Investigating addition and subtraction strategies of Grade 3 learners within the context of a Maths Club

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Declaration

I declare that this research report is my own work. It is submitted for the Degree of Master of Education in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Signature

Date: 18 June 2013

Abstract

This study was concerned with any shifts that could be seen in the use of addition and subtraction strategies by Grade (Gr) 3 learners, and the link between these strategies and resources made available to the learners within the context of the Maths Club (MC). The critical questions that I sought to answer were, a) What addition and subtraction strategies do learners display at the start of maths club?; b) What addition and subtraction strategies and resources does maths club suggest to learners?, c) What shifts in addition and subtraction strategies can be seen during the course of the maths club?

My conceptual framework which I later used as an analytical framework was drawn from various literature reviewed for the study with the work of Carpenter, Fennema, Franke, and Levi (1999) being the foreground. The framework consists of modeling strategies, counting strategies and number facts. Progression within and across strategies forms a large part of the exploration of the study.

The sample used for the study was six Gr 3 learners together with 2 Student Teachers (ST) conducting lessons within the MC. Data collected during the sequence of the 7 videotaped maths club sessions involving the 6-learner group held across the year formed the dataset for my analysis. Various data sources were used for this study, with videotaping forming the key data source. Field notes with feedback from the student teachers were also used. Lesson plans and workbooks used within the MC sessions were collected and analysed.

Findings show that learners relied on concrete strategies with 'unit counting' and 'counting from' dominating. Learners seemed to be using resources based on their availability. Where resources were used there did not seem to be any promotion of their use from concrete strategies to more sophisticated abstract strategies, partly because of the way ST introduced the resources to the learners. Though there was evidence of the use of abstract strategies, there was no evidence of a linear progression from concrete to abstract strategies as the learners often reverted to concrete strategies.

The implications of this study are that learners at grade 3 level need to be helped to progress from concrete use of strategies to more abstract use. One of the ways to help

learners with this progression appears to be the incorporation of appropriate resources into the execution of strategies which implies that teachers need to be much more careful in their planning of what resources are available, when and in what sequence. In addition to the incorporation of appropriate resources emphasis on structure such as 10 structure on a bead string or beans for example would need to be a focus so that learners may be able to shift from *modeling of* a situation to *modeling for* representation of mathematical reasoning.

Dedication

To my beloved mother, Nocawa Zikhona Takane, my late father Bonisile Samson Hantibi, my uncle Vusumzi Takane, the entire Takane family and the children of Africa.

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List of Abbreviations

SA	South Africa
Gr	Grade
MC	Maths Club
FP	Foundation Phase
ANA	Annual National Assessments
ST	Student Teacher
B Ed	Bachelor of Education
WMC-P	Wits Maths Connect Primary Project
PPW	Part-Part Whole
CAPS	Curriculum Assessment Policy Statement
MCM	Maths Club Money

Glossary

ADDEND	An item being added to another item
COMMUTATIVITY	Swapping the numbers around and still get the same answer when adding
SUBITIZING	Visually immediate apprehension and labelling of small numbers

Chapter 1 - Introduction

1.1 Introduction

In the context of South Africa (SA) as in many other countries, Mathematics Education has received a lot of attention and has drawn a lot of interest from researchers. In recent years, this attention has also spread to primary education which has been described as somewhat neglected in SA especially when it comes to research (Venkat, Adler, Rollnick, Setati, & Vhurumuku, 2009). Most research is also focused on investigating teachers and teaching strategies. It has been noted that there is very little research on learners and learning strategies in the South African context (Ensor et al, 2009). Ensor et al. (2009) further notes that 'it is deeply problematic that a significant corpus exists in Britain, Europe and the USA on number acquisition from birth to well into the primary years, and in the absence of locally-based research we have been obliged to draw on this corpus to understand what is going on, and going wrong, in the early years of education in SA' (p. 9). It is on the basis of this context that I saw a need to research South African learners' understandings of early number ideas. This study is about the addition and subtraction strategies acquired and used by Grade 3 (Gr) learners in the context of a Maths Club (MC). In this chapter I begin by discussing the problem statement, rationale for my study and research questions.

1.2 Problem Statement and Rationale

Empirical evidence reveals that there are a number of challenges facing learners and teachers in our education system in South African classrooms. Evidence shows that there has been poor performance in our schools. As much as this problem is as a result of many different factors (Fleisch, 2008), it can be deduced that teachers are not able to give individual attention to learners. One implication in primary school is that Foundation Phase (FP) teachers are not able to judge very closely what mathematical strategies related to number concepts learners apply, whether the learners are acquiring strategies introduced to them and whether there are shifts in their learning over time.

In the statement of the release of the Annual National Assessments (ANA) Results for 2011 the Minister of Basic Education, Mrs Angie Motshekga reveals the results administered from the tests written by Gr 3 and Gr 6 learners. The results reveal that in Gr 3, the national average performance of learners in numeracy is 28%. The average performance of the learners in Gauteng primary schools is 30% (DBE, 2011). From these results it is clearly evident that we have a crisis in the area of mathematics education especially in primary schools.

Research findings provide more evidence of a 'crisis' in primary education. For example Fleisch (2008) notes that at the end of 2001 Gr 3 learners taking the tests in elementary mathematics achieved an average score of 30 per cent on a range of numeracy tasks. By 2007 this had increased to 35 per cent. Moreover Gr 3 mathematics results in 2006 showed that more than 60% of Gr 3 learners were performing below the expected level for literacy and numeracy. Within this learner performance Ensor et al. (2009) found that students remained highly dependent on concrete strategies for solving number problems at Gr 3 level. These authors describe concrete strategies as those involving tally counting in ones. This dependence on use of concrete strategies is also evident in baseline data collected at the school where this study is located. Therefore, children's strategies for working on addition and subtraction problems form the central focus of the proposed study.

In the literature (Anghileri, 2006; Carpenter & Moser, 1984; Denvir & Brown, 1986b; Wright, Martand, & Stafford, 2006) there are several investigations of addition and subtraction strategies used by FP children. I have chosen to focus on addition and subtraction because there is substantial evidence that South African FP learners are making insufficient progress towards more sophisticated and efficient strategies in this area. I investigate addition and subtraction strategies in a MC pilot study being trialled in one Johannesburg primary school. This MC involved pre-service Bachelor of Education

(B Ed) teachers taking a second year methodology course that ran alongside an elective primary maths content course called Concepts and Literacy in Mathematics (CLM). This MC has made addition and subtraction their focus since it was evident from the beginning that it was an area where the learners had foundational problems.

Carpenter and Moser (1984) in their study of addition and subtraction concepts in FP have found that children are capable of modeling a problem presented to them. These 'direct models', which are applicable to some addition and subtraction problems, involve using physical objects or fingers to directly model the action or relationship described in a problem, and Carpenter & Moser label this as a useful initial strategy at the entry level. According to Carpenter et al. (1999) progression involves shifts from concrete counting strategies (usually based on counting all the objects in direct modeling) to more sophisticated counting strategies. According to Ensor et al. (2009) counting strategies lay the foundation for learning addition and subtraction. 'With experience, and the acquisition of a growing repertoire of number facts, learners develop the competence to compute without a reliance on counting strategies' (Ensor et al., 2009, p.12). The South African evidence points to too few learners successfully making these shifts.

This problem in the context of number learning in the FP has led to my study which has sought to explore children's strategies for answering addition and subtraction problems over the course of the year in the context of a MC. Within the study I have sought to explore shifts in the strategies that the children could use related to strategies that were suggested to them in the MC. Details of the MC in which my study is located are presented below.

1.3 The Study Context

The problem discussed above led to the design of a pilot MC project that linked primary mathematics B Ed teacher education at the university I am studying in with work happening in the context of the Wits Maths Connect Primary Project (WMC-P). The broader focus of the WMC-P is professional development of teachers with the aim of improving teaching and learning of mathematics. The school in which the MC was conducted is one of the 10 schools in which the WMC-P is being conducted and this school was chosen because it is within the proximity of the university which makes it easily accessible. Additionally, the Head of FP in the school was keen and agreeable for her school to be involved in the MC project. The aim of the MC was to pilot a school-based elective *methodology course* [1]for second year B Ed student teachers (n=22) through gaining insights from working to support the number learning of a group of 45 Gr 3 learners in the school. The MC model involved a pair of B Ed students working with small groups of learners over 45 minute after school sessions across the university academic year. For most sessions the course co-ordinator designed tasks to be used in the MC sessions and discussed these with the B Ed student teachers who then worked on them with the learners. On some occasions the B Ed student teachers designed their own tasks for use with the learners in the MC. The MC co-ordinator decided that the groups would remain the same throughout the course of the MC in order to allow student teachers to build relationships with the learners and keep track of learning. The only time the group were split was when there was a game activity that required all learners to move around and associate with other groups. My study focused on addition and subtraction strategies used by one group of 6 learners over the course of the year working with 2 student teachers. These 45 learners in the MC were chosen through negotiations with the Principal and Head of FP since the MC could not include the whole cohort in Gr 3 due to lack of resources and capacity. The school decided to open the MC to learners in the top half of attainment in Numeracy, based on Gr 2 Numeracy performance. My study sought to explore the addition and subtraction strategies used over the course of the year by the selected group of 6 learners. The aim of the study was to investigate any changes that could be seen in the Gr 3 learners' addition and subtraction strategies across the MC year, and the main research question related to the changes that could be seen in the grade 3 learners' addition and subtraction strategies over time.

[1] A course that offered opportunities for the student teachers to apply what they were learning about teaching in the context of a school, rather than in a university-based course

1.4 Critical Questions

The critical questions that I sought to answer were:

1. What addition and subtraction strategies do learners display at the start of Maths Club?

2. What addition and subtraction strategies and resources does Maths Club suggest to learners?

3. What shifts (if any) in addition and subtraction strategies can be seen during the course of the Maths Club?

1.5 Outline of the thesis by Chapter

In this chapter (Chapter 1) I have discussed the problem statement, rationale, context of the study, and critical questions, now I outline the thesis by chapter.

In Chapter 2 I discuss the literature reviewed on addition and subtraction with the focus on strategies. I discuss addition and subtraction strategies as presented in literature. I also discuss the conceptual framework which is formulated from the concepts emerging from literature.

In Chapter 3 I discuss the research methodology. I provide details of how the data was collected, organised and analysed. In Chapter 4 I discuss the analysis of the data and findings.

In Chapter 5 I discuss the conclusions based on the findings and provide recommendations for researcher, policy makers and practice.

1.6 Summary

In this chapter I have detailed the introduction to this study, the problem statement and rationale for embarking on this study, and the study context. Lastly I have outlined the chapters and what will be discussed in each chapter.

Chapter 2 - Literature Review

2.1 Addition and Subtraction

'Addition is an operation that makes a sum out of two known addends and subtraction is an operation that makes an addend out of a known sum and a known addend' (Fuson, 1992, p.244). Addition and subtraction are said to be intricately related (Bowie, 2012). For example a learner could be presented with a problem such as: *Sue had 6 apples. Bob gave her some more. Now Sue has 9 apples. How many did Bob give her?* The number sentence of this problem could be written as either $6 + ? = 9$ or $9 - 6 = ?$ Anghileri (2006) argues that operations of addition and subtraction can be presented through practices relating to combining and partitioning collections of objects. She elaborates that concepts related to these operations begin to be formalized when the results can be expressed using particular (mathematical) language, and later symbolic format. Children need to have an understanding that addition and subtraction symbols can be associated with meaningful words (Anghileri, 2006). For example Anghileri (2006) provides words such as 'add', 'plus', 'together make', 'take away', 'minus', 'subtract' and 'difference between' and notes that they give some indication of the diversity of language that needs to be mastered in children's developing understanding of context and situations that will later be symbolised. Carpenter et al. (1999) found that 'there are important distinctions between different types of addition and subtraction problems, which are reflected in the way that children think about and solve them' (p. 2). They identify four basic classes of problems: *Join*, *Separate, Part-Part-Whole (PPW)*, and *Compare* in which various problem types can be varied depending on the Unknown items in the problem presented. Askew (2012) refers to these classes of problems as 'roots' of addition and subtraction. For both Join and Separate category problems, there are distinct types: *Result Unknown*, *Change Unknown*, and *Start Unknown* which can be generated by varying the *unknown* (Carpenter et al., 1999). For *PPW* the distinct problem types are *Whole Unknown* and *Part Unknown*, and for *Compare* problems they are *Difference Unknown*, *Compared Set Unknown* and *Referent Unknown* (Carpenter et al., 1999). Carpenter et al. (1999) point out the problem types themselves vary in difficulty, but also note that progression can be associated with the different counting strategies used to solve specific problems. In the next section, I detail the increasing sophistication of different counting strategies in the context of addition and subtraction.

2.2 Addition and Subtraction Strategies

From a range of literature a coherent identification of addition and subtraction strategies emerges (Anghileri, 2006, 2007; Carpenter et al., 1999; Carpenter & Moser, 1984; Denvir & Brown, 1986a; Steffe, von Glasersfeld, Richards, & Cobb, 1983; Wright, Martand, Stafford, & Stanger, 2002). Carpenter et al. (1999) group these strategies according to the following categories: Direct Modeling strategies, Counting strategies and Number Facts strategies. They further describe the strategies and their relation to problems. Direct Modeling strategies are referred to as *Joining All*, *Joining To*, *Separating From*, *Separating To*, *Matching* and *Trial and Error*. Secondly, they describe Counting strategies as *Counting On From First*, *Counting On From Larger*, *Counting On To*, *Counting Down* and *Counting Down To*. Lastly they describe Number Fact strategies as strategies used by children when they recall numbers and then use them to derive facts in a given problem. Carpenter et al. (1999) submit that 'over time, *Direct Modeling* strategies are replaced by the use of *Counting Strategies,* and finally most children come to rely on number facts' (p.26). Once learners are able to achieve this shift they would have mastered FP numeracy. Ensor et al. (2009) introduces interlinked aspects to mastering Foundation Phase numeracy as: progression in acquiring the number concept, the shift from concrete to abstract reasoning, and relatedly, the move from counting to calculating. In the next section I provide two figures where I outline a summary of children's solution strategies as described and detailed by Carpenter et al. (1999) above.

