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Spectra of the Excited Giant Gravitons From the Two loop Dilatation Operator

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Declaration

I declare that the work contained in this thesis is my work and that any work done previously by others or by myself has been acknowledged and referenced accordingly. This thesis is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg and it has not been submitted before at any other tertiary institution.

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06 June 2016

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Abstract

The AdS/CFT correspondence is a conjectured exact duality between type $\mathbf{II}$B string theory on the $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ Super Yang-Mills theory, a conformal field theory (CFT), on the boundary of the AdS space. A specific observable of the CFT, which can be read from the two point correlation function, is the anomalous dimension. In this dissertation we will compute spectra of anomalous dimensions of excited giant gravitons up to two loops in a specific limit. We are interested in the anomalous dimensions because the AdS/CFT correspondence associates them with energies of states in quantum gravity. We study operators constructed using n Z fields and m Y fields with $n << m$. In this case $\frac{m}{n}$ is a small parameter. At the leading order in $\frac{m}{n}$ and at large N, the problem of determining the anomalous dimensions can be mapped into the dynamics of m non-interacting magnons. The subleading terms at two loops, computed for the first time in this dissertation, induce interactions between the magnons. Even after including this new correction, we find the BPS operators remain BPS.

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1. Introduction

An important outstanding problem in theoretical physics is to provide a non-perturbative definition of string theory valid in all super selection sectors or compactifications and for all values of the string coupling and all other moduli. There is no guarantee that a single definition exists. Indeed, there may be profound reasons (related to duality) for why string theory will always be known as a collection of partial descriptions that may, at best, locally be patched together. However, the construction of this non-perturbative definition if it exists, would allow one to approach questions about the size of the string landscape and the existence of a vacuum that matches the observed Universe in a completely rigorous mathematical way. It would also give some appreciation into what string theory is. Gaining some insight into whether or not this definition exists and how it may be constructed, is the primary motivation for the research of this dissertation.

A particularly promising approach to this question is provided by the gauge / gravity duality and more generally, by holography [1]. The AdS / CFT correspondence [2], [3], [4] provides a concrete example. Indeed, using the original example of the AdS/CFT correspondence, we can define a class of quantum type IIB string theories: those that are embedded in spacetimes which are asymptotically $AdS_5 \times S^5$ with background five form flux. The definition for this class of string theories is in terms of the highly symmetric superconformal four dimensional $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $U(N)$. The definition provides a one-to-one and onto mapping between states of the quantum gravity and quantum operators of the gauge theory. In the mapping we must find all the objects known to string theory, perturbative and non-perturbative. Thus, for example, in addition to the perturbative spectrum of closed strings, we must find D-branes and their open string excitations, as well as other spacetime geometries, living in the gauge theory.

It is well known that the quantization of the closed string in $AdS_5 \times S^5$ is a non-trivial problem. This is a consequence of the complexity and non-linearity of the equations of motion of the sigma model. This difficulty is however, completely resolved using the integrability of the string sigma model which is a powerful tool to solve the model and determine the spectrum of closed string states [5]. By now, using integrability, the string sigma model has been solved for a large class of closed string backgrounds in terms of algebraic curves in a complex manifold. Quantum fluctuations of these algebraic curves provide information about the excited closed string states. For a nice review see [6].

The spectrum of $\mathcal{N} = 4$ super Yang-Mills theory is computed by the eigenvalues of the dilatation operator. This spectrum has been determined exactly in the planar limit. The key ingredient is again integrability: the dilatation operator can be identified with the Hamiltonian of an integrable spin chain [7] . By now there is overwhelming evidence that planar $\mathcal{N} = 4$ super Yang-Mills theory is integrable to all orders in perturbation theory and the spectrum of anomalous dimensions is calculated by a Bethe ansatz for an integrable $\mathrm{psu}(2,2|4)$ spin chain. In the thermodynamic limit the solutions to the Bethe ansatz of the gauge theory are the algebraic curves of the closed string. This proves that we should identify the planar limit of $\mathcal{N} = 4$ super Yang-Mills theory with classical string theory on $AdS_5 \times S^5$. Given our long term objective of constructing a non-perturbative definition of string theory, it is natural to ask whether integrability carries over to non-perturbative sectors of the correspondence. For non-maximal giant graviton branes the boundary Yang-Baxter equations are not satisfied [8] which indicates that integrability is probably not present.

Even in the absence of integrability for the open string it is possible that there are other structures that replace it. These additional structures would give powerful insights into the non-perturbative dynamics of the string theory. The search for these possible structures motivates the research questions posed by

this study. Concretely, in this dissertation we will compute certain subleading contributions to the two loop dilatation operator. The leading contributions at one loop and two loops have been computed [9], [10], [11] and it is now clear that the leading contribution corresponds to a dilute gass of non-interacting magnons [8]. The terms we compute correspond to the first interactions of the magnons, and hence these are important to establish the character of the interacting magnon system.

The first hints of the gauge / gravity duality appeared in 't Hooft's work demonstrating that Yang-Mills theories admit a large N expansion [12]. The perturbative sector of the string theory is captured by summing planar ribbon diagrams. In this sector, states of the quantum gravity map into quantum field theory operators that have a bare dimension which grows at most as the square root of N as we take N to infinity. Ribbon diagrams of higher genus encode $1/N$ corrections. This provides convincing evidence that ribbon graphs are to be identified with string worldsheets and $1/N$ is to be identified with the string coupling. The non-perturbative sector of the string theory maps into operators whose dimensions always grows faster than the square root of N as we take N to infinity. This implies that the planar diagrams no longer dominate the large N expansion and the usual organization of the $1/N$ expansion in terms of the genus of ribbon graphs breaks down. Standard large N expansion methods are no longer useful and a new approach to the problem must be developed. The approach taken in the investigation of this proposal makes heavy use of group representation theory that has manifested a rich interplay between combinatorics, geometry and topology. In one example of this approach a basis for the local operators of the theory, the so-called restricted Schur polynomials, are constructed. The construction of these operators uses intertwining maps that map between representations of subgroups of the symmetric group. Observables of the gauge theory, including the dilatation operator, can be expressed as simple expressions in terms of these intertwining operators. This approach allows one to exhibit remarkable and, at first sight, completely unexpected simplifications. For example, the computation of the spectrum of anomalous dimensions of operators dual to excited giant graviton branes is reduced to the problem of diagonalizing collections of decoupled harmonic oscillators [9]. The origin of these oscillators in the dual gravitational picture can be traced back to magnon excitations of the half BPS sector [8].

In the remainder of the first chapter, we review additional background material which motivates and informs this study. In the second chapter of this dissertation, the relevant methods and techniques from group representation theory are reviewed. Chapter three explains how these methods are used to evaluate the one loop dilatation operator and chapter 4 does the same for the two loop dilatation operator. In chapter 5 we discuss our results and in chapter 6 we conclude.

1.1 Anti-de Sitter (AdS) spacetime

Low energy gravity is described by Einstein's theory of general relativity (GR). This theory describes the gravitational interaction in terms of the curvature of spacetime. Curvature can be zero, positive or negative. The spacetime with a constant negative curvature $R < 0$ is called anti-de Sitter spacetime, or simply AdS spacetime [13]. In this space the sum of the interior angles of a triangle, will be less than 180 degrees. AdS spacetime has a boundary. According to the AdS/CFT correspondence, quantum gravity in AdS spacetime is completely equivalent to the dynamics of a conformal field theory which lives on the boundary of the spacetime. In order to properly appreciate the correspondence, we need to know the basics of the geometry of AdS space. In this section we will review these basics.

The AdS space is an exact solution of Einstein's field equations [13], with Lorentzian signature $(-++\cdots+)$. In 5 dimensions, we can visualize this geometry as the hyperboloid. We consider a spacetime with two timelike dimensions, with co-ordinates X^0 and X^5 and four spatial dimensions with co-ordinates

$\vec{X}$. Anti-de Sitter space AdS_5 is the surface

$$-(X^0)^2+(X^1)^2+\cdots+(X^4)^2-(X^5)^2=-R^2, \tag{1.1}$$

embedded in this six dimensional space. R is the radius of curvature and it is a constant.

The metric of the AdS space is written in different forms, depending on the choice of co-ordinate system. The metric of the six dimensional embedding space is

$$ds^2=-d(X^0)^2+d(X^1)^2+d(X^2)^2+d(X^3)^2+d(X^4)^2-d(X^5)^2. \tag{1.2}$$

We can introduce global co-ordinates to solve (1.1). Global co-ordinates are defined by

$$\begin{aligned} X^0&=R\cosh\rho\cos\tau, \quad X^5=R\cosh\rho\sin\tau,\\ X^1&=R\sinh\rho\cos\theta, \quad X^2=R\sinh\rho\sin\theta\cos\phi,\\ X^3&=R\sinh\rho\sin\theta\sin\phi\cos\psi, \quad X^4=R\sinh\rho\sin\theta\sin\phi\sin\psi, \end{aligned} \tag{1.3}$$

where ρ is dimensionless, $\rho\geq 0$, $0\leq\theta\leq\pi$, $0\leq\phi\leq 2\pi$, $0\leq\psi\leq\pi$ and $0\leq\tau\leq\infty$. Global co-ordinates cover the entire hyperboloid. The metric induced on the hyperboloid is

$$ds^2=\eta_{ij}dX^idX^j=\eta_{ij}\frac{\partial X^i}{\partial\zeta^\alpha}\frac{\partial X^j}{\partial\zeta^\beta}d\zeta^\alpha d\zeta^\beta, \tag{1.4}$$

where $i,j=0,1,\cdots,5$, $\alpha,\ \beta=0,1,\cdots,4$, and $\zeta^0=\rho$, $\zeta^1=\tau$, $\zeta^2=\theta$, $\zeta^3=\phi$ and $\zeta^4=\psi$. We call $d\theta^2+\sinh^2\phi\left(d\phi^2+\sin^2\psi d\psi^2\right)=d\Omega_3^2$ which is the metric on the unit three sphere. The induced metric is then

$$ds^2=R^2\left(-\cosh^2\rho d\tau^2+d\rho^2+\sinh^2\rho d\Omega_3^2\right). \tag{1.5}$$

which is the metric of AdS_5.

The second theory participating in the duality, conformal field theory (CFT), is a quantum field theory that is invariant under conformal symmetry. Conformal symmetry is given by the Poincaré group, special conformal transformations and dilatations. The conformal transformation is a transformation that preserves the angle between any two vectors but not necessarily the sizes of the vectors. In next section we will discuss these types of transformations in detail.

1.2 $SO(4,2)$, the conformal group

The conformal group is the group of conformal transformations. A conformal transformation of the coordinates is an invertible mapping $x\to x'$, which leaves the metric tensor invariant up to a scale[14]

$$g'_{\mu\nu}(x')=\Lambda(x)g_{\mu\nu}(x).$$

The conformal group is a Lie group. As for any Lie group, the elements of the conformal group depend smoothly on the set of parameters of the conformal transformations. There are four displacements (a^μ) parametrizing translations, six dimensionless parameters $(\omega^{\mu\nu})$ parametrizing Lorentz boosts and rotations, one dimensionless parameter parametrizing scaling and four inverse length parameters, parametrizing special conformal transformations. By studying infinitesimal group transformations, we

arrive at the Lie algebra. It is good enough to study infinitesimal transformations, since finite transformations can be realized by composing an infinite number N of infinitesimal transformations. To illustrate this point, study translations in one dimension. Under an infinitesimal displacement

$$f(x) \to f(x+\epsilon) = \left(1 + \epsilon \frac{d}{dx}\right) f(x).$$

We can write this as

$$f(x+\epsilon) = (1 + i\epsilon P) f(x)$$

where

$$P = -i\frac{d}{dx}.$$

We say that P is the generator of translations. There is one generator for every parameter in the Lie group. Thus the conformal group has 15 generators.

Lets get a finite transformation by composing N translations, each by an amount $\epsilon = \frac{a}{N}$. We take the limit $N \to \infty$ with a held fixed. Set

$$u = \lim_{N\to\infty} \left(1 + \frac{a}{N}\frac{d}{dx}\right)^N.$$

Note that

$$\begin{aligned}
\log(u) &= N \log\left(1 + \frac{a}{N}\frac{d}{dx}\right) \\
&= N\left(\frac{a}{N}\frac{d}{dx} + O\left(\frac{1}{N^2}\right)\right) \\
&= a\frac{d}{dx} + O\left(\frac{1}{N}\right).
\end{aligned}$$

Thus, in the $N \to \infty$ limit we have

$$\log(u) = a\frac{d}{dx} \quad \Rightarrow u = e^{a\frac{d}{dx}}$$

Now, using the Taylor expansion we find

$$e^{a\frac{d}{dx}} f(x) = \left(1 + a\frac{d}{dx} + \frac{a^2}{2!}\frac{d^2}{dx^2} + \cdots\right) f(x) = f(x+a) + O(a^2)$$

which proves that, we have managed to construct a finite translation by composing an infinite number of infinitesimal translations. We can define the translation operator in one dimension as

$$T(a) = e^{iaP}.$$

The relation given above is perfectly general: group elements are always obtained by exponentiating elements of the Lie algebra. We can generalize this result to $d-$ dimensional spacetime as follows. We

change $a \to a^\mu$, $x \to X^\mu$ and $\frac{d}{dx} \to \frac{\partial}{\partial X^\mu}$. Then we can write $T_{a^\mu} = e^{ia^\nu P_\nu}$, where μ, $\nu = 0, 1, \cdots d-1$. Under a translation

$$T_{a^\mu} : X^\mu \to X'^\mu = X^\mu + a^\mu. \tag{1.6}$$

Now consider infinitesimal Lorentz transformations (rotations and boost), under which

$$X^\mu \to X'^\mu = X^\mu + i\omega^\mu_\nu X^\nu = \left(1 + i\omega^\rho_\sigma L^\sigma_\rho\right) X^\mu \tag{1.7}$$

where L^μ_ν are the generators of rotations and boosts

$$L_{\mu\nu} = i(x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu})$$

and the parameters define an anti-symmetric tensor

$$\omega_{\mu\nu} = -\omega_{\nu\mu}.$$

To motivate the form of the above generators, consider the rotations. To be concrete we will discuss a rotation around the x axis in detail and then state the general result which is an obvious generalization. A rotation about the x axis of a four vector X^μ can be written as

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}.$$

For an infinitesimal rotation, set $\theta = \epsilon$ so that $\cos\epsilon = 1 - \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{4!} - \cdots$ and $\sin\epsilon = \epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} - \cdots$. Keeping only the linear order in ϵ, we find

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\epsilon \\ 0 & 0 & \epsilon & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

or, equivalently

$$t' = t, \quad x' = x, \quad y' = y + \epsilon z, \quad z' = z - \epsilon y. \tag{1.8}$$

It is now clear that this transformation is generated by the following matrix $L_x = (-ix_3\frac{\partial}{\partial x_2} + ix_2\frac{\partial}{\partial x_3})$, which is the angular momentum from non-relativistic quantum mechanics. Using the same logic, we find the generators of rotations around y, z axes are

$$L_y = (-ix_3\frac{\partial}{\partial x_1} + ix_1\frac{\partial}{\partial x_3}), \qquad L_z = (-ix_1\frac{\partial}{\partial x_2} + ix_2\frac{\partial}{\partial x_1}). \tag{1.9}$$

A boost along the x direction is implemented by the following matrix

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}.$$

Infinitesimally, we can write $\phi = \epsilon$, $\cosh\epsilon = 1 + \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{4!} + \cdots$ and $\sinh\epsilon = \epsilon + \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} + \cdots$. To obtain the generator we only need to work to first order in ϵ, so that we find

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -\epsilon & 0 & 0 \\ -\epsilon & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}.$$

This can also be written as

$$t' = t - \epsilon x, \quad x' = x - \epsilon t, \quad y' = y, \quad z' = z. \tag{1.10}$$

This result implies that the generator of boosts along the $x-$ direction is given by $M_{01} = (-ix_1\frac{\partial}{\partial x_0} + ix_0\frac{\partial}{\partial x_1})$. By the same logic we can write the generators of boosts along the y, z directions as follows

$$M_{02} = (-ix_2\frac{\partial}{\partial x_0} + ix_0\frac{\partial}{\partial x_2}), \qquad M_{03} = (-ix_3\frac{\partial}{\partial x_0} + ix_0\frac{\partial}{\partial x_3}). \tag{1.11}$$

Next we would like to obtain the generator of scale transformations, which is the dilatation operator. A finite scale transformation acts on the co-ordinate X^μ as follows

$$X^\mu \to X'^\mu = e^a X^\mu$$

The dilatation is a scaling of a vector X^μ, and it is defined as follows

$$X'^\mu \to e^a X^\mu$$

Infinitesimally, using the Taylor expansion around the identity group element, we find

$$\begin{aligned} X'^\mu &= (1+\epsilon)X^\mu, \\ X^\mu + \epsilon X^\mu &= X^\mu - iDX^\mu. \end{aligned}$$

The above equation implies that

$$\epsilon X^\mu = -iDX^\mu \Rightarrow D = i\epsilon\partial, \tag{1.12}$$

where the operator $-iD$ is the dilatation generator.

Finally, to obtain the generator of the special conformal transformation, it is useful to consider a discrete conformal transformation, known as inversion, defined by:

$$X^\mu \to X'^\mu = \frac{X^\mu}{X^2}.$$

The special conformal transformations can be constructed by performing a translation, preceded and proceeded by an inversion:

$$X_\mu \to X'^\mu = \frac{X^\mu}{X^2} \to \frac{X^\mu}{b^\mu} + b^\mu = \frac{\frac{X^\mu}{X^2} + b^\mu}{(\frac{X^\mu}{X^2} + b^\mu)^2} = \frac{X^\mu + b^\mu X^2}{1 + b^2 + 2b \cdot X} = X^\mu + (b^\mu X^2 - 2(b \cdot X)X^\mu). \tag{1.13}$$

From the above expression we easily obtain the following generator of special conformal transformations

$$iX^2 \frac{\partial}{\partial X^\mu} - 2iX_\mu X \cdot \frac{\partial}{\partial X}.$$

There are an infinite number of group elements in the conformal group $SO(4,2)$ but only 15 generators in the conformal algebra $\mathfrak{so}(4,2)$. Thus, it is much simple to study the Lie algebra than it is to study the Lie group. To proceed, we need to explain what replaces that group multiplication structure, at the level of the algebra. To develop this connection, we can write the group multiplication law as

$$g(x'_1, \cdots, x'_n)g(x_1, \cdots, x_n) = g(f_1(x'_1, \cdots, x'_{n,1}, \cdots, x_n), \cdots f_n((x'_1, \cdots, x'_{n,1}, \cdots, x_n)). \tag{1.14}$$

The parameters x_i label the group elements. They could be a translation or any other parameter of the $SO(4,2)$ group elements. The functions $f_1(x'_1, \cdots, x'_{n,1}, \cdots, x_n)$ are smooth functions of the coordinates x_i and the x'_i . We will choose our labels so that

$$e = g(x_1 = 0, x_2 = 0, \cdots, x_n = 0),$$

with e the identity of the group. The above equation implies that

$$f_i(x_1, \cdots, x_n, 0, \cdots, 0) = f_i(0, \cdots, 0, x_1, \cdots, x_n) = x_i.$$

Thus, to second order in the x_i and x'_i we have

$$f_i(x_1, x_2, \cdots, x_{n-1}, x_n) = x_i + x'_i + c_{abc}x_b x_c + \cdots \tag{1.15}$$

Expand our group elements, around the identity, to second order in the x_as to obtain

$$g = 1 + ix_a T_a + x_b x_c T_{bc} + \cdots \tag{1.16}$$

In this expression, repeated indices are summed as usual. T_a are the generators of the group. T_{bc} is contracted with a symmetric object $(x_b x_c)$ so that only its symmetric piece contributes and we have $T_{bc} = T_{cb}$. Thus, from now on we assume that T_{bc} is a symmetric object. The group multiplication law can now be written as

$$\begin{aligned} &\left(1 + ix'_a T_a + \frac{1}{2}x'_b x'_c T_{bc} + \cdots\right) \times \left(1 + ix_d T_d + \frac{1}{2}x_e x_f T_{ef} + \cdots\right) \\ &= 1 + ix'_a T_a + ix_a T_a + \frac{1}{2}x_a x_b T_{ab} + \frac{1}{2}x'_a x'_b T_{ab} - x'_a x_b T_a T_b + \cdots \end{aligned} \tag{1.17}$$

From equations (1.14) and (1.15) we obtain

$$\begin{aligned} &1 + if_a(x_1, \cdots, x'_n)T_a + \frac{1}{2}f_a(x_1, \cdots, x'_n)f_b(x_1, \cdots, x_n)T_{ab} \\ &= 1 + i(x_a + x'_a + c_{abc}x'_b x_c + \cdots)T_a + \frac{1}{2}(x_b + x'_b + \cdots)(x_a + x'_a \cdots)T_{ab} \\ &= 1 + ix'_a T_a + ix_a T_a + \frac{1}{2}x_a x_b T_{ab} + \frac{1}{2}x'_a x'_b T_{ab} + x_a x'_b T_{ab} + iT_a c_{abc} x_b x'_c + \cdots \end{aligned} \tag{1.18}$$

By comparing equation (1.17) and equation (1.18) learn that

$$T_{ba} = -T_a T_b - iT_e c_{eab}.$$

We have already argued that T_{ab} is a symmetric object, so that for consistency we must require $T_{ab} - T_{ba} = 0$. This implies that

$$T_a T_b - T_b T_a = -iT_e(c_{eab} - c_{eba}) \equiv if_{abe}T_e.$$

Thus, the generators obey the following algebra

$$[T_a, T_b] = -if_{abc}T_c. \tag{1.19}$$

The generators of all Lie algebras must obey equation (1.19), which is called the Lie algebra of the Lie group. Notice that if the generators are hermittian, the commutator $[T_a, T_b]$ is antihermittian. For the right hand side of the algebra to be antihermittian, the f_{abc} must be real numbers. These numbers are called the structure constants of the Lie algebra. The structure constants can be computed using any representation, they are a property of the group and not of a specific representation.

Using the generators we obtained above, we can now compute the $\mathfrak{so}(4,2)$ algebra. We find that the generators of $SO(4,2)$ obey the following commutation relations

$$\begin{gathered} [D, P_\mu] = iP_\mu, \qquad [D, K_\mu] = -iK_\mu, \\ [K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}), \quad [K_\mu, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu), \\ [P_\rho, L_{\mu\nu}] = 2i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu), \quad [L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}). \end{gathered} \tag{1.20}$$

In fact, these relations define the Lie algebra of the conformal transformations [14].

Conformal symmetry implies powerful constraints on the correlation functions of the theory. The correlation function between two fields $\mathcal{O}_i$ that have a definite scaling dimension $\triangle_i$ takes the form

$$\left\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \right\rangle = \frac{\delta_{\triangle_1 \triangle_2}}{|x_1 - x_2|^{2\triangle_1}}.$$

The dimensions $\triangle_i$ are the eigenvalues of the dilatation operator D. They admit an expansion in power of $\hbar$. Computing the eigenvalues of D in a particular limit of $\mathcal{N} = 4$ super Yang Mills theory is one of the main goals of this dissertation. To understand why the dimensions of operators are an interesting quantity, we need to relate them to quantities in the dual quantum gravity. That is goal of the next section.

1.3 Relating energies in quantum gravity to dimensions in CFT.

The field theory lives on the background of AdS space. Looking back at the AdS_5 metric given in equation (1.5), we see that the boundary (realized at $\rho \to \infty$) has metric

$$ds^2 = \frac{R^2 e^{2\rho}}{4}\Big(- d\tau^2 + d\Omega_3^2\Big),$$

which is the metric of $\mathbb{R} \times S^3$. The theory is invariant under time translations $\tau \to \tau + a$ and the corresponding conserved quantity is the energy of states in the quantum gravity theory.

The (Euclidean) CFT is defined on $\mathbb{R}^4$ with the metric

$$ds^2 = dr^2 + r^2 d\Omega_3^2.$$

Scaling $r \to e^{\lambda} r$ is a symmetry of the theory. The corresponding conserved quantities are the scaling dimensions of the states (or operators). By changing co-ordinates from r to $\tau = \ln(r)$ we find

$$ds^2 = e^{2\tau}\big(d\tau^2 + d\Omega_3^2\big).$$

Now, performing a scaling (which is a symmetry of the theory) we obtain

$$ds^2 = d\tau^2 + d\Omega_3^2$$

which is $\mathbb{R} \times S^3$. Wick rotating we find

$$ds^2 = -d\tau^2 + d\Omega_3^2$$

which matches the metric of the boundary of AdS_5. Notice that translating τ induces a scaling of r, since under $\tau \to \tau + a$ we have

$$r = e^{\tau} \to e^{\tau + a} = e^a \cdot e^{\tau} = e^a \cdot r.$$

Identifying the corresponding conserved quantities tells us that energies in quantum gravity map into scaling dimensions in the CFT. Thus, the computation of dimensions we carry out is teaching us about the energy spectrum of quantum gravity. In the next section we introduce the specific CFT that plays a role in the AdS/CFT correspondence.

1.4 $\mathcal{N} = 4$ SYM Theory

$\mathcal{N} = 4$ SYM is the unique $4-$dimensional quantum filed theory that contains the maximum supersymmetry theory without including the gravitational interaction. In order to derive the Lagrangian of $\mathcal{N} = 4$ super Yang-Mills theory it is convenient to start from $\mathcal{N} = 1$ super Yang-Mills theory in ten dimensions and then dimensionally reduce it to four dimensions. If we do this we obtain the following Lagrangian [15]

$$\begin{aligned}\mathcal{L} =& \frac{N}{4\lambda} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2}\sum_{i=1}^{3} (D_\mu A_i)^a (D^\mu A_i)^a + \frac{1}{2}\sum_{i=1}^{3} (D_\mu B_i)^a (D^\mu B_i)^a - V(A_i, B_j) \\ & - \frac{i}{2}\left(\bar{\psi}\right)^a \gamma^\mu \left(D_\mu\right)^a - \frac{\lambda}{2N} f^{abc} \bar{\psi}^a \alpha^i A^{bi} \psi^c - i\frac{\lambda}{2N} f^{abc} \bar{\psi}^a \beta^j \gamma_5 B^{bj} \psi^c\end{aligned} \tag{1.21}$$

where the potential is:

$$V(A_i,B_j)=\frac{g_{YM}^2}{4}f^{abc}A_i^bA_j^cf^{afg}A_i^fA_j^g+\frac{g_{YM}^2}{4}f^{abc}B_i^bB_j^cf^{afg}B_i^fB_j^g+\frac{g_{YM}^2}{2}f^{abc}A_i^bB_j^cf^{afg}A_i^fB_j^g.$$

It contains a gluon field, four Majorana spinors and six real scalars. They all transform according to the adjoint representation of the gauge group. The $4-$dimensional internal matrices α and β satisfy the following algebra[15]:

$$\{\alpha^i,\alpha^j\}=\{\beta^i,\beta^j\}=2\delta^{ij},\qquad \left[\alpha^i,\beta^j\right]=0,$$

and

$$\left[\alpha^i,\alpha^j\right]=-2\epsilon^{ijk}\alpha^k,\qquad \left[\beta^i,\beta^j\right]=-2\epsilon^{ijk}\beta^k.$$

They can be chosen as follows

$$\alpha^i\equiv\bar{\eta}^i_{AB}=\delta_{iA}\delta_{B4}-\delta_{iB}\delta_{A4}+\epsilon_{iAB4}\quad \beta^i\equiv\bar{\eta}^i_{AB}=\delta_{iA}\delta_{B4}-\delta_{iB}\delta_{A4}+\epsilon_{iAB4}$$

where A,B are four dimensional indices.

$\mathcal{N}=4$ super Yang-Mills theory is invariant under an internal $SU(4)$ symmetry group that is an $R-$symmetry. The generators of spacetime symmetries do not commute with the generators of the spacetime symmetry group. The generators of global symmetries do commute with the spacetime symmetry group. The generators of an $R-$symmetry commute with the bosonic spacetime symmetry but do not commute with the fermionic spacetime symmetry generators. In order to manifestly see this invariance it is convenient to introduce the field:

$$\bar{\Phi}_{AB}=\frac{1}{2\sqrt{2}}\left[\eta^i_{AB}A_i-\bar{\eta}^i_{AB}B_i\right].$$

This antisymmetric field transforms according to the vector representation of $SO(6)$. By rewriting the Lagrangian in equation (1.21) in terms of Φ and in terms of the Weyl spinors, that transform according to the 4 of $SU(4)$, we find

$$\begin{aligned}\mathcal{L}=&-\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu}+\left(D_\mu\bar{\Phi}_{AB}\right)_a\left(D_\mu\Phi^{AB}\right)_a-i\psi_a^{\alpha A}\sigma^\mu_{\alpha\dot{\alpha}}\left(D_\mu\bar{\psi}^{\dot{\alpha}}_A\right)_a\\&-\frac{\lambda}{N}{}_{YM}f^{abc}\Phi_b^{AB}\Phi_c^{CD}f^{ade}\bar{\Phi}^d_{AB}\bar{\Phi}^e_{CD}-\frac{\lambda}{N}\sqrt{2}f^{abc}\left[\psi_a^{\alpha A}\bar{\Phi}^b_{AB}\psi^{cB}_\alpha+\bar{\psi}^a_{\dot{\alpha}A}\Phi_b^{AB}\bar{\psi}^{\dot{\alpha}}_{cB}\right],\end{aligned}\tag{1.22}$$

where λ is the interaction coupling. λ is called the t'Hooft coupling. If we consider a large N limit, we are lead naturally to an expansion in $\frac{1}{N}$ [12]. The Yang-Mills coupling is related to the t'Hooft coupling by

$$g^2_{YM}=\frac{\lambda}{N}.$$

The Lagrangian is manifestly invariant under the $SU(4)$ R-symmetry transformations:

$$\psi^A_\alpha\to U^A_B\psi^B_\alpha,\qquad \bar{\psi}_{\dot{\alpha}A}\to(U)^A_B\bar{\psi}^B_{\bar{\alpha}},$$

and

$$\Phi^{AB} \to U^A_C \Phi^{CD} (U^T)^B_D, \quad \bar{\Phi}_{AB} \to (U)^C_A \bar{\Phi}_{CD} (U^\dagger)^D_B,$$

where U is a unitary matrix $(UU^\dagger) = 1$.

The four-dimensional quantum theory is conformal invariant and it is also invariant under the same number of supersymmetry transformations. We conclude that $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions is invariant under 32 supersymmetries[15].

We can trade the antisymmetric $SO(4)$ index AB for the $SO(6)$ vector index i. The kinetic term becomes

$$\frac{1}{4}\mathsf{Tr}(F_{\mu\nu}, F^{\mu\nu}) + \frac{1}{2}\sum_i \mathsf{Tr}(D_\mu \phi_i, D^\mu \phi_i).$$

Thus the bosonic part of the Lagrangian reduces to that of a gauge field and six adjoint-valued scalars in 4 dimensions, along with the potential

$$V = -\frac{1}{4}\sum_{i,j} \mathsf{Tr}([\phi_i, \phi_j], [\phi_i, \phi_j]).$$

Next we need to look at the fermions. We can write the fermionic term as

$$\frac{i}{2} = \mathsf{Tr}(\bar{\Lambda}, \Gamma^\mu D_\mu \Lambda) + \mathsf{Tr}(\bar{\Lambda}, \Gamma^i [\phi_i, \Lambda]).$$

The second term is a Yukawa-type term in 4 dimensions. The fermions will play no role in what follows and we do not consider them further.

In any quantum field theory, to compute the two point function, we need to perform a sum over Feynman diagrams. We are studying operators composed using a very large number of fundamental fields, so that performing this sum is a very complicated problem. To solve it we need to develop new technology. Our approach to solving this problem, is to develop methods that exploit the full power of group representation theory. Since this plays such an important role in our approach, in the next chapter we will give a brief introduction to these methods, before explaining how they can be used to solve our problem.

Chapters 2 and 3 of this dissertation review existing results. The material in chapters 4, 5 and 6 is new and has submitted to the arXiv (see preprint arXiv: 1512.05019) and to Physical Review D for publication.

2. Mathematical Tools

In this chapter we will discuss different aspects of group theory and the relevant representation theory. These results are the foundations of our study, and this discussion will introduce our conventions and notation.

2.1 Basic Concepts from Group Theory

A group is a set $\mathcal{G}$ with an operation $\circ$ called group composition. The group composition law produces a new group element g given two group elements. The group composition law is associative. The group contains a unique identity element with respect to group composition. Finally, for every element in the group there is a unique inverse element.

The order of the group $|\mathcal{G}|$ is defined as the number of elements in the group. The order of the group can be finite, infinite but countable or infinite and uncountable[16].

A subgroup $\mathcal{B}$ of the group $\mathcal{G}$ is any subset of $\mathcal{G}$ that is itself a group. If the order of $\mathcal{B}$ is less than the order of $\mathcal{G}$, then we have a proper subgroup.

2.2 The Symmetric Group S_n

The Symmetric Group S_n is the set of *permutations* of n elements. A permutation is simply a way of rearranging the members of a set. If we have n distinct objects, then the order of this group is $n!$.

In order to prove that the set of permutations is a group, we can view a permutation as a bijective map from the set of objects to itself. The set of all permutations on a set is then the set of all bijective maps on the set. Group composition is given by map composition. Since map composition is associative, the group composition law is associative. The identity map is a bijective map from the set of objects to itself, so the identity is an element of the group. Finally, after permuting the elements in a set, we can always bring them back to their original order with a bijective map so that for each permutation, the inverse permutation belongs to the group. Thus the set of permutations satisfy all the conditions required to be a group. S_n is a non-commutative group.

Notation for Permutations

To describe a given permutation, we could list the elements of the set being permuted in the first row, and their image under the permutation in the second row. For example, a permutation of the set $[1,2,3,4,5,6]$ denoted by σ, could be described as

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 3 & 5 & 4 & 1 \end{bmatrix}. \tag{2.1}$$

An equivalent description of this permutation is given by $\sigma(1) = 2$, $\sigma(2) = 6$, $\sigma(3) = 3$, $\sigma(4) = 5$, $\sigma(5) = 4$ and $\sigma(6) = 1$. Another way of writing the same permutation is to change how the elements in the first row are ordered

$$\begin{bmatrix} 6 & 3 & 1 & 5 & 4 & 2 \\ 1 & 3 & 2 & 4 & 5 & 6 \end{bmatrix}. \tag{2.2}$$

Equations (2.1) and (2.2) specify the same permutation. The identity permutation is given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}.$$

Here $\sigma(i) = i$ where $i = 1, 2, \cdots, 6$. The permutation (2.1) can also be expressed graphically as shown in figure 2.1.

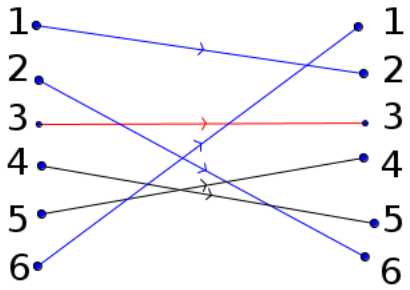


Figure 2.1: A graphical illustration of a permutation.

2.2.1 Cycle notation. A more convenient notation is obtained by expressing the permutation as product of cycles, which are the orbits of the permutation. For example, for the permutation given in equation (2.1) we get

$$\sigma = (126)(3)(45).$$

In the above equation, the permutation has 3 orbits, each called a cycle. An orbit that has length i is called an $i-$cycle. In our example σ is composed of a $3-$cycle (126), a $1-$cycle (3) and a $2-$cycle (45). The cycle lists the objects in the order they appear as the orbit is traversed. This notation will play an important role in our calculations.

The vector space V_N. Let V_N denote the N dimensional vector space. This space has basis given by the N basis vectors

$$|e^1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad |e^2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad |e^N\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Each vector has N components. The symmetric group S_n has an action on the tenser product of n copies of V_N. To develop this idea in a simple but non-trivial setting, consider $m = 3$. In this case we define an action of S_3 on $V_N \otimes V_N \otimes V_N \equiv V_N^{\otimes 3}$. A basis for $V_N^{\otimes 3}$ is provided by the N^3 vectors

$$|e^i\rangle \otimes |e^j\rangle \otimes \left|e^k\right\rangle \quad i, j, k = 1, \cdots, N$$

each of which has N^3 components. To refer to the components of these vectors, we use the notation

$$(e^i)^{i_1} (e^j)^{i_2} (e^k)^{i_3}$$

In component notation, permutations are represented as

$$\sigma^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_n} = \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \cdots \delta^{i_n}_{j_{\sigma(n)}}.$$

For example,

$$(123)^{i_1 i_2 i_3}_{j_1 j_2 j_3} (e^i)^{j_1} (e^j)^{j_2} (e^k)^{j_3} = \delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1} (e^i)^{j_1} (e^j)^{j_2} (e^k)^{j_3} = (e^j)^{i_1} (e^k)^{i_2} (e^i)^{i_3}.$$

This can also be written as

$$(123) \left|e^i\right\rangle \otimes \left|e^j\right\rangle \otimes \left|e^k\right\rangle = \left|e^j\right\rangle \otimes \left|e^k\right\rangle \otimes \left|e^i\right\rangle .$$

A very important point to which we should pay attention, is the multiplication rule for permutations in this representation. The rule is

$$(\sigma_1)^{i_1 \cdots i_n}_{j_1 \cdots j_n} (\sigma_2)^{j_1 \cdots j_n}_{k_1 \cdots k_n} = (\sigma_2 \cdot \sigma_1)^{i_1 \cdots i_n}_{k_1 \cdots k_n}$$

where $\sigma_2 \cdot \sigma_1$ is the usual multiplication between cycles of the symmetric group. Thus, for example,

$$(12) \cdot (23) = (123)$$

but

$$(12)^{i_1 i_2 i_3}_{j_1 j_2 j_3} \cdot (23)^{j_1 j_2 j_3}_{k_1 k_2 k_3} = (132)^{i_1 i_2 i_3}_{k_1 k_2 k_3}.$$

2.3 Matrix representations of finite groups

Let G be a finite group, i.e $|G|$ is finite and let V be an N-dimensional vector space over a field F. The set of invertible $N \times N$ matrices, $GL(N, F)$, is a group which maps $V \to V$. Note that the group $GL(N, F)$ is not finite. In what follows we will usually not specify the field F.

