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Stirling cycle engine design and optimisation

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an approximation of reality, and in no sense can computer simulation ever replace physical experimentation, and certainly not theoretical analysis. Theoretical analysis should be used to guide simulation efforts; the large number of variables precludes the simulation as an effective design tool without theoretical guidance, particularly during the preliminary design phase. Though simulation may give the user a feel for a particular machine, it is not conducive to developing a broad understanding of the behaviour of the class of machines. Only theoretical analysis addresses this issue. Once the engine configuration and geometry are roughly defined the simulation technique is very powerful in refining the performance estimates.

Kirkley's coupled analysis [Ki63, Ki65] was the first effort to account for irreversibilities within the cycle with any analysis. Allowance was made for non-isothermal working spaces, pressure drop losses and imperfect regeneration. His model was simplified in that he assumed a single pressure drop discontinuity in the centre of the regenerator space. For the pressure loss calculation, the temperature of the cooler and its adjacent half were at a constant lower temperature and the heater and its adjacent half were at a constant higher temperature. The working space temperatures were allowed to vary due to non-isothermal processes. The actual pressure drop equation was derived empirically from steady flow tests. For the thermodynamic calculations, the cooler and heater were assumed to be at their respective constant lower and higher temperatures and the regenerator was assumed to have a linear temperature profile. The system was described in terms of differential equations and solved numerically. Reasonable correlation with an experimental engine was obtained when the friction factor was increased 1.8 times (the engine

speed was 300 RPM). It will be shown later (§4) that an increase of the friction factor in oscillatory flow has been predicted theoretically by Trikha [Tr75].

Finkelstein [Fi62, Fi64, Fi67.1, Fi67.2, Fi75] has presented five papers on the analysis of Stirling cycle machines in which allowance has been made for non-isothermal processes, pressure drop losses and imperfect heat transfer and regeneration. He has always approached the problem by treating the machine as a complete and continuous entity. Only his last paper (which is his most complete effort) will be discussed here.

Finkelstein's work is based on the standard integral forms of the energy and continuity equations. A momentum equation as such has not been used. Instead he assumes that the flow resistance is concentrated at the inlet/exit of the working spaces and that the mass flow is proportional to the product of the mean pressure and pressure gradient. Berchowitz [Be78] has investigated this assumption in detail and has shown that it is likely to cause significant error, particularly if the flow is turbulent. Finkelstein was the first to use conditional nodal temperatures (described in Appendix D) which are now used in many advanced simulations due to their stabilizing effect on the numerical solution. Computational time for this model was reported to be twelve minutes on an IBM 360/65 machine for one cycle.

The limitations of Finkelstein's model were first addressed by Urieli [Ur77, UR77] while under the supervision of Professor Rallis at the University of the Witwatersrand. Further improvements were made by

Berchowitz [Be78, BR78, BU79] while also working under Professor Rallis. This work is described in detail in Appendix D and is therefore not repeated here. To date, the basic physical model developed at the University of the Witwatersrand has not been improved, though various numerical schemes have been used to accelerate the solution. In this thesis, therefore, it will be assumed that the Witwatersrand model is representative of the state of development of simulation methods.

Other relevant research which does not fit into the broad categories defined in this section will be introduced in the later sections as appropriate.

1.6 Purpose of this Study

Since design and optimisation are the central theme of this work, emphasis has been placed on theoretical analysis or, as it has been called, second order analysis. In the coming chapters, simple cycle analysis will be used to demonstrate some basic concepts about the thermodynamic operation of Stirling engines. The Schmidt analysis will be developed in full and extended. Certain aspects of the Schmidt analysis are very useful in the initial design and layout of a machine and allow one to obtain closed-form solutions to the dynamics of free-piston engines.

Preliminary heat exchanger optimisation based on minimisation of the rate of entropy production will be developed. Special effort has been made to develop equations that describe how various losses degenerate the ideal

performance. The basic cycle used here is the ideal adiabatic cycle for which a semi-closed form solution has been developed. Three machines for which there are relatively complete data will be used to validate the analysis.

Scaling techniques, first suggested by Gedeon [GeB1], will also be discussed. This type of analysis involves the setting up of non-dimensional groups which define the set of all Stirling engines thus allowing one to configure a new engine from a known existing one. The advantage of this approach is that once one has access to a well operating and performing machine, it is then possible to design machines by simple formulae to almost any power level. Usually analysis would still be used to validate the scaled configuration. The saving of time by not having to embark upon extensive optimisation studies is usually significant.

Numerical simulation will be discussed but not applied. Owing to the large number of variables in the simulation it is not possible to present generic optimisation charts nor is it necessary. Second order analysis will be shown to present a good start for an optimum configuration which may then be improved by successive simulation efforts.

2 SIMPLE CYCLE ANALYSIS OR FIRST ORDER ANALYSIS

2.1 Background

The major assumption of simple cycle analysis is that the working gas is at every instant at the same average state throughout the engine. This assumption allows one to analyse the cycle by assuming discrete thermodynamic processes and in so doing, obtain closed-form results. The first publication of this type of analysis seems to have been by Professor Wood of the Stevens Institute of Technology in 1894 [Wo1894]. This approach appears to have been largely ignored until 1975, when it was essentially rediscovered by Professor C J Rallis [RU77, Wa80]. The version by Rallis is far more general than that of Wood, particularly in the treatment of regenerator effectiveness and the effects of adiabatic expansion and compression. Recently G T Reader and C Hooper have taken simple cycle analysis to what must surely be its logical conclusion [RH83].

The principal value of this type of analysis is its ability to indicate in a simple manner the broader characteristics of the cycle. The actual performance is, however, not determined with any reasonable accuracy owing to the limitation of the assumption concerning the average state of the gas. Simple cycle analysis may be looked upon as a first order evaluation in that it is usually exercised in the preliminary stages of investigation in order to obtain a qualitative appreciation of the processes in the cycle.

2.2 Ideal Stirling Cycle

Referring to Figure 1.2 it is assumed that from state 1 to state 2 the gas is entirely at the cold temperature and is being compressed isothermally. State 2 to state 3 represents constant volume heating through the regenerator and state 3 to 4 represents the entire mass of gas being expanded isothermally at the hot temperature. Finally the gas is cooled at constant volume from state 4 to 1 by depositing its heat in the regenerator. Figure 2.1 shows both the pV and Ts diagrams which correspond to these processes. For the sake of interest the Carnot cycle is also shown between the same volumes, pressures and temperatures (cycle 1-2'-3-4'-1). It is clear that the Stirling cycle exhibits considerably more cyclic work than the classic Carnot cycle for the same cycle limits.

