only solve something which is an equation, meaning that you need to find a value which can be replaced in those variables in order to make it, um, 'equal-to' to 0. But in this case he suggests simplifying, meaning that you have to put the like terms together, and so forth. So*

maybe it goes back to the teacher of not over-emphasizing certain terminologies in algebra." (Pair 7, turn 4). [Extract 8.3]

As indicated in extract 8.3, student-teachers' *repertoire* in locating the source of error in teaching emphasis points to how the teacher could not have emphasized key terms and concepts in school algebra. These terms and concepts include the instructions "simplify" which applies to an open algebraic expression, and "solve" which applies to a linear equation. Their argument is that if such terms and concepts are not emphasized enough by teachers then learners might end up confusing "simplifying" an open algebraic expression for "solving" a linear equation. This is because they do not understand the meaning of an open algebraic expression such as $2x+5+3x-7$ and a linear equation such as $2x+5+3x-7=0$ and the instructions associated with them such as simplify and solve, respectively. This suggests that it is not the learner's fault but that the teacher has not been introducing expressions in the right way.

Therefore, student-teachers spoke with the voice of the official discourse of school algebra in terms of the processes of simplifying an open algebraic expression and solving a linear equation. As for form of practice, student-teachers are oriented towards teaching and absences. They construct the teachers as having absences in their teaching, which in turn result in absences in learners. Moreover, locating the source of error in teaching resonates with teacher-educators' as well as student-teachers' discourses and are discussed in Chapters 6 and 7, respectively.

- Error stems from teaching sequence

In locating the error in teaching sequence, student-teachers talked of two issues: the sequencing of algebraic topics during teaching between open algebraic expressions and linear equations (Pairs 5 and 7); and learners' familiarity with solving linear equations in such a way that whenever they see x, they want to find its value, hence equating the open algebraic expression to zero (Pairs 1, 3, 4, 5 and 6).

"*Maybe even the sequencing of the, the topics in algebra. If they just started with equations, simple equation solving, that's when they go to expressions because you would think that even these expressions are equations. Maybe the sequencing should be expressions are taught first before one moves to equations"* (Pair 7, turn 6).

"You cannot start with solving equations before simplifying expressions because in solving equations there's also simplification" (Pair 7, turn 16).

"*I was just trying to measure another source of this problem. It may be that some teachers think adding like terms or simplifying by adding like terms they think it's something that pupils can do they know and maybe they may skip teaching that and go straight to equations"* (Pair 7, turn 14) [Extract 8.4]

"*... when they started dealing with equations of one variable and so on, so to them maybe they must have done quite enough of such and all that was in their minds is that when you deal with equations of one variable, all they were looking for was ... the value of x and so on. So for them, looking at this, first thing in their mind ''equal-to' 0', all of those equations, most of them they are something 'equal-to' something, and since there is nothing, you put a 0"* (Pair 1, turn 46)

"*Well normally the pupils would have known how to solve the equations. So to them an expression of this nature, what will come to mind is actually solving because they have learned about solving for x, because in most cases the questions which come, or in class we normally say solve for x. So when there's an expression which is written in terms of x, they will always want to solve for x. …"* (Pair 6, turn 17)

"… if given an expression like the way it has been given, they will think that this problem is actually not complete because it has no equal sign. So that's why they have to put that 'equal-to' zero. That's when they will say that now this thing is complete. I can nicely solve for x because they are used to solving equations which have two parts – the right side and the left side." (Pair 6, turn 19). [Extract 8.5]

As indicated in extracts 8.4 and 8.5, student-teachers *repertoires* in locating the sources of error in teaching sequence points to two issues, namely, ordering or in Ryan and Williams (2007) terms, overgeneralization, respectively. In reference to ordering, student-teachers pointed to that the teaching of equations could have superseded the teaching of open algebraic expressions, and hence learners viewing expressions as equations. They argue that the skill of simplifying expressions would aid solving equations, and hence it is necessary to focus on simplification of expressions before solving equations. Related to this source of the problem is that student-teachers also pointed out that the teachers could have assumed that if they taught equations only, learners would easily simplify given an open expression. Sequencing subject matter appropriately is one of the issues teacher-educators raised and is discussed in Chapter 6, and now student-teachers are raising a similar issue.

As for the aspect of overgeneralization, student-teachers talk pointed to the issues of met-before and met-after as discussed in the *reservoir*. In terms of met-after, student-teachers

pointed to that the source of the problem was that the sight of x compelled learners to solve. The student-teachers, in Hart et al. (1981, p. 108) terms, pointed to the issue of learners seeing x as "letter as specific unknown" that needs to be operated on and eventually evaluated. This suggests overgeneralization of using processes for solving linear equations over simplifying open algebraic expressions. There is interference between learners' knowledge of solving linear equations and their remembering of simplifying open algebraic expressions, hence met-after. As for met-before, in Wagner and Parker (1993) terms, student-teachers pointed to how learners could have seen the equal sign as an operation sign "write-the-answer". For learners to write-the- answer in this case, which is finding the value of x, they decided to equate the expression to zero. This suggests that the overgeneralization is due to early closure that is the need to produce an answer, which is the value of x. What student-teachers did not mention here, which Wagner and Parker pointed to is that this view could be as a result of what learners experienced in arithmetic where writing the answer was the norm, hence met-before.

In the case of ordering, student-teachers spoke with the voice of the official discourse of school algebra in that simplifying open algebraic expressions should be taught first before solving linear equations, and indicated in the school algebra curriculum in Chapter 2. As for form of practice, student-teachers are oriented towards teaching and absences. Teachers have been criticized for failing to make decisions on what to introduce first to learners between simplifying expressions and solving equations. They were viewed as introducing solving linear equations before simplifying open expressions, or completely ignored the latter and instead focused on the former. This in turn could have resulted in absences in learners. It was not the learner's fault here but it was that the sequencing had not been orderly. The student-teachers see what is absent in the learners is as a result of what could be absent in the curriculum. Moreover, criteria of the official discourse of the process of simplifying open expressions were made explicit when student-teachers talked about "*simplifying by adding like terms*".

As for the case of overgeneralization, student-teachers spoke with the voice of the unofficial discourse. Their descriptions of the sources of error as a result of overgeneralization were in terms of met-before or met-after, although they did not mention these terms. This suggests that their talk was in their everyday professional experience of teaching and learning. In terms of form of practice, student-teachers are oriented towards learner and presences. They

pointed to how learners' knowledge of related concepts such as solving a linear equation, and equal sign as an operation sign 'write-the-answer' could have had influence on how learners simplified open algebraic expressions. Moreover, criteria of the official discourse of simplifying open algebraic expressions are neither implicit nor explicit, as it is just not talked about when referring to overgeneralization as the source of error.

- Error stems from interpretation problem

In locating the error in a problem of interpretation, student-teachers talked of two issues: how learners could have had problems with the interpretation of concepts such as a linear equation and an open algebraic expression (Pairs 1, 2, 3, and 6) and the interpretation of the implication ($\Rightarrow$) symbol (Pair 7).

> *"The concepts of mathematics, interpretation of the concepts, which one is an equation, which one is just an expression or which one is just a statement and so on, ... Not understood at all. It's a misinterpretation differentiating from just an expression and an equation, what we mean an equation is*" (Pair 1, turns 29 and 33).
>
> "*Well, when we say an expression, in mathematics it's a, in this case it's, it may mean any mathematical statement – that's just the basic meaning first. But then whenever we include the equal sign to that mathematical statement, we make it an equation. ... It means the left hand side equals the right hand side. Something on the left hand side must 'equal-to' something on the right hand side ...*" (Pair 1, turns 35 and 41).
>
> "*It could be, ja it could be their understanding of the word 'simplify' because with some pupils when they hear 'simplify' they think it's evaluate or find the value. That's how they interpret it, they interpret it. That's how they understand it in other words*" (Pair 6, turn 15) [Extract 8.6]
>
> "*Then the other thing is when the teacher is maybe demonstrating on the board when you are on simplification, they use the symbol, for example, what we have here is 2x + 5 + 3x -7, when it goes to the next stage they use this symbol 'which implies' which is 5x – 2. Then they continue using that symbol. This symbol is interpreted to mean 'equal-to'. This could be what leads to such a problem. They may not understand what the teacher is doing as he is moving from one stage to the next, using the symbol 'implies that', the first stage implies the second stage and so on. In the end they may confuse this to be ''equal-to''. When they are given an expression, they will equate to zero and solve*" (Pair 7, turn 23) [Extract 8.7]

As indicated in extracts 8.6 and 8.7, student-teachers *repertoires* pointed to a problem of interpretation between an open algebraic expression and a linear equation, and between an 'implication' symbol and an ''equal-to'' (=) symbol, respectively. In terms of a problem of

interpretation between an open algebraic expression and a linear equation, student-teachers pointed to how difficult it could have been for learners to understand the difference in meaning between the two concepts. Student-teachers talked of how both are mathematical statements, except that the linear equation is equated to something. In Wagner and Parker (1993) terms, student-teachers' explanation of the ''equal-to'' sign for a linear equation is described in a relational way in that the right hand side is 'equal-to' the left hand side. Moreover, student-teachers also pointed to that there could have been a tendency among learners to interpret 'simplify' as requiring them to 'solve' for a letter which is unknown.

Therefore, student-teachers spoke with the voice of the official discourse of school algebra in describing the difference between an open algebraic expression and a linear equation. In terms of form of practice, student-teachers are oriented towards mathematics (school algebra) and absences. Learners did not understand the difference between the concept of an open algebraic expression and that of a linear equation. Moreover, student-teachers make implicit criteria of the official discourse of simplifying an open algebraic expression since they only pointed to how it is a mathematical statement which has not been equated to anything. There is no mention of the process of carrying out the simplification.

As for a problem of interpretation between an implication symbol and an 'equal-to' symbol, student-teachers pointed to how the teachers could have demonstrated to their learners that the implication symbol in the process of simplifying is an indication of a shift from one stage to another. But this use of the implication symbol could have been misinterpreted by learners to mean 'equal-to', hence equating the expression to zero and solving for *x*. It is interesting to note that the student-teachers' description of how teachers could have used the implication symbol is also problematic. The underlying mathematical meaning of the implication symbol is that it is used in comparing two propositions. For example, if P implies Q where P is x =2 and Q is x^2 = 4, then $x = 2 \Rightarrow x^2 = 4$, and if P is true, then Q is also true. Alternatively, if P implies Q and P is $(y-0)=2(x-1)$ when Q is $y=2x-2$, then $(y-0)=2(x-1) \Rightarrow y=2x-2$, and if P is true then Q is also true.

Therefore, to say $2x+5+3x-7 \Rightarrow 5x-2$ is mathematically problematic since both sentences are not propositions as there is insufficient information to warrant truthfulness for each. This suggests that how the implication symbol is used in school is a misapplication because teachers know that $2x+5+3x-7 \Rightarrow 5x-2$, hence reflecting the misuse in school

mathematical practices in Zambia[19]. Teachers cannot be blamed for doing this if it is part of the school culture. However, for student-teachers to have suggested that learners could have had a problem of interpretation between implication symbol and 'equal-to' symbol, is exhibiting an error of their own. I could not further interpret this in terms of voice and orientation.

8.2.4 Decisions about remediation student-teachers suggested

Student-teachers talked about two ways in which the error recognized in Scenario 1 could be remediated. The two ways included emphasizing key concepts or terms during teaching; and encouraging learners to practice simplifying expressions more often with the former being the most dominant.

- Emphasizing key concepts or terms during teaching

In suggesting emphasizing key concepts during teaching, seven pairs (pairs 1, 2, 3, 4, 5, 6, and 7), talked of the importance of explaining the meaning of key concepts or terms so that the difference between an open algebraic expression and a linear equation is emphasised. This relates to one of the sources of the error earlier discussed of the teacher not emphasizing key concepts.

> "*I think one way of helping the learners is to emphasize the difference between expressions and literally the equations, or in this case the one which was given was an expression. Now the learners started what, solving an equation. So it's just important for you to explain the difference between an expression and the equation, so much so that they don't mix the two, because once they mix then they're wrong"* (Pair 1, turn 54).
>
> "*If I was the teacher, I think other than just writing to say 'simplify the expression', I think even verbally I can tell them to say we are not doing this, but we just want to make these complicated expressions simpler. We are not solving them whilst finding the values for the given letters, but we just want to make them simpler"* (Pair 4, turn 36).
>
> *"... it would actually help to do one where you simplify and say if we were to solve for it this is what would be there. There has to be an equal sign because then that's the sure way of saying we need to, the x to be 'equal-to' something. So I think I would*

[19] Here I am drawing on my experience that this is a common practice among most school mathematics teachers in Zambia, but I do not know elsewhere.

start with the simplification part. I'd simplify it or maybe with them we do the simplification part." (Pair 4, turn 37).

***I**:Can you give examples for that?* (Pair 4, turn 38)

"Ya, the same one of 2x + 5 + 3x – 7 on one side. So I say, 'Let's simplify this together.' Then we simplify it. So it will come to um 5x +, I mean -, -2. Then I say the same one if I was to equate it to something then I say we solve for x. Suppose I equate it to 0, we solve for x, what do we get? Then at some point we wanna get x 'equal-to' something. Then I say, OK, on this side what we are doing is we are solving for x. On this side what we are doing is we are simplifying, and not really solving for x" (Pair 4, turn 39). [Extract 8.8]

As indicated in extract 8.8, student-teachers' *repertoires* in suggesting emphasizing key concepts or terms during teaching as one of the remediating strategies pointed to two main ways on how the distinction between an expression and an equation can be made. One way would be to tell the learners that what they are doing by solving for *x* is wrong, and the correct way is to simplify by expressing the given open algebraic expression in its simplest form. In Borasi's (1987) terms, this suggests the contested strategy of simply explaining the same topic over again in that it does not get errors fixed, and this is discussed in Section 3.3.3. The other way is to contrast open algebraic expressions with what they are not, that is linear equations so that learners can discern the critical features of an expression. This in Ryan and Williams (2007) terms suggests creating a "cognitive conflict" from the psychological perspective by using examples and counter-examples. They argue that this is one way of addressing learners' errors as a result of prototypical thinking. An example of error as a result of prototypical thinking would be such as one explained above in section 8.2.3 where sight of *x* compels learners to solve for *x*. However, constructivists argue that strategies for remediating learners' errors should go beyond just explaining correct ideas to learners and creating a "cognitive conflict" (Sasman, Linchevski, Olivier, & Liebenberg, 1998).

Student-teachers spoke with the voice of the official discourse of school algebra in expressing the need for teachers to emphasize what it means to simplify an open algebraic expression, and to solve a linear equation. In terms of form of practice, student-teachers are oriented towards teacher and presences[20]. They point to that teachers ought to incorporate in their

[20] As indicated in Chapter 5, orientation towards teacher and presences is about the reasoning that teachers use in deciding how to respond based on learners' understanding.

teaching explanations on the distinction between an open algebraic expression and a linear equation. Their argument is that such a strategy will enable learners to develop understanding of what simplifying an open algebraic expression means, and not confuse it with solving a linear equation. Moreover, criteria of what it means to simplify an open algebraic expression and solve a linear equation are made explicit.

- Provide learners with practice exercises

Only pair 7 talked of the importance of giving learners opportunity to practice more often on how to simplify open algebraic expressions.

> *"I think it's practice. If they practice more in the simplification of algebraic expressions and then ..."* (Pair 7, turn 18).
>
> "***I:*** *What do you mean when you say 'practice'? You give them 20 questions?*" (Pair 7, turn 19)
>
> *"It's not necessarily giving them 20 questions at a go, it's basically the practice. They can do 4 questions today, another day they if they are given a period in which they practice so that when they go to equations they will never confuse expressions and equations because the knowledge on simplification was consolidated"* (Pair 7, turn 20). [Extract 8.9]

As indicated in extract 8.9, student-teachers' *repertoire* pointed to that to remediate the error recognized in the scenario, learners should, at appropriate time intervals, practice simplifying of algebraic expressions. The teachers ought to provide their learners with such opportunities. It is like the "practice makes perfect" analogy in that if learners consistently practice simplifying of algebraic expressions, the concept will be consolidated, and hence they will not confuse expressions for equations because more time has been invested in practicing. Put differently, in Arcavi's (1995, p. 149) terms, it is related to "the harmful advocacy of the drill-and-practice rationale of learning mathematics where the concern is: Let's take care of the drill, meanings will emerge from practice". Doerr & Wood (2004) say that student-teachers come to the initial teacher training with such understanding. Borasi (1987) would argue that giving learners additional practice exercises does not get errors fixed but rather view errors as *springboards for inquiry*, and this is discussed in Sections 3.3.3 and 3.3.5.

Constructivists would argue that the strategy is short term in that it addresses the error directly and not the reason for the error made. In Skemp's (1976) terms, this strategy suggests developing in learners instrumental ways of reasoning in that if learners do more exercises

they will acquire the rules for simplifying algebraic expressions without necessarily understanding the underlying meanings to the rules. The concern here is with learner productions rather than learner thinking, and the orientation is inclined to whether the learners have got the answers right. Although Arcavi (1995) problematizes the issue of giving learners practice exercises as a strategy of addressing learner errors, he agrees with Sfard in seeing potential of such practice in enhancing the learning of school mathematics if done with a different foci as described in Section 3.3.3. Therefore, it is interesting how student-teachers suggest the issue of practice exercises although in a limiting way.

However, student-teachers spoke with the voice of the official discourse of school algebra in that the curriculum texts (for example textbooks) do provide a range of practice exercises which learners could engage with so as to enhance their understanding of concepts. In terms of form of practice, student-teachers are oriented towards teacher and presences. The teachers ought to be incorporating in their teaching opportunities for their learners to practice simplification of open algebraic expressions. Their argument is that if learners practiced more often they would consolidate their understanding of concepts in such a way that confusing them would not be an issue because they know what they are doing. In talking about practice exercises, student-teachers make implicit criteria of the official discourse of the process of simplifying open algebraic expressions.

8.2.5 Summary and conclusion of student-teachers' repertoires on Scenario 1

As can be seen in Table 19, in column 1, all the eight pairs of student-teachers that were interviewed are numbered from 1 to 8. Positionings are described for each of the steps of carrying out error analysis, which are recognize/misrecognize the error (column 2), explanation for the sources of error (column 4), and suggested remediation strategies (column 6). The positionings are in terms of "voice" and "form of practice" as shown in columns 3, 5 and 7. In terms of "voice", it is either they speak the official discourse of school algebra (OD) or unofficial discourse (UD) which is in terms of the everyday professional experience of teaching and learning (E) or discursive or theoretical (D/T – in the specialised language of mathematics education research). As for "form of practice" the orientation is either towards teaching and absences (TA) or learner and presences (LP) or mathematics and absences (MA) or teacher and presences (TP). The design of the tables is consistent throughout the scenarios under discussion in both Chapters 8 and 9, and they resemble Table 19.

Table 19: Synopsis of student-teachers' reasoning processes to the expression-equation problem

Pairs	Recognition (R)/ Misrecognition(M)		Positionings		Explanation for the sources of error					Positionings						Remediation		Positionings	
	R	M	Voice of ...	Form of practice	Teaching Emphasis (concepts and terms)	Teaching sequence			Interpretation problem (expression and equation√; implication symbol√)	Voice of ...			Form of practice Oriented towards:			Emphasis in teaching (distinction between concepts and terms)	Practice exercises	Voice of ...	Form of practice
			OD	OMA		Ordering	Overgeneralization			OD	UD		TA	LP	MA			OD	OTP
						(equations before expressions)	**Met-before** (equal sign as an operation sign write-the-answer)	**Met-after** (solving linear equations)			E	D/T							
1	√		√	√	√		√	√	√	√√	√√		√	√√	√	√		√	√
2	√								√							√			
3	√							√	√							√			
4	√				√			√								√			
5	√					√		√		√			√			√			
6	√						√	√	√							√			
7	√				√	√			√							√	√	√	√
8		√																	

As can be seen in column 2, under recognition 7 out of the 8 pairs of student-teachers interviewed did recognize the error in the scenario while 1 pair misrecognised it. The seven pairs who recognised the error stated that the error was as a result of learners equating the open algebraic expression to zero. In recognizing the error, positionings in column 3 are that the 7 pairs spoke with the voice of the official discourse (OD) of school algebra in terms of what simplifying an open algebraic expression and solving a linear equation means. The former is about expressing in its simplest form and the latter is about finding the value of *x*. As for form of practice, student-teachers are oriented towards mathematics (school algebra) and what the learners did wrong with simplifying an open algebraic expression is that they equated to zero and solved for *x*, hence absences (OMA).

Three sources of the error were identified and are presented in column 4, namely, teaching sequence, interpretation problem and teaching emphasis listed in the order starting with the most dominant. With reference to teaching sequence, student-teachers pointed to two issues, which are ordering (pairs 5 and 7) and overgeneralization (met-before: pairs 1 and 6; met-after: pairs 1, 3, 4, 5 and 6). In terms of ordering, student-teachers spoke with the voice of the official discourse (OD) of school algebra in that simplifying open algebraic expressions is first introduced to learners before solving linear equations (see column 5). As for form of practice, student-teachers are oriented towards teaching and absences (TA). The teachers were criticized for maybe having taught linear equations before introducing their learners to simplifying open algebraic expressions. Moreover, student-teachers said that teachers could have completely ignored in their teaching to introduce simplifying open algebraic expressions and only focused on solving linear equations.

As for overgeneralization, student-teachers pointed to issues of met-before and met-after without necessarily using these terms. This suggests that their talk was in their everyday professional experience of teaching and learning (see UD-E in column 5). In referring to met-before, student-teachers talked of how learners could have seen an ‘equal-to’ sign as an operation sign ‘write-the-answer’ sign, and hence equated the expression to zero to enable them solve for *x*. As for met-after, student-teachers pointed to how learners’ learning of solving linear equations could have interfered with learners’ remembering of simplifying open algebraic expressions. The issues of met-before and met-after suggest an orientation towards learner and presences (see LP in column 5). Learners were not seen as empty vessels but that they had prior learning, though constraining, of ‘equal-to’ sign as operation sign

‘write-the-answer’ sign and solving linear equations, which they came with to the learning of simplifying open algebraic expressions.

The second dominant source of error is a problem of interpretation, and pairs 1, 2, 3 and 6 pointed to the issue of expression and equation, and pair 7 pointed to the issue of implication symbol. Student-teachers spoke with the voice of the official discourse of school of algebra in explaining the difference between an open algebraic expression and a linear equation (see OD in column 5). In terms of form of practice, student-teachers are oriented towards mathematics (school algebra) and absences (see MA in column 5). Learners were criticized for interpreting an open algebraic expression wrongly. They could not have understood how an open algebraic expression is different from a linear equation. This suggests that the error is as a result of learning, that it is a problem with learners’ understanding.

The third source of error is located in teaching emphasis, and pairs 1, 4 and 7 referred to this issue. Student-teachers spoke with the voice of the official discourse of school algebra in referring to concepts and terms such as simplify, open algebraic expression, solve, and equation (see OD in column 5). In terms of form of practice, student-teachers are oriented towards teaching and absences (see TA in column 5). Teachers have been criticized for not emphasizing in their teaching key terms or concepts such as simplify, open algebraic expression, solve, and linear equation.

What is interesting is that only three pairs located source of the error in teaching, but when it came to suggesting remediating strategies (column 6), all the seven pairs pointed to how emphasis in teaching was critical. Only one pair suggested the importance of practice exercises. As can be seen in column 7, in suggesting remediating strategies, student-teachers spoke with the voice of the official discourse of school algebra (OD). They talked of the process and what it means to simplify open algebraic expressions and linear equations; and exercises that are made available for learners to practice.

In terms of form of practice, student-teachers are oriented towards teacher and presences (see OTP in column 7). Teachers ought to incorporate in their teaching descriptions of both the difference between simplifying an open algebraic expression and solving a linear equation, including the required processes. This, student-teachers argue would enable learners make distinctions between particular concepts and the related terms used. Moreover, student-

teachers pointed to how teachers need to incorporate in their teaching practice exercises for the consolidation of concepts in learners.

For some student-teachers to have suggested emphasis in teaching as one of the remediating strategies when the source of error was not located in teaching suggests that they are distancing themselves from teachers taking responsibility, not of teaching but of learners' errors. Moreover, as previously in teacher-educators' discourses, the transmission view of teaching and learning is also dominant. Student-teachers are saying that learners will learn if the teacher emphasizes key concepts by explaining and contrasting, sequence their instructions appropriately, and give their learners frequent practice exercises. If learning does not occur on the part of the learner then it means that the teacher did not do what is expected of him or her. Student-teachers also recruit or take-up from teacher-educators' discourse and their initial discourses an understanding that sources of errors are in teaching, and the importance of sequencing subject matter appropriately. Therefore, their talk is in the everyday professional experience of teaching and learning.

This conception of learning contradicts the earlier issues discussed of overgeneralization, and a problem of interpretation where learners were seen to come with some understanding, though partial, to the learning of simplifying open algebraic expressions. This suggests that student-teachers' conceptions of learning are contradictory, which is quite interesting for this scenario. It would be interesting to see what happens with the other scenarios. As for the outlier, the question I would ask is: What does it mean when student-teachers continue to hold similar misconceptions well into their final year of study?

8.3 SCENARIO 2: The conjoining problem

The original fragment was used to illuminate teacher Benny's strategy of how he taught simplification of open algebraic expressions to his learners, and how he was unaware of learners' tendency to conjoin or 'finish' open expressions, and it reads as follows:

> Teacher Benny writes the expression $3m + 2 + 2m$ on the board and asks: 'What does this equal to?' Confirming immediately with the rule: 'Add the numbers separately and add the letters separately.' Then he suggests coloring the 'numbers': $3m + \mathbf{2} + 2m$, and writes $5m + 2$. A student asks: 'And what now?' Another student suggests: '$7m$'. The teacher (rather surprised by this answer) says: 'No! $5m + 2$ does *not* equal $7m$.' And he repeats the rule again: 'The rule is: add the numbers separately and add the

letters separately.' Then he gives the students another example and colors the (free) numbers: $4a + \mathbf{5} - 2a + \mathbf{7}$. The teacher emphasizes the rule by dictating it to the students and asking them to repeat it out loud. The rest of the lesson is devoted to work on similar exercises. The students continue to experience difficulties. For example, towards the end of the lesson, a student asks: 'Why doesn't $10 + 2b$ equal $12b$?' The teacher does not respond to the question. (Tirosh, Even, & Robinson, 1998, p. 55)

The scenario as I constructed it reads as follows:

Teacher Benny engaged his class in learning algebraic expressions.

He worked with the rule "add the numbers separately and add letters separately"

Benny: Asks what $3m+2+2m$ equals?

Benny: Reads the rule.

Benny: Suggests colouring the 'numbers' and writes $5m+2$.

L_1: And what now?

L_2: $7m$

Benny: No $5m+2$ does not equal $7m$, and repeats the rule again.

Benny: Gives another example $4a + \mathbf{5} - 2a + \mathbf{7}$, colours free numbers, dictates the rule to the class and then asks the class to repeat the rule.

Learners: Work on similar exercises and continue experiencing difficulties.

Source: Adapted from Tirosh, Even and Robinson, 1998, also reported in Even and Tirosh, 2002

In this scenario teacher Benny worked with his learners on simplification of expressions and in particular $3m+2+2m$. The learners were not convinced that $5m+2$ is the simplified form of the given expression despite the teacher using the two strategies to explain that of the rule and that of coloring. Instead they further simplified $5m+2$ and obtained $7m$. The scenario still resembles the original extract, except for some rephrasing and editing in some places. I have named the scenario "*the conjoining problem*" and student-teachers' interviews were based on it.

8.3.1 A Constructivist/Sociocultural interpretation of Scenario 2 and sources of error

- *What is the possible reservoir from the mathematics education literature?*

The mathematical question provided for in scenario 2 is an open algebraic expression, which requires simplifying. The operations involved in simplifying expressions are adding or subtracting of like terms, which could result in a simplified form of an algebraic expression. To facilitate this process, teacher Benny decided to work with two strategies at the same time, that of the rule "add the numbers separately and add letters separately" and colouring free numbers. As described by Tirosh et al. (1998), the dominant issue in this scenario is that of learners' tendency to conjoin or "finish" open expressions. They suggest that this is one of the well documented difficulties learners experience when learning algebraic expressions. The authors are aware that the phrasing of the rule by teacher Benny is a problem because $5m+2$ could yield $7+m$ or $7m$ among several variations that could arise. Moreover, teacher Benny as a novice teacher was not aware that learners had a tendency to conjoin or "finish" open expressions as shown in the interviews and in the planning of his lessons. As a result, when learners gave an incorrect answer, he stated that it was wrong and emphasized the rule. According to Tirosh et al. (op cit), teacher Benny's approach symbolizes some version of "collecting like terms" method in simplifying algebraic expressions. However, there was some resistance from his learners to accept an expression that had a '+' operational symbol between terms, for example $5m+2$, as a final answer.

As discussed in Section 8.2.1, two broader dominant ways of explaining the tendency by learners to conjoin or 'finish' open expressions from a constructivist/sociocultural perspective are identified, namely overgeneralization (Ryan & Williams, 2007) and operational thinking (Sfard & Linchevski, 1994). According to Even & Tirosh (2002), the issue of overgeneralization could be described in three specific ways. Firstly, there is a relationship between conjoining and adding. This suggests overgeneralizing from wanting to use conventions for conjoining on adding terms in open expressions. The example here is that provided by Stacey & MacGregor (1994) that learners could draw on their knowledge from other school subjects, such as Chemistry where $C + O_2 \rightarrow CO_2$, to their work on algebraic symbols, hence an issue of met-before.

Secondly, in natural language as indicated by Tall & Thomas (1991), there is a relationship between the meanings of *and* and *plus*. This also suggests overgeneralization "to consider *ab*

to mean the same as *a* + *b* because the term *ab* is read as *a and b* and interpreted as *a* + *b*" (Even & Tirosh, 2002, p. 221). The overgeneralization, in Lima & Tall (2008) terms could also be considered as a result of met-after. There is a relationship between multiplication and addition of terms. This suggests that learners could relate operations meant for multiplication of terms, for example $5m \times 2 = 10m$, to the addition of terms ($5m + 2 = 7m$). That is, since numbers are being multiplied; one can also go ahead and add numbers. This means the tendency by learners to overgeneralize new learning (i.e. multiplication of terms) over remembering previous learning (i.e. addition of terms).

Thirdly, there is a relationship between arithmetic and algebraic expressions, hence overgeneralization from wanting to use operations for solving arithmetic expressions to simplifying open algebraic expressions, met-before in Lima & Tall (2008) terms. This according to Booth (1988) is as a result of cognitive difficulties in learners to accept lack of closure and see open expressions as something further needing to be done (i.e. seeing open expressions as incomplete). Moreover, there is an expectation that a "final single termed answer" be obtained. This could be because of interpreting the operational symbol '+' in arithmetic terms as actions further needing to be performed, hence conjoining the terms. In Hart et al. (1981, p. 108) terms, the issue of conjoining could also be described as "letter not used". This suggests overgeneralization of number into algebra where learners would see meaning in the numbers, for example 5 and 2 in the expression $5m + 2$, and then 'properly' combine but leave the letter as it is or completely ignore it, hence getting 7m or 7 as the answer, respectively. In Herscovics & Linchevski (1994) terms, the notion of '*cognitive gap*' is used to describe the difficulties learners experience as a result of the movement from arithmetic to algebra.

The tendency by learners to conjoin or 'finish' open expressions can also be explained operationally in terms of process-object duality (Sfard & Linchevski, 1994). The expression $5m + 2$ is usually seen by learners as a process to be performed "adding five times *m* and 2", that is requiring carrying out the addition operation to come up with one thing 7*m*. This is similar to seeing '+' as actions to be performed as already discussed. It is not seen as an object "sum of the product of 5 and *m*, and 2". Therefore, the learners could not have attained an object conception of the open algebraic expression $5m + 2$. They are not able to

reify, that is failure to move from seeing an open expression as operational to seeing its structure.

Drawing from the mathematics education literature, it can be shown that the possible *reservoir* from the mathematics education literature in describing the conjoining problem from the constructivist/cognitive perspective could be broadly in two ways: as a result of overgeneralization and as a result of inflexible operational thinking. Overgeneralization includes conversions of conjoining used in other subjects or mathematics topics; the meaning of *and* and *plus* in natural language; and using operations for solving arithmetic expressions due to met-before, early closure, seeing '+' as operational, and letter not used. The term *cognitive gap* is also recognized to describe this difficulty in moving from arithmetic to algebra. Within algebra, overgeneralization in terms of met-after of multiplication of terms over addition of terms could also be another possible way of describing the conjoining problem. Operationally, the conjoining problem is described in terms of the process-object duality, and conclusions are made that learners could not have attained the object conception of the open algebraic expression. These could be ways of explaining the error in this scenario. Therefore, what *repertoires* of the conjoining problem do the student-teachers bring to this *reservoir* and how do these relate to their initial expressed views on LMT and to teacher-educators' discourses.

8.3.2 Summary and conclusion of student-teachers' repertoires on Scenario 2

The process of analysis follows that of Sections 8.2.2, 8.2.3, and 8.2.4 and the detail, in terms of extracts and evidence is located in appendix F. Here, the summary and conclusion of analysis follows. As already mentioned in Section 8.2.5, Table 19 is the same in design as Table 20 that follows.

Table 20: Synopsis of student-teachers' reasoning processes to the conjoining problem

Pairs	Recognition (R) of error	Positionings		Explanation for the sources of error							Positionings							Remediation	Positionings		
		Voice of …	Form of practice	Teaching emphasis (TE)				Teaching sequence (TS)			Voice of …			Form of practice: Oriented towards …				Emphasis in teaching	Voice of …		Form of practice: Oriented towards …
		OD/UD(E)	OMA	Rule: not clear√; ambiguous√	Elaboration (DM of term √;recognition of like terms√; concrete objects√	Teaching not interactive	Inadequate probing	Ordering	Overgeneralization		OD	UD		TA	MA	LP	Same rule (definition of term) √; different rule (concrete objects√, language√); meaning of '+' in algebra√	OD	UD	TP
									Met-before: Add or subtract nos.√; Sets√	Met-after: Multiply algebra terms		E	D/T						E	
1	√	√	√	√√	√√	√		√	√√		√√	√√		√√√		√	√√		√	√√√√
2	√			√√	√√				√								√√	√		
3	√			√√	√√		√		√								√			
4	√			√√	√√				√								√			
5	√			√	√			√									√			
6	√			√	√√		√	√	√	√							√			
7	√			√	√√												√ √	√		
8	√			√√	√√												√			

As shown in table 20, all the 8 pairs recognized the error in the scenario. They provided varied explanations for the sources of error and made suggestions for remediating strategies. Their *repertoires* in recognizing the error pointed to the method of collecting like terms. They talked of how learners were not able to distinguish between like and unlike terms. In recognizing the error, they spoke with voice of the official and unofficial discourse (OD/UD). In terms of voice of the official discourse (OD) of school algebra, they pointed to rules that govern basic processes of algebra in that what gets added when simplifying open algebraic expressions are like terms.

As for unofficial discourse (UD), student-teachers' descriptions of their recognition of the error pointed to terms developed in mathematics education literature and discussed in the *reservoir* such as conjoining or 'finish' open expressions, 'letter not used', seeking closure, and seeing '+' as operational. However, the student-teachers did not use the exact terms, again an indication that they drew their resources from their everyday professional experience of teaching and learning (E). In terms of form of practice, student-teachers are oriented towards mathematics and absences (OMA). They blamed learners for what they did wrong with simplifying open algebraic expressions in that they added unlike terms to come up with one single term as the simplified form, contrary to the rules of mathematics. Moreover, criteria of the official discourse of simplifying open algebraic expressions are explicit as student-teachers stated that it is about identifying and grouping the like terms and adding them.

