$$g(w)=c\int_0^\infty \exp\left\{-\frac{n}{2\beta^2}\ln^2 Z-\tfrac{1}{2}wZ\right\}(wz)^{k/2-1}\,dZ,\quad 0<w<\infty \tag{3.4.12}$$

$= 0$ elsewhere

where $C = \dfrac{\sqrt{\frac{n}{\beta^2}}}{\Gamma(k/2)2^{k/2}\sqrt{2\pi}}$

and g is the pdf of W.

To find an exact confidence interval for u_x/u_y, let

$$w = \frac{2e^{\alpha}m\bar{y}}{u_y\bar{x}_G} \tag{3.4.13}$$

Then,

$$\Pr\{a<w<b\} = P\left\{\frac{e^{\beta^2/2}a\bar{x}_G}{2m\bar{y}} < \frac{u_x}{u_y} < \frac{e^{\beta^2/2}b\bar{x}_G}{2m\bar{y}}\right\} = P, \tag{3.4.14}$$

which implies a 100p% confidence interval for A is

$$\frac{1}{1+B_1} < A < \frac{1}{1+B_2}$$

where $B_1 = \dfrac{e^{\beta^2/2}b\bar{x}_G}{2m\bar{y}}$

and $B_2 = \dfrac{e^{\beta^2/2}a\bar{x}_G}{2m\bar{y}}$

A further paper by Gray and Schucany (1969), extended the results of Gray and Lewis (1967), by developing a test of hypothesis, based on the assumption of lognormal repair times and exponential failure times.

Given that the parameter β^2 of the lognormally distributed repair time is known, and since the distribution of $W=2e^{\alpha}\bar{Y}/u_y\bar{X}$ (see equation 3.4.13) is also known, then it is possible to find a and b such that

$$\Pr\left\{\frac{2e^{\alpha}m\overline{Y}}{u_y\overline{X}} < b_\gamma\right\} = \gamma \tag{3.4.15}$$

Hence, under the null hypothesis H_o,

$$\Pr\left\{\frac{2m\overline{Y}(1-Ao)}{\overline{X}\exp\{\beta^2/2\}\,A_o} < b_\gamma \,/\, H_o\right\} = \gamma$$

$$\text{or } \Pr\left\{\frac{\overline{X}}{\overline{Y}} > \frac{2m}{b_\gamma\exp\{\beta^2/2\}}\left(\frac{1-A_o}{A_o}\right) / H_o\right\} = \gamma \tag{3.4.16}$$

Therefore, a test with critical region of size γ is obtained by calculating the test statistic $(\overline{x}/\overline{y})$ and rejecting H_o whenever

$$\frac{\overline{x}}{\overline{y}} > \frac{2m}{b_\gamma\exp\{\beta^2/2\}}\left(\frac{1-A_o}{A_o}\right) \tag{3.4.17}$$

The power function for testing H_o versus H_1 is given by

$$\Pr\left\{\frac{2m\overline{Y}}{\overline{X}\exp\{\beta^2/2\}}\left(\frac{1-A_o}{A_o}\right) < b_\gamma \,/\, A=A_1\right\} = 1-B(A_1) \tag{3.4.18}$$

where $B(A_1)$ is the type II error for the specific alternative $A=A_1$.

3.5 Jackknifing the availability estimate :

The only paper in the literature referring to this topic was presented by Gaver and Chu (1979). They studied the behaviour of the standard jackknife method for estimating system availability, with the emphasis on forming confidence intervals for availability. Exponential, gamma and long tailed h distributions were used for typical uptimes and downtimes for the one component and two components in parallel cases. The investigation was carried out using Monte-Carlo simulations with a basic comparison between the jackknife method and Thompson's method. In the one component case, the jackknife method was found to be the more robust, in terms of confidence interval coverage, especially when the long tailed h distribution was used for the uptimes. However, no results were presented where an improvement in coverage coincided with a shorter mean length and smaller variance of length. In the two components in parallel case, the jackknife method again gave very robust results in terms of coverage, average length and variance of length. As yet, no parametric method for confidence interval testing has been developed in the two or more component case. Finally, all the results presented in the paper corresponded to confidence interval estimation and the accuracy of the availability estimates in terms of bias, sample variance or mean square error were not discussed.

A summary of the theoretical methods used in Gaver and Chu's paper is as follows:

Let U be a typical uptime for a one component system, and D a typical downtime. If U and D are random variables with finite expectation $E\{U\}$ and $E\{D\}$, then the long run availability is

$$\boxed{A = \lim_{t\to\infty} \Pr\{\text{unit up at time } t\} = \frac{E\{U\}}{E\{U\} + E\{D\}} ,} \qquad (3.5.1)$$

under a wide range of assumptions concerning the sequences $\{U_i, i=1,2,\ldots\}$ and $\{D_i, i=1,2\ldots\}$ of successive up and downtime copies of U and D.

Suppose that the observations on up and downtimes from a sample size n, are $u_1, u_2, \ldots, u_n$ and $d_1, d_2, \ldots, d_n$ respectively. Assuming no parametric forms for the distributions of U and D it is then possible to obtain an availability estimate by

$$\boxed{\hat{A} = \frac{\bar{u}}{\bar{u} + \bar{d}} ,} \qquad (3.5.2)$$

where as usual the bar denotes the sample average (the estimate $\hat{A}$ will be referred to as the 'standard' estimate in the remainder of this chapter).

For the jackknife method, Gaver and Chu recommend a log-logistic transformation of the availability estimate

$$\text{i.e. } \ln\left(\frac{\hat{A}}{1-\hat{A}}\right) = \ln(\bar{u}) - \ln(\bar{d}) = Z \tag{3.5.3}$$

The quantity Z will tend to be symmetrically distributed and therefore be amenable to valid jackknifing ; see Mosteller and Tukey (1977), and Miller (1974).

The next step in the jackknife method is to recompute Z by leaving out, successively, the n pairs of sample values $(u_1,d_1),(u_2,d_2),\ldots,(u_n,d_n)$; the j^{th} version of Z is then

$$Z_{-j} = \ln\{(n-1)^{-1} \sum_{i\neq j} u_i - \ln\{(n-1)^{-1} \sum_{i\neq j} d_i\}$$

$$= \ln\{ \sum_{i\neq j} u_i - \ln\{ \sum_{i\neq j} d_i\} \text{ , } j=1,2,\ldots,n \tag{3.5.4}$$

The pseudovalues of Z are defined to be:

$$Z_j = nZ-(n-1)Z_{-j} \text{ , } j=1,2,\ldots,n$$

Then, the mean and variance of the pseudovalues are:

$$\hat{Z} = \frac{1}{n} \sum_{j=1}^{n} Z_j \text{ and } S_z^2 = \frac{1}{n-1} \sum_{j=1}^{n} (Z_j-\hat{Z})^2 \tag{3.5.5}$$

Thus, the jackknife availability estimate is therefore defined as

$$\hat{A}_{JK} = \frac{e^{\hat{Z}}}{1+e^{\hat{Z}}} \tag{3.5.6}$$

Confidence limits on the transformed value Z are found by treating the pseudovalues as independent sample values and using the Student's t statistic

$$\text{i.e. upper limit} = H_\alpha = \hat{Z}+t_{1-\alpha/2}(n-1)\cdot\sqrt{\frac{S_Z^2}{n}}$$

$$\text{lower limit} = L_\alpha = \hat{Z}-t_{1-\alpha/2}(n-1)\cdot\sqrt{\frac{S_Z^2}{n}} \qquad (3.5.7)$$

where $t_{1-\alpha/2}(n-1)$ is the $(1-\alpha/2)100\%$ quantile of the Student's t with $(n-1)$ degrees of freedom. Translating these intervals to apply to availability one can obtain the non-symmetric $(1-\alpha)100\%$ confidence interval

$$\frac{e^{L_\alpha}}{1+e^{L_\alpha}} \leq A \leq \frac{e^{H_\alpha}}{1+e^{H_\alpha}} \qquad (3.5.8)$$

Next, consider a two component system where the components are arranged in parallel. Hence, the system is only unavailable when both components are simulataneously unavailable.

If A_i is the availability of the i^{th} component, then assuming independence, the system unavailability is defined as:

$$\bar{A} = \prod_{i=1}^{2} \bar{A}_i = \prod_{i=1}^{2} \frac{E\{D_i\}}{E\{U_i\}+E\{D_i\}} \qquad (3.5.9)$$

Then, the j^{th} jackknife pseudovalue of $\ln \bar{A}_i$ is computed as follows:

$$I_{ij} = (\ln\bar{A}_i) = n\ln\bar{A}_i-(n-1)(\ln\bar{A}_i)_{-j}$$

$$= n\{-\ln(1+e^{Z_i})\} -(n-1)\{-\ln(1+e^{Z_{i,-j}})\} \qquad (3.5.10)$$

where Z_i is the 'Z' of equation (3.5.3), computed using data from the i^{th} component.

The sample mean and variance of the i^{th} component (log unavailability) pseudovalues are calculated as:

$$I_i = \frac{1}{n_i}\sum_{j=1}^{n_j} I_{ij}\ , \quad S_i^2 = \frac{1}{n_i-1}\sum_{j=1}^{n_j}(I_{ij}-I_i)^2 \qquad (3.5.11)$$

These are used to establish the upper and lower confidence intervals on ln.A as follows:

$$H_\alpha = I + t_{1-\alpha/2}\left(\sum_{i=1}^{2} n_i - 2\right) S$$

and (3.5.12)

$$L_\alpha = I - t_{1-\alpha/2}\left(\sum_{i=1}^{2} n_i - 2\right) S$$

where, as before, $t_{1-\alpha/2}(m)$ the $(1-\alpha/2)100\%$ percentile of the Student t distribution with m degrees of freedom. Translating these intervals to unavailability leads to:

$$e^{L\alpha} \leqslant \tilde{A} \leqslant e^{H\alpha} \qquad (3.5.13)$$

3.6 Monte-Carlo simulation studies : New results on the availability ratio

This section describes the extensive simulation studies that were undertaken in new areas of research regarding the non-parametric estimation of the availability parameter. Three different p.d.f.'s were generally used to generate the uptime or downtime deviates for each simulation run, namely the exponential, gamma and long tailed h distribution. The Weibull distribution was also considered for some specific runs. The long tailed h and Weibull distributions are defined as follows:

$$\frac{(1-h)^2 X e^{hX}}{\lambda} \sim \text{Long tailed h}$$

where X is exponentially distributed with E(X)=1, and h is the tail stretch parameter (0<h<1), and $\begin{cases} \frac{A}{B} x^{A-1} \exp(\frac{-x^{A}}{B}) & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases} \sim$ Weibul

where A and B are the parameters such that $A,B > 0 \cdot 0$.

