Schur Polynomials, Restricted Schur Polynomials and the AdS/CFT Correspondence

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Declaration

I declare that all results presented in this thesis are original except where reference is made to the work of others. The following is the list of my original works discussed in this thesis.

1. R. Bhattacharyya, R. de Mello Koch, M. Stephanou, "Exact Multi-Restricted Schur Polynomial Correlators," (2008), Journal of High Energy Physics, Volume 2008, Number 6, arXiv:hep-th/0805.3025.

2. R. de Mello Koch, N. Ives, M. Stephanou, "Correlators in Non-Trivial Backgrounds," (2008), Physical Review D, Volume 79, Issue 2, ID 026004, arXiv:hep-th/0810.4041.

3. R. de Mello Koch, T. Dey, N. Ives, M. Stephanou, "Correlators of Operators with a Large R-Charge," (2009), Journal of High Energy Physics, Volume 2009, Number 8, arXiv:hep-th/0905.2273.

4. R. de Mello Koch, T. Dey, N. Ives, M. Stephanou,"Hints of Integrability Beyond the Planar Limit: Nontrivial Backgrounds," (2009), Journal of High Energy Physics, Volume 2010, Number 1, arXiv:hep-th/0911.0967.

These papers are references [1], [2], [3], [4] in the bibliography and constitute the subject matter of sections 7, 8, 9, 10 (along with associated appendices A, B, C, D) respectively. This thesis is being submitted for the degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

In this thesis we introduce novel technology pertaining to the calculation of certain classes of correlation functions in the half BPS, near BPS and multi-matrix sectors (particularly the $SU(2)$ sector) of $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory using the Schur polynomial and restricted Schur polynomial bases. This technology allows the detailed exploration of a host of physical questions within the AdS/CFT correspondence. We first treat multi-point correlators of restricted Schur polynomials which constitute a particularly convenient basis for the $SU(2)$ and general multi-matrix sectors of $\mathcal{N} = 4$ SYM. We introduce a product rule for the restricted Schur polynomials, which, together with the two point correlator result of arXiv:0801.2061 allows us to compute exact multi-point correlators, in the free field theory limit. This facilitates the exploration of physical questions such as elucidating the appropriate degrees of freedom for a perturbative description of quantum gravity in the sectors under consideration.

We then treat the calculation of correlators of operators in the presence of large background operators with a $\mathcal{R}$-charge of $O(N^2)$ that are dual to asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$ backgrounds. Evaluating these correlators is a hard problem in general since the planar approximation fails. In this thesis we develop general techniques, known as the "cutting rules", that allow the computation of such correlators. Specializing to the LLM annulus geometry allows a number of concrete results to be derived. We then study the perturbation theory of these correlators and identify new perturbative expansion parameters replacing $1/N$. Motivated by these results we explore new BMN-type sectors.

Finally, we treat the problem of computing the anomalous dimensions of a class of (nearly) half BPS operators with a large $\mathcal{R}$-charge. This problem is reduced to the problem of diagonalizing a Cuntz oscillator chain. Non-planar corrections must be summed to correctly construct the Cuntz oscillator dynamics. We explore whether these non-planar corrections that account for backreaction from the heavy operator rather than quantum corrections in the dual gravitational theory also spoil integrability as is generally the case with quantum corrections. We find a limit in which our Cuntz chain continues to admit extra conserved charges suggesting that integrability might survive.

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1 Introduction

Quantizing gravity is one of the greatest open challenges of contemporary theoretical physics. Einstein's general theory of relativity (GR) marked a dramatic shift in our thinking about gravity and provided a unified framework in which to treat space, time, matter and energy. Although very successful in describing physics on large scales, GR breaks down when a sufficiently large amount of matter is amassed in a very small volume of space. Naïve attempts to reconcile GR with the theory governing physics on small scales, namely quantum field theory (QFT), failed due to the fact that the QFT resulting from attempting to quantize gravity (whether it is pure gravity or gravity coupled to matter) is non-renormalizable. Superstring theory is a quantum theory that does succeed in incorporating gravity in a consistent way and thus provides an opportunity to explore a true theory of quantum gravity. In addition to being a candidate for a theory unifying all the known forces of nature, it is a theory in which beautiful mathematical structure has been revealed and profound physical connections established. Perhaps the most powerful of these is the deep and fascinating link between string theory and a class of quantum fields theories that was elucidated by Maldacena in the form of the AdS/CFT correspondence [5].

The AdS/CFT correspondence is a conjectured duality between a string theory defined on a certain space (the product of Anti de-Sitter space and a compact manifold) and a conformal field theory on the boundary of this space. This conjecture is a statement about the full dynamical equivalence of the two sides of the correspondence and goes beyond a kinematical equivalence. The fact that the string theory is defined on a space with a different dimensionality to that of the gauge theory can be related to one of the profound physical ideas that accompany the conjecture - the holographic principle [6, 7]. The other deep physical idea related to the AdS/CFT conjecture is the proposal that gauge theories at large N are equivalent to string theories as proposed by 't Hooft [8]. The correspondence is a strong/weak coupling duality, and relates strongly coupled field theory to string theory with small curvature corrections and vice versa. Naturally this makes testing the duality more difficult. However, a wealth of evidence has been obtained in support of the conjecture, with new and novel tests continually being sought. Under the assumption that the conjecture is indeed correct, it provides a fresh perspective with new and powerful tools to re-examine long standing open problems in physics. We can study the gauge theory and investigate fascinating questions in the dual gravitational theory such as the information loss paradox and the dynamical emergence of spacetime or we could consider suitable systems in the gravitational theory to learn about strongly coupled quantum field theories. For example, this approach has recently been applied in studying critical phenomena in condensed matter systems, particularly superconductors (for recent and accessible introductions see [9, 10]). In order to understand, test and ultimately apply the correspondence with maximum effect, it seems clear that in addition to a clear dictionary relating the two sides of the correspondence we need the ability to perform dynamical calculations in both the string theory and the gauge theory. Developing some of the required technology to perform such calculations on the field theory side of the correspondence is a major focus of this thesis.

The original form of the AdS/CFT correspondence relates type IIB string theory defined on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the maximally supersymmetric four dimensional gauge theory, $\mathcal{N} = 4$ Super Yang-Mills [11] defined on the boundary of this space. The $\mathcal{N} = 4$ Super Yang-Mills ($\mathcal{N} = 4$ SYM) action contains six real scalar fields transforming in the adjoint of $U(N)$. Operators can be built from three complex matrix fields constructed from complex linear combinations of these six scalar fields. The half BPS sector (the sector in which gauge invariant operators are constructed from a single complex matrix field) of $\mathcal{N} = 4$ SYM has been a particularly fruitful laboratory in which to establish the aforementioned dictionary. Half of the supersymmetries of the gauge theory are preserved in this sector. In the half BPS sector, the identification across the duality for objects in the gravitational theory such as gravitons and giant gravitons (D-branes which are extended in the S^5 or AdS_5 space of the AdS_5xS^5 background, see [13, 14]) has been established. In addition, the map between half BPS operators and new classical background geometries on the gravity side has been elucidated [15]. In the near BPS sector (where operators are predominantly composed of a single complex matrix field), operators dual to strings have been identified [16].

Gravitons are dual to gauge invariant operators in the gauge theory which are single traces of $O(1)$ matrix fields [5]. Operators which are products of such traces with the same total number of fields but different numbers of traces are orthogonal in the large N limit and can be identified with multi-particle graviton states. The number of traces in the operator is identified with particle number. The orthogonality of operators constructed from products of traces fails when they are comprised of $O(\sqrt{N})$ matrix fields and it is no longer possible to consider these operators as dual to graviton states. Instead, in the near BPS sector, they are dual to strings [16]. Finally, giant gravitons are dual to multi-trace operators comprised of $O(N)$ matrix fields known as Schur polynomials [17, 18]. The changing interpretation of objects in the dual gravitational theory as the $\mathcal{R}$-charge (a conserved charge associated with the $\mathcal{R}$-symmetry of $\mathcal{N}=4$ SYM - it corresponds to the number of matrix fields for these operators) of the gauge theory operators changes can be seen by considering the results of [12], [13]. Following the analysis of [13] the radius of an object, r, in $\mathrm{AdS}_5 \times \mathrm{S}^5$ is related to its momentum, L by

$$r = \sqrt{\frac{L}{N}} R. \tag{1.1}$$

Here R is the radius of curvature of the AdS space and N is the number of units of five form flux. Combining this with the identification of parameters across the correspondence, we see that the size of the object in the gravitational theory dual to a gauge theory operator with $\mathcal{R}$-charge L, is:

$$r = \sqrt{\frac{L}{N}} (4\pi)^{\frac{1}{4}} N^{\frac{1}{4}} (g_{YM})^{\frac{1}{2}} l_s, \tag{1.2}$$

where N is now identified as the rank of the gauge group $U(N)$, g_{YM} is the gauge theory coupling and l_s is the characteristic string length. We are considering the large N limit on the gauge theory side and the gauge theory coupling g_{YM} is taken to be fixed and very small. Operators comprised of $O(1)$ matrix fields can therefore be seen to be dual to point like objects in the gravitational theory. Operators comprised of $O(\sqrt{N})$ matrix fields are dual to objects of fixed size in units of the string length. Finally, the size of the gravitational objects dual to operators comprised of $O(N)$ fields is on the order of the radius of curvature of the AdS space and thus are naturally identified with giant gravitons.

The Schur polynomials constitute a very useful basis for the half BPS sector of $\mathcal{N} = 4$ SYM. In mathematics, these polynomials labeled by Young diagrams have particular relevance in symmetric and unitary group theory. Schur polynomials are in fact characters of irreducible representations of $SU(N)$. The particular utility of the Schur polynomial basis arises due to some special properties. Firstly, the two point function of two Schur polynomials is known to all orders in $1/N$ and moreover is diagonal in the Young diagram labels [17, 19]. Secondly, Schur polynomials satisfy a product rule and thus the calculation of higher point functions can be reduced to evaluating linear combinations of known two point functions. This product rule refers to the fact that the product of two Schur polynomials can be expressed as a linear combination of Schur polynomials with certain non-negative integral coefficients known as Littlewood-Richardson numbers. In addition to this, the Young diagram labels seem to encapsulate some interesting physics.

One initial example of this is that, in terms of the giant gravitons, the Young diagram labels of the dual Schur polynomials encode an effect known as the stringy exclusion principle [20, 21]. Sphere giants (D3-branes wrapping an $S^3 \subset S^5$ of the AdS_5xS^5 background) are identified with Schur polynomials labeled by Young diagrams consisting of a single column (corresponding to fully anti-symmetric representations), where the length of the column is identified with the angular momentum of the giant. This captures the bound on the size of the sphere giants (they can only expand to the radius of the S^5 and thus following [13] can have an angular momentum of at most N)

since the maximum number of boxes that can be placed in a single column is precisely N.

As mentioned previously, Schur polynomials of $O(N^2)$ fields are dual to new classical background geometries in the string theory. One way to understand that these large background operators are dual to new geometries is to consider that the spectrum of conformal dimensions of operators in $\mathcal{N}=4$ SYM maps into the energy spectrum of string states in $\mathrm{AdS}_5\times\mathrm{S}^5$. Thus operators with a very large $\mathcal{R}$-charge are dual to very heavy states in the string theory and back reaction leads to a new geometry that is only asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$. The Young diagrams labeling these background Schur polynomials correspond to boundary conditions determining the metric of these new geometries, known as LLM geometries[15]. The LLM geometries are half BPS solutions of type IIB string theory which are asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$. More concretely, the half BPS sector of $\mathcal{N}=4$ SYM admits a description in terms of free fermions; the eigenvalue dynamics of the matrix field from which half BPS operators are built is captured by the dynamics of N non-interacting fermions in a harmonic oscillator potential [17, 22]. The Young diagram labeling the Schur polynomial maps into a particular Fermion energy occupation (configuration in the Fermionic phase space). These configurations in phase space can be represented by the black and white coloring of a two dimensional plane, where the black regions correspond to energy levels which are filled (a black ring corresponds to a single occupied energy level) and white regions correspond to energy levels which are unoccupied. This free fermion phase space is identified with the LLM plane which is a plane in the 10-dimensional LLM geometry located at $y=0$ and spanned by x_1, x_2[15]. The LLM plane constitutes the boundary condition determining the LLM geometry. The vacuum case in which the LLM plane corresponds to $\mathrm{AdS}_5\times\mathrm{S}^5$ is a black circular droplet of area πN. The fact that Schur polynomials comprised of $O(N^2)$ fields correspond to new background geometries that are only asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$ can be seen by considering the LLM plane. It is only when the Schur polynomial contains $O(N^2)$ fields that the Fermion energy occupation it describes can be altered sufficiently to constitute a departure from $\mathrm{AdS}_5 \times \mathrm{S}^5$ and not simply small fluctuations of the original background. The LLM plane configurations described by the background Schur polynomials are rotationally symmetric. For example, the annulus background (which will play a large role in the research presented in this thesis) corresponds to a Schur polynomial labeled by a Young diagram with N rows, and M columns where M is $O(N)$. This describes the LLM plane consisting of a black annulus with inner radius $\sqrt{M}$ and outer radius $\sqrt{M+N}$.

A fascinating question in the context of the AdS/CFT correspondence is how the local physics of the ten dimensional string theory is encoded in the dual hologram. The Schur polynomial description of the half BPS sector yields tantalizing clues as to how aspects of the geometry and dynamics of the string theory emerge in the gauge theory. The Young diagram labels of the Schur polynomials in particular seem to encapsulate some of the local dynamics of the string theory. Concretely, localized excitations of the backgrounds described by large Schur polynomials can be defined via a modified form of the Littlewood-Richardson rule that only adds blocks to a particular region of the Young diagram[23]. On the string theory side these excitations are localized in the radial direction of the plane constituting the boundary condition for the metric (they are smeared out in the angular direction). The localization of graviton, string and giant graviton excitations from the gauge theory perspective as well as a discussion on the possible significance of Young diagrams in this context is presented in [23]. See also [24].

Progressing beyond the half BPS sector of $\mathcal{N} = 4$ SYM to the two matrix sector ($SU(2)$ sector) and to general multi-matrix models we are presented with a choice of a number of possible bases to organize the degrees of freedom [25, 26, 27], all of which diagonalize the free field theory two point function. The most natural basis in which to consider open-string excitations of giant gravitons is the restricted Schur basis [28, 30, 31, 32, 33]. This basis generalizes and extends the Schur polynomial basis. In developing and exploring the restricted Schur basis and its associated computational technology, new mathematics is required. In particular, in defining the restricted Schur polynomial, the group theoretic concept of a character has to be extended to the restricted character which corresponds to only tracing over a subspace of the carrier space of an irreducible representation of the symmetric group. In the multi-matrix sector, we can define generalized versions of the graviton operators discussed previously built from $O(1)$ matrix fields. Restricted Schur polynomials are appropriate in defining giant graviton operators built from $O(N)$ matrix fields and multi-charge background operators built from $O(N^2)$ matrix fields.

Of particular interest is the ability to calculate higher point correlators of restricted Schur polynomials since, in general, correlators fully capture the dynamics of the gauge theory. The approach followed in the research presented here is to generalize the procedure applicable to Schur polynomials, i.e. to introduce a product rule for restricted Schur polynomials and then reduce the calculation of higher point functions to evaluating linear combinations of two point functions, the result of which is known. In order to define this product rule, the notion of a Littlewood Richardson number has to be generalized to a restricted Littlewood Richardson number. To calculate the restricted Littlewood Richardson numbers it is necessary to be able to calculate arbitrary restricted characters of irreducible representations of S_n, motivating further study of the restricted characters. With the technology to calculate higher point functions of restricted Schur polynomials, a number of intriguing physical questions become open to exploration. At finite N, the gauge theory captures quantum mechanical aspects of the dual string theory. At low enough energies, gravitons furnish a suitable set of degrees of freedom to establish a perturbative quantum gravity description and we would expect more fundamental degrees of freedom to be appropriate at high energies. In [34], three point functions of half BPS operators dual to gravitons were used to establish that gravitons cease to be weakly interacting degrees of freedom beyond the energy scale of $N^{\frac{1}{2}}$ ($O(\sqrt{N})$ fields comprising the operators). Moreover, they identified the half BPS operators dual to sphere giants as being weakly interacting beyond this energy scale and thus identified the more fundamental, weakly interacting degrees of freedom to employ in a perturbative quantum gravity description at high energies. Following the single matrix analysis performed by [34] we seek to determine if, in the more general case of multi-matrix operators, gravitons are no longer appropriate degrees of freedom for a perturbative description of quantum gravity beyond a certain energy scale. Further, we seek to determine if the multi-matrix generalization of Sphere giants are also weakly interacting degrees of freedom at high energies as was the case in the half BPS sector. Finally, we utilize the restricted Schur technology to link the restricted Schur polynomials to the single matrix Schur polynomials and in doing so obtain a physical interpretation for both sets of operators.

In addition to the technology allowing the calculation of higher point restricted Schur correlators, we develop technology that enables us to compute correlators of operators in the presence of the large background Schur polynomial operators comprised of $O(N^2)$ fields. In the half BPS sector these background operators are dual to the LLM geometries described previously. We also go beyond the half BPS sector in some special cases by deriving some results for correlators of operators in the presence of certain multi-charge backgrounds described by restricted Schur polynomials.

In evaluating the gauge theory correlators of operators in the presence of these large background operators, contractions between the operator considered and the background operator cannot be neglected. The planar approximation fails in this case due to the build up of enormous combinatoric factors in front of the non-planar diagrams. In the present work, we argue that including these contractions is the gauge theory accounting of the new classical background geometry in the string theory. This is motivated by the fact that these contractions only need to be taken into account precisely in the regime that the dual state is heavy enough to significantly deform our original geometry. We also explore what it means to have background operators with what at first would appear to be open string excitations. Naturally, from the string theory perspective, we would only expect open string excitations to exist in the context of excited giant gravitons i.e. open strings must end on D-branes. Further, given that our background geometry arises from a very heavy object, no longer having an interpretation in terms of giant gravitons, we would expect only closed string excitations to appear. We seek evidence that the dual gauge theory description actually incorporates these expectations.

Evaluating the result of these correlators in practice would at first seem very difficult if not intractable. We present a framework in which evaluating such correlators is reduced to the problem of evaluating the action of derivative operators on Schur polynomials. The actions of these derivative operators are then related - via operations that we term cutting rules - to the known actions of a specific set of derivative operators. This approach turns out to be powerful and general. Finally, we derive a concrete set of results in the specific case of the annulus background geometry dual to a Schur polynomial for which the results of applying the cutting rules are particularly simple. We find that, at least in this case, the presence of the background is captured by simply rescaling the propagators representing the contraction of fields in the correlator in the absence of the background Schur polynomial, thus confirming some of the results of [23].

The cutting rules described above not only make previously intractable calculations possible to perform, they are also entirely general in that they can be applied for any background Schur polynomial. These calculations are still fairly computationally intensive to perform however. A natural question to ask is whether this generality can be traded for more efficiency in evaluating such correlators in the case of special backgrounds. The usefulness of this is not only in reducing the computational intensity of the calculations but also in more clearly revealing interesting mathematical and physical properties of the results. It turns out that more efficient techniques can indeed be applied in the special case of the annulus background (as well as for extensions to multi-ring and multi-charge backgrounds). By applying Schur polynomial technology we show that for half BPS operators that can be rewritten in the

Schur polynomial basis, the result of correlators in the annulus background in a theory with gauge group $U(N)$ is equivalent to vacuum correlators in a theory with the gauge group $U(N+M)$ where M is an $O(N)$ number corresponding to the number of columns in the Young diagram labeling the background Schur polynomial. In effect, all N dependence is replaced by $N+M$ dependence. This is an exact identification and thus we have found a new expansion parameter $1/(N+M)$ (replacing $1/N$) which renders the perturbation theory sensible once more. This reorganization is a remarkable one and represents resumming an infinite number of diagrams. A comparison to Supergravity utilizing holographic renormalization is used to verify the correctness of these results. Further, we explore if this reorganization is valid beyond the half BPS sector. To this end, we perform a leading order and exact analysis of near BPS correlators (correlators of operators predominantly comprised of a single matrix field with a small number of another complex matrix field) to determine if they also admit an expansion in $1/(N+M)$.

The AdS/CFT correspondence is a strong/weak coupling duality and thus in the regime in which curvature corrections in the string theory are suppressed (when the radius of $\mathrm{AdS}_5 \times \mathrm{S}^5$ is much larger than the string length), the 't Hooft coupling of the dual gauge theory is large and thus perturbation theory cannot be directly applied in the case of operators that are not protected by supersymmetry. As shown by BMN [16] however, there are sectors of the gauge theory in which an effective coupling can be identified which can be fixed to some small value, even as the 't Hooft coupling is sent to infinity, allowing perturbative gauge theory results to be compared with the string theory. Utilizing restricted Schur technology, we identify a BMN-like sector for two-charge background operators motivated by the identification of effective genus counting parameters as discussed above.

The conformal (scaling) dimension of an operator in $\mathcal{N}=4$ SYM is an important quantity since, in addition to completely determining the spacetime scaling of two and three point correlators in the gauge theory, the conformal dimensions of operators map into energies of string states in the context of the AdS/CFT correspondence. As an aside, constructing a test of the correspondence utilizing this identification would be frustrated in general by the strong/weak coupling nature of the duality and inherent difficulties in evaluating both the string spectrum and the spectrum of conformal dimensions in the gauge theory. However, these difficulties were deftly circumvented by BMN who performed a successful quantitative comparison of the spectra.

The scaling dimension of operators can be defined by the two point correlation function. For conformal operators the two point function has the form:

$$\langle \mathcal{O}_\alpha(x)\mathcal{O}_\beta(0)\rangle = \frac{\delta_{\alpha\beta}}{|x|^{2\Delta_\alpha}}.$$

Note however that, it is only the class of operators possessing a definite scaling dimension (conformal operators) that have a diagonal two point function. In general there is the problem of mixing and the two point function is a non-diagonal matrix. The conformal dimension is comprised of the classical scaling dimension and the anomalous dimension (the quantum correction to the classical dimension). The classical scaling dimension of an operator is simply equal to the sum of the scaling dimensions of the constituent fields comprising the operator. Determining the anomalous dimension contribution to the scaling dimension is far more involved however and involves renormalization. Specifically, in the first works in this area, it involved determining a renormalization factor that renders a chosen correlator finite to some order in the perturbation theory. The matrix of anomalous dimensions was then determined in the usual way from the renormalization factor. A far more efficient way of determining the scaling dimensions of operators was introduced in [35] via the introduction of a dilatation operator which has the following action on a conformal operator

$$[D, \mathcal{O}_\alpha] = \Delta_a \mathcal{O}_\alpha.$$

The dilatation operator admits a perturbative expansion in g_{YM}^2

$$D = \sum_{k=0}^{\infty} \left(\frac{g_{YM}^2}{16\pi^2} \right)^k D_{2k}.$$

The k-loop dilatation generator, D_{2k}, is determined once, eliminating the need for a case by case evaluation of renormalized two-point correlators to determine anomalous dimensions. Acting the k-loop dilatation generator on some basis of states yields a dilatation matrix. Diagonalizing this matrix gives the conformal operators in the basis of the original set of states (corresponding to the eigenvectors of the dilatation matrix up to a normalization factor) and the k-loop anomalous dimensions corresponding to the eigenvalues of the matrix.

Determining the spectrum of scaling dimensions of general classes of operators in $\mathcal{N} = 4$ SYM is, in general, a highly non-trivial task however. Fortunately, there is strong evidence to suggest that the planar limit of $\mathcal{N} = 4$ SYM is exactly integrable. This would mean that the matrix of anomalous dimensions can be diagonalized exactly using the Bethe ansatz. Initial indications of this were obtained in [36] where the one-loop mixing matrix for anomalous dimensions of scalar operators was identified with the Hamiltonian of an integrable vector $so(6)$ spin chain. These one-loop results were extended in [37] to include all operators in $\mathcal{N} = 4$ SYM. In addition, integrability has been demonstrated in the $su(2)$ sector to two and three loops [35]. Further evidence for exact planar integrability of $\mathcal{N} = 4$ SYM is furnished by [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50]. Integrability does not in general survive non-planar corrections associated with quantum corrections (finite N corrections). A natural question to ask however is if it survives non-planar corrections owing to the presence of large operators containing $O(N^2)$ fields (dual to new background geometries as discussed above). In this case, non-planar contributions are no longer suppressed due to the overpowering combinatoric factors multiplying the non-planar diagrams. In order to attempt to answer this question, we define an effective dilatation operator. The effective dilatation operator captures the presence of the large background operator and diagonalizing its action will yield the conformal dimensions of operators in these new backgrounds.

A spin chain description is not appropriate for the effective dilatation operator since its action allows the number of fields in each trace to change dynamically due to the non-planar corrections. The Cuntz chain description, introduced in [51], is appropriate however. In mapping trace operators to a particular Cuntz chain state we reinterpret the fields in the trace. One of the matrix fields sets up a fixed lattice which the other matrix field populates. In this way, we obtain particles (described by Cuntz oscillators) hopping on a fixed size lattice. In the Cuntz chain description, the number of particles populating the lattice is dynamical. Once expressed in the Cuntz chain language the effective dilatation operator for the annulus background (labeled by a Young diagram of N rows and M columns) was studied for hints of integrability. The presence of an infinite number of conserved charges in the new system would certainly be convincing evidence that integrability does indeed survive. We find that taking a particular limit of our effective Cuntz chain Hamiltonian (the large M limit) does indeed furnish an infinite number of higher conserved charges to all loops. Another limit that warrants exploration is the semi-classical limit of the Cuntz chain. The semi-classical limit is obtained by taking the number of lattice sites to infinity, $L \sim \sqrt{N} \to \infty$ and $\lambda \to \infty$, holding $\frac{\lambda}{L^2}$ fixed and putting each site into a coherent state. To develop some insight into this limit, we study the semi-classical limit of the undeformed case of the Cuntz chain and determine how integrability is established. We then apply these insights to the annulus background and search for signs of integrability. We also explore the integrability of a semi-classical system in which the large background Schur polynomial corresponds to the LLM boundary condition identified with the Superstar geometry [52, 74].

In section 2 we briefly discuss the motivation for the AdS/CFT correspondence and state the relationship between the parameters of the gauge theory and those of the string theory. In section 3 we review pertinent Schur polynomial technology. The relationship between Schur polynomials and giant gravitons is briefly considered in section 4. We then review relevant restricted Schur polynomial technology for distinguishable matrix words and indistinguishable matrix words in sections 5 and 6 respectively. In section 7 we introduce a product rule for restricted Schur polynomials and discuss the evaluation of restricted characters relevant to calculating the restricted Littlewood-Richardson numbers. In addition, we discuss physical applications of the new techniques for evaluating multi-point correlators of restricted Schur polynomials. In section 8 we introduce the cutting rules which are used to calculate correlators of operators with a large $\mathcal{R}$-charge. The questions of how the gauge theory encodes the backreaction of new background geometries and how to correctly interpret what appear to be open string excitations of the new backgrounds are answered in this section. In section 9 we explore new perturbative expansion parameters for correlators of operators with a large $\mathcal{R}$-charge along with some fascinating physical implications. In section 10 we treat the evaluation of anomalous dimensions of operators with a large $\mathcal{R}$-charge and investigate the question of integrability in the presence of non-planar corrections due to backreaction. We discuss our results in section 11. Finally, we collect some relevant technical details associated with the above sections in the appendices A, B, C and D.

2 The AdS/CFT Correspondence

2.1 Motivation

The AdS/CFT correspondence is a conjectured duality between a theory incorporating gravity and one that does not incorporate gravity [5]. The original form of the AdS/CFT correspondence as elucidated by Maldacena relates type IIB string theory defined on $AdS_5 \times S^5$ and four dimensional $\mathcal{N} = 4$ Super Yang-Mills theory. The primary motivation for the correspondence is obtained by considering a system of N parallel, coincident (or near coincident) D3 branes in Type IIB string theory and then taking a low energy limit. This system of D3 branes admits two possible descriptions. The first description entails considering the D3 branes as massive charged objects acting as a source for supergravity fields (we thus consider a supergravity solution carrying D3 brane charge). Naturally, this description can only be trusted at large N and large 't Hooft coupling where the background becomes approximately flat. The second description, views the D3 branes as boundary conditions for open strings (i.e. open strings end on D-branes). It is noteworthy that in the first description, only closed strings appear whereas in the second description both open and closed strings appear.

Consider the first description, the metric for the D3 brane then has the form:

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \left(1 + \frac{R^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2).$$

Now, the low energy limit yields very long wavelength supergravity modes propagating in the bulk region where space is approximately flat (space is flat as a consequence of the fact that N and the 't Hooft coupling are large). In addition, all the modes of Type IIB string theory close to the stack of branes will be red-shifted due to the very large gravitational potential and thus appear to be shifted to low energy (from the perspective of an external observer). The supergravity modes do not interact with the D3 branes since their wavelengths are much larger than the gravitational size of the branes in the limit considered. Also, the modes very near to the horizon find it increasingly difficult to escape the gravitational potential and return to the bulk. The near horizon geometry of the D3 branes is given by taking the limit $r << R$:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_2^2 + dx_3^2) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5^2.$$

This is nothing but the metric of AdS_5xS^5 space. Thus, in this description, we have two decoupled sectors in the theory - long wavelength supergravity modes freely propagating in flat 10 dimensional Minkowski space and all the modes of Type IIB string theory in the AdS_5xS^5 geometry.

The second description, involves both open and closed strings in the Type IIB string theory in which the open strings end on D-branes. The closed string states in the bulk are described in the low energy limit by Type IIB Supergravity whereas the open string states are described in the low energy limit by $\mathcal{N} = 4$ SYM theory. In this low energy limit, the interaction Lagrangian mixing the bulk and brane sectors falls away and the two sectors again decouple, with the supergravity theory in the bulk becoming free. Thus, the two decoupled sectors are now 3+1 dimensional $\mathcal{N} = 4$ SYM theory and long wavelength supergravity modes propagating in the bulk.

In both descriptions, one of the decoupled sectors of the theory is that of long wavelength supergravity modes propagating in flat 10 dimensional Minkowski space. Since the two descriptions and their low energy physics should be equivalent, this leads us to conclude that the other sector of the descriptions should also match. Hence, the conjecture: Type IIB superstring theory on AdS_5xS^5 is dual to $\mathcal{N} = 4$ 3+1 dimensional SYM theory.

2.2 Identification of Parameters in AdS/CFT

The mapping of parameters of the string theory into those of the gauge theory (and vice versa) is as follows:

$$g_s = g_{YM}^2, \tag{2.1}$$

$$\left(\frac{R}{l_s}\right)^4 = 4\pi g_{YM}^2 N = 4\pi\lambda. \tag{2.2}$$

The parameters in the Type IIB string theory have the following description: g_s is the string coupling constant and l_s is the string length. The parameter R is the radius of curvature of the AdS_5 and S^5 spaces constituting the AdS_5xS^5 geometry. The parameters relating to the $\mathcal{N} = 4$, 3+1 dimensional SYM theory with gauge group $U(N)$ are N and g_{YM}^2, the coupling constant. The above equations explicitly demonstrate the strong-weak coupling nature of the duality. The string theory is tractable when the string coupling constant is small (which corresponds to small g_{YM}^2) and the string length is much smaller than the radius of curvature of the space (corresponding to large 't Hooft coupling, λ). This is due to the fact that when the string coupling, g_s is small higher order terms in the loop expansion can be neglected. Further, when evaluating the leading term in the loop expansion, $l_s << R$ (i.e. λ large) implies that curvature corrections can be dropped. In contrast the gauge theory would naïvely appear to be tractable only when λ is small. There are however sectors of the gauge theory where field theory results can be expressed perturbatively, even at large λ as demonstrated by BMN [16]. These results are expressed in terms of a new perturbative expansion parameter, an effective 't Hooft coupling (associated with an effective genus counting parameter replacing $\frac{1}{N}$), which can be fixed to some small value even as λ is sent to infinity. Thus, in certain sectors it is possible to compare perturbative gauge theory results to results from the string theory. In section 9.1.5 we identify new BMN-like sectors in which an effective genus counting parameter and effective 't Hooft coupling is determined.

3 Schur Polynomials

The relevant part of the Lagrangian of $\mathcal{N} = 4$ SYM in four dimensions is:

$$\mathcal{L} = \frac{1}{g^2}\mathrm{tr}\left\{ D_\mu \Phi_i D^\mu \Phi_i + \frac{1}{2}[\Phi_i, \Phi_j]^2 + \dots \right\},$$

where we have suppressed terms involving fermions or gauge bosons - in this thesis we only consider the scalar fields. This Lagrangian contains six real scalar fields, Φ_i ($i = 1...6$), transforming in the adjoint of $U(N)$. We can construct the following complex linear combinations of the real scalar fields:

$$Z = \Phi_1 + i\Phi_2,\ Y = \Phi_3 + i\Phi_4,\ X = \Phi_5 + i\Phi_6.$$

One can consider $\mathcal{N} = 4$ SYM defined on R^4 or on $R \times S^3$ ($R \times S^3$ is conformally equivalent to R^4). If considering the theory on R^4 we have two point correlators of the complex matrix fields for which the space-time dependence is determined by conformal invariance and is diagonal in the color indices:

$$\left\langle Z_{ij}\left(x_1\right) Z^\dagger_{kl}\left(x_2\right)\right\rangle \propto \frac{\delta_{il}\delta_{jk}}{\left|x_1 - x_2\right|^2}.$$

From this point forward we consider the two point functions (3.1). Any space-time dependent or constant factors have been suppressed as they play no role in the technology reviewed below or developed in later sections. Moreover, these factors can be trivially re-instated in the final results if so desired.

$$\left\langle Z_{ij} Z^\dagger_{kl}\right\rangle = \delta_{il}\delta_{jk},\ \left\langle Y_{ij} Y^\dagger_{kl}\right\rangle = \delta_{il}\delta_{jk},\ \left\langle X_{ij} X^\dagger_{kl}\right\rangle = \delta_{il}\delta_{jk}. \tag{3.1}$$

There are a number of valid bases in which the gauge invariant operators in the half BPS sector (constructed from a single matrix, Z for example) of $\mathcal{N} = 4$ SYM can be expressed. These include the trace basis, the sub-determinant basis or the Schur polynomial basis. As discussed in the introduction, the Schur polynomial basis is a particularly useful basis in which to express operators in the half BPS sector. This can be motivated both mathematically and physically. Firstly, the two point correlators of Schur polynomials are known to all orders in $1/N$ and are diagonal in the Young diagram labels of the Schur polynomials. Secondly, the existence of a product rule for Schur polynomials allows higher point correlators to be evaluated using the two point correlator result. Physically, the Schur polynomial operators facilitate a natural identification of operators in the gauge theory corresponding to giant gravitons and new background geometries on the string theory side of the AdS/CFT correspondence.

The Schur polynomial is defined as follows:

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} \dots Z^{i_{n-1}}_{i\sigma(n-1)} Z^{i_n}_{i\sigma(n)}.$$

The label R is a Young diagram of n boxes. Young diagrams of n boxes are in one-to-one correspondence with the irreducible representations of the symmetric group S_n and thus a Schur polynomial labeled by R is associated with a particular irreducible representation of the symmetric group. The factor $\chi_R(\sigma)$ is the character of $\sigma \in S_n$ in the irreducible representation R (the trace of the associated matrix representing σ in the irreducible representation R). Schur polynomials also have an interpretation in Unitary group theory. Specifically, the Schur polynomial $\chi_R(U)$ is the character of $U \in SU(N)$ in the irreducible representation R.

Since this is a half BPS operator, its classical scaling dimension receives no corrections and is simply equal to the number of matrix fields comprising the operator, $\Delta = n$. Also, as a half BPS operator, the $\mathcal{R}$ - charge, $J = \Delta = n$.

For example, explicit expressions for all the possible Schur polynomials for n=1, 2, 3 are:

$$
\begin{aligned}
\chi_{\Box} &= \mathrm{Tr}\, Z, \\
\chi_{\Box\Box} &= \frac{1}{2}\left((\mathrm{Tr}\, Z)^2 + \mathrm{Tr}\, Z^2\right), \\
\chi_{\substack{\Box\\\Box}} &= \frac{1}{2}\left((\mathrm{Tr}\, Z)^2 - \mathrm{Tr}\, Z^2\right), \\
\chi_{\substack{\Box\\\Box\\\Box}} &= \frac{1}{6}\left((\mathrm{Tr}\, Z)^3 - 3\,(\mathrm{Tr}\, Z)\left(\mathrm{Tr}\, Z^2\right) + 2\mathrm{Tr}\, Z^3\right), \\
\chi_{\substack{\Box\Box\\\Box}} &= \frac{1}{3}\left((\mathrm{Tr}\, Z)^3 - \mathrm{Tr}\, Z^3\right), \\
\chi_{\Box\Box\Box} &= \frac{1}{6}\left((\mathrm{Tr}\, Z)^3 + 3\,(\mathrm{Tr}\, Z)\left(\mathrm{Tr}\, Z^2\right) + 2\mathrm{Tr}\, Z^3\right).
\end{aligned}
$$

Obtaining the explicit form of a Schur polynomial involves determining the character of each group element in irreducible representation R as well as the associated multi-trace factor appearing in each term of the Schur polynomial. The procedure can be considerably simplified in realizing that all group elements with a certain cycle structure (a cycle of a particular length or products of cycles of particular lengths) belong to the same conjugacy class and have the same character for a given irreducible representation. Moreover it is clear that all multi-trace factors are equal for group elements belonging to a particular conjugacy class. One way to obtain the required characters is to explicitly construct the matrices representing the group elements of S_n in irreducible representation R. A description of an algorithm for building these representations is given in [55] for example. Alternatively, the strand diagram technology introduced in [33] (summarized in section 5.1) furnishes a neat, graphical way to determine the irreducible representations of S_n.

Fortunately however, the Schur polynomial technology we utilize does not necessitate treating the polynomials at the level of individual terms and instead the techniques are stated in terms of the Young diagram labels. In other words the Young diagram labels (possibly with a multiplicity label as is the case with the restricted Schur polynomials) encode all the requisite information.

3.1 Two Point Correlators

In [17] the following result for the exact two point correlation function of Schur polynomials in the free field theory limit was obtained:

$$\langle \chi_R(Z)\, \chi_S\left(Z^{\dagger}\right) \rangle = \delta_{RS} f_R.$$

The correlator is diagonal in the Young diagram labels i.e. it is only non-zero when $R = S$. In this result, the quantity f_R denotes the product of what are termed weights of the boxes of the Young diagram labeling the Schur polynomial (these are not Dynkin weights). The weight of a box in the ith row and jth column is given by $N - i + j$ and thus the product of weights of all the boxes comprising the young diagram is:

$$f_R = \prod_{i,j} (N - i + j).$$

Figure 1 illustrates the assignment of weights to boxes of a Young diagram.

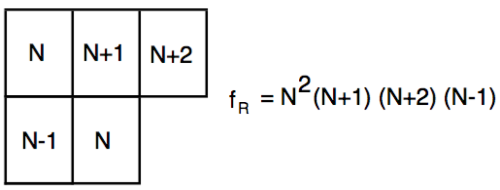


Figure 1: Illustration of weights of boxes of a Young diagram.

The result in [17] was obtained by exploiting the link between the symmetric group and the unitary group and the quantity f_R appears in the dimension of the irreducible representation of the unitary group labeled by R, $\mathrm{Dim}_N(R)$,

$$\mathrm{Dim}_N(R) = \frac{f_R}{\mathrm{Hooks}_R}.$$

Here Hooks_R is the product of the hook lengths for each box in the Young diagram R. The hook length of a particular box can be obtained by drawing an elbow (hook) in the box and counting how many boxes this elbow passes through. An example of a box with a hook length of four is illustrated in figure 2.

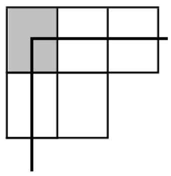


Figure 2: The shaded box has a hook length of 4.

An example of the application of this result is:

$$\langle \chi_{\tiny\yng(3,1)}(Z)\,\chi_{\tiny\yng(3,1)}\big(Z^\dagger\big)\rangle = N(N^2-1)(N+2).$$

3.2 Product Rule

The Schur polynomial $\chi_R(U)$ gives the character of $U \in SU(N)$ in the $SU(N)$ irreducible representation labeled by Young diagram R. The representation obtained by taking the direct product of two irreducible $SU(N)$ representations R_1 and R_2 is in general reducible. The number of times that irreducible representation T appears is given by the Littlewood-Richardson number $f_{R_1\,R_2;T}$. Using the fact that the Schur polynomials compute characters, it is clear that

$$\chi_{R_1}(Z)\chi_{R_2}(Z) = \sum_T f_{R_1\,R_2;T}\chi_T(Z).$$

Using this product rule, it is easy to compute multipoint functions in terms of two point functions. For example, for three point functions we have

$$\langle \chi_{R_1}(Z)\chi_{R_2}(Z)\chi_S(Z)^\dagger\rangle = \sum_T f_{R_1\,R_2;T}\langle\chi_T(Z)\chi_S(Z)^\dagger\rangle\,,$$

and for four point functions

$$\begin{aligned}\langle \chi_{R_1}(Z)\chi_{R_2}(Z)\chi_{R_3}(Z)\chi_S(Z)^\dagger\rangle &= \sum_T f_{R_1\,R_2;T}\langle\chi_T(Z)\chi_{R_3}(Z)\chi_S(Z)^\dagger\rangle \\ &= \sum_T\sum_U f_{R_1\,R_2;T}f_{R_3\,T;U}\langle\chi_U(Z)\chi_S(Z)^\dagger\rangle\,.\end{aligned}$$

Since we are not explicitly displaying the spacetime dependence, it is important to note that if there are more than one of both holomorphic and antiholomorphic operators at different spacetime points then the spacetime dependence does not simply factor out of the above correlators and an analysis of the spacetime dependence has to be performed (this is known as a non-extremal correlator). In the case of extremal correlators, the above results are in perfect agreement with the exact computations of [17],[19]. It is also clear that we can compute n-point functions knowing only the product rule and the two point functions.

4 Schur Polynomials and Giant Gravitons

In [17] Schur polynomials comprised of $O(N)$ fields were proposed as the natural gauge theory duals to half BPS giant gravitons. These giant gravitons are half BPS D3-branes which are extended in the S^5 or AdS_5 space of the AdS_5xS^5 background [13, 14]. Sphere giant gravitons are D3-branes wrapping an $S^3 \subset S^5$ of the AdS_5xS^5 background, whereas AdS giant gravitons wrap an $S^3 \subset AdS_5$ of the AdS_5xS^5 background. The giant gravitons are gravitons propagating in the bulk that have been spatially extended (blown up) as a consequence of the Myers effect [12]. The magnitude of the enlargement is determined by the angular momentum of the giant graviton [13] through equation (1.1). In the case of the sphere giant gravitons expanding into the S^5 space, the gravitons can only expand until they reach the maximum radius possible - that of the S^5 itself, R. This implies an upper bound on angular momentum of the giant, corresponding to N. The Schur polynomials encode this effect, known as the Stringy Exclusion Principle [20, 21], in the gauge theory. If we identify Schur polynomials labeled by Young diagrams of a single column (the fully anti-symmetric representation) with sphere giant gravitons then this cutoff in angular momentum is manifest. This is due to the fact that the number of boxes in the column is identified with the angular momentum of the giant and there can be at most N rows in a Young diagram characterizing an irreducible representation of $U(N)$. For a giant graviton expanded in the AdS_5 space of the AdS_5xS^5 background, we see that identification with a Schur polynomial labeled by a Young diagram consisting of a single row (the fully symmetric representation) is also consistent with the dual gravitational theory. This identification does not place a limit on the size of the giant in AdS_5 space and rightly so (the giant's size should be unbounded in this case) but it does place a limit on the number of AdS giants in that the Young diagram can have at most N rows and thus N AdS giants. This is consistent with the gravitational theory in that one unit of five form flux is lost when passing through each AdS giant. Thus, if there are N or more AdS giants the flux at the center of the AdS space will become zero or negative. A positive five form flux is required to support an AdS giant however. Given the identification of a sphere giant graviton with a Schur polynomial labeled by a Young diagram which is a column of $O(N)$ boxes, it seems natural to identify a Young diagram with $O(1)$ columns in which each column has $O(N)$ boxes as a bound state of sphere giants. Similarly, we can identify a Schur polynomial labeled by a Young diagram of $O(1)$ rows each containing $O(N)$ boxes with a bound state of AdS giants.

5 Restricted Schur Polynomials - Distinguishable Matrix Words

Restricted Schur polynomials are the multi-matrix generalization of the single matrix Schur polynomials described above. Restricted Schur polynomials are constructed from more than one type of matrix field (Z and Y for example), individually or combined into matrix words. These matrix words are simply products of matrix fields, termed letters, which could, in principle, include the complex matrix fields, fermions or covariant derivatives of these fields. To be identified with an open string excitation, the matrix word must be comprised of $O(\sqrt{N})$ letters. In the case of distinguishable matrix words, these matrix words are all distinct. This is the first case that we will consider. The case with indistinguishable matrix fields or words will be reviewed in section 6.

The restricted Schur basis permits a very natural field theory description of open-string excitations of giant gravitons across the AdS/CFT correspondence [28, 30, 31, 32, 33]. On the string theory side, exciting a giant graviton corresponds to attaching open strings to the giant. In the dual field theory, the Schur polynomial operators described previously are extended to include open strings in their description in the following way. Firstly, for each open string attached to the giant we replace a matrix Z in the Schur polynomial by an open string word denoted by $(W^a)^i_j$. Secondly, the overall coefficient of the polynomial is modified from $\frac{1}{n!}$ to $\frac{1}{(n-k)!}$, where k is the number of open strings attached. Finally, the conventional trace operation used in obtaining the character is replaced with an operation known as a restricted trace, yielding what is termed a restricted character. This concept will be elaborated upon in section 5.1 below. The resultant operator is known as a restricted Schur polynomial:

$$\chi^{(k)}_{R,R_1}(Z, W^{(1)}, \ldots, W^{(k)}) = \frac{1}{(n-k)!} \sum_{\sigma \in S_n} Tr_{R_1}(\Gamma_R(\sigma)) Tr(\sigma Z^{\otimes n-k} W^{(k)} \ldots W^{(1)}),$$

$$Tr(\sigma Z^{\otimes n-k} W^{(k)} \dots W^{(1)}) = Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} \dots Z^{i_{n-k}}_{i\sigma(n-k)} (W^{(k)})^{i_{n-k+1}}_{i\sigma(n-k+1)} \dots (W^{(1)})^{i_n}_{i\sigma(n)}.$$

In this definition, R_1 is an irreducible representation of S_{n-k} and we therefore associate it with a Young diagram with n-k boxes. For the identification of the restricted Schur polynomial with an excited giant graviton to be sensible, n must be O(N) and the number of open string words, k must be O(1). In the above definition, $W^{(1)} \dots W^{(k)}$ are all distinct.

5.1 Restricted Characters

A restricted character is given by taking a restricted trace of a group element. By a restricted trace, we mean that we don't trace over the whole carrier space on which the group acts; we trace only over a subspace

$$\chi_{R,R_1}(\sigma) = \mathrm{Tr}_{R_1}(\Gamma_R(\sigma)).$$

R is an irreducible representation of S_n; we can think of R as a Young diagram with n boxes. The subspace R_1 is the carrier space of a subgroup of S_n. Consequently, a convenient way to specify which subspace of the full space we consider, is by knocking boxes off the Young diagram R; the smaller Young diagram is R_1. Finally, we also need to consider restricted characters in which the row and column indices are traced over different subspaces. In this case, we compute

$$\chi_{R,R_1R_2}(\sigma) = \mathrm{Tr}_{R_1R_2}(\Gamma_R(\sigma))$$

by summing the row index over R_1 and the column index over R_2. This requires that we have an isomorphism between R_1 and R_2 because we need to correlate the row and column indices in the sum. This isomorphism amounts to a choice of basis and is specified by requiring for σ in the subgroup of which R_1 and R_2 are irreducible representations, we have $\Gamma_{R_1}(\sigma) = \Gamma_{R_2}(\sigma)$. We represent these subspaces graphically by drawing R as a Young diagram and placing two labels in each box to be dropped. If a total of m boxes are to be dropped the labels run from 1 to m. To get the row (column) subspace R_1 (R_2) drop boxes from R according to the upper (lower) index in each box.

Consider an irreducible representation of S_5, corresponding to R = $\yng(2,2,1)$ for example. The following are examples of the graphical notation used to indicate tracing over the row and column indices of one particular subspace (the subspace identified by the particular chain of subductions of R where the box labeled 1 is removed first, then 2):

$$\chi_{\ytableaushort{{}{},{}1,2}}(\sigma), \qquad \chi_{\ytableaushort{{}{},{}2,1}}(\sigma).$$

The restricted Schur polynomials associated with restricted characters of this form are identified with systems of excited giant gravitons provided the number of Z's comprising the operator is $O(N)$ and the number of open string words (each comprised of $O(\sqrt{N})$ fields) is $O(1)$. In this case, the operators defined using such restricted characters are identified with bound states of excited giant gravitons in which any given open string has endpoints attached to a single giant graviton.

Below are examples of the graphical notation used to indicate tracing row and column indices over different subspaces:

$$\chi_{\ytableaushort{{}{},{}{\substack{1\\2}},{\substack{2\\1}}}}(\sigma), \qquad \chi_{\ytableaushort{{}{},{}{\substack{2\\1}},{\substack{1\\2}}}}(\sigma)$$

Again, the restricted Schur polynomials associated with restricted characters of the above form are identified with systems of excited giant gravitons provided we are in the regime described above. The operators defined using restricted characters of this form (tracing over off-diagonal blocks of $\Gamma_R(\sigma)$) are identified with bound states of excited giant gravitons in which we have open strings stretching between the giant gravitons, which we henceforth refer to as stretched string states. The pairs of labels in each box of the Young diagram are naturally identified as Chan-Paton factors for open strings.

In [33] general formulas for restricted characters were obtained. We briefly review these methods below.

The general algorithm used to compute restricted characters has three steps:

- Decompose the group element whose trace is to be computed into a product of two cycles of the form $\Gamma_R\left((i, i+1)\right)$. Insert a complete set of states between each factor.
- The only non-zero matrix elements of each $\Gamma_R\left((i, i+1)\right)$ factor, are obtained when the order of boxes dropped to obtain the carrier space of the bra matches the order of boxes dropped to obtain the carrier space of the ket, except for the $(n-i+1)^{\text{th}}$ and $(n-i)^{\text{th}}$ boxes, whose order can be swapped.
- The known value of the matrix elements for precisely the two cases arising in the previous point are plugged in to get the value of the restricted character.

A very convenient way to implement this algorithm is by using *strand diagrams* [33]. If, after factorizing the group element as described in the first point above, p indices are involved, we draw a picture with p columns. The columns are populated by labeled strands - each strand represents one of the boxes that are to be dropped from the Young diagram. Label the strands by the upper index in the box. The box corresponding to the strand that appears in the first column is to be dropped first; the box corresponding to the strand in the second column is to be dropped second and so on. The strands are ordered at the top of the diagram, according to the order in which they must be dropped to get the row index. The strands are ordered at the bottom of the diagram according to the column index. The strands move from the top of the diagram to the bottom of the diagram, without breaking, so that strand ends at the top connect to the corresponding strand ends at the bottom. To connect the strands (which in general are in a different order at the top and bottom of the diagram) we need to weave the strands, thereby allowing them to swap columns. The allowed swaps depends on the specific group element whose trace we are computing. To determine the allowed swaps, write the group element as a product of cycles of the form $(i, i+1)$ i.e. decompose the group element into a product of adjacent transpositions. Each cycle $(i, i+1)$ is drawn as a box which straddles the columns $n-i$ and $n-i+1$ where our original group element belongs to S_n. Boxes for cycles on the right in the decomposition into adjacent transpositions are drawn above boxes for cycles on the left. When the strands pass through a box, they may do so without swapping or by swapping columns. Each box is associated with a factor. Imagine that the strands passing through the box, reading from left to right, are labeled n and m. The weights associated with these boxes are c_n and c_m respectively. If the strands do not swap inside the box the factor for the box is

$$f_{\text{no swap}} = \frac{1}{c_n - c_m}.$$

If the strands do swap inside the box, the factor is

$$f_{\text{swap}} = \sqrt{1 - \frac{1}{(c_n - c_m)^2}}.$$

Denote the product of the factors, one from each box, by F. We have

$$\mathrm{Tr}_{R_1,R_2}\Big(\Gamma_R(\sigma)\Big) = \sum_i F_i \mathrm{dim}_{R_1},$$

where the index i runs over all possible paths consistent with the boundary conditions.

With a little thought, the astute reader should be able to convince herself that this graphical rule is nothing but a convenient representation of the algorithm given above. We end with an example. The character

$$\chi_1 = \mathrm{Tr}_{\begin{ytableau} \ & \ & {}^{1}_{3} \\ \ & {}^{2}_{1} \\ {}^{3}_{2} \end{ytableau}}\Big(\Gamma_{\ydiagram{3,2,1}}\big((6,4)\big)\Big).$$

is represented by the strand diagram of figure 3. To obtain this strand dia-

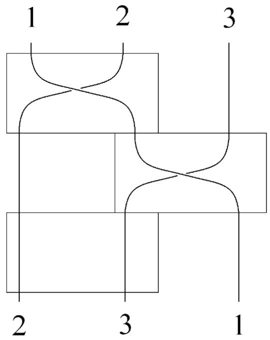


Figure 3: Strand diagram for calculating restricted character of $(6,4)$.

gram write $(6,4) = (6,5)(4,5)(6,5)$. The factors for the upper most, middle and lower most boxes are $\sqrt{1-\frac{1}{(c_1-c_2)^2}}$, $\sqrt{1-\frac{1}{(c_1-c_3)^2}}$, and $\frac{1}{c_2-c_3}$ respectively. Thus,

$$\begin{aligned} \chi_1 &= \sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_1-c_3)^2}}\frac{1}{c_2-c_3}\dim_{\ydiagram{2,1}} \\ &= 2\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_1-c_3)^2}}\frac{1}{c_2-c_3}. \end{aligned}$$

For further details and more examples, see [33].

5.2 Two Point Correlators

The result of evaluating the two point correlators of restricted Schur polynomials with distinguishable open string words was derived in [31]. These results are valid as long as the number of Z's comprising the restricted Schur polynomial operator is less than $O(N^2)$ and the number of Z's in any open string word is $O(1)$. In this regime contractions between the open string word and the rest of the operator can be neglected. In the case where the number of Z's comprising the restricted Schur polynomial operator is $O(N^2)$, these contractions can no longer be neglected. Indeed, the interpretation of the restricted Schur polynomial operators changes and they are now identified with new classical backgrounds with string excitations on the string theory side of the AdS/CFT correspondence. The technology to evaluate correlators in this regime is developed in section 8, along with an interpretation for the contractions that can no longer be neglected. Further, we shed light on the apparent conundrum presented by what appear to be open string excitations in the absence of giant gravitons.

In the language of [31], for each open string word the most general form that the two point function can take is:

$$\langle (W^a)^i_j (W^{a\dagger})^k_l \rangle = F_0^a \delta^i_l \delta^k_j + F_1^a \delta^i_j \delta^k_l. \tag{5.1}$$

Where F_0^a and F_1^a are dependent on the precise composition of the open string word. In evaluating the result of correlators involving multiple open string words, contractions which mix four or more words (corresponding to string interactions) are dropped and only pairwise contractions are considered. Moreover, it is assumed that:

$$\langle (W^a)^i_j (W^{b\dagger})^k_l \rangle \propto \delta^{ab}$$

which will mean that only a single pairing contributes.

Before stating the result of the correlator, we need to introduce some notation and requisite technology, namely the reduction operation which corresponds to the action of $D_W \equiv \frac{d}{dW_i^i}$ on a restricted Schur polynomial. The result of this operation, along with subtleties that may require the application of what is termed the subgroup swap rule are discussed in some detail in the next section.