Definition: A matrix representation of a group G acting on a vector space V, denoted by $\Gamma(g)$, is a map from the group to a set of matrices such that,

$$\Gamma : G \to GL(N)$$

$$g \to \Gamma(g).$$

This map must respect the group composition law. Mathematically this can be stated as

$$\Gamma_R(g_1 \circ g_2) = \Gamma_R(g_1)\Gamma_R(g_2). \tag{2.3}$$

On the left hand side of (2.3), we are composing two group elements g_1 and g_2 using the group composition law. On the right hand side of (2.3), we are multiplying matrices with the usual matrix multiplication. This implies $\Gamma(g^{-1}) = [\Gamma(g)]^{-1}$ and $\Gamma(\mathbb{1}) = I_{N\times N}$ where $\mathbb{1}$ is the identity group element and $I_{N\times N}$ is the $N \times N$ identity matrix.

2.3.1 Equivalent and Inequivalent Representations. If we change the basis of the vector space that the matrices representing the group elements act on, the matrices $\Gamma(g)$ of the representation are also transformed. Under a change of basis by a non-singular matrix S, we find the matrices representing the group transform as[16]

$$\Gamma'(g) = S\Gamma(g)S^{-1}.$$

The representations $\Gamma'(g)$ and $\Gamma(g)$ are called equivalent. Two representations that are not related in this way are inequivalent.

2.3.2 Types of Representations. There are two types of representation, reducible and irreducible. We will discuss both of them.

Reducible Representation: Γ_R is a reducible representation if it is equivalent to a representation with block diagonal matrices. Equivalently, the matrix representation of a group $\mathcal{G}$ is said to be reducible if there exist non-trivial subspaces that are invariant under the action of the group. In this case, it is always possible to choose a basis so that the matrices representing the group elements are block diagonal,

$$\Gamma(g) = \begin{pmatrix} \Gamma_1(g) & 0 \\ 0 & \Gamma_2(g) \end{pmatrix} \qquad \forall g \in \mathcal{G}.$$

Thus the matrix representation can be decomposed into the direct sum of two representation matrices as follows

$$\Gamma(g) = \Gamma_1(g) \oplus \Gamma_2(g). \tag{2.4}$$

Any representation that can be reduced to the form (2.4) is called completely reducible. We will never consider representations that are not completely reducible in this study.

Irreducible representation (irrep):

Any representation that is not reducible is called irreducible.

2.3.3 Operations Defined on Representations. Given one or more matrix representations, we can define two operations that can be used to produce new matrix representations. To illustrate these operations, let Γ_1 and Γ_2 be two representations of a group $\mathcal{G}$ acting on vector spaces V_1 and V_2 of dimension n_1 and n_2 respectively. Denote the corresponding basis vectors $|a_i\rangle$ and $|b_j\rangle$. *The Direct Sum of Representations* is defined as follows: $\Gamma_1 \oplus \Gamma_2$ acting on $V_1 \oplus V_2$, has dimension $n_1 + n_2$. The matrices of the direct sum representation are[16]

$$\Gamma(g) = \begin{bmatrix} \Gamma_1(g) & 0 \\ 0 & \Gamma_2(g) \end{bmatrix}.$$

This representation is reducible.

The Tensor Product of Representations is defined as follows: $\Gamma_1(g) \otimes \Gamma_2(g)$, acting on $V_1 \times V_2$, has dimension $n_1 \cdot n_2$. A basis for $V_1 \times V_2$ is given by $|a_i\rangle \otimes |b_j\rangle$ and the matrices of the tensor product representation are given by[16]

$$(\Gamma(g))_{im\ jn} = (\Gamma_1(g))_{ij} \cdot (\Gamma_2(g))_{mn}.$$

The tensor product will usually produce a reducible representation. The resulting representation can be decomposed into a direct sum of irreducible representations, using standard Clebsch-Gordon techniques.

2.4 The Matrix Representations of S_n

The matrix representations of S_n make use of Young diagrams, which play an important role in this study. The complete set of inequivalent and irreducible representations of the symmetric group S_n can be labeled by Young diagrams R with n boxes. These representations are sets of matrices $\Gamma_R(\sigma)$ which act on a vector space V. We start by defining what a Young diagram is.

2.4.1 Young diagrams. Let n be a strictly positive integer and let $R = (R_1, R_2, \cdots, R_i)$ be a partition of n denoted by $R \vdash n$. A Young diagram R is a finite collection of n boxes, arranged in left-justified rows such that the rows lengths are weakly decreasing, that is, each row length is equal to or less than the row above it. An example of a valid Young when $n = 3$ and with particular partition $R = (2, 1)$ is given in the left diagram in figure 2.2. An invalid Young diagram is shown in the right diagram of figure 2.2.

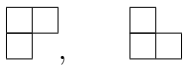


Figure 2.2: A valid and an invalid Young diagram.

2.4.2 Hook length h. In the discussion below we define the dimension of an irreducible representation using the notion of a hook length. We can associate a hook length to each box in the Young diagram. The hook length associated to box i in a Young diagram R, is the number of boxes below box i in the same column plus the number of boxes to the right of box i in the same row plus 1. For an example see equation (2.6). The number in each box is the hook length associated to that box.

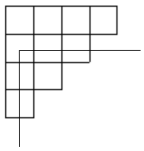


Figure 2.3: The hook length of a box in a Young diagram. For the box shown $h = 5$.

2.4.3 The Dimension d. In a previous subsection we have seen that a Young diagram $R \vdash n$ labels a finite dimensional vector space. Since $R \vdash n$ has the structure of a finite dimensional vector space, we can define a representation of the permutation group S_n on $R \vdash n$. We say that a Young diagram $R \vdash n$ labels irreps of S_n . We will see in the next subsection how the irreducible matrix representation of a permutation in S_n is defined. For now, we are interested in the dimension of the vector space associated to $R \vdash n$ denoted by d_R. The dimension d_R of an irreducible representation, labeled by Young diagram R, is given by the order of the symmetric group divided by the product of hook lengths,

$$d_R = \frac{n!}{\prod_{l \in R} hook(l)} = \frac{n!}{hooks_R}. \tag{2.5}$$

Here's an example of how we use the above formula

$$\text{hook lengths} = \begin{array}{|c|c|c|c|}\hline 8&5&3&1\\\hline 5&3&1\\\cline{1-3} 3&1\\\cline{1-2} 1\\\cline{1-1}\end{array} \quad \rightarrow \quad d_{\begin{array}{|c|c|c|c|}\hline &&&\\\hline &&\\\cline{1-3} &\\\cline{1-2} \\\cline{1-1}\end{array}} = \frac{n!}{hooks_{\begin{array}{|c|c|c|c|}\hline &&&\\\hline &&\\\cline{1-3} &\\\cline{1-2} \\\cline{1-1}\end{array}}} = \frac{10!}{8\cdot 5^2\cdot 3^3\cdot 1} = 672. \tag{2.6}$$

2.4.4 Content c_i. Besides the hook length described above, we can associate to each box in a Young diagram R, a second number, called the content of the box. For a box in row i and column j the content of the box is defined to be $j-i$. The content can be positive, negative or zero. This is an example showing the content of a Young diagram

$$\begin{array}{|c|c|c|}\hline 0&1&2\\\hline -1&0\\\cline{1-2} -2\\\cline{1-1}\end{array}$$

Figure 2.4: Example of the content a Young diagram.

2.4.5 Young-Yamanouchi Symbol. This symbol is given by filling the boxes of a Young diagram with integers, such that each box is assigned a unique integer. For a diagram with n boxes we use the integers $1, 2, \ldots n$. The entries of each column and row are strictly decreasing as we move towards the bottom of the diagram and towards the right of the diagram. The possible Young-Yamanouchi symbols for a diagram with $n=3$ boxes are shown in figure 2.5. The number of Young-Yamanouchi states for a given Young diagram is equal to the dimension of the irreducible representation labeled by the Young diagram. This is our first hint that Young-Yamanouchi states label the basis vectors of the representation.

$$\begin{array}{|c|c|c|}\hline 3&2&1\\\hline\end{array}, \quad \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}. \tag{2.7}$$

Figure 2.5: Example of Young-Yamanouchi states for $n=3$.

Here are examples of some illegal Young-Yamanouchi states.

$$\begin{array}{|c|c|c|}\hline 2&3&1\\\hline\end{array}, \quad \begin{array}{|c|}\hline 3\\\hline 1\\\hline 2\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}.$$

Figure 2.6: Example of forbidden Young-Yamanouchi states.

For any Young diagram the number of allowed labels is equal to the dimension of the irreducible representation. Each label corresponds to a vector in the basis for the carrier space. This basis is

orthonormal. For example

$$\left\langle \begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array} \middle| \begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array} \right\rangle = 1, \qquad \left\langle \begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array} \middle| \begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array} \right\rangle = 0. \tag{2.8}$$

2.4.6 Branching rule. Given a representation R of S_n, how can we decompose this into irreducible representations under an S_{n-1} subgroup? The answer to this question motivates the Young-Yamanouchi states describing the representations of the symmetric group. Consider a Young diagram $R \vdash n, n \geq 2$. In general there is more than one Young diagram $R \vdash (n-1)$ that can be subduced from R after removing one box. The branching rule is

$$R = \bigoplus_{R'} R', \tag{2.9}$$

The right hand side in equation (2.9) is a direct sum of the Young diagrams $R' \vdash (n-1)$ subduced from $R \vdash n$ by removing one box from R. To illustrate the branching rule, we will give an example which decomposes an irreducible representation of S_4 into irreducible representations of S_3.

Consider the irreducible representation of S_4 labelled by the Young diagram $\begin{array}{|c|c|c|}\hline\end{array}$. There are three Young-Yamanouchi states in this representation,

$$\begin{array}{|c|c|c|c|}\hline 4&3&2\\\hline 1\\\cline{1-1}\end{array}, \qquad \begin{array}{|c|c|c|c|}\hline 4&3&1\\\hline 2\\\cline{1-1}\end{array}, \qquad \begin{array}{|c|c|c|c|}\hline 4&2&1\\\hline 3\\\cline{1-1}\end{array}.$$

We will consider the subgroup $S_3 \subset S_4$ consisting of those permutations that leave box 1 fixed. Concretely, our subgroup is

$$S_3 = \{\mathbb{1}, (23), (24), (34), (234), (243)\}$$

Since there are no permutations that can move the position of box 1 in the Young-Yamanouchi symbol, it is clear that these states span two irreducible representation of S_3 , that is, the states

$$\begin{array}{|c|c|c|c|}\hline 4&3&1\\\hline 2\\\cline{1-1}\end{array}, \qquad \begin{array}{|c|c|c|c|}\hline 4&2&1\\\hline 3\\\cline{1-1}\end{array},$$

will mix with each other but they will not mix with

$$\begin{array}{|c|c|c|c|}\hline 4&3&2\\\hline 1\\\cline{1-1}\end{array}.$$

Summarizing, after the restriction we find

$$\begin{array}{|c|c|c|}\hline \\\hline \\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|}\hline \\\hline \\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline \\\hline\end{array}. \tag{2.10}$$

Since the Young diagrams in equation (2.10) label a finite dimensional vector spaces, the dimension of the left hand side must equal the dimension of the right hand side. We know that $d_{\begin{array}{|c|c|c|}\hline\end{array}} = 3$, so that the dimension of the right hand side must also be 3. The right hand side is a direct sum of two different vector spaces. Hence, we must show that

$$d_{\begin{array}{|c|c|}\hline\\\hline\\\cline{1-1}\end{array}} + d_{\begin{array}{|c|c|c|}\hline\end{array}} = 3.$$

We have,

$$\begin{array}{|c|c|}\hline\end{array} : hooks_{\begin{array}{|c|c|}\hline\\\hline\\\cline{1-1}\end{array}} = 3 \cdot 1 \cdot 1 \Rightarrow d_{\begin{array}{|c|c|}\hline\\\hline\\\cline{1-1}\end{array}} = \frac{3!}{3 \cdot 1 \cdot 1} = 2$$

$$\begin{array}{|c|c|c|}\hline\end{array} : hooks_{\begin{array}{|c|c|c|}\hline\end{array}} = 3 \cdot 2 \cdot 1 \Rightarrow d_{\begin{array}{|c|c|c|}\hline\end{array}} = \frac{3!}{3 \cdot 2 \cdot 1} = 1$$

which proves that $d_{\begin{array}{|c|c|}\hline\\\hline\\\cline{1-1}\end{array}} = d_{\begin{array}{|c|c|}\hline\\\hline\\\cline{1-1}\end{array}} + d_{\begin{array}{|c|c|c|}\hline\end{array}}$.

2.4.7 Young's Orthogonal Representation. Lets σ_1, $\sigma_2 \in S_n$, and $\Gamma_R(\sigma_1)$, $\Gamma_R(\sigma_2)$ be matrix representations of the elements in S_n labelled by the same Young diagram R. Then

$$\Gamma_R(\sigma_1) \cdot \Gamma_R(\sigma_2) = \Gamma_R(\sigma_1 \circ \sigma_2)\,. \tag{2.11}$$

Recall that a matrix representation is a map from elements of the symmetric group to some matrix algebra and that this map preserves the composition law of the group. The matrix representing an element of the symmetric group can be specified by giving its action on the Young-Yamonouchi basis. Since the adjacent permutations can be used to generate the symmetric group, we only need a formula for the action of adjacent permutations. Let $\widehat{R}$ be a specific labeled Young-Yamanouchi state and let $\widehat{R}_{ij}$ denote the same state, but with the boxes i and j swapped. The rule for the action of the group elements on the basis vectors of the carrier space is

$$\Gamma_R((i,i+1))|\widehat{R}\rangle = \frac{1}{c_i - c_{i+1}}|\widehat{R}\rangle + \sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}}|\widehat{R}_{i,i+1}\rangle, \tag{2.12}$$

where c_i is the content of the box labeled i.

Now we will give an example of how to construct the matrix representation using equation (2.12). Consider the permutation group $S_3 = \{\mathbb{1}, (12), (13), (23), (123), (132)\}$ and the Young diagram $\ydiagram{2,1}$. Then the only possible states are

$$\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}\,, \qquad \begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}\,.$$

and the content of R is

$$\begin{array}{|c|c|}\hline 0&1\\\hline -1\\\cline{1-1}\end{array}\,,$$

The matrix representation of the identity $\mathbb{1} \in S_3$ is

$$\Gamma_{\ydiagram{2,1}}(\mathbb{1}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Accordingly to our earlier definition in equation (2.12) we have

$$(12)\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array} = \frac{1}{(-1-1)}\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array} + \sqrt{1 - \frac{1}{(-1-1)^2}}\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array} = -\frac{1}{2}\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array} + \frac{\sqrt{3}}{2}\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}\,,$$

$$(12)\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array} = \frac{1}{(1-(-1))}\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array} + \sqrt{1 - \frac{1}{(1-(-1))^2}}\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array} = \frac{1}{2}\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array} + \frac{\sqrt{3}}{2}\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}\,.$$

Hence,

$$\Gamma_{\ydiagram{2,1}}((12)) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Next we have

$$(23)\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array} = \frac{1}{(1-0)}\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array} + \sqrt{1 - \frac{1}{(1-0)^2}}\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}\,,$$

$$(23)\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array} = \frac{1}{(-1-0)}\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array} + \sqrt{1-\frac{1}{(-1-0)^2}}\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array} = -\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}.$$

The matrix representation of the permutation (23) is thus given by

$$\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((23)) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

We can check this matrix representation by noting

$$\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12))\cdot\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12)) = \Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12)(12)) = \Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}(\mathbb{1}) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

and

$$\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12)^{-1}) = \Big(\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12))\Big)^{-1} = \begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} = \Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12)).$$

We can construct the matrix representation of the remaining elements of S_3 by combining the two group elements (12), (23) using the group composition law. Then $(13) = (12)(23)(12)$, $(123) = (12)(23)$ and $(132) = (23)(12)$. We have,

$$\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((13)) = \Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12)(23)(12)) = \Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12))\cdot\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((23))\cdot\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12)) = \begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix},$$

and for (123),

$$\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((123)) = \Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12)(23)) = \Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12))\cdot\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((23)) = \begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}.$$

Finally for (132), we have

$$\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((132)) = \Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((23)(12)) = \Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((23))\cdot\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((12)) = \begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}.$$

Using the six 2×2 matrices above, we can check that $\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}$ is indeed an irreducible representation of S_3 on the vector space spanned by $\{\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}\}$.

If we consider the subgroup $S_2 = \{\mathbb{1}, (23)\}$ of S_3, by using the branching rule,

$$\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline \ &\ \\\hline\end{array}\oplus\begin{array}{|c|}\hline \ \\\hline \ \\\cline{1-1}\end{array},$$

we must have

$$\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((23)) = \Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline\end{array}}((23))\oplus\Gamma_{\begin{array}{|c|}\hline \ \\\hline \ \\\cline{1-1}\end{array}}((23)) = \begin{bmatrix}\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline\end{array}}((23)) & 0\\ 0 & \Gamma_{\begin{array}{|c|}\hline \ \\\hline \ \\\cline{1-1}\end{array}}((23))\end{bmatrix},$$

A quick calculation using the definition of Γ_R shows that $\Gamma_{\begin{array}{|c|}\hline \ \\\hline \ \\\cline{1-1}\end{array}}((23)) = -1$ and $\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline\end{array}}((23)) = 1$. Comparing to our result above

$$\Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array}}((23)) = \Gamma_{\begin{array}{|c|c|}\hline \ &\ \\\hline\end{array}}((23))\oplus\Gamma_{\begin{array}{|c|}\hline \ \\\hline \ \\\cline{1-1}\end{array}}((23)) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},$$

we have perfect agreement.

2.5 The Matrix Representations of $U(N)$

Schur–Weyl duality is an important tool in our work. Consider the fundamental representation of $U(N)$. The matrices in this representation are $N \times N$ matrices which act on an N dimensional vector space V. Schur-Weyl duality arises when we consider the tensor product of n copies of V denoted $V^{\otimes n}$. It is a consequence of the fact that the action of the unitary group on $V^{\otimes n}$ (given by the tensor product representation) commutes with the action of S_n (which acts by permuting the copies in the tensor product). Schur-Weyl duality is a deep relation between the representation theory of the symmetric group and the general linear group. Note that the unitary group $U(N)$ is a subgroup of the general linear group. One consequence of Schur-Weyl duality, is that we can also use Young diagrams to label irreducible representations of the unitary group $U(N)$ and even to label the states in the representation.

Our first goal is to demonstrate that the action of $U(N)$ on $V^{\otimes n}$ commutes with the action of S_n on $V^{\otimes n}$. The action of σ on $V^{\otimes n}$ is given by

$$\sigma : \quad V_p^{\otimes n} \to V_p^{\otimes n}$$

$$\sigma\Big(\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(n)\Big) = \vec{v}(\sigma(1)) \otimes \vec{v}(\sigma(2)) \otimes \cdots \otimes \vec{v}(\sigma(n)).$$

In the above equation σ moves the vector in the ith slot to the $\sigma(i)^{\text{th}}$ slot without changing the value of the vector. The action of the unitary group element U on $V^{\otimes n}$ is given by

$$U : \quad V_p^{\otimes n} \to V_p^{\otimes n}$$

$$U\left(\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \overrightarrow{v}(n)\right) = D(U)\vec{v}(1) \otimes D(U)\vec{v}(2) \otimes \cdots \otimes D(U)\vec{v}(n),$$

where $D(U)$ is the $N \times N$ unitary matrix representing group element U. To see that the these two actions commute, consider

$$\begin{aligned} U \cdot (\sigma \cdot (\vec{v}(1) \otimes \cdots \otimes \vec{v}(n))) &= U \cdot (\vec{v}\sigma((1)) \otimes \cdots \otimes \vec{v}\sigma((n))) \\ &= D(U)\vec{v}\sigma((1)) \otimes \cdots \otimes D(U)\vec{v}\sigma((n)) \\ &= \sigma \cdot (D(U)\vec{v}(1) \otimes \cdots \otimes D(U)\vec{v}(n)) \\ &= \sigma \cdot (U \cdot (\vec{v}(1) \otimes \cdots \otimes \vec{v}(n)))\,. \end{aligned} \tag{2.13}$$

The above equation implies

$$U\left(\sigma(v^{\otimes n})\right) - \sigma\left(U(v^{\otimes n})\right) = 0 \Rightarrow (U\sigma - \sigma U)\, v^{\otimes n} = 0, \tag{2.14}$$

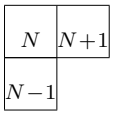
$$[U, \sigma] = 0,$$

which demonstrates that σ and U commute. As a consequence of this symmetry, states in a definite S_n representation are also in a definite $U(N)$ representation. Hence, they must be simultaneously diagonalizable on the N^n-dimensional vector space $V^{\otimes n}$. Therefore, σ and U must share the same eigenvectors.

2.5.1 The dimension Dim_R . An irreducible representation of $U(N)$ is labelled by a Young diagram R. The dimension of this representation depends on the hook lengths introduced above, as well as a new set of numbers called the factors. The factors are again easily read from the Young diagram. The factor of a box in the Young diagram defined to be N plus the content of the box. The product of all the factors in Young diagram R is denoted by f_R. The dimension Dim_R of an irreducible representation R of $U(N)$ is

$$\mathsf{Dim}_R = \frac{f_R}{\prod_{i\in R} hooks(i)}.$$

For example the Young diagram $R = \ydiagram{2,1}$, has the following factors



$$f_{\ydiagram{2,1}} = N(N+1)(N-1).$$

Thus the dimension $\mathsf{Dim}_{\ydiagram{2,1}}$ is

$$\mathsf{Dim}_{\ydiagram{2,1}} = \frac{N(N+1)(N-1)}{3\cdot 1\cdot 1} = \frac{N^3-N}{3} \tag{2.15}$$

In the table below, we list all possible Young diagrams with $n = 4$ boxes, as well as the product of the hook lengths and the dimensions of the S_4 and $U(N)$ irreducible representations that they label.

$R \vdash n$	$hooks_R$	d_R	Dim_R
$\ydiagram{4}$	24	1	$\frac{N(N+1)(N+2)(N+3)}{24}$
$\ydiagram{3,1}$	8	3	$\frac{N(N+2)(N^2-1)}{8}$
$\ydiagram{2,2}$	12	2	$\frac{N^2(N^2-1)}{12}$
$\ydiagram{2,1,1}$	8	3	$\frac{N(N^2-1)(N-2)}{8}$
$\ydiagram{1,1,1,1}$	24	1	$\frac{N(N-1)(N-2)(N-3)}{24}$

Summing the product of the S_n dimension and the $U(N)$ dimension we learn that

$$\sum_R^{R\vdash 4} d_R \mathsf{Dim}_R = \frac{N(N+1)(N+2)(N+3)}{24} + 3\frac{N(N^2-1)(N+2)}{8} + 2\frac{N^2(N^2-1)}{12} + 3\frac{N(N^2-1)(N-2)}{8} + \frac{N(N-1)(N-2)(N-3)}{24} = N^4 = \mathsf{dim}(V^{\otimes 4}).$$

In general, for Young diagrams with n boxes, we find

$$\mathsf{dim}(V^{\otimes n}) = \sum_R^{R\vdash n} d_R \cdot \mathsf{Dim}_R.$$

This is a consequence of Shur-Weyl duality and it gives a convincing check that every state in $V^{\otimes n}$ is uniquely labeled by giving an S_n state label and a $U(N)$ state label. It is a convincing check of the fact the σ and U share the same eigenvectors.

2.5.2 Gelfand-Tsetlin Patterns. Gelfand and Tsetlin have introduced a powerful labeling for the basis states of any irreducible representation of $U(N)$. The inequivalent irreducible representations of $GL(N,C)$ are given by specifying a sequence of N integers, called the weights

$$m = (m_{1,N}, m_{2,N}, \cdots m_{N,N}). \tag{2.16}$$

The weights satisfy the condition $m_{l,N} \geq m_{l+1,N}$. This sequence of integers can be identified with the row lengths of a Young diagram R, that has no more than N rows. Indeed, there is a Young diagram labeling the irreducible representations of $GL(N,C)$. If we restrict to the $GL(N-1,C)$ subgroup, our representation decomposes into a sum of $GL(N-1,C)$ irreducible representations. For the new irreducible representations of $GL(N-1,C)$ we can define the highest weight as

$$m' = (m_{1,N-1}, m_{2,N-1}, \cdots, m_{N-1,N-1}). \tag{2.17}$$

We can carry on repeating the above procedure, and restrict to ever smaller subgroups until we reach $GL(1,C)$.

For each state in the carrier space of the original group $GL(N,C)$,there is a unique labeling in terms of the representations which appear as the above sequence of restrictions is carried out. This unique pattern of the weights for each irreducible representation can be assembled as a triangular arrangement of integers as follows[9]

$$M = \begin{bmatrix} m_{1N} & & m_{2N} & & \cdots & & m_{N-1N-1} & & m_{NN} \\ & m_{1,N-1} & & m_{2,N-1} & & \cdots & & m_{N-1,N-1} & \\ & & \cdots & & \cdots & & \cdots & & \\ & & & m_{12} & & m_{22} & & & \\ & & & & m_{11} & & & & \end{bmatrix}. \tag{2.18}$$

An example to illustrate the construction of Gelfand-Testlin patterns is in order. Consider the $U(3)$ representation described by the Young diagram $\square\square$. For $N=3$ and using equation (2.15), this representation is $8-$dimensional and the states of the representation are labeled by $8-$different Gelfand-Tsetlin patterns

$$\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}$$
$$\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix}.$$

The first row gives the weights that specify the irreducible representation of the state. Lower rows are required to satisfy the betweenness condition, $m_{l,N} \geq m_{l,N-1} \geq m_{l+1,N}$. Vectors labeled by two distinct patterns are orthogonal and the dimension of the irreducible representation is equal to the total number of distinct Gelfand-Tsetlin patterns that can be defined. Finally, this discussion is relevant to $U(N)$ representation theory because the restriction of any irreducible representation of $GL(N,C)$ onto the subgroup $U(N)$ is also irreducible implying that the carrier space of the irreducible representations of $U(N)$ and $GL(N,C)$ share the same basis[9].

The basis states of a $u(N)$ irreducible representation can also be identified by labeling a given Young diagram R. The result is called a semi-standard Young diagram. Each box contains a single integer between 1 and N inclusive such that the numbers in each row of boxes weakly increase from left to right and the numbers in each column strictly increase from top to bottom. The number of semi-standard Young diagrams is equal to the number of Gelfand-Tsetlin patterns.

2.5.3 Clebsch-Gordon Coefficients. Let R and S be two irreducible representations of the unitary group $U(N)$. The tensor product of these representations decompose into a direct sum of irreducible components as follows

$$R \otimes S = \sum_T T. \tag{2.19}$$

A particular irreducible representation T can appear more than once in the product of $R \otimes S$. For this reason we must, in general, introduce a multiplicity index for T in this decomposition. In our applications we will deal only with the case that one of the representations in the product has weight $(1, 0, \cdots, 0)$ and in this special case, all multiplicities are equal to 1. Thus, we will not need a multiplicity index. A basis for the tensor product is provided by the states[9]

$$|\mathbf{m}_R, \mathbf{M}_R; \mathbf{m}_S, \mathbf{M}_S\rangle,$$

where $\mathbf{m}_R$ is the weight of the representation and $\mathbf{M}_R$ is the Gelfand-Tsetlin pattern for a particular state in the carrier space of irrep R. Alternatively, the basis is given by a collection of bases for the irreducible representations T appearing in the sum on the right hand side of equation (2.19). The Clebsch-Gordon coefficients supply the transformation matrix between these two bases

$$\langle \mathbf{m}_R, \mathbf{M}_R; \mathbf{m}_S, \mathbf{M}_S | \mathbf{m}_T, \mathbf{M}_T \rangle. \tag{2.20}$$

An efficient approach towards computation of the Clebsch-Gordan coefficients is to write a recursion relation which expresses the the coefficients of $U(N)$ in terms of the Clebsch-Gordon coefficients of $U(N-1)$. Concretely

$$\langle \mathbf{m}_N, \mathbf{M}; \mathbf{m}'_N, \mathbf{M}' | \mathbf{m}''_N, \mathbf{M}'' \rangle = \left(\begin{array}{cc|c} \mathbf{m}_N & \mathbf{m}'_N & \mathbf{m}''_N \\ \mathbf{m}_{N-1} & \mathbf{m}'_{N-1} & \mathbf{m}''_{N-1} \end{array} \right) \langle \mathbf{m}_{N-1}, \mathbf{M}_1; \mathbf{m}'_{N-1}, \mathbf{M}'_1 | \mathbf{m}''_{N-1}, \mathbf{M}''_1 \rangle. \tag{2.21}$$

On the right hand side we have the Clebsch-Gordan coefficients of the group $U(N-1)$ and on the left hand side we have the Clebsch-Gordan coefficients of the group $U(N)$. The weights $\mathbf{m}_N, \mathbf{m}'_N, \mathbf{m}''_N$ label irreducible representations of $U(N)$, while the weights $\mathbf{m}_{N-1}, \mathbf{m}'_{N-1}, \mathbf{m}''_{N-1}$ label irreducible representations of $U(N-1)$. The Gelfand-Tsetlin patterns $\mathbf{M}_1, \mathbf{M}'_1$ and $\mathbf{M}''_1$ are obtained from M, M' and M$''$ respectively, by removing the first row. Thus, the weights $\mathbf{m}_{N-1}$,$\mathbf{m}'_{N-1}$, $\mathbf{m}''_{N-1}$ correspond to the second rows in $\mathbf{M}, \mathbf{M}'$ and $\mathbf{M}''$. The coefficients $\left(\begin{array}{cc|c} \mathbf{m}_N & \mathbf{m}'_N & \mathbf{m}''_N \\ \mathbf{m}_{N-1} & \mathbf{m}'_{N-1} & \mathbf{m}''_{N-1} \end{array} \right)$ are called the scalar factors of the Clebsch-Gordan coefficients $\langle \mathbf{m}_N, \mathbf{M}; \mathbf{m}'_N, \mathbf{M}' | \mathbf{m}''_N, \mathbf{M}'' \rangle$. Applying the above factorization to the chain of subgroups referenced by the Gelfand-Tsetlin pattern, we obtain[9]

$$\begin{aligned} \langle \mathbf{m}_N, \mathbf{M}; \mathbf{m}'_N, \mathbf{M}' | \mathbf{m}''_N, \mathbf{M}'' \rangle = & \left(\begin{array}{cc|c} \mathbf{m}_N & \mathbf{m}'_N & \mathbf{m}''_N \\ \mathbf{m}_{N-1} & \mathbf{m}'_{N-1} & \mathbf{m}''_{N-1} \end{array} \right) \left(\begin{array}{cc|c} \mathbf{m}_{N-1} & \mathbf{m}'_{N-1} & \mathbf{m}''_{N-1} \\ \mathbf{m}_{N-2} & \mathbf{m}'_{N-2} & \mathbf{m}''_{N-2} \end{array} \right) \\ & \left(\begin{array}{cc|c} \mathbf{m}_{N-2} & \mathbf{m}'_{N-2} & \mathbf{m}''_{N-2} \\ \mathbf{m}_{N-3} & \mathbf{m}'_{N-3} & \mathbf{m}''_{N-3} \end{array} \right) \cdots \end{aligned} \tag{2.22}$$

Thus, as demonstrated, the Clebsch-Gordan coefficients can be written as a product of scalar factors. There is a selection rule for the Clebsch-Gordan coefficients. The Clebsch-Gordan coefficients vanish unless

$$\sum_{i=1}^{j} m_{ij} + \sum_{i=1}^{j} m'_{ij} = \sum_{i=1}^{j} m''_{ij} \qquad j = 1, 2, 3 \cdots N. \tag{2.23}$$

As mentioned above, the only Clebsch-Gordan coefficients that will be needed for our applications come from taking the product of some general representation with the fundamental representation[9].

2.6 Schur Polynomials and Restricted Schur polynomials

In this study we will consider operators built from two complex scalar fields Y and Z

$$Y = \phi_1 + i\phi_2, \qquad\qquad Z = \phi_3 + i\phi_4,$$

where the ϕ_i, $i = 1 \cdots 6$ are bosonic scalar fields. These scalars take their values in the Lie algebra $\mathfrak{u}(N)$ of the gauge group of $\mathcal{N} = 4$ super Yang Mills theory. The collection of operators built using only Y and Z is called the $SU(2)$ sector of the theory. We will consider a specific basis of local operators for the sector of the theory we are interested in. The basis is given by restricted Schur polynomials which are homogeneous polynomials of degree n in Z and of degree m in Y. We take $m \sim O(N)$ and $n \sim O(N)$ so that these operators have a large R-charge.

In the following subsections we will define the Schur polynomial and the restricted Schur polynomial.

Schur Polynomials are defined as follows

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \dots Z^{i_{n-1}}_{i_{\sigma(n-1)}} Z^{i_n}_{i_{\sigma(n)}}. \tag{2.24}$$

In the above formula, $\chi_R(\sigma)$ is the character of group element σ in the representation R. The character is the trace of the matrix representing the group element

$$\chi_R(\sigma) = \mathrm{tr}\big(\Gamma_R(\sigma)\big).$$

There is one gauge invariant operator built out of the Z fields for every conjugacy class of the group. Since the number of conjugacy classes is equal to the number of irreducible representations, the number of Schur polynomials is equal to the number of gauge invariant operators. The Schur polynomials provide a basis for the local gauge invariant operators of the theory, built using only the Z field.

Restricted Schur Polynomials In this subsection we will discuss restricted Schur polynomials, which are defined as follows

$$\chi_{R,(r,s)_{\alpha\beta}}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s)_{\alpha\beta}}(\sigma) Y^{i_1}_{i_{\sigma(1)}} \dots Y^{i_m}_{i_{\sigma(m)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}} \dots Z^{i_{m+n}}_{i_{\sigma(m+n)}}.$$

In the above formula, $\chi_{R,(r,s)_{\alpha\beta}}(\sigma)$ is a restricted character. As before, χ is the trace of the matrix representing $\sigma \in S_{n+m}$ in the irreducible representation R. The difference is that the trace is taken over the indices belonging to the subspace carrying an irreducible representation of the subgroup $S_n \times S_m \subset S_{n+m}$. The specific irreducible representation of $S_n \times S_m$ we consider is denoted by (r, s). The indices

α and β are multiplicity indices. When restricting from S_{n+m} to $S_n \times S_m \subset S_{n+m}$ a specific irreducible representation can be subduced multiple times, with each copy carried by a different subspace of the carrier space of R. The multiplicity indices resolve these different subspaces. The restricted Schur polynomials provide a basis for the local gauge invariant operators built using both Z and Y fields.

Now that we have introduced the gauge theory operators of interest to us we will describe their interpretation in the dual string theory. The article [17] argued that Schur polynomials built using order N Z fields correspond to half BPS giant gravitons. These giant gravitons are D3-branes which wrap a three sphere living either in the S^5 (when the Schur polynomial is labeled by a Young diagram with a long column) or in the AdS_5 space (when the Schur polynomial is labeled by a single long row). Since the D3 brane has a worldvolume that has the topology of a sphere, they have a vanishing monopole charge but do carry a dipole charge. A giant graviton is a graviton, propagating in spacetime that has spatially extended (blown up) as a consequence of polarization by the external five form field strength. The force produced by polarization is a Lorentz-like force, so the size of the giant is determined by its velocity and hence the angular momentum of the graviton. Giant gravitons that wrap a three sphere in S^5 are bounded by the size of the S^5 and so their angular momentum is bounded. This matches the fact that the number of boxes in the first column of a Young diagram is bounded by N. To get an intuitive idea of a giant graviton, the reader can imagine a rubber sheet. The giant graviton will vibrate and oscillate in the way a rubber sheet will. These excited giant graviton states can be described by attaching open strings to the half BPS giant graviton. For a detailed discussion of the giant graviton solutions see [18], [19], [20].

A Schur polynomial with p long rows describes a system of p giant gravitons that wrap a sphere contained in the AdS_5 space. A Schur polynomial with p long columns describes a system of p giant gravitons that wrap a sphere contained in the S^5 space. A restricted Schur polynomial with p long rows describes a system of p excited giant gravitons that wrap a sphere contained in the AdS_5 spacetime. A restricted Schur polynomial with p long columns describes a system of p excited giant gravitons that wrap a sphere contained in the S^5 space. For more details see [21], [22], [23]

This completes the discussion of the background mathematical tools that will be used in this study. In chapter 3 we will review how these methods are used to evaluate the action of the one loop dilatation operator. In chapter 4 we will use the same methods to compute the action of the two loop dilatation operator.

3. Review of the One loop Dilatation Generator

In this chapter we are going to review the acting of the one loop dilatation operator acting on restricted Schur polynomials labelled by Young diagrams with p long rows.