For each simulation run, a 1000 samples were generated and the parameter values were such that $E(U)=100$ and $E(D)=1$, in the cases where the Weibull distribution was not used. The accuracy of the availability estimate was expressed in terms of bias, sample variance and mean square error. The 95 and 99% levels were considered for the confidence limits and their robustness were determined in terms of coverage, average length and variance in length. In terms of robust confidence intervals, the primary concern was to achieve good coverage for the parameter value. Having achieved the first objective, the second concern was to achieve a short average length and a small variance in length.

Three points should be noted regarding the simulation results presented in this chapter, namely:

(i) the standard method refers to the standard availability estimate using Thompson's method for confidence limit estimation

(ii) the jackknife method using improved t quantiles refers to Hinkley's method as described in Section 1.15.

(iii) in the one component case, the bootstrap estimate was calculated without a log-logistic transformation of the estimate. Transformation of the estimate for the bootstrap method did not produce better results and are therefore not presented.

(unless stated otherwise, the number of bootstrap iterations was 120 in each case).

The following simulation studies were undertaken:

(i) a comparison between the standard method and the jackknife and bootstrap methods in the one component case using either exponential, gamma or long tailed h distributions for the uptimes and downtimes. Three different sample sizes of 10, 15 and 20 were considered. The results of the simulation study are summarised in Tables 3.6.1-3.6.6 respectively. A comparison of the different methods, in terms of coverage and average length is presented in graphical form in Figures 3.6.1-3.6.6 respectively. The conclusions of the study are as follows:

a. the jackknife estimator has a smaller bias than the standard estimator at the expense of a very small increase of the mean square error.

b. the bootstrap estimator has a larger bias than either the standard or jackknife estimator.

c. the jackknife method using t quantiles and the bootstrap method are trustworthy in establishing confidence limits. In fact, the bootstrap method generally outperforms the jackknife method based on the criteria of better coverage, shorter average lengths and smaller variances of length.

d. the standard method does not produce trustworthy confidence limits using the long tailed h distribution. This result is in general agreement with Gaver and Chu (1979).

e. using the 'improved' t quantile method for the jackknife estimate produces a slightly better coverage than using ordinary t quantiles but at the expense of longer average lengths and larger variances in length. However, the 'improved' method still does not produce better results than the bootstrap method.

(ii) a comparison between the standard method and the jackknife and bootstrap methods in the one component case using either mixed distributions or contaminated distributions. For the mixed distribution, two different distributions, one of size 14 and one of size 6, were considered to form an overall sample of size 20. In the contaminated distribution case, a probability value of 0.1 with a corresponding multiplying factor

Table 3.6.1 : Bias, Sample Variance, Mean Square Error for Various Availability Estimation Methods in the One Component Case

		Standard Method	Jackknife Method	Bootstrap Method
	Bias	-0.0009	-0.0005	-0.0019
A	Sample Variance	1.95×10^{-5}	1.86×10^{-5}	2.23×10^{-5}
	Mean Square Error	2.03×10^{-5}	1.89×10^{-5}	2.66×10^{-5}
	Bias	-0.0005	-0.0003	-0.0012
B	Sample Variance	1.37×10^{-5}	1.31×10^{-5}	1.50×10^{-5}
	Mean Square Error	1.39×10^{-5}	1.32×10^{-5}	1.64×10^{-5}
	Bias	-0.0004	-0.0003	-0.0009
C	Sample Variance	1.03×10^{-5}	1.03×10^{-5}	1.09×10^{-5}
	Mean Square Error	1.05×10^{-5}	1.03×10^{-5}	1.18×10^{-5}

Uptime Deviates $U \sim$ Exponential, $\lambda = 0.01$;

Downtime Deviates $D \sim$ Gamma, $\mu = 1$, $k = 3$;

Sample Size A=10, B=15, C=20

Table 3.6.2 : Sample Confidence Intervals for Various Availability Estimation Methods in the One Component Case

			Standard Method	Jackknife method using t quantiles	Jackknife method using improved t quantiles	Bootstrap method using normal quantiles
A	Coverage	95%	98.4	94.8	94.9	96.3
		99%	99.8	98.9	98.9	99.5
	Average Length	95%	2.17×10^{-2}	1.94×10^{-2}	2.45×10^{-2}	2.02×10^{-2}
		99%	3.15×10^{-2}	3.20×10^{-2}	5.98×10^{-2}	2.91×10^{-2}
	Variance of Length	95%	6.52×10^{-5}	1.06×10^{-4}	9.32×10^{-4}	1.11×10^{-4}
		99%	1.35×10^{-4}	3.88×10^{-3}	1.55×10^{-2}	2.23×10^{-4}
B	Coverage	95%	98.1	94.1	94.3	94.6
		99%	99.9	98.8	98.9	98.8
	Average Length	95%	1.63×10^{-2}	1.39×10^{-2}	1.59×10^{-2}	1.31×10^{-2}
		99%	2.29×10^{-2}	2.06×10^{-2}	3.05×10^{-2}	1.72×10^{-2}
	Variance of Length	95%	2.59×10^{-5}	3.09×10^{-5}	1.36×10^{-4}	2.50×10^{-5}
		99%	5.05×10^{-5}	7.85×10^{-5}	3.48×10^{-3}	4.31×10^{-5}
C	Coverage	95%	97.7	94.2	94.8	95.6
		99%	100.0	98.2	98.2	98.9
	Average Length	95%	1.37×10^{-2}	1.15×10^{-2}	1.26×10^{-2}	1.17×10^{-2}
		99%	1.89×10^{-2}	1.64×10^{-2}	2.08×10^{-2}	1.60×10^{-2}
	Variance of Length	95%	1.31×10^{-5}	1.46×10^{-5}	3.28×10^{-5}	1.34×10^{-5}
		99%	2.49×10^{-5}	3.20×10^{-5}	5.88×10^{-4}	2.49×10^{-5}

Uptime Deviates $U \sim$ Exponential, $\lambda=0.01$;

Downtime Deviates $D \sim$ Gamma, $\mu=1$, $k=3$;

Sample Size A=10, B=15, C=20

Table 3.6.3 : Bias, Sample Variance, Mean Square Error for Various Availability Estimation Methods in the One Component Case

		Standard Method	Jackknife Method	Bootstrap Method
	Bias	-0.0021	-0.0015	-0.0039
A	Sample Variance	4.39×10^{-5}	4.36×10^{-5}	5.00×10^{-5}
	Mean Square Error	4.84×10^{-5}	4.36×10^{-5}	6.55×10^{-5}
	Bias	-0.0013	-0.0008	-0.0025
B	Sample Variance	2.65×10^{-5}	2.71×10^{-5}	2.83×10^{-5}
	Mean Square Error	2.66×10^{-5}	2.78×10^{-5}	3.45×10^{-5}
	Bias	-0.0010	-0.0006	-0.0019
C	Sample Variance	1.94×10^{-5}	1.98×10^{-5}	1.99×10^{-5}
	Mean Square Error	2.04×10^{-5}	2.02×10^{-5}	2.38×10^{-5}

Uptime Deviates $U \sim$ Long tailed h, $\lambda=0{,}01$, $h=0{,}2$;

Downtime Deviates $D \sim$ Gamma, $\mu=1$, $k=2$;

Sample Size : A=10, B=15, C=20

Table 3.6.4 : Sample confidence inervals for Various Availability Estimation Methods in the One Component Case :

			Standard Method	Jackknife method using t quantiles	Jackknife method using improved t quantiles	Bootstrap method using normal quantiles
A	Coverage	95%	91.3	91.3	92.3	93.9
		99%	97.5	97.1	97.6	98.2
	Average Length	95%	2.41×10^{-2}	2.88×10^{-2}	5.41×10^{-2}	2.56×10^{-2}
		99%	3.49×10^{-2}	5.36×10^{-2}	1.53×10^{-1}	3.36×10^{-2}
	Variance of Length	95%	1.54×10^{-4}	3.01×10^{-4}	9.05×10^{-3}	2.07×10^{-4}
		99%	3.16×10^{-4}	2.07×10^{-3}	6.23×10^{-2}	3.58×10^{-4}
B	Coverage	95%	91.4	93.2	93.4	94.0
		99%	97.3	97.1	97.5	98.6
	Average Length	95%	1.74×10^{-2}	2.03×10^{-2}	3.61×10^{-2}	1.88×10^{-2}
		99%	2.44×10^{-2}	3.28×10^{-2}	1.06×10^{-1}	2.47×10^{-2}
	Variance of Length	95%	5.58×10^{-5}	9.94×10^{-5}	4.28×10^{-3}	6.90×10^{-5}
		99%	1.08×10^{-4}	4.69×10^{-4}	4.57×10^{-2}	1.19×10^{-4}
C	Coverage	95%	90.6	91.8	93.3	93.5
		99%	97.1	97.6	98.1	98.3
	Average Length	95%	1.44×10^{-2}	1.66×10^{-2}	2.67×10^{-2}	1.57×10^{-2}
		99%	1.98×10^{-2}	2.51×10^{-2}	8.05×10^{-2}	2.06×10^{-2}
	Variance of Length	95%	2.80×10^{-5}	4.44×10^{-5}	1.72×10^{-3}	3.26×10^{-5}
		99%	5.29×10^{-5}	1.53×10^{-4}	3.38×10^{-2}	5.64×10^{-5}

Uptime Deviates $U \sim$ Long tailed h, $\lambda=0.01$, $h=0.2$;

Downtime Deviates $D \sim$ Gamma, $\mu=1$, $k=2$;

Sample size A=10, B=15, C=20

Table 3.6.5 : Bias, Sample Variance, Mean Square Error for Various Availability Estimation Methods in the One Component Case :

		Standard Method	Jackknife Method	Bootstrap Method
	Bias	-0.0021	-0.0018	-0.0039
A	Sample Variance	5.33×10^{-5}	5.55×10^{-5}	6.20×10^{-5}
	Mean Square Error	5.79×10^{-5}	5.88×10^{-5}	7.74×10^{-5}
	Bias	-0.0013	-0.0011	-0.0025
B	Sample Variance	3.16×10^{-5}	3.26×10^{-5}	3.44×10^{-5}
	Mean Square Error	3.33×10^{-5}	3.37×10^{-5}	4.08×10^{-5}
	Bias	-0.0011	-0.0008	-0.0020
C	Sample Variance	2.23×10^{-5}	2.30×10^{-5}	2.42×10^{-5}
	Mean Square Error	2.38×10^{-5}	2.30×10^{-5}	2.81×10^{-5}

Uptime Deviates $U \sim$ Long tailed h, $\lambda=0.01$, $h=0.2$;

Downtime Deviates $D \sim$ Exponential, $\mu=1$;

Sample Size : A=10, B=15, C=20

Table 3.6.6 : Sample Confidence Intervals for Various Availability Estimation Methods in the One Component Case :