5.2.1 Reduction Rule

In this section we will consider the action of

$$\mathrm{Tr}\left(\frac{d}{dZ}\right) \equiv D_Z, \quad \text{and} \quad \mathrm{Tr}\left(\frac{d}{dW^{(k)}}\right) \equiv D_{W^{(k)}},$$

on restricted Schur polynomials. We call these "reductions" of the restricted Schur polynomial because the action of the operators removes boxes from the Young diagram label of the polynomial. The action of $D_{W^{(k)}}$ on a restricted Schur polynomial has been worked out in [31] and is very simply stated (provided we are reducing with respect to the word associated with the index to be fixed first in restriction). It removes the box associated with $W^{(k)}$ (the box labeled by k in the Young diagram) and multiplies the resultant polynomial by the weight of the removed box (the definition of the weight of a box is provided in section 3.1).

For example:

$$D_{W^{(1)}}\chi_{\young(~~1,~~)} = (N+2)\chi_{\young(~~,~~)}.$$

One subtlety to consider is that if we have a stretched string state involving the open string with the smallest label and we reduce with respect to that open string word then the result of the reduction vanishes i.e.

$$D_{W^{(1)}}\chi_{\young(~~{\substack{1\\2}},~{\substack{2\\1}})} = 0.$$

However, it is not true in general that:

$$D_{W^{(2)}}\chi_{\ytableaushort{{}{}{\substack{1\\2}},{}{\substack{2\\1}}}} = 0.$$

Correctly evaluating the result of this reduction requires the application of the subgroup swap rule of [31], reviewed in the section 5.2.2.

Acting D_Z on an ordinary Schur polynomial produces all Schur polynomials that can be obtained by removing a single box[57, 31]. Each of the polynomials produced are multiplied by the weight of the removed box. For example:

$$D_Z\chi_{\yng(3,1)} = (N+2)\chi_{\yng(2,1)} + (N-1)\chi_{\yng(3)}$$

Finally, we will evaluate the action of D_Z on a restricted Schur polynomial. By explicitly evaluating the derivative, we have

$$\begin{aligned} \frac{d}{dZ^a_a}\chi^{(1)}_{R,R_1}(Z,W) &= \frac{1}{(n-2)!}\sum_{\sigma\in S_n}\mathrm{Tr}\,(\Gamma_R(\sigma))\,Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}\delta^{i_{n-1}}_{i_{\sigma(n-1)}}W^{i_n}_{i_{\sigma(n)}} \\ &= D_X\sum_\alpha \chi^{(2)}_{R,T_\alpha}(Z,X,W), \end{aligned} \tag{5.2}$$

where in the restricted Schur polynomial $\chi^{(2)}_{R,T_\alpha}(Z,X,W)$, W is associated with the box that must be removed from R to obtain R_1 and X is associated with the box that must be removed from R_1 to obtain T_α. In this last formula, the representations T_α are all representations that can be obtained by removing a single box from R_1, so that

$$R_1 = \oplus_\alpha T_\alpha.$$

The reduction with respect to X in ((5.2)) is now easily computed using the subgroup swap rule.

5.2.2 Subgroup Swap Rule

Consider the definition provided previously for a restricted Schur polynomial with k distinguishable open string words, $W^{(1)}...W^{(k)}$. In order to reduce with respect to the open string which does not have the smallest label we need to apply the subgroup swap rule. Thus far, we have assumed that in restricting to the $S_{n-k}\otimes(S_1)^k$ subgroup we have first restricted to the $S_{n-1}\otimes(S_1)$ subgroup that leaves the index of $W^{(1)}$ (i.e. n) inert, then further restricted to the $S_{n-2}\otimes(S_1)^2$ subgroup that leaves the index of $W^{(2)}$ (i.e. n-1) inert and so forth. This is denoted as:

$$\chi_{R,R^{(k)}}|_1|_2\cdots|_k.$$

If however, we were to first restrict to the subgroup that leaves the index of $W^{(2)}$ inert and then further restrict to the subgroup that leaves the index of $W^{(1)}$ inert, we would in general obtain a different polynomial which is denoted:

$$\chi_{R,R^{(k)}}|_2|_1\cdots|_k.$$

These two polynomials are related through the subgroup swap rule of [31]. In general, the subgroup swap rule relates any two restricted Schur polynomials that differ in the interchange of two adjacent indices indicating the order of restriction. Of course any ordering of the indices can be related to any other ordering by successive applications of the subgroup swap rule.

Returning to the importance of the subgroup swap rule to the reduction rule, note that the reduction rule only yields the particularly simple result of removing the appropriate box of the Young diagram labeling the restricted Schur polynomial and multiplying by the weight of the removed box if we are reducing with respect to the index that appears first in the order of restrictions. If not, we must apply the subgroup swap rule until the appropriate index appears first.

To state the rule we utilize the graphical notation introduced previously (in specifying the restricted character). Note that:

$$\chi_{\begin{array}{|c|c|}\hline & \\ \hline & 1 \\ \hline 2 \\ \cline{1-1}\end{array}}\Big|_1\Big|_2 \equiv \chi_{\begin{array}{|c|c|}\hline & \\ \hline & {}^{1}_{1} \\ \hline {}^{2}_{2} \\ \cline{1-1}\end{array}}\Big|_1\Big|_2.$$

Now, we denote the weight of the box with the index that is fixed first (before the swap) in the upper left hand corner as c_1^U. We denote the weight of the box with the index that is fixed first (before the swap) in the lower right hand corner as c_1^L. The weight of the box with the index that is fixed second (before the swap) in the upper left hand corner is denoted as c_2^U. The weight of the box with the index that is fixed second (before the swap) in the lower right hand corner is denoted as c_2^U. We define the following swap factors associated with swapping a pair of upper or lower indices:

$$S^U = \frac{1}{c_1^U - c_2^U}, \qquad S^L = \frac{1}{c_1^L - c_2^L}.$$

We define the following no swap factors associated with leaving a pair of upper or lower indices inert:

$$N^U = \sqrt{1 - \frac{1}{(c_1^U - c_2^U)^2}}, \qquad N^L = \sqrt{1 - \frac{1}{(c_1^L - c_2^L)^2}}.$$

The subgroup swap rule states that the result of swapping the order of two adjacent indices indicating the order of restriction is captured by implementing all possible swaps of the upper and lower labels corresponding to those indices and including the appropriate swap or no swap factors. For example:

$$
\begin{aligned}
\chi_{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & 1 \\ \hline \ & \ & \substack{2\\3} \\ \cline{1-3} \substack{3\\2} \\ \cline{1-1}\end{array}}|_1|_3|_2 &= N^U N^L \chi_{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & 1 \\ \hline \ & \ & \substack{2\\3} \\ \cline{1-3} \substack{3\\2} \\ \cline{1-1}\end{array}}|_3|_1|_2 + N^U S^L \chi_{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \substack{1\\3} \\ \hline \ & \ & \substack{2\\1} \\ \cline{1-3} \substack{3\\2} \\ \cline{1-1}\end{array}}|_3|_1|_2 \\
&+ S^U N^L \chi_{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \substack{3\\1} \\ \hline \ & \ & \substack{2\\3} \\ \cline{1-3} \substack{1\\2} \\ \cline{1-1}\end{array}}|_3|_1|_2 + S^U S^L \chi_{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & 3 \\ \hline \ & \ & \substack{2\\1} \\ \cline{1-3} \substack{1\\2} \\ \cline{1-1}\end{array}}|_3|_1|_2 \\
&= \frac{3\sqrt{2}}{5} \chi_{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & 1 \\ \hline \ & \ & \substack{2\\3} \\ \cline{1-3} \substack{3\\2} \\ \cline{1-1}\end{array}}|_3|_1|_2 + \frac{\sqrt{6}}{5} \chi_{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \substack{1\\3} \\ \hline \ & \ & \substack{2\\1} \\ \cline{1-3} \substack{3\\2} \\ \cline{1-1}\end{array}}|_3|_1|_2 \\
&+ \frac{\sqrt{3}}{10} \chi_{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \substack{3\\1} \\ \hline \ & \ & \substack{2\\3} \\ \cline{1-3} \substack{1\\2} \\ \cline{1-1}\end{array}}|_3|_1|_2 + \frac{1}{10} \chi_{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & 3 \\ \hline \ & \ & \substack{2\\1} \\ \cline{1-3} \substack{1\\2} \\ \cline{1-1}\end{array}}|_3|_1|_2 .
\end{aligned}
$$

Where:

$$
N^U = \sqrt{1 - \frac{1}{(c_1^U - c_3^U)^2}} = \frac{2\sqrt{6}}{5}, \quad N^L = \sqrt{1 - \frac{1}{(c_1^L - c_3^L)^2}} = \frac{\sqrt{3}}{2},
$$

$$
S^U = \frac{1}{(c_1^U - c_3^U)} = \frac{1}{5}, \quad S^L = \frac{1}{(c_1^L - c_3^L)} = \frac{1}{2}.
$$

An example of a reduction operation requiring the application of the subgroup swap rule is:

$$
\begin{aligned}
D_{W^{(3)}}\chi_{\young(\ \ \ 1,\ \ {{}^{2}_{3}},{{}^{3}_{2}})}\big|_{1}\big|_{3}\big|_{2} &= \frac{3\sqrt{2}}{5}D_{W^{(3)}}\chi_{\young(\ \ \ 1,\ \ {{}^{2}_{3}},{{}^{3}_{2}})}\big|_{3}\big|_{1}\big|_{2} + \frac{\sqrt{6}}{5}D_{W^{(3)}}\chi_{\young(\ \ \ {{}^{1}_{3}},\ \ {{}^{2}_{1}},{{}^{3}_{2}})}\big|_{3}\big|_{1}\big|_{2} \\
&+ \frac{\sqrt{3}}{10}D_{W^{(3)}}\chi_{\young(\ \ \ {{}^{3}_{1}},\ \ {{}^{2}_{3}},{{}^{1}_{2}})}\big|_{3}\big|_{1}\big|_{2} + \frac{1}{10}D_{W^{(3)}}\chi_{\young(\ \ \ 3,\ \ {{}^{2}_{1}},{{}^{1}_{2}})}\big|_{3}\big|_{1}\big|_{2} \\
&= 0+0+0+\frac{1}{10}(N+3)\chi_{\young(\ \ \ ,\ \ {{}^{2}_{1}},{{}^{1}_{2}})}\big|_{1}\big|_{2} \\
&= \frac{1}{10}(N+3)\chi_{\young(\ \ \ ,\ \ {{}^{2}_{1}},{{}^{1}_{2}})}\big|_{1}\big|_{2}.
\end{aligned}
$$

5.2.3 Two Point Correlator Result and Example

Now, given the form of the open string two point function ((5.1)), it is easy to see that upon Wick contracting a particular open string word, the correlator can be split into a sum of terms, one with a coefficient of F_0^a and the other with a coefficient of F_1^a. The first of these contributions corresponds to implementing the delta function index structure associated with F_0^a in the open string two point function. The latter contribution corresponds to implementing the delta function index structure associated with F_1^a in the open string two point function. An algorithm for keeping track of and ultimately evaluating the resultant contributions of both types for one or more open string words was worked out in [31]. The F_0^a contribution is said to correspond to the case where the open string word W^a has been 'glued'. The F_1^a contribution turns out to correspond to taking a reduction with respect to the open string word W^a. For restricted Schur correlators involving multiple open string words, the F_0 and F_1 type contributions from each open string must be evaluated, in what is essentially a recursive procedure that is applied until we are left with correlators in which all strings are glued or no strings remain after successive reductions. In the latter case, the two point correlator result for ordinary Schur polynomials can be applied. In the case of correlators involving one or more glued strings, the following result ([31]) is applied:

For restricted Schur polynomials involving restricted traces where we trace over the row and column indices of a particular subspace:

$$\langle \chi_{R,R^{(n)}} \chi^{\dagger}_{S,S^{(n)}} \rangle|_{\text{glued}} = \frac{(\text{Hooks})_R}{(\text{Hooks})_{R_n}} f_R \delta_{R\to R^{(n)},S\to S^{(n)}}.$$

For restricted Schur polynomials involving restricted traces where we trace over row and column indices of different subspaces:

$$\langle \chi_{R,R^{(n)}T^{(n)}} \chi^{\dagger}_{S,S^{(n)}U^{(n)}} \rangle|_{\text{glued}} = \frac{(\text{Hooks})_R}{(\text{Hooks})_{R_n}} f_R \delta_{R\to (R^{(n)}T^{(n)}),S\to (S^{(n)}U^{(n)})}.$$

These results are applicable for n glued strings. The delta function $\delta_{R\to R^{(n)},S\to S^{(n)}}$ indicates that, in addition to R matching S and $R^{(n)}$ matching $S^{(n)}$, all the irreducible representations appearing in intermediate stages of subducing $R^{(n)}$ from R must precisely match the irreducible representations appearing in intermediate stages of subducing $S^{(n)}$ from S. Similarly for $\delta_{R\to (R^{(n)}T^{(n)}),S\to (S^{(n)}U^{(n)})}$.

For example, bringing it all together (note that if an open string is glued it is denoted graphically by an arrow above the associated open string label in the Young diagram) :

$$
\begin{aligned}
\langle \chi_{\ytableaushort{{}1,{},2}} \chi^{\dagger}_{\ytableaushort{{}1,{},2}} \rangle &= F^1_0 \langle \chi_{\ytableaushort{{}{\overrightarrow{1}},{},2}} \chi^{\dagger}_{\ytableaushort{{}{\overrightarrow{1}},{},2}} \rangle + F^1_1 \langle D_{W^{(1)}} \chi_{\ytableaushort{{}1,{},2}} (D_{W^{(1)}} \chi_{\ytableaushort{{}1,{},2}})^{\dagger} \rangle \\
&= F^1_0 F^2_0 \langle \chi_{\ytableaushort{{}{\overrightarrow{1}},{},{\overrightarrow{2}}}} \chi^{\dagger}_{\ytableaushort{{}{\overrightarrow{1}},{},{\overrightarrow{2}}}} \rangle + F^1_0 F^2_1 \langle D_{W^{(2)}} \chi_{\ytableaushort{{}{\overrightarrow{1}},{},2}} (D_{W^{(2)}} \chi_{\ytableaushort{{}{\overrightarrow{1}},{},2}})^{\dagger} \rangle \\
&+ F^1_1 F^2_0 \langle D_{W^{(1)}} \chi_{\ytableaushort{{}1,{},{\overrightarrow{2}}}} (D_{W^{(1)}} \chi_{\ytableaushort{{}1,{},{\overrightarrow{2}}}})^{\dagger} \rangle \\
&+ F^1_1 F^2_1 \langle D_{W^{(2)}} D_{W^{(1)}} \chi_{\ytableaushort{{}1,{},2}} (D_{W^{(2)}} D_{W^{(1)}} \chi_{\ytableaushort{{}1,{},2}})^{\dagger} \rangle \\
&= F^1_0 F^2_0 (4N(N^2-1)(N-2)) \\
&+ F^1_0 F^2_1 (\frac{32}{27} N(N^2-1)(N-2)^2 + \frac{1}{27} N(N^2-1)(N+1)(N-2)) \\
&+ F^1_1 F^2_0 (3N(N^2-1)(N+1)(N-2)) \\
&+ F^1_1 F^2_1 (N(N^2-1)(N+1)(N-2)^2).
\end{aligned}
$$

6 Restricted Schur Polynomials - Indistinguishable Matrix Words

We now consider the case where the matrix words comprising the restricted Schur polynomial are indistinguishable. For example we could have an operator constructed from two of the Higgs fields, Z and X say. The definition of the restricted Schur polynomial in this case is[27]:

$$\chi_{R,R_\alpha}(Z,X) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\sigma)) \mathrm{Tr}(\sigma Z^{\otimes n} \otimes X^{\otimes m}),$$

$$\mathrm{Tr}(\sigma Z^{\otimes n} \otimes X^{\otimes m}) = Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} \dots Z^{i_n}_{i\sigma(n)} X^{i_{n+1}}_{i\sigma(n+1)} \dots X^{i_{n+m}}_{i\sigma(n+m)}.$$

In this definition, R is an irreducible representation of S_{n+m} and is associated with a Young diagram of $n+m$ boxes. R_α is an irreducible representation of $S_n \times S_m$ and is therefore specified by two Young diagrams, one comprised of n boxes and the other of m boxes ($R_\alpha \equiv (r_{\alpha 1}, r_{\alpha 2})$). The $S_n \times S_m$ subgroup is the subgroup for which S_n acts on the n indices of the Z's and S_m acts on the m indices of the X's. Again, $\mathrm{Tr}_{R_\alpha}(\Gamma_R(\sigma))$ indicates taking a restricted trace of the group element $\sigma \in S_{n+m}$ in the irreducible representation R. The restricted trace operation corresponds to only tracing over the subspace corresponding to R_α. Note that this definition is easily generalized to more than two types of indistinguishable matrix comprising the restricted Schur polynomial operator.

6.1 Two Point Correlators

In [27] the following result for the two point correlation function of the restricted Schur polynomials was obtained:

$$
\begin{aligned}
\langle \chi_{R,(r_{\alpha_1},r_{\alpha_2})}\chi^{\dagger}_{S,(s_{\beta_1},s_{\beta_2})}\rangle &= \delta_{RS}\delta_{r_{\alpha_1}s_{\beta_1}}\delta_{r_{\alpha_2}s_{\beta_2}}\frac{(\text{hooks})_R}{(\text{hooks})_{R_\alpha}}f_R \\
&= \delta_{RS}\delta_{r_{\alpha_1}s_{\beta_1}}\delta_{r_{\alpha_2}s_{\beta_2}}\frac{(\text{hooks})_R}{(\text{hooks})_{r_{\alpha_1}}(\text{hooks})_{r_{\alpha_2}}}f_R\,.
\end{aligned}
$$

One of the most useful features of the Schur polynomials was the fact that higher point correlators could be calculated using only the two point function and a product rule. Given the two point correlator for restricted Schur polynomials stated above, a product rule allowing higher point functions to be calculated would be a very powerful tool. This product rule is derived in the next section and applied to studying the strength of graviton interactions for the case of single trace operators built out of more than one type of matrix field. This allows us to determine whether the identification of such traces with graviton degrees of freedom is only sensible when the number of fields comprising the operator is $O(1)$ as was the case in the half BPS sector. In addition, the product rule facilitates an interpretation of the multi-matrix restricted Schur polynomials in terms of ordinary Schur polynomials.

7 A Product Rule for Restricted Schur Polynomials

In this section we will argue that the restricted Schur polynomials satisfy a simple product rule. In section 7.1 we will introduce the concept of the dual character which will be be used in defining the product rule in section 7.2. We focus on polynomials built using two complex Higgs fields, X and Z. The analogous product rule for the Schur polynomials is determined by the Littlewood-Richardson numbers. For this reason, we call the numbers that enter our rule *restricted Littlewood-Richardson numbers*. In section 7.3 we develop techniques that can be used to evaluate the restricted Littlewood-Richardson numbers. Essentially, we perform a change of basis so that in the new basis we can compute the restricted characters using the strand diagram techniques developed in [33]. These methods are not very efficient and one is not able, in general, to effectively deal with restricted Schur polynomials which have both a large number of X fields and a large number of Z fields. There are however special cases for which we can obtain explicit results. These results are used in section 7.4 to describe some aspects of the fully quantum mechanical bulk gravity. In particular, we provide evidence that the giant gravitons (restricted Schur polynomials for the totally antisymmetric representations) furnish suitable high energy degrees of freedom, extending the half BPS studies of [34]. In section 7.5 we present the evidence for our proposal that the composite operator $\chi_{R,(r_n,r_m)}(Z,X)$ is constructed from the half BPS "partons" $\chi_{r_n}(Z)$ and $\chi_{r_m}(X)$. Finally, we collect some relevant technical details in appendix A.

7.1 Dual Characters

To begin, we review Appendix I of [31]. If two permutations σ, τ satisfy

$$\mathrm{Tr}\,(\sigma Z^{\otimes n} \otimes X^{\otimes m}) = \mathrm{Tr}\,(\tau Z^{\otimes n} \otimes X^{\otimes m}),$$

we say they are *restricted conjugate*. Restricted conjugate is an equivalence relation. Clearly two elements σ and τ are restricted conjugate if they satisfy

$$\sigma = \mu^{-1}\tau\mu,$$

for some $\mu \in S_n \times S_m$. Denote the number of restricted conjugate classes by $N(n,m)$ and let $n^\sigma(n,m)$ denote the number of elements in the restricted conjugate class with representative σ. $N(n,m)$ is also equal to the number of restricted Schur polynomials $\chi_{R,(r_n,r_m)}$ with R an irreducible representation of S_{n+m} and (r_n,r_m) an irreducible representation of $S_n \times S_m$. The equality between the number of restricted conjugate classes $N(n,m)$ and the total number of labels $R,(r_n,r_m)$ generalizes the familiar equality for the symmetric group of the number of conjugacy classes and the number of irreducible representations. Introduce the matrix

$$(M^{-1})_{\sigma\tau} = \sum_{(R,(r_n,r_m))} \chi_{R,(r_n,r_m)}(\sigma)\chi_{R,(r_n,r_m)}(\tau), \tag{7.1}$$

where the sum on the right hand side runs over all possible labels $(R,(r_n,r_m))$. The restricted character is defined by[31]

$$\chi_{R,(r_n,r_m)}(\sigma) = \mathrm{Tr}_{\,(r_n,r_m)}\left(\Gamma_R(\sigma)\right).$$

Define the *dual restricted character* (which we denote by a superscript) by

$$\chi^{R,(r_n,r_m)}(\sigma) = \frac{n!m!}{n^\sigma(n,m)} \sum_{[\tau]_r} M_{\sigma\tau}\chi_{R,(r_n,r_m)}(\tau).$$

The indices σ and τ that appear in ((7.1)) run over the restricted conjugacy classes. Denote the restricted conjugacy class with representative ψ by $[\psi]_r$. It is clear that

$$\sum_{R,(r_n,r_m)} \chi^{R,(r_n,r_m)}(\sigma)\chi_{R,(r_n,r_m)}(\rho) = n!m!\delta([\sigma]_r[\rho]_r). \tag{7.2}$$

In appendix A.2 we derive a formula for the dual restricted character in terms of the restricted character. To correctly state this formula, we need to spell out both the row and the column indices that are summed in the trace. For an "on the diagonal block" trace, the column and row indices that are summed come from the same carrier space and we are summing diagonal elements of the matrix; for an "off the diagonal block" trace, the column and row indices that are traced come from distinct carrier spaces so that we are summing off diagonal elements of the matrix[1]. Indicating both row and column indices of the restricted trace, the dual character is given by

$$\chi^{R,((r_n,r_m),(s_n,s_m))}(\tau) = \frac{d_R n! m!}{d_{r_n} d_{r_m} (n+m)!} \chi_{R,((s_n,s_m),(r_n,r_m))}(\tau). \tag{7.3}$$

[1] See [30],[31] for more details.

7.2 Product Rule

We will now define what we call *restricted Littlewood-Richardson numbers*. The restricted Littlewood-Richardson numbers determine the product rule for restricted Schur polynomials in exactly the same way that the Littlewood-Richardson numbers determine the product rule for Schur polynomials. Let R_1 be an irreducible representation of $S_{n_1+m_1}$ and let (r_{n_1}, r_{m_1}) be an irreducible representation of $S_{n_1} \times S_{m_1}$. Let R_2 be an irreducible representation of $S_{n_2+m_2}$ and let (r_{n_2}, r_{m_2}) be an irreducible representation of $S_{n_2} \times S_{m_2}$. Finally, let R_{1+2} be an irreducible representation of $S_{n_1+n_2+m_1+m_2}$ and let $(r_{n_{1+2}}, r_{m_{1+2}})$ be an irreducible representation of $S_{n_1+n_2} \times S_{m_1+m_2}$. The restricted Littlewood-Richardson numbers are defined by

$$f^{R_{1+2},(r_{n_{1+2}},r_{m_{1+2}})}_{R_1,(r_{n_1},r_{m_1})\,R_2,(r_{n_2},r_{m_2})} = \frac{1}{n_1!n_2!m_1!m_2!} \tag{7.4}$$

$$\times \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}} \chi_{R_1,(r_{n_1},r_{m_1})}(\sigma_1)\chi_{R_2,(r_{n_2},r_{m_2})}(\sigma_2)\chi^{R_{1+2},(r_{n_{1+2}},r_{m_{1+2}})}(\sigma_1 \circ \sigma_2).$$

To streamline the notation, from now on we will replace the composite label $R_i, (r_{n_i}, r_{m_i})$ simply by $\{i\}$ and we define $n_{12} \equiv n_1 + n_2$, $m_{12} \equiv m_1 + m_2$. With the new streamlined notation we write

$$f^{\{1+2\}}_{\{1\}\,\{2\}} = \frac{1}{n_1!n_2!m_1!m_2!} \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}} \chi_{\{1\}}(\sigma_1)\chi_{\{2\}}(\sigma_2)\chi^{\{1+2\}}(\sigma_1 \circ \sigma_2).$$

The restricted Schur product rule says

$$\chi_{\{1\}}(Z,X)\chi_{\{2\}}(Z,X) = \sum_{\{1+2\}} f^{\{1+2\}}_{\{1\}\,\{2\}}\chi_{\{1+2\}}(Z,X). \tag{7.5}$$

A few comments are in order. The restricted Schur polynomial $\chi_{\{1+2\}}(Z,X)$ is given by

$$\chi_{\{1+2\}}(Z,X) = \frac{1}{n_{12}!m_{12}!} \sum_{\rho \in S_{n_{12}+m_{12}}} \mathrm{Tr}_{\,(r_{n_{1+2}},r_{m_{1+2}})}\left(\Gamma_{R_{1+2}}(\rho)\right) \mathrm{Tr}\,(\rho Z^{\otimes n_{12}} \otimes X^{\otimes m_{12}}).$$

$(r_{n_{1+2}}, r_{m_{1+2}})$ is an irreducible representation of the $S_{n_{12}} \times S_{m_{12}}$ subgroup which permutes indices of the Zs amongst each other and the indices of the Xs amongst each other. The representation R_1 is an irreducible representation of the $S_{n_1+m_1}$ subgroup that acts on the first n_1 Zs and the first m_1 Xs; the representation R_2 is an irreducible representation of the $S_{n_2+m_2}$ subgroup that acts on the remaining n_2 Zs and m_2 Xs. To demonstrate the restricted Schur product rule, consider

$$\sum_{\{1+2\}} f_{\{1\}\,\{2\}}^{\{1+2\}} \chi_{\{1+2\}}(Z,X) = \frac{1}{n_1!n_2!m_1!m_2!}$$

$$\times \sum_{\{1+2\}} \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}} \chi_{\{1\}}(\sigma_1)\chi_{\{2\}}(\sigma_2)\chi^{\{1+2\}}(\sigma_1 \circ \sigma_2)$$

$$\times \frac{1}{n_{12}!m_{12}!} \sum_{\rho \in S_{n_{12}+m_{12}}} \mathrm{Tr}_{\,(r_{n_{1+2}}, r_{m_{1+2}})} \left(\Gamma_{R_{1+2}}(\rho)\right) \mathrm{Tr}\,(\rho Z^{\otimes n_1+n_2} \otimes X^{\otimes m_1+m_2}).$$

After using ((7.2)) we obtain

$$\sum_{\{1+2\}} f_{\{1\}\,\{2\}}^{\{1+2\}} \chi_{\{1+2\}}(Z,X) = \frac{1}{n_1!n_2!m_1!m_2!} \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}} \sum_{\rho \in S_{n_{12}+m_{12}}} \chi_{\{1\}}(\sigma_1)\chi_{\{2\}}(\sigma_2) \times$$

$$\times \delta(\left[\sigma_1 \circ \sigma_2\right]_r \left[\rho\right]_r) \mathrm{Tr}\,(\rho Z^{\otimes n_{12}} \otimes X^{\otimes m_{12}})$$

$$= \frac{1}{n_1!n_2!m_1!m_2!} \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}} \chi_{\{1\}}(\sigma_1)\chi_{\{2\}}(\sigma_2) \mathrm{Tr}\,(\sigma_1 \circ \sigma_2 Z^{\otimes n_{12}} \otimes X^{\otimes m_{12}})$$

$$= \frac{1}{n_1!m_1!} \sum_{\sigma_1 \in S_{n_1+m_1}} \chi_{\{1\}}(\sigma_1) \mathrm{Tr}\,(\sigma_1 Z^{\otimes n_1} \otimes X^{\otimes m_1}) \frac{1}{n_2!m_2!} \sum_{\sigma_2 \in S_{n_2+m_2}} \chi_{\{2\}}(\sigma_2) \mathrm{Tr}\,(\sigma_2 Z^{\otimes n_2} \otimes X^{\otimes m_2})$$

$$= \chi_{\{1\}}(Z,X)\chi_{\{2\}}(Z,X),$$

which proves the product rule.

Using the explicit formula for the dual character ((7.3)) leads to the formula

$$f_{R_1,(r_{n_1},r_{m_1})\,R_2,(r_{n_2},r_{m_2})}^{R_{1+2},(r_{n_{1+2}},r_{m_{1+2}})(s_{n_{1+2}},s_{m_{1+2}})} = \frac{n_{12}!m_{12}!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!} \times$$

$$\frac{d_{R_{1+2}}}{d_{r_{n_{1+2}}} d_{r_{m_{1+2}}}} \sum_{\sigma_1 \in S_{n_1+m_1}} \sum_{\sigma_2 \in S_{n_2+m_2}} \chi_{R_1,(r_{n_1},r_{m_1})}(\sigma_1) \chi_{R_2,(r_{n_2},r_{m_2})}(\sigma_2) \times$$
$$\chi_{R_{1+2},(s_{n_{1+2}},s_{m_{1+2}})(r_{n_{1+2}},r_{m_{1+2}})}(\sigma_1 \circ \sigma_2) \tag{7.6}$$

for the restricted Littlewood-Richardson numbers.

7.3 Computation of the Restricted Littlewood-Richardson Numbers

In this section we will develop rules that will allow us to compute the restricted characters needed to evaluate the restricted Littlewood Richardson numbers. A general diagrammatic method, *strand diagrams*, to compute restricted characters in the case that the polynomial is built from a single matrix has been developed in [33]. In this section we would like to develop methods that are powerful enough to allow the computation of restricted characters for polynomials built out of both Z and X. It is enough to compute the characters of two cycles, since any element can be decomposed into a product of two cycles. It is precisely this fact that was exploited to build the strand diagrams of [33]. In the next two sections the character of arbitrary two cycles for an on the diagonal block restriction and then an off the diagonal block restriction are computed. Finally, some example computations of restricted Littlewood-Richardson coefficients are discussed.

7.3.1 On the diagonal restricted characters of two cycles

We need two pieces of information: first, we will introduce three Casimirs that will be particularly useful. Second, we will argue that all characters for two cycles which do not belong to $S_n \times S_m$ are equal. Taken together, these two facts will allow us to compute the restricted character of an arbitrary two cycle.

Let the first n indices be associated with the Z matrices and the next m indices be associated with the X fields. Greek indices run over $\alpha = 1, 2, ..., m + n$. Indices from the start of the alphabet run over $a = 1, 2, ..., n$. Indices from the middle of the alphabet run over $i = n + 1, n + 2, ..., n + m$. The operator

$$\hat{O}_{n+m} = \sum_{\alpha<\beta=1}^{n+m} (\alpha\beta)$$

is a Casimir of S_{n+m}. When acting on an irreducible representation described by a Young Diagram R with r_i boxes in row i and c_j boxes in column j it takes the value

$$\hat{O}_{n+m}|R\rangle = \big[\sum_i \frac{r_i(r_i-1)}{2} - \sum_j \frac{c_j(c_j-1)}{2}\big]|R\rangle.$$

The operators

$$\hat{O}_n = \sum_{a<b=1}^{n} (ab), \qquad \hat{O}_m = \sum_{i<j=n+1}^{n+m} (ij),$$

are Casimirs of $S_n \times S_m$. Consider the $S_n \times S_m$ irreducible representation R, described by Young diagrams r_n (for the Zs) and r_m (for the Xs). r_n has $r_{1,i}$ boxes in row i and $c_{1,j}$ boxes in column j; r_m has $r_{2,i}$ boxes in row i and $c_{2,j}$ boxes in column j. These Casimirs take the values

$$\hat{O}_n|(r_n,r_m)\rangle = \big[\sum_i \frac{r_{1,i}(r_{1,i}-1)}{2} - \sum_j \frac{c_{1,j}(c_{1,j}-1)}{2}\big]|(r_n,r_m)\rangle,$$

$$\hat{O}_m|(r_n,r_m)\rangle = \big[\sum_i \frac{r_{2,i}(r_{2,i}-1)}{2} - \sum_j \frac{c_{2,j}(c_{2,j}-1)}{2}\big]|(r_n,r_m)\rangle.$$

If a cycle belongs to $S_n \times S_m$ it has the form (ab) or (ij). In this case

$$\mathrm{Tr}_{(r_n,r_m)}((ab)) = \mathrm{Tr}_{r_n}((ab))d_{r_m}, \quad \text{and} \quad \mathrm{Tr}_{(r_n,r_m)}((ij)) = \mathrm{Tr}_{r_m}((ij))d_{r_n}.$$

We can calculate $\mathrm{Tr}_{r_n}((ab))$ and $\mathrm{Tr}_{r_m}((ij))$ using the results of [33]. We will now argue that all characters for two cycles which do not belong to $S_n \times S_m$ are equal. Any two such cycles can be related as

$$\begin{aligned}
\Gamma_R\big((aj)\big) &= \Gamma_R\big((ab)\big)\Gamma_R\big((jl)\big)\Gamma_R\big((bl)\big)\Gamma_R\big((jl)\big)\Gamma_R\big((ab)\big) \\
&= \Gamma_R\big((ab)\big)^{-1}\Gamma_R\big((jl)\big)^{-1}\Gamma_R\big((bl)\big)\Gamma_R\big((jl)\big)\Gamma_R\big((ab)\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Tr}_{(r_n,r_m)}\Big(\Gamma_R\big((aj)\big)\Big) &= \mathrm{Tr}_{(r_n,r_m)}\Big(\Gamma_R\big((ab)\big)^{-1}\Gamma_R\big((jl)\big)^{-1}\Gamma_R\big((bl)\big)\Gamma_R\big((jl)\big)\Gamma_R\big((ab)\big)\Big)\\
&= \mathrm{Tr}_{(r_n,r_m)}\Big(\Gamma_{R_\gamma}\big((ab)\big)^{-1}\Gamma_{R_\gamma}\big((jl)\big)^{-1}\Gamma_R\big((bl)\big)\Gamma_{R_\gamma}\big((jl)\big)\Gamma_{R_\gamma}\big((ab)\big)\Big)\\
&= \mathrm{Tr}_{(r_n,r_m)}\Big(\Gamma_R\big((bl)\big)\Big).
\end{aligned}$$

Clearly, all characters for two cycles which do not belong to $S_n \times S_m$ are equal.

Taken together, these two facts imply that (the cycle (aj) does not belong to $S_n \times S_m$)

$$\begin{aligned}
\mathrm{Tr}_{(r_n,r_m)}\Big(\Gamma_R\big((aj)\big)\Big) &= \frac{1}{nm}\sum_{i=n+1}^{n+m}\sum_{a=1}^{n}\mathrm{Tr}_{(r_n,r_m)}\Big(\Gamma_R\big((ia)\big)\Big)\\
&= \frac{1}{nm}\mathrm{Tr}_{(r_n,r_m)}(\hat{O}_{n+m}-\hat{O}_n-\hat{O}_m). \qquad (7.7)
\end{aligned}$$

The eigenvalue of the Casimir operator $\hat{O}_{n+m}-\hat{O}_n-\hat{O}_m$ can be written as the sum of the weights of R minus the sum of the weights of r_n minus the sum of the weights of r_m. This observation allows us to derive much simpler formulas for the case that r_m say, contains only a few boxes. Indeed, we can imagine that (r_n, r_m) was formed by peeling boxes off R to leave r_n and then combining the peeled boxes to form r_m. In this case, the eigenvalue of $\hat{O}_{n+m}-\hat{O}_n-\hat{O}_m$ is given by the sum of weights of the boxes peeled off R minus the sum of the weights of the boxes in r_m. We will now give the simplified versions of ((7.7)) for the cases that $m = 1, 2$ or 3.

Restricted character for $m = 1$: In the following formula, R is an irreducible representation of S_{n+1} and r_n is an irreducible representation of S_n of dimension d_{r_n}. A single box must be removed from R to obtain r_n. Denote the weight of the box that must be removed by c_1. The simplified character formula is

$$\mathrm{Tr}_{(r_n,\square)}\Big(\Gamma_R\big((aj)\big)\Big) = \frac{c_1 - N}{n}d_{r_n}.$$

Restricted character for $m = 2$: In the following formula, R is an irreducible representation of S_{n+2} and r_n is an irreducible representation of S_n of dimension d_{r_n}. Two boxes must be removed from R to obtain r_n. Denote the weights of the boxes that must be removed by c_1 and c_2. The simplified character formulas are

$$\mathrm{Tr}_{\,(r_n,\square\square)}\Big(\Gamma_R\big((aj)\big)\Big) = \frac{c_1 + c_2 - 2N - 1}{2n} d_{r_n},$$

$$\mathrm{Tr}_{\,(r_n,\substack{\square\\\square})}\Big(\Gamma_R\big((aj)\big)\Big) = \frac{c_1 + c_2 - 2N + 1}{2n} d_{r_n}.$$

Restricted character for $m = 3$: In the following formula, R is an irreducible representation of S_{n+3} and r_n is an irreducible representation of S_n of dimension d_{r_n}. Three boxes must be removed from R to obtain r_n. Denote the weights of the boxes that must be removed by c_1, c_2 and c_3. The simplified character formulas are

$$\mathrm{Tr}_{\,(r_n,\square\square\square)}\Big(\Gamma_R\big((aj)\big)\Big) = \frac{c_1 + c_2 + c_3 - 3N - 3}{3n} d_{r_n},$$

$$\mathrm{Tr}_{\,(r_n,\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix})}\Big(\Gamma_R\big((aj)\big)\Big) = 2\frac{c_1 + c_2 + c_3 - 3N}{3n} d_{r_n},$$

$$\mathrm{Tr}_{\,(r_n,\substack{\square\\\square\\\square})}\Big(\Gamma_R\big((aj)\big)\Big) = \frac{c_1 + c_2 + c_3 - 3N + 3}{3n} d_{r_n}.$$

7.3.2 Off the diagonal restricted characters of two cycles

Before we can compute generic restricted characters, we need to compute traces over off the diagonal blocks: $\mathrm{Tr}_{\,(r_n,r_m),(s_n,s_m)}\Big(\Gamma_R\big((\alpha\beta)\big)\Big)$, where (r_n, r_m) and (s_n, s_m) are distinct $S_n \times S_m$ representations. This character clearly vanishes if $(\alpha\beta)$ belongs to the $S_n \times S_m$ subgroup. What about the mn two cycles that do not belong to $S_n \times S_m$? For concreteness, consider the computation of $\mathrm{Tr}_{\,(r_n,r_m)(s_n,s_m)}\Big(\Gamma_R\big((ai)\big)\Big)$. Let $(r_n, r_m)'_I$ denote the complete set of irreducible representations of the $S_{n-1} \times S_{m-1}$ subgroup subduced by (r_n, r_m), that is $\oplus_I\, (r_n, r_m)'_I = (r_n, r_m)$. The $S_{n-1} \times S_{m-1}$ subgroup of interest is obtained by keeping all elements of $S_n\times S_m$ that hold indices a and i fixed. The representations (r_n,r_m) and (s_n,s_m) have the same shape[2] so that we can establish a bijective map between their bases. We will assume that this bijective map is the identity, which we can always arrange by a suitable choice of basis. This choice of basis ensures that when we subduce to the $S_{n-1}\times S_{m-1}$ subgroup we have

$$\mathrm{Tr}_{\,(r_n,r_m)(s_n,s_m)}\Big(\Gamma_R\big((ai)\big)\Big)=\sum_I \mathrm{Tr}_{\,(r_n,r_m)'_I(s_n,s_m)'_I}\Big(\Gamma_R\big((ai)\big)\Big).$$

We will now provide further insight into this formula. It is straight forward to prove that

$$\langle (r_n,r_m)'_I,i|\Gamma_R\big((ai)\big)\Big)|(s_n,s_m)'_I\rangle$$

vanishes unless $(r_n,r_m)'_I$ and $(s_n,s_m)'_I$ have the same shape. Introduce the Casimirs

$$\hat{O}_{n-1}=\sum_{c<b=1\neq a}^{n}(cb),\qquad \hat{O}_{m-1}=\sum_{k<j=n+1\neq i}^{n+m}(kj).$$

Denote the eigenvalue of Casimir $\hat{O}_{n-1}$ for the $(r_n,r_m)'_I$ representation by $\lambda_r^{(n-1)}$ and the eigenvalue of Casimir $\hat{O}_{m-1}$ for the $(r_n,r_m)'_I$ representation by $\lambda_r^{(m-1)}$. Clearly, we have

$$\begin{aligned}\lambda_r^{(n-1)}\langle (r_n,r_m)'_I,i|\Gamma_R\big((ai)\big)|(s_n,s_m)'_I\rangle &= \langle (r_n,r_m)'_I,i|\hat{O}_{n-1}\Gamma_R\big((ai)\big)|(s_n,s_m)'_I\rangle\\ &= \langle (r_n,r_m)'_I,i|\Gamma_R\big((ai)\big)\hat{O}_{n-1}|(s_n,s_m)'_I\rangle\\ &= \lambda_s^{(n-1)}\langle (r_n,r_m)'_I,i|\Gamma_R\big((ai)\big)|(s_n,s_m)'_I\rangle,\end{aligned}$$

$$\begin{aligned}\lambda_r^{(m-1)}\langle (r_n,r_m)'_I,i|\Gamma_R\big((ai)\big)|(s_n,s_m)'_I\rangle &= \langle (r_n,r_m)'_I,i|\hat{O}_{m-1}\Gamma_R\big((ai)\big)|(s_n,s_m)'_I\rangle\\ &= \langle (r_n,r_m)'_I,i|\Gamma_R\big((ai)\big)\hat{O}_{m-1}|(s_n,s_m)'_I\rangle\\ &= \lambda_s^{(m-1)}\langle (r_n,r_m)'_I,i|\Gamma_R\big((ai)\big)|(s_n,s_m)'_I\rangle,\end{aligned}$$

[2]It is only when (r_n,r_m) and (s_n,s_m) have the same shape that the trace $\mathrm{Tr}_{\,(r_n,r_m)(s_n,s_m)}$ has any meaning.

which implies that

$$\langle (r_n,r_m)'_I, i|\Gamma_R\big((ai)\big)|(s_n,s_m)'_I\rangle \propto \delta_{\lambda_r^{(n-1)}\,\lambda_s^{(n-1)}}\delta_{\lambda_r^{(m-1)}\,\lambda_s^{(m-1)}}.$$

We can repeat this with the complete set of Casimirs of $S_{n-1}\times S_{m-1}$, allowing us to conclude that

$$\langle (r_n,r_m)'_I, i|\Gamma_R\big((ai)\big)|(s_n,s_m)'_I\rangle \propto \delta_{(r_n,r_m)'_I\,(s_n,s_m)'_I}.$$

$\delta_{(r_n,r_m)'_I\,(s_n,s_m)'_I}$ is equal to 1 if $(r_n,r_m)'_I$ and $(s_n,s_m)'_I$ have the same shape. Of course, this is just a consequence of Schur's Lemma: since $\Gamma_R\big((ai)\big)$ commutes with all of the elements of the $S_{n-1}\times S_{m-1}$ subgroup, it is proportional to the identity when acting on any irreducible representation of $S_{n-1}\times S_{m-1}$. Thus, (the label i inside the ket labels the carrier space state)

$$\langle (r_n,r_m)'_I, j|\Gamma_R\big((ai)\big)|(r_n,r_m)'_I, i\rangle = \eta_{(r_n,r_m)'_I}\delta_{ij}\,.$$

If we allow $\Gamma_R\big((ai)\big)$ to act in the full carrier space of R, this equation is modified to (the sum is over all irreducible representations (t_n,t_m) that can subduce a $(t_n,t_m)'_K$ of the same shape as $(r_n,r_m)'_I$; the $\delta_{(r_n,r_m)'_I\,(s_n,s_m)'_J}$ on the last line is there to remind the reader that a non-zero result is obtained only if $(r_n,r_m)'_I$ and $(s_n,s_m)'_J$ have the same shape)

$$\begin{aligned}\langle (s_n,s_m)'_J, j|\Gamma_R\big((ai)\big)|(r_n,r_m)'_I, i\rangle &= \sum_{(t_n,t_m)'_K}\eta^{(t_n,t_m)'_K}_{(r_n,r_m)'_I}\langle (s_n,s_m)'_J, j|(t_n,t_m)'_K, i\rangle\\ &= \eta^{(s_n,s_m)'_J}_{(r_n,r_m)'_I}\delta_{(r_n,r_m)'_I\,(s_n,s_m)'_J}\delta_{ij}\end{aligned}$$

Although $(s_n,s_m)'_J$ and $(r_n,r_m)'_I$ have the same shape, they may be distinct representations in which case they were subduced by *different* $S_n\times S_m$ representations. These matrix elements are needed to provide a complete generalization of strand diagrams. This has an important implication: in the present case, the most general non-vanishing strand diagrams will not only record the reordering of boxes in the row and column states, it will also allow the shape of the row state $S_n\times S_m$ representation to reorder itself (by the movement of a single box in r_n and/or a single box in r_m) into the column state $S_n \times S_m$ representation.

The trace we are interested in can also be expressed in terms of $\eta^{(r_n,r_m)'_I}_{(r_n,r_m)'_J}$. Indeed, it is clear that

$$
\begin{aligned}
\mathrm{Tr}_{\,(r_n,r_m)(s_n,s_m)}\Big(\Gamma_R\big((ai)\big)\Big) &= \sum_I \mathrm{Tr}_{\,(r_n,r_m)'_I(s_n,s_m)'_I}\Big(\Gamma_R\big((ai)\big)\Big) \\
&= \sum_I \eta^{(r_n,r_m)'_I}_{(r_n,r_m)'_I} d_{(r_n,r_m)'_I}.
\end{aligned}
$$

$d_{(r_n,r_m)'_I}$ is the dimension of the irreducible representation $(r_n, r_m)'_I$. Since (r_n, r_m) and (s_n, s_m) have the same shape, we will not need to worry about the extra complication of changing the $S_n \times S_m$ representation between the row and column states.

To determine the restricted character we need, we now only need to fix the $\eta^{(r_n,r_m)'_I}_{(r_n,r_m)'_I}$. The most direct way we have found to do this proceeds by determining the explicit change of basis from a natural basis of S_{n+m} to a natural basis for the $S_n \times S_m$ subgroup. We will illustrate the method with an example: using the methods of the last subsection, we easily find

$$
\mathrm{Tr}_{\yng(2,1),\yng(2)}\left(\Gamma_{\yng(3,1,1)}\big((3,4)\big)\right) = -\frac{1}{3}.
$$

To extract $\yng(2,1)$ from $\yng(3,1,1)$ we need to pull off the last box in the first row and the last box in the first column. They can be pulled off in any order, so that we can write (i labels the states in the $\yng(2,1)$ carrier space)

$$
|\yng(3,1,1);\yng(2,1)\,i,\yng(2)\rangle = \alpha|\young(\ \ 1,\ ,2)\,i\rangle + \beta|\young(\ \ 2,\ ,1)\,i\rangle. \tag{7.8}
$$

Because the above state is normalized, we need $\alpha^2 + \beta^2 = 1$. Next, using appendix D.2 of [31] we can write

$$
\Gamma_{\yng(3,1,1)}((45))\,|\yng(3,1,1);\yng(2,1)\,i,\yng(2)\rangle = |\yng(3,1,1);\yng(2,1)\,i,\yng(2)\rangle,
$$

$$\Gamma_{\tiny\yng(3,1,1)}((45))\left[\alpha\left|\tiny\yng(2,1)\,{}^{\boxed{1}}_{\boxed{2}}\; i\right\rangle+\beta\left|\tiny\yng(2,1)\,{}^{\boxed{2}}_{\boxed{1}}\; i\right\rangle\right]=\left(\frac{\alpha}{4}+\beta\frac{\sqrt{15}}{4}\right)\left|\tiny\yng(2,1)\,{}^{\boxed{1}}_{\boxed{2}}\; i\right\rangle+\left(-\frac{\beta}{4}+\alpha\frac{\sqrt{15}}{4}\right)\left|\tiny\yng(2,1)\,{}^{\boxed{2}}_{\boxed{1}}\; i\right\rangle.$$

This then implies two equations

$$\frac{\alpha}{4}+\beta\frac{\sqrt{15}}{4}=\alpha,\qquad -\frac{\beta}{4}+\alpha\frac{\sqrt{15}}{4}=\beta.$$

They are not independent (as expected) and imply $\alpha=\sqrt{\frac{5}{3}}\beta$ so that

$$\alpha=\sqrt{\frac{5}{8}},\qquad \beta=\sqrt{\frac{3}{8}}.$$

It is now straight forward to use the standard strand diagram techniques of [33] to verify that

$$\sum_i\left(\sqrt{\frac{5}{8}}\left\langle\tiny\yng(2,1)\,{}^{\boxed{1}}_{\boxed{2}}\; i\right|+\sqrt{\frac{3}{8}}\left\langle\tiny\yng(2,1)\,{}^{\boxed{2}}_{\boxed{1}}\; i\right|\right)\Gamma_{\tiny\yng(3,1,1)}\big((3,4)\big)\left(\sqrt{\frac{5}{8}}\left|\tiny\yng(2,1)\,{}^{\boxed{1}}_{\boxed{2}}\; i\right\rangle+\sqrt{\frac{3}{8}}\left|\tiny\yng(2,1)\,{}^{\boxed{2}}_{\boxed{1}}\; i\right\rangle\right)=-\frac{1}{3}.$$

Although we have computed an on the diagonal character, it is clear that once the relationship ((7.8)) is established, it can be used to compute off the diagonal block characters by employing standard strand diagrams.

Summary of the logic: Using the action of 2-cycles from the S_m subgroup - something we already know in both bases - we have been able to determine the explicit change of basis from a natural basis of S_{n+m} to a natural basis for the $S_n\times S_m$ subgroup. This has enabled us to compute the restricted character of the two cycle $(n,n+1)$ which "straddles" S_n and S_m. Thus, we can now compute the restricted characters of all two cycles of the form $(i,i+1)$, which is all that is needed to compute the restricted character of a general group element.

We will now argue that, as long as the number of boxes in r_m is small, we can find formulas for the above change of basis for any representation r_n. Removing the m boxes (used to build r_m) from the S_{n+m} representation R gives a set of states that carry an index for the carrier space of r_n. The coefficient describing the change of basis is clearly independent of this index. Thus, in what follows, the carrier space index of representation r_n plays no role and is hence suppressed.

Restricted character of $(n, n+1)$ *for* $m = 1$*:* Denote the two labels of the restricted trace by $R, (r_n, \square)$ and $R, (s_n, \square)$. R is a Young diagram with $n+1$ boxes; r_n and s_n are both Young diagrams with n boxes; they are both obtained by removing a single box from R. Denote the weight of the box that must be removed from R to obtain r_n by c_r; denote the weight of the box that must be removed from R to obtain s_n by c_s. For an off the diagonal block, $c_r \neq c_s$. Let R'' denote the Young diagram obtained when both boxes are removed. It is now straight forward to show that

$$\mathrm{Tr}_{(r_n,\square)(s_n,\square)}\left(\Gamma_R\Big((n,n+1)\Big)\right) = \sqrt{1 - \frac{1}{(c_r - c_s)^2}}\, d_{R''},$$

where $d_{R''}$ is the dimension of R''.

Restricted character of $(n, n+1)$ *for* $m = 2$*:* The two possible types of restricted Schur labels are $R, (r_n, \ydiagram{2})$ and $R, (r_n, \ydiagram{1,1})$. R is a Young diagram with $n+2$ boxes; r_n is a Young diagram with n boxes. r_n is obtained from R by removing two boxes from R. We say that box a is above box b if the weight of box a is greater than the weight of box b. Denote the weights of the two boxes as c_1 and c_2, such that the box with weight c_1 is above the box with weight c_2. The relations between the natural S_{n+2} basis and the natural $S_n \times S_2$ bases are (although we have written these relations using a particular Young diagram, they are true in general; on the right hand side, the box with label a is to be removed first)

$$|\ydiagram{3,1,1}\,;\ydiagram{2,1}\, i, \ydiagram{2}\rangle = \sqrt{\frac{c_1 - c_2 + 1}{2(c_1 - c_2)}}\,|\begin{ytableau} \, & \, & a \\ \, \\ b \end{ytableau}\, i\rangle + \sqrt{\frac{c_1 - c_2 - 1}{2(c_1 - c_2)}}\,|\begin{ytableau} \, & \, & b \\ \, \\ a \end{ytableau}\, i\rangle,$$

$$|\ydiagram{3,1,1}\,;\ydiagram{2,1}\, i, \ydiagram{1,1}\rangle = \sqrt{\frac{c_1 - c_2 - 1}{2(c_1 - c_2)}}\,|\begin{ytableau} \, & \, & a \\ \, \\ b \end{ytableau}\, i\rangle - \sqrt{\frac{c_1 - c_2 + 1}{2(c_1 - c_2)}}\,|\begin{ytableau} \, & \, & b \\ \, \\ a \end{ytableau}\, i\rangle.$$

Notice that these states are orthogonal as they must be. It is now straight forward to obtain any particular character we want by employing standard strand diagram methods.

We will now consider a particular example. The labels for the restricted trace are $R,(r_n,\ydiagram{2})$ and $R,(s_n,\ydiagram{2})$. To obtain a non-zero off the diagonal restricted character, one of the boxes removed from R to obtain r_n must be in the same position as one of the boxes removed from R to obtain s_n. Assume that the common box has weight c_1. Denote the weight of the second box that must be removed to obtain r_n by c_2 and denote the weight of the second box that must be removed to obtain s_n by c_2^*. Denote the Young diagram obtained by removing all three boxes from R by R'''. It is straight forward to show that

$$\mathrm{Tr}_{(r_n,\ydiagram{2})(s_n,\ydiagram{2})}\Big(\Gamma_R\Big((n,n+1)\Big)\Big)=\sqrt{\frac{c_1-c_2+1}{2(c_1-c_2)}}\sqrt{\frac{c_1-c_2^*+1}{2(c_1-c_2^*)}}\sqrt{1-\frac{1}{(c_2-c_2^*)^2}}d_{R'''}.$$

Restricted character of $(n,n+1)$ *for* $m=3$*:* We will discuss this example in some detail because it will provide the key to obtaining general results. We will consider the case in which the three boxes to be removed have no sides in common. In this case, the three possible types of restricted Schur labels are $R,(r_n,\ydiagram{3})$, $R,(r_n,\ydiagram{2,1})$ and $R,(r_n,\ydiagram{1,1,1})$. R is a Young diagram with $n+3$ boxes; r_n is a Young diagram with n boxes. r_n is obtained from R by removing three boxes from R. Denote the weights of the boxes to be removed by c_1, c_2 and c_3. The box with weight c_1 lies above the boxes with weights c_2 and c_3; the box with weight c_2 lies above the box with weight c_3. Consider the expansion (on the right hand side, the box with label a is to be removed first the box with label b second and the box with label c third)

$$\begin{aligned}\Big|\ydiagram{3,2,1};\ydiagram{2,1}\,\ydiagram{3}\Big\rangle &= \alpha_{123}\Big|\ytableaushort{{}{}a,{}b,c}\Big\rangle + \alpha_{132}\Big|\ytableaushort{{}{}a,{}c,b}\Big\rangle + \alpha_{213}\Big|\ytableaushort{{}{}b,{}a,c}\Big\rangle\\ &+\alpha_{231}\Big|\ytableaushort{{}{}b,{}c,a}\Big\rangle + \alpha_{312}\Big|\ytableaushort{{}{}c,{}a,b}\Big\rangle + \alpha_{321}\Big|\ytableaushort{{}{}c,{}b,a}\Big\rangle .\end{aligned}$$

Notice that the subscripts encode the order in which boxes are to be dropped.