3.1 Interpretation of the Dilatation Generator

In any relativistic quantum field theory, the fields of the theory transform in a representation of the Poincaré group

$$\phi_A(X) \to \phi'_A(X') = \Gamma_A^B(g)\phi_B(X)$$

where $\Gamma_A^B(g)$ is a representation of the group element g. For the scalar field as an example, $\Gamma_A(g) = 1 \forall g$. The index A runs over the states in the representation, which are labeled by a pair of spins. The representation itself is labeled by the mass of the particle and the spin of the particle. In a conformal field theory, we must specify additional information, which specifies how the field operator transforms under a scaling transformation

$$X \to X' = \lambda X.$$

A field of definite scaling dimension $\triangle$ transforms as

$$\phi'_{A\triangle}(X) = \lambda^{-\triangle}\phi_{A\triangle}(\lambda^{-1}X)$$

where $\triangle$ is known as the dimension of the operator $\phi_{A\triangle}$. All fields in the conformal field theory are massless. The representations of the conformal group are therefore labeled by the spin of the particle and its scaling dimension. The dimension of $\phi_{A\triangle}$ can be defined equivalently as:

$$[D, \phi_{A\triangle}] = -i \triangle \phi_{A\triangle}$$

i.e. as an eigenvalue of the dilatation operator.

We can expand the operator D and the eigenvalue $\triangle$ in a series by treating $\hbar$ as small parameter. The result is

$$D = D_0 + \hbar D_1 + \hbar^2 D_2 + \cdots$$
$$\triangle = \triangle_0 + \hbar \triangle_1 + \hbar^2 \triangle_2 + \cdots$$

We call D_0 the tree level dilatation generator, D_1 the one loop dilatation generator, D_2 the two loop dilatation generator and so on. We call $\triangle_0$ the classical dimension, $\triangle_1$ the one loop anomalous dimension, $\triangle_2$ the two loop anomalous dimension and so on. In this study we are considering the action of D_1 and D_2 on restricted Schur polynomials.

3.2 The dilatation generator at one loop

The dilatation operator at one loop in the $SU(2)$ sector of the theory is [24]

$$D = -g_{YM}^2 \mathsf{Tr}\left([Y,Z]\,[\partial_Y,\partial_Z]\right).$$

When D acts on a restricted Schur polynomial, the result is [25]

$$D\chi_{R(r,s)_{\alpha\beta}} = \sum_{T(t,u)_{\mu\nu}} M_{R(r,s)_{\alpha\beta},T(t,u)_{\mu\nu}}\chi_{T(t,u)_{\mu\nu}} \tag{3.1}$$

where

$$\begin{aligned} M_{R(r,s)_{\alpha\beta},T(t,u)_{\mu\nu}} = &- g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T mn}{d_{R'} d_t d_u (m+n)} \mathsf{Tr}\Big([\Gamma_R((1,m+1)), P_{R\to(r,s)_{\alpha\beta}}] I_{R'T'} \times \\ &[\Gamma_T((1,m+1)), P_{T\to(t,u)_{\mu\nu}}] I_{T'R'}\Big). \end{aligned}$$

R' is obtained by dropping a box from R. Similarly, T' is obtained by dropping a box from T. Clearly then, R' and T' label irreducible representations of S_{n+m-1}. In the above expression, $c_{RR'}$ is the factor of the corner box removed from Young diagram R to obtain R'. $I_{T'R'}$ is an intertwiner, which is a map from the irreducible representation labeled by R' to the irreducible representation labeled by T'. Schur's lemma implies that this map is only non-zero if R' and T' have the same shape. $P_{R\to(r,s)_{\alpha\beta}}$ and $P_{T\to(t,u)_{\mu\nu}}$ are operators that project us from an irreducible representation of S_{n+m} into a subspace corresponding to an irreducible representation of $S_n \times S_m$.

To evaluate the dilatation operator we need to simplify the above expression. This will be accomplished using the *displaced corners approximation*, which we will describe in detail below. First however, we will explain how equation (3.1) for the dilatation generator is derived.

3.2.1 Action of the Dilatation Generator on Restricted Schur Polynomials. After displaying indices explicitly, the dilatation generator is

$$D = -g_{YM}^2 [Y,Z]^i_j \left[\frac{\partial}{\partial Y^k_j}\frac{\partial}{\partial Z^i_k} - \frac{\partial}{\partial Z^k_j}\frac{\partial}{\partial Y^i_k}\right]. \tag{3.2}$$

We can now act on a restricted Schur polynomial,

$$\begin{aligned} D\chi_{R(r,s)_{\alpha\beta}}(\sigma Y^{\otimes m} Z^{\otimes n}) = -g_{YM}^2 [Y,Z]^i_j \left(\frac{d}{dZ^k_j}\frac{d}{dY^i_k} - \frac{d}{dY^k_j}\frac{d}{dZ^i_k}\right) \frac{1}{n!m!} \sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)_{\alpha\beta}}(\sigma) \times \\ \left(Y^{i_1}_{i_{\sigma(1)}} \cdots Y^{i_m}_{i_{\sigma(m)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}} \cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\right). \end{aligned} \tag{3.3}$$

To evaluate the above expression recall that

$$\frac{\partial Z^i_j}{\partial Z^a_b} = \delta^i_a \delta^b_j.$$

We can now compute the derivatives appearing in the above commutator (3.2) as follows:

Acting with $\frac{\partial}{\partial Z_k^i}$ we find

$$\frac{\partial\left(Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\right)}{\partial Z^i_k}=\Bigg(\left(\delta^{i_{m+1}}_i\delta^k_{i_{\sigma(m+1)}}Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\right)+\left(Z^{i_{m+1}}_{i_{\sigma(m+1)}}\delta^{i_{m+2}}_i\delta^k_{i_{\sigma(m+2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\right)\cdots \tag{3.4}$$

$$+Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{m+n-1}}_{i_{\sigma(m+n-1)}}\delta^{i_{m+n}}_i\delta^k_{i_{\sigma(m+n)}}\Bigg).$$

We can simplify the above equation by converting terms of the form $Z^{i_{m+1}}_{i_{\sigma(m+1)}}\delta^{i_{m+2}}_i\delta^k_{i_{\sigma(m+2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}$ to $\delta^{i_{m+1}}_i\delta^k_{i_{\sigma(m+1)}}Z^{i_{m+2}}_{i_{\sigma(m+2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}$. This conversion is achieved by using

$$\prod_{a=1}^n V^{i_a}_{i_{\sigma(a)}}=\prod_{a=1}^n V^{i_\tau(a)}_{i_{\tau(\sigma(a))}}=\prod_{a=1}^n V^{i_{\sigma^{-1}(a)}}_{i_a}, \tag{3.5}$$

$$\prod_{a=1}^n \delta^{i_a}_{i_{\sigma(a)}}=\prod_{a=1}^n \delta^{i_\psi(a)}_{i_{\psi(\sigma(a))}}=\prod_{a=1}^n \delta^{i_{\sigma^{-1}(a)}}_{i_a} \tag{3.6}$$

where $\tau\in S_{n+m}$. The idea is now to perform a change of variables, replacing σ by a new permutation. After this change of variables, the sum in (3.4) will collapse from n terms to a single term.

Take the second term in (3.4) as an example. We will convert $(m+1)$ to $(m+2)$ in the second term in equation (3.4) and $\delta^{i_{m+2}}_{i_{\sigma(m+2)}}$ to $\delta^{i_{m+1}}_{i_{\sigma(m+1)}}$ by changing variables as $\sigma=(m+1,m+2)\tau(m+1,m+2)$. This change of variable has a trivial effect on the range of summation, because we are summing over all $\sigma\in S_{n+m}$. To see this we will argue that the sum over σ and τ both run over $(n+m)!$ distinct elements. This implies they must both run over S_{n+m}. To argue that both sums run over $(n+m)!$ distinct elements, we will argue that if two elements τ_1 and τ_2 are distinct after the change of variables, they were distinct before the change of variables. The argument is as follows

$$\tau_1\neq\tau_2\Rightarrow(m+1,m+2)\tau_1\neq(m+1,m+2)\tau_2$$
$$\Rightarrow(m+1,m+2)\tau_1(m+1,m+2)\neq(m+1,m+2)\tau_2(m+1,m+2),$$

thus

$$\sigma_1\neq\sigma_2.$$

We also need to consider the effect of this change of variables on the restricted character. Since in our change of variables we have conjugated by an element of $S_n\times S_m$, and since

$$\chi_{R(r,s)_{\alpha\beta}}(\sigma)=\mathsf{Tr}_{(r,s)_{\alpha\beta}}\left(\Gamma_R(\sigma)\right).$$

we have

$$\chi_{R(r,s)_{\alpha\beta}}(\rho\sigma\rho^{-1})=\chi_{R(r,s)_{\alpha\beta}}(\sigma), \tag{3.7}$$

Indeed, for any $\rho\in S_n\times S_m$ we have

$$\mathsf{Tr}_{(r,s)_{\alpha\beta}}\left(\Gamma_R(\rho)\Gamma_R(\sigma)\Gamma_R(\rho^{-1})\right)=\mathsf{Tr}\left(P_{R\to(r,s)_{\alpha\beta}}\Gamma_R(\rho)\Gamma_R(\sigma)\Gamma_R(\rho^{-1})\right)$$

and thus, as consequence of the cyclicity of the trace, we find

$$\begin{aligned}\mathsf{Tr}_{(r,s)_{\alpha\beta}}\big(\Gamma_R(\rho)\Gamma_R(\sigma)\Gamma_R(\rho^{-1})\big) &= \mathsf{Tr}\big(\Gamma_R(\rho^{-1})P_{R,(r,s)_{\alpha\beta}}\Gamma_R(\rho)\Gamma_R(\sigma)\big)\\ &= \mathsf{Tr}\big(P_{R,(r,s)_{\alpha\beta}}\Gamma_R(\rho^{-1})\Gamma_R(\rho)\Gamma_R(\sigma)\big)\\ &= \mathsf{Tr}\big(P_{R,(r,s)_{\alpha\beta}}\Gamma_R(\sigma)\big).\end{aligned} \tag{3.8}$$

In equation (3.8) we have used the fact that

$$\Gamma_R(\rho)P_{R(r,s)_{\alpha\beta}} = P_{R(r,s)_{\alpha\beta}}\Gamma_R(\rho) \tag{3.9}$$

This deserves a little explanation. Assume that $R \vdash m+n$ decomposes under the $S_n \times S_m$ subgroup as follows

$$\Gamma_R(\sigma) = \lambda_1 \oplus \lambda_1 \oplus \lambda_2 \oplus \lambda_3$$

where $\lambda_i \quad i = 1, 2, \cdots$ are irreducible matrix representations of $S_n \times S_m$. Concretely, $\Gamma_R(\sigma)$ has the following block diagonal form and the intertwining operator can be written in matrix form as follows,

$$\Gamma_R(\rho) = \begin{bmatrix} \lambda_1^{\alpha} & 0 & 0 & 0 \\ 0 & \lambda_1^{\beta} & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}, \qquad \mathsf{P}_{R,(r,s)_{\alpha\beta}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have illustrated an example for which $\alpha \neq \beta$ so that the non-zero block of the projection operator is not on the diagonal. The LHS of equation (3.9) is as follows

$$\Gamma_R(\rho)\mathsf{P}_{R,(r,s)_{\alpha\beta}} = \begin{bmatrix} \lambda_1^{\alpha} & 0 & 0 & 0 \\ 0 & \lambda_1^{\beta} & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.10}$$

The RHS of equation (3.9) can be written as

$$\mathsf{P}_{R,(r,s)_{\alpha\beta}}\Gamma_R(\rho) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1^{\alpha} & 0 & 0 & 0 \\ 0 & \lambda_1^{\beta} & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.11}$$

This demonstrates how (3.9) is true for all $\rho \in S_m \times S_n$ subgroup. Also, it is now clear why ρ had to be an element of the subgroup: only in this case is $\Gamma_R(\rho)$ block diagonal. We follow exactly the same process to convert $Z^{i_{(m+1)}}_{i_{\sigma(m+1)}} Z^{i_{(m+2)}}_{i_{\sigma(m+2)}} \delta^{i_{(m+3)}}_{i} \delta^{k}_{\sigma(m+3)}$ to $\delta^{i_{(m+1)}}_{i} \delta^{k}_{\sigma(m+1)} Z^{i_{(m+2)}}_{i_{\sigma(m+2)}} Z^{i_{(m+3)}}_{i_{\sigma(m+3)}}$. For the change of variables use $\sigma = (m+1, m+3)\tau(m+1, m+3)$. We keep going, up to $Z^{i_{(m+n-1)}}_{i_{\sigma(m+n-1)}} \delta^{i_{(m+n)}}_{i} \delta^{k}_{\sigma(m+n)}$ for which we use $\sigma = (m+1, m+n)\tau(m+1, m+n)$. With the above substitutions we effectively get

$$\frac{\partial \left(Z^{i_{m+1}}_{i_{\sigma(m+1)}} \cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}} \right)}{\partial Z^{i}_{k}} = n \left(\delta^{i_{(m+1)}}_{i} \delta^{k}_{i_{\sigma(m+1)}} Z^{i_{m+2}}_{i_{\sigma(m+2)}} \cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}} \right) \tag{3.12}$$

For the $\frac{\partial\left(Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\right)}{\partial Z^k_j}$ derivative, after applying the same process, we find we can set

$$\frac{\partial\left(Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\right)}{\partial Z^k_j}=n\left(\delta^{i_1}_k\delta^j_{i_{\sigma(1)}}Z^{i_{m+2}}_{\sigma(m+2)}\cdots Z^{i_{m+n}}_{\sigma(m+n)}\right)\tag{3.13}$$

For the derivatives with respect to Y we can use the same logic to get the following results,

$$\frac{\partial\left(Y^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\right)}{\partial Y^i_k}=m\left(\delta^{i_1}_i\delta^k_{i_{\sigma(1)}}Y^{i_2}_{\sigma(2)}\cdots Y^{i_m}_{\sigma(m)}\right)\tag{3.14}$$

and,

$$\frac{\partial\left(Y^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\right)}{\partial Y^k_j}=m\left(\delta^{i_1}_k\delta^j_{i_{\sigma(1)}}Y^{i_2}_{\sigma(2)}\cdots Y^{i_m}_{\sigma(n)}\right).\tag{3.15}$$

Consider now the second derivatives, given by $\frac{\partial}{\partial Y^k_j}\left[\frac{\partial\left(Y^{\otimes m}Z^{\otimes n}\right)}{\partial Z^i_k}\right]$ and $\frac{\partial}{\partial Z^k_j}\left[\frac{\partial\left(Y^{\otimes m}Z^{\otimes n}\right)}{\partial Y^i_k}\right]$. Write the first of these as follows

$$\frac{\partial}{\partial Y^k_j}\left[\frac{\partial\left(Y^{\otimes m}Z^{\otimes n}\right)}{\partial Z^i_k}\right]=\frac{\partial}{\partial Y^k_j}\left[Y^{\otimes m}\frac{\partial\left(Z^{\otimes n}\right)}{\partial Z^i_k}\right]=\left(\frac{\partial Y^{\otimes m}}{\partial Y^k_j}\right)\left(\frac{\partial Z^{\otimes n}}{\partial Z^i_k}\right).\tag{3.16}$$

Using the result we obtained in equation (3.16), it is a simple matter to find, after summing repeated indices

$$\frac{\partial}{\partial Y^k_j}\left[\frac{\partial\left(Y^{\otimes m}Z^{\otimes n}\right)}{\partial Z^i_k}\right]=mn\left[\delta^{i_1}_{i_{\sigma(m+1)}}\delta^{i_{m+1}}_{i_{\sigma(1)}}\right]\left(Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\cdot Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{n+m}}_{i_{\sigma(m+n)}}\right).$$

Similarly, we find

$$\frac{\partial}{\partial Z^k_j}\left[\frac{\partial\left(Y^{\otimes m}Z^{\otimes n}\right)}{\partial Y^i_k}\right]=mn\left[\delta^{i_{\sigma(1)}}_{i_{\sigma(m+1)}}\delta^{i_{m+1}}_{i_{\sigma(1)}}\right]\left(Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\cdot Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{n+m}}_{i_{\sigma(m+n)}}\right).$$

Taking the difference of these results yields

$$\frac{\partial}{\partial Y^k_j}\left[\frac{\left(Y^{\otimes m}Z^{\otimes n}\right)}{\partial Z^i_k}\right]-\frac{\partial}{\partial Z^k_j}\left[\frac{\left(Y^{\otimes m}Z^{\otimes n}\right)}{\partial Y^i_k}\right]=mn\left(Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\cdot Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{n+m}}_{i_{\sigma(m+n)}}\right)$$
$$\left[\delta^{i_{m+1}}_{i_{\sigma(1)}}\delta^{i_1}_{i_{m+1}}-\delta^{i_1}_{i_{\sigma(m+1)}}\delta^{i_{m+1}}_{i_1}\right].$$

Using this result, after a little algebraic manipulation, equation (3.3) becomes

$$D\chi_{R(r,s)_{\alpha\beta}}\left(Y^{\otimes m}Z^{\otimes n}\right)=\frac{1}{(n-1)!(m-1)!}\sum_{\sigma\in S_{n+m}}\chi_{R(r,s)_{\alpha\beta}}(\sigma)\tag{3.17}$$
$$\left(Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\cdot Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{n+m}}_{i_{\sigma(m+n)}}\right)\times\left[\delta^{i_{m+1}}_{i_\sigma(1)}[Z,Y]^{i_1}_{i_{\sigma(m+1)}}-\delta^{i_1}_{i_{\sigma(m+1)}}[Z,Y]^{i_{m+1}}_{i_{\sigma(1)}}\right].$$

Observe that in the terms in square brackets, the upper indices and lower indices do not match. To match the indices, we act on the upper index of the first term with $(1,m+1)$ which is accomplished by transforming the summation variable $\sigma\to(m+1,1)\sigma$. With this transformation we have

$$\chi_{R(r,s)_{\alpha\beta}}\left((m+1,1)\sigma\right)\left(Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\cdot Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{n+m}}_{i_{\sigma(m+n)}}\right)\times\delta^{i_{m+1}}_{i_{(m+1,1)\sigma(1)}}[Z,Y]^{i_1}_{i_{(m+1,1)\sigma(m+1)}},$$

$$=\chi_{R(r,s)_{\alpha\beta}}\left((m+1,1)\sigma\right)\left(Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\cdot Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{n+m}}_{i_{\sigma(m+n)}}\right)\times\delta^{i_1}_{i_\sigma(1)}[Z,Y]^{i_{m+1}}_{i_{\sigma(m+1)}}. \tag{3.18}$$

Likewise we act on the lower index of the second term by changing the summation variable as $\sigma\to\sigma(m+1,1)$, to obtain

$$\chi_{R(r,s)_{\alpha\beta}}\left(\sigma(m+1,1)\right)\left(Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\cdot Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{n+m}}_{i_{\sigma(m+n)}}\right)\times\delta^{i_1}_{i_\sigma(m+1,1)(1)}[Z,Y]^{i_{m+1}}_{i_{\sigma(m+1,1)(m+1)}},$$

$$=\chi_{R(r,s)_{\alpha\beta}}\left((m+1,1)\sigma\right)\times Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\cdot Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{n+m}}_{i_{\sigma(m+n)}}\times\delta^{i_1}_{i_\sigma(1)}[Z,Y]^{i_{m+1}}_{i_{\sigma(m+1)}}. \tag{3.19}$$

The action of the dilatation operator is now

$$\begin{aligned}D\chi_{R(r,s)_{\alpha\beta}}\left(Y^{\otimes m}Z^{\otimes n}\right)=\frac{1}{(n-1)!(m-1)!}\sum_{\sigma\in S_{n+m}}\chi_{R(r,s)_{\alpha\beta}}\left([\sigma,(m+1,1)]\right)\times\\ Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\cdot Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{n+m}}_{i_{\sigma(m+n)}}\times\delta^{i_1}_{i_\sigma(1)}[ZY-YZ]^{i_{m+1}}_{i_{\sigma(m+1)}}.\end{aligned} \tag{3.20}$$

To proceed further we will rewrite $Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\cdot Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{n+m}}_{i_{\sigma(m+n)}}(ZY)^{i_{m+1}}_{i_{\sigma(m+1)}}$ as a trace. This will allow us to express the result in terms of restricted Schur polynomials. Towards this end consider the contribution from the first term in square brackets in (3.20)

$$\begin{aligned}Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}(ZY)^{i_{m+1}}_{i_{\sigma(m+1)}}Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}&=Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+1}}_{i_\sigma(1)}Y^{i_1}_{i_{\sigma(m+1)}}Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\\&=Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+1}}_{i_{\sigma(1,m+1)(1)}}Y^{i_1}_{i_{\sigma(1,m+1)(m+1)}}Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\\&=Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+1}}_{i_{\sigma(m+1)}}Y^{i_1}_{i_{\sigma(1)}}Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\\&=Y^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\\&=\mathsf{Tr}\left(\sigma(1,m+1)Y^{\otimes m}\otimes Z^{\otimes n}\right).\end{aligned}$$

The contribution from the second term in square brackets in (3.20) can be written using the same trick, by acting on the lower index, which is accomplished by changing $\sigma\to(1,m+1)\sigma$. The result is

$$\begin{aligned}1\otimes Y^{i_2}_{i_{\sigma(2)}}(YZ)^{i_{m+1}}_{i_{\sigma(m+1)}}Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}&=Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Y^{i_{m+1}}_{i_{\sigma(1)}}Z^{i_1}_{i_{\sigma(m+1)}}Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\\&=Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Y^{i_{m+1}}_{i_{(1,m+1)\sigma(1)}}Z^{i_1}_{i_{(1,m+1)\sigma(m+1)}}Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\\&=Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Y^{i_{(1,m+1)(m+1)}}_{i_{\sigma(1)}}Z^{i_{(1,m+1)(1)}}_{i_{\sigma(m+1)}}Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\\&=Y^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\\&=\mathsf{Tr}\left((1,m+1)\sigma Y^{\otimes m}\otimes Z^{\otimes n}\right).\end{aligned}$$

Therefore we can write

$$\mathsf{Tr}\left([\sigma,(1,m+1)]\,Y^{\otimes m}Z^{\otimes n}\right)=Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\cdot Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{n+m}}_{i_{\sigma(m+n)}}[Z,Y]^{i_{m+1}}_{i_{\sigma(m+1)}},\tag{3.21}$$

allowing us to find the following action for D

$$\mathsf{D}\chi_{R,(r,s)_{\alpha\beta}}\left(Y^{\otimes m}Z^{\otimes n}\right)=\frac{1}{(n-1)!(m-1)!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)_{\alpha\beta}}\left([(1,m+1),\sigma]\right)\mathsf{Tr}\left([\sigma,(1,m+1)]\,Y^{\otimes m}Z^{\otimes n}\right)\delta^{i_1}_{i_{\sigma(1)}}.\tag{3.22}$$

To simplify the above expression, we write the group S_{n+m} in terms of the subgroup S_{n+m-1} and the left cosets of this subgroup. The subgroup is defined by $\sigma(1)=1$ and the decomposition we use is

$$S_{n+m}=S_{n+m-1}\oplus(12)S_{n+m-1}\oplus(13)S_{n+m-1}\oplus\cdots\oplus(1,m+n)S_{n+m-1}.$$

A little manipulation now shows that

$$\begin{aligned}\mathsf{D}\chi_{R,(r,s)_{\alpha\beta}}\left(Y^{\otimes m}Z^{\otimes n}\right)=&\frac{1}{(n-1)!(m-1)!}\sum_{\sigma\in S_{n+m-1}}\sum_{i=1}^{m+n}\chi_{R,(r,s)_{\alpha\beta}}\left([(1,m+1),(1,i)\sigma]\right)\times\\&\mathsf{Tr}\left([(1,i)\sigma,(1,m+1)]\,Y^{\otimes m}Z^{\otimes n}\right)\delta^{i_1}_{i_{\sigma(1)}}\\=&\frac{1}{(n-1)!(m-1)!}\sum_{\sigma\in S_{n+m-1}}\chi_{R,(r,s)_{\alpha\beta}}\left(\left[(1,m+1),\left(N+\sum_{i=2}^{m+n}(1,i)\right)\sigma\right]\right)\times\\&\mathsf{Tr}\left([\sigma,(1,m+1)]\,Y^{\otimes m}Z^{\otimes n}\right).\end{aligned}\tag{3.23}$$

Consider the expression $[N+(12)+(13)\cdots(1,m+n-1)]$ which can be written as follows

$$C_{n+m}-C_{n+m-1}=\sum_{i=1,i<j\leq n+m}^{n+m}(i,j)-\sum_{i=2,i<j\leq n+m}^{n+m}(i,j)=N+(12)+(13)\cdots(1,m+n-1)$$

where C_{n+m} and C_{n+m-1} are Casimirs of S_{n+m} and S_{n+m-1}. The value of these Casimirs when acting on a state belonging to a representation labeled by a Young diagram R is given by

$$C_{n+m}|R,a\rangle=\left(\sum_i(r_i-1)r_i-\sum_j(c_j-1)c_j\right)|R,a\rangle$$

where r_i are the lengths of the rows of R and c_j are the lengths of the columns of R. Recall the branching rule

$$R=\bigoplus_{R'}R'$$

With the branching rule in mind, it is clear that

$$\Gamma_R(N+C_{m+n})=(N+\lambda_R)\times\mathbb{1}_{d_R\times d_R}=\bigoplus_{R'}(N+\lambda_R)\mathbb{1}_{d_{R'}\times d_{R'}},\tag{3.24}$$

$$\Gamma_R(N+C_{m+n-1})=\bigoplus_{R'}(N+\lambda_{R'})\mathbb{1}_{d_{R'}\times d_{R'}}.\tag{3.25}$$

The values of the Casimir eigenvalues are [22]

$$\lambda_R = \sum_i \frac{r_i(r_i-1)}{2} - \sum_j \frac{c_j(c_j-1)}{2}, \qquad \lambda_{R'} = \sum_{i'} \frac{r'_{i'}(r'_{i'}-1)}{2} - \sum_{j'} \frac{c'_{j'}(c'_{j'}-1)}{2}. \tag{3.26}$$

Using these eigenvalues it is straight forward to verify that

$$\begin{aligned} c_{RR'} = N + \lambda_R - \lambda_{R'} &= \left[\sum_i \frac{r_i(r_i-1)}{2} - \sum_j \frac{c_j(c_j-1)}{2}\right] - \left[\sum_{i'} \frac{r'_{i'}(r'_{i'}-1)}{2} - \sum_{j'} \frac{c'_{j'}(c'_{j'}-1)}{2}\right] \\ &= \left[\sum_i \frac{r_i(r_i-1)}{2} - \sum_{i'} \frac{r'_{i'}(r'_{i'}-1)}{2}\right] - \left[\sum_j \frac{c_j(c_j-1)}{2} - \sum_{j'} \frac{c'_{j'}(c'_{j'}-1)}{2}\right] \end{aligned}$$

where $c_{RR'}$ is the factor of box that must be removed to go from R to R'. Therefore equation (3.23) becomes

$$\begin{aligned} \mathsf{D}\chi_{R,(r,s)_{\alpha\beta}}\left(Y^{\otimes m}Z^{\otimes n}\right) = &\sum_{R'} c_{RR'} \frac{1}{(n-1)!(m-1)!} \sum_{\sigma\in S_{n+m-1}} \chi_{R,(r,s)_{\alpha\beta}}\left([\Gamma_R((1,m+1)),\Gamma_{R'}(\sigma)]\right)\times \\ &\mathsf{Tr}\left([\sigma,(1,m+1)]\,Y^{\otimes m}Z^{\otimes n}\right). \end{aligned} \tag{3.27}$$

Now, using the identity [26]

$$\mathsf{Tr}\left(\sigma Y^{\otimes m}Z^{\otimes m}\right) = \sum_{T,(t,u)} \frac{d_T n!m!}{d_t d_u (m+n)!}\chi_{T,(t,u)}(\sigma^{-1})\chi_{T,(t,u)}(YZ),$$

we find

$$\begin{aligned} D\chi_{R,(r,s)}(YZ) = &\sum_{R'} \frac{c_{RR'}d_T n!m!}{(m+n)!d_t d_u (n-1)!(m-1)!} \sum_{\sigma\in S_{m+n-1}} \mathsf{Tr}_{(r,s)_{\alpha\beta}}\left[\Gamma_R((1,m+1)),\Gamma_{R'}((\sigma))\right]\times \\ &\mathsf{Tr}_{(t,u)}\left(\Gamma_T(\sigma^{-1})\right)\chi_{T,(t,u)}(YZ). \end{aligned} \tag{3.28}$$

The sum over σ is now easily carried out by using the fundamental orthogonality theorem. A little care is needed since we are summing over S_{n+m-1} but representations of S_{n+m} appear. Recall that the fundamental orthogonality relation says

$$\sum_{\sigma\in\mathcal{G}} \Gamma_R(\sigma)_{ab}\Gamma_S(\sigma^{-1})_{cd} = \frac{|\mathcal{G}|}{d_R}\delta_{RS}\delta_{ad}\delta_{cb}. \tag{3.29}$$

Restricting to the S_{n+m-1} subgroup, $\Gamma_{R(r,s)_{ab}}$ takes the following block diagonal form, as does $\Gamma_S(\sigma^{-1})_{cd}$ which assumes the form

$$\Gamma_R(\sigma)_{ab} = \left[\begin{array}{c|c|c|c} R & 0 & 0 & 0 \\ \hline 0 & R & 0 & 0 \\ \hline 0 & 0 & S & 0 \\ \hline 0 & 0 & 0 & S \end{array}\right]_{ab}, \qquad \Gamma_S(\sigma^{-1})_{cd} = \left[\begin{array}{c|cc} R=S & 0 & 0 \\ \hline & 0 & 0 \\ \hline 0\ 0 & & S \\ 0\ 0 & & \end{array}\right]_{cd}.$$

Applying the fundamental orthogonality relation in matrix form, (here we assume that the representations that contribute are $R' = R_1$ and $S' = S_1$)

$$\left[\begin{array}{c|c|c|c} R_1 & 0 & 0 & 0 \\ \hline 0 & R_2 & 0 & 0 \\ \hline 0 & 0 & S_1 & 0 \\ \hline 0 & 0 & 0 & S_2 \end{array}\right]_{ab} \left[\begin{array}{cc|cc} & & 0 & 0 \\ R_1 = S_1 & & 0 & 0 \\ \hline 0 & 0 & & \\ 0 & 0 & S_2 & \end{array}\right]_{cd} = \frac{(m+n-1)!}{d_{R'}} \delta_{R'S'} \left[\begin{array}{c|c|c|c} 0 & \mathbb{1} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]_{ad} \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline \mathbb{1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]_{cb} \tag{3.30}$$

$$I_{R'T'} = \left[\begin{array}{c|c|c|c} 0 & \mathbb{1} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]_{ad}, \qquad I_{T'R'} = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline \mathbb{1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]_{cb}.$$

To summarize, in terms of

$$M_{R,(r,s);T(t,u)} = -g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T mn}{d_{R'} d_t d_u (m+n)} \mathsf{Tr}\left([\Gamma_R((1,m+1)), P_{R\to}(r,s)] I_{R'T'} [\Gamma_T((1,m+1)), P_{T\to}(t,u)] I_{T'R'}\right) \tag{3.31}$$

we finally have

$$D\chi_{R(r,s)} = \sum_{T,(t,u)} M_{R,(r,s);T(t,u)} \chi_{R(r,s)}(YZ).$$

We would like to rewrite this in terms of operators with normalized two point functions. The two point correlation function is

$$\langle \chi_{R,(r,s)_{\alpha\beta}}(Y,Z), \chi_{T,(t,u)}(Y,Z)^\dagger \rangle = \delta_{R,(r,s);T,(t,u)} \frac{f_R hooks_R}{hooks_r hook_s}, \tag{3.32}$$

so that a normalised operator (denoted with an $\mathcal{O}$) is given by

$$\chi_{R,(r,s)}(Y,Z) = \sqrt{\frac{f_R hooks_R}{hooks_r hook_s}} \mathcal{O}_{R,(r,s)}(Y,Z). \tag{3.33}$$

The dilatation operator, acting on normalized operators, is

$$D\mathcal{O}_{R,(r,s)}(Y,Z) = \sum_{T,(t,u)} N_{R,(r,s);T,(t,u)} \mathcal{O}_{T,(t,u)}(Y,Z), \tag{3.34}$$

where,

$$N_{R,(r,s);T,(t,u)} = \sum_{R'} \frac{c_{RR'} d_T mn}{d_t d_u (m+n)} \sqrt{\frac{f_T hooks_T hooks_r hoooks_s}{f_R hooks_R hooks_t hoooks_u}} \times \tag{3.35}$$
$$\mathsf{Tr}\left([\Gamma_R((1,m+1)), P_{R\to}(r,s)] I_{R'T'} [\Gamma_T((1,m+1)), P_{T\to}(t,u)] I_{T'R'}\right).$$

This completes our review of the derivation of the action of the dilatation operator on the $SU(2)$ sector. Evaluating this expression is a highly non-trivial task. We need to introduce an approximation scheme to accomplish this and this is the task we turn to next.

3.2.2 Displaced Corners Approximations. The key difficulty in evaluating matrix elements of the dilatation operator is in writing explicit expressions for the intertwining operators $P_{R\to(r,s)\alpha\beta}$. The displaced corners approximation allows us to write concrete formulas for these intertwiners.

The displaced corners approximation holds when the difference in the lengths of any two rows in the Young diagram is order N. To ensure that this is the case, we take $m \ll n$. Recall that the action of adjacent permutations on Young-Yamanouchi symbols is given in terms of the difference in the content of the two boxes being swapped. In the displaced corners approximation this difference is equal to 1 if the boxes are in the same row and $O(N)$ if the boxes are in different rows. Thus, in this limit the action of the symmetric group becomes particularly simple. One consequence of this simplified action is that for a Young diagram with p rows, there is a $U(p)$ group acting which commutes with the action of S_m. To demonstrate this, one needs to associate each removed box with a vector in a p dimensional vector space V_p. In this way, the m removed boxes associated with the Y's define a vector in $V_p^{\otimes m}$. The trace over $R \oplus T$ in the formula for the matrix elements of the dilatation operator now factorize into a trace over $r \oplus t$ times a trace over $V_p^{\otimes m}$. The bulk of the work is in evaluating the trace over $V_p^{\otimes m}$. This trace can now be evaluated using the methods developed in[9]. It is possible and useful to write the intertwining maps in terms of the basis for the fundamental representation of the Lie algebra $u(p)$ given by $(E_{ij})_{ab} = \delta_{ia}\delta_{jb}$ and obeying[11]

$$E_{ij}E_{kl} = \delta_{jk}E_{il}. \tag{3.36}$$

Note that E_{ij} is a matrix with zero entries everywhere except for the element in the ith row and jth column which is 1. Each E_{ij} acts on the space V appearing in the tensor product $V^{\otimes m}$. To define operator that acts on $V^{\otimes m}$ we take a tensor product of E_{ij} with the identity $\mathbb{1}$ acting on V. For example

$$E_{ij}^{(2)} = \mathbb{1} \otimes E_{ij} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

gives an operator that applies E_{ij} to the second V in $V^{\otimes p}$ and the identity to all other V's in $V^{\otimes p}$. There are p factors on the RHS of the above equation. Although there is a lot more that could be said at this point, this is enough detail for what we want to do. A box removed from row i is associated to a vector v_i, which is an eigenstate of E_{ii} with eigenvalue 1. If we remove a box from row i of R to obtain R', and a box from row j of T to obtain T', assuming that R' and T' have the same shape, we have

$$I_{R'T'} = E_{ij}^{(1)}, \qquad I_{T'R'} = E_{ji}^{(1)} \tag{3.37}$$

for the intertwiner. Denote the number of boxes in row i of R by r_i and the number of boxes that must be removed from row i of R to obtain row i of r by m_i. We collect the m_i into a vector $\vec{m}$ and say that we remove $\vec{m}$ from R to obtain r. We are now ready to evaluate the trace over $V_p^{\otimes m}$. Assume that we remove the box from row i of R to obtain R', and from row j of T to obtain T'. From (3.37) we see that these terms have a coefficient of $\sqrt{c_{RR'}c_{TT'}} = \sqrt{(N+r_i)(N+r_j)}$. The intertwining maps are $I_{T'R'} = E_{ji}^{(1)}$ and $I_{R'T'} = E_{ij}^{(1)}$. We remove $\vec{m}$ from R to obtain r and $\vec{n}$ from T to obtain t. We will now explain how to evaluate the traces that appear in the matrix elements of the dilatation operator, using the displaced corners approximation. We need to write $\Gamma_R((1, m+1))$ in term of the E_{ij}^a as follows

$$\Gamma_R((1, m+1) = E_{ab}^{(1)}E_{ba}^{(m+1)} \tag{3.38}$$

and similarly for $\Gamma_T((1, m+1))$. The projection operators are written as

$$P_{R\to(r,s)_{\alpha\beta}} = \mathbb{1}_r \otimes p_{s_{\alpha\beta}}, \qquad P_{T\to(t,u)_{\mu\nu}} = \mathbb{1}_t \otimes p_{u_{\mu\nu}}. \tag{3.39}$$

3.2.3 Dilatation Operator Coefficient. In this subsection we explain how to evaluate the coefficient in equation (3.35) in the large N limit. Use the relation between the dimension and the hooks from equation (2.5), with simple algebraical manipulations, to get

$$\begin{aligned}\frac{d_T nm}{d_{R'} d_t d_u (n+m)}\sqrt{\frac{h_T h_r h_s}{h_R h_t h_u}} &= \frac{(m+n)!}{h_T} \times \frac{mn h_{R'} h_t h_u}{(m+n-1)!m!n!(m+n)}\sqrt{\frac{h_T h_r h_s}{h_R h_t h_u}} \\ &= \sqrt{\frac{h_t h_r}{h_T h_R}} \times \sqrt{h_u h_s}\frac{1}{(m-1)!(n-1)!} \times h_{R'}.\end{aligned} \tag{3.40}$$

In what follows, T_1 is the length of row 1 of T and R_2 is the length of row 2 of R. We use a similar notation for the length of the rows of r and t. In the displaced comes approximation we can set

$$\begin{aligned}R_i &= r_i\Big(1 + O\left(\frac{m}{n}\right)\Big), \\ T_i &= t_i\Big(1 + O\left(\frac{m}{n}\right)\Big).\end{aligned}$$

To leading order at large N we can write

$$\sqrt{\frac{h_r h_t}{h_R h_T}} = \frac{1}{\sqrt{T_1^{m_1} T_2^{m_2} R_1^{m_1} R_2^{m_2}}}.$$

Further, again at large N, we have $T_1 = R_1$ and $T_2 = R_2$ so that the above result becomes

$$\sqrt{\frac{h_r h_t}{h_R h_T}} = \frac{1}{R_1^{m_1} R_2^{m_2}}. \tag{3.41}$$

When computing the trace in equation (3.35) we will pick up a factor of $d_{r'}$. If we drop a box from row i of R to obtain R', then we also drop a box from row i of r to obtain r'[9]. Using the same idea, we can easily compute $\frac{h_{R'}}{h_{r'}}$, to find

$$\frac{h_{R'}}{h_{r'}} = R_1^{m_1} R_2^{m_2}. \tag{3.42}$$

Combining equations (3.41) and (3.42) we now find

$$\sqrt{\frac{h_r h_t}{h_R h_T}}\frac{h_{R'}}{h_{r'}} = 1.$$

The N dependence of the coefficient is

$$c_{RR'}\sqrt{\frac{f_T}{f_R}} = \sqrt{c_{RR'} c_{TT'}}. \tag{3.43}$$

3.2.4 Evaluating Traces. In this section we will discuses the evaluation of the trace appearing in equation (3.35). We start by writing this trace as a trace over $m+1$ slots (all the Y slots plus one Z slot) times a trace over $n-1$ slots (the remaining Z slots). The trace over the $n-1$ slots is over the carrier space R^{m+1} which is described by a Young diagram that can be obtained by removing $m+1$ boxes from R, or equivalently by removing one box from r or equivalently by removing one box from t - these all give the same Young diagram describing R^{m+1}. R^{m+1} has different shapes depending on where the $(m+1)^{\text{th}}$ box is removed. The results from the last subsection clearly imply that the dimension

of the symmetric group representation R^{m+1}, denoted $d_{R^{m+1}}$, depends on the details of this shape. If the $(m+1)^{\text{th}}$ box is removed from row i, denote this dimension by $d^i_{R^{m+1}}$. Our general strategy is to trace over the last Z slot (the $(m+1)^{\text{th}}$ slot) which then leaves a trace over $V_p^{\otimes m}$. This trace is then evaluated using elementary $U(p)$ representation theory.