The first figure (figure 2.2) provides a hierarchical view of these strategies in relation to problem classes and types. The first figure is extracted from Carpenter et al. (1999) and in the second figure (figure 2.3) I use ideas adapted from Carpenter et al. (1999). I firstly briefly discuss figure 2.2. On top of the solid line are classes of problems and problem types. The arrows are pointing to the strategies related to a specific problem class and type. In the next row below is what can be referred to as less sophisticated strategies (Joining All, Separating From, Joining To, Matching, Trial and Error) which are called *direct modeling* strategies. Below direct modelling strategies are *counting strategies* (*Count On From First*, *Count On From Larger*, *Counting Down*, *Counting Down To*, *Trial and Error)*. Below 'counting strategies' are 'number facts' which are referred to as flexible choice of strategies also known as 'derived facts' and 'recalled facts'. As mentioned above, there is a progression within and across the strategies: *Direct Modeling* strategies are less sophisticated, *Counting Strategies* are more sophisticated than *Direct Modeling* strategies, and *Number Facts* are more sophisticated than *Counting Strategies*.

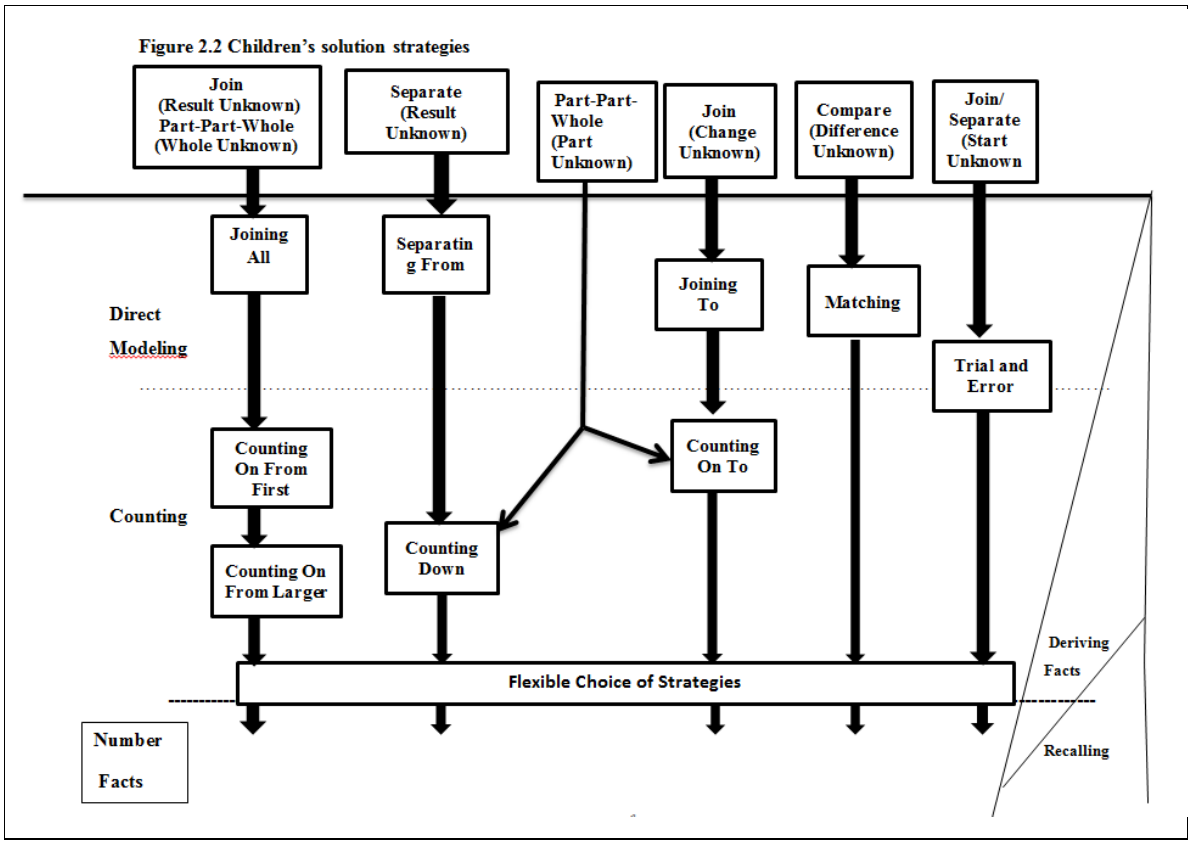

Figure 2.2 Children's solution strategies

In table 2.3 below I present examples to show progression of the strategies discussed. The layout of the table is such that the *Strategies* on the right hand side of the table are described in relation to the *Addition and Subtraction Problems*. Carpenter et al. (1999) have identified what they refer to as four basic *classes of problems*: *Join, Separate, Part-*

Part-Whole, and Compare. Below table 2.3 I then present a brief description of the columns.

Table 2.3 Examples of Children's Solution Strategies in relation to Addition and Subtraction Problems

Addition and Subtraction Problems				Stategies		
Classes of Problems	Types of Problems	Examples: Word Sentences	Associated Number Sentences	Direct Modeling Strategies	Counting Strategies	Number Fact Strategies
Join	Result Unknown	Robin had 2 toy cars. Her parents gave her 5 more cars for her birthday. How many toy cars did she have then?	2 + 5 = ?	***Joining All***	***Counting On from First*** ***Counting On From Larger***	Recalled Facts and Derived Facts
	Change Unknown	Robin had 5 toy cars. Her parents her some more toy cars for her birthday. Then she had 7 toy cars. How many toy cars did Robin's parents give her for her birthday?	5 + ? = 7	***Joining To***	***Counting On To***	
	Start Unknown	Robin had some toy cars. Her parents gave her 2 more toy cars for her birthday. Then she had 7 toy cars. How many toy cars did Robin have before her birthday?	? + 2 = 7	**Trial and Error**	Less likely to be used by children	
Separate	Result Unknown	Collen had 8 guppies. She gave 3 to Roger. How many guppies does Colleen have left?	8 - 3 = ?	***Separating From***	***Counting Down***	
	Change Unknown	Collen had 8 guppies. She gave some guppies to Roger. Then she had 5 guppies left. How many guppies did Colleen have to start with?	8 - ? = 5	***Separating To***	***Counting Down To***	
	Start Unknown	Collen had some guppies. She gave 3 guppies to Roger. Then she had 5 guppies left. How many guppies did Colleen have to start with?	? - 3 = 5	**Trial and Error**	Less likely to be used by children	
Part-Part Whole	Whole Unkown	6 boys and 4 girls were playing soccer. How many children were playing soccer?	6 + 4 = ?	No commonly used strategy corresponding to the action or relationship described in the problem.	No commonly used strategy corresponding to the action or relationship described in the problem.	
	Part Unknown	10 children were playing soccer. 6 were boys and the rest were girls. How many girls were playing soccer?	10 - 6 = ? Or 6 + ? = 10 Or ? + 6 = 10	No commonly used strategy corresponding to the action or relationship described in the problem.	No commonly used strategy corresponding to the action or relationship described in the problem.	
Compare	Difference Unknown	Mark has 3 mice. Joy has 7 mice. Joy has how many more mice than Mark?	3 + ? = 7 Or 7 - 3 =? Or ? + 3 = 7	***Matching***	No commonly used strategy corresponding to the action or relationship described in the problem.	
	Compared Set Unknown	Mark has 3 mice. Joy has 4 more mice than Mark? How many mice does Joy have?	3 + ? = 7 Or 7 - 3 =? Or ? + 3 = 7	No commonly used strategy corresponding to the action or relationship described in the problem.	No commonly used strategy corresponding to the action or relationship described in the problem.	
	Referent Unknown	Joy has 7 mice. She has 4 more mice than Mark. How many mice does Mark have?	4 + ? = 7 Or ? + 4 = 7 Or 7 - 4 = ?	No commonly used strategy corresponding to the action or relationship described in the problem.	No commonly used strategy corresponding to the action or relationship described in the problem.	

2.2.1 Classes of Problems

According to Carpenter et al. (1999) *Join* and *Separate* problems involve action. In *Join* problems elements are added to a given set, and in *Separate* problems elements are removed from a given set. '*PPW* and *Compare* problems do not involve actions' (Carpenter et al., 1999, p.7). These authors state that *PPW* problems involve a relationship between a set and its two subsets, and *Compare* problems involve comparison between two disjoint sets.

2.2.2 Types of Problems

Within each class Carpenter et al. (1999) have identified what they refer to as distinct types of problems with items that can be distinguished as *Unknown*. They further elaborate that each of these types represents a different problem to young children and children use different strategies to solve them and they also vary in difficulty (Carpenter et al., 1999). They distinguish problem types for Join and Separate problems as *Result Unknown*, *Change Unknown, Start Unknown.* The first two Join and Separate problems are amenable to direct modelling whilst the *Start Unknown* problems are not (Carpenter et al., 1999). This makes *Start Unknown* problems harder than the other two types in the initial stages. Part-Part-Whole problems are distinguished as *Whole Unknown* and *Part Unknown*; and Compare problems as *Difference Unknown*, *Compared Set Unknown*, and *Referent Unknown.* A finer grained analysis of difficulty in problem types is linked to strategies and discussed later.

2.2.3 Types of Problems, Word Sentences and Number Sentences

Both word sentences and number sentences can be used to represent distinctions among problem types (Carpenter et al., 1999). Table 2.3 above displays examples of word sentences represented in number sentences. Carpenter, Moser, and Bebout (1988) argue that 'although writing number sentences to represent addition and subtraction situations may not be necessary to solve simple problems with small numbers, it is a first step in learning to represent problems mathematically' (p.345). 'Before children receive formal instruction in addition and subtraction, they can analyse and solve simple addition and subtraction word problems by directly representing with physical objects or counting strategies modeling the action or relationships described in a problem, however the modeling and counting strategies that children use to solve simple problems with relatively small numbers are too cumbersome to be effective with more complex problems or problems with large numbers' (Carpenter et al., 1988, p.345).

There is therefore agreement that more sophisticated strategies are necessary in order to solve more complex problems or problems with large numbers (Thompson, 1999b). Carpenter, Hiebert, and Moser (1983) found that children who are taught to represent word problems with number sentences in standard form (a + b=? and a - b=?) do not struggle to represent problems describing simple joining and separating actions, but they struggle to represent the join-change unknown problems usually represented as (a + ?=b) or (a - ?=b). According to Carpenter et al. (1999) the representation of number sentences is particularly useful with Join and Separate problems. For example (extracted from table 2.3 above) 2 + 5 = ? and 8 – 3 = ? correspond to the three quantities ('2' & '8'- Start Unknown, '5' & '3' - Change Unknown, '?' - result unknown). They further state that 'as with the word problems, any of the terms can be *unknown*, yielding a number sentence that corresponds to a particular *Join* and *Separate* problem' (Carpenter et al., 1999, p.10) With regards to *PPW* and *Compare* problems they insert that it is not possible to make a clear correspondence between each of the problems and the number sentences, as the number sentences end up being represented as *Join* and *Separate* problems. This seems to suggest that it would be important, in the context of *Join* and *Separate* problems, for children to learn how to set up the number sentence associated with the direct model (which has already given them the answer). In the context of the *PPW* where direct models are harder to apply, the direction of working might then be *problem > number sentence >direct modelling strategy* to solve the problem. This is also reflected in (figure 2.2) presented and discussed above where *Join* (Result Unknown) and *PPW* (Whole Unknown) are associated together (see figure 2.2). Also for example (extracted from table 2.3 above) a *PPW* problem is presented as follows:

6 boys and 4 girls were playing soccer, how many children were playing?

This problem can be represented in the form of a number sentence as 6 + 4 =? And a *Compare* problem (also extracted from table 2.2 above) is presented as follows:

Mark has 3 mice. Joy has 7 mice. Joy has how many more mice than Mark?

This problem can be represented in the form of number sentence in the following ways:

3 + ? = 7 or 7 - 3 =?

2.2.4 Direct Modeling Strategies

Carpenter et al. (1999) describe *Direct Modeling* as 'distinguished by the child's explicit physical representation of each quantity in a problem and the action or relationship involving those quantities before counting the resulting set' (p. 22). van der Walle, Karp, and Bay-Williams (2010) describe direct modelling as 'the use of manipulatives or drawing along with counting to represent directly the meaning of an operation or story problem'. What is emerging from these descriptions is that children model problem situations directly and that in the process there is a manipulation of objects by children. Carpenter et al. (1999) identifies direct modelling strategies as *Joining All*, *Joining To*, *Separating From*, *Separating To*, *Matching* and *Trial and Error*. Next I describe in detail the application of these strategies in relation to word problems.

It must be noted that Carpenter et al. (1999) discuss strategies in relation to word problems, while other literature discuss strategies in relation to abstract problems. There is agreement amongst various literature that strategies can be identified as *direct modeling*, *counting strategies* and *number facts*. I begin by detailing modeling strategies as described earlier.

2.2.4.1 Joining All

Word Problem: Robin had 2 toy cars. Her parents gave her 5 more cars for her birthday. How many toy cars did she have then?

A set of 2 objects and a set of 5 objects are constructed. The sets are joined and the union of the two sets is counted. Askew (1998b) refers to this strategy as 'Count All' and offers an example that 'In order for a child to add (5 + 4), a child will count out five fingers or counters, count out another four, and then recount them all, starting with the first finger of the five' (p.51). Beckmann (2005) discusses the direct modeling, count all strategy and explicitly refers to this strategy as 'level 1'.

2.2.4.2 Joining To

Word Problem: Robin had 5 toy cars. Her parents give her some more toy cars for her birthday. Then she had 7 toy cars. How many toy cars did Robin's parents give her for her birthday?

A set of 5 objects is constructed. Objects are added to this set until there is a total of 7 objects. The answer is found by counting the number of objects added. This strategy requires advanced planning which 'Joining All' does not and children can use different colours to differentiate between counters (Carpenter et al., 1999). Children can also use memorisation to keep the number of objects added in a situation where the objects are the same colour.