By considering the action of the cycle of $(n+2, n+3)$ on the above expression, we obtain 6 equations. Only three of these are independent; they read

$$\alpha_{123} = \frac{1}{c_1 - c_2}\alpha_{123} + \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\alpha_{213},$$

$$\alpha_{132} = \frac{1}{c_1 - c_3}\alpha_{132} + \sqrt{1 - \frac{1}{(c_1 - c_3)^2}}\alpha_{231},$$

$$\alpha_{321} = \frac{1}{c_3 - c_2}\alpha_{321} + \sqrt{1 - \frac{1}{(c_3 - c_2)^2}}\alpha_{312}.$$

Similarly, by considering the action of the cycle $(n+1, n+2)$ we obtain the following three independent equations

$$\alpha_{123} = \frac{1}{c_2 - c_3}\alpha_{123} + \sqrt{1 - \frac{1}{(c_2 - c_3)^2}}\alpha_{132},$$

$$\alpha_{213} = \frac{1}{c_1 - c_3}\alpha_{213} + \sqrt{1 - \frac{1}{(c_1 - c_3)^2}}\alpha_{312},$$

$$\alpha_{321} = \frac{1}{c_2 - c_1}\alpha_{321} + \sqrt{1 - \frac{1}{(c_2 - c_1)^2}}\alpha_{231}.$$

These six equations can be written in a very compact form: let p denote the subscript of a particular coefficient, i.e. a particular ordering of the three numbers 1, 2 and 3. Define the action of the cycle $(i, i+1)$ on p as follows: all numbers not equal to i or $i+1$ stay where they are; the numbers i and $i+1$ swap positions. Thus, $(1, 2) \cdot 123 = 213$. We will also index the entries of p by $i = 1, 2, 3$. Thus, for $p = 231$, we have $p(1) = 2$, $p(2) = 3$ and $p(3) = 1$. The six equations above can now be written as

$$\alpha_p = \frac{1}{c_{p^{-1}(i)} - c_{p^{-1}(i+1)}}\alpha_p + \sqrt{1 - \frac{1}{(c_{p^{-1}(i)} - c_{p^{-1}(i+1)})^2}}\alpha_{(i,i+1)\cdot p}, \qquad i = 1, 2, ..., m-1. \tag{7.9}$$

Simple algebra gives

$$\alpha_p = \sqrt{\frac{c_{p^{-1}(i)} - c_{p^{-1}(i+1)} + 1}{c_{p^{-1}(i)} - c_{p^{-1}(i+1)} - 1}}\alpha_{(i,i+1)\cdot p}\,.$$

These equations completely determine the unknown coefficients. Indeed, each subscript is a particular ordering of the three numbers 1, 2 and 3. By using the first set of equations we can swap the positions of 1 and 2; by using the second set we can swap the positions of 2 and 3. Using these two operations we can relate any coefficient to any other. Thus, the normalization condition

$$\alpha_{123}^2 + \alpha_{132}^2 + \alpha_{213}^2 + \alpha_{231}^2 + \alpha_{312}^2 + \alpha_{321}^2 = 1$$

can be written completely in terms of α_{123} (say). This determines α_{123} and hence all of the coefficients. Solving for α_{123} we obtain

$$\alpha_{123} = \sqrt{\frac{(c_1 - c_2 + 1)(c_1 - c_3 + 1)(c_2 - c_3 + 1)}{6(c_1 - c_2)(c_1 - c_3)(c_2 - c_3)}}.$$

These conclusions are rather general: the equations following from applications of two cycles of the form $(n+i, n+i+1)$, $i = 1, 2, ..., m-1$ are given by ((7.9)) for any state on the right hand side with label $R, (r_n, (m)\,)$ where (m) is the completely symmetric representation. For any state on the right hand side, the equations following from application of the two cycles $(i, i+1)$, $i = 1, 2, ..., m-1$ together with the normalization condition determines the expansion coefficients uniquely. The resulting solution is

$$\alpha_{123\cdots m} = \sqrt{\frac{1}{m!}\prod_{i=1}^{m-1}\left[\prod_{j=i+1}^{m}\frac{c_i - c_j + 1}{c_i - c_j}\right]}.$$

It is now straight forward to compute restricted characters for any restricted traces with labels $R, (r_n, \square\!\square\!\square)$.

Next, consider restricted traces with label $R, (r_n, \begin{smallmatrix}\square\\ \square\end{smallmatrix})$. Things work exactly as for the case just considered. The relevant expansion is

$$\begin{aligned}
|\,\ydiagram{3,2,1};\ydiagram{2,1}\,\ydiagram{1,1}\,\rangle &= \beta_{123}|\begin{ytableau} \; & \; & a \\ \; & b \\ c \end{ytableau}\rangle \;+\; \beta_{132}|\begin{ytableau} \; & \; & a \\ \; & c \\ b \end{ytableau}\rangle + \beta_{213}|\begin{ytableau} \; & \; & b \\ \; & a \\ c \end{ytableau}\rangle \\
&+\beta_{231}|\begin{ytableau} \; & \; & b \\ \; & c \\ a \end{ytableau}\rangle \;+\; \beta_{312}|\begin{ytableau} \; & \; & c \\ \; & a \\ b \end{ytableau}\rangle + \beta_{321}|\begin{ytableau} \; & \; & c \\ \; & b \\ a \end{ytableau}\rangle\,.
\end{aligned}$$

We will write the results of our analysis for general m. It is straight forward to obtain

$$-\beta_p = \frac{1}{c_{p^{-1}(i)} - c_{p^{-1}(i+1)}}\beta_p + \sqrt{1 - \frac{1}{(c_{p^{-1}(i)} - c_{p^{-1}(i+1)})^2}}\beta_{(i,i+1)\cdot p}, \qquad i = 1, 2, ..., m-1, \tag{7.10}$$

and hence

$$\beta_p = -\sqrt{\frac{c_{p^{-1}(i)} - c_{p^{-1}(i+1)} - 1}{c_{p^{-1}(i)} - c_{p^{-1}(i+1)} + 1}}\beta_{(i,i+1)\cdot p}.$$

Solving these equations together with the normalization condition gives

$$\beta_{123\cdots m}\sqrt{\frac{1}{m!}\prod_{i=1}^{m-1}\left[\prod_{j=i+1}^{m}\frac{c_i - c_j - 1}{c_i - c_j}\right]}.$$

As a check of our phase conventions, we have checked that α_p and β_p define orthogonal vectors.

We have not been able to treat the last case, restricted traces with labels $R, (r_n, \ydiagram{2,1})$, for general r_m. Further, the solution we have obtained is not unique, because the representation $\ydiagram{2,1}$ appears twice in the outer product $\square \times \square \times \square$. We have tried to use the freedom we have to choose the simplest possible solution. Consider the expansion

$$\begin{aligned}
|\,\ydiagram{3,2,1};\ydiagram{2,1}\,\ydiagram{2,1}\,i\rangle^a &= \gamma^a_{123\,i}|\begin{ytableau} \; & \; & a \\ \; & b \\ c \end{ytableau}\rangle \;+\; \gamma^a_{132\,i}|\begin{ytableau} \; & \; & a \\ \; & c \\ b \end{ytableau}\rangle + \gamma^a_{213\,i}|\begin{ytableau} \; & \; & b \\ \; & a \\ c \end{ytableau}\rangle \\
&+\gamma^a_{231\,i}|\begin{ytableau} \; & \; & b \\ \; & c \\ a \end{ytableau}\rangle \;+\; \gamma^a_{312\,i}|\begin{ytableau} \; & \; & c \\ \; & a \\ b \end{ytableau}\rangle + \gamma^a_{321\,i}|\begin{ytableau} \; & \; & c \\ \; & b \\ a \end{ytableau}\rangle\,.
\end{aligned}$$

a is a multiplicity label - it takes the values $1, 2$. i is a label for states in the carrier space of r_m; it takes one of the two values

$$i = 1 \leftrightarrow \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}\,, \qquad i = 2 \leftrightarrow \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,.$$

A straight forward (but tedious) computation now determines

$$\gamma^1_{123\,1} = -\frac{(c_{12}-2)\sqrt{(c_{23}+1)(c_{23}-1)(c_{13}+1)}}{d}, \qquad \gamma^1_{132\,1} = \sqrt{\frac{c_{23}-1}{c_{23}+1}}\gamma^1_{123\,1},$$

$$\gamma^1_{213\,1} = 2\frac{\sqrt{(c_{12}-1)(c_{12}+1)(c_{23}+1)(c_{23}-1)(c_{13}+1)}}{d}, \qquad \gamma^1_{312\,1} = \sqrt{\frac{c_{13}-1}{c_{13}+1}}\gamma^1_{213\,1},$$

$$\gamma^1_{231\,1} = -\frac{(c_{12}+1)(c_{23}+2)\sqrt{(c_{13}-1)}}{d}, \qquad \gamma^1_{321\,1} = \sqrt{\frac{c_{12}-1}{c_{12}+1}}\gamma^1_{231\,1},$$

$$\gamma^1_{123\,2} = \frac{c_{12}\sqrt{3(c_{23}+1)(c_{23}-1)(c_{13}+1)}}{d}, \qquad \gamma^1_{132\,2} = -\sqrt{\frac{c_{23}+1}{c_{23}-1}}\gamma^1_{123\,2},$$

$$\gamma^1_{213\,2} = 0, \qquad \gamma^1_{312\,2} = 0,$$

$$\gamma^1_{231\,2} = -\frac{(c_{12}-1)c_{23}\sqrt{3(c_{13}-1)}}{d}, \qquad \gamma^1_{321\,2} = -\sqrt{\frac{c_{12}+1}{c_{12}-1}}\gamma^1_{231\,2},$$

$$\gamma^2_{123\,1} = \frac{(c_{23}+1)\sqrt{3(c_{12}+1)(c_{12}-1)(c_{13}-1)}}{d}, \qquad \gamma^2_{132\,1} = \sqrt{\frac{c_{23}-1}{c_{23}+1}}\gamma^2_{123\,1},$$

$$\gamma^2_{213\,1} = -\frac{(c_{13}+1)\sqrt{3(c_{13}-1)}}{d}, \qquad \gamma^2_{312\,1} = \sqrt{\frac{c_{13}-1}{c_{13}+1}}\gamma^2_{213\,1},$$

$$\gamma^2_{231\,1} = -\frac{\sqrt{3(c_{12}+1)(c_{12}-1)(c_{23}+1)(c_{23}-1)(c_{13}+1)}}{d}, \qquad \gamma^2_{321\,1} = \sqrt{\frac{c_{12}-1}{c_{12}+1}}\gamma^2_{231\,1},$$

$$\gamma^2_{123\,2} = \frac{(c_{23}-1)\sqrt{(c_{12}+1)(c_{12}-1)(c_{13}-1)}}{d}, \qquad \gamma^2_{132\,2} = -\sqrt{\frac{c_{23}+1}{c_{23}-1}}\gamma^2_{123\,2},$$

$$\gamma^2_{213\,2} = \frac{(2c_{12}c_{23}+c_{12}-c_{23}+1)\sqrt{c_{13}-1}}{d}, \qquad \gamma^2_{312\,2} = -\sqrt{\frac{c_{13}+1}{c_{13}-1}}\gamma^2_{213\,2},$$

$$\gamma^2_{231\,2} = \frac{\sqrt{(c_{12}+1)(c_{12}-1)(c_{23}+1)(c_{23}-1)(c_{13}+1)}}{d}, \qquad \gamma^2_{321\,2} = -\sqrt{\frac{c_{12}+1}{c_{12}-1}}\gamma^2_{231\,2},$$

where $c_{ij} \equiv c_i - c_j$ and

$$d = \sqrt{6(c_1 - c_2)(c_2 - c_3)(c_1 - c_3)(2(c_1 - c_2)(c_2 - c_3) - 2c_2 + c_3 + c_1 + 1)}.$$

As a partial check of our phase conventions we have verified that

$$\{ |\,\yng(3,2,1) ; \yng(2,1)\,\yng(3) \rangle \,,\; |\,\yng(3,2,1) ; \yng(2,1)\,\yng(1,1,1) \rangle \,,\; |\,\yng(3,2,1) ; \yng(2,1)\,\yng(2,1)\, i \rangle^a \}$$

provides an orthonormal basis. This does not yet provide a complete treatment of the $m = 3$ case, because we have not yet considered the case that the three boxes removed share common sides. The generalization to this case is straight forward, using the methods we developed in this section.

7.3.3 Some example computations of restricted Littlewood-Richardson Coefficients

In this subsection we will discuss the computation of the restricted Littlewood-Richardson coefficients that will be needed to study graviton interactions in the next section. The first case we wish to consider are representations where R, r_n and r_m are totally antisymmetric. In the case of a single matrix, these correlators are dual to branes that have expanded in the S^5 of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry. Since $n + m$ is cut off at N it is natural to conjecture that the same interpretation holds even when the operator is built using two complex Higgs fields. As an example, for $n = 2$ and $m = 3$ we are discussing the restricted representation with labels

$$R = \yng(1,1,1,1,1) \equiv (1^5), \qquad (r_n, r_m) = (\yng(1,1,1), \yng(1,1)) \equiv (1^3, 1^2).$$

This particular class of restricted characters are particularly easy to compute: because the dimension of the representation and the dimensions of the restriction are equal, the restriction is trivial and we have ($\epsilon(\sigma)$ is 1 if σ is an odd permutation and 0 if σ is an even permutation)

$$\chi_{(1^{n+m}),(1^n,1^m)}(\sigma)=\chi_{(1^{n+m})}(\sigma)=(-1)^{\epsilon(\sigma)},$$

and

$$\chi^{(1^{n+m}),(1^n,1^m)}(\sigma)=\frac{n!m!}{(n+m)!}\chi_{1^{n+m}}(\sigma).$$

Consequently, for the case under consideration, we can express the restricted Littlewood-Richardson numbers in terms of the Littlewood-Richardson numbers as

$$f^{1^{n_1+n_2+m_1+m_2},(1^{n_1+n_2},1^{m_1+m_2})}_{1^{n_1+m_1},(1^{n_1},1^{m_1})\,1^{n_2+m_2},(1^{n_2},1^{m_2})}=\frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!}f_{1^{n_1+m_1}\,1^{n_2+m_2}\,1^{n_1+n_2+m_1+m_2}},$$

where we have used the formula

$$f_{RST}=\frac{1}{n_R!n_S!}\sum_{\alpha_1\in S_R}\sum_{\alpha_2\in S_S}\chi_R(\sigma_1)\chi_S(\sigma_2)\chi_T(\sigma_1\circ\sigma_2).$$

It is satisfying that upon setting $m_1=m_2=0$ our restricted Littlewood-Richardson numbers reproduce the standard Littlewood-Richardson numbers. Rules for the computation of the Littlewood-Richardson coefficients can be found, for example, in [68]. For the case under consideration here, the Littlewood-Richardson number is 1, so that

$$f^{1^{n_1+n_2+m_1+m_2},(1^{n_1+n_2},1^{m_1+m_2})}_{1^{n_1+m_1},(1^{n_1},1^{m_1})\,1^{n_2+m_2},(1^{n_2},1^{m_2})}=\frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!}.$$

Notice that these restricted Littlewood-Richardson numbers are all ≤ 1.

A second case of interest, are representations where R, r_n and r_m are totally symmetric. The corresponding single matrix correlators were conjectured to be dual to branes that have expanded in the AdS$_5$ of the AdS$_5\times$S^5 geometry. A beautiful test of this conjecture was given in [69]. We again conjecture that the same interpretation holds even when the operator is built using two complex Higgs fields. Again, this class of restricted characters are easy to compute because the dimension of the representation and the dimension of the restriction are equal. Exactly as above, we can express the restricted Littlewood-Richardson numbers in terms of the Littlewood-Richardson numbers as

$$
\begin{aligned}
f^{(n_{12}+m_{12}),((n_1+n_2),(m_1+m_2))}_{(n_1+m_1),((n_1),(m_1))\,(n_2+m_2),((n_2),(m_2))} &= \frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!} f_{(n_1+m_1)\,(n_2+m_2)\,(n_{12}+m_{12})} \\
&= \frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!},
\end{aligned}
$$

where we have used the fact that in this case the Littlewood-Richardson number is again 1. Again, upon setting $m_1 = m_2 = 0$ our restricted Littlewood-Richardson numbers reproduce the standard Littlewood-Richardson numbers.

Another case in which general results can be obtained, involves multiplying a polynomial in Z ($\chi_{R_1}(Z)$ with R_1 an irreducible representation of S_n) by a polynomial in X ($\chi_{R_2}(X)$ with R_2 an irreducible representation of S_m). The relevant restricted Littlewood-Richardson numbers are simply

$$
\begin{aligned}
f^{R,(r_n,r_m)}_{R_1,(R_1,\cdot)\,R_2,(\cdot,R_2)} &= \frac{d_R}{(n+m)!d_{r_n}d_{r_m}} \sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_m} \chi_{R_1}(\sigma_1)\chi_{R_2}(\sigma_2)\chi_{R,(r_n,r_m)}(\sigma_1\circ\sigma_2) \\
&= \frac{d_R}{(n+m)!d_{r_n}d_{r_m}} \sum_{\sigma_1\in S_n} \chi_{R_1}(\sigma_1)\chi_{r_n}(\sigma_1) \sum_{\sigma_2\in S_m} \chi_{R_2}(\sigma_2)\chi_{r_m}(\sigma_2) \\
&= \frac{n!m!}{(n+m)!}\frac{d_R}{d_{r_n}d_{r_m}}\delta_{R_1 r_n}\delta_{R_2 r_m}\,.
\end{aligned}
$$

We have not explicitly indicated a multiplicity label for r_n and r_m. This label is needed if there is more than one way to subduce (r_n, r_m) from R.

Finally, if we consider products of polynomials for which the total number of Xs participating is small, we can construct the explicit change of basis developed in the previous subsection. With this change of basis in hand, all restricted characters appearing in the formula for the restricted Littlewood-Richardson numbers can be computed using standard strand diagram techniques.

7.4 Graviton Interactions

A description of perturbative quantum gravity phrased in terms of graviton degrees of freedom is expected to fail at high enough energy, when the gravitons become strongly interacting. A perturbative description of the high energy physics needs to be phrased in terms of new degrees of freedom that are weakly interacting at high energy. The exact computations of [34], performed in the half BPS sector of IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ show that the sphere giant gravitons provide satisfactory degrees of freedom for a high energy description. The half BPS dynamics is captured by the dynamics of a single matrix. In this section, using the methods we have developed above, we will extend the results of [34] beyond the one matrix sector. Our conclusions are consistent with those of [34].

To estimate where the graviton description breaks down, we will estimate where the two point functions differ appreciably from their planar limit. This difference indicates that non-planar corrections are no longer suppressed and this implies that the large N orthogonality of the trace basis fails. The trace basis is dual, at low energies, to the supergravity Fock space. The fact that orthogonality fails tells us that there are non-negligible interactions mixing the graviton Fock space states; these interactions will invalidate perturbation theory. If we build traces that contain only Xs or only Zs, we know that orthogonality fails when we have $O(\sqrt{N})$ fields in each trace[66, 67]. Is this conclusion still valid when we consider traces containing both Xs and Zs? The computations of [66, 67] used the fact that the combinatorics associated with the Wick contractions can be computed using a zero dimensional complex matrix model. These techniques do not have an easy extension to the mixed traces that interest us. A more powerful alternative technique (employing symmetric group theory methods) to compute the single trace correlators was developed in [19]. Using the logic of [19], we are able to compute mixed trace correlators. The basic idea is that using the results of [27], we know how to compute mixed correlators in the restricted Schur basis. To get correlators of traces we simply need to change from the restricted Schur basis to the trace basis. This change of basis is easily accomplished using dual characters. Any single trace operator can be written as

$$\mathrm{Tr}\,(\sigma_c Z^{\otimes n}\otimes X^{\otimes m}) = \sum_{R,(r_n,r_m)} \chi^{R,(r_n,r_m)}(\sigma_c)\chi_{R,(r_n,r_m)}(Z,X),$$

where σ_c is an $n+m$ cycle. For concreteness we will consider the trace $\mathrm{Tr}\,(Z^n X^m)$ which corresponds to $\sigma_c = (1\,2\,3\,\cdots\,m+n-2\,m+n-1\,m+n)$, up to conjugation by an element of $S_n\times S_m$. The computation of the relevant restricted characters is carried out in appendix A.4. We find that this character vanishes unless r_n and r_m are both hooks - Young diagrams with at most a single column whose length is >1. Denote the number of rows in r_n by s_n+1 and the number of rows in r_m by s_m+1. To subduce (r_n,r_m), the Young diagram R must have at most two column with lengths >2. We again imagine that boxes are removed from R to compose r_m. The boxes that remain form r_n. The result of appendix A.4 says

$$\chi_{R,(r_n,r_m)}(\sigma_c) = \frac{(-1)^{s_n+s_m}}{mn}\left(\sum_i c_i - mN - \frac{m(m-2s_m-1)}{2}\right),$$

where the sum on i runs over all boxes removed from R to produce r_m. Thus, the dual character is

$$\begin{aligned}\chi^{R,(r_n,r_m)}(\sigma_c) &= \frac{(\mathrm{hooks})_{r_n}(\mathrm{hooks})_{r_m}}{(\mathrm{hooks})_R}\frac{(-1)^{s_n+s_m}}{mn}\left(\sum_i c_i - mN - \frac{m(m-2s_m-1)}{2}\right)\\ &= s_n!(n-s_n-1)!s_m!(m-s_m-1)!(-1)^{s_n+s_m}\times\mathcal{F},\end{aligned}$$

where the R dependent factor $\mathcal{F}$ is given by

$$\mathcal{F} = \frac{\left(\sum_i c_i - mN - \frac{m(m-2s_m-1)}{2}\right)}{(\mathrm{hooks})_R}.$$

Using the two point functions of [27] it is now straight forward to obtain

$$\langle \mathrm{Tr}\,(\sigma_c Z^{\otimes n}\otimes X^{\otimes m})^\dagger \mathrm{Tr}\,(\sigma_c Z^{\otimes n}\otimes X^{\otimes m})\rangle = \sum_{R,(r_n,r_m)} \frac{s_m!(m-s_m-1)!s_n!(n-s_n-1)!}{mn}$$

$$\times \mathrm{Dim}_R \left(\sum_i c_i - mN - \frac{m(m-2s_m-1)}{2} \right)^2 ,$$

where Dim_R is the dimension of $SU(N)$ representation R and the sum runs over all labels such that r_n and r_m are hooks. Denote the number of boxes in the first row of R by r_1 and the number of boxes in the second row of R by r_2. Denote the number of boxes in the first column of R by c_1 and the number of boxes in the second column of R by c_2. It is straight forward to see that

$$r_2 = m+n-s_m-s_n-r_1+1, \qquad c_2 = s_n+s_m-3-c_1, \qquad r_1+r_2+c_1+c_2 = n+m+4.$$

Summing over $R, (r_n, r_m)$ can be replaced by a sum over s_n, s_m, r_1 and c_1. The value of the sum is

$$\langle \mathrm{Tr}\,(\sigma_c Z^{\otimes n} \otimes X^{\otimes m})^\dagger \mathrm{Tr}\,(\sigma_c Z^{\otimes n} \otimes X^{\otimes m}) \rangle = \frac{N^2}{(n+1)(m+1)(N^2-1)} \times$$

$$\begin{aligned}
&\left(\frac{1}{(n+2)(m+2)} \left[\frac{(N+n+1)!}{N!} - \frac{(N-1)!}{(N-n-2)!} \right] \left[\frac{(N+m+1)!}{N!} - \frac{(N-1)!}{(N-m-2)!} \right] \right. \\
- \quad & \frac{1}{N(m+2)} \left[\frac{(N+n)!}{N!} - \frac{(N-1)!}{(N-n-1)!} \right] \left[\frac{(N+m+1)!}{N!} - \frac{(N-1)!}{(N-m-2)!} \right] \\
- \quad & \frac{1}{N(n+2)} \left[\frac{(N+n+1)!}{N!} - \frac{(N-1)!}{(N-n-2)!} \right] \left[\frac{(N+m)!}{N!} - \frac{(N-1)!}{(N-m-1)!} \right] \\
+ \quad & \left. \left[\frac{(N+n)!}{N!} - \frac{(N-1)!}{(N-n-1)!} \right] \left[\frac{(N+m)!}{N!} - \frac{(N-1)!}{(N-m-1)!} \right] \right) .
\end{aligned}$$

The above result is exact (i.e. to all orders in N) in the free field theory limit. Following [34] we make use of the identity

$$\frac{(N+p_1)!}{(N-p_0-1)!} = N^{p_0+p_1+1} \exp \left[\frac{1}{2N}(p_1-p_0)(p_0+p_1+1) + O(N^{-2}) \right] ,$$

valid when $p_0 \ll N$ and $p_1 \ll N$, to explore the behavior of our two point function. We find an appreciable difference from the planar answer (which is N^{n+m}) when n^2/N or m^2/N (or both) are held fixed. The leading behavior of our two point correlation function is

$$\langle \mathrm{Tr}\,(\sigma_c Z^{\otimes n}\otimes X^{\otimes m})^\dagger \mathrm{Tr}\,(\sigma_c Z^{\otimes n}\otimes X^{\otimes m})\rangle = \frac{4N^{n+m+2}}{nm}\sinh\frac{n^2}{2N}\sinh\frac{m^2}{2N}.$$

We have dropped terms of order $\frac{m}{N}$ and $\frac{n}{N}$. No finite order calculation in $1/N$ will reproduce this result - in this range of values for m, n perturbation theory breaks down. This suggests that, even when more than one matrix species participates in the trace, the validity of perturbation theory in terms of gravitons (that is single trace operators) breaks down when there are $O(\sqrt{N})$ matrices in the trace. The cautious reader might object that we have not built operators that correspond to chiral primaries of the SYM and hence that our operators are not dual to gravitons. This is indeed true. However, the effects we study arise because the large number of non-planar contractions possible over power the usual N^{-2} suppression of higher genus effects. Thus, this effect is really only sensitive to the number of fields appearing in each trace and not the specific details of how these fields are distributed.

We will now compute the three point function for three restricted Schur polynomials. In what follows we will always employ the multi particle normalization

$$\Gamma(1;2|1+2) = \frac{\langle \chi_{\{1\}}\chi_{\{2\}}\chi^\dagger_{\{1+2\}}\rangle}{||\chi_{\{1\}}||\,||\chi_{\{2\}}||\,||\chi_{\{1+2\}}||}$$

of [61]. Consider first the case that each restricted Schur has totally antisymmetric labels i.e. labels of the form $1^{n+m}, (1^n, 1^m)$. This corresponds to the interaction of three sphere giant gravitons. In this case it is straight forward, using the results of section 7.2 and 7.3.3 and of [27] to obtain

$$\begin{aligned}\Gamma(1;2|1+2) &= \frac{\langle \chi_{1^{n_1+m_1},(1^{n_1},1^{m_1})}\chi_{1^{n_2+m_2},(1^{n_2},1^{m_2})}\chi^\dagger_{1^{n_{12}+m_{12}},(1^{n_{12}},1^{m_{12}})}\rangle}{||\chi_{1^{n_1+m_1},(1^{n_1},1^{m_1})}||\,||\chi_{1^{n_2+m_2},(1^{n_2},1^{m_2})}||\,||\chi_{1^{n_{12}+m_{12}},(1^{n_{12}},1^{m_{12}})}||}\\ &= \sqrt{\frac{(N-n_1-m_1)!(N-n_2-m_2)!}{N!(N-n_{12}-m_{12})!}}\sqrt{\frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!}}.\end{aligned}$$

We have written this result as a product of two square root factors. The first factor on the last line has the same form as the one matrix result. If $m_1 =$

$m_2 = 0$, this factor is identically equal to the correlation function computed in the one matrix case. The growth of this term for different values of n_1+m_1 and n_2+m_2 has been considered in detail in [17, 34]. For both n_1+m_1 and n_2+m_2 fixed as $N \to \infty$ we find that[17, 34] $\Gamma(1;2|1+2) \sim O(1)$, so that these restricted Schur polynomials do not provide weakly coupled degrees of freedom for long wavelength (low energy) modes. For $\frac{n_1+m_1}{N}$ and $\frac{n_2+m_2}{N}$ fixed and small in the limit $N \to \infty$ we find that[17, 34] $\Gamma(1;2|1+2) \sim e^{-\alpha N}$ with α positive and $O(1)$: these restricted Schur polynomials provide weakly coupled degrees of freedom for high energy modes. The second factor is always ≤ 1. To see this, consider the binomial expansion of

$$(1+x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k, \qquad \text{where} \qquad \binom{m}{k} = \frac{m!}{k!(m-k)!}.$$

By comparing the coefficient of x^{r+s} coming from the expansion of $(1+x)^m$ times the expansion of $(1+x)^n$ to the coefficient of x^{r+s} coming from the expansion of $(1+x)^{m+n}$, we learn that

$$\binom{m}{r}\binom{n}{s} \quad + \quad \text{non negative integers} \quad = \binom{m+n}{r+s}.$$

Thus

$$\frac{\binom{m}{r}\binom{n}{s}}{\binom{m+n}{r+s}} \leq 1,$$

which proves that the second factor is ≤ 1. Notice that when $m_1 = m_2 = 0$, the second factor is identically equal to 1 so that our result correctly reduces to the one matrix result of [17]. In summary, in the multimatrix sector we find the giant graviton degrees of freedom are weakly coupled at high energy, consistent with the conclusions of [34].

Finally, it is equally easy to compute the correlation function in the case that all three giants interacting are in completely symmetric representations. This corresponds to the interaction of three AdS giant gravitons. The result is

$$\begin{aligned}\Gamma(1;2|1+2) &= \frac{\langle \chi_{(n_1+m_1),((n_1),(m_1))}\chi_{(n_2+m_2),((n_2),(m_2))}\chi^\dagger_{(n_{12}+m_{12}),((n_{12}),(m_{12}))}\rangle}{||\chi_{(n_1+m_1),((n_1),(m_1))}||\,||\chi_{(n_2+m_2),((n_2),(m_2))}||\,||\chi_{(n_{12}+m_{12}),((n_{12}),(m_{12}))}||} \\ &= \sqrt{\frac{(N+n_{12}+m_{12}-1)!(N-1)!}{(N+m_1+n_1-1)!(N+n_2+m_2-1)!}}\sqrt{\frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!}}\,.\end{aligned}$$

It is again easy to verify that if we set $m_1 = 0 = m_2$, we recover the one matrix result of [17]. For $\frac{n_1+m_1}{N}$ and $\frac{n_2+m_2}{N}$ fixed and small in the limit $N \to \infty$, we find $\Gamma(1;2|1+2) \sim e^{\alpha N}$ with α positive and $O(1)$ suggesting that these restricted Schur polynomials do not provide weakly coupled degrees of freedom for high energy modes.

7.5 Interpretation of Restricted Schur Polynomials

The restricted Schur polynomial $\chi_{R,(r_n,r_m)}(Z,X)$ has three labels R, r_n and r_m. The label r_n is naturally associated to the Zs and the label r_m to the Xs. It seems natural to think that the composite operator $\chi_{R,(r_n,r_m)}(Z,X)$ is constructed from the half BPS "partons" $\chi_{r_n}(Z)$ and $\chi_{r_m}(X)$. In this section we will see if we can gather some evidence that $\chi_{R,(r_n,r_m)}(Z,X)$ indeed has a partonic structure.

Before considering the parton structure of the restricted Schur polynomials, we recall a related relevant problem: the parton structure of hadrons[70]. At high energy, above about 10 GeV, proton-proton scattering produces a large number of pions, with momenta mainly collinear with the collision axis. The probability of producing a pion with a large component of momentum transverse to the collision axis is exponentially suppressed with the value of the transverse momentum. A picture of the proton as a loosely bound assemblage of partons was consistent with the observed data. It is at high energies, where the partons become weakly interacting (thanks to asymptotic freedom) that they are visible in the experimental results. At low energies the constituents are strongly interacting and the parton structure of the proton is not visible.

Is there an analogous limit in which we expect the constituents of the restricted Schur polynomial are weakly interacting? Do we see evidence for a partonic structure in this limit?

There are two distinct ways in which we could obtain a weakly interacting system. The parameter which controls the interactions among our proposed constituents is $\frac{1}{N}$. We could thus imagine taking $N \to \infty$ and systematically expanding our (exact) correlator results, keeping only the leading order. Alternatively, we could ask if there are situations where we expect the constituents are naturally weakly interacting. We will follow this second approach. Interactions between membranes are mediated by the open strings ending on the membrane's worldvolume; in general the string and the membrane can exchange momentum, that is, the string can exert a force on the membrane. For the special case of a nearly maximal sphere giant[3], this interaction is highly suppressed and the open strings attached to a maximal giant do not exert a force on it [51, 71, 72]. Thus, to get weakly interacting partons, we consider a bound state built from two partons, one of which is a boundstate of nearly maximal sphere giants. In what follows, we take r_n to be a Young diagram with c columns and p rows with c a number of $O(1)$ and $N-p$ a number of $O(1)$. From the results of section 7.3.3 we know that

$$\chi_{r_n}(Z)\chi_{r_m}(X) = \frac{n!m!}{(n+m)!d_{r_n}d_{r_m}}\sum_R\sum_i d_R\chi_{R,(r_n^{(i)},r_m^{(i)})}(Z,X),$$

where the sum is over all representations R that can subduce (r_n, r_m) and (i) is a multiplicity label distinguishing the different (r_n, r_m) representations that can be subduced. In general we would not expect such operators to be protected by supersymmetry. If we normalize the above operators so that they have a unit two point function we obtain

$$\hat{\chi}_{r_n}(Z)\hat{\chi}_{r_m}(X) = \sum_R\sum_i\sqrt{\frac{\mathrm{Dim}_R}{\mathrm{Dim}_{r_n}\mathrm{Dim}_{r_m}}}\hat{\chi}_{R,(r_n^{(i)},r_m^{(i)})}(Z,X), \tag{7.11}$$

where a hat denotes operators with unit two point function and Dim_R denotes the dimension of the $SU(N)$ irreducible representation labeled by R. It is simple to check that there is a single Young diagram R that dominates the sum on the right hand side; it has the form displayed in figure 4 below.

Indeed, any boxes stacked below r_n must be antisymmetrized with the other indices in the same column of the Young diagram; since the number of boxes already in the column is p, the index corresponding to the boxes stacked below r_n can only take $N-p=O(1)$ values. In addition, one divides by a big symmetry factor (it is $O(N)$). In contrast to this, the indices corresponding to boxes stacked next to r_n are symmetrized with boxes appearing in the same row. These can take $O(N)$ values. Since the number of boxes in the row is $O(1)$ one divides by an extra symmetry factor of $O(1)$. There is a

[3]To be more precise, we consider a giant graviton which carries momentum p with $N-p$ a number of $O(1)$.

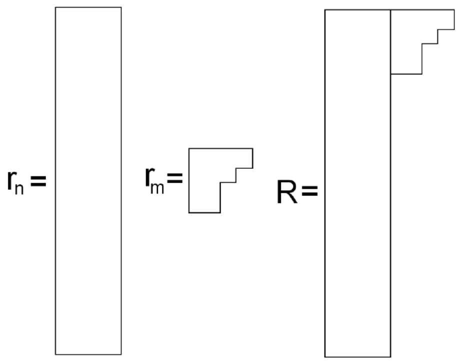


Figure 4: The Young diagram which dominates (7.11) is obtained by stacking r_m next to r_n as shown.

single way to subduce this dominant R so that there is no need for the i index. It is now clear that the leading correction to this term is of order $\frac{1}{N}$. Lets now consider the interaction of $\hat{\chi}_{R,(r_{n_1},r_{m_1})}(Z,X)$ and $\hat{\chi}_{S,(s_{n_2},s_{m_2})}(Z,X)$, where both r_{n_1} and s_{n_2} are boundstates of nearly maximal sphere giants. This is the analog of scattering two protons. Using (7.11) it is now straight forward to see that

$$\chi_{R,(r_{n_1},r_{m_1})}(Z,X)\chi_{S,(s_{n_2},s_{m_2})}(Z,X) = \frac{(\text{hooks})_R(\text{hooks})_S}{(\text{hooks})_{r_{n_1}}(\text{hooks})_{r_{m_1}}(\text{hooks})_{s_{n_2}}(\text{hooks})_{s_{m_2}}}\times$$

$$\times\sum_{t_{n_{12}}}\sum_{t_{m_{12}}}\sum_{T}\sum_{i}\frac{n_{12}!m_{12}!d_T}{(n_{12}+m_{12})!d_{t_{n_{12}}}d_{t_{m_{12}}}}f_{r_{n_1}s_{n_2}t_{n_{12}}}f_{r_{m_1}s_{m_2}t_{m_{12}}}\chi_{T,(t^{(i)}_{n_{12}},t^{(i)}_{m_{12}})}(Z,X)\,.$$

Even in the large N limit, the right hand side is a sum over a number of terms - many possible states can be formed as a result of the interaction of our two composites. Note however that the detailed structure of this interaction, in particular which representations $t_{n_{12}}$ and $t_{m_{12}}$ appear, is determined by the Littlewood-Richardson coefficients $f_{r_{n_1}s_{n_2}t_{n_1+n_2}}$ and $f_{r_{m_1}s_{m_2}t_{m_1+m_2}}$. These Littlewood-Richardson coefficients give the detailed form of the interaction between the half BPS partons

$$\chi_{r_{n_1}}(Z)\chi_{s_{n_2}}(Z) = \sum_{t_{n_1+n_2}} f_{r_{n_1}s_{n_2}t_{n_1+n_2}}\chi_{t_{n_1+n_2}}(Z),$$

$$\chi_{r_{m_1}}(X)\chi_{s_{m_2}}(X) = \sum_{t_{m_1+m_2}} f_{r_{m_1}s_{m_2}t_{m_1+m_2}}\chi_{t_{m_1+m_2}}(X).$$

Thus, the interaction of the composites is as a result of interactions between the partons. The picture that has emerged is very similar to the parton structure of hadrons described above.

8 Correlators of Operators with a Large R-Charge: The Cutting Rules

As discussed in the introduction, half BPS operators comprised of $O(N^2)$ fields are dual to new background geometries on the string theory side of the AdS/CFT correspondence. A natural way to explore the physics of these new geometries, is to compute correlation functions in the presence of the operator creating the new background. In this way operators with different string theory duals can be used to probe the background. Since the operator creating the background has $O(N^2)$ fields, this task is non trivial and in general, we require new techniques to evaluate such correlators. The focus of this section will be to develop such techniques.

For the special case of operators built only from Z or from $Z^\dagger$ [23] has shown that these correlators can be computed at leading order using the known product rule and two point function of Schur polynomials[17]. In section 9 we make use of the same inputs (product rule and two point function of Schur polynomials) and arrive at a simply stated rule allowing us to establish an exact relation between correlators in the trivial background and in the annulus background. This has some intriguing implications pertaining to a new expansion parameter for the perturbation theory of such correlators.

In [23] it was shown how to define operators in the super Yang-Mills theory dual to gravitons that are local in the bulk[4] of the dual quantum gravity. The definition of these local operators was in terms of a modified product rule, which is a refinement of the usual Littlewood-Richardson rule. When using the usual Littlewood-Richardson rule, to take the product $\Box \times R$, the single box would be added to all possible rows of the Young diagram R as long as R with the box added is again a legal Young diagram. In contrast to this, the local operators only add boxes to a specific location in the Young diagram. Thus, for example, we can define a local operator that would only add a box to the first row. We label these local operators by the location

[4] More precisely, they are local in the radial direction of the LLM plane and are located at $y = 0$ - i.e. on the LLM plane. They are s-waves on both S^3s in the geometry and are smeared along the ϕ coordinate of the LLM plane. See [23].

on the Young diagram to which they would add (in the case of acting with $\mathrm{Tr}\, Z$) or remove (in the case of $\mathrm{Tr}\, \frac{d}{dZ}$) boxes. These locations are labeled as a_i (for inward point corners) and b_i (for outward pointing corners) with i increasing as you move along the edge of the Young diagram from the upper right towards the lower left. See Figure 5 for an example of our labeling. Correlators of these local operators can be computed using the modified product rule[23]. Local operators built with $O(1)$ fields, that do not mix Z and $Z^\dagger$ are dual to gravitons; they are half BPS probes.

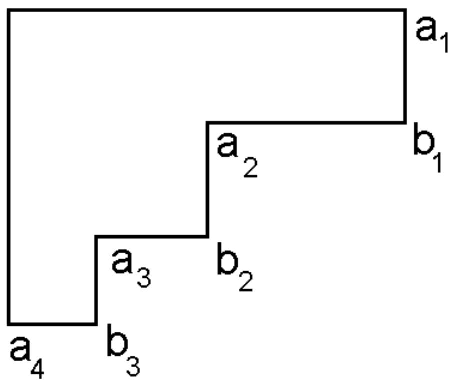


Figure 5: This figure illustrates our labeling of the corners of a Young diagram.

Probing the background with an operator that is not half BPS gives much richer information. In this case we have two natural possibilities: we can excite the background by attaching an open string to obtain a restricted Schur polynomial along the lines of [30, 31, 32, 33], or we could probe the new background with closed strings[78, 79, 23]. The interpretation of the open string excitation is not at all obvious. When the $\mathcal{R}$-charge of the operator to which the string is attached is $O(N)$, we know that the excitation indeed behaves like an open string attached to a giant graviton [28, 30, 51, 71, 69, 31, 32, 33]. These excited giant graviton operators are the restricted Schur polynomials. In this case the backreaction of the giant graviton can be neglected and the system is well described as a giant graviton, with open strings attached, moving in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry. This is nothing like the situation we study in this section. When the operator to which the open string is attached has an $\mathcal{R}$-charge of $O(N^2)$ it deforms the geometry - it is not a surface on which open strings can end, it is a new classical geometry: a new metric with some background fluxes. Our results clearly show that there is nothing special about how the endpoints of the string interact; they behave just like the bulk of the string. This is a clear demonstration that there is no brane on which string endpoints end[5]: the operator which is being excited is not a membrane; its a new geometry. To arrive at this conclusion, we need to compute correlators of traces that mix Z and $Z^\dagger$.

To probe the geometry with a closed string, one needs to compute correlators of single trace operators of the form

$$\mathrm{Tr}\,(Y Z_{a_i}^{n_1} Y Z_{a_i}^{n_2} Y \cdots Y Z_{a_i}^{n_L}).$$

This closed string is localized at the corner a_i in the geometry. Because these operators are nearly BPS their anomalous dimensions receive only a small correction and we can safely work to one loop. By studying this correction, we can obtain geometric information about the new background [78, 79, 23] indicating that this probe is indeed a valuable source of information about the geometry. The Wick contraction of the Y fields is straight forward because there are no Ys in the operator which creates the new background. After Wick contracting the Y fields, we are left with the problem of computing correlators of traces that mix Z and $Z^\dagger$.

These mixed correlators cannot be computed using the modified product rule. In [23] it was conjectured that these mixed correlators can be computed using modified ribbon diagrams. The modification simply amounts to rescaling the old propagator by c/N, where c is the weight of the box added to the background Young diagram by the (local) operator.

In this section we develop techniques that allow the direct computation of these correlation functions. Our results are in perfect agreement with the conjecture of [23]. Although we have focused on half BPS backgrounds our results will certainly be applicable more generally. In situations in which

[5]Of course, it is possible to excite giant gravitons on these geometries, in which case open strings excitations do appear. The perturbative string spectrum contains no open strings.

backreaction can be ignored, we have already developed techniques for computing the correlation functions of restricted Schur polynomials [31, 32, 33]. In these cases contractions between fields belonging to open string words and the remaining fields in the restricted Schur, make a subleading contribution in a systematic large N expansion. We will argue that back reaction in the gauge theory is accounted for by including these contractions. Our approach to computing these extra contributions starts by noting that the two point correlator (we suppress spacetime dependence which plays no role in these considerations)

$$\left\langle (Z^\dagger)^k_l Z^i_j \right\rangle = \delta^i_l \delta^k_j,$$

is reproduced by identifying

$$(Z^\dagger)^k_l \leftrightarrow \frac{d}{dZ^l_k}.$$

In this way, the contributions to a correlation function of two restricted Schur polynomials coming from contractions between Zs that belong to the open string and Zs that belong to the brane, can be written as a differential operator acting on the restricted Schur polynomials. This differential operator will in general, contain a product of derivatives with respect to the open string words as well as derivatives with respect to Z and $Z^\dagger$. We give a rule for "cutting" any such product up into eight basic types of derivatives and then derive simple formulas for the action of these derivatives. In this way, we can compute arbitrary mixed trace correlators, in any background, to any order in a systematic large N expansion. By specializing to the annulus geometry, we find significant simplifications allowing us to prove the modified ribbon rule of [23]. We then consider LLM geometries that correspond to a set of well separated concentric rings. The rings give a picture of the eigenvalue density of Z[23]: the eigenvalues split into well separated clumps. In the large N and large 't Hooft coupling limit the off diagonal modes connecting eigenvalues in different rings will be very heavy and decouple. Thus, Z becomes block diagonal with the number of blocks matching the number of rings. Recycling the annulus result then gives us a more general proof of the modified ribbon rule. This section is arranged as follows: In the next subsection, we consider "open string excitations" of the annulus background. The treatment of closed string excitations then follows, with no extra work. In section 8.2 we generalize our results to backgrounds which correspond to a set of concentric rings. In appendix B we collect some technical details and examples.

8.1 Backreaction: Annulus Geometry

The calculation of two point correlation functions of restricted Schur polynomials with open strings attached has been studied in [30, 31, 32, 33]. In these studies, contractions between fields in the open string and fields in the operator representing the brane were neglected. In the present considerations, the number of fields in the restricted Schur polynomial is $O(N^2)$. Operators with $\mathcal{R}$ charge of $O(N^2)$ are dual to new geometries, so that the back reaction of the operator must be taken into account. In section 8.1.1 we will argue that the contractions between fields in the open string word and the remaining fields in the operator can no longer be neglected. This is how the backreaction of the operator on the geometry is accounted for in the gauge theory. The open string words that we consider will use Z and Y as letters. To compute correlators in the large N limit, it is useful to treat the Ys as defining a lattice populated by Zs. The Zs themselves can be represented by Cuntz oscillators, which simply keep track of the planar contractions. In this way the problem of computing anomalous dimensions of operators becomes the problem of computing the spectrum of a Cuntz oscillator Hamiltonian. In section 8.1.2 we will argue that the net effect of the backreaction is to produce a scaling of the Cuntz oscillators, in agreement with [23]. A special case of this result was first obtained in [79], for an LLM geometry with annulus boundary condition on the LLM plane. In section 8.1.3 we will show that the open string endpoints behave exactly like the bulk of the string. We will further argue that the "open string" excitations are best thought of as closed strings propagating on a new background. Finally, in section 8.1.4 we consider probing the new backgrounds with closed strings.

8.1.1 Brane/Sring Contractions

To simplify the presentation of our methods, we will study an operator labeled by a rectangular Young diagram with N_1 rows and M_1 columns. Denote the irreducible representation of $S_{N_1 M_1}$ that this Young diagram corresponds to by R. We will consider exciting this BPS operator by attaching a single open string. The open string word has to be associated to the box in the[6] N_1th row and M_1th column, since this is the only box that can be removed to leave a valid Young diagram. Denote the irreducible representation of $S_{N_1 M_1 - 1}$ obtained by removing the box associated to the open string by R_1. The operator we study is

$$\begin{aligned}
\chi^{(1)}_{R,R_1}(Z,W) &= \frac{1}{(n-1)!}\sum_{\sigma\in S_n} \mathrm{Tr}_{R_1}\left(\Gamma_R(\sigma)\right) Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}} W^{i_n}_{i_{\sigma(n)}} \\
&\equiv \mathcal{F}(R,R_1)^a_b \, W^b_a. \qquad (8.1)
\end{aligned}$$

For concreteness, consider an open string with a single impurity

$$W^j_i = (Y^{n_1} Z Y^{J-n_1})^j_i.$$

We assume that J is $O(\sqrt{N})$ with $g_2 = \frac{J^2}{N} \ll 1$ so that when contracting the open string words we need only sum planar diagrams[66, 67]. The correlation function we wish to compute is (attach the same open string word to both operators)

$$I_{R\,R_1,S\,S_1} = \left\langle \chi^{(1)}_{R,R_1} \chi^{(1)}_{S,S_1}{}^{\dagger} \right\rangle .$$

We will seperate the computation of this correlator into two pieces: $I^{(0)}_{R\,R_1,S\,S_1}$ obtained by neglecting contractions between the impurity in the open string word and fields in the $\mathcal{F}^a_b$ piece of the operator and $I^{(1)}_{R\,R_1,S\,S_1}$ obtained by contracting the impurity in the open string word with a field in the $\mathcal{F}^a_b$ piece of the operator.

First, consider $I^{(0)}_{R\,R_1,S\,S_1}$. Using the results of [31], we find

$$I^{(0)}_{R\,R_1,S\,S_1} = N_1 M_1 N^J f_R \left(1 + O(g_2^2)\right) \delta_{RS}\delta_{R_1 S_1}. \qquad (8.2)$$

[6] The row closest to the top is the first row; the leftmost column is the first column.

Associate a weight $N - j + i$ to the box in the ith column and jth row of R. f_R is the product of the weights of the Young diagram R (see section 3.1). In the language of [31] only the F_0 contraction of the open strings contribute in the large N limit of the correlator (8.2). If $R \neq S$ but $R_1 = S_1$, then in the language of [31], the only contribution comes from the F_1 contraction of the open string words. It is straight forward to consider this case using our methods, although we do not do so here.

Next, consider $I^{(1)}_{R\,R_1,S\,S_1}$. After contracting all of the Y fields in W with the $Y^\dagger$ fields in $W^\dagger$, contract a Z in W with a $Z^\dagger$ in $\mathcal{F}^\dagger$ and a $Z^\dagger$ in $W^\dagger$ with a Z in $\mathcal{F}$. We obtain

$$\begin{aligned} I^{(1)}_{R\,R_1,S\,S_1} &= N^{J-2}\left\langle \frac{d\mathcal{F}(R,R_1)^a_b}{dZ^c_d}\frac{d(\mathcal{F}(S,S_1)^\dagger)^b_a}{d(Z^\dagger)^d_c}\right\rangle \\ &= N^{J-2}\left\langle \frac{d}{dZ^c_d}\frac{d}{d(Z^\dagger)^d_c}\frac{d}{dW^b_a}\frac{d}{d(W^\dagger)^a_b}\chi^{(1)}_{R,R_1}\chi^{(1)}_{S,S_1}{}^\dagger\right\rangle \\ &= N^{J-2}\left\langle \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi^{(1)}_{R,R_1}\chi^{(1)}_{S,S_1}{}^\dagger\right\rangle . \end{aligned}$$

We will now introduce a convenient graphical notation. The derivative operator that we need to consider is determined by the fields from the open strings that are contracted with $\mathcal{F}$ and $\mathcal{F}^\dagger$. Our notation keeps track of these fields and gives a simple picture from which we can read off the relevant derivative operator. We denote $\mathcal{F}$ and $\mathcal{F}^\dagger$ by open ellipses, with a single index line entering the ellipse and a single index line leaving the ellipse. We do not draw the fields in $\mathcal{F}$ and $\mathcal{F}^\dagger$ or their contractions. The contractions of fields in the open string words are drawn using the usual ribbon diagram (also called "fat graph" or "double line") representation. The Y contractions are given by filled ribbons. The Z contractions are empty ribbons. Fields left uncontracted in the diagram are to be contracted with the fields in $\mathcal{F}$ and $\mathcal{F}^\dagger$. The graphical representation of the two terms we have considered are given in figure 6.

To read the derivative operator from the diagram, replace each upper "open stub" (= uncontracted Z field) by a derivative with respect to $Z^\dagger$, each lower "open stub" (= uncontracted $Z^\dagger$ field) by a derivative with respect to

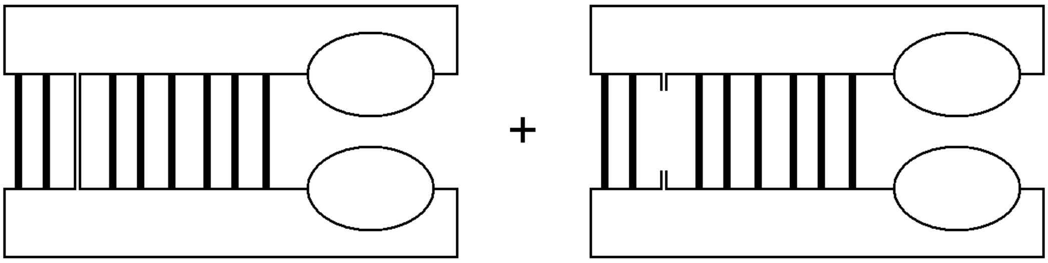


Figure 6: The graphical representation of $I^{(0)}_{RR_1,SS_1}$ and $I^{(1)}_{RR_1,SS_1}$. We have set $n_1 = 2$ and $J = 8$.

Z, the upper ellipse by a derivative with respect to the open string word W and the lower ellipse by a derivative with respect to the open string word $W^\dagger$. All derivatives in the same index loop are in the same trace.

In general, when we have many impurities in the open string word, we may have multiple contractions between fields belonging to the open strings and fields in $\mathcal{F}$ and $\mathcal{F}^\dagger$. In all of these cases we will be able to write these contributions as the expectation value of a derivative operator acting on $\chi^{(1)}_{R,R_1}\chi^{(1)}_{S,S_1}{}^\dagger$. The precise structure of the derivative operator will depend on the details of the specific contractions we consider. As another example, if the reader translates the diagram shown in figure 7, she should obtain

$$\left\langle \mathrm{Tr}\left(\frac{d^2}{dZ^2}\right)\mathrm{Tr}\left(\frac{d^2}{dZ^{\dagger 2}}\right)\mathrm{Tr}\left(\frac{d^3}{dZ^3}\frac{d^3}{dZ^{\dagger 3}}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi^{(1)}_{R,R_1}\chi^{(1)}_{S,S_1}{}^\dagger\right\rangle .$$

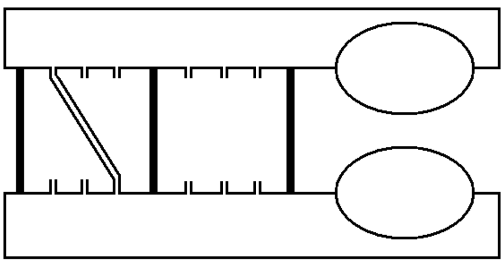


Figure 7: The graphical representation of one term contributing to the correlator $\left\langle \chi^{(1)}_{R,R_1}\chi^{(1)}_{S,S_1}{}^\dagger \right\rangle$. The open string word $W = YZ^3YZ^3Y$.

To get the full set of contributions to the correlator we need to draw all distinct diagrams allowed such that all possible connections of solid ribbons are included, and all possible combinations of connections of hollow ribbons as well as disconnected stumps are included.

The fact that we can account for contractions between fields in the open string words and fields in $\mathcal{F}$ or $\mathcal{F}^\dagger$ as a derivative operator acting on the restricted Schur polynomials is a useful observation because, in general, we can break an arbitrary derivative operator into a product of eight basic types of derivatives, as shown in appendix B.1. We call this process "cutting". The first cutting rule allows us to cut single derivatives out of any given trace[7] to leave a product of alternating holomorphic and antiholomorphic derivatives. The second rule allows us to cut a product of a holomorphic and an antiholomorphic derivative out of a given trace. In both cases the restricted Schur polynomial is modified by inclusion of an extra factor in the restricted character. The reader can consult appendix B.1 for the details. The action of these basic derivatives on a general restricted Schur polynomial, is described by the simple formulas collected in appendix B.2 along with the reduction rule reviewed in section 5.2.1. After applying the cutting and then the reduction rules, it is straight forward to obtain

$$I^{(1)}_{R\,R_1,S\,S_1} = N^{J-2}(N_1M_1)^2 f_R \left(1 + O(g_2^2)\right) . \tag{8.3}$$

Comparing (8.2) and (8.3), we see that the contraction between the impurity in the open string and fields in $\mathcal{F}(R, R_1)$ and $\mathcal{F}(S, S_1)^\dagger$ need only be taken into account when N_1M_1 is $O(N^2)$. This is precisely the regime in which the operator is dual to a very heavy state whose back reaction on the original $\mathrm{AdS}_5\times\mathrm{S}^5$ space produces a new geometry, so it is natural to interpret these contractions as the field theory accounting of the back reaction of the heavy state: by including these contractions, the string "interacts with the back reacted geometry". This is the key result of this section, and although we have only illustrated it in a simple example the conclusion is general.