Similar to Scenario 1, in explaining reasons for the error, student-teachers pointed to two broader categories, namely, teaching emphasis and teaching sequence where the former is dominant. In terms of source of error being as a result of teaching emphasis, student-teachers spoke with voice of the official and unofficial discourse in terms of mathematical rules and their elaboration, and teaching approaches, respectively. In talking about mathematical rules and their elaboration, student-teachers pointed to how the rule teacher Benny used was inadequate to enable learners distinguish between like and unlike terms. They suggested how the definitional meaning (DM) of a term ($5m = m + m + m + m + m$), ways of recognizing like terms (terms with same variable raised to the same power), and use of concrete objects could have been useful ways of elaborating the rule. However, use of concrete objects to represent letters in school algebra is contested in mathematics education literature as it tends to obscure the meaning of letters. According to Tirosh et al. (1998, p. 55), this is referred to as the 'fruit

salad' teaching approaches, and is discussed in Section 3.2.3. This suggests that student-teachers exhibited an error of their own because 5m means m + m + m+ m + m and not 5 spoons which 'means' spoon + spoon + spoon + spoon + spoon. Moreover, Hart et al. (1981, p. 104) reported that this is one of the problems learners in their early learning of algebra experience, hence the notion of "letter used as an object", and is discussed in Section 3.4.3. Therefore, using concrete objects as a resource would also create other problems.

As for unofficial discourse, student-teachers talked of how dominance of teacher talk, and inappropriate teacher response to learner answers could inhibit learning. They explained that what gets lost when the lesson is dominated by the teacher is access to what the learners are thinking about teacher explanations. Therefore, similar to teacher-educators' and their initial discourses, student-teachers argue for an interactive approach to teaching as it provides opportunity for the teacher to assess learner learning including their misconceptions and seeks ways of remediating before making progress in teaching. This, they said, could be made possible by encouraging learners to talk and ask questions based on teacher explanations.

In referring to inappropriate teacher responses to learner answers, student-teachers pointed to the importance of probing learners' responses so as to access learner reasoning processes. They suggested that if teacher Benny had done so, he was going to realize that there was a problem in the way learners were interpreting the rule, and maybe could have thought of other ways of elaborating. Teacher-educators also talked of the issue of probing learners' answers. Therefore, in talking about the effects of teacher dominance and not probing learners' responses, student-teachers again drew on their everyday professional experience of teaching and learning.

As for form of practice, student-teachers are oriented towards teaching and absences (TA). Learners could not distinguish between like terms and unlike terms because the teacher did not pay attention to the mathematical rule and its elaboration, and to the teaching approaches. The rule lacked clarity and was ambiguous, and there was no adequate elaboration. A sense of absences is also realized in these student-teachers in suggesting use of concrete objects as one of the ways of elaborating the issue of like terms and unlike terms. Moreover, the teaching could have been dominated by teacher talk and the teacher could not have probed learners' responses. This again suggests a transmission view of teaching and learning in that if teachers use rules that are clear, unambiguous and elaborated, make their lessons

interactive and probe learners' answers then learners will certainly learn. Therefore, student-teachers view of teaching and learning is in their everyday professional experience of teaching and learning than in the language of research. However, criteria of the official discourse of simplifying open algebraic expressions are explicit in terms of how like terms could be recognized using the definitional meaning and the notion that they have same variables that are raised to the same power.

The second source of error is that it is as a result of the teaching sequence talked about in terms of ordering and overgeneralization. In referring to ordering, which also translates into locating the error in teaching, student-teachers spoke with voice of the official discourse of school algebra in that recognition of like terms is first introduced to learners before addition or multiplication of terms. As for form of practice, student-teachers are oriented towards teaching and absences. Learners made the error because the teacher could not have ensured that learners understand the concept of like terms before addition or multiplication of terms. They argued that this is because the recognition of like terms informs how the addition and multiplication of terms is carried out. Therefore, similar to student-teachers' initial discourses, this is about teachers ensuring that learners understand the prerequisite basic concepts prior to the concept being introduced. Teacher-educators talked of the importance of sequencing subject matter appropriately.

Moreover, student-teachers make implicit criteria of simplifying open algebraic expressions since focus was only on the sequencing of concepts. For example, they talked of introducing identification of like terms before addition or multiplication. Locating the source of error in ordering, hence in teaching further suggests a transmission view of teaching and learning in that if concepts are introduced to learners in the order in which they inform each other, then learning is more certain. A similar conclusion as that for teaching emphasis being a source of error is made here in that they are equating sequencing of concepts to learning, as though it is direct. Therefore, student-teachers draw resources from their everyday professional experience of teaching and learning.

In referring to overgeneralization, hence locating the source of error in learning, student-teachers spoke with voice of the unofficial discourse. They pointed to how previously learned or newly learned concepts could have influenced learning of a particular concept. This is an issue of met-before or met-after, respectively. In terms of met-before, they talked of how the concept of finding the number of elements in a set, and how ordinary numbers are added or

subtracted to come up with a single digit could have influenced how learners simplified open algebraic expressions. As for met-after, student-teachers talked of how latter learning of multiplication of terms could have affected the remembering of addition of terms. Although student-teachers' descriptions of the source of error pointed to overgeneralization in general, and met-before or met-after in particular, as previously in Scenario 1, they did not use these terms which have been developed in the mathematics education literature. This suggests that they drew their resources from their everyday professional experience of teaching and learning. This is a further indication that these terms are not given specific focus in these student-teachers' mathematics education courses.

As for form of practice, student-teachers are oriented towards learner and presences (LP). They do not see learners as empty vessels, but that they come with knowledge of finding the number of elements in a given set, adding or subtracting ordinary numbers, and multiplication of terms, though constraining, to the learning of simplification of open algebraic expressions. However, criteria of how to deal with like or unlike terms in simplifying open algebraic expressions are implicit when referring to overgeneralization as one of the sources of error.

In suggesting remediating strategies of the error, student-teachers pointed to one broad aspect, that of emphasis in teaching. Emphasis in teaching was also talked about in Scenario 1. Three ways of how this could be done were suggested. They talked of the need for teachers to use the same rule or change the rule but in both cases with some elaboration on the meaning of like and unlike terms. In terms of elaboration, they suggested use of the definitional meaning of a term in one case and use of concrete objects (although contested as argued) and learners' language other than English in the other. The third suggested way was the need for teachers to emphasize the meaning of addition of terms in algebra as it is different when dealing with ordinary numbers.

In suggesting ways of how teachers would help their learners develop the required algebraic thinking in terms of simplifying open expressions, student-teachers spoke with voice of the official discourse (OD) or voice of the unofficial discourse (UD). In terms of voice of the official discourse of school algebra, they pointed to the definitional meaning of a term as an elaboration to the rule. As for voice of the unofficial discourse, student-teachers pointed to the issue of code-switching as an instructional resource without necessarily using the term. Therefore, they drew the resources from their everyday professional experience of teaching

and learning. As for form of practice, student-teachers were oriented towards teacher and presences (TP). They pointed to how teachers ought to be incorporating in their teaching mathematical rules which are clearly elaborated to avoid any ambiguity or interpretation problems. Moreover, teachers ought to explain how the operational view of '+' fails to work in simplifying open algebraic expressions as it involves like and unlike terms. However, as already indicated, student-teachers exhibited absences in further suggesting use of concrete objects as one of the ways of elaborating the rule concerned with simplifying open algebraic expressions. Criteria of the official discourse of simplifying open expressions are explicit as student-teachers pointed to how it involves a consideration of like and unlike terms, and how the operation '+' is implicated.

These three suggested remediating ways are also closely linked to some of the explanations provided for the sources of error where concerns about the rule being unclear or ambiguous were raised and how this could have been improved. Now student-teachers here point to how the teachers ought to be emphasizing key concepts or issues during their teaching. This is a further indication that a transmission view of teaching and learning is prevalent in that if teachers elaborate the rules they use, and emphasize the different role '+' plays pertaining to simplifying open algebraic expressions, then learning will take place. It is so direct that they are equating teaching to learning, and further away from the language of research.

In conclusion, student-teachers' *repertoires* here too reflect contradictions to: while the transmission view of teaching and learning is so dominant, there is also an aspect of not seeing learners as empty vessels. In the former case, if learning does not occur on the part of learners then it means that teachers did not do what is expected of them. They used rules that lacked clarity and were ambiguous, lessons were dominated by teacher talk, learners' responses were not probed, and sequencing of concepts was inappropriate. In the latter case, there is an understanding that learners tend to apply their knowledge of mathematics inappropriately to seemingly similar contexts. They had knowledge of finding the number of elements given a set, adding or subtracting ordinary numbers, and multiplying algebraic terms, which they applied inappropriately to the simplification of open algebraic expressions to get a single lettered term. Moreover, in both instances, they drew resources from their everyday professional experience of teaching and learning and from the official discourse of school algebra. It is like the contradiction realized in analyzing student-teachers' reasoning

processes to the expression-equation problem is prevalent since it has turned out to be similar in scenario 2. It will be interesting to see what happens with scenario 3.

8.4 SCENARIO 3: The quadratic-linear equation problem

In this scenario, all the eight pairs of student-teachers interviewed identified the error that learners made. They provided varied explanations for the error and made suggestions on remediating strategies. These are summarized and discussed as in Scenario 2 with the table and some conclusions. The detail in terms of extracts and evidence is located in Appendix G. Before focusing on summary and conclusion of student-teachers' *repertoires* on Scenario 3, I present some discussion on the scenario, which I have named "the quadratic-linear equation problem".

The scenario read as follows:

A grade 10 teacher asked learners to solve the equation $2x^2 = 6x$ on the board.

L_1: Divides both sides of the equation by 2 and obtains $x^2 = 3x$.
Then divides both sides by x and gets $x = 3$

L_2: You cannot divide both sides by x

L_1: If you can divide both sides by 2, why can't you divide by x?

Adapted from Learning Mathematical Knowledge for Teaching, 2004

In this scenario, a grade 10 teacher worked with his learners on solving quadratic equations and in particular $2x^2 = 6x$. The learners debated among themselves on whether it was right to divide the expression by x. Two opposing views emerged from this debate. The one is that since the expression could be divided by 2, it could also be divided by x. The other is that it was not appropriate to divide the expression by x. Before discussing student-teachers' talk of Scenario 3, I turn to its interpretation in terms of the mathematics education literature, and

more specifically from the constructivist/sociocultural perspective. The LMT items by Hill et al., from which Scenario 3 has been adapted, were used to evaluate teachers' SMK[21].

8.4.1 A Constructivist/Sociocultural interpretation of Scenario 3 and sources of error

- *What is the possible reservoir from the mathematics education literature?*

The mathematical question provided for in Scenario 3 is quadratic in its form, it is a quadratic function and therefore we should have two solutions. A quadratic equation either has two distinct or equal real roots, or two imaginary roots. These two values of x could be obtained using any of the four methods where appropriate, namely, factorization, completing the square, formula, and graphical. Therefore, to solve the equation $2x^2 = 6x$, we have:

$$2x^2 = 6x$$

$2x^2 - 6x = 0$ [Adding -6x, the additive inverse of 6x on both sides of the equation]

$2x(x-3) = 0$ [Factoring out 2x as a common factor]

Either $2x = 0$ or $x - 3 = 0$ [Equating each factor to zero]

$x = 0$ $\qquad$ $x = 3$ [Solving for x]

Two issues could be going on in the background in Scenario 3. The one issue is that there is misrecognition of the quadratic form, and the other issue is on understanding the division by a variable, and that the variable is actually an element of the set of real numbers and therefore it includes zero. As a result, the quadratic equation is reduced to a linear form, and the saying that you cannot divide by x signals that x could be any value, including zero.

From the constructivist/sociocultural perspective, Scenario 3 too can be explained as a result of a tendency by learners to overgeneralize (Ryan & Williams, 2007) and in terms of operational thinking (Sfard & Linchevski, 1994). The issue of overgeneralization could be described in two specific ways. On the one hand, what is going on here is the relationship between linear and quadratic equations. That is, overgeneralizing from using operations for

[21] For confidentiality, I could not use the item exactly as it is but we have permission in the QUANTUM project to use the items, so I adapted it. Adler & Patahuddin (in press) used this as one of the items but masked it, in their study that investigated recontextualizing items that measure MKfT into scenario based interviews.

solving linear equations to solving quadratic equations, and so the effect of met-before. On the other hand, what is going on here is the relationship between number and variable. Following Hart et al. (1981), the issue of overgeneralization could be looked at in terms of treating a variable as a given number (when L_1 stated that "If you can divide both sides by 2, why can't you divide by x?") and not as a specific unknown number requiring to be operated on. In Hart et al. (1981, p. 105) terms, this description could be associated with the issue of "*letter evaluated* where children avoid having to operate on a specific unknown, in this case by simply giving the unknown a value". This is reported as one of the problems learners in their early learning of algebra experience. Therefore, the overgeneralization is from number into algebra, this too is an issue of met-before.

In terms of operational thinking, the error might also be explained in terms of the learner not having attained the process-object conception of roots of a quadratic equation (Sfard & Linchevski, 1994). As a process, learners see the equation '$2x^2 = 6x$' as one requiring the operations for solving linear equations, therefore, dividing by 2 and then x to get $x = 3$. They do not see the equation as quadratic in form and thus obtaining two values of x using the operations for solving quadratic equations. These operations include factorization, completing the square, and formula. Therefore, learners have inadequate operational conception of solving a quadratic equation. Moreover, they also have inadequate structural conception of the quadratic equation '$2x^2 = 6x$' in that it is a parabola cutting the X-axis at two points, and at these two points, the value of y is zero, hence unable to reify a quadratic equation.

A possible *reservoir* drawn from the mathematics education literature pertaining to ways of explaining the sources of error in the scenario has been described in terms of overgeneralization and process-object duality. In terms of overgeneralization, it is an issue of met-before, thus using operations for solving linear equation to solving quadratic equations, and overgeneralization of number into algebra. In terms of process-object duality, it is about inadequate conception of operational and structural thinking. What follows is the analysis of student-teachers' *repertoires* and how they relate to the described *reservoir*, their initial expressed views on errors and misconceptions and to teacher-educators' discourses.

8.4.2 Summary and conclusion of student-teachers' repertoires on Scenario 3

Table 21 too follows exactly the same structure as in Tables 19 and 20.

Table 21: Synopsis of student-teachers' reasoning processes to the quadratic-linear equation problem

Pairs	Recognition	Positionings		Explanation for the sources of error			Positionings						Remediation		Positionings		
		Voice of …	Form of practice	Teaching sequence	Teaching emphasis	Interpretation problem	Voice of …			Form of practice: Oriented towards …			Emphasis in teaching	Practice exercises	Voice of …		Form of practice
		OD/UD	OMA	Overgeneralization	Key issues of linear and quadratics√, focused on one method√, work in simplest terms√	Learners lacked methods and rules	OD	UD		TA	MA	LP	Methods and rules	Methods and rules to gain familiarity	OD	UD	OTP
				Met-before: Solve linear equations				E	D/T								
1	√	√	√	√	√		√	√		√		√	√		√		√
2	√			√									√				
3	√			√									√				
4	√				√	√	√√			√	√		√	√	√		√
5	√			√	√	√							√				
6	√				√								√	√			
7	√			√	√√	√	√			√			√				
8	√			√									√				

In summary, as indicated in table 21, all the 8 pairs of student-teachers interviewed did recognize the error in the scenario, and stated that the error was as a result of learners not distinguishing between linear and quadratic equations. The problem was recognized operationally and structurally as described in the *reservoir*. Operationally, student-teachers said that learners were not aware that they needed to solve the quadratic equation to get two values of *x*. They talked of how division by *x* meant giving the letter a value resulting in only one value of *x* to be solved, hence an issue of "letter evaluated". Student-teachers' argument is that it is inappropriate to divide both sides of the equation by *x*, which is being solved for. As for the problem to be identified structurally, student-teachers referred to learners' inability to realize the graphical representation of the quadratic equation in that it will cut the X-axis at two points, hence required to get two values of *x*.

In recognizing the error in the scenario, student-teachers spoke with voice of the official discourse (OD) or unofficial discourse (UD). In terms of voice of the official discourse of school algebra student-teachers pointed to the characteristics of a quadratic equation both operationally and structurally, and how these are different from a linear equation. In terms of voice of the unofficial discourse of school algebra, student-teachers descriptions of their recognition of the error pointed to the problem with the operational and structural conception of a quadratic equation, and "letter evaluated" without necessarily mentioning these terms developed in mathematics education literature. This suggests that student-teachers drew resources from their everyday professional experience of teaching and learning, an indication that these terms are not given specific focus in these student-teachers mathematics education courses. As for form of practice, student-teachers are oriented towards mathematics and absences (MA). What learners have done wrong with solving quadratic equations is that they have divided both sides of the equation by *x* reducing it to a linear equation, hence obtaining one value of *x* instead of two. Criteria of the official discourse of solving a quadratic equation are explicit in that it is about finding the two values of *x*, although the method of doing so is implicit.

In explaining the sources of error, as in Scenarios 1 and 2, three major categories emerged from the analysis, namely teaching sequence, teaching emphasis, and a problem of interpretation, and are arranged in the order of dominance. In locating the source of error in the teaching sequence, student-teachers talked of the issue of learners' knowledge of solving linear equation interfering with their learning of solving quadratic equations. This suggests

again, as described in the *reservoir*, an issue of overgeneralization, and in particular met-before. Therefore, student-teachers spoke with voice of the unofficial discourse (UD) by referring to the issue of overgeneralization and met-before in particular. For them to have not mentioned these terms in the descriptions they gave suggests that they drew resources from their everyday professional experience of teaching and learning. As for form of practice, student-teachers are oriented towards learner and presences (LP). They see learners coming with some knowledge of solving linear equations to the learning of solving quadratic equations. Criteria of the official discourse of solving quadratic expressions are implicit in the student-teachers' talk when referring to the issue of overgeneralization in that there is mention of the two *x* values to be obtained without necessarily explaining the methods to be used.

In locating the source of error in teaching, student-teachers *repertoires* pointed to three aspects: Firstly, not emphasizing key points or issues in teaching linear and quadratic equations, including the meaning of division by zero which is also discussed in the *reservoir*. Teachers tend not to emphasize meanings of linear and quadratic equations in terms of the number of solutions that are supposed to be obtained after carrying out the manipulation. For example, a linear equation would only have one solution, a quadratic equation would have two solutions, and a cubic equation would have three solutions. Student-teachers argue that if this emphasis was made by the teacher, learners would not further divide the quadratic equation by *x* because they would realize that doing so means losing one solution. Moreover, in a situation where the *x* which is dividing both sides of the quadratic equation was zero, the student-teachers suggest that the teacher could not have explained to learners the meaning of division by zero. This would have been made possible by drawing from the history of the development of mathematics.

Secondly, student-teachers indicated that the teacher could have restricted the teaching to only one method of solving quadratic equations other than the graphical method. They argue that if learners had been introduced to the graphical method, they would have realized that the graph of the quadratic equation cuts the X – axis at two points. Therefore, getting one value of *x* would not have made sense. Thirdly, student-teachers also point to that the tendency by teachers to encourage their learners to work with expressions in their simplest form in the process of solving by identifying common factors seems to be a good mathematical practice. However, they suggest that if this practice is misapplied, it could result in errors as is the case

with Scenario 3. Student-teachers also indicated that the issue of working with expressions in their simplest form is synonymous to another mathematical rule that states that for you not to change the meaning of the equation, whatever you do on the left hand side do the same on the right hand side.

Student-teachers spoke with voice of the official discourse (OD) of school algebra in terms of what it means to solve linear and quadratic equations in terms of number of roots to be obtained and their graphical representations. As for form of practice, student-teachers are oriented towards teaching and absences (TA). Learners could not distinguish between a linear equation and a quadratic equation because the teacher did not emphasize their meaning in terms of the number of roots to be obtained, restricted teaching to only one method, and dealt with expressions in their simplest terms. Criteria of official discourse of simplifying quadratic equations are explicit in that two roots would be obtained using either the factorization or graphical methods.

In talking about the source of error being as a result of a problem of interpretation, student-teachers' *repertoire* pointed to learners not being knowledgeable about the methods and rules that are required to solve quadratic equations, hence their misrecognition of quadratic equations. Student-teachers pointed to that if learners knew the methods and rules required, they would have realized that they needed to find two solutions, and dividing by x would not have been helpful as it would reduce the quadratic equation to a linear and only yield one solution. Student-teachers suggested that it would have been helpful if learners had used methods such as the quadratic formula, completing the square, and graphical.

Student-teachers spoke with voice of the official discourse (OD) of school algebra by relating to the methods and rules associated with solving quadratic equations. In terms of form of practice, student-teachers are oriented towards mathematics and absences (MA). Learners are being blamed for not knowing the methods and rules required to solve quadratic equations, hence locating the reason for the error in learning. Criteria of the official discourse of solving quadratic equations are explicit in that student-teachers refer to the methods that would be appropriate, and the two solutions expected to be found.

As in Scenario 1, two major categories are identified in Scenario 3 as ways that would enable learners making the error recognized develop the required algebraic thinking. These include emphasis in teaching (teacher emphasizing procedure or key concepts), and practice exercises

(teacher encouraging learners to practice solving), and the former is dominant. In referring to the emphasis in teaching, student-teachers' *repertoires* are in two ways. Firstly, is that the teacher should make explicit in the question what method of solving the learners should use so that the issue of dividing by *x* does not arise as it changes the meaning of the given expression. The methods of solving quadratic equations mentioned included factorization, formula, and completing the square; and for Scenario 3 factorization method is opted out because it is easily applicable to the type of equation given. Moreover, student-teachers pointed to that the three methods of solving quadratic equations outlined are unique and cannot be used to solve linear equations. Conversely, ways of solving linear equations cannot be used to solve quadratic equations.

Secondly, is that the teacher should emphasize the importance of expressing the given quadratic equation in its standard or general form ($y = ax^2 + bx + c$, or $ax^2 + bx + c = 0$). For the quadratic equation in the scenario, the general form would be $x^2 - 3x = 0$ where *c* in this case is zero. Moreover, they pointed to that for a quadratic equation, it should be emphasized that two solutions are to be obtained. This is because of the nature of the expression in that the highest power of *x* is two. These two values of *x* could also be represented graphically as they are points on the X-axis where *y* = 0. The suggested remediating strategies so far discussed relate to two of the sources of error earlier raised, that of teacher having not emphasized key concepts or issues during teaching of linear and quadratic equations, and that of learners having a problem of interpretation in terms of methods and rules. Now the student-teachers are suggesting that the teachers ought to emphasize key concepts and issues during teaching of quadratic equations. As argued in Scenario 1, in that not every pair pointed to teaching emphasis as a source of error but when it came to suggesting remediating strategies they all talked of emphasis in teaching is again an indication of teacher not taking responsibility of learner errors but of teaching.

In suggesting emphasis in teaching as one of the remediating strategies, student-teachers spoke with voice of the official discourse (OD) of school algebra in terms of indicating methods of solving quadratic equations, and expressing quadratic equations in the standard or general form before solving for the two values of *x*, which could also be graphically represented as values of x when y = 0. As for form of practice, student-teachers are oriented towards teacher and presences (TP). They pointed to how the teachers ought to be thinking of incorporating in their teaching specific methods their learners should use in solving quadratic

equations, and that quadratic equations should always be expressed in standard or general form before solving for the two values of *x*. Moreover, they should also explain to their learners that strategies for solving quadratic equations are different from those for solving linear equations, and how dividing by *x* is problematic. In suggesting the possible remediating strategies, student-teachers make explicit criteria for solving quadratic equations in terms of expressing them in standard form and using the appropriate methods to get the two values of *x*.

In referring to practice exercises, student-teachers' *repertoires* are in two ways. Firstly, that through practice the learners would learn to obey the rules of solving quadratic equations such as writing the quadratic equation in standard form; and that of not dividing the quadratic equation by the variable of which they are tasked to solve for. Secondly, that the teacher would ensure that learners practice more of solving quadratic equations so that they become familiar with the process of how to go about solving them even in challenging circumstances using methods such as factorization, formula, or completing the square. Student-teachers suggesting practice exercises as one of the remediating strategies could be linked to one of the sources of error, namely, a problem of interpretation. They pointed to how learners did not demonstrate understanding of methods and rules associated with solving quadratic equations, and now teachers have to ensure that their learners practice more often so that they become familiar with the methods and rules.

In referring to practice exercises, student-teachers spoke with voice of the official discourse (OD) of school algebra in terms of methods (factorization, formula, completing square) and rules (expressing in standard form, and not dividing by variable of which you are solving for) of solving quadratic equations. As for form of practice, student-teachers are oriented towards teacher and presences (TP). They pointed to that teachers ought to think of incorporating in their teaching opportunity for their learners to practice solving quadratic equations more often so that they become familiar with the methods and rules. Moreover, student-teachers make explicit criteria of solving quadratic equations in terms of the methods and rules associated with the process as they make reference to practice exercises.

In conclusion, a transmission view of teaching and learning is reinforced in that if teachers emphasize the procedures in solving quadratic equations in terms of methods and rules, and make their learners practice these procedures, then learning will occur. Therefore, student-teachers' view of teaching and learning is in their everyday professional experience. What too

remains stable is the interesting contradictory view that learners are not tabular razors when reference is made to overgeneralization and in particular met-before. Although the mathematics education course for these student-teachers is shaped by the theory of constructivism/cognition and elements are present in their discourses, they come out of it with a strong transmission view of teaching and learning. At one level it is not surprising it is such a dominant discourse but at another level they are trying to engage with the view that teaching does not equal learning. Another interesting thing is that student-teachers are not able to describe such a situation using concepts developed in mathematics education literature such as overgeneralization. This is too an indication that focus on errors is an implicit focus in these student-teachers mathematics education courses.

8.5 Conclusion to Chapter 8

Across the three scenarios, all the eight pairs of student-teachers interviewed recognized the errors. The only misrecognition was in Scenario 1, and this happened to only one pair. In recognizing the errors, student-teachers' positionings are largely in terms of voice of the official discourse of school algebra, and orientation is towards mathematics and absences. Three main sources of errors are also identified across the three scenarios, which are: teaching emphasis, teaching sequence (Ordering/overgeneralization), and a problem of interpretation. Positionings in terms of voice is largely in the unofficial discourse, and more specifically in the everyday professional knowledge of teaching and learning, and orientation is also largely towards teaching and absences or learner and presences. Moreover, student-teachers do recruit or take-up from teacher-educators' discourses and from their initial realizations of LMT that sources of errors stem from teaching and/or ordering. In terms of possible remediation strategies, emphasis in teaching and practice exercises were suggested, and the former being dominant. Positionings are largely in voice of the official discourse of school algebra, and orientation is towards teacher and presences.

The main finding of student-teachers' discourses on scenarios on learner errors is that they hold contradictory realizations about teaching and learning, that is, the transmission view where teaching is equated to learning, and that learners are not empty vessels but have prior learning, and the former is the most dominant. The transmission view of teaching and learning also suggests a practical accomplishment. In terms of the problem being mathematical, student-teachers' discourses are largely focused on the inadequate conception of process and less on the inadequate conception of structure of algebraic expressions.

Moreover, that student-teachers do not speak in the language of research is an indication that focus on learner errors is not given explicit focus in these student-teachers' mathematics education courses. This finding reinforces the earlier one that LMT is weakly classified and framed, hence implicit messages being relayed. Criteria of what counts as carrying out error analysis is weak. It will be interesting to see what happens to student-teachers' realizations in Chapter 9 when focus is on scenario based interviews on learner errors in algebra elicited from learners' own working.

CHAPTER 9

9 STUDENT-TEACHERS' DISCOURSES OF ERRORS ELICITED FROM LEARNERS' OWN WORKING (SCENARIOS 4, 5 and 6)

9.1 Introduction

The analysis presented in this Chapter is on three scenarios on common learner errors in school algebra reported in the specialized field of mathematics education research but elicited from learners' own working on algebraic problems. Details of how these scenarios were chosen are discussed in Section 5.4.1.3 of this thesis. The difference between the scenarios discussed in Chapter 8 and these three in this Chapter is that the latter are presented in learners' own handwriting. These were actual learner productions selected to provoke discussion on LMT. The scenarios included *the expression-expansion by index problem*, *the expression-expansion by number problem*, and *the quadratic equation-factors problem.* As described in Section 5.4.1.3, the algebraic activity in the expression-expansion by number problem is generational while in the other two scenarios, it is transformational. The interview questions I engaged with the pairs of student-teachers on these scenarios were similar to the ones used in the scenarios presented in Chapter 8. These were designed to generate *repertoires* on the student-teachers' recognition or misrecognition of the errors in the scenarios, and the *how* of their explanations for the errors in terms of the likely sources and suggested remediating strategies.

At the beginning of each discussion of the scenario, there is an analysis to ascertain a possible *reservoir* of explanations from the mathematics education literature. Similar to the analysis presented in Chapter 8, these explanations are drawn from a 'constructivist/sociocultural' perspective and are broadly referred to in terms of overgeneralization and process-object duality/proceptual thinking. Therefore, in discussing student-teachers' *repertoires*, ways in which their talk relates to the *reservoir* were included. Also included in the analysis of each scenario is a preamble of the nature of the mathematical problem, how the solution is obtained, and what the error is.

In analyzing student-teachers' *repertoires*, I also took into consideration their positionings by focusing on three issues: (1) Discourse: Whether they spoke with voice of the official or unofficial discourse, (2) Form of practice: whether they are oriented towards learner/teacher and presences, or oriented towards mathematics/teaching and absences, and (3) School

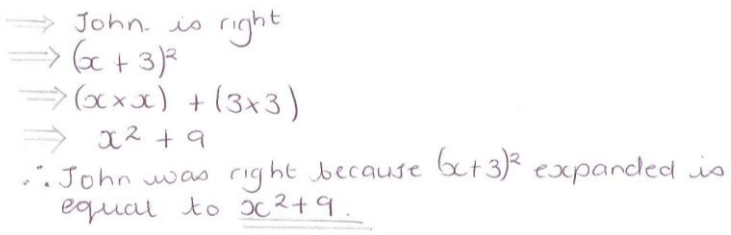
mathematics: Whether the student-teachers made explicit or implicit the criteria of the official discourse of school mathematics (school algebra).

Therefore, a similar frame as used in Chapters 6, 7 and 8 in terms of the "what" and "how" of engaging with LMT was used. As stated in Section 8.1, I was looking for the criteria in use, and whether these related to "is this a teaching problem?", "is this a learning problem?", "is this a mathematical problem?", or "is this a curriculum problem?" Student-teachers' discourses of errors elicited from learners' own working is predominantly a reinforcement of the findings in Chapter 8. They show that student-teachers hold contradictory views about teaching and learning: The transmission view where teaching is equated to learning, hence a practical accomplishment; and that learners are not empty vessels but have prior learning. In terms of the problem being mathematical, student-teachers' discourses are largely focused on inadequate conception of the process and less on the inadequate conception of structure of algebraic expressions. Moreover, that student-teachers do not speak in the language of research but everyday professional experience of teaching and learning is indications that focus on learner errors is not given explicit focus in these student-teachers' mathematics education courses.

The following discussions focus on scenarios 4, 5, and 6 upon which the above findings were based, and I am following the same structure as I did in Chapter 8. I am going to discuss Scenario 4 in detail, and Scenarios 5 and 6 in summary with the detailed analysis in terms of evidence and extracts given in Appendices H and I, respectively. In analyzing all the three scenarios, similar major categories of errors emerged although informed differently based on the mathematics in focus.

9.2 SCENARIO 4: The expression-expansion by index problem

The scenario read as follows:

> In Grade 9B at a school in Zambia, learners were asked to expand $(x + 3)^2$.
>
> Mary said it is $x^2 + 6x + 9$, and John said it is $x^2 + 9$.
>
> Who is right, Mary or John? Explain your answer.
>
> 
>
>
> Source: Adapted from Vermeulen, 2007

Vermeulen (2007) reports on the teachers' familiarity with learners' tendency to distribute the power over the sum of terms in an expression, for example $(x + y)^2 = x^2 + y^2$. This idea was then adapted to come up with the task in scenario 4 which learners in Zambian secondary schools worked with. Learners were asked to decide who was right between Mary and John for the expansion of $(x + 3)^2$ where Mary had written that it was $x^2 + 6x + 9$ and John had written $x^2 + 9$. They were also asked to explain their answer. The most prominent response was that John was right because $(x + 3)^2$ when expanded is $(x \times x) + (3 \times 3)$ giving $x^2 + 9$. Therefore, student-teachers' interviews were based on this learner's response as a typical example.

9.2.1 A Constructivist/Sociocultural interpretation of Scenario 4 and sources of error

- *What is the possible reservoir from the mathematics education literature?*

In the scenario, the mathematical question in focus is a binomial $(x + 3)$ which when squared, the result is the expression $(x + 3)^2$ whose expansion is

$= (x + 3)(x + 3)$ [Meaning of $(x + 3)^2$]

$= x(x + 3) + 3(x + 3)$ [Applying the distributive law]

$= x^2 + 3x + 3x + 9$ [Applying the distributive law]

$= x^2 + 6x + 9$ [Adding like terms and get an equivalent expression of the expansion of $(x + 3)^2$]

However, for the learners to have chosen $x^2 + 9$ as the correct answer signifies misrecognition of the distributive property. They were using the distributive law for raising to a power in the way they would use it to multiply by a number.

From the perspective of a constructivist/sociocultural, this error too could be explained as a result of overgeneralization (Ryan & Williams, 2007; Vermeulen, 2007) or operational thinking (Sfard & Linchevski, 1994). In terms of overgeneralization, there are at least two ways in which this can happen but the literature only talks about overgeneralization of the distributive property. The one explanation is that there is a relationship between expanding binomials such as $(x + 3)^2$ and expanding products of terms such as $(3x)^2$. This suggests that operations for expanding a product of terms raised to a power (index of a product in this case), for example $(3x)^2 = 3^2x^2$, have been used to expand the binomial, for example $(x + 3)^2 = x^2 + 3^2$. The learners know how to expand product of terms raised to a power, and are using this to expand the sum of terms raised to a power, hence overgeneralization of the law of exponents for products to a power. In Lima and Tall (2008) terms, the overgeneralization here could be described as an issue of met-after in that knowledge of the expansion of a product of terms raised to a power using the law of exponents is interfering with expanding a binomial.

The second explanation is that there is a relationship between squaring a binomial $(x + 3)^2$ and expanding the binomial by multiplying by two, that is, $2(x+3) = 2x+6$ therefore, distributing the power of 2 over (x + 3), that is, $(x+3)^2 = x^2 + 3^2$. They are distributing the power 2 over addition and confusing it to say actually the distributive law only holds for multiplication over addition, hence overgeneralization of the distributive property. The overgeneralization could be described as an issue met-before in that the learners know the distributive law of multiplication over addition but have applied it inappropriately on squaring a binomial.

The error can also be explained in terms of process-object duality (Sfard & Linchevski, 1994). As a process, learners are unable to see that operationally $(x + 3)^2$ means 'adding 3 to x

and then multiplying it by itself. As an object, learners are unable to see structure in $(x + 3)^2$ in that it is 'the square of the sum of x and 3'.

I have shown that the possible *reservoir* in explaining the sources of error pertaining to the expression-expansion by index problem include the effect of met-after or met-before; and inadequate conception of operational or structural thinking. What follows is a discussion of student-teachers' *repertoires* and how this relates to the established possible *reservoir*, teacher-educators' discourses, and their initial expressed views on LMT.

9.2.2 Student-teachers' recognition/misrecognition of the error

All the 8 pairs of student-teachers interviewed identified the error in Scenario 4. In Jacobs et al. (2010) terms, they attended to learners' strategy by stating that learners have no knowledge of or do not understand what it means to square a binomial.

> *I think on this one, what the learner was thinking aha is I think he thought in term, he thought of it in such a way that since x plus 3 is squared then the two can go directly into x so that x become squared x squared and 3 become squared 3 squared and maybe by this time they would have learnt indices so the learner knew that x squared is the same as x times x and 3 squared is the same as 3 uh by 3 which was giving this learner x squared plus 9* (Pair 3, turn 2).
>
> *If you are to look at this question, x plus three everything squared, may be they don't know what this square is because the moment you look at that what was supposed to come to their mind that here we are multiplying x plus three by itself, x by x plus three, uh huh, plus 3 by x plus three.*(Pair 8, turn 3) [Extract 9.1]

As indicated in extract 9.1, student-teachers' main *repertoire* in recognizing the error in the scenario is that learners cannot distinguish between squaring a term or number and squaring a binomial. Student-teachers pointed to that learners' interpretation of $(x + 3)^2$ is thought of in terms of expressing the expression $(x + 3)$ as a sum of squares by distributing the power 2 over x and 3 in the brackets. Their recognition of the error is based on their knowledge that $(x + 3)^2$ means multiplying $(x + 3)$ by itself.