			Standard Method	Jackknife method using t quantiles	Jackknife method using improved t quantiles	Bootstrap method using normal quantiles
A	Coverage	95%	88.5	93.9	94.6	93.5
		99%	96.7	98.1	98.2	97.8
	Average Length	95%	2.40×10^{-2}	3.50×10^{-2}	6.02×10^{-2}	2.80×10^{-2}
		99%	3.48×10^{-2}	6.63×10^{-2}	1.64×10^{-1}	3.68×10^{-2}
	Variance of Length	95%	1.88×10^{-4}	5.51×10^{-4}	8.68×10^{-3}	2.69×10^{-4}
		99%	3.83×10^{-4}	2.89×10^{-3}	5.64×10^{-2}	4.64×10^{-4}
B	Coverage	95%	88.7	93.8	95.2	93.9
		99%	95.9	98.0	98.3	97.6
	Average Length	95%	1.75×10^{-2}	2.37×10^{-2}	3.70×10^{-2}	2.05×10^{-2}
		99%	2.46×10^{-2}	3.88×10^{-2}	9.79×10^{-2}	2.70×10^{-2}
	Variance of Length	95%	6.70×10^{-5}	1.53×10^{-4}	3.30×10^{-3}	8.89×10^{-5}
		99%	1.30×10^{-4}	5.93×10^{-4}	3.10×10^{-2}	1.53×10^{-4}
C	Coverage	95%	86.5	92.8	94.4	93.6
		99%	95.6	98.1	98.5	97.8
	Average Length	95%	1.44×10^{-2}	1.91×10^{-2}	2.83×10^{-2}	1.80×10^{-2}
		99%	1.99×10^{-2}	2.92×10^{-2}	7.51×10^{-2}	2.26×10^{-2}
	Variance of Length	95%	3.36×10^{-5}	7.00×10^{-5}	1.53×10^{-3}	4.69×10^{-5}
		99%	6.36×10^{-5}	2.09×10^{-4}	2.56×10^{-2}	8.10×10^{-5}

Uptime Deviates $U \sim$ Long tailed h, $\lambda=0.01$, $h=0.2$;

Downtime Deviates $D \sim$ Exponential, $\mu=1$;

Sample size : A=10, B=15, C=20

Figure 3.6.1 : 95% Confidence Intervals in the One Component Case

(U ~ Exponential, λ=0.01; D~Gamma, μ=1, k=3)

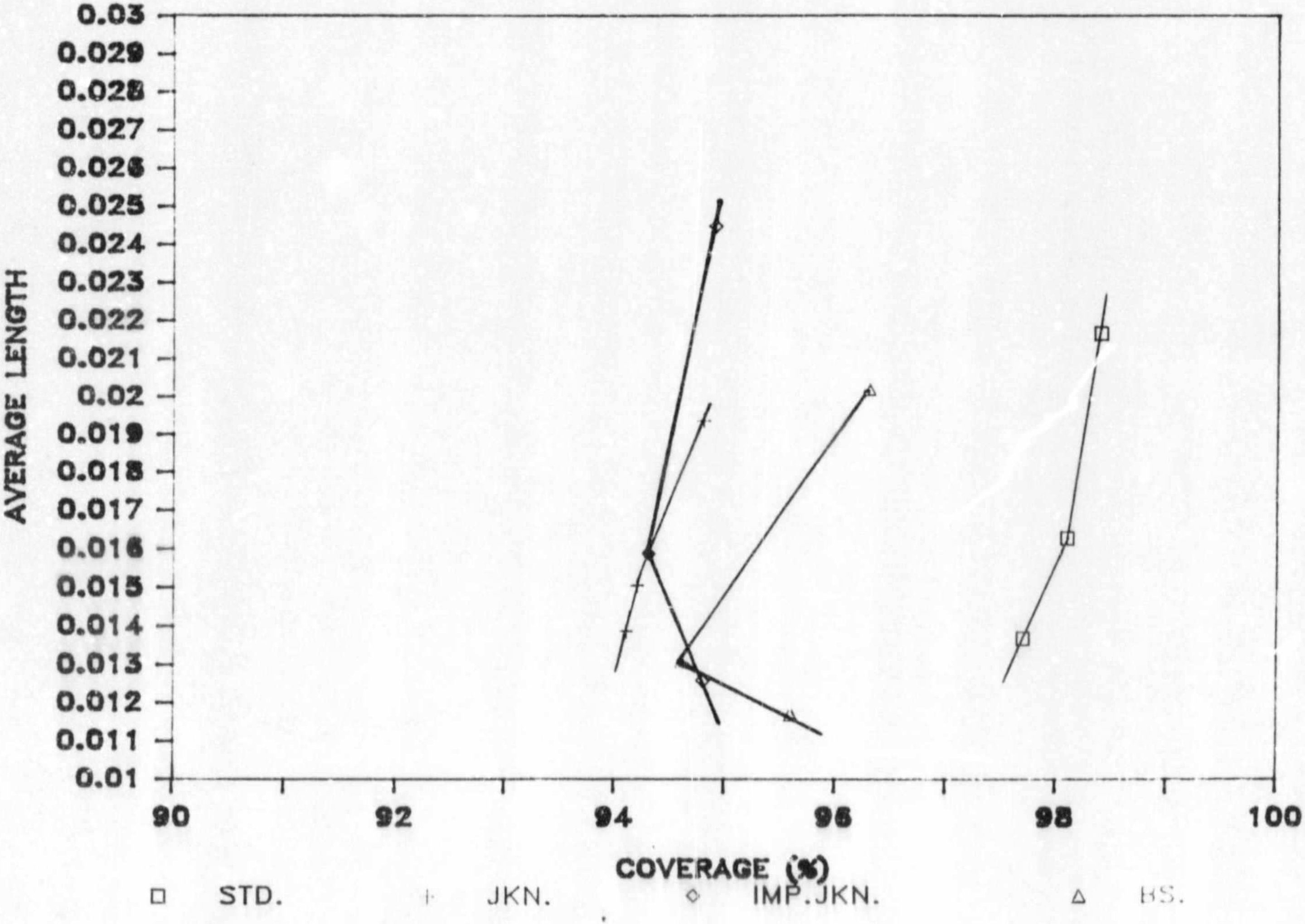


Figure 3.6.2 : 99% Confidence Intervals in the One Component Case

(U ~ Exponential, λ=0.01; D ~ Gamma, μ=1, k=3)

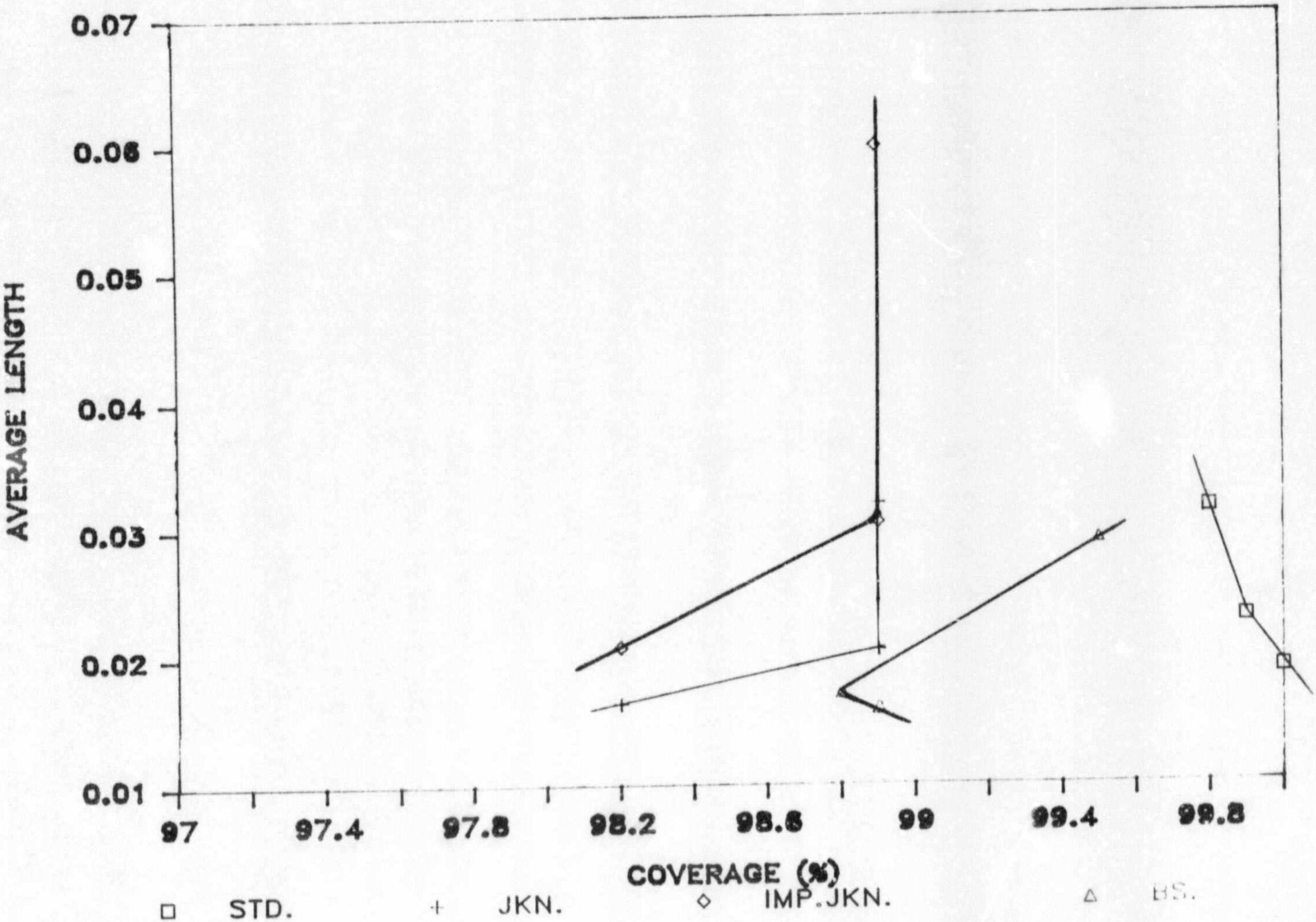


Figure 3.6.3 : 95% Confidence Intervals in the One Component Case

(U ~ Long tailed h, λ=0,01, h=0.2;D~Gamma, μ=1, k=2)