Summary: *The contractions between fields in the open string word and the*

[7] To cut a holomorphic (antiholomorphic) derivative out of the trace the derivative on its left must also be holomorphic (antiholomorphic).

remaining fields in the operator need only be taken into account when the number of fields in the operator creating the background is $O(N^2)$. These contractions are the field theory accounting of the back reaction of this heavy state.

8.1.2 Modified Cuntz Oscillators

In this section, we will set $N_1 = N$ and $M_1 = M$. This corresponds to taking an annulus boundary condition for the dual LLM geometry. In this case, we can have excitations of the two edges of the annulus[23]: by acting with Z_a we add boxes to the upper right corner of the Young diagram (corresponding to the outer edge of the annulus) and by acting with $\frac{d}{dZ_b}$ we erode boxes from the lower right corner (corresponding to the inner edge of the annulus). There is a huge simplification that arises for the annulus: we can simply replace the local operators Z_a and $\frac{d}{dZ_b}$ by Z and $\frac{d}{dZ}$. This is simply because Z is unable to add boxes anywhere except the first few rows and $\frac{d}{dZ}$ is unable to remove boxes from anywhere except the last few rows. Our open string lives at the outer edge of the annulus which implies that R_1 is a rectangle with M columns and N rows and R has one extra box in the first row, giving $M + 1$ boxes in the first row. The simplest situation in which to illustrate our result is to consider open string excitations that have multiple impurities at a single site. For 9 impurities, the open string word is $W^i_j = (Y(Z)^9Y)^i_j$. We will get contributions from contracting n impurities in the open string with fields in $\mathcal{F}, \mathcal{F}^\dagger$ for $n = 0, 1, 2, ..., 9$. There are $9!/(n!(9 - n)!)$ distinct contractions for a given n. The specific details of the contractions matters. For example, in the case that $n = 4$, if none of the impurities in the open string that are contracted with $\mathcal{F}, \mathcal{F}^\dagger$ are adjacent (see figure 8), we obtain the following contribution (this formula is correct to leading order at large N)

$$N^2 \left\langle \left[\mathrm{Tr} \left(\frac{d}{dZ} \frac{d}{dZ^\dagger} \right) \right]^4 \mathrm{Tr} \left(\frac{d}{dW} \frac{d}{dW^\dagger} \right) \chi^{(1)}_{R,R_1} (\chi^{(1)}_{R,R_1})^\dagger \right\rangle = (MN)^4 N^2 \frac{M+N}{N} f_R \left(1 + O(g_2^2)\right). \tag{8.4}$$

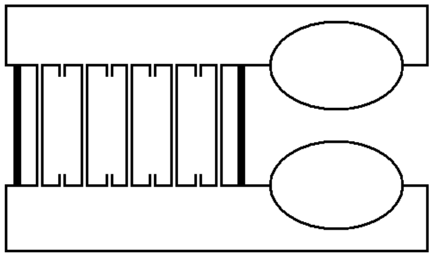


Figure 8: The diagram giving the contribution in (8.4)

Now consider the contribution coming from the term with all four impurities adjacent (see figure 9)

$$\begin{aligned} I_4 &= N^5 \left\langle \mathrm{Tr}\left(\left[\frac{d}{dZ}\right]^4 \left[\frac{d}{dZ^\dagger}\right]^4\right) \chi^{(1)}_{R,R_1} (\chi^{(1)}_{R,R_1})^\dagger \right\rangle \\ &= N^5 \left\langle \mathrm{Tr}\left(\frac{d}{dX_4}\frac{d}{dX_4^\dagger}\right) \prod_{i=1}^{3} \mathrm{Tr}\left(\frac{d}{dX_i}\right) \mathrm{Tr}\left(\frac{d}{dX_i^\dagger}\right) \chi^{(1,4)}_{R,R_1;P} (\chi^{(1,4)}_{R,R_1;P})^\dagger \right\rangle, \end{aligned} \tag{8.5}$$

where

$$P = (n-4, n-3, n-2, n-1).$$

To obtain this expression, we have used the methods of appendix B.1 to decompose the derivative operator into a product of basic types. This can now be evaluated using the methods developed in appendix B.2 and the technology reviewed in section 5. The details of some similar example calculations are summarized in appendix B.3. Although the details are completely different to the (8.4) calculation, we find *exactly the same result*

$$I_4 = N^6 M^4 \frac{M+N}{N} f_R \left(1 + O(g_2^2)\right). \tag{8.6}$$

This is general: if we have p impurities at a site, the contribution to the correlator coming from all $p!/(n!(p-n)!)$ contractions between n impurities on the open string and the fields in $\mathcal{F}, \mathcal{F}^\dagger$ are all the same size. Further, it is

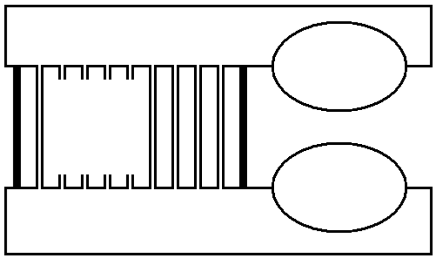


Figure 9: The diagram giving the contribution in (8.6)

now straight forward to check that each of the terms contributing when we have n impurities in the open string words contracting with fields in $\mathcal{F}$ and $\mathcal{F}^\dagger$, gives

$$N^{p+1}\left(\frac{M}{N}\right)^n \frac{M+N}{N} f_R$$

and therefore that for our example with $p = 9$

$$\langle \chi^{(1)}_{R,R_1} \chi^{(1)}_{R,R_1}{}^\dagger \rangle = \sum_{n=0}^{9} N^{10}\left(\frac{M}{N}\right)^n \frac{M+N}{N} f_R \frac{9!}{n!(9-n)!} = N^9(M+N) f_R \left(1+\frac{M}{N}\right)^9 .$$

If we had p impurities in the site, we would have obtained

$$\langle \chi^{(1)}_{R,R_1} \chi^{(1)}_{R,R_1}{}^\dagger \rangle = (M+N) N^p f_R \left(1+\frac{M}{N}\right)^p . \tag{8.7}$$

Open string words are most appropriately described by a Cuntz chain due to the dynamical nature of the number of fields in the word [51]. This is contrasted with the spin chain description of operators dual to closed strings where the number of fields comprising the operators is fixed. As discussed in section 8.1, in the Cuntz chain description we treat the Y fields as defining a lattice populated by the Z fields (impurities). We thus obtain particles described by Cuntz oscillators hopping on a lattice of fixed size where the number of particles populating the lattice is dynamical. It is noteworthy that the coherent state expectation value of the Cuntz chain Hamiltonian for an open string attached to a giant graviton reproduces the open string action for an open string attached to a sphere giant [51, 71] and an AdS giant [69] in $\mathrm{AdS}_5\times\mathrm{S}^5$ and for an open string attached to a sphere giant in a deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ background [108]. The ordinary Cuntz oscillators satisfy (we associate one of these oscillators to each site of the chain):

$$aa^\dagger = 1, \qquad a^\dagger a = 1 - |0\rangle\langle 0|.$$

Adding an extra impurity in the open string word corresponds to applying another Cuntz oscillator to the state. Clearly, in view of (8.7), the correct way to account for the background is to rescale the Cuntz oscillators describing the impurities in the open string

$$aa^\dagger = \left(1+\frac{M}{N}\right), \qquad a^\dagger a = \left(1+\frac{M}{N}\right)(1 - |0\rangle\langle 0|)\,.$$

The factor $1+\frac{M}{N}$ is $\frac{c}{N}$ with c the weight of boxes in the upper right hand region of the Young diagram.

We can give this calculation a slightly different interpretation which will allow us to state the general result: after contracting the Y fields planarly, the above correlator can be viewed as the expectation value of a product of single trace operators, in the new background

$$\left\langle \chi^{(1)}_{R,R_1}(Z,W)(\chi^{(1)}_{R,R_1}(Z,W))^\dagger \right\rangle \;=\; \frac{\mathrm{Hooks}_R}{\mathrm{Hooks}_{R_1}}\frac{f_R}{f_{R_1}}\left\langle \mathrm{Tr}\,((Z_a)^n(Z_a^\dagger)^n)\chi_{R_1}(Z)(\chi_{R_1}(Z))^\dagger \right\rangle .$$

So that,

$$\frac{\left\langle \chi^{(1)}_{R,R_1}(Z,W)(\chi^{(1)}_{R,R_1}(Z,W))^\dagger \right\rangle}{f_R} = \frac{N+M}{N}\frac{\left\langle \mathrm{Tr}\,((Z_a)^n(Z_a^\dagger)^n)\chi_{R_1}(Z)(\chi_{R_1}(Z))^\dagger \right\rangle}{f_{R_1}},$$

i.e. these correlators are related by a constant factor independent of n. These fields only add boxes in the first few rows, i.e. in the upper right region of the Young diagram. They are thus local operators according to [23]. Thus, when computing correlators in the annulus background, we can reproduce the above result by using free field theory, after rescaling all propagators by

$\frac{c}{N}$ where c is the weight of the added boxes. Below we will show how this generalizes for an LLM background comprised of concentric annuli.

In appendix B.5 we give a rigorous derivation of this result. We also compute the expectation value of

$$O = \langle \mathrm{Tr}\left(\frac{d^n}{dZ^n}\frac{d^n}{d(Z^\dagger)^n}\right)\rangle\,,$$

for the annulus background.

Summary: *In the annulus background the original matrix Z is a local operator in the sense that it only adds boxes in the first few rows of the Young diagram. The derivative $\frac{d}{dZ}$ is also a local operator in the sense that it only removes boxes from the last few rows of the Young diagram. To compute correlation functions of these local operators one uses ribbon diagrams, where each ribbon carries an extra factor of $\frac{c}{N}$ where c is the weight of the boxes added or removed by the local operator.*

8.1.3 Tying up loose ends

Since we are considering open string excitations, we need to pay some attention to the end point interactions. General methods to determine the interactions for a single string[32] or for multistrings[33] are known. The strength of this interaction is given by $\sqrt{\frac{c}{N}}$. Consider a string built using $L+1$ Ys. These Ys form a lattice on which the Zs hop. The Hamiltonian for the string takes the form (this endpoint interaction assumes that the open string is attached to a single brane and not a boundstate of branes - see [32])

$$H = 2\lambda\sum_{l=1}^{L} a_l^\dagger a_l - \lambda\sum_{l=1}^{L-1}(a_l^\dagger a_{l+1} + a_l a_{l+1}^\dagger) + \sqrt{\frac{c}{N}}\lambda(a_1 + a_1^\dagger + a_L + a_L^\dagger). \quad (8.8)$$

Here c is the weight of the box occupied by the open string.

If the $\mathcal{R}$-charge of the background is $O(1)$ or $O(\sqrt{N})$ the operator we are studying is dual to a graviton or a string, but not a brane. In this case, the weight of the box occupied by the open string is $O(N)$, so that $\sqrt{\frac{c}{N}} = 1$.

Further, the Cuntz oscillators satisfy

$$a_l a_l^\dagger = 1, \qquad a_l^\dagger a_l = 1 - |0\rangle\langle 0|.$$

This implies that hopping onto and off of the string is no different from hopping between bulk sites. This implies that the end point dynamics is not special: the end points are not "stuck to a brane". Our string is a closed string, not an open string. If we now consider the case of an operator with an $\mathcal{R}$ charge of $O(N)$ and further that the operator has $O(1)$ rows (or O(1) columns), then the open string is attached to a box with a weight of αN with $\alpha = O(1)$. The Cuntz oscillators are unchanged. This implies that the end point dynamics is special: hopping onto and off of the string has a weight $\sqrt{\alpha}$. In this case, we do indeed have an open string excitation, as has been verified in [28, 51, 71, 69, 32]. Finally, consider the case of interest to us here, when the operator has an $\mathcal{R}$-charge of $O(N^2)$ and all edges with a length of $O(N)$. This requirement on the length of all edges is needed if the operator is to correspond to a regular LLM geometry. [8] In this case, the Cuntz oscillators are modified to

$$a_l a_l^\dagger = \frac{c}{N}, \qquad a_l^\dagger a_l = \frac{c}{N}\left(1 - |0\rangle\langle 0|\right).$$

To make all dependence on the weight of the box occupied by the open string explicit, use the rescaled oscillators $a_l = \sqrt{\frac{c}{N}}\tilde{a}_l$. In terms of these oscillators

$$H = \frac{c}{N}\left(2\lambda\sum_{l=1}^{L}\tilde{a}_l^\dagger\tilde{a}_l - \lambda\sum_{l=1}^{L-1}(\tilde{a}_l^\dagger\tilde{a}_{l+1} + \tilde{a}_l\tilde{a}_{l+1}^\dagger) + \lambda(\tilde{a}_1 + \tilde{a}_1^\dagger + \tilde{a}_L + \tilde{a}_L^\dagger)\right), \tag{8.9}$$
$$\tilde{a}_l\tilde{a}_l^\dagger = 1, \qquad \tilde{a}_l^\dagger\tilde{a}_l = 1 - |0\rangle\langle 0|.$$

Once again there is nothing special about the string endpoints which behave

[8] Indeed, a rectangular Young diagram with M columns and N rows, plus one more column with αN boxes with $\alpha = O(1)$ corresponds to a finite size D3-brane on the back reacted LLM geometry. This D3 will admit open string excitations. We are considering operators dual to geometries without any D3-branes which is achieved precisely by our restriction that all edges have a length of $O(N)$.

exactly like the bulk of the string! The astute reader might object that hopping in the bulk is between two sites of the string which is different to hopping off of and onto the string, which is what happens at the string endpoints. This is simply an artifact of how we have split the restricted Schur polynomial into a string plus background. Indeed as Zs hop off the string, extra boxes are added to the Young diagram. One could rather describe these extra boxes as impurities in an $L+1$th site of the string. For example, if there are no extra boxes in the Young digram, no Zs can hop onto the string; with the new interpretation we would say that the $L+1$th site is empty and hence nothing can hop out of this site. There are two facts that make this reinterpretation possible:

- Each time we add a Z in the open string word, we get an extra index loop giving an extra N and an extra $\frac{c}{N}$ from the extra (rescaled ribbon) propagator, giving a total extra factor of c. By adding an extra box, the factor of the product of the weights (f_R) in the restricted Schur correlation function has an extra factor of c. Thus adding a box or an impurity contributes the same factor.

- We deal with Cuntz oscillators, that is, distinguishable particles. Thus, there are no extra $n!$ type normalizations that appear for n bosons. Corresponding to this, the correlators of the restricted Schur polynomials is proportional to 1 if the Young diagrams participating have the same shape, and to 0 otherwise. (See Appendix B.6 for a detailed matching.)

This again suggests that the excitation is best thought of as a closed string and not an open string. This has an appealing interpretation: the operator we are exciting has an $\mathcal{R}$-charge of $O(N^2)$. It does not correspond to a brane, but rather to a new geometry. In this case we do not expect to see any open string excitations in the spectrum. It is satisfying that this is indeed the case.

Summary: *The dynamics of the string "endpoints" is identical to the dynamics of the bulk of the string. The excitation behaves like a closed string,* *not an open string. This is expected since the operator being excited is dual to a new background and not a brane.*

8.1.4 Back Reaction: Closed Strings

To consider closed strings we should probe the geometry with a single trace operator

$$\mathcal{O} = \mathrm{Tr}\,(YZ^{n_1}YZ^{n_2}Y\cdots YZ^{n_L}).$$

The leading large N contribution to this correlator is given by contracting the Y fields planarly. The above correlator then becomes the expectation value of a product of single trace operators, in the new background. This has been computed above.

8.2 Backreaction: Multi Rings

In this section we will consider LLM geometries that correspond to a set of well separated thick rings. The background with three rings would for example, be described by a Young diagram with N rows and the same shape as the one in Figure 5. The black rings can be viewed as a picture of the eigenvalue density of Z[23]. Thus, the eigenvalues will split into three well separated clumps. In the limit that we expect a classical geometry to emerge (large N and large 't Hooft coupling) the off diagonal modes connecting these three subsectors will be very heavy and decouple. We expect that, when studying almost BPS states, the effect of these modes on the dynamics can be neglected. There is no reason to neglect off diagonal modes connecting eigenvalues in the same sector. Thus, for our purposes, we can replace Z by a block diagonal matrix with the number of blocks matching the number of clumps of eigenvalues. If clump i contains N_i eigenvalues it corresponds to an $N_i \times N_i$ block.

A geometry with M rings can thus be considered as an M matrix model. The matrices Z_i are $N_i \times N_i$ dimensional matrices, where clump i contains N_i eigenvalues. Acting with $\mathrm{Tr}\,(Z_i)$ will only add boxes to the rows corresponding to ring i[23]. These boxes have weight c_i. The matrices are not interacting so that we actually have M one matrix models. Each of these matrix models has an annulus background - one described by a Young diagram with N_i rows. To make sure that the eigenvalues localize correctly into the multi-ring geometry, one needs to ensure that the weight of the boxes in the rightmost column match the weights of the corresponding boxes in the original Young diagram. This follows because the weights give the radius squared of the position of the corresponding eigenvalue on the LLM plane[23]. Note that we are not just projecting the eigenvalues. Indeed, for block i we integrate over the full set of N_i^2 matrix elements. We can now easily recycle the results of Appendix B.5 to obtain

$$\frac{\langle \chi_B \chi_B^\dagger \mathrm{Tr}\,(Z_i^n Z_i^{\dagger\, n}) \rangle}{\langle \chi_B \chi_B^\dagger \rangle} = N_i c_i^n \,,$$

$$\frac{\langle \chi_B \chi_B^\dagger \mathrm{Tr}\left(\frac{d^n}{dZ_i^n} \frac{d^n}{dZ_i^{\dagger\, n}} \right) \rangle}{\langle \chi_B \chi_B^\dagger \rangle} = N_i c_i^n \, ,$$

where c_i are the weights of the boxes added or removed, respectively. The computations of these correlators is one of the main results of this section.

Summary: *In the multi-ring LLM background the original matrix Z breaks into local blocks Z_i, which are $N_i \times N_i$ dimensional matrices, where clump i contains N_i eigenvalues. To compute correlation functions of these local operators one uses ribbon diagrams, where each ribbon carries an extra factor of $\frac{c}{N}$ where c is the weight of the boxes added or removed by the local operator and one includes a factor of $\frac{N_i}{N}$ for each trace in the local operator.*

As a nontrivial consequence of our result, note that the net affect of the background on the Cuntz Hamiltonian (8.9) is simply to scale the Cuntz oscillators by $\frac{c}{N}$.

9 Correlators of Operators with a Large R-Charge: A New Expansion Parameter

In the previous section we introduced techniques that facilitated calculating the result of restricted Schur polynomial correlators in the case where the restricted Schur polynomial operators were comprised of $O(N^2)$ fields. These techniques were then applied to correlators of operators in the presence of large background Schur polynomials comprised of $O(N^2)$ fields (dual to LLM geometries[15, 74]). The techniques, based on the cutting rules, are entirely general and could be applied to any half BPS background operator or even some special cases of non BPS backgrounds, in the free field theory limit. Specializing to the half BPS annulus background where the calculations simplify considerably, some concrete results were derived. These results included a demonstration that when the number of fields in the operator creating the background is $O(N^2)$, contractions which would have been dropped in a systematic large N expansion when the operator was comprised of $O(N)$ fields must now be included. This is the field theory accounting of the back reaction of the heavy state on the string theory side. In addition evidence was provided from the field theory side that open strings attached to restricted Schur polynomials of $O(N^2)$ fields are in fact closed strings propagating on a new background geometry. In this section, we again consider the problem of computing correlators of heavy operators. In particular, we study the perturbation theory of such correlators.

The operators we have in mind are a small perturbation of a BPS operator which has $\mathcal{R}$-charge $\sim$ conformal dimension $\sim N^2$. In the language of [80] we study *almost* BPS operators. To solve this problem one must reorganize perturbation theory by resumming an infinite number of diagrams. The need for this reorganization is that for these operators non-planar diagrams can't be neglected[18]. A similar situation is provided by the BMN sector[16] of $\mathcal{N} = 4$ super Yang-Mills theory. In this case one considers operators with $\mathcal{R}$-charge $J \sim O(\sqrt{N})$. One finds the usual $\frac{1}{N}$ expansion parameter is replaced by a new expansion parameter equal to $\frac{J^2}{N}$[66, 67]. Thus, one must take $\frac{J^2}{N} \ll 1$ to suppress non-planar diagrams.

The question we ask and answer in this section is:

Is there a reorganization of the $\frac{1}{N}$ expansion for almost BPS operators which have $\mathcal{R}$-charge of $O(N^2)$ and if so, what is the expansion parameter?

We focus on almost BPS operators since we want to extrapolate our computations to strong coupling, where we can compare with the dual string theory.

By now it is clear that a particularly useful way to describe the half BPS sector is to use the Schur polynomials[17, 19, 22]. Among the operators we consider, are small perturbations of a Schur polynomial $\chi_B(Z)$ with B a Young diagram that has M columns and N rows, and M is $O(N)$. This corresponds to an LLM geometry[15] with boundary condition that is an annulus as encountered in section 8. The inner radius of the annulus is $\propto \sqrt{M}$ and the outer radius is $\propto \sqrt{M+N}$. We demonstrate, in section 9.1.1, that there is a reorganization of the perturbation theory in the half BPS sector and that the new expansion parameter is $\frac{1}{M+N}$. In section 9.1.2 we confirm this answer by studying the holography of the IIB supergravity in the relevant LLM geometry. We then generalize these results to multi-ring LLM geometries (in section 9.1.3), backgrounds with more than one charge (in sections 9.1.4 and 9.1.5) and beyond the BPS sector (in section 9.2). The Schwinger-Dyson equations provide a very powerful approach to the computation of correlators in the annulus background. We present these details in appendix C.

9.1 Half BPS Sector

Schur polynomials provide a very convenient reorganization of the half BPS sector. This is due to the fact that their two point function is known to all orders in $\frac{1}{N}$[17, 19] and that they satisfy a product rule allowing computation of exact n-point correlators using only two-point functions (see section 3 for a summary of the results we use). In this section we will use this Schur technology to provide an answer, in the half BPS sector, to the question posed above.

$\mathcal{N} = 4$ super Yang-Mills theory has 6 scalars ϕ_i transforming in the adjoint of the gauge group and in the **6** of the $SU(4)_{\mathcal{R}}$ symmetry. We shall use the complex combinations

$$Z = \phi_1 + i\phi_2, \qquad Y = \phi_3 + i\phi_4, \qquad X = \phi_5 + i\phi_6,$$

in what follows. We focus on the contributions coming from the color combinatorics; we drop all spacetime dependence from two point correlators[9], that is, we use the two point functions

$$\left\langle Z_{ij} Z^\dagger_{kl} \right\rangle = \left\langle Y_{ij} Y^\dagger_{kl} \right\rangle = \left\langle X_{ij} X^\dagger_{kl} \right\rangle = \delta_{il}\delta_{jk} \, . \tag{9.1}$$

Since we are not explicitly displaying the spacetime dependences, it is important to point out that all holomorphic operators are inserted at a specific spacetime event and all anti-holomorphic operators are inserted at a second spacetime event. These correlators are called extremal correlators; there are non-renormalization theorems protecting these correlators for half BPS operators.[81, 82].

When we talk about a half BPS operator in what follows, we mean an operator built only from Zs. These operators will not break any further supersymmetries beyond those broken by the background itself. We obtain almost BPS operators by sprinkling Ys and Xs in the operator.

9.1.1 Super Yang-Mills Amplitudes

The exact computation of multi-trace correlators at zero coupling is most easily carried out by expressing the multi-trace operators of interest in terms of Schur polynomials

$$\prod_i \mathrm{Tr}\,(Z^{n_i}) = \sum_R \alpha_R \chi_R(Z), \qquad \prod_j \mathrm{Tr}\,((Z^\dagger)^{m_j}) = \sum_R \beta_R \chi_R(Z^\dagger). \tag{9.2}$$

The coefficients α_R and β_R appearing in the above expansion have no dependence on N. It will be useful first to compute these correlators with a trivial

[9] It is simple to reinstate the spacetime dependence in the final result.

background. A little Schur magic now gives (see section 3)

$$\begin{aligned}\mathcal{A}(\{n_i;m_j\},N)\equiv\left\langle\prod_{i,j}\mathrm{Tr}\,(Z^{n_i})\mathrm{Tr}\,((Z^\dagger)^{m_j})\right\rangle &= \sum_{R,T}\alpha_R\beta_T\left\langle\chi_R(Z)\chi_T(Z^\dagger)\right\rangle\\ &= \sum_R\alpha_R\beta_R f_R\,. \end{aligned}\tag{9.3}$$

In this last equation f_R is the product of the weights[10] (one for each box in the Young diagram) for Young diagram R.

In what follows, B denotes the rectangular Young diagram with N rows and M columns. We will often refer to B as the annulus background because the corresponding LLM geometry is obtained by taking the LLM boundary condition to be a black annulus on a white plane. The expectation value of an operator O in background B is given by

$$\langle O\rangle_B\equiv\frac{\left\langle\chi_B(Z)\chi_B(Z^\dagger)O\right\rangle}{\langle\chi_B(Z)\chi_B(Z^\dagger)\rangle}\,.\tag{9.4}$$

Our normalization is chosen so that the expectation value of the identity is 1. Recall that B is a Young diagram with N rows and M columns. The product rule satisfied by the Schur polynomials can be viewed as a consequence of the fact that the Schur polynomials themselves are characters of $SU(N)$ and that (i) the character of a direct product of representations is just the product of the characters and (ii) the character of a given reducible representation is equal to the sum of characters of the irreducible representations appearing in the reducible representation. In the product

$$\chi_R(Z)\chi_S(Z)=\sum_T g_{RST}\chi_T(Z),$$

the Littlewood-Richardson number g_{RST} counts the number of times irreducible representation T appears in the direct product of the irreducible representations R and S. Since we choose our background B to be a Young diagram with M columns and N rows, it is a singlet of $SU(N)$ and conse-

[10]Recall that the box in row i and column j has a weight $N+j-i$.

quently we have the product shown in figure 10.

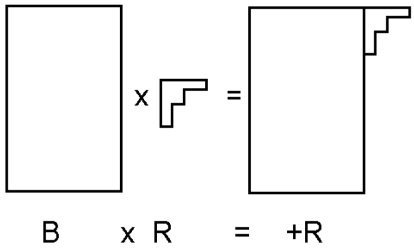


Figure 10: The product rule used to compute correlators in background B. This figure defines $+R$.

Using this product rule, the correlator in the annulus background is

$$\begin{aligned}\mathcal{A}_B(\{n_i;m_j\},N)\equiv\left\langle\prod_{i,j}\mathrm{Tr}\,(Z^{n_i})\mathrm{Tr}\,(Z^\dagger)^{m_j}\right\rangle_B &= \sum_{R,T}\alpha_R\beta_T\frac{\langle\chi_B(Z)\chi_R(Z)\chi_B(Z^\dagger)\chi_T(Z^\dagger)\rangle}{f_B}\\ &= \sum_{R,T}\alpha_R\beta_T\frac{\langle\chi_{+R}(Z)\chi_{+T}(Z^\dagger)\rangle}{f_B}\\ &= \sum_{R}\alpha_R\beta_R\frac{f_{+R}}{f_B}\,. \end{aligned}\tag{9.5}$$

Recall that f_{+R} is the product of the weights appearing in Young diagram $+R$ and f_B is the product of the weights appearing in Young diagram B. All of the weights appearing in the product f_B are repeated in the product f_{+R} so that after canceling common factors f_{+R}/f_B is simply equal to the product of the weights of the extra boxes stacked to the right of B to form $+R$. Thus, f_{+R}/f_B is obtained from f_R by replacing $N\to N+M$. Comparing (9.5) and (9.3) we find

$$\mathcal{A}_B(\{n_i;m_j\},N)=\mathcal{A}(\{n_i;m_j\},M+N).\tag{9.6}$$

We know that the correlator $\mathcal{A}(\{n_i;m_j\},N)$ admits an expansion in $\frac{1}{N}$; (9.6) tells us that $\mathcal{A}_B(\{n_i;m_j\},N)$ admits an expansion in $\frac{1}{N+M}$. If we assume that $\sum_i n_i\sim O(1)$, we obtain correlators of operators that are dual to gravitons. Thus, for gravitons in the background B $\frac{1}{N+M}$ clearly plays the role of a loop expansion parameter. *Thus, we learn that there is a reorganization of the* $\frac{1}{N}$ *expansion for these correlators in the annulus background B and further,* *that the new expansion parameter is* $\frac{1}{N+M}$. Our relation (9.6) implies that the only effect of the background is to shift $N \to N + M$.

One can easily check that this relation (9.6) is not a property of the full theory. Although a slight modification of (9.6) does allow us to relate the one point functions[11] $\langle \mathrm{Tr}\,(Z^n Z^{\dagger\,n})\rangle$ and $\langle \mathrm{Tr}\,(Z^n Z^{\dagger\,n})\rangle_B$, we have not worked out a relation between amplitudes in general. Our interest in the one point functions $\langle \mathrm{Tr}\,(Z^n Z^{\dagger\,n})\rangle_B$ is that they appear in the intermediate steps of the computation of correlators of BMN-like probes of the annulus so that we manage to obtain a simple relation between the trivial background and the annulus background for both the BPS and near-BPS sectors of the theory.

The amplitudes $\mathcal{A}(\{n_i; m_j\}, M+N)$ are the amplitudes of a theory with a gauge group of rank $N+M$ and no background. In the LLM language, the boundary condition for this geometry is simply a black disk of radius $\sqrt{N+M}$. In our theory (with gauge group of rank N) the background B corresponds to an annulus with inner radius $\sqrt{M}$ and outer radius $\sqrt{M+N}$. Thus, one way to interpret the relation (9.6) is that *any* of the half BPS probes considered above are unable to detect the hole in the middle of the annulus. We have not considered probes built using $\frac{d}{dZ}$ instead of Z; these will detect the hole. Indeed, acting with traces of Z on the background $\chi_B(Z)$ produces a new Schur polynomial that has extra boxes stacked adjacent to the upper right hand corner of B - this corner maps to the outer edge of the annulus defining the LLM boundary condition. Acting with $\frac{d}{dZ}$ erodes corners from the lower right corner of B - this corner maps to the inner edge of the annulus. See [23] for further details. Probes built using $\frac{d}{dZ}$ instead of Z are also half BPS probes.

We have related amplitudes in the gauge theory with a background of M giant gravitons and gauge group $U(N)$ to amplitudes in the gauge theory with trivial background and gauge group $U(N+M)$. This is reminiscent of the infrared duality proposed in [83] which exchanges the rank of the gauge group and the number of giant gravitons. In fact, (9.6) tells us that half BPS correlators computed in the $U(N)$ gauge theory in the background of M giant gravitons are exactly equal to the same half BPS correlators computed

[11] Again computed in the theory with gauge group $U(N+M)$.

in the $U(M)$ gauge theory in the background of N giant gravitons. Since these correlators are extremal and hence not renormalized, our computations give the value of these correlators in the deep infrared limit of the gauge theory and thus seem to provide nontrivial support for the duality proposed in [83]. Note however, that correlators of operators built using $\frac{d}{dZ}$ (which are also extremal) will not agree. This is not obviously in conflict with the proposed duality of [83]. In [83] near extremal black holes are considered. These can be understood as a condensate of giant gravitons[52] represented in $\mathcal{N} = 4$ super Yang-Mills theory by a Schur polynomial corresponding to a triangular Young diagram [74]. The LLM boundary condition for our background is a black annulus and $\frac{d}{dZ}$ correlators explore the inner edge of the annulus. The LLM boundary condition dual to the Schur polynomial with triangular Young diagram label is a gray disk; there is no inner edge. It would be interesting to see how many of our results for the black annulus boundary condition can be generalized to the gray disk boundary condition. This generalization is nontrivial.

The main evidence given in [83] for the proposed duality was an exact correspondence between the gravitational entropy formulae of small and large charge solutions. We can give arguments, in the free field theory, that suggest that the entropy of the state formed by N_g condensed giant gravitons in the theory obtained by taking the near-horizon limit of N D3-branes is equal to the entropy of the state formed by N condensed giant gravitons in the theory obtained by taking the near-horizon limit of N_g D3-branes. This provides further evidence for the duality of [83] in a completely different region of parameter space to that probed by supergravity. According to[84] the near extremal black holes in AdS space can be obtained by attaching open string excitations to the Schur polynomials with triangular Young diagram labels. If we are very close to extremality, the number of open string excitations attached is much smaller compared to the total number of fields in the operator. In this situation instead of attaching open strings by adding boxes to the original triangular label, it should be a good approximation to fix the tableau shape and replace some boxes (in arbitrary places on the Young diagram) with open strings. Assuming that the state obtained by deleting the open strings is again a typical state, i.e. again a triangle we have something like the situation given in figure 11. To compute the entropy associated with these states, we need to count the number of different ways of attaching the open string defects. This is given by the number of different ways we can pull the occupied boxes off the triangular Young diagram. We don't know exactly how to compute this number. However, it is easy to argue that this number is invariant under flipping the Young diagram so that the two states in figure 11 are swapped: this follows from two facts (i) the counting problem is constrained by the rule that we can pull boxes off in any order as long as after each box is pulled off we continue to have a valid Young diagram[12] and (ii) a shape that is (is not) a valid Young diagram before the flip is (is not) a valid Young diagram after the flip. Thus, the entropy of these two states agree.

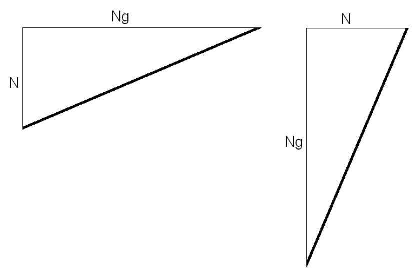


Figure 11: The typical state of N_g condensed giant gravitons in the theory with gauge group $U(N)$ (on the left) and the typical state of N condensed giant gravitons in the theory with gauge group $U(N_g)$ (on the right). The black stripe represents the boxes occupied by open string defects.

A few comments about the above flip of the Young diagram are in order. The half BPS sector of $\mathcal{N} = 4$ super Yang-Mills theory can be mapped to the quantum Hall system with filling factor equal to one [85, 86]. In the half BPS sector the above flip of the Young diagram is a Z_2 symmetry that exchanges particles and holes[85, 86, 87]. If one employs a continuum (field theory) description of the quantum Hall fluid, the resulting fluid is incompressible so that the corresponding continuum Lagrangian has a gauge invariance under

[12]This is required because removing each box must define a valid subduction. See[30, 31, 32, 33].

area preserving diffeomorphisms[88]. Small fluctuations of the density, for a fluid of charged particles in a background magnetic field, are well described by a Chern-Simons theory; the $U(1)$ gauge invariance is nothing but the area preserving diffeomorphisms for the small fluctuations[88]. The fluid description correctly captures the long distance physics of the quantum Hall effect. It does not capture the fact that, since the fluid is described by N electrons, it has an intrinsic granular structure. One can capture this granular structure by using a Chern-Simons matrix model description[88]. In this Chern-Simons matrix model description the above Z_2 symmetry is nothing but level/rank duality of the Chern-Simons matrix model[89, 90].

One puzzling feature of the duality of [83] is the fact that it mixes giant gravitons and the branes (called background branes in [83]) whose near horizon geometry is the AdS space we work in. There is a big difference between the background branes and the giant gravitons. Indeed, in string theory the background branes carry a net RR-charge; giant gravitons carry no net charge - they are dipoles. In the dual super Yang-Mills theory, changing the number of background branes changes the rank of the gauge group i.e. the number of fields we integrate over when performing a path integral quantization. Changing the number of giant gravitons leaves the rank of the gauge group unchanged, but it does change the background i.e. the integrand we use when performing a path integral quantization. If the duality of [83] is correct, we need to understand how, in this case, changing the integrand has exactly the same effect as changing the number of variables over which we integrate. Computing half BPS correlators in the annulus geometry gives a nice toy model in which to explore this issue. This is because we can reduce the whole problem to eigenvalue dynamics in zero dimensions. Consider the computation of the correlator $\left\langle \mathrm{Tr}\,(Z)\mathrm{Tr}\,(Z^\dagger)\right\rangle$ in the trivial vacuum of the $U(N+M)$ theory (Δ is usual the Van der Monde determinant)

$$\int \prod_{i=1}^{M+N} dz_i d\bar{z}_i \Delta(z)\Delta(\bar{z}) \sum_{j=1}^{N+M} z_j \sum_{k=1}^{N+M} \bar{z}_k \, e^{-\sum_{i=1}^{N+M} z_i\bar{z}_i} \,. \tag{9.7}$$

The same correlator in the $U(N)$ theory with a background of M giant gravitons is

$$\int \prod_{i=1}^{N} dz_i d\bar{z}_i \Delta(z)\Delta(\bar{z}) \prod_{l=1}^{N} (z_l \bar{z}_l)^M \sum_{j=1}^{N} z_j \sum_{k=1}^{N} \bar{z}_k \, e^{-\sum_{i=1}^{N} z_i \bar{z}_i} . \qquad (9.8)$$

We already know that (9.7) and (9.8) give the same result. The reason why the two agree is now evident: in (9.7) we integrate over an extra M variables; these extra contributions add to give a larger result than that obtained for the same correlator in the $U(N)$ theory. In (9.8) the extra factor in the integrand implies that the integrand now peaks at larger values for the $|z_l|$; this again gives a larger result than that obtained for the same correlator in the $U(N)$ theory in the trivial vacuum. Our computation gives a simple picture of how, in this case, changing the integrand has exactly the same effect as changing the number of variables over which we integrate. It also suggests that the duality of [83] might be derived by starting with a suitable $U(N+N_g)$ theory and (i) integrating out the degrees of freedom associated with N_g colors to get a $U(N)$ gauge theory with a background given by a condensate of N_g giant gravitons or (ii) integrating out the degrees of freedom associated with N colors to get a $U(N_g)$ gauge theory with a background given by a condensate of N giant gravitons.

9.1.2 Supergravity Amplitudes

According to the AdS/CFT correspondence, correlation functions can be computed in the strong coupling limit of $\mathcal{N} = 4$ super Yang-Mills theory using the dual hologram, which is type IIB supergravity. The paper [91], has given a powerful general approach to holography in the LLM backgrounds, generalizing and extending the Coulomb branch analysis of [92, 93]. The formalism given in [91] is an application of the general method of [94] which employs the method of holographic renormalization[95]. An important result of [91] was the demonstration that the asymptotics of the LLM solutions correctly encode the vacuum expectation values of all single trace half BPS operators to the leading order in the large N expansion. *Can we reproduce the results of the last section using holography in the LLM background?*

One case of interest to us is that of graviton three point functions, in the background created by a heavy operator. We have computed these correlation functions in the free field theory; supergravity will reproduce these correlators in the strong coupling limit. We will now argue that it is natural to expect that these two results will agree since we can argue that there is a non-renormalization theorem protecting the three point functions of interest to us. The logic of this argument is very similar to the argument of [91] which argued that the generic one point function in the LLM background is protected. The three point functions we are interested in

$$\Big\langle \mathrm{Tr}(Z^n)\mathrm{Tr}(Z^m)\mathrm{Tr}(Z^{\dagger\, m+n})\Big\rangle_B$$

can of course, also be understood as a ratio of (higher point) correlators in the original trivial background

$$\frac{\Big\langle \chi_B(Z)\chi_B(Z^\dagger)\mathrm{Tr}(Z^n)\mathrm{Tr}(Z^m)\mathrm{Tr}(Z^{\dagger\, m+n})\Big\rangle}{\langle \chi_B(Z)\chi_B(Z^\dagger)\rangle}\,.$$

The correlators appearing in the above expressions are extremal correlators in the language of [96]. The computations of [96] show that at the leading order in large N, the extremal correlators take the same value at large 't Hooft coupling as in the free field theory and hence it is natural to expect that all extremal correlators are not renormalized[13].

To extract three point functions using the methods of [91], we would need to solve the equations of motion, to quadratic order, around the LLM solution. We have not managed to do this. For the case of extremal correlators, there is some room for optimism: the analysis of [96] argued that for extremal correlators, the bulk cubic supergravity coupling vanishes and the entire contribution comes from a boundary term. The LLM geometries are asymptotically AdS so that one might have hoped that there was a simple explanation of the results of the previous section. We have not found one. Note also that the fact that the bulk cubic supergravity coupling vanishes in

[13]Note that the supergravity result suggests that the planar contribution is not renormalized. Here we are using the stronger conjecture [96] which claims the non-renormalization for any N. Our results provide further evidence for this stronger conjecture.

the AdS_5 background need not imply that it continues to vanish in the LLM background.

Rather than pursuing the computation of three (and higher) point functions directly, we have found it simpler to relate the three point functions we'd like to compute to one point functions, since these have already been obtained in [91]. For concreteness, consider the three point function (B is again the annulus diagram)

$$\left\langle \mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^{\dagger\,4})\right\rangle_B$$

which can also be written as the one point function of $\mathrm{Tr}(Z^2)$ in the normalized state

$$|\Phi\rangle = \mathcal{N}(\mathrm{Tr}\,(Z^2) + \mathrm{Tr}\,(Z^4))\chi_B(Z)|0\rangle\,.$$

$\mathcal{N}$ is a normalization factor. Now, summing only planar diagrams

$$\left\langle \mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^{\dagger\,4})\right\rangle = 16N^3$$

so that according to our result of the previous section

$$\left\langle \mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^{\dagger\,4})\right\rangle_B = 16(N+M)^3\,.$$

The supergravity one point function is computed with a normalized state $|\Phi\rangle$; the above three point function is computed with a different normalization. Noting that (we are again using results from the previous section)[14]

$$\left\langle (\mathrm{Tr}\,(Z^2) + \mathrm{Tr}\,(Z^4))(\mathrm{Tr}\,(Z^{\dagger\,2}) + \mathrm{Tr}\,(Z^{\dagger\,4}))\chi_B(Z)\chi_B(Z^\dagger)\right\rangle$$

$$= \left[4(N+M)^4 + 22(N+M)^2\right] f_B = 4(N+M)^4 f_B \left(1 + O((N+M)^{-2})\right)\,,$$

we easily find

$$\langle\Phi|\mathrm{Tr}(Z^2)|\Phi\rangle = \frac{4}{N+M}\,.$$

This is the result we would like to reproduce from supergravity.

[14] Note that since $|\Phi\rangle$ has two additive components the one-point function has additional cross terms; these however evaluate to zero leaving only the term we wish to compute.

We will summarize the main steps involved in extracting correlators of local operators in the boundary gauge theory, from the ten dimensional asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$ solutions of Type IIB supergravity. For all the details and further references consult [91]. Given an asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$ solution of Type IIB supergravity, one must first systematically reduce the ten dimensional solution to a solution of five dimensional gravity coupled to an infinite number of five dimensional fields. If this reduction is performed in terms of gauge invariant variables (as it is in [91]), it does not matter in which gauge the original solution is supplied. This reduction is then supplemented with a non-linear field redefinition, performed in such a way that the equations of motion in terms of the redefined fields can be obtained by minimizing a local five dimensional action. One can then apply the usual holographic rules to compute the correlation function of boundary operators from the bulk five dimensional description. To obtain the renormalized correlation functions in the gauge theory one must appropriately renormalize the bulk gravitational action using holographic renormalization. The supergravity field that couples to the boundary operator that we are interested in ($\mathrm{Tr}(Z^2)$) is a mixture of (components of the) metric on the S^5 and the four form RR potential on the S^5. It is denoted by S^{22} in [91]. The one point function we want can now be obtained by variation of the renormalized on shell supergravity action with respect to the boundary value of S^{22}. The result is [91][15]

$$\langle \mathcal{O}_{S^{22}} \rangle = N^2 \int r^3 \rho\, e^{2i\phi} dr d\phi\,,$$

where ρ is the boundary condition for the LLM geometry. This boundary condition is most easily determined by exploiting the free fermion description of the half BPS sector of $\mathcal{N}=4$ super Yang-Mills theory[17, 22]. To make the transition to the free fermion description, rewrite the state $|\Phi\rangle$ in terms

[15]This differs by a factor of $\sqrt{2}\pi^2$ from formula (3.60) of [91] because we are using a different normalization. Further, since we have dropped all spacetime dependence, we have also dropped a factor of e^{2it}.

of Schur polynomials. This is easily accomplished using

$$\mathrm{Tr}(Z^2) = \chi_{\yng(2)}(Z) - \chi_{\yng(1,1)}(Z)\,,$$

$$\mathrm{Tr}(Z^4) = \chi_{\yng(4)}(Z) - \chi_{\yng(3,1)}(Z) + \chi_{\yng(2,1,1)}(Z) - \chi_{\yng(1,1,1,1)}(Z)\,,$$

and the product rule for Schur polynomials. The Schur polynomials correspond to energy eigenkets of the free fermions. The energies of the free fermions in the state are

$$E_i = \lambda_i + N - i + 1 \qquad i = 1, ..., N,$$

where λ_i is the number of boxes in the ith row of the Young diagram label for the Schur polynomial. Using this interpretation it is straightforward to obtain the fermion phase space density. The density we obtain in this way is

$$\rho = \rho_1 + \cos(2\phi)\rho_2\,,$$

where

$$\rho_2 = \frac{1}{2\pi(N+M)^4}\frac{e^{-Nr^2}}{(N+M-1)!}\Big[(Nr^2)^{N+M+2} + (N+M-1)(Nr^2)^{N+M+1}$$
$$-(N+M)(N+M+1)(N+M-1)(N+M-2)(Nr^2)^{N+M-2}$$
$$-(N+M)(N+M-1)^2(N+M-2)(N+M-3)(Nr^2)^{N+M-3}\Big]\,.$$

We will not need ρ_1 in what follows. As already noted in [91], the state $|\Phi\rangle$ which is a superposition of a small number (for us, 6) of Schur polynomials, does not give rise to a smooth supergravity solution. Indeed, for a regular supergravity solution, $\pi\rho$ should only take the values $\{0,1\}$; this is not the case for the above ρ. It is now straightforward to check that

$$\langle \mathcal{O}_{S^{22}} \rangle = \frac{4}{N+M}$$

which is the result we wanted to demonstrate.

9.1.3 Multi Ring Backgrounds

Using the results of [23] and section 8 we can generalize our results for the annulus to a general multi ring LLM geometry. The Young diagrams of these geometries are not a rectangle, but rather are shapes with more than 4 edges and all edges have a length of order N boxes. In a geometry with m thick rings, local graviton operators were defined in [23]. When taking a product of the Young diagram of the background (B_{ring}) with the Young diagram of the probe (R), one removes boxes from the probe Young diagram and adds them, at all possible positions, to B_{ring} (respecting the usual rules for multiplying Young diagrams). The local graviton operators are defined by giving their product rule: the boxes removed from R can only be added at a specific location on B_{ring}. This definition is precise enough to allow the computation of correlators; see [23] for more details. We can also give a rough constructive definition of the local graviton operators: since we consider a boundary condition with m thick rings, the eigenvalue distribution in the dual matrix model will split into m well separated clumps. In the limit that we expect a classical geometry to emerge (large N and large 't Hooft coupling) the off diagonal modes connecting these m subsectors will be very heavy and decouple. We expect that, when studying almost BPS states, the effect of these modes on the dynamics can be neglected. There is no reason to neglect off diagonal modes connecting eigenvalues in the same sector. Z can thus be replaced by a block diagonal matrix with m blocks. If clump i contains N_i eigenvalues it corresponds to an $N_i \times N_i$ block. To construct a local graviton operator, do not use the full matrix Z, but rather use one of the blocks Z_i.

It is now straight forward to verify that, if ring i has outer radius $\sqrt{N+M_i}$ then we again get an exact relation

$$\mathcal{A}_B(\{n_i; m_j\}, N) = \mathcal{A}(\{n_i; m_j\}, M_i + N), \tag{9.9}$$

for correlators $\mathcal{A}_B(\{n_i; m_j\}, N)$ of gravitons localized at edge i. Thus, the gravitons at the edge of each ring have their own expansion parameter equal to $\frac{1}{N+M_i}$.

Finally, consider a background obtained by exciting k_2 sphere giants; each giant is assumed to carry a momentum $k_3 < N$. The corresponding LLM geometry will be a black disk (of area $N - k_3$) surrounded by a white annulus of area k_2 which is itself surrounded by a black annulus of area k_3 (this boundary condition is shown in figure 3 of [86]). From the results of this section we know that excitations of the central black droplet could be described using either a $U(N)$ theory with a background of k_2 sphere giants or using a $U(N - k_3)$ theory. This is closely related to remarks made in [86], arrived at using a totally different approach.

9.1.4 Backgrounds with > 1 Charges

In this section we would like to study the zero coupling limit of backgrounds made from two matrices Z and Y. There are a number of bases which we can employ for this study. The basis described in [25] builds operators with definite quantum numbers for any desired global symmetry groups acting on the matrices. The basis of [26] uses the Brauer algebra to build correlators involving Z and $Z^\dagger$; this basis seems to be the most natural for exploring brane/anti-brane systems. The basis of [27] most directly allows one to consider open string excitations[28, 29, 30, 31, 32, 33] of the operator; this is the restricted Schur basis. These three bases do not coincide and a detailed link between them has been discussed in [97]. Casimirs that distinguish these bases have been given in [98]. Finding a spacetime interpretation of these Casimirs is a promising approach towards understanding the dual description of these bases.

In section 7 a particularly simple set of restricted Schur polynomials were identified; these are the operators that we will focus on in this section. The operator is built using NM_1 Z fields and NM_2 Y fields. The restricted Schur polynomial has two labels $\chi_{R,(r_1,r_2)}$. R is an irreducible representation of $S_{N(M_1+M_2)}$. We take R to be a Young diagram with $M_1 + M_2$ columns and N rows. (r_1, r_2) is an irreducible representation of the $S_{NM_1} \times S_{NM_2}$ subgroup. We take r_1 to be a Young diagram with M_1 columns and N rows and r_2 to be a Young diagram with M_2 columns and N rows. We take both

M_1 and M_2 to be $O(N)$. For this particular restricted Schur polynomial we have

$$\chi_{R,(r_1,r_2)}(Z,Y) = \frac{\text{hooks}_R}{\text{hooks}_{r_1}\,\text{hooks}_{r_2}}\chi_{r_1}(Z)\chi_{r_2}(Y)\,,$$

where $\chi_{r_1}(Z)$ and $\chi_{r_2}(Y)$ are standard Schur polynomials. The simplicity of this background is a consequence of the fact that the above operator factorizes. These operators are not BPS or even near-BPS. Indeed, in appendix C.3 we compute the anomalous dimension of these operators, which is given by

$$\Delta = N(M_1+M_2) + 4\lambda M_1 M_2\,, \qquad \lambda = N g_{YM}^2.$$

Thus, the results of this section can not be extrapolated to strong coupling. It is still interesting to ask if, at weak coupling, we can reorganize the usual $\frac{1}{N}$ expansion.

After normalizing so that the identity has an expectation value of 1, we find

$$\langle O\rangle_{(r_1,r_2)} = \frac{\left\langle \chi_{r_1}(Z)\chi_{r_1}(Z^\dagger)\chi_{r_2}(Y)\chi_{r_2}(Y^\dagger)O\right\rangle}{\langle \chi_{r_1}(Z)\chi_{r_1}(Z^\dagger)\chi_{r_2}(Y)\chi_{r_2}(Y^\dagger)\rangle}\,.$$

Using exactly the same methods as were used in section 9.1.1, we find that the complete effect of the background on correlators of operators built using only Zs with operators built using only $Z^\dagger$s is to replace $N \to N+M_1$. In addition, we find that the complete effect of the background on correlators of operators built using only Ys with operators built using only $Y^\dagger$s is to replace $N \to N+M_2$. Thus, these two types of observables admit a different reorganization of the $1/N$ expansion; the small parameter of the two expansions is not the same.

9.1.5 A Generalization of the BMN Limit for Two Charge Backgrounds

The AdS/CFT correspondence forces us to take the limit of large ‘t Hooft coupling λ, if we are to suppress curvature corrections in the string theory. It is thus a strong/weak coupling duality. This naive observation is evaded by the remarkable result of BMN[16]: the expansion parameter of the super Yang-Mills theory may be suppressed by large quantum numbers for “almost

BPS operators" [80]. The existence of double scaling BMN-like limits provides a very rich class of tractable problems. In this section we would like to explore the proposed BMN-like sectors discovered in [99][16]. The work [99] itself was an extension of [84]. The existence of these limits may be very relevant to understanding the gauge theory description of near-extremal charged black holes in AdS_5. Using the technology we have developed, it should be possible to study these limits.

We use the same two-charge background that we used in the previous subsection. We will however take $M_1 = O(\sqrt{N})$ and $M_2 = O(\sqrt{N})$ with the 't Hooft coupling λ large but fixed[17]. In this case, the charges $J_1 = NM_1 \sim O(N^{\frac{3}{2}})$ and $J_2 = NM_1 \sim O(N^{\frac{3}{2}})$, whilst the difference (see appendix C.3)

$$\Delta - J_1 - J_2 = 4\lambda M_1 M_2 \sim N$$

so that[18]

$$\eta \equiv \frac{\Delta - J_1 - J_2}{J_1 + J_2} \sim N^{-\frac{1}{2}} \to 0\,.$$

Our background (plus open string excitations which do not modify this scaling) is a near BPS operator, in the language of [80]. These results are in perfect agreement with the supergravity arguments of [99] - the strong and weak coupling results match. The operator we have obtained is a "near quarter BPS" state. The fact that the anomalous dimension is proportional to $M_1 M_2$ strongly suggests that it arises from the dynamics of the open strings stretching from the stack of M_1 giants (described by $\chi_{r_1}(Z)$) to the stack of M_2 giants (described by $\chi_{r_2}(Y)$).

Let us now recall a few well known facts. 't Hooft[8] has already made the beautiful observation that the perturbative expansion of a matrix model

[16] We thank Shahin Sheikh-Jabbari for extremely useful correspondence, which lead to the results of this subsection.

[17] This is not quite the same limit as that described in [99]; in [99] the limit with g_{YM}^2 fixed is considered.

[18] We kept the 't Hooft coupling fixed in order to obtain this scaling for η, which matches precisely what one obtains in the BMN limit.

can be written in the form

$$\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda)\,, \tag{9.10}$$

where $f_g(\lambda)$ is a polynomial in the 't Hooft coupling - there is no additional N dependence apart from the multiplying factor of N^{2-2g}. Here, $\frac{1}{N}$ plays the role of a genus counting parameter. In the BMN set up, one learnt that there is a new effective genus counting parameter $g_2 = \frac{J^2}{N}$ replacing $\frac{1}{N}$, by studying the two point functions of BMN gauge invariant operators (each has angular momentum J). In terms of this effective genus counting parameter, the effective 't Hooft coupling is $g_{YM}^2/g_2 = g_{YM}^2 N/J^2$. Can we repeat this analysis for the operators considered here? Towards this end, we set $M_1 = M_2 = M$ and compute the two point function

$$I_2 = \langle \chi_{r_1}(Z)^\dagger \chi_{r_2}(Y)^\dagger \chi_{r_1}(Z)\chi_{r_2}(Y)\rangle = \left(\frac{G_2(N+M+1)}{G_2(N+1)G_2(M+1)}\right)^2 ,$$

where $G_2(n+1)$ is the Barnes function defined by ($\Gamma(z)$ is the Gamma function)

$$G_2(z+1) = \Gamma(z)G_2(z)\,, \qquad G_2(n+1) = \prod_{k=1}^{n-1} k!\,.$$

We will now set $N = \alpha M^2$ where α is a number of $O(1)$ in the large N limit. The Barnes function has the asymptotic expansion

$$\log G_2(N+1) = \frac{N^2}{2}\log(N) - \frac{1}{12}\log(N) - \frac{3}{4}N^2 + \frac{N}{2}\log(2\pi) + \zeta'(-1) + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)N^{2g-2}}\,,$$

where B_{2g} are the Bernoulli numbers and $\zeta'(-1)$ is the derivative of the Riemann zeta function evaluated at -1. Using this asymptotic expansion, it is clear that I_2 can be split into a product $F_{\text{non-pert}}F_{\text{pert}}$, where $F_{\text{non-pert}}$ is a non-perturbative piece that can not be expanded in $\frac{1}{M}$ and F_{pert} is a factor that does admit an expansion in $\frac{1}{M}$[19]. This suggests that we should identify

[19] This factorization is not unique; it will be fixed by the physics of the problem. We do not yet know how to do this.