2.2.4.3 Separating From

Word Problem: Colleen had 8 guppies. She gave 3 to Roger. How many guppies does Colleen have left?

A set of 8 objects is constructed. 3 objects are removed. The answer is the number of remaining objects. Beckmann (2005) refers to this direct modelling strategy as *'take away'* and categorizes it as 'level 1'.

2.2.4.4 Separating To

Word Problem: Colleen had 8 guppies. She gave some guppies to Roger. Then she had 5 guppies left. How many guppies did Colleen give to Roger?

A set of 8 objects is counted out. Objects are removed from it until the number of objects remaining is equal to 5. The answer is the number of objects removed.

2.2.4.5 Matching

Word Problem: Mark has 3 mice. Joy has 7 mice. Joy has how many more mice than Mark?

A set of 3 objects and a set of 7 objects are matched 1-to-1 until one set is used up. The answer is the number of unmatched objects remaining in the larger set. Anghileri (2006) submits that a skill required in this strategy is for a child to be able to check that both objects have a 'starting point' from which the quantities can be viewed. She cautions that if children use unequal objects in the different rows they might be misled in thinking that the row with shorter or smaller objects has a smaller quantity.

2.2.4.6 Trial and Error

Word Problem: Colleen had some guppies. She gave 3 guppies to Roger. Then she had 5 guppies left. How many guppies did Colleen have to start with?

A set of objects is constructed. A set of 3 objects is added to the set, and the resulting set is counted. If the final count is 5, then the number of objects in the initial set is the answer. If it is not 5, a different initial set is tried.

Carpenter et al. (1999, p.26) argue that 'direct modelers are not consistently successful in solving all problems that can be modelled due to some problems being more difficult to model than others'. According to Carpenter et al. (1999) this causes no problems with the *Joining All* and *Separating From* strategies, but it can cause difficulties in *Joining To*. They present the following example to support their argument:

> *Robin had 5 toy cars. How many more toy cars does she have to get for her birthday to make 9 toy cars?*
>
> "Nick makes a set of 5 cubes and then adds 4 more cubes to the set counting, "6, 7, 8, 9," as he adds the cubes. He is not careful to keep the new cubes separate, so when he finishes adding them, he cannot distinguish them from the original set of 5 cubes. He looks confused for a moment and then counts the entire set of 9 cubes and responds, '9'…" (Carpenter et al., 1999, p.27*)*

Their conclusion from this example is that Nick models the action in the problem, however does not realize that he needs to keep separate the five cubes in the initial set and the four cubes that he adds. As a result he has no way of figuring out how many cubes he added. The implication for this conclusion is that the *Joining To* modeling strategy is more difficult to apply than *Joining All*. Carpenter et al. (1999) state that most direct modelers have difficulty solving *Start Unknown* problems with understanding because the initial set is unknown. They further insert that 'the only alternative for modeling the problem is *Trial and Error*, and *Trial and Error* is a difficult strategy for most direct modelers to apply' (p.27). They further maintain that physical objects a child may use (fingers, counters, tally marks) are used to represent objects in the problem' (p. 23). van der Walle et al. (2010) maintain that children using direct modeling will soon transfer their ideas to methods that do not rely on materials or counting. They further argue that the direct-modeling phase provides a necessary background of ideas and that the developmental strategies provide children who are not ready for more efficient methods a way to explore the same problems as classmates who have progressed beyond this stage. Children should therefore not be pushed prematurely to abandon manipulative approaches, though at some point it is important for them to be encouraged to progress from *modeling strategies* to *counting strategies* which are more sophisticated. In the next section I detail counting strategies and present word problems and descriptions of what strategy is used in relation to the problem.

2.2.5 Counting Strategies

As discussed earlier, and supported by the CAPS (Curriculum Assessment Policy Statement), at Gr 3 learners must be able to shift from using concrete counting strategies to more sophisticated strategies (DBE, 2011a). This concurs with Carpenter's (1999) view that as children progress they replace concrete *Direct Modeling* strategies with more efficient *Counting Strategies* (Carpenter et al., 1999). According to Carpenter et al. (1999) the use of *Counting Strategies* marks an important stage in the development of number concepts.

Carpenter et al. (1999) further suggest that counting strategies are more efficient and abstract than modelling strategies. They have identified counting strategies as *Counting On From First*, *Counting On From Larger, Counting On To, Counting Down, Counting Down To*.

2.2.5.1 Counting On From First

Word Problem: Robin had 2 toy cars. Her parents gave her 5 more toy cars for her birthday. How many toy cars did she have then?

The counting begins with 2 and continues on 5 more counts. The answer is the last number in the counting sequence. Thompson (1999a) has also identified this strategy and further elaborates that in order for children to *count on* they must realise that the number sequence can be 'broken' and that counting can begin at any point in the sequence. Beckmann (2005) refers to this strategy as just a *count on* strategy though her description and example fits that of a *count on from first* strategy.

2.2.5.2 Counting On From Larger

Word Problem: Robin had 2 toy cars. Her parents gave her 5 more cars for her birthday. How many toy cars did she have then?

The counting sequence begins with 5 and continues on 2 more counts. The answer is the last number in the counting sequence. In using the strategy children require an awareness of commutativity. Fingers are usually used to keep track of the steps incremented in the counting sequence. It is more efficient in making the increment counting process shorter.

2.2.5.3 Counting On To

Word Problem: Robin had 5 toy cars. Her parents gave her some more toy cars for her birthday. Then she had 7 toy cars. How many toy cars did Robin's parents give her for her birthday?

A forward counting sequence starts from 5 and continues until 7 is reached. The answer is the number of counting words in the sequence.

2.2.5.4 Counting Down

Word Problem: Colleen had 8 guppies. She gave 3 to Roger. How many guppies does Colleen have left?

A backward counting sequence is initiated from 8. The sequence continues for 3 more counts. The last number in the counting sequence is the answer.

2.2.5.5 Counting Down To

Word Problem: Colleen had 8 guppies. She gave some guppies to Roger. Then she had 5 guppies left. How many guppies did Colleen give to Roger?

A backward counting sequence starts from 8 continues until 5 is reached. The answer is the number of words in the counting sequence. Though children may fall back to using direct *Modeling Strategies*, most children come to rely on the *Counting On* strategies and not all children use *Counting Down* consistently due to the difficulty in counting backwards (Carpenter et al., 1999). This suggests that the *Counting Down* strategies are more sophisticated than *Counting On* strategies.

Carpenter et al. (1999) posit that when using a counting strategy, a child fundamentally recognizes that they do not have to actually construct and count sets, rather they can find the answer by focusing on the counting sequence itself. These authors further state that 'counting strategies generally involve some sort of simultaneous double counting, and the physical objects a child may use (fingers, counters, tally marks) are used to keep track of counts rather than to represent objects in the problem' (p. 23). As children progress they are expected to shift from counting to non-counting strategies or what other research refers to as *number fact* or *calculating strategies* (Thompson, 1999a).

2.2.6 Number Facts

Carpenter et al. (1999) submit that children learn number facts both in and out of school and apply the knowledge in problem solving. They maintain that children often use a small set of memorized facts to derive solutions for problems involving other number combinations. Children use *Derived Facts* solutions which according to Carpenter et al. (1999) are based on understanding relations between numbers. For example if a child is asked to solve 6 + 8 =? They can shout out immediately that the answer is 14 and if asked how they got the answer, their response could be that they knew that 6 + 6 is 12 and adding 2 then results in 14. In this case the child has used a derived fact from a recalled number fact. Carpenter et al. (1999) suggest that counting strategies and derived facts are relatively efficient strategies for solving problems. They also argue that being given the opportunity to solve many problems with strategies they have invented, enables children to eventually learn most number facts at a recall level. Other researchers (Askew, 1998a; Beckmann, 2005; Thompson, 1999a, 1999b) have expanded derived facts since they argue that at this level children use known facts to find related facts. According to Thompson (1999a, p.2) 'children initially work on 'numbers to 20', which involves gradual acquisition of addition and subtraction bonds…'In addition he describes the next stage as working with 'numbers 20 to 100' which include two-digit additions and subtractions and argues that two-digit addition and subtraction should be treated differently from single-digit work. Askew (1998b) concurs with this argument and refers to children able to deduce number facts up to 20 as 'Key Stage 1' (level 1), and those confident to mentally add or subtract two-digit numbers he refers to as 'Key Stage 2 (level 2) children'. Askew (1998b) describes further strategies which he says can be explored further by children once they have begun to build a repertoire of known facts and have simple strategies for deriving facts. Following are these strategies as described by Askew (1998a, p.52).

- *'knowing the complements to 10: the pairs of numbers that add to 10*
  2 + 8, 3 + 7 and so on
- *being able to add 10 instantly to a single digit*
- *being able to rapidly add 9 by*
  adding 10 and take off one or take off one and add 10
- *knowing the doubles to 10 + 10*
- *use the known double facts to rapidly find near doubles*
  6 + 7 is one more than 6 + 6
- *being able to 'bridge through 10'*
  use knowledge of complements to partition a number to make bond to 10 for example,
  8 + 5 = 8 + 2 + 3
- *using the commutative law*
  2 + 8 = 8 + 2

Thompson (1999a) refers to derived strategies as calculating strategies and describes them in table 2.2.6 below, through linked examples and strategies:

Table 2.2.6 Calculating Strategies

Example in Abstract Form	**Strategy**	**Description**
a) 18-9	Doubles fact (subtraction)	The child knows that 9 and 9 makes 18, so the answer is 9.
b) 8 + 5	Near-doubles (addition)	The child knows that 5 + 5 is 10, add the extra 3 to get 13.
c) 9 – 5	Near-doubles (subtraction)	Child knows that 10 take away 5 is 5 and 9 is one down from 10 so they minus one from 5 to get 4.
d) 7 - 3	Subtraction as inverse of addition	The child knows that 3 and 4 is 7 so they take 3 away.
e) 6 + 7	Using fives	The child identifies that 6 is 5 + 1 and 7 is 5 + 2, so adds the two 5's and additional 2 and 1 to make 13.
f) 8 + 6	Bridging through ten (addition)	The child takes the first number (8) and counts up to 10: 8 + 2=10; then counts on what is left over 8 + 2 + 4=14
g) 12 ─ 4	Bridging through ten (subtraction)	Similarly to bridging through ten in addition; the child takes the first number (12) and subtracts to get to 10: 12─2=10; and then takes away what is left: 12─2─2=8. This strategy requires awareness of partitioning and complements of 10.
h) 9 + 5	Compensation	The child knows that 10 + 5 is 15, and 9 is one less than 10, so take away 1 to get 14.
i) 7 + 9=6 +10	Balancing	In using this strategy a child must have an idea of 'equivalence' (Thompson, 1999a, p.4)

(Thompson, 1999a, p.4) concludes certain strategies will prove to be most useful for later work and these strategies are:

- *Bridging up and down through ten (using complements in ten);*
- *Partitioning single digit number (7 is 5 + 2 or 4 + 3)*
- *Compensation (adding 9 by adding 10 and subtracting 1)*

Thompson and Smith (1999) suggest there are four main two-digit addition and subtraction strategies used by children for mental calculation. These strategies are:

- Partitioning
    - It is reported that in England and USA this is the most common strategy (Beishuizen & Anghileri, 1998; Thompson, 1999b). Thompson (1999b) further elaborates that this strategy is called *partitioning* or *'split'* method because the numbers being added or subtracted are both partitioned into multiples of ten and ones. He also informs that in research literature this strategy is known as 1010 (ten-ten, TT) strategy (Askew, 1998b; Beishuizen & Anghileri, 1998) . Ellemor-Collins, Wright, and Lewis (2007) have identified this strategy as *collections-based* or *split* strategy. Klein, Beishuizen, and Treffers (1998) refer to this strategy as decomposition or regrouping procedures. Examples representing both addition and subtraction are provided by Thompson (1999b, p.24):

> Sammy is asked to solve (63 + 56), Sammy's response is:
> *'119…I added the sixty and fifty first and then added three and six'*

In this example Sammy has split the 63 into 60 and 3; then split the 56 into 50 and 6; then has added 60 and 50 together and 3 and 6 together; and then has added the two subtotals together (110 + 9) to get the answer which is 119.

> Rebecca is asked to solve (68-32), and her response is:
>
> *'Thirty six…I took away thirty from sixty and then took away two from eight'*.

Rebecca has split the 68 and the 32 in a similar way as Sammy; she has calculated two separate subtractions and then combined the two answers to get 36.

Thompson (1999b) cautions that the problem with this strategy is that it breaks down with subtraction as it requires regrouping, for example if a learner is asked to solve 23─17 they will minus 10 from 20 then they will have to add 3 to 10 to make 13 then subtract 7 to get 6. They could also make an error of subtracting 3 from 7 which could lead to getting 14 (wrong answer) as their answer. Klein et al. (1998) found that split (1010) strategies led to difficulty in developing independence from concrete materials and procedural confusion.

- Sequencing
    - Thompson (1999b) inserts that this strategy is not frequently used in The Netherlands. Beishuizen and Anghileri (1998) concur with Thompson and give their reasoning to be that building up patterns of 10-jumps is an obstacle for many children. Thompson (1999b) further elaborates that in this strategy 'one of the numbers in the calculation is retained as a whole, and chunks of the number are added to or subtracted from it' (p. 25). Ellemor-Collins et al. (2007) has identified this strategy as the sequence-based or jump strategy. Thompson (1999b) explains that this strategy 'is described as a jump strategy because it can be easily represented – practically or mentally – on a number line, where the procedure starts at one of the numbers and progresses towards the answer in jumps along the line where conveniently-sized chunks of the second number are added or subtracted' (p.25). (Beishuizen, Van Putten, & Van Mulken, 1997; Klein et al., 1998) found that jump (N10) strategy resulted in fewer errors and enabled making efficient computation choices. Examples provided by (Thompson, 1999b, p.25) are as follows:

Paul is asked to solve (55 + 42), and his response is:

’97…I added the 40 to the 55 and that made 95…and another two is 97’

Sarah is asked to solve (54 - 27), and her response is:

’27…I knew that 54 take away 20 is 34 and then you need to take another 7 off…well I took the 7…I did it by because I know 3 and 4 equals 7 because the 4 made it down to 30 then you just have to take another 3 off’.