In recognizing the error learners made in the scenario, student-teachers spoke with voice of the official discourse of what it means to expand a binomial by squaring. It involves multiplying the binomial by itself and then applying the distributive property. In terms of form of practice, student-teachers are oriented towards mathematics (distributive property) and absences, that is, what the learners have done wrong with squaring a binomial is that they

have distributed the power 2 over each term in the expression (x + 3) to get the 'equivalent' expression x^2 + 9. Criteria of the official discourse of school algebra (distributive property) are made explicit.

9.2.3 Student-teachers' explanation of the sources of error

Two major categories of the sources of the error recognised in the scenario emerged from the analysis. These were that the error stems from a "problem of interpretation" and "the teaching sequence". Each source of error is discussed:

- Error stems from interpretation problem

In discussing the source of error being as a result of learners' experiencing problems in interpretation, two issues emerged: one concerns the definitional meaning of $(x + 3)^2$ (Pairs 3, 4, 5, 7 and 8) and the other was about using shortcuts (direct expansion) in expanding the square of a binomial (Pair 8).

> *Uhaa I think aha maybe the the issue of expansion was not well understood by the learners because when you've get a number like aha or maybe we say a to the power 3 this is the same as a times a times a. So in this case what the learner was supposed to do for him or her not to make a mistake was to expand this in this very form which is x plus 3 and then multiply it by x plus 3. So I think the the whole issue here was aha the expansion but they did not understand the the concept of expanding they did not understand it well that's why they did it in this way* (Pair 3, turn 4)
>
> *Maybe this was as a result of shortcuts in mathematics. Now where there's expansion and simplify, it's better the expression x plus 3 in brackets squared is written, before you start even expanding it's written the way it's supposed to be written. What it means x plus 3 in brackets squared, what does this expression…if pupils can understand what this expression stands for, then that's when they can be told to expand.*
>
> ***Interviewer:*** *What does it stand for?*
>
> *To rewrite this expression, this expression it means, x plus 3 has been multiplied by itself. Which is x plus 3 in brackets, x plus 3…there are two x plus 3's here…expressions multiplied by themselves. That's what it means. So if they can understand what this means, then that's when they can proceed on expansion. But it's like they just took it literally that x and 3 have to be squared. The idea of power and multiplication is like…they got confused. That's the way I looked at it.* (Pair 7, turns 2, 3 and 4)

> *… these questions are straightforward and teachers emphasize when it comes to how to expand.* (Pair 8, turn 7) [Extract 9.2]
>
> *Because the only mistake which one can make is when you are expanding direct. If they were trying to expand it direct, that is where one can make a mistake, but still more even teachers do emphasize that when you are expanding direct, you just say x squared plus x times three, that will be three x times two, that is six x plus three squared that is nine.* (Pair 8, turn 9) [Extract 9.3]

As shown in extracts 9.2 and 9.3, in locating the error in learners experiencing problems with interpretation, student-teachers' *repertoires* point to the meaning of $(x + 3)^2$, and what it means to expand $(x + 3)^2$ directly, respectively. In referring to the meaning of $(x + 3)^2$, student-teachers said that learners seemed not to recognize that it is about multiplying $(x + 3)$ by $(x + 3)$ before proceeding with the expansion, synonymous to a^3 which means a x a x a. Therefore, it is about what the power 2 means in terms of multiplying the base which in this case is $(x + 3)$. The other issue is that learners experience problems with what it means to expand $(x + 3)^2$ directly. They do not seem to figure out that it is about squaring the first term plus double the product of the two terms plus square the second term. The student-teachers also point to that this is despite teachers emphasizing meanings. Expanding $(x + 3)^2$ directly suggests shortcuts to expanding a binomial since the underlying meaning of the process is not dealt with.

Therefore, student-teachers spoke with voice of the official discourse of school algebra in terms of the meaning of $(x + 3)^2$ and what it means to expand directly. In terms of form of practice, they are oriented towards mathematics and absences. The learners are being criticized for not demonstrating understanding of the meaning of expanding algebraic expressions despite their teachers making emphasis. This suggests that if teachers emphasize these meanings in their teaching, there should be no reason why learners should be experiencing such interpretation problems. Moreover, criteria of the official discourse of expanding algebraic expressions are explicit.

- Error stems from teaching sequence

In locating the error in the teaching sequence, student-teachers talked about the influence knowledge of related concepts has on learning of the expansion of $(x + 3)^2$, hence overgeneralization. Three aspects were identified pertaining to this issue with the dominant one being the expansion of terms (pairs 1, 2, 4, 5, 6, 7, and 8). This was talked about in terms

of inappropriate application of the law of indices and confusion between multiplication and addition in expanding a term and an expression. The other two are addition of like terms (pair 8), and multiplication as repeated addition (pairs 2 and 7). Below are extracts on these three issues outlined in this section.

> *Well! Here these pupils … Ok! This pupil who pointed at John having given the correct answer. What was going on in the mind of this pupil was that all he/she thought of was application of the law of index, which states that if you have a number or lets say two numbers (AxB) raised to the power of n this can also be simplified or expanded in the form* A^n *x* B^n*. Now this rule, this law of indices was misinterpreted here, because all the pupil saw was [Ok!] I have two numbers or two terms inside the brackets and (I've got) it is raised to the power 2. I can distribute the index throughout inside. What the pupil did not consider was the operation in the terms inside the bracket. Inside the bracket we have +, so because we have + inside the bracket that distorts everything, that law of index cannot apply here. Now the pupil already had that … ok that law of index and this is what it is. So just went straight out and gave that as a solution. Now on the final we had the solution as* x^2 *+ 9. So it was the application of the law of indices to algebraic expression which does not match.* (Pair 6, turn 2)

> *And the other part of their thinking was…what separates the x and the 3 (between?) the plus sign? So the plus sign could have been confused with the multiplication sign,… . Because here you can only push in the 2 if it was 3x inside, 3x inside, then that be squared, as in 3x in brackets squared. Then the two can go in because you are only multiplying 3 times x…you are only multiplying 3 by x, so you can push in the 2…*

> ***Interviewer:*** *So how would that be? If you apply the power 2?*

> *If you have the 3x then you apply the power 2. Then you can push in the two of 3 squared and then the x also squared. So you have 9x squared. Which is the same as 3x multiplied by 3x. If the sign there was multiplication, their reasoning was going to be correct. But then you have addition separating the two numbers. So then you just have to write the two expressions independently, so you say x in the same brackets, x plus 3, again the x plus 3 in that order, then you come to expanding.*

> ***Interviewer:*** *So you are saying that thinking which applies to when the expression is multiplication, it's the one which could have let them…*

> *Not just multiplication, could even be division.*

Which could have led to them pushing in the 2.

So they would have understood this in terms of multiplication. No wonder they pushed in the 2. (Pair 4, turns 5 – 11) [Extract 9.4]

It is like what came to their mind was you multiply or you add like terms together. So, quite okay that they may have expanded in this way, x by three then x by three then they said you multiply like terms together, then unlike terms together, grouping of like terms together because first they said x by x, x times x, said this is x times x then plus like terms together, three times three, okay three times three. They said x times x that is x squared plus three times three that is nine. The issue of putting like terms together, may be that is the one confusing. (Pair 8, turn 38) [Extract 9.5]

Multiplication is also added addition

I: What do you mean?

(laughter) its also added addition (inaudible) if you're given for example 3 times 2, I remember at one time when I taught multiplication to my pupils I was simply telling them to say oh this 3 I would attribute it to as a as sets and then the 2 I would say these are the elements that are contained in a in a set, then from there I would say there are 3, I will come up with the 3 sets and then in those 3 sets I will indicate to say oh there are 2 elements so I would come up with elements maybe 2 stones or apples or whatever and even the next one there are 2 stones, the next one there are 2 elements there, then from there I would say when you add these 3 sets of elements you say oh here there okay you put these uhm elements or the 3 sets in one of the sets, then that should be able to give you the product of 3 times 2 which is 6, so in short I've uhm 2 plus 2 plus 2 is equal to 6, no wonder why I said oh multiplication is the same as added addition (laughter)

I: (laughter) so you think that would have led to the learners thinking that when you are adding, you're also multiplying.

I think they they are trying to confuse the two maybe, that's the more reason why probably instead of adding they sometimes multiply and multiplying they will add. (Pair 2, turns 26 – 30)

… The problem is how we operate numbers, expression, where we use the addition and multiplication. It's like where sometimes where we're supposed to add, the learners multiply. Where we are supposed to multiply, the learners add. Because in this expression it's like the learner was just multiplying. That's why they had to get

John as the correct answer. … So the biggest problem maybe in this task was how to differentiate or when to use addition in algebraic expression as per required, and how to use multiplication in algebraic expression as per required. (Pair 7, turn 9) [Extract 9.6]

In Ryan and Williams (2007) terms, student-teachers' *repertoires* in locating the source of error in the teaching sequence too points to the issue of overgeneralization although there is no mention of this term in their talk. Three relationships were identified pertaining to this issue. Firstly, as shown in extract 9.4, there is a relationship between expansion of a term and expansion of an expression. This suggests applying inappropriately operations for expanding terms to expanding of expressions. Examples as indicated in the extract include using the product of an index rule from a school topic 'indices' which states that $(A \times B)^n = A^n \times B^n$, hence disregarding the role of '+' operation in the expansion of $(x + 3)^2$. Alternatively, "*the plus sign could have been confused with the multiplication sign*" (Pair 4, turn 5) as the power 2 can only be distributed if the term being expanded was $3x$ and not $(x + 3)$ to get $9x^2$, which is the same as $3x \times 3x$. This finding is similar to one of the issues of overgeneralization of the law of exponents for products to a power discussed in the *reservoir*. In Lima and Tall (2008) terms, the issue of product of an index rule could be described as met-after as it interferes with remembering of the expansion of expressions. Similarly, the confusion between addition and multiplication operations could be described as met-before as the expansion of a term by multiplying $3x$ by itself is misapplied on the expansion of an expression.

Secondly, as shown in extract 9.5, there is a relationship between grouping of like terms and expansion of expressions. This suggests overgeneralization of the method of collection of like terms over expansion of expressions. In this case, it is about getting the meaning of $(x + 3)^2$ correctly as $(x + 3)(x + 3)$ and then misapplying the method of collecting like terms as $(x \times x) + (3 \times 3)$ to get $x^2 + 9$ as the answer. In Lima and Tall (2008) terms, this is an issue of met-before since what is known about collection of like terms is inappropriately applied to expanding expressions.

Thirdly, as shown in extract 9.6, there is a relationship between the notion of multiplication as repeated addition and the expansion of expressions. This suggests overgeneralization of multiplication as repeated addition over expansion of expressions. In this case, since 3×2 expressed as repeated addition is $2 + 2 + 2$ where you have three sets containing two elements in each. This idea is applied inappropriately when dealing with expansion of expressions by

thinking that multiplication could mean addition and vice versa. For example, $(x + 3)(x + 3)$ to mean $(x \times x) + (3 \times 3)$. The issue here is when addition or multiplication can be used in the appropriate manner when dealing with the expansion of algebraic expressions. In Lima and Tall (2008) terms, the overgeneralization is forward that is met-before.

Since the student-teachers pointed to that the sources of the error could be as a result of overgeneralization, again they spoke with voice of the unofficial discourse and specifically in the everyday professional knowledge of teaching and learning. They stated that knowledge of related mathematical concepts such as expanding a term, adding like terms, and multiplication as repeated addition could influence how learners think about a particular concept (expanding an expression raised to a power) in ways that would result in errors. Their talk was not in the language of research where terms such as overgeneralization and in particular met-before or met-after could be used. In terms of form of practice, the student-teachers are oriented towards learners and presences. They are not criticizing the learners for overgeneralizing but instead they see learners as coming with some understanding to the learning of expansion of expressions. However, criteria of the official discourse are implicit since no detail is made reference to by the student-teachers on the process of squaring a binomial (or on how the algebraic expressions should be expanded) when talking about issues of overgeneralization.

9.2.4 Decisions about remediation student-teachers suggested

Two ways in which student-teachers suggested they would remediate the error recognized in the scenario were identified. The dominant strategy proposed was the importance of teachers to emphasize key concepts or issues during their teaching while the other strategy was that teachers should ensure that they provide their learners with practice exercises.

- Emphasizing key concepts or issues during teaching

Student-teachers talked of emphasizing the meaning of a number to the power of another number, and then extending this to the meaning of a term or expression to the power of a number. Moreover, they also talked of teachers' need to ensure that their learners do mathematics logically rather than mechanically. To enhance this, student-teachers talked about the need for teachers to encourage their learners to exercise patience and follow the right procedures or principles when solving mathematical problems. This was talked about in

similar ways across all the 8 pairs that were interviewed. Extracts pertaining to this issue are presented in the following sections:

> *You have to go to the definition first. What does it say in terms of these expanding, such terms? How would…first of all you have to know that if you have been given 'a' square or three or its raised to the power four, you have to make sure that you multiply just the same as two to the power of four. This means that multiplying two four times. So, even in this way, it is just the same as multiplying x plus three two times. And how do we multiply x plus three two times? You just expand it, just like this and then you say x…you open the brackets, x plus three, then plus three, open the brackets x plus three, then from here you will be able to get the correct answer.* (Pair 8, turn 35)
>
> *If they say x plus 3 to the power 2 is equal or sorry open brackets x plus 3 you close the brackets to the power 2 is the same as x plus 3 multiplied by x plus 3, once they know that I think that problem may not arise and possibly also relate you can relate with other topics in, in mathematics like x plus 3, this expression x plus 3 and x times 3, the two are different, so you should also stress or emphasize on that point and say if there is this there's multiplication here uh we do it this way and if there's addition we do it this way, maybe they will be able to understand properly.* (Pair 2, turns 23)
>
> *So I think as teachers we need to show the difference, or to explain or to emphasize this point that in multiplication and addition… Again using the actual numbers I think can help in differentiating between the two, to show that these two cannot give the same result.* (Pair 7, turns 5 and 7) [Extract 9.7]
>
> *May be, one way of doing it may, is to make an emphasis on doing mathematics with logic, not doing it mechanically because once you see it then you start working in a certain way like that […] So, it is mechanical because the expansion was squaring and then they do it. You must follow the logic behind it. When it is x plus three squared, it should be x plus three brackets then x plus three brackets. So, you must take the logic of each and every question that you are given, not just straight away, start doing it because the way of squaring that way, no.* (Pair 1, turn 14)
>
> *And again in this I aha it was supposed to, okay it has to be emphasized that short cuts they are not all that useful in mathematics because a learner has just seen that x plus 3 then raised to power 2, him thinks, seeing this just went on direct to say aha maybe it might be the same if I just multiply x two times and then 3 two times so the the issue of short cuts again not trying to do it the the the the correct way which seems to be longer.* (Pair 3, turn 8) [Extract 9.8]
>
> *I think aha like we have just said aha we aha we I would tell them that mathematics definitely you have to be patient for you to come up with the right thing and you have to follow the right procedures the aha right principles which are guiding a certain uh*

a certain thing which you are given so if it means expanding you should expand like the way it was x plus 3 to the power 2, you should multiply this thing twice and then do your arithmetic there and maybe you may come up with the right answer and then the issue of like terms again must be emphasized again in this one aha and maybe in addition to try to to try to give examples where as a teacher you demonstrate that in as much as certain things can be done direct but you try by all means to work out examples step by step so that they can see that for every step there was something done and it was logical unlike just going straight to the answer so that even if you can give them an exercise when marking you do not emphasize on the final answer but you emphasize on the steps that were taken were they taking the correct steps to arrive to the answer which they are giving. (Pair 3, turn 10) [Extract 9.9]

There is an indication in extract 9.7 that student-teachers' *repertoire* towards how they would help their learners develop the required algebraic thinking pertaining to the error identified in the scenario is in the definitional meaning of raising a term or an expression to a power. They point to how important it is for the teacher to emphasize that $(x + 3)^2$ means $(x + 3)$ times $(x + 3)$ and how the distributive law is implied in that x is distributed over $(x + 3)$ and 3 is also distributed over $(x + 3)$ to get an equivalent expression. Moreover, student-teachers pointed to how important it is for the teacher to emphasize the difference that arises between $(x + 3)^2$ and $(3x)^2$, and how using numbers could also help. In Ryan and Williams (2007) terms this suggests cognitive conflict as one of the remediating strategies where learners are introduced to a concept in focus together with what they are confusing it with. The issue of cognitive conflict was also raised in Scenario 1.

As shown in extract 9.8, in focusing on the meaning of $(x + 3)^2$, student-teachers also pointed to how important it is for teachers to emphasize to their learners the essence of doing mathematics logically rather than mechanically. They suggest that the way learners expanded $(x + 3)^2$ by distributing the power 2 to each term could be described as working in a mechanical way. There was no logic involved since the mathematical meaning of $(x + 3)^2$ was not engaged with. As indicated in the student-teachers' initial expressed views on LMT, the issue of working mechanically is also talked about in terms of using shortcuts and how constrained they tend to be when solving mathematical problems.

The issue of shortcuts is further addressed in extract 9.9 in which student-teachers pointed to the need for teachers to inform their learners that doing mathematics requires patience to come up with the correct answer since it involves use of appropriate procedures and principles. For example, the guiding procedures and principles in expanding $(x + 3)^2$ include

multiplying (x + 3) by itself, distributing over, and adding like terms for one to get the equivalent expression. They also pointed to that although direct expansion is inevitable in this situation; teachers ought to focus on the issue of showing all the necessary steps to get to the answer. They argued that doing so could instill in learners a sense of logic required. This in Skemp's (1976) terms suggests teaching for relational understanding rather than instrumental understanding. Teacher educators, as shown in their interviews, also highly recommended this way of focus when teaching.

Therefore, in suggesting that the teachers ought to emphasize key concepts or issues during their teaching as one of the remediating strategies, student-teachers spoke with voice of the official discourse of school algebra in terms of the definitional meaning of expanding terms or expressions raised to a power. In terms of form of practice, student-teachers are oriented towards teacher and presences. They pointed to how the teachers ought to be incorporating in their teaching: definitional meaning of expanding terms or expressions raised to a power and this includes the required procedures and principles, and how using numbers could help establish this difference. This is meant to make learners aware of the importance of doing mathematics logically rather than mechanically and that this requires patience on the part of their learners so that they can get the expansions right. This suggests that teachers ought to be incorporating in their teaching aspects that would help in addressing what learners do wrong with expanding expressions raised to a power (squaring a binomial). Therefore, criteria of the official discourse of expanding expressions raised to a power are explicit.

- Providing learners with practice exercises

In suggesting that teachers ought to provide their learners with practice exercises, student-teachers also talked of the importance of teachers to exercise patience with them in the process. However, only Pair 4 talked about this remediating strategy.

> *Now that comes with practice. They really have to be sure that they understand the meaning of square or any positive power. Then you can say, if we have a problem, so to say, then you can push in the three. Because you're only saying that each term will come as many times as that power. Because it is a product, so each term of that product will come as many times as that power. But where you have an addition, then it is the same expression that is coming that number of times, not independent terms or independent numbers. But that really comes with practice; they should do as many examples as you can give them, and make sure that you are patient with them.* (Pair 4, turn 13) [Extract 9.10]

Student-teachers' *repertoire* in suggesting more practice points that learners have to indulge in practicing. Therefore, it is mandated upon the teachers to provide such practice to their learners by giving them many exercises to do. In the process of practicing, student-teachers suggested that the teachers should exercise patience with their learners. The issue of practice exercises is reinforced here as it was also raised in Scenarios 1 and 3. This included how it resonates with the argument by Arcavi (1995) and Borasi (1987) that if learner errors are to be addressed, a different focus is required in terms of kinds of practices other than literal additional practice of the solution process to a number of similar mathematical problems.

In suggesting that teachers provide learners with more exercises to practice, student-teachers spoke with the voice of the official discourse of the procedures and principles required for expanding terms or expressions raised to any power. In terms of form of practice, the orientation is towards teacher and presences in that the teachers ought to be incorporating in their teaching more exercises for their learners to practice, and teachers ought to have patience with their learners in the process. Student-teachers' argument is that doing so would enable learners develop the required procedures and principles of expanding terms or expressions raised to a power. Moreover, criteria are explicit of the official discourse of expanding terms or expressions.

9.2.5 Summary and conclusion of student-teachers' repertoires on Scenario 4

As already mentioned in Section 8.2.5, the mediation of Table 19 provided for applies to all summary tables on scenarios in both Chapters 8 and 9.

Table 22: Synopsis of student-teachers' reasoning processes to the expression-expansion by index problem

Pairs	Recognition (R)/ misrecognition (M) of error	Positionings		Explanation for the sources of error					Positionings						Remediation		Positionings	
		Voice of …	Form of practice	Teaching Emphasis	Teaching Sequence		Interpretation problem		Voice of …			Form of practice Oriented towards			Emphasis in teaching	Practice exercises	Voice of …	Form of practice
	R	O D	O M A		Overgeneralization		D M	S	O D	U D		T A	L P	M A	DM of expanding term or expression √; Use numbers to show difference √; logic (appropriate procedures and principles) vs mechanical (Shortcuts)√;		OD	OTP
					Met-before (expanding a term√/like terms√/multiplication as repeated addition√)	Met-after (Indices)				E	D/T							
1	√	√	√		√					√			√		√ √		√√	√√
2	√				√ √					√			√		√			
3	√						√		√					√	√ √			
4	√				√		√								√	√		
5	√				√		√								√			
6	√					√				√			√		√			
7	√				√ √		√								√ √		√	√
8	√				√	√	√	√	√	√			√	√	√			

In Table 22, the analysis for *the expression-expansion by index problem* has shown that all the eight pairs interviewed recognized the error in the scenario. They talked of how learners had problems distinguishing between expansion of expressions (squaring a binomial) and expansion of terms as shown in their explanations for the choice of $x^2 + 9$ as the equivalent expression to $(x + 3)^2$. In recognizing the error, student-teachers spoke with voice of the official discourse (OD) of school algebra by explaining how the expansion is carried out. In terms of form of practice, student-teachers are oriented towards mathematics (expansion of expressions) and absences (OMA) in that learners have distributed the power 2 to each term of the expression (x + 3) as though they are expanding a term.

As observed, when student-teachers explained the possible sources of the error, they did not point to the error stemming from teaching in that the teacher could not have emphasized certain concepts or terms concerning the expansion of terms and expressions. Instead, they located the sources of error in learners experiencing problems with interpretation of the definitional meaning (DM) of $(x + 3)^2$ and the shortcut (S) of using direct expansion. They indicated that this is despite the teacher emphasizing. The student-teachers are therefore oriented towards mathematics, hence constructing their learners as having absences in that they cannot interpret the meaning of $(x + 3)^2$ and the process of how to expand directly. This suggests that student-teachers are not equating teaching to learning in that if the teacher has taught, then learning should happen.

As also observed, student-teachers do realize that learners come with competencies to the learning of expanding expressions such as expanding terms, addition of like terms, and multiplication as repeated addition, hence oriented towards learner and presences (LP). This is talked about in terms of overgeneralization backwards or forward, that is met-after or met-before, respectively. Student-teachers' descriptions point to these terms although no mention of the terms is made. This too suggests that focus on learner errors is not given explicit focus in these student-teachers' mathematics education courses. Student-teachers spoke with voice of unofficial discourse (UD) in relation to the everyday professional experience of what it means to teach and learn and not in the language of research.

In making suggestions on what remediating strategies they would put in place to help learners develop the required algebraic thinking, student-teachers spoke with voice of the official discourse (OD) in terms of the definitional meaning of expanding terms or expressions. In terms of form of practice, student-teachers are oriented towards teacher and presences (TP).

They talked of the importance of teachers emphasizing the definitional meaning (DM) of expansions of terms and expressions. This includes the procedures and principles involved in coming up with the equivalent expressions, and how numbers could be used to show, for example, the difference between squaring either a monomial or a binomial. They suggest that if teachers emphasize the definitional meaning of expansion of terms and expressions, learners would realize that mathematics is logical rather than mechanical. Student-teachers argue that the realization that mathematics is logical and not mechanical develops with practice involving learners working out many exercises. In explaining the issue of seeing mathematics as logic and not mechanic, student-teachers recruit or take-up from teacher-educators' discourses and their initial expressed views the importance of developing relational understanding in learners by explaining each step taken in solving mathematical problems and how use of shortcuts impede learning.

To suggest emphasis in teaching and practice exercises as remediating strategies when the sources of error were not located in teaching emphasis suggests a contradictory view to the earlier view that teaching does not equal learning. Moreover, as argued in Section 8.2.5 (Scenario 1) not locating the source of error in teaching also suggests that student-teachers are distancing themselves from teachers taking responsibility, not of teaching, but of learners' errors. Here again they have a transmission view of teaching and learning in that if teachers emphasize concepts during teaching and give their learners a lot of exercises for practice, learning will occur. It is important to note that if scenarios are presented with learners' own working, error stemming from the reason that the teacher could not have emphasized key concepts or terms during teaching is not suggested as one of the sources. It will be interesting to see what happens with Scenarios 5 and 6 that follow.

9.3 SCENARIO 5: The expression-expansion by number problem

The scenario read as follows:

Multiply n + 5 by 4



Source: Adopted from Hart et al., 1981

As indicated above, the question 'Multiply n + 5 by 4' taken from the Concepts in Science and Mathematics Studies (CSMS) was used without making any changes. Learners (14 year olds) that were involved in the CSMS study in London responded to this problem in four different ways:

(1) 17 % gave the correct answer as $4n + 20$ or $4(n + 5)$;

(2) 19 % gave $4n + 5$ or $4 \times n + 5$ as the answer;

(3) 31% gave n + 20 as the answer; and

(4) 15 % gave 20 as the answer.

It is interesting to note that 65 % of learners' responses were incorrect, an indication that early learning of algebra is not so straight forward even though the questions seem to be deceptively simple. However, the question in discussion was categorised as a level 4 activity in terms of difficulty. Similarly, for my study, the most common incorrect responses from the Zambian learners (14 year olds) were $n + 20$ and $20n$. These are again in learners' own handwriting as typical examples as shown in Scenario 5, and they were used in the interviews with student-teachers.

9.3.1 A Constructivist/Sociocultural interpretation of Scenario 5 and sources of error

- *What is the possible reservoir from the mathematics education literature?*

As discussed in Section 5.4.1.3, the algebraic activity in Scenario 5 is generational. The mathematical question in focus is a binomial expression $n + 5$ which can be expanded by multiplying by a factor 4 to obtain an equivalent expression $4n + 20$. The process is referred to as the distributive law of multiplication over addition where each term in the expression $n + 5$ is multiplied by the same factor 4 while maintaining the addition operation. As discussed in Section 3.4.3 of this thesis, according to Hart et al. (1981) the letter *n* ought to be interpreted as a specific unknown. For the learners to have constructed and interpreted 'multiply $n + 5$ by 4' to be '$n + 5$ x 4' or 'n + (5 x 4)' and getting $n + 20$ in one instance and a further $20n$ in another signifies a misrecognition by learners of the mathematical task in question. The misrecognition is that 5 has been multiplied by 4 to get 20 which has been added to n to get '$n + 20$' and a further '$20n$' obtained is as a result of 'adding' 20 to *n*.

These two errors are identified in mathematics education literature as some of the typical learners' errors in the early learning of algebra or further learning, which most teachers of mathematics are aware (Vermeulen, 2007). The first error ($20n$) is called the conjoining or early closure problem and the second one ($n + 20$) is called the "incomplete application of the distributive property" (Vermeulen, 2007, p. 15). The focus here is on the latter error as the former has already been discussed comprehensively in section 8.3. Moreover, student-teachers gave similar explanations for the sources of the error such as seeing '+' as operational compelling learners to add and come up with one thing, and the thought that since you can multiply 20 x *n* to get $20n$, you can also add $n + 20$ to get $20n$. The question is how would the error due to incomplete application of the distributive property, which here I have called the expression-expansion by number problem, be explained in constructivist/sociocultural terms?

The error under discussion could too be looked at from a constructivist/sociocultural perspective in two ways, that is, overgeneralization (Ryan & Williams, 2007) and operational thinking (Sfard & Linchevski, 1994). In terms of overgeneralization, three relationships are identified. Firstly, there is a relationship between multiplying an expression such as (n + 5) by a number and adding a number to a term such as $3n + 4$. There is an aspect of direct translation of the question where the operation 'x 4' has been attached to the expression as a

whole (n + 5 x 4) as though they were adding 4 to 3*n* to get 3*n* + 4 (Hart et al., 1981). Learners know how to add a number onto a term when asked to 'add 4 onto 3n' and are using this idea to model the statement 'multiply n + 5 by 4', hence overgeneralization of adding a number to a term. This could also be categorized as a modelling error (Ryan & Williams, 2007) as discussed in Section 3.3.3 of the literature review.

Secondly, a relationship is also identified between use of brackets and multiplication, and use of brackets and addition. Learners know how to carry out the multiplication first before addition given '*n* + 5 x 4' and they are using this idea to 'multiply *n* + 5 by 4', hence overgeneralization of the use of brackets. Thirdly, a relationship is established between identification of like terms, and in this case numbers, and multiplication of an expression by a number. Hart et al. (1981) explain that learners could have seen meaning in the numbers 4 and 5 which they 'properly' multiplied, and the letter was left without doing anything with it. This suggests overgeneralization of like terms over multiplication. Therefore, the expressions *n* + 20 just like 20*n* could also be categorized as "letter not used" (Hart et al., 1981, pg 108). In Lima and Tall (2008) terms, the three descriptions of overgeneralization discussed above could also be an indication of met-before.

The error can also be explained in terms of inadequate conception of the process-object duality (Sfard & Linchevski, 1994) and what is emphasized more is the process. As a process, learners are unable to see that operationally (*n* + 5) x 4 means 'adding 5 to *n* and then multiplying by 4. As an object, learners are unable to see structure in (*n* + 5) x 4 in that it is 'the product of the sum of *n* and 5, and 4'. Moreover, Hart et al. (1981) argue that the structure of (*n* + 5) x 4 is complex, hence categorizing it as a level 4 task. As discussed in Section 3.4.3, Watson (2009) would also argue that the error could be as a result of not treating letters and numbers as symbols in a structure when used together.

I have so far discussed that possible explanations for the sources of error in Scenario 5 are largely in terms of overgeneralization, in particular, met-before, and inadequate process-object duality. In terms of overgeneralization, knowledge of adding a number to a term, collecting like terms, and use of brackets when several operations are involved could have interfered with the appropriate application of the distributive property. The descriptions pertaining to process-object duality suggest that learners worked operationally while making errors because they cannot see the structure. Therefore, what *repertoires* of the expression-

expansion by number problem do the student-teachers bring to this *reservoir* and how do these relate to their initial expressed views on LMT and to teacher-educators' discourses?

9.3.2 Summary and conclusion of student-teachers' repertoires on Scenario 5

As in Chapter 8, the process of analysis follows that of Sections 9.2.2, 9.2.3 and 9.2.4, and the detail, in terms of extracts and evidence is located in Appendix H. Here, summary and conclusion of the analysis follows with Table 23 assuming a similar design as mediated for Table 19.

Table 23: Synopsis of student-teachers' reasoning processes to the expression-expansion by number problem

Pairs	Recognition (R)/ Misrecognition (M) of error	Positionings		Explanation for the sources of error			Positionings						Remediation		Positionings	
	R	Voice of …	Form of practice	Teaching emphasis (n + 5 as one quantity √; expanding algebraic expressions √)	Teaching sequence	Interpretation problem (Direct translation)	Voice of …			Form of practice Oriented towards …			Emphasis in teaching (n + 5 as one quantity bracketed and distributive law applied √; y = n + 5 √)	Practice exercises (On distributive law and how brackets are implicated)	Voice of …	Form of practice
		OD	OMA		Overgeneralization		OD	UD		TA	LP	MA			OD	OTP
					Met-before (like terms √; BODMAS √;			E	D/T							
1	√	√	√		√ √	√	√	√ √			√ √	√	√		√	√
2	√			√	√	√	√			√			√ √		√	√
3	√				√ √								√	√	√	√
4	√			√ √	√ √	√	√			√			√	√		
5	√				√ √	√							√ √	√		
6	√				√								√			
7	√				√	√							√	√		
8	√			√	√ √								√			

Analysis of *the expression – expansion by number problem* as summarized in Table 23 has shown that all the 8 pairs interviewed recognized the error in the scenario. In recognizing the error in Scenario 5, they spoke with voice of the official discourse (OD) of school algebra in that the error was due to inappropriate completion of the distributive property. The student-teachers talked of how the learners could not have considered $n + 5$ as one quantity, which should be expanded by multiplying out by 4 to get an equivalent expression $4n + 20$. In terms of form of practice, student-teachers are oriented towards mathematics (expansion of a binomial by a number) and absences (OMA) in that what the learners did wrong with the distributive property is to multiply only 5 by 4 and got $n + 20$ and a further $20n$ as equivalent expressions. Moreover, student-teachers pointed to how the letter *n* was not operated on resulting in the problem being categorized as 'letter not used' as similarly discussed in the *reservoir*. As already stated, the issue of conjoining or early closure was extensively discussed in Section 8.3; therefore, learners getting a further $20n$ as an equivalent expression did not form part of this discussion.

Three major sources of the error were identified, namely, teaching sequence, interpretation problem and teaching emphasis. These are arranged in the order starting with the most prevalent. In locating the error in teaching sequence, student-teachers pointed to the error being as a result of *overgeneralization* of like terms and BODMAS. This suggests that learners' prior learning of collecting like terms and BODMAS could have interfered with their appropriate application of the distributive law, hence the effect of *met-before*. In referring to the issue of collecting like terms, student-teachers pointed to how learners could have considered 5 and 4 in their mathematical model '$n + 5 \times 4$' to be like terms, which they ended up multiplying. As for *n*, it was not taken into consideration since there was nothing else in *n* to multiply with, hence getting $n + 20$. This means that learners have knowledge of collecting like terms but are applying it inappropriately when expanding a binomial by a number using the distributive property. As for the issue of BODMAS, student-teachers pointed to how learners are aware that the operation multiplication is carried out first before addition. This could have made the learners enclose inappropriately '5 x 4' in brackets in their modeled expression 'n + 5 x 4' instead of '$n + 5$', hence overgeneralization of brackets over multiplication. Therefore, student-teachers' *repertoires* of the overgeneralization of collecting like terms and BODMAS (met-before) and the issue of ignoring the *n* ('letter not used') are similar to what was discussed in the *reservoir*.

It is interesting to note that student-teachers' descriptions pointed to the issue of *overgeneralization*, in particular, *met-before*; and *letter not used* without necessarily making mention of these terms. Two conclusions can be made from the above discussion. The first conclusion is that the student-teachers spoke with the voice of the unofficial discourse in that their descriptions of locating the error in prior learning were drawn from the everyday professional experience of what it means to teach and learn (UD-E). They did not use terms such as *overgeneralization* or *met-before* or *letter not used* which are in the language of research [discursive/theoretical terms (D/T) developed in the specialized field of mathematics education research]. The other conclusion is that since the language of research was not used, it suggests that errors and misconceptions are not given specific focus in these student-teachers' mathematics education courses. In terms of form of practice, student-teachers are oriented towards learner and presences (LP). They do not view learners as empty vessels but that they have such knowledge as of collecting like terms and BODMAS, which they bring to the learning of expanding a binomial by a number. Moreover, in talking of overgeneralization, criteria of the official discourse of school algebra in terms of the distributive property are implicit. Student-teachers do not explicitly point to the process of carrying out the distributive property or its underlying meaning in 'multiplying $n + 5$ by 4'.

In locating the error in interpretation, student-teachers spoke with voice of the official discourse (OD) of the process of expanding a binomial by a number for them to realize that what learners did is an issue of direct translation. In terms of form of practice, student-teachers are oriented towards mathematics (distributive property) and absences (MA). What the learners did wrong with the distributive property was that they had replaced 'by' in the statement 'Multiply $n + 5$ by 4' with the multiplication operation for the expression to read '$n + 5 \times 4$'. This suggests that the mathematical problem was expressed in the form it was phrased without necessarily taking into consideration the process or the meaning of the distributive property. In talking about the problem of interpretation, student-teachers do not make explicit criteria of the distributive property.