Referring to Figure 2.1, the isothermal specific work is given by

$$w = R[T_3 \ln(V_4/V_3) - T_1 \ln(V_1/V_2)] \qquad (2.1)$$

defining the volume ratio

$$r = V_4/V_3 = V_1/V_2 = V_{max}/V_{min} \qquad (2.2)$$

and the temperature ratio

$$\tau = T_3/T_1 = T_h/T_k \qquad (2.3)$$

where T_h and T_k are respectively the hot and cold end absolute temperatures.

We obtain:

$$w = R\,T_1(\tau - 1)\ln r \qquad (2.4)$$

The specific work output may be represented by the indicated mean effective pressure

$$\text{imep} = w/\text{stroke volume} = R\,T_1(\tau - 1)\ln r\,/(V_1 - V_2) \qquad (2.5)$$

or in dimensionless form with respect to the pressure at state 1 as

$$\zeta = \text{imep}/p_1 = r(\tau - 1)\ln r/(r - 1) \qquad (2.6)$$

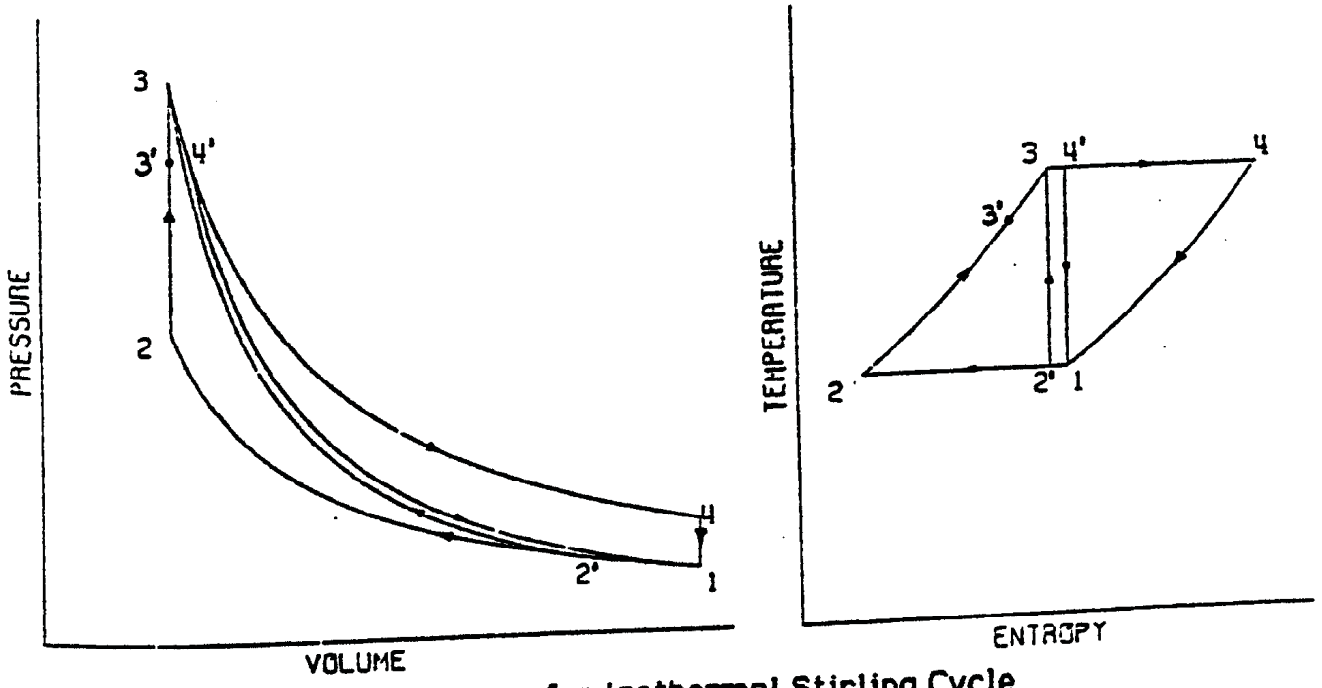


Figure 2.1 pv and Ts Diagrams for Isothermal Stirling Cycle

If regeneration is imperfect, then the constant volume heating process will not bring the working gas fully to state 3, but rather to a lower

temperature at state 3'. The extra energy required to bring the gas from state 3' to state 3 must be supplied externally. Thus, together with the heat supplied during the expansion process (ie, from state 3 to 4), the externally supplied heat is:

$$q_{ext} = c_v(T_3 - T_{3'}) + R\,T_3 \ln r \tag{2.7}$$

Regenerator effectiveness is defined as the actual heat transferred unidirectionally between the gas and regenerator matrix divided by the heat transferred unidirectionally in an ideal regenerator. Since these processes occur at constant volume, the regenerator effectiveness is given by

$$\epsilon = (T_{3'} - T_1)/(T_3 - T_1) \tag{2.8}$$

Eliminating $T_{3'}$ between (2.7) and (2.8) yields:

$$q_{ext} = [R\,T_1/(\gamma - 1)][(1 - \epsilon)(\tau - 1) + (\gamma - 1)\tau \ln r] \tag{2.9}$$

Thermal efficiency is defined as

$$\eta = w/q_{ext} \tag{2.10}$$

From (2.4) and (2.9), this becomes

$$\eta = (\gamma - 1)(\tau - 1)\ln r/[(\gamma - 1)\tau \ln r + (1 - \epsilon)(\tau - 1)] \tag{2.11}$$

$$\eta = (\gamma - 1)(\tau - 1)\ln r/[(\gamma - 1)\tau \ln r + (1 - \epsilon)(\tau - 1)] \qquad (2.11)$$

Figure 2.2 shows the thermal efficiency and dimensionless specific work plotted as functions of the compression ratio for a range of regenerator effectiveness. The temperature ratio is taken at 2.86 and the specific heat ratio at 1.4. Note that at a regenerator effectiveness of unity, the thermal efficiency reduces to the Carnot efficiency.