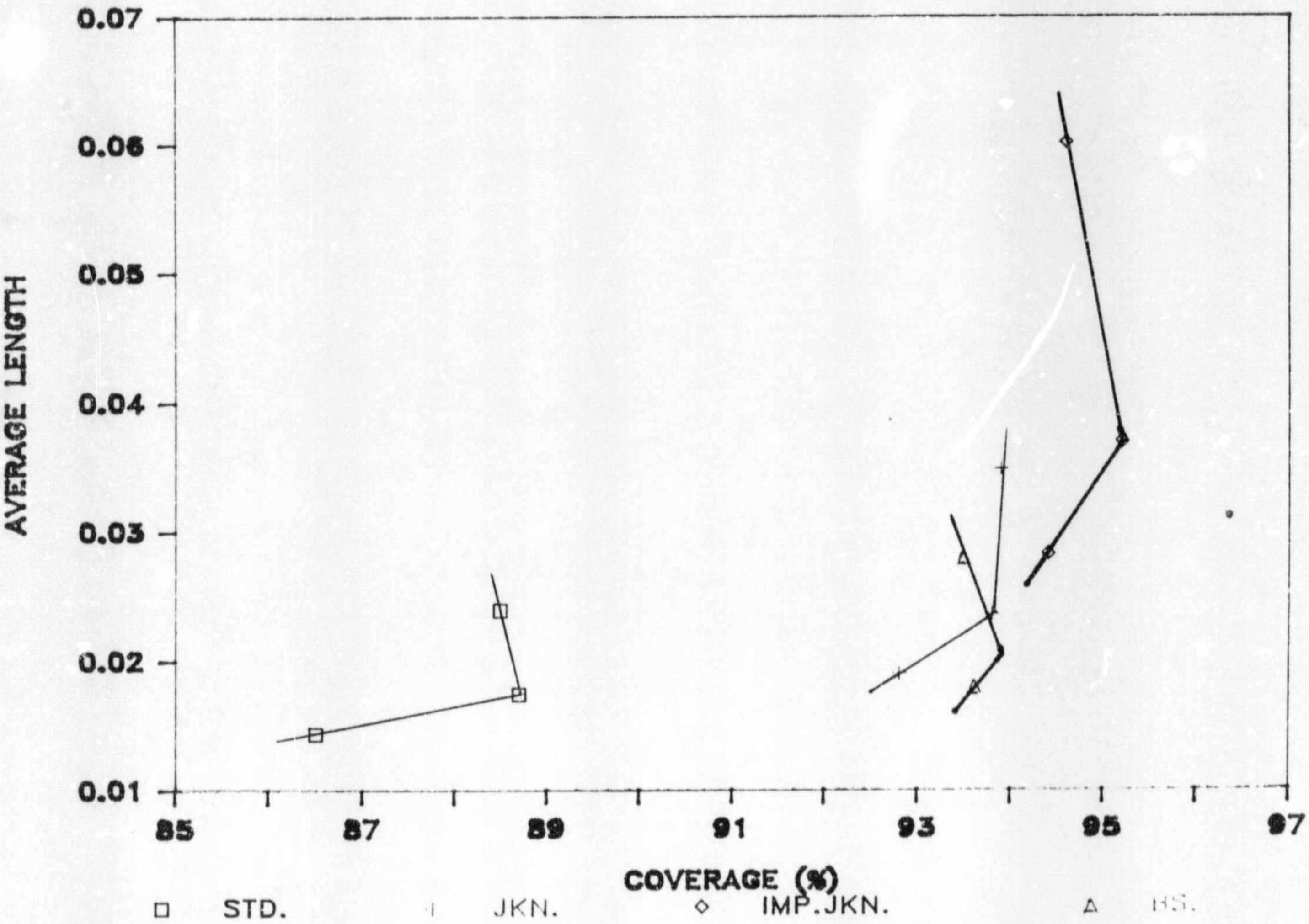


Figure 3.6.4 : 99% Confidence Intervals in the One Component Case

(U ~ Long tailed h, λ=0.01, h=0.2; D~Gamma, μ=1, k=2)

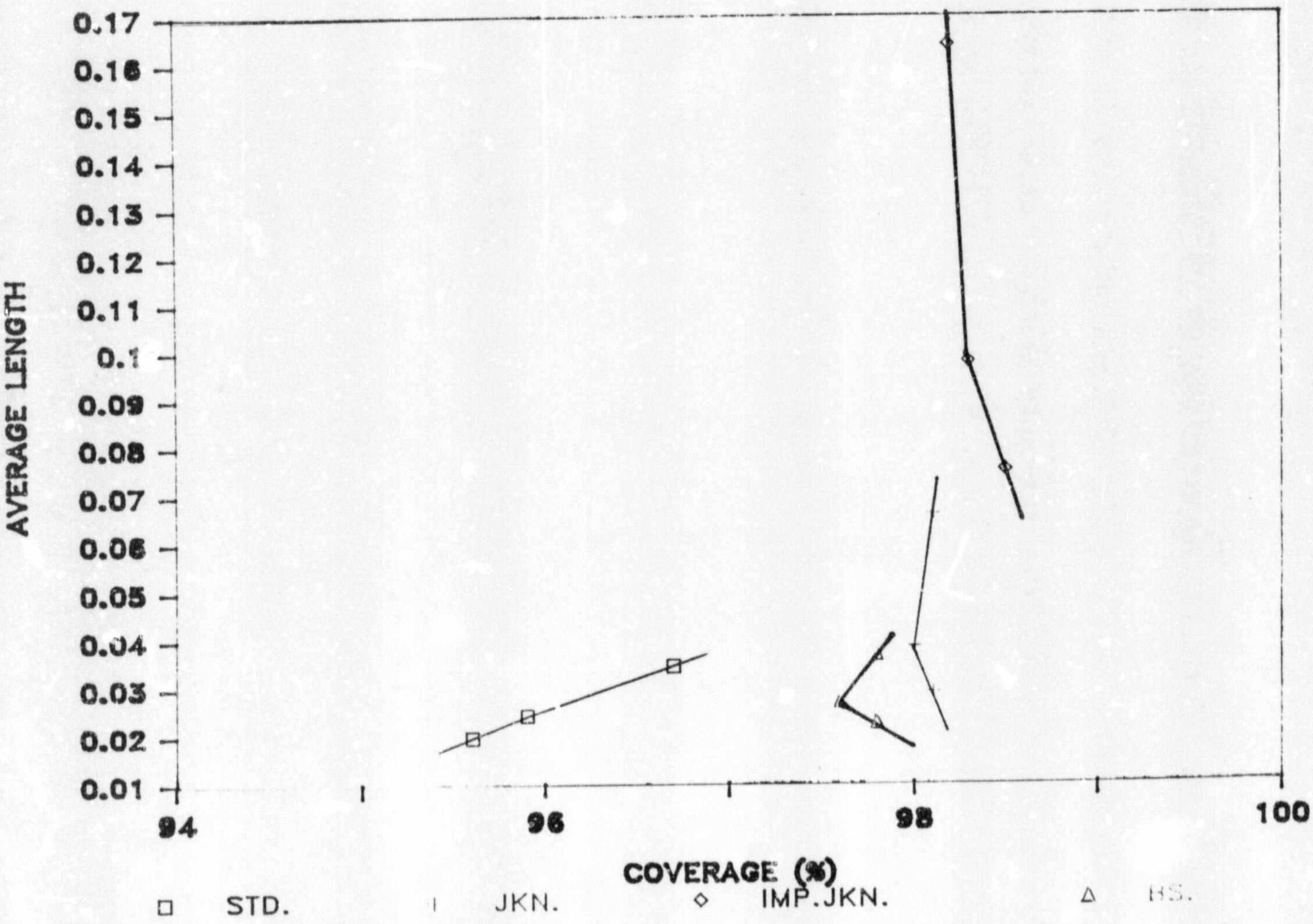


Figure 3.6.5 : 95% Confidence Intervals in the Ome Component Case

(U ~ Long tailed h, λ=0.01, h=0.2; D~Exponential, μ=1)