In locating the error in teaching emphasis, student-teachers spoke with voice of the official discourse (OD) of school algebra in terms of the distributive property and how brackets are implicated. As for form of practice, student-teachers are oriented towards teaching and absences (TA). They point to how the learners did not complete the application of the distributive property because the teachers could not have emphasized in their teaching that

the binomial 'n + 5' is one quantity or thing. This one quantity or thing should be bracketed and each term multiplied out by another quantity, and in this case 4. Moreover, the student-teachers stated that the teachers could also have not emphasized in their teaching that the mathematical process at hand was expanding a binomial by a number, hence the error. This suggests that student-teachers are equating teaching to learning in that learners' errors are as a result of teachers not emphasizing key concepts or issues during their teaching. Therefore, if teachers emphasize then learners will not make mistakes, hence a transmission view of teaching and learning. Again, as indicated in Section 8.5, student-teachers recruit or take-up from teacher-educators' discourses and their own initial discourses on LMT that teaching is a possible source of errors. However, student-teachers make explicit criteria of the official discourse of the process of expanding the binomial 'n + 5' by 4 in locating the error in teaching.

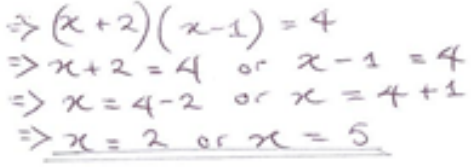
It is also interesting to note that in locating the source of error in teaching emphasis, only three pairs pointed to the issues raised. But when suggesting possible remediating strategies, all the 8 pairs are oriented towards teacher and presences (OTP). They pointed to how teachers ought to be incorporating in their teaching that 'n + 5' is one quantity which should be bracketed and distributed over by 4. To further emphasize the issue of one quantity, student-teachers pointed to how the teachers could be incorporating in their teaching that the one quantity could be represented by a single variable such as y to convince their learners that the mathematical task is about 4y. Thereafter, the y can be substituted by 'n + 5' and then apply the distributive property.

Moreover, student-teachers pointed to how the teachers ought to be incorporating in their teaching giving their learners more and varied exercises on the distributive property for them to practice. They suggested that the varied exercises could include questions that already have brackets and those that are in form of statements such as the one in the scenario so that learners become familiar with the distributive property. Therefore, student-teachers spoke in voice of the official discourse (OD) of school algebra in terms of the process of carrying out the distributive property but not on its definitional meaning; and the importance of practice exercises. As argued in Scenarios 1 and 4, in referring to the issue of emphasis in teaching what is reinforced here is that student-teachers are pointing to that a teacher is responsible not of learner errors but of teaching, hence equating teaching to learning. The issue of practice exercises is also reinforced here as it was also raised in Scenarios 1, 3 and 4.

9.4 SCENARIO 6: The quadratic equation-factors problem

The scenario read as follows:

Solve for x in the equation $(x - 1)(x + 2) = 4$



Source: Adapted from Bell, 1995

As indicated in Scenario 6, the mathematical question in focus was adapted from Bell (1995). He reported on Lee & Wheeler's (1987) study on general algebraic strategies. Lee and Wheeler (op cit) were concerned with the algebraic thinking of 15 year old learners, and in particular their conceptions of generalization and justification. Of interest to my study among the 12 problems that were used to test 354 learners followed by 25 individual interviews is the following question:

> *What are the main differences between the following two algebraic statements?*
>
> $(x-1)(x+2)=4$
>
> $(x-1)(x+2)=x^2+x-2$ (Bell, 1995, p. 44)

In providing solutions to such problems, the expectation from learners was that they needed to use algebraic language in their reasoning about the situations provided. The conclusion was that learners treated both statements as equations to solve. None of the learners thought of one statement as being an identity and the other as being true for two particular values of *x*. This finding by Bell (1995) suggested dominance of manipulation over reasoning.

For my study, the first statement '$(x-1)(x+2)=4$' was used and asked to selected 15 – 16 year old learners in Zambia to solve for *x*, hence the adaption as this was not the primary focus of what the statement was intended for as already indicated above. The choice of this particular problem was based on my interest to see what strategies of manipulation the learners would use to solve a non standard quadratic equation such as $(x-1)(x+2)=4$. The

most prominent error the learners made in solving for *x* in the quadratic equation was that they equated each factor to 4 and stated that *x* was either equal to 2 or 5, as shown in Scenario 6. This learner's response was used as a typical example and student-teachers' interviews were based on this scenario in terms of their analysis of learners' conceptions. However, it is not possible to conclude that the learners' way of solving the quadratic equation exhibited in the scenario is similar to what learners in Lee and Wheeler's study did in carrying out the manipulation. This is because no detail is provided on how they engaged with both statements as equations to be solved.

9.4.1 A Constructivist/Sociocultural interpretation of Scenario 6 and sources of error

- *What is the possible reservoir from the mathematics education literature?*

As already indicated in Section 9.4, the mathematical question in Scenario 6 is a product of two binomials equated to 4, hence a quadratic equation. This can be talked about in two ways, which are the axiomatic reasoning and the functional interpretation. The one way, which is the axiomatic reasoning is that in order to solve for *x*, the quadratic equation needs to be transformed into the form $a.b = 0$ because if $a.b = 0$, it can be concluded that either $a = 0$ or $b = 0$ or both are zero, hence "*the zero factor property or rule or principle*". What is in focus here is finding the roots of a quadratic equation. This suggests that the property of products of two factors equals zero is the key mathematical idea that underlies the quadratic equation in the scenario.

In applying *the zero factor property*, $(x-1)(x+2) = 4$ has to be expressed in standard form, which is $ax^2 + bx + c = 0$ and then apply the factorization method. If the quadratic equation cannot be expressed in the form $a.b = 0$, then other methods such as the quadratic formula or completing the square could be used so that we get it into this form. Moreover, if $a.b = 4$ there is nothing that one can say about *a* or *b* in terms of making any claims about the values of the factors as they would not be mathematically justified, hence the problem here. However, it is important to check if the values of *x* obtained after solving the quadratic equation are correct. This could be done by substituting each of the values of *x* in the original quadratic equation to see if it is satisfied, if not then one would trace back in their working to identify where they could have made a mistake.

Now expressing the quadratic equation $(x-1)(x+2)=4$ in the form $a.b=0$, we have:

$(x-1)(x+2)=4$

$x(x+2)-1(x+2)=4$ [Applying the distributive law]

$x^2+2x-x-2=4$ [Applying the distributive law]

$x^2+x-2=4$ [Adding like terms]

$x^2+x-2-4=0$ [Adding -4 the additive inverse of 4 on both sides of the equation]

$x^2+x-6=0$ [Adding like terms, hence standard form of quadratic equation]

Using the factorization method:

Product is -6, sum is 1 and the two factors which when multiplied will give - 6 and when added will give 1 are 3 and -2.

$x^2+3x-2x-6=0$ [Expressing the term in *x* as a sum of the factors]

$x(x+3)-2(x+3)=0$ [Factoring out a common factor]

$(x+3)(x-2)=0$ [Factoring out a common factor]

Either $x+3=0$ or $x-2=0$ [Equating each factor to zero since either or both of the factors must be zero]

$x=-3$ $x=2$ [Adding -3, additive inverse of 3; or adding 2, additive inverse of -2 on both sides of each equation, respectively]

From the solution, it is clear that this type of quadratic equation requires application of the distributive law, addition of like terms and expressing the quadratic equation in the standard form before applying the factorization method. For learners to have equated each of the factors (*x*-1) and (*x*+2) to 4 when solving for *x* in the expression $(x-1)(x+2)=4$ is an indication that they do not know the meaning of *the zero factor property* and its applicability. I have shown above how the quadratic equation $(x-1)(x+2)=4$ could be solved by using the factorization method. It is interesting to note that one of the values of *x* learners got (*x* = 2) was able to satisfy the given quadratic equation despite using a method which cannot be mathematically justified in similar ways as *the zero factor property*.

The other way the quadratic equation in the scenario can be talked about is in terms of the functional interpretation. The functional interpretation requires an understanding that the non-standard quadratic equation $(x-1)(x+2)=4$ could be considered as a quadratic function in the form $y=ax^2+bx+c$. If this quadratic function is expressed in standard form '$ax^2+bx+c=0$', then graphically we find the *x*-intercepts where the graph, which is a parabola in this case, intersects with the X-axis. The intersection happens when y = 0, and the *x*-intercepts are the values of *x* that are obtained by solving the standard quadratic equation, and these should satisfy the original quadratic equation, hence the solutions. Therefore, solving for the *x*-intercepts takes us back to the axiomatic reasoning already discussed if the factorization method has to be used.

Two ways of explaining this error again from a constructivist/sociocultural perspective are identified, namely, overgeneralization (Ryan & Williams, 2007) or operational thinking (Sfard & Linchevski, 1994). In terms of overgeneralization, there is a relationship between *the zero factor property* such as $(x-1)(x+2)=0$ and a non-standard quadratic equation such as $(x-1)(x+2)=4$. This suggests that *the zero factor property*, for example, where either $(x-1)=0$ or $(x+2)=0$ or both factors are equal to zero, has been applied inappropriately to solve for *x* in the non-standard quadratic equation, for example, where either $(x-1)=4$ or $(x+2)=4$. The learners know how to solve for *x* by taking into consideration *the zero factor property* using the factorization method when given a standard quadratic equation. The problem arises when they are given a non-standard quadratic equation with factors $[(x-1)(x+2)=4]$. They are seduced into applying *the zero factor property* because it looks so much like *a.b* = 0, which implies that either one of the two factors is zero or both are equal to zero. This is an indication of overgeneralization of *the zero factor property* when using the factorization method, hence the effect of met-before. What learners already know about *the zero factor property* when solving a standard quadratic equation using the factorization method, they tend to overgeneralize forward when required to solve for *x* given a non-standard quadratic equation with factors.

The error can also be explained in terms of inadequate process-object duality (Sfard & Linchevski, 1994) with process dominating. As a process, learners are unable to see operationally that the non-standard quadratic equation $[(x-1)(x+2)=4]$ has to be re-factorized by expanding the binomials, collecting like terms, and expressing it in standard

form. Only then would they apply the factorization method for it to assume the form $a.b = 0$. This is aimed at finding two particular values of x, which will satisfy the given quadratic equation $(x-1)(x+2)=4$. As an object, two issues arise. The first one is that learners see the product of two factors equal to 4 being similar to *the zero factor property*, which is the product of two factors equal to zero. They are seduced into seeing the same structure in the two, that is $a.b = 4$ and $a.b = 0$ without realizing that $a.b = 4$ is not mathematically justified. This suggests that learners' focus is more on the process rather than the underlying meaning of *the zero factor property*, which is why the error proliferates. The second issue is that structurally learners cannot see that they are finding the x-intercepts of the function $y = ax^2 + bx + c$, where $y = 0$. The x-intercepts are the values of x where the graph of the quadratic function, which happens to be a parabola in this case, intersects with the X-axis and at these points $y = 0$.

I have so far discussed that possible explanations for the sources of error in Scenario 6 are largely in terms of overgeneralization, in particular, met-before, and inadequate process-object duality. The overgeneralization is of *the zero factor property* over non-standard quadratic equation, in particular the issue of met-before. The descriptions pertaining to process-object duality suggest that learners worked operationally while making errors because they cannot see the structure in the axiomatic reasoning and the functional interpretation. Therefore, what *repertoires* of the quadratic equation-factors problem do the student-teachers bring to this *reservoir* and how do these relate to their initial expressed views on LMT and to teacher-educators' discourses?

9.4.2 Summary and conclusion of student-teachers' repertoires on Scenario 6

As stated in Section 9.3.2, Table 24 too follows exactly the same structure as the tables before on scenarios. Here, I present the summary and conclusion of student-teachers' *repertoires* on Scenario 6 with the detail including extracts and evidence provided for in Appendix I.

Table 24: Synopsis of student-teachers' reasoning processes to the quadratic equation-factors problem

Pairs	Recognition (R) / misrecognition (M) of error		Positionings		Explanation for the sources of error				Positionings						Remediation		Positionings	
	R	M	Voice of …	Form of practice	Teaching emphasis (Focused on standard quadratic equations and ignored non-standard ones √, Shortcuts √		Teaching Sequence Overgeneralization Met-before (solving a standard quadratic equation using factorization method)	Interpretation problem (Quadratic equation as already factorized)	Voice			Form of practice Oriented towards …			Emphasis in teaching (Procedure of factorizing non-standard quadratic equations√; graphical meaning of solving for *x* √)	Practice exercises (Variety in terms of forms of non-standard quadratic equations)	Voice of …	Form of practice
			OD	OMA					OD	UD		TA	LP	MA			OD	OTP
										E	D/T							
1	√		√	√	√				√			√			√		√	√
2	√							√	√					√	√		√	√
3	√						√	√		√			√		√			
4	√				√		√								√ √			
5	√						√								√			
6	√							√							√			
7	√				√	√			√			√			√	√	√	√
8		√																

As shown in Table 24, the analysis of the quadratic equation-factors problem indicates that of the 8 pairs interviewed, 7 pairs recognized (R) the error in the scenario, and only 1 pair did not, hence misrecognition (M) of the error. It is interesting to note that this is the same pair that misrecognized the error in Scenario 1. Those who recognized the error talked of how learners had problems distinguishing between solving a standard quadratic equation and a non-standard one using the factorization method. This is because learners equated factors in the non-standard quadratic equation '$(x-1)(x+2)=4$' to 4. Student-teachers pointed to that the underlying meaning of expressing the quadratic equation in standard form is that it should in the process assume the form $a.b = 0$, where either a or b is equal to zero or both are equal to zero. In recognizing the error, student-teachers spoke with voice of the official discourse (OD) of school algebra in terms of the process required in solving quadratic equations using factorization method and the underlying meaning in terms of the *zero factor property*. In terms of form of practice, student-teachers are oriented towards mathematics and absences (OMA). Focus is on factorizing non-standard quadratic equations, and what the learners are doing wrong is that they are equating each factor to 4 as though it was a standard quadratic equation such as $(x-1)(x+2)=0$. Moreover, student-teachers make explicit criteria of the process of solving non-standard quadratic equations in that key is expressing it in standard form so that it assumes the form $a.b = 0$.

However, those who misrecognized the error displayed themselves one of the discursive errors, and as a result I could not further explore their discourses about errors and misconceptions. They talked of how they saw nothing wrong with what the learners did since the quadratic equation was already factorized. Moreover, student-teachers' justification of what learners did in the scenario was attributed to that they saw lack of specification in the questioning of the task in the scenario. They suggested that it should have read: 'solve for the values of x which satisfy the given equation'. Since it read 'solve for x' then either 2 or 5 which learners obtained are justified irrespective of whether they satisfy the quadratic equation $(x-1)(x+2)=4$ or not.

Three broader categories of sources of the error were identified, namely, teaching sequence, teaching emphasis, and interpretation problem. Firstly, in terms of the source of error being as a result of the teaching sequence, student-teachers talked of how the learners' knowledge of solving standard quadratic equations using the factorization method could have been an overgeneralization over solving non-standard quadratic equation as given in the scenario.

What is interesting about how the student-teachers talked about overgeneralization is that it is different from how it has been discussed in the *reservoir*. In the *reservoir*, the overgeneralization is from the product of two factors equal to zero ($a.b = 0$) to the product of two factors equal to any number other than zero ($a.b = 4$). It is not from the standard form to the non-standard form as student-teachers indicated. Since student-teachers talked of that the quadratic equation $(x-1)(x+2)=4$ is not in the standard form, their discourse is about the process. It is about the procedures we need to follow to solve but not what underlies why we need to get it in the form $a.b = 0$.

However, it should be noted that when the error was located in the learner (prior learning), student-teachers' explanations pointed to overgeneralization and more specifically met-before without necessarily mentioning the terms. This suggests that student-teachers spoke with voice of the unofficial discourse in terms of the everyday professional experience of what it means to teach and learn (UD-E). This is a further indication that errors and misconceptions are not given specific focus in these student-teachers' mathematics education courses. Moreover, student-teachers are oriented towards learner and presences (LP), an indication that learners are not seen as empty vessels but have prior learning, hence recruiting from teacher-educators' discoursed the importance of prior knowledge. Student-teachers point to how learners have some knowledge of how to factorize standard quadratic equations which they had come with to the learning of factorizing non-standard quadratic equation. In Smith et al. (1993) terms, student-teachers recognize the important role prior knowledge, though constraining, plays in the process of learning.

Secondly, in relating the source of error to the teaching emphasis, student-teachers talked of how the teachers could have focused on factorizing standard quadratic equations and ignored the non-standard ones. This could result in learners factorizing non-standard quadratic equations as though they were factorizing standard quadratic equations because they have had no experience of factorizing non-standard ones. Therefore, the emphasis here is on the standard form and not because of *the zero factor property*, which is the underlying mathematics. Moreover, student-teachers talked of how teachers could have emphasized to their learners the importance of using shortcuts in mathematics. As a result, learners could have thought that what they did was the fastest way of factorizing non-standard quadratic equations. Locating source of error in teaching is too a reinforcement of teacher-educators' discourses and student-teachers' initial discourses on LMT. The issue of shortcuts raised here

contradicts student-teachers' earlier initial discourses on LMT in that they talked of how use of shortcuts in teaching should be discouraged if teaching for understanding is to be enhanced.

In locating the error in teaching, student-teachers spoke with voice of the official discourse (OD) of school algebra in terms of the process of solving standard and non-standard quadratic equations using the factorization method. In orienting themselves towards teaching and absences (TA), student-teachers equate teaching to learning, in that if learners are taught how to factorize non-standard quadratic equations and how not to use shortcuts then learning should happen. This suggests a transmission view of teaching and learning.

Thirdly, student-teachers talked of the source of error being as a result of an interpretation problem in that learners could have thought that the quadratic equation had already been factorized and what was remaining is to equate the factors to 4 and get either values of *x*. As a result, the learners missed the underlying mathematics, which is *the zero factor property*. In locating the error in learning, student-teachers spoke with voice of the official discourse (OD) of school algebra in terms of the process of factorizing non-standard quadratic equations. Student-teachers pointed to that the process involves expansion, collection of like terms, and expression of the quadratic equation in standard form before applying the factorization method. In terms of form of practice, the student-teachers are oriented towards mathematics and absences (MA). What learners are doing wrong with the mathematics is to think that the quadratic equation has already been factorized. This suggests that as discussed in the *reservoir*, student-teachers are pointing to that learners have inadequate operational conception of solving a quadratic equation of the form $(x-1)(x+2)=4$ using the factorization method.

It is interesting to note that as in Scenario 5, in explaining the source of the error as stemming from teaching emphasis; only 3 pairs located the error in teaching but when suggesting remediating strategies all the 7 pairs pointed to the importance of the teacher emphasizing key concepts or issues during teaching. This again suggests that student-teachers are also pointing to that teachers are responsible, not of learner errors but of teaching. They talked of the importance of emphasizing procedure of factorizing non-standard quadratic equations in that key is expressing them in standard form; and the graphical representation of what solving for *x* means. They pointed to that if teachers can represent the graphical meaning of a

quadratic equation and emphasize that we are finding the x-intercepts where the graph is intersecting with the X-axis (values of x where $y = 0$), then learners will always remember to express quadratic equations of any form in standard form. Student-teachers' argument is that learners understand the mathematics better when you use practical situations. In reference to the *reservoir*, focus here is on the process and the underlying functional interpretation, and not on the axiomatic reasoning (*the zero factor property*). Student-teachers also talked of the need for learners to experience variety in terms of non-standard quadratic equations to avoid any confusion that might arise as a result of focusing on only the standard form.

Therefore, student-teachers spoke with voice of the official discourse (OD) of school algebra in terms of what it means to solve quadratic equations of different forms. As for form of practice, they are oriented towards teacher and presences (OTP) in terms of what teachers ought to be incorporating in their teaching in terms of process, functional interpretation, and varied practice exercises for the purposes of learner learning. This suggests that teaching is crucial to working with learners' errors and misconceptions, hence reinforcing the transmission view of teaching and learning. Moreover, in Sfard and Linchevski's (1994) terms, student-teachers' remediation suggestions point to the importance of developing in learners both operational and structural thinking in solving quadratic equations. As in Scenarios 1, 3, 4 and 5, the issue of practice exercises is reinforced.

9.5 Conclusion to Chapter 9

The analysis has shown that again across all the three scenarios (4, 5 and 6), student-teachers attended to learners' strategy in that largely they recognized the errors. Only 1 pair could not recognize the error in Scenario 6. It is interesting to note that this is the same pair that misrecognized the error that was presented in Scenario 1 of Chapter 8. Those who recognized the errors pointed to how the learners could not square a binomial, complete the application of the distributive law, and apply the zero factor property. In recognizing the errors, they all spoke with voice of the official discourse of school algebra in terms of the processes required in squaring a binomial, applying the distributive property and applying the zero factor property. As for form of practice, they are oriented towards mathematics (school algebra) and what learners are doing wrong with squaring a binomial, applying the distributive and the zero factor property, hence absences. With squaring the binomial, the learners were distributing the power to each term, for example, $(x+3)^2 = x^2 + 3^2$. In terms of applying the distributive property, learners when asked to 'Multiply $n + 5$ by 4' only multiplied 5 by 4 and

not n by 4. In terms of the zero factor property, learners equated each factor in $(x-1)(x+2)=4$ to 4 defeating the underlying mathematics that if a . b = 0, then either a =0 or b = 0 or both are equal to zero. Therefore, in recognizing the errors, student-teachers make explicit the criteria of the official discourse of school algebra. The conclusion is that student-teachers' *repertoires* in recognizing the errors in the scenarios depended on their knowledge of school algebra involved, and largely in terms of the process conception.

The pair that misrecognized the error in Scenario 6 talked of how they saw no problem with what the learners did. They pointed to how there was no specification in the question that required learners to solve for x values which should both satisfy the quadratic equation. Therefore, even if x = 2 satisfied the quadratic equation and x = 5 did not, the student-teachers could still see nothing wrong with what the learners had done. Since student-teachers misrecognized the error, I could not make any further analysis of their discourses on learners' errors and misconceptions.

Across all the three scenarios, the sources of errors that student-teachers pointed to were again largely in three broad categories, namely, teaching emphasis, teaching sequence, and the problem with interpretation. Among these three sources, the most dominant was teaching sequence, in particular, *overgeneralization* forward, hence *met-before*. They talked of how learners' knowledge of expanding a term, adding like terms, and multiplication as repeated addition could have interfered with the learning of squaring a binomial. They also talked of how learners' new knowledge of the law of exponents for products to a power could have interfered with learners' remembering of expanding square of a binomial, hence an issue of overgeneralization backwards, in particular, *met-after*.

In terms of incomplete application of the distributive property, the issue was of *met-before*. Student-teachers saw learners as having knowledge of collecting like terms, and BODMAS where multiplication is carried out first before other operations and applied this inappropriately to the distributive property. As for the zero factor property, the issue of *met-before* also arose. Student-teachers saw learners as coming with knowledge of solving a standard quadratic equation using the method of factorization. There is an indication here that student-teachers *repertoires* suggest an orientation towards learner and presences. They do not view learners as empty vessels but that they have knowledge of prior learning, though constraining, which they bring to the learning of other related concepts.

Moreover, across all the three scenarios, student-teachers spoke with voice of the unofficial discourse. Their descriptions of the sources of errors due to teaching sequence pointed to *overgeneralization*, in particular, *met-before* to a larger extent and *met-after* to a lesser extent. Interestingly, they did not use these discursive or theoretical terms developed in mathematics education literature. Therefore, student-teachers' *repertoires* were in the everyday professional experience of what it means to teach and learn. The conclusion therefore is that there is no specific focus on errors and misconceptions in these student-teachers' mathematics education courses.

The second dominant source of errors was the problem of interpretation. In squaring a binomial, student-teachers pointed to how learners were not able to interpret its definitional meaning in that it meant (x + 3) multiplied by itself. Moreover, student-teachers talked of how learners could not interpret the shortcut involved in carrying out direct expansion. They also pointed to that this was despite the teachers emphasizing. In terms of applying the distributive property, student-teachers pointed to how learners were not able to interpret 'Multiply n + 5 by 4'. Instead they interpreted directly by replacing 'by' in the statement with the multiplication operation to get 'n + 5 x 4'. In terms of the zero factor property, student-teachers pointed to how the learners interpreted $(x-1)(x+2)=4$ to have already been factorized and all that was remaining was to get either values of x.

Therefore, student-teachers' *repertoires* in locating the errors in the problem of interpretation suggest an orientation towards mathematics (school algebra) and absences. Learners are blamed for misinterpreting squaring a binomial, the distributive property, and the zero factor property. Moreover, student-teachers spoke with the voice of the official discourse of school algebra. It is interesting to note that when student-teachers talked about the definitional meaning of squaring a binomial and directly expanding a binomial they said this was despite teachers emphasizing. This suggests that student-teachers are equating teaching to learning in that if the teacher has taught, then learning should occur.

The third and less dominant source of errors was located in teaching emphasis. It is interesting to note that when student-teachers talked about the issue of squaring a binomial, they did not locate the source of error in teaching. They located the source of error in teaching when talking about the issues of the distributive property and the zero factor property although sparingly. In terms of the incomplete application of the distributive

property, student-teachers pointed to that the teachers could not have emphasized that $n + 5$ is one quantity, or alerted their learners that the mathematical task at hand was expanding algebraic expressions. In talking about the zero factor property, student-teachers pointed to how the teachers could have emphasized the process of solving standard quadratic equations using the method of factorization and ignored the non-standard ones. Moreover, they talked of how the teachers could have emphasized the importance of carrying out shortcuts in mathematics. As a result, when learners saw $(x-1)(x+2)=4$, the shortcut to them was to straightaway equate each factor to 4 and find either values of x.

Therefore, student-teachers' *repertoires* in locating the errors in teaching emphasis suggest an orientation towards teaching and absences. Teachers are criticized for not emphasizing key concepts or issues during teaching. Moreover, student-teachers spoke with the voice of the official discourse of school algebra in terms of what the teacher should have emphasized for learners to get the mathematics right. The conclusion is that student-teachers are equating teaching to learning, in that if teacher emphasizes then learners will learn, hence a transmission view of teaching and learning. This is a contradiction to the earlier conclusion on the problem of interpretation where teaching is not equated to learning.

Across all the three scenarios, student-teachers pointed to two possible remediating strategies, namely, emphasis in teaching, and practice examples with the dominant one being the former. In referring to the issue of squaring a binomial, student-teachers talked of how teachers ought to be emphasizing in their teaching the definitional meaning of expanding a term or expression and possible use of numbers to show the difference. They also pointed to how teachers ought to inform their learners that doing mathematics is about following the logic in terms of appropriate procedures and principles and not mechanical such as using shortcuts. In referring to the issue of incomplete application of the distributive property, student-teachers talked of how the teachers ought to be emphasizing in their teaching that: n + 5 is one quantity or thing, which should be bracketed and the distributive law applied by multiplying by 4, and that the issue of one thing could be represented by $y = n + 5$, hence $4y$. In referring to the issue of the zero factor property, student-teachers pointed to how teachers ought to be incorporating in their teaching procedures for factorizing non-standard quadratic equations and its graphical meaning of solving for x.

Therefore, student-teachers' *repertoires* suggest an orientation towards teacher and presences, and they spoke with voice of the official discourse of school algebra in terms of emphasizing in teaching procedures and principles. What is interesting about this dominant suggested strategy is that in talking about the issue of squaring a binomial, student-teachers did not locate the source of error in teaching emphasis. For the other two, that is the issue of the distributive property and the zero factor property, locating the source of error in teaching emphasis was also done sparingly. When it came to suggesting possible remediating strategies, across all the three scenarios, all pairs pointed to how the teachers ought to incorporate in their teaching key procedures and principles. On one hand, a similar conclusion could be made here that student-teachers have a transmission view of teaching and learning in that if teachers emphasize, then learners will learn. It is so direct in that teaching equals learning, hence teachers largely taking responsibility of teaching and not learner errors. Moreover, student-teachers are contradicting their earlier view of teaching not equal to learning which was identified when they located the source of errors in the problem with interpretation, especially in relation to the issue of squaring a binomial.

On the other hand, while the issue of teaching emphasis as a source of error was very dominant in Chapter 8 and as recruited from teacher-educators' discourses and student-teachers' initial discourses on LMT, it is less dominant in this Chapter. Therefore, the conclusion here although an inference, would be that when the scenario starts to look like student-teachers' own learners, then they distance themselves somewhat, not from their job of teaching but from the responsibility of the learners' errors. Moreover, this suggests that when working with student-teachers on learner errors, getting to the basis of some issues become obscured if actual learner productions are not dealt with. It therefore opens up the question, what kinds of records of practice engaged with in pre-service teacher education will make a difference?

The less dominant remediating strategy student-teachers pointed to as already indicated was that the teachers ought to be incorporating in their teaching many and varied exercises for their learners to practice. They said that this is the major way learners could become familiar with squaring a binomial, and applying the distributive and the zero factor properties. Therefore, student-teachers' *repertoires* suggest an orientation towards teacher and presences. They also spoke with the official voice of school algebra. This suggests that student-teachers are aware of the role exercises play in the teaching and learning of

mathematics although in a limiting way as it involves giving learners additional practice exercises.

As in Chapter 8, the main finding here is that student-teachers hold contradictory realizations about teaching and learning. They see learners as coming with prior learning to the teaching and learning situation as well as equating teaching to learning, hence a transmission view; and the former is the most dominant. The transmission view of teaching and learning again suggests that if emphasis is made in teaching and learners practice related tasks then learning is certain, hence a practical accomplishment. Student-teachers' discourses on error analysis are also largely in terms of process conceptions which are rather faulty and sparingly do they say the fault is due to not seeing structure in the algebraic expressions. Moreover, student-teachers also speak in the everyday professional knowledge of teaching and learning and not in the language of research. These findings again suggest that error analysis is not given explicit focus in these student-teachers' mathematics education courses, hence LMT is weakly classified and framed. Implicit messages are being relayed on what counts as criteria for carrying out error analysis.

CHAPTER 10

10 CONCLUSION AND RECOMMENDATIONS

10.1 Introduction

In this chapter, I discuss lessons learned from my study by referring to the findings and discussing them in relation to the implications for policy, research and practice. Included in the discussion are the research questions and how my study has attempted to answer them. My experience in dealing with LMT as a secondary mathematics teacher and now a mathematics teacher-educator; policy in Zambia on knowledge base for teaching, and the relation to the specialized field of mathematics education research made me embark on this study. The interest in my study was to explore and understand Zambian student-teachers' preparedness to participate in the discourse of engaging with LMT, particularly in school algebra, as they drew towards the end of their four years teacher education programme. Exploring student-teachers' preparedness would have been incomplete if I did not establish teacher-educators' privileged selections of the what and how of LMT, that is, what teacher-educators say they make available to their student-teachers as LMT, where in the mathematics education courses this is focused on, and how they position student-teachers/teachers, learners and the curriculum in their talk. This was based on the assumption that focusing on the discourse of and about engaging with LMT in teacher education is of much benefit to teaching and learning in general, and algebra in particular.

In exploring student-teachers' preparedness to participate in the discourse of LMT, focus was on their recognition and realizations, and this was in two ways. Firstly, a similar structure as for teacher-educators was used to interview student-teachers to establish what they said LMT is, and their positioning of teachers/student-teachers/teacher-educators, learners and the curriculum. In analyzing student-teachers' discourses, I also focused on how their talk relates to teacher-educators' in terms of the messages they recruit or take-up; and how this might be explained. Secondly, I focused on student-teachers' descriptions and explanations of common learner errors in school algebra. The learner errors were designed in scenario form and drawn from the specialized field of mathematics education research, and those elicited from learners' own working. Positionings and why their discourses are as they are were also explored. Moreover, the relationship between student-teachers' talk on scenarios and teacher-

educators' discourses and initial student-teachers' discourse were also explored in terms of messages that they recruit or take-up and how these might be explained.

In this conclusion, I provide a summary of the theory that informed my study and the analytic resources involved in analyzing the data. I also include in the conclusion findings, insights, and surprises. Methodological and theoretical insights gained are also discussed, hence contributing to the growing field of knowledge in mathematics teacher education in two ways, which are: what, how and with what effects a focus on LMT is occurring in practice; and the field understanding of mathematical knowledge for teaching algebra and the interpretation and use of learner thinking in particular. I also raise questions for further research in relation to LMT.

10.2 Framing of my study and analytic resources drawn from the literature

My study which is a qualitative case study is framed by Bernstein's (1982, 1996, 2000) theory of the pedagogic device whose sole purpose is to evaluate. I was concerned with evaluation criteria pertaining to the discourse of engaging with LMT, hence my unit of analysis, drawn from Adler & Davis (2006) and Davis et al. (2007), an evaluative event. My unit was structured by statements in form of texts about LMT. For my study, I had broad interviews with teacher-educators and student-teachers as conversations, and then scenario-based interviews with student-teachers. I have argued in Section 5.5 that the way Adler & Davis (op cit) used the notion of an evaluative event when focus was on assessment tasks is similar to how I have focused on student-teachers' participation in the scenarios. Each scenario was viewed as a new event because when scenarios were discussed with student-teachers, they had to legitimate what they said based on what learners had done. My interest then was to establish criteria the student-teachers gave for why learners make errors as exemplified in the scenarios. As a result, each analysis of student-teachers' interviews on scenarios was considered a piece of text that attracted evaluation. The evaluation was in terms of a three-stage process for carrying out error analysis, which following Peng & Luo (2009) and Jacobs et al. (2010) I re-described them as identify, explain and remediate.

I have also argued that I did not describe broad interviews with teacher-educators and student-teachers as events in the way Adler & Davis (2006) and Davis et al. (2007) talked about an event as they were concerned with pedagogy while my concern were conversations.

My assumption was that within these conversations aspects of pedagogy in relation to LMT would also be discussed. Here too, judgments would be made about what LMT is and what it is not, and why; and criteria transmitted. Therefore, event in this case for my study was recognized differently. Since teacher-educators and student-teachers did legitimate the descriptions of what entails the discourse of engaging with LMT, I considered a piece of text that described what engaging with LMT is and how it is justified as an event. The recognition was in terms of how their talk resonated with any one of Even & Tirosh (2002) broader categories of what entails the discourse of engaging with LMT and positionings. Three major categories of engaging with LMT drawn from Even & Tirosh (op cit) included developing in learners both instrumental and relational understanding, focus on learner errors, and creating an environment where teacher can listen to learners. These emerged from initial engagement with the data, and then were used to systematically interpret it. Therefore, in both cases (broad interviews with teacher-educators and student-teachers, and then scenario-based interviews with student-teachers) I was interested in how judgments are made about what the discourse of engaging with LMT is, and so the criteria.

This interest in my study influenced my use and re-description of *an evaluative event* as a unit of analysis in teacher education research but used it differently in that I used it on interviews. A focus on what the teacher-educators or student-teachers said is an inevitable limitation to the study, as has been discussed in Section 5.8. I did not follow student-teachers into schools – and so do not have data to understand how they would deal with LMT in the classroom situation as the context would be different with different effects. Further research would be needed to understand how student-teachers would work with LMT as a resource for teaching and learning in the classroom situation.

In Bernstein's (2000) terms, the notions of *reservoir* and *repertoire* were also used in describing and discussing student-teachers' discourses on scenarios. For each of the six scenarios, a possible *reservoir* was established on how the error would be described in the specialized field of mathematics education research. In Ryan & Williams' (2007) terms, the *reservoir* included modelling error, prototypical error, error as a result of overgeneralization, and error as a result of inadequate process-object conception [proceptual thinking in Watson's (2009) terms or process-object duality in Sfard & Linchevski's (1994) terms]. Error as a result of overgeneralization was re-described, in Lima & Tall (2008) terms, as an issue of *met-before* or *met-after*. Other concepts for explaining sources of learner error in Hart et al.