The box removed from R to obtain R' is removed from the bth row of R and the box removed from T to obtain T' is removed from the ath row of T . After tracing over the $n-1$ Z slots associated to R^{m+1} (this produces a factor of $d^b_{R^{m+1}}$), multiplying the symmetric group elements $(1, m+1)$ with the intertwiner and then tracing over the $(m+1)^{\text{th}}$ slot we are left with a trace over $V_p^{\otimes m}$. To perform this final trace, we will use the result from subsection 2.5 which allows to us write the trace in terms of Clebsch-Gorden coefficient of $U(p)$.

3.2.5 The Spin Chain Descriptions. In this subsection we will focus on the case that R has two rows. In this case we can define a map from labeled Young diagrams onto spin chain states. This allows us to convert our dilatation operator expression into a particularly simple form. This form is an eigenvalue value problem, which we can solve explicitly. Each box in the Young diagram corresponds to a spin. A box in the first row is a spin up state, denote by 1. A box in the second row is a spin down state, denoted by 0. For example,

$$\begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & 4 & 2 & 1 \\ \hline \ & \ & 5 & 3 \\ \cline{1-4}\end{array} \leftrightarrow |11010\rangle. \tag{3.44}$$

A basis for the spin chain states is given by

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \quad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}. \tag{3.45}$$

The operators

$$a = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \quad a^\dagger = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} \tag{3.46}$$

act on these states as follows

$$\begin{aligned} a^\dagger|0\rangle = |1\rangle \quad a^\dagger|1\rangle = 0 \\ a|1\rangle = |0\rangle \quad a|0\rangle = 0. \end{aligned} \tag{3.47}$$

Here is an example of a spin chain state, for a spin chain with 5 lattice sites

$$|11011\rangle = |1\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle. \tag{3.48}$$

Using

$$a|1\rangle = |0\rangle \tag{3.49}$$

we can rewrite the spin chain state as

$$|11011\rangle = |1\rangle \otimes |1\rangle \otimes a|1\rangle \otimes |1\rangle \otimes |1\rangle. \tag{3.50}$$

We can write the operators a, $a^\dagger$ in terms of the Pauli matrices σ^1, σ^2 and σ^3 as follows,

$$a = \frac{1}{2}(\sigma^1 + i\sigma^2), \quad a^\dagger = \frac{1}{2}(\sigma^1 - i\sigma^2), \quad aa^\dagger = \frac{1}{2}(\mathbb{1} - \sigma^3) \tag{3.51}$$

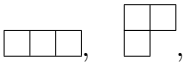
where

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and the total spin of the system is defined as follows [1]

$$J^1 = \sum_{p=1}^{L} \frac{1}{2}\sigma_p^1, \quad J^2 = \sum_{p=1}^{L} \frac{1}{2}\sigma_p^2, \quad J^3 = \sum_{p=1}^{L} \frac{1}{2}\sigma_p^3 \tag{3.52}$$
$$\vec{J}\cdot\vec{J} = J^1J^1 + J^2J^2 + J^3J^3.$$

We have written the formulas for a spin chain with a total of L lattice sites. Here capital letters refer to operators and small letters refer to eigenvalues. We can illustrate the connection between Young diagrams and spin chain states using a specific example and then state the general rule. Consider the case of 3 spins. The impurities can be organized into any irreducible representation corresponding to a Young diagram with 3 blocks and two rows. The possible irreducible representations are



(3.53)

and they have dimensions 1 and 2 respectively. Coupling 3 spin $\frac{1}{2}$ particles, using the usual rule for the addition of angular momentum, we have

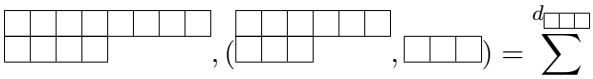
$$\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = \frac{3}{2}\oplus\frac{1}{2}\oplus\frac{1}{2}. \tag{3.54}$$

From equation (3.54) we notice that the number of possible Young diagrams for the impurities matches the number of angular momentum multiplets and likewise the degeneracy of each multiplet matches the dimension of the S_3 irreducible representation. These connections are a consequence of the Schur-Weyl duality between the symmetric groups and the unitary groups[27]. Notice that in the example there is no need for multiplicity labels. This is true in general for operators labeled by Young diagrams that have no more than two rows. Consequently, we drop the multiplicity labels from the restricted Schur polynomial in what follows.

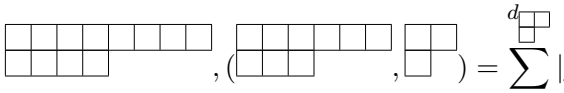
Now we consider a general case corresponding to the restricted Schur polynomial $\chi_{R,(r,s)}$. The $J^2 = j(j+1)$ quantum number tells us the shape of the Young diagram s. If number of boxes in the first row is s_1 and number of boxes in the second row is s_2, then $j = \frac{s_1-s_2}{2}$. The j^3 quantum number tells us how many impurities sit in the first and second rows of R, that is, it tells us how many boxes must be removed from each row of R to obtain r. Denote the number of impurities in the first and the second row by m_1 and m_2 respectively. We have

$$j^3 = \frac{m_1 - m_2}{2}. \tag{3.55}$$

The following examples show the projection operators appearing in the restricted Schur polynomials, written in terms of the j, j^3 states



$$= \sum_{i=1}^{d} |j = \frac{3}{2}, j^3 = \frac{1}{2}, i\rangle\langle j = \frac{3}{2}, j^3 = \frac{1}{2}, i| \tag{3.56}$$



$$= \sum_{i=1}^{d} |j = \frac{1}{2}, j^3 = \frac{1}{2}, i\rangle\langle j = \frac{1}{2}, j^3 = \frac{1}{2}, i|.$$

[1] The explicit description and calculation of the spin chain operators can be found in [27].

In the above, i labels all the states with the displayed (j, j^3) quantum numbers; it runs from 1 to the dimension of the irreducible representation organizing the impurities. The irreducible representation that organizes these $SU(2)$ multiplicities is the representation of S_m that the projection operator defining the restricted Schur polynomials projects to. These results represent a general construction of the projection operators used to define the restricted Schur polynomials.

When we evaluate the traces in equation (3.35) we need to consider the action of $\Gamma_R((1, m+1))$. To do this we need to add another spin site to our chain. $\Gamma_R((1, m+1))$ can then be taken to act on the first and $m+1$ th site of the spin chain. We could allow $\Gamma_R((1, m+1))$ to act on any of the first m sites and the $(m+1)$ th site of the spin chain without changing the final result[27]. Including this extra site our projectors become,

$$P_{R,(r,s)} = \sum_i |j, j^3, i\rangle\langle j, j^3, i| \otimes 1 = \sum_i |j, j^3, i\rangle\langle j, j^3, i| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)\,. \tag{3.57}$$

We can now evaluate the trace in equation (3.35),

$$\begin{aligned}
&\mathrm{Tr}\Big(\Big[\Gamma_R((1,m+1)), P_{R\to(r,s)}\Big] I_{R'T'} \Big[\Gamma_T((1,m+1)), P_{T\to(r,s)}\Big] I_{T'R'}\Big) \qquad (3.58)\\
&= \mathrm{Tr}\Big(\Gamma_R((1,m+1)) P_{R\to(r,s)} I_{R'T'} \Gamma_T((1,m+1)) P_{T\to(t,u)} I_{T'R'}\Big) - \mathrm{Tr}\Big(\Gamma_R((1,m+1)) P_{R\to(r,s)} I_{R'T'}\\
&\times P_{T\to(t,u)} \Gamma_T((1,m+1)) I_{T'R'}\Big) - \mathrm{Tr}\Big(P_{R\to(r,s)} \Gamma_R((1,m+1)) I_{R'T'} \Gamma_T((1,m+1)) P_{T\to(t,u)} I_{T'R'}\Big) +\\
&\mathrm{Tr}\Big(P_{R\to(r,s)} \Gamma_R((1,m+1)) I_{R'T'} P_{T\to(t,u)} \Gamma_T((1,m+1)) I_{T'R'}\Big)
\end{aligned}$$

Using equations (3.37), (3.38) and (3.39) we find

$$\begin{aligned}
&\mathrm{Tr}\Big(\Gamma_R((1,m+1)) P_{R\to(r,s)} I_{R'T'} \Gamma_T((1,m+1)) P_{T\to(t,u)} I_{T'R'}\Big) \qquad (3.59)\\
&= \mathrm{Tr}\Big(E^{(1)}_{ab} E^{(m+1)}_{ba} \mathbb{1}_r \otimes P_s E^{(1)}_{ij} E^{(1)}_{cd} E^{(m+1)}_{dc} \mathbb{1}_t \otimes P_u E^{(1)}_{ji}\Big)\\
&= \mathrm{Tr}\Big(\delta_{ia} E^{(1)}_{jb} E^{(m+1)}_{ba} \mathbb{1}_r \otimes P_s \delta_{jc} E^{(1)}_{id} E^{(m+1)}_{dc} \mathbb{1}_t \otimes P_u\Big)\\
&= \mathrm{Tr}\Big(E^{(1)}_{jb} P_s E^{(1)}_{id} P_u\Big)\, \mathrm{Tr}\Big(E^{(m+1)}_{bi} \mathbb{1}_r E^{(m+1)}_{dj} \mathbb{1}_t\Big)\\
&= \mathrm{Tr}\Big(E^{(1)}_{jj} P_s E^{(1)}_{ii} P_u\Big) dr'_i.
\end{aligned}$$

In the following calculation we set $i = 2$ and $j = 1$. In this case $E^1_{jj} = E^1_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $E^1_{ii} = E^1_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Here we have chosen $R = T$ as an example

$$R, (r,s) \to (r, j, j^3) \Rightarrow s \to j, j^3, \qquad T, (t,u) \to (r, j, j^3) \Rightarrow u \to j, j^3.$$

Decompose the states appearing in our projector as

$$\begin{aligned}
|jj^3; I_S\rangle &= \sqrt{\frac{j+j^3}{2j}} \left|j - \frac{1}{2}, j^3 - \frac{1}{2}, I_{s'_1}; \frac{1}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{j-j^3}{2j}} \left|j - \frac{1}{2}, j^3 + \frac{1}{2}, I_{s'_2}; \frac{1}{2}\frac{-1}{2}\right\rangle \qquad (3.60)\\
&- \sqrt{\frac{j-j^3+1}{2(j+1)}} \left|j + \frac{1}{2}, j^3 - \frac{1}{2}, I_{s'_1}; \frac{1}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{j+j^3+1}{2(j+1)}} \left|j + \frac{1}{2}, j^3 + \frac{1}{2}, I_{s'_2}; \frac{1}{2}\frac{-1}{2}\right\rangle. \qquad (3.61)
\end{aligned}$$

This decomposition can be written as a linear combination of the states

$$|\psi_s, I_s\rangle \equiv \sqrt{\frac{j+j^3}{2j}}\left|j-\frac{1}{2}, j^3-\frac{1}{2}, I_{s_1'}; \frac{1}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{j-j^3}{2j}}\left|j-\frac{1}{2}, j^3+\frac{1}{2}, I_{s_2'}; \frac{1}{2}\frac{-1}{2}\right\rangle, \tag{3.62}$$
$$|\phi_s, I_s\rangle \equiv -\sqrt{\frac{j-j^3+1}{2(j+1)}}\left|j+\frac{1}{2}, j^3-\frac{1}{2}, I_{s_1'}; \frac{1}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{j+j^3+1}{2(j+1)}}\left|j+\frac{1}{2}, j^3+\frac{1}{2}, I_{s_2'}; \frac{1}{2}\frac{-1}{2}\right\rangle.$$

We will also need

$$|\phi_u, I_u\rangle \equiv \sqrt{\frac{j+j^3}{2j}}\left|j+\frac{1}{2}, j^3-\frac{1}{2}, I_{u_1'}; \frac{1}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{j-j^3}{2j}}\left|j+\frac{1}{2}, j^3+\frac{1}{2}, I_{u_2'}; \frac{1}{2}\frac{-1}{2}\right\rangle, \tag{3.63}$$
$$|\psi_u, I_u\rangle \equiv -\sqrt{\frac{j-j^3+1}{2(j+1)}}\left|j+\frac{3}{2}, j^3-\frac{1}{2}, I_{u_1'}; \frac{1}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{j+j^3+1}{2(j+1)}}\left|j+\frac{3}{2}, j^3+\frac{1}{2}, I_{u_2'}; \frac{1}{2}\frac{-1}{2}\right\rangle.$$

To obtain these result we have made use of the following $SU(2)$ Clebsch-Gordon coefficients

$$\left\langle j-\frac{1}{2}, j^3-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right| |j, j^3\rangle = \sqrt{\frac{j+j^3}{2j}}, \quad \left\langle j+\frac{1}{2}, j^3-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right| |j, j^3\rangle = -\sqrt{\frac{j-j^3+1}{2(j+1)}}, \tag{3.64}$$
$$\left\langle j-\frac{1}{2}, j^3+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\right| |j, j^3\rangle = \sqrt{\frac{j-j^3}{2j}}, \quad \left\langle j+\frac{1}{2}, j^3+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\right| |j, j^3\rangle = \sqrt{\frac{j+j^3+1}{2(j+1)}}$$

We can rewrite the projectors as follows,

$$P_s = \sum_{I_s} \left(|\psi_s, I_s\rangle \langle\psi_s, I_s| + |\phi_s, I_s\rangle \langle\phi_s, I_s|\right). \tag{3.65}$$
$$P_u = \sum_{I_u} \left(|\psi_u, I_u\rangle \langle\psi_u, I_u| + |\phi_u, I_u\rangle \langle\phi_u, I_u|\right).$$

In above equation we sum over all possible Young-Yamanouchi symbols, indexed by I_s. Using these projectors, we find

$$\mathrm{Tr}\left(E_{11}^{(1)} P_s E_{22}^{(1)} P_u\right) = \langle\psi_u, I_u| E_{11}^{(1)} |\psi_s, I_s\rangle \langle\psi_s, I_s| E_{22}^{(1)} |\psi_u, I_u\rangle + \langle\phi_u, I_u| E_{11}^{(1)} |\phi_s, I_s\rangle \langle\phi_s, I_s| E_{22}^{(1)} |\phi_u, I_u\rangle \tag{3.66}$$
$$= \left(\frac{j+j^3}{2j}\frac{j-j^3}{2j}\right) ds_1' + \left(\frac{j+j^3+1}{2(j+1)}\frac{j-j^3+1}{2(j+1)}\right) ds_2'. \tag{3.67}$$

where

$$\langle\psi_u, I_u| = (|\psi_u, I_u\rangle)^\dagger \equiv \left\langle j+\frac{3}{2}, j^3-\frac{1}{2}, I_{u_1'}; \frac{1}{2}\frac{1}{2}\right| \sqrt{\frac{j-j^3+1}{2(j+1)}} + \left\langle j+\frac{1}{2}, j^3+\frac{1}{2}, I_{u_2'}; \frac{1}{2}\frac{-1}{2}\right| \sqrt{\frac{j-j^3+}{2j}}$$
$$\langle\phi_u, I_u| = (|\phi_u, I_u\rangle)^\dagger \equiv -\left\langle j+\frac{3}{2}, j^3-\frac{1}{2}, I_{u_1'}; \frac{1}{2}\frac{1}{2}\right| \sqrt{\frac{j-j^3}{2j}} + \left\langle j+\frac{1}{2}, j^3+\frac{1}{2}, I_{u_2'}; \frac{1}{2}\frac{-1}{2}\right| \sqrt{\frac{j-j^3}{2j}}.$$

We also need to evaluate $\frac{h_s}{h_{s'_2}}$. We will compute this value using a toy example from which we can extract the general formula. The toy example is

$$\frac{h_s}{h_{s'_1}} = \frac{h_u}{h_{s'_1}} = \frac{\begin{array}{|c|c|c|c|c|c|c|c|}\hline 9&8&7&6&4&3&2&1\\\hline\end{array}\;\begin{array}{|c|c|c|c|}\hline 4&3&2&1\\\hline\end{array}}{\begin{array}{|c|c|c|c|c|c|c|}\hline 8&7&6&5&3&2&1\\\hline\end{array}\;\begin{array}{|c|c|c|c|}\hline 4&3&2&1\\\hline\end{array}} = \frac{4.9}{5} = \frac{j(m+2j+2)}{(2j+1)}, \tag{3.68}$$

$$\frac{h_s}{h_{s'_2}} = \frac{h_u}{h_{s'_2}} = \frac{\begin{array}{|c|c|c|c|c|c|c|c|}\hline 9&8&7&6&4&3&2&1\\\hline\end{array}\;\begin{array}{|c|c|c|c|}\hline 4&3&2&1\\\hline\end{array}}{\begin{array}{|c|c|c|c|c|c|c|c|}\hline 9&8&7&5&4&3&2&1\\\hline\end{array}\;\begin{array}{|c|c|c|}\hline 3&2&1\\\hline\end{array}} = \frac{6.4}{5} = \frac{(j+1)(m-2j)}{2j+1}. \tag{3.69}$$

We have given the general formula for the final result. Putting everything together we can write the action of the dilatation operator on restricted Schur polynomials as follows

$$\begin{aligned} DO(r,j,j^3) =& g_{YM}^2 \left[-\frac{1}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)} \right) \Delta O(r,j,j^3) + \right. \\ & \sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}} \left(\frac{(j+j^3+1)(j-j^3-1)}{2(j+1)} \right) \Delta O(r,(j+1),j^3) + \\ & \left. \sqrt{\frac{(m+2j+2)(m-2j)}{(2j+1)(2j-1)}} \left(\frac{(j+j^3)(j-j^3)}{2j} \right) \Delta O(r,(j-1),j^3) \right], \end{aligned} \tag{3.70}$$

where

$$\begin{aligned} \Delta O(r,j,j^3) = & \sqrt{(N+R_1)(N+R_2-2)} O(r+1,j,j^3) + O(r-1,j+1,j^3) \\ & - (2N+R_1+R_2-2) O(r,j,j^3). \end{aligned}$$

This completes our review of the one loop dilatation operator. Using the same methods, we will compute the action of the two loop dilatation operator in the next chapter. This is the beginning of the novel results obtained in this study.

4. Two Loop Dilatation Generator

In this chapter we will discuss the two loop dilatation operator. Assuming that the number of Z fields (n) is very much greater than the number of Y fields (m), we find that there is a leading contribution and a subleading contribution. The subleading contribution is supressed by a relative factor of $\frac{m}{n}$. The leading term includes two derivatives with respect to Z and one with respect to Y. It has been studied in [11]. This computation is reviewed below. The subleading term includes two derivatives with respect to Y and one with respect to Z. This term has not been considered before and evaluating it is one of the main accomplishments of this dissertation.

The two loop dilatation generator is [24],

$$\begin{aligned} D_4 = &-2g^2 : \mathrm{Tr}\left(\left[[Y,Z],\frac{\partial}{\partial Z}\right]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],Z\right]\right) : \\ &-2g^2 : \mathrm{Tr}\left(\left[[Y,Z],\frac{\partial}{\partial Y}\right]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],Y\right]\right) : \\ &-2g^2 : \mathrm{Tr}\left([[Y,Z],T^a]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],T^a\right]\right) : \end{aligned} \tag{4.1}$$

where :: denotes normal ordering, indicating that derivatives do not act on fields within the ::, and

$$g = \frac{g_{YM}^2}{16\pi^2}.$$

4.1 The calculation of the terms involving Two Z Derivatives

The leading contribution to the dilatation operator is from the term

$$: \mathrm{Tr}\left(\left[[Y,Z],\frac{\partial}{\partial Z}\right]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],Z\right]\right) : . \tag{4.2}$$

Acting on a restricted Schur polynomial, we obtain

$$:\mathrm{Tr}\left(\left[[Y,Z],\frac{\partial}{\partial Z}\right]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],Z\right]\right) : \frac{1}{m!n!}\sum_{\sigma\in S_{n+m}}\chi_{R(r,s)_{\alpha\beta}}(\sigma)\left(Y^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\right) \tag{4.3}$$

$$\begin{aligned} =: \mathrm{Tr}\Bigg((YZ-ZY)\frac{\partial}{\partial Z}\left(\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right)Z-(YZ-ZY)\frac{\partial}{\partial Z}Z\left(\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right) \\ -\frac{\partial}{\partial Z}(YZ-ZY)\left(\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right)Z+\frac{\partial}{\partial Z}(YZ-ZY)Z\left(\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right)\Bigg) : \\ \times\frac{1}{m!n!}\sum_{\sigma\in S_{n+m}}\chi_{R(r,s)_{\alpha\beta}}(\sigma)\left(Y^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\right). \end{aligned}$$

One of the terms contributing to the above expression is

$$\left(\frac{\partial}{\partial Z_a^b}\left(\frac{\partial}{\partial Y_b^e}\frac{\partial}{\partial Z_e^c}-\frac{\partial}{\partial Z_b^e}\frac{\partial}{\partial Y_e^c}\right)\left(Z_d^c(YZ-ZY)_a^d\right)\right)\chi_{R(r,s)_{\alpha\beta}}(Y,Z) \tag{4.4}$$
$$=\frac{1}{(m-2)!(n-1)!}\sum_{\sigma\in S_{m+n}}\chi_{R(r,s)_{\alpha\beta}}(\sigma)\Big[[(ZYZ-ZZY)]^{i_{m+1}}_{i_{\sigma(m+2)}}(\delta^{i_{m+2}}_{i_{\sigma(1)}}\delta^{i_1}_{i\sigma(m+1)})$$
$$-[(ZYZ-ZZY)]^{i_1}_{i_{\sigma(m+2)}}(\delta^{i_{m+1}}_{i_{\sigma(1)}}\delta^{i_{m+2}}_{i_{\sigma(m+1)}})\Big]\left(Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+3}}_{i_{\sigma(m+3)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\right)\chi_{R,(r,s)_{\alpha\beta}}\Big(Y,Z\Big).$$

We can simplify the first and second terms by a change of variable σ. For the first term set $\sigma=(1,m+2)\tau(m+1,m+2)$. For the second term set $\sigma=(1,m+1)\tau(m+1,m+2)$. Apply the above change of variables and define $D'^1=\left(\frac{\partial}{\partial Z_a^b}\left(\frac{\partial}{\partial Y_b^e}\frac{\partial}{\partial Z_e^c}-\frac{\partial}{\partial Z_b^e}\frac{\partial}{\partial Y_e^c}\right)\left(Z_d^c(YZ-ZY)_a^d\right)\right)$ to get

$$D'^1\chi_{R(r,s)_{\alpha\beta}}(Y,Z)=\frac{1}{(m-2)!(n-1)!}\sum_{\tau\in S_{m+n}}\chi_{R(r,s)_{\alpha\beta}}\Big((1,m+2)\tau(m+1,m+2) \tag{4.5}$$
$$-(1,m+1)\tau(m+1,m+2)\Big)\delta^{i_1}_{i_{\tau(1)}}Y^{i_{m+2}}_{i_{\tau(m+2)}}Y^{i_3}_{i_{\tau(3)}}\cdots Y^{i_m}_{i_{\tau(m)}}$$
$$(ZYZ-ZZY)^{i_{m+1}}_{i_{\tau(m+1)}}\delta^{i_{m+2}}_{i_{\tau(m+2)}}Z^{i_{m+3}}_{i_{\tau(m+3)}}\cdots Z^{i_{m+n}}_{i_{\tau(m+n)}}.$$

Next, by replacing $\tau\to(2,m+2)\tau(2,m+2)$, we find

$$D'^1\chi_{R(r,s)_{\alpha\beta}}(Y,Z)=\frac{1}{(m-2)!(n-1)!}\sum_{\tau\in S_{m+n}}\chi_{R(r,s)_{\alpha\beta}}\Big((1,m+2,2)\tau(2,m+2,m+1) \tag{4.6}$$
$$-(1,m+1)(2,m+2)\tau(2,m+2,m+1)\Big)\delta^{i_1}_{i_{\tau(1)}}\delta^{i_2}_{i_{\tau(2)}}Y^{i_3}_{i_{\tau(3)}}\cdots Y^{i_m}_{i_{\tau(m)}}$$
$$(ZYZ-ZZY)^{i_{m+1}}_{i_{\tau(m+1)}}Y^{i_{m+2}}_{i_{\tau(m+2)}}Z^{i_{m+3}}_{i_{\tau(m+3)}}\cdots Z^{i_{m+n}}_{i_{\tau(m+n)}}.$$

Focus on the product of fields in the above expression. Recalling that we only sum over permutations with $\tau(1)=1$ and $\tau(2)=2$, we can write the product of fields as

$$\mathrm{Tr}\left((2,m+2)\tau(2,m+2)\left[(1,m+1,2)-(1,2,m+1)\right]Y^{\otimes 2}\otimes Y^{\otimes m-2}\otimes Z^{\otimes 2}\otimes Z^{\otimes n-2}\right) \tag{4.7}$$
$$=\mathrm{Tr}\left((2,m+2)\tau(2,m+2)\left[(1,m+1,2)-(1,2,m+1)\right]Y^{\otimes m}Z^{\otimes n}\right).$$

We then obtain

$$D'^1\chi_{R(r,s)_{\alpha\beta}}(Y,Z)=\frac{1}{(m-1)!(n-2)!}\sum_{\tau\in S_{m+n}}\chi_{R(r,s)_{\alpha\beta}}\Big((1,m+2,2)\tau(2,m+2,m+1)- \tag{4.8}$$
$$(1,m+1)(2,m+2)\tau(2,m+2,m+1)\Big)\mathrm{Tr}\Big(\left[(2,m+2)\tau(m+2,2,m+1)\right.$$
$$\left.-(2,m+2)\tau(m+2,2,m+1,1)\right]Y^{\otimes m}Z^{\otimes n}\Big).$$

We can reduce the sum over S_{n+m} to sum over S_{n+m-2} using the same steps that lead to equation (3.27) in section 3. The sum over σ is now easily carried out by using the fundamental orthogonality

theorem. A little care is needed since we are summing over S_{n+m-2} but representations of S_{n+m} appear. Recall that the fundamental orthogonality relation is

$$\sum_{\sigma\in\mathcal{G}}\Gamma_R(\sigma)_{ab}\Gamma_S(\sigma^{-1})_{cd}=\frac{|\mathcal{G}|}{d_R}\delta_{RS}\delta_{ad}\delta_{cb}. \tag{4.9}$$

Keeping in mind the discussion after equation (3.29) in section 3, we finally obtain

$$\begin{aligned}
&D'^1\chi_{R(r,s)_{\alpha\beta}}(YZ)\frac{d_T mn(n-1)}{d_t d_u(m+n)!}\sum_{R'}c_{RR'}c_{R'R''}\Big[\mathsf{Tr}\big(I_{T''R''}(2,m+2,m+1)P_{R\to(r,s)_{\alpha\beta}}(1,m+2,2)\\
&I_{R''T''}(2,m+2)P_{T\to(t,u)_{\gamma\delta}}(m+2,2,1,m+1)\big)-\mathsf{Tr}\big(I_{T''R''}(2,m+2,m+1)P_{R\to(r,s)_{\alpha\beta}}(1,m+2,2)\\
&I_{R''T''}(2,m+2)P_{T\to(t,u)_{\gamma\delta}}(m+2,2,m+1,1)\big)-\mathsf{Tr}\big(I_{T''R''}(2,m+2,m+1)P_{R\to(r,s)_{\alpha\beta}}(1,m+1)(m+2,2)\\
&I_{R''T''}(2,m+2)P_{T\to(t,u)_{\gamma\delta}}(m+2,2,1,m+1)\big)+\mathsf{Tr}\Big(I_{T''R''}(2,m+2,m+1)P_{R\to(r,s)_{\alpha\beta}}(1,m+1)(m+2,2)\\
&I_{R''T''}(2,m+2)P_{T\to(t,u)_{\gamma\delta}}(m+2,2,m+1,1)\big)\Big].
\end{aligned}\tag{4.10}$$

To obtain representations of the subgroups S_{m+n-2} from the representation of the group S_{m+n} we need to drop two boxes. We will use the following conventions,

$$R\quad\overset{i}{\to}\quad R'\overset{j}{\to}\quad R'',\qquad T\quad\overset{k}{\to}\quad T'\overset{l}{\to}\quad T'',$$

where we have indicated the row from which the box must be dropped to obtain the subgroup representation. With these conventions, and with the discussion of section 3 in mind, we can write the intertwiners $I_{T''R''}$, $I_{R''T''}$ as

$$I_{T''R''}=E^{(1)}_{ki}E^{(2)}_{lj}E^{(m+1)}_{tt}E^{(m+2)}_{uu},\qquad I_{R''T''}=E^{(1)}_{ik}E^{(2)}_{jl}E^{(m+1)}_{rr}E^{(m+2)}_{ss}. \tag{4.11}$$

Note that in the above expressions repeated indices are summed. To compute the action of the permutations on the intertwiner act on the rows of the intertwiner with the permutation and act on the columns of the intertwiner with the inverse permutation. We find

$$(m+2,2,1,m+1)I_{T''R''}(2,m+2,m+1)=E^{(1)}_{li}E^{(m+1)}_{kj}, \tag{4.12}$$

$$(m+2,2,m+1,1)I_{T''R''}(2,m+2,m+1)=E^{(1)}_{ti}E^{(m+1)}_{lj}E^{(m+2)}_{kt}, \tag{4.13}$$

$$(1,m+2,2)I_{R''T''}(2,m+2)=E^{(1)}_{jk}E^{(m+2)}_{il}, \tag{4.14}$$

$$(1,m+1)(2,m+2)I_{R''T''}(2,m+2)=E^{(1)}_{rk}E^{(m+1)}_{ir}E^{(m+2)}_{jl}. \tag{4.15}$$

Recall the definition of the intertwining map

$$P_{R\to(r,s)_{\alpha\beta}}=p_{R\to(r,s)_{\alpha\beta}}\otimes\mathbb{1}_r. \tag{4.16}$$

In the above, $P_{R\to(r,s)_{\alpha\beta}}$ acts in the carrier space of R, $p_{R\to(r,s)_{\alpha\beta}}$ acts in $V^{\otimes m}$ and $\mathbb{1}_r$ is the identity in the r subspace of R. We are now in a position to evaluate the traces appearing in (4.10). For the first

term, the result is only non-zero when $k = j$ and $l = i$. Let $\vec{m}$ be a vector which records how many boxes must be dropped from R to obtain r. The trace is only non-zero if the vector describing how to go from R to r is equal to the vector describing how to go from T to t. We summarize this by saying that for this term, $\vec{m}$ is conserved. The representations which participate are

$$\begin{array}{cccccccc} R & \xrightarrow{i} & R' & \xrightarrow{j} & R'', & r & \xrightarrow{j} & r' \xrightarrow{i} r'', \\ T & \xrightarrow{j} & T' & \xrightarrow{i} & T'', & t & \xrightarrow{j} & t' \xrightarrow{i} t''. \end{array}$$

The result from the first trace in (4.10) is

$$\begin{aligned} & c_{R,R'_i} c_{R'_i,R''_{ik}} \mathsf{Tr}\left(E^{(1)}_{li} E^{(m+1)}_{kj} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{jk} E^{(m+2)}_{il} p_{T\to(t,u)_{\gamma\delta}}\right) \\ & = c_{R,R'_i} c_{R'_i,R''_{ik}} dr''_{ik} \mathsf{Tr}\left(E^{(1)}_{ii} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{kk} p_{T\to(t,u)_{\gamma\delta}}\right). \end{aligned} \tag{4.17}$$

For the second trace in (4.10) we only obtain a non-zero result if $j = k = l$, $i = t$ and $\vec{m}$ is conserved. The representations which participate are

$$\begin{array}{cccccccc} R & \xrightarrow{i} & R' & \xrightarrow{j} & R'', & r & \xrightarrow{j} & r' \xrightarrow{i} r'', \\ T & \xrightarrow{j} & T' & \xrightarrow{j} & T'', & t & \xrightarrow{j} & t' \xrightarrow{j} t''. \end{array}$$

The second trace evaluates to

$$\begin{aligned} & - c_{R,R'} c_{R',R''} \mathsf{Tr}\left(E^{(1)}_{ti} E^{(m+1)}_{lj} E^{(m+2)}_{kt} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{jk} E^{(m+2)}_{il} p_{T\to(t,u)_{\gamma\delta}}\right) \\ & = -c_{R,R'_i} c_{R'_i,R''_{ik}} dr''_{ik} \mathsf{Tr}\left(E^{(1)}_{ii} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{kk} p_{T\to(t,u)_{\gamma\delta}}\right). \end{aligned} \tag{4.18}$$

The third trace is only non-zero if $i = j = l$, $k = r$ and $\vec{m}$ is again conserved. The representations which participate are

$$\begin{array}{cccccccc} R & \xrightarrow{i} & R' & \xrightarrow{i} & R'', & r & \xrightarrow{i} & r' \xrightarrow{i} r'', \\ T & \xrightarrow{k} & T' & \xrightarrow{i} & T'', & t & \xrightarrow{k} & t' \xrightarrow{i} t''. \end{array}$$

The third trace evaluates to

$$\begin{aligned} & - c_{R,R'} c_{R',R''} \mathsf{Tr}\left(E^{(1)}_{li} E^{(m+1)}_{kj} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{rk} E^{(m+1)}_{ir} E^{(m+2)}_{jl} p_{T\to(t,u)_{\gamma\delta}}\right) \\ & = -c_{R,R'_i} c_{R'_i,R''_{ii}} dr''_{ii} \mathsf{Tr}\left(E^{(1)}_{ii} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{kk} p_{T\to(t,u)_{\gamma\delta}}\right). \end{aligned} \tag{4.19}$$

The last trace is only non-zero if $i = j = t$, $k = l = r$ and $\vec{m}$ is again conserved. The representations which participate are

$$\begin{array}{cccccccc} R & \xrightarrow{i} & R' & \xrightarrow{i} & R'', & r & \xrightarrow{i} & r' \xrightarrow{i} r'', \\ T & \xrightarrow{k} & T' & \xrightarrow{k} & T'', & t & \xrightarrow{k} & t' \xrightarrow{k} t'', \end{array}$$

and the trace evaluates to

$$c_{R,R'}c_{R',R''}\mathsf{Tr}\left(E^{(1)}_{ti}E^{(m+1)}_{lj}E^{(m+2)}_{kt}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{rk}E^{(m+1)}_{ir}E^{(m+2)}_{jl}p_{T\to(t,u)_{\gamma\delta}}\right) \tag{4.20}$$
$$= c_{R,R'_i}c_{R'_i,R''_{ii}}dr''_{ii}\mathsf{Tr}\left(E^{(1)}_{ii}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{kk}p_{T\to(t,u)_{\gamma\delta}}\right).$$

Next, we consider the term

$$-:\mathsf{Tr}\left([Y,Z]\frac{\partial}{\partial Z}Z\left[\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right]\right):\chi_{R,(r,s)_{\alpha\beta}}(Y,Z)\equiv D'^2\chi_{R,(r,s)_{\alpha\beta}}(Y,Z). \tag{4.21}$$

Using the same techniques as for the first term we considered, we obtain

$$D'^2\chi_{R,(r,s)_{\alpha\beta}}(Y,Z)=-\frac{d_T mn(m-1)}{d_t d_u(m+n)!}\sum_{R'}c_{RR'}\mathsf{Tr}\Big(I_{T'R'}(m+2,m+1,1)P_{R\to(r,s)_{\alpha\beta}} \tag{4.22}$$
$$\times I_{R'T'}(1,m+1)P_{T\to(t,u)_{\gamma\delta}}-I_{T'R'}(m+2,m+1,1)P_{R\to(r,s)_{\alpha\beta}}I_{R'T'}P_{T\to(t,u)_{\gamma\delta}}(1,m+1)$$
$$-I_{T'R'}(m+1,m+2)P_{R\to(r,s)_{\alpha\beta}}(1,m+1)I_{R'T'}(1,m+1)P_{T\to(t,u)_{\gamma\delta}}+I_{T'R'}(m+1,m+2)$$
$$\times P_{R\to(r,s)_{\alpha\beta}}(1,m+1)I_{R'T'}(1,m+1)P_{T\to(t,u)_{\gamma\delta}}(1,m+1)\Big).$$