In his attempt to solve the problem, Paul has split the 42 into 40 and 2; he has added the 40 onto the 55 to get 95; and he has added on the two to get the correct answer 97. Sarah has calculated (54-20); has partitioned the remaining 7 into a 4 and a 3 (because of the 4 in the 34); subtracted the 4 from the 34 to give her a handy multiple of ten; and then used her knowledge of complements in ten to subtract the remaining 3 from the 30 (since 10-3 is 7 then 30-3 is 27).

- Mixed method
    - This strategy combines the first stages of the partitioning method with the later stages of sequencing strategy. For example Nicholas was asked to solve (37 + 45), his response was:

’82…I added the 40 and the 30 which made 70…and then added the 5 which made 75…and then added the 7 which made 82’. (p.25)

Nicholas started by splitting the two numbers and then finding the sum (or the difference) as he would if using the partitioning strategy. He then sequentially combines each units digit separately with their interim totals.

Laura was asked to solve (68-32) and her response was:

’36…I knew 60 take away 30 is 30 and add the 8 on is 38…and then you take the 2 from the 8 which is 36’.

Laura also starts by splitting the two numbers and then finding the sum. After sequentially combining each units digit separately with their interim totals, Laura adds the 8 and then subtracts the 2. Thompson (1999b, p.25) argues that ‘the basic partitioning strategy needs to be modified in order to deal with subtractions like 85-37, but the mixed method does not, as the following example shows’:

Afzal is asked to work out (54─27) and the response is:

’27…I took 50 from…I took 20 from 50…which gave me 30…then I added the 4 and took away the 7’.

According to Thompson (1999b) Afzal’s calculation procedure shows excellent number sense. He has subtracted the multiples of 10 to get 30 and has then added back the 4 to get 34 before subtracting 7. This method is known as the *split-jump* method or *10S* (partitioning followed by sequencing).

- Compensation
  - A strategy which involves adding or subtracting a number larger than the number specified in the calculation-usually the next higher multiple of 10- and then modifying the answer by ‘compensating’ for the extra bit added or subtracted. For example:

    Lauren is asked to solve (46 + 39), her response is:

    ’85…Well, I said that was 40…so 40 plus 46 is 86 and you’ve got to take 1 away’.

    Sarah is asked to solve (86-39), her response is:

    ’47…I added the 39 up to 40…I took 40 away from 86 is 46…and then added another 1…Well, I knew that the units wouldn’t change…it would just be the 10’s…and 8 take away 4 is 4’.

    Both children have treated 39 as 40. Lauren has then added the 40 to the 46 and then taken 1 away, whereas Sarah has subtracted it from 86 and then added 1 on. Thompson (1999b) suggest that this strategy would appear to demand a high level of confidence with numbers and a good implicit understanding of the inverse relationship between operations. Also the user of this strategy needs to be aware that adding 1 more than required must be compensated for by taking 1 away from the answer, similarly, subtracting one more than necessary implies adding 1 onto the answer.

- ‘Complementary addition’ (as known in England and useful for difference problems)
    - The Dutch refer to this strategy as ‘adding to 10’ (*A*10) (Thompson, 1999b). Thompson (1999b) further elaborates that this strategy is mostly used by adults rather than by children, but it is successfully taught in some European countries where work on difference problems is used to stimulate its use. The strategy is more effective if the numbers in a problem are close together for example:

      James was asked to solve (73─68), and his response was:

      ‘Five…I added 2 onto 68…so then that made 70, and then I added another 3 on’.

      James has noticed the proximity of the two numbers and despite the subtraction sign has had the confidence to find the difference using complementary addition (Thompson, 1999b). Thompson (1999b) further elaborates that for numbers with a larger gap it is often difficult to keep a tally of the numbers added on without writing something down.

In solving addition and subtraction problems, what seems to clearly be emerging is the dominance of counting strategies which suggests that counting strategies have as their foundation, the concept of what it means to count. Thus in the next section I discuss counting strategies by reviewing the principles of counting.

2.3 Counting Principles

Gelman and Gallistel (1986) have identified counting principles which they suggest need to be in place for children to be successfully mastering counting. These principles are detailed below:

- ***The 1-1 principle***: This principle entails two concepts working simultaneously which are partitioning and tagging or marking. Partitioning is being able to differentiate what has been counted and what has not. Tagging is assigning markers to each item in a collection. These two concepts must work together and can be done concretely with objects or mentally.
- ***The stable-order principle***: The understanding of the sequencing of numbers, which often begins as rote counting. In addition to assigning markers to items in a collection, 'children need to recognise that the tags themselves are organised in a repeatable, stable order' (Ensor et al., 2009, p.9).
- ***The cardinal principle:*** Understanding that a number has numerosity and is itself an object which can then be manipulated. For a child's later number reasoning it is important to note that 'a number such as 5 encapsulates numerosity (counting items 1 through to 5) but also that 5 represents the total number of items, and becomes an object which can be manipulated'(Ensor et al., 2009, p.10). This manipulation can also be related to using for example '10' as a benchmark which according to McIntosh, Reys, and Reys (1992) provides essential mental referents for thinking about numbers. They further elaborate 'benchmarks are often used to judge the size of an answer or to round a number so that it is easier to mentally process' (p. 6). This principle involves skill and pattern recognition as children use number words *'two', 'three', 'four'* before they identify them within the structure of the counting system (Anghileri, 2006). Anghileri (2006) further submits that understanding that '10' can be both '10' individual items and the same time a '10' that is counted as one '10' is necessary for understanding place value.
- ***The abstraction principle***: The awareness of the different sets of items that can be counted namely: "perceptual units (which can be seen), figural items (items not present, but recallable – for example the number of dwellers in a home), motor units (movements like steps or handclaps), verbal units (utterances of number words and abstract units" (Steffe et al., 1983)

Gelman and Gallistel (1986) refer to 1-1, stable order, and cardinal principles as *how-to-count* principles as they refer directly to the action of counting. The abstraction principle is referred to as the what-to-count principle because of the different units that can be counted. The last order irrelevance principle is a requirement for a child to understand and be able to use counting strategies (Gelman & Gallistel, 1986). Ensor et al. (2009) state that these principles are usually mastered over a period of time, and are needed to work with numbers as representations of numerosity. Other important aspects that emerge from literature that are described as important skills related to addition and subtraction strategies are subitizing and place value. Next I discuss these skills.

2.4 Subitizing

Revkin, Piazza, Izard, Cohen, and Dehaene (2008, p.345) describes subitizing as 'the ability to enumerate a small group of four or fewer objects fast and accurately'. In Fuson (1992) subitizing is described as a visually immediate apprehension and labelling of small numbers. She further relates subitizing to perceptual counting and argues that perceptual counting provides an early basis for addition and also plays a role in the more advanced conceptual levels of addition and subtraction, for children will mostly first exhibit some conceptual advance by using perceptual methods. According to Anghileri (2006) subitizing is when patterns or special arrangements of objects are instantly recognized. She further inserts that 'there is no counting involved but a spatial arrangement becomes familiar and it is consistently associated with the number word (p.25). This skill can be related to the abstraction principle discussed above.

2.5 Place Value and Standard Algorithms

Carpenter, Franke, Jacobs, Fennema, and Empson (1998) differentiate between standard algorithms and invented strategies. They state that 'standard algorithms are quite far removed from their conceptual underpinnings while invented strategies are generally derived directly from the underlying multi-digit concepts' (Carpenter et al., 1998, p.5). They further elaborate that with standard addition and subtraction algorithms, numbers are aligned so that the ones, tens, hundreds and bigger digits can be added in columns, but no reference is necessarily made that addition involves the same unit (adding ones together, tens together etc). Most strategies on the contrary, label the units being combined for example if children are asked to add 20 and 13, it must be clear that they are adding ‘20’ and ‘10’ or two tens and 1 ten, rather than adding two numbers that appear in the same column. Fuson (1992) reports that research on addition and subtraction provided evidence that U.S. children demonstrated inadequate place-value and multi-digit subtraction knowledge in grade 3. She further argues that this inability of third graders to understand or use multi-digit concepts affects multi-digit addition and subtraction in that it results in learners ‘aligning numbers on the left instead of by their positional values (i.e., on the right) in order to add or subtract them’(p. 262). They have found that in addition many grade 3 learners identify the ‘1’ traded over to tens or hundreds column as a one and not as a ten or a hundred. Various literature advocates for children to be allowed to work out their own strategies and procedures so that they can make meaning of place value (Beishuizen & Anghileri, 1998; Fuson et al., 1997). Making meaning and understanding mathematics involves progression from practical experiences to talking about these experiences (Anghileri, 2006). These practical experiences involve use of various physical representations. In the next section I discuss use of physical representations in solving addition and subtraction problems.

2.6 Use of physical representations in addition and subtraction

The idea of using manipulative materials is so that learners can develop their mental imagery. Beishuizen (1993) asserts that concrete, manipulative materials have long been in use as popular learning aids in primary education, but their effects have not been acknowledged. Anghileri (2006) states that if children begin with manipulation of real objects, sorting and rearranging different collections, they are introduced to patterns that will be identified with number words. She further elaborates that before children can operate with abstract numbers, they will learn to ‘model’ situations with their fingers or some form of apparatus that can represent the real objects. Anghileri (2006) reports that beads and cubes are introduced in classrooms because they can be linked together and so have the potential to represent numbers in a structured way. ‘Cubes are sometimes favoured because they can be used not only to model different numbers, but also to show the place value structure of the number system in ‘tens’ and ‘units’ (Anghileri, 2006, p.9). Beishuizen and Anghileri (1998) contend that coloured beads structured in tens on a string should be introduced, followed by a marked number line because ‘in this way mental representation and mental activation are stimulated more than through the modelling support of arithmetic blocks where passive reading-off behaviour has been one of the negative side-effects in the past' (p.526).

Anghileri (1996) states that various versions of the number line can be used to model a counting sequence. According to Beishuizen and Anghileri (1998) a number line is an old model, but new in its empty format. The introduction of the 'empty format'[2] is a means to promote the flexibility of mental strategies which has received a higher priority (Beishuizen, 1993). The idea of using physical representations as discussed earlier is to promote a shift from using concrete strategies to more sophisticated abstract procedures which are often efficient through the ten-structure. Beishuizen (1993) found that Dutch children faced difficulty in using the 'N10' jumps strategy on a number line, and therefore revert to the '1010' strategy. Askew and Brown (2003) suggest that the empty number line needs to be structured carefully in order to provide a useful model for addition and subtraction. They advocate for use of the most appropriate representations but mention that the dominant culture in the UK is to offer a wide range of representations while other cultures focus on a more limited range of representations. They maintain that 'working in a structured and systematic way with a limited but effective set of representations may be more helpful than offering children a wide range of representations.'(p. 13). This implies that teachers should have an awareness of the most appropriate representations which can help learners progress from a concrete to a more abstract way of solving addition and subtraction problems. Fuson (1992) argues that the availability of multi-units in certain materials (for example, base-ten blocks or bead string) also does not mean that a given child uses conceptual multi-units for those materials. For example, a child may not see a ten-unit as made up of ten single unit items even if the child uses the verbal label "ten" for that block or bead string. Carpenter et al. (1999) asserts that while using a counting strategy, physical objects a child may use are used to keep track of counts rather than to represent objects in the problem, and this would suggest that a child is using the object at a more sophisticated level (Gravemeijer,

[2] 'empty format' number line refers to a number line that is not labelled with numbers, when using a number line to solve an addition problem for example, learners are expected to only put the numbers being added on the 'empty' number line

1994). Klein et al. (1998) suggests that ‘a structured bead string should be used as an introductory model for the empty number line’ (p.446).

‘This bead string has 100 beads, ordered following the ten structure: 10 red beads followed by 10 white beads followed by 10 red beads and so on (see Figure 2.6.1 - used to show how a coloured bead string would look like). This structure helps students find a given number and familiarizes children with the positioning of numbers up to 100 and the quantities the numbers represent. The tens can serve as a point of reference in two ways: For example, there are 6 tens in 64 and there are almost 8 tens in 79. After children work with the bead string, the number line can be introduced as a model of the bead string (see Figure 2.6.1). By using the empty number line, children can extend their counting strategies and raise the sophistication level of their strategies from counting by ones to counting by tens to counting by multiples of ten.’ (Klein et al., 1998, p. 446)

Figure 2.6.1 Example of a structured bead string

These authors further provide four reasons for using the empty number line:

a) *The empty number line is well-suited to link up with informal solution procedures because of the linear character of the number line.*
b) *The second reason for using the empty number line is that it provides the opportunity to raise the level of the students' activity.*
c) *The third reason for using the empty number line is its natural and transparent character.*
d) *Students using the empty number line are cognitively involved in their actions.’ (p.447)*

(Gravemeijer & Stephan, 2002) state that the primary goal is to ensure that students eventually, can mentally model the way they would act with the bead string, bearing in mind that a mark does not just refer to a position on the bead string but signifies a quantity, the number of beads up to that position. For example if a child was asked to add 24 and 13 on a bead string, they would have to realise where the first mark is are 24 beads which represent the quantity ‘24’, and where the last mark is are 37 beads which represents the quantity ‘37’ which is a result of having added 13 to 24 (see figure 2.6.2 below).