(1981) terms relate to learners' interpretation of letters, and for my study these included *letter evaluated, letter not used, letter used as an object,* and *letter used as a specific unknown*. Student-teachers' explanations for the sources of learner errors formed what in Bernstein's (2000) terms I have referred to as shared *repertoires*. Student-teachers *repertoires* were discussed in relation to the possible *reservoir,* initial student-teachers' discourses on LMT and teacher-educators' discourses. Resources drawn from Davis et al. (2007) also included the distinction between sensibility (a practical accomplishment) and intelligibility, in relation to teacher education. These concepts enabled the explanation of whether the discourse of engaging with LMT dealt with in mathematics teacher education is in the realm of the sensible and/or intelligible.

Bernstein's (2000) notions of classification and framing were used to describe the social structuring of the discourse of engaging with LMT in the mathematics education curriculum in terms of what the teacher-educators and student-teachers said it is and the 'teaching' and 'learning' of it, respectively. I also used Morgan et al. (2002) further elaboration of Bernstein's (op cit) notions of classification and framing in referring to teacher-educators' and student-teachers' positionings of teachers/student-teachers/teacher-educators, learners and the curriculum in terms of absences and presences. 'Absences' for my study meant when the talk pointed to what was thought to be not paid attention to in terms of teaching and learning with reference to teachers/student-teachers/teacher-educators, or learners, or the curriculum. For Morgan et al. (op cit) absences referred to teachers' consistent use of resources (assessment guidelines) that were provided; hence my use of absences is a re-description. In terms of 'presences', I followed a similar description Morgan et al. (op cit) used. They referred to presences when teachers' concern was on the quality of learners' thinking exhibited. In my study, I referred to 'presences' when the talk was in solidarity with what teachers/student-teachers/teacher-educators, or learners, or the curriculum are able to do or address.

The notion of positionings was also used when further focus on LMT was on scenarios on common learner errors described in literature and those that were elicited from learners' own working. The positionings were described in terms of 'voice' and 'form of practice' with some re-descriptions and extensions. 'Voice' for my study related to whether student-teachers spoke in the official discourse of school algebra or unofficial discourse. Voice(s) of unofficial discourse was further categorized into two, that is, either everyday professional knowledge of

teaching and learning discourses, or discursive/theoretical discourses by drawing on terms or concepts developed in the specialized field of mathematics education research. As for Morgan et al.'s (2002), teachers were positioned either speaking voice of official discourse of evaluation or voice(s) of other discourses. This is an indication that Morgan et al. (op cit) do not identify voice(s) of unofficial with everyday, but to be unofficial in the sense of other discourses which are not official sources (of school mathematics as informed by the ORF). Unofficial sources could include specialized mathematics education discourses or specialized mathematics education research discourses or other discourses. My identification of unofficial with 'everyday' is a further elaboration. Morgan et al. (op cit) identifies generality of teachers' commentaries to be either localized or specialized judgments. For my study, in the event that student-teachers are positioned in terms of voice(s) of unofficial discourse; their commentaries would be either everyday professional knowledge of teaching and learning discourses (localized), or discursive/theoretical discourses by drawing on terms or concepts developed in mathematics education literature (specialized in the sense of a vertical discourse). Reference to 'everyday' in this context refers to localized everyday discourses amongst student-teachers, hence implicit in the sense of a horizontal discourse. For example, if student-teachers talked of how dominance of teacher talk and inappropriate teacher response to learner answers could inhibit learning, I would identify this commentary with localized everyday discourse amongst teachers. If in the commentary, student-teachers make reference to the IRE interactions, I would categorize it under discursive/theoretical discourse.

In relation to 'form of practice', for my study, student-teachers were oriented towards either teaching and absences or learner and presences or mathematics and absences or teacher and presences. Similar to Morgan et al.'s (op cit) model were orientations towards learner and presences or mathematics and absences; and orientations towards teaching and absences or teacher and presences are my own extensions.

That I bring Bernstein's social theory to bear on what and how of LMT in terms of what teacher-educators and student-teachers say LMT is and how they teach or learn for it as well as in terms of what student-teacher recognize as the error and how of the explanations for the sources of error and suggested remediating strategies is in itself a theoretical and methodological contribution. In particular, methodologically, I have extended the use of the notion of *an evaluative event* as a unit of analysis on interviews unlike before when it was used on assessment tasks and instruction in teacher education. Theoretically, I have shown

how my study has been an attempt towards developing the internal and external languages of description concerning the discourse of engaging with LMT, an issue of potential future development.

10.3 Findings, insights and surprises

The analysis has shown that teacher-educators' privileged selections of what entails LMT is weakly classified and framed, hence implicit messages being relayed to student-teachers. Whether teacher-educators' focus is on developing in learners both relational and instrumental understanding, or learner errors, or creating an environment where teacher can listen to learners, there is a range of mixed messages being relayed, hence messages within them are spread out. That is, the criteria for what counts as LMT are weak because they are spread out. Moreover, it is also evident that across all the three major categories, LMT is a practical accomplishment as principles that would guide discussions around it are not so clear. LMT is talked about when focus is on principles that guide discussions on topics/courses in the mathematics education curriculum. The indication here is that the privileging by the teacher-educators while in these three domains, some of the big discourses in the specialized fields of mathematics education research and mathematics education are filtering through but in a very weak way.

Teacher-educators' discourses emphasize on how LMT as relational understanding is about working with mathematics as processes involving arguing, evidencing, systematic thinking, conceptual thinking, different strategies of getting to answer, and not just giving answers. They also talked about error as part of construction of knowledge involving prior knowledge, error as genuine, and analysis of error involving recognition of error and suggesting possible remediating strategies although constrained by time and teacher not recognizing occurrence of error. Focus is also on a classroom pedagogy that elicits learner thinking involving learners' explanations of their responses and questioning each other, consideration of their background knowledge, and application of knowledge in different contexts. However, there is an absence of discussion of sources of error outside of these being a problem of teaching. In terms of 'teaching' for LMT, teacher-educators' expectation is that through the practice of teaching such as peer teaching or school teaching practice, student-teachers demonstrate that they: plan lessons adequately, are aware of different routes to an answer, are not concerned about curriculum coverage for examination purposes, reflect on their teaching, sequence

subject matter appropriately, probe learner answers, ask questions from the onset, see learners as individuals, and facilitate learning.

In terms of positioning of student-teachers/teachers/teacher-educators or learners or the curriculum, teacher-educators as well as student-teachers hold contradictory views in referring to what is there pertaining to LMT as absences or presences. Teacher-educators see absences in student-teachers in terms of the dispositions outlined above largely due to absences in the teacher education curriculum in that not much time is given to school teaching practice and is programmed towards the end of the training, and that there is no in-depth focus on specific school mathematics. Moreover, absences in student-teachers are as a result of them being de-motivated due to large classes of learners who are also de-motivated. Teacher-educators also see presences in learners in that they have different strategies of getting to an answer which teachers do not utilize or that they have limited and not wrong mathematical conceptions. In encouraging student-teachers to assume dispositions involving probing learner answers, asking questions from the onset, seeing learners as individuals, and facilitating learning, positioning of learners is largely in terms of presences. Presences would also be realized in student-teachers if theory developed out of practice. Where LMT is taught is distributed across topics/course(s), an indication that it has no specific focus. Teacher-educators pointed to how LMT was discussed when focus was on, for example, aims and objectives of teaching mathematics, lesson planning, sequencing instructions, problem solving, assessment, strategies for teaching, school mathematics, and a foundation mathematics course for teachers.

As expected, student-teachers' realization of what entails LMT is also weakly classified and framed but they were able to recruit or take-up some of the messages from their teacher-educators with further elaboration in some cases. Similar to teacher-educators, student-teachers also talked of LMT as relational understanding involving strategies of getting to an answer, and conceptual thinking. Focus on learner errors was also referred to in terms of strategies for carrying out error analysis involving recognition of error and suggesting possible remediating strategies outside of which they locate source of error in teaching. In relation to classroom pedagogy that elicits learner thinking, student-teachers referred to how learners' thought processes could be accessed through facilitation. While teacher-educators encourage student-teachers to assume the role of a facilitator in the process of teaching and learning, they do not explain how this can be done and yet student-teachers do so. Student-

teachers' elaboration of facilitation as a way of working with learners' thought processes involve teacher awareness that learners have different learning abilities that need to be embraced despite large class sizes and time constraints. This could be made possible by the teacher: providing different work for slow and fast learners, exercising patience with learners' learning, and having good questioning techniques. In referring to strategies of getting to an answer, teacher-educators referred to the need for a teacher to be knowledgeable about multiple routes to an answer for the benefit of learner learning while student-teachers elaborate ways of working with learners' mathematical routes to answers.

As for positionings, discourse of absences dominated with some discourse of presences in student-teachers/teachers/teacher-educators or learners or the curriculum. Student-teachers see absences in themselves for not being familiar with some school mathematical problems in terms of: methods of solutions, explaining why some steps are taken, thinking of necessary basic concepts, and adjusting their thinking to that of learners. Absences realized in student-teachers are due to: absences in teacher-educators for not explaining to student-teachers why some steps in solving school mathematics are taken, context such as large class size and time constraints, absences in the teacher education curriculum for not having a book on misconceptions in school mathematics. Moreover, similar to teacher-educators' discourses, student-teachers pointed to how in teacher education there is no specific focus on school mathematics topics, and that school teaching practice is left to the end of the programme. Presences in student-teachers would be due to: understanding multiple routes to answers, and how such understanding would result in learners being creative, bringing experiences of learner errors and dealing with large classes, and awareness of the role of facilitation.

Student-teachers' realizations on scenario-based interviews with specific focus on common learner errors in school algebra are that largely, they were able to recognize learner errors except for one pair who could not recognize the error in scenarios 1 and 6. I have therefore, questioned what this would mean for learner learning if student-teachers cannot recognize such common learner errors as they near the end of their four year programme. In recognizing the errors in the scenarios, student-teachers largely spoke with voice of the official discourse of school algebra and were oriented towards mathematics and absences. Across all the six scenarios, three dominant explanations for the sources of error emerged which are: teaching emphasis, teaching sequence (ordering, overgeneralization – met-before or met-after), and a problem of interpretation. Student-teachers recruit or take-up from their teacher-educators'

discourses the understanding that sources of learner errors are in teaching in terms of emphasizing and ordering terms and concepts. In explaining the sources of error, student-teachers spoke either in voice of the official discourse of school algebra or unofficial discourse (everyday professional knowledge of teaching and learning). Orientation was either towards teaching and absences or learner and presences or mathematics and absences. Two main categories also emerged as possible remediation strategies, which are emphasis in teaching and practice exercises. In suggesting possible remediation strategies, student-teachers spoke with voice of the official discourse of school algebra and were oriented towards teacher and presences.

10.4 Reflection on the study and findings

In this sort of work, I can see the deficit discourses prevail in teacher-educators' or student-teachers' discourses when what they say about LMT can be described in terms of absences. Doerr (2004) says the problem with research that has focused on teachers' knowledge of algebra (teachers' SMK and PCK) is that it portrays a teacher who is deficient, but I have argued that my study is a departure from such focus to exploring what it is student-teachers know and their interpretation of the discourse of engaging with LMT. What I do in my study is bring analytic tools which Bernstein (2000) ably provides that can describe deficiencies teacher-educators or student-teachers see in each other including making reference to learners and the curriculum, and they are contradictory. For my study, 'absences' was referred to when teacher-educators or student-teachers talk pointed to what was thought to be not paid attention to in terms of teaching and/or learning with reference to teacher-educators/student-teachers/teachers or learners or the curriculum. I am not saying that teacher-educators or student-teachers are deficient but that what they say LMT is and the 'teaching' or 'learning' of it, they position the teacher/student-teacher/teacher-educator or learner or the curriculum in particular ways, one of which is in terms of absences.

Getting around the deficit was inevitable given the context of my study and that it was not primary focus. Teacher-educators or student-teachers seeing deficiencies is not surprising as it is persuasive since student-teachers are learning how to become teachers of mathematics while teacher-educators' role is to teach them to become so. For teacher-educators, seeing deficiencies make them reflect on what it is in terms of teacher education curriculum could be improved to enhance the preparation of teachers. As for student-teachers, seeing deficiencies

make them be aware of what they have to pay attention to in the process of learning how to teach school mathematics.

It is surprising that student-teachers hold contradictory views of teaching and learning despite constructivism being the theory espoused in their mathematics education courses. They equate teaching to learning; hence a transmission view as well as have the understanding that learners are not empty vessels but come with prior learning to the teaching and learning situation. This suggests that focus on LMT in terms of the nature of errors and strategies for carrying out error analysis is not a principled focus in these student-teachers' mathematics education courses. This is further confirmed in that student-teachers spoke in the everyday professional knowledge of teaching and learning.

It is also surprising that when scenarios described and discussed in Chapter 8 had the teacher involved, the dominant discourse for the sources of errors was located in teaching and so similar to the teacher-educators' discourses on LMT. As for scenarios that were elicited from learners' own working and discussed in Chapter 9, the dominant reason for the errors was in teaching sequence and more specifically overgeneralization (met-before/met-after). What is interesting is that the same interview questions were asked for scenarios discussed in both Chapters 8 and 9. Student-teachers were asked to explain their thinking of learner reasoning that was exhibited in each of the six scenarios. Following Segall's (2004) argument and discussed in Section 3.2, it follows that the difference in the design of scenarios in Chapter 8 to those in Chapter 9 invited student-teachers to participate in the discourse of engaging with LMT in particular ways. This suggests that the scenarios acted as pedagogical invitations for learning (or inquiry).

I have therefore argued that when the scenarios start to look like their own learners, student-teachers distance themselves somewhat, not from their job of teaching but from the responsibility of learners' errors. I then raised a question pertaining to the kinds of records of practice engaged with in pre-service teacher education that will make a difference if one is to get at the basis of some issues. I asked the question because my study has shown that when working with student-teachers on learner errors, getting to the basis of some issues become obscured if actual learner productions are not dealt with. Further research would also be needed to establish student-teachers' discourses of LMT when focus is on other records of practice such as watching a video of a teacher introducing algebra to secondary school learners or reading research.

From the aforesaid, I therefore argue that there is a way in which LMT ought to be explicitly structured in teacher education, for example, to include use of discursive resources from research. Like in the Manor project reported by Even & Tirosh (2002), explicit structuring of LMT would mean that the implicit messages would develop into more formal, deliberate, and explicit knowledge. Teacher-educators or student-teachers would then not only draw from everyday professional knowledge of teaching and learning but also from discursive resources developed in the specialized field of mathematics education research. I therefore argue together with Davis et al. (2007) that there is a way 'intelligibility' and 'sensibility' should regulate each other to build the knowledge base of LMT and the learning of it worthwhile for the benefit of the learner. Doing so would enable student-teachers to build on and interpret their experience-based knowledge of LMT using research-based knowledge. This would in turn indicate that LMT is not simply a practical accomplishment, but it has a knowledge base, and so discursive resources that can be made available for teacher-educators or student-teachers to work with, hence principled reproduction of LMT. Therefore, more work is needed on how LMT would be described if it has to be explicitly structured. This would include a longitudinal study as a follow-up on how teacher-educators and student-teachers work with the discourse of engaging with LMT in their lectures.

10.5 Setback in the analysis process

In trying to describe and discuss teacher-educators' and student-teachers' discourses of what they thought LMT is and how they 'teach' or 'learn' for it, respectively, I suffered what could be considered a setback. The process of analysis was not as linear and neat as portrayed since the organization of a large amount of qualitative data required conceptual thought and this takes time. In analyzing the data, I worked with a framework that seemed to be so rigid in the sense that I could see and understand what was going on but it was blocking others from seeing. My initial framework appeared useful for the early data I examined, but then did not expand sufficiently for the full data set. I thus revisited my early work and reconceptualized my framework so that it was more expansive and able to communicate productively with others. This included some re-consideration of the theoretical framing of the study, and the development of a robust frame so that the full data set could be uniformly analyzed leading to greater clarity in observing regularities across the data.

The reconceptualization of my framework also enabled me re-describe evaluation criteria for the legitimation of the discourse of engaging with LMT for teacher-educators, student-

teachers and how this compares with the field of mathematics education. This comparison has resulted in a re-description in some cases, hence an attempt towards the development of a strong language of description. The revisiting of some of the data through the refined lens has been important in developing a clear and coherent analysis and argument from my empirical work. Doing this was helpful because I was able to see different kinds of presences where I thought there were absences.

10.6 Recommendations for further research, policy and practice

I make the following recommendations for further research, policy and practice in relation to the findings of my study.

Recommendations for further research:

- Observe teacher-educators and student-teachers in their lectures as they interact on the discourse of engaging with LMT in the topics of mathematics education courses they said it is focused on to establish if similar findings would arise.

- Follow-up student-teachers in schools to establish how LMT is mediated in the classroom resonate with the findings, specifically, whether they put into practice what they say LMT is. While I make this recommendation, I take cognizance of Ensor's (2001) work which showed that it is not easy to transfer learning on a course into a new context, the school.

- Research on how the discourse of engaging with LMT could be structured if it has to be explicitly focused on, hence necessary to develop a strongly classified discursive base within the specialized fields of mathematics education research and mathematics education.

- Conduct a similar study at the beginning of the pre-service programme and mid-way the programme to establish if similar results would be found or there will be shifts in teacher-educators' privileged selections and student-teachers' realizations, hence broaden understanding of my findings.

- Broaden records of practice or discursive resources to include, for example, a teacher introducing algebra to learners and/or student-teachers reading research papers on LMT, respectively to establish if similar results would be obtained.

- The effect on the findings if focus of the study was extended to include mathematicians from the school of natural sciences who offer pure mathematics courses (75% of the programme) to student-teachers, and education teacher-educators from the school of education offering education courses (psychology and sociology of education).

- Establish reasons for why student-teachers hold contradictory views of teaching and learning when the theory espoused in their mathematics education courses is constructivism.

Recommendations for practice:

- Since the discourse of engaging with LMT is crucial in the preparation of teachers of mathematics and in the work of teaching, its focus should not be left to individual lecturers to decide but be explicitly outlined in the mathematics education curriculum and informed by discursive resources developed in the specialized field of mathematics education research and mathematics education.

- A book on common learner errors could be written to cover an extensive range in school algebra as well as other school topics, and this could be used as a reference book in the mathematics education courses.

- School teaching practice should be given adequate time and not left to the end of the programme so that issues of LMT can ably be dealt with during and after teaching practice.

- Dealing with LMT could be enhanced by including in the mathematics education courses in-depth focus on a selection of specific school topics.

- Since teacher education is also concerned with making known to student-teachers the curriculum intended in schools, teacher-educators should also focus on new topics introduced in the school curriculum especially if student-teachers have never had experience with in school so that they can produce teachers who are responsive to the needs of the school curriculum.

- Provide student-teachers with discursive resources on which LMT could be built so that explicit messages are transmitted otherwise student-teachers will base their

interpretations of LMT on what they know based on their experiences while in school or what their lecturers portray as 'best' practices, hence missing the criteria for practices that entail LMT.

Recommendations for policy:

- One of the guidelines that could be made available to teacher-educators is that, in teacher preparation, they should relate research to practice. This I suppose would enhance student-teachers' learning of LMT.

10.7 Concluding remarks

In this chapter, I have explained how the study was set and analyzed to describe and discuss what is recognized and focused on as the discourse of engaging with LMT generally, and crucially in algebra that underlies progress towards further studies. In describing the analytic resources that were used, I have discussed how my study has made a theoretical and methodological contribution to the specialized field of mathematics education research's understanding of MfT in relation to the discourse of engaging with LMT. A reflection of challenges I encountered in the process of coming up with a language of description to describe all the data sets has also been provided. I have shown that in a mathematics education programme where the discourse of engaging with LMT is weakly classified and framed, learning is inevitable although contradictory and spread out. This suggests invisible pedagogic practices in that the principles that regulate the discourse of engaging with LMT are implicit resulting in evaluation criteria being implicit to the acquirer. I have therefore argued that strengthening the learning is what is needed. I have also explained why some of the descriptions of teacher-educators' and student-teachers' positionings were in terms of absences. Ultimately, both teacher-educators and student-teachers see one of the significance of LMT as "*teaching learners mathematics as opposed to teaching mathematics*" (Extract 6.10 and Extract 7.23).

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12 APPENDICES

12.1 Appendix A: Teacher-educators' interview guide

This interview is designed for teacher-educators currently teaching mathematics education courses to student-teachers in the BScEd, BAEd, and BEd(Secondary) programmes in Zambia. As you are aware, this interview is as a result of the research which is focused on student-teachers' knowledge and understanding of and about the discourse of engaging with learner mathematical thinking, hence their preparedness for the tasks of teaching. This discourse implies dealing with learners' errors, misconceptions and difficulties that they experience in the learning of mathematics. Related to this are other instances that bring the image of the learner in focus. Teacher-educators themselves are very much aware of the importance of knowing mathematics in order to teach it well, hence what it means for the discourse in mention. The purpose of this interview is to explore with teacher-educators what and how they deal with the discourse in relation to selection, sequencing, pacing, and criteria, hence control over the social base.

- Tell me about yourself and experience in teaching mathematics education courses.

  -When did you start teaching? At what levels?

  -How long have you been teaching mathematics education courses in the University?

  -What difference have you found between teaching in a school and in teacher education?

  Probe: Their contents?

  Ways of communicating?

- Who are your student-teachers? What grade levels are they being prepared to teach? What kind of schools are they being prepared for?

  Probe: Their backgrounds/foregrounds?

  Why you think they are on the programme?

  How are they informing or shaping the mathematics education courses you offer?

- Mathematics teachers have to do a lot of things like explain content, mark learners work, make diagrams clear on the board, explain a procedure, design a test, mark homework, etc. Apart from what I have mentioned, what other aspects of the work of teaching do you think your student-teachers are prepared for?

  Probe: Which ones do you think they are well prepared for?

  Why do you think they are well prepared?

  Which ones do you think they are not well prepared for?

  Why do you think they are not well prepared?

- Think about your teaching; what do you think are the most significant moments in teaching?

- One of the things (you did not mention) that teachers do a lot is to understand why the learners are doing what they are doing. When they are going wrong. Why some are getting things right and some are getting things wrong. How do you work with this?

  Probe: How did you prepare them for that?

How might you prepare them for that?

- Some researchers have indicated that if observations are to be made on whether learning has occurred, moments that count are discontinuities, the jumps in the learning process. What are your experiences or your views on this?

  Probe: Since you are preparing student teachers for the work of teaching mathematics, how do you focus on learners' particular mathematical misconceptions, errors or difficulties in your courses?

  Is there a specific topic in your courses that deal with this?

  Why do you think it is important to focus on this in your courses?

- How do you make selections on what should be in focus in terms of dealing with learners' misconceptions, errors or difficulties?

  Probe: Why these selections?

- How is the way you discuss learners' mathematical misconceptions, errors or difficulties relate to developing teachers who are competent in teaching mathematics?

  Probe: Do you experience situations during the course of your teaching, when student teachers make reference to learners' misconceptions, errors or difficulties unexpectedly in responding to classroom discussions? Give an example.

- Give me an example of a typical lecture session where your student-teachers are engaged with what learners would do right or wrong?

  Probe: What were the problems that you met and how did you solve them?

  What was the importance of doing that?

  How did your student-teachers respond?

  If you were given a chance to teach this aspect again, what would you add to your lesson and why?

- In your assessment activities, give me an example of how you provide opportunities for your student-teachers to engage with learners' misconceptions, errors or difficulties?

  Probe: Is the way learners' misconceptions, errors or difficulties dealt with in the programme in any way relate to teachers' classroom practices? How are these connections made and why?

- As you focus on learners' misconceptions, errors or difficulties, how do you deal with a situation in your teaching when the student-teachers cannot grasp what you are trying to explain? First give me an example. What do you do when that happens?

  Probe: If the problem is as a result of a gap in student teachers' understanding, do you side track from your original planning of presentation of concepts in order to fill the gap and thereafter proceed with the earlier plans?

- One of the difficulties in any teaching is covering what we intend. So talk to me about when you are focusing on learners' misconceptions, errors or difficulties in your teaching, do you feel your lessons go fast or do you feel they go slow? What determines that?

Probe: Do you cover what you plan? If yes, how? If no, why?

- How do you convey to your student-teachers of what you expect of them in responding to tasks involving learners' misconceptions, errors or difficulties whether during lecture sessions or in assignments?

  Probe: Are they given any opportunity to decide on how to respond to tasks? How and why?

- Apart from dealing with learners' misconceptions, errors or difficulties, what other activities do you deal with in your lessons that make reference to learners in school?

  Probe: Give me examples of how you deal with each of the activities you have mentioned.

  How do the student teachers respond to such activities?

  How do you think these types of activities will help your student-teachers develop into teachers competent to do their work of teaching mathematics in school?

- Anything else that you think I have left out in relation to dealing with learners' misconceptions, errors or difficulties and any issues which bring a school learner into discussion in your lessons you think is important and you wish to share with me?

12.2 Appendix B: Student-teachers' interview guide

This interview is designed for student-teachers currently pursuing their BScEd, BAEd, and BEd(Secondary) programmes in Zambia. As you are aware, this interview is as a result of the research which is focused on student-teachers' knowledge and understanding of and about the discourse of engaging with learner mathematical thinking, hence their preparedness for the tasks of teaching. This discourse involves dealing with learners' errors, misconceptions and difficulties that they experience in the learning of mathematics. Related to this are other instances that bring the image of the learner in focus. The purpose of this interview is to explore with student-teachers what and how they orientate themselves towards the discourse.

- Tell me about yourself and why you decided to do a programme in mathematics teacher education?

  -What inspired you to undertake the programme?

  -How have your expectations been met now that you are nearing the end of the programme?

  Probe: Anything else you think should have been done but was not in terms of helping you develop into a competent mathematics teacher?

- Can you share with me some of the things that teachers have to do in their work of teaching?

  - Which ones do you think have been adequately addressed in teacher education? And why?

  - Which ones do you think have not been adequately addressed in teacher education? And why?

- One of the things (which you have/have not mentioned) which teachers have to do is understand learners' varied ways of thinking in terms of misconceptions, errors, difficulties or creativity. Some of these expected and some unexpected. Moreover, some researchers have indicated that if observations are to be made on whether learning has occurred, moments that count are its discontinuities, the jumps in the learning process.

  - What are your experiences or your views on this?

  - How prepared are you for this?

  - In your responses to course activities, do you make reference to what learners would do and what they would not do? How is this done?

    Probe: In your responses to the activities you deal with in the programme involving learners' misconceptions, errors, difficulties or creativity, do you in any way relate to teachers' classroom practices? How are these connections made and why?

  - What do you think is required of you to competently engage with learners' misconceptions, errors, difficulties or creativity?

  Probe: How does this relate to your understanding of particular school topics?

  Your understanding of learners' various solutions to a mathematical problem?

  Your knowledge on how and why learners come up with such solutions?

  Your knowledge between the standard ways and the non standard ways?

Your knowledge of the conceptions underlying all the different ways?

-Give me examples of how you dealt with each of the above in your programme?

Probe: What were the problems that you met and how did you solve them?

What was the importance of doing that?

If you were given a chance to learn this aspect of focusing on learners' misconceptions, errors, difficulties or creativity again, what would you work towards refining and why?

-How might you have been prepared?

- Do you have any school mathematics topics that you have dealt with in your courses directly?

Probe: Give examples

Explain more on this topic, how was it taught?

In your own opinion, how useful is it for you in terms of how you would teach in secondary school?

-Did you experience a situation when you had problems with solving school mathematical problems?

Probe: How did this experience inform you as a student teacher and a mathematics teacher to be?

- Do you think it is useful for you to deal with learners' mathematical misconceptions, errors, difficulties, or creativity in teacher education?

Probe: Explain with examples how this is important or not important?

- Have you ever encountered any difficulties when you engaged with learners' mathematical misconceptions, errors, difficulties, or creativity?

-What are those difficulties?

-How did you deal with them or overcome them?

-Do you think it would help to experience such difficulties at teacher education level – how?

Probe: Any other difficulties that you encountered? Which one was the most difficult? What did you do to overcome the difficulty?

- Apart from dealing with learners' misconceptions, errors, difficulties, or creativity, what other activities do you deal with in your lessons that make reference to learners in school?

Probe: Give me examples of how you deal with each of the activities you have mentioned.

How do you respond to such activities?

How do you think these types of activities will help you develop into teachers competent to do your work of teaching mathematics in school?

- How would you hope to put into practice the notion of engaging with learner mathematical thinking in your teaching?

  Probe: In your opinion, do you think this is feasible or not?

  What challenges do you think you would encounter? How and why do you think so?

  Explain with examples on how you would rise to the challenges you have identified?

- What is it that you think is important with regard to engaging with learner mathematical thinking that you think your programme has not addressed?

  Probe: How do you think the outlined importance mentioned above could be addressed in mathematics teacher education?

- Anything else that you think I have left out that would enrich our discussion?

12.3 Appendix C: Student-teachers' scenario-based interview guide

This interview is designed for student-teachers currently pursuing their BScEd, BAEd, and BEd(Secondary) programmes in Zambia. As you are aware, this interview is as a result of the research which is focused on student-teachers' knowledge and understanding of and about the discourse of engaging with learner mathematical thinking, hence their preparedness for the tasks of teaching. This discourse involves dealing with learners' errors, misconceptions and difficulties that they experience in the learning of mathematics. Related to this are other instances that bring the image of the learner in focus. The purpose of this interview is to explore with student-teachers what and how they are able to talk about learners' thinking and sense making related to algebraic thinking illustrated in the scenarios.

- How do you interpret what learners do in this scenario? Is it right or wrong?

  Explain?

  Probe: Are you familiar with what the learners are doing in solving the tasks?

- How do you explain and justify what the learners are doing right or wrong?

  Probe: What do you think are the sources of what learners are doing right or wrong in the tasks?

- How do you relate what learners are doing right or wrong to key concepts in the school curriculum?

- How do you interpret what other teachers do with what learners do right or wrong?

  Probe: If you were the teacher in this classroom, how different or similar were you going to manage the situation?

- What representations, in relation to the tasks, do you propose to help learners develop the required algebraic thinking?

  Probe: How would you help such learners develop the required algebraic thinking in relation to the tasks?

- What do you think about these tasks and what you have experienced in relation to: you as a mathematics student-teacher, the learners you are going to teach, and your mathematics education courses in teacher education?

12.3.1 Scenario 1

A Grade 8 teacher noticed that when learners were asked to simplify the expression

$2x + 5 + 3x – 7$, they did the following:

$2x + 5 + 3x – 7 = 0$

$5x - 2 = 0$

$5x = 2$

$x = \frac{2}{5}$

Source: Adapted from Wagner & Parker, 1993

i. Identify learner thinking in the worked out solution.

ii. What is the source of their thinking?

iii. How would you help such learners overcome their difficulty?

12.3.2 Scenario 2

Teacher Benny engaged his class in learning algebraic expressions.

He worked with the rule "add the numbers separately and add letters separately"

Benny: Asks what $3m+2+2m$ equals?

Benny: Reads the rule.

Benny: Suggests colouring the 'numbers' and writes $5m+2$.

L_1: And what now?

L_2: $7m$

Benny: No $5m+2$ does not equal $7m$, and repeats the rule again.

Benny: Gives another example $4a + \mathbf{5} - 2a + \mathbf{7}$, colours free numbers, dictates the rule to the class and then asks the class to repeat the rule.

Learners: Work on similar exercises and continue experiencing difficulties.

Source: Adapted from Tirosh, Even and Robinson, 1998, also reported in Even and Tirosh, 2002

Assume you are the teacher in this class, how would you explain learners' understanding that has developed?

12.3.3 Scenario 3

A grade 10 teacher asked learners to solve the equation $2x^2 = 6x$ on the board.

L_1: Divides both sides of the equation by 2 and obtains $x^2 = 3x$.
Then divides both sides by x and gets $x = 3$

L_2: You cannot divide both sides by x

L_1: If you can divide both sides by 2, why can't you divide by x?

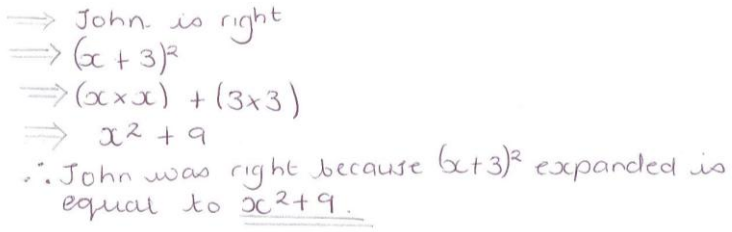
Adapted from Learning Mathematical Knowledge for Teaching, 2004

12.3.4 Scenario 4

In Grade 9B at a school in Zambia, learners were asked to expand $(x + 3)^2$.

Mary said it is $x^2 + 6x + 9$, and John said it is $x^2 + 9$.

Who is right, Mary or John? Explain your answer.



Source: Adapted from Vermeulen, 2007

12.3.5 Scenario 5

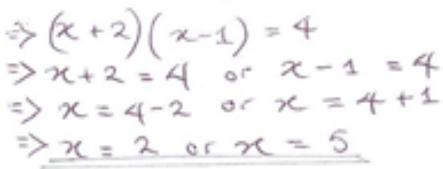
Multiply n + 5 by 4



Source: Adopted from Hart et al., 1981

12.3.6 Scenario 6

Solve for x in the equation $(x – 1)(x + 2) = 4$



Source: Adapted from Bell, 1995

12.4 Appendix D:Detailed illustration of coding and analysis of three events

Titus' discourse of LMT						
Excerpt	**Where**	**What**	**How in terms of positionings and 'teaching'**			
		LMT	**T or ST**	**C**	**L**	**Teaching**
I: Oh. So one of the things you did not mention that teachers do a lot is to understand why the learners are doing what they are doing. T: OK I: When they are going wrong, why some are getting things right and some are getting things wrong. How do you work with this? T: **(Extract 6.50)** Well I think even at that time I used to understand what learners were doing because for example when I took over the two Grade 12, Form 5 classes at the beginning of my teaching they said, 'No, mathematics is difficult, we don't want to learn anything.' But I discovered that their former teacher was a very hard person. She shouted at them. She would say all sorts of things and that made them lose interest. But I explained to them, I showed them that it was not very difficult. But I should say my only looking at, my understanding of how to use mathematics, children's mathematics thinking is not something that I used to think about very much as a teacher, it's something that I think about very much now. In fact I have been influencing, I attended two training projects by the Ministry of Education. In Ndola for example, there was a group of what we call Resource Centre Co-ordinators			Teachers rarely think about learners' mathematical thinking as a resource for teaching. **A**		Learners find mathematics hard to learn because of the nature of their teachers and how they handle their learners to an extent of losing interest in mathematics. See absences in learners as a result of nature of Teachers. **A**	

I: OK T: Ja, from all provincial Resource Centre Co-ordinators and district from the half of Zambia to the north, the copper belt, they all gathered in Ndola – about 5 provinces – and I talked about how they can use children's thinking in their teaching of mathematics. I: And how? T: It was very well taken and they were very surprised that in their own teaching they have not, they do not usually think that children know something already and that it's possible to get to what they know already and use it in shaping your lessons and so on. They didn't do that. We also met at Andrew's Motel, Lusaka for the other groups, the other provinces that were not part of the Ndola one. I: OK T: So they were surprised, they… One thing I discovered is that the idea of discussion is usually not part of teaching. I: From them? T: From them I: From their perspective? T: Yes, from their perspective. I: OK		LEM - Using learners' prior knowledge as a resource for teaching (PKRT) Env- Discussion as part of teaching (DPT)	There is a sense of absences in teachers in terms of working with learners' prior knowledge in their lessons. **A**			

T: That discussion was not part of teaching. And whereas a child can answer a question when you ask a question, a direct question, and the child answers, they were not asking the child to explain why the child thought the answer was correct. You see, if you asked a question in class you say, 'What is this and this?' and then the child gives you an answer and then you say, 'Very good. Sit down.' The rest of the group in the class will wonder why that answer was correct. I: Mmm T: They may not know even why it was correct, but if a child is asked to explain and encourage the others to question that person, those were the things I was dealing with. So I wasn't thinking so much about how children learn, what they understand really, when I was teaching in secondary school. It's only now that I look back and, and I can see that, and that's what I try to do with my students here. What I tell them usually is that the most important thing is not for you to learn the content of the course, my aim is to get you to start thinking like teachers do. If you start thinking like a teacher, then you can do almost anything. I: So what, what does it mean to 'think like a teacher'? T: To think like a teacher means to understand what children already know and to think about, 'How do I get them to understand this? How do I develop their knowledge in this area?' Ja. Whatever you are going to teach, what do they already know about this? How can I connect them to this? So this is what I, that's what I tell them that they should develop, they should learn to think like a teacher, not to think like somebody else who's not a teacher. *(chuckles)*		Env – DPT – EQ LEM - PKRT	Sense of absences in teachers in terms of seeking explanations from learners for their answers. **A** There is a sense of absences in teachers in terms of thinking about learner learning. A See absences in student teachers thinking like teachers. It is a skill that needs to be learned.			