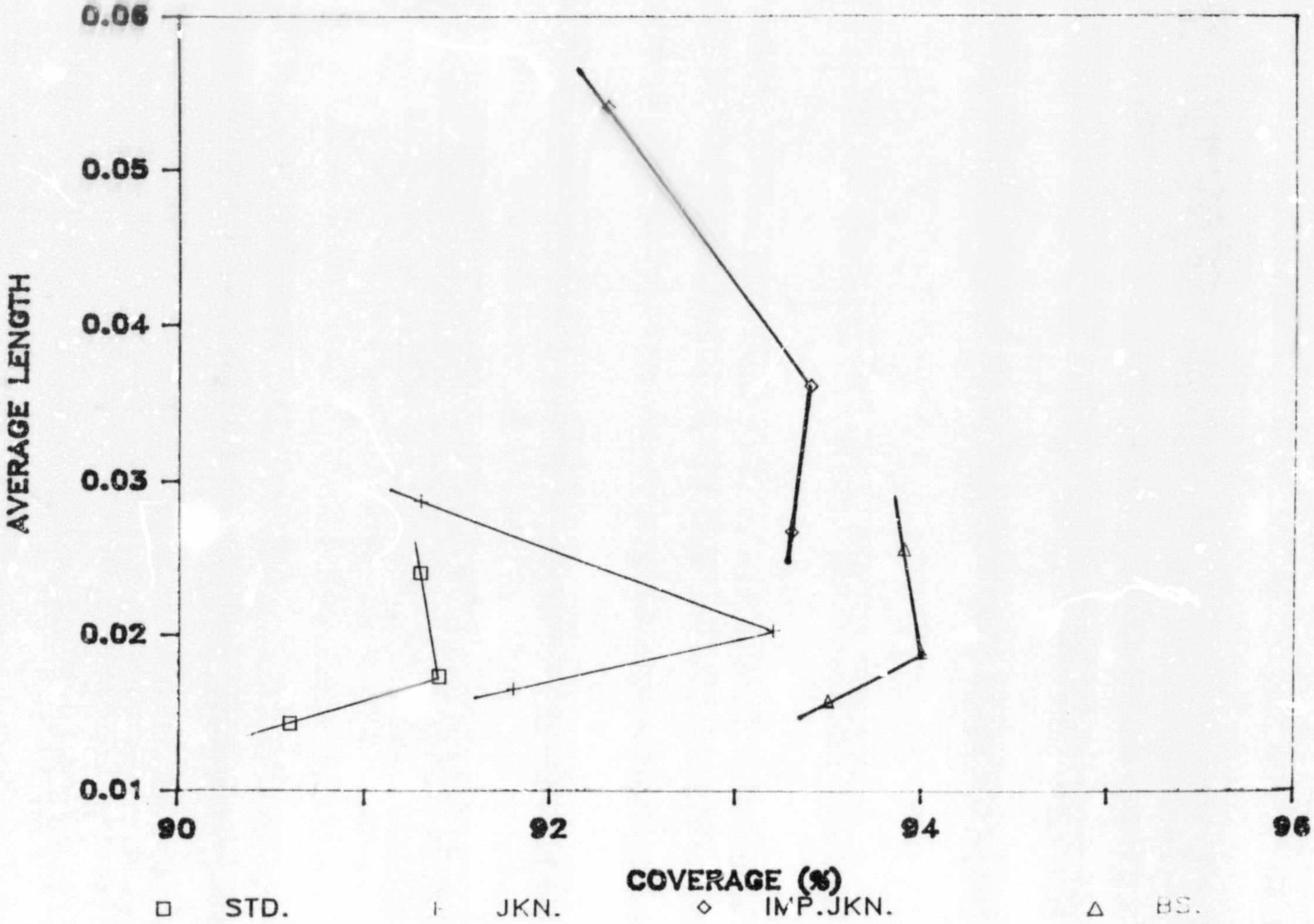
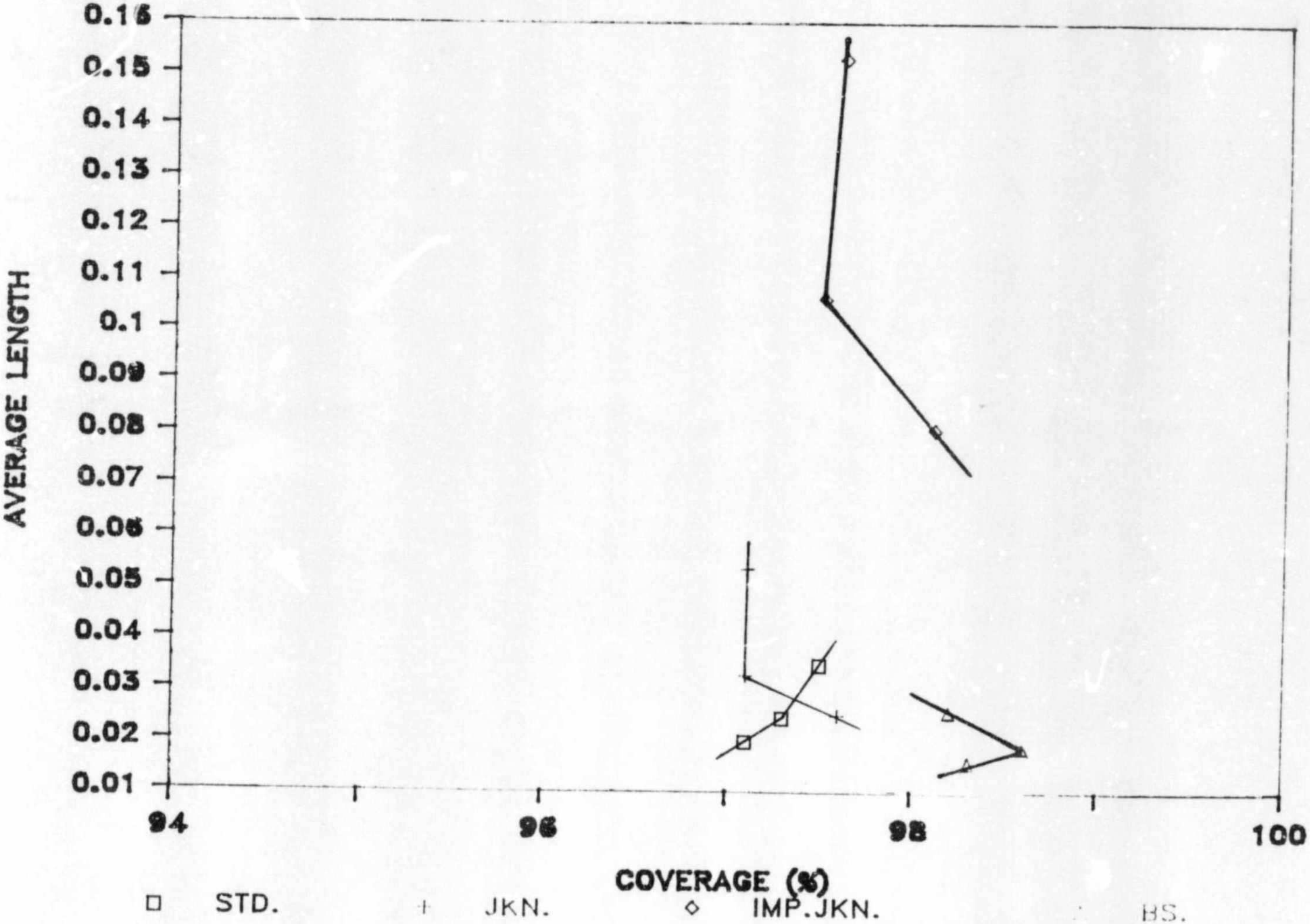


Figure 3.6.6 : 99% Confidence Intervals in the One Component Case

(U~ Long tailed h, $\lambda=0.01$, $h=0.2$; D~Exponential, $\mu=1$)



of 3 were considered. The results of the simulation study are summarised in Tables 3.6.7 and 3.6.8 and in Tables 3.6.9 and 3.6.10, respectively, and the conclusions are as follows:

a. the results are in general agreement with the first simulation study undertaken

b. the bootstrap method produces shorter average lengths and smaller variances in length than either of the jackknife methods. However, there are some occasions when the jackknife methods produce slightly better coverage than the bootstrap method.

(iii) a comparison between the standard method and the jackknife and bootstrap methods in the one component case using the Weibull distribution. The results of the simulation study are summarised in Tables 3.6.11 and 3.6.12 and the conclusions are as follows :

a. the standard estimator has a smaller bias than either the jackknife or bootstrap estimator.

b. only the jackknife method using t quantiles or Hinkley's 'improved' t quantiles produced trustworthy confidence limits .

c. the standard method produced very poor confidence limits and should not be used in the Weibull case .

d. the bootstrap method produced modest confidence limits although the average length and variance of length were again better than either of the jackknife methods .

(iv) a variation on the jackknife estimator, known as the minimum jackknife risk estimator (see Frangos and Stone, 1984), was considered to determine whether an improvement on the availability estimate in the one component case could be attained. The new estimator is defined as follows:

$$\text{Let } \hat{\mu}(\alpha)=\alpha\hat{\mu}_o+(1-\alpha)\hat{\mu}_1 \text{ , where } 0<\alpha<1 \tag{3.6.1}$$

Table 3.6.7 : Bias, Sample Variance, Mean Square Error Using Mixed Distributions in the One Component Case :

		Standard Method	Jackknife Method	Bootstrap Method
A	Bias	-0.0006	-0.0005	-0.0012
	Sample Variance	1.57×10^{-5}	1.62×10^{-5}	1.68×10^{-5}
	Mean Square Error	1.61×10^{-5}	1.65×10^{-5}	1.83×10^{-5}
B	Bias	-0.0004	-0.0004	-0.0009
	Sample Variance	1.21×10^{-5}	1.23×10^{-5}	1.34×10^{-5}
	Mean Square Error	1.23×10^{-5}	1.24×10^{-5}	1.42×10^{-5}
C	Bias	-0.0008	-0.0007	-0.0015
	Sample Variance	1.85×10^{-5}	1.95×10^{-5}	1.91×10^{-5}
	Mean Square Error	1.92×10^{-5}	2.00×10^{-5}	2.14×10^{-5}

A : Exponential, $\lambda=0.01$ ($n_1=14$), Long tailed h, $\lambda=0.01$, $h=0.2$ ($n_2=6$):

Exponential, $\mu=1$:

B : Exponential, $\lambda=0.01$;

Exponential, $\mu=1$ ($n_1=14$), Gamma, $\mu=1$, $k=2$ ($n_2=6$);

C : Long tailed h, $\lambda=0.01$, $h=0.2$ ($n_1=14$), Gamma, $\mu=0.01$, $k=2$ ($n_2=6$);

Exponential, $\mu=1$;

Table 3.6.8 : Sample Confidence Intervals Using Mixed Distributions in the One Component Case :

			Standard Method	Jackknife method using t quantiles	Jackknife method using improved t quantiles	Bootstrap method using normal quantiles
A	Coverage	95%	93.0	94.4	94.9	94.4
		99%	98.1	98.7	99.0	97.4
	Average Length	95%	139×10^{2}	1.59×10^{-2}	1.96×10^{-2}	1.44×10^{-2}
		99%	1.92×10^{-2}	2.36×10^{-2}	4.11×10^{-2}	1.89×10^{-2}
	Variance of Length	95%	2.24×10^{-5}	4.12×10^{-5}	4.69×10^{-4}	2.69×10^{-5}
		99%	4.24×10^{-5}	1.07×10^{-5}	9.39×10^{-3}	4.65×10^{-5}
B	Coverage	95%	96.0	94.5	95.0	94.1
		99%	99.5	99.0	99.2	97.5
	Average Length	95%	1.36×10^{-2}	1.40×10^{-2}	1.52×10^{-2}	1.27×10^{-2}
		99%	1.88×10^{-2}	2.03×10^{-2}	2.46×10^{-2}	1.67×10^{-2}
	Variance of Length	95%	1.68×10^{-5}	2.86×10^{-5}	4.78×10^{-5}	1.89×10^{-5}
		99%	3.17×10^{-5}	6.48×10^{-5}	2.61×10^{-5}	3.27×10^{-5}
C	Coverage	95%	90.5	93.3	94.2	94.2
		99%	96.5	98.3	98.8	97.5
	Average Length	95%	1.42×10^{-2}	1.69×10^{-2}	2.31×10^{-2}	1.50×10^{-2}
		99%	2.00×10^{-2}	2.54×10^{-2}	5.90×10^{-2}	2.00×10^{-2}
	Variance of Length	95%	2.67×10^{-5}	4.53×10^{-5}	9.25×10^{-4}	2.98×10^{-5}
		99%	5.06×10^{-5}	1.31×10^{-4}	1.88×10^{-2}	5.15×10^{-5}

A : Exponential, $\lambda=0.01$ ($n_1=14$), Long tailed h, $\lambda=0.01$, $h=0.2$ ($n_2=6$);

Exponential, $\mu=1$;

B : Exponential, $\lambda=0.01$;

Exponential, $\mu=1$ ($n_1=14$), Gamma, $\mu=1$, $k=2$ ($n_2=6$);

C : Long tailed h, $\lambda=0.01$, $h=0.2$ ($n_1=14$), Gamma $\lambda=0.01$, $k=2$ ($n_2=6$);

Exponential, $\mu=1$;

Table 3.6.9 : Bias, Sample Variance, Mean Square Error Using Contaminated Distributions in the One Component Case :

		Standard Method	Jackknife Method	Bootstrap Method
A	Bias	-0.0004	-0.0003	-0.0010
	Sample Variance	1.62×10^{-5}	1.61×10^{-5}	1.78×10^{-5}
	Mean Square Error	1.64×10^{-5}	1.62×10^{-5}	1.88×10^{-5}
B	Bias	-0.0012	-0.0009	-0.0020
	Sample Variance	1.99×10^{-5}	2.06×10^{-5}	2.10×10^{-5}
	Mean Square Error	2.12×10^{-5}	2.15×10^{-5}	2.51×10^{-5}
C	Bias	-0.0011	-0.0008	-0.0022
	Sample Variance	2.94×10^{-5}	3.05×10^{-5}	3.15×10^{-5}
	Mean Square Error	3.07×10^{-5}	3.11×10^{-5}	3.63×10^{-5}

A : Exponential, $\lambda=0.01$

Gamme, $\mu=1$, $k=3$, $p=0.9$, $mf=3$;