$g_2 = \frac{1}{M} \sim \frac{1}{\sqrt{N}}$. In the BMN case only even powers of g_2 appear in the genus expansion; here we can not be sure if only even powers of g_2 appear. This depends on precisely how we factorize I_2 into $F_{\rm non-pert} F_{\rm pert}$. Using this genus expansion, we would expect an effective 't Hooft coupling[20]

$$\tilde{\lambda} = g_{YM}^2 M = \frac{1}{\sqrt{\alpha}} g_{YM}^2 \sqrt{N}.$$

Keeping this effective 't Hooft coupling fixed but arbitrarily small, one finds

$$\Delta = 2NM\left(1 + 2\tilde{\lambda}\right) = 2\alpha M^3 \left(1 + 2\tilde{\lambda}\right).$$

This looks like a polynomial is $\tilde{\lambda}$ times some power of M. Clearly, in view of (9.10), $\tilde{\lambda}$ is indeed the correct 't Hooft coupling. To properly specify our state of intersecting giants, we need to specify the three Young diagrams which would label the restricted Schur polynomial of the giant system. Clearly, with our simple choice it seems that we have correctly obtained the correct Yang-Mills operator to describe the decoupling limits of [84, 99] to one loop. It would be interesting to compute higher loop corrections and verify that one does indeed continue to obtain a polynomial in $\tilde{\lambda}$ times M^3. If extra M dependence does appear, it might be a sign that our simple guess for the background needs to be corrected at $O(\tilde{\lambda})$. This is not completely unexpected: these operators are not protected so one would expect corrections in general. Finally, holding $\tilde{\lambda}$ fixed we find the 't Hooft coupling $\lambda = g_{YM}^2 N = \tilde{\lambda}\sqrt{\alpha N}$ goes to infinity. Thus, even though we have large λ the one loop correction can be made arbitrarily small. We are thus optimistic that, similar to what happens in the BMN limit, it will be possible to compare perturbative field theory results to results obtained from the dual gravitational description.

The genus counting parameter $g_2 = \frac{1}{M}$ and the effective 't Hooft coupling $\tilde{\lambda} = g_{YM}^2 M$ are exactly the parameters we would expect for the description of the large M limit of a $U(M)$ gauge theory. This is rather natural: we have on the order of M giant gravitons whose worldvolume theory at low energy will

[20]With this scaling, which is different to what we had above, η does not scale with N, but can be made arbitrarily small.

be a $U(M)$ gauge theory. This provides strong field theory evidence that the sector of the theory identified in [84, 99] is indeed captured by the dynamics of open string defects distributed on a boundstate of intersecting giant gravitons. Its natural to think that the decoupling limits of [84, 99] capture the decoupled low energy world volume theory of the intersecting giant gravitons in the same way that Maldacena's limit[5] captures the decoupled low energy world volume theory of N D3 branes. Our explicit computations show that the low energy world volume theory of the giant gravitons is weakly coupled even when the original $\mathcal{N} = 4$ super Yang-Mills theory is strongly coupled.

9.2 Beyond the Half BPS Sector

In this section we would like to determine the sector of the theory in which our reorganization of perturbation theory is valid. First, the near BPS operators we are interested in are BMN-like gauge invariant operators

$$\mathcal{O}(\{n\}) = \mathrm{Tr}\,(YZ^{n_1}YZ^{n_2}YZ^{n_3}Y\cdots YZ^{n_L}).$$

To compute the two point correlator $\left\langle \mathcal{O}(\{n\})\mathcal{O}^\dagger(\{n\})\right\rangle_B$ we start by contracting the Y fields planarly. After performing the Y contractions we need to compute

$$\left\langle \prod_i \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\, n_i}) \right\rangle_B . \tag{9.11}$$

We need to verify that the above amplitude admits an expansion with parameter $\frac{1}{(N+M)}$ to demonstrate that our reorganization of perturbation theory does indeed apply to the half BPS and almost BPS sectors.

We will demonstrate that a relation very similar to (9.6) holds for the leading contribution to the one point functions (9.11). We will then derive an exact relation. The first property we use is factorization (valid to leading order and if $\sum_i n_i \sim O(1)$)

$$\left\langle \prod_i \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\, n_i}) \right\rangle_B = \prod_i \left\langle \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\, n_i}) \right\rangle_B .$$

Factorization in the annulus background has been discussed in [23] and in Appendix C.1. Thus, we need only consider

$$\left\langle \mathrm{Tr}\,(Z^{n}Z^{\dagger\, n}) \right\rangle_B = \left\langle \mathrm{Tr}\,(Z^{n}Y)\mathrm{Tr}\,(Z^{\dagger\, n}Y^\dagger) \right\rangle_B .$$

This computation can be completed in exactly the same way as the computation we tackled in section 9.1.1. We start by writing the gauge invariant operator of interest in terms of restricted Schur polynomials

$$\mathcal{O}(n) = \mathrm{Tr}\,(Z^nY) = \sum_{(R,R')} \alpha_{(R,R')}\chi_{(R,R')}(Z,Y). \tag{9.12}$$

R is a Young diagram with $n+1$ boxes and R' a Young diagram with n boxes. The expansion coefficients $\alpha_{R,R'}$ are independent of N. The two point function of restricted Schur polynomials has been computed in [31, 27]. Using these results, we find

$$\mathcal{C}(n,N) = \left\langle \mathcal{O}(n)\mathcal{O}^\dagger(n)\right\rangle = \sum_{(R,R')} \alpha^2_{R,R'} \frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}} f_R\,.$$

The factor $\mathrm{hooks}_R/\mathrm{hooks}_{R'}$ does not depend on N. To compute these amplitudes in the annulus background, we need the analog of the product rule given in figure 10. The product rule for restricted Schur polynomials (section 7) was used in appendix B.5 to compute precisely the restricted Littlewood-Richardson number needed here. Again, only a single term enters in the product

$$\chi_B(Z)\chi_{(R,R')}(Z,Y) = g_{B\,(R,R')\,(+R,+R')}\ \chi_{(+R,+R')}(Z,Y).$$

$+R'$ is obtained from $+R$ by dropping a single box. This restricted Littlewood-Richardson number was computed in appendix B.5 from its definition; this involves a summation over restricted characters. There is however, a much simpler way to obtain this result. Once one has established the form $\chi_B(Z)\chi_{(R,R')}(Z,Y) = g_{B\,(R,R')\,(+R,+R')}\ \chi_{(+R,+R')}(Z,Y)$, reducing both sides with respect to Y gives[21]

$$c_{R,R'}\chi_B(Z)\chi_{R'}(Z) = c_{R,R'}\chi_{+R'}(Z) = c_{+R,+R'}g_{B\,(R,R')\,(+R,+R')}\ \chi_{+R'}(Z),$$

where we have used the product rule of section 9.1.1: $\chi_B(Z)\chi_{R'}(Z) = \chi_{+R'}(Z)$ and where $c_{T,T'}$ is the weight of the box that must be dropped from T to obtain T'. Thus,

$$g_{B\,(R,R')\,(+R,+R')} = \frac{c_{R,R'}}{c_{+R,+R'}}.$$

This formula is exact. Although it is possible to compute things exactly, we will also make good use of the leading large $N+M$ version of our results.

[21] To reduce with respect to Y we take the derivative $\mathrm{Tr}\,\frac{d}{dY}$. A formula for the reduction of Schur polynomials is given in [57]; a formula for the reduction of restricted Schur polynomials is given in [31]. The reduction rule is reviewed in section 5.2.1.

For this reason we start with a large $N+M$ analysis and then give exact results.

Large N Analysis: To leading order in large $N+M$ we have $c_{R,R'}=N$ and $c_{+R,+R'}=N+M$ which reproduces the known result of appendix B.5. It is now straight forward to find

$$\begin{aligned}\mathcal{C}_B(n,N) &= \left\langle \mathcal{O}(n)\mathcal{O}^\dagger(n)\right\rangle_B \\ &= \sum_{(R,R')} \alpha^2_{R,R'} \left(\frac{N}{N+M}\right)^2 \frac{\text{hooks}_{+R}}{\text{hooks}_{+R'}} \frac{f_{+R}}{f_B} \\ &= \left(\frac{N}{N+M}\right) \sum_{(R,R')} \alpha^2_{R,R'} \frac{\text{hooks}_R}{\text{hooks}_{R'}} \frac{f_{+R}}{f_B} \,. \end{aligned} \tag{9.13}$$

Arguing exactly as we did in section 9.1.1 we find

$$\mathcal{C}_B(n,N) = \left(\frac{N}{N+M}\right) \mathcal{C}(n,N+M)\,. \tag{9.14}$$

This is very similar to, but not quite the same as (9.6). Note also that (9.14) was only derived at leading order; (9.6) is however exact. Some insight into these relations can be obtained by noticing that at leading order in N we have

$$\left\langle \text{Tr}\,(Z^n)\text{Tr}\,(Z^{\dagger\, n})\right\rangle = N^n, \qquad \left\langle \text{Tr}\,(Z^n Z^{\dagger\, n})\right\rangle = N^{n+1}.$$

Thus, as long as we restrict ourselves to the leading order, (9.6) and (9.14) can be restated as follows: consider an operator $O(Z,Z^\dagger)$ which is a product of factors of the form (9.2) or (9.12). Then

$$\left\langle O(Z,Z^\dagger)\right\rangle_B = \left\langle O\left(\sqrt{\frac{N+M}{N}}Z, \sqrt{\frac{N+M}{N}}Z^\dagger\right)\right\rangle\,. \tag{9.15}$$

At leading order, the only effect of the background is to rescale Z. This observation will be useful below.

Exact Analysis: The box $c_{RR'}$ has a weight $N+p$ where p is a number of

$O(1)$. Then

$$g_{B\,(R,R')\,(+R,+R')} = \frac{N+p}{N+M+p}, \qquad \frac{\text{hooks}_{+R}}{\text{hooks}_{+R'}} = \frac{N+M+p}{N+p}\frac{\text{hooks}_R}{\text{hooks}_{R'}}\,,$$

so that

$$\left\langle \text{Tr}\,(Z^nY)\text{Tr}\,(Z^{\dagger\,n}Y^\dagger)\right\rangle_B = \sum_{(R,R')} \frac{N+p}{N+M+p}\frac{\text{hooks}_R}{\text{hooks}_{R'}}\frac{1}{(n+1)^2}\frac{f_{+R}}{f_B}$$

$$= \sum_{(R,R')} \frac{\text{hooks}_R}{\text{hooks}_{R'}}\frac{1}{(n+1)^2}\frac{f_{+R}}{f_B} - M\sum_{(R,R')}\frac{1}{N+M+p}\frac{\text{hooks}_R}{\text{hooks}_{R'}}\frac{1}{(n+1)^2}\frac{f_{+R}}{f_B}$$

$$= \sum_{(R,R')} \frac{\text{hooks}_R}{\text{hooks}_{R'}}\frac{1}{(n+1)^2}\frac{f_{+R}}{f_B} - M\sum_{(R,R')}\frac{\text{hooks}_R}{\text{hooks}_{R'}}\frac{1}{(n+1)^2}\frac{f_{+R'}}{f_B}\,.$$

In the above sums, R runs over all hook Young diagrams with $n+1$ boxes (diagrams with at most one column containing more than one box; see appendix B.5) and R' runs over all possible ways of removing a box from the hook (there are usually 2 possible ways). The first term in this last expression is nothing but $\mathcal{C}(n, N+M)$. Now, it is a simple matter to verify that

$$\sum_R \frac{\text{hooks}_R}{\text{hooks}_{R'}}\frac{1}{(n+1)^2} = \sum_R \frac{d_{R'}}{d_R}\frac{1}{n+1} = \frac{1}{n+1}\frac{n+1}{n} = \frac{1}{n}. \tag{9.16}$$

The first equality follows from the formula for an irreducible representation of S_n: $d_T = \frac{n!}{\text{hooks}_T}$; the second formula follows upon inserting the explicit expression for the dimensions of the hook Young diagrams R and R'. Thus, the coefficient of $f_{+R'}/f_B$ is $1/n$. Next consider

$$\mathcal{C}(n-1, N+M) = \sum_{(R',R'')} \frac{\text{hooks}_{R'}}{\text{hooks}_{R''}}\frac{1}{n^2}\frac{f_{+R'}}{f_B} = \sum_{(R',R'')}\frac{d_{R''}}{d_{R'}}\frac{1}{n}\frac{f_{+R'}}{f_B}\,. \tag{9.17}$$

In the above sum, R' runs over all hook Young diagrams with n boxes and R'' runs over all possible ways of removing a box from this hook (there are 2 possible ways). The coefficient of $f_{+R'}/f_B$ in this last sum is

$$\sum_{R''} \frac{d_{R''}}{d_{R'}} \frac{1}{n} = \frac{1}{n} \,.$$

This follows because we sum over all possible subductions R'' of R'. This proves that (9.16) and (9.17) are identical and hence we find the exact relation

$$\mathcal{C}_B(n,N) = \mathcal{C}(n,N+M) - M\mathcal{C}(n-1,N+M)\,. \tag{9.18}$$

We have explicitly checked that it holds for $n = 1,2,3,4,5,6$. $\mathcal{C}(n,N+M)$ admits an expansions in $\frac{1}{M+N}$; thus, we have demonstrated that the amplitudes $\mathcal{C}_B(n,N)$ admit an expansion in $\frac{1}{M+N}$, albeit with the above extra M dependence in the expansion. M is the number of maximal giant gravitons making up the background; it is not surprising that there are amplitudes that have some additional dependence on M. The point is that all N dependence in our amplitude has, according to (9.18), been replaced by an $N+M$ dependence which is perfectly consistent with what we found in the half BPS sector.

Given the above success one may ask if we can consider more general correlators and to work beyond the leading order. This more general analysis would tell us the class of operators for which our reorganization works, that is, the full class of operators that can be expanded in $\frac{1}{M+N}$. Recall that the Schwinger-Dyson equations of a theory determine the correlation functions of the theory. Thus, one possible approach to our problem (now that the above analysis has suggested what to search for), is to demonstrate the replacement $N \to M+N$ at the level of the Schwinger-Dyson equations. This demonstration turns out to be straightforward. Start in the trivial background; consider first the simple Schwinger-Dyson equation[22]

$$0 = \int \left[dZ dZ^\dagger\right] \frac{d}{dZ_{ij}} \left((Z^{n+1} Z^{\dagger\, n})_{ij} e^{-S} \right) \,.$$

[22]Since we compute correlators using (9.1), to obtain Schwinger-Dyson equations that determine our correlators we must consider a zero dimensional matrix model with action $S = \mathrm{Tr}\,(ZZ^\dagger)$.

Performing the derivative we find

$$\left\langle \mathrm{Tr}\,(Z^{n+1}Z^{\dagger\,n+1})\right\rangle = N\left\langle \mathrm{Tr}\,(Z^{n}Z^{\dagger\,n})\right\rangle + \sum_{r=1}^{n}\left\langle \mathrm{Tr}\,(Z^{r})\mathrm{Tr}\,(Z^{n-r}Z^{\dagger\,n})\right\rangle \,.$$

An easy computation (see Appendix C.1) shows that, in the background B, the above Schwinger-Dyson equation becomes

$$\left\langle \mathrm{Tr}\,(Z^{n+1}Z^{\dagger\,n+1})\right\rangle_B = (N{+}M)\left\langle \mathrm{Tr}\,(Z^{n}Z^{\dagger\,n})\right\rangle_B + \sum_{r=1}^{n}\left\langle \mathrm{Tr}\,(Z^{r})\mathrm{Tr}\,(Z^{n-r}Z^{\dagger\,n})\right\rangle_B \,.$$

Clearly, the net effect of the background is to replace $N \to N+M$ in perfect harmony with (9.6). This conclusion is rather general: the net effect of the background, at the level of the Schwinger-Dyson equations, is the replacement $N \to N+M$ suggesting that (9.6) should hold for all correlators of operators built using only Z and $Z^\dagger$. This conclusion is too quick. To determine the correlators, we should imagine solving these equations iteratively; the value of $\left\langle \mathrm{Tr}\,(Z^{n+1}Z^{\dagger\,n+1})\right\rangle_B$ will thus depend on $\left\langle \mathrm{Tr}\,(Z^{n}Z^{\dagger\,n})\right\rangle_B$. We should start the process with the equation that follows for $n=0$

$$\left\langle \mathrm{Tr}\,(ZZ^{\dagger})\right\rangle_B = (N+M)N\,.$$

The N multiplying $(M+N)$ on the right hand side comes from $\mathrm{Tr}\,(1)$. This introduces an N dependence, which is not replaced by $M+N$. This result is in perfect agreement with (9.14). By carefully analyzing the Schwinger-Dyson equations for the half BPS gauge invariant operators, one can verify that a similar problem does not occur for these operators, which is consistent with (9.6). The above departure from a pure $M+N$ dependence is rather mild and one may hope that there is a simple generalization of our results.

We are claiming that we have a new $\frac{1}{M+N}$ expansion parameter. Why not simply replace $M+N \to (\mu+1)N$ with $\mu = \frac{M}{N}$ so that we have the usual $\frac{1}{N}$ expansion with μ dependent coefficients? There are (at least) two reasons to reject this proposal

- The loop expansion parameter in the half BPS sector is $\frac{1}{M+N}$; the expansion coefficients have no additional M or N dependence. Since this sector includes gravitons, we should identify $\frac{1}{M+N}$ as the $\hbar$ for the graviton interactions. Indeed, a $\frac{1}{N}$ expansion with μ dependent coefficients is an expansion whose coefficients change as N is changed, indicating additional $\hbar$ dependence in these coefficients.

- Our relation (9.6) is exact and holds for any value of M. As the size of M changes the character of the expansion changes and it can be misleading to think that fluctuations are controlled by $\frac{1}{N}$. If $M = O(1)$, $M + N$ can be replaced by N and we have the usual $\frac{1}{N}$ expansion as expected - this is the theory in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. In this case, $\mu = O(\frac{1}{N})$ and the coefficients again become N independent. When $M = O(N)$, $M + N$ is itself of order N, and $\frac{1}{N}$ continues to control the size of fluctuations. This is the $\frac{1}{M+N}$ expansion we found above - for example the theory in the LLM annulus background. In this case, $\mu = O(1)$, and the coefficients themselves are of order 1. When $M = O(N^2)$ we can replace $M + N$ by M so that our $\frac{1}{M+N}$ expansion effectively becomes a $\frac{1}{M}$ expansion. The correlators are dominated by contractions with the background and fluctuations are much smaller than in the usual $\frac{1}{N}$ expansion: they are controlled by $\frac{1}{M} \sim \frac{1}{N^2}$. In this case, $\mu = O(N)$ so that the coefficients are unusually small - of size $\frac{1}{N}$. Particularly in the last case, it would be absurd to suggest that we have a $\frac{1}{N}$ expansion parameter.

This completes our demonstration that the BPS and near-BPS sectors of the theory admit a $\frac{1}{M+N}$ expansion.

A study of the closed strings probing the LLM annulus background has been completed in [78, 79, 23] and in section 8. One can describe the gauge invariant operators $O(\{n\})$ in terms of Cuntz oscillators (representing the Zs) populating a lattice (formed from the Ys) as in [51, 71, 69, 31, 32, 33]. The article [79] in particular, pointed out that the net effect of the background, on the one loop dilatation operator, was to rescale the Cuntz oscillators. This rescaling is nothing but the relation we have found in (9.15)! This rescaling could also be accounted for by using a new ‘t Hooft coupling

$$\lambda_{\text{new}} = g_{YM}^2(N + M).$$

This again looks natural: we have the inverse of the effective genus counting parameter times g_{YM}^2. The net effect of the background on the one loop anomalous dimension operator (in this sector) is to replace $\lambda \to \lambda_{\text{new}}$. In the next section we investigate the effect of the background on the anomalous dimension operator at two loops.

10 Anomalous Dimensions of Operators with a large R Charge and Integrability

In sections 8 and 9 we have developed techniques to systematically study operators with an $\mathcal{R}$-charge of $O(N^2)$. In particular, we have developed general techniques, termed cutting rules, that allow us to calculate correlators of operators in the presence of general LLM background operators comprised of $O(N^2)$ fields, in the free field theory limit. We have also developed significant insight into the properties of correlators of operators in the presence of the half BPS annulus background, in the half BPS and near BPS sectors. These correlators cannot be treated using the standard $1/N$ perturbative expansion since the usual suppression of non-planar diagrams is, in this case, overpowered by huge combinatoric factors[18]. Instead, we have obtained evidence that the appropriate expansion parameter is $1/(N+M)$ where M is an $O(N)$ number describing the number of columns of the Young diagram labeling the annulus background operator. We will now study the anomalous dimensions of operators with classical dimension of order N^2 and investigate the question of integrability in non-trivial backgrounds.

There is by now an impressive body of work suggesting that *planar* $\mathcal{N}=4$ super Yang-Mills theory is *exactly integrable*. This would be very fortunate indeed, since it would mean the problem of computing the spectrum of all possible scaling dimensions of the gauge theory can be solved exactly, in the large N limit, by employing a Bethe ansatz. This has been established for the complete set of possible operators at one loop in[36, 37] and to two and three loops in the su(2) sector[35]. Given these results it is natural to guess that integrability extends to all orders in perturbation theory and perhaps even to the non-perturbative level[35]. There is now mounting evidence that this guess is correct [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50].

The idea of spin chain parity played a central role in the discovery of the planar two and three loop integrability[35]. Acting on a single trace operator, parity simply reverses the order of fields inside the trace. The dilatation operator commutes with parity, so that as we would expect, the dilatation eigenstates are also parity eigenstates. In addition, [35] found that eigenstates with opposite parity were degenerate - this was quite unexpected. This degeneracy can be explained by the existence of a higher conserved charge (U_2 in the notation of [35]) that commutes with the dilatation operator but anticommutes with parity. Its also possible (and useful) to put this argument on its head: one can interpret the degeneracy of states with opposite parity as evidence for the existence of further conserved quantities. When non-planar corrections are taken into account, parity remains a good quantum number but the degeneracy is lifted (see [35] for a discussion of the $\mathcal{N} = 4$ super Yang-Mills case and [101] for a very relevant discussion in the context of the ABJM theory). Although this only proves that the standard construction of conserved charges does not work away from the strict planar limit, it does suggests that integrability might not survive away from this limit. Clearly an important question is

Does integrability survive non-planar corrections?

In this section we will explicitly describe a situation in which we do sum (in fact, an infinite number of) non-planar corrections. The non-planar corrections that we consider arise because we are interested in computing the anomalous dimensions of operators whose classical dimension is of order N^2. Further, we collect some evidence that the resulting dynamics remains integrable. This suggests that, at least in certain situations, the answer to the above question is positive.

The specific operators we study are spelt out in detail in section 10.1.1. In section 10.1.2 we give the dilatation operator to two loops. To obtain this result, ribbon diagrams with arbitrarily large genus are summed. In section 10.1.3 we study the action of our dilatation operator. The action of this dilatation operator is *not easily formulated in a spin chain language because, nonplanar corrections allow the number of fields within each trace to change*. This translates into a spin chain with a variable lattice length. A convenient reformulation of the problem, in terms of a Cuntz lattice, was given in [51]. In the reformulation, particles (described by Cuntz oscillators) hop on a lattice of fixed size. The fact that the size of the spin chain lattice was dynamical now translates into the fact that the total number of particles populating the Cuntz lattice is dynamical. We give the Cuntz oscillator description of the dilatation operator for the class of operators we study in section 10.1.4. In section 10.2 we start to look for signs that our Cuntz chain is integrable. To start, we rewrite the spin chain (in the su(2) sector) of [36, 37, 35] in the Cuntz oscillator language. In particular, we write the conserved charge U_2 of[35] in terms of Cuntz oscillators. We have verified, using this expression for U_2, that U_2 does not commute with the Cuntz chain Hamiltonian corresponding to the annulus geometry. Another way explore the integrability of the original model is to study the semi-classical limit of the spin chain, which can be matched to the low energy limit of the principal chiral model. It is known that the principal chiral model is integrable. We can show that the Cuntz chain corresponding to the spin chain of [36, 37, 35] is indeed equivalent to the low energy limit of the principal chiral model - the spin chain and the Cuntz chain simply correspond to different choices of gauge. We give the explicit form of the gauge transformation relating the two in section 10.2.2. In section 10.2.3 we study the large M limit of our Hamiltonian and argue that we can indeed write down higher conserved charges. This suggests that integrability survives in this limit. In appendix D we present some technical details relevant to this section.

10.1 Two Loop Cuntz Chain of the LLM Background

$\mathcal{N}=4$ super Yang-Mills theory has 6 scalars ϕ_i transforming in the adjoint of the gauge group and in the **6** of the $SU(4)_{\mathcal{R}}$ symmetry. We shall use the complex combinations

$$Z=\phi_1+i\phi_2, \qquad Y=\phi_3+i\phi_4, \qquad X=\phi_5+i\phi_6, \tag{10.1}$$

in what follows. All operators that we study are built using only Z and Y; they belong to the su(2) sector of the theory. Many of the expressions that we write involve traces over Zs, Ys and derivatives of them. To avoid confusion, we will now spell out the index structure of a few expressions

$$\mathrm{Tr}\left(Z\frac{\partial}{\partial Z}\right) = Z_{ij}\frac{\partial}{\partial Z_{ij}}\,, \tag{10.2}$$

$$\mathrm{Tr}\left(ZY\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right) = Z_{ij}Y_{jk}\frac{\partial}{\partial Z_{lk}}\frac{\partial}{\partial Y_{il}}\,. \tag{10.3}$$

10.1.1 Annulus Background

Schur polynomials provide a very convenient reorganization of the half BPS sector of the $\mathcal{N}=4$ super Yang-Mills theory due to the properties outlined in section 3. The half BPS sector of the theory can be reduced to the dynamics of the eigenvalues of Z, which is the dynamics of N non interacting fermions in an external harmonic oscillator potential[17, 22]. The half BPS sector of operators with $\mathcal{R}$ charge of order N^2, are dual to solutions of type IIB supergravity - the LLM geometries[15]. For a careful discussion of precisely what aspects of the eigenvalue dynamics the supergravity captures and vice versa, see [91]. For a recent discussion of the macroscopic description of the dual geometry see [102]. It is by now well known that the space of the LLM geometries is given by a black and white coloring of a two dimensional plane[15, 74]; this colored plane is isomorphic to the fermionic phase space of the eigenvalues of Z. The Schur polynomials correspond to geometries that are invariant under rotations in this plane.

The Schur polynomials are labeled by Young diagrams. In what follows, we will be interested in $\chi_B(Z)$ with B a Young diagram that has N rows and M columns. We take M to be of order N. Note that we can also express

$$\chi_B(Z) = (\det(Z))^M\ . \tag{10.4}$$

This implies that

$$\frac{\partial}{\partial Z_{ij}}\chi_B(Z) = M(Z^{-1})_{ji}\chi_B(Z)\,, \tag{10.5}$$

a formula that we will make good use of below. The coloring describing the dual LLM geometry is a black annulus. The inner white disk has an area of

$\frac{M\pi}{N}$ whilst the black annulus itself has an area of π in units that assign an area of $\frac{\pi}{N}$ to each fermion state in phase space.

10.1.2 Two Loop Effective Dilatation Operator

The two loop dilatation operator, in the su(2) sector, has been computed in [35]. Using the conventions of [35], the dilatation operator can be expanded as

$$D=\sum_{k=0}^{\infty}\left(\frac{g_{YM}^2}{16\pi^2}\right)^k D_{2k}=\sum_{k=0}^{\infty}g^{2k}D_{2k}\,, \tag{10.6}$$

where the tree level, one loop and two loop contributions are

$$D_0=\mathrm{Tr}\left(Z\frac{\partial}{\partial Z}\right)+\mathrm{Tr}\left(Y\frac{\partial}{\partial Y}\right)\,, \tag{10.7}$$

$$D_2=-2:\mathrm{Tr}\left([Z,Y]\left[\frac{\partial}{\partial Z},\frac{\partial}{\partial Y}\right]\right):, \tag{10.8}$$

$$\begin{aligned}D_4=&-2:\mathrm{Tr}\left(\left[[Y,Z],\frac{\partial}{\partial Z}\right]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],Z\right]\right):\\&-2:\mathrm{Tr}\left(\left[[Y,Z],\frac{\partial}{\partial Y}\right]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],Y\right]\right):\\&-2:\mathrm{Tr}\left([[Y,Z],T^a]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],T^a\right]\right):\,.\end{aligned} \tag{10.9}$$

The normal ordering symbols here indicate that derivatives within the normal ordering symbols do not act on fields inside the normal ordering symbols.

We allow this dilatation operator to act on an operator built mainly from Zs with a few "impurities" = Ys added. For most of our study we explicitly display formulas for operators with two or three impurities. Our final formulas are however, completely general, covering the case that we have $O(1)$ impurities. Adapting the notation of [35] we define

$$\mathcal{O}_B(p,J_0,J_1,...,J_k)\equiv\chi_B(Z)\mathcal{O}_p^{J_0;J_1,\ldots,J_k}\equiv\chi_B(Z)\mathrm{Tr}\left(YZ^pYZ^{J_0-p}\right)\prod_{i=1}^{k}\mathrm{Tr}\,Z^{J_i}\,, \tag{10.10}$$

$$\mathcal{Q}_B(J_0,J_1,J_2,...,J_k)\equiv\chi_B(Z)\mathcal{Q}^{J_0,J_1;J_2,...,J_k}\equiv\chi_B(Z)\mathrm{Tr}\,(YZ^{J_0})\mathrm{Tr}\,(YZ^{J_1})\prod_{i=2}^{k}\mathrm{Tr}\,Z^{J_i}\,, \tag{10.11}$$

where $\chi_B(Z)$ is the operator creating the background, which was defined in the previous section. Our strategy is to define an effective dilatation operator D_{eff} as

$$D\left(\chi_B(Z)\mathcal{O}_p^{J_0;J_1,...,J_k}\right)=\chi_B(Z)D_{\text{eff}}\mathcal{O}_p^{J_0;J_1,...,J_k}\,. \tag{10.12}$$

Diagonalizing the action of D_{eff} on the gauge invariant operators $\mathcal{O}_p^{J_0;J_1,...,J_k}$ is clearly equivalent to diagonalizing the action of D on $\mathcal{O}_B(p,J_0,J_1,...,J_k)$. It is natural to interpret D_{eff} as the dilatation operator for the LLM background, that is, for the theory that is deformed by the insertion of $\chi_B(Z)\chi_B(Z^\dagger)$ in the path integral. Notice that we can write

$$D_{\text{eff}}=\frac{1}{\chi_B(Z)}D\chi_B(Z)\,. \tag{10.13}$$

This formula remains correct even after replacing $\chi_B(Z)$ by any other operator creating the background, which depends only on Z. Ultimately, we will restrict ourselves to the large $M+N$ limit. To capture this limit, as explained in section 9, one needs to re-sum an infinite number of nonplanar diagrams; ribbon diagrams with arbitrarily large genus are summed. *This limit is certainly not the planar limit of $\mathcal{N}=4$ super Yang Mills theory.*

The crucial observation needed in the computation of D_{eff} is that

$$\frac{\partial}{\partial Z_{ij}}\left(\chi_B(Z)\mathcal{O}_p^{J_0;J_1,...,J_k}\right)=\chi_B(Z)\left(\frac{\partial}{\partial Z_{ij}}+M(Z^{-1})_{ji}\right)\mathcal{O}_p^{J_0;J_1,...,J_k}\,. \tag{10.14}$$

Repeated application of this formula gives

$$D_{0\text{ eff}}=D_0+MN\,, \tag{10.15}$$

$$D_{2\text{ eff}}=D_2-2M\mathrm{Tr}\left(\left(ZYZ^{-1}+Z^{-1}YZ-2Y\right)\frac{\partial}{\partial Y}\right)\,, \tag{10.16}$$

$$
\begin{aligned}
D_{4\,\rm eff} &= D_4 + 4NM\mathrm{Tr}\left(\left(ZYZ^{-1}+Z^{-1}YZ-2Y\right)\frac{\partial}{\partial Y}\right)\\
&+ 2M:\mathrm{Tr}\,[Z,Y]\left[Z^{-1},\left[Z,\left[\frac{\partial}{\partial Z},\frac{\partial}{\partial Y}\right]\right]\right]:+2M:\mathrm{Tr}\,[Z,Y]\left[\frac{\partial}{\partial Z},\left[Z,\left[Z^{-1},\frac{\partial}{\partial Y}\right]\right]\right]:\\
&+ 2M:\mathrm{Tr}\,[Z,Y]\left[\frac{\partial}{\partial Y},\left[Y,\left[Z^{-1},\frac{\partial}{\partial Y}\right]\right]\right]:\\
&- 2M^2\mathrm{Tr}\left(\left(Z^2YZ^{-2}-4ZYZ^{-1}+6Y-4Z^{-1}YZ+Z^{-2}YZ^2\right)\frac{\partial}{\partial Y}\right)
\end{aligned}
\tag{10.17}
$$

for the tree level, one loop and two loop contributions to $D_{\rm eff}$. The formula for $D_{0\,\rm eff}$ has a straight forward interpretation - the dimension of the gauge invariant operator $\mathcal{O}_p^{J_0;J_1,\ldots,J_k}$ is shifted by MN due to the presence of the background, which has dimension MN. The answer for $D_{2\,\rm eff}$ has already been obtained and discussed in [78, 79, 23] and section 8.

10.1.3 Action of the Two Loop Effective Dilatation Operator

A useful observation made in [35], is that, when acting with $g^2D_2+g^4D_4$ on the generic two impurity gauge invariant operators $\mathcal{O}_p^{J_0;J_1,\ldots,J_k}$ and $\mathcal{Q}^{J_0,J_1;J_2,\ldots,J_k}$ operators of type $\mathcal{Q}^{J_0,J_1;J_2,\ldots,J_k}$ are never produced. This is easy to understand: acting with $g^2D_2+g^4D_4$ always inserts a commutator $\big[Y,Z\big]$ into a trace; this trace vanishes unless it contains another Y. This observation generalizes: when acting with $g^2D_{2\,\rm eff}+g^4D_{4\,\rm eff}$ on the generic two impurity gauge invariant operators $\mathcal{O}_p^{J_0;J_1,\ldots,J_k}$ and $\mathcal{Q}^{J_0,J_1;J_2,\ldots,J_k}$ operators of type $\mathcal{Q}^{J_0,J_1;J_2,\ldots,J_k}$ are never produced. This follows because both

$$
\mathrm{Tr}\left(\left(ZYZ^{-1}+Z^{-1}YZ-2Y\right)\frac{\partial}{\partial Y}\right) \tag{10.18}
$$

and

$$
\mathrm{Tr}\left(\left(Z^2YZ^{-2}-4ZYZ^{-1}+6Y-4Z^{-1}YZ+Z^{-2}YZ^2\right)\frac{\partial}{\partial Y}\right) \tag{10.19}
$$

annihilate gauge invariant operators containing a single Y and because the remaining terms in $g^2D_{2\,\rm eff}+g^4D_{4\,\rm eff}$ always insert a $\big[Y,Z\big]$ into a trace. In what follows we will study the anomalous dimensions of the operators

$\mathcal{O}_B(p, J_0, J_1, ..., J_k)$. Clearly, from the observation we just made, we do not need to consider the operators $\mathcal{Q}_B(J_0, J_1, J_2, ..., J_k)$ to do this.

Consider the action of $D_{2\text{ eff}}$ on $\mathcal{O}_p^{J_0;J_1,...,J_k}$. Following [35] we can break $D_{2\text{ eff}}$ into three pieces

$$D_{2\text{ eff}} = D_{2,0\text{ eff}} + D_{2,+\text{ eff}} + D_{2,-\text{ eff}}\ , \tag{10.20}$$

where $D_{2,0\text{ eff}}$ preserves the number of traces in $\mathcal{O}_p^{J_0;J_1,...,J_k}$, $D_{2,+\text{ eff}}$ increases (by 1) the number of traces and $D_{2,-\text{ eff}}$ decreases (by 1) the number of traces. From (10.16) it is clear that the additional term proportional to M can only contribute to $D_{2,0\text{ eff}}$. The terms $D_{2,+\text{ eff}}$ and $D_{2,-\text{ eff}}$ which involve gauge invariant operator splitting and joining will not contribute at the leading order; they will be important when computing, for example, subleading corrections to the leading $M+N$ limit. The additional contributions proportional to M in (10.16) have an important effect: they imply that p and $J_0 - p$ can be negative. Thus, we need to consider gauge invariant operators in which we populate the two "gaps between the Ys" with both positive and negative powers of Z. This implies that p in $\mathcal{O}_p^{J_0;J_1,...,J_k}$ is completely unrestricted.

It is easy to write down an exact expression for the action of $D_{2\text{ eff}}$

$$\begin{aligned} D_{2,0\text{ eff}}\mathcal{O}_p^{J_0;J_1,...,J_k} &= -4A_1\Big(\mathcal{O}_{p+1}^{J_0;J_1,...,J_k} \\ &-\mathcal{O}_p^{J_0;J_1,...,J_k}\Big) - 4A_2\Big(\mathcal{O}_{p-1}^{J_0;J_1,...,J_k} \\ &-\mathcal{O}_p^{J_0;J_1,...,J_k}\Big)\ , \end{aligned} \tag{10.21}$$

$$\begin{aligned} D_{2,+\text{ eff}}\mathcal{O}_p^{J_0;J_1,...,J_k} &= 4\sum_{J_{k+1}=1}^{p-1}\Big(\mathcal{O}_{p-J_{k+1}}^{J_0-J_{k+1};J_1,...,J_k,J_{k+1}} \\ &-\mathcal{O}_{p-J_{k+1}-1}^{J_0-J_{k+1};J_1,...,J_k,J_{k+1}}\Big) - 4\sum_{J_{k+1}=1}^{J_0-p-1}\Big(\mathcal{O}_{p+1}^{J_0-J_{k+1};J_1,...,J_k,J_{k+1}} \\ &- \mathcal{O}_p^{J_0-J_{k+1};J_1,...,J_k,J_{k+1}}\Big)\ , \end{aligned} \tag{10.22}$$

$$D_{2,-\ \mathrm{eff}}\mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4\sum_{i=1}^{k} J_i\Big(\mathcal{O}_{p+J_i}^{J_0+J_i;J_1,..,\hat{J}_i,..,J_k}$$

$$-\mathcal{O}_{p+J_i-1}^{J_0+J_i;J_1,..,\hat{J}_i,..,J_k}\Big) - 4\sum_{i=1}^{k} J_i\Big(\mathcal{O}_{p+1}^{J_0+J_i;J_1,..,\hat{J}_i,..,J_k}$$

$$- \mathcal{O}_{p}^{J_0+J_i;J_1,..,\hat{J}_i,..,J_k}\Big)\,, \tag{10.23}$$

where in the last expression hatted variables are removed from the argument of $\mathcal{O}$ and

$$\begin{aligned} A_1 &= M+N \qquad J_0 > p \\ &= M \qquad \text{otherwise} \end{aligned} \tag{10.24}$$

$$\begin{aligned} A_2 &= M+N \qquad p > 0 \\ &= M \qquad \text{otherwise}\,. \end{aligned} \tag{10.25}$$

Again motivated by [35] we can write

$$D_{4\ \mathrm{eff}} = -\frac{1}{4}(D_{2\ \mathrm{eff}})^2 + \delta D_{4\ \mathrm{eff}} \tag{10.26}$$

where

$$\delta D_{4\ \mathrm{eff}} = 2 : \mathrm{Tr}\ [Z,Y]\left[\frac{d}{dY},\left[Y,\left[\frac{d}{dZ},\frac{d}{dY}\right]\right]\right] : +2M : \mathrm{Tr}\ [Z,Y]\left[\frac{d}{dY},\left[Y,\left[Z^{-1},\frac{d}{dY}\right]\right]\right] : \,.$$

We can decompose $\delta D_{4\ \mathrm{eff}}$ as

$$\delta D_{4\ \mathrm{eff}} = \delta D_{4,0\ \mathrm{eff}} + \delta D_{4,+\ \mathrm{eff}} + \delta D_{4,-\ \mathrm{eff}} + \delta D_{4,+-\ \mathrm{eff}} + \delta D_{4,--\ \mathrm{eff}} + \delta D_{4,++\ \mathrm{eff}}\,. \tag{10.27}$$

The number of pluses/minuses in the subscripts on the right hand side of this last expression tell us how many traces are added/removed by the action of that particular term. $\delta D_{4,+-\ \mathrm{eff}}$ adds and removes a trace and hence it comes from summing higher genus ribbon diagrams which contribute to the trace conserving piece of $\delta D_{4\ \mathrm{eff}}$. It is again easy to write down exact expressions for the action of these terms. These expressions are given in Appendix D.3.

10.1.4 Leading $M+N$ Limit

To extract the leading terms at large N (recall that we take M to be of order N) we need to rewrite the action of $D_{2\,\mathrm{eff}}$ obtained in the last subsection in terms of normalized gauge invariant operators, that is, gauge invariant operators that have a suitably normalized two point function. The relevant two point correlators have been computed in Appendix D.1. The result is most easily written in terms of a Cuntz oscillator chain. We will now focus on gauge invariant operators that have $k=0$. If we relax the restriction to two impurities (which we do from now on), the gauge invariant operators we study have the form

$$\mathcal{O}_B(\{p\},J_0)\equiv\chi_B(Z)\mathrm{Tr}\,(YZ^{p_1}YZ^{p_2}\cdots YZ^{p_n})\,,\qquad\sum_{i=1}^{n}p_i=J_0\,. \tag{10.28}$$

To translate this gauge invariant operator into a Cuntz chain state, we associate a site of the Cuntz chain with each of the gaps between the Ys. The above gauge invariant operator defines a state in a Cuntz chain with n sites. Further, the number of Zs in each site gives the occupation number of that site. Finally, we require that operators with a normalized (free field) two point function map into normalized Cuntz chain states. This last point deserves a few comments. The spacetime dependence of the free field correlators we compute is trivially determined by the bare dimension of the operator. Thus we can compute all correlators in zero dimensions. In this zero dimensional model, each two point function is just a number. An operator with a normalized two point function is one for which this number is one. This map between two point functions and the norm of states is of course nothing but the usual state operator correspondence.

The dilatation operator allows the Zs to hop between sites of the Cuntz chain. At leading order in large $M+N$, the action of $D_{2\,\mathrm{eff}}$ is given entirely by $D_{2,0\,\mathrm{eff}}$. Using the correspondence given in the last equation of Appendix D.1, the operator equation (10.21) can be translated into an equation for the action of $D_{2\,\mathrm{eff}}$ on the Cuntz lattice. If all Cuntz lattice occupation numbers are positive then, since $D_{2\,\mathrm{eff}}$ lowers each occupation number by at most 1, acting with $D_{2\,\mathrm{eff}}$ can't change the value $p_- = 0$ where p_- is negative the sum of all the negative occupation numbers. This implies that all gauge invariant operators have the same leading two point function and hence, when acting on a Cuntz lattice with, for example, two sites ($\lambda = g^2 N$)

$$g^2 D_{2,0\ \mathrm{eff}} |\{p_1, p_2\}\rangle = -4\lambda \frac{N+M}{N} \left(|\{p_1+1, p_2-1\}\rangle - 2|\{p_1, p_2\}\rangle + |\{p_1-1, p_2+1\}\rangle \right) . \tag{10.29}$$

Similarly, if all Cuntz lattice occupation numbers are negative then, since $D_{2\,\mathrm{eff}}$ raises each occupation number by at most 1, acting with $D_{2\,\mathrm{eff}}$ again can't change the value $p_- = \sum_i p_i$. Again all gauge invariant operators have the same leading two point function and hence, when acting on a Cuntz lattice with, for example, two sites

$$g^2 D_{2,0\ \mathrm{eff}} |\{p_1, p_2\}\rangle = -4\lambda \frac{M}{N} \left(|\{p_1+1, p_2-1\}\rangle - 2|\{p_1, p_2\}\rangle + |\{p_1-1, p_2+1\}\rangle \right) . \tag{10.30}$$

Finally, consider for example the case that $D_{2,0\ \mathrm{eff}}$ acts on a lattice with two sites and occupation numbers $p_1 = 0$, $p_2 = 2$. In this case, the normalization of the gauge invariant operators is not the same: three terms have $p_- = 0$ and one has $p_- = 1$. Taking this into account gives

$$\begin{aligned} g^2 D_{2,0\ \mathrm{eff}} |\{0, 2\}\rangle &= -4\lambda \frac{M+N}{N} \left(|\{1,1\}\rangle - |\{0,2\}\rangle \right) \\ &- 4\lambda \frac{M}{N} \left(\frac{\sqrt{M+N}}{\sqrt{M}} |\{-1,3\}\rangle - |\{0,2\}\rangle \right) . \end{aligned} \tag{10.31}$$

There is a nice convenient way to summarize these results, see [78, 79, 23] and section 8. We will introduce Cuntz oscillators which satisfy the algebra (we associate one of these oscillators to each site of the chain)

$$a^\dagger a = \frac{M}{N} + \theta(\hat{n}+1) - |0\rangle\langle 0| \,, \qquad a a^\dagger = \frac{M}{N} + \theta(\hat{n}+1) \,, \tag{10.32}$$

with $\hat{n}$ the number operator. Notice that when $M = 0$ we have only positive occupation numbers so that the above relations reduce to the usual ones

$$a^\dagger a = 1 - |0\rangle\langle 0|\,, \qquad aa^\dagger = 1\,. \tag{10.33}$$

We can also define these oscillators by giving their action on states of good particle number

$$a|n\rangle = \sqrt{1+\frac{M}{N}}|n-1\rangle \qquad n>0 \tag{10.34}$$

$$a|n\rangle = \sqrt{\frac{M}{N}}|n-1\rangle \qquad n\le 0\,. \tag{10.35}$$

In terms of these Cuntz oscillators we have

$$g^2 D_{2\ \mathrm{eff}} = 2\lambda\sum_{l=1}^{L}(a_l^\dagger - a_{l+1}^\dagger)(a_l - a_{l+1})\,. \tag{10.36}$$

There is a rather direct way to extract (part of the) geometry of the dual LLM solution from this Cuntz oscillator description, see [78, 79, 23] and section 8. To see this, consider the coherent state

$$|z\rangle = \sum_{n=-\infty}^{0}\left(\frac{N}{M}\right)^{\frac{n}{2}} z^n|n\rangle + \sum_{n=1}^{\infty}\left(\frac{N}{M+N}\right)^{\frac{n}{2}} z^n|n\rangle\,. \tag{10.37}$$

The norm of this state

$$\langle z|z\rangle = \sum_{n=0}^{\infty}\frac{M^n}{N^n|z|^{2n}} + \sum_{n=1}^{\infty}\frac{N^n|z|^{2n}}{(M+N)^n} \tag{10.38}$$

is only finite if $\frac{M}{N}\le |z|^2 \le \frac{M+N}{N}$, that is, $|z|^2$ must lie within the annulus. Clearly z is a complex coordinate for the LLM plane.

This one loop result is intriguing: the effect of the background has been completely accounted for by simply modifying the Cuntz oscillator algebra. This is a remarkably simple change. It is natural to ask

Is the net effect of the background (even at higher loops) completely accounted for, by simply modifying the Cuntz oscillator algebra?

We do not have a complete answer to this question. Computing the form of the two loop answer ($D_{4\ \mathrm{eff}}$) is already a rather involved task. We have however studied this question for two sites. In this case

$$g^4 D_{4\ \mathrm{eff}} = -\frac{1}{4}(g^2 D_{2\ \mathrm{eff}})^2 + g^4 \delta D_{4\ \mathrm{eff}} \tag{10.39}$$

where

$$g^4 \delta D_{4\ \mathrm{eff}} = 4\lambda^2 \sum_{l=1}^{2} \Big(a_l^\dagger \big[a_{l+1}, a_{l+1}^\dagger\big] a_{l+1} + a_l^\dagger a_{l+1} \big[a_l, a_l^\dagger\big]$$
$$-a_l^\dagger a_l \big[a_{l+1}, a_{l+1}^\dagger\big] - a_l^\dagger \big[a_l, a_l^\dagger\big] a_l \Big) \ . \tag{10.40}$$

Thus, at two loops for a Cuntz chain with two sites we find that, again, the net effect of the background is to modify the Cuntz oscillator algebra.

10.2 Conservation Laws

In the last section we have explained how to extract an effective dilatation operator. Diagonalizing this effective dilatation operator will give the spectrum of anomalous dimensions for a class of operators whose classical dimension is of order N^2. Non-planar diagrams have to be included to obtain this dilatation operator. One consequence of this is that a spin chain description is no longer useful and we have instead passed to a Cuntz lattice description. In this section we want to answer two questions:

- The Cuntz lattice description can be employed even before a background is introduced. What is the translation of the conserved quantities of the original spin chain into the Cuntz lattice language?
- Is there any evidence that the effective dilatation operator obtained in the presence of the LLM annulus background is integrable?

10.2.1 U_2 rewritten in the Cuntz Oscillator Language

Before we obtain the conserved charge U_2 in the Cuntz oscillator language, its useful to first write it as a differential operator acting on gauge invariant operators. We will use this expression when we search for a corresponding conserved charge in the nontrivial background. It is a straight forward exercise to find that the planar action of

$$U_2 = \mathrm{Tr}\left((YZZ - ZYZ)\frac{d}{dY}\frac{d}{dZ}\frac{d}{dZ}\right) + \mathrm{Tr}\left((ZZY - YZZ)\frac{d}{dZ}\frac{d}{dY}\frac{d}{dZ}\right)$$
$$+\mathrm{Tr}\left((ZYZ - ZZY)\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dY}\right) + \mathrm{Tr}\left((ZYY - YZY)\frac{d}{dZ}\frac{d}{dY}\frac{d}{dY}\right)$$
$$+\mathrm{Tr}\left((YYZ - ZYY)\frac{d}{dY}\frac{d}{dZ}\frac{d}{dY}\right) + \mathrm{Tr}\left((YZY - YYZ)\frac{d}{dY}\frac{d}{dY}\frac{d}{dZ}\right), \tag{10.41}$$

on single trace gauge invariant operators matches the action of U_2 on the spin chain. To illustrate how to obtain a Cuntz oscillator representation, consider the first term above: it acts as

$$ZZY \to YZZ - ZYZ\,. \tag{10.42}$$

This term can be represented in terms of Cuntz oscillators as

$$\sum_{l=1}^{L}(a^\dagger_{l+1}a^\dagger_{l+1} - a^\dagger_l a^\dagger_{l+1})a_l a_l\,. \tag{10.43}$$

The second term in the sum should not be truncated to $a^\dagger_{l+1}a_l$ since we have to make sure that there are at least 2 Zs in lattice site l. The result for U_2 is

$$\begin{aligned} U_2 = \sum_{l=1}^{L}\Big(&(a^\dagger_{l+1}a^\dagger_{l+1} - a^\dagger_l a^\dagger_{l+1})a_l a_l + (a^\dagger_l a^\dagger_l - a^\dagger_{l+1}a^\dagger_{l+1})a_{l+1}a_l \\ &+(a^\dagger_l a^\dagger_{l-1} - a^\dagger_{l-1}a^\dagger_{l-1})a_l a_l + (a^\dagger_{l-1} - a^\dagger_l)a_{l+1}\big[a_l, a^\dagger_l\big] \\ &+ (a^\dagger_{l+1} - a^\dagger_{l-1})\big[a_l, a^\dagger_l\big]a_l + (a^\dagger_l - a^\dagger_{l+1})a_{l-1}\big[a_l, a^\dagger_l\big]\Big)\,. \end{aligned} \tag{10.44}$$

Using this procedure it is straight forward to write down the Cuntz oscillator representation for any of the conserved charges of the spin chain.

It is tedious but straight forward to compute the commutator of U_2 given above and $D_{2\,\mathrm{eff}}$ - they do not commute. It seems natural to consider the operator

$$U_{2\,\mathrm{eff}} = \frac{1}{\chi_B(Z)}U_2\chi_B(Z)\,. \tag{10.45}$$

It is a simple matter to compute $U_{2\,\mathrm{eff}}$ using the above expression for U_2 as a differential operator acting on gauge invariant operators. Again, $D_{2\,\mathrm{eff}}$ and $U_{2\,\mathrm{eff}}$ do not commute. However, in the leading large M limit the two do commute suggesting that this may be an interesting limit of $D_{2\,\mathrm{eff}}$. This is explored in detail in section 10.2.3 below.

10.2.2 Classical Limit

The original spin chain description of the dilatation operator can be replaced by a sigma model in the limit of a large number of sites. This sigma model precisely matches the Polyakov action describing the propagation of closed strings in $AdS_5 \times S^5$, in a particular limit[103, 104, 105]. This has also been extended to other examples of the AdS/CFT correspondence[51, 106, 107, 108, 71]. The Cuntz oscillator description of the dilatation operator is simply an alternative language in the undeformed background; when we consider the deformed background the spin chain description is not convenient, so that in this case it is best to use the Cuntz oscillator description. We can obtain a semiclassical description of the Cuntz chain by again considering a large number of sites[71]. In this subsection we will provide further insight into the relation between the spin chain and Cuntz oscillator descriptions by showing that in the dual string theory the two descriptions are simply related by a change of worldsheet gauge choice.

The semi-classical limit of the Cuntz chain is obtained by taking $L \sim \sqrt{N} \to \infty$, $\lambda \to \infty$ holding $\frac{\lambda}{L^2}$ fixed and by putting each lattice site into a coherent state (we are discussing the undeformed theory so there are no negatively occupied sites)

$$|z\rangle = \sqrt{1-|z|^2}\sum_{n=0}^{\infty} z^n |n\rangle\,, \qquad |n\rangle = (a^\dagger)^n|0\rangle\,. \tag{10.46}$$

The coherent state parameter of the l^{th} site is traded for a radius and an angle $z_l = r_l e^{i\phi_l}$. The action is given, as usual, by

$$S = \int dt \left(i\langle Z|\frac{d}{dt}|Z\rangle - \langle Z|D|Z\rangle \right) \qquad |Z\rangle = \prod_l |z_l\rangle\,. \tag{10.47}$$

After trading the sum over l for an integral over σ, the action is

$$S = -L\int dt \int_0^1 d\sigma \left(\frac{r^2\dot{\phi}}{1-r^2} + \frac{\lambda}{L^2}(r'^2 + r^2\phi'^2) \right)\,. \tag{10.48}$$

For a detailed derivation the reader could consult [71]. We would like to show how this Cuntz chain sigma model can be recovered from the standard string sigma model action on $R \times S^3$.

A string moving on $R \times S^3$ can be described by the principal chiral model with su(2) valued currents

$$j_\tau = g^{-1}\frac{\partial g}{\partial \tau}\,, \qquad j_\sigma = g^{-1}\frac{\partial g}{\partial \sigma}\,, \tag{10.49}$$

where

$$g = \begin{bmatrix} Z & iY \\ i\bar{Y} & \bar{Z} \end{bmatrix} \in \mathrm{SU}(2)\,, \tag{10.50}$$

and Z and Y are the coordinates of a sphere

$$|Z|^2 + |Y|^2 = 1\,. \tag{10.51}$$

We are choosing to employ a principal chiral model description since this description manifests the integrability of the model. Parametrize the sphere coordinates as follows

$$Z = re^{i(\kappa\tau - \phi)}\,, \qquad Y = \sqrt{1-r^2}e^{i(\varphi+\kappa\tau)}\,. \tag{10.52}$$

The equations of motion

$$\partial_\tau j_\tau - \partial_\sigma j_\sigma = 0 \tag{10.53}$$

can be obtained from the Polyakov action in conformal gauge and after fixing the residual conformal diffeomorphism freedom by choosing $t = \kappa\tau$. In this gauge, energy is homogeneously distributed along σ. To obtain the low energy limit, we take $\kappa \to \infty$ holding $\kappa\dot{r}$, $\kappa\dot{\phi}$ and $\kappa\dot{\varphi}$ fixed. It is precisely in this gauge and in this limit that[103] matched the semiclassical limit of the one loop spin chain to the string sigma model. The Lagrangian in this limit becomes

$$\mathcal{L} = -\frac{1}{4}(j_\tau^2 - j_\sigma^2) = r^2\kappa\dot{\phi} - (1-r^2)\kappa\dot{\varphi} + \frac{1}{2}r^2\phi'^2 + \frac{1}{2}(1-r^2)\varphi'^2 + \frac{1}{2}\frac{r'^2}{1-r^2}\,.$$

The equations of motion following from this action needs to be supplemented by the usual Virasoro constraints, which in this limit are $j_\tau^2 + j_\sigma^2 = 0$ and

$$\kappa\varphi'(1-r^2)^2 + \kappa\phi' r^2(r^2-1) + O(1) = 0\,. \tag{10.54}$$

The Cuntz sigma model (10.48) does not contain the field φ. Thus, it should be eliminated before we can expect to obtain agreement. Integrate by parts to obtain

$$\mathcal{L} = r^2\kappa\dot{\phi} - \varphi\frac{\partial}{\partial\tau}(1-r^2)\kappa + \frac{1}{2}r^2\phi'^2 + \frac{1}{2}(1-r^2)\varphi'^2 + \frac{1}{2}\frac{r'^2}{1-r^2}\,. \qquad (10.55)$$

Now, using the φ equation of motion (to rewrite the coefficients of φ in the action) and the (square of the) Virasoro constraint (10.54) we find

$$\mathcal{L} = r^2\kappa\dot{\phi} + \frac{1}{2}r^2\phi'^2 + \frac{1}{2}\frac{r'^2}{1-r^2} - \frac{1}{2}\frac{r^4}{1-r^2}\phi'^2\,. \qquad (10.56)$$

This does not agree with the result (10.48).