I: OK T: Anyway, that's what I try to do.						
I: Some researchers have indicated that if observations are to be made on whether learning has occurred, moments that count are discontinuities, the jumps in the learning process. So what are your experiences or your views on this? T: Ja, the discontinuities I think relate to the fact that children know something, but you start them at a certain point. You don't link what they know with what you are doing so there is a what we call a seamless transition from lack of knowledge to being knowledgeable. That… I: It's referring to the learner. T: The learner? I: Yes, whereby there's no smooth path of understanding. They have some T: some problems, ja I: some problems in moving from understanding from this level to the T: to the next level I: Yes. So there's a, there's a gap in between in their understanding. Or maybe we should say the way they have understood the concepts is not in the way it is supposed to be understood mathematically. T: Yes						

I: Those are the jumps T: OK I: To them they are, that's how they've understood the issue. T: Yes, I think I: But you as a teacher you'll see a problem with their way of understanding. T: **(Extract 6.51)** Yes I think certainly from my own, from the time when I began teaching I've always been very careful with the way I sequence the work, sequencing the subject matter. When subject matter is sequenced appropriately (I'm not saying correctly) there's no such thing as sequencing correctly but appropriately with regard to what children already know there will not be any gaps in knowledge because you'll be able to pick them from the level where they are and bring them up to a certain level of understanding. So personally I also try to tell my students here and I try to demonstrate. And even when I go to talk, observe students on school experience, I see a lot of those discontinuities. I: In the learners or the student teacher? T: In the teaching and you can see it from the blank looks on the faces of the learners, you can say, 'Something here, they are not quite sure.' And since… I: Do you have an example to share?						PKRT is realized through sequencing subject matter appropriately (**SSMA**) PKRT is realized through practice of teaching. **PT – TP**

T: Ah, I can't remember clearly anything, but what I do remember clearly, that when somebody was teaching he said something and then the learners went something like, 'Ah' they, he didn't get it but their mouths opened and I could see that something was wrong, but the person did not. Now unfortunately I cannot get back because I wish I had thought about it earlier. But I've done a lot of observation of teaching and I always see these students start something but they move on. For example, here's an example. Somebody is teaching a class and then uses a term which they assume that students know, but students don't know it. They will just use a term and not define it. Now when… I: For example? T: Well for example something like perimeter. And maybe I can even go down to primary school. One day I observed a, I left this place; I went to showground where they train primary school teachers. And I was observing, I joined the group of lecturers from there who went to observe somebody who was teaching. Now this person was using English *(chuckles)* and he was teaching a science lesson. (Unfortunately it was a science lesson.) It was about a bicycle. So he talked about a bicycle in English for a long time and left the bicycle outside. So he was drawing this and so on. And at one point he asked a question in English and asked again. Nobody raised his or her hand up. But when he changed it to a Zambian language Chitonga and asked, then all the hands went up. Then later that's when he brought the bicycle and said, 'This is what I was talking about.' So I said, 'But this, there is some problem here. You were talking about wheels,' (when we were talking to him later), 'You were talking about wheels and you			Sense of absences in student teachers in terms of knowledge gaps/PKRT during teaching practice. **A**			

were doing, you were drawing wheels, why didn't you come with the bicycle and say, "This is what we mean by a wheel" and so on, so that when, after you've talked about that then you could draw a bicycle and students would understand.' But he started by drawing something and talking to them in English, then later brought a bicycle in and… But in secondary teaching, I think mainly it's students who use new words without defining them and they assume that everyone understands them, but *(chuckles)* usually they don't. I: What about the situation where sequencing has been well done and the teaching has been well delivered but the learners do demonstrate some, their understanding, the way they have understood what has been taught seem to have some problems here and there. So how do you explain such a situation? T: I do remember someone at Luanshya Girls who did not understand what a determinant is. Not a determinant, but ja, a determinant of a 2 by 2 matrix. He thought that a determinant of a 2 by 2 matrix is the number you get, you know, when you work out the determinant itself, then you divide that number by, by the sum of the numbers inside. So somehow he would always write a determinant as something, as a fraction. He, what he was thinking was that since determinants are calculated when you want to get an inverse so the final determinant is always the determinant itself divided by some number. Now students were taught that. Now I thought that was conceptually wrong. So a person like that can teach and people will still be confused at the end. Now I used the word 'delivered'. I: Oh, the teacher is the one who had the problem?			There is a sense of absences in teaching in terms of conceptual understanding and how this impacts on learner learning. **A**			

T: The teacher had a problem so even if he taught well students would still be, have difficulty. I: OK T: Now you use the word 'deliver'. Well in fact I don't use the word 'deliver' anymore I: *(laughs)* T: because delivering to me sounds like picking a bunch of knowledge and then giving it to the students I: giving it to the students, OK. T: What I look at is helping students to develop I: conceptual understanding T: conceptual understanding, ja. I: Mmm T: So I don't even use the word 'impart'. Whenever I find it in one of the assignments I circle it. I: You circle it T: Yes, and say ? I: You don't 'impart'. T: Ja, you don't 'impart' knowledge, you don't deliver knowledge, but you teach. And teaching involves an interactive process whereby students tell you what they know, you learn from them and you						

also tell them what you know and so on. But you were asking about I: situations where T: where the lesson has been taught I: Ja T: but students are not clear about something. I: Ja, some students still, their understanding is still limited in that the text they are producing have some difficulties. T: Yes, but they should know that sequencing is never a correct *(chuckles)* there's no correct sequence for everyone. So even in a small group of students you are sequencing for the average student, maybe for the majority of them I: Ja T: but there could be some students whose own understanding of things is maybe some what we may call slow learners (if that terms means to applicable) *(chuckles)* who may require a more systematic approach to the issue of, or who may require a slower explanation. So even when you have taught a class and you have sequenced appropriately you will still not be able to reach everyone. There may still be some people who will not understand. You're asking whether what could be the problem. The problem could be maybe in the sequencing or in the teacher's own understanding. It may also be that some students simply should not be in that class.						If learners experience difficulties it means there is a problem with the sequencing (SSMA), teacher knowledge (TK), or that such

Maybe they belong elsewhere. I'm not very sure. I: OK *(laughs)* T: Ja. You could sequence something correctly but not make use of their understanding, the understanding they already have. So even if you sequence it correctly and finish your teaching you have to lecture to them, they may not still have understood anything. And it is also possible that during the lecture, during the lesson you may be asking questions, you ask students questions and then when the answer comes, like I saw in the primary school, you clap for this one. So they clap. When they clap they don't know why they're clapping, they know it's correct but they don't know I: why is it correct T: why is it correct? So you can sequence things correctly and teach your lesson and yet some students might not understand fully what has happened because those who answered questions answered them and you never probed the answers, you never tried to justify, to ask them to **justify their thinking**. So that leaves room for *(chuckles)* for some incorrect thinking, ja.		**LEM- PKRT** **Env-DPT-E**	There is a sense of absences in teachers in terms of sequencing if learners' prior knowledge and explanations for their answers are not taken into consideration. **A**			learners are not supposed to learn mathematics. SSMA includes taking into consideration learner prior knowledge (PKRT) and learner explanations for the answers they provide to teacher questions (EAPTQ). **EAPTQ- JT**

I: So since you are preparing student teachers for the work of teaching mathematics, how do you focus on learners' particular mathematical misconceptions, errors or difficulties in your courses? T: Actually I've been talking about, I've been talking to students about not using the term 'misconceptions' or 'errors'. *(chuckles)* The term that I've adopted now is 'alternative conceptions'. **(Extract 6.52)** When a student, when they are teaching students if they ask a question and a student gives a wrong answer, 'wrong' (in inverted commas) I: Why in inverted commas? T: Because that's their understanding. That's their own understanding of what is happening, the teacher should try to ask the student to explain things. In fact what I've been advising students to do is you don't necessarily need to say, 'That is wrong. That answer is wrong.' What you do is you ask the student to explain and then allow others to question that student. That student will actually discover on his or her own that 'my earlier answer was wrong' and revise it without you necessarily saying it's wrong. So in handling alternate conceptions I always ask student teachers to always probe. You could say, for example, you ask a question and then somebody answers in a very strange fashion. You could say something like, 'I've never thought of that as an answer, tell me, why do you think that's an answer?' Or you could say, 'That's a different way of thinking' or maybe, I don't know *(laughs)* 'That's something that I've never thought about before. Tell me, why do you think this is the thing?' And then the child will explain. So that's how I ask them to handle what we might call misconceptions.		Env-DPT-E -Q	Teachers do not have to refer to learners' answers as wrong as they will eventually discover on their own when the space for discussion is opened up.			Env-DPL-E-Q is learned through probing learner answers (PLA)

I: OK T: Ja. I call them 'alternative conceptions' so that teachers should always want to listen to even a wrong answer because to probe a wrong answer means eventually you understand why that person thinks it's correct and after the other students have chipped in, there's what we call 'negotiation of meaning' *(chuckles)* I: Mmm T: The student will learn that, 'Oh, so my own thinking was actually not correct.' So she adjusts her own thinking. But teachers don't need to say, 'That is wrong.' In my case I found out that it is not very necessary any more. I: OK. So T: Unless it's a joke and the student is just saying something. I: Is there a specific topic in your courses that focuses on alternate conceptions? T: Um, a specific topic? I: Topic, yes T: In my courses? I: Yes, which deals with alternate conceptions?			Through probing learners' answers, it is no longer necessary for a teacher to point out that learners' answers are wrong as learners will realize on their own and adjust their thinking.			Through probing teacher will eventually understand learners' thinking processes and come to a common understanding through discussion. (PLA)
T: (Extract 6.53) I think problem solving is a good example. You know when students, in problem solving you normally want to, not so much looking for	Problem solving	MRR-FA-RA				

the answer, but the route to the answer. And you'll find that there are different ways in which students think. My own study now is focused on what we call 'mental calculation'. 'Mental calculation' means students work out an answer, sometimes not necessarily, they may not be, they may not even have written it down, but what I've been trying to do is when a student works out an answer mentally, ask them to show you how they worked it out. I: OK T: You will find out that there are different ways to get to a correct answer. And that's the thing that I try to do. So problem solving is a very good example of where students will bring their own ways of thinking. I: Can you give me one example of the mental mathematics where students have come up with varying ways of finding the answer? T: For example I was dealing with a Grade 2 class. If you said 29 – 18, for example, students, some students will say '29 is very close to 30 and 18 is close to 20. Why don't I just say 30 – 20 and then I take away, I add 2 and take away 1?' *(laughs)* I: OK. T: If you say 30 – 20 you get 10 I: 10 T: Add 2 I: 12	Problem solving	**MRR-RA**			There is a sense of presences in learners in terms of coming with some understanding to the learning situation.	

T: Take away 1 I: 11 T: 29 – 18 is 11 *(chuckles)* I: OK *(laughs)* T: So *(laughs)* so this is how children sometimes work. Now the problem is when children get into school we want to teach them directly the procedures which have been refined already. What we are supposed to do is to take what they already know and eventually show them that actually what they are doing can be done more efficiently using the new procedures that have been developed already. 29 – 18 you write 29 on top and then – 18 below. I: Mmm T: Ja. But students don't usually work things like that. They find things in their own way and then if you don't understand what they are doing, if you don't try to ask them to explain to you what they are thinking, then you'll go away and say, 'It's wrong. She can't even explain the answer.' It's just *(laughs)*. So anyway, that's one of the things I've been doing.			There is a sense of absences in teachers in terms of working with learners' already existing strategies before they are introduced to conventional ones. **A**		There is a sense of presences in terms of strategies to solutions to mathematical problems learners come with to a lesson. **P**	MRR-RA is learned by teachers knowing that learners have their own strategies of working out mathematical problems and it is the teacher's role to ask for learner explanations of their reasoning. (ALER)

12.5 Appendix E: The ethics clearance letter, and examples of information letters and consent forms

12.5.1 The ethics clearance letter

Wits School of Education

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Tel: +27 (11 717-3007 • Fax: +27 (11 717-3009 • E-mail: enquiries@soe.wits.ac.za • Website: www.wits.ac.za

STUDENT NUMBER: 381478
Protocol: 2009ECE66

15 August 2009

Mrs PP Nalube
WSoE

Dear Mrs. Nalube

Application for Ethics Clearance: Doctor of Philosophy

The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate has considered your application for ethics clearance for your proposal entitled:

Student Teachers Learning Mathematics for Teaching: Learner Thinking and Sense Making in Algebra

The following comments were made:

- It is stated that the research will be stored in a locked cupboard, but it does not say for how long. The time limit for the storage of the raw data is 3-5 years and this should be stated.

Recommendation:

Clearance is granted.

Yours sincerely

Matsie Mabeta
Wits School of Education

Cc Supervisor: Prof. J Adler (via email)

12.5.2 Examples of information letters and consent forms

To: The Mathematics Education Teacher Educators, School of Education

From: Mrs. Patricia P. Nalube—Doctoral Student (University of the Witwatersrand)

Date: July 2009

Subject: **Invitation to Participate in a Research Study on Learner Mathematical Thinking**

My name is Patricia P. Nalube, a PhD student in the department of Mathematics Education at the University of the Witwatersrand, South Africa. I am writing this letter to invite you to participate in my doctoral research study, which is investigating student teachers' knowledge and understanding of learner mathematical thinking. The focus of the study, entitled *Student Teachers Learning Mathematics for Teaching: Learner Thinking and Sense Making in Algebra*, is to establish student teachers' preparedness to engage with learner mathematical thinking in general, and algebra in particular, in relation to the intended mathematics teacher education curriculum. The aim of the study is not to evaluate the Zambian teacher education programmes but rather to explore and explain how the discourse of and about learner mathematical thinking is structured in the mathematics teacher education curriculum as well as how the student teachers participate in the discourse.

As a participant in the study, you will be invited to take part in the data collection process in two ways. Firstly, you will be requested to kindly provide copies of the curriculum, course, and assessment materials used in the courses that you are currently teaching. These materials are to assist in a document analysis which is aimed at providing the context in which the discourse of and about engaging with learner mathematical thinking occurs in teacher education. Secondly, you will be invited to participate in an individual interview in which I will try to find out how, in your courses, you engage student teachers in the discourse of and about learner mathematical thinking.

This study will take place between August 2009 and January 2010. The individual interview will be about 45 to 60 minutes long and will be audio taped. Please be assured that your participation in this study is voluntary and that you may withdraw at any time. Below are detailed steps that will be taken to ensure anonymity and confidentiality of your participation in the study:

- Your name or that of your institution, and other features that might render you recognisable, will not be used in the final research report
- The raw data which may contain your name will be stored in a locked cabinet, and will be destroyed when no longer required
- Only researchers directly involved in my research study, such as my project supervisor and members of my supervisory committee, will see the data, for example notes from interviews and transcripts of the audiotapes.

Once the research has been completed, a brief summary of the findings, and subsequently, the whole thesis, will be made available to your institution. It is also possible that findings will be presented at academic conferences and published in national and international academic journals. Finally, I would like to take this opportunity to invite you, if you are interested, to participate in any collaborative work that may develop from the study in terms of sharing knowledge and expertise through publications and initiation of further research activities.

If you agree to take part in this study, please read through and sign the attached consent form for data collection.

Should you require any further information about the study, please do not hesitate to contact me or my supervisor at the contact details below.

Mrs. Patricia P. Nalube

(PhD Student and Researcher)

Ph: +27-11-717-3410 or +27-8335-27350

Email: Patricia.Nalube@wits.ac.za

Professor Jill Adler

(Project Supervisor)

Ph: +27-11-717-73413

Email: Jill.Adler@wits.ac.za

Yours Sincerely,

_____

Mrs. Patricia P. Nalube

MATHEMATICS EDUCATION TEACHER EDUCATOR CONSENT FORM TO PARTICIPATE IN THE LEARNER MATHEMATICAL THINKING RESEARCH STUDY

I, (please print)_____**,** a mathematics education teacher educator in the Department of Mathematics Education at (specify University) _____, agree to participate in a research study entitled *Student Teachers Learning Mathematics for Teaching: Learner Thinking and Sense Making in Algebra*, conducted by Mrs. Patricia P. Nalube, for her doctoral research study, with specific focus on student teachers' knowledge and understanding of learner mathematical thinking.

I am aware that the research process may involve:

- An analysis of curricular and assessment documents used in mathematics education courses that I am currently teaching (please tick) Yes ☐ No ☐
- An individual interview, which will be audio taped (please tick) Yes ☐ No ☐

I am satisfied that the aims of study and my role in the study have been explained at the beginning of the project and that there will be ongoing discussion during the process of data collection.

I am also aware that protection to anonymity and confidentiality has been guaranteed and that I can withdraw from participating in the study at any time.

_____ _____

Signature **DATE**

To: The Mathematics Education Student Teachers, School of Education

From: Mrs. Patricia P. Nalube—Doctoral Student (University of the Witwatersrand)

Date: July 2009

Subject: **Invitation to Participate in a Research Study on Learner Mathematical Thinking**

My name is Patricia P. Nalube, a PhD student in the department of Mathematics Education at the University of the Witwatersrand, South Africa. I am writing this letter to invite you to participate in my doctoral research study, which is investigating student teachers' knowledge and understanding of learner mathematical thinking. The focus of the study, entitled *Student Teachers Learning Mathematics for Teaching: Learner Thinking and Sense Making in Algebra*, is to establish student teachers' preparedness to engage with learner mathematical thinking in general, and algebra in particular, in relation to the intended mathematics teacher education curriculum. The aim of the study is not to evaluate the student teachers' mathematical performance but rather to explore and explain how the discourse of and about learner mathematical thinking is structured in the mathematics teacher education curriculum as well as how the student teachers participate in the discourse.

As a participant in the study, you will be invited to take part in the data collection process in two ways. Firstly, you will be requested to voluntarily take part in three individual interviews. The first interview is to establish how you may orientate yourself to the discourse of and about learner mathematical thinking. The second interview will be to ascertain how you may engage in the discourse of and about learner mathematical thinking in solving algebraic problems. In the third interview you will be invited to reflect on your experiences of working on the assigned algebraic tasks and the ways you will have attended to the notion of learner mathematical thinking. Secondly, you will be invited to participate in three focus group discussions in which you, together with other student teachers, will be discussing different ways of engaging learners in deep mathematical thinking about a selected algebraic task.

This study will take place between August 2009 and January 2010. The individual interviews will be about 45 to 60 minutes long each and will be audio taped. The focus group discussions will be two to three hours long and will be audio and video taped. Ideas from journal articles on how one may attend to learner mathematical thinking will be shared and made available for reflective discussions. Also, the focus group will watch a video tape on different ways of introducing algebra to learners. This will be followed by a group discussion on what learner mathematical thinking is entailed in the teaching and learning of algebra.

Please be assured that your participation in this study is voluntary and that you may withdraw from the study at any time. Below are detailed steps that will be taken to ensure anonymity and confidentiality of your participation in the study:

- Your name or that of your institution, and other features that might render you recognisable, will not be used in the final research report
- The raw data which may contain your name will be stored in a locked cabinet, and will be destroyed when no longer required
- Only researchers directly involved in my research study, such as my project supervisor and members of my supervisory committee, will see the data, for example notes from interviews and transcripts of the audio and video tapes.

Once the research has been completed, a brief summary of the findings, and subsequently, the whole thesis, will be made available to your institution. It is also possible that findings of this study will be presented at academic conferences and published in national and international academic journals.

If you agree to take part in this study, please read through and sign the attached consent form for data collection.

Should you require any further information about the study, please do not hesitate to contact me or my supervisor at the contact details below.

Mrs. Patricia P. Nalube

(PhD Student and Researcher)

Ph: +27-11-717-3410 or +27-8335-27350

Email: Patricia.Nalube@wits.ac.za

Professor Jill Adler

(Project Supervisor)

Ph: +27-11-717-73413

Email: Jill.Adler@wits.ac.za

Yours Sincerely,

_____

Mrs. Patricia P. Nalube

MATHEMATICS EDUCATION STUDENT TEACHER CONSENT FORM TO PARTICIPATE IN THE MATHEMATICAL LEARNER THINKING RESEARCH STUDY

I, (please print)_____**,** a student teacher in the Department of Mathematics Education at (specify University) _____, agree to participate in a research study entitled *Student Teachers Learning Mathematics for Teaching: Learner Thinking and Sense Making in Algebra*, conducted by Mrs. Patricia P. Nalube, for her doctoral research study, with specific focus on student teachers' knowledge and understanding of learner mathematical thinking.

I am aware that the research process may involve:

- Three individual interviews which will be audio taped (please tick) Yes ☐
  No ☐
- Four focus group discussions, which will be video/audio taped (please tick)
  Yes ☐
  No ☐
- Analysis of my teaching practice file (please tick) Yes ☐
  No ☐

I am satisfied that the aims of study and my role in the study have been explained at the beginning of the project and that there will be ongoing discussion during the process of data collection.

I am also aware that protection to anonymity and confidentiality has been guaranteed and that I can withdraw from participating in the study at any time.

_____ _____

Signature **DATE**

INSTITUTIONAL CONSENT FORM FOR THE SCHOOLS' PARTICIPATION IN THE RESEARCH

I, (please print)_____ in the capacity or on behalf of the **Ministry of Education in Zambia,** give permission to Mrs. Patricia P. Nalube to conduct her doctoral research study entitled *Student Teachers Learning Mathematics for Teaching: Learner Thinking and Sense Making in Algebra*, in selected schools, with specific focus on student teachers' knowledge and understanding of learner mathematical thinking.

I am aware that the research process will engage learners in mathematical tasks where they will be solving algebraic problems and that the purpose of the study is not to assess their mathematical performance.

I am satisfied that the aims of study and the role of the institution in the study have been explained at the beginning of the project and that there will be ongoing discussion during the process of data collection.

I am also aware that the participation of the learners in the study is voluntary and that any participant can withdraw from the study at any time.

_____ _____

Signature **DATE**

INSTITUTIONAL CONSENT FORM FOR A SCHOOL TO PARTICIPATE IN THE RESEARCH

I, (full name)_____ the Head Teacher of (name of school) _____ give permission to Mrs. Patricia P. Nalube to conduct her doctoral research study entitled *Student Teachers Learning Mathematics for Teaching: Learner Thinking and Sense Making in Algebra*, in the school mentioned above, with specific focus on student teachers' knowledge and understanding of learner mathematical thinking.

I am aware that the research process will engage learners in mathematical tasks where they will be solving algebraic problems and that the purpose of the study is not to assess their mathematical performance.

I am satisfied that the aims of study and the role of the institution in the study have been explained at the beginning of the project and that there will be ongoing discussion during the process of data collection.

I am also aware that the participation of the learners in the study is voluntary and that any participant can withdraw from the study at any time.

_____ _____

Signature **DATE**

PARENTS' CONSENT FORM TO ALLOW CHILDREN TO PARTICIPATE IN THE RESEARCH

I, (name of parent or guardian)_____ parent or guardian to (name of child)_____ give permission to Mrs. Patricia P. Nalube to engage my child in her doctoral research study entitled *Student Teachers Learning Mathematics for Teaching: Learner Thinking and Sense Making in Algebra*, with specific focus on student teachers' knowledge and understanding of learner mathematical thinking.

I understand that the purpose of the study is not to assess my child's mathematical performance. I am aware that the research process will engage my child in mathematical tasks where he or she will be solving algebraic problems. I am also aware that data will be collected from my child's mathematics exercise book.

I am satisfied that the aims of study and the role my child will play in the study have been explained at the beginning of the project.

I am also aware that the participation of my child in the study is voluntary and that he or she can withdraw from the study at any time.

_____ _____

Signature **DATE**

12.6 Appendix F: Analysis of student-teachers' talk on Scenario 2

8.3.2. Student-teachers recognition/misrecognition of the error

Across all the eight pairs, student-teachers recognized the error in Scenario 2.

> **Extract 8.10**
>
> *So with Learner 2 it's like he doesn't really understand what is happening; and he went on to add the 5m + 2 because in the 5m there's a number and also 2 is a number. It is wrong because free numbers cannot be added to the coefficients of some variable. That is not in the rules of mathematics, in mathematics. I'm saying, I'm saying a free number, 2 in this case, cannot be added to the coefficient of a variable, which is 5 in this case* (Pair 5, turn 10, 12, 14). *It was because he thought this expression is still being simplified further* (Pair 8, turn 39).
>
> "*... the main problem is ... going back to the first few rules about basic processes of algebra. Add the like terms and do not add the unlike terms. Just the simple part of it: All like terms group them together. Identify which ones are the like terms. Identify that is what really is mostly a problem to pupils*" (Pair 1, turn 47).

As typified in extract 8.10, across all the 8 pairs, their main *repertoire* in recognizing the error in the scenario pointed to the problem with the method of 'collecting like terms' in some form or another. The problem for the strategy was learners not distinguishing like terms from unlike terms, or collapsing an expression into one whole. Although they did not use terms such as conjoin or 'finish' open expressions, 'letter not used', seeking closure, and seeing '+' as operational from the *reservoir*, their description of the problem relates to such notions. So their *repertoire* in describing the problem in the scenario is in the knower mode, which I have defined here as drawing from their everyday professional knowledge of teaching and learning. This is because they did not use discursive resources developed in the mathematics education literature, hence, in Morgan et al. (2002) terms, they spoke with voice of the unofficial discourse (UD).

Moreover, in recognizing the error, in Morgan et al. (2002) terms student-teachers also spoke with the voice of the official discourse (OD) of school mathematics and the issue of rules that govern the basic processes of algebra. For example, in expressing open algebraic expressions into their simplest form, only like terms are added. As discussed in section 2.3.1 of this thesis, in the school mathematics curriculum there is emphasis on developing in learners the ability to manipulate algebraic symbols and carry out operations. The learners were being blamed for what they had done wrong with open algebraic expressions in that they had added unlike terms to come up with one single termed answer contrary to the rules of mathematics. This suggests that student-teachers orientation is towards school mathematics with a focus on absences. Moreover, the criteria of the official discourse of simplifying open algebraic expressions are explicit as student-teachers stated that it is about identifying and adding the like terms.

8.3.3. Student-teachers' explanation of the sources of the conjoining error

In explaining reasons for the error, two major categories emerged from the analysis, namely, error as a result of teaching emphasis, and error as a result of teaching sequence with the dominant discourse being the former. In referring to teaching emphasis as a source of error, student-teachers talked about how problematic teacher Benny's rule was in simplifying open algebraic expressions. As for teaching sequence, two issues arose. The first issue is that student-teachers talked of how the ordering of what to focus on first before introducing learners to simplifying open algebraic expressions could have been a problem. The second issue is that learners' knowledge of other concepts could have interfered with the simplifying of open algebraic expressions.

- Error stems from teaching emphasis

Although all the 8 pairs of student-teachers located the source of the error recognized in the scenario in teaching, they did so in three different ways. Firstly, all the 8 pairs talked of unclear instructions in terms of the rule that teacher Benny used. Pairs 1, 2, 3, 4 and 8 expressed dissatisfaction with the rule because it was ambiguous. They talked of how it would result in other interpretation problems and yield answers other than 7*m*. The issue of the rule was also talked about in three ways that is in terms of the meaning of 5*m*, in the definitional meaning of a term (Pair 5); ways of recognizing like terms (Pairs 1, 2, 3, 4, 6, 7 and 8); and using concrete objects (Pairs 1, 2, 3, 4, 6, 7 and 8) to reinforce the meaning of a term. In all these three ways, the problem is that the teacher could not have emphasized the definitional meaning of a term and appropriate ways of recognizing like terms as can be observed in the ambiguity of the rule. So the way student-teachers emphasized recognition of like terms is different with some of them even bringing their own errors to bear on it. I evidence this in the four extracts below.

Extract 8.11

I think it comes to a point where the rule given is what they have and they're acting on that rule or information, maybe. Looking at Learner 1 I think his justification can come from that there isn't so much information given on what really, exactly is happening. Then on Learner 2 there's also a little bit of justification in the sense that he doesn't, the rule doesn't come out clearly on what should be done. Like for example when he says, 'Add the numbers separately I give an example, a not linear example, what is there is 5m. 5 is a number then m is a letter. And then there's also +2 there. So Learner 2 was also thinking, 'OK there's 5 and there's 2, I think I can go like that.' So there's a little bit of the justification both in Learner 1 and Learner 2, coming from the statement given. And that's the rule. (Pair 5, turn 16).

... Because even me I'll go by what the learners were doing. I will simply add the 3 + 2 + 2 then I will get 7 and then I'll add the letters separately as well, which is m and m. And in this case the only difference here is that the learner got m, but for me I would have gotten 2m. so 2m + 7 would have given me 9m. Except that there's a difference here in the way…The learner just he looked at m and he added m. Just added the numbers separately, he added 3 + 2 + 2(Pair 2, turn 17) … *Suppose there was also w instead of only m, so to them they would still misinterpret it. ...* (Pair 1, turn 72).

Extract 8.12

And then I think what the learners are not understanding is that 5m is just another way of writing m + m + m five times. m added to itself five times. They do not understand that rule. So this 5 is not just like a number, a free number, as 2 is. They are not understanding that kind of, it wasn't explained properly. They're going on adding the 5 to the 2. (Pair 5, turn 17)

Extract 8.13

... How do you determine like terms? You look at the variables. The same variables raised to the same value as in the same power, they are common, I mean they are like. You add them. But in this case this instruction is simply talking about adding numbers separately and adding letters separately. Now there are some letters that are, there are some numbers that are co-efficient to letters, how do you add them? It's not clear (Pair 6, turn 17).

Extract 8.14

"*… it is better first of all before you introduce to them when you're dealing with letters, to first of all deal with them using the things that they usually use – spoons and forks – you know you use them, the things they like using in their everyday life*". *At least almost everyone has an idea of how a spoon is so if you tell them, you bring maybe 10 spoons and say, 'OK 3 spoons, 2 spoons. And bring a fork. Add these. How many spoons are there? How many*

forks are there?' You'll be able to realize that OK they will be, get it right from that point. And then you begin to develop from there. You maybe deal with fruits and stuff. And from there you can now go back to the letters. And then once we understand the letters you would see that now … (Pair 1, turn 47).

The second and third sources of error that I categorized as being located in teaching are talked about in terms of dominance of teacher talk resulting in inadequate assessment of learners' understanding (Pair 1); and inappropriate teacher response to learners' problems (Pairs 3 and 6). The student-teachers assert that if the teachers could identify learners' shortcomings, they would find out and explain to learners why they are making the errors. The focus in both is on what teachers have to pay attention to, in general, in the process of teaching and learning. Teachers should make their classrooms interactive, and they should attend to problems their learners experience in the teaching process in ways that enhance learning. extracts 8.15 and 8.16 pertaining to the two issues raised are presented:.

Extract 8.15

"*And when you are teaching your pupils avoid just you yourself talking. Be able to make an interactive kind where you are able to see their thinking also because sometimes you may think you have explained to them when actually they haven't gotten it. So one of the sources is actually maybe you the teacher not being able to assess, you know, correctly even if you have explained it's not a guarantee that they have learnt. You might have explained in detail fully to a pupil, but that's not a guarantee that they have learnt. Maybe the point is try to go with them as you are explaining. Move with them. Find out from their side, let them ask questions. And from there you are able to explain every other misconception that they still have. And after you've cleared the dust that's when you can say, 'Now we are safe, we can move on to the next part'* (Pair 1, turn 47).

Extract 8.16

So now but looking at this teacher's response because he responded to the pupils' solutions, here as teachers we need to look at pupils as people who are operating in a limited environment. OK, these learners are operating in a limited environment. They don't have all the information as to how these expressions should be operated on, OK. Now they've been given this instruction and they give the solution according to their understanding and the teacher just comes in and says, 'No', OK. No, 5m + 2 does not equal 7m. Then he repeats the rule again of which they are carrying their own understanding. Now I believe here this, I see it this way. This teacher should have tried to rephrase his rule looking at, because he should have looked into the reasoning of the pupils, what has made them reason this way? OK. So he should have been able to point out obviously maybe it is the rule that I've given which is not giving a clear picture of how to simplify algebraic expressions. Instead of saying, 'No,' he should have said, 'OK, in that way we don't have to do this, or we don't have to operate them like this.' In other words this rule, then we explain the rules, this rule means this, this, this, this because it's like the rule is too abstract for the level of the pupils. But OK, as for me, I would be able to point out to say, 'OK, what is meant here is this.' But here we are looking at the pupil who is meeting this topic for the first time. (Pair 6, turn 28).

Ja, I think the teacher's response in this case was, was not appropriate in that he, I think it was not necessary for him to give another example after noticing the problem which was there. I think he was supposed to go back now to explain at each level why they made mistakes, explaining to them what was supposed to have been done and not what they did, so that before going to give them more examples they understand the mistakes which they have made while they've been working the previous example, unlike giving them more examples which just probably confused them more. (Pair 3, turn 35)

Ja, and again like he has said, before moving on to giving them another example, the teacher would have asked the pupil or the learner who gave the answer to be 7m why he did it like that, unlike just giving the response that, 'No, 5m plus 2 does not 'equal-to' 7m' and repeats the rule. It was not necessary for him to do this. All what was needed for the teacher was to ask learner 2 as in why he did what he did and then try to definitely for learner 2 to come up with this answer was wrong in one way or the other. So it is from there once the teacher asks why the learner did it, did it in this way, that's when he was going to know where to put emphasis on and the remedy he can give to such a learner. (Pair 3, turn 36).

Student-teachers *repertoire* in locating the sources of error in teaching emphasis is ultimately in two ways, according to the discourses of mathematical principles and teaching principles. In terms of the discourse of mathematical principles, as indicated in extracts 8.11, 8.12, 8.13 and 8.14, student-teachers talk pointed to that there was a problem with the teacher's use of the rule in his instructions as it did not reflect the meaning of like and unlike terms. Moreover, student-teachers just like Tirosh et al. (1998) raised similar concerns pertaining to the rule the teacher used in that it would result in multiple interpretations whereby 9*m* other than 7*m* or 7 + *m* would also be one of the answers. They further pointed to that an interpretation problem would still arise in a situation where an open algebraic expression has a combination of letters other than just *m*.

As a result of the rule, student-teachers suggested that more information was necessary in terms of the definitional meaning of a term (5*m* means *m* + *m* + *m* + *m* + *m*) and the recognition of like terms with focus on the same variable and the power it is raised to (terms with same variable raised to the same power are like terms). A further suggestion was that using concrete objects from learners' immediate environment as a first step during teaching of like and unlike terms could enhance learning when letters are introduced. The issue of concrete objects is exhibiting part of the error in that they do not understand that 5*m* means *m* + *m* + *m* + *m* + *m* and not spoon + spoon + spoon + spoon + spoon. As described in section 3.3.3 of this thesis, according to Bell (1995) and Tirosh et al. (1998), this approach, known in mathematics education literature as the 'fruit salad' is contested in the teaching and learning of algebra as it tends to obscure the meaning of letters. Furthermore, Hart et al. (1981, pg 104) would describe this problem as "letter used as an object". So this is going to create just as much problems as what the problem is happens to be one of their own resources. Therefore, suggesting that this fundamental problem in algebra is not being dealt with in these student-teachers mathematics education courses.

In terms of the discourse of teaching principles, as indicated in extract 8.15, student-teachers talked about teaching being dominated by teacher talk, they pointed to that what gets lost is access to what it is the learners are thinking about teacher explanations. They argue that teaching that is interactive provides opportunity for the teacher to assess learner learning including their misconceptions and seek ways of remediating before making progress in teaching. One of the ways they suggest of how the teaching could be interactive is by encouraging learners to talk and ask questions based on teacher explanations.