Sample size = 20

B : Long tailed h, $\lambda=0.01$, $h=0.2$, $p=0.9$, $mf=3$;

Exponential, $\mu=1$;

Sample size = 20

C : Long tailed h, $\lambda=0.01$, $h=0.2$

Gamma, $\mu=1$, $k=2$, $p=0.9$, $mf=3$;

Sample size = 20

Table 3.6.10 : Sample Confidence Intervals Using Contaminated Distributions in the One Component Case :

			Standard Method	Jackknife method using t quantiles	Jackknife method using improved t quantiles	Bootstrap method using normal quantiles
A	Coverage	95%	96.9	94.3	94.5	94.1
		99%	99.7	98.3	98.7	97.0
	Average Length	95%	1.62×10^{-2}	1.53×10^{-2}	1.70×10^{-2}	1.41×10^{-2}
		99%	2.23×10^{-2}	2.21×10^{-2}	2.76×10^{-2}	1.85×10^{-2}
	Variance of Length	95%	2.28×10^{-5}	4.22×10^{-5}	7.09×10^{-5}	2.87×10^{-5}
		99%	4.30×10^{-5}	9.48×10^{-5}	3.65×10^{-5}	4.95×10^{-5}
B	Coverage	95%	85.0	91.9	93.7	95.1
		99%	93.6	97.3	[illegible]	98.3
	Average Length	95%	1.25×10^{-2}	1.68×10^{-2}	2.56×10^{-2}	1.54×10^{-2}
		99%	1.72×10^{-2}	2.58×10^{-2}	7.44×10^{-2}	2.03×10^{-2}
	Variance of Length	95%	2.95×10^{-2}	5.71×10^{-5}	1.28×10^{-3}	3.87×10^{-5}
		99%	5.59×10^{-5}	1.69×10^{-4}	2.77×10^{-2}	6.67×10^{-5}
C	Coverage	95%	87.6	92.8	94.3	94.5
		99%	96.2	97.9	98.4	98.3
	Average Length	95%	1.70×10^{-2}	2.16×10^{-2}	3.36×10^{-2}	1.97×10^{-2}
		99%	2.35×10^{-2}	3.29×10^{-2}	8.74×10^{-2}	2.59×10^{-2}
	Variance of Length	95%	4.45×10^{-5}	1.14×10^{-4}	2.35×10^{-2}	7.09×10^{-5}
		99%	8.38×10^{-5}	3.55×10^{-4}	3.02×10^{-2}	1.22×10^{-4}

A : Exponential, $\lambda=0.01$;
Gamma, $\mu=1$, $k=3$, $p=0.9$, $mf=3$;
Sample size = 20

B : Long tailed h, $\lambda=0.1$, $h=0.2$, $p=0.9$, $mf=3$;
Exponential, $\mu=1$;
Sample size = 20

C : Long tailed h, $\lambda=0.01$, $h=02$;
Gamma, $\mu=1$, $k=2$, $p=0.9$, $mf=3$;
Sample size = 20

Table 3.6.11 : Bias, Sample Variance, Mean Square Error Using the Weibull Distributions in the One Component Case :

		Standard Method	Jackknife Method	Bootstrap Method
	Bias	-0.0004	-0.0030	-0.0007
A	Sample Variance	1.07×10^{-4}	1.81×10^{-4}	1.09×10^{-4}
	Mean Square Error	1.07×10^{-4}	1.90×10^{-4}	1.09×10^{-4}
	Bias	-0.0003	-0.0009	-0.0005
B	Sample Variance	1.19×10^{-5}	1.83×10^{-5}	1.25×10^{-5}
	Mean Square Error	1.20×10^{-5}	1.91×10^{-5}	1.28×10^{-5}
	Bias	0.0259	0.0315	0.0148
C	Sample Variance	5.13×10^{-3}	5.16×10^{-3}	5.09×10^{-3}
	Mean Square Error	5.80×10^{-3}	6.15×10^{-3}	5.31×10^{-3}

A : Gamma, $\lambda=0.01$, $k=2$; Weibull, $\mu=0.5$, $A=1.0$, $B=0.5$;
Sample size = 20

B : Exponential, $\lambda=0.01$, Weibull, $\mu=2.0$, $A=2.0$, $B=0.5$;
Sample size = 20

C : Weibull, $\lambda=0.3$, $A=0.5$, $B=0.6$, Exponential, $\mu=1.0$;
Sample size = 20

Table 3.6.12 : Sample Confidence Intervals Using the Weibull Distributions in the One Component Case :

			Standard Method	Jackknife method using t quantiles	Jackknife method using improved t quantiles	Bootstrap method using normal quantiles
A	Coverage	95%	76.3	91.8	93.3	82.5
		99%	87.1	96.4	97.6	88.1
	Average Length	95%	2.59×10^{-2}	6.48×10^{-2}	1.22×10^{-1}	3.42×10^{-2}
		99%	3.56×10^{-2}	1.01×10^{-2}	2.48×10^{-1}	4.49×10^{-2}
	Variance of Length	95%	1.63×10^{-4}	6.15×10^{-3}	3.40×10^{-2}	4.88×10^{-4}
		99%	3.03×10^{-4}	1.37×10^{-2}	9.2[illegible]$\times10^{-2}$	8.43×10^{-4}
B	Coverage	95%	74.5	91.6	93.1	85.9
		99%	85.7	96.7	97.8	91.6
	Average Length	95%	6.98×10^{-3}	2.06×10^{-2}	4.84×10^{-2}	1.05×10^{-2}
		99%	9.65×10^{-3}	3.53×10^{-2}	1.36×10^{-2}	1.38×10^{-2}
	Variance of Length	95%	1.61×10^{-5}	1.42×10^{-3}	1.46×10^{-2}	5.10×10^{-5}
		99%	3.06×10^{-5}	4.32×10^{-3}	6.70×10^{-2}	8.81×10^{-5}
C	Coverage	95%	82.2	92.3	94.2	88.3
		99%	93.8	98.0	98.6	94.5
	Average Length	95%	2.11×10^{-1}	2.83×10^{-1}	3.25×10^{-1}	2.64×10^{-1}
		99%	2.78×10^{-1}	3.83×10^{-1}	4.67×10^{-1}	3.47×10^{-1}
	Variance of Length	95%	1.89×10^{-3}	6.50×10^{-3}	1.58×10^{-2}	4.89×10^{-3}
		99%	2.99×10^{-3}	1.05×10^{-2}	3.00×10^{-2}	8.45×10^{-3}

A : Gamma, $\lambda=0.01$, $k=2$, Weibull, $\mu=0.5$, $A=1.0$, $B=0.5$;
Sample size = 20

B : Exponential, $\lambda=1$; Weibull, $\mu=20$, $A=2.0$, $B=0.5$;
Sample size = 20

C : Weibull, $\lambda=0.3$, $A=0.5$, $B=0.6$; Exponential, $\mu=1.0$;
Sample size = 20

and $\hat{\mu}_o$ and $\hat{\mu}_1$ are the availability estimates using the standard and jackknife method respectively.

Then

$$E\{\hat{u}(\alpha)\} = \mu$$

and

$$\text{var}\ \{\hat{\mu}(\alpha)\} = \alpha^2 \text{var}(\hat{\mu}_o) + 2\alpha(1-\alpha)\text{cov}(\hat{\mu}_o,\hat{\mu}_1) + (1-\alpha)^2\ \text{var}(\hat{\mu}_1) \qquad (3.6.2)$$

Let α_m be the value of α that minimises (3.6.2).

i.e. Differentiating (3.6.2) w.r.t. α and equating to zero, leads to:

$$\alpha_m = \frac{\text{var}(\hat{\mu}_1) - \text{cov}(\hat{\mu}_o,\hat{\mu}_1)}{\text{var}(\hat{\mu}_o) - 2\text{cov}(\hat{\mu}_o,\hat{\mu}_1) + \text{var}(\hat{\mu}_1)} \qquad (3.6.3)$$

Then, the minimum jackknife risk estimator, as defined by Frangos and Stone, $\hat{\mu}_{mjr} = \alpha_m \hat{\mu}_o + (1-\alpha_m)\hat{\mu}_1$ (3.6.4)

Now consider the case where the standard variance estimate is unknown and the jackknife variance estimator is then used for this value i.e. $\text{var}(\hat{\mu}_o) = \text{var}(\hat{\mu}_1)$

Equation (3.6.3) then simply reduces to the form $\alpha_m = 0.5$

Hence, $\hat{\mu}_{mjr} = 0.5(\hat{\mu}_o + \hat{\mu}_1)$ (3.6.5)

Next, let $\hat{\sigma}_{mjr}$ denotes the jackknife estimator of the variance of $\hat{\mu}_{mjr}$.
Construct pseudovalues $\hat{p}_i = n\hat{\mu}_{mjr} - (n-1)\hat{\mu}^{-i}_{mjr}$, for $i=1\ldots,n$.

Hence

$$\hat{\sigma}_{mjr} = \frac{1}{(n-1)} \sum_{i=1}^{n} (p_i - \bar{\hat{p}})^2 \text{ , where } \bar{\hat{p}}_i = \frac{\sum_{i=1}^{n} \hat{p}_i}{n} \qquad (3.6.6)$$

Therefore,

$$\frac{\hat{\mu}_{mjr}-\mu}{\sqrt{\hat{\sigma}_{mjr}}} \sim \text{Student-t or normal dsn depending on the size of n, with (n-1) degrees of freedom.}$$

The results of the simulation study using the estimator are summarised in Tables 3.6.13 and 3.6.14. If these results are compared with those from Tables 3.6.1-3.6.6, for the corresponding sample sizes, the following conclusions can be made:

a. the minimum jackknife risk estimate has a smaller bias than either the standard, jackknife or bootstrap estimate, with a corresponding marginally smaller mean square error.

b. the minimum jackknife risk method is trustworthy in establishing confidence limits. It compares favourably with either of the jackknife methods although the bootstrap is still the best of all methods.

In the second case, the variance estimators for the standard and jackknife methods are calculated using bootstrap repetitions as follows:

Given a sample $\{X_1,\ldots,X_n\}$, the availability estimates are calculated using the standard method and jackknife method, respectively i.e. $\hat{\mu}_o$ and $\hat{\mu}_1$

Next, generate B independent samples (B=120) from the original sample. For each sample, calculate the bootstrap pseudovalues, $\hat{\mu}_o^{*i}$, $\hat{\mu}_1^{*i}$, $i=1,\ldots,B$ respectively.