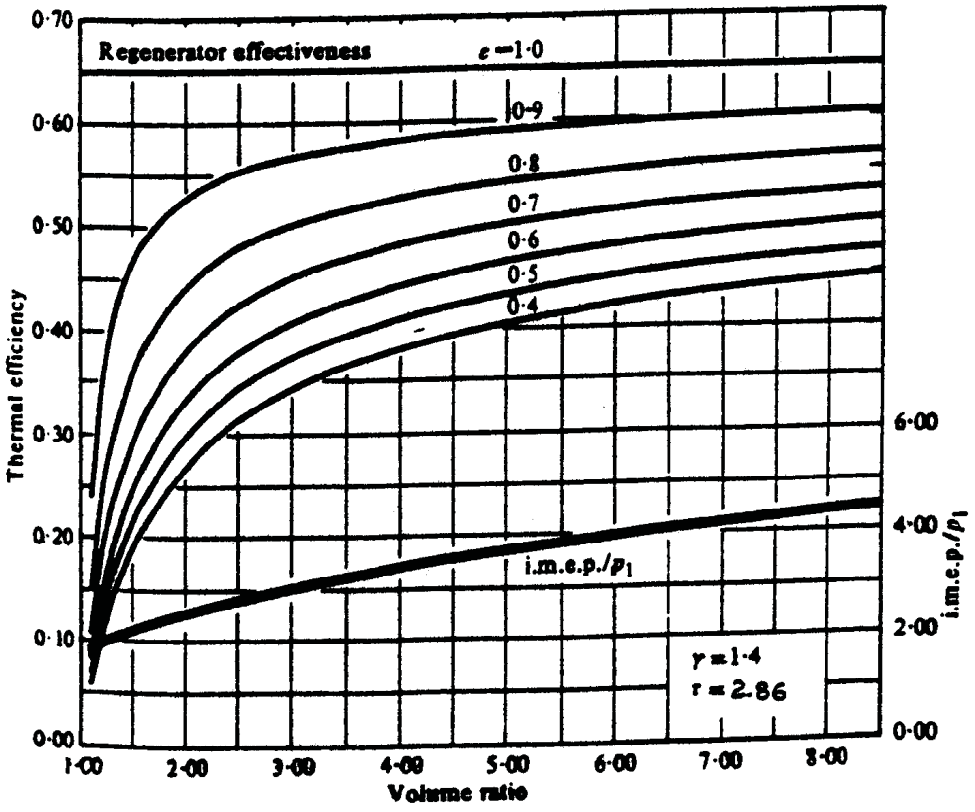


Figure 2.2 Ideal Stirling Cycle

Typically, the volume ratio for practical engines varies between about 1.2 for higher pressure helium engines to about 1.4 for lower pressure air engines. Figure 2.2 does not suggest that low volume ratios are optimum, either in work or efficiency.

develop a simple model that might be more representative of the actual cycle [RU77]. In particular, he felt that the assumption of isothermal expansion or compression was unreasonable owing to the poor heat transfer in the working spaces of practical machines. It is more reasonable to expect that the expansion and compression processes occur polytropically and in the limit approach adiabatic conditions. Professor Rallis therefore proposed an ideal cycle in which all heat transfer occurred under constant volume conditions and expansion or compression was adiabatic. This cycle is generally referred to as the ideal pseudo-Stirling cycle or adiabatic Stirling cycle.

2.3 Ideal Adiabatic Stirling Cycle

Referring to Figure 2.3, the compression process 1 to 2 is now adiabatic. After compression the working gas is transferred at constant volume firstly through the cooler where its pressure drops to 2" then through the regenerator where its pressure rises to 3' and finally through the heater where the pressure further rises to 3. After expansion a similar process occurs in the reverse order, ie, first through the heater, then regenerator, then cooler.

Following a similar procedure as before, the specific work and external heat supplied are now given by

$$w = c_v T_1[\tau[1 - (1/r^{\gamma-1})] - (r^{\gamma-1} - 1)] \qquad (2.12)$$

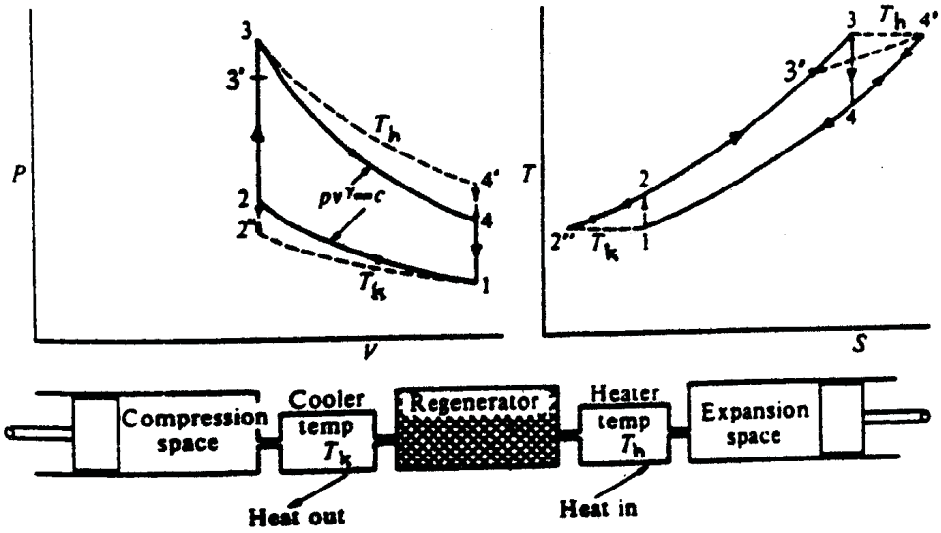


Figure 2.3 pv **and** Ts **Diagrams for Pseudo-Stirling Cycle**

$$q_{ext} = c_V T_1[(\tau - 1)(1 - \epsilon) + \tau[1 - (1/r^{\gamma - 1})]] \qquad (2.13)$$

From which

$$\zeta = r(\tau[1 - (1/r^{\gamma - 1})] - r^{\gamma - 1} + 1)/[(\gamma - 1)(r - 1)] \qquad (2.14)$$

$$\eta = (\tau[1 - (1/r^{\gamma - 1})] - r^{\gamma - 1} + 1)/[(\tau - 1)(1 - \epsilon) + \tau[1 - (1/r^{\gamma - 1})]] \qquad (2.15)$$

Equations (2.14) and (2.15) are plotted in Figure 2.4 against volume ratio and regenerator effectiveness. Here, unlike the ideal Stirling cycle model, both thermal efficiency and specific work exhibit maxima at fairly low compression ratios. The optimum value of compression ratio is a function of regenerator effectiveness. In practical engines it is not uncommon for

regenerator effectiveness to be in excess of 95%. For effectivenesses in this range the cycle analysis suggests that the optimum volume ratio will be less than two, as a general rule, this would appear to agree with practice. Thus, from a cycle analysis point of view, the pseudo-Stirling appears to more accurately describe the thermodynamic cycle of practical machines. If this is true, then it is not reasonable to say that the ideal efficiency of a practical Stirling cycle is the Carnot efficiency.