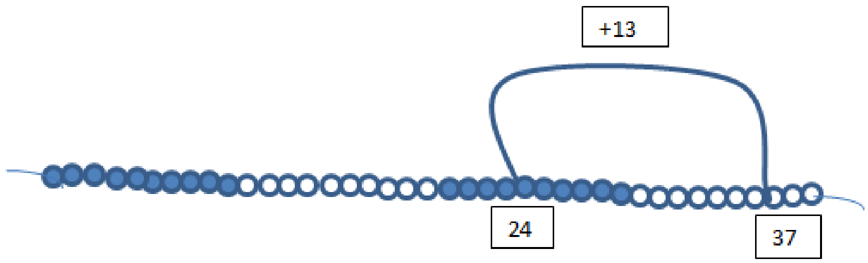

Figure 2.6.2 Demonstrating 24 + 13 on the bead string

Modeling on an empty number line would look like either of the figures below:

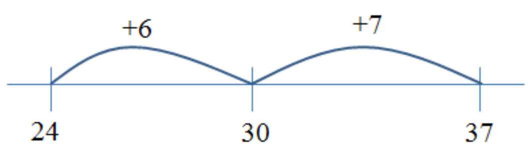

Figure 2.6.3 Using a partitioning strategy

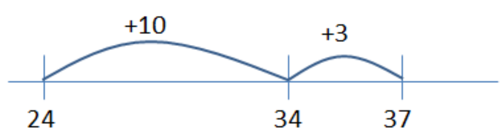

Figure 2.6.4 Using a sequencing strategy

(Gravemeijer & Stephan, 2002; Klein et al., 1998) argue that since another principle of RME (Real Mathematics Education) is that a model should not only be a *model of a situation* (for instance a context problem) but should also become a *model for representing* mathematical solutions, the empty number line satisfies these requirements because it not only allows students to express and communicate their own solution procedures but also facilitates those solution procedures. Marking the steps on the number line functions as a kind of scaffolding: it shows which part of the operation has been carried out and what remains to be done.

According to Gravemeijer and Stephan (2002) the idea of using models is that the models will help students re(invent) the more formal mathematics that is aimed for. They further elaborate that the empty number line must be viewed as a means of expressing solution methods rather than a measurement instrument. Gravemeijer and Stephan (2002) submits that when the transition from *model of* to *model for* takes place, the students will no longer need the support of a model for their more formal mathematical reasoning. More formal mathematical reasoning involves use of traditional algorithms or recognition of place value which has been discussed earlier.

2.7 Summary of the Conceptual Framework

In this chapter I have discussed literature on addition and subtraction, strategies in relation thereof, and skills that feed into the strategies. Table 2.7 below provides a summarised conceptual framework which is based on the literature reviewed in this chapter and provides the relevant categories and indicators with which to look at the data in Chapter 4. I now provide a brief explanation of the categories used in Table 2.7. The 'Level' represents the progression of strategies as discussed by Carpenter et al. (1999). For example children at level 1 use *Modeling Strategies*, at level 2 they would use *Counting Strategies* and at level 3 they would use *Number Fact* strategies. 'Problem Type' indicates that learners could either be presented with word or abstract problems to solve. 'Addition and Subtraction Strategies' represent strategies which learners might use to solve either word or abstract problems. 'Description' depicts what the learners would tend to do when solving a problem. For example, at level 1 learners tend to construct the first set and then the second set when asked to solve a join problem. 'Counting Principles' represent the type of counting principle associated with a specific level. 'Counting Principles' are also hierarchical as discussed earlier. Lastly, 'Physical Representations' represent the nature of objects used at a specific level for a specific strategy used by a learner to solve a problem.

Table 2.7 Addition and Subtraction Strategies

Level	Problem Type	Addition and Subtraction Strategies	Description	Counting Principles	Physical Representations
1	Word or Abstract	**Direct Modeling Strategies** • Joining All • Count All • Joining To • Separating From • Separating To • Matching • Trial & Error	Dependent on counting procedures with tangible objects Construction of first set in additive and subtractive tasks	1-1 principle	Counting objects need to be accessible perceptually (Wright et al, 2006) Using fingers sequentially while counting aloud reflects sophistication Uses manipulative objects to model situations
2	Word or Abstract	**Counting Strategies** • Count On From First • Count On from Larger • Counting On To • Counting Down	'Keeps first addend in the head' then counts on second addend. No construction of first set	Stable order principle and cardinality principle Abstraction principle	Can use either concrete objects or fingers to count on Children can keep a physical tally of number words (Thompson, 1999) Facility with finger patterns is required (Wright et al, 2006) Uses manipulative objects to model for mathematical reasoning
3	Word or Abstract	**Number Facts** For numbers between 1 and 20 • Doubles fact (subtraction) • Near-doubles (addition and subtraction) • Subtraction as inverse of addition • Using fives • Bridging through ten (addition and subtraction) • Compensation • Balancing For Numbers between 20 and 100 • Partitioning (TT) • Sequencing (NT) • Mixed Methods • Compensation • Complementary addition	Addends are viewed as objects which can be manipulated using recalled or derived facts	Abstraction principle Order Irrelevance principle	Child sees the relationship between two numbers, can visualise and apply the appropriate derived fact Manipulative objects are replaced by mental reasoning

2.8 Summary

In this chapter I have discussed addition and subtraction and have highlighted that amongst other dynamics discussed they are intricately related. I have then discussed addition and subtraction strategies and have found that these strategies are discussed using different terminology in the literature. What has emerged is that there is progress within and across the strategies and the progression promotes a shift from using concrete strategies to more sophisticated strategies. Skills that feed into these strategies are also discussed focusing on subitizing and place value. Considering that addition and subtraction strategies entail counting in them, it was fitting to discuss principles of counting. Lastly, use of physical representatives also formed part of my discussion since there is evidence in literature that learners in Gr 3 are still relying on concrete material for solving addition and subtraction problems. There is also advocacy for an awareness of the limitations of some of the strategies which implies that teachers would have to devise ways to deal with these limitations in ways that help learners use more efficient addition and subtraction strategies.

Chapter 3 - Methodology

3.1 Approach and Method

The approach of this study is qualitative because this study is an exploration of a phenomenon (Creswell, 2012). Creswell (2012) outlines characteristics of qualitative research including the following that are relevant to my study:

a) **Natural setting** - data is collected at the site where participants experience the issue or problem under study as opposed to bringing individuals into a lab. In the case of my study, the natural setting is the school since I am exploring addition and subtraction strategies of school-going Gr 3 learners.

b) **Researcher as key instrument** - qualitative researchers collect data themselves through examining documents, observing behaviour, or interviewing participants instead of using instruments designed by other researchers. I have collected data myself as a researcher through various sources which will be mentioned later in the study.

c) **Multiple sources of data -** qualitative researchers generally gather forms of data, such as interviews, observations, and documents, rather than rely on a single source. In my study I have used a video tape, field notes, lesson plans and workbooks of learners.

d) **Inductive data analysis** - qualitative researchers build their patterns, categories, and themes from the bottom up, by organizing the data into more increasingly more abstract units of information. This inductive process illustrates working back and forth between the themes and the database until the researcher has established a comprehensive set of themes. In my study, whilst I formulated an analytical framework that provides me with categories for my analysis, I also use a bottom up strategy of analysis as other categories emerged from empirical data which were not included in my analytical framework.

e) **Interpretive** - qualitative research is a form of interpretive inquiry in which researchers make an interpretation of what they can see, hear, and understand. In

> my study I make interpretations of what I can see, hear and understand from video data, field-notes, workbooks, and lesson plans.

The method of this study is a case study. Creswell (2009) describes a case study as,

> 'A qualitative strategy in which the researcher explores in depth a program, event, activity, process, or one or more individuals. The case(s) are bounded by time and activity, and researchers collect detailed information using a variety of data collection procedures over a sustained period of time' (p. 227).

According to Bell (1987) a case study is an opportunity for one aspect of a problem to be studied in some depth within limited timescale. Cohen and Manion (1980) assert that unlike the experimenter who manipulates variables to determine their causal significance or the surveyor who asks standardised questions of large, representative samples of individuals, the case study researcher naturally observes the characteristics of an individual unit - a child, a group, a class, a school, or a community. The process then allows for deep probing and intensive analysis of the phenomena being studied. What I have discussed above justifies my choice of a case study since I will be exploring a 'case' of a small group of 8 comprised of two student teachers and six Gr 3 learners, where each learner becomes a 'case' as I tell the story of the trajectory of each learner across tasks and across the year. I have chosen a case study method because it has allowed me to investigate the learners involved in depth within their real-life contexts (Yin, 2009). One of the advantages of using a case study is that it explores issues that may lead to further research. Though one of the critiques of a case study is that results are not easily generalizable, quoted in Bell (2005), (Bassey 1981: 85) states that:

> 'An important criterion for judging the merit of a case study is the extent to which the details are sufficient and appropriate for a teacher working in a similar situation to relate his decision making to that described in the case study. The relatability of a case study is more important than its generalizability'. (p.11)

He further elaborates that 'if case studies are carried out systematically and critically, if they are aimed at the improvement of education, if they are relatable, and if by publication of the findings they extend the boundaries of existing knowledge, then they are valid forms of educational research' (p.12)

In this study I investigate addition and subtraction strategies which are acquired and applied across grades throughout primary and secondary education. Though the findings of the study might not be generalized due to the small sample of this study and the focus being a group in a school, the results relating to the topic can be related to other contexts and may also contribute to the body of knowledge.

3.2 Setting

As stated earlier, because this study is located within the WMC-P broader project, the school where the maths club was conducted had already been identified based on two factors: the proximity from the university which allowed for easy access and the relationship with the Head of FP in the school who agreed for her school to be part of the MC project. This particular school where the study is based was an 'ex-model C' functional government primary school with learners who come from different socio economic backgrounds. It is a quintile 5 (least poor) school, with a school fee of R7000 per annum and class sizes of 40 learners per class.

3.3 Sample

Six grade 3 learners together with 2 student teachers conducting lessons within the mentioned group were observed. The age range of the learners is between 9 and 10 years old, and the learners' mother tongue is not English which is the language of learning and teaching. The group was chosen based firstly on the commitment of the student teachers as observed not only in the MC but also in their Wits course lectures which I also observed. Secondly, both student teachers expressed willingness to allow me to focus on the learner teaching and the learning of the group in the context of the MC. The lessons and activities carried out in the maths club were the same for all groups, and were discussed with the students in the content course associated with this school-based methodology course.

3.4 Procedures

Data was collected during the sequence of the maths club sessions which took place once a week after school. The initial duration for the lessons was 45 minutes but the duration of most lessons was about 38 minutes due to organisational issues such as accessing the hall and getting the learners organised. Of the 13 MC lessons that were anticipated to be observed, only 10 were observed as some were cancelled by the school organisers, of the 10 only 7 were video captured as all learners were mixed in the other 3 due to the nature of the tasks. The number of lessons reflects the weeks that were available in the academic year working around practicum sessions for all B Ed students and public holidays. Various data sources were used for this study. Observation data through videotaping formed the key data source that allowed analysis that could be used to answer the first research question: What addition and subtraction strategies do learners display at the start of the Maths Club? Field notes with feedback from the student teachers were also used and so were the learners' workbooks with exercises the learners worked on as their homework.

To answer the second question which is: 'What addition and subtraction strategies are suggested by the Maths Club?' The lesson plans and worksheets used within the MC sessions were collected and analysed. Observation data through video was also collected, as this provided insights into the ways in which planned tasks were enacted by the student teachers involved, which reflected an overlap to differing degrees with task intentions, and especially so given that they were interpreted by pre-service teachers.

Lastly, to answer the third question which is: What shifts can be seen in the learners' addition and subtraction strategies during the course of the Maths Club?, observation data through video was used and allowed me to access in-depth understanding and application of the learners' addition and subtraction strategies. This observation has provided insights into inconsistences in the learners' choice of a specific mathematical strategy – a feature I elaborate on in the analysis.

The idea was to capture all learners at once while solving problems of the same task so that I would be able to see what strategies each learner is using for the same task given to all the learners. During collection of data it became impossible to capture all learners at the same time. This based on various reasons which are firstly, that the learners only exhibited strategies when the teacher was next to them asking them to show or explain their process of working out their problems. Secondly, learners mostly just wrote answers without showing strategies. Due to the setup of the MC with the participation of other learners, it became very noisy such that I had to be very close to the teacher and learner while they discussed the strategies applied and therefore could not capture other learners at the same time. It must therefore be noted that observational data analysed was based on the learners working on different tasks at different times, even whilst all written work was collected and photocopied. Lastly, since the focus of my study was investigating strategies I focused on data that reflected learners use of strategies during activities and largely ignored other mathematical aspects such as learners getting a correct answer or not.

3.5 Process of analysis

Data is analysed according to Creswell's (2012) commonly used 6 steps of analysing qualitative data. These steps are paraphrased as follows:

1) Collection of data (video, workbooks, field notes, lesson plans)
2) Preparation of data for analysis (transcription of video data and field notes)
3) Reading through of data to obtain a general sense of material
4) Coding of data (locate text segments and assign a code label to them)
5) Coding of the text for Themes to Be Used in the research report
6) Coding of the text for description to be used in the research report.

All relevant information pertaining to the context of the case include unanticipated factors related to addition and subtraction strategies are recorded in order to provide a holistic view for readers and also to indicate the conditions under which claims are made as well as to ensure that all reporting is unbiased and accurate (Leedy, 2001). As mentioned above, an interpretive method of analysis is used to summarize what has been heard in terms of common words, phrases, themes of patterns that aid an understanding and interpretation of that which is emerging (Creswell, 2012; Maree, 2007).

3.6 Reliability, Validity, and Generalizability

3.6.1 Reliability

Bell (2005) describes reliability in general as 'the extent to which a test or procedure produces similar results under constant conditions on all occasions' (p.117). While Creswell (2009) describes qualitative reliability as 'indicating that the researcher's approach is consistent across different researchers and different projects' (p.190**).** In the case of my study I had intended to use learners' workbooks and field notes with the idea of testing reliability of the main instrument which was videotaping. Learners' workbooks contained addition and subtraction problems which learners were expected to solve and strategies which they used (if they used any) would help me compare the consistency of their use of strategies in comparison to what is reflected on the video. Field notes provided general feedback from the student teachers which helped me with the generalization of the results of my study but restricted me from making specific claims regarding each specific learner in the sample. I have also found that my approach is consistent across different researchers see (Beishuizen, 1993; Cooper, Heirdsfield, & Irons, 1995; Denvir & Brown, 1986a, 1986b; Ellemor-Collins & Wright, 2009; Wright, 1994). These researchers also adopted a qualitative case study approach when exploring addition and subtraction strategies of foundation phase learners. They also use data collection instruments such as video taping, learner's written work, interviews, lesson plans. Though the main instrument for most researchers appears to be interviews I was not able to do interviews due to the time limitation of my study and I found videotaping to be most appropriate for my study.