Now, calculate the bootstrap estimates :

$$\hat{\mu}_o^{*} = \frac{1}{B}\sum_{i=1}^{B}\frac{\hat{\mu}_o^{*i}}{B}$$

$$\text{and } \hat{\mu}_1^{*} = \frac{1}{B}\sum_{i=1}^{B}\frac{\hat{\mu}_1^{*i}}{B} \qquad (3.6.7)$$

and corresponding variance estimates :

Table 3.6.13 : Bias, Sample Variance, Mean Square Error Using Minimum Jackknife Risk Estimator (Standard Variance Estimate Unknown) :

		Minimum Jackknife Risk Method
A	Bias	-0.0003
	Sample Variance	1.82×10^{-5}
	Mean Square Error	1.83×10^{-5}
B	Bias	-0.0005
	Sample Variance	2.64×10^{-5}
	Mean Square Error	2.65×10^{-5}
C	Bias	-0.0005
	Sample Variance	2.25×10^{-5}
	Mean Square Error	2.27×10^{-5}

A : Exponential, $\lambda=0.01$; Gamma, $\mu=1$, $k=3$; Sample Size = 10

B : Long tailed h, $\lambda=0.02$, $h=0.2$; Gamma, $\mu=1$, $k=2$; Sample Size = 15

C : Long tailed h, $\lambda=0.01$, $h=0.2$; Exponential, $\mu=1$; Sample Size = 20

Table 3.6.14 : Sample Confidence Intervals Using Minimum Jackknife Risk Estimator (Standard Variance Estimate Unknown) :

			Minimum Jackknife Risk Method
A	Coverage	95%	92.8
		99%	97.4
	Average Length	95%	1.84×10^{-2}
		99%	2.65×10^{-2}
	Variance of Length	95%	1.27×10^{-4}
		99%	2.63×10^{-4}
B	Coverage	95%	93.7
		99%	98.5
	Average Length	95%	2.09×10^{-2}
		99%	2.91×10^{-2}
	Variance of Length	95%	1.91×10^{-4}
		99%	3.67×10^{-4}
C	Coverage	95%	93.3
		99%	97.2
	Average Length	95%	1.88×10^{-2}
		99%	2.57×10^{-2}
	Variance of Length	95%	9.96×10^{-5}
		99%	1.86×10^{-4}

A : Exponential, λ=0.01 ; Gamma, μ=1, k=3 ; Sample Size = 10

B : Long tailed h, λ=0.01, h=0.2 ; Gamma, μ=1, k=2 ; Sample Size = 15

C : Long tailed h, λ=0.01, h=0.2 ; Exponential, μ=1 ; Sample Size = 20

$$\mathrm{var}(\hat{\mu}_o) = \sum_{i=1}^{B} \frac{(\hat{\mu}_o^{*i} - \mu_o^*)^2}{(B-1)} \tag{3.6.8}$$

and $\mathrm{var}(\hat{\mu}_1) = \sum_{i=1}^{B} \frac{(\hat{\mu}_1^{*i} - \mu_1^*)^2}{(B-1)}$

The covariance estimator, $\mathrm{cov}(\hat{\mu}_o, \hat{\mu}_1) = \sum_{i=1}^{B} \frac{(\hat{\mu}_o^{*i} - \mu_o^*)(\hat{\mu}_1^{*i} - \mu_1^*)}{(B-1)}$ (3.6.9)

Hence, calculate $\alpha_{mjr} = \frac{\mathrm{var}(\hat{\mu}_1) - \mathrm{cov}(\hat{\mu}_o \hat{\mu}_1)}{\mathrm{var}(\hat{\mu}_o) - 2\mathrm{cov}(\hat{\mu}_o, \hat{\mu}_1) + \mathrm{var}(\hat{\mu}_1)}$

Therefore,

$$\hat{\mu}_{mjr} = \alpha_{mjr} (\hat{\mu}_o) + (1 - \alpha_{mjr}) \hat{\mu}_1 \tag{3.6.10}$$

The minimum jackknife risk variance estimator is calculated as follows:

Delete the j^{th} observation from the original sample

i.e. $\{ X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n \}$

Then, repeat the previous steps and create the jackknife pseudovalues for the minimum jackknife risk estimator, μ_{mjr}^{-j} , $j = 1, \ldots, n$

Then,

$$\bar{\mu}_{mjr}^{-j} = \frac{1}{n} \sum_{i=1}^{n} \mu_{mjr}^{-j} \tag{3.6.11}$$

and variance estimator, $\mathrm{var}(\hat{\mu}_{mjr}) = \frac{1}{n(n-1)} \sum_{i=1}^{n} (\mu_{mjr}^{-j} - \bar{\mu}_{mjr}^{-j})^2$ (3.6.12)

The confidence intervals are determined by

$$\frac{\hat{\mu}_{mjr} - \mu}{\sqrt{\mathrm{var}(\hat{\mu}_{mjr})}} \sim t_{n-1} \text{ or } N(0,1)$$

depending on the size of n.

Using the same uptime and downtime distributions as before, the resu[illegible] shown in Table 3.6.15. It will be noticed that the confidence limi[illegible]s are not given. The calculation of the minimum jackknife va[illegible]e estimator was very time consuming in terms of C.P.U.S. Using a reduced simulation run size of 100, the initial confidence limit results showed no improvement from the first case and, consequently, the computer program was modified not to calculate the minimum jackknife risk variance estimate. If the results are compared to Tables 3.6.1, 3.6.3, 3.6.5 and 3.6.13 respectively, for the corresponding sample sizes, the main conclusion to be drawn, is that this method produced a marginal improvement in mean square error than any of the other methods.

(v) The final simulation study in the one component case concerned advanced bootstrap versions for estimating standard errors and confidence limits, i.e. the percentile method and the bias corrected percentile method. A description of these methods is given in Section 2.9. These two non-parametric methods were evaluated using a small sample size N=10, and the number of bootstrap repetitions was set at 400. The results are shown in Table 3.6.16. Both the percentile and bias corrected percentile method give very similar results. The methods only give modest confidence interval coverage when compared to the bootstrap method using normal quantiles. Although they did produce shorter average lengths and smaller variances in length, the results were generally disappointing and, overall, were not an improvement on the standard bootstrap method using normal quantiles.

(vi) The last simulation study concerned more than one component in the system. In particular, the jackknife and bootstrap methods were applied to a two components in series system.

Suppose the two components are arranged in series so that, in order for the system to be available, both must be simulataneously available. If A_i is the availability of the i^{th} component, then, assuming independence, the system availability is seen to be:

Table 3.6.15 : Bias, Sample Variance, Mean Square Error Using Minimum Jackknife Risk Estimator

		Minimum Jackknife Risk Method
A	Bias	-0.0003
	Sample Variance	1.72×10^{-5}
	Mean Square Error	1.73×10^{-5}
B	Bias	-0.0009
	Sample Variance	2.52×10^{-5}
	Mean Square Error	2.61×10^{-5}
C	Bias	-0.0009
	Sample Variance	2.14×10^{-5}
	Mean Square Error	2.20×10^{-5}

A : Exponential, λ=0.01 ; Gamma, μ=1, k=3 ; Sample Size = 10

B : Long tailed h, λ=0.01, h=0.2 ; Gamma, μ=1, k=2 ; Sample Size = 15

C : Long tailed h, λ=0.01, h=0.2 ; Exponential, μ=1, Sample Size = 20

Table 3.6.16 : Sample Confidence Intervals Using Variations in the Bootstrap Method :

			Bootstrap Method using normal quantiles	Percentile Method	Bias Corrected Percentile Method
A	Coverage	95%	96.3	89.9	90.6
		99%	99.5	97.0	96.6
	Average Length	95%	2.02×10^{-2}	1.67×10^{-2}	1.62×10^{-2}
		99%	2.91×10^{-2}	2.40×10^{-2}	2.41×10^{-2}
	Variance of Length	95%	1.11×10^{-4}	7.54×10^{-5}	6.71×10^{-5}
		99%	2.23×10^{-4}	1.87×10^{-4}	1.83×10^{-5}
B	Coverage	95%	93.9	86.4	86.5
		99%	98.2	93.6	93.4
	Average Length	95%	2.56×10^{-2}	2.41×10^{-2}	2.33×10^{-2}
		99%	3.36×10^{-2}	3.11×10^{-2}	3.01×10^{-2}
	Variance of Length	95%	2.07×10^{-4}	1.74×10^{-4}	1.57×10^{-4}
		99%	3.58×10^{-4}	3.43×10^{-4}	3.36×10^{-4}
C	Coverage	95%	93.5	89.3	89.4
		99%	97.8	95.7	95.8
	Average Length	95%	2.80×10^{-2}	2.65×10^{-2}	2.61×10^{-2}
		99%	3.68×10^{-2}	3.82×10^{-2}	3.88×10^{-2}
	Variance of Length	95%	2.69×10^{-2}	2.32×10^{-4}	2.25×10^{-4}
		99%	5.64×10^{-2}	5.42×10^{-4}	5.54×10^{-4}

A : Exponential, λ=0.01, Gamma, μ=1, k=3 ; Sample Size = 10

B : Long tailed h, λ=0.01, h=0.2 ; Gamma, μ=1, k=2 ; Sample Size = 10

C : Long tailed h, λ=0.01, h=0.2 ; Exponential, μ=1 ; Sample Size = 10

$$A = \prod_{i=1}^{2} A_i = \prod_{i=1}^{2} \frac{E\{U_i\}}{E\{U_i\} + E\{D_i\}} \tag{3.6.12}$$

The jackknife pseudovalues of ln A_i are computed in a similar manner, as described in (3.5.10). For the bootstrap method, the bootstrap repetitions of ln A are computed and the bootstrap estimator is then calculated in the normal manner. The log transformation is necessary in the two component case, otherwise, the bootstrap variance estimator, while it can be calculated for each component, cannot be determined for the system as a whole. Also, Thompson's method and Hinkley's method using 'improved' t quantiles are only valid for the one component case and are therefore not presented in this study.

Using different distributional forms for the uptimes and downtimes for each of the two components, the results are shown in Tables 3.6.17 and 3.6.18. The conclusions are as follows:

a. the jackknife estimator has the smallest bias and mean square error than the standard and bootstrap estimators.
b. the bootstrap estimator has the largest bias and mean square error than the other two estimators
c. the jackknife and bootstrap method are trustworthy in establishing confidence limits. In fact, as in the one component case from the first simulation study, the bootstrap outperforms the jackknife method, based on the criteria of better coverage, shorter average lengths and smaller variances of length.

4.5 **Summary of simulation results** :

The following conclusions can be made regarding the results presented in the previous section:

(i) the jackknife estimator generally produced a smaller bias than the standard or bootstrap estimators at the expense of a small increase in mean square error or, as in the two component case, with a smaller mean square error as well.