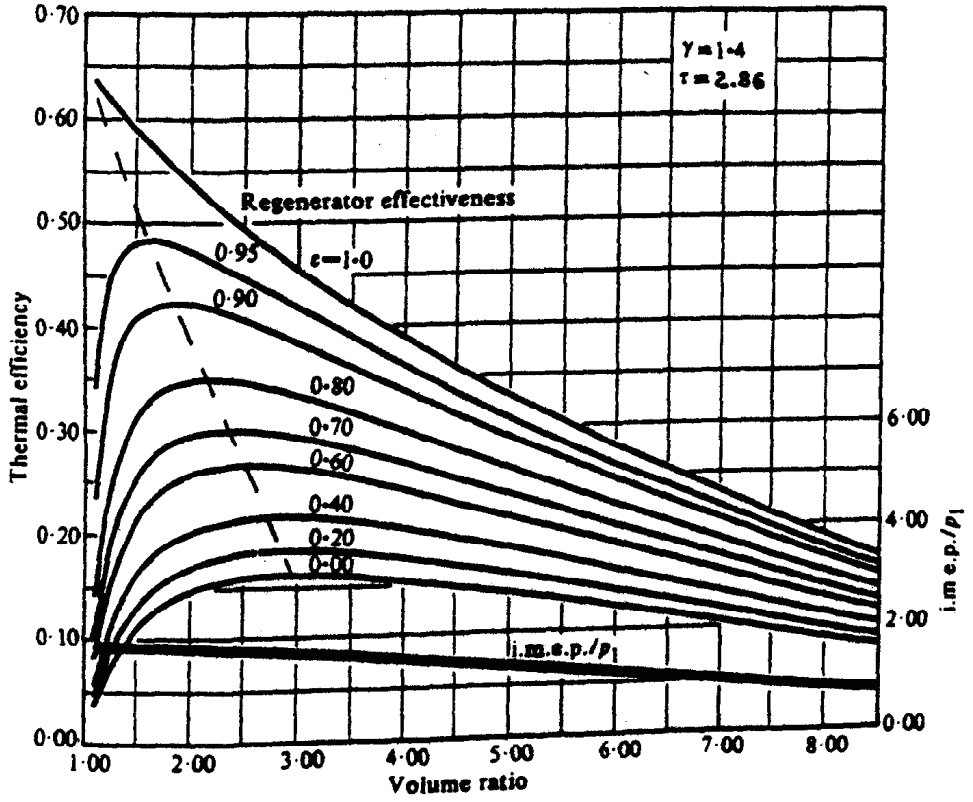


Figure 2.4 Pseudo-Stirling Cycle

A further observation from Figure 2.3 is that the cooling of the gas after compression, and reheating it after expansion, introduce irreversibilities into the cycle for no advantage in power. The reason being that these two processes do not change the enclosed area in the pv diagram yet they introduce additional temperature differentials. Finkelstein pointed this out

in his PhD thesis in 1952 and suggested the use of porting to avoid the reheating and recooling processes [Fi53] (due to an oversight, the physical embodiment of Finkelstein's ported cycle would not have operated as he had envisioned). Rallis *et al* analysed a ported constant volume regenerative cycle and showed that porting could improve the thermal efficiency, particularly for engines with regenerators of lower effectiveness [RU77, Wa80]. Rallis also included some possible engine configurations that would realise the cycle. As far as is known, the ported constant volume regenerative cycle has not been studied in any greater detail than this first preliminary analysis. Generally it is felt that the inclusion of ports, no matter how simple, is an added complication to the mechanical system for too small a pay-off. But until this cycle is looked at in more detail, this opinion remains unsubstantiated.

2.4 Application

In applying simple cycle analysis to a machine it is necessary to calculate the total mass of the working gas. If leakage past the seals is assumed to be small and is neglected, then obtaining the working gas mass is a simple application of the ideal gas law (assuming, of course, that the pressure, volumes and temperatures are known). Unfortunately, engine performance so calculated is hopelessly overestimated. As was noted before, this lack of correlation is probably attributable mainly to the assumption of uniform state. If the working medium is considered to be continuous, then it is not too implausible to define an effective mass of working gas which when subjected to the simple cycle will produce the observed power. The problem would then be to derive this effective working gas mass. Since the Schmidt

analysis (see §3) does account for the distribution of mass and temperature in an otherwise ideal engine, one possibility is to equate the work given by the simple Stirling cycle to the Schmidt cycle work, ie Equations (2.4) and (3.36). This gives

$$Rm_e T_k(\tau - 1)\ln r = W_{Schmidt} \tag{2.16}$$

or

$$m_e = W_{Schmidt} / [R T_k(\tau - 1)\ln r] \tag{2.17}$$

This procedure calibrates the simple cycle at the condition of perfect regeneration. Imperfect regeneration will have some effect on the mass of gas in the regenerator. If, however, the regenerator is fairly efficient, then this effect should be small and negligible. As an approximation, m_e as given by (2.17) should give improved results for the ideal adiabatic cycle too.

The effective mass method allows the simple cycle analysis to account for another broad characteristic of Stirling engines, namely the effect of dead or unswept volumes. As the dead volume increases with respect to total volume, m_e decreases thus reducing the cyclic work.

For quantitative prediction, the simple cycle is still limited by the determination of m_e and the assumption of discrete processes. It can

therefore be concluded that while simple cycle analyses indicate general features of the thermodynamics of Stirling engines, their immediate use in design and sizing is too restrictive.

3 ISOTHERMAL ANALYSIS

3.1 Background

Much like the Otto and Diesel cycles have become the classic cycles to describe the spark and compression ignition engines, the cycle described by Schmidt in 1871 has become the classic Stirling cycle. This has come about more because the Schmidt analysis is mathematically tractable rather than through its ability to predict the real cycle.