The disagreement is not surprising: the sigma model (10.56) is written in a gauge in which energy is distributed homogeneously along the string; the sigma model (10.48) corresponds to a gauge in which p_φ (= angular momentum conjugate to φ) is distributed homogeneously along the string. In the Cuntz chain, only Ys mark lattice sites; in the usual spin chain both Ys and Zs mark lattice sites. Consequently to go from the σ_{cc} coordinate of the Cuntz chain to the σ_{sc} coordinate of the spin chain, we need to "add the Zs back in"

$$\sigma_{sc} = \sigma_{cc} + \int_0^{\sigma_{cc}} n_z(\sigma')d\sigma' = \sigma_{cc} + \int_0^{\sigma_{cc}} \frac{r^2}{1-r^2}d\sigma'\,, \qquad \tau_{sc} = \tau_{cc}\,. \quad (10.57)$$

In this last equation, $n_z(\sigma') = r^2/(1-r^2)$ is the expected number of Cuntz particles (= number of Zs) at σ'. It is now straight forward to compute

$$\frac{\partial\sigma_{sc}}{\partial\sigma_{cc}} = \frac{1}{1-r^2}, \qquad \frac{\partial\tau_{sc}}{\partial\tau_{cc}} = 1, \qquad \frac{\partial\tau_{sc}}{\partial\sigma_{cc}} = 0\,, \qquad (10.58)$$

$$\frac{\partial\sigma_{sc}}{\partial\tau_{cc}} = -2r^2\frac{\partial\phi}{\partial\sigma_{cc}} = -2\frac{r^2}{1-r^2}\frac{\partial\phi}{\partial\sigma_{sc}}\,. \qquad (10.59)$$

The integrability condition

$$\frac{\partial}{\partial\tau_{cc}}\frac{\partial}{\partial\sigma_{cc}}\sigma_{sc} = \frac{\partial}{\partial\tau_{cc}}\frac{1}{1-r^2} = \frac{\partial}{\partial\sigma_{cc}}\frac{\partial}{\partial\tau_{cc}}\sigma_{sc} = \frac{\partial}{\partial\sigma_{cc}}\left(-2r^2\frac{\partial\phi}{\partial\sigma_{cc}}\right) \qquad (10.60)$$

is nothing but the ϕ equation of motion derived from (10.48). It is now a straight forward exercise to verify that after this change of coordinates (10.48) and (10.56) match perfectly.

The Cuntz oscillator Hamiltonian (10.48) can be written as

$$S = -L \int dt \int_0^1 d\sigma \left(n_z(\sigma)\dot{\phi} + \frac{\lambda}{L^2}(r'^2 + r^2\phi'^2) \right) . \tag{10.61}$$

The advantage of this rewriting is that it holds in general - that is, both for the undeformed and deformed backgrounds. Inserting the explicit expression for the expected number of Cuntz particles in the deformed background we find

$$S = -L \int dt \int_0^1 d\sigma \left(\left[\frac{r^2}{1 + \frac{M}{N} - r^2} - \frac{\frac{M}{N}}{\frac{M}{N} - r^2} \right] \dot{\phi} + \frac{\lambda}{L^2}(r'^2 + r^2\phi'^2) \right) . \tag{10.62}$$

If one drops either of the two terms in square braces one obtains a model which can be related to the low energy limit of the principal chiral model and hence is the low energy limit of an integrable model. The physical interpretation of such a truncation is clear: keeping only the first term corresponds to focusing on fluctuations localized at the outer edge of the annulus; keeping only the second term corresponds to focusing on fluctuations localized at the inner edge of the annulus. For these classes of fluctuations, it seems that the dynamics is integrable. It would be interesting to establish if the full model is integrable or not.

There is a very natural generalization to multi ring LLM geometries, corresponding to backgrounds created by Schur polynomials labeled by a Young diagram with more than 4 edges and each edge with a length of $O(N)$. In this case $n_z(\sigma)$ is a sum of terms, one for each edge. Restricting to fluctuations localized about a particular edge again gives a model which can be related to the low energy limit of the principal chiral model and hence is the low energy limit of an integrable model. These localized excitations have been constructed in [23].

The superstar geometry[52] has been related to an LLM geometry with boundary condition given by a sequence of concentric alternating black and white rings[74]. Rings of the same color have the same area and the total area of the black rings is π. As mentioned above, to construct $n_z(\sigma)$ we need to sum a term for each edge of the multi-ring geometry. We will consider a geometry which corresponds to the Young diagram shown in figure 12 with $n_1, n_2 << N$. In this case, we sum a very large number of terms and hence may use the Euler-Maclaurin formula to rewrite the sum as an integral. Carrying this integral out we find (we dropped an additive constant that will not contribute to the equations of motion)

$$n_z(\sigma) = \frac{\alpha}{\alpha+\beta}\frac{r^2}{1+\frac{M}{N}-r^2} \tag{10.63}$$

where

$$\alpha = \frac{n_1}{N}, \qquad \beta = \frac{n_2}{N}\,. \tag{10.64}$$

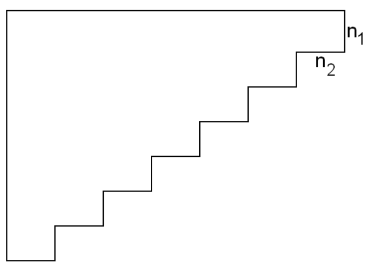


Figure 12: The Young diagram corresponding to the superstar geometry.

Thus, the semiclassical limit of the Cuntz chain model is

$$S = -L\int dt \int_0^1 d\sigma \left(\frac{\alpha}{\alpha+\beta}\frac{r^2}{1+\frac{M}{N}-r^2}\dot{\phi} + \frac{\lambda}{L^2}(r'^2 + r^2\phi'^2) \right) . \tag{10.65}$$

After rescaling t we can recover the action (10.48) up to an overall multiplicative constant. Thus, the model can again be related to the low energy limit of an integrable model.

10.2.3 Integrability in the Large M Limit

In this subsection we will consider the large M limit, that is, we take $M, N \to \infty$ and in addition, the ratio $\frac{N}{M} \to 0$. In this limit we suppress all $\frac{N}{M}$ dependence. The dilatation operator for the sector we consider, after subtracting the classical dimension out, can be written as

$$D = D\left(Z, Y, \frac{d}{dZ}, \frac{d}{dY}\right) . \tag{10.66}$$

To get the large M limit of D_{eff} (we denote this operator by $\tilde{D}_{\text{eff}}$) we should simply replace $\frac{d}{dZ}$ by MZ^{-1} in the above expression to obtain

$$\tilde{D}_{\text{eff}} = \tilde{D}_{\text{eff}}\left(Z, Y, MZ^{-1}, \frac{d}{dY}\right) . \tag{10.67}$$

Expand this operator as

$$\tilde{D}_{\text{eff}} = \sum_n \tilde{D}_{\text{eff n}} \tag{10.68}$$

where $\tilde{D}_{\text{eff n}}$ has a total of n derivatives with respect to Y. From the general structure of a connected planar l-loop vertex we know that D will act on $l + 1$ adjacent sites; thus, it will contain $l + 1$ derivatives. The leading contribution in the large M limit would come from terms which have all $l+1$ derivatives acting on Z and these are replaced to give an $M^{l+1}\text{Tr}\,(Z^{-l-1})$. This leading term is captured in $\tilde{D}_{\text{eff}\,0}$, which up to an arbitrary coefficient, is now determined. Since the dimension of $\text{Tr}\,(Z^J)$ is not corrected, it must be annihilated by D and hence, in the large M limit it must be annihilated by $\tilde{D}_{\text{eff}\,0}$. This implies that $\tilde{D}_{\text{eff}\,0}$ vanishes. Thus, the leading contribution will in fact come from the $\tilde{D}_{\text{eff}\,1}$. The l-loop term will thus have a $(g^2M)^l$ dependence - this is the dependence that the present argument captures[23]. Since D is dimensionless, and preserves the total number of Zs and the total

[23] Of course, our large M limit is a double scaling limit in which we take $M \to \infty$, $g^2 \to 0$ holding g^2M fixed. This limit is the natural one: our effective genus counting parameter is $\frac{1}{M}$ so that $\lambda = g^2M$ is the obvious definition of the 't Hooft coupling. See section 9 for further details.

number of Ys, it is clear that, to leading order at large M

$$\tilde{D}_{\text{eff}} = \tilde{D}_{\text{eff}\,1} = \sum_n c_n \text{Tr}\left(Z^n Y Z^{-n} \frac{d}{dY}\right) \tag{10.69}$$

where the c_n depend on $g^2 M$. It is trivial to see that

$$\left[\text{Tr}\left(Z^{-n} Y Z^n \frac{d}{dY}\right), \text{Tr}\left(Z^{-m} Y Z^m \frac{d}{dY}\right)\right] = 0\,, \tag{10.70}$$

so that

$$\left[\tilde{D}_{\text{eff}}, \text{Tr}\left(Z^{-m} Y Z^m \frac{d}{dY}\right)\right] = 0\,, \tag{10.71}$$

which clearly demonstrates an infinite number of conserved quantities to all loops.

What is the physical meaning of this limit? Recall that the dilatation operator can be read from the two point functions of the theory. Restricting to operators constructed from scalar fields only is, in general, not possible due to operator mixing. However, it is possible to show[35] that it is consistent to restrict to operators built using only the two complex fields Y and Z. In this case, we can compute the two point functions in a reduced model comprising of only the Y and Z matrices. Interpreted in this way, the dilatation operator can be understood as implementing the Wick contractions associated with the F-term vertex. For example, consider the combination

$$\frac{d}{dZ} + MZ^{-1} \tag{10.72}$$

which replaces $\frac{d}{dZ}$ in transforming the undeformed into the deformed Cuntz chain. The $\frac{d}{dZ}$ term represents a contraction between the vertex and a $Z^\dagger$ in one of the fields whose two point function we are computing; to see this connection it is useful to remember that

$$\langle Z^\dagger_{ij} Z_{kl}\rangle = \delta_{jk}\delta_{il} = \frac{d}{dZ_{ji}} Z_{kl}\,. \tag{10.73}$$

In contrast to this, the MZ^{-1} term represents a contraction between the vertex and a $Z^\dagger$ in the operator representing the background. In the large M limit, the contractions with the background completely dominate as compared to contractions with fields belonging to the operators of the two point functions we are computing. One can think that the matrices entering into the operators are "bits of a string". In the limit that we consider, the different bits in the string do not interact with each other - they interact only with the background. We would indeed expect the dynamics to simplify in this limit.

In the large M limit, the action of the Cuntz chain (10.62) becomes

$$S = -L \int dt \int_0^1 d\sigma \left(-\dot{\phi} + \frac{\lambda}{L^2}(r'^2 + r^2 \phi'^2) \right) . \tag{10.74}$$

Since $\dot{\phi}$ is a total derivative, all time derivatives drop out of the equations of motion. This implies that the dynamics becomes trivial which is indeed consistent with integrability. It is rather interesting that there is a class of operators in $\mathcal{N} = 4$ super Yang-Mills theory that have such a simple description.

In terms of the dual LLM boundary condition, the large M limit corresponds to an annulus with a large radius and fixed area, so that the annulus is becoming very thin.

11 Discussion

In this thesis we have introduced new and powerful technology facilitating the computation of certain classes of correlators in the half BPS, near BPS and multi-matrix sectors (particularly the $SU(2)$ sector) of $\mathcal{N} = 4$ SYM using the Schur polynomial and restricted Schur polynomial bases. The Schur polynomial basis provides a particularly useful reorganization of the half BPS sector of $\mathcal{N} = 4$ SYM due to certain special properties. The restricted Schur polynomials extend these properties to the multi-matrix sector and in doing so prove themselves to be a very useful reorganization in this more general setting. Firstly the restricted Schur polynomials, built from two or more types of matrices once again exactly diagonalize the free two point function[27].

In section 7 we demonstrated that the restricted Schur polynomials also satisfy a simple product rule, generalizing the rule known for Schur polynomials. Using this rule it is now possible to perform computations of higher point correlation functions of restricted Schur polynomials. We obtained explicit formulas for three point functions of a large class of restricted Schur polynomials which are built using antisymmetric or symmetric representations. Using these results we have been able to argue that restricted Schur polynomials built using the antisymmetric representations provide a suitable set of degrees of freedom for the description of perturbative quantum gravity, in agreement with the conclusions of [34] and extending the explicit computations performed there.

A key building block appearing in the restricted Schur polynomial is the restricted character. The restricted character is a generalization of the usual character, which plays a central role in group theory. In section 7 there is a development of the restricted character theory parallel to the character theory of finite groups. The notions of a conjugacy class and of a dual character, have been generalized to the notions of restricted conjugacy class and dual restricted characters. These generalizations lead to a set of orthogonality relations satisfied by the restricted characters. In this way we have ultimately obtained the generalization of the Littlewood-Richardson coefficient.

By studying the interaction of two restricted Schur polynomials we have suggested a physical interpretation for the labels of the restricted Schur polynomial: the composite operator $\chi_{R,(r_n,r_m)}(Z,X)$ is constructed from the half BPS "partons" $\chi_{r_n}(Z)$ and $\chi_{r_m}(X)$. Specifically, we have identified a composite operator in which the constituent partons are weakly interacting. The interaction of two such composites is largely determined by the interactions of the partons. This is analogous to the shower of pions produced when two hadrons interact; the pion shower can be attributed to the parton-parton interactions between pairs of partons belonging to different hadrons. In cases when the partons are strongly interacting (for example, at low energy), they are not visible and a partonic description of hadrons is not useful. In a similar way, we expect that when the constituent half BPS partons are strongly interacting, this might not be a useful interpretation of the restricted Schur polynomials. The picture of a composite comprised of half BPS partons seems to be similar to the proposal of [73] that every supersymmetric four dimensional black hole of finite area can be split up into microstates made of primitive half BPS "atoms". The idea that each half BPS state should be treated as an independent parton matches nicely with the picture that has emerged for half BPS states in AdS_5[15, 74]. Finally, in [75] eighth and quarter BPS states were obtained by putting together half BPS dual giants; making contact with that work may provide insight into identifying which restricted Schur polynomials correspond to quarter or eighth BPS states.

Although we focused on the case of two matrices in section 7, the extension to more matrices is straight forward as described in Appendix A.3. It would be interesting to work out some explicit examples with more than two matrices.

The Littlewood-Richardson numbers have many interpretations: as coefficients in the decomposition of tensor products into irreducible GL_n modules, as coefficients in the decomposition of skew Specht modules into irreducibles, as coefficients in the decomposition of S_n representations induced from Young subgroups and as intersection numbers in the Schubert calculus on a Grassmannian[76]. How much of this generalizes for restricted Littlewood-Richardson numbers? Finally, our work contains some general directions for the study of multi-matrix models. The eigenvalue based techniques that were so useful for the description of one matrix models do not seem to have an easy generalization for multi matrices. For one matrix models, the eigenvalues provide a set of $O(N)$ variables; since fluctuations about the large N configuration are $O(N^{-2})$, these provide a suitable set of variables for the implementation of a saddle point approximation. In general, it would be incorrect to assume that the matrices of a multimatrix model can be simultaneously diagonalized[24]. Thus, at best it seems that one needs to keep the eigenvalues of each matrix, plus unitary matrices which encode how one goes between the bases in which a particular matrix is diagonal. This gives a total of $O(N^2)$ variables, so that this does not provide a promising starting point for a saddle point analysis. In contrast to this, using the technology of [31, 25, 26, 27], we can give a rather detailed description of the free multi-matrix models. One loop results for the super Yang-Mills F term also have a description in this framework[32, 33, 62]. It would be interesting to develop these methods further by developing efficient techniques for extracting large N results and for managing more general potentials.

Utilizing the Schur polynomial and restricted Schur polynomial technology, the next challenge addressed was that of calculating the correlators of operators with an $\mathcal{R}$-charge of $O(N^2)$. These operators are dual to states that have a large mass and consequently back reaction on the dual geometry is important. Schur polynomials comprised of $O(N^2)$ fields are dual to new background geometries that are only asymptotically $\mathrm{AdS}_5\times \mathrm{S}^5$ known as LLM geometries. Calculating correlators of certain operators in the presence of these large background Schur polynomials corresponds to elucidating the dual string theory dynamics on these new geometries with different probes e.g. gravitons, strings or giant gravitons. In general computing these correlators is highly non-trivial due to the fact that non-planar diagrams are no longer suppressed: although they come with the usual $\frac{1}{N^2}$ suppressions, these are overpowered by huge combinatoric factors and the usual $\frac{1}{N}$ expansion breaks down. In section 8 we introduced techniques, known as the

[24]An exception to this is the sector of the theory comprising of the chiral ring at strong coupling - see [77].

cutting rules, that allowed the calculation of correlators of restricted Schur polynomials when the background has $O(N^2)$ fields and correlators of generic operators in the presence of background Schur polynomials.

The backgrounds we have considered are LLM geometries that correspond to a set of concentric rings. We have probed these backgrounds with operators corresponding to both open strings and closed strings. Contractions between fields in the string words and fields in the operator creating the background need only be taken into account when the number of fields in the operator creating the background is $O(N^2)$; these contractions are the field theory accounting of the back reaction on the geometry. From the results of [23], we know that in the new background we can break the original matrix Z into "local pieces", Z_i, which add boxes at specific locations on the Young diagram. In section 8 we have given a precise definition for this decomposition: the original Z matrix decomposes into a block diagonal matrix. There is a block for each ring. The dimension N_i of the blocks is equal to the number of eigenvalues in each ring. These blocks are the Z_i. To compute correlation functions of these local operators, use the usual free field theory ribbon diagrams, but each ribbon now carries an extra factor of $\frac{c}{N}$ with c the weight of the boxes added by the local operator. The complete effect of the background is the extra $\frac{c}{N}$ factor now carried by each propagator, in perfect agreement with [23]. This is a considerable simplification.

Our study of open string excitations shows that the dynamics of the string endpoints is identical to the dynamics of the bulk of the string. Open string excitations of the operators with an $\mathcal{R}$-charge of $O(N^2)$ behave like a closed string; there are no open string excitations with their special end point interactions: in the new background we have only closed string excitations. This is expected since the operator being excited is dual to a new geometry and not a brane.

Finally, the techniques we have developed here are equally applicable to the computation of correlators in the presence of the multi-matrix operators of [25, 26, 27, 98].

In section 9 we delved deeper into the perturbation theory of correlators of operators whose classical dimension is $O(N^2)$ and demonstrated that although the standard $1/N$ expansion breaks down, it is possible to reorganize the expansion and we have identified the small parameters that control these expansions. These results were neatly captured by surprisingly simple relations between correlators in the trivial background and correlators in the background of our heavy operator.

Another interesting result that we have obtained, is that in the multi ring backgrounds and in the multi matrix backgrounds there was more than just one coupling. At first site this may seem puzzling, since the dilaton is a constant. To get some insight into what is going on, consider QCD. At high energies QCD is well described by a Lagrangian of quarks and gluons together with the coupling g_{YM}^2. At low energies, the coupling grows and one needs - somehow - to reorganize the theory of quarks and gluons into a low energy effective theory. The semi-classical objects in this low energy theory will be protons, neutrons, pions,... and there will be many possible coupling constants telling us how these semi-classical objects interact. The relevance of this story is that for the correlators we have studied, the planar approximation breaks down and one again needs to reorganize the theory. In section 9 we have managed to explicitly perform this reorganization; the objects that we find that have different couplings are naturally interpreted as different semi-classical objects in the effective theory.

There will be effects that are non-perturbative in the new expansion parameter. An example of a non-perturbative amplitude is the transition from a maximal sphere giant graviton ($\chi_{[1^N]}(Z)$; $[1^N]$ denotes a Young diagram with one column containing N boxes) into KK gravitons with identical angular momentum $J << N$. The amplitude is given by[17, 61]

$$\frac{\left|\langle\chi_{[1^N]}(Z^\dagger)(\mathrm{Tr}\,(Z^J))^{N/J}\rangle\right|^2}{\langle\chi_{[1^N]}(Z^\dagger)\chi_{[1^N]}(Z)\rangle\ \langle\mathrm{Tr}\,(Z^{\dagger J})\mathrm{Tr}\,(Z^J)\rangle^{N/J}} \sim (2\pi)^{\frac{1}{2}}e^{-\frac{1}{g}-\frac{1}{2}\log(g)-(1/(gJ))\log(J)}$$

where $g = 1/N$. This is non-perturbative in $1/N$. To get the corresponding amplitude in the annulus, using Schur technology we can again argue that we simply need to replace $N \to N + M$, so that the transition amplitude is non-perturbative in $1/(M + N)$.

The techniques of section 9 allow us to reorganize the $1/N$ expansion for classes of observables in backgrounds described by Schur polynomials labeled by Young diagrams whose edges are all $O(N)$. This does not exhaust the operators that are dual to classical geometries. Indeed, the Schur polynomials correspond to spacetimes whose LLM boundary condition preserve a rotational symmetry on the plane in which the LLM boundary condition is specified. This is a small subset of the possible half BPS LLM geometries. Further, even amongst those LLM geometries with a rotationally invariant boundary condition, it is likely that we have only dealt with a subset of the geometries that can occur. Indeed, consider a triangular Young diagram T such that its longest column has N boxes, and each column has one box less than the column to its left. Thus our Young diagram has N columns in total. We have tested, in a number of cases, that factorization of the expectation value

$$\langle \mathcal{O} \rangle_T \equiv \frac{\langle \mathcal{O} \chi_T(Z) \chi_T(Z^\dagger) \rangle}{\langle \chi_T(Z) \chi_T(Z^\dagger) \rangle}$$

holds. This suggests that $\chi_T(Z)$ is again dual to a classical geometry. We have not yet understood how to reorganize the large N expansion in this case.

One very interesting class of excitations of an operator with a very large classical dimension, is obtained by attaching "open string defects" to the operator representing the background. Recent evidence[84, 99] suggests that these open string excited operators are dual to black hole micro states. Consequently, we expect that black hole microstate dynamics is captured in the dynamics of these operators. Our results (see in particular (9.15)) suggest that it may well be possible to write down spin chain descriptions for the open string excitations. Towards this end, we have explored the proposed BMN-type sectors discovered in [99]. The existence of these limits is sure to play an important role in understanding the gauge theory description of near-extremal charged black holes in AdS_5. Taking $M_1 = O(\sqrt{N})$ and $M_2 = O(\sqrt{N})$ with the 't Hooft coupling λ large but fixed, the charges J_1

and J_2 scale as $O(N^{\frac{3}{2}})$, whilst the difference

$$\frac{\Delta - J_1 - J_2}{J_1 + J_2} \sim N^{-\frac{1}{2}} \to 0$$

in perfect agreement with the supergravity arguments of [99]. Thus, strong and weak coupling results agree - as expected for a BMN-like limit. We have further argued that if we hold $M_1 = M_2 = O(\sqrt{N})$ fixed we find an effective genus counting parameter $g_2 = \frac{1}{M}$ and an effective 't Hooft coupling $\tilde{\lambda} = g_{YM}^2 M$. Keeping the effective 't Hooft coupling arbitrarily small, the one loop correction to the anomalous dimension can be made arbitrarily small, even though λ is sent to infinity in the limit. This is similar to what happens in the BMN limit suggesting that it will be possible to compare perturbative field theory results to results obtained from the dual gravitational description. The above effective 't Hooft coupling and genus counting parameters are naturally identified with those of the gauge theory living on the world volume of our system of intersecting giant gravitons. It would be very interesting to determine the dimensionality of this theory: is it a 2d gauge theory as the results of [84, 99] imply? Of course, the operators we have considered here are presumably too simple to describe the black hole microstates; for that one needs to consider triangular Young diagrams [74]. The triangular Young diagrams have a large number of corners which implies a large number of possible excitations of the operator. This is a significant increase in complexity as compared to the studies in section 9.

Finally, studying the rectangular Young diagrams that we have considered here allowed us to investigate the anomalous dimensions of operators with a classical dimension of $O(N^2)$. The problem of computing the anomalous dimensions of operators with a large ($O(N^2)$) $\mathcal{R}$-charge, addressed in section 10, corresponds to a generalization, in the dual gravitational description, to string dynamics in spacetimes that are only asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$. The problem can again be reduced to diagonalizing a Hamiltonian. In the $\mathrm{AdS}_5\times\mathrm{S}^5$ spacetime, this Hamiltonian was an integrable spin chain. As a consequence of the fact that our strings can exchange angular momentum with the background, our Hamiltonian describes Cuntz particles hopping on a lattice. In the gauge theory description, the terms in the dilatation operator that allow the strings to exchange angular momentum with the background arise from summing (an infinite number of) non-planar corrections. It is surprisingly straight forward to write down very explicit expressions for the relevant Cuntz chain Hamiltonians.

A natural question to ask is if our Cuntz chain Hamiltonians correspond to integrable systems. We don't know. However, we have given some evidence that the large M limit of our Hamiltonian does admit higher conserved charges and that certain localized semiclassical excitations are described by the low energy limit of a principal chiral model, so an optimist would indeed conjecture that our Cuntz chain Hamiltonian is integrable.

A Appendix - Product Rule for Restricted Schur Polynomials

A.1 Identities and notation

The polynomials we construct are built out of two matrices, X and Z. We will typically use an n to denote the number of Zs in the polynomial and an m to denote the number of Xs in the polynomial. The polynomial is built using an irreducible representation of the symmetric group S_{n+m} that permutes the indices of the Z and X fields amongst each other. We will denote the Young diagrams labeling representations of the symmetric group acting on both Xs and Zs using capital letters. The indices of the matrix representing an element of this group will be denoted by capital Greek letters. Thus, the elements of the matrix representing σ in the S_{n+m} irreducible representation R will be denoted by $\left[\Gamma_R(\sigma)\right]_{\Psi\Phi}$. The irreducible representations of the S_n subgroup that acts only on the indices of the Z fields will be denoted by a small letter with a subscript n; matrix elements of this representation will be indexed using letters from the start of the alphabet. Thus, the elements of the matrix representing σ in the S_n irreducible representation r_n will be denoted by $\left[\Gamma_{r_n}(\sigma)\right]_{ab}$. The irreducible representations of the S_m subgroup that acts only on the indices of the X fields will be denoted by a small letter with a subscript m; matrix elements of this representation will be indexed using letters from the middle of the alphabet. Thus, the elements of the matrix representing σ in the S_m irreducible representation r_m will be denoted by $\left[\Gamma_{r_m}(\sigma)\right]_{ij}$. Finally, irreducible representations of the $S_n \times S_m$ subgroup that permute the indices of the Xs and permute the indices of the Zs is denoted by (r_n, r_m).

The delta function $\delta(\sigma)$ is 1 if σ is the identity and zero otherwise. In this appendix the $\delta(\cdot)$ is usually summed with a summand that is a class function. In addition, we usually have a delta function defined on the classes, i.e. we have $\delta(\left[\sigma\right]_r \left[\tau\right]_r)$ instead of $\delta(\sigma\tau)$. The relation between these two is easily established: Let $F(\big[\sigma]_r)$ be any class function. By comparing

$$\sum_{\sigma \in S_n} F(\big[\sigma]_r)\delta(\big[\sigma\big]_r\big[\tau\big]_r) \qquad \text{and} \qquad \sum_{\sigma \in S_n} F(\big[\sigma]_r)\delta(\sigma\tau)$$

we find that, when summing the delta function multiplied by any class function, over the whole group, we can freely replace

$$\frac{\delta(\big[\sigma\big]_r\big[\tau\big]_r)}{n^{\sigma}_{R,(r_n,r_m)}} \leftrightarrow \delta(\sigma\tau).$$

A.2 A Formula for the restricted Littlewood-Richardson numbers

In this appendix we will employ a bra/ket notation. In what follows, R is an irreducible representation of S_{n+m} and (r_n, r_m) is an irreducible representation of $S_n \times S_m$. We label the irreducible representations of $S_n \times S_m$ by a Young diagram r_n with n boxes (which labels an irreducible representation of S_n) and a Young diagram r_m with m boxes (which labels an irreducible representation of S_m). For states belonging to the carrier space of R we write $|R, \Gamma\rangle$. For states belonging to the carrier space of (r_n, r_m) we write labels acted on by the S_n and S_m subgroups separately $|r_n, a; r_m, i\rangle$. For example, we write

$$\big[\Gamma_R(\alpha)\big]_{\Lambda\Psi} = \langle R, \Lambda|\alpha|R, \Psi\rangle.$$

For a trace over an "on the diagonal block" (see [30],[31] for an explanation of this terminology) we write

$$\mathrm{Tr}_{\,(r_n,r_m)}(\Gamma_R(\sigma)) = \sum_{i,a}\sum_{\Lambda\Theta}\langle r_n, a; r_m, i|R, \Lambda\rangle\big[\Gamma_R(\sigma)\big]_{\Lambda\Theta}\langle R, \Theta|r_n, a; r_m, i\rangle.$$

For an "off the diagonal block" we write

$$\mathrm{Tr}_{\,(r_n,r_m),(s_n,s_m)}(\Gamma_R(\sigma)) = \sum_{i,a}\sum_{\Lambda\Theta}\langle r_n, a; r_m, i|R, \Lambda\rangle\big[\Gamma_R(\sigma)\big]_{\Lambda\Theta}\langle R, \Theta|s_n, a; s_m, i\rangle.$$

For the labels of the off the diagonal block, we need (r_n, r_m) to have the same shape as (s_n, s_m). This means that r_n and s_n as well as s_m and r_m have the same shape.

For $\alpha_1 \in S_n$ and $\alpha_2 \in S_m$ we have

$$\begin{aligned}
\big[\Gamma_R(\alpha_1\circ\alpha_2)\big]_{\Psi\Theta} &= \langle R,\Psi|\alpha_1\circ\alpha_2|R,\Theta\rangle \\
&= \sum_{r_n,r_m,a,i}\ \sum_{t_n,t_m,b,j}\langle R,\Psi|r_n,a;r_m,i\rangle\langle r_n,a;r_m,i|\alpha_1\circ\alpha_2|t_n,b;t_m,j\rangle \\
&\qquad \times\langle t_n,b;t_m,j|R,\Theta\rangle \\
&= \sum_{r_n,r_m,a,i}\ \sum_{b,j}\langle R,\Psi|r_n,a;r_m,i\rangle\big[\Gamma_{r_n}(\alpha_1)\big]_{ab}\big[\Gamma_{r_m}(\alpha_2)\big]_{ij} \\
&\qquad \times\langle r_n,b;r_m,j|R,\Theta\rangle\,.
\end{aligned}$$

To get to the last line we have used the fact that if $\alpha_1 \in S_n$ and $\alpha_2 \in S_m$ then $\langle r_n,a;r_m,i|\alpha_1\circ\alpha_2|t_n,b;t_m,j\rangle \propto \delta_{r_nt_n}\delta_{r_mt_m}$. Use this identity to compute

$$\begin{aligned}
&\sum_{\alpha_1\in S_n}\ \sum_{\alpha_2\in S_m}\big[\Gamma_R(\alpha_1\circ\alpha_2)\big]_{\Psi\Theta}\big[\Gamma_{r_n}(\alpha_1^{-1})\big]_{ab}\big[\Gamma_{r_m}(\alpha_2^{-1})\big]_{ij} \\
=\ & \sum_{\alpha_1\,,\alpha_2}\sum_{t_nt_m}\sum_{c\,d\,k\,l}\langle R,\Psi|t_n,c;t_m,k\rangle\big[\Gamma_{t_n}(\alpha_1)\big]_{cd}\big[\Gamma_{t_m}(\alpha_2)\big]_{kl}\langle t_n,d;t_m,l|R,\Theta\rangle\,\times \\
\times\ & \big[\Gamma_{r_n}(\alpha_1^{-1})\big]_{ab}\big[\Gamma_{r_m}(\alpha_2^{-1})\big]_{ij} \\
=\ & \frac{n!m!}{d_{r_n}d_{r_m}}\sum_i\langle R,\Psi|r_n^{(i)},b;r_m^{(i)},j\rangle\langle r_n^{(i)},a;r_m^{(i)},i|R,\Theta\rangle\,.
\end{aligned}$$

To obtain this result we have used the fundamental orthogonality relation

$$\sum_{\alpha\in\mathcal{G}}\big[\Gamma_R(\alpha)\big]_{ab}\big[\Gamma_S(\alpha^{-1})\big]_{cd} = \frac{g}{d_R}\delta_{RS}\delta_{ad}\delta_{bc},$$

with d_R the dimension of irreducible representation R and g the order of $\mathcal{G}$. A given $S_n\times S_m$ irreducible representation may be subduced more than once by an S_{n+m} irreducible representation R. When applying this relation to the sums over α_1 and α_2, we will get a contribution from all representations whose S_n and S_m labels match. The index i in $(r_n^{(i)},r_m^{(i)})$ runs over the complete set of identical $S_n\times S_m$ irreducible representations. This last identity will be used to argue that the dual character is

$$\chi^{R,R_\gamma}(\tau) = \chi^{R,(r_n^{(i)},r_m^{(i)})(r_n^{(j)},r_m^{(j)})}(\tau) = \frac{d_Rn!m!}{d_{r_n}d_{r_m}(n+m)!}\chi_{R,(r_n^{(j)},r_m^{(j)})(r_n^{(i)},r_m^{(i)})}(\tau).$$

To verify this, we compute

$$
\begin{aligned}
& \sum_{R,R_\gamma} \chi^{R,R_\gamma}(\tau)\chi_{R,R_\gamma}(\sigma) \\
= & \sum_{R} \sum_{r_n^{(i)},r_m^{(i)}} \sum_{r_n^{(j)},r_m^{(j)}} \chi^{R,(r_n^{(i)},r_m^{(i)})(r_n^{(j)},r_m^{(j)})}(\tau)\chi_{R,(r_n^{(i)},r_m^{(i)})(r_n^{(j)},r_m^{(j)})}(\sigma) \\
= & \sum_{R} \sum_{r_n^{(i)},r_m^{(i)}} \sum_{r_n^{(j)},r_m^{(j)}} \sum_{\Theta,\Psi,\Gamma,\Sigma} \sum_{a,b,i,j} \\
& \quad \Gamma_R(\sigma)_{\Theta\Psi}\Gamma_R(\tau)_{\Gamma\Sigma}\langle r_n^{(i)},a;r_m^{(i)},i|R,\Theta\rangle\langle R,\Psi|r_n^{(j)},a;r_m^{(j)},i\rangle \\
& \quad \times\langle r_n^{(j)},b;r_m^{(j)},j|R,\Gamma\rangle\langle R,\Sigma|r_n^{(i)},b;r_m^{(i)},j\rangle\frac{d_R n!m!}{d_{r_n}d_{r_m}(n+m)!} \\
= & \sum_{R}\sum_{r_n,r_m}\sum_{\Theta,\Psi,\Gamma,\Sigma}\sum_{a,b,i,j}\Gamma_R(\sigma)_{\Theta\Psi}\Gamma_R(\tau)_{\Gamma\Sigma}\frac{d_R n!m!}{d_{r_n}d_{r_m}(n+m)!}\left(\frac{d_{r_n}d_{r_m}}{n!m!}\right)^2 \\
& \quad \times\sum_{\alpha_1,\alpha_2}\left[\Gamma_R(\alpha_1\circ\alpha_2)\right]_{\Sigma\Theta}\left[\Gamma_{r_n}(\alpha_1^{-1})\right]_{ba}\left[\Gamma_{r_m}(\alpha_2^{-1})\right]_{ji} \\
& \quad \times\sum_{\beta_1,\beta_2}\left[\Gamma_R(\beta_1\circ\beta_2)\right]_{\Psi\Gamma}\left[\Gamma_{r_n}(\beta_1^{-1})\right]_{ab}\left[\Gamma_{r_m}(\beta_2^{-1})\right]_{ij} \\
= & \sum_{R}\sum_{\alpha_1,\alpha_2}\Gamma_R(\sigma)_{\Theta\Psi}\Gamma_R(\tau)_{\Gamma\Sigma}\frac{d_R}{(n+m)!}\left[\Gamma_R(\alpha_1\circ\alpha_2)\right]_{\Sigma\Theta}\left[\Gamma_R(\alpha_1^{-1}\circ\alpha_2^{-1})\right]_{\Psi\Gamma} \\
= & \sum_{R}\sum_{\alpha_1,\alpha_2}\chi_R(\sigma(\alpha_1^{-1}\circ\alpha_2^{-1})\tau(\alpha_1\circ\alpha_2))\frac{d_R}{(n+m)!} \\
= & \; n!m!\delta(\left[\sigma\right]_r\left[\tau\right]_r),
\end{aligned}
$$

which demonstrates the result. In the above proof we have made use of the formula

$$\sum_R \frac{d_R}{g}\chi_R(\sigma)=\delta(\sigma).$$

Clearly then,

$$f^{T,(t_n^{(i)},t_m^{(i)})(t_n^{(j)},t_m^{(j)})}_{R,R_\alpha\,S,S_\beta}=\frac{1}{n_1!n_2!m_1!m_2!}\sum_{\sigma_1\in S_{n_1+m_1}}\sum_{\sigma_2\in S_{n_2+m_2}}$$

$$\frac{d_T(n_1+n_2)!(m_1+m_2)!}{d_{t_n}d_{t_m}(n_1+n_2+m_1+m_2)!}\chi_{R,R_\alpha}(\sigma_1)\chi_{S,S_\beta}(\sigma_2)\chi_{T,(t_n^{(j)},t_m^{(j)})(t_n^{(i)},t_m^{(i)})}(\sigma_1\circ\sigma_2).$$

A.3 More than 2 Matrices

Consider a matrix model with d species of matrices. R is an irreducible representation of $S_{n_1+n_2+\cdots+n_d}$ and $(r_{n_1}, r_{n_2}, \cdots, r_{n_d})$ is an irreducible representation of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_d}$. The Young diagram r_{n_i} has n_i boxes; it labels an irreducible representation of S_{n_i}. It is straight forward to show that

$$\prod_{i=1}^{d} \sum_{\alpha_i \in S_{n_i}} \left[\Gamma_R(\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_d)\right]_{\Psi\Theta} \left[\Gamma_{r_{n_1}}(\alpha_1^{-1})\right]_{a_1 b_1} \cdots \left[\Gamma_{r_{n_d}}(\alpha_d^{-1})\right]_{a_d b_d}$$

$$= \prod_{i=1}^{d} \frac{n_i!}{d_{r_{n_i}}} \sum_j \langle R, \Psi | r_{n_1}^{(j)}, b_1; r_{n_2}^{(j)}, b_2; \cdots r_{n_d}^{(j)}, b_d \rangle \langle r_{n_1}^{(j)}, a_1; r_{n_2}^{(j)} a_2; \cdots ; r_{n_d}^{(j)} a_d | R, \Theta \rangle \, .$$

This last identity can again be used to argue that the dual character is

$$\chi^{R,(r_{n_1}^{(i)}, r_{n_2}^{(i)}, \cdots, r_{n_d}^{(i)})(r_{n_1}^{(j)}, r_{n_2}^{(j)}, \cdots, r_{n_d}^{(j)})}(\tau) = \prod_{k=1}^{d} \frac{n_k!}{d_{r_k}} \frac{d_R}{(n_1+n_2+\cdots+n_k)!} \chi_{R,(r_{n_1}^{(j)}, r_{n_2}^{(j)}, \cdots, r_{n_d}^{(j)})(r_{n_1}^{(i)}, r_{n_2}^{(i)}, \cdots, r_{n_d}^{(i)})}(\tau).$$

Clearly then, (R is an irreducible representation of $S_{n_1+n_2+\cdots+n_d}$; R_α is an irreducible representation of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_d}$; S is an irreducible representation of $S_{m_1+m_2+\cdots+m_d}$; S_β is an irreducible representation of $S_{m_1} \times S_{m_2} \times \cdots \times S_{m_d}$; T is an irreducible representation of $S_{n_1+n_2+\cdots+n_d+m_1+m_2+\cdots+m_d}$; $(r_{n_1}^{(i)}, r_{n_2}^{(i)}, \cdots, r_{n_d}^{(i)})$ and $(r_{n_1}^{(j)}, r_{n_2}^{(j)}, \cdots, r_{n_d}^{(j)})$ have the same shape and are both irreducible representations of $S_{n_1+m_1} \times S_{n_2+m_2} \times \cdots \times S_{n_d+m_d}$)

$$f_{R,R_\alpha\, S,S_\beta}^{T,(r_{n_1}^{(i)}, r_{n_2}^{(i)}, \cdots, r_{n_d}^{(i)})(r_{n_1}^{(j)}, r_{n_2}^{(j)}, \cdots, r_{n_d}^{(j)})} = \prod_{i=1}^{d} \frac{1}{n_i! m_i!} \sum_{\sigma_1 \in S_{n_1+n_2+\cdots+n_d}} \sum_{\sigma_2 \in S_{m_1+m_2+\cdots+m_d}}$$

$$\prod_{k=1}^{d} \frac{(n_k+m_k)!}{d_{r_k}} \frac{d_R}{(n_1+n_2+\cdots+n_k+m_1+m_2+\cdots+m_k)!} \times$$

$$\times \chi_{R,R_\alpha}(\sigma_1) \chi_{S,S_\beta}(\sigma_2) \chi_{T,(r_{n_1}^{(j)}, r_{n_2}^{(j)}, \cdots, r_{n_d}^{(j)})(r_{n_1}^{(i)}, r_{n_2}^{(i)}, \cdots, r_{n_d}^{(i)})}(\tau).$$

A.4 Restricted Character Computations

In this appendix we will consider the computation of the restricted character, for arbitrary representations, on the $m+n$ cycle $\sigma_c = (1,2,3,\cdots,m+n)$. This character is needed to evaluate the two point functions used in section 7.4. Representations are labeled by a Young diagram. To specify a Young diagram, we will list the number of boxes in each row. In this appendix, the Young diagrams, called "hooks" in [19] will play an important role. Listing the number of boxes in each row, the hook diagrams are always of the form $(n+m-s,1,1,\cdots,1)$. Recall that the symmetric group character $\chi_R(\sigma_c)$ is $(-1)^s$ if R is a hook and zero otherwise. Let us verify this formula, using strand diagrams, for the hook $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$. There are two possible ways of removing the three boxes

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \;.$$

Using the decomposition

$$(123) = (12)(23),$$

the strand diagram gives

$$\begin{aligned} \frac{1}{c_1-c_2}\frac{1}{c_2-c_3} &= -\frac{1}{2}\times 1 \quad \text{for} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \\ &= \frac{1}{2}\times -1 \quad \text{for} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \;. \end{aligned} \tag{A.1}$$

The sum of these contributions is -1 as it should be. Refer to a particular order of removing the boxes from the Young diagram as a *path* through the Young diagram. The strand diagram computation for the character of the $m+n$ cycle σ_c in the hook representation $(n+m-s,1,1,\cdots,1)$ implies that

$$\sum_{\text{paths}}\prod_{i=1}^{m+n-1}\frac{1}{c_i-c_{i+1}} = (-1)^s.$$

First we establish that the restricted character of the $m+n$ cycle σ_c with representation $(n+m-s,1,1,\cdots,1)$, and representation of the restriction obtained either by removing the box from the last or the first row. In this case, don't sum all paths - only sum paths that start from the last or first row respectively. Denote these two sums graphically as

$$\begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1} \ \\ \cline{1-1}\end{array} = \sum_{\text{paths starting in row 1}} \prod_{i=1}^{m+n-1} \frac{1}{c_i - c_{i+1}},$$

$$\begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} = \sum_{\text{paths starting in row s+1}} \prod_{i=1}^{m+n-1} \frac{1}{c_i - c_{i+1}}.$$

The result we will establish says that (recall that there are $m+n$ boxes in the hook)

$$\begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1} \ \\ \cline{1-1}\end{array} = (-1)^s \frac{m+n-1-s}{m+n-1},$$

$$\begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} = (-1)^s \frac{s}{m+n-1}.$$

Assuming this result is true for a hook with $m+n$ boxes and $s+1$ rows, it is straight forward to prove it is true for a hook with $m+n+1$ boxes and $s+1$ or $s+2$ rows. Indeed, for $m+n+1$ boxes in the hook and $m+n+1-s$

boxes in the first row (so that the hook has $s+1$ rows), we have

$$
\begin{aligned}
\begin{ytableau} \; & \; & \; & \; & \; & \; & 1 \\ \; \\ \; \\ \; \end{ytableau}
&= \begin{ytableau} \; & \; & \; & \; & \; & 2 & 1 \\ \; \\ \; \\ \; \end{ytableau}
+ \begin{ytableau} \; & \; & \; & \; & \; & \; & 1 \\ \; \\ \; \\ 2 \end{ytableau} \\
&= \begin{ytableau} \; & \; & \; & \; & \; & 1 \\ \; \\ \; \\ \; \end{ytableau}
+ \frac{1}{m+n} \begin{ytableau} \; & \; & \; & \; & \; & \; \\ \; \\ \; \\ 1 \end{ytableau} \\
&= \frac{m+n-s-1}{m+n-1}(-1)^s + \frac{1}{m+n}\frac{s}{m+n-1}(-1)^s \\
&= \frac{m+n-s}{m+n}(-1)^s \,.
\end{aligned}
$$

For $m+n+1$ boxes in the hook and $m+n-s$ boxes in the first row (so that the hook has $s+2$ rows), we have

$$
\begin{aligned}
\begin{ytableau} \; & \; & \; & \; & \; & \; \\ \; \\ \; \\ \; \\ 1 \end{ytableau}
&= \begin{ytableau} \; & \; & \; & \; & \; & 2 \\ \; \\ \; \\ \; \\ 1 \end{ytableau}
+ \begin{ytableau} \; & \; & \; & \; & \; & \; \\ \; \\ \; \\ 2 \\ 1 \end{ytableau} \\
&= -\frac{1}{m+n} \begin{ytableau} \; & \; & \; & \; & \; & 1 \\ \; \\ \; \\ \; \end{ytableau}
- \begin{ytableau} \; & \; & \; & \; & \; & \; \\ \; \\ \; \\ 1 \end{ytableau} \\
&= -\frac{1}{m+n}\frac{m+n-s-1}{m+n-1}(-1)^s - \frac{s}{m+n-1}(-1)^s \\
&= \frac{s+1}{m+n}(-1)^{s+1} \,.
\end{aligned}
$$

Thus, to establish the result it is enough to show that

$$
\begin{ytableau} \; & 1 \\ \; \end{ytableau} = -\frac{1}{2} = \begin{ytableau} \; & \; \\ 1 \end{ytableau} \,,
$$

which has already been demonstrated in (A.1) above.

We are now ready to tackle the computation of $\mathrm{Tr}_{(r_n,r_m)}\left(\Gamma_R(\sigma_c)\right)$. We again use a strand diagram technique, which amounts to summing over all paths and decomposing σ_c into a product of two cycles. The values of all two cycles $(i, i+1)$ *except for* $(n, n+1)$ are given by $(c_i - c_{i+1})^{-1}$ with the weights c_i read off the paths, coming from r_n for $i \leq n-1$ or from r_m for $i > n$. If we strip off the boxes belonging to r_n first, we can factor out a term which equals the character of $(1, 2, \cdots, n)$ in irreducible representation r_n so that $\mathrm{Tr}_{(r_n, r_m)}(\Gamma_R(\sigma_c))$ vanishes unless r_n is a hook. Stripping off the boxes that belong to r_m first, allows us to conclude that $\mathrm{Tr}_{(r_n, r_m)}(\Gamma_R(\sigma_c))$ again vanishes unless r_m is a hook. Let the hook associated with r_n have n boxes, arranged as $(n - s_n, 1, 1, \cdots, 1)$ and let the hook associated with r_m have m boxes, arranged as $(m - s_m, 1, 1, \cdots, 1)$. Given that both r_n and r_m have to be hooks, what are the allowed values of R? Further, given these values, what is $\mathrm{Tr}_{(r_n, r_m)}(\Gamma_R(\sigma_c))$? The allowed values of R fit into four possible cases. In what follows we will list the structure of the $R, (r_n, r_m)$ label for these four cases.

Case 1: r_m contained in r_n with no overlap: To denote the structure of this case we display R together with an x in the boxes which are removed to give r_m

$$\begin{ytableau} \, & \, & \, & \, \\ \, & x & x \\ \, & x \\ \, \end{ytableau} \quad .$$

In general, the last box removed from r_m does not correspond to a specific box in R. In this case the number $c_n - c_{n+1}$ appearing in the usual strand diagram computations is not well defined - the strand diagrams techniques of [33] are not applicable. However, for case 1, the box in the second row and second column is the last box of r_m that is removed. Thus, $c_n - c_{n+1}$ has a definite value and it is simple to compute the value of the $(n, n+1)$ cycle. Using the formulas given above, it is straight forward to verify that

$$\begin{aligned} \mathrm{Tr}_{(r_n, r_m)}(\Gamma_R(\sigma_c)) &= -(-1)^{s_m} \frac{1}{s_n} \frac{s_n (-1)^{s_n}}{n-1} + (-1)^{s_m} \frac{1}{n - s_n - 1} \frac{(n - s_n - 1)(-1)^{s_n}}{n-1} \\ &= 0. \end{aligned}$$

Case 2: r_m contained in r_n with row overlap: R together with an x in the boxes which are removed to give r_m is displayed below:

$$\begin{ytableau} \ & \ & \ & \ & x \\ \ & x & x \\ \ & x \\ \ \end{ytableau} \; .$$

In this case, the box in the second row and second column or the last marked box in the first row, is the last box of r_m that is removed. The argument for case 1 does not easily generalize to case 2 (or cases 3 and 4).

Case 3: r_m contained in r_n with column overlap: R together with an x in the boxes which are removed to give r_m is displayed below:

$$\begin{ytableau} \ & \ & \ & \ \\ \ & x & x \\ \ & x \\ \ \\ x \end{ytableau}$$

Case 4: r_m contained in r_n with row and column overlap: R together with an x in the boxes which are removed to give r_m is displayed below:

$$\begin{ytableau} \ & \ & \ & \ & x \\ \ & x & x \\ \ & x \\ \ \\ x \end{ytableau}$$

We will now give an argument applicable to all four cases. As a nontrivial check of our general formula, we will verify that it predicts zero for case 1. Decompose our $n+m$ cycle σ_c as

$$\sigma_c = (1, 2, ..., n)(n, n+1)(n+1, n+2, ..., n+m).$$

Using the $S_n \times S_m$ symmetry enjoyed by our restricted character, we can replace

$$(1,2,...,n) \to \mathcal{C}_n = \frac{\sum_{\text{n cycles}}(i_1,i_2,...,i_n)}{(n-1)!},$$

$$(n+1,n+2,...,n+m) \to \mathcal{C}_m = \frac{\sum_{\text{m cycles}}(i_1,i_2,...,i_m)}{(m-1)!}.$$

$\mathcal{C}_n$ is a sum over the $(n-1)!$ n-cycles in S_n; $\mathcal{C}_m$ is a sum over the $(m-1)!$ m-cycles in S_m. It is clear that $\mathcal{C}_n$ and $\mathcal{C}_m$ are Casimirs of the (r_n, r_m) representation. Using the known characters of the hooks, it is straight forward to see that

$$\mathrm{Tr}\,(\mathcal{C}_n) = (-1)^{s_n}, \qquad \mathrm{Tr}\,(\mathcal{C}_m) = (-1)^{s_m},$$

so that these two Casimirs have eigenvalue

$$\mathcal{C}_n = \frac{(-1)^{s_n}}{d_{r_n}}, \qquad \mathcal{C}_m = \frac{(-1)^{s_m}}{d_{r_m}}.$$

Thus, we now have

$$\begin{aligned}
\mathrm{Tr}_{\,(r_n,r_m)}\,(\Gamma_R(\sigma_c)) &= \mathrm{Tr}_{\,(r_n,r_m)}\,(\mathcal{C}_n(n,n+1)\mathcal{C}_m) \\
&= \frac{(-1)^{s_n}}{d_{r_n}}\frac{(-1)^{s_m}}{d_{r_m}}\mathrm{Tr}_{\,(r_n,r_m)}\,((n,n+1)) \\
&= \frac{(-1)^{s_n+s_m}}{mn}\left(\sum_i c_i - mN - \frac{m(m-2s_m-1)}{2}\right)
\end{aligned}$$

where to get to the last line we have used formula (7.7). The sum in the last line is over the weights in R which are peeled off and recombined to produce r_m. Notice that to evaluate these hooks we have not needed off the diagonal block traces of the $(n, n+1)$ character, which is why we are able to obtain a simple and general formula. In addition, this formula is in perfect agreement with the result obtained above for case 1.

B Appendix - Correlators of Operators with a Large R-Charge: The Cutting Rules

B.1 Decomposing Derivative Operators

As argued in section 8.1.1, the contributions to a correlation function of two restricted Schur polynomials, coming from contractions between Zs that belong to the open string and Zs that belong to the brane, can be written as a differential operator acting on the restricted Schur polynomials. In this appendix we will show that any such string of derivatives can be written in terms of eight basic types of derivatives, acting on modified restricted Schur polynomials. This result is a useful one because it is possible to work out general formulas for the action of these eight basic derivative types on the modified restricted Schur polynomials. We will illustrate the basic procedure with an example, leaving a statement of the general result for the next section. In section B.1.3 we show some examples of the use of the cutting rules.