Closely linked to this interpretation, as indicated in extract 8.16, is that student-teachers pointed to that the teacher did not pay attention to establishing why the learners reasoned in the way they did instead of insisting on emphasizing the rule of which learners had a different understanding. They suggested that doing so would have made the teacher realize that maybe there was something wrong with the way learners were interpreting the rule and therefore needed further elaboration. Especially that the learners were encountering simplifying expressions for the first time. In Even and Tirosh's (2002) and Titus (one of the teacher educators that participated in this study) terms, encouraging a classroom atmosphere where learners can ask questions based on teacher explanations and the teacher asking learners to explain their reasoning to the answers they provide, are some of the ways of accessing learner thinking. Moreover, Kenneth, one of the mathematics education teacher educators pointed to these as ways that would develop in learners thinking that is a mathematical way of reasoning.

In locating the sources of the error in teaching, according to Morgan et al. (2002) student-teachers spoke with the voice of the official and unofficial discourses in terms of mathematical principles and teaching principles, respectively. In terms of official discourse, student-teachers talked of how the rule was inadequate to enable learners distinguish between like terms and unlike terms, and how the definitional meaning and recognition of a term could have assisted. As for unofficial discourse, student-teachers talked of how dominance of teacher talk and inappropriate teacher response to learner answers could inhibit learning. So they drew on their everyday professional experience of teaching and learning. They exhibit an orientation towards learners and describe what is present in them as a result of absences in the teaching in terms of mathematical and teaching principles. In doing so, they pointed to how justified the learners were in using the rule, which they thought lacked detail, as the only source of information provided for by their teacher, and hence coming up with a single lettered term. Moreover, the resource of using concrete objects from learners' immediate environment further shows absences in these student-teachers mathematics education courses. Student-teachers were also oriented towards teaching and absences in that they blamed teachers for not making their teaching interactive and for not probing their learners' answers. Criteria of the official discourse of simplifying open algebraic expressions are explicit in terms of how like terms could be recognized using the definitional meaning and the notion that they have the same variables that are raised to the same power, as resources.

- Error stems from teaching sequence

Two sources of the error emerged in this category. The one source is that the student-teachers talked of how the teacher could not have ensured that the learners understand the concept of like terms before dealing with addition and multiplication of terms (Pairs 1, 5, and 6). The other source is that the error could be as a result of previously already learned concepts (Pairs 1, 2, 3, 4, and 6) or newly learned concepts in the school curriculum (Pair 6). In this case, student-teachers talked of the concept of sets and the number system where a single digit could be the answer or the misapplication of multiplication of terms over addition of terms.

Extract 8.17

I think the concepts in the school curriculum are related… There is a concept of common factors or like terms before they do the addition and the multiplication. I think those concepts are related. And failure to understand the other might lead to a misunderstanding of, of this.

***I:** So assuming you are a teacher of this class, how would you explain what learners have responded to?*

I think the concept of addition, multiplication is one of those that come as early as grade 1, so as a teacher of this class I would question if it wasn't me who started with them, what has been happening before I came? (Pair 5, turns 21, 23, 24 and 25).

Extract 8.18

… Like the concept of sets as in how many elements are there. You get that as in one set, let's say in set A there are 3 mangoes + 2 mangoes + 2 apples, how many elements are there in A? They would definitely say there are 7 elements (Pair 1, turn 55). *They would not say, 'No there are 5 elements plus 2' but they would just say 'There are 7 elements there'. So maybe the other related, but it's from the set point of view and you're dealing with sets* (Pair 1, turn 59).

I: Any other? You got something?

Ja. Whenever we are adding numbers whether we are dealing with the number system with subtraction and addition, whereas the … we always get a single digit as a solution.

So in this case where there's a 5m + 2 the pupils, rather the learners, would want to get a single answer. So it would be right for them to say 5m + 2 will still be 7m because whenever they're adding the whole numbers they will need to find only 1 solution, not maybe part of it and the other part (Pair 1, turns 60, 61, 63 and 65).

… if you are adding, then you should come up with the total. So whether it's letters mixed with numbers, they will just add them and possibly come with what they came up with, with what learner 2 said (Pair 4, turn 43).

…when it comes to multiplication 5m x 2 the answer will be 10m. So in such a situation when it comes to adding 5m + 2 … then you say it will be, you collect the like terms, the pupils find it very difficult to understand because when they multiply it must be combined 5m x 2 it will be 10m. So that source of confusion is also there which needs to be explained (Pair 6, turn 29).

Student-teachers *repertoires* in locating the source of error in teaching sequence are in two ways, namely ordering (sequence) and overgeneralization. In the case of ordering (sequencing), although implicit, there is an indication in extract 8.17 that the learners added the unlike terms to come up with a single-termed answer because their teacher could not have ensured that they understand basic concepts such as identifying like terms prior to addition and multiplication. In Morgan et al. (2002) terms, student-teachers spoke with the voice of the official discourse of school algebra in terms of sequencing in that recognition of like terms should be introduced first before addition and multiplication of terms. In terms of form of practice, student-teachers are oriented towards teaching and absences. So absences realized in learners of adding unlike terms are as a result of absences in teaching of not ensuring that learners understand the basic concept of recognizing like terms prior to addition and multiplication of terms. Moreover, criteria of the official discourse of simplifying open algebraic expressions are implicit in talking about the ordering.

As for overgeneralization, there is an indication in extract 8.18 that the possible sources of error pertaining to the issue of coming up with a single – termed answer are due to learners' knowledge of previous concepts of sets (finding the number of elements in a given set); and number system (addition and subtraction of whole numbers) being misapplied on new learning (addition of terms). As pair 4 has pointed to, this could be because of seeing '+' as requiring further actions of finding the total, hence seeing '+' as operational as described in the *reservoir*. Moreover, the source of error could also be as a result of learners' learning of new concepts (multiplication of terms) being confused with remembering of previous knowledge (addition of terms). In Ryan and Williams (2007) terms, the explanations the student-teachers provided for the sources of error relate to the issue of overgeralization forward in the one case and overgeneralization backwards in the other. So in Lima and Tall's (2008) words, the one is referred to as met-before and the other met-after. Student-teachers explanations align with these terms described in mathematics education literature even though no names were assigned.

Therefore, in Morgan et al. (2002) terms, in locating the source of error in learning, student-teachers spoke with the voice of the unofficial discourse by referring to overgeneralization, and in particular met-before or met-after. They drew on their everyday professional experience of teaching and learning since they lacked these discursive resources developed, and described in the *reservoir*. In terms of form of practice, student-teachers were oriented towards learner and presences. They saw learners coming with some understanding of addition or subtraction of numbers, or multiplication of terms to the learning of simplifying open algebraic expressions, hence drawing on earlier or new related school mathematics concepts. However, criteria of how to deal with like and unlike terms in simplifying open algebraic expressions is implicitly referred to when referring to overgeneralization.

8.3.4. Decisions about remediation student-teachers suggested on the conjoining problem

Only one way of how student-teachers thought they would remediate the error recognized in the scenario was identified. They said that it was important for the teachers to emphasize key concepts or issues during teaching.

- Emphasize key concepts or issues during teaching

In reference to this broad category three suggestions were identified, and these included: work with same rule with some elaboration on definitional meaning of a term (Pairs 2 and 5); work with different rule with some elaboration on meaning of like terms using concrete objects (Pairs 1, 2, 3, 4, 6, 7, and 8) or familiar language (Pair 1); and teacher emphasize what addition means when it comes to algebra (Pairs 5 and 7). As can be seen, the most dominant discourse among these suggested remediating ways was the need to change the rule and elaborate using concrete objects. These are discussed by providing evidence from the empirical data.

Extract 8.19

I would start by the addition, multiplication of letters first. .... So I would say m + m = 2m. Then I can do this even with other letters just saying, just putting an emphasis that these are letters and not numbers. So when we say 2m what we mean is m + m which is 2m. Probably I would even extend to the multiplication, like m x *m = m^2. Such kind of a thing. Then I'll move on to the numbers. Probably the numbers are introduced in the early stages but I will also talk about the numbers, the addition of numbers, like 1 + 1 = 2, 2 + 2 = 4. Then when it comes to combination of these two do not mix up the concept.* ***The concept of adding letters alone and adding of numbers alone still stands****. So when I say 3m + 2 +2m then I am saying m + m + m + 2 + m + m. Then I will get the m + m + m + m + m to give me 5m. And then the 2 remains because it does not have any other number to add to or subtract to* (Pair 5, turns 28 and 30).

Extract 8.20

"*... I would now establish certain rules, like first thing is group the like terms. Before you do anything group the like terms together, and then secondly add the like terms. Now in that way I must first of all have explained what we mean by like terms.*" (Pair 1, turn 87)

"*so in giving rules we should be trying to be as precise as possible so that it does not give a lot of meanings to the pupils, and a lot of interpretations*" (Pair 4, turn 112).

I will do first use the concrete ideas...Then if you say you get the number of girls and the number of boys you say "How many girls are there and how many boys are there? (Pair 7, turn 17)*" Then maybe we even use animals say "2 dogs + 5 dogs* (Pair 7, turn 21)*". It's important also to give an example where you cannot add say you say "2 dogs + 3 cats". From there we can now go to the abstract things like* (Pair 7, turn 25).

Extract 8.21

Because that's a problem of maybe the English part from them, so that is why sometimes maybe the other part is to look at the language we are familiar with first of all and then build it from there (Pair 1, turn 91).

If I find in the people from Eastern Province I would say, 'OK guys the first thing is'kusankha vo palana' (choosing similar things), 'vamene vili vo palana ndiye vo yika pamodzi' (things which are similar are the ones you put together) first. By 'vopalana' (similar) I mean not that they should be 'visankhala teti' (they will be like this), no, 2 and 5 since 'sivo palana' (they are not similar) I mean numbers, letters 'nama letters yo palana' (with letters that are

similar). So after 'mwaviyika pamodzi paja ndiye manje mu yamba uchita ma' (putting them together now you start doing) addition. So from that point maybe if I explain to them from the language they easily interpret and then now I take it to English. Then from that point they would be able to know that oh 'nivo palana' (those similar) first (Pair 1, turn 93).

Extract 8.22

But once you come to algebra the teacher should, should make sure that he emphasizes or she emphasizes that in addition of algebra it is different from our usual numbers because in algebra we deal with like terms and unlike terms (Pair 7, turn 35).

Student-teachers repertoire towards how they would help their learners develop the required algebraic thinking in relation to simplifying open expressions was in three ways as already stated. Firstly, as indicated in extract 8.19, student-teachers suggested upholding the rule that teacher Benny used but with some elaboration on the definitional meaning of a term. They pointed to how they would remind their learners that adding numbers meant , for example, 2 + 2 =4 while adding letters meant $m + m = 2m$ or vice versa. Therefore, if these two aspects were used in simplifying the open expression $3m + 2 + 2m$, it would mean m + m + m + 2 + m + m, and applying the rule you get $5m + 2$ as letters and numbers are added separately.

The second suggested remediating strategy, as indicated in extract 8.20 and 8.21 was that they would change the rule teacher Benny used to 'Group the like terms together and then add them'. By using this rule, student-teachers thought it would address the issue of ambiguity as learners try to grapple with the meaning and interpretation of the rule. To elaborate this rule in terms of what is meant by like or unlike terms; student-teachers suggested using concrete objects from learners' immediate environment, or learners' familiar language other than English. The use of learners' language other than English could be associated with Adler's (2001) issue of code-switching as an instructional resource in the teaching and learning of mathematics.

The third suggested remediating strategy, as indicated in extract 8.22, is that student-teachers pointed to the need for teachers to emphasize that addition of terms in algebra is different from adding ordinary numbers as it involves taking into consideration like and unlike terms.

These three suggested remediating ways are also closely linked to some of the explanations provided for the sources of error as discussed in section 8.3.3 where concerns about the rule being unclear or ambiguous were raised and how this could have been improved. The ways in which the rule could have been improved were by using the definitional meaning of a term, recognizing like terms by identifying terms with the same variable raised to the same power, and using concrete objects (although contested as argued). The student-teachers here pointed to the need for teachers to use the same rule or change the rule but in both cases with some elaboration on the meaning of like and unlike terms. They suggested use of the definitional meaning in one case and use of concrete objects (although contested as argued) and learners' language other than English in the other. Moreover, as for overgeneralization of addition of numbers (seeing '+' as operational), student-teachers suggested the need for teachers to emphasize the meaning of addition of terms in algebra as it is different when dealing with ordinary numbers.

In suggesting ways of how teachers would help their learners develop the required algebraic thinking in terms of simplifying open expressions, student-teachers spoke with voice of the official discourse or voice of the unofficial discourse. In terms of voice of the official discourse of school algebra, they pointed to the definitional meaning of a term as an elaboration to the rule. As for voice of the unofficial discourse, student-teachers pointed to the issue of code-switching as an instructional resource without necessarily using the term. So they drew from their everyday professional experience of teaching and learning.

As for form of practice, student-teachers were oriented towards teacher and presences. They pointed to how teachers ought to incorporate in their teaching mathematical rules which are clearly elaborated to avoid any ambiguity or interpretation problems. Moreover, teachers ought to explain how the operational view of '+' fails to work in simplifying open algebraic expressions as it involves like and unlike terms. However, student-teachers exhibited absences in further suggesting use of concrete objects as one of the ways of elaborating the rule concerned with simplifying open algebraic expressions. Criteria of the official discourse of simplifying open expressions are explicit as student-teachers pointed to how it involves a consideration of like and unlike terms, and how the operation '+' is implicated.

12.7 Appendix G: Analysis of student-teachers' talk on Scenario 3

8.4.2. Student-teachers recognition/misrecognition of the error

Across all the eight pairs of student-teachers interviewed pertaining to scenario 3, in Jacobs et al. (2010) terms, they attended to learner strategy by identifying the problem in the solution provided by the learner.

Extract 8.23

"*… dividing by x means that some solution will be lost because the quadratic will now reduce to a linear form. That's why it was wrong for, to divide by x*" (Pair 7, turn 13).

"*… this Learner 1 was just thinking of a linear equation and not knowing the characteristics of a quadratic equation"* (Pair 2, turn 25).

"*In the first place it was OK – the first step there it was OK. $2x^2 = 6x$. You can find out what is common and divide both sides of the equation by 2 there and you get $x^2 = 3x$. Now at that stage you are looking for the value of x so there is no way you can divide. Since this is an equation there is no way you can divide both sides of the equation by x.*

… So we are interested in that variable x. As a result we have to solve for that variable. Now if you just divide both sides of the equation by x then you only remain with the one value of x which is a 3. But where have you taken the other value of x?" (Pair 2 turns 8 and 22)

"*Now this is a quadratic expression. If you are to graph this you are supposed to have something like this which you, which you'll be able to cut the X axis on two points. Now since we only have one value, how is it going to be sketched? It means that it will only be, it will only cut one, one point at this one. It won't pass through, it will be a straight line. It will only pass through x = 3, unlike being a curve or something like this. So this, this one is not correct. Learner 1 was wrong because this is a quadratic expression and you are supposed to find two values of which the graph is cutting the X axis*" (Pair 8, turn 30).

As shown in extract 8.23, student-teachers pointed to that the problem for the strategy was that the learners could not distinguish between linear and quadratic equations. The problem was recognized, in Sfard & Linchevski (1994) terms, both operationally and structurally. In terms of the problem being identified operationally, the indication in student-teachers talk is that the learners were not aware that they needed to solve the quadratic equation to get two values of *x*. Dividing by *x* meant giving the letter a value resulting in only one value of *x* to be solved. This is similar to the issue of "letter evaluated" described in the *reservoir*. Student-teachers' argument is that it is inappropriate to divide both sides of the equation by *x*, for which you are solving for. Moreover, for the problem to be identified structurally, the indication in student-teachers talk is that learners could not figure out the graphical representation of the quadratic equation where the graph cuts the X – axis at two points. This thinking could have made learners realize that they needed to get two values of *x*. All the 8 pairs pointed to this problem in some form or another.

In recognizing the error in the scenario, student-teachers spoke with voice of the official discourse or unofficial discourse. In terms of voice of the official discourse of school algebra student-teachers pointed to the characteristics of a quadratic equation both operationally and structurally, and how these are different from a linear equation. In terms of voice of the unofficial discourse, student-teachers descriptions of their recognition of the error pointed to the problem with the operational and structural view of a quadratic equation, and "letter evaluated" without necessarily mentioning these terms developed in mathematics education literature. This suggests that student-teachers drew resources from their everyday professional experience of teaching and learning, an indication that these terms are not given specific focus in these student-teachers mathematics education courses.

As for form of practice, student-teachers were oriented towards mathematics and absences. What learners have done wrong with solving quadratic equations is that they have divided both sides of the equation by *x* reducing it to a linear equation, hence obtaining one value of *x* instead of two. Criteria of the official discourse of solving a quadratic equation are explicit in that it is about finding the two values of *x*, although the method of doing so is implicit.

8.4.3. Student-teachers' explanation of the sources of error on the quadratic-linear equation problem

In explaining possible reasons for the error, three major categories emerged from the analysis, namely, teaching sequence, teaching emphasis, and a problem of interpretation. Interestingly for this scenario, the dominant source of error was teaching sequence as 6 out of 8 pairs of student-teachers talked of this although some also talked of other sources. In what follows, I describe these three sources.

- Error stems from teaching sequence

In locating the source of error in teaching sequence, all the six pairs (pairs 1, 2, 3, 5, 7, and 8) talked of the issue of learners' knowledge of solving linear equation interfering with their learning of solving quadratic equations. I provide pair 5's talk as a typical example.

Extract 8.24

"*Like let's assume they were just doing linear equations, then after that they start the quadratic equations. So once we expose this as a new concept they would always want you to go back and do the same thing they were doing when they were solving for x in the linear equations. (Pair 5 turn 31)*

I: Can you give an example?

Like the way you prepared the questions, like let's say 5x – 2x or 5x – 2 = 0. Solve for x. We are talking about the straight line which has only probably one solution. Now when you come to the quadratic equations which have two solutions given to a pupil he would want to use the same concept he got from the equation of a straight line which has one solution on the quadratic equation. And that's where the problem comes. So given a quadratic equation he wants to make it the linear equation or a straight line for him just to come up with the value of x" (Pair 5, turns 31, 32 and 33).

In Ryan and Williams (2007) terms, student-teachers *repertoire* in locating the error in learning pointed to the issue of overgeneralization. Student-teachers explanation of the source of error pertaining to overgeneralization is that the descriptions they provided are in resonance with the term although no mention of the term was made. As indicated in extract 8.24, there is a relationship between solving a linear equation and solving a quadratic equation. This suggests applying inappropriately operations for solving a linear equation on solving a quadratic equation. More specifically, as described in the reservoir, the overgeneralization is forward in that learners' knowledge of linear equations tends to interfere with new learning of quadratic equations. In Lima and Tall (2008) terms, this is the effect of met-before where learners tend to work towards getting one value of *x* as if they were solving linear equations when the focus is on quadratic equations requiring two values of *x*.

Therefore, student-teachers spoke with voice of the unofficial discourse by referring to the issue of overgeneralization and met-before in particular. For them to have not mentioned these terms in the descriptions they gave suggests that they drew resources from their everyday professional experience of teaching and learning. As for form of practice, student-teachers were oriented towards learner and presences. They saw learners coming with some knowledge of solving linear equations to the learning of solving quadratic equations. Criteria of the official discourse of solving quadratic expressions are

implicit in that there is mention of the two *x* values to be obtained without necessarily explaining the methods to be used when referring to the issue of overgeneralization.

- Error stems from teaching emphasis

In locating the source of error in teaching, three aspects are identified, namely, not emphasizing key points or issues in teaching linear and quadratic equations, including the meaning of division by zero (pairs 1 and 7); restricting the teaching of quadratic equations to only one method of solving (Pair 7); and dealing with expressions in their simplest terms (Pairs 4, 5 and 6). I discuss these in turn.

Extract 8.25

"*Because I think it also* ***gets back to how teacher present this without emphasizing certain key points*** *because this maybe may come like the solving of equations. But in solving of equations definitely there are different types of key questions. They are what are known as the simple questions or linear questions – linear meaning one solution only. It's, it's, it's the one solutioned question with just an x. But when you say* x^2 *definitely the* x^2 *itself means that there are two answers to this question, or there are two solutions, there are two roots to this equation. Similarly if you had to the power 3 already what will come in my mind is that whenever I'm solving this one I should, I should come up with 3 roots or 3 solutions. So in this case the pupils like they got disturbed between a linear equation which is a single root equation and a quadratic which is a double or a two root equation. So I think it gets back to the teacher. What these, what different types of equations mean. Just looking at them what does this equation tells you to do or what do you expect out of this equation? Just looking at it, it does*" (Pair 7, turn 11).

"*Maybe if the teacher did emphasize that a quadratic results in two solutions and therefore dividing by x in this situation would mean losing one solution because you are not going to do this*" (Pair 7, turn 13).

"*I think it gets back to how teachers present work.*

Where you, we take casual, we take things casually. We don't, we don't explain what this, because these are abstract things. So sometimes when we don't explain things which look to be simple to teachers they are not as simple as they may be to pupils" (Pair 7, turns 15 and 17).

"*So I don't know if this teacher emphasized on the type of equation because I think in secondary school we just look at two types of equations – linear and, and quadratic. And what do they mean when they say linear, what does it mean? I think all those things are supposed to be emphasized*" (Pair 7, turn 64).

Extract 8.26

"*…but then dividing by an x which for x we don't know, is a problem because the x can be a 0*" (Pair 1, turn 34).

"*When there's a 0 in the denominator then the function or the expression in the real number system is undefined*" (Pair 1, turn 16).

"*Yes. So I know those are just the concepts that mostly which are not explained by teachers dividing a 0, they only explain you can divide a number by any number. And yet they forget the 0 concept. And the 0 concept really has a bearing on the interpretation of the mathematical concept and that is why even the history of mathematics is also needed in this case when you are dealing with the 0 part, looking at how, the concept of a 0 itself was developed and the need for a 0 and the rules behind a 0 part and, you know, why multiplying*

any number by a 0 gives you 0 and so on. All those. Why can't we divide by a 0 and so on. It's because when we divide by a 0 it's like as we are drawing very close to 0 we are dividing into a number, we are getting a very bigger number.

Now suppose we reached actually 0 itself, would we know what that number is? And that number is the number now which after some years in history they came up with the infinite number, symbol, not a number – the infinite symbol which was just a symbol symbolizing well a number, as far as we can go we are gone, but we can't go beyond this" (Pair 1, turn 18).

Extract 8.27

Another source of it could be restricting oneself to just one method of solving these equations. If the teacher had also used a graphical way, pupils would be able to see that the, the graph would cut the X-axis at 2 points so that there will be two solutions to this, just as the linear equation would cut the X-axis at one point, meaning there would be only 1 such. Then getting the only one solution to a quadratic equation would not make sense if the child explained using graphical method (Pair 7, turn 18).

Extract 8.28

"*It is because as I remember also being taught, when you are solving you just look for what can go into the left expression and on the, in the right expression. So upon looking at it he saw that…*

It's like instead of trying to simplify as much as possible so that you are just dealing with expressions in simple terms. So here, so that your 2 can go here, even on the other side. Then again you saw that x can go on the left and on the right and he came up with 3" (Pair 4, turns 49 and 51).

"*I think another rule in maths that the learner was trying to use was that when you have an equation what happens on the right side of an equation has to happen, happens on the left side of an equation. If you add the 2 to the x^2 on this side of the, on the left a 2 had to be added on the right and nothing changes. So he was trying to think in those lines and then divide on this side by x, even on the other side by x*". (Pair 5, turn 5)

In locating the source of error in teaching, student-teachers repertoires pointed to three aspects as already mentioned. Firstly, as indicated in extract 8.25, student-teachers pointed to that there is a tendency among teachers to take for granted seemingly easy concepts which are abstract but are not necessarily easy for learners. As a result teachers tend not to emphasize meanings of linear and quadratic equations in terms of the number of solutions that are supposed to be obtained after carrying out the manipulation. For example, a linear equation would only have one solution, a quadratic equation would have two solutions, and a cubic equation would have three solutions. They argue that if this emphasis was made by the teacher, learners would not further divide the quadratic equation by *x* because they would realize that doing so means losing one solution. As indicated in extract 8.26, in a situation where the *x* which is dividing both sides of the quadratic equation was zero, the student-teachers suggest that the teacher could not have explained to learners the meaning of division by zero. This would have been made possible by drawing from the history of the development of mathematics.

Secondly, as indicated in extract 8.27, student-teachers indicated that the teacher could have introduced learners to only one method of solving the quadratic equation other than the graphical method. They argue that if learners had been introduced to the graphical method, they would have realized that the graph of the quadratic equation cuts the X – axis at two points. Therefore, getting one value of *x* would not have made sense. Thirdly, as indicated in extract 8.28, student-teachers also point to that the tendency by teachers to encourage their learners to work with expressions in their simplest form in the process of solving by identifying common factors seems to be a good mathematical practice. However, they suggest that if this practice is misapplied, it could result in

errors as is the case with scenario 3. Student-teachers also indicated that the issue of working with expressions in their simplest form is synonymous to another mathematical rule that states that for you not to change the meaning of the equation, whatever you do on the left hand side do the same on the right hand side.

Student-teachers spoke with voice of the official discourse of school algebra in terms of what it means to solve linear and quadratic equations in terms of number of roots to be obtained and their graphical representations. As for form of practice, student-teachers were oriented towards teaching and absences. Learners could not distinguish between a linear equation and a quadratic equation because the teacher did not emphasize their meaning in terms of the number of roots to be obtained, restricted teaching to only one method, and dealt with expressions in their simplest terms. Criteria of official discourse of simplifying quadratic equations are explicit in that two roots would be obtained using either the factorization or graphical methods.

- Error stems from interpretation problem

In talking about the source of error being as a result of a problem of interpretation, student-teachers talked of how learners could have not been aware of which methods of solving quadratic equations to use. They lacked rules for solving quadratic equations, hence their misrecognition of quadratic equations.

Extract 8.29

"*Um I think it's a lack of using correct methods of doing, of solving quadratic equations*" (Pair 7, turn 60).

"*… Now I think the reason why this one did this is because of* ***lack of some rules on how to solve the same equations because I don't think their teacher taught them to divide through by the letters as he did.*** *But since he saw that it can divide on the left and on the right, he was tempted to do this*" *(Pair 4, turn 47).*

"*I think with Learner 1 what was in his mind was that this quadratic equation has only one solution, which would only have one solution. That's why he went on even to dividing by x.* ***But if he had in mind that a quadratic equation always has two solutions he could have not done that.*** *Probably, probably he would have used that quadratic, that formula negative b, b plus/minus the square root of* $b^2 - 4ac$ *everything over 2a for him to get to two solutions.*

I think there's, there's so many ways, there's completing the square, the graphical method and then the formula method which was mentioned there before, before the other one (I just can't remember what it's called)" *(Pair 5, turns 25 and 27).*

In locating the reason for the error in learning, student-teachers repertoire pointed to learners not being knowledgeable about the methods and rules that are required to solve quadratic equations. They pointed to that if they knew the methods and rules required, they would have realized that they needed to find two solutions, and dividing by *x* would not have been helpful as it would reduce the quadratic equation to a linear and only yield one solution. Student-teachers suggested that it would have been helpful if learners had used methods such as the quadratic formula, completing the square, and graphical.

Student-teachers spoke with voice of the official discourse of school algebra by relating to the methods and rules associated with solving quadratic equations. In terms of form of practice, student-teachers were oriented towards mathematics and absences. Learners were being blamed for not knowing the methods and rules required to solve quadratic equations, hence misrecognition of

quadratic equations. Criteria of the official discourse of solving quadratic equations are explicit in referring to the methods that would be appropriate, and the two solutions expected to be found.

8.4.4. Decisions about remediation student-teachers suggested on the quadratic-linear equation problem

Across all the eight pairs of student-teachers interviewed, two major categories were identified as ways that would enable learners making the error recognized in the scenario develop the required algebraic thinking. These included emphasis in teaching (teacher emphasizing procedure or key concepts), and practice exercises (teacher encouraging learners to practice solving). Between the two the former was the most dominant since all the eight pairs talked of this strategy while the latter was less dominant as only two pairs (pairs 4 and 6) made reference to the strategy. I discuss these two categories in turn.

- Emphasis in teaching (Emphasize procedure/key concepts during teaching)

In referring to the need for teachers to emphasize procedure or key concepts during teaching, student-teachers talked of two issues. These included indicating which methods should be used in solving quadratic equations, and expressing quadratic equations in the standard or general form and getting two values of *x*.

Extract 8.30

"… Well definitely here um looking at the learners that are meeting this topic for the first time I think I would put an emphasis, had it been me setting the question, I would put an emphasis using a quadratic method, not formula, a quadratic method. In quadratic method you can use the formula, you can use factorization, you can use completing the square.

… In the first place I would make it a point: you don't divide or you don't, ja, you don't divide variables. By dividing variables you are changing the meaning of the function of the equation of the expression altogether, OK. So now here since this is a quadratic function the best you can do (because it is a very simple quadratic equation) you factorize. … So you have $x^2 - 3x = 0$. *Then you factor out the x you have* $x(x-3)$ *then there, since there it's a product of its factors then definitely either one of them is 0. So if* $x = 0$ *then in the other case x will be 'equal-to' 3. So you have your solutions*" (Pair 6, turns 17 and 46)

"… you cannot solve quadratic equations as if you are solving linear equations … as those three methods they cannot apply on linear (referring to factorization, formula, and completing the square methods)" (Pair 8, turn 52). "*… you cannot apply what you do in linear equations to what you are supposed to do in quadratic equations because that one is an equation of another degree* " (Pair 8, turn 56).

Extract 8.31

… So normally the child, the teacher should emphasize that a question like this one must be written in a standard way of a quadratic equation. That is, this must now be like $x^2 - 3x = 0$. *In this case see the constant is 'equal-to' 0 and because x is raised to the power 2, like my colleague says, you expect 2 solutions. … .*

Yes here normally the emphasis must be that the children must always remember that the quadratic equation is of the form $y = ax^2 + bx + c$. *Or let me say* $ax^2 + bx + c = 0$. *So in this case they must always write whatever they are given in terms of that standard way of a quadratic equation so that it looks like that"* (Pair 6, turns 50 and 60).

"… So maybe in this case we can say let's try to plot this graph. So as they plot they will see that the graph maybe will cut the X axis at two points. Then you tell that where this graph is

cutting the X axis those are the solutions because the y at that axis, at those two points is zero. So those two points are the values which can be put. Those are the only two values which can be put as the solutions in order to get the value zero" (Pair 7, turn 86).

As shown in extract 8.30, student-teachers repertoires in suggesting ways of helping their learners develop the required algebraic thinking pertaining to the error recognized in the scenario is in two ways as already mentioned. Firstly, is that the teacher should make explicit in the question what method of solving the learners should use so that the issue of dividing by *x* does not arise as it changes the meaning of the given expression. The methods of solving quadratic equations mentioned included factorization, formula, and completing the square; and for this scenario factorization method is opted out because it is easily applicable to the type of equation given. Moreover, student-teachers pointed to that the three methods of solving quadratic equations outlined are unique and cannot be used to solve linear equations. Conversely, ways of solving linear equations cannot be used to solve quadratic equations.

Secondly, as indicated in extract 8.31, is that the teacher should emphasize the importance of expressing the given quadratic equation in its standard or general form ($y = ax^2 + bx + c$, or $ax^2 + bx + c = 0$). For the quadratic equation in the scenario, the general form would be $x^2 - 3x = 0$ where *c* in this case is zero. Moreover, they pointed to that for a quadratic equation, it should be emphasized that two solutions are to be obtained. This is because of the nature of the expression in that the highest power of *x* is two. These two values of x could also be represented graphically as they are points on the X-axis where $y = 0$. The suggested remediating strategies so far discussed relate to two of the sources of error earlier raised, that of teacher having not emphasized key concepts or issues during teaching of linear and quadratic equations, and that of learners having a problem of interpretation in terms of methods and rules. Now the student-teachers are suggesting that the teachers ought to emphasize key concepts and issues during teaching of quadratic equations.

In suggesting emphasis in teaching as one of the remediating strategies, student-teachers spoke with voice of the official discourse of school algebra in terms of indicating methods of solving quadratic equations and expressing them in the standard or general form before solving for the two values of *x*, which could also be graphically represented as values of x when y = 0. As for form of practice, student-teachers were oriented towards teacher and presences. They pointed to how the teachers ought to be thinking of incorporating in their teaching specific methods their learners should use in solving quadratic equations, and that quadratic equations should always be expressed in standard or general form before solving for the two values of *x*. Moreover, they should also explain to their learners that strategies for solving quadratic equations are different from those for solving linear equations, and how dividing by *x* is problematic. In suggesting the possible remediating strategies, student-teachers make explicit criteria for solving quadratic equations in terms of expressing them in standard form and using the appropriate methods to get the two values of *x*.

- Practice exercises (Encourage learners to practice solving quadratic equations more often)

The other remediating strategy student-teachers suggested other than emphasis in teaching is that teachers ought to encourage their learners to practice solving quadratic equations more often so that they become familiar with the methods and rules.

Extract 8.32

"*... But basically to,* ***to try and have a good practice of this is by obeying the rule*** *of just taking, I mean adding the negative of that and so that is on the one side, everything on the one side and then zero on this side. That's the best way. That's the best approach.*

They should be very comfortable with all the rules and the rule that was missing here is that one of not dividing by the variable because you are solving for the variable. So it was not clear to Learner 1 when they…" (Pair 4, turns 62 and 94).

> "*… but I would make sure that at least* ***they involve themselves too much into practice*** *so that whenever a quadratic expression appears like this they are* ***cognizant with all the processes of solving quadratic functions with quadratic methods like factorization, using the formula or completing the square***. ***So I would make them put them into practice.*** *Each and every time helping them out to understand whenever I involve a challenging question then we get there*" (Pair 6, turn 52).

As indicated in extract 8.32, in suggesting that learners practice more often, student-teachers' *repertoires* point to the issue of practice in two ways. Firstly, that through practice the learners would learn to obey the rules of solving quadratic equations such as writing the quadratic equation in standard form; and that of not dividing the quadratic equation by the variable of which they are tasked to solve for. Secondly, that the teacher would ensure that learners practice more of solving quadratic equations so that they become familiar with the process of how to go about solving them even in challenging circumstances using methods such as factorization, formula, or completing the square. Student-teachers suggesting practice exercises as one of the remediating strategies could be linked to one of the sources of error, namely a problem of interpretation. They pointed to how learners did not demonstrate understanding of methods and rules associated with solving quadratic equations, and now teachers have to ensure that their learners practice more often so that they become familiar with the methods and rules.

In referring to practice exercises, student-teachers spoke with voice of the official discourse of school algebra in terms of methods (factorization, formula, completing square) and rules (expressing in standard form, and not dividing by variable of which you are solving for) of solving quadratic equations. As for form of practice, student-teachers were oriented towards teacher and presences. They pointed to that teachers ought to think of incorporating in their teaching opportunity for their learners to practice solving quadratic equations more often so that they become familiar with the methods and rules. Moreover, student-teachers make explicit criteria of solving quadratic equations in terms of the methods and rules associated with the process as they make reference to practice exercises.

12.8 Appendix H: Analysis of student-teachers' talk on Scenario 5

9.3.2. Student-teachers' recognition/misrecognition of the error

Across all the 8 pairs, in Jacobs et al. (2010) terms, student-teachers attended to learners' strategy by identifying the error in the scenario. They talked in similar ways about the issue of the distributive law.