3.6.2 Validity

According to Creswell (2009) qualitative validity 'means that the researcher checks for accuracy of the findings by employing certain procedures' (p.190). Bell (2005) argues that if an item is unreliable, then it must also be invalid, but a reliable item might not necessarily be valid. In other words, it could produce consistent responses on all occasions, but not be measuring what it is supposed to measure. In my study I have ensured validity by using different sources of data collection (Creswell, 2009) which are videotaping, learner workbooks, field notes. Videotaping allowed to capture phenomena of interest such as what the learners were uttering, writing and acting while solving addition and subtraction problems. Some of the limitations were that the learners could not be captured at the same time while enacting the tasks since the capturing had to focus on discussions between teacher and learners so as to determine what strategies the learner was using. Also the limitation of using workbooks was that the learners did not reflect what kind of strategies they were using while solving problems, but inferences could be made from the pattern of correct or incorrect answers. The limitation of field notes was that during discussions with the student teachers the feedback was general and learner name were not mentioned, but the feedback helped to provide an overall idea of what kind of strategies learners were using while solving addition and subtraction problems.

3.6.3 Generalizability

Since my sample only involves a small group of learners in one school, the findings of my study cannot be generalized. The study exposes ways of exploring aspects of number concepts which can be adapted for relevance to a certain setting.

3.7 Ethics

To deal with ethical considerations, firstly information and consent letters were written and handed over to the participants assuring confidentiality and anonymity in all reporting. Concerning data collected through documents I stipulated that I would ensure that the information is stored and locked up in a safe place where others will not have access. With regards to video data, coding of the school and teacher/learner names is used to ensure anonymity and confidentiality. Information has been kept in a safe place in the Wits Maths Connect department.

3.8 Summary

In this chapter, I have discussed the approach and the methodological tools used to organise data. I have mentioned that the approach is qualitative and the method is a case study of 6 learners and each learner is a case. I have also discussed the setting of the study and reasons for choosing the setting which includes proximity and willingness of HOD for their school to participate. Including reliability, validity and generalizability in the discussion was important. In the next chapter I carry out the analysis of data.

Chapter 4 - Analysis

4.1 Findings and Analysis

In this chapter I present my findings and analysis and respond to key questions asked in this study which are:

- What addition and subtraction strategies do learners display at the start of Maths Club?
- What addition and subtraction strategies and resources does Maths Club suggest to learners?
- What shifts (if any) in addition and subtraction strategies can be seen during the course of the Maths Club?

The analytical framework formulated from key ideas emerging from literature in Chapter 2 is used to analyse the data.

I begin this chapter by providing an explanation of how the data is categorized and presented. This will be followed by an explanation of the format that is used to present the findings, namely, tables, extracts from transcripts and an explanation of these. The rest of the chapter will focus on the presentation of the data analysis.

4.2 Presentation and Categorization of Data

After collection, the video data was transcribed and amounts of data were placed into tracking categories that related to the models and strategies drawn from across the literature review and the empirical space. The categories related to the tasks that were worked on by learners with a focus on their strategies and use of models. Given that the Maths Club model was based on pre-service teachers working with groups of learners, there were quite frequent suggestions of strategies to use by these teachers – commentary on these interventions adds texture to my answers to the second research question: “What addition and subtraction strategies does Maths Club suggest to learners?” Across my analysis, I base my comments on predominant learner strategies on tasks based on learners having solved two to three related questions. Instances where only one sum within a task was attempted are sometimes commented on, but I do not draw conclusions about learner strategy in these instances due to the limited extent of the data in these cases.

Common themes that linked the transcribed data to the literature on counting strategies relating to addition and subtraction across the learners were chosen and coded as follows:

Table 4.2.1 Themes, Codes and Descriptions

Strategy	**Description**
Unit Counting (UC)	Where learners count in ones
Count All (CA)	Where learners recount a set of combined objects
Count On (COF)	Where learners count on from first number
Count On from Larger (COL)	Where learners count on from the large number
Count Down From (CDF)	Where learners count down from minuend
Count Down To (CDT)	Where learners count down to subtrahend
Separate From (SF)	Where learners remove objects from a constructed set to get the answer
Count Up (CU)	Where a learner counts up from the subtrahend to the minuend to get the answer
Forward Counting (FC)	Where learners count forward to 'take away' the subtrahend from the minuend
Standard Algorithm (SA)	Where a column method is used to solve problems
Recalled Fact Immediate Answer (RFIA)	Where an immediate answer is given without counting involved
Recalled Fact Other (RFO)	As stipulated in the analytical framework, recalled fact will be categorised based on the RF strategy used

In addition to the coding above related to strategies are skills found that feed into the strategies that also emerge from the literature (Fuson, 1992; Revkin et al., 2008; Wright et al., 2002) and become a focus of interest for discussion. There appears to be an argument regarding how to classify these aspects since some literature (Thompson, 1999b; van der Walle et al., 2010) refers to them as strategies while in other literature they are discussed and described as skills. In this study I have chosen to also classify these aspects as strategies since it will allow coherence in the analysis discussions. These strategies are as follows:

Table 4.2.2 Themes, Codes and Descriptions

Strategy	Description
Subitizing (SU)	Where learners put counters in any patterned form and where learners enumerate sets rapidly and accurately
Grouped Counting (GC)	Where learners count in groups of 2, 5, 10 etc
Partitioning (PA) or Breaking Down (BD)	Where learners break down numbers into tens and ones for example or work with partitions of 5/10
Place Value (PV)	Where there is an awareness of a value based on a number's position

Given the prevalence of student teacher intervention and use of physical representations when attempting problem solving in my data, I have added Student Teacher Intervention (STI) and physical representation (PR) as codes to note where a learner strategy was linked to a Teacher Intervention and to note the use of physical representation in attempting to solve a problem. The codes are as follows:

Table 4.2.3 Themes, Codes and Descriptions

Student Teacher Intervention-STI	Where the teacher interrupts and intervenes either with probing, questioning, 'telling' learners what to do or offers a strategy
Physical Representation-PR	Where any form of physical representation is used

These themes above also feed into my analytical framework that was formulated from key ideas that emerged from literature as discussed in Chapter 2.

As mentioned earlier this study is a case study with a sample of 6 learners and each learner is a case. The findings are presented according to each learner with the aim being to present a story of each learner across the tasks and across the year. Due to time limitations (45 minutes per session) and the level of the learners (relying on concrete strategies to solve problems), one lesson most of the time stretched over to the next session (Monday), hence only 7 sessions are reflected instead of 13 which was the number of overall sessions of the Maths Club. The report focuses on 4 of the 6 learners in the focal group. One learner dropped out of the Maths Club; of the remaining five learners, four gave 'rich' data in terms of oral and written communication, and thus, this study focuses on data from these four learners. The fifth learner whose data is not included was a very weak learner, who much of the time, and especially in the beginning, did not attempt to solve any of the set problems. If the teacher was not with her she would just stare at her work, and when the teacher was around she was only be able to solve a problem with the intervention of the teacher. It is also important to note that except for one learner, all the learners in the sample missed at least one lesson due to transport problems.

This analysis involves a detailed description using the analytical framework formulated from key ideas that have emerged from literature, and presented in Chapter 2. In order to structure the analysis in this chapter I present the overview of the Maths Club in the form of a table. In the table below I begin with the summary of each session. In the summary I include the focus of the session. This was taken from the design of the session and the worksheet that was given by the Maths Club co-ordinator and discussed with the students in the session and explicitly stated as the intended focus of the tasks for the session. It should also be noted that some of the examples used were spontaneously given by the student teachers primarily as ways of either scaffolding or assessing understanding. In the table, I detail both the separate tasks and the examples that fell within each one. In several instances the focal learners did not complete the tasks. Thus, when I make the distinction between task and activity, activity focuses on tasks that were engaged with by learners. In table 4.2.4 below I provide a summary of which examples were engaged with by the focal learner. So in summary the table presents the following information:

Table 4.2.4 Summary of Maths Club Overview

Session	**Focus**	**Task**	**Activity**	**Resources**	**Strategies emerging per learner**
Indicates which of the 7 sessions being analysed	Indicates the intended focus of the work prescribed for the lesson	Details the task presented to the learner including subtasks and the examples which fall under each task	Indicates which task/subtask/example each learner was working with	Indicates resources available and used during enactment of tasks	Details strategies emerging from the examples the learner was working with

The resources available and used are noted because these emerged as an important focus within the strategies the learners enacted, with resources sometimes used in contradictory ways. This element therefore represents a post hoc addition to the analysis, rather than a literature led category (although I have noted in the literature the resources advocated and ways they should be used within early addition and subtraction). Literature relating strategies to the use of resources within mathematics education is therefore drawn into my analysis of the data. Lastly the last column indicates summarised strategies as they are central in the discussion. Following the table I exemplify the dominant strategies used by individual learners or commonly across all four. What needs to be noted is that learners exhibited a mixture of strategies while solving one problem, therefore in some instances one example is used to exemplify more than one strategy. It is also important to note that each learner is named with anonymised initials: NT, LY, NK, KT, and in the transcript extracts used I use ST to refer to the Student Teacher. In instances where learners took part in game activities, in the analysis discussion their actions are described and no transcripts are extracted since there was no interaction with either the teacher or other learners due to the nature of the games. Examples used to describe strategies reflected by learners when solving problems are extracted from the transcript verbatim where there was an interaction with the teacher.

Following the summary tables, I provide a more in depth analysis of each learner's use of strategies related to resources across the year, and comment on any progressions seen within these aspects. I present the tables per session beginning with session 1 and each learner strategy analysis. The detail below answers my second research question, 'What addition and subtraction strategies and resources are offered by the Maths Club?'

4.3 Analysis of Learners' Strategies Emerging in Session 1

Table 4.3 Summary of Session 1

Session	Focus	Task	Activity	Resources	Learner Strategies			
1	Solving problems by counting using groups of 10's and 1's Use of beans and bead string	**Task 1** 1.1) 12 + 14 1.2) 23 + 24 1.3) 24 + 35 1.4) 18 + 62 1.5) 27 + 12 1.6) 10 + 19 + 21 1.7) 35 + 24 1.8) 24 + 23 1.9) 14 + 12	**Task 1:** **NT**: 1.1 **LY**: 1.6 **NK**: 1.9 **KT**: No activity	Beans, Bead string, Fingers	**NT:** PA RF(IA)	**LY:** SU UC GC in 2's RF(IA)	**NK:** GC in 2's SU RF(IA)	**KT:** No Activity
		Subtask 1: What do you notice with numbers a) 1 & 9, b) 2 & 8, and c) 3 & 7?	**Subtask 1** LY: a, b, c NK: a, b, c					
		Task 2 2.1 Lerato bought 10 sweets on Monday; she then buys 20 sweets on Tuesday. How many sweets has she bought in total? 2.2 Thandi ate 30 jelly beans during first break; she eats another	**Task 2:** **NT:** No activity **LY**: No activity **NK**: 2.1, 2.2, 2.3 **KT**: 2.5	Beans, Bead string, Fingers			TI GC in 10's, RFIA UC	GC in 10's, SU FC RFIA UC

		10 during second break. How many jelly beans has she eaten altogether? 2.3 Tammy bought 72 marbles. She lost 24. How many does she have left? 2.4 Nelly's mom baked 100 biscuits. Nelly and her brother ate a total of 32 biscuits. How many biscuits are left? 2.5 Brenda has 45 crayons. She gives 33 to her friend. How many crayons does Brenda have left?						

4.3.1 Partitioning (PA) and Recalled Fact Immediate Answer (RFIA)

RFIA appears to be a dominant strategy across all the 4 learners. NT is the only learner that exhibits a PA strategy. I exemplify these strategies by using NT's example where both strategies are exhibited:

(In attempting to solve ***12 + 14*** NT counted 14 beans in ones, put them in a heap. She separated 10 beans from 14 beans, counted verbally "1,2,3…10", put 10 beans in a heap, counted 12 beans in the same manner, separated 10 beans from 12 counting verbally in ones in the same manner as before. There were now 4 heaps separated from each other '2,4,10,10'). ST intervened:

ST : How many tens have you added and what do they give you?
NT: 20
ST : And when you add the left overs it gives you?...
NT: (counted on from 20, started adding the 2 beans by counting verbally) "21,22", then added 4 beans counting verbally) "23,24,25,26".

NT exhibits the PA strategy by splitting of both 12 and 14 into tens and ones and put the beans in a heap. NK exhibits the same strategy similarly to NT while LY exhibits the same strategy but puts the beans in a patterned form. And she exhibits the RFIA by giving an immediate answer of '20' when asked what she gets when adding the two tens. The other 3 learners exhibit the RFIA strategy in the same manner as NT.

4.3.2 Subitizing (SU), Unit Counting (UC), Grouped Counting (GC) in 2's

SU and UC is dominant across 3 of the 4 learners and GC in 2's and GC in 10's are used by only 2 of the learners.

I exemplify these strategies using LY's example and all the other learners exhibit the strategy in the same manner as LY except for KT who uses a bead string to group count in 2's rather than beans:

10 + 19 + 21

ST: "Show me how you got the answer to number 6, '10 + 19 + 21', use your beads to show me".
LY: (Counted out 10 beans verbally from a heap of beans, "2,4,6,8,10", (put them in a patterned form so that they looked like this):

Figure 4.3.2 Evidence of a Subitizing strategy

To add 19 and 21 using the beans LY exhibits the same strategy but when adding 9 of 19 she reverts to counting in ones, but still organises the beans in a pattern such that at the end the picture of the pattern is as follows:

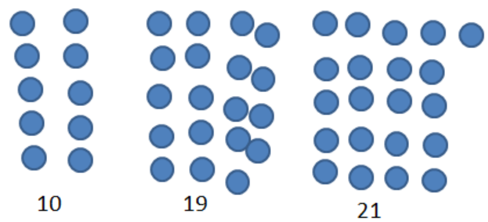

Figure 4.3.3 Consistent Evidence of a Subitizing strategy

4.3.3 Grouped Counting (GC) in 10's and Forward Counting (FC)

Brenda has 45 crayons. She gives 33 to her friend. How many crayons does Brenda have left?