B.1.1 Warm Up

The example we study is

$$\begin{aligned} I_2 \;&=\; \left(\frac{d}{dZ^c_c}\right)\left(\frac{d}{dZ^d_e}\frac{d}{dZ^e_f}\frac{d}{d(Z^\dagger)^f_d}\right)\left(\frac{d}{dZ^g_h}\frac{d}{d(Z^\dagger)^h_g}\right)\left(\frac{d}{d(Z^\dagger)^k_l}\frac{d}{d(Z^\dagger)^l_k}\right)\times \\ &\times\; \left(\frac{d}{dZ^a_b}\frac{d}{dW^b_a}\right)\left(\frac{d}{d(Z^\dagger)^m_n}\frac{d}{d(W^\dagger)^n_m}\right)\chi^{(1)}_{R,R_1}(Z,W)\left(\chi^{(1)}_{R,R_1}(Z,W)\right)^\dagger . \end{aligned}$$

Using the notations of (8.1), computing the derivatives with respect to the open string words gives

$$I_2 = \left(\frac{d}{dZ^c_c}\right)\left(\frac{d}{dZ^d_e}\frac{d}{dZ^e_f}\frac{d}{d(Z^\dagger)^f_d}\right)\left(\frac{d}{dZ^g_h}\frac{d}{d(Z^\dagger)^h_g}\right)\left(\frac{d}{d(Z^\dagger)^k_l}\frac{d}{d(Z^\dagger)^l_k}\right)\frac{d\mathcal{F}^a_b}{dZ^a_b}\frac{d(\mathcal{F}^\dagger)^m_n}{d(Z^\dagger)^m_n}.$$

Computing the remaining derivatives and summing over repeated indices, we easily obtain

$$\begin{aligned} I_2 &= \left[\frac{1}{(n-6)!}\right]^2 \sum_{\sigma\in S_n}\sum_{\tau\in S_n} \mathrm{Tr}_{R_1}\left(\Gamma_R(\sigma)\right)\mathrm{Tr}_{R_1}\left(\Gamma_R(\tau)\right)^* \times \\ &\times \; Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-6}}_{i_{\sigma(n-6)}}(Z^\dagger)^{j_1}_{j_{\tau(1)}}\cdots (Z^\dagger)^{j_{n-6}}_{j_{\tau(n-6)}}\delta^{i_n}_{i_{\sigma(n-1)}}\delta^{i_{n-1}}_{i_{\sigma(n)}}\delta^{i_{n-2}}_{j_{\tau(n-4)}} \times \\ &\times \; \delta^{j_{n-4}}_{i_{\sigma(n-2)}}\delta^{i_{n-3}}_{i_{\sigma(n-4)}}\delta^{j_{n-5}}_{i_{\sigma(n-3)}}\delta^{i_{n-4}}_{j_{\tau(n-5)}}\delta^{i_{n-5}}_{i_{\sigma(n-5)}}\delta^{j_n}_{j_{\tau(n-1)}}\delta^{j_{n-1}}_{j_{\tau(n)}}\delta^{j_{n-2}}_{j_{\tau(n-3)}}\delta^{j_{n-3}}_{j_{\tau(n-2)}} . \end{aligned} \tag{B.1}$$

Now, define the permutations

$$P = (n, n-1)(n-3, n-4), \qquad Q = (n, n-1)(n-2, n-3).$$

Further, set

$$\sigma = \psi P, \qquad \tau = \lambda Q.$$

Changing variables in the above sums (B.1) from σ to ψ and from τ to λ we find

$$\begin{aligned} I_2 = & \left[\frac{1}{(n-6)!}\right]^2 \sum_{\psi\in S_n}\sum_{\lambda\in S_n} \mathrm{Tr}_{R_1}\left(\Gamma_R(\psi P)\right)\mathrm{Tr}_{R_1}\left(\Gamma_R(\lambda Q)\right)^* \times \\ \times \; & Z^{i_1}_{i_{\psi(1)}}\cdots Z^{i_{n-6}}_{i_{\psi(n-6)}}(Z^\dagger)^{j_1}_{j_{\lambda(1)}}\cdots (Z^\dagger)^{j_{n-6}}_{j_{\lambda(n-6)}}\delta^{i_n}_{i_{\psi(n)}}\delta^{i_{n-1}}_{i_{\psi(n-1)}}\delta^{i_{n-3}}_{i_{\psi(n-3)}} \times \\ \times \; & \delta^{i_{n-5}}_{i_{\psi(n-5)}}\delta^{j_n}_{j_{\lambda(n)}}\delta^{j_{n-1}}_{j_{\lambda(n-1)}}\delta^{j_{n-2}}_{j_{\lambda(n-2)}}\delta^{j_{n-3}}_{j_{\lambda(n-3)}}\delta^{i_{n-2}}_{j_{\lambda(n-4)}}\delta^{j_{n-4}}_{i_{\psi(n-2)}}\delta^{j_{n-5}}_{i_{\psi(n-4)}}\delta^{i_{n-4}}_{j_{\lambda(n-5)}} . \end{aligned}$$

The reason why we made the change of variables from σ and τ to λ and ψ is now clear: in (B.1) Kronecker deltas with two i indices or two j indices did not have the property that the upper index was related to the lower index by permutation; after the change of variables, all such Kronecker deltas do have this property. This is useful, because a Kronecker delta with this property is produced by acting on the restricted Schur polynomial with the trace of a derivative. One is tempted to replace all such Kronecker deltas with indices i_j $j < n$ by the trace of a derivative with respect to Z; this is not quite correct. As an example, $\delta^{i_{n-1}}_{i_{\psi(n-1)}}$ in the last expression above is obtained by differentiating only $Z^{i_{n-1}}_{i_{\psi(n-1)}}$ - the trace of a derivative with respect to Z will generate this term as well as terms that come from acting on every single other Z in the polynomial. Further, due to the presence of P and Q it really does make a difference which Z is differentiated. This is, however, easily overcome: we can replace $Z^{i_{n-1}}_{i_{\psi(n-1)}}$ by a new matrix $X^{i_{n-1}}_{i_{\psi(n-1)}}$ so that $\delta^{i_{n-1}}_{i_{\psi(n-1)}}$ can safely be replaced by the trace of a derivative with respect to X. We call these new matrices "open string place holders". It is easy to see that I_2 now takes the form

$$I_2 = \mathrm{Tr}\,\frac{d}{dX_1}\mathrm{Tr}\,\frac{d}{dX_3}\mathrm{Tr}\,\frac{d}{dX_5}\mathrm{Tr}\,\frac{d}{dW}\mathrm{Tr}\,\frac{d}{dX_1^\dagger}\mathrm{Tr}\,\frac{d}{dX_2^\dagger}\mathrm{Tr}\,\frac{d}{dX_3^\dagger}\mathrm{Tr}\,\frac{d}{dW^\dagger}\mathrm{Tr}\,\frac{d}{dX_2}\frac{d}{dX_4^\dagger}\times$$

$$\times\mathrm{Tr}\,\frac{d}{dX_4}\frac{d}{dX_5^\dagger}\chi^{(1,5)}_{R,R_1;P}(Z,W)(\chi^{(1,5)}_{R,R_1;Q}(Z,W))^\dagger,$$

where we have introduced the new notation

$$\chi^{(1,m)}_{R,R_1;\Lambda}(Z,W) \equiv \frac{1}{(n-1)!}\sum_{\sigma\in S_n}\mathrm{Tr}\,_{R_1}\left(\Gamma_R(\sigma\Lambda)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-m-1}}_{i_{\sigma(n-m-1)}}\prod_{k=1}^{m} X^{i_{n-k}}_{i_{\sigma(n-k)}} W^{i_n}_{i_{\sigma(n)}}\,.$$

In this formula Λ is any element of the symmetric group. Thus, the original derivative operator has been decomposed into a product of basic operations as advertised. The Schur polynomial has been modified by the inclusion of a new factor (Λ in the last equation) inside the trace; we call this factor the *trace insertion*. Since the trace insertion is a new factor in the trace, our notation includes the trace insertion after the existing trace labels.

B.1.2 General Rule

In this section we give general rules for decomposing a differential operator into a product of basic operations. The full set of basic operations is

$$\mathrm{Tr}\left(\frac{d}{dZ}\right),\quad \mathrm{Tr}\left(\frac{d}{dW}\right),\quad \mathrm{Tr}\left(\frac{d}{dZ^\dagger}\right),\quad \mathrm{Tr}\left(\frac{d}{dW^\dagger}\right),\quad \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right),$$

$$\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dZ^\dagger}\right),\quad \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dW^\dagger}\right)\quad\text{and}\quad \mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right).$$

The first four differential operators are simply reductions. We call the last four operators "mixed derivatives".

A general rule must give a recipe for reading off the trace insertion and product of basic operations (the new derivative operator) from any differential operator to be dissected. Of course, it is just a summary of what happens when one performs the analog of the $\sigma, \tau \to \psi, \lambda$ change of variables of the last section.

In this section, we assume that the open string word is associated with the nth index i_n as in (8.1). In what follows we will switch to an obvious matrix notation, illustrated in the following example

$$\frac{d}{dZ^a_b}\frac{d}{dW^b_c}\frac{d}{d(Z^\dagger)^c_d}\frac{d}{d(W^\dagger)^d_a} \to (DD_W D^\dagger D^\dagger_W)\,.$$

Terms within a single bracket are traced. We start by giving each of the derivatives with respect to W or Z a label, counting down from n. D_W is given the label n. We then give each of the derivatives with respect to $W^\dagger$ or $Z^\dagger$ a label, again counting down from n. $D^\dagger_W$ is given the label n. As an example, the operator

$$(D)(DDD^\dagger)(DD^\dagger)(D^\dagger D^\dagger)(DD_W)(D^\dagger D^\dagger_W)$$

is labelled as follows (the labels for D, D_W appear above the operator; the labels for $D^\dagger, D^\dagger_W$ appear below the operator)

$$\begin{array}{cccccccccccc}
n-5 & n-4 & n-3 & \cdot & n-2 & \cdot & \cdot & \cdot & n-1 & n & \cdot & \cdot \\
(D) & (D & D & D^\dagger) & (D & D^\dagger) & (D^\dagger & D^\dagger) & (D & D_W) & (D^\dagger & D^\dagger_W) \\
\cdot & \cdot & \cdot & n-5 & \cdot & n-4 & n-3 & n-2 & \cdot & \cdot & n-1 & n
\end{array}$$

The Z derivatives with labels will be replaced with open string place holders. There are two cutting rules:

First cutting rule: If, within any given trace, D (or any other holomorphic derivative) has another holomorphic derivative to its left, it can be removed from the trace and placed into its own trace. The two cycle which swaps the label of D and the label of its neighbor on the left is added, on the left, to the trace insertion of the holomorphic Schur polynomial. If, within any given trace, $D^\dagger$ (or any other antiholomorphic derivative) has another antiholomorphic derivative to its left, it can be removed from the trace and placed into its own trace. The two cycle which swaps the label of $D^\dagger$ and the label of its neighbor on the left is added, on the left, to the trace insertion of the antiholomorphic Schur polynomial.

Second cutting rule: If within any given trace $DD^\dagger$ (or any other product of a holomorphic with an antiholomorphic derivative) has a second $DD^\dagger$ (or any other product of a holomorphic with an antiholomorphic derivative) to its right, then the "middle two" derivatives can be removed from the existing trace and placed into their own trace. The two cycle which swaps the labels of the two holomorphic derivatives is added, on the left, to the trace insertion of the holomorphic Schur polynomial. If within any given trace $D^\dagger D$ (or any other product of an antiholomorphic with a holomorphic derivative) has a second $D^\dagger D$ (or any other product of an antiholomorphic with a holomorphic derivative) to its right, then the "middle two" derivatives can be removed from the existing trace and placed into their own trace. The two cycle which swaps the labels of the two antiholomorphic derivatives is added, on the left, to the trace insertion of the antiholomorphic Schur polynomial.

We have stated the rules using the terms "holomorphic/antiholomorphic" derivative. Stated in this way, the rule are valid even if there is more than one open string attached to the restricted Schur polynomial. Any derivatives cut out of the product, with respect to Z or $Z^\dagger$ are replaced by derivatives with respect to open string place holders.

B.1.3 Examples

In this appendix we give some examples of how the cutting rules are used. This is done so that the reader can test that she understands how to correctly apply the rules. The operator

$$\mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dW}\right)$$

becomes

$$\mathrm{Tr}\left(\frac{d}{dX_3}\right)\mathrm{Tr}\left(\frac{d}{dX_2}\right)\mathrm{Tr}\left(\frac{d}{dX_1}\right)\mathrm{Tr}\left(\frac{d}{dW}\right).$$

The antiholomorphic trace insertion is 1; the holomorphic trace insertion is $(n-3, n-2, n-1, n)$. The operator

$$\mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\right)$$

becomes

$$\mathrm{Tr}\left(\frac{d}{dX_3}\frac{d}{dX_2^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dX_2}\frac{d}{dX_1^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dX_1}\frac{d}{dX_3^\dagger}\right).$$

The antiholomorphic trace insertion is 1; the holomorphic trace insertion is $(n-3, n-2)(n-1, n-2)$. By cycling a derivative around the operator we have dissected can be written as

$$\mathrm{Tr}\left(\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\right)$$

Cutting this operator up gives a holomorphic trace insertion of 1 and a non-trivial antiholomorphic trace insertion. Clearly the result of cutting is not unique. Of course, these different dissections all lead to the same value for the correlation function.

B.2 Mixed Derivative Rules

In this appendix we will explain how to evaluate

$$\left\langle \left[\hat{O} \chi^{(1)}_{R,R_1}(Z,W)(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger \right] \right\rangle$$

in free field theory, in the case that $\hat{O}$ is one of the mixed derivative operators. All the arguments in this appendix are unchanged if a trace insertion factor is included.

B.2.1 $\hat{O} = \mathbf{Tr}\,\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right)$

Consider the Schwinger-Dyson equation

$$0 = \int dZ dZ^\dagger dY dY^\dagger \frac{d}{dZ^a_b}\left(\chi^{(1)}_{R,R_1}(Z,W)\left[\frac{d}{d(Z^\dagger)^b_a}(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger\right] e^{-S}\right),$$

where

$$S = \mathrm{Tr}\,(ZZ^\dagger + YY^\dagger).$$

The Schwinger-Dyson equation implies

$$\begin{aligned} \left\langle \left[\mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right) \chi^{(1)}_{R,R_1}(Z,W)(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger \right] \right\rangle &= \Big\langle \chi^{(1)}_{R,R_1}(Z,W)(Z^\dagger)^b_a \\ &\times \left[\frac{d}{d(Z^\dagger)^b_a}(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger\right] \Big\rangle \\ &= n_{Z^\dagger} \left\langle \chi^{(1)}_{R,R_1}(Z,W)(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger \right\rangle \end{aligned}$$

where $n_{Z^\dagger}$ is the number of $Z^\dagger$ matrices appearing in $(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger$. The correlator $\langle \chi^{(1)}_{R,R_1}(Z,W)(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger \rangle$ is now easily evaluated using the results of [31].

B.2.2 $\hat{O} = \textbf{Tr}\,\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)$

This operator simply "contracts" the two open string words - it picks out the F_0 contribution to the correlator in the language of [31]. Thus,

$$\left\langle \mathrm{Tr}\,\left(\frac{d}{dW'}\frac{d}{dW^\dagger}\right)\chi^{(1)}_{R,R_1}(Z,W')(\chi^{(1)}_{S,S_1}(Z,W))^\dagger\right\rangle$$

is simply equal to the coefficient of the F_0 contribution to the correlator

$$\left\langle \chi^{(1)}_{R,R_1}(Z,W')(\chi^{(1)}_{S,S_1}(Z,W))^\dagger\right\rangle\,.$$

B.2.3 $\hat{O} = \textbf{Tr}\,\left(\frac{d}{dZ}\frac{d}{dW^\dagger}\right)$

Explicitly performing the derivative with respect to Z in

$$I = \frac{d}{dZ^e_d}\chi^{(1)}_{R,R_1}(Z,W)\frac{d}{d(W^\dagger)^d_e}(\chi^{(1)}(Z,W))^\dagger$$

we obtain

$$\begin{aligned} I &= \frac{1}{(n-2)!}\sum_{\sigma\in S_n}\mathrm{Tr}_{\,R_1}\left(\Gamma_R(\sigma)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}\delta^{i_{n-1}}_{e}\delta^{d}_{i_{\sigma(n-1)}}W^{i_n}_{i_{\sigma(n)}}\frac{d}{d(W^\dagger)^d_e}(\chi^{(1)}(Z,W))^\dagger \\ &= \frac{1}{(n-2)!}\frac{d}{dX^e_d}\sum_{\sigma\in S_n}\mathrm{Tr}_{\,R_1}\left(\Gamma_R(\sigma)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}X^{i_{n-1}}_{i_{\sigma(n-1)}}W^{i_n}_{i_{\sigma(n)}}\frac{d}{d(W^\dagger)^d_e}(\chi^{(1)}(Z,W))^\dagger\,. \end{aligned}$$

If we now introduce the representations T_α defined by removing a single box from R_1, so that

$$R_1 = \oplus_\alpha T_\alpha,$$

we obtain

$$I = \sum_\alpha \frac{d}{dX^e_d}\chi^{(2)}_{R,T_\alpha}(Z,X,W)\frac{d}{d(W^\dagger)^d_e}(\chi^{(1)}(Z,W))^\dagger,$$

where in the restricted Schur polynomial $\tilde{\chi}^{(2)}_{R,T_\alpha}(Z,X,W)$, W is associated with the box that must be removed from R to obtain R_1 and X is associated with the box that must be removed from R_1 to obtain T_α. After using the subgroup swap rule of [31] to swap X and W, this correlator can be evaluated exactly as in the previous subsection.

B.2.4 $\hat{O} = \mathbf{Tr}\left(\frac{d}{dW}\frac{d}{dZ^\dagger}\right)$

The evaluation of this term is essentially the same as the term treated in the last subsection.

B.3 Example Correlator

In this appendix we give the details of the computation of a correlator of the type considered in section 8.1.2

$$I_{RR_1,RR_1} = \left\langle \chi^{(1)}_{R,R_1} (\chi^{(1)}_{R,R_1})^\dagger \right\rangle \,.$$

We deal with three impurities in the open string $W^i_j = (YZ^3Y)^i_j$ and take R_1 to be the rectangular Young diagram with N rows and M columns with $M = O(N)$. R is given by adding a box in the upper right hand corner, i.e. in the first row. This example is already involved enough to nicely illustrate the use of our technology.

No Brane/String Contractions: This contribution comes from the diagram given below.

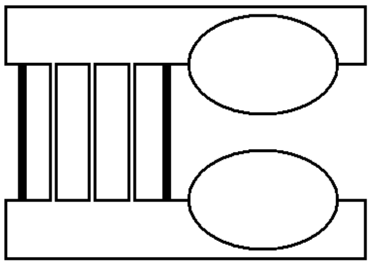


Figure 13: The contribution with no brane/string contractions.

Using the rules of [31] we easily obtain, at leading order in a large N expansion

$$I^{(0)}_{RR_1,RR_1} = N^4 \frac{\text{hooks}_R}{\text{hooks}_{R'}} f_R = N^3(M+N) f_R \,.$$

One Brane/String Contraction: This contribution comes from the three diagrams given below.

All three diagrams give the same contribution. We do not need to use our cutting rules yet; we do use the results of Appendices B.2.1 and B.2.2.

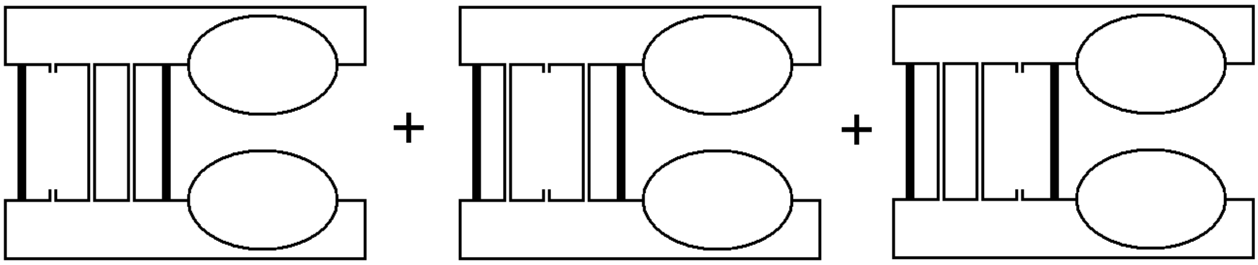


Figure 14: The contribution with one brane/string contraction.

The result is

$$I^{(1)}_{RR_1,RR_1} = 3N^2 \left\langle \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right) \mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right) \chi^{(1)}_{R,R_1}(\chi^{(1)}_{R,R_1})^\dagger \right\rangle$$

$$= 3N^2(MN)\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}} f_R = 3MN^2(M+N)f_R\,.$$

Two Brane/String Contractions: This contribution comes from the three diagrams given below.

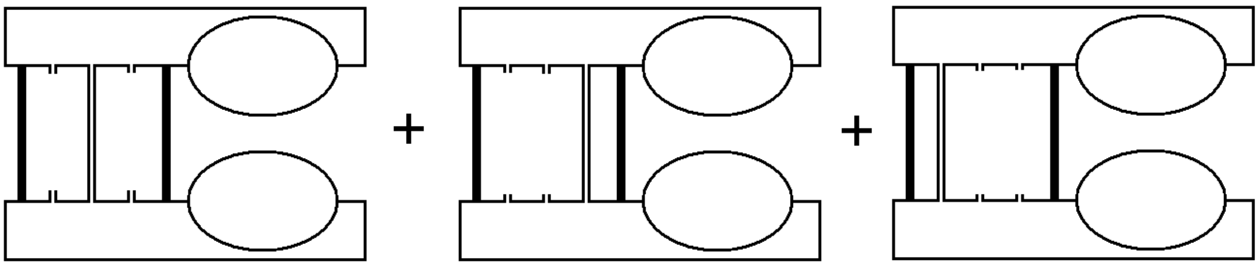


Figure 15: The contribution with two brane/string contractions.

The first diagram is the simplest to evaluate. We can again do it without using the cutting rules. The result is

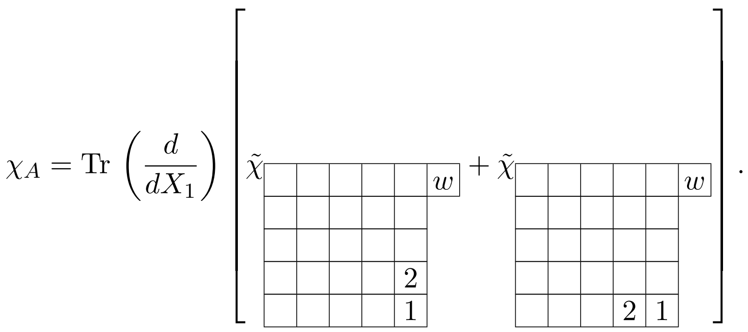
$$\left\langle \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right)^2 \mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right) \chi^{(1)}_{R,R_1}(\chi^{(1)}_{R,R_1})^\dagger \right\rangle$$

$$= (MN)^2\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}} f_R = M^2N(M+N)f_R\,.$$

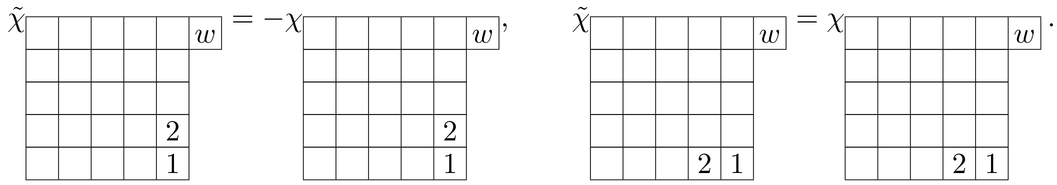
The evaluation of the second and third diagrams are exactly the same. Consider the second diagram. We need to evaluate

$$N \left\langle \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi^{(1)}_{R,R_1}(\chi^{(1)}_{R,R_1})^\dagger \right\rangle .$$

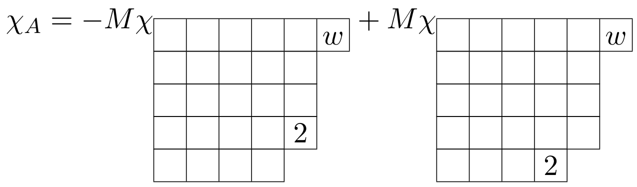
We now need to use our cutting rules and the associated open string holders. We start to use the graphical notation that draws the Young diagram, with the open string word (w) and the open string place holders (1 and 2) on R (see section 5). We draw R_1 with 5 rows and 5 columns, but our results hold for general M and N. After cutting we have to evaluate



The tilde on χ is to denote the fact that there is a trace insertion factor of $(n-1, n-2)$ arising from the cutting. Using strand diagrams we can eliminate the trace insertion factors for each term



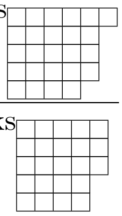
After using the subgroup swap rule to swap w and X_1, we can compute the reduction to obtain (there are some terms that arise from swapping w and 1; these are however $O\left(\frac{1}{N^2}\right)$ so they can be dropped to leading order in N)

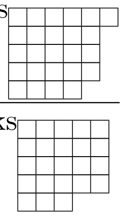


To get the contribution from the second diagram, we now simply need to compute

$$N\left\langle \mathrm{Tr}\left(\frac{d}{dX_2}\frac{d}{dX_2^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi_A\chi_A^\dagger\right\rangle .$$

To obtain this, we need to use





$$h_1 = \frac{\text{hooks}}{\text{hooks}} = \frac{M(M+N)}{2}, \qquad h_2 = \frac{\text{hooks}}{\text{hooks}} = \frac{M(M+N)}{2}.$$

It is now straight forward to obtain

$$N\left\langle \mathrm{Tr}\left(\frac{d}{dX_2}\frac{d}{dX_2^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi_A\chi_A^\dagger\right\rangle = NMf_R\,(h_1+h_2)$$

$$= NM^2(M+N)f_R\,.$$

Notice that although the computation for diagram 2 was completely different to the computation for diagram 1, they give exactly the same result. As already mentioned, the third diagram gives exactly the same contribution as the second so that

$$I^{(2)}_{RR_1,RR_1} = 3M^2N(M+N)f_R\,.$$

Three Brane/String Contractions: This contribution comes from the diagram given below.

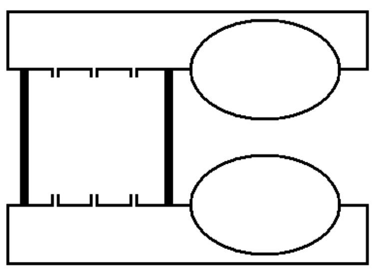
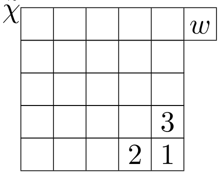
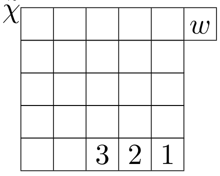
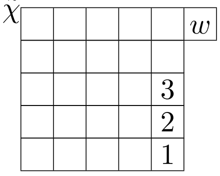
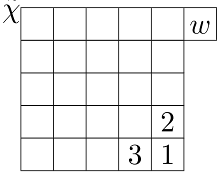


Figure 16: The contribution with three brane/string contractions.

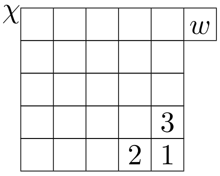
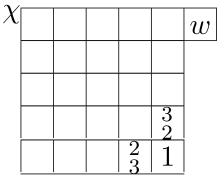
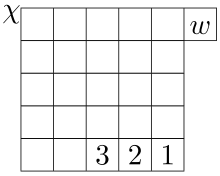
For this contribution we need to evaluate

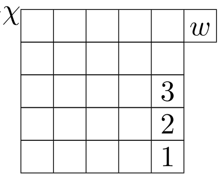
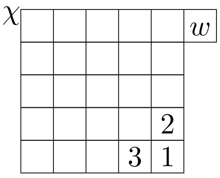
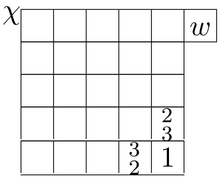
$$\left\langle \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ^\dagger}\frac{d}{dZ^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi^{(1)}_{R,R_1}(\chi^{(1)}_{R,R_1})^\dagger\right\rangle .$$

We cut two holomorphic derivatives and two antiholomorphic derivatives out of the trace. Thus, we will need a total of three open string place holders; the trace insertion factor is $(n-3,n-2)(n-1,n-2)$. To recover the trace over R_1 we again need to sum over all ways of distributing the open string place holders. The result is

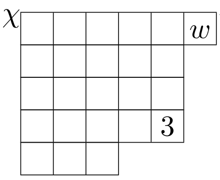
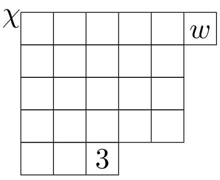
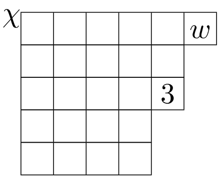
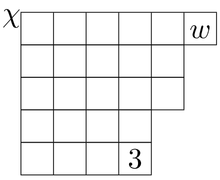
$\tilde{\chi}$  $+\tilde{\chi}$  $+\tilde{\chi}$  $+\tilde{\chi}$  .

After accounting for the trace insertion factor, we obtain

$-\frac{1}{2}\chi$  $+\frac{\sqrt{3}}{2}\chi$  $+\chi$ 

$+\chi$  $-\frac{1}{2}\chi$  $-\frac{\sqrt{3}}{2}\chi$  .

We now need to use the subgroup swap rule so that we can reduce with respect to X_1 and X_2. There is again a dramatic simplification because the terms in which the location of w changes are suppressed at large N. The result after reducing is

$\chi_A = -\frac{M^2}{2}\chi$  $+M^2\chi$  $+M^2\chi$  $-\frac{M^2}{2}\chi$  .

To get the contribution from the three brane/string contractions, we now need to compute

$$\left\langle \chi_A \chi_A^\dagger \right\rangle = M^3(M+N)f_R.$$

To get this we used

$$\frac{\mathrm{hooks}_{\yng(6,5,5,5,3)}}{\mathrm{hooks}_{\yng(5,5,5,4,3)}} = \frac{2M(M+N)}{3}, \qquad \frac{\mathrm{hooks}_{\yng(6,5,5,5,3)}}{\mathrm{hooks}_{\yng(5,5,5,5,2)}} = \frac{M(M+N)}{3},$$

$$\frac{\mathrm{hooks}_{\yng(6,5,5,4,4)}}{\mathrm{hooks}_{\yng(5,5,5,4,3)}} = \frac{2M(M+N)}{3}, \qquad \frac{\mathrm{hooks}_{\yng(6,5,5,4,4)}}{\mathrm{hooks}_{\yng(5,5,4,4,4)}} = \frac{M(M+N)}{3}.$$

Putting things together, we have

$$I_{RR_1,RR_1} = (N^3+3MN^2+3M^2N+M^3)(M+N)f_R = \left(1+\frac{M}{N}\right)^3 N^3(M+N)f_R\,.$$

B.4 A Further Correlator Example

In this appendix we will present a detailed calculation of

$$\frac{\langle \chi_B\,(Z)\,\mathrm{Tr}\,(\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ^\dagger})\chi_B\left(Z^\dagger\right)\rangle}{\langle \chi_B\,(Z)\,\chi_B\,(Z^\dagger)\rangle},$$

using the cutting rules. Note that this term corresponds to the contribution to the correlator

$$\frac{\langle \chi_B\,(Z)\,\mathrm{Tr}\,(Z^\dagger Z Z^\dagger Z Z^\dagger Z^\dagger Z Z)\chi_B\left(Z^\dagger\right)\rangle}{\langle \chi_B\,(Z)\,\chi_B\,(Z^\dagger)\rangle},$$

where all fields in the operator $\mathrm{Tr}\,(Z^\dagger Z Z^\dagger Z Z^\dagger Z^\dagger Z Z)$ are contracted with the background, χ_B.

A summary of the procedure utilized in calculating a correlator of this type using the cutting rules is as follows:

- The derivative operator is cut up into a product of derivatives of the basic types acting on restricted Schur polynomials as outlined in appendix B.1. This generates trace insertion factors that appear in the restricted characters of the restricted Schur polynomials.
- The trace insertion factors are evaluated using strand diagrams (reviewed in section 5.1) and we are left with a linear combination of restricted Schur polynomials with no trace insertion factor present.
- The reductions (if any are generated by cutting up the original derivative operator) are now evaluated (section 5.2.1).
- Following the reductions we now evaluate the resultant correlator utilizing the results of [31] (reviewed in section 5.2). In order to apply these results we need to subgroup swap the labels of the holomorphic (or anti-holomorphic) restricted Schur polynomials in such a way that the appropriate open string words are glued together. For example, suppose we had the following derivative operator:
$$\mathrm{Tr}\,(\frac{d}{dX_1}\frac{d}{dX_2^\dagger})\mathrm{Tr}\,(\frac{d}{dX_2}\frac{d}{dX_1^\dagger}).$$

We would need to change the order of restriction of either the holomorphic or anti-holomorphic restricted Schur polynomials such that:

$$\chi_{R,R^{(2)}}|_1|_2 \to \chi_{R,R^{(2)}}|_2|_1.$$

Then for all those pairs of holomorphic and anti-holomorphic restricted Schur polynomials for which the chain of subductions match a contribution to the correlator of

$$\frac{\mathrm{Hooks}_R}{\mathrm{Hooks}_{R^{(n)}}} f_R,$$

is obtained.

Now, cutting up our operator of interest we obtain the following:

$$\mathrm{Tr}\,(\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ^\dagger})$$

$$\to \mathrm{Tr}\,(\frac{d}{dX_4}\frac{d}{dX_2^\dagger})\mathrm{Tr}\,(\frac{d}{dX_3}\frac{d}{dX_4^\dagger})\mathrm{Tr}\,(\frac{d}{dX_2}\frac{d}{dX_3^\dagger})\mathrm{Tr}\,(\frac{d}{dX_1})\mathrm{Tr}\,(\frac{d}{dX_1^\dagger}),$$

with the trace insertion factors stated below.

Holomorphic trace insertion factor: $(n-2, n-3)(n-1, n-2)(n, n-1)$,

anti-holomorphic trace insertion factor: $(n, n-1)$.

In order to simplify the procedure we take advantage of the freedom we have in relabeling the open string word placeholders (relabeling is an automorphism of the operator defined by the derivative operator acting on the background Schur polynomial). By applying a suitable relabeling we can avoid the need to apply the subgroup swap rule in evaluating reductions or in the gluing of open string words. The relabeling we perform is outlined below.

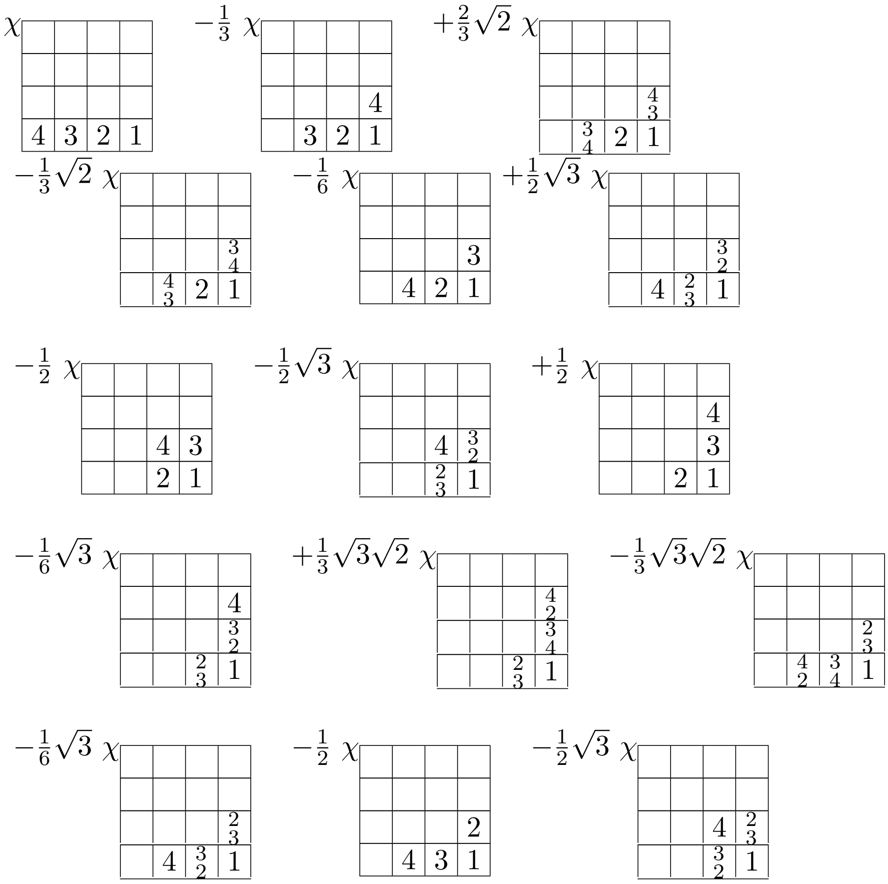
- Relabel the open string word placeholders such that our operator is as follows:

$$\mathrm{Tr}\,(\frac{d}{dX_4}\frac{d}{dX_4^\dagger})\mathrm{Tr}\,(\frac{d}{dX_3}\frac{d}{dX_3^\dagger})\mathrm{Tr}\,(\frac{d}{dX_2}\frac{d}{dX_2^\dagger})\mathrm{Tr}\,(\frac{d}{dX_1})\mathrm{Tr}\,(\frac{d}{dX_1^\dagger}).$$

- Now, send $n \to n$, $n-1 \to n-3$, $n-2 \to n-1$, $n-3 \to n-2$ in the anti-holomorphic restricted Schur polynomials with trace insertion factors so that no subgroup swaps are necessary. The only effect of this is that the anti-holomorphic trace insertion factor is changed to $(n, n-3)$.

We can now apply strand diagrams to evaluate the trace insertion factors and then directly evaluate the reductions and gluings. Note that as in appendix B.3 we draw the Young diagram B with a certain specific number of rows and columns (in this case, the number of rows and columns is 4) but the results hold for general N and M.

The holomorphic restricted Schur Polynomials after evaluating the trace insertions are as follows:



$$+\tfrac{1}{2}\,\chi_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & 4 & 2 \\ \hline \ & & 3 & 1 \\ \hline \end{array}} \quad +\tfrac{1}{2}\sqrt{3}\,\chi_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & 4 \\ \hline \ & & & \substack{2\\3} \\ \hline \ & & \substack{3\\2} & 1 \\ \hline \end{array}} \quad +\tfrac{1}{6}\,\chi_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & 4 \\ \hline \ & & & 2 \\ \hline \ & & 3 & 1 \\ \hline \end{array}}$$

$$-\tfrac{1}{3}\sqrt{2}\,\chi_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \substack{4\\3} \\ \hline \ & & & 2 \\ \hline \ & & \substack{3\\4} & 1 \\ \hline \end{array}} \quad +\tfrac{2}{3}\sqrt{2}\,\chi_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \substack{3\\4} \\ \hline \ & & & 2 \\ \hline \ & & \substack{4\\3} & 1 \\ \hline \end{array}} \quad +\tfrac{1}{3}\,\chi_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & 3 \\ \hline \ & & & 2 \\ \hline \ & & 4 & 1 \\ \hline \end{array}}$$

$$-\,\chi_{\begin{array}{|c|c|c|c|}\hline \ & & & 4 \\ \hline \ & & & 3 \\ \hline \ & & & 2 \\ \hline \ & & & 1 \\ \hline \end{array}}.$$

The anti-holomorphic restricted Schur Polynomials after evaluating the trace insertions are as follows:

$$\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & & \\ \hline 4 & 3 & 2 & 1 \\ \hline \end{array}} \quad -\tfrac{1}{3}\,\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & & 4 \\ \hline \ & 3 & 2 & 1 \\ \hline \end{array}} \quad -\tfrac{1}{3}\sqrt{2}\,\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & & \substack{4\\3} \\ \hline \ & \substack{3\\4} & 2 & 1 \\ \hline \end{array}} \quad -\tfrac{1}{3}\sqrt{3}\sqrt{2}\,\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & & \substack{4\\3} \\ \hline \ & \substack{3\\2} & \substack{2\\4} & 1 \\ \hline \end{array}} \quad -\tfrac{1}{3}\sqrt{2}$$

$$\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & & \substack{3\\4} \\ \hline \ & \substack{4\\3} & 2 & 1 \\ \hline \end{array}} \quad +\tfrac{5}{6}\,\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & & 3 \\ \hline \ & 4 & 2 & 1 \\ \hline \end{array}} \quad -\tfrac{1}{6}\sqrt{3}\,\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & & \substack{3\\2} \\ \hline \ & 4 & \substack{2\\3} & 1 \\ \hline \end{array}} \quad -\tfrac{1}{2}\,\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & 4 & 3 \\ \hline \ & & 2 & 1 \\ \hline \end{array}} \quad +\tfrac{1}{2}\sqrt{3}$$

$$\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & 4 & \substack{3\\2} \\ \hline \ & & \substack{2\\3} & 1 \\ \hline \end{array}} \quad -\tfrac{1}{2}\,\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & 4 \\ \hline \ & & & 3 \\ \hline \ & & 2 & 1 \\ \hline \end{array}} \quad -\tfrac{1}{6}\sqrt{3}\,\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & 4 \\ \hline \ & & & \substack{3\\2} \\ \hline \ & & \substack{2\\3} & 1 \\ \hline \end{array}} \quad +\tfrac{1}{3}\sqrt{3}\sqrt{2}\,\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \substack{4\\2} \\ \hline \ & & & \substack{3\\4} \\ \hline \ & & \substack{2\\3} & 1 \\ \hline \end{array}} \quad -\tfrac{1}{3}\sqrt{3}\sqrt{2}$$

$$\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & & \substack{2\\3} \\ \hline \ & \substack{4\\2} & \substack{3\\4} & 1 \\ \hline \end{array}} \quad -\tfrac{1}{6}\sqrt{3}\,\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & & \substack{2\\3} \\ \hline \ & 4 & \substack{3\\2} & 1 \\ \hline \end{array}} \quad +\tfrac{1}{2}\,\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & & 2 \\ \hline \ & 4 & 3 & 1 \\ \hline \end{array}} \quad +\tfrac{1}{2}\sqrt{3}\,\chi^{\dagger}_{\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & 4 & \substack{2\\3} \\ \hline \ & & \substack{3\\2} & 1 \\ \hline \end{array}} \quad +\tfrac{1}{2}$$

$$
\chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & 4 & 2 \\ \ & \ & 3 & 1 \end{ytableau}}
-\tfrac{1}{6}\sqrt{3}\,\chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & 4 \\ \ & \ & \ & \substack{2\\3} \\ \ & \ & \substack{3\\2} & 1 \end{ytableau}}
-\tfrac{5}{6}\,\chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & 4 \\ \ & \ & \ & 2 \\ \ & \ & 3 & 1 \end{ytableau}}
-\tfrac{1}{3}\sqrt{2}\,\chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \substack{4\\3} \\ \ & \ & \ & 2 \\ \ & \ & \substack{3\\4} & 1 \end{ytableau}}
+\tfrac{1}{3}\sqrt{3}\sqrt{2}
$$

$$
\chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \substack{3\\2} \\ \ & \ & \ & \substack{2\\4} \\ \ & \ & \substack{4\\3} & 1 \end{ytableau}}
-\tfrac{1}{3}\sqrt{2}\,\chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \substack{3\\4} \\ \ & \ & \ & 2 \\ \ & \ & \substack{4\\3} & 1 \end{ytableau}}
+\tfrac{1}{3}\,\chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & 3 \\ \ & \ & \ & 2 \\ \ & \ & 4 & 1 \end{ytableau}}
-\,\chi^\dagger_{\begin{ytableau} \ & \ & \ & 4 \\ \ & \ & \ & 3 \\ \ & \ & \ & 2 \\ \ & \ & \ & 1 \end{ytableau}}.
$$

Evaluating the reduction acting on the holomorphic restricted Schur polynomials yields:

$$
M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \ \\ 4 & 3 & 2 \end{ytableau}}
-\tfrac{1}{3}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & 4 \\ \ & 3 & 2 \end{ytableau}}
+\tfrac{2}{3}\sqrt{2}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \substack{4\\3} \\ \ & \substack{3\\4} & 2 \end{ytableau}}
-\tfrac{1}{3}\sqrt{2}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \substack{3\\4} \\ \ & \substack{4\\3} & 2 \end{ytableau}}
$$

$$
-\tfrac{1}{6}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & 3 \\ \ & 4 & 2 \end{ytableau}}
+\tfrac{1}{2}\sqrt{3}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \substack{3\\2} \\ \ & 4 & \substack{2\\3} \end{ytableau}}
-\tfrac{1}{2}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & 4 & 3 \\ \ & \ & 2 \end{ytableau}}
-\tfrac{1}{2}\sqrt{3}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & 4 & \substack{3\\2} \\ \ & \ & \substack{2\\3} \end{ytableau}}
$$

$$
+\tfrac{1}{2}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & 4 \\ \ & \ & \ & 3 \\ \ & \ & 2 \end{ytableau}}
-\tfrac{1}{6}\sqrt{3}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & 4 \\ \ & \ & \ & \substack{3\\2} \\ \ & \ & \substack{2\\3} \end{ytableau}}
+\tfrac{1}{3}\sqrt{3}\sqrt{2}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \substack{4\\2} \\ \ & \ & \ & \substack{3\\4} \\ \ & \ & \substack{2\\3} \end{ytableau}}
-\tfrac{1}{3}\sqrt{3}\sqrt{2}M
$$

$$
\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \substack{2\\3} \\ \ & \substack{4\\2} & \substack{3\\4} \end{ytableau}}
-\tfrac{1}{6}\sqrt{3}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \substack{2\\3} \\ \ & 4 & \substack{3\\2} \end{ytableau}}
-\tfrac{1}{2}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & 2 \\ \ & 4 & 3 \end{ytableau}}
-\tfrac{1}{2}\sqrt{3}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & 4 & \substack{2\\3} \\ \ & \ & \substack{3\\2} \end{ytableau}}
$$

$$
+\tfrac{1}{2}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & 4 & 2 \\ \ & \ & 3 \end{ytableau}}
+\tfrac{1}{2}\sqrt{3}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & 4 \\ \ & \ & \ & \substack{2\\3} \\ \ & \ & \substack{3\\2} \end{ytableau}}
+\tfrac{1}{6}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & 4 \\ \ & \ & \ & 2 \\ \ & \ & 3 \end{ytableau}}
-\tfrac{1}{3}\sqrt{2}M\,\chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \substack{4\\3} \\ \ & \ & \ & 2 \\ \ & \ & \substack{3\\4} \end{ytableau}}
$$

$$+\tfrac{2}{3}\sqrt{2}M\ \chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \substack{3\\4} \\ \ & \ & \ & 2 \\ \ & \ & \substack{4\\3} \end{ytableau}} \quad +\tfrac{1}{3}M\ \chi_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & 3 \\ \ & \ & \ & 2 \\ \ & \ & 4 \end{ytableau}} \quad -M\ \chi_{\begin{ytableau} \ & \ & \ & 4 \\ \ & \ & \ & 3 \\ \ & \ & \ & 2 \\ \ & \ & \ \end{ytableau}}.$$

Evaluating the reduction acting on the anti-holomorphic Schur polynomials yields:

$$M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \ \\ 4 & 3 & 2 \end{ytableau}} \quad -\tfrac{1}{3}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & 4 \\ \ & 3 & 2 \end{ytableau}} \quad -\tfrac{1}{3}\sqrt{2}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \substack{4\\3} \\ \ & \substack{3\\4} & 2 \end{ytableau}} \quad -\tfrac{1}{3}\sqrt{3}\sqrt{2}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \substack{4\\3} \\ \ & \substack{3\\2} & \substack{2\\4} \end{ytableau}}$$

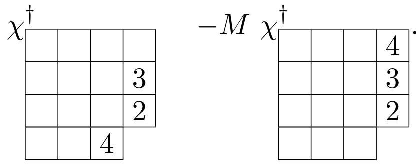
$$-\tfrac{1}{3}\sqrt{2}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \substack{3\\4} \\ \ & \substack{4\\3} & 2 \end{ytableau}} \quad +\tfrac{5}{6}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & 3 \\ \ & 4 & 2 \end{ytableau}} \quad -\tfrac{1}{6}\sqrt{3}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \substack{3\\2} \\ \ & 4 & \substack{2\\3} \end{ytableau}} \quad -\tfrac{1}{2}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & 4 & 3 \\ \ & \ & 2 \end{ytableau}}$$

$$+\tfrac{1}{2}\sqrt{3}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & 4 & \substack{3\\2} \\ \ & \ & \substack{2\\3} \end{ytableau}} \quad -\tfrac{1}{2}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & 4 \\ \ & \ & \ & 3 \\ \ & \ & 2 \end{ytableau}} \quad -\tfrac{1}{6}\sqrt{3}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & 4 \\ \ & \ & \ & \substack{3\\2} \\ \ & \ & \substack{2\\3} \end{ytableau}} \quad +\tfrac{1}{3}\sqrt{3}\sqrt{2}M$$

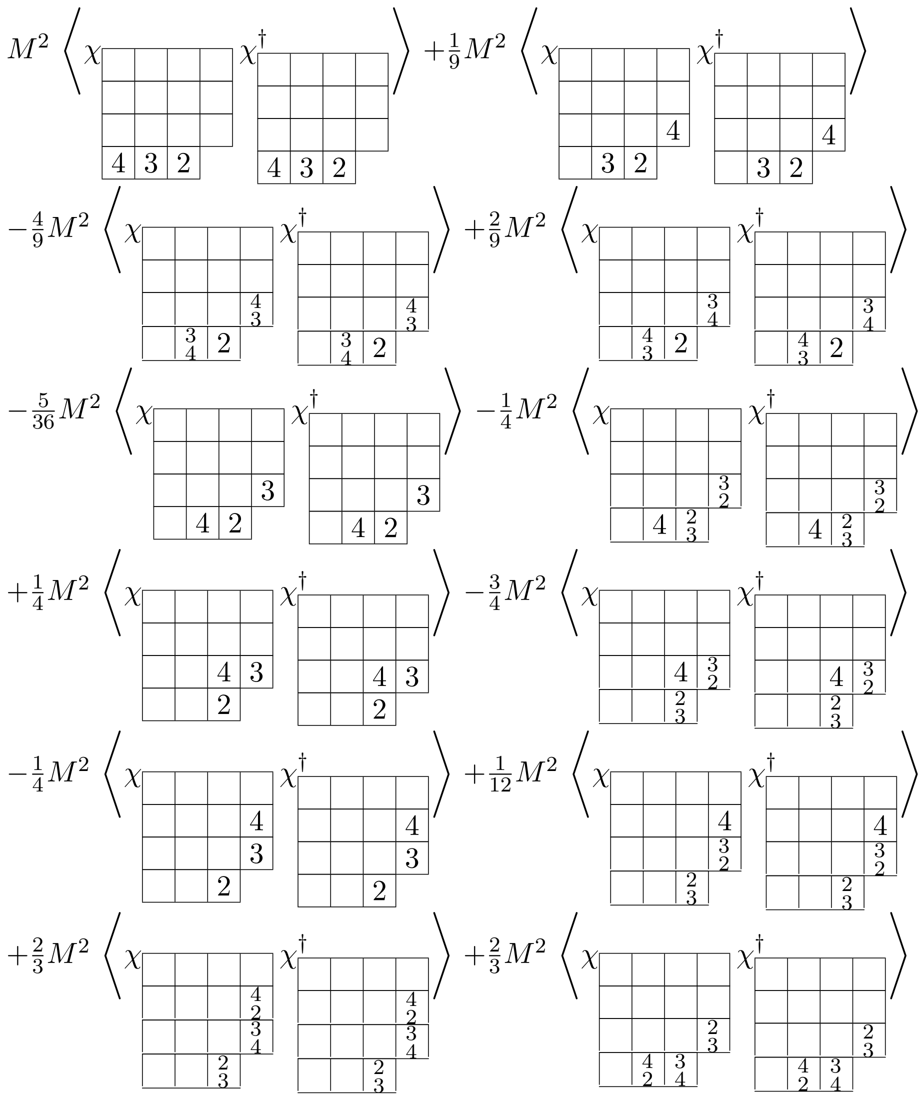
$$\chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \substack{4\\2} \\ \ & \ & \ & \substack{3\\4} \\ \ & \ & \substack{2\\3} \end{ytableau}} \quad -\tfrac{1}{3}\sqrt{3}\sqrt{2}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \substack{2\\3} \\ \ & \substack{4\\2} & \substack{3\\4} \end{ytableau}} \quad -\tfrac{1}{6}\sqrt{3}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \substack{2\\3} \\ \ & 4 & \substack{3\\2} \end{ytableau}} \quad +\tfrac{1}{2}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & 2 \\ \ & 4 & 3 \end{ytableau}}$$

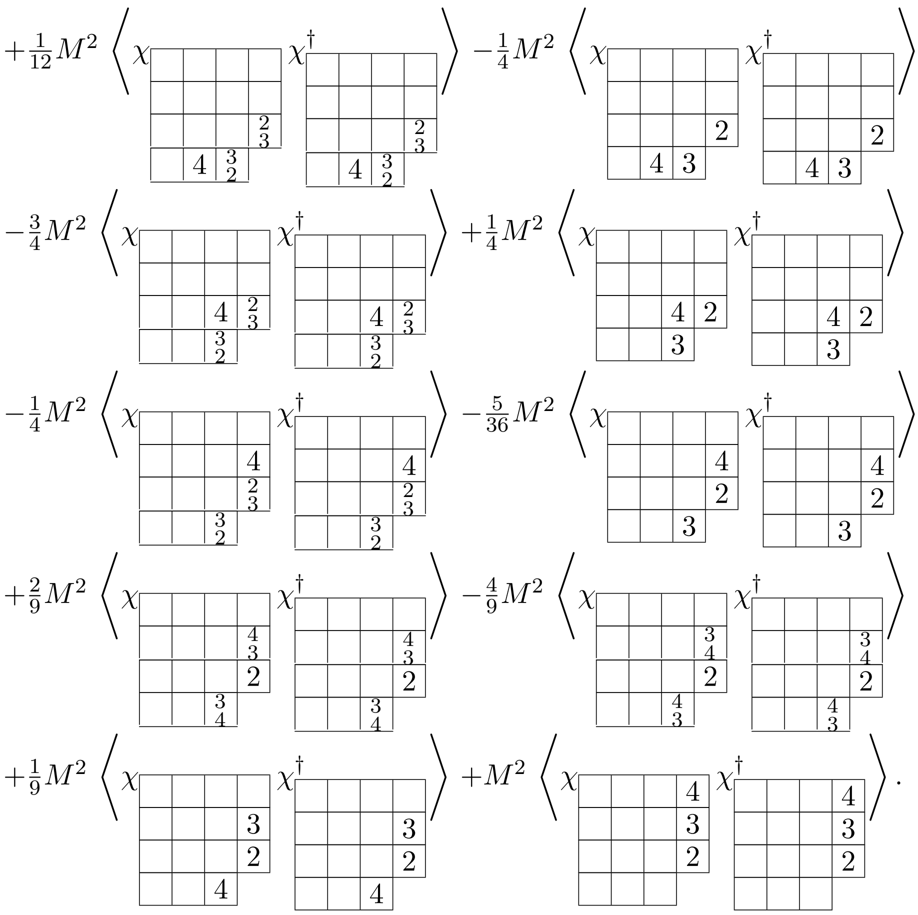
$$+\tfrac{1}{2}\sqrt{3}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & 4 & \substack{2\\3} \\ \ & \ & \substack{3\\2} \end{ytableau}} \quad +\tfrac{1}{2}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & 4 & 2 \\ \ & \ & 3 \end{ytableau}} \quad -\tfrac{1}{6}\sqrt{3}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & 4 \\ \ & \ & \ & \substack{2\\3} \\ \ & \ & \substack{3\\2} \end{ytableau}} \quad -\tfrac{5}{6}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & 4 \\ \ & \ & \ & 2 \\ \ & \ & 3 \end{ytableau}}$$

$$-\tfrac{1}{3}\sqrt{2}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \substack{4\\3} \\ \ & \ & \ & 2 \\ \ & \ & \substack{3\\4} \end{ytableau}} \quad +\tfrac{1}{3}\sqrt{3}\sqrt{2}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \substack{3\\2} \\ \ & \ & \ & \substack{2\\4} \\ \ & \ & \substack{4\\3} \end{ytableau}} \quad -\tfrac{1}{3}\sqrt{2}M\ \chi^\dagger_{\begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \substack{3\\4} \\ \ & \ & \ & 2 \\ \ & \ & \substack{4\\3} \end{ytableau}} \quad +\tfrac{1}{3}M$$

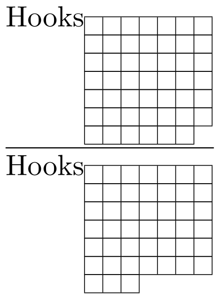


Finally, we need to evaluate the resultant correlators of restricted Schur polynomials:





Utilizing the following results for the hooks factors (note that again the Young diagrams below represent a rectangular block of N rows and M columns with boxes removed in the lower right hand corner):



$$= \frac{(M-1)(M-2)(M-3)(N+1)(N+2)(N+3)}{24},$$

$$\frac{\text{Hooks}\,\ydiagram{7,7,7,7,7,7,6}}{\text{Hooks}\,\ydiagram{7,7,7,7,7,6,4}} = \frac{(M-1)(M-2)(M+1)(N-1)(N+1)(N+2)}{8},$$

$$\frac{\text{Hooks}\,\ydiagram{7,7,7,7,7,7,6}}{\text{Hooks}\,\ydiagram{7,7,7,6,6,6,6}} = \frac{(M+1)(M+2)(M+3)(N-1)(N-2)(N-3)}{24},$$

$$\frac{\text{Hooks}\,\ydiagram{7,7,7,7,7,7,6}}{\text{Hooks}\,\ydiagram{7,7,7,7,7,5,5}} = \frac{(M)(M+1)(M-1)(N)(N+1)(N-1)}{12},$$

$$\frac{\text{Hooks}\,\ydiagram{7,7,7,7,7,7,6}}{\text{Hooks}\,\ydiagram{7,7,7,7,6,6,5}} = \frac{(M-1)(M+1)(M+2)(N-1)(N+1)(N-2)}{8},$$

we obtain the following final result:

$$\frac{\langle \chi_B\left(Z\right) \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ^\dagger}\right)\chi_B\left(Z^\dagger\right)\rangle}{\langle \chi_B\left(Z\right)\chi_B\left(Z^\dagger\right)\rangle}$$

$$= -3M - 3N^2M - 3M^3 + N^3M^2 + 10NM^2 - 3N^2M^3 + NM^4.$$

B.5 General Results for the Annulus

In this appendix we consider a background $\chi_B(Z)$ where B is a Young diagram with M columns and N rows. We will compute the two correlators

$$I_1 = \frac{\langle \chi_B \chi_B^\dagger \mathrm{Tr}\,(Z^n Z^{\dagger\, n})\rangle}{\langle \chi_B \chi_B^\dagger\rangle},$$

and

$$I_2 = \frac{\langle \mathrm{Tr}\,\left(\frac{d^n}{dZ^n}\frac{d^n}{dZ^{\dagger\, n}}\right)\chi_B \chi_B^\dagger\rangle}{\langle \chi_B \chi_B^\dagger\rangle},$$

in the large N limit.