The action of the permutations on the intertwiners give the following results

$$(1,m+1)I_{T'R'}(1,m+2,m+1)=E^{(1)}_{tu}E^{(m+1)}_{ki}E^{(m+2)}_{ut}, \tag{4.23}$$
$$I_{T'R'}(m+2,m+1)=E^{(1)}_{ki}E^{(m+1)}_{tu}E^{(m+2)}_{ut}, \tag{4.24}$$
$$(1,m+1)I_{R'T'}(m+1,m+2)=E^{(1)}_{ti}E^{(m+1)}_{ku}E^{(m+2)}_{ut}, \tag{4.25}$$
$$I_{R'T'}(1,m+1)=E^{(1)}_{ir}E^{(m+1)}_{rk}, \tag{4.26}$$
$$(1,m+1)I_{R'T'}(1,m+1)=E^{(m+1)}_{ik}, \tag{4.27}$$
$$(1,m+1)I_{R'T'}=E^{(1)}_{rk}E^{(m+1)}_{ir}. \tag{4.28}$$

Preforming the trace, we find that the first term in (4.22) is only non-zero if $k=u=t$, $i=r$ and $\vec{m}$ is conserved. The representations which play a role are

$$R \xrightarrow{i} R', \qquad r \xrightarrow{i} r' \xrightarrow{k} r'',$$
$$T \xrightarrow{k} T', \qquad t \xrightarrow{k} t' \xrightarrow{k} t''.$$

The trace evaluates to

$$c_{R,R'}\mathsf{Tr}\left(E^{(1)}_{ku}E^{(m+1)}_{ti}E^{(m+2)}_{ut}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{ir}E^{(m+1)}_{rk}p_{T\to(t,u)_{\gamma\delta}}\right) \tag{4.29}$$
$$= c_{R,R'_i}dr''_{ik}\mathsf{Tr}\left(E^{(1)}_{kk}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{ii}p_{T\to(t,u)_{\gamma\delta}}\right).$$

The second term is only non-zero if $i=k$, $t=u$ and $\vec{m}$ is conserved. The representations that play a role are

$$\begin{array}{cccccccc} R & \stackrel{i}{\to} & R', & r & \stackrel{i}{\to} & r' & \stackrel{t}{\to} & r'', \\ T & \stackrel{i}{\to} & T', & t & \stackrel{i}{\to} & t' & \stackrel{t}{\to} & t''. \end{array}$$

Performing the trace we find

$$\begin{aligned} &- c_{R,R'}\mathsf{Tr}\left(E^{(1)}_{tu}E^{(m+1)}_{ki}E^{(m+2)}_{ut}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{ik}p_{T\to(t,u)_{\gamma\delta}}\right) \\ &= -c_{R,R'_i}dr''_{it}\mathsf{Tr}\left(E^{(1)}_{tt}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{ii}p_{T\to(t,u)_{\gamma\delta}}\right). \end{aligned} \tag{4.30}$$

After taking the trace we find that the third term is only non-vanishing if $i=k=t=u$ and $\vec{m}$ is conserved. The representations that play a role are

$$\begin{array}{cccccccc} R & \stackrel{i}{\to} & R', & r & \stackrel{i}{\to} & r' & \stackrel{i}{\to} & r'', \\ T & \stackrel{i}{\to} & T', & t & \stackrel{i}{\to} & t' & \stackrel{i}{\to} & t''. \end{array}$$

After tracing we have

$$\begin{aligned} &- c_{R,R'}\mathsf{Tr}\left(E^{(1)}_{ki}E^{(m+1)}_{tu}E^{(m+2)}_{ut}p_{R\to(r,s)_{\alpha\beta}}E^{(m+1)}_{lk}p_{T\to(t,u)_{\gamma\delta}}\right) \\ &= -c_{R,R'_i}dr''_{ii}\mathsf{Tr}\left(E^{(1)}_{ii}p_{R\to(r,s)_{\alpha\gamma}}\right)\delta_{R,(r,s)T,(t,u)}\delta_{\beta\delta}. \end{aligned} \tag{4.31}$$

The fourth term is only non-vanishing if $i=t=u$, $k=r$ and $\vec{m}$ is conserved. The representations that participate are

$$\begin{array}{cccccccc} R & \stackrel{i}{\to} & R', & r & \stackrel{i}{\to} & r' & \stackrel{i}{\to} & r'', \\ T & \stackrel{k}{\to} & T', & t & \stackrel{k}{\to} & t' & \stackrel{i}{\to} & t''. \end{array}$$

The trace evaluates to

$$\begin{aligned} &c_{R,R'}\mathsf{Tr}\left(E^{(1)}_{ti}E^{(m+1)}_{ku}E^{(m+2)}_{ut}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{rk}E^{(m+1)}_{ir}p_{T\to(t,u)_{\gamma\delta}}\right) \\ &= c_{R,R'_i}dr''_{ii}\mathsf{Tr}\left(E^{(1)}_{ii}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{kk}p_{T\to(t,u)_{\gamma\delta}}\right). \end{aligned} \tag{4.32}$$

The next term we consider is

$$-:\mathsf{Tr}\left(Z\frac{\partial}{\partial Z}[Y,Z]\left[\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right]\right):\chi_{R,(r,s)_{\alpha\beta}}(Y,Z)\equiv D'^3\chi_{R,(r,s)_{\alpha\beta}}(Y,Z). \tag{4.33}$$

Arguing as above, we find

$$\begin{aligned} &D'^3\chi_{R,(r,s)_{\alpha\beta}}(Y^{\otimes m}\otimes Z^{\otimes n}) = -\frac{d_T mn(m-1)}{d_t d_u(m+n)!}\sum_{R'}c_{RR'}\mathsf{Tr}\Big(I_{T'R'}(1,m+1,m+2)P_{R\to(r,s)_{\alpha\beta}}I_{R'T'} \\ &(1,m+1)P_{T\to(t,u)_{\gamma\delta}} - I_{T'R'}(1,m+1,m+2)P_{R\to(r,s)_{\alpha\beta}}I_{R'T'}P_{T\to(t,u)_{\gamma\delta}}(1,m+1) - I_{T'R'} \\ &(m+1,m+2)P_{R\to(r,s)_{\alpha\beta}}(1,m+2)I_{R'T'}(1,m+1)P_{T\to(t,u)_{\gamma\delta}} + I_{T'R'}(m+1,m+2) \\ &P_{R\to(r,s)_{\alpha\beta}}(1,m+2)I_{R'T'}P_{T\to(t,u)_{\gamma\delta}}(1,m+1)\Big). \end{aligned} \tag{4.34}$$

To compute the right hand side of the above equation, we need to evaluate

$$I_{T'R'}(1,m+1,m+2) = E^{(1)}_{kt}E^{(m+1)}_{tu}E^{(m+2)}_{ui}, \tag{4.35}$$
$$(1,m+1)I_{T'R'}(1,m+1,m+2) = E^{(1)}_{ku}E^{(m+2)}_{ui}, \tag{4.36}$$
$$I_{T'R'}(m+2,m+1) = E^{(1)}_{ki}E^{(m+1)}_{tu}E^{(m+2)}_{ut}, \tag{4.37}$$
$$(1,m+1)I_{T'R'}(m+2,m+1) = E^{(1)}_{ti}E^{(m+1)}_{ku}E^{(m+2)}_{ut}, \tag{4.38}$$
$$I_{R'T'}(1,m+1) = E^{(1)}_{ir}E^{(m+1)}_{rk}, \tag{4.39}$$
$$(1,m+2)I_{R'T'}(1,m+1) = E^{(1)}_{rs}E^{(m+1)}_{rk}E^{(m+2)}_{is}, \tag{4.40}$$
$$(1,m+2)I_{R'T'} = E^{(1)}_{sk}E^{(m+2)}_{is}. \tag{4.41}$$

There are again four traces to compute. The first trace is only non-zero if $i=r=u$, $k=t$ and $\vec{m}$ is conserved. The representations we use are

$$\begin{array}{cccccccc} R & \stackrel{i}{\to} & R', & r & \stackrel{i}{\to} & r' & \stackrel{i}{\to} & r'', \\ T & \stackrel{k}{\to} & T', & t & \stackrel{k}{\to} & t' & \stackrel{i}{\to} & t''. \end{array}$$

The value of this term is

$$\begin{aligned} &c_{R,R'}\mathsf{Tr}\left(E^{(1)}_{kt}E^{(m+1)}_{tu}E^{(m+2)}_{ui}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{ir}E^{(m+1)}_{rk}p_{T\to(t,u)_{\gamma\delta}}\right) \\ &= c_{R,R'_i}dr''_{ii}\mathsf{Tr}\left(E^{(1)}_{kk}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{ii}p_{T\to(t,u)_{\gamma\delta}}\right). \end{aligned} \tag{4.42}$$

For a non-zero result for the second trace, we need $i=k=u$ and $\vec{m}$ is conserved. The representations which participate are

$$\begin{array}{cccccccc} R & \stackrel{i}{\to} & R', & r & \stackrel{i}{\to} & r' & \stackrel{i}{\to} & r'', \\ T & \stackrel{i}{\to} & T', & t & \stackrel{i}{\to} & t' & \stackrel{i}{\to} & t''. \end{array}$$

And this term evaluates to

$$\begin{aligned} &-c_{R,R'}\mathsf{Tr}\left(E^{(1)}_{tt}E^{(m+1)}_{ku}E^{(m+2)}_{ui}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{ik}p_{T\to(t,u)_{\gamma\delta}}\right) \\ &= -c_{R,R'_i}dr''_{ii}\mathsf{Tr}\left(E^{(1)}_{ii}p_{R\to(r,s)_{\delta\beta}}\right)\delta_{R,(r,s);T,(t,u)}\delta_{\alpha\gamma}. \end{aligned} \tag{4.43}$$

The third trace is non-zero if and only if $s=u=r$, $i=k=t$ and $\vec{m}$ is conserved. The representations which play a role are

$$\begin{array}{cccccccc} R & \stackrel{i}{\to} & R', & r & \stackrel{s}{\to} & r' & \stackrel{i}{\to} & r'', \\ T & \stackrel{i}{\to} & T', & t & \stackrel{i}{\to} & t' & \stackrel{s}{\to} & t''. \end{array}$$

The third trace evaluates to

$$
\begin{aligned}
&- c_{R,R'}\mathsf{Tr}\left(E^{(1)}_{ki}E^{(m+1)}_{tu}E^{(m+2)}_{ut}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{sr}E^{(m+1)}_{rk}E^{(m+2)}_{is}p_{T\to(t,u)_{\gamma\delta}}\right) \qquad (4.44)\\
&= -c_{R,R'_i}dr''_{is}\mathsf{Tr}\left(E^{(1)}_{ii}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{ss}p_{T\to(t,u)_{\delta\gamma}}\right).
\end{aligned}
$$

Finally, for the fourth trace we get a non-zero result if $k=s=u$, $i=t$ and $\vec{m}$ is conserved. When evaluating this term, the representations we use are

$$
\begin{array}{cccccccc}
R & \stackrel{i}{\to} & R', & r & \stackrel{k}{\to} & r' & \stackrel{i}{\to} & r'',\\
T & \stackrel{k}{\to} & T', & t & \stackrel{k}{\to} & t' & \stackrel{k}{\to} & t''.
\end{array}
$$

The forth term evaluates to

$$
\begin{aligned}
&c_{R,R'}\mathsf{Tr}\left(E^{(1)}_{ti}E^{(m+1)}_{ku}E^{(m+2)}_{ut}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{sk}E^{(m+1)}_{is}p_{T\to(t,u)_{\gamma\delta}}\right) \qquad (4.45)\\
&= c_{R,R'_i}dr''_{ki}\mathsf{Tr}\left(E^{(1)}_{ii}p_{R\to(r,s)_{\alpha\beta}}E^{(1)}_{kk}p_{T\to(t,u)_{\gamma\delta}}\right).
\end{aligned}
$$

The last term we need to consider is

$$
:\mathsf{Tr}\left((YZ-ZY)Z\left(\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right)\frac{\partial}{\partial Z}\right):\chi_{R(r,s)_{\alpha\beta}}(Y,Z)\equiv D'^4\chi_{R(r,s)_{\alpha\beta}}(Y,Z). \qquad (4.46)
$$

After some work we obtain

$$
\begin{aligned}
D'^4\chi_{R(r,s)_{\alpha\beta}}(Y^{\otimes m}\otimes Z^{\otimes n}) &= \frac{d_T mn(n-1)}{d_t d_u(m+n)!}\sum_{R'}c_{RR'}c_{R'R''}\Big[\mathsf{Tr}\big(I_{T''R''}(2,m+2)(1,m+1)P_{R\to(r,s)_{\alpha\beta}} \qquad (4.47)\\
&(2,m+2)(1,2)I_{R''T''}(1,m+1)(2,m+2)P_{T\to(t,u)_{\gamma\delta}}(m+2,2,m+1)\big)-\mathsf{Tr}\big(I_{T''R''}(1,m+1)(2,m+2)\\
&P_{R\to(r,s)_{\alpha\beta}}(1,m+2,2)I_{R''T''}(m+1,2,m+2)P_{T\to(t,u)_{\gamma\delta}}(m+2,2,1)\big)-\mathsf{Tr}\big(I_{T''R''}(2,m+2,m+1)\\
&P_{R\to(r,s)_{\alpha\beta}}(1,m+2,2)I_{R''T''}(1,m+1)(2,m+2)P_{T\to(t,u)_{\gamma\delta}}(m+2,2,m+1)\big)+\mathsf{Tr}\big(I_{T''R''}(2,m+2,m+1)\\
&P_{R\to(r,s)_{\alpha\beta}}(1,m+2,2)I_{R''T''}(m+1,2,m+2)P_{T\to(t,u)_{\gamma\delta}}(m+2,2,1)\big)\Big].
\end{aligned}
$$

To evaluate this expression, we need the following results

$$
(m+2,2,m+1)I_{T''R''}(1,m+1)(2,m+2)=E^{(1)}_{kt}E^{(m+1)}_{li}E^{(m+2)}_{tj}, \qquad (4.48)
$$

$$
(m+2,2,1)I_{T''R''}(1,m+1)(2,m+2)=E^{(1)}_{lt}E^{(m+1)}_{ti}E^{(m+2)}_{kj}, \qquad (4.49)
$$

$$
(1,m+2,2)I_{T''R''}(2,m+2,m+1)=E^{(1)}_{li}E^{(m+1)}_{tj}E^{(m+2)}_{kt}, \qquad (4.50)
$$

$$
(m+2,2,m+1)I_{T''R''}(2,m+2,m+1)=E^{(1)}_{ki}E^{(m+1)}_{lj}, \qquad (4.51)
$$

$$(1, m+2, 2) I_{R''T''}(1, m+1)(2, m+2) = E^{(1)}_{jr} E^{(m+1)}_{rk} E^{(m+2)}_{il}, \tag{4.52}$$

$$(1, m+2, 2) I_{R''T''}(m+1, 2, m+2) = E^{(1)}_{jk} E^{(m+1)}_{rl} E^{(m+2)}_{ir}. \tag{4.53}$$

To compute the evaluation of the action of D'^4 we again need to evaluate four traces. To get a non-zero result for the first term we need $i = j$, $k = l = t$ and again $\vec{m}$ is conserved. The representations that appear are

$$\begin{array}{cccccccccc} R & \xrightarrow{i} & R' & \xrightarrow{i} & R'', & & r & \xrightarrow{i} & r' & \xrightarrow{i} & r'', \\ T & \xrightarrow{k} & T' & \xrightarrow{k} & T'', & & t & \xrightarrow{k} & t' & \xrightarrow{k} & t''. \end{array}$$

For the first term in (4.47) we now find

$$\begin{aligned} & c_{R,R'} c_{R',R''} \mathsf{Tr}\left(E^{(1)}_{kt} E^{(m+1)}_{li} E^{(m+2)}_{tj} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{jr} E^{(m+1)}_{rk} E^{(m+2)}_{il} p_{T\to(t,u)_{\gamma\delta}} \right) \\ & = c_{R,R'_i} c_{R'_i,R''_i} dr''_{ii} \mathsf{Tr}\left(E^{(1)}_{kk} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{ii} p_{T\to(t,u)_{\gamma\delta}} \right). \end{aligned} \tag{4.54}$$

The second trace is non-zero if $i = j = k = r$, $l = t$ and $\vec{m}$ is conserved. The representations needed to evaluate the trace are

$$\begin{array}{cccccccccc} R & \xrightarrow{i} & R' & \xrightarrow{i} & R'', & & r & \xrightarrow{i} & r' & \xrightarrow{i} & r'', \\ T & \xrightarrow{i} & T' & \xrightarrow{l} & T'', & & t & \xrightarrow{l} & t' & \xrightarrow{i} & t''. \end{array}$$

We obtain

$$\begin{aligned} & - c_{R,R'} c_{R',R''} \mathsf{Tr}\left(E^{(1)}_{lt} E^{(m+2)}_{kj} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{jk} p_{T\to(t,u)_{\gamma\delta}} \right) \\ & = -c_{R,R'_i} c_{R'_i,R''_i} dr''_{ii} \mathsf{Tr}\left(E^{(1)}_{ll} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{ii} p_{T\to(t,u)_{\gamma\delta}} \right). \end{aligned} \tag{4.55}$$

The third term is only non-zero if $i = j = k = r$, $l = t$ and $\vec{m}$ is conserved. The representations used in evaluating this term are

$$\begin{array}{cccccccccc} R & \xrightarrow{i} & R' & \xrightarrow{i} & R'', & & r & \xrightarrow{i} & r' & \xrightarrow{i} & r'', \\ T & \xrightarrow{l} & T' & \xrightarrow{l} & T'', & & t & \xrightarrow{l} & t' & \xrightarrow{l} & t''. \end{array}$$

We obtain

$$\begin{aligned} & - c_{R,R'} c_{R',R''} \mathsf{Tr}\left(E^{(1)}_{lt} E^{(m+1)}_{ti} E^{(m+2)}_{kj} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{jk} E^{(m+1)}_{rl} E^{(m+2)}_{ir} p_{T\to(t,u)_{\gamma\delta}} \right) \\ & = -c_{R,R'_i} c_{R'_i,R''_i} dr''_{ii} \mathsf{Tr}\left(E^{(1)}_{ll} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{ii} p_{T\to(t,u)_{\gamma\delta}} \right). \end{aligned} \tag{4.56}$$

To obtain a non-zero result for the fourth term, we must set $i = l = t$, $j = k = r$ and $\vec{m}$ is conserved. The representation we make use of are

$$\begin{aligned} R &\xrightarrow{i} R' \xrightarrow{j} R'', \qquad r \xrightarrow{j} r' \xrightarrow{i} r'', \\ T &\xrightarrow{j} T' \xrightarrow{i} T'', \qquad t \xrightarrow{i} t' \xrightarrow{j} t''. \end{aligned}$$

We obtain

$$\begin{aligned} & c_{R,R'} c_{R',R''} \mathsf{Tr}\left(E^{(1)}_{li} E^{(m+1)}_{tj} E^{(m+2)}_{kt} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{jk} E^{(m+1)}_{rl} E^{(m+2)}_{ir} p_{T\to(t,u)_{\gamma\delta}}\right) \\ & = c_{R,R'_i} c_{R'_i,R''_i} dr''_{ij} \mathsf{Tr}\left(E^{(1)}_{ii} p_{R\to(r,s)_{\alpha\beta}} E^{(1)}_{kk} p_{T\to(t,u)_{\gamma\delta}}\right). \end{aligned} \tag{4.57}$$

4.2 Collecting the final result

We are now ready to collect the final result. We will organize the answer by how many boxes move between R and T. We will do things concretely for 2 rows since this make the discussion most transparent. Denote the row lengths of R and T by R_i and T_i. There are a total of 5 distinct terms. We can write these in terms of an operator Δ that acts on the labels R and r of the normalized restricted Schur polynomial. We have already seen that all terms we evaluated conserve $\vec{m}$. As a consequence, R and r change in exactly the same way. Terms Δ^{--} and Δ^{++} allow two boxes to move between R and T. The terms Δ^{+} and Δ^{-} allow a single box to move between R and T. However, since these terms still come from the action of two derivatives, we know that two boxes in R will be involved, and these may belong to the same or different rows. Thus we will split Δ^{+} and Δ^{-} into two parts. The term Δ^{0} does not change the shape of R

Contributions from (4.4) and (4.46):

Term Δ^{--} : $R_1 = T_1+2$, $R_2 = T_2-2$. From (4.20) we get a contribution of $\mathsf{Tr}(E^{(1)}_{11} p_{R,(r,s)_{\alpha\beta}} E^{(1)}_{22} p_{T,(t,u)_{\delta\gamma}})$ to this term. From (4.54) we get a contribution of $\mathsf{Tr}(E^{(1)}_{22} p_{R,(r,s)\alpha\beta} E^{(1)}_{11} p_{T,(t,u)\gamma\delta})$. These contributions are hermittian conjugates of each other expected since the dilatation operator is Hermitian. For this term

$$\begin{aligned} c_{RR'} &= (N + R_1 - 1), \qquad c_{R'R''} = (N + R_1 - 2), \\ c_{TT'} &= (N + R_2), \qquad c_{T'T''} = (N + R_2 - 1). \end{aligned} \tag{4.58}$$

Term Δ^{++} : $R_1 = T_1-2$, $R_2 = T_2+2$. From (4.20) we get a contribution $\mathsf{Tr}(E^{(1)}_{22} p_{R,(r,s)_{\alpha\beta}} E^{(1)}_{11} p_{T,(t,u)\delta\gamma})$. From (4.54) we get $\mathsf{Tr}(E^{(1)}_{11} p_{R,(r,s)_{\alpha\beta}} E^{(1)}_{22} p_{T,(t,u)\delta\gamma})$. These contributions are hermittian conjugates of each other as we anticipated. For this term

$$\begin{aligned} c_{RR'} &= (N + R_2 - 2), \qquad c_{R'R''} = (N + R_2 - 3), \\ c_{TT'} &= (N + R_1 + 1), \qquad c_{T'T''} = (N + R_1). \end{aligned} \tag{4.59}$$

Term Δ^{-} : $R_1 = T_1 + 1$, $R_2 = T_2 - 1$. Remove one box from row 1 of R and one box from row 2 of R. From (4.18) we get $\mathsf{Tr}(E^{(1)}_{11} p_{R,(r,s)\alpha\beta} E^{(1)}_{22} p_{T,(t,u)\delta\gamma})$. From (4.55) we get $\mathsf{Tr}(E^{(1)}_{22} p_{R,(r,s)\alpha\beta} E^{(1)}_{11} p_{T,(t,u)\delta\gamma})$. These contributions are hermittian conjugates of each other as we anticipated. For this term

$$c_{RR'} = (N + R_2 - 2), \qquad c_{R'R''} = (N + R_2 - 1), \tag{4.60}$$
$$c_{TT'} = (N + R_2 - 1), \qquad c_{T'T''} = (N + R_1 - 2).$$

Term Δ^- : $R_1 = T_1 + 1$, $R_2 = T_2 - 1$. Remove two boxes from row 1 of R. From (4.19) we get $\mathrm{Tr}(E^{(1)}_{11} p_{R,(r,s)\alpha\beta} E^{(1)}_{22} p_{T,(t,u)\delta\gamma})$. From (4.55) we get $\mathrm{Tr}(E^{(1)}_{22} p_{R,(r,s)\alpha\beta} E^{(1)}_{11} p_{T,(t,u)\delta\gamma})$. These contributions are hermittian conjugates of each other as we anticipated. For this term

$$c_{RR'} = (N + R_1 - 1), \qquad c_{R'R''} = (N + R_1 - 2), \tag{4.61}$$
$$c_{TT'} = (N + R_1 - 2), \qquad c_{T'T''} = (N + R_2 - 1).$$

Term Δ^+ : $R_1 = T_1 - 1$, $R_2 = T_2 + 1$. Remove one box from row 1 of R and one box from row 2 of R. From (4.18) we get $\mathrm{Tr}(E^{(1)}_{22} p_{R,(r,s)\alpha\beta} E^{(1)}_{11} p_{T,(t,u)\delta\gamma})$. From (4.56) we get $\mathrm{Tr}(E^{(1)}_{11} p_{R,(r,s)\alpha\beta} E^{(1)}_{22} p_{T,(t,u)\delta\gamma})$. These contributions are hermittian conjugates of each other as we anticipated. For this term

$$c_{RR'} = (N + R_2 - 2), \qquad c_{R'R''} = (N + R_1 - 1), \tag{4.62}$$
$$c_{TT'} = (N + R_1), \qquad c_{T'T''} = (N + R_1 - 1).$$

Term Δ^+ : $R_1 = T_1 - 1$, $R_2 = T_2 + 1$. Remove two boxes from row 2 of R. From (4.19) we get $\mathrm{Tr}(E^{(1)}_{22} p_{R,(r,s)\alpha\beta} E^{(1)}_{11} p_{T,(t,u)\delta\gamma})$. From (4.55) we get $\mathrm{Tr}(E^{(1)}_{11} p_{R,(r,s)\alpha\beta} E^{(1)}_{22} p_{T,(t,u)\delta\gamma})$. These contributions are hermittian conjugates of each other as we anticipated. For this term

$$c_{RR'} = (N + R_2 - 2), \qquad c_{R'R''} = (N + R_2 - 3), \tag{4.63}$$
$$c_{TT'} = (N + R_1), \qquad c_{T'T''} = (N + R_2 - 3).$$

Term Δ^0 : $R_1 = T_1$, $R_2 = T_2$. Remove two boxes from row 1 or row 2 of R. We find contributions from (4.17), (4.18), (4.19), (4.20), (4.54), (4.55), (4.56) and (4.57) which sum to zero. If we remove one box from row 1 and one box from row 2, we get a non-zero result. From (4.17) we get $\mathrm{Tr}(E^{(1)}_{11} p_{R,(r,s)\alpha\beta} E^{(1)}_{22} p_{T,(t,u)\delta\gamma})$ and $\mathrm{Tr}(E^{(1)}_{22} p_{R,(r,s)\alpha\beta} E^{(1)}_{11} p_{T,(t,u)\delta\gamma}$. From (4.57) we get $\mathrm{Tr}(E^{(1)}_{11} p_{R,(r,s)\alpha\beta}$ $E^{(1)}_{22} p_{T,(t,u)\delta\gamma})$ and $\mathrm{Tr}(E^{(1)}_{22} p_{R,(r,s)\alpha\beta} E^{(1)}_{11} p_{T,(t,u)\delta\gamma})$ These contributions are hermittian conjugates of each other as we anticipated. For this term

$$c_{RR'} = c_{TT'} = (N + R_1 - 1), \qquad c_{R'R''} = c_{T'T''} = (N + R_2 - 2). \tag{4.64}$$

We need to include the normalization factor

$$\mathcal{N} = \sqrt{\frac{f_T d_R d_t d_u}{f_R d_T d_r d_s}}, \tag{4.65}$$

and the coefficient

$$\mathcal{C} = \frac{d_T mn(n-1) d r'' c_{RR'} c_{R'R''}}{d_t d_u d_{R''}(n+m)(n+m-1)}. \tag{4.66}$$

The product of $\mathcal{N}$ and C simplifies to

$$\sqrt{c_{RR'}c_{R'R''}c_{TT'}c_{T'T''}}\frac{m}{d_u d_s} \tag{4.67}$$

at large N.

Contributions from (4.21) and (4.33):

Term Δ^- : $R_1 = T_1 + 1$, $R_2 = T_2 - 1$. From (4.54) we get $\mathrm{Tr}(E^{(1)}_{22}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{12}}$. From (4.57) we get $\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{22}p_{T,(t,u)_{\delta\gamma}})d_{r''_{12}}$. From (4.29) we get $\mathrm{Tr}(E^{(1)}_{22}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{11}}$. From (4.32) we get $\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{22}p_{T,(t,u)_{\delta\gamma}})d_{r''_{12}}$. Summing these terms we find

$$\mathrm{Tr}(E^{(1)}_{22}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}}) + \mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{22}p_{T,(t,u)_{\delta\gamma}}d_{r'_1} \tag{4.68}$$

For these terms

$$c_{RR'} = N + R_1 - 1, \qquad c_{TT'} = N + R_2 - 1. \tag{4.69}$$

Term Δ^+ : $R_1 = T_1 - 1$, $R_2 = T_2 + 1$. From(4.54) we get $\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{22}p_{T,(t,u)_{\delta\gamma}})d_{r''_{12}}$. From (4.57) we get $\mathrm{Tr}(E^{(1)}_{22}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{22}}$. From (4.29) we get $\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{22}p_{T,(t,u)_{\delta\gamma}})d_{r''_{22}}$. From (4.32) we get $\mathrm{Tr}(E^{(1)}_{22}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{12}}$. Summing these terms we find

$$\mathrm{Tr}(E^{(1)}_{22}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}}) + \mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{22}p_{T,(t,u)_{\delta\gamma}}d_{r'_2}. \tag{4.70}$$

For these terms

$$c_{RR'} = N + R_2 - 2, \qquad c_{TT'} = N + R_1. \tag{4.71}$$

Term Δ^0 : $R_1 = T_1$, $R_2 = T_2$. Remove the box from the first row of R. From (4.54) we get $\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{11}}$. From (4.55) we get $-\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{11}}$, and $-\mathrm{Tr}(E^{(1)}_{22}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{12}}$. From (4.56) we get $-\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{11}}$, and $-\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{11}}$. From (4.57) we get $-\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{11}}$. From (4.29) we get $-\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{11}}$, and $-\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{22}p_{T,(t,u)_{\delta\gamma}})d_{r''_{11}}$.

From (4.30) we get $-\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{22}p_{T,(t,u)_{\delta\gamma}})d_{r''_{12}}$, and $-\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{11}}$. From (4.31) we get $\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{11}}$. From (4.32) we get $\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})d_{r''_{11}}$. There are some cancellations. The final sum gives

$$-d_{r''_1}\left(\mathrm{Tr}(E^{(1)}_{11}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{22}p_{T,(t,u)_{\delta\gamma}}) + \mathrm{Tr}(E^{(1)}_{22}p_{R,(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T,(t,u)_{\delta\gamma}})\right).$$

For these terms in the calculation we have

$$c_{RR'} = c_{TT'} = N + R_1 - 1. \tag{4.72}$$

Term Δ^0 : $R_1 = T_1$, $R_2 = T_2$. Remove the box from the second row of R. Again many terms contribute and there are again some cancellations. The final sum gives

$$-d_{r_2''}\left(\mathsf{Tr}(E_{11}^{(1)}p_{R,(r,s)_{\alpha\beta}}E_{22}^{(1)}p_{T,(t,u)_{\delta\gamma}})+\mathsf{Tr}(E_{22}^{(1)}p_{R,(r,s)_{\alpha\beta}}E_{11}^{(1)}p_{T,(t,u)_{\delta\gamma}})\right).$$

For these terms

$$c_{RR'}=c_{TT'}=N+R_2-2. \tag{4.73}$$

Again we need to include a normalization factor from equation (4.65), and the coefficient

$$\mathcal{C}=\frac{d_T mn(n-1)dr^{'}c_{RR'}}{d_t d_u d_{R'}(n+m)(n+m-1)}. \tag{4.74}$$

Their product simplifies, at large N, to

$$\sqrt{c_{RR'}c_{TT'}}\frac{m(n-1)}{d_u d_s}. \tag{4.75}$$

We can obtain the final answer can summing these contributions. The result is

$$:\mathsf{Tr}\Big(\Big[[Y,Z],\frac{\partial}{\partial Z}\Big]\Big[[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}],Z\Big]\Big):\chi_{R,(r,s)_{\alpha\beta}}=(-2g_{YM}^2)\frac{m}{\sqrt{d_s d_u}}\Big(\mathsf{Tr}(E_{11}^{(1)}p_{R(r,s)_{\alpha\beta}}E_{22}^{(1)}p_{T(t,u)_{\delta\gamma}})+ \tag{4.76}$$

$$\mathsf{Tr}(E_{22}^{(1)}p_{R(r,s)_{\alpha\beta}}E_{11}^{(1)}p_{T(t,u)_{\delta\gamma}})\Big)\times\Big(\sqrt{(N+R_1-1)(N+R_1-2)(N+R_2-1)(N+R_2)}\Delta^{--}$$
$$+\sqrt{(N+R_1)(N+R_1+1)(N+R_2-2)(N+R_2-3)}\Delta^{++}-\sqrt{(N+R_1-1)(N+R_2-1)}$$
$$(2N-3)\Delta^{-}-\sqrt{(N+R_2-2)(N+R_1)}(2N-3)\Delta^{+}+\big[2(N+R_1-1)(N+R_2-2)$$
$$-(n-1)(2N+n-3)\big]\Delta^{0}\Big).$$

where $n=r_1+r_2$.