Table 3.6.17 : Bias, Sample Variance, Mean Square Error for Various Availability Estimation Methods in the Two Components in Series Case :

		Standard Method	Jackknife Method	Bootstrap Method
	Bias	-0.0033	0.0002	-0.0063
A	Sample Variance	1.01×10^{-4}	8.55×10^{-5}	1.21×10^{-4}
	Mean Square Error	1.12×10^{-4}	8.55×10^{-5}	1.62×10^{-4}
	Bias	-0.0049	-0.0003	-0.0085
B	Sample Variance	1.49×10^{-4}	1.53×10^{-4}	1.67×10^{-4}
	Mean Square Error	1.73×10^{-4}	1.53×10^{-4}	2.40×10^{-4}
	Bias	-0.0015	0.0001	-0.003
C	Sample Variance	5.46×10^{-5}	5.00×10^{-5}	6.05×10^{-5}
	Mean Square Error	5.67×10^{-5}	5.00×10^{-5}	6.88×10^{-5}

A : Uptimes : Exponential, $\lambda_1=0.01$; Exponential $\lambda_2=0.02$;
Downtimes : Gamma, $\mu_1=1.0$, k=2 ; Gamma, $\mu_2=1.0$, k=3 ;
Sample Sizes = 10

B : Uptimes : Long tailed h, $\lambda_1=0.01$, h=0.2 ; Long tailed h, $\lambda_2=0.02$, h=0.2;
Downtimes : Gamma, $\mu_1=1.0$, k=2 ; Exponential, $\mu_2=1.0$;
Sample Sizes = 15

C : Uptimes : Exponential, $\lambda_1=0.01$; Exponential, $\lambda_2=0.02$;
Downtimes : Exponential, $\mu_1=1.0$; Exponential, $\mu_2=1.0$;
Sample Sizes = 20

Table 3.6.18 : Sample Confidence Intervals for Non-Parametric Methods in the Two Components in Series Case :

			Jackknife Method	Bootstrap Method
A	Coverage	95%	95.4	95.5
		99%	98.8	99.4
	Average Length	95%	4.16×10^{-2}	4.00×10^{-2}
		99%	5.69×10^{-2}	5.26×10^{-2}
	Variance of Length	95%	4.38×10^{-4}	3.26×10^{-4}
		99%	8.24×10^{-4}	5.64×10^{-4}
B	Coverage	95%	94.4	94.8
		99%	98.8	98.9
	Average Length	95%	4.90×10^{-2}	4.60×10^{-2}
		99%	6.62×10^{-2}	6.05×10^{-2}
	Variance of Length	95%	$4.92 \times 10^{-}$[illegible]	3.40×10^{-4}
		99%	8.97×10^{-4}	5.88×10^{-4}
C	Coverage	95%	93.6	95.0
		99%	97.5	99.0
	Average Length	95%	2.97×10^{-2}	2.90×10^{-2}
		99%	3.98×10^{-2}	3.81×10^{-2}
	Variance of Length	95%	9.10×10^{-5}	7.39×10^{-5}
		99%	1.63×10^{-4}	1.28×10^{-4}

A : Uptimes : Exponential, $\lambda_1=0.01$; Exponential, $\lambda_2=0.02$;
Downtimes : Gamma, $\mu_1=1.0$, $k=2$; Gamma, $\mu_2=1.0$, $k=3$;
Sample Sizes = 10

B : Uptimes : Long tailed h, $\lambda_1=0.01$, $h=0.2$; Long tailed h, $\lambda_2=0.02$, $h=0.2$;
Downtimes : Gamma, $\mu_1=1.0$, $k=2$; Exponential, $\mu_2=1.0$;
Sample Sizes = 15

C : Uptimes : Exponential, $\lambda_1=0.01$; Exponential, $\lambda_2=0.02$;
Downtimes : Exponential, $\mu_1=1.0$; Exponential , $\mu_2=1.0$;
Sample Sizes = 20

(ii) the two minimum jackknife risk estimators improved on the ordinary jackknife estimate in terms of bias and mean square error.

(iii) the two jackknife methods, using t quantiles and 'improved' t quantiles were the most robust methods in terms of confidence interval estimation

(iv) the bootstrap method, using normal quantiles, outperformed the jackknife methods, in terms of coverage, average length and variance of length except in the case where the Weibull distribution was used

(v) the standard method was generally untrustworthy in producing confidence limits

(vi) the percentile and bias corrected percentile methods did not enhance the existing bootstrap method in terms of interval estimation.

CHAPTER 4

JACKKNIFE AND BOOTSTRAP INTERVAL ESTIMATION IN NON-LINEAR REGRESSION MODELS

4.1 Introduction

In many practical applications, it is frequently necessary to fit an algebraic function to an observed relationship between two or more variables. A 'least squares' method which minimises the sum of the squares of the errors, is used to fit models which are linear in the parameters and are of the type:

$$Y=\beta_o+\beta_1 Z_1+\beta_2 Z_2+\ldots+\beta_p Z_p+\varepsilon \qquad (4.1.1)$$

where the Z_i can represent any function of the independent variables $X_1, X_2, \ldots, X_k$ and ε is a random variable with mean zero and constant variance σ^2, unknown and independent of Z. However, while equation (4.1.1) can represent a wide variety of relationships, there are many situations in which a model of this form is not appropriate. When a researcher is led to a model of nonlinear form in the parameters, it is usually preferable to fit such a model whenever possible, rather than to fit an alternative, perhaps less realistic, linear model.

There are many mathematical models that arise in the scientific area that contain one or more non-linear parameters. Non-linear regression models are of considerable practical importance in a variety of applications, and these have been well-documented in two papers, one by Gallant (1975) and the other by Kanemasu (1972).

The general form of the non-linear model is:

$$Y=f(x,\theta)+e, \qquad (4.1.2)$$

where f is a non-linear function of the unknown parameter $\theta=(\theta_1,\theta_2,\ldots,\theta_p)'$ and e is a random variable, with mean 0 and variance σ^2, unknown and independent of x. A central problem to statistical inference in non-linear regression models is the interval estimation of θ. Conventional methods for the interval estimation of θ depend upon norma ity assumptions for e and linearizing approximations to a regression surface which is a non-linear function of θ. The validity of these assumptions is an area of research which will be investigated in a latter section of this chapter.

There are several currently employed methods available for estimating the parameters of a non-linear system. Three of the more common methods are the linearisation, the steepest descent and the Marquardt's compromise method. A brief description of each method is as follows:

(i) The linearisation (or Taylor series) method uses the results of linear least squares in a succession of stages (see Draper and Smith (1966)). Suppose the model is postulated having the form:

$$Y_u=f(x_u,\theta)+\varepsilon_u \tag{4.1.3}$$

where $x_u = (x_{1u},x_{2u},\ldots,x_{ku})'$

and ε_u is the u^{th} error, $u=1,2,\ldots,k$

Let $\theta_{10},\theta_{20},\ldots,\theta_{po}$ be initial values for the parameters $\theta_1,\theta_2\ldots,\theta_p$, which are derived either by intelligent guesses or preliminary estimates, based on the information that is available. By carrying out a Taylor series expansion of $f(x_u,\theta)$ about the point θ_o, where $\theta_o=(\theta_{10},\theta_{20},\ldots,\theta_{p0})'$, then $f(x_u,\theta)$ approximates to:

$$f(x_u,\theta)=f(x_u,\theta_o) + \sum_{i=1}^{P} \left\{\frac{df(x_u,\theta)}{\theta_i}\right\}_{\theta=\theta_o} (\theta_i-\theta_{io}) \tag{4.1.4}$$

Next set $f_u^\circ = f(x_u, \theta_o)$

$$\beta_i^\circ = \theta_i - \theta_{io}$$

$$Z_{iu}^\circ = \left\{\frac{\partial f(x_u, \theta)}{\partial \theta_i}\right\}_{\theta=\theta_o}$$

then, the model (4.1.4) approximates to the form:

$$Y_u - f_u^\circ = \sum_{i=1}^{p} b_i^\circ Z_{iu}^\circ + \varepsilon_u \qquad (u=1,2,\ldots,n) \qquad (4.1.5)$$

Now, the parameters β_i° , $i=1,2,\ldots,p$ can be estimated by linear least squares theory. The vector b_o will therefore minimise the sum of squares:

$$SS(\theta) = \sum_{u=1}^{n} \{Y_u - f(x_u, \theta_o) - \sum_{i=1}^{P} \beta_i^\circ Z_{iu}^\circ\}^2 \qquad (4.1.6)$$

with respect to the β_i° , $i=1,2,\ldots,p$, where $\beta_i^\circ = \theta_i - \theta_{io}$. Writing $b_i^\circ = \theta_{i1} - \theta_{io}$, then the θ_{i1}, $i=1,2,\ldots,p$ can be considered as the revised best estimator of θ.

An iterative process than takes place where the revised estimators θ_{i1}, replace the values θ_{io}, and the procedure described by equations (4.1.4) to (4.1.6) is then repeated. This will produce another set of estimators θ_{i2} , and so on. This process is continued until the solution converges. The test for convergence is , given successive iterations j, (j+1),

$$|\{\theta_{i(j+1)} - \theta_{ij}\}/\theta_{ij}| < \delta, \ i=1,2,\ldots,p \qquad (4.1.7)$$

where δ is a prespecified quantity. (e.g. 1.0×10^{-6}).

(ii) The steepest descent method uses an iterative approach to minimise the sum of squares function, $S(\theta)$, which is defined in the usual manner

$$\text{i.e.} \quad S(\theta) = \sum_{u=1}^{n} \{Y_u - f(X_u, \theta)\}^2 \qquad (4.1.8)$$

The basic method involves moving from an initial point θ_o, along the vector with components

$$-\frac{\partial S(\theta)}{\partial \theta_1}, \quad -\frac{\partial S(\theta)}{\partial \theta_2}, \ldots, \quad -\frac{\partial S(\theta)}{\partial \theta_p},$$

whose values change continuously as the path is followed. This involves estimating the vector slope components at various places on the surface $S(\theta)$ by fitting planer approximating functions.

The procedure is described as follows (see Davies, 1954):

Start in one particular region of the parameter space. Then, make several runs by selecting n combinations of levels of $\theta_1, \theta_2, \ldots, \theta_p$ and, on each occasion evaluate $S(\theta)$. Using the observed values of $\theta_1, \theta_2, \ldots, \theta_p$ as the independent variables and the evaluated $S(\theta)$ values as the dependent value, fit the following model by least squares method:

$$\text{"Observed } S(\theta)\text{"} = \beta_o + \sum_{i=1}^{p} \beta_i(\theta_i - \bar{\theta}_i)/S_i + \varepsilon \qquad (4.1.9)$$

where $\bar{\theta}_i$ = mean of the levels θ_{iu}

and S_i = a scaling factor such that $\sum_{u=1}^{n} (\theta_{iu} - \bar{\theta}_i)^2/S_i^2$ is a constant.

The negative of the estimated coefficients

$-b_1, -b_2, \ldots, -b_p$

indicate the direction of steepest descent. The maximum decrease in $S(\theta)$ is obtained by moving along the line which contains points such that

$(\theta_i - \bar{\theta}_i)/S_i$ is proportional to $-\lambda b_i$

The path of steepest descent then containts points $(\theta_1, \theta_2, \ldots, \theta_p)$ such that

$$\frac{(\theta_i - \bar{\theta}_i)}{S_i} = -\lambda b_i,$$