(KT counted 45 beads, moving each group of 10 in one go): "10,20,30,40", *then counted* "1,2,3,4,5" adding 5 beads to 40). (Looked at the teacher) "Then I minus 33"
ST: "Why do you minus 33?"
KT: "Because she gave away 33"
ST: "Now we have to take away 33"
KT: (Counted 33 beads in the same manner as when counting 45, but counts on from 30) "31,32,33"
ST: "Now how many are left?"
KT: Counts left overs in ones, "1,2,3,4…" (teacher stopped her)

GC in 10's by KT is reflected as she counts in groups of tens "10, 20…" using the beads. FC is reflected as she subtracts 33 beans by counting forward, then what is left between 33 and 45 would be the answer which she counts in ones.

4.3.4 Concluding Summary of Session 1

In this session combined strategies were used within and across learners and tasks. Though the focus of the session was to solve addition and subtraction problems using groups of tens and ones or partitioning there was very little evidence of exhibiting this strategy. There is evidence of RFIA strategy across all learners but there is also a reflection of a lot of counting strategies. All learners used beans to represent objects, and organising beans in heaps rather than in patterned form suggested a lack of possession of a SU skill which helps learners to rapidly enumerate a set. The use of the bead string by KT also did not seem to help her with the awareness of the 10 structure and number bonds of 10. Instead she used the bead string to represent objects – which is also reflected in her reverting to unit counting.

4.4 Analysis of Learners' Strategies Emerging in Session 2

Table 4.4 Summary of Session 2

Session	**Focus**	**Task**	**Activity**	**Resources**	**Learner Strategies**			
					NT	**LY**	**NK**	**KT**
2	Game: Number bonds of multiples of 10. Pair up numbers to make up multiples of 10	Pair up any number bonds and multiples of 10 within the number range 1-30 Learners pair up numbers that are on cards which had been placed on tables throughout the room.	**NT**: 14 + '6', 16 + '4' **LY**: 27 + 28 **NK**: 27 + '3', 21 + 9 **KT**: 1 + 9	Number cards ranging between 1 and 30 Fingers	COF UC	UC COF	UC COF RFIA	RFIA COF UC

4.4.1 Count On from First (COF) and Unit Counting (UC)

COF is the most dominant strategy across all 4 learners. Since there was evidence of unit counting while exhibiting the CO strategies these will be discussed together. I exemplify the COF strategy by providing an example from NT, all the other learners exhibit the strategies in the same manner as NT:

(In pairing '14 + 6' NT grabbed a card from the table with the number 14 then *counted on from 14* using her fingers to keep track of counting verbally: "15,16,17,18,19,20". She shouted "6 ")

4.4.2 Recalled Fact Immediate Answer (RFIA)

NK and KT exhibit this strategy and an example from KT is provided below:

(KT first grabbed a number card from the table with a 1 and immediately grabbed another card with a 9 to make up 10).

4.3.3 Concluding Summary of Session 2

In this session same as in the session above, the focus of the session was to pair up number bonds of multiples of 10 and the purpose of the number cards was to promote immediate enumeration of numbers or recalled fact strategies. Only NK and KT were able to pair up numbers without counting. In most instances in this session lack of awareness of number bonds of multiples of 20 is evident as learners count using their fingers to find numbers that pair up with one other. The most dominating strategy in this session is UC while exhibiting a COF strategy. Though using fingers is associated with concrete strategies, in this session fingers were used in a more sophisticated manner as they were used to keep track of counting.

4.5 Analysis of Learners' strategies in session 3

Table 4.5 Summary of Session 3

Session	Focus	Task	Activity	Resources	Learner Strategies			
					<u>**NT**</u>	<u>**LY**</u>	<u>**NK**</u>	<u>**KT**</u>
3	Addition and subtraction problems. The ten-structure/number bonds of 10. Work with beans and bead strings to solve addition and subtraction problems	The numbers in this task are given randomly by the teacher: 1) 10 + 10 2) 58-20 3) 20 + 10 4) 50-3 5) 30-10 6) 98-60 7) 40-3 8) 28 + 17 9) 60-30 10) 50-3 11) 38 + 12 12) 12 + 38	**NT**: 1, 2, 3, 4, 5, 6, 7 **LY**: 8, 9, 11, 12 **NK**: 7, 8, 9 **KT**: 2, 8, 9, 10,	Fingers, Beans, Bead String Paper and Pencil-(student teacher is the one randomly writing down number sentences which are solved by learners).	GC in 10's, GC in 2's, SU FC	GC in 10's GC in 2's SU RFIA UC CDF SF	GC in 3's SU FC SF	GC in 10's GC in 2's SU FC SF

In this task a bead string is used as a resource and the word 'add' is used for moving or bringing beads together, and the word 'take away' is used for subtraction problems when separating beads from each other or moving the subtrahend from the minuend to get the answer.

4.5.1 Grouped Counting (GC) in 10's, Grouped Counting (GC) in 2's, Subitizing (SU)

These strategies are dominant across all 4 learners. I provide NK's example to exemplify the strategies:

28 + 17
(NK counted 28 out loud on the bead string): "10,20" then said "8" (moving 8 beads in one go), then she started counting in 2's to add 17). (ST intervened).

In this example NK exhibits all 3 strategies and other 3 learners demonstrate them in the same manner as NK.

4.5.2 Separate From (SF)

LY exhibits a CDF strategy in a subtraction problem, the example is as follows:

60-30
LY: (Counted 60 beads on bead string in tens out loud: "10,20,30,40,50,60" (pointing to a group of tens with each count). Then said "I take away 30" (LY counted verbally in tens moving the beads to the opposite direction (separating 30 beads from 60): "10,20,30" then said "the answer is 30".

4.5.3 Forward Counting (FC)

Contrary to LY, KT and NT exhibit a FC strategy in attempting to solve a subtraction problem. The example is as follows:

58-20
(KT counted 58 beans out loud in tens in one go) "10,20,30,40,50", (then 'added' 8 beads in one go 58) (teacher intervened)

ST: Now you need to take away 20
KG: (counted forward in tens subtracting 20 beads from 58) "10,20". (KG then counted
the 38 beads left in tens out loud): "10,20,30" (as she started to add 8 beans counting in 2's (student teacher intervened).

4.5.4 Concluding Summary of Session 3

It seems as though when the learners are asked to subtract 'friendly[3]' numbers (60-30), they are able to separate from or take away, but when asked to solve a problem with other numbers 58-20 they use a forward count strategy which is a less sophisticated strategy

[3] A 'friendly' number is any number in multiples of 10 which allows easy manipulation for learners

since it requires counting first the subtrahend, and then the remainder which is the answer. Only KT reflected use of this less sophisticated strategy in other instances for example when subtracting 33 from 45 in session 1, and the rest of the learners only reflected one instance. Therefore a strong claim cannot be made in their case. The bead string is a dominant resource in this session and the teacher forces the learners to use it when solving problems. Though beans and fingers are available, they are not used by the learners. Concrete counting strategies are dominant in this session and vary according to the problems being solved. It appears as if the availability of the bead string in this session promotes concrete counting rather than mental strategies.

From the next session onwards is a reflection of tasks and activities that were conducted in the second part of the year. It is important to note this as I will be looking for any shifts that can be seen as the learners progressed with the sessions.

4.6 Analysis of Learners' strategies in Session 4

Table 4.6 Summary of Session 4

Session	Focus	Task	Activity	Resources	Learner Strategies			
					NT	**LY**	**NK**	**KT**
4	'Bingo' Game: Number bonds of 20	Pair up numbers to make number bonds of 20. The student teacher calls out a number, learners pair up that number to make up 20 Learners scratch off their strip of paper the number they are pairing with the teacher's if they have it.	**NT**: 13 + 7 9 + 11 **LY**: Absent **NK**: 14 + 6 16 + 4 9 + 11 5 + 15 **KT**: 7 + 13 0 + 20	Strips of paper with 5 random numbers between 1 and 20 invented by learners, Fingers	COF UC	NA	COF UC RFIA	UC COF RFIA

It is important to note that in the activities in this session, the first addend is the number being called by the teacher, and the second addend is the number being paired up by the learner, for example (13 + 7) means a teacher called out 13 and the learner added 7 to make up 20.

In this session, only 3 learners participated. COF and UC are dominant across all learners. RFIA is exhibited by 2 of the 3 learners.

4.6.1 Count On From First (COF) and Unit Counting (UC)

I use NT's example to exemplify the strategies:

13 + 7
Teacher calls 13
NT **counted on from** 13, verbally using her fingers, she called out 13 twice ("keeping it in her head"), then she counted on out loud: "14,15,16….20" (she held out 7 fingers - fingers are used to represent counts in this instance). NK and KT exhibit the strategies in the same manner as NT.

4.6.2 Recalled Fact Immediate Answer (RFIA)

NK and KT exhibit this strategy. I provide an example from NK:

5 + 15
Teacher calls '5'
NK immediately puts up her hand and shouts '15' without counting. She scratches 15 off her strip of paper.

0 + 20
Teacher calls "0"
KT immediately shouts out '20' (without counting) she scratches out the '20' on the strip of paper.

4.6.3 Concluding Summary of Session 4

In this session what is interesting is that it appears as though it was easier for NK and KT to exhibit recalled fact strategies when dealing with 'friendly numbers' or multiples of 5 and 10 because in other examples they exhibited counting strategies as mentioned earlier. Fingers are used to keep track of counting and their use is dominant across learners and tasks. In this session there is evidence of a combination of strategies used based on the nature of the problem presented to them.

4.7 Analysis of Learner's strategies in session 5

Table 4.7 Summary of Session 5

Session	Focus	Task	Activity	Resources	Learner Strategies			
					NT	**LY**	**NK**	**KT**
5	Solving Join and Separate Problems • Emphasizing structure of tens and ones. • Linking addition and subtraction to story problems • Subtraction as take and difference • Place Value-making sense of standard column algorithm	**Task 1:** Solve the following addition and subtraction word problems. 1.1 Anna has 6 sweets. Babalwa gives her 9 more. How many more sweets does Anna have now? 1.2 Patricia has 15 sweets. Puleng has 6 sweets. How many sweets do they have altogether? 1.3 Dudu has 7 sweets. How many more does she need to have 13 sweets? 1.4 Kelebogile has 4 sweets. How many more does she need so she has 18 sweets? 1.5 Gogo has 19 sweets. She eats 6 of them. How many does she have left? 1.6 Jabu has 18 sweets. He eats some of them. There are 10 left. How many did He eat? 1.7 Lulu has 5 sweets. Maria has 9 sweets. How many more sweets does Maria have than Lulu? 1.8 Maru has 8 sweets. Nandipha has 13 sweets. How many more sweets does Nandipha have than Maru? 1.9 Thato has 12 sweets. Thaba has 18 sweets. How many more sweets must Thato get to have as many as Thaba?	**Task 1** **NT:** 1.7, 1.8 **LY**: Absent **NK**: 1.1 to 1.7 **KT:** 1.1 to 1.7	Fingers, Beans, Pen and Paper (worksheets-learners were asked to write their answers next to the questions)	SU, SF, UC, COF, CU	NA	CU UC COF CA	RFIA SF UC CDF GC in 2's CDT Matching

		1.10 Vuyo has 23 sweets. Wandile has 16 sweets. How many fewer does Vuyo have than Wandile?						
		Task 2 **Homework** 2.1 Darius has 9 sweets. Ephraim has 8 sweets. How many do they have altogether? 2.2 Patience has 24 sweets. She give 7 to her friend. How many does she have now? 2.3 Linda has 17 sweets. She gives some to her friend. Now she has 12 sweets left. How many did she give to her friend? 2.4 Kelebogile has some sweets. She gives 4 to her friend. Now she has 11 sweets. How many did she have to start with? 2.5 Patricia has 15 sweets. Puleng has 6 sweets. How many sweets must Patricia eat so she has the same number of sweets as Puleng? 2.6 Joshua has 12 sweets. Phetiwe has 15 sweets. How many more sweets does Phetiwe have than Joshua? 2.7 Sizwe has 23 sweets. Thabo has 5 less sweets than Sizwe. How many sweets does Thabo have?	**Task 2** **Homework** **NT:** No activity **LY**: Absent **NK**: 2.1, 2.2, 2.4, 2.5, 2.7, (random 23-10) **KT**: 2.6, 2.7, (random 23-10)					

4.7.1 Count On From First (COF) and Unit Counting (UC)

In this session, COF and UC are dominant across all 3 learners though other strategies seem to be exhibited by individuals. All learners exhibit these strategies in the same manner. I exemplify the strategies by providing NT's example below:

Patricia has 15 sweets. Puleng has 6 sweets. How many sweets do they have altogether? (Join-Result unknown)

NT put her pencil down, prepared her fingers, then said, verbally '15', '15' (called '15' twice, then she **counted on from** '15'…16,17,18,19,20,21', she looked up and said) '21','21' (same as above). She wrote the number sentence as 15 + 6 = 21.

4.7.2 Separate From (SF), Count Down To (CDT) and Matching

NT and KT exhibit a SF strategy. I present an example from KT:

Joshua has 12 sweets. Phetiwe has 15 sweets. How many more sweets does Phetiwe have than Joshua? (Compare-Difference Unknown)

KT: (counted out 12 beans verbally in 2's, put in a heap, then counted out 15 beans in the same manner, put in a separate heap). (From the heap of 15 beans KT counted down to 12 in 1's out loud) "15… 14,13,12". (Then she separated 3 beans from the heap of 15 beans and said "So Phetiwe has 3 sweets more than Joshua"). The picture looked as follows:

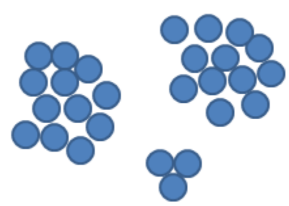

Figure 4.7.2 Evidence of a Matching strategy

4.7.3 Count UP (CU)

NT and NK exhibit a CU strategy when solving a *Compare-difference unknown* problem.