4.3 The calculation of the terms involving Two Y Derivatives

In order to compute the terms involving two Y derivatives, we will proceed as we did above. We need to evaluate

$$:\mathsf{Tr}\Big(\Big[[Y,Z],\frac{\partial}{\partial Y}\Big]\Big[\Big[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\Big],Y\Big]\Big):\chi_{R(r,s)_{\alpha\beta}}(\sigma)(Y,Z)=:\mathsf{Tr}\Big[(YZ-ZY)\frac{\partial}{\partial Y}(\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y})Y$$
$$-(YZ-ZY)\frac{\partial}{\partial Y}Y(\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y})-\frac{\partial}{\partial Y}(YZ-ZY)(\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y})Y$$
$$+\frac{\partial}{\partial Y}(YZ-ZY)Y(\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y})\Big]:\chi_{R(r,s)_{\alpha\beta}}(\sigma)(Y,Z). \tag{4.77}$$

Define

$$D^1\chi_{R,(r,s)_{\alpha\beta}}(Y,Z)=\Big(\frac{\partial}{\partial Y_a^b}\Big(\frac{\partial}{\partial Y_b^e}\frac{\partial}{\partial Z_e^c}-\frac{\partial}{\partial Z_b^e}\frac{\partial}{\partial Y_e^c}\Big)(YYZ-YZY)_a^c\Big)\chi_{R,(r,s)_{\alpha\beta}}(Y,Z).$$

Arguing exactly as we have done above, we find

$$D^1\chi_{R(r,s)_{\alpha\beta}}(Y,Z) = M^1_{R,(r,s);T,(t,u)_{\gamma\delta}}\chi_{T,(t,u)_{\gamma\delta}}(Y,Z) \tag{4.78}$$

where

$$\begin{aligned} M^1_{R,(r,s);T(t,u)_{\gamma\delta}} =& \frac{d_T m(m-1)n}{d_t d_u (m+n)!}\sum_{T,(t,u)_{\gamma,\delta}}\sum_{\sigma\in S_{n+m-2}}\sum_{R'} c_{RR'}c_{R'R''} \\ & \mathsf{Tr}_{(r,s)_{\alpha\beta}}\left(\Gamma_R(\sigma(1,2,m+1)-(1,m+1)\sigma(2,m+1))\times\right. \\ & \left.\mathsf{Tr}_{(t,u)_{\gamma\delta}}\left(\Gamma_T((1,2)\sigma^{-1}(2,m+1)-(2,m+1)\sigma^{-1}(1,m+1))\right)\right). \end{aligned} \tag{4.79}$$

The sum over σ can be done by using the fundamental orthogonality relation. Working in terms of normalized operators we find

$$D^1\mathcal{O}_{R,(r,s)_{\alpha\beta}}(YZ) = \sum_{T,(t,u)_{\gamma\delta}} N^1_{R,(r,s)_{\alpha\beta};T,(t,u)_{\gamma\delta}}\mathcal{O}_{T,(t,u)_{\gamma\delta}}(YZ) \tag{4.80}$$

where

$$\begin{aligned} N^1_{R,(r,s);T(t,u)} =& \sum_{R''T''}\delta_{R''T''}\frac{d_T mn(m-1)c_{RR'}c_{R'R''}}{d_t d_u(m+n)!}\sqrt{\frac{f_T \mathsf{hooks}_T \mathsf{hooks}_r \mathsf{hooks}_s}{f_R \mathsf{hooks}_R \mathsf{hooks}_t \mathsf{hooks}_u}} \\ & \mathsf{Tr}\left(I_{T'R'}\left[\Gamma_R((1,2,m+1)P_{R(r,s)_{\alpha\beta}}-\Gamma_R((2,m+1)P_{R(r,s)_{\alpha\beta}}\Gamma_R((1,m+1))\right]\right. \\ & \left.\times I_{R'T'}\left[\Gamma_T((2,m+1)P_{T(t,u)_{\gamma\delta}}\Gamma_T((1,2)-\Gamma_T((1,m+1)P_{T(t,u)_{\gamma\delta}}\Gamma_T((2,m+1))\right]\right), \end{aligned} \tag{4.81}$$

which we can be split into the following four terms

$$\begin{aligned} N^1_{R,(r,s);T,(t,u)} =& \sum_{R''T''}\delta_{R''T''}\frac{d_T mn(m-1)c_{RR'}c_{R'R''}}{d_t d_u(m+n)!}\sqrt{\frac{f_T \mathsf{hooks}_T \mathsf{hooks}_r \mathsf{hooks}_s}{f_R \mathsf{hooks}_R \mathsf{hooks}_t \mathsf{hooks}_u}} \\ & \mathsf{Tr}\Big(I_{T''R''}\Gamma_R(2,m+1,1)P_{R(r,s)_{\alpha\beta}}I_{R''T''}\Gamma_T(2,m+1)P_{T(t,u)_{\gamma\delta}}\Gamma_T(1,2) \\ & -I_{T''R''}\Gamma_R(2,m+1,1)P_{R(r,s)_{\alpha\beta}}I_{R''T''}\Gamma_T(1,m+1)P_{T(t,u)_{\gamma\delta}}\Gamma_T(2,m+1) \\ & -I_{T''R''}\Gamma_R(2,m+1)P_{R(r,s)_{\alpha\beta}}\Gamma_R(1,m+1)I_{R''T''}\Gamma_T(2,m+1)P_{T(t,u)_{\gamma\delta}}\Gamma_T(1,2) \\ & +I_{T''R''}\Gamma_R(2,m+1)P_{R(r,s)_{\alpha\beta}}\Gamma_R(1,m+1)I_{R''T''}\Gamma_T(1,m+1)P_{T(t,u)_{\gamma\delta}}\Gamma_T(2,m+1)\Big). \end{aligned} \tag{4.82}$$

To obtain the representations of the subgroups of S_{n+m-2} from the representations R and T of S_{m+n}, we need to drop two boxes. Our convention for these representations is as follows

$$R \quad \stackrel{i}{\to} \quad R' \stackrel{j}{\to} \quad R'', \qquad T \quad \stackrel{k}{\to} \quad T' \stackrel{l}{\to} \quad T''.$$

With these conventions we can write the intertwiners $I_{T''R''}$, $I_{R''T''}$ as

$$I_{T''R''} = E^{(1)}_{ki}E^{(2)}_{lj}E^{(m+1)}_{tt}, \qquad I_{R''T''} = E^{(1)}_{ik}E^{(2)}_{jl}E^{(m+1)}_{rr}. \tag{4.83}$$

Note that in the above expressions repeated indices are summed. We again need to consider the action of permutations on the intertwiners. The results we need are

$$\Gamma_T(1,2)I_{T''R''}\Gamma_R(2,m+1,1) = E^{(1)}_{lj}E^{(2)}_{kt}E^{(m+1)}_{ti}, \tag{4.84}$$
$$I_{R''T''}\Gamma_T(2,m+1) = E^{(1)}_{ik}E^{(2)}_{jr}E^{(m+1)}_{rl}, \tag{4.85}$$
$$\Gamma_T(1,m+1)I_{T''R''}\Gamma_R(2,m+1,1) = E^{(1)}_{tj}E^{(2)}_{lt}E^{(m+1)}_{ki}, \tag{4.86}$$
$$I_{R''T''}\Gamma_T(2,m+1) = E^{(1)}_{ik}E^{(2)}_{jr}E^{(m+1)}_{rl}, \tag{4.87}$$
$$\Gamma_T(1,2)I_{T''R''}\Gamma_R(2,m+1) = E^{(1)}_{li}E^{(2)}_{kt}E^{(m+1)}_{tj}, \tag{4.88}$$
$$\Gamma_R(1,m+1)I_{R''T''}\Gamma_T(2,m+1) = E^{(1)}_{rk}E^{(2)}_{jr}E^{(m+1)}_{il}, \tag{4.89}$$
$$\Gamma_T(1,m+1)I_{T''R''}\Gamma_R(2,m+1) = E^{(1)}_{ti}E^{(2)}_{lt}E^{(m+1)}_{kj}, \tag{4.90}$$
$$\Gamma_R(1,1+m)I_{R''T''}\Gamma_T(2,m+1) = E^{(1)}_{rk}E^{(2)}_{jr}E^{(m+1)}_{il}. \tag{4.91}$$

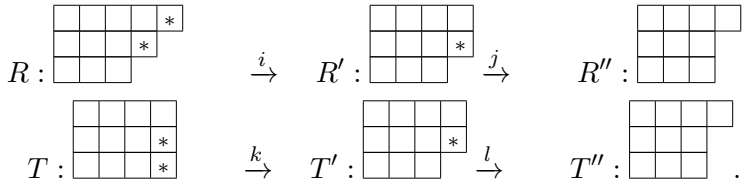
Recall the definition of the projector

$$P_{R\to(r,s)_{\alpha\beta}} = p_{R\to(r,s)_{\alpha\beta}} \otimes \mathbb{1}_r. \tag{4.92}$$

In the above, $P_{R\to(r,s)_{\alpha\beta}}$ acts in the carrier space of R, $p_{R\to(r,s)_{\alpha\beta}}$ acts in $V^{\otimes m}$ and $\mathbb{1}_r$ is the identity in the r subspace of R. In contrast to the leading term, we will see that in general $\vec{m}$ is no longer conserved. It is helpful to discuss a concrete example. Below we will find contributions where the following representations participate

$$\begin{array}{ccccccccc} R & \stackrel{i}{\to} & R' & \stackrel{j}{\to} & R'', & \quad r & \stackrel{i}{\to} & r', \\ T & \stackrel{k}{\to} & T' & \stackrel{l}{\to} & T'', & \quad t & \stackrel{l}{\to} & t'. \end{array}$$

A graphical example of the restriction is (here $i=1,\ j=2,\ k=3$ and $l=2$)



(4.93)

The discussion is most easily carried out by representing each Young diagram by a vector. The vector $\vec{R}$ representing Young diagram R has components which give the row length of R. Using unit vectors $\hat{i}$ and $\hat{j}$ we can write

$$\vec{R}' = \vec{R} - \hat{i} \qquad \text{and} \qquad \vec{R}'' = \vec{R} - \hat{i} - \hat{j}.$$

Similarly

$$\vec{T}' = \vec{T} - \hat{k} \qquad \text{and} \qquad \vec{T}'' = \vec{T} - \hat{k} - \hat{l}.$$

In this notation we can write

$$\begin{aligned} \vec{m} = \vec{R} - \vec{r} =& \vec{R}'' + \hat{i} + \hat{j} - (\vec{r}\,' + \hat{i}) \\ =& \vec{R}'' - \vec{r}\,' + \hat{j}. \end{aligned}$$

Similarly

$$\begin{aligned}\vec{n} = \vec{T} - \vec{t} =& \vec{T}'' + \hat{k} + \hat{l} - (\vec{t}\,' + \hat{l}) \\ =& \vec{T}'' - \vec{t}\,' + \hat{k}.\end{aligned}$$

Now, we know that the sum over the S_{n+m-2} subgroup sets $\vec{T}'' = \vec{R}''$ and the trace over the Z slots sets $\vec{r}\,'' = \vec{t}\,''$. Thus

$$\vec{m} - \vec{n} = \hat{j} - \hat{k}.$$

Consequently, wherever $\hat{j} \neq \hat{k}$ we see that $\vec{m}$ is not conserved.

Consider the trace in the first term in (4.82). We only get a non-zero answer if $i = r$ and $l = t$. The representations which participate are

$$\begin{array}{cccccccc} R & \xrightarrow{i} & R' & \xrightarrow{j} & R'', & r & \xrightarrow{i} & r', \\ T & \xrightarrow{k} & T' & \xrightarrow{l} & T'', & t & \xrightarrow{l} & t'. \end{array}$$

The trace is now easily evaluated.

$$\begin{aligned} & c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{lj}E^{(2)}_{kt}E^{(m+1)}_{ti}p_s \otimes \mathbb{1}_r E^{(1)}_{ik}E^{(2)}_{jr}E^{(m+1)}_{rl}p_u \otimes \mathbb{1}_t\right) \\ &= c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{lj}E^{(2)}_{kt}p_s E^{(1)}_{ik}E^{(2)}_{jr}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{ti} \otimes \mathbb{1}_{r'}E^{(m+1)}_{rl} \otimes \mathbb{1}_{t'}\right) \\ &= c_{RR'_i}c_{R'_iR''_{ij}}dr'_i\mathsf{Tr}\left(E^{(1)}_{lj}E^{(2)}_{kl}p_s E^{(1)}_{ik}E^{(2)}_{ji}p_u\right). \end{aligned} \tag{4.94}$$

This term does not necessarily conserve $\vec{m}$ since

$$\Delta\vec{m} = \hat{k} - \hat{j},$$

and we need not have $\vec{k} = \vec{j}$ for a non-zero result. The second term in (4.82) is only non-zero if $i = r$ and $k = l$. The representations which are relevant are

$$\begin{array}{cccccccc} R & \xrightarrow{i} & R' & \xrightarrow{j} & R'', & r & \xrightarrow{i} & r', \\ T & \xrightarrow{k} & T' & \xrightarrow{k} & T'', & t & \xrightarrow{k} & t'. \end{array}$$

The trace evaluates to

$$\begin{aligned} & -c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{tj}E^{(2)}_{lt}E^{(m+1)}_{ki}p_s \otimes \mathbb{1}_r E^{(1)}_{ik}E^{(2)}_{jr}E^{(m+1)}_{rl}p_u \otimes \mathbb{1}_t\right) \\ &= -c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{tj}E^{(2)}_{lt}p_s E^{(1)}_{ik}E^{(2)}_{jr}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{ki} \otimes \mathbb{1}_{r'}E^{(m+1)}_{rl} \otimes \mathbb{1}_{t'}\right) \\ &= -c_{RR'_i}c_{R'_iR''_{ij}}dr'_i\mathsf{Tr}\left(E^{(1)}_{tj}E^{(2)}_{kt}p_s E^{(1)}_{ik}E^{(2)}_{ji}p_u\right). \end{aligned} \tag{4.95}$$

For this term we find

$$\Delta\vec{m} = \hat{k} - \hat{j}.$$

Again, $\vec{m}$ is not necessarily conserved. The third trace sets $i = j$ and $l = t$. The representations which are involved are

$$\begin{array}{ccccccc} R & \stackrel{i}{\to} & R' & \stackrel{i}{\to} & R'', & r & \stackrel{i}{\to} & r', \\ T & \stackrel{k}{\to} & T' & \stackrel{l}{\to} & T'', & t & \stackrel{l}{\to} & t'. \end{array}$$

The trace evaluates to

$$\begin{aligned} & - c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{li}E^{(2)}_{kt}E^{(m+1)}_{tj}p_s \otimes \mathbb{1}_r E^{(1)}_{rk}E^{(2)}_{jr}E^{(m+1)}_{il}p_u \otimes \mathbb{1}_t\right) \\ & = -c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{li}E^{(2)}_{kt}p_s E^{(1)}_{rk}E^{(2)}_{jr}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{tj} \otimes \mathbb{1}_{r'}E^{(m+1)}_{il} \otimes \mathbb{1}_{t'}\right) \\ & = -c_{RR'_i}c_{R'_iR''_{ii}}dr'_i\mathsf{Tr}\left(E^{(1)}_{li}E^{(2)}_{kl}p_s E^{(1)}_{rk}E^{(2)}_{ik}p_u\right). \end{aligned} \tag{4.96}$$

For this term we again see that $\vec{m}$ need not be conserved

$$\Delta\vec{m} = \hat{k} - \hat{i}.$$

Finally, for the fourth trace $i = l$ and

$$\begin{array}{ccccccc} R & \stackrel{i}{\to} & R' & \stackrel{j}{\to} & R'', & r & \stackrel{i}{\to} & r', \\ T & \stackrel{k}{\to} & T' & \stackrel{i}{\to} & T'', & t & \stackrel{i}{\to} & t'. \end{array}$$

This term gives

$$\begin{aligned} & c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{rk}E^{(2)}_{jr}E^{(m+1)}_{il}p_s \otimes \mathbb{1}_r E^{(1)}_{rk}E^{(2)}_{jr}E^{(m+1)}_{il}p_u \otimes \mathbb{1}_t\right) \\ & = c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{rk}E^{(2)}_{jr}p_s E^{(1)}_{rk}E^{(2)}_{jr}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{il} \otimes \mathbb{1}_{r'}E^{(m+1)}_{il} \otimes \mathbb{1}_{t'}\right) \\ & = c_{RR'_i}c_{R'_iR''_{ii}}dr'_i\mathsf{Tr}\left(E^{(1)}_{rk}E^{(2)}_{jr}p_s E^{(1)}_{rk}E^{(2)}_{jr}p_u\right), \end{aligned} \tag{4.97}$$

and we have (again $\vec{m}$ need not be conserved)

$$\Delta\vec{m} = \hat{k} - \hat{j}.$$

We can now consider the second term in (4.77)

$$- : \mathsf{Tr}\Big([Y,Z]\frac{\partial}{\partial Y}Y\left[\frac{\partial}{\partial Y}\frac{\partial}{\partial Z} - \frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right]\Big) : \chi_{R(r,s)_{\alpha\beta}}(Y,Z). \tag{4.98}$$

Define

$$D^2 = - : \mathsf{Tr}\Big([Y,Z]\frac{\partial}{\partial Y}Y\left[\frac{\partial}{\partial Y}\frac{\partial}{\partial Z} - \frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right]\Big) : .$$

Working as we did above, we find

$$D^2\mathcal{O}_{R,(r,s)_{\alpha\beta}}(Y,Z) = \sum_{T,(t,u)_{\gamma\delta}} N^2_{R,(r,s)_{\alpha\beta};T,(t,u)_{\gamma\delta}}\mathcal{O}_{T,(t,u)_{\gamma\delta}}(Y,Z) \tag{4.99}$$

where

$$N^2_{R,(r,s);T,(t,u)} = \sum_{R'T'} \delta_{R'T'} \frac{d_T mn(m-1)c_{RR'}}{d_t d_u (m+n)!} \tag{4.100}$$
$$\mathsf{Tr}\left(I_{T'R'}\left[\Gamma_R((2,m+1,1)P_{R(r,s)_{\alpha\beta}} - \Gamma_R((2,m+1)P_{R(r,s)_{\alpha\beta}}\Gamma_R((1,m+1))\right]\right.$$
$$\left.\times I_{R'T'}\left[\Gamma_T((1,m+1)P_{T(t,u)_{\gamma\delta}} - P_{T(t,u)_{\gamma\delta}}\Gamma_T((1,m+1))\right]\right).$$

In the calculation that follows we will need to make use of representations of S_{n+m-1}, and representations of S_{n-1}. Our conventions for these representations are

$$R \xrightarrow{i} R', \qquad r \xrightarrow{i} r',$$
$$T \xrightarrow{k} T', \qquad t \xrightarrow{k} t'.$$

We can write the intertwiners $I_{T'R'}$ and $I_{R'T'}$ as

$$I_{T'R'} = E^{(1)}_{ki}E^{(2)}_{ll}E^{(m+1)}_{tt}, \qquad I_{R'T'} = E^{(1)}_{ik}E^{(2)}_{jj}E^{(m+1)}_{rr}. \tag{4.101}$$

To evaluate (4.100), we need to make use of the following results

$$I_{T'R'}\Gamma_R(2,m+1,1) = E^{(1)}_{kl}E^{(2)}_{lt}E^{(m+1)}_{ti}, \tag{4.102}$$
$$I_{R'T'}\Gamma_T(1,m+1) = E^{(1)}_{ir}E^{(2)}_{jj}E^{(m+1)}_{rk}, \tag{4.103}$$
$$\Gamma_T(1,m+1)I_{T'R'}\Gamma_R(2,m+1,1) = E^{(1)}_{tl}E^{(2)}_{lt}E^{(m+1)}_{ki}, \tag{4.104}$$
$$I_{R'T'} = E^{(1)}_{ik}E^{(2)}_{jj}E^{(m+1)}_{rr}, \tag{4.105}$$
$$I_{T'R'}\Gamma_R(2,m+1) = E^{(1)}_{ki}E^{(2)}_{lt}E^{(m+1)}_{tl}, \tag{4.106}$$
$$\Gamma_R(1,m+1)I_{R'T'}\Gamma_T(1,m+1) = E^{(1)}_{rr}E^{(2)}_{jj}E^{(m+1)}_{ik}, \tag{4.107}$$
$$\Gamma_T(1,m+1)I_{T'R'}\Gamma_R(2,m+1) = E^{(1)}_{ti}E^{(2)}_{lt}E^{(m+1)}_{kl}, \tag{4.108}$$
$$\Gamma_R(1,1+m)I_{R'T'} = E^{(1)}_{rk}E^{(2)}_{jj}E^{(m+1)}_{ir}. \tag{4.109}$$

There are four terms appearing in (4.100). Performing the trace in the first term sets $i = r$ and $k = t$. After tracing we have

$$c_{RR'}\mathsf{Tr}\left(E^{(1)}_{kl}E^{(2)}_{lt}E^{(m+1)}_{ti}p_s \otimes \mathbb{1}_r E^{(1)}_{ir}E^{(m+1)}_{rk}p_u \otimes \mathbb{1}_t\right) \tag{4.110}$$
$$= c_{RR'}\mathsf{Tr}\left(E^{(1)}_{kl}E^{(2)}_{lt}p_s E^{(1)}_{ir}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{ti} \otimes \mathbb{1}_{r'}E^{(m+1)}_{rk} \otimes \mathbb{1}_{t'}\right)$$
$$= c_{RR'_i}d'_i\mathsf{Tr}\left(E^{(1)}_{kl}E^{(2)}_{lk}p_s E^{(1)}_{ii}p_u\right).$$

The trace in the second term sets $i = k$. For this term we find

$$-c_{RR'}\mathsf{Tr}\left(E^{(1)}_{tl}E^{(2)}_{lt}E^{(m+1)}_{ki}p_s \otimes \mathbb{1}_r E^{(1)}_{ik}p_u \otimes \mathbb{1}_t\right) \tag{4.111}$$
$$= -c_{RR'}\mathsf{Tr}\left(E^{(1)}_{tl}E^{(2)}_{lt}p_s E^{(1)}_{ik}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{ki} \otimes \mathbb{1}_{r'}\mathbb{1}_{t'}\right)$$
$$= -c_{RR'_i}d'_i\mathsf{Tr}\left(E^{(1)}_{tl}E^{(2)}_{lt}p_s E^{(1)}_{ii}p_u\right).$$

For a non-zero result for the third term we must have $i = l$, $k = t$. The answer after tracing is

$$\begin{aligned} & - c_{RR'} \mathsf{Tr}\left(E^{(1)}_{ki} E^{(2)}_{lt} E^{(m+1)}_{tl} p_s \otimes \mathbb{1}_r E^{(m+1)}_{ik} p_u \otimes \mathbb{1}_t\right) \\ & = -c_{RR'} \mathsf{Tr}\left(E^{(1)}_{ki} E^{(2)}_{lt} p_s p_u\right) \mathsf{Tr}\left(E^{(m+1)}_{tl} \otimes \mathbb{1}_{r'} E^{(m+1)}_{ik} \mathbb{1}_{t'}\right) \\ & = -c_{RR'_i} dr'_i \mathsf{Tr}\left(E^{(1)}_{ki} E^{(2)}_{ik} p_s p_u\right). \end{aligned} \tag{4.112}$$

Finally, the trace in the fourth term sets $i = l$, $k = r$. This term gives

$$\begin{aligned} & c_{RR'} \mathsf{Tr}\left(E^{(1)}_{ti} E^{(2)}_{lt} E^{(m+1)}_{kl} p_s \otimes \mathbb{1}_r E^{(1)}_{rk} E^{(m+1)}_{ir} p_u \otimes \mathbb{1}_t\right) \\ & = c_{RR'} \mathsf{Tr}\left(E^{(1)}_{ti} E^{(2)}_{lt} p_s E^{(1)}_{rk} p_u\right) \mathsf{Tr}\left(E^{(m+1)}_{kl} \otimes \mathbb{1}_{r'} E^{(m+1)}_{ir} \mathbb{1}_{t'}\right) \\ & = c_{RR'_i} dr'_i \mathsf{Tr}\left(E^{(1)}_{ti} E^{(2)}_{it} p_s E^{(1)}_{kk} p_u\right). \end{aligned} \tag{4.113}$$

Now, we will discuss the third term in equation (4.77), which is given by

$$- : \mathsf{Tr}\left(\frac{\partial}{\partial Y}[Y,Z]\left[\frac{\partial}{\partial Y}\frac{\partial}{\partial Z} - \frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right] Y\right) : \chi_{R,(r,s)_{\alpha\beta}}(Y,Z). \tag{4.114}$$

Define

$$D^3 = - : \mathsf{Tr}\left(Y \frac{\partial}{\partial Y}[Y,Z]\left[\frac{\partial}{\partial Y}\frac{\partial}{\partial Z} - \frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right]\right) : .$$

A straightforward but tedious computation gives

$$D^3 \mathcal{O}_{R,(r,s)_{\alpha\beta}}(YZ) = \sum_{T,(t,u)_{\gamma\delta}} N^3_{R,(r,s)_{\alpha\beta};T,(t,u)_{\gamma\delta}} \mathcal{O}_{T,(t,u)_{\gamma\delta}}(YZ) \tag{4.115}$$

where

$$\begin{aligned} N^3_{R,(r,s)_{\alpha\beta};T(t,u)_{\gamma\delta}} = & \sum_{R'} \delta_{R'T'} \frac{d_T mn(m-1) c_{RR'}}{d_t d_u (m+n)!} \sqrt{\frac{f_T \mathsf{hooks}_T \mathsf{hooks}_r \mathsf{hooks}_s}{f_R \mathsf{hooks}_R \mathsf{hooks}_t \mathsf{hooks}_u}} \\ & \mathsf{Tr}\Big(I_{T'R'}\Big[\Gamma_R((1,2,m+1)) P_{R(r,s)_{\alpha\beta}} \Gamma_R((1,2,m+1)) - \Gamma_R((12) P_{R(r,s)_{\alpha\beta}} \Gamma_R((1,m+1))\Big] \\ & I_{R'T'}\Big[\Gamma_T((1,m+1)) P_{T(t,u)_{\gamma\delta}} - P_{T(t,u)_{\gamma\delta}} \Gamma_T((1,m+1))\Big]\Big). \end{aligned} \tag{4.116}$$

To evaluate the trace in (4.116) we will need the following results

$$I_{T'R'} \Gamma_R((1,2,m+1)) = E^{(1)}_{kl} E^{(2)}_{lt} E^{(m+1)}_{ti}, \tag{4.117}$$

$$\Gamma_R((1,2,m+1)) I_{R'T'} \Gamma_T(1,m+1) = E^{(2)}_{ij} E^{(m+1)}_{jk}, \tag{4.118}$$

$$\Gamma_T((1,m+1)) I_{T'R'} \Gamma_R((1,2,m+1)) = E^{(1)}_{tl} E^{(2)}_{lt} E^{(m+1)}_{ki}, \tag{4.119}$$

$$\Gamma_R((1,2,m+1)) I_{R'T'} = E^{(1)}_{rk} E^{(2)}_{ij} E^{(m+1)}_{jr}, \tag{4.120}$$

$$I_{T'R'} \Gamma_R((12)) = E^{(1)}_{kl} E^{(2)}_{li}, \tag{4.121}$$

$$\Gamma_R((1,m+1,2)) I_{R'T'} \Gamma_T((1,m+1)) = E^{(m+1)}_{ik}, \tag{4.122}$$

$$\Gamma_T((1,m+1))I_{T'R'}\Gamma_R((12)) = E^{(1)}_{tl}E^{(2)}_{li}E^{(m+1)}_{kt}, \tag{4.123}$$

$$\Gamma_R((1,m+1))I_{R'T'} = E^{(1)}_{rk}E^{(2)}_{ir}. \tag{4.124}$$

The relevant projector is given in equation (4.92) and acts in the carrier space of R, $p_{R\to(r,s)_{\alpha\beta}}$ acts in $V^{\otimes m}$ and $\mathbb{1}_r$ is the identity in the r subspace of R. Then from equation (4.117) we find $i=j$, $k=t$ in which case the first term in (4.116) gives

$$\begin{aligned} &c_{RR'}\mathsf{Tr}\left(E^{(1)}_{kl}E^{(2)}_{lt}E^{(m+1)}_{ti}p_s\otimes\mathbb{1}_rE^{(2)}_{ij}E^{(m+1)}_{jk}p_u\otimes\mathbb{1}_t\right) \\ &= c_{RR'}\mathsf{Tr}\left(E^{(1)}_{kl}E^{(2)}_{lt}p_sE^{(2)}_{ij}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{ti}\otimes\mathbb{1}_{r'}E^{(m+1)}_{jk}\mathbb{1}_{t'}\right) \\ &= c_{RR'_i}dr'_i\mathsf{Tr}\left(E^{(1)}_{kl}E^{(2)}_{lk}p_sE^{(1)}_{ii}p_u\right). \end{aligned} \tag{4.125}$$

For the second term in (4.116), the trace sets $i=j$, $k=r$. This term evaluates to

$$\begin{aligned} &-c_{RR'}\mathsf{Tr}\left(E^{(1)}_{tl}E^{(2)}_{lt}E^{(m+1)}_{ki}p_s\otimes\mathbb{1}_rE^{(1)}_{rk}E^{(2)}_{ij}E^{(m+1)}_{jr}p_u\otimes\mathbb{1}_t\right) \\ &= -c_{RR'}\mathsf{Tr}\left(E^{(1)}_{tl}E^{(2)}_{lt}p_sE^{(1)}_{rk}E^{(2)}_{ij}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{ki}\otimes\mathbb{1}_{r'}E^{(m+1)}_{jr}\mathbb{1}_{t'}\right) \\ &= -c_{RR'_i}dr'_i\mathsf{Tr}\left(E^{(1)}_{tl}E^{(2)}_{lt}p_sE^{(1)}_{kk}E^{(2)}_{ii}p_u\right). \end{aligned} \tag{4.126}$$

For the third term in (4.116) $i=k$ and j^3 is conserved. We find

$$\begin{aligned} &-c_{RR'}\mathsf{Tr}\left(E^{(1)}_{kl}E^{(2)}_{li}p_s\otimes\mathbb{1}_rE^{(m+1)}_{ik}p_u\otimes\mathbb{1}_t\right) \\ &= -c_{RR'}\mathsf{Tr}\left(E^{(1)}_{kl}E^{(2)}_{li}p_sp_u\right)\mathsf{Tr}\left(E^{(m+1)}_{ik}\otimes\mathbb{1}_{t'}\right) \\ &= -c_{RR'_i}dr'_i\mathsf{Tr}\left(E^{(1)}_{il}E^{(2)}_{li}p_sp_u\right). \end{aligned} \tag{4.127}$$

Finally, for the last term in (4.116) we find $i=t$, $k=r$ and this term yields

$$\begin{aligned} &c_{RR'}\mathsf{Tr}\left(E^{(1)}_{tl}E^{(2)}_{li}E^{(m+1)}_{kt}p_s\otimes\mathbb{1}_rE^{(1)}_{rk}E^{(m+1)}_{ir}p_u\otimes\mathbb{1}_t\right) \\ &= c_{RR'}\mathsf{Tr}\left(E^{(1)}_{tl}E^{(2)}_{li}p_sE^{(1)}_{rk}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{kt}\otimes\mathbb{1}_{r'}E^{(m+1)}_{ir}\mathbb{1}_{t'}\right) \\ &= c_{RR'_i}dr'_i\mathsf{Tr}\left(E^{(1)}_{il}E^{(2)}_{li}p_sE^{(1)}_{kk}p_u\right). \end{aligned} \tag{4.128}$$

Finally, we will discuss the last term in equation (4.77), using the same techniques that we used above. The last term is

$$:\mathsf{Tr}\left((YZ-ZY)Y\left(\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right)\frac{\partial}{\partial Y}\right):\chi_{R(r,s)_{\alpha\beta}}(Y,Z). \tag{4.129}$$

Define

$$D^4 =: \mathsf{Tr}\left((YZ-ZY)Y\left(\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}-\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right)\frac{\partial}{\partial Y}\right):.$$

Acting on correctly normalized operators we find.

$$D^4\mathcal{O}_{R,(r,s)_{\alpha\beta}}(Y,Z)=\sum_{T,(t,u)_{\gamma\delta}} N^4_{R,(r,s)_{\alpha\beta};T,(t,u)_{\gamma\delta}}\mathcal{O}_{T,(t,u)_{\gamma\delta}}(Y,Z) \tag{4.130}$$

where

$$\begin{aligned} N^4_{R,(r,s);T(t,u)} =&\delta_{R'T'}\sum_{R'T'}\frac{d_T mn(m-1)c_{RR'}c_{R'R''}}{d_t d_u(m+n)!}\sqrt{\frac{f_T \mathsf{hooks}_T \mathsf{hooks}_r \mathsf{hooks}_s}{f_R \mathsf{hooks}_R \mathsf{hooks}_t \mathsf{hooks}_u}} \\ &\mathsf{Tr}\left(I_{T'R'}\left[\Gamma_R((2,m+1)P_{R(r,s)}\Gamma_R((1,m+1)-\Gamma_R((12))P_{R(r,s)}\Gamma_R((1,m+1))\right]\right. \\ &\left.\times I_{R'T'}\left[\Gamma_T((1,m+1)P_{T(t,u)}\Gamma_T((2,m+1)-\Gamma_T((12)P_{T(t,u)}\Gamma_T((1,m+1))\right]\right). \end{aligned} \tag{4.131}$$

We will make use of the results

$$\Gamma_T((2,m+1))I_{T''R''}\Gamma_R((2,m+1)) = E^{(1)}_{ti}E^{(2)}_{lt}E^{(m+1)}_{kj}, \tag{4.132}$$

$$\Gamma_R((1,m+1))I_{R''T''}\Gamma_T(12) = E^{(1)}_{rl}E^{(2)}_{jk}E^{(m+1)}_{ir}, \tag{4.133}$$

$$\Gamma_T((1,m+1))I_{T''R''}\Gamma_R((2,m+1)) = E^{(1)}_{ti}E^{(2)}_{lt}E^{(m+1)}_{kj}, \tag{4.134}$$

$$\Gamma_R((1,1+m))I_{R''T''}\Gamma_T((12)) = E^{(1)}_{rl}E^{(2)}_{jk}E^{(m+1)}_{ir}, \tag{4.135}$$

$$\Gamma_T((2,m+1))I_{T''R''}\Gamma_R((12)) = E^{(1)}_{kj}E^{(2)}_{ti}E^{(m+1)}_{lt}, \tag{4.136}$$

$$(\Gamma_R((1,m+1))I_{R''T''}\Gamma_T((1,m+1)) = E^{(2)}_{jl}E^{(m+1)}_{ik}, \tag{4.137}$$

$$\Gamma_T((1,m+1))I_{T''R''}\Gamma_R((12)) = E^{(1)}_{tj}E^{(2)}_{li}E^{(m+1)}_{kt}, \tag{4.138}$$

$$\Gamma_R((1,m+1))I_{R''T''}\Gamma_T((12)) = E^{(1)}_{rl}E^{(2)}_{jk}E^{(m+1)}_{ir}. \tag{4.139}$$

The first term in (4.131) gives

$$\begin{aligned} &c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{ki}E^{(m+1)}_{lj}p_s\otimes\mathbb{1}_r E^{(2)}_{jl}E^{(m+1)}_{ik}p_u\otimes\mathbb{1}_t\right) \\ &= c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{ki}p_sE^{(2)}_{jl}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{lj}\otimes\mathbb{1}_{r'}E^{(}m+1)_{ik}\otimes\mathbb{1}_{t'}\right) \\ &= c_{RR'_i}c_{R'_iR''_{ii}}dr'_i\mathsf{Tr}\left(E^{(1)}_{ki}p_sE^{(2)}_{ik}p_u\right). \end{aligned} \tag{4.140}$$

This term changes $\vec{m}$ by

$$\Delta\vec{m}=\hat{k}-\hat{i}.$$

The second term changes $\vec{m}$ by

$$\Delta\vec{m}=\hat{l}-\hat{i}.$$

It evaluates to

$$\begin{aligned} &- c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{ti}E^{(2)}_{lt}E^{(m+1)}_{kj}p_s\otimes\mathbb{1}_r E^{(2)}_{jl}E^{(m+1)}_{ik}p_u\otimes\mathbb{1}_t\right) \\ &= -c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{ti}E^{(2)}_{lt}p_sE^{(2)}_{jl}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{kj}\otimes\mathbb{1}_{r'}E^(m+1)_{ik}\otimes\mathbb{1}_{t'}\right) \\ &= -c_{RR'_i}c_{R'_iR''_{ii}}dr'_i\mathsf{Tr}\left(E^{(1)}_{ti}E^{(2)}_{lt}p_sE^{(2)}_{il}p_u\right). \end{aligned} \tag{4.141}$$

The third term changes $\vec{m}$ by

$$\Delta\vec{m} = \hat{k} - \hat{j},$$

and after tracing we find

$$\begin{aligned} &- c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{kj}E^{(2)}_{ti}E^{(m+1)}_{lt}p_s\otimes\mathbb{1}_r E^{(2)}_{jl}E^{(m+1)}_{ik}p_u\otimes\mathbb{1}_t\right) \\ &= -c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{kj}E^{(2)}_{ti}p_sE^{(2)}_{jl}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{lt}\otimes\mathbb{1}_{r'}E^(m+1)_{ik}\otimes\mathbb{1}_{t'}\right) \\ &= -c_{RR'_i}c_{R'_iR''_{ii}}dr'_i\mathsf{Tr}\left(E^{(1)}_{kj}E^{(2)}_{ii}p_sE^{(2)}_{jk}p_u\right). \end{aligned} \tag{4.142}$$

For the fourth term

$$\Delta\vec{m} = \hat{l} - \hat{j}$$

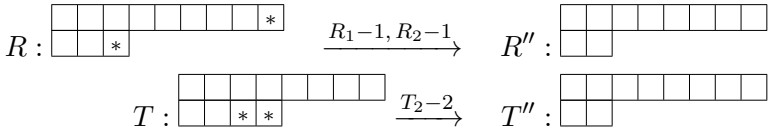
and

$$\begin{aligned} &c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{tj}E^{(2)}_{li}E^{(m+1)}_{kt}p_s\otimes\mathbb{1}_r E^{(1)}_{rl}E^{(2)}_{jk}E^{(m+1)}_{ir}p_u\otimes\mathbb{1}_t\right) \\ &= c_{RR'}c_{R'R''}\mathsf{Tr}\left(E^{(1)}_{tj}E^{(2)}_{li}p_sE^{(1)}_{rl}E^{(2)}_{jk}p_u\right)\mathsf{Tr}\left(E^{(m+1)}_{kt}\otimes\mathbb{1}_{r'}E^(m+1)_{ir}\otimes\mathbb{1}_{t'}\right) \\ &= c_{RR'_i}c_{R'_iR''_{ii}}dr'_i\mathsf{Tr}\left(E^{(1)}_{ij}E^{(2)}_{li}p_sE^{(1)}_{kl}E^{(2)}_{jk}p_u\right). \end{aligned} \tag{4.143}$$

4.4 Collecting the final result

We now need to collect the final result. We will organize the answer by how many boxes move between R and T. We will do things concretely for 2 rows to make the discussion as simple as possible. Denote the row lengths of R and T by R_i and T_i respectively.

4.4.1 Contribution from D^1 and D^4. Note that there is no contribution corresponding to terms like Δ^{++} and Δ^{--}. The term Δ^-_a arises when we remove the two boxes from different rows, so we have two possibilities. The first one is when we remove one box from the first row and one from the second row in diagram R to obtain R'' and remove two boxes from the second row in diagram T to obtain T''. In this case, $R_1 = T_1 + 1$ and $R_2 = T_2 - 1$.



(4.144)

From above diagrams we can write the factors $c_{RR'}$, $c_{R'R''}$ and $c_{TT'}\,c_{T'T''}$ as a function of N plus the content of the relevant boxes as

$$c_{RR'_1} = N + R_1 - 1, \quad c_{R'_1R''_{12}} = N + R_2 - 2, \tag{4.145}$$

$$c_{TT'_2} = N + R_2 - 2, \quad c_{T'_2T''_{22}} = N + R_2 - 2. \tag{4.146}$$

Thus, $c_{RR'_1}$, $c_{R'_1R''_{12}} = (N + R_1 - 1)(N + R_2 - 2)$, and

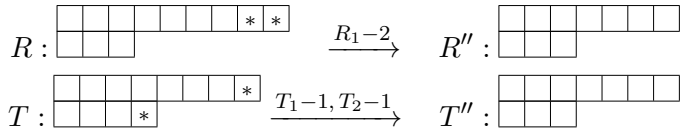
$$c_{RR'_1}c_{R'_1R''_{12}}\sqrt{\frac{f_T}{f_R}} = (N + R_2 - 2)\sqrt{(N + R_1 - 1)(N + R_2 - 1)}. \tag{4.147}$$

The contributing terms are (4.94), (4.95), (4.142) and (4.143). After some simplification we find

$$c_{RR'_1}c_{R'_1R''_{12}}dr'_1\Big(-\mathsf{Tr}(E^{(1)}_{12}E^{(2)}_{21}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{12}E^{(2)}_{21}p_{T(t,s)_{\gamma\delta}}) + \mathsf{Tr}(E^{(1)}_{12}E^{(2)}_{21}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{22}E^{(2)}_{22}p_{T(t,s)_{\gamma\delta}}) \tag{4.148}$$

$$-\mathsf{Tr}(E^{(1)}_{22}E^{(2)}_{11}p_{R(r,s)_{\alpha\beta}}E^{(2)}_{22}p_{T(t,s)_{\gamma\delta}})\Big).$$

The second possibility Δ_b^- arises when we remove two boxes from row one of the diagram R to obtain R'' and remove one box from the first row and one box from the second row in diagram T to obtain T''. We can write the length R_1 as function of T_1, $R_1 = T_1 + 1$ and likewise for R_2 which is $R_2 = T_2 - 1$.