The major assumptions of Schmidt's analysis is that the gas in the working spaces and attendant heat exchangers is at the respective constant upper and lower cycle temperatures, and that regeneration is perfect. This implies that the gas is isothermal everywhere but with a spatial temperature distribution as indicated in Figure 3.1. In order to obtain closed-form solutions, Schmidt also assumed that the volumes of the working spaces vary sinusoidally.

Apart from the fact that the Schmidt analysis produces convenient to manipulate closed-form solutions, it is generally fairly accurate in its prediction of cycle power (if based on mean working gas temperatures). This is of great advantage when preliminary engine sizing and geometric optimisation is required. On the other hand, the analysis is well near useless when it becomes necessary to predict efficiency and external heat transfer.

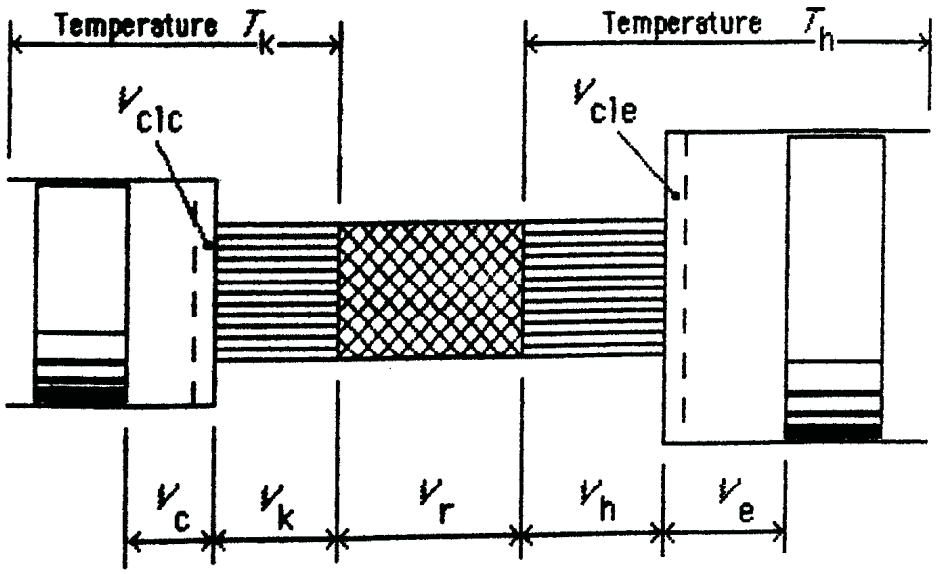


Figure 3.1 Temperature Distribution for Isothermal Analysis

Schmidt's isothermal assumption makes it possible to generate a simple result for the working gas pressure variation for arbitrary volume variations. This result may then be used to investigate how different drive systems affect the output power [UB84]. Unfortunately, it becomes considerably more difficult to derive closed-form solutions to performance for other than sinusoidal volume variations. Also, the effort in obtaining these closed-form solutions is not really warranted since the general trends assuming sinusoidal motions are valid for all other coventional drives. For example, the optimum volume phase angle and volume ratio, or the dependence of power on charge pressure, and so on, are all directly evident from the sinusoidal case, and, within the limitation of the isothermal approximation, well near generic for all drive systems.

3.2 The Isothermal Model

The assumption of isothermal working spaces and heat exchangers implies that the heat exchangers (including the regenerator) are perfectly effective. In addition to this the following assumptions are also made:

1) There is no spatial pressure drop.

2) The mass of working fluid is constant, therefore there is no leakage.

3) The gas is ideal.

4) The speed of the machine is constant.

5) Cyclic steady state is established.

6) Gas kinetic and potential energy is neglected.

The notation used throughout this work is that suggested by Urieli [UB84] and is as follows. The engine model is configured as a five component serially connected model, the five parts being compression space c, cooler k, regenerator r, heater h and expansion space e. It is tacitly assumed that all engines considered in this work can be reduced to this five component model. Therefore, any interconnecting ducts, say between the compression space and cooler, will be included in the compression space clearance volume. Each component is considered as a homogeneous entity, or cell, the gas therein being represented by its instantaneous mass m, absolute temperature T, volume V and pressure p. Each of these properties are in turn identified with a particular cell by means of a cell subscript, thus for example m_k is the mass of gas in the cooler cell and V_e is the volume of the expansion space. In this ideal case it is assumed that there is no pressure drop, thus p is not subscripted and represents the instantaneous pressure throughout the system.

3.3 Analysis

The total mass of working gas in the machine is constant, hence:

$$m_t = m_c + m_k + m_r + m_h + m_e \tag{3.1}$$

Using the ideal gas law:

$$m = pV/(RT) \tag{3.2}$$

Equation (3.1) may be written:

$$m_t = p(V_c/T_k + V_k/T_k + V_r/T_r + V_h/T_h + V_e/T_h)/R \tag{3.3}$$

where T_r is some effective regenerator temperature which remains to be determined.

Consider Figure 3.2. The temperature distribution in the regenerator is assumed to be linear, thus:

$$T_r(x) = (T_h - T_k)x/L_r + T_k \tag{3.4}$$

$$\therefore m_r = A_r L_r/R \int_0^{L_r} p[(T_h - T_k)x + T_k L_r]^{-1} dx \tag{3.5}$$

$$= pV_r \ln(T_h/T_k)/[R(T_h - T_k)] \tag{3.6}$$

By comparing this result to Equation (3.2), it can be seen that T_r is given by the log mean temperature.

$$T_r = \ln(T_h/T_k)/(T_h - T_k) \tag{3.7}$$

Substituting Equation (3.7) into Equation (3.3) and solving for p:

$$p = m_t R[V_c/T_k + V_k/T_k + V_r \ln(T_h/T_k)/(T_h - T_k) + V_h/T_h + V_e/T_h]^{-1} \tag{3.8}$$

Equation (3.8) gives the working gas pressure variation for arbitrary volume variations in V_c and V_e.