Extract 9.11

Uh the first one the learner did not understand, he doesn't know the meaning of how to translate any, any piece of information in a statement form into a mathematical sentence and because the question says multiply n plus 5 which is an expression by 4 and in this case this person just wrote n plus 5 as it is then he multiplied by 4 then he did not

I: Instead of what

Instead of introducing the brackets

I: How do you introduce the brackets

n plus 5 is just a single quantity in this case, you consider it as a single quantity as a result you open and then close the brackets to show that this first of all you should deal with what is in the brackets then you multiply by 4 but this is not uh this is this was not the case with this person here so what he did was just he left out the brackets and when you you leave out the brackets then it will give you a different interpretation and the interpretation that he got was n, n plus 5 multiplied by 4 and when you simplify this you get n plus 20 which is correct but it wasn't correct because of the mistake that he or she made by removing the brackets or by not treating n plus 5 as a single quantity

I: The second response

(laughter) the second response it was the same thing, there this person now just looked at the order of operations you have n plus 5 times 4 so which mathematical operation between addition and multiplication comes first its multiplication, introduced brackets around 5 and 4 and then simplified what was in the brackets there and got 20 then added 20 to what was there to n, thereafter, multiplied, messed it up by multiplying n plus 20 to get 20n

(Pair 2, turns 2 – 8)

Extract 9.12

This is multiplication where we are given addition and multiplied by a numeral. So the answer in the first place is wrong. Why? The learner lacks the idea of distribution. Because there are two terms n plus 5 to be multiplied by 4.

So in part A, this one the learner maybe did not understand the question. Because it said multiply n plus 5 by 4. So the learner what he did was just to multiply numerals on their own and do away with the letter. Already there's no distributive law, it's not applied.

(Pair 7, turns 2 and 4)

It is evident here that student-teachers main repertoire in recognizing the error in the scenario is about learners' inappropriate completion of the distributive law due to their misrecognition of the $n + 5$ as one quantity, which should be enclosed in brackets and then each term multiplied by 4. Instead, the learners decided to multiply only 5 by 4 to get 20, resulting in $n + 20$ and $20n$ as an equivalent expression to $(n + 5) \times 4$. Student-teachers also pointed out how problematic it is to interpret the

mathematical sentence as $n + 5 \times 4$ as it completely drifts one away from the intended purpose. Moreover, considering this interpretation brings in focus the issue of order of operations (BODMAS), hence compelling learners to carry out the multiplication of numbers first. In a way, just as discussed in the reservoir, student-teachers do talk about 'letter not used' when they make reference to learners multiplying numbers 5 and 4 only without operating on *n*.

In recognizing the error, student-teachers also spoke with the official voice of school mathematics in terms of the method of using the distributive law of multiplication over addition where each term in the binomial $(n + 5)$ should be expanded by 4. In terms of the form of practice, the orientation is towards mathematics and absences in that learners did not recognize the importance of using the distributive property as they opted to multiply the numbers only. This suggests that the student-teachers make explicit the criteria of the official discourse of school mathematics in terms of using the distributive law of multiplication over addition operationally.

9.3.3. Student-teachers' explanations of the sources of error

Three major categories of the sources of the error recognised in the scenario have emerged from the analysis. The dominant one being that the error stems from teaching sequence, followed by that the error stems from an interpretation problem and lastly that the error stems from the teaching emphasis.

- Error stems from teaching sequence

In this broader category, student-teachers talked about how knowledge of related school mathematics concepts influences the learning of expanding an expression by a number. These included knowledge of like terms (Pairs 1, 3, 4, 5, 6, 7, and 8), and knowledge of BODMAS (Pairs 1, 2, 3, 4, 5, and 8). Student-teachers talked about these two issues in similar ways.

Extract 9.13

… So I think what could have been going on in this pupil's mind was the consideration of like terms when you are adding but this time, this pupil applied this to multiplication. Obviously that is why this pupil had to multiply 5 by 4 without considering n, ok, because when you are adding, you only add like terms so he or she looked at 5 and 4 as constants, so you can only multiply 5 and 4 according to this pupil. But in this case, this is multiplication, so you cannot use that law or that rule. 4 has to be distributed throughout the whole expression. (Pair 6, turn 2)

… I have seen so far I can tell that learners have difficulty in dealing with letters, especially if you've got something unknown just letters, letter a, letter x. They've got difficulties with those things because like if you have n plus five…when you are multiplying this by four, now for them they think you are not multiplying the whole lot of n plus five, you are just multiplying five by four. … Now they think if…because n is not a number and five is a value so you are supposed to multiply five by four because these are values. And then you are going to get twenty which will be twenty now plus n…this is the mistake which was going on in this task. (Pair 8, turn 8)

Extract 9.14

… The second one, he started very well. n plus five times four, he said n plus now applied BODMAS but wrongly. He went on to say n plus four, then plus, then n plus four times five that is twenty. …

… I still emphasise that and this idea of using brackets…this was a very good idea she has applied here, he or she. But the problem is she has applied it wrongly. This was a good idea. We don't just put brackets when it is multiplication. No! as long as you have been given a quantity which is multiplied by a certain figure, you are supposed to put, if that quantity has

division or has got multiplication or whatever, and it has got two or more terms, it is better you put brackets. Then that one which has got maybe one number just leave it even without brackets. Just multiply by that. (Pair 8, turns 2 and 18)

In locating the error in teaching sequence, two relationships are identified pertaining to student-teachers' repertoires. Firstly, there is a relationship between the method of collecting like terms and multiplication of terms. As shown in extract 9.13, student-teachers point to that learners are aware of how to deal with the method of collecting like terms when working with algebraic expressions but this has been misapplied on the multiplication of expressions. In this case, learners could have considered 5 and 4 in their mathematical model 'n + 5 x 4' to be like terms which they ended up multiplying, and n was not taken into consideration since there was nothing in n to multiply with, hence getting n + 20. This description, in Ryan and Williams (2007) terms, suggests overgeneralization of like terms over expanding of expressions. In particular, in Lima and Tall (2008) terms, this is about met-before where learners' knowledge of like terms is interfering with the learning of expanding expressions by a number. The issue of like terms is similar to what was described in the reservoir, and to what student-teachers raised earlier on in scenario 4 pertaining to the expression – expansion by index problem. Moreover, there is also awareness among student-teachers that learners experience difficulties with letters in their learning of algebra from the number of scenarios so far engaged with. Here they also talk of the issue of ignoring the n, which in Hart et al. (1981, pg 108) terms suggests "letter not used".

The second relationship is between use of brackets and multiplication. As shown in extract 9.14, student-teachers pointed to that learners are aware that the operation multiplication is carried out first before addition. This could have made the learners enclose '5 x 4' in brackets in their modeled expression 'n + 5 x 4'. Student-teachers also pointed to that although the use of brackets tended to be good in this context, it is inappropriate given that they were supposed to 'multiply n + 5 by 4'. This is so because brackets are not only applicable to carrying out multiplication first but also to other expressions with two or more terms when dealing with expansions. In Ryan and Williams (2007) terms, this suggests overgeneralization of brackets over multiplication. In particular, in Lima and Tall (2008) terms it is an issue of met-before where learners' knowledge of carrying out multiplication first before addition is interfering with the expansion of expressions by a number in terms of using brackets.

In pointing to overgeneralization as one of the sources of error in the scenario, student-teachers spoke with the voice of the unofficial discourse in terms of how prior knowledge, though constraining in Smith et al. (1993) terms, influences the learning of related concepts. In this case it is the knowledge of like terms and BODMAS which could have had an effect on the expanding of expressions by a factor to get an equivalent expression. So their descriptions are in the professional experience of what it means to teach and not discursive as terms such as overgeneralization and met – before are not mentioned. In terms of form of practice, student-teachers are oriented towards learner and presences. They do not see learners as empty vessels to be filled but that they come with competencies of like terms and BODMAS to the learning of expanding expressions by a factor, hence a competence mode. However, criteria of the official discourse on expanding expressions by a factor are implicit since no detailed explanations are provided as they discuss the issue of overgeneralization.

- Error stems from interpretation problem

Student-teachers (Pairs 1, 2, 4, 5, and 7) also talked of the source of the error in the scenario being as a result of direct interpretation of the mathematical statement.

Extract 9.15

They decided to write it in this form because basically just pick on n plus 5 as it is, when you are multiplying it by 4 you introduce the times sign. So it is so direct an interpretation that

then he just picks on n plus 5, then the 'by' is the multiplication, and then you have a 4 there. That was the direct translation of this statement. Just like when our learner is asked maybe to say, divide…maybe you ask our learners to say, divide 40 by half. So they take it so direct to say 40 to be divided by half, should be 20. Because you are dividing it by half, we are getting…we are halving 40. I don't know whether that term is ok but we are getting half-half. It's in parts. So divide 40 by half, they take it that 40 to be divided by half should be 40 divided by 2, which is 20. In other words, when in arithmetic we says it is 40 over half. Where the half is inverted to the number 2. So you have 40 times 2 over one, which is 80. So dividing 40 by half doesn't give us something less than 40 but something greater than 40. So the interpretation here was so direct. (Pair 4, turn 7)

In locating the problem in the direct interpretation, student-teachers repertoire points to how in the statement 'Multiply n + 5 by 4' the 'by' was directly replaced by the multiplication operation to read as 'n + 5 x 4'. So the mathematical problem was expressed in the form it was phrased, hence learners misinterpreting the definitional meaning of 'Multiply n + 5 by 4'. Student-teachers assert that the problem of direct interpretation has its roots in arithmetic where when learners are asked, for example, to divide 40 by ½ their response would immediately be 20. Hardly would they realize that it is supposed to be 80.

Moreover, in locating the problem in learning (direct interpretation), student-teachers spoke with the voice of the official discourse of what it means to expand a binomial for them to have realized that what the learners have done is an issue of direct interpretation. In terms of form of practice, they are oriented towards mathematics and absences. The learners are being criticized for directly replacing 'by' with the multiplication operation in the statement 'Multiply n + 5 by 4' and get 'n + 5 x 4'. However, criteria of the distributive property are implicit.

- Error stems from teaching emphasis

In locating the source of error in teaching, student-teachers (Pairs 2, 4 and 8) talked of how the issue of seeing 'n + 5' as one quantity could not have been emphasized.

Extract 9.16

No, they were not taught to say if you are given an expression not a single expression but an expression involving an algebraic expression involving more than one term that should be treated as a single quantity so they never understood it like

Int: But how sure are you that they were not taught because they had covered the algebra syllabus

It's the the work that is telling me that probably they didn't know because if they had known the they should put the brackets around

(Pair 2, turns 11, 12 and 13)

Again maybe when being taught algebra and the quantities in algebra, maybe their teacher did not emphasize that if at all you have a minus or a plus in between a variable and a number that should be considered as one thing. Now due to lack of that knowledge the pupil had to do what he had done here. I think it's because of the sign.

*Even on **'expanding algebraic expressions'** this carries forth from that. Because when you're expanding you are basically taking the quantity that is outside of brackets through every term that is there inside. So basically this can be emphasized from expanding the expressions, where you take n plus 5 as an expression, and then 4 as a quantity I would say, because 4 run through n plus 5 in the order of multiplication. So maybe when you refer to **'expanding algebraic expressions'**, it should really help our learners.*

(Pair 4, turns 4 and 5)

So in short here, I would say the teacher was supposed to emphasize on the issue of how to apply the BODMAS brackets first and what you should solve. Like in here, they were supposed to understand if you find the question of that nature multiply n plus five, make sure that you put that in brackets then you multiply by four. Meaning that four will be a factor. (Pair 8, turn 2)

As indicated in extract 9.16, in locating the source of the error in teaching emphasis, student-teachers' repertoire point to two issues. The first issue is that the teacher could not have emphasized that an expression with more than one term could be considered as one quantity or one thing (for example n + 5) which should be enclosed in brackets and multiplied out by another quantity such as 4. Related to this is that the teacher could not have emphasized that the process to be carried out in such a mathematical problem is "*expanding algebraic expressions*" (Pair 4, turn 5).

In locating the problem in teaching emphasis, following Morgan et al. (2002), student-teachers spoke with the voice of the official discourse of school mathematics in terms of expanding expressions and appropriate use of brackets. As for form of practice, student-teachers are orientated towards teaching and absences. Teachers are being criticized for not focusing on key issues during their teaching. The learners are not aware that an expression with more than one term could be considered as one quantity, hence enclosed in brackets, and that can be expanded by another quantity because teachers did not emphasize. However, student-teachers make explicit criteria of the official discourse in terms of the process of expanding algebraic expressions.

9.3.4. Decisions about remediation suggested by student-teachers

Two broader categories of how student-teachers would remediate the error recognized in the scenario are that the teacher ought to emphasize key concepts or issues during teaching; and that the teacher should give their learners more similar practice examples. The former suggestion is the dominant one since all the pairs' talk aligned with it while the later suggestion was less dominant since only four pairs' (pairs 3, 4, 5 and 7) talk aligned with it. the two suggested remediation strategies are discussed in the following sections.

- Emphasizing key concepts or issues during teaching

In suggesting that the teachers should emphasize key concepts or issues during teaching, student-teachers talked about the appropriate use of brackets and seeing an expression $n + 5$ as one quantity, and how the distributive law of a number over an expression is applied.

Extract 9.17

I think ah uh uh we have to make sure that they understand that when we are multiplying an expression by a number then that number is going to multiply each uh each term or each digit in the expression yeah by that number the number is going to multiply the whole lot of the things in the expression, in short before they even go any further they should know that this n plus 5 is one and therefore if we are multiplying it should be n plus 5 and then putting the brackets to show that this is one and then we multiply by 4 and when we do this this one the 4 is going to multiply ah the n there and then we add and then we multiply the 4 by the 5 to get our 20 (Pair 3, turn 13)

I think there its uh its its very important that the teacher makes sure that the learners understand the concept of commutative, associative and distributive laws and I think that is the very point that these learners were missing they did not know that actually the multiplication of a number to an expression is distributive. (Pair 3, turn 17)

First of all for me madam they should know that each time they are given an algebraic expression which involves more than one term they should be treating it as a single quantity so in short we are simply saying if I'm to assume that n plus 5 is equal to if if one can assume to say n plus 5 is equal to y so this y is taken to be a single quantity then you are simply saying y times or y multiply yes y times 4, since it says multiply n plus 5 since it is equal to y, multiply y by 4 which will give us 4y, but when you substitute y by n plus 5 into 4y it will give you uhuu 4n plus 20. (Pair 2, turn 23)

As indicated in extract 9.17, student-teachers' repertoire towards how they would help their learners develop the required algebraic thinking pertaining to the error recognized in the scenario is in the distributive property of multiplication over addition. They point to how important it is for teachers to emphasize that the binomial $n + 5$ is one thing or quantity which should be enclosed in brackets and each term multiplied out by 4. Moreover, for learners to see that $n + 5$ is one thing or quantity, student-teachers suggested that the teachers could let $y = n + 5$. This implies that you are multiplying y by 4 which gives 4y, and substituting y which is equal to $n + 5$ in 4y yields $4n + 20$. Therefore, student-teachers' remediating strategy is in the process. It is in the procedures required to apply the distributive property given the multiplication of a binomial by a number.

In suggesting that teachers ought to emphasize key concepts or issues during their teaching as one of the broader categories of the remediating strategies, student-teachers spoke with the voice of the official discourse of the process of applying the distributive property of multiplication over addition. In terms of form of practice, student-teachers are oriented towards teacher and presences. They point to how the teachers ought to be incorporating in their teaching that n + 5 is one quantity which should be bracketed and the distributive law applied when multiplying by 4 so as to enhance learner learning. Moreover, the one quantity could be substituted by one variable say *y* so that learners can see that you are multiplying out each term in *y* by 4. So criteria of the official discourse of school algebra in terms of the process of carrying out the distributive property are explicit and the underlying meaning of 4(n + 5) is implicit.

- Giving learners more similar practice exercises

Student-teachers said that the major way learners could become familiar with the distributive law is for the teachers to provide their learners with similar practice exercises on how brackets are applied

Extract 9.18

… do a lot of practice with the use of BODMAS. (Pair 4, turn 23)

Like what I said, I think when you see that these common mistakes are so many, try to give them a lot of examples. (Pair 5, turn 24)

Another way that we can help them, if we can as teachers work with learners on practise, they practise on both types of questions. So that which uses brackets and that which uses just a statement, like multiply this by that. If we have examples of this type and that of brackets, then definitely next time they will meet this type they won't change, and just multiply a number by a number. (Pair 7, turn 18)

As indicated in extract 9.18, student-teachers' repertoire towards remediating the error in the scenario is about providing learners with a lot and variety of practice examples. This includes those that require interpretation and those that already have brackets in their form. They point to how important it is for teachers to work with their learners on such tasks so that they become familiar with the distributive property irrespective of the form in which the question is asked. This suggests that practice is critical to the enhancement of learner learning.

In suggesting that teachers ought to give their learners more similar practice examples once they realize that their learners are experiencing problems with the distributive property, student-teachers spoke with the voice of the official discourse of school algebra. This is so because in the school algebra curriculum variety of tasks are provided for learners to practice the distributive property. As for form of practice, student-teachers are oriented towards teacher and presences. Teachers ought to incorporate in their teaching more and variety of practice examples on application of the distributive property to enhance learners' learning. However, criteria of the official discourse in terms of both process and meaning of the distributive property are neither implicit nor explicit in talking about the importance of practice examples.

12.9 Appendix I: Analysis of student-teachers' talk on Scenario 6

9.4.2. Student-teachers' recognition/misrecognition of the error

Pairs 1 – 7 of the student-teachers interviewed were able to identify the error in scenario 6 but pair 8 could not. In Jacob et al. (2010) terms, the 7 pairs attended to learners' strategy by stating that learners do not know what it means, given a non – standard quadratic equation, to solve for *x* using the factorization method.

Extract 9.19

M: The learners' understanding in this task is a problem of, ... Because on the part of quadratic expressions when are we allowed to do that? It is when on the other side it is equal to zero. That is when we can say these in brackets would be equal to that or that. Now if it is not equal to zero, then we must first of all expand what is in brackets and then go back to quadratic equations, how we solve them. Are they factorizable, later on by taking the like terms together and grouping all of them together making sure that one side of the equation is zero. In that way, we can solve it so that in this case like I try to pass through it to be x minus two multiplied by x plus three equals zero, which means either x minus two is equal to zero or x plus three is equal to zero so that x is two or x is negative three. (Pair 1, turns 2 and 8)

I: Oh, so you found x as being two or negative three?

M: Yes.

I: So we have an x is equal to two there.

M: Of course there is an x equals two, we cannot take this method to be strictly correct, because if we try to change anything on another one that would be similar in the same way. I don't think I would emphasize to say it would be same value that we get. Somehow somewhere there will be a change.

I: What do you mean?

M: What I mean is, this two of course there is a chance that a wrong answer could be found to be the exact answer that the teacher wanted but the method you have used is a wrong method, anyway. (Pair 1, turns 9 – 14)

B: ... when we were introduced to quadratic equations I remember very well, that at a point where you have factorized and the other side is zero, then you can take them to say either this part is equal to zero or the other part will be zero. And now that is the background of where we say a variable times another variable equal zero is the same…is either one variable is zero or the other one is zero, or vice versa. But, one of them could be zero, the other one, or both of them could zero. That is why we take either that to be zero or the other one to be zero because there will be a situation where may be the other one is not zero and may be two, so that whenever we say two times zero it will still give us zero. (Pair 1, turn 23)

Student-teachers' main repertoire in recognizing the error in the scenario is that learners cannot distinguish what it means to solve for *x* using the factorization method in a standard quadratic equation and a non – standard one. Their argument is that learners' process used in the scenario would be justified if for example $(x-1)(x+2)$ was equal to zero and not 4. Their recognition of the error is based on their knowledge of what it means to solve for *x* using the factorization method, given a quadratic equation in any form. Key to the method is that the quadratic equation should be expressed in standard form so that it eventually assumes the form $a.b = 0$ where either *a* or *b* is equal to zero or both are equal to zero. Student-teachers also point to how they would not accept $x = 2$ as one of the answers learners found, though correct, because of the wrong method used.

In recognizing the error learners made in the scenario, student-teachers spoke with the official voice of school algebra by emphasizing the process required when factorizing quadratic equations and the underlying meaning of equating the factors to zero. In terms of form of practice, student-teachers are oriented towards mathematics and that learners did not express the given quadratic equation $(x-1)(x+2)=4$ in standard form before solving for *x* by factorization method, hence absences. Moreover, student-teachers make explicit criteria of what it means to solve quadratic equations of any form using factorization method in that key is equating the expression to zero.

Similar to scenario 1, it is interesting to note that pair 8 is again an outlier as they did not recognize the error in scenario 6, hence a misrecognition. They talked of how they could not see any problem in what the learners had done.

Extract 9.20

First the learner actually didn't…the work she did…the work he or she did was okay, this was okay. If you have or if you are multiplying two…if you have an equation of this nature which is a quadratic equation but it has been factorized, say that this other part of the quadratic equation, x minus one, is equal to four or this other part, x plus two is equal to four, since these are the factors of the quadratic expression, equation I mean. So, from those two you can solve for x.

No, according to this question, this answer is correct. Now, there are some questions whereby they specify [that] find the values for x which satisfy the following expression. Now, in this case, you are just solving for x. So, this x is equal to five is the solution. They are not saying you should find the value for x which satisfies this equation. No, we are solving for x in this equation. So this answer is correct according to me. (Pair 8, turns 2 and 21)

Pair 8's repertoire in misrecognizing the error as indicated in extract 9.20 is that they see nothing wrong with what learners have done in the scenario. This is so because the student-teachers see the quadratic equation $(x-1)(x+2)=4$ as having already been factorized; therefore, necessary to equate each factor to 4 and get the two values of *x*, which in this case are 2 and 5. Their argument is that since there is no specification in the given question on that the values of *x* they find should both satisfy the equation, it is acceptable in this case for learners to have found $x = 2$ or $x = 5$. Similar to what learners did in the scenario, this suggests that pair 8 does not seem to know what it means to solve a quadratic equation of the form $(x-1)(x+2)=4$ using the factorization method. Since the student-teachers displayed themselves as one of the discursive errors, it was not possible for me to explore further their discourses about learner errors and misconceptions.

9.4.3. Student-teachers' explanation of sources of error in Scenario 6

Three major categories of the sources of the error recognized in the scenario emerged from the analysis. These are that the error stems from teaching sequence, teaching emphasis, and interpretation problem.

- Error stems from teaching sequence

In locating the error in the teaching sequence, student-teachers talked about the influence knowledge of factorizing a standard quadratic equation such as $(x-1)(x+2)=0$ might have on factorizing a non – standard one such as $(x-1)(x+2)=4$ (Pairs 3, 4 and 5).

Extract 9.21

It must have been that may be the learner might have come across an expression that…let me say, x plus a in brackets then x plus b in brackets equal to zero. And then because it would say

x plus a is equal to zero or x plus b is equal to zero, which is…which gives you the values of x as either negative a or negative b. The learners didn't realize that…I don't know. May be they thought the way you handle this situation that is illustrated, when you have a zero on the other side of the equation is the same way you handle it if you have a non-zero variable, non-zero number. If you have any other number, you handle it in the similar way as you handle it when you have a zero, they could have been thinking in those lines. (Pair 5, turn 2)

In Ryan and Williams (2007) terms, student-teachers' repertoire in locating the error in teaching sequence as indicated in extract 9.21 points to the issue of overgeneralization although no mention of the word is made in their talk. There is a relationship between factorizing quadratic equations already expressed in standard form and those expressed in a non standard form. This points to that learners have knowledge of what it means to factorize a standard quadratic equation such as $(x+a)(x+b)=0$ where each factor is equated to zero such that *x* is either -a or –b. in this case, the learners are using the knowledge to factorize a quadratic equation of the form $(x-1)(x+2)=4$ because they think the same operations apply if factors are equated to any other number other than zero, hence overgeneralization. In Lima and Tall (2008) terms, this is an issue of met-before where knowledge of factorizing a standard quadratic equation is misapplied on the factorization of a non – standard quadratic equation.

Since the student-teachers are pointing to that the source of the error could be as a result of overgeneralization, and in particular met-before, they spoke with the voice of the unofficial discourse. They talked of how knowledge of related concepts such as factorizing standard quadratic equations could have influenced how learners dealt with factorizing a non – standard quadratic equation such as the one in the scenario, and hence resulting in the error. In terms of form of practice, student-teachers are oriented towards learners and presences. They are not criticizing the learners for overgeneralization but instead they see learners as coming with some knowledge of how to factorize standard quadratic equations to the learning of factorizing non – standard quadratic equations. However, criteria of the official discourse of school algebra in terms of how a non – standard quadratic equation $(x-1)(x+2)=4$ could be factorized to solve for *x* are implicit when talking about generalization as one of the sources of the error.

- Error stems from teaching emphasis

In locating the source of the error in the teaching emphasis, two issues arose. The first one is that the teacher could have focused more on standard quadratic equations and ignored the non – standard ones (Pairs 1, 4 and 7) while the other is about the consequences of working with shortcuts in mathematics (Pair 7).

Extract 9.22

… so suppose the teacher did not stress the point that suppose you have another number here, you take it to the other side, and you remain with a zero. So, the learner could have thought that other numbers would hold. So if you can have x minus one, multiplied by x plus 2, in those brackets, equals zero, then if you can still say x minus one is equal to zero or x plus 2 is equal to zero, it should still hold for other numbers, that is if the teacher has not stressed that point. So our learner here is taking it from that stance, from that position, to say, then he will still have this equated to 4, the other one also equated to 4. Then he goes on and finds those answers. (Pair 4, turn 2)

I think here, the learner did this because of teachers giving the same type of problems. Where x plus 2 multiplied by x minus one, is equal to zero. All the quadratic equations that the child was given was equated to zero. As a result, next time when he gives such a question, the child will be thinking that this time this is equal to 4, can be done the same way. That's why the

learner did this. Otherwise they are supposed to expand, then rearrange, so that we have a quadratic equation equal to zero. Then factorize and then… (Pair 7, turn 10)

Extract 9.23

These are the results of what we call shortcuts in mathematics. They say, the way teachers we normally emphasize the fastest way of doing mathematics, sometimes we just look at the question, and then the answer is there. So, I think the learner here was confused. When we apply the fastest way of evaluating an algebraic expression in this instance, it is only true when it is x minus one in brackets, x plus 2 in brackets, equals to zero. But not when there's a number. The only way to find this is first you expand, then equal it to that number. Then you follow the usual procedure. (Pair 7, turn 4)

As shown in extracts 9.22 and 9.23, in locating the error in the teaching emphasis, student-teachers' repertoires point to that the teacher could not have emphasized how to solve a non – standard quadratic equation using factorization method, and the tendency by teachers to encourage their learners to use shortcuts, respectively. In referring to the teaching not being focused on how to solve non-standard quadratic equations using factorizing method, student-teachers pointed to how the teacher could not have stressed the importance of ensuring that the standard form is taken into consideration. In this case by expanding and rearranging so that the quadratic equation is equated to zero before carrying out the factorization. This suggests that the learners are not able to apply the appropriate approach for solving for *x*, given a non – standard quadratic equation, using the factorization method because the teacher did not teach them on how to do so. As a result the learners concluded that if the factorization method can work for $(x-1)(x+2)=0$, then it should also work for $(x-1)(x+2)=4$. This relationship is similar to the issue of overgeneralization, and in particular met – before, earlier discussed on error stemming from teaching sequence.

As for the tendency by teachers to encourage their learners to use shortcuts, student-teachers pointed to that this could have confused the learners as they thought of quick ways on how they could solve for *x*. So the quick way for the learners to solve for *x* was to equate each factor in the non – standard quadratic equation '$(x-1)(x+2)=4$' to 4 instead of applying the appropriate process.

In Morgan et al. (2002) terms, in locating the error in teaching emphasis, student-teachers spoke with the voice of the official discourse of school algebra in terms of factorizing quadratic equations using factorization method. They point to how important it is for teachers to expose their learners to different forms of quadratic equations and how the method of factorization is implicated. They argue that doing so would not leave learners to make inappropriate inferences. As for form of practice, student-teachers are oriented towards teaching and absences. The teacher is being criticized for not emphasizing on how non – standard quadratic equations should be solved using the factorization method and for emphasizing on shortcuts when solving mathematical problems. They argue that this could have resulted in learners not being aware of the need to express the given quadratic equation '$(x-1)(x+2)=4$' in standard form by expanding and rearranging before carrying out the factorization method, and that shortcuts do not help in such a situation. However, student-teachers make explicit criteria of how non – standard quadratic equations such as '$(x-1)(x+2)=4$' could be solved using the factorization method.

- Error stems from interpretation problem

In locating the error in the interpretation problem, student-teachers (Pairs 2, 3 and 6) talked of how learners could have thought that the non – standard quadratic equation was already factorized.

Extract 9.24

... Now this pupil just thought it wise that since I have the ... the terms, since ... if I factorized which ever stage I am at, I'll simply equate the factors: each factor I have this side I will equate it to the constant on the other side. But in this case this expression is not simplified. So the first step was needed to simplify this equation then from there now you equate whatever the values you are going to find. Then you solve because the method the pupil thought to use was factorization, so there was need to follow the right procedure of factorization.

So just to add on, I think that quadratic equation ... the standard form is $ax^2 + bx + c = 0$. *So a situation like this one, like he has put it, the child was supposed, first of all, to expand the terms on the left-hand-side (LHS) and then adding the like terms together so that this child can come up with the standard form of quadratic equation and then after you get that standard form of quadratic equation that's when you can use either factorization method or (what is that) the formula method.*

Here, like he has put it, it is because they think this thing is already factorized, for them they know that if you factorize then you just say either this one is equal to that or that is equal to that. So they didn't want to suffer (laughter)

That's ... that's the thing , that's why I said they were avoiding work because obviously, one of the pupils could have thought to say: "Mmm! But this is not 0, because in a standard form if you factorise this has to be 0. Now ... Er! ... but again this is factorized so obviously this question ... our teacher made this simple for us, so let's just go guys" then they started going.(Pair 6, turns 4, 5, 7 and 10)

In locating the source of the error in the interpretation problem, student-teachers' repertoire, as shown in extract 9.24, is that learners saw the non – standard quadratic equation $(x-1)(x+2)=4$ as already having been factorized for them. The learners thought that all that was required for them to do was to equate the factors (x – 1) or (x +2) to 4. This suggests that learners misinterpreted the procedure for solving a quadratic equation of the form given using the factorization method. They did not recognize the need to express the non – standard quadratic equation as a standard one by expanding and adding like terms before solving for x using the factorization method. They also did not recognize that factorization method requires factors to be equated to zero and not any other number.

In Morgan et al. (2002) terms, in locating the error in learning (interpretation problem) student-teachers spoke with the voice of the official discourse of school algebra in terms of what it means to factorize a non – standard quadratic equation of the form $(x-1)(x+2)=4$. It requires being expressed in standard form by expanding and adding the like terms before applying the factorization method. As for form of practice, student-teachers are oriented towards mathematics and absences. The learners are being criticized for thinking that the non – standard quadratic equation was already factorized and could not think of ways of expressing it in standard form before carrying out the factorization method. However, criteria of the official discourse of solving a non – standard quadratic equation using the factorization method are made explicit.

9.4.4. Decisions about remediation student-teachers suggest

Two ways of how student-teachers talked of how they would remediate the error in the scenario were identified. The dominant strategy proposed was that the teachers ought to emphasize key concepts or issues during their teaching. The second strategy was that the teachers ought to provide learners with practice exercises.

- Emphasizing key concepts or issues during teaching

In referring to emphasizing key concepts or issues during teaching, student-teachers talked of two issues: (1) emphasizing procedure in terms of expressing non – standard quadratic equations in standard form before application of the factorization method (Pairs 1, 3, 4, 6, and 7); and (2) graphical representation of a quadratic equation (Pairs 2, 4 and 5).

Extract 9.25

It is the logical way. I think our pupils must be taught how to handle such first of all. We should tell them to say number one, when it is of this form, make sure that on one side it is zero and then what is on the other side is completely factorized then you can apply this method. So, the logical way of how to do it, you give them you show your pupils, you emphasize it and in that way the pupils will be able to learn and see how they would apply it … .(Pair 1, turn 27)

… solving quadratic equation, the emphasis should be put that always the right-hand-side must be 0 and then all these other terms must be put on the left hand side.

Following the standard form of quadratic equation.

And then from there that's when you can solve either by using the quadratic formula or the factorization method. (Pair 6, turns 12, 13 and 14)

Extract 9.26

I think as he stated it is always very necessary like such equations as that one, to as much as possible go back to what the meaning is. So when you say you have a quadratic equation, it's not just basically an equation like that. Tell them where that is coming from. So you say state, in other words, mostly our learners understand more with the practical things, where they can see why they are manipulating these numbers. They are not just saying this goes the other side, that goes there, and why is that happening. So you say, ok, if you have a quadratic equation in this form, ax squared plus bx plus c is equal to zero, basically what you are saying is that you are finding the values of x which will make the expression zero, the value of y. So you have the XOY- plane, you interpret on that plane. If you can interpret it on that plane it would be very easy for the learners to follow through and say, that is why we are saying we should equate to zero because we want to get all the values of x where this is cutting the x axis…as in the x - intercepts where y is zero. So basically just as much as possible, bring it back to the sketching because that is where you follow through. If they can see some sketches somewhere and you justify it from those sketches then it is…and basically even work with them the way they are working and let them strike a difference between the two workings. And unlike just discarding it to say this is wrong, it is better you try and say, ok, in that case, let's now put it on y equals to 4 and see what comes out. (Pair 4, turn 27)

As shown in extract 9.25, student-teachers' repertoire points to how the teachers ought to emphasize by showing their learners that the logical way to solve non – standard quadratic equations is to ensure that they are expressed in standard form. Only then would they apply the factorization or quadratic formula method. Their argument is that if the teacher emphasized, then learning will occur on the part of the learners as they will be able to apply the correct procedure. Moreover, as shown in extract 9.26, to emphasize the issue of expressing the non – standard quadratic equation into a standard form, use of graphical representation was suggested. They argue that graphical representation of a quadratic equation would provide a practical interpretation to learners on the meaning of solving for x given a quadratic equation. Solving for x means finding the x values where the parabola cuts the X – axis where $y = 0$. For learners to see the difference between their working in the scenario and what the teachers would be emphasizing, student-teachers pointed to that it would be necessary for teachers to

contrast this meaning with the graphical representation of $(x-1)(x+2)=4$, that is when $y = 4$. Drawing from the reservoir, in Sfard and Linchevski (1994) terms, this suggests the process – object duality in terms of the teacher ensuring that learners develop both operational and structural view of a quadratic equation. This means that given a quadratic equation of any form, the teachers ought to ensure that their learners know how to solve for *x* and why they are doing so.

In suggesting that teachers ought to emphasize in their teaching procedure of factorizing non – standard quadratic equations and the graphical representation of the meaning of solving for *x*, student-teachers spoke with the voice of the official discourse of school algebra. As for form of practice, they are oriented towards teacher and presences in that the teacher ought to incorporatingin their teaching procedure for factorizing non – standard quadratic equations and what it means graphically to solve for *x*. Therefore, student-teachers make explicit criteria of the official discourse of the meaning of solving for *x* given a non – standard quadratic equation.

- Providing learners with practice exercises

In suggesting that teachers ought to give their learners practice exercises, student-teachers talked of the need to introduce learners to variety in terms of forms of quadratic equations. Only pair 7 pointed to this issue.

> **Extract 9.27**
>
> *… Similar questions normally give confusion. For instance, if I write zero, suppose…ok, it was, x minus one in brackets x plus 2 equals zero. Suppose it was zero equals x minus one and x plus 2, certain pupils will have problems too. Because they are used that the letters the expressions will be to the left hand side. And what we add should be…what we equate it to should be on the right hand side.*
>
> *I: So in short what are you trying to say?*
>
> *The teacher should give a variety of questions because from the look of things this teacher was always giving questions such that an expression equals to zero. …* (Pair 7, turns 23, 24 and 25)

As shown in extract 9.27, student-teachers' repertoire points to providing learners with variety in terms of standard and non – standard quadratic equations. Their argument is that such exposure would enable learners become familiar with different forms in which quadratic equations could be presented, hence reduction in the confusion that would arise. They pointed to that the danger in focusing only on standard quadratic equations is that given non – standard quadratic equations learners would tend to solve them as though they were solving standard quadratic equations as has been the case in the scenario.

Therefore, in talking about teachers providing to their learners variety in terms of different forms of quadratic equations, student-teachers spoke with the voice of the official discourse of school algebra. For example, the textbook makes available different forms of quadratic equations for learners to use for practice. As for form of practice, student-teachers are oriented towards teachers and presences in that the teachers ought to be incorporating in their teaching different forms of quadratic equations as practice exercises. Moreover, in talking about different forms of quadratic equations, student-teachers make implicit criteria of how these quadratic equations are supposed to be solved.