(4.149)

From above diagrams we can read off the following factors,

$$c_{RR'_1} = N + R_1 - 1, \quad c_{R'_1R''_{11}} = N + R_1 - 2, \tag{4.150}$$

$$c_{TT'_1} = N + R_1 - 2, \quad c_{T'_1T''_{12}} = N + R_2 - 1. \tag{4.151}$$

Thus $c_{RR'_1}c_{R'_1R''_{11}} = (N + R_1 - 1)(N + R_1 - 1)$, and

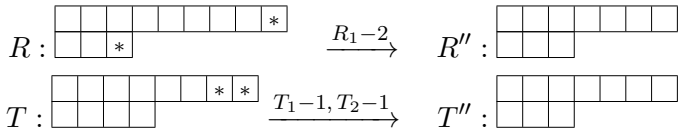
$$c_{RR'_{11}}c_{R'_1R''_{11}}\sqrt{\frac{f_T}{f_R}} = (N + R_1 - 2)\sqrt{(N + R_1 - 1)(N + R_2 - 1)}. \tag{4.152}$$

The contributing terms are (4.94), (4.96), (4.141) and (4.143). After some simplification we have

$$c_{RR'_1}c_{R'_1R''_{11}}dr'_1\Big(-\mathsf{Tr}(E^{(1)}_{21}E^{(2)}_{12}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{21}E^{(2)}_{12}p_{T(t,s)_{\gamma\delta}}) - \mathsf{Tr}(E^{(1)}_{11}E^{(2)}_{11}p_{R(r,s)_{\alpha\beta}}E^{(2)}_{11}p_{T(t,s)_{\gamma\delta}})$$

$$-\mathsf{Tr}(E^{(1)}_{21}E^{(2)}_{12}p_{R(r,s)_{\alpha\beta}}E^{(2)}_{11}p_{T(t,s)_{\gamma\delta}}) + \mathsf{Tr}(E^{(1)}_{11}E^{(2)}_{11}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{21}E^{(2)}_{12}p_{T(t,s)_{\gamma\delta}})\Big). \tag{4.153}$$

The term Δ_a^+ arises when we remove two boxes from the first row in diagram R to obtain R'' and remove one box the first row and one box from the second row in diagram T to obtain T''. We can write the length R_1 as a function of T_1, as $R_1 = T_1 - 1$ and likewise for R_2 which is $R_2 = T_2 + 1$.



(4.154)

We read off the factors as follows

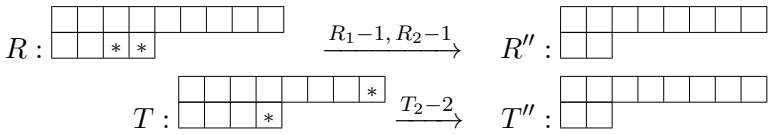
$$c_{RR'_2} = N + R_2 - 2, \qquad c_{R'_2 R''_{21}} = N + R_1 - 1, \tag{4.155}$$

$$c_{TT'_1} = N + R_1 - 2, \qquad c_{T'_1 T''_{11}} = N + R_1 - 3. \tag{4.156}$$

Thus, $c_{RR'_2} c_{R'_2 R''_{21}} = (N + R_1 - 1)(N + R_1 - 2)$. The contributing terms are (4.94), (4.95), (4.142) and (4.143). After some simplification we have

$$c_{RR'_1} c_{R'_1 R''_{11}} dr'_1 \Big(-\mathsf{Tr}(E^{(1)}_{21} E^{(2)}_{12} p_{R(r,s)_{\alpha\beta}} E^{(1)}_{21} E^{(2)}_{12} p_{T(t,s)_{\gamma\delta}}) - \mathsf{Tr}(E^{(1)}_{11} E^{(2)}_{22} p_{R(r,s)_{\alpha\beta}} E^{(2)}_{11} p_{T(t,s)_{\gamma\delta}}) + \mathsf{Tr}(E^{(1)}_{21} E^{(2)}_{12} p_{R(r,s)_{\alpha\beta}} E^{(1)}_{11} E^{(2)}_{11} p_{T(t,s)_{\gamma\delta}}) \Big) . \tag{4.157}$$

The possibility Δ^+_b arises when we remove one box from the first row and one box from the second row of the diagram R to obtain R'' and remove two boxes from the first row in diagram T to obtain T''. We can write the length R_1 as a function of T_1 as $R_1 = T_1 - 1$ and likewise for R_2 which is $R_2 = T_2 + 1$.



(4.158)

From above diagrams we find

$$c_{RR'} = N + R_2 - 2, \quad c_{R'R''} = N + R_2 - 3, \tag{4.159}$$

$$c_{TT'} = N + R_1, \quad c_{T'T''} = N + R_2 - 3. \tag{4.160}$$

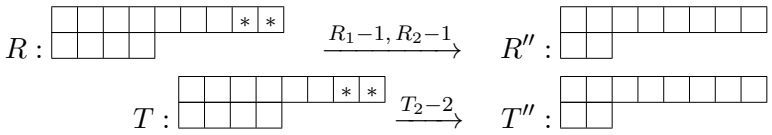
Thus, $c_{RR'} c_{R'R''} = (N + R_2 - 2)(N + R_2 - 3)$, and

$$c_{RR'} c_{R'R''} \sqrt{\frac{f_T}{f_R}} = (N + R_2 - 3)\sqrt{(N + R_1)(N + R_2 - 2)}. \tag{4.161}$$

The contributing terms are (4.94), (4.96), (4.141) and (4.143). After some simplification we find

$$c_{RR'_2} c_{R'_2 R''_{22}} dr'_2 \Big(-\mathsf{Tr}(E^{(1)}_{12} E^{(2)}_{21} p_{R(r,s)_{\alpha\beta}} E^{(1)}_{12} E^{(2)}_{21} p_{T(t,s)_{\gamma\delta}}) - \mathsf{Tr}(E^{(1)}_{12} E^{(2)}_{21} p_{R(r,s)_{\alpha\beta}} E^{(2)}_{22} p_{T(t,s)_{\gamma\delta}}) - \mathsf{Tr}(E^{(1)}_{22} E^{(2)}_{22} p_{R(r,s)_{\alpha\beta}} E^{(2)}_{22} p_{T(t,s)_{\gamma\delta}}) + \mathsf{Tr}(E^{(1)}_{22} E^{(2)}_{22} p_{R(r,s)_{\alpha\beta}} E^{(1)}_{12} E^{(2)}_{21} p_{T(t,s)_{\gamma\delta}}) \Big) . \tag{4.162}$$

The Δ^0_a term arises when we remove two boxes from row one of the diagram R to obtain R''. We can write the length R_1 as a function of T_1 as $R_1 = T_1$ and likewise for R_2 which is $R_2 = T_2$.



(4.163)

From above diagrams we can see that

$$c_{RR'_1} = c_{TT'_1} = N + R_1 - 1, \quad c_{T'_1T''_{11}} = c_{T'_1T''_{11}} = N + R_1 - 2. \tag{4.164}$$

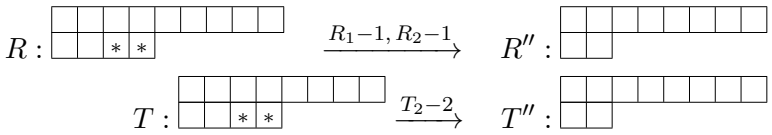
Thus, $c_{RR'}c_{R'R''} = (N + R_1 - 1)(N + R_1 - 2)$, and

$$c_{RR'_1}c_{R'_1R''_{11}}\sqrt{\frac{f_T}{f_R}} = (N + R_1 - 1)(N + R_1 - 2). \tag{4.165}$$

The contributing terms are (4.95), (4.96), (4.97), (4.111), (4.128), (4.140), (4.141), (4.142) and (4.143). After some simplification we have

$$\begin{aligned} c_{RR'_1}c_{R'_1R''_{11}}dr'_1 \Big(&-\mathsf{Tr}(E^{(1)}_{11}E^{(2)}_{11}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{11}E^{(2)}_{11}p_{T(t,s)_{\gamma\delta}}) + \mathsf{Tr}(E^{(1)}_{21}E^{(2)}_{12}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{21}E^{(2)}_{12}p_{T(t,s)_{\gamma\delta}}) \\ &- \mathsf{Tr}(E^{(1)}_{21}E^{(2)}_{12}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{11}E^{(2)}_{11}p_{T(t,s)_{\gamma\delta}}) - \mathsf{Tr}(E^{(1)}_{11}E^{(2)}_{11}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{21}E^{(2)}_{12}p_{T(t,s)_{\gamma\delta}}) \\ &- 2\mathsf{Tr}(E^{(1)}_{11}E^{(2)}_{11}p_{R(r,s)_{\alpha\beta}}E^{(2)}_{11}p_{T(t,s)_{\gamma\delta}}) - \mathsf{Tr}(E^{(1)}_{21}E^{(2)}_{12}p_{R(r,s)_{\alpha\beta}}E^{(2)}_{11}p_{T(t,s)_{\gamma\delta}}) \\ &+\mathsf{Tr}(E^{(1)}_{11}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{11}p_{T(t,s)_{\gamma\delta}})\Big). \end{aligned} \tag{4.166}$$

The Δ^0_b term arises when we remove two boxes from row one of the diagram R to obtain R''. We can write the length R_1 as a function of T_1 as $R_1 = T_1$ and likewise for R_2 which is $R_2 = T_2$.



(4.167)

From above diagrams we find

$$c_{RR'_2} = c_{TT'_2} = N + R_2 - 2, \quad c_{R'_2R''_2} = c_{T'_2T''_{22}} = N + R_2 - 3. \tag{4.168}$$

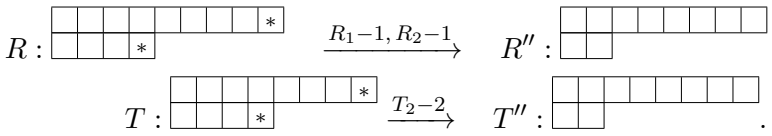
Thus, $c_{RR'}c_{R'R''} = (N + R_2 - 2)(N + R_2 - 3)$ and

$$c_{RR'_2}c_{R'_2R''_{22}}\sqrt{\frac{f_T}{f_R}} = (N + R_2 - 2)(N + R_2 - 3). \tag{4.169}$$

The contributing terms are (4.94), (4.95),(4.97), (4.140), (4.141), (4.142) and (4.143). After some simplification we have

$$\begin{aligned} c_{RR'_1}c_{R'_1R''_{11}}dr'_2 \Big(&-\mathsf{Tr}(E^{(1)}_{12}E^{(2)}_{21}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{12}E^{(2)}_{21}p_{T(t,s)_{\gamma\delta}}) + \mathsf{Tr}(E^{(1)}_{22}E^{(2)}_{22}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{22}E^{(2)}_{22}p_{T(t,s)_{\gamma\delta}}) \\ &- \mathsf{Tr}(E^{(1)}_{12}E^{(2)}_{21}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{22}E^{(2)}_{22}p_{T(t,s)_{\gamma\delta}}) - \mathsf{Tr}(E^{(1)}_{22}E^{(2)}_{22}p_{R(r,s)_{\alpha\beta}}E^{(1)}_{12}E^{(2)}_{21}p_{T(t,s)_{\gamma\delta}}) \\ &- 2\mathsf{Tr}(E^{(1)}_{22}E^{(2)}_{22}p_{R(r,s)_{\alpha\beta}}E^{(2)}_{22}p_{T(t,s)_{\gamma\delta}}) - \mathsf{Tr}(E^{(1)}_{12}E^{(2)}_{21}p_{R(r,s)_{\alpha\beta}}E^{(2)}_{22}p_{T(t,s)_{\gamma\delta}}) \\ &+\mathsf{Tr}(E^{(1)}_{22}p_{R(r,s)_{\alpha\beta}}E^{(2)}_{22}p_{T(t,s)_{\gamma\delta}})\Big). \end{aligned} \tag{4.170}$$

The Δ_c^0 representation is obtained when we remove one box from row one of the diagram R to obtain R''. We can write the length R_1 as a function of T_1, $R_1 = T_1$ and likewise for R_2 which is $R_2 = T_2$



(4.171)

From above diagrams we read off

$$c_{RR_1'} = c_{TT_1'} = N + R_1 - 1, \quad c_{R_1'R_{12}''} = c_{T_1'T_{12}''} = N + R_2 - 2. \tag{4.172}$$

Thus, $c_{RR_1'}c_{R_1'R_{12}''} = (N + R_2 - 1)(N + R_2 - 2)$, and

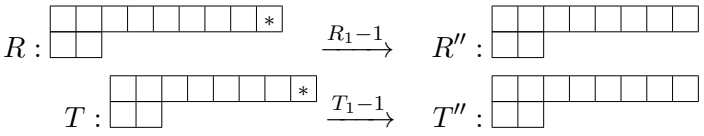
$$c_{RR_1'}c_{R_1'R_{12}''}\sqrt{\frac{f_T}{f_R}} = (N + R_1 - 1)(N + R_2 - 2). \tag{4.173}$$

The contributing terms are (4.94), (4.95),(4.96), (4.97), (4.126) and (4.143). After some simplification we have

$$\begin{aligned} c_{RR_1'}c_{R_1'R_{12}''}dr_1' \Big(&2\mathsf{Tr}(E_{12}^{(1)}E_{21}^{(2)}p_{R(r,s)_{\alpha\beta}}E_{12}^{(1)}E_{21}^{(2)}p_{T(t,s)_{\gamma\delta}}) + 2\mathsf{Tr}(E_{21}^{(1)}E_{12}^{(2)}p_{R(r,s)_{\alpha\beta}}E_{21}^{(1)}E_{12}^{(2)}p_{T(t,s)_{\gamma\delta}}) \\ &+ \mathsf{Tr}(E_{11}^{(1)}E_{11}^{(2)}p_{R(r,s)_{\alpha\beta}}E_{11}^{(1)}E_{11}^{(2)}p_{T(t,s)_{\gamma\delta}}) + \mathsf{Tr}(E_{22}^{(1)}E_{22}^{(2)}p_{R(r,s)_{\alpha\beta}}E_{22}^{(1)}E_{22}^{(2)}p_{T(t,s)_{\gamma\delta}}) \\ &+ \mathsf{Tr}(E_{11}^{(1)}E_{21}^{(2)}p_{R(r,s)_{\alpha\beta}}E_{22}^{(1)}E_{12}^{(2)}p_{T(t,s)_{\gamma\delta}}) + \mathsf{Tr}(E_{21}^{(1)}E_{22}^{(2)}p_{R(r,s)_{\alpha\beta}}E_{12}^{(1)}E_{11}^{(2)}p_{T(t,s)_{\gamma\delta}}) \\ &+\mathsf{Tr}(E_{12}^{(1)}E_{11}^{(2)}p_{R(r,s)_{\alpha\beta}}E_{21}^{(1)}E_{22}^{(2)}p_{T(t,s)_{\gamma\delta}})\Big). \end{aligned} \tag{4.174}$$

4.4.2 Contribution from terms (4.98) and (4.114). By requiring the conservation of j^3 and the extra condition $r' = t'$ we learn that there is no contribution for Δ'^- and Δ'^+.

For $\Delta_a'^0$: $R_1 = T_1$, $R_2 = T_2$



(4.175)

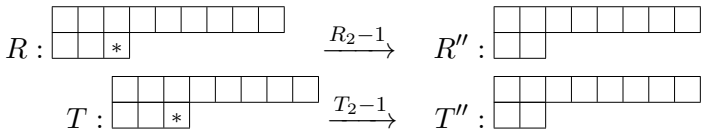
We compute the factors as follows

$$c_{RR'} = c_{TT'} = N + R_1 - 1\, c_{RR'}\sqrt{\frac{f_T}{F_R}} = N + R_1 - 1. \tag{4.176}$$

The contributing terms are (4.112) and (4.127) and so we have,

$$\begin{aligned} c_{RR_1'}dr_1' \Big(&\mathsf{Tr}(E_{11}^{(1)}E_{11}^{(2)}p_{R(r,s)_{\alpha\beta}}E_{11}^{(1)}p_{T(t,s)_{\gamma\delta}}) + \mathsf{Tr}(E_{21}^{(1)}E_{12}^{(2)}p_{R(r,s)_{\alpha\beta}}E_{11}^{(1)}p_{T(t,s)_{\gamma\delta}}) \\ &+\mathsf{Tr}(E_{12}^{(1)}E_{21}^{(2)}p_{R(r,s)_{\alpha\beta}}E_{11}^{(1)}p_{T(t,s)_{\gamma\delta}}) + \mathsf{Tr}(E_{22}^{(1)}E_{22}^{(2)}p_{R(r,s)_{\alpha\beta}}E_{11}^{(1)}p_{T(t,s)_{\gamma\delta}})\Big). \end{aligned} \tag{4.177}$$

For $\Delta_b^{\prime 0}$: $R_1 = T_1$, $R_2 = T_2$



(4.178)

We compute the factors as follows

$$c_{RR'_2} = c_{TT'_2} = N + R_2 - 2\, c_{RR'_2} \sqrt{\frac{f_T}{F_R}} = (N + R_1 - 2). \tag{4.179}$$

The contributing terms are (4.112) and (4.127) and so we have,

$$\begin{aligned} c_{RR'_1} dr'_1 \Big(&\mathsf{Tr}(E_{11}^{(1)} E_{11}^{(2)} p_{R(r,s)_{\alpha\beta}} E_{22}^{(1)} p_{T(t,s)_{\gamma\delta}}) + \mathsf{Tr}(E_{21}^{(1)} E_{12}^{(2)} p_{R(r,s)_{\alpha\beta}} E_{22}^{(1)} p_{T(t,s)_{\gamma\delta}}) \\ &+ \mathsf{Tr}(E_{12}^{(1)} E_{21}^{(2)} p_{R(r,s)_{\alpha\beta}} E_{22}^{(1)} p_{T(t,s)_{\gamma\delta}}) + \mathsf{Tr}(E_{22}^{(1)} E_{22}^{(2)} p_{R(r,s)_{\alpha\beta}} E_{11}^{(1)} p_{T(t,s)_{\gamma\delta}}) \Big). \end{aligned} \tag{4.180}$$

4.4.3 Computing coefficients. We compute the following coefficient using the displaced corners approximation

$$\frac{d_T mn(m-1)! d_{r'}}{d_{R''} d_t d_u (m+n)(m+n-1)} \sqrt{\frac{\mathsf{hooks}_T \mathsf{hooks}_r \mathsf{hooks}_s}{\mathsf{hooks}_R \mathsf{hooks}_t \mathsf{hooks}_u}}. \tag{4.181}$$

Using

$$d_T = \frac{(m+n)!}{\mathsf{hooks}_T}, \quad d_{R''} = \frac{(m+n-2)!}{\mathsf{hooks}_{R''}}, \quad d_t = \frac{n!}{\mathsf{hooks}_t}, \quad d_u = \frac{m!}{\mathsf{hooks}_u}, \quad d_{r'} = \frac{(n-1)!}{\mathsf{hooks}_{r'}} \tag{4.182}$$

we find

$$\frac{d_T mn(m-1)! d_{r'}}{d_{R''} d_t d_u (m+n)(m+n-1)} \sqrt{\frac{\mathsf{hooks}_T \mathsf{hooks}_r \mathsf{hooks}_s}{\mathsf{hooks}_R \mathsf{hooks}_t \mathsf{hooks}_u}} = \frac{m(m-1)}{\sqrt{d_u d_s}}. \tag{4.183}$$

We also need

$$\frac{d_T mn(m-1) d_{r'}}{d_{R''} d_t d_u (m+n)(m+n-1)} \sqrt{\frac{\mathsf{hooks}_T \mathsf{hooks}_r \mathsf{hooks}_s}{\mathsf{hooks}_R \mathsf{hooks}_t \mathsf{hooks}_u}} = \frac{m(m-1)}{(n+m-1)} \frac{1}{\sqrt{d_u d_s}}. \tag{4.184}$$

The final answer, obtained by summing all of the above contributions is

$$\begin{aligned} D_4^{(2)} O_{R,(r,s)\alpha\beta}(Z,Y) \;=\; &\sum_{T,(t,u)\mu\nu} \Big(M^{(a)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} - M^{(b)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} - M^{(c)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} \\ &+ M^{(d)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} \Big) O_{T,(t,u)\mu\nu}(Z,Y). \end{aligned} \tag{4.185}$$

where

$$
\begin{aligned}
M^{(a)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} &= \sum_{R'R''} \delta_{R''T''} \frac{d_T m(m-1)nc_{RR'}c_{R'R''}}{d_{R''}d_t d_u(m+n)(m+n-1)} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_R \text{hooks}_R \text{hooks}_t \text{hooks}_u}}\, d_{r'_i}\delta_{r'_i t'_l} \\
&\times \Bigg[\text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{kj} E^{(2)}_{ll} p_{s\alpha\beta} E^{(1)}_{ii} E^{(2)}_{jk} p_{u\mu\nu} \right) - \text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{kj} p_{s\alpha\beta} E^{(1)}_{ii} E^{(2)}_{jk} p_{u\mu\nu} \right) \delta_{kl} \\
&\quad - \text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{ki} E^{(2)}_{ll} p_{s\alpha\beta} E^{(2)}_{ik} p_{u\mu\nu} \right) \delta_{ij} + \text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{ki} p_{s\alpha\beta} E^{(2)}_{ik} p_{u\mu\nu} \right) \delta_{ij}\delta_{kl} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
M^{(b)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} &= \sum_{R'} \delta_{R'T'} \frac{d_T m(m-1)nc_{RR'}}{d_{R'}d_t d_u(m+n)} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_R \text{hooks}_R \text{hooks}_t \text{hooks}_u}}\, d_{r'_i}\delta_{r'_i t'_k} \\
&\times \Bigg[\text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{ka} E^{(2)}_{ak} p_{s\alpha\beta} E^{(1)}_{ii} p_{u\mu\nu} \right) - \text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{ba} E^{(2)}_{ab} p_{s\alpha\beta} E^{(1)}_{ii} p_{u\mu\nu} \right) \delta_{ik} \\
&\quad - \text{Tr}_{V_p^{\otimes m}} \left(p_{u\mu\nu} E^{(1)}_{ki} E^{(2)}_{ik} p_{s\alpha\beta} \right) + \text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{bi} E^{(2)}_{ib} p_{s\alpha\beta} E^{(1)}_{kk} p_{u\mu\nu} \right) \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
M^{(c)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} &= \sum_{R'} \delta_{R'T'} \frac{d_T m(m-1)nc_{RR'}}{d_{R'}d_t d_u(m+n)} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_R \text{hooks}_R \text{hooks}_t \text{hooks}_u}}\, d_{r'_i}\delta_{r'_i t'_k} \\
&\times \Bigg[\text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{kk} p_{s\alpha\beta} E^{(1)}_{ic} E^{(2)}_{ci} p_{u\mu\nu} \right) - \text{Tr}_{V_p^{\otimes m}} \left(p_{s\alpha\beta} E^{(1)}_{ik} E^{(2)}_{ki} p_{u\mu\nu} \right) \\
&\quad - \text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{ii} p_{s\alpha\beta} E^{(1)}_{ab} E^{(2)}_{ba} p_{u\mu\nu} \right) \delta_{ik} + \text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{ii} p_{s\alpha\beta} E^{(1)}_{ck} E^{(2)}_{kc} p_{u\mu\nu} \right) \Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
M^{(d)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} &= \sum_{R'R''} \delta_{R''T''} \frac{d_T m(m-1)nc_{RR'}c_{R'R''}}{d_{R''}d_t d_u(m+n)(m+n-1)} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_R \text{hooks}_R \text{hooks}_t \text{hooks}_u}}\, d_{r'_i}\delta_{r'_i t'_k} \\
&\times \Bigg[\text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{ka} E^{(2)}_{ai} p_{s\alpha\beta} E^{(1)}_{ib} E^{(2)}_{bk} p_{u\mu\nu} \right) \delta_{ij}\delta_{kl} - \text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{la} E^{(2)}_{ai} p_{s\alpha\beta} E^{(1)}_{ik} E^{(2)}_{kl} p_{u\mu\nu} \right) \delta_{ij} \\
&\quad - \text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{ki} E^{(2)}_{ij} p_{s\alpha\beta} E^{(1)}_{jb} E^{(2)}_{bk} p_{u\mu\nu} \right) \delta_{lk} + \text{Tr}_{V_p^{\otimes m}} \left(E^{(1)}_{li} E^{(2)}_{ij} p_{s\alpha\beta} E^{(1)}_{jk} E^{(2)}_{kl} p_{u\mu\nu} \right) \Bigg].
\end{aligned}
$$

From now on we use the notation $D_4^{(2)}$ to denote the subleading contribution to the two loop dilatation operator.

To complete the evaluation of $D_4^{(2)}$ we need to perform the trace over $V_p^{\otimes m}$. To carry this out we need to construct the projectors $p_{s_{\alpha\beta}}$ and $p_{u_{\mu\nu}}$ and then allow the $E^{(A)}_{ij}$ to act on them. This is all possible and only involves $U(p)$ representation theory. However, a much simpler procedure is to change basis, from the restricted Schur polynomials to Gauss graph operators. These operator diagonalize the leading contributions to the dilatation operator, so that this change of basis is also very useful when we consider computing anomalous dimensions. The next chapter explains the new basis and provides all the background needed to complete the evaluation of the action of $D_4^{(2)}$.

5. Tracing over $V_p^{\otimes m}$

To complete the evaluation of the action of the two loop dilatation operator, we need to perform the the trace over $V_p^{\otimes m}$. This chapter explain how these traces are computed and it discuses the results we obtain after preforming these traces.

5.1 Double Coset Background

The double coset ansatz was formulated in [10], by diaganolizing the one loop dilatation operator. In this section we will review those aspects of [10] that are crucial for our study.

At the most basic level, the double coset ansatz follows from the fact that there are two ways to decompose $V_p^{\otimes m}$. To start, refine $V_p^{\otimes m}$ by the $U(1)^p$ charges which are measured by E_{ii} as follows

$$V_p = \oplus_{i=1}^{p} V_i. \tag{5.1}$$

The vector space V_i is a one-dimensional space. It is spanned by the eigenstate of E_{ii} with eigenvalue one. Consequently if $v_i \in V_i$ we have

$$\begin{aligned} E_{ii} v_j &= \delta_{ij} v_i, \\ E_{ij} v_k &= \delta_{jk} v_i. \end{aligned} \tag{5.2}$$

In the restricted Schur polynomial construction of [9] for long rows, a state in V_i corresponds to a Y-box in the i'th row. The $U(1)$ charges of a restricted Schur polynomial can be collected into the vector $\vec{m}$, which corresponds to a vector with m_1 copies of v_1 , m_2 copies of v_2 etc.

$$|\bar{v}, \vec{m}\rangle \equiv |v_1^{\otimes m_1} \otimes v_2^{\otimes m_2} \otimes \cdots v_p^{\otimes m_p}\rangle. \tag{5.3}$$

A general state with these charges is given by acting with a permutation

$$|v_\sigma\rangle \equiv \sigma |v_1^{\otimes m_1} \otimes v_2^{\otimes m_2} \otimes \cdots v_p^{\otimes m_p}\rangle \tag{5.4}$$

where

$$\sigma |v_{i_1} \otimes \cdots \otimes v_{i_p}\rangle = |v_{i_{\sigma(1)}} \otimes \cdots \otimes v_{i_{\sigma(p)}}\rangle. \tag{5.5}$$

This description enjoys a symmetry under

$$H = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_p} \tag{5.6}$$

and as a consequence, not all σ give independent vectors

$$|v_\sigma\rangle = |v_{\sigma\gamma}\rangle \tag{5.7}$$

if $\gamma \in H$.

The restricted Schur polynomials are organized by reduction multiplicities of $U(p)$ to $U(1)^p$, which are counted by the Kotska numbers and resolved by the Gelfand-Tsetlin patterns. It is possible to prove the equality of Kotska numbers and the branching multiplicity of $S_m \to H$. This is a very direct indication that there are two possible ways to organize the local operators of the theory.

We can develop the steps above at the level of a basis for $V_p^{\otimes m}$. In terms of the branching coefficients, defined by

$$\frac{1}{|H|}\sum_{\gamma\in H}\Gamma^{(s)}_{ik}(\gamma)=\sum_{\mu}B^{s\to 1_H}_{i\mu}B^{s\to 1_H}_{k\mu} \tag{5.8}$$

we have

$$|\vec{m},s,\mu;i\rangle\equiv\sum_j B^{s\to 1_H}_{j\mu}|v_{s,i,j}\rangle=\sum_j B^{s\to 1_H}_{j\mu}\sum_{\sigma\in S_m}\Gamma^{(s)}_{ij}(\sigma)|v_\sigma\rangle. \tag{5.9}$$

The μ index is a multiplicity for reduction of S_m into H. We also have

$$\langle\vec{m},u,\nu;j|=\frac{d_u}{m!|H|}\sum_{\tau\in S_m}\langle\bar{v},\vec{m}|\tau^{-1}\Gamma^{(u)}_{jk}(\tau)B^{u\to 1_H}_{k\nu} \tag{5.10}$$

which ensures the correct normalization

$$\langle\vec{m},u,\nu;j|\vec{n},s,\mu;i\rangle=\delta_{\vec{m}\vec{n}}\delta_{us}\delta_{ji}\delta_{\mu\nu}. \tag{5.11}$$

Finally, the group-theoretic coefficients

$$C^s_{\mu_1\mu_2}(\tau)=|H|\sqrt{\frac{d_s}{m!}}\Gamma^{(s)}_{kl}(\tau)B^{s\to 1_H}_{k\mu_1}B^{s\to 1_H}_{l\mu_2} \tag{5.12}$$

provide an orthogonal transformation between double coset elements labeled by σ and the restricted Schur polynomials labeled by an irreducible representation $s\vdash m$ and a pair of multiplicities μ_1,μ_2.

5.2 How to compute traces

In this section we will compute the traces needed to evaluate the action of the two loop dilatation operator in the Gauss graph basis. The generic form of the trace we need to evaluate is

$$\mathrm{Tr}(Ap_{s\mu\nu}Bp_{u\gamma\delta}) \tag{5.13}$$

with A and B any arbitrary product of the $E^{(A)}_{ij}$s. If we are able to compute this trace, we are able to evaluate the action of any differential operator that does not change the number of Z or Y fields, on the Gauss graph operator in the displaced corners approximation. This therefore provides a general method to exploit the simplifications of the large N limit, for this class of operators.

The Fourier transform we want to consider maps between functions labeled by an irreducible representation s and a pair of multiplicity labels and functions that take values on the double coset $H\setminus S_m/H$. We can choose a permutation σ to represent each class of the coset $\big[\sigma\big]$. The transform is then

$$\tilde{f}(\big[\sigma\big])=\sum_{s,\alpha,\beta}\Gamma_s(\sigma)_{ab}B^{s\to 1_H}_{a\alpha}B^{s\to 1_H}_{b\beta}f(s,\alpha,\beta) \tag{5.14}$$

For further details the reader is referred to [10].

5.2.1 Projector transformed. In this section we will Fourier transform the intertwining map used to define the restricted Schur polynomial. The projector that participates in the trace (5.13) can be expressed as

$$p_{s\mu\nu} = \sum_a |\vec{m}, s, \mu, a\rangle\langle \vec{m}, s, \nu, a| \tag{5.15}$$

We will make use of the relations

$$\langle \vec{m}, t, \nu, j| = \frac{d_t}{m!|H|} \sum_{\tau\in S_m} \langle \bar{v}, \vec{m}|\tau^{-1}\Gamma^{(t)}_{jk}(\tau) B^{t\to 1_H}_{k\nu} \tag{5.16}$$

and

$$|\vec{m}, s, \mu, i\rangle = \sum_{\sigma\in S_m} \Gamma^{(s)}_{il}(\sigma) B^{s\to 1_H}_{l\mu} \sigma|\bar{v}, \vec{m}\rangle \tag{5.17}$$

as well as

$$\langle \bar{v}, \vec{m}|\sigma|\bar{v}, \vec{m}\rangle = \sum_{\gamma\in H} \delta(\sigma\gamma) \tag{5.18}$$

and

$$\sum_\nu B^{s\to 1_H}_{c\nu} B^{s\to 1_H}_{d\nu} = \frac{1}{|H|} \sum_{\gamma\in H} \Gamma^{(s)}(\gamma)_{cd}. \tag{5.19}$$

We will use the notation $|v_\sigma\rangle = \sigma|\bar{v}\rangle$. It is then simple to show that

$$\begin{aligned} \sum_{s\,\mu\,\nu\,l\,m} p_{s\mu\nu} B^{s\to 1_H}_{m\mu} B^{s\to 1_H}_{l\nu} \Gamma^{(s)}_{ml}(\sigma) &= \sum_{s\,\mu\,\nu\,l\,m\,a} |\vec{m}, s, \mu, a\rangle\langle \vec{m}, s, \nu, a| B^{s\to 1_H}_{m\mu} B^{s\to 1_H}_{l\nu} \Gamma^{(s)}_{ml}(\sigma) \\ &= \frac{1}{|H|^3} \sum_\psi \sum_{\tau\in S_m} \sum_{\gamma_1,\gamma_2\in H} |v_\psi\rangle\langle v_\tau|\delta(\tau^{-1}\psi\gamma_2\sigma\gamma_1) \end{aligned} \tag{5.20}$$

This last equation implies that the permutation applied to the ket and the permutation applied to the bra are related by multiplication by a permutation representing the double coset element.

5.2.2 Summing over H. We consider an intertwining map $p_{s\mu\nu}$ built on the state $|\bar{v}_1, \vec{m}_1\rangle$ with symmetry group H_1 and intertwining map $p_{t\gamma\delta}$ built on the state $|\bar{v}_2, \vec{m}_2\rangle$ with symmetry group H_2. We make no assumptions about H_1 and H_2. In general they will be different groups and hence we Fourier transform $p_{s\mu\nu}$ and $p_{t\gamma\delta}$ to different double cosets. Using the result obtained above, the Fourier transform of

$$\mathrm{Tr}(A p_{s\mu\nu} B p_{u\gamma\delta}) \tag{5.21}$$

is

$$\begin{aligned} T &= \frac{1}{|H_1|^3|H_2|^3} \sum_{\gamma_i\in H_i} \sum_{\tau_i\in H_i} \sum_{\psi_i\in S_m} \langle v_2|\gamma_2\sigma_2^{-1}\tau_2\psi_2^{-1} A\psi_1|v_1\rangle\langle v_1|\gamma_1\sigma_1^{-1}\tau_1\psi_1^{-1} B\psi_2|v_2\rangle \\ &= \frac{1}{|H_1|^3|H_2|^3} \sum_{\gamma_i\in H_i} \sum_{\tau_i\in H_i} \sum_{\psi_i\in S_m} \langle v_2|\gamma_2\sigma_2^{-1}\tau_2\psi_2^{-1} A\psi_1|v_1\rangle\langle v_2|\psi_2^{-1} B^T\psi_1\tau_1\sigma_1\gamma_1|v_1\rangle. \end{aligned} \tag{5.22}$$

To get to the last line, we used the fact that the matrix element $\langle v_1|\gamma_1\sigma_1^{-1}\tau_1\psi_1^{-1}B\psi_2|v_2\rangle$ is a real number and the permutations are represented by matrices with real elements. To make the discussion concrete, it is useful to make a specific choice for A and B. This will allow us to illustrate the argument in a very concrete setting. In the end we will state the general result. Choose, for example

$$A = E^{(1)}_{ki}E^{(2)}_{ij}, \qquad B = E^{(1)}_{jl}E^{(2)}_{lk}. \tag{5.23}$$

Using the facts that

$$\begin{array}{rcll} \gamma|v_1\rangle &=& |v_1\rangle & \forall\gamma\in H_1 \\ \beta|v_2\rangle &=& |v_2\rangle & \forall\beta\in H_2 \\ \psi^{-1}E^{(a)}_{qr}\psi &=& E^{\psi(a)}_{qr} & \forall\psi\in S_m \end{array} \tag{5.24}$$

we readily find

$$\begin{aligned} T &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_2|\sigma_2^{-1}E^{\psi_2(1)}_{ki}E^{\psi_2(2)}_{ij}\psi_1|v_1\rangle\langle v_2|E^{\psi_2(1)}_{lj}E^{\psi_2(2)}_{kl}\psi_1\sigma_1|v_1\rangle \\ &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_2|\sigma_2^{-1}\psi_2^{-1}A\psi_2\psi_1|v_1\rangle\langle v_2|\psi_2^{-1}B^T\psi_2\psi_1\sigma_1|v_1\rangle. \end{aligned} \tag{5.25}$$

where on the last line we have written the general result. Our next task is to compute the sums over ψ_1 and ψ_2.

5.2.3 Summing over S_m. In this subsection we consider two increasing difficult examples before we state the general result. The first example is closely related to the trace needed to obtain the one loop dilatation operator. Since we know the result of this trace, this example is a nice test of our ideas. The second example is a simple but non-trivial example which will illustrate how the general case works. In the following section we will quote the result for the general case.