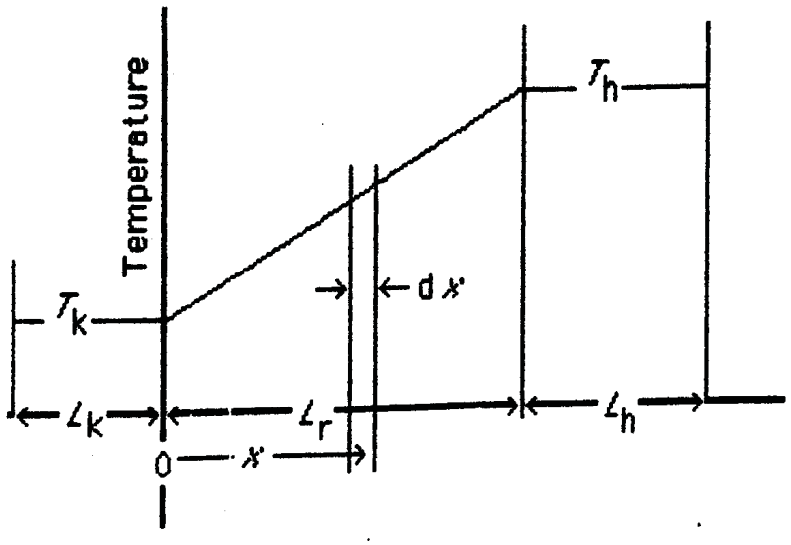


Figure 3.2 Ideal Regenerator Temperature Profile

Work is done on the surroundings by virtue of the varying volumes of the working spaces V_c and V_e. The total work done by the engine is therefore

the sum of the work done by the compression and expansion spaces:

$$W = \oint p\,\mathrm{d}V_c + \oint p\,\mathrm{d}V_e = \oint p(\mathrm{d}V_c/\mathrm{d}\theta + \mathrm{d}V_e/\mathrm{d}\theta)\mathrm{d}\theta \qquad (3.9)$$

where θ is the crank angle.

Heat transfer from the environment to the working gas occurs at the cold and hot temperatures T_k and T_h, respectively. Typically, Stirling engines will have a separate heater and cooler as indicated in Figure 3.1. To investigate the heat transfer to these spaces it is necessary to consider the working gas energy equation.

A generalised cell is shown in Figure 3.3 which may either be reduced to a working space cell or heat exchanger cell. The variables w_i and T_i refer to gas flowing into the cell and w_o and T_o refer to gas flowing out of the cell. The derivative operator is denoted by D, thus for example, Dm refers to the mass derivative $\mathrm{d}m/\mathrm{d}t$.

The word statement of the energy equation for the working gas in the generalised cell is as follows:

$$\left\{\begin{matrix}\text{rate of heat}\\ \text{transfer into}\\ \text{the cell}\end{matrix}\right\} + \left\{\begin{matrix}\text{net enthalpy}\\ \text{convected into}\\ \text{the cell}\end{matrix}\right\} = \left\{\begin{matrix}\text{rate of work}\\ \text{done on the}\\ \text{surroundings}\end{matrix}\right\} + \left\{\begin{matrix}\text{rate of increase}\\ \text{of internal energy}\\ \text{in the cell}\end{matrix}\right\}$$

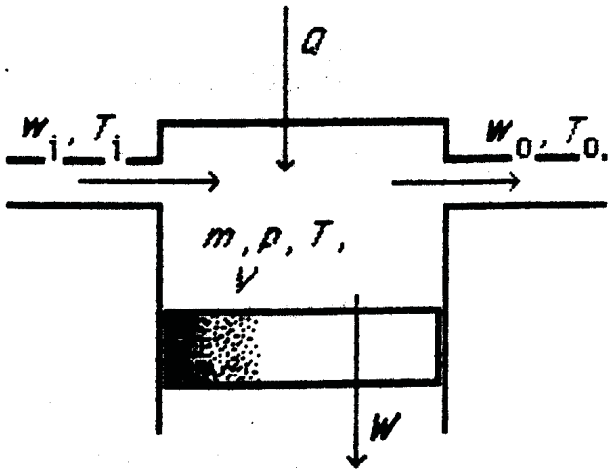


Figure 3.3 Generalised Cell

Mathematically the word statement becomes:

$$D Q + (c_p T_i w_i - c_p T_o w_o) = D W + c_v D(m T) \qquad (3.10)$$

where c_p and c_v are respectively the specific heat capacity of the gas at constant pressure and constant volume.

Equation (3.10) is the well known classical form of the energy equation for non-steady flow. It is discussed in greater detail in Appendix D, and in the form given, gas momentum effects have been ignored.

Applying Equation (3.10) to a constant volume isothermal heat exchanger, it is clear that $D W = 0$ and $T_i = T_o$. Equation (3.10) thus reduces to:

$$D Q = c_p T (w_o - w_i) + c_v T D m \qquad (3.11)$$

From mass conservation, $(w_o - w_i)$ is simply the net rate of mass accumulation within the cell, Equation (3.11) thus simplifies to:

$$D Q = T(c_v - c_p) D m \tag{3.12}$$

Since the cyclic change of mass in the volume is zero, we have:

$$Q = T(c_v - c_p) \oint d m = 0 \tag{3.13}$$

This implies that there is no net heat transfer to the isothermal heat exchangers. This rather startling result is resolved when applying the energy equation to the working spaces. By similar arguments it is simple to show that

$$Q = \oint p \, d V = W \tag{3.14}$$

Thus all the external heat transfer for the isothermal cycle occurs across the boundaries of the working spaces.

To investigate the instantaneous heat transfer rate, the convected enthalpy and internal energy terms need to be included as follows:

$$D Q = D W + c_v D(mT) - c_p T_i w_i \tag{3.15}$$

Since the rate of change of cell mass must equal the mass flow into the

cell:

$$\mathrm{D}m = w_i \tag{3.16}$$

Noting that temperature is constant and that $\mathrm{D}W = p\mathrm{D}V$, (3.16) is substituted into (3.15) to give

$$\mathrm{D}Q = p\mathrm{D}V + (c_v - c_p)T\mathrm{D}m \tag{3.17}$$

From the ideal gas law, the rate of change of mass is given by

$$\mathrm{D}m = (p\mathrm{D}V + V\mathrm{D}p)/(RT) \tag{3.18}$$

and since the working medium is a gas, the following result also holds:

$$c_p - c_v = R \tag{3.19}$$

Substituting (3.19) and (3.18) into (3.17) to obtain the final form for the heat exchanger instantaneous heat transfer:

$$\mathrm{D}Q = -V\mathrm{D}p$$

which for each working space becomes

$$\mathrm{D}Q_c = -V_c\mathrm{D}p \tag{3.20}$$

$$\mathrm{D}Q_e = -V_e \mathrm{D}p \tag{3.21}$$

By applying Equation (3.10) to the regenerator, the following result is obtained in a similar manner:

$$\mathrm{D}Q_r = V_r c_v \mathrm{D}p/R - c_p(T_k w_{kr} - T_h w_{rh}) \tag{3.22}$$

where w_{kr} and w_{rh} are respectively the mass flow across the cooler/regenerator and regenerator/heater interfaces. By applying mass balances over the heat exchanger and associated working space, we have:

$$w_{kr} = -\mathrm{D}m_c - \mathrm{D}m_k$$

$$w_{rh} = \mathrm{D}m_e + \mathrm{D}m_h$$

which from (3.18) becomes:

$$w_{kr} = -[\mathrm{D}p(V_k + V_c) + p\,\mathrm{D}V_c]/(RT_k) \tag{3.23}$$

$$w_{rh} = [\mathrm{D}p(V_h + V_e) + p\,\mathrm{D}V_e]/(RT_h) \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.22):

$$D\dot{Q}_r = V_r c_v Dp/R + c_p[Dp(V_k + V_c + V_h + V_e) + p(D V_c + D V_e)]/R \tag{3.25}$$

which is the final form of the regenerator instantaneous heat transfer.

The ideal instantaneous heat transfer as calculated by (3.20) and (3.21) has previously been compared to the instantaneous heat transfer that might be expected to occur in real machines [Be78]. Figure 3.4 compares third order simulation results for the instantaneous heat transfer to the isothermal analysis. For the machine used in this comparison, the working spaces were essentially adiabatic and were simulated as such. The simulation heat transfer rates apply therefore to the separate cooler and heater heat exchangers while the isothermal heat transfer rates are evaluated at the working spaces. From Figure 3.4 it can be seen that the isothermal heat rates bear little resemblance in form to the heat rates calculated by simulation. This leads to the cyclic heat transfer requirements being poorly estimated by the isothermal analysis (as might be expected).

Following a similar comparison for the instantaneous heat transferred in the regenerator (Figure 3.5), it can be seen that the isothermal model and simulation are reasonably close in magnitude and form. This agreement is of consequence since it allows for a simple calculation for the heat transfer in a practical regenerator. This feature of the isothermal analysis will be extended later.

It is of interest to note here, that the heat transfer in a Stirling engine regenerator is due to two distinct mechanisms: that due to enthalpy trans-

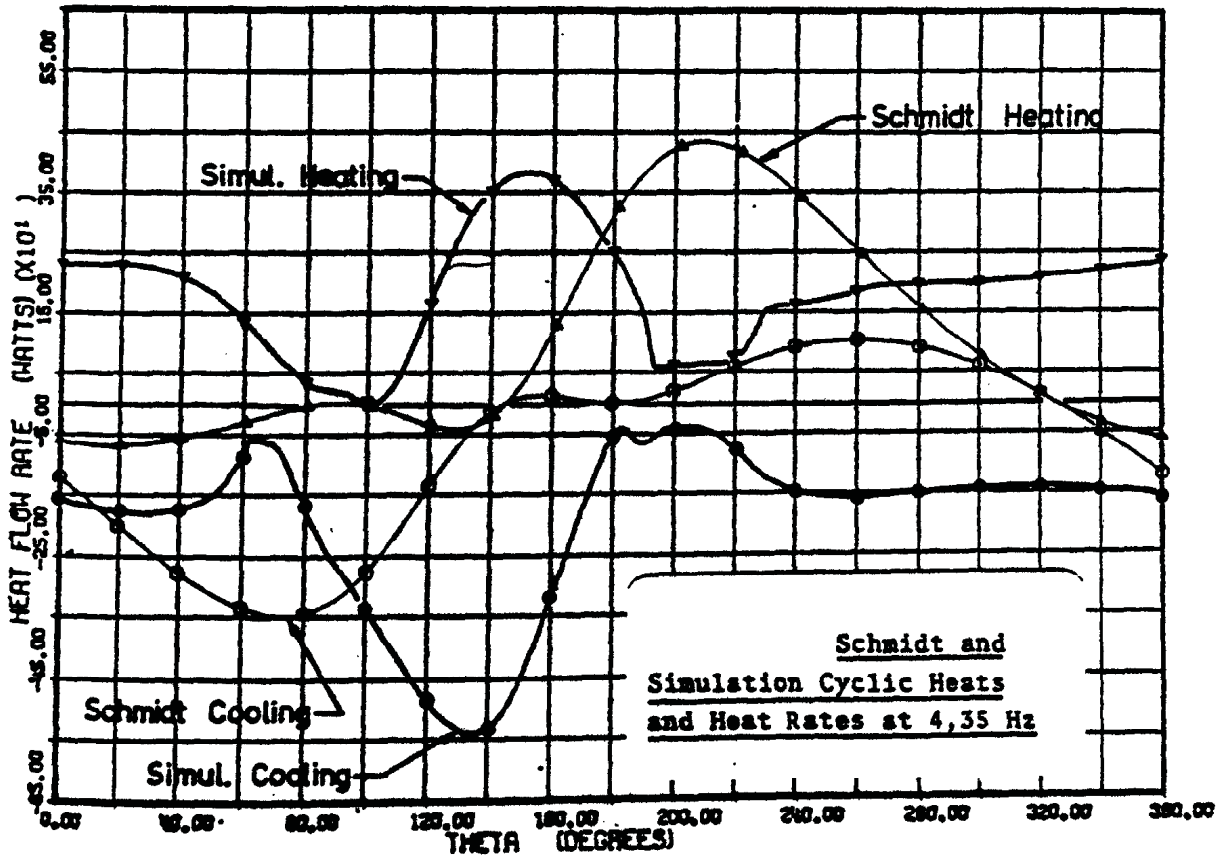


Figure 3.4 Isothermal and Simulation Cyclic Heat Rate [Be78]