B.5.1 Computation of I_1

We will make use of a dummy field D which does not interact with Z and has a two point function

$$\left\langle (D^\dagger)^k_l D^i_j \right\rangle = \delta^i_l \delta^k_j,$$

Including D does not change the value of any normalized correlation functions of operators built only out of Z and $Z^\dagger$. In particular, it does not change the value of I_1. Using D, we can rewrite

$$I_1 = \frac{\langle \chi_B \chi_B^\dagger \mathrm{Tr}\,(Z^n D)\mathrm{Tr}\,(D^\dagger Z^{\dagger\, n})\rangle}{\langle \chi_B \chi_B^\dagger\rangle}.$$

This is a useful step, because after using the identities

$$\mathrm{Tr}\,(Z^n D) = \frac{1}{n+1} D_{ij}\frac{d}{dZ_{ij}}\mathrm{Tr}\,(Z^{n+1}),$$

and (this identity was proved in Appendix 6 of [19])

$$\mathrm{Tr}\,(Z^{n+1}) = \sum_{s=0}^{n} (-1)^s \chi_{(n+1-s,1^s)}(Z),$$

where $(n+1-s, 1^s)$ denotes a Young diagram with $s+1$ rows; the first row has $n+1-s$ boxes and all remaining rows have one box, we can write $\mathrm{Tr}\,(Z^n D)$ as a sum of restricted Schur polynomials

$$\mathrm{Tr}\,(Z^n D) = \frac{1}{n+1} \sum_{s=0}^{n} \sum_{h_s} (-1)^s \chi_{(n+1-s,1^s),h_s}(Z,D),$$

where h_s is an irreducible representation of S_n. The sum over h_s is a sum over all possible S_n irreducible representations that can be subduced from the S_{n+1} representation $(n+1-s, 1^s)$. To proceed, we would like to evaluate the product

$$\chi_B(Z)\chi_{(n+1-s,1^s),h_s}(Z,D) \tag{B.2}$$

for any s and h_s. This product can be computed using the restricted Littlewood-Richardson rule derived in section 7. The difficult part of this computation entails evaluating the restricted Littlewood-Richardson numbers, which include the sum

$$\sum_{\sigma_1 \in S_{NM}} \sum_{\sigma_2 \in S_{n+1}} \chi_B(\sigma_1)\chi_{(n+1-s,1^s),h_s}(\sigma_2)\chi_{R,R'}(\sigma_1 \circ \sigma_2).$$

To evaluate this sum, note that both

$$\frac{d_B}{NM!} \sum_{\sigma_1 \in S_{NM}} \chi_B(\sigma_1)\sigma_1 \qquad \text{and} \qquad \frac{d_{(n+1-s,1^s)}}{(n+1)!} \sum_{\sigma_2 \in S_{n+1}} \chi_{(n+1-s,1^s),h_s}(\sigma_2)\sigma_2$$

are projection operators. Thus, the sum we need to compute is simply the partial trace (over (R, R')) of the direct product of two projectors. In general this is not a very useful observation because one can't choose a basis which is both simultaneously a basis of B and $((n+1-s, 1^s), h_s)$ on the one hand and (R, R') on the other. However, for the case we consider here a simultaneous basis can indeed be chosen as we now explain.

The above sum is needed to compute the coefficient of the term $\chi_{R,R'}(Z, D)$ appearing in the product (B.2). Since B has N rows, we can only stack $((n+1-s, 1^s), h_s)$ as a complete Young diagram, to the right of B; denote this new Young diagram by $(+(n+1-s, 1^s), +h_s)$. To see that this is the case, note that we could start with (R, R') which is an irreducible representation of S_{NM+n+1} and keep restricting to smaller and smaller subgroups, by freezing the indices that S_{n+1} acts on. Doing $n+1$ restrictions we have the subgroup S_{NM} and we must have reduced (R, R') to B. This forces (R, R') to be $(+(n+1-s, 1^s), +h_s)$ and it provides a simultaneous basis for B and $((n+1-s, 1^s), h_s)$ and for $(+(n+1-s, 1^s), +h_s)$.

It is now straight forward to see that

$$\frac{d_B}{NM!}\frac{d_{(n+1-s,1^s)}}{(n+1)!}\sum_{\sigma_1\in S_{NM}}\sum_{\sigma_2\in S_{n+1}}\chi_B(\sigma_1)\chi_{(n+1-s,1^s),h_s}(\sigma_2)\chi_{+(n+1-s,1^s),+h_s}(\sigma_1\circ\sigma_2)=d_{h_s}d_B,$$

where the right hand side is nothing but the dimension of the space that we traced over. Consequently,

$$\sum_{\sigma_1\in S_{NM}}\sum_{\sigma_2\in S_{n+1}}\chi_B(\sigma_1)\chi_{(n+1-s,1^s),h_s}(\sigma_2)\chi_{R,R'}(\sigma_1\circ\sigma_2)=(n+1)!NM!\frac{d_{h_s}}{d_{(n+1-s,1^s)}}.$$

Some straightforward manipulations now give

$$\chi_B(Z)\mathrm{Tr}\,(Z^nD)=\frac{1}{n+1}\frac{N}{N+M}\sum_{s=0}^{n}\sum_{h_s}(-1)^s\chi_{+(n+1-s,1^s),+h_s}(Z,D).$$

Thus, we have reduced the computation of I_1 to the computation of a two point function which is easily performed (we keep only the leading term at large N)

$$\begin{aligned}
I_1 &= \frac{\langle\chi_B\chi_B^\dagger\mathrm{Tr}\,(Z^nZ^{\dagger\,n})\rangle}{\langle\chi_B\chi_B^\dagger\rangle}\\
&= \frac{1}{f_B}\frac{1}{(n+1)^2}\frac{N^2}{(N+M)^2}\langle\sum_{s=0}^{n}\sum_{h_s}(-1)^s\chi_{+(n+1-s,1^s),+h_s}(Z,D)\sum_{t=0}^{n}\sum_{h_t}(-1)^t\chi_{+(n+1-t,1^t),+h_s}(Z,D)^\dagger\rangle\\
&= \frac{1}{(n+1)^2}N(M+N)^n\sum_{s=0}^{n}\sum_{h_s}\frac{(\mathrm{hooks})_{(n+1-s,1^s)}}{(\mathrm{hooks})_{h_s}}\\
&= N(M+N)^n\,.
\end{aligned}$$

B.5.2 Computation of I_2

It is clear that we can write

$$I_2=\frac{\langle\chi_B\chi_B^\dagger:\mathrm{Tr}\,(Z^nZ^{\dagger\,n}):\rangle}{\langle\chi_B\chi_B^\dagger\rangle}\equiv\langle:\mathrm{Tr}\,(Z^nZ^{\dagger\,n}):\rangle_B,$$

where $:O:$ denotes the normal ordering of O. Thus, we can obtain I_2 from I_1 by subtracting all terms with an odd number of self contractions (contractions between two fields in $\mathrm{Tr}\,(Z^nZ^{\dagger\,n})$) from I_1 and adding back all the terms with an even number of self contractions. The term with one self contraction, for example, gives

$$\sum_{r=1}^{n}\langle\mathrm{Tr}\,(Z^{n-r}(Z^\dagger)^{n-r})\mathrm{Tr}\,(Z^{r-1}(Z^\dagger)^{r-1})\rangle_B=\sum_{r=1}^{n}\langle\mathrm{Tr}\,(Z^{n-r}(Z^\dagger)^{n-r})\rangle_B\langle\mathrm{Tr}\,(Z^{r-1}(Z^\dagger)^{r-1})\rangle_B$$

$$=\sum_{r=1}^{n}N(M+N)^{n-r}N(M+N)^{r-1}=nN^2(M+N)^{n-1}.$$

To obtain this result we made use of large N factorization and the result of the previous subsection. A very similar argument gives

$$\frac{n!}{c!(n-c)!}N^{1+c}(N+M)^{n-c}$$

for the term with c self contractions. Thus

$$\begin{aligned}
I_2 &= N(N+M)^n-\sum_{c=1}^{n}\frac{n!}{c!(n-c)!}(-N)^{1+c}(N+M)^{n-c}\\
&= N\sum_{c=0}^{n}\frac{n!}{c!(n-c)!}(-N)^{c}(N+M)^{n-c}\\
&= N(N+M-N)^n\\
&= NM^n\,.
\end{aligned}$$

B.6 Last Site Dictionary

In this section we will explain how to translate between a "closed string" description of the operator

$$w = \mathrm{Tr}\,(YZ^{n_1}YZ^{n_2}Y\cdots YZ^{n_L})$$

and an "open string" description

$$\sum_{R,R'} \alpha_{R,R'}\chi^{(1)}_{R,R'}(Z,w) \qquad w^i_j = (YZ^{n_1}YZ^{n_2}Y\cdots YZ^{n_{L-1}}Y)^i_j,$$

where in this second description the last site is described by the Young diagrams R, R'. One simply makes repeated use of the identity

$$\chi^{(1)}_{R,R'}(Z,w) - \chi_{R'}(Z)\mathrm{Tr}\,(w) = \sum_\alpha \frac{1}{d_{R''_\alpha}}\mathrm{Tr}_{\,R''_\alpha}(\Gamma_R\,[(n,n-1)])\chi^{(1)}_{R',R''_\alpha}(Z,Zw).$$

which was derived in [32]. The second term on the LHS in the above identity does not contribute at large N. Start from

$$\chi^{(1)}_{\ydiagram{1},\cdot}(Z,Z^{n_L}w) \equiv \mathrm{Tr}\,(Z^{n_L}w),$$

and use the identity to pull Z's off Z^{n_L} and onto the Young diagram R. For example, for $n_L = 1, 2, 3$ we have

$$\mathrm{Tr}\,(Zw) = \frac{1}{2}\left(\chi_{\ytableaushort{{}w}} - \chi_{\ytableaushort{{},w}}\right),$$

$$\mathrm{Tr}\,(Z^2w) = \frac{1}{3}\left(\chi_{\ytableaushort{{}{}w}} - \chi_{\ytableaushort{{}{},w}} - \chi_{\ytableaushort{{}w,{}}} + \chi_{\ytableaushort{{},{},w}}\right),$$

$$\mathrm{Tr}\,(Z^3w) = \frac{1}{4}\left(\chi_{\ytableaushort{{}{}{}w}} - \chi_{\ytableaushort{{}{}{},w}} - \chi_{\ytableaushort{{}{}w,{}}} + \chi_{\ytableaushort{{}{},{},w}} + \chi_{\ytableaushort{{}w,{},{}}} - \chi_{\ytableaushort{{},{},{},w}}\right).$$

These formulas are exact.

C Appendix - Correlators of Operators with a Large R-Charge: A New Expansion Parameter

C.1 Schwinger-Dyson Equations in the Annulus Background

The Schwinger-Dyson equations provide a powerful approach to computing correlators in the annulus background. They are far more computationally efficient than the approach based on cutting rules developed in section 8. The advantage of the cutting rules are their generality: the cutting rules work in any background. In deriving the Schwinger-Dyson equation, a crucial observation is that for the background we consider (recall that B is a Young diagram with M columns and N rows)

$$\chi_B(Z) = \det(Z)^M \, .$$

We make repeated use of this fact and consequently, the results of this appendix apply only to the annulus background.

C.1.1 Schwinger-Dyson Equations

Start by considering

$$0 = \int \left[dZdZ^\dagger\right] \frac{d}{dZ_{ij}} \left((Z^{n+1}Z^{\dagger\, n})_{ij}\chi_B(Z)\chi_B(Z^\dagger)e^{-S}\right) .$$

Carrying out the derivative is straightforward, except perhaps for the term obtained when the derivative acts on the background. To evaluate this term, note that

$$\frac{d}{dZ_{ij}}\chi_B(Z) = \frac{d}{dZ_{ij}} \det(Z)^M = M(Z^{-1})_{ji} \det(Z)^M \, .$$

Thus, this term contributes

$$M \left\langle \mathrm{Tr}\,(Z^n Z^{\dagger\, n}) \right\rangle_B$$

to the Schwinger-Dyson equation. Next, focus on the term obtained by acting on the first Z in $(Z^{n+1}Z^{\dagger\,n})_{ij}$ which gives

$$N\left\langle \mathrm{Tr}\,(Z^n Z^{\dagger\,n})\right\rangle_B\ ;$$

these two terms combine to give the claimed $N\to M+N$ replacement in the Schwinger-Dyson equations. Writing out all of the terms we have

$$\left\langle \mathrm{Tr}\,(Z^{n+1}Z^{\dagger\,n+1})\right\rangle_B = (N+M)\left\langle \mathrm{Tr}\,(Z^n Z^{\dagger\,n})\right\rangle_B + \sum_{r=1}^{n}\left\langle \mathrm{Tr}\,(Z^r)\mathrm{Tr}\,(Z^{n-r}Z^{\dagger\,n})\right\rangle_B\ .$$

A slightly more general Schwinger-Dyson equation which we found useful in the computation of correlators reads

$$0=\int\left[dZdZ^\dagger\right]\frac{d}{dZ_{ij}}\left((Z^mZ^{\dagger\,n})_{ij}\mathrm{Tr}\,(Z^{n+1-m})\chi_B(Z)\chi_B(Z^\dagger)e^{-S}\right)$$

which is easily seen to give

$$\left\langle \mathrm{Tr}\,(Z^{\dagger\,n+1}Z^m)\mathrm{Tr}\,(Z^{n+1-m})\right\rangle_B = \sum_{r=1}^{m-1}\left\langle \mathrm{Tr}\,(Z^{\dagger\,n}Z^{m-r-1})\mathrm{Tr}\,(Z^r)\mathrm{Tr}\,(Z^{n+1-m})\right\rangle_B$$

$$+(N+M)\left\langle \mathrm{Tr}\,(Z^{\dagger\,n}Z^{m-1})\mathrm{Tr}\,(Z^{n+1-m})\right\rangle_B + (n+1-m)\left\langle \mathrm{Tr}\,(Z^nZ^{\dagger\,n})\right\rangle_B\ .$$

This starting point could easily be generalized to

$$0=\int\left[dZdZ^\dagger\right]\frac{d}{dZ_{ij}}\left((Z^mZ^{\dagger\,n})_{ij}\mathrm{Tr}\,(Z^{n+1-m})O(Z,Z^\dagger)\chi_B(Z)\chi_B(Z^\dagger)e^{-S}\right)$$

where $O(Z,Z^\dagger)$ is any gauge invariant operator.

Computing correlators is now straightforward. We can obtain all correlators of the form $\left\langle\prod_{i,j}\mathrm{Tr}\,(Z^{n_i})\mathrm{Tr}\,((Z^\dagger)^{m_j})\right\rangle_B$ using (9.6). Using these in the above Schwinger-Dyson equations, we can easily determine the correlators $\left\langle\prod_i \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B$ which are of relevance for the near-BPS sector of the theory.

C.1.2 Testing Factorization

Now that we can use the Schwinger-Dyson equations to compute correlators exactly, we can answer some interesting questions. One obvious question is if the annulus geometry provides a good background. For this to be the case, we need to have a factorization of the background expectation values of gauge invariant observables. This implies that a single saddle point is dominating the gauge theory path integral. By the gauge theory/gravity correspondence, this saddle point represents a particular space-time geometry in gravity, that is, a classical spacetime has emerged. One nice general result follows from

$$0 = \int \left[dZdZ^\dagger\right] \frac{d}{dZ_{ij}} \left(Z_{ij} \left[\mathrm{Tr}\,(ZZ^\dagger)\right]^n \chi_B(Z)\chi_B(Z^\dagger) e^{-S} \right)$$

which implies that

$$\left\langle \left[\mathrm{Tr}\,(ZZ^\dagger)\right]^{n+1} \right\rangle_B = (N^2 + MN + n) \left\langle \left[\mathrm{Tr}\,(ZZ^\dagger)\right]^{n} \right\rangle_B .$$

This recursion relation is easily solved to give

$$\left\langle \left[\mathrm{Tr}\,(ZZ^\dagger)\right]^{n+1} \right\rangle_B = \prod_{i=0}^{n} (N^2 + MN + i)\,.$$

Keeping only the leading order[25], we have

$$\left\langle \left[\mathrm{Tr}\,(ZZ^\dagger)\right]^{n+1} \right\rangle_B = (N^2 + MN)^{n+1} = \left\langle \mathrm{Tr}\,(ZZ^\dagger) \right\rangle_B^{n+1} ,$$

demonstrating factorization for these amplitudes. We can easily generalize this by considering

$$0 = \int \left[dZdZ^\dagger\right] \frac{d}{dZ_{ij}} \left(Z_{ij} \mathrm{Tr}\,(Z^p Z^{\dagger p}) \left[\mathrm{Tr}\,(ZZ^\dagger)\right]^n \chi_B(Z)\chi_B(Z^\dagger) e^{-S} \right)$$

which implies

$$\left\langle \left[\mathrm{Tr}\,(ZZ^\dagger)\right]^{n+1} \mathrm{Tr}\,(Z^p Z^{\dagger p}) \right\rangle_B = (N^2 + MN + n + p) \left\langle \left[\mathrm{Tr}\,(ZZ^\dagger)\right]^{n} \mathrm{Tr}\,(Z^p Z^{\dagger p}) \right\rangle_B .$$

Once again this is easy to solve, giving

$$\left\langle \left[\mathrm{Tr}\,(ZZ^\dagger)\right]^{n+1} \mathrm{Tr}\,(Z^p Z^{\dagger p}) \right\rangle_B = \prod_{i=0}^{n} (N^2 + MN + i + p) \left\langle \mathrm{Tr}\,(Z^p Z^{\dagger p}) \right\rangle_B ,$$

[25] Of course, n and p (used below) are $O(1)$.

which becomes, at the leading order,

$$\begin{aligned}\left\langle\left[\mathrm{Tr}\,(ZZ^\dagger)\right]^{n+1}\mathrm{Tr}\,(Z^pZ^{\dagger p})\right\rangle_B &= (N^2+MN)^{n+1}\left\langle\mathrm{Tr}\,(Z^pZ^{\dagger p})\right\rangle_B\\ &= \left\langle\mathrm{Tr}\,(Z^pZ^{\dagger p})\right\rangle_B\left\langle\mathrm{Tr}\,(ZZ^\dagger)\right\rangle_B^{n+1},\end{aligned}$$

again demonstrating factorization.

Above we have been careful to compute things to all orders. If we simply assume factorization and keep only the leading order, we get additional information about the leading behavior of various gauge invariant operators. For example,

$$\begin{aligned}\left\langle\mathrm{Tr}\,(Z^{p+1}Z^{\dagger\,p+1})\prod_i\mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B &= (N{+}M)\left\langle\mathrm{Tr}\,(Z^pZ^{\dagger\,p})\prod_i\mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B\\ &+\sum_{r=1}^{p}\left\langle\mathrm{Tr}\,(Z^{p-r}Z^{\dagger\,p})\mathrm{Tr}\,(Z^r)\prod_i\mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B\\ &+\sum_j\sum_{r=0}^{n_j-1}\left\langle\mathrm{Tr}\,(Z^{r+p+1}Z^{\dagger\,p}Z^{n_j-r-1}Z^{\dagger\,n_j})\prod_{i\neq j}\mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B,\end{aligned}$$

becomes, after assuming factorization

$$\begin{aligned}\left\langle\mathrm{Tr}\,(Z^{p+1}Z^{\dagger\,p+1})\right\rangle_B\prod_i\left\langle\mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B &= (N{+}M)\left\langle\mathrm{Tr}\,(Z^pZ^{\dagger\,p})\right\rangle_B\prod_i\left\langle\mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B\\ &+\sum_j\sum_{r=0}^{n_j-1}\left\langle\mathrm{Tr}\,(Z^{r+p+1}Z^{\dagger\,p}Z^{n_j-r-1}Z^{\dagger\,n_j})\right\rangle_B\prod_{i\neq j}\left\langle\mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B,\end{aligned}$$

The second term of the right hand side is subleading compared to the first term because (i) the first term is multiplied by $(N+M)$ and (ii) the second term has one less trace in it. Thus it may be dropped to give

$$\left\langle\mathrm{Tr}\,(Z^{p+1}Z^{\dagger\,p+1})\right\rangle_B\prod_i\left\langle\mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B = (N{+}M)\left\langle\mathrm{Tr}\,(Z^pZ^{\dagger\,p})\right\rangle_B\prod_i\left\langle\mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B.$$

Iterating this relation, the above result is clearly equivalent to

$$\left\langle\mathrm{Tr}\,(Z^qZ^{\dagger\,q})\right\rangle_B = N(N+M)^q\,.$$

which we know is correct.

C.1.3 Testing the Cutting Rules

In section 8 a method to compute general correlators in any arbitrary LLM background was given. Now that we have an efficient way to compute correlators in the annulus background we can ask: Do the cutting rule methods really work? In this appendix we will compute a specific correlator, first using the Schwinger-Dyson equations and then using the cutting rules. We have found complete agreement between the cutting rule result and the result from the Schwinger-Dyson equations for any correlator we have computed. The cutting rules work.

Starting from

$$0 = \int \left[dZdZ^\dagger\right] \frac{d}{dZ_{ij}} \left((Z^{\dagger\, n} Z^m Z^{\dagger\, p} Z^{n+p-m+1})_{ij} \chi_B(Z)\chi_B(Z^\dagger) e^{-S}\right)$$

we obtain

$$\left\langle \mathrm{Tr}\,(Z^{\dagger n+1} Z^m Z^{\dagger p} Z^{n+p-m+1}) \right\rangle_B = \sum_{r=0}^{m-1} \left\langle \mathrm{Tr}\,(Z^{\dagger n} Z^r) \mathrm{Tr}\,(Z^{\dagger p} Z^{n+p-r}) \right\rangle_B$$

$$+ \sum_{r=0}^{n+p-m-1} \left\langle \mathrm{Tr}\,(Z^{\dagger n} Z^m Z^{\dagger p} Z^r) \mathrm{Tr}\,(Z^{n+p-m-r}) \right\rangle_B + (N+M) \left\langle \mathrm{Tr}\,(Z^{\dagger n} Z^m Z^{\dagger p} Z^{n+p-m}) \right\rangle_B .$$

Setting $n = 0$, $m = 1$ and $p = 1$ we have

$$\left\langle \mathrm{Tr}\,(Z^\dagger Z Z^\dagger Z) \right\rangle_B = (2N + M) \left\langle \mathrm{Tr}\,(Z^\dagger Z) \right\rangle_B = (2N + M)N(N + M) .$$

We will summarize the cutting rule computation; for more details the reader should consult section 8. Evaluating this correlator using cutting rules, there are four contributions: the term with no contractions with the background gives $2N^3$. The terms coming from contracting one Z in the gauge invariant operator with a $Z^\dagger$ in $\chi_B(Z^\dagger)$ give

$$4N \left\langle \mathrm{Tr}\, \left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right) \chi_B(Z)\chi_B(Z^\dagger) \right\rangle = 4N^2 M.$$

The terms coming from contracting both Zs in the gauge invariant operator with $Z^\dagger$s in $\chi_B(Z^\dagger)$ give

$$\left\langle \mathrm{Tr}\, \left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\right) \chi_B(Z)\chi_B(Z^\dagger) \right\rangle = M^2 N - N^2 M.$$

To evaluate this last contribution we had to cut the trace of four derivatives into a product of two traces, each containing two derivatives. This is accompanied by a nontrivial trace insertion factor that we evaluated in representation B. Summing these terms we have

$$2N^3 + 4N^2M + M^2N - N^2M = (2N + M)N(N + M)\,.$$

C.2 Schwinger-Dyson Equations for > 1 Charge Background

The background of interest in this appendix is (recall that r_1 is a rectangular Young diagram with N rows and M_1 columns and r_2 is a rectangular Young diagram with N rows and M_2 columns)

$$\chi_{r_1}(Z)\chi_{r_2}(Y) = (\det(Z))^{M_1}(\det(Y))^{M_2} \, .$$

The Schwinger-Dyson equations continue to provide a powerful approach to correlator computations, when we consider this background built using more than one matrix. In this appendix we will give a few example computations. Consider the identity[26]

$$0 = \int \left[dZdZ^\dagger dY dY^\dagger\right] \frac{d}{dZ_{ij}} \left((Z^n Y^m Y^{\dagger\, m} Z^{\dagger\, n-1})_{ij} \chi_{r_1}(Z)\chi_{r_1}(Z^\dagger)\chi_{r_2}(Y)\chi_{r_2}(Y^\dagger) e^{-S} \right)$$

which leads to the following Schwinger-Dyson equation

$$\left\langle \mathrm{Tr}\,(Z^n Y^m Y^{\dagger\, m} Z^{\dagger\, n}) \right\rangle_{(r_1,r_2)} = (N+M_1) \left\langle \mathrm{Tr}\,(Z^{n-1} Y^m Y^{\dagger\, m} Z^{\dagger\, n-1}) \right\rangle_{(r_1,r_2)}$$
$$+ \sum_{r=1}^{n-1} \left\langle \mathrm{Tr}\,(Z^r) \mathrm{Tr}\,(Z^{n-1-r} Y^m Y^{\dagger\, m} Z^{\dagger\, n-1}) \right\rangle_{(r_1,r_2)} \, .$$

If we had been working in the trivial vacuum, the only difference would have been to replace $N+M_1$ in the above equation by N. One way to think about the above Schwinger-Dyson equation is that to go from the left hand side to the right hand side, we perform one of the Z Wick contractions. To obtain the equation that follows when we perform a Y Wick contraction, start with the identity

$$0 = \int \left[dZdZ^\dagger dY dY^\dagger\right] \frac{d}{dY_{ij}} \left((Y^n Z^m Z^{\dagger\, m} Y^{\dagger\, n-1})_{ij} \chi_{r_1}(Z)\chi_{r_1}(Z^\dagger)\chi_{r_2}(Y)\chi_{r_2}(Y^\dagger) e^{-S} \right)$$

which leads to the following Schwinger-Dyson equation

$$\left\langle \mathrm{Tr}\,(Y^n Z^m Z^{\dagger\, m} Y^{\dagger\, n}) \right\rangle_{(r_1,r_2)} = (N+M_2) \left\langle \mathrm{Tr}\,(Y^{n-1} Z^m Z^{\dagger\, m} Y^{\dagger\, n-1}) \right\rangle_{(r_1,r_2)}$$

[26] The action $S = \mathrm{Tr}\,(ZZ^\dagger) + \mathrm{Tr}\,(YY^\dagger)$.

$$+\sum_{r=1}^{n-1}\Big\langle \mathrm{Tr}\,(Y^r)\mathrm{Tr}\,(Y^{n-1-r}Z^m Z^{\dagger\,m}Y^{\dagger\,n-1})\Big\rangle_{(r_1,r_2)}\,.$$

If we had been working in the trivial vacuum, the only difference would have been to replace $N+M_2$ in the above equation by N. This structure is parallel to the structure we found for backgrounds constructed using a single matrix: in this case we have found that to reproduce correlators of operators built only using Zs or $Z^\dagger$s (Ys or $Y^\dagger$s) we simply replace $N\to N+M_1$ ($N\to N+M_2$). This structure is again emerging at the level of the Schwinger-Dyson equations.

Consider the last Schwinger-Dyson equation given above. In the large N limit the first term on the right hand side gives the leading contribution. The second term has one more trace in it whilst the first term is multiplied by $(N+M_2)$. Naive counting of powers of N would suggest that these two terms are of the same order. However, the leading contribution to the second term $\langle \mathrm{Tr}\,(Y^r)\rangle_{(r_1,r_2)}\,\langle \mathrm{Tr}\,(Y^{n-1-r}Z^m Z^{\dagger\,m}Y^{\dagger\,n-1})\rangle_{(r_1,r_2)}$ vanishes. Dropping the second term and iterating we find

$$\Big\langle \mathrm{Tr}\,(Y^n Z^m Z^{\dagger\,m}Y^{\dagger\,n})\Big\rangle_{(r_1,r_2)}=(N+M_2)^n\Big\langle \mathrm{Tr}\,(Z^m Z^{\dagger\,m})\Big\rangle_{(r_1,r_2)}=N(N+M_1)^m(N+M_2)^n\,. \tag{C.1}$$

This looks very similar to the relation (9.15); indeed, we can write

$$\Big\langle O(Z,Z^\dagger,Y,Y^\dagger)\Big\rangle_{(r_1,r_2)}=\left\langle O\left(\sqrt{\frac{N+M_1}{N}}Z,\sqrt{\frac{N+M_1}{N}}Z^\dagger,\sqrt{\frac{N+M_2}{N}}Y,\sqrt{\frac{N+M_2}{N}}Y^\dagger\right)\right\rangle\,.$$

Relations of this type would again be very useful in deriving spin chains for gauge invariant operators in this two matrix background.

Finally, although we have already argued that there are large ‘t Hooft coupling corrections to the background, it is still interesting to ask if factorization holds. Such backgrounds are naturally interpreted as classical backgrounds that receive curvature corrections. For operators that do not mix Zs and Ys in the same trace, correlators factorize into a Z correlator times a Y correlator. Using the results of Appendix C.1 we clearly have factorization in this case. For operators with mixed traces, a bit more work is needed. We will give some examples in which factorization is clear. Consider

$$0=\int\left[dZdZ^\dagger dYdY^\dagger\right]\frac{d}{dY_{ij}}\Big((Y^{n+1}Z^mZ^{\dagger\,m}Y^{\dagger\,n})_{ij}\times$$

$$\times \prod_a \mathrm{Tr}\,(Z^{n_a} Z^{\dagger n_a}) \chi_{r_1}(Z)\chi_{r_1}(Z^\dagger)\chi_{r_2}(Y)\chi_{r_2}(Y^\dagger) e^{-S}\Bigg)$$

which implies

$$\left\langle \mathrm{Tr}\,(Y^{\dagger\, n} Y^n Z^m Z^{\dagger m}) \prod_a \mathrm{Tr}\,(Z^{n_a} Z^{\dagger n_a}) \right\rangle_{(r_1,r_2)}$$

$$= (N+M_2) \left\langle \mathrm{Tr}\,(Y^{\dagger\, n-1} Y^{n-1} Z^m Z^{\dagger m}) \prod_a \mathrm{Tr}\,(Z^{n_a} Z^{\dagger n_a}) \right\rangle_{(r_1,r_2)}$$

$$+ \sum_{r=1}^{n-1} \left\langle \mathrm{Tr}\,(Y^r) \mathrm{Tr}\,(Y^{n-1-r} Z^m Z^{\dagger\, m} Y^{\dagger\, n-1}) \prod_a \mathrm{Tr}\,(Z^{n_a} Z^{\dagger n_a}) \right\rangle_{(r_1,r_2)} .$$

In the large N limit the first term on the right hand side gives the leading contribution so that we can drop the second term. Iterating, we find

$$\left\langle \mathrm{Tr}\,(Y^{\dagger\, n} Y^n Z^m Z^{\dagger m}) \prod_a \mathrm{Tr}\,(Z^{n_a} Z^{\dagger n_a}) \right\rangle_{(r_1,r_2)} = (N+M_2)^n \left\langle \mathrm{Tr}\,(Z^m Z^{\dagger m}) \prod_a \mathrm{Tr}\,(Z^{n_a} Z^{\dagger n_a}) \right\rangle_{(r_1,r_2)}$$

$$= (N+M_2)^n \left\langle \mathrm{Tr}\,(Z^m Z^{\dagger m}) \right\rangle_{(r_1,r_2)} \prod_a \left\langle \mathrm{Tr}\,(Z^{n_a} Z^{\dagger n_a}) \right\rangle_{(r_1,r_2)} ,$$

where to get the last equality we used factorization of the $Z, Z^\dagger$ correlators. Now, use (C.1) to identify $(N+M_2)^n \left\langle \mathrm{Tr}\,(Z^m Z^{\dagger m}) \right\rangle$ as $\left\langle \mathrm{Tr}\,(Y^{\dagger n} Y^n Z^m Z^{\dagger m}) \right\rangle$ in the last line above so that

$$\left\langle \mathrm{Tr}\,(Y^{\dagger n} Y^n Z^m Z^{\dagger m}) \prod_a \mathrm{Tr}\,(Z^{n_a} Z^{\dagger n_a}) \right\rangle_{(r_1,r_2)}$$

$$= \left\langle \mathrm{Tr}\,(Y^{\dagger n} Y^n Z^m Z^{\dagger m}) \right\rangle_{(r_1,r_2)} \prod_a \left\langle \mathrm{Tr}\,(Z^{n_a} Z^{\dagger n_a}) \right\rangle_{(r_1,r_2)} .$$

We can give a rather general argument for factorization: consider

$$0 = \int \left[dZ dZ^\dagger dY dY^\dagger\right] \frac{d}{dY_{ij}} \left((Y^{p+1} Z^n Z^{\dagger\, n} Y^{\dagger\, p})_{ij} \mathcal{O} \chi_{r_1}(Z)\chi_{r_1}(Z^\dagger)\chi_{r_2}(Y)\chi_{r_2}(Y^\dagger) e^{-S}\right)$$

where $\mathcal{O}$ is any gauge invariant operator. This implies

$$\left\langle \mathrm{Tr}\,(Y^{\dagger\, p+1} Y^{p+1} Z^n Z^{\dagger n}) \mathcal{O} \right\rangle_{(r_1,r_2)} = (N+M_2) \left\langle \mathrm{Tr}\,(Y^{\dagger\, p} Y^p Z^n Z^{\dagger n}) \mathcal{O} \right\rangle_{(r_1,r_2)}$$

$$+\sum_{r=1}^{p}\left\langle \mathrm{Tr}\,(Y^r)\mathrm{Tr}\,(Y^{p-r}Z^m Z^{\dagger\,m}Y^{\dagger\,p})\mathcal{O}\right\rangle_{(r_1,r_2)}+\left\langle (Y^{p+1}Z^n Z^{\dagger\,n}Y^{\dagger\,p})_{ij}\frac{d}{dY_{ij}}\mathcal{O}\right\rangle_{(r_1,r_2)} .$$

The second term on the right hand side can be dropped - it vanishes at leading order. The third term on the right hand side can also be dropped - it represents a trace joining term, so that it has one less trace than the first term. If we now rescale $Y \to \frac{N}{N+M_2}Y$ and $Z \to \frac{N}{N+M_1}Z$ we recover the Schwinger-Dyson equations of the theory in the trivial vacuum (after dropping the same two terms justified with the same two reasons). We know that factorization was a property of the old Schwinger-Dyson equations so that we have just learnt that it is a property of the new Schwinger-Dyson equations too.

C.3 Anomalous Dimension for > 1 Charge Background

In this section we will explain how to compute the one loop anomalous dimension of the backgrounds considered in section 9.1.4.

C.3.1 An Identity: Excited Giant Correlators

Correlation functions of restricted Schur polynomials have been computed in [31] (see also [32, 33]). The logic in these computations is first to contract the open string words and then to compute the remaining contractions. In this section we will obtain a formula that describes the result of contracting all fields *except* the open string words. Although we derive our formula for the case of one string attached, it is simple to extend it to the general case. The formula we are after says

$$\left\langle \chi^{(1)}_{R,R'}(Z,W)\chi^{(1)}_{R,R'}(Z^\dagger,W^\dagger)\right\rangle = A\left\langle \mathrm{Tr}\,(WW^\dagger)\right\rangle + B\left\langle \mathrm{Tr}\,(W)\mathrm{Tr}\,(W^\dagger)\right\rangle\,.$$

Recall[31] that the allowed index structure for open string word two point functions is

$$\left\langle W^i_j (W^\dagger)^k_l\right\rangle = \delta^i_l\delta^k_j F_0 + \delta^i_j\delta^k_l F_1\,.$$

Thus,

$$\left\langle \chi^{(1)}_{R,R'}(Z,W)\chi^{(1)}_{R,R'}(Z^\dagger,W^\dagger)\right\rangle = A(N^2F_0+NF_1)+B(N^2F_1+NF_0)\,.$$

From the technology developed in [31], we also know that

$$\left\langle \chi^{(1)}_{R,R'}(Z,W)\chi^{(1)}_{R,R'}(Z^\dagger,W^\dagger)\right\rangle = \frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}} f_R F_0 + c_{RR'} f_R F_1\,.$$

It is now trivial to find

$$A = \left(\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}}N^2 - c_{RR'}N\right)\frac{f_R}{N^4-N^2}\,,$$

$$B = \left(N^2 c_{RR'} - N\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}}\right)\frac{f_R}{N^4-N^2}\,.$$

C.3.2 Leading Contribution to Background Correlator

The formulas we write in this section will not be general; we are considering the backgrounds r_1 and r_2 of section 9.1.4. With a little extra effort one could be general. We are interested in computing the normalized correlation function

$$\Big\langle \chi_{r_1}(Z^\dagger)\chi_{r_2}(Y^\dagger)\chi_{r_1}(Z)\chi_{r_2}(Y) \Big\rangle \,,$$

to one loop. According to Appendix B of [67], the D-term, self energy and gluon exchange cancel at one loop order (using techniques of [100]), so to this order we only need to consider the contributions from the F-term. Towards this end we will now evaluate

$$I_1 = \Big\langle \chi_{r_1}(Z^\dagger)\chi_{r_2}(Y^\dagger)\chi_{r_1}(Z)\chi_{r_2}(Y)\mathrm{Tr}\left(\left[Z,Y\right]\left[Z^\dagger,Y^\dagger\right]\right) \Big\rangle \,.$$

The tricky part of this computation is the evaluation of the color combinatoric factor. To do this evaluation we can work in zero dimensions. Since we drop self energy corrections, the F-term is normal ordered and hence the above correlator can be written as

$$\Big\langle \mathrm{Tr}\left(\left[\frac{\partial}{\partial Z},\frac{\partial}{\partial Y}\right]\left[\frac{\partial}{\partial Z^\dagger},\frac{\partial}{\partial Y^\dagger}\right]\right)\chi_{r_1}(Z^\dagger)\chi_{r_2}(Y^\dagger)\chi_{r_1}(Z)\chi_{r_2}(Y) \Big\rangle \,.$$

This can be rewritten, using dummy open string variables, as

$$\Big\langle \mathrm{Tr}\left(\left[\frac{\partial}{\partial W},\frac{\partial}{\partial V}\right]\left[\frac{\partial}{\partial W^\dagger},\frac{\partial}{\partial V^\dagger}\right]\right)\chi^{(1)}_{r_1,r_1'}(Z^\dagger,W^\dagger)\chi^{(1)}_{r_2,r_2'}(Y^\dagger,V^\dagger)\chi^{(1)}_{r_1,r_1'}(Z,W)\chi^{(1)}_{r_2,r_2'}(Y,V) \Big\rangle \,.$$

For both r_1 and r_2 there is only one way possible to attach the open string. Using the results of the previous subsection we now obtain

$$\mathrm{Tr}\left(\left[\frac{\partial}{\partial W},\frac{\partial}{\partial V}\right]\left[\frac{\partial}{\partial W^\dagger},\frac{\partial}{\partial V^\dagger}\right]\right)\Big(A_1\mathrm{Tr}\,(WW^\dagger)+B_1\mathrm{Tr}\,(W)\mathrm{Tr}\,(W^\dagger)\Big)\Big(A_2\mathrm{Tr}\,(VV^\dagger)+B_2\mathrm{Tr}\,(V)\mathrm{Tr}\,(V^\dagger)\Big)\,,$$

where

$$A_1 = \frac{M_1}{N}f_{r_1}\,,\quad A_2 = \frac{M_2}{N}f_{r_2}\,,\quad B_i = 0\,.$$

It is a simple matter to find

$$\frac{I_1}{f_{r_1}f_{r_2}} = -2NM_1M_2 + 2\frac{M_1M_2}{N}\,.$$

The leading contribution to this correlator comes from the terms

$$\mathrm{Tr}\,(ZYY^{\dagger}Z^{\dagger}) + \mathrm{Tr}\,(YZZ^{\dagger}Y^{\dagger})\,.$$

This computation will allow us, in the next subsection, to identify and evaluate the leading contribution to the one loop anomalous dimension.

C.3.3 Leading Contribution to the One Loop Dilatation Operator

The $O(g_{YM}^0)$ contribution to the anomalous dimension

$$D_0 = \mathrm{Tr}\left(Z\frac{\partial}{\partial Z}\right) + \mathrm{Tr}\left(Y\frac{\partial}{\partial Y}\right)$$

gives $\Delta_0 = NM_1 + NM_2$. To obtain the leading piece of the $O(g_{YM}^2)$ contribution to the anomalous dimension, which we have identified in the previous subsection, replace

$$D_1 = -2g_{YM}^2\mathrm{Tr}\,:\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right]\Big[Y,Z\Big]:\rightarrow 2g_{YM}^2\mathrm{Tr}\left(ZY\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}\right)+2g_{YM}^2\mathrm{Tr}\left(YZ\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right)\,.$$

The normal ordering symbols here indicate that derivatives within the normal ordering symbols do not act on fields inside the normal ordering symbols. It is now straightforward to argue that

$$\left(2g_{YM}^2\mathrm{Tr}\left(ZY\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}\right) + 2g_{YM}^2\mathrm{Tr}\left(YZ\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right)\right)\chi_{r_1}(Z)\chi_{r_2}(Y)$$
$$= 4g_{YM}^2NM_1M_2\chi_{r_1}(Z)\chi_{r_2}(Y)\,.$$

We are interested in two cases:

- Both M_1 and M_2 are $O(N)$. Holding $g_{YM}^2N = \lambda$ fixed and large (which is the regime in which we expect that we can trust the dual geometry), we find a one loop correction of $O(N^2)$ times λ to the tree level value which is itself $O(N^2)$. At large N the tree level and one loop results are of the same order.
- Both M_1 and M_2 are $O(\sqrt{N})$. Holding $g_{YM}^2N = \lambda$ fixed and large, we find a one loop correction of $O(N)$ times λ to the tree level value which is itself $O(N^{3/2})$. At large N the tree level result dominates the one loop correction.

Notice that if we take M_1 to be $O(N)$ and keep M_2 to be $O(1)$ or if we take M_2 to be $O(N)$ and keep M_1 to be $O(1)$, the one loop correction becomes negligible as compared to the tree level value, as we would expect. It is also interesting to note that our operator is an eigenoperator of D_1, so that at one loop and at large N it does not mix with other operators.

D Appendix - Anomalous Dimensions of Operators with a large R Charge and Integrability

D.1 Two Point Functions

In this Appendix we will compute the two point functions used in section 10.1. We only want these two point functions to the leading order in an $M+N$ expansion. We will use f_B to denote the product of the weights of Young diagram B. It is straight forward to obtain

$$f_B = \frac{G_2(N+M+1)}{G_2(N+1)G_2(M+1)}\,, \tag{D.1}$$

where $G_2(n+1)$ is the Barnes function defined by ($\Gamma(z)$ is the Gamma function)

$$G_2(z+1) = \Gamma(z)G_2(z)\,. \tag{D.2}$$

In particular, for an integer $z=n$ we have

$$G_2(n+1) = \prod_{k=1}^{n-1} k!\,. \tag{D.3}$$

First consider the free field theory two point function

$$\left\langle \chi_B(Z)\chi_B(Z^\dagger)\mathrm{Tr}\,(Y^2 Z^{J_0})\mathrm{Tr}\,(Y^2 Z^{J_0})^\dagger \right\rangle. \tag{D.4}$$

According to section 8, this two point function is given, at the leading order in a large $M+N$ expansion, by f_B times $\left(\frac{M+N}{N}\right)^{J_0}$ times the free field theory two point function in the trivial background

$$\left\langle \mathrm{Tr}\,(Y^2 Z^{J_0})\mathrm{Tr}\,(Y^2 Z^{J_0})^\dagger \right\rangle = N^{J_0+2} + O(N^{J_0})\,. \tag{D.5}$$

Thus,

$$\left\langle \chi_B(Z)\chi_B(Z^\dagger)\mathrm{Tr}\,(Y^2 Z^{J_0})\mathrm{Tr}\,(Y^2 Z^{J_0})^\dagger \right\rangle = f_B(N+M)^{J_0}N^2\,. \tag{D.6}$$

Arguing in exactly the same way, we find

$$\left\langle \chi_B(Z)\chi_B(Z^\dagger)\mathrm{Tr}\,(YZ^nYZ^{J_0-n})\mathrm{Tr}\,(YZ^mYZ^{J_0-m})^\dagger \right\rangle = f_B(N+M)^{J_0}N^2\delta_{mn}\,. \tag{D.7}$$

Next, consider

$$\left\langle \chi^{(1)}_{B,B'}(Z,YZ^{J_0+1}Y)\chi^{(1)}_{B,B'}(Z,YZ^{J_0+1}Y)^\dagger \right\rangle \tag{D.8}$$

where

$$\chi^{(1)}_{B,B'}(Z,YZ^{J_0+1}Y) = \left(YZ^{J_0+1}Y\right)_{ij}\frac{\partial}{\partial Z_{ij}}\chi_B(Z)\,. \tag{D.9}$$

These correlators have been computed in [31]. The result is

$$\left\langle \chi^{(1)}_{B,B'}(Z,W)\chi^{(1)}_{B,B'}(Z^\dagger,W^\dagger) \right\rangle = \frac{\mathrm{hooks}_B}{\mathrm{hooks}_{B'}} f_B F_0 + c_{BB'} f_B F_1\,. \tag{D.10}$$

We find $c_{BB'} = M$ and $\frac{\mathrm{hooks}_B}{\mathrm{hooks}_{B'}} = MN$. Further, for the open string word $W = YZ^{J_0+1}Y$ we find that at the leading order

$$F_0 = N^{J_0+2}\left(\frac{M+N}{N}\right)^{J_0+1}, \qquad F_1 \sim N^{J_0-1}\left(\frac{M+N}{N}\right)^{J_0+1}, \tag{D.11}$$

so that, to the leading order we have

$$\left\langle \chi^{(1)}_{B,B'}(Z,YZ^{J_0+1}Y)\chi^{(1)}_{B,B'}(Z,YZ^{J_0+1}Y)^\dagger \right\rangle = f_B M N^2 (M+N)^{J_0+1}\,. \tag{D.12}$$

This correlator result can also be written as

$$\left\langle \chi_B(Z)\chi_B(Z^\dagger)\mathrm{Tr}\,(YZ^{-1}YZ^{J_0+1})\mathrm{Tr}\,(YZ^{-1}YZ^{J_0+1})^\dagger \right\rangle = f_B\frac{1}{M}(N+M)^{J_0+1}N^2\,. \tag{D.13}$$

Again arguing in exactly the same way, we find

$$\left\langle \chi_B(Z)\chi_B(Z^\dagger)\mathrm{Tr}\,(YZ^{-n}YZ^{J_0+n})\mathrm{Tr}\,(YZ^{-m}YZ^{J_0+m})^\dagger \right\rangle = f_B\frac{1}{M^n}(N+M)^{J_0+n}N^2\delta_{mn}\,. \tag{D.14}$$

Notice that at large enough M that $\frac{N}{M}$ can be neglected, all the gauge invariant operators considered have exactly the same two point function.

Finally, by using the methods of this Appendix, we can obtain the general result

$$\langle \mathcal{O}_B(\{p\}, J_0) \mathcal{O}_B(\{p\}, J_0)^\dagger \rangle = N^2 f_B \frac{(M+N)^{J_0+p_-}}{M^{p_-}}\,, \tag{D.15}$$

where $\{p\}$ denotes the occupation numbers of the Cuntz chain and p_- is negative the sum of all the negative occupation numbers. Thus, we have the correspondence

$$\mathcal{O}_B(\{p\}, J_0) \leftrightarrow \sqrt{N^2 f_B \frac{(M+N)^{J_0+p_-}}{M^{p_-}}} |\{p\}\rangle\,, \tag{D.16}$$

between operators and normalized Cuntz lattice states.

D.2 More on the Cuntz Chain

To specify the general Cuntz chain model (10.61) one needs to specify the expected number of Cuntz particles $n_z(\sigma)$. Given $n_z(\sigma)$, what is the corresponding supergravity background? Using the results of [78] as well as (10.61), the metric on the $y = 0$ plane and the circle along which the string moves (parametrized by[27] φ) can be written as

$$ds^2 = -h^{-2}(Dt)^2 + h^2 dz d\bar{z} + h^{-2} d\varphi^2\,, \qquad Dt = dt - \frac{1}{2} i\bar{V} dz + \frac{1}{2} iV d\bar{z}\,, \tag{D.17}$$

$$V = \frac{n_z}{\bar{z}}, \qquad h^4 = \frac{\partial V}{\partial z}, \qquad z = re^{i\phi}\,. \tag{D.18}$$

[27] The angular momentum along this circle is due to the Y fields appearing in the gauge invariant operator dual to the string.

D.3 Explicit Expressions for the Two Loop Dilatation Operator

In these expressions hatted indices are again to be dropped, $\theta(x) = 1$ if $x > 0$ and vanishes otherwise. To obtain this result we have assumed $J_0 > 0$, $J_0 - p \geq 0$ and $p \geq 0$ - assumptions which can easily be relaxed if need be.

$$\delta D_{4,0\ \text{eff}} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4(\mathcal{O}_1^{J_0;J_1,\ldots,J_k} - \mathcal{O}_0^{J_0;J_1,\ldots,J_k}) \times \\ \times N(N+M)(\delta_{p=0} + \delta_{p=J_0} - \delta_{p=1} - \delta_{p=J_0-1})\,, \tag{D.19}$$

$$\delta D_{4,+\ \text{eff}} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4(M+2N)\theta(p-1) \times \\ \times (\mathcal{O}_0^{J_0-p+1;J_1,\ldots,J_k,p-1} - \mathcal{O}_1^{J_0-p+1;J_1,\ldots,J_k,p-1}) \\ +4(M+2N)\theta(J_0-p-1)(\mathcal{O}_0^{p+1;J_1,\ldots,J_k,J_0-p-1} - \mathcal{O}_1^{p+1;J_1,\ldots,J_k,J_0-p-1}) \\ +4(N+M)\theta(p)(\mathcal{O}_1^{J_0-p;J_1,\ldots,J_k,p} - \mathcal{O}_0^{J_0-p;J_1,\ldots,J_k,p}) \\ +4(N+M)\theta(J_0-p)(\mathcal{O}_1^{p;J_1,\ldots,J_k,J_0-p} - \mathcal{O}_0^{p;J_1,\ldots,J_k,J_0-p}) \\ + \sum_{s=1}^{J_0-1} 4N(\delta_{p=0} + \delta_{p=J_0})(\mathcal{O}_1^{J_0-s;J_1,\ldots,J_k,s} - \mathcal{O}_0^{J_0-s;J_1,\ldots,J_k,s})\,, \tag{D.20}$$

$$\delta D_{4,-\ \text{eff}} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4N \sum_{j=1}^{k} J_j\,(\delta_{p=0} + \delta_{p=J_0}) \times \\ \times (\mathcal{O}_1^{J_0+J_j;J_1,\ldots,\hat{J}_j,\ldots,J_k} - \mathcal{O}_0^{J_0+J_j;J_1,\ldots,\hat{J}_j,\ldots,J_k}) \\ - 4N(\delta_{p=0} + \delta_{p=J_0}) \sum_{i=1}^{k} \delta_{J_i=1}(\mathcal{O}_1^{J_0+1;J_1,\ldots,\hat{J}_i,\ldots,J_k} - \mathcal{O}_0^{J_0+1;J_1,\ldots,\hat{J}_i,\ldots,J_k})\,, \tag{D.21}$$

$$\delta D_{4,+-\ \text{eff}} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4\theta(p) \sum_{i=1}^{k} J_i(\mathcal{O}_1^{J_0+J_i-p;J_1,\ldots,\hat{J}_i,\ldots,J_k,p} \\ -\mathcal{O}_0^{J_0+J_i-p;J_1,\ldots,\hat{J}_i,\ldots,J_k,p}) + 4\theta(J_0-p) \sum_{i=1}^{k} J_i(\mathcal{O}_1^{J_i+p;J_1,\ldots,\hat{J}_i,\ldots,J_k,J_0-p}$$

$$-\mathcal{O}_0^{J_i+p;J_1,\ldots,\hat{J}_i,\ldots,J_k,J_0-p}) - 4\theta(J_i-1)\sum_{i=1}^{k} J_i(\mathcal{O}_1^{p+1;J_1,\ldots,\hat{J}_i,\ldots,J_k,J_0+J_i-p-1}$$

$$-\mathcal{O}_0^{p+1;J_1,\ldots,\hat{J}_i,\ldots,J_k,J_0+J_i-p-1}) - 4\theta(J_i-1)\sum_{i=1}^{k} J_i(\mathcal{O}_1^{J_0-p+1;J_1,\ldots,\hat{J}_i,\ldots,J_k,J_i+p-1}$$

$$- \mathcal{O}_0^{J_0-p+1;J_1,\ldots,\hat{J}_i,\ldots,J_k,J_i+p-1})\,, \tag{D.22}$$

$$\delta D_{4,++\ \mathrm{eff}}\mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4\sum_{r=1}^{p-2}(\mathcal{O}_0^{J_0-p+1;J_1,\ldots,J_k,r,p-r-1}$$

$$-\mathcal{O}_1^{J_0-p+1;J_1,\ldots,J_k,r,p-r-1}) + 4\sum_{r=1}^{J_0-p-2}(\mathcal{O}_0^{p+1;J_1,\ldots,J_k,r,J_0-p-r-1}$$

$$-\mathcal{O}_1^{p+1;J_1,\ldots,J_k,r,J_0-p-r-1}) + 4\theta(p)\sum_{s=1}^{J_0-p-1}(\mathcal{O}_1^{J_0-p-s;J_1,\ldots,J_k,s,p}$$

$$-\mathcal{O}_0^{J_0-p-s;J_1,\ldots,J_k,s,p}) + 4\theta(J_0-p)\sum_{s=1}^{p-1}(\mathcal{O}_1^{p-s;J_1,\ldots,J_k,s,J_0-p}$$

$$- \mathcal{O}_0^{p-s;J_1,\ldots,J_k,s,J_0-p})\,, \tag{D.23}$$

$$\delta D_{4,--\ \mathrm{eff}}\mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 0\,. \tag{D.24}$$

Setting $M = 0$ in the above expressions gives exact agreement with Appendix E of [35] except for the last term in our expression for $\delta D_{4,-}$. The extra term that we have ensures that no joinings between the trace with the Ys and a trace without Ys and a single Z can occur. In this case, the $\frac{d}{dZ_{ij}}$ in $\delta D_{4\ \mathrm{eff}}$ acts on $\mathrm{Tr}\,(Z)$ to produce δ_{ij}. This vanishes because $\frac{d}{dZ_{ij}}$ appears inside a commutator; the extra term is needed.

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