First Evaluation

Choose $A = E^{(1)}_{ab}$ and $B = E^{(1)}_{ba}$ to get

$$\begin{aligned} T &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_2|\sigma_2^{-1}E^{\psi_2(1)}_{ab}\psi_1|v_1\rangle\langle v_2|E^{\psi_2(1)}_{ab}\psi_1\sigma_1|v_1\rangle \\ &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_1|\psi_1^{-1}E^{\psi_2(1)}_{ba}\sigma_2|v_2\rangle\langle v_2|E^{\psi_2(1)}_{ab}\psi_1\sigma_1|v_1\rangle. \end{aligned} \tag{5.26}$$

We must turn a b vector in $|v_1\rangle$ into an a vector (and possibly permute) to get $|v_2\rangle$. Since the ordering of the slots in $|v_1\rangle$ and $|v_2\rangle$ is arbitrary, we can remove this possible permutation by declaring

$$|v_2\rangle = E^{(1)}_{ab}|v_1\rangle, \qquad |v_1\rangle = E^{(1)}_{ba}|v_2\rangle. \tag{5.27}$$

Thus (in the computation below we denote by S^1_a (S^2_a) the set of all slots in $|v_1\rangle$ ($|v_2\rangle$) that are filled with an a vector)

$$\begin{aligned} T &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_1|\psi_1^{-1}E^{\psi_2(1)}_{ba}\sigma_2E^{(1)}_{ab}|v_1\rangle\langle v_1|E^{(1)}_{ba}E^{\psi_2(1)}_{ab}\psi_1\sigma_1|v_1\rangle \\ &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_1|\psi_1^{-1}\sigma_2E^{\sigma_2\psi_2(1)}_{ba}E^{(1)}_{ab}|v_1\rangle\langle v_1|E^{(1)}_{ba}E^{\psi_2(1)}_{ab}\psi_1\sigma_1|v_1\rangle \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\sum_{\gamma_i\in H_1}\delta\Big(\gamma_1\psi_1^{-1}\sigma_2(1,\sigma_2\psi_2(1))\Big)\delta\Big(\gamma_2\sigma_1^{-1}\psi_1^{-1}(1,\psi_2(1))\Big)\\
&\quad\times\sum_{x\in S_a^2}\delta(\sigma_2\psi_2(1),x)\sum_{y\in S_a^2}\delta(\psi_2(1),y)\\
&= \frac{1}{|H_1||H_2|}\sum_{\psi_2\in S_m}\sum_{\gamma_i\in H_1}\delta\Big(\gamma_1\sigma_1\gamma_2(1,\psi_2(1))\sigma_2(1,\sigma_2\psi_2(1))\Big)\\
&\quad\times\sum_{x\in S_a^2}\delta(\sigma_2\psi_2(1),x)\sum_{y\in S_a^2}\delta(\psi_2(1),y).
\end{aligned}
\tag{5.28}
$$

Now consider the final sum over ψ_2. $\psi_2(1)$ is the start point of an oriented edge in Gauss graph σ_2 and $\sigma_2\psi_2(1)$ is the end point of the same edge. The delta functions on the last line ensure that both endpoints of this string are attached to node a in the Gauss graph. This is swapped with the edge labeled 1 (i.e. the edge in the first slot) and compared to σ_1. According to (5.27), the edge in the first slot of $|v_1\rangle$ is attached to node b. Thus the above sum is ensuring that when a closed loop on node a of σ_2 is removed and reattached to node b of σ_2 we get σ_1. The above sum is non-zero only when σ_1 and σ_2 are related in this way. The deltas only fix $\psi(1)$, so summing over S_m the remaining "unfixed" piece of ψ_2 gives $(m-1)!$. The first delta will, as usual, give the norm of Gauss graph σ_1 and we will get a non-zero contribution whenever $\psi_2(1)$ is one of the values in S_a^2. There are $n_{aa}(\sigma_2)$ possible values. Thus, when T is non-zero it takes the value

$$
T = (m-1)!n_{aa}(\sigma_2)|O(\sigma_1)|^2 \tag{5.29}
$$

where we have assumed that both σ_1 and σ_2 have a total of p nodes and we denote the number of oriented line segments stretching from node k to node l of σ by $n_{kl}(\sigma)$. We have denoted the "norm" of the Gauss graph operator by $|O(\sigma_1)|^2$. This is the value of the two point function of the Gauss operator

$$
|O(\sigma_1)|^2 = \prod_{i=1}^{p} n_{ii}(\sigma_1)! \prod_{k,l=1,l\neq k}^{p} n_{kl}(\sigma_1)! = \langle O_{R,r}(\sigma_1)^\dagger O_{R,r}(\sigma_1)\rangle. \tag{5.30}
$$

The value of the trace (5.29) is in perfect agreement with the known result[28].

Second evaluation

For the second example we consider, we choose

$$
A = E^{(1)}_{ki}E^{(2)}_{ij}, \qquad B = E^{(1)}_{jl}E^{(2)}_{lk}. \tag{5.31}
$$

There is some freedom in the placement of the indices on A and B. To see why this is the case, recall that we are evaluating the Fourier transform of

$$
\mathrm{Tr}(Ap_{s\mu\nu}Bp_{u\gamma\delta}). \tag{5.32}
$$

The intertwining maps $p_{s\mu\nu}$ and $p_{u\gamma\delta}$ commute with any element of S_m. Consequently we have

$$
\mathrm{Tr}(Ap_{s\mu\nu}Bp_{u\gamma\delta}) = \mathrm{Tr}(A\,\sigma\,\sigma^{-1}\,p_{s\mu\nu}Bp_{u\gamma\delta}) = \mathrm{Tr}(A\,\sigma\,p_{s\mu\nu}\,\sigma^{-1}\,Bp_{u\gamma\delta}) \tag{5.33}
$$

where σ is any element in S_m. Choosing $\sigma = (12)$ and using the representation $(12) = E^{(1)}_{ab}E^{(2)}_{ba}$ where a and b are summed from 1 to p, as well as the product rule

$$
E^{(A)}_{ab}E^{(B)}_{cd} = E^{(B)}_{cd}E^{(A)}_{ab}, \qquad A\neq B, \qquad E^{(A)}_{ab}E^{(A)}_{cd} = \delta_{bc}E^{(A)}_{ad}, \tag{5.34}
$$

we find

$$A(12) = E^{(1)}_{ki}E^{(2)}_{ij}(12) = E^{(1)}_{ki}E^{(2)}_{ij}E^{(1)}_{ab}E^{(2)}_{ba} = E^{(1)}_{kj}E^{(2)}_{ii}, \tag{5.35}$$

$$(12)B = (12)E^{(1)}_{jl}E^{(2)}_{lk} = E^{(1)}_{ab}E^{(2)}_{ba}E^{(1)}_{jl}E^{(2)}_{lk} = E^{(1)}_{ll}E^{(2)}_{jk}. \tag{5.36}$$

This implies that we can rather consider $A = E^{(1)}_{kj}E^{(2)}_{ii}$ and $B = E^{(1)}_{ll}E^{(2)}_{jk}$ without changing the value of the trace. In this case we have (argue as we did above and use $|v_2\rangle = E^{(1)}_{kj}|v_1\rangle$)

$$\begin{aligned} T &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_2|\sigma_2^{-1}E^{\psi_2(1)}_{kj}E^{\psi(2)}_{ii}\psi_1|v_1\rangle\langle v_2|E^{\psi_2(1)}_{ll}E^{\psi(2)}_{kj}\psi_1\sigma_1|v_1\rangle \\ &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_1|\psi_1^{-1}E^{\psi_2(1)}_{jk}E^{\psi(2)}_{ii}\sigma_2|v_2\rangle\langle v_2|E^{\psi_2(1)}_{ll}E^{\psi(2)}_{kj}\psi_1\sigma_1|v_1\rangle. \end{aligned} \tag{5.37}$$

Notice that when $E^{\psi(2)}_{ii}$ acts on $|v_2\rangle$, it does not change the identity of any of the vectors appearing in $|v_2\rangle$. On the other hand, $E^{\psi_2(1)}_{jk}$ will turn an e_k into an e_j. Thus, again up to an arbitrary permutation which we can always remove, we must have

$$|v_2\rangle = E^{(1)}_{kj}|v_1\rangle \tag{5.38}$$

The trace now takes the value

$$\begin{aligned} T &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_1|\psi_1^{-1}E^{\psi_2(1)}_{jk}E^{\psi(2)}_{ii}\sigma_2E^{(1)}_{kj}|v_1\rangle\langle v_1|E^{(1)}_{jk}E^{\psi_2(1)}_{ll}E^{\psi(2)}_{kj}\psi_1\sigma_1|v_1\rangle \\ &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_1|\psi_1^{-1}\sigma_2E^{\sigma_2\psi_2(1)}_{jk}E^{\sigma_2\psi(2)}_{ii}E^{(1)}_{kj}|v_1\rangle\langle v_1|E^{(1)}_{jk}E^{\psi_2(1)}_{ll}E^{\psi(2)}_{kj}\psi_1\sigma_1|v_1\rangle \\ &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\sum_{\gamma_i\in H_1}\delta\Big(\gamma_1\psi_1^{-1}\sigma_2(\sigma_2\psi_2(1),1)\Big)\delta\Big(\gamma_2\sigma_1^{-1}\psi_1^{-1}(\psi_2(2),1)\Big) \\ &\quad\times\sum_{x\in S^2_l}\delta(x,\psi_2(1))\sum_{y\in S^2_k}\delta(y,\psi_2(2))\sum_{w\in S^2_k}\delta(w,\sigma_2\psi_2(1))\sum_{v\in S^2_i}\delta(v,\sigma_2\psi_2(2)) \\ &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\sum_{\gamma_i\in H_1}\delta\Big(\gamma_1\sigma_1\gamma_2(1,\psi_2(2))\sigma_2(\sigma_2\psi_2(1),1)\Big) \\ &\quad\times\sum_{x\in S^2_l}\delta(x,\psi_2(1))\sum_{y\in S^2_k}\delta(y,\psi_2(2))\sum_{w\in S^2_k}\delta(w,\sigma_2\psi_2(1))\sum_{v\in S^2_i}\delta(v,\sigma_2\psi_2(2)) \\ &= \frac{(m-2)!}{|H_1||H_2|}n_{lk}(\sigma_2)n_{ki}(\sigma_2)\prod_{q=1}^{p}n_{qq}(\sigma_1)!\prod_{r,s=1,\,r\neq s}^{p}n_{rs}(\sigma_1)! \end{aligned} \tag{5.39}$$

whenever it is non-zero.

5.2.4 General Result. Recall that both T and R have p long rows or columns. For the general result we consider

$$\begin{aligned} &\sum_{s\,\mu\,\nu\,l\,m}\sum_{u\,\gamma\,\delta\,n\,p}\mathrm{Tr}(Ap_{s\mu\nu}Bp_{u\gamma\delta})B^{s\to 1_H}_{m\mu}B^{s\to 1_H}_{l\nu}\Gamma^{(s)}_{ml}(\sigma_1)B^{u\to 1_H}_{p\gamma}B^{u\to 1_H}_{n\delta}\Gamma^{(u)}_{pn}(\sigma_2) \\ &\qquad = \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_2|\sigma_2^{-1}\left(\psi_2^{-1}A\psi_2\right)\psi_1|v_1\rangle\langle v_2|\left(\psi_2^{-1}B^T\psi_2\right)\psi_1\sigma_1|v_1\rangle \end{aligned} \tag{5.40}$$

where A and B are each products of a collection of the $E_{ab}^{(\alpha)}$s with $1 \leq a, b \leq p$ and $1 \leq \alpha \leq m$. We say that $E_{ab}^{(\alpha)}$ occupies slot α. The sum over ψ_2 sums over the possible choices for the slots into which we place the factors of $E_{ab}^{(\alpha)}$ in A and B. Thus, the specific slots chosen for the factors in A and B are arbitrary - we must simply respect the relative ordering of factors in A and B, i.e. factors sharing the same slot in one labeling share the same slot in all labelings. The sum over ψ_1 ensures that the relative labeling of the vectors appearing in $|v_1\rangle$ and $|v_2\rangle$ is arbitrary. Thus, the specific labeling of the directed edges in the Gauss graph is arbitrary which ensures that the above sum is indeed defined on the relevant double cosets to which σ_1 and σ_2 belong. There are two pieces of information that we need to read from σ_1, σ_2, A and B:

1 When is the sum nonzero?

2 What is the value of the sum?

It is simplest to begin with the second question first. Towards this end, consider the expressions for A and B. After using the algebra for the $E_{ab}^{(\alpha)}$ if needed, we know that at most a single $E_{ab}^{(\alpha)}$ acts per slot in both A and B. By inserting factors of

$$\sum_{a=1}^{p} E_{aa}^{(\alpha)} = \mathbf{1} \tag{5.41}$$

if necessary, we can ensure that the same set of occupied slots appears in A and B. For concreteness, assume that q slots are occupied in both. Use i_α^r (i_α^c) to denote the row (column) indices of the E_{ab} in the αth slot in B and use j_α^r (j_α^c) to denote the row (column) indices of the E_{ab} in the αth slot in A. Thanks to the lessons we have learned from the examples treated above, when the sum is non-zero it is given by

$$\frac{1}{|H_1||H_2|} \sum_{\psi_i \in S_m} \langle v_2|\sigma_2^{-1}\left(\psi_2^{-1} A \psi_2\right)\psi_1|v_1\rangle\langle v_2|\left(\psi_2^{-1} B^T \psi_2\right)\psi_1\sigma_1|v_1\rangle$$
$$= \frac{(m-q)!|O(\sigma_1)|^2}{|H_1||H_2|} \prod_{\alpha=1}^{q} n_{i_\alpha^c j_\alpha^r}(\sigma_2). \tag{5.42}$$

If any particular $n_{ij}(\sigma_2)$ appears more than once, each new factor in the product is to be reduced by 1. For example, $n_{12}(\sigma_2)^3$ would be replaced by $n_{12}(\sigma_2)(n_{12}(\sigma_2)-1)(n_{12}(\sigma_2)-2)$. By taking the transpose of (5.42), the value of the sum is not changed because it is a real number. However, the roles of σ_1 and σ_2, as well as of A and B are reversed on the LHS of (5.42). Consequently, we must also have

$$\frac{1}{|H_1||H_2|} \sum_{\psi_i \in S_m} \langle v_2|\sigma_2^{-1}\left(\psi_2^{-1} A \psi_2\right)\psi_1|v_1\rangle\langle v_2|\left(\psi_2^{-1} B^T \psi_2\right)\psi_1\sigma_1|v_1\rangle$$
$$= \frac{(m-q)!|O(\sigma_2)|^2}{|H_1||H_2|} \prod_{\alpha=1}^{q} n_{j_\alpha^c i_\alpha^r}(\sigma_1) \tag{5.43}$$

The equality of (5.42) and (5.43) defines our delta function. We find that $\delta_{AB}([\sigma_1][\sigma_2]) = 1$ if

$$\frac{(m-q)!|O(\sigma_1)|^2}{|H_1||H_2|} \prod_{\alpha=1}^{q} n_{i_\alpha^c j_\alpha^r}(\sigma_2) = \frac{(m-q)!|O(\sigma_2)|^2}{|H_1||H_2|} \prod_{\alpha=1}^{q} n_{j_\alpha^c i_\alpha^r}(\sigma_1) \tag{5.44}$$

and it is zero otherwise.

We will sketch how the general result is proved. First, even if A and B straddle $q < m$ slots, by using (5.41) we can always introduce further $E^{(\alpha)}$s so that all m slots are straddled. Thus, without loss of generality we can now focus on the $q = m$ case. In this case, it is easy to prove that if $\prod_{\alpha=1}^{q} n_{i^c_\alpha j^r_\alpha}(\sigma_2)$ is non-zero, it is given by

$$\prod_{\alpha=1}^{q} n_{i^c_\alpha j^r_\alpha}(\sigma_2) = |O(\sigma_2)|^2 \tag{5.45}$$

which proves the result. Similarly, if $\prod_{\alpha=1}^{q} n_{j^c_\alpha i^r_\alpha}(\sigma_1)$ is non-zero, it is equal to

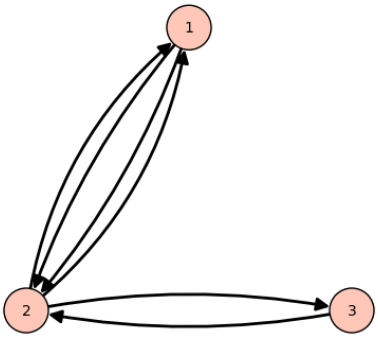
$$\prod_{\alpha=1}^{q} n_{j^c_\alpha i^r_\alpha}(\sigma_1) = |O(\sigma_1)|^2 \tag{5.46}$$

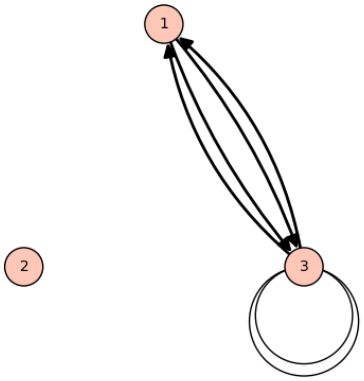
which again proves the general result.

5.2.5 Illustration of the General Result. To illustrate the formula derived in the previous section consider computing the trace for the case that

$$\begin{aligned} A &= E^{(1)}_{31} E^{(2)}_{11} E^{(3)}_{32} E^{(4)}_{12} E^{(5)}_{32} E^{(6)}_{33}, \\ B &= E^{(1)}_{23} E^{(2)}_{23} E^{(3)}_{11} E^{(4)}_{13} E^{(5)}_{31} E^{(6)}_{23}. \end{aligned} \tag{5.47}$$

For our example we have $m = 6$, $q = 6$ and $p = 3$, so that $(m - q)! = 1$. We choose σ_1 and σ_2 as illustrated below

$\sigma_1 =$ 

$\sigma_2 =$ 

,

From these Gauss graphs we easily read off $H_1 = S_2 \times S_3$ and $H_2 = S_4 \times S_2$. Consequently $|H_1| = 12$ and $|H_2| = 48$. If we choose the permutation

$$\sigma_1 = (13)(24)(56) \in H_1 \setminus S_6 / H_1 \tag{5.48}$$

to represent the first Gauss graph, then we can choose the first factor in H_1 to permute 1 and 2 and the second factor to permute 3,4 and 5. If we choose the permutation

$$\sigma_2 = (12)(45) \tag{5.49}$$

to represent the second Gauss graph, then we can choose the first factor to permute 2 and 5 and the last factor to permute 1, 3, 4 and 6.

From the row indices of A given by the ordered set $\{3,1,3,1,3,3\}$, and the column indices of B given by the ordered set $\{3,3,1,3,1,3\}$, we read off

$$\begin{aligned}\prod_{\alpha=1}^{q} n_{i_\alpha^c j_\alpha^r}(\sigma_2) &= n_{33}(\sigma_2)n_{31}(\sigma_2)n_{13}(\sigma_2)n_{31}(\sigma_2)n_{13}(\sigma_2)n_{33}(\sigma_2)\\ \to n_{33}(\sigma_2)(n_{33}(\sigma_2)-1)n_{31}(\sigma_2)(n_{31}(\sigma_2)-1)n_{13}(\sigma_2)(n_{13}(\sigma_2)-1)&\\ = |O(\sigma_2)|^2 = 8&\end{aligned} \tag{5.50}$$

which indicates that the sum may indeed be non-zero. From the column indices of A given by the ordered set $\{1,1,2,2,2,3\}$, and the row indices of B given by the ordered set $\{2,2,1,1,3,2\}$, we read off

$$\begin{aligned}\prod_{\alpha=1}^{q} n_{j_\alpha^c i_\alpha^r}(\sigma_1) &= n_{12}(\sigma_1)n_{12}(\sigma_1)n_{21}(\sigma_1)n_{21}(\sigma_1)n_{23}(\sigma_1)n_{32}(\sigma_1)\\ \to n_{12}(\sigma_1)(n_{12}(\sigma_1)-1)n_{21}(\sigma_1)(n_{21}(\sigma_1)-1)n_{23}(\sigma_1)n_{32}(\sigma_1)&\\ = |O(\sigma_1)|^2 = 4&\end{aligned} \tag{5.51}$$

which indicates that the sum is indeed non-zero. We finally obtain

$$\frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_2|\sigma_2^{-1}\left(\psi_2^{-1}A\psi_2\right)\psi_1|v_1\rangle\langle v_2|\left(\psi_2^{-1}B^T\psi_2\right)\psi_1\sigma_1|v_1\rangle = \frac{32}{|H_1||H_2|} = \frac{1}{18}. \tag{5.52}$$

5.3 Final result for the subleading contribution to the two loop dilatation operator

To compute the remaining trace in (4.185) over $V_p^{\otimes m}$ we will again move to the Gauss graph basis. Using the results described above, we now find

$$D_4 O_{R,r}(\sigma_1) = \sum_{T,t,\sigma_2}\left(M^{1,\sigma_1,\sigma_2}_{T,t;R,r} - M^{2,\sigma_1,\sigma_2}_{T,t;R,r} - M^{3,\sigma_1,\sigma_2}_{T,t;R,r} + M^{4,\sigma_1,\sigma_2}_{T,t;R,r}\right)O_{T,t}(\sigma_2), \tag{5.53}$$

where

$$\begin{aligned}M^{1,\sigma_1,\sigma_2}_{T,t;R,r} &= \sum_{R',R''}\delta_{R''T''}\delta_{r'_i t'_l}\sqrt{\frac{c_{RR'}c_{R'R''}c_{TT'}c_{T'T''}}{R_j T_k}}\,|O_{R,r}(\sigma_1)|^2\,\delta_{AB}\left([\sigma_1],[\sigma_2]\right)\\ &\quad\times\Big[n_{ik}(\sigma_2)n_{kl}(\sigma_2) - \delta_{kl}\sum_{a=1}^{p}n_{ik}(\sigma_2)n_{ka}(\sigma_2) - \delta_{ij}\sum_{b=1}^{p}n_{bk}(\sigma_2)n_{kl}(\sigma_2)\\ &\qquad+\delta_{ij}\delta_{kl}\sum_{a,b=1}^{p}n_{bk}(\sigma_2)n_{ka}(\sigma_2)\Big],\end{aligned}$$

$$\begin{aligned}M^{2,\sigma_1,\sigma_2}_{T,t;R,r} &= \sum_{R'}\delta_{R'T'}\delta_{r'_i t'_k}\sqrt{c_{RR'}c_{TT'}}\,|O_{R,r}(\sigma_1)|^2\,\delta_{AB}\left([\sigma_1],[\sigma_2]\right)\\ &\quad\times\Big[\sum_{a,b=1}^{p}n_{ik}(\sigma_2)n_{ba}(\sigma_2) - \delta_{ik}\sum_{a,b,c=1}^{p}n_{ib}(\sigma_2)n_{ca}(\sigma_2) - \sum_{a,b=1}^{p}n_{ak}(\sigma_2)n_{bi}(\sigma_2)\end{aligned}$$

$$+\sum_{a,b=1}^{p} n_{kb}(\sigma_2)n_{ai}(\sigma_2)\Big],$$

$$\begin{aligned} M^{3,\sigma_1,\sigma_2}_{T,t;R,r} &= \sum_{R'} \delta_{R'T'}\delta_{r'_i t'_k}\sqrt{c_{RR'}c_{TT'}}\,|O_{R,r}(\sigma_1)|^2\,\delta_{AB}\left([\sigma_1],[\sigma_2]\right) \\ &\times\Big[\sum_{a,b=1}^{p} n_{bk}(\sigma_2)n_{ia}(\sigma_2) - \sum_{a,b=1}^{p} n_{ka}(\sigma_2)n_{ib}(\sigma_2) - \delta_{ik}\sum_{a,b,c=1}^{p} n_{bi}(\sigma_2)n_{ac}(\sigma_2) \\ &+\sum_{a,b=1}^{p} n_{ki}(\sigma_2)n_{ba}(\sigma_2)\Big], \end{aligned} \tag{5.54}$$

and

$$\begin{aligned} M^{4,\sigma_1,\sigma_2}_{T,t;R,r} &= \sum_{R',R''} \delta_{R''T''}\delta_{r'_i t'_k}\sqrt{\frac{c_{RR'}c_{R'R''}c_{TT'}c_{T'T''}}{R_j T_l}}\,|O_{R,r}(\sigma_1)|^2\,\delta_{AB}\left([\sigma_1],[\sigma_2]\right) \\ &\times\Big[\delta_{ij}\delta_{kl}\sum_{a,b=1}^{p} n_{bl}(\sigma_2)n_{la}(\sigma_2) - \delta_{ij}\sum_{a=1}^{p} n_{kl}(\sigma_2)n_{la}(\sigma_2) - \delta_{kl}\sum_{b=1}^{p} n_{bl}(\sigma_2)n_{li}(\sigma_2) \\ &+n_{kl}(\sigma_2)n_{li}(\sigma_2)\Big]. \end{aligned} \tag{5.55}$$

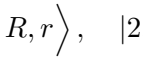
Notice that both $M^{1,\sigma_1,\sigma_2}_{T,t;R,r}$ and $M^{4,\sigma_1,\sigma_2}_{T,t;R,r}$ depend on the length of the rows of the Young diagrams R and T that participate. Since these lengths determine the angular momentum of the giants, they determine the radius to which the giants will expand. This is the first dependence of the anomalous dimensions on the geometry of the giant graviton.

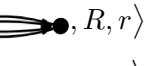
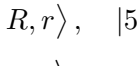
To illustrate the form of operator mixing captured in the above formulas, it is helpful to consider a specific example. This is the content of the next section.

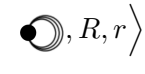
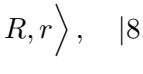
5.4 Example: A 2 Giant Graviton Boundstate with 4 Strings Attached

In this section we will consider the simplest nontrivial system that exhibits the general structure of the subleading operator mixing problem. This problem is a system of two giant gravitons with four strings attached. As we have explained, there are $\frac{m}{n}$ corrections to the anomalous dimension from the one loop contribution as well as from the leading terms at two loops. These corrections do not induce extra operator mixing, so that the Gauss graph operators continue to have a good anomlaous dimension. The only extra operator mixing comes from the subleading contibution we have just evaluated, and this is what we aim to explore.

Each element of the Gauss graph basis is labeled by two Young diagrams R, r as well as a Gauss graph. We will number the states according to their Gauss graph labeling as shown below

$|1,R,r\rangle =$  $,R,r\rangle$, $|2,R,r\rangle =$  $,R,r\rangle$, $|3,R,r\rangle =$  $,R,r\rangle$,

$|4,R,r\rangle =$  $,R,r\rangle$, $|5,R,r\rangle =$  $,R,r\rangle$, $|6,R,r\rangle =$  $,R,r\rangle$,

$|7,R,r\rangle =$  $,R,r\rangle$, $|8,R,r\rangle =$  $,R,r\rangle$, $|9,R,r\rangle =$  $,R,r\rangle$.

The states $|i\rangle$ for $i = 5, 6, 7, 8, 9$ are BPS at the leading order in $\frac{m}{n}$. The subleading corrections to the anomalous dimension coming from one loop, as well as from the leading terms at two loops is a multiplicative order $\frac{m}{n}$ correction and vanishes because the leading order anomalous dimension vanishes. Evaluating the subleading order contribution, we find

$$D_4^{(2)} |i, R, r\rangle = 0 \qquad i = 5, 6, 7, 8, 9 \tag{5.56}$$

so that the states that are BPS at the leading order do not receive a subleading correction. This is not peculiar to the example we consider and is to be expected generally, since for the BPS states we have $n_{ab}(\sigma_2) = 0$ for $a \neq b$.

The state $|4\rangle$ also does not mix with other states. However, for this state we have a nontrivial correction to the eigenvalue since

$$\begin{aligned}
D_4^{(2)} |4, R, r\rangle = \sum_{T,t} 64 \Bigg[& \delta_{RT} \left(\frac{\delta_{r_1' t_1'}}{R_2} + \frac{\delta_{r_2' t_2'}}{R_1} \right) \delta_{R_{12}'' T_{12}''} (N + R_1 - 1)(N + R_2 - 2) \\
& + \delta_{RT} \frac{\delta_{r_1' t_1'}}{R_1} \delta_{R_{11}'' T_{11}''} (N + R_1 - 1)(N + R_1 - 2) + \delta_{RT} \frac{\delta_{r_2' t_2'}}{R_2} \delta_{R_{22}'' T_{22}''} (N + R_2 - 2)(N + R_2 - 3) \\
& - \delta_{RT} \delta_{r_1' t_1'} \delta_{R_1' T_1'} (N + R_1 - 1) - \delta_{RT} \delta_{r_2' t_2'} \delta_{R_2' T_2'} (N + R_2 - 2) \\
& - \delta_{R_1' T_2'} \frac{\delta_{r_1' t_2'}}{\sqrt{R_1 (R_1 - 1)}} \delta_{R_{11}'' T_{12}''} (N + R_1 - 2) \sqrt{(N + R_1 - 1)(N + R_2 - 1)} \\
& - \delta_{R_2' T_1'} \frac{\delta_{r_2' t_1'}}{\sqrt{R_1 (R_1 + 1)}} \delta_{R_{12}'' T_{11}''} (N + R_1 - 1) \sqrt{(N + R_1)(N + R_2 - 2)} \\
& - \delta_{R_2' T_1'} \frac{\delta_{r_2' t_1'}}{\sqrt{R_2 (R_2 - 1)}} \delta_{R_{22}'' T_{12}''} (N + R_2 - 3) \sqrt{(N + R_1)(N + R_2 - 2)} \\
& - \delta_{R_1', T_2'} \frac{\delta_{r_1' t_2'}}{\sqrt{R_2 (R_2 + 1)}} \delta_{R_{12}'' T_{22}''} (N + R_2 - 2) \sqrt{(N + R_1 - 1)(N + R2 - 1)} \\
& + \delta_{R_1' T_2'} \delta_{r_1' t_2'} \sqrt{(N + R_1 - 1)(N + R_2 - 1)} + \delta_{r_2' t_1'} \delta_{R_2' T_1'} \sqrt{(N + R_1)(N + R_2 - 2)} \Bigg] |4, T, t\rangle \,.
\end{aligned} \tag{5.57}$$

The remaining states, $|i\rangle$ with $i = 1, 2, 3$ mix under the action of $D_4^{(2)}$. Using a matrix notation

$$D_4^{(2)} |i, R, r\rangle = \sum_{T,t} (\mathcal{D}_4^{(2)})_{ij} |j, T, t\rangle \qquad i, j = 1, 2, 3 \tag{5.58}$$

the action of $D_4^{(2)}$ in this subspace is given by

$$\mathcal{D}_4^{(2)} = A \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix} + B \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + C \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + C^\dagger \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{5.59}$$

where the coefficients A, B and C are

$$A = \delta_{RT} \left(\frac{\delta_{r_1' t_1'}}{R_2} + \frac{\delta_{r_2' t_2'}}{R_1} \right) \delta_{R_{12}'' T_{12}''} (N + R_1 - 1)(N + R_2 - 2)$$

$$
\begin{aligned}
&+ \quad \delta_{RT}\frac{\delta_{r_1't_1'}}{R_1}\delta_{R''_{11}T''_{11}}(N+R_1-1)(N+R_1-2)\\
&+ \quad \delta_{RT}\frac{\delta_{r_2't_2'}}{R_2}\delta_{R''_{22}T''_{22}}(N+R_2-2)(N+R_2-3)\\
&- \quad 2\delta_{RT}\delta_{r_1't_1'}\delta_{R'_1T'_1}(N+R_1-1)-2\delta_{RT}\delta_{r_2't_2'}\delta_{R'_2T'_2}(N+R_2-2)\\
&- \quad \delta_{R'_1T'_2}\frac{\delta_{r_1't_2'}}{\sqrt{R_1(R_1-1)}}\delta_{R''_{11}T''_{12}}(N+R_1-2)\sqrt{(N+R_1-1)(N+R_2-1)}\\
&- \quad \delta_{R'_2T'_1}\frac{\delta_{r_2't_1'}}{\sqrt{R_1(R_1+1)}}\delta_{R''_{12}T''_{11}}(N+R_1-1)\sqrt{(N+R_1)(N+R_2-2)}\\
&- \quad \delta_{R'_2T'_1}\frac{\delta_{r_2't_1'}}{\sqrt{R_2(R_2-1)}}\delta_{R''_{22}T''_{12}}(N+R_2-3)\sqrt{(N+R_1)(N+R_2-2)}\\
&- \quad \delta_{R'_1T'_2}\frac{\delta_{r_1't_2'}}{\sqrt{R_2(R_2+1)}}\delta_{R''_{12}T''_{22}}(N+R_2-2)\sqrt{(N+R_1-1)(N+R_2-1)}\\
&+ \quad 2\delta_{r_2't_1'}\delta_{R'_2T'_1}\sqrt{(N+R_1)(N+R_2-2)}+2\delta_{r_1't_2'}\delta_{R'_1T'_2}\sqrt{(N+R_1-1)(N+R_2-1)}
\end{aligned}
\tag{5.60}
$$

$$
B \quad = \quad -8\delta_{r_2't_1'}\delta_{RT}\delta_{R''_{12}T''_{12}}\sqrt{\frac{(N+R_1-1)(N+R_2-1)}{R_1R_2}} \tag{5.61}
$$

$$
\begin{aligned}
C \quad &= \quad 8\frac{\delta_{r_1't_2'}}{\sqrt{R_1(R_2+2)}}\delta_{R''_{11}T''_{22}}\sqrt{(N+R_1-1)(N+R_1-2)(N+R_2-1)(N+R_2)}\\
&- \quad 8\delta_{R'_1T'_2}\frac{\delta_{r_1't_1'}}{\sqrt{R_1(R_2+1)}}\delta_{R''_{11}T''_{12}}(N+R_1-2)\sqrt{(N+R_1-1)(N+R_2-1)}\\
&- \quad 8\delta_{R'_1T'_2}\frac{\delta_{r_2't_2'}}{\sqrt{R_1(R_2+1)}}\delta_{R''_{12}T''_{22}}(N+R_2-2)\sqrt{(N+R_1-1)(N+R_2-1)}
\end{aligned}
\tag{5.62}
$$

$$
\begin{aligned}
C^\dagger \quad &= \quad 8\frac{\delta_{r_2't_1'}}{\sqrt{R_2(R_1+2)}}\delta_{R''_{22}T''_{11}}\sqrt{(N+R_1)(N+R_1+1)(N+R_2-2)(N+R_2-3)}\\
&- \quad 8\delta_{R'_2T'_1}\frac{\delta_{r_1't_1'}}{\sqrt{R_2,(R_1+1)}}\delta_{R''_{12}T''_{11}}(N+R_1-1)\sqrt{(N+R_1)(N+R_2-2)}\\
&- \quad 8\delta_{R'_2T'_1}\frac{\delta_{r_2't_2'}}{\sqrt{R_2(R_1+1)}}\delta_{R''_{22}T''_{12}}(N+R_2-3)\sqrt{(N+R_1)(N+R_2-2)}.
\end{aligned}
\tag{5.63}
$$

The coefficients A, B and C are operators that have a nontrivial action of the R, r labels of the Gauss graph operators. It is straightforward to check that the matrix coefficients of these operators do not commute and hence they are not simulataneously diagonalizable. This implies that the action of the dilatation operator not longer factorizes into commuting actions on the Z and Y fields. It is this failure of factorization that we intended to demonstrate.

6. Conclusions

Our results have a number of interesting features that deserve comment. In the $\frac{m}{n} = 0$ limit, the action of the dilatation operator factorizes into an action on the Z fields and an action on the Y fields. The subleading correction has spoiled this factorization of the dilatation operator. This is rather natural: in the limit $\frac{m}{n} = 0$ we consider a giant graviton built with an infinite number ($n = \infty$) of Z fields, so that the backreaction of the magnons on the giant graviton can be neglected. Without backreaction, we expect the dyamics of the giant is completely decoupled from the dynamics of the magnons and this is the root of the factorized action of the dilatation operator. By adding the first correction, we are saying that n is large but not infinite. In this situation, although back reaction is small, it is not zero. The magnons will now provide a small perturbation to the dyamics of the giant, the action of the dilatation operator will no longer factorize into an action on the giant (i.e. on the Zs) times an action on the magnons (i.e. on the Ys).

The subleading correction spoiled the factorization of the dilatation operator by introducing further operator mixing. Another interesting result of our analysis, is that the subleading corrections did not induce extra mixing for the BPS operators. Indeed, after accounting for the complete $\frac{m}{n}$ correction to two loops, we found our BPS operators remain uncorrected and continue to have a vanishing anomalous dimension. Although our computation is performed in a specific example, we argued that we expect this conclusion to be general since for the BPS operators we have $n_{ab}(\sigma) = 0$ for $a \neq b$. Looking at the result (5.53), it is clear that vanishing $n_{ab}(\sigma)$ implies a vanishing action of $D_4^{(2)}$.

The form of the action of the dilatation operator implies that when the correction to the anomalous dimension is non-zero it will depend on the length of the rows of the Young diagrams labeling the operator. Since these lengths determine the angular momentum of the giants, they determine the radius to which the giants will expand. This implies that the anomalous dimensions start to depend on the geometry of the giant graviton.

The dynamics of open strings on the worldvolume of a giant graviton is expected to give rise to a Yang-Mills theory at low energy. The lightest mode of the open strings attached to the giant becomes the gauge boson of the theory. This suggests that within $\mathcal{N} = 4$ super Yang-Mills theory, we should see classes of operators whose dynamics is captured by a new emergent gauge theory. The acronym emergent is particularly apt in this case because this new Yang-Mills theory will be local on a space that is distinct from the space of the original spacetime of the $\mathcal{N} = 4$ super Yang-Mills theory. The gauge symmetry which determines the interactions of the theory is a local symmetry with the respect to this new space and the time of the original spacetime. The space of the emergent Yang-Mills theory is the worldvolume of the giant graviton, which itself is built from the Zs. So this space and an associated local gauge invariance is to emerge from the dynamics of the Z matrices in the large N limit. For the operators dual to giant gravitons studied in this article, it is natural to think that the magnons themselves will become the gauge bosons. Indeed, the allowed state space of the magnons is parametrized by a double coset. The structure of this double coset is determined by the expected Gauss Law of the emergent gauge theory. To really understand the mechanism behind this emergence it is important that we get a good handle on how the magnons interact. It is by studying these interactions that we may hope to recognize the Yang-Mills theory that must emerge. In this article we have computed the first of these interactions and we have developed tools that allow us to study these interactions in general.

In this thesis we have explored a two giant graviton boundstate with four strings attached. We have found that the only extra operator mixing coming from the subleading contribution $D_4^{(2)}$. We have

developed a powerful technique to evaluate the most general trace we could encounter, which is much more general than the traces that appear at two loops. These results will be useful in further studies of the dynamics of Gauss graph operators.

This work may be extended to three-loop level, where we expect to find further corrections to the one and two loop results. It is important to confirm that the results and conclusions reached at two loop are insensitive and still valid (that is under control at two (and higher) loops.

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