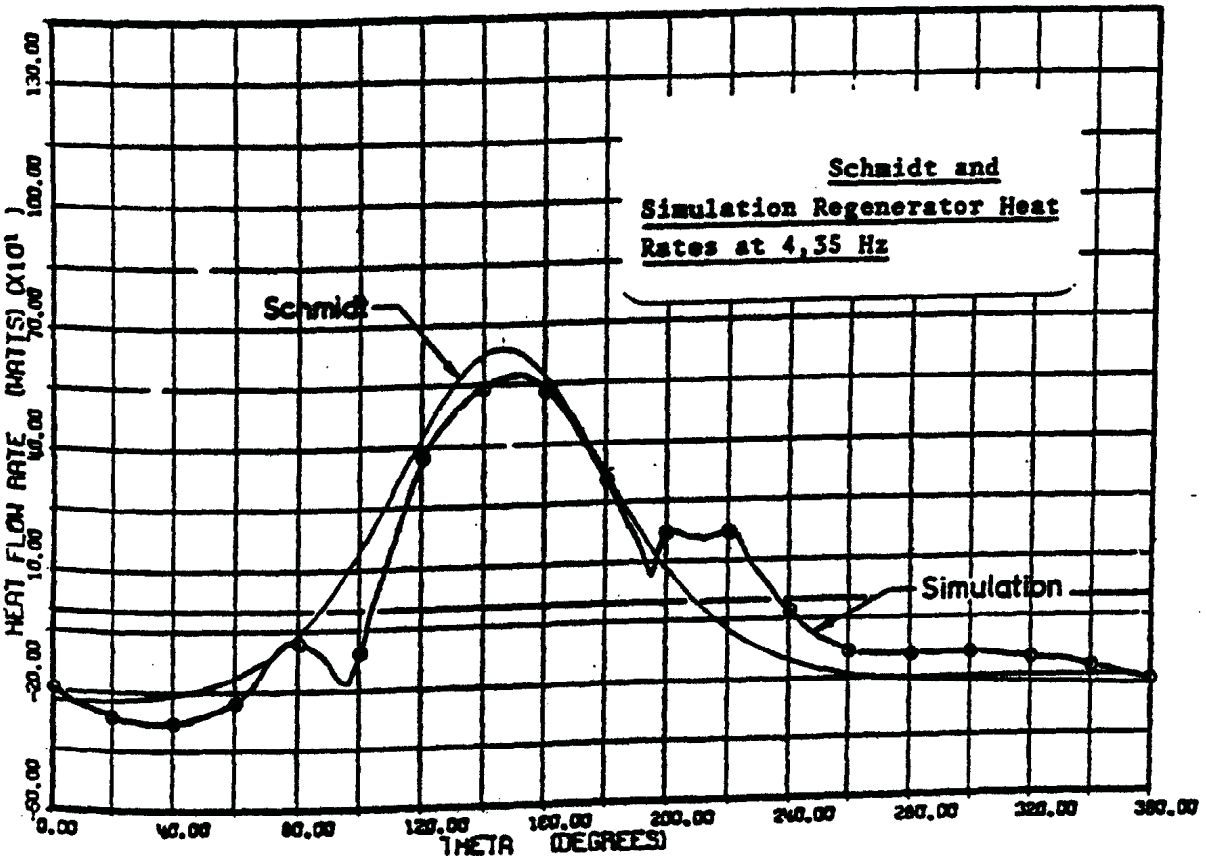


Figure 3.5 Isothermal and Simulation Regenerator Heat Rates [Be78]

port and that due to pressure variations of the working fluid. These two mechanisms may be identified in Equation (3.25) as the second and first terms on the right hand side respectively. Conventional recuperator theory only addresses the enthalpy part of the energy transfer and is therefore not suitable for application to Stirling engine regenerators. Both the enthalpy and pressure terms are of similar order (Creswick noted this in 1965 [Cr65]).

The final set of pertinent equations is defined in Table 3.1. Notice that sinusoidal volume variations for V_c and V_e have been chosen. This is only by way of example; any volume variations could have been defined. V_{clc} and V_{swc} are the clearance and swept volumes of the compression space, V_{cle} and V_{swe} of the expansion space and α is the phase angle advance of the expansion to compression volume variations. The independent variable is time t. The crankangle θ is given by

$$\theta = \omega t \tag{3.26}$$

The solution of these equations may be carried out analytically or by numerical integration. Urieli and Berchowitz have presented numerical solutions for various drive systems [UB84], however, as pointed out previously, it is sufficiently instructive to consider only the case of sinusoidal volume variations. For this case there is the fairly simple Schmidt closed-form solutions [Sc1871].

Table 3.1 Ideal Isothermal Model Set of Equations

$$\theta = \omega t$$

$$V_c = V_{clc} + 0.5\, V_{swc}(1 + \cos\theta)$$

$$V_e = V_{cle} + 0.5\, V_{swe}[1 + \cos(\theta + \alpha)]$$

Volumes

$$D\,V_c = -0.5\omega\, V_{swc}\sin\theta$$

$$D\,V_e = -0.5\omega\, V_{swe}\sin(\theta + \alpha)$$

$$p = m_t R/[\,V_c/T_k + V_k/T_k + V_r \ln(T_h/T_k)/(T_h - T_k) + V_h/T_h + V_e/T_h]$$

Pressure

$$Dp = -m_t R(D\,V_c/T_k + D\,V_e/T_h)/[(V_c + V_k)/T_k + V_r \ln(T_h/T_k)/(T_h - T_k) + (V_h + V_e)/T_h]^2$$

$$D\,W = p(D\,V_c + D\,V_e)$$

Work

$$DQ_c = -V_c Dp$$

Energy

$$DQ_r = V_r c_v Dp/R + c_p[Dp(V_k + V_c + V_h + V_e) + p(D\,V_c + D\,V_e)]/R$$

$$DQ_e = -V_e Dp$$

3.4 Schmidt Closed-Form Solutions

Referring to Table 3.1, substitution of the sinusoidal volume variations (V_c and V_e) into the result for pressure yields the following after simplification:

$$p = \frac{m_t R}{0.5(V_{swe}\cos\alpha/T_h + V_{swc}/T_k)\cos\theta - 0.5\,V_{swe}\sin\alpha\sin\theta/T_h + S} \tag{3.27}$$

where

$$S = 0.5(V_{swc}/T_k + V_{swe}/T_h) + (V_k + V_{clc})/T_k + V_r\ln(T_h/T_k)/(T_h - T_k) + (V_h + V_{cle})/T_h$$

Using the trigonometric substitution shown in Figure 3.6, Equation (3.27) may be written as follows:

$$p = m_t R/[a\cos(\beta + \theta) + S] \tag{3.28}$$

where

$$a = 0.5[(V_{swe}\cos\alpha/T_h + V_{swc}/T_k)^2 + (V_{swe}\sin\alpha/T_h)^2]^{1/2}$$

and

$$\beta = \tan^{-1}[(V_{swe}\sin\alpha/T_h)/(V_{swe}\cos\alpha/T_h + V_{swc}/T_k)]$$