Dudu has 7 sweets. How many more does she need to have 13 sweets? (Compare-difference unknown)

(NT started by counting out a heap of beans and counted on from 7 verbally: '8,9,10,11,12,13' (stopped at 13, and had counted out 6 beans). NT did not write a full number sentence, but she wrote on her paper: ‘3. 6. (‘3’ represents ‘number 3’ and ‘6’ represents the answer). NK exhibits the strategy in the same manner as NT.

4.7.4 Count Down From (CDF)

KT exhibits a CDF strategy seen in the example below:

Sizwe has 23 sweets. Thabo has 5 less sweets than Sizwe. How many sweets does Thabo have? (Compare-Difference Unknown)

ST: What's the easier way to do question 7?
KT: The easier way is to say 23 minus 5
ST: To count down?
KT: Yes, I start from 23 going down
ST: How do you do it?
KT: (raised a finger with each count while counting down from 23 in 1's out loud) “23…22,21,20,19,18”

4.7.5 Count All (CA)

NK exhibited this strategy in the example below:

Sizwe has 23 sweets. Thabo has 5 less sweets than Sizwe. How many sweets does Thabo have?

ST: How did you get 18?
NK: “I said 23 minus 5 (prepared beans to count with and started counting in ones): “1,2…20” (she put them in a heap). “Then I said 1,2,3,4,5” (NK separated 5 beans from the heap of 20 and put the 5 beans in a heap and counted the remaining beans in ones): “1,2,3…15” (there are 15 beans remaining, she stopped and stared at the beans)
ST: How many beans do you have there altogether?
NK: (counted all the beans again starting with the heap of 5 to the heap of 15): “1,2,3,4…20”(she added 3 more beans by counting on from 20, adding to the heap of 15 beans such that she has now a heap of 18 beans): “21,22,23”. (She counted all the beans in the heap of 18 beans in 1’s): “1,2,3…18”

4.7.6 Concluding Summary of Session 5

A combination of strategies is evident as per sessions discussed earlier. UC appears to be dominant across learners and tasks. KT reflects sophistication as she exhibits a matching strategy however, though she reflects sophistication in the manner she exhibits the strategy, organising the beans in a patterned form would have provided a better sense of enumeration and would have eliminated the process of counting. Resources made available to learners were used in a combined manner depending to what was available at the point of solving a problem. For example fingers were used to keep track of counting.

In some instances such as KT's case there was an awareness of representation of quantities using the beans, and in some instances beans were used to represent objects.

4.8 Analysis of Learners' Strategies in Session 6

Table 4.7 Summary of Session 6

Session	Focus	Task	Activity	Resources	Learner Strategies			
					NT	**LY**	**NK**	**KT**
6	Solving Join, Separate, Compare Problems a) Place Value b) Connections between words, actions, pictures, symbols	**Task 1** Solve the following addition and subtraction word problems using 'Maths Club Money': 1.1 I have R23 and you have R45. How much money do we have altogether? 1.2 I have R34 and you have R52. How much money do we have altogether? 1.3 I have R36 and you have R25. How much money do we have altogether? 1.4 I have R27 and you have R18. How much money do we have altogether? 1.5 You have R67 and you give me R23. How much money do you have left? 1.6 You have R43 and you give me R11. How much money do you have left? 1.7 You have R42 and you give me R28. How much money do you have left? 1.8 You have R56 and you give me R39. How much money do you have left? 1.9 I have R46 and you have R54. How much more money do you have than me? 1.10 I have R35 and you have R52. How much more money do you have?	**Task 1** **NT**: 1.4, 1.5, 1.8, 1.10. **LY**: 1.1, 1.4 **NK**: 1.1, 1.2, 1.3, 1.4, 1.8 **KT**: 1.2, 1.3, 1.4, 1.5	Fingers, R10 and R1 notes , pen, paper	PA UC COF SU GC in 10's	PA RFIA	PA RFIA PV SA UC COF JT	PA UC COF
		Task 2 **Homework** 2.1 I have R24 and you have R45. How much do we have altogether? 2.2 I have R46 and I give you R13. How much I do I have left? 2.3 I have R57 and I earn another R64. How much I have now? 2.4 I have R53 and I buy a book for R24. How much do I have now?	**Task 2** **Homework** **NT**: No activity **LY**: 2.1, 2.2, 2.7 **NK**: No activity **KT**: No activity	R10 and R1 notes , pen, paper				

		2.5 I have R37 and you have R61. How much more money do you have than me?							
		Task 3 **Hundreds** 3.1 I have R133 and you have R245. How much money do we have altogether? 3.2 I have R236 and you have R357. How much money do we have altogether? 3.3 You have R267 and you give me R123. How much money do you have left? 3.4 You have R267 and you give me R138. How much money do you have left? 3.5 You have R267 and you give me R173. How much money do you have left? 3.6 You have R267 and you give me R178. How much money do you have left? 3.7 You have R235 and I have R230. How much more money do you have than me? 3.8 You have R235 and I have R228. How much more money do you have than me? 3.9 You have R235 and I have R123. How much money do you have than me? 3.10 You have R235 and I have R157. How much more money do you have than me?	**Task 3** **Hundreds** **NT:** 3.1, 3.2, 3.3, 3.4, **LY:** Absent **NK:** Absent **KT:** Absent	R10 and R1 notes , pen, paper					

4.8.1 Partitioning (PA)

This strategy is dominant across all 4 learners and all learners exhibit this strategy in the same manner. This strategy was exhibited when some of the learners used the ‘maths club money’ [4] to represent the 10’s and 1’s. It was also used when some of the learners used a formal writing method. Examples are presented below:

Example 1: Using the ‘Maths Club Money’

I have R27 and you have R18. How much money do we have altogether? (Join-Result Unknown)

ST: “How did you get R45 as the answer? Did you use the cards?”
NT: “Yes”, (taking the ‘R10’ cards) “I said 2 tens” (she put 2 ‘R10’ cards on the table) “and 1 ten” (she put 1 ‘R10’ card on the table), “and then I count 7” (unit counted 7 ‘R1’ cards out loud) “1,2,3,4,..7” (she put them in a heap, she then unit counted 8 ‘R1’ cards). Started counting on from 30, counting in 1's from the 7 ‘R1’cards, 31,32,33 (Student Teacher intervened)

Example 2: Using a formal writing method

I have R27 and you have R18. How much money do we have altogether? (Join-Result Unknown)
LY wrote down the number sentence: 20 + 10 + 7 + 8, added tens together, then added units together (see figure 4.7.2.2)

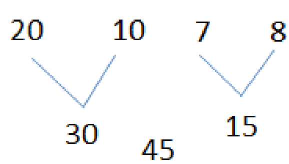

Figure 4.8.1 Evidence of a Partitioning Strategy

4.8.2 Count On From (COF) and Unit Counting (UC), Standard Algorithm (SA)

The COF and UC strategies are dominant across 3 of the 4 learners. The SA strategy was exhibited by NK and KT of the learners. I present two examples to demonstrate how the strategies were exhibited:

Example 1: Using the column method
I have R27 and you have R18. How much money do we have altogether?

[4] ‘Maths Club Money was used as a resource to help learners using column addition or subtraction to get an idea of place value, examples are provided in the appendices.

NK used the ‘column addition method’, she wrote down her sum as follows:

$$\begin{array}{r} {}^{1}2\,7 \\ \underline{1\,8} \\ 4\,5 \end{array}$$

Figure 4.8.2 Evidence of a Standard Algorithm method

NK started by adding the units 7 and 8, counted 7 R1 verbally in ones, then counted on from “8, 9, 10…15” (used the R1 cards as she counted on). NK put the answer 5 underneath the 'units column', then put the 1 representing a ten next to the 2 on the 'tens' column. She added 2 plus 1 plus 1 in the 'tens column' by unit counting the ‘R1’ cards and got 4, the answer was 45.

Example 2: Using Maths Club Money

I have R34 and you have R52. How much money do we have altogether? (Join-Result Unknown)

(KT Wrote down the number sentence as R34 + R52=
(She counted 8 ‘R10’ cards in a heap and got R80, she then counted on from R80 out loud) “81,82…86”, she looked at the teacher and said “86”, then she wrote R86 as her answer).

4.8.3 Recalled Fact Immediate Answer (RFIA)

LY and NK exhibited these strategies. I present example from each of them since they used different approaches but used the same strategy.

NK’s example: *I have R23 and you have R45. How much money do we have altogether? (Join-Result Unknown)*

NK wrote her sum as follows:
R68
R20 + R40=R60
R5 + R3= R8
(NK first wrote R20 + R40 = 60, below she wrote R5 + R3=R8, then she wrote her answer R68 at the top). So R68 as answer rests on some RF – for R20 +R40 and R5 + R3.

LY’s example: *I have 23 and you have R45. How much money do we have altogether? (Join-Result Unknown)*

LY: (wrote down the number sentence: 40 + 20 + 3 + 5-left out the ‘R’ that indicates that she is working with money, she indicated that she is GROUPING tens together and units together, added 40 + 20 and got 60, then added 3 + 5 and got 8, added 60 plus 8 and got 68 [see figure 4.7.2.1 below]).

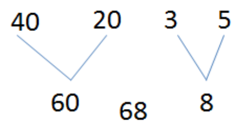

Figure 4.8.3 Evidence of a Recalled Fact strategy

4.8.4 Subitizing (SU) and Grouped Counting (GC) in 10's

These strategies were exhibited by one learner of the three learners and the example is as follows:

You have R56 and you give me R39. How much money do you have left? (Separate-Result Unknown)

ST: "So what must you do?"
NT: "You take away R39 from R56"
ST: "I wanna see your answer"
NT: (counted 5 'R10' cards and put them in a patterned form, one below the other, counting out loud as she outlined cards) "10,20,30,40,50". (She counted 6 'R1' cards adding them onto the 5 'R10' cards. The picture that results from NT's process is as below:

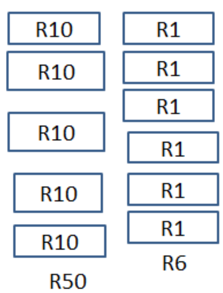

Figure 4.8.4 Subitizing using MC money

NT then removed 3 'R10' cards and put them on the side and was left with R26: 2 'R10' cards and 6 'R1'cards). Then she still has to subtract R9 from R26 and teacher asks NT to replace one R10 with 10 R1 cards. Then she removed 9 R1 cards and was left with R17.

4.8.5 Concluding Summary of Session 6

In this session LY exhibits uses the same approach (written method) to solve the problems and she exhibits a RFIA and PA strategies with no counting involved at all.

While with NK, what is noted is that when she uses a written approach she exhibits the same strategies and does not involve counting at all, but when she uses a column method as per figure 4.7.3.1 above, then she reverts back to counting strategies. It is also interesting that with the written method, both NT and NK do not use the ‘maths club money’, but when using the column method the ‘maths club money’ is used. Also, it seems as though SU is associated with certain objects as it was easier for NT to put the ‘maths club money’ into a patterned form than it was with the beans. Though the R1 cards were made available to promote breaking up and making up a R10, it appears as though it promoted concrete counting strategies since the learner had to count in 1’s when breaking and making up a R10. It appears as though because there is not an easy way of escaping counting in ones when R1 cards or objects that have one or single units (for example bead strings) are made available to learners to solve problems in spite of an awareness of bonds of 10 that have been seen in the context of other tasks and resources. This suggests that the approach used to solve the problem is localised within the kind of problem being solved and or the physical representation or resource that is available. Also it appears as though the availability of some resources such as beans and bead string promote use of concrete counting strategies.

4.9 Analysis of Learners’ strategies in session 7

Table 4.9 Summary of Session 7

Session	Focus	Task	Activity	Resources	Learner Strategies			
					NT	**LY**	**NK**	**KT**
7	Game: Mental computational strategies	Solve addition and subtraction problems on a piece of paper as fast as possible. 1. 9 +3 2. 20 + 4 3. 18 + 7 4. 17 + 6 5. 18 + 14 6. 26-6 7. 23 + 46 8. 20-8 9. 47-32 10. 62-28 11. 19-17 12. 18-14 13. 17-4	**NT**: 1,2,3,4,5, 8 **LY**: 6, 7, 8, 10 NK: Absent KT: Absent	Fingers, Pen and Paper	UC COF	UC CDF RF PA GC in 10’s	NA	NA

4.9.1 Count On From First (COF), Unit Counting (UC), Count Down From (CDF)

UC is dominant across the 2 learners. UC is associated with COF as observed in the sessions discussed above. UC and COF is exhibited similarly to the previous sessions discussed. I exemplify the CDF strategy which has not been discussed previously as follows:

20-8
(LY held her hands with her fingers facing upwards, she counted in ones while counting down from 20 moving one finger with each count. She was counting verbally but very softly. She wrote down 12 as her answer).

4.9.2 Recalled Fact Immediate Answer (RFIA), Partitioning (PA), Grouped Counting (GC) in 10's

These strategies were exhibited by LY similarly to the sessions already discussed. In one example in this session LY reverts back to counting using fingers, though fingers are used to keep track of counting while previously she had exhibited a RFIA strategy throughout session 6. She does exhibit a RFIA when giving "60" as an answer, but when adding units she reverts back to counting.

23+46
ST: "What is 20 plus 40?"
LY: "60"
ST: "What is 3 + 6?"
LY: (used fingers to count on from larger, she counted out loud) "7,8,9" (wrote 9 next to 60)
ST: "What's 60 plus 9"
LY: "69" (wrote her answer down)

4.9.3 Concluding Summary of Session 7

UC is dominant in this session across learners though other strategies are also used. There continues to be a combination and a back and forth use of strategies within examples and sessions. Similarly to other sessions fingers are used to keep track of counting.

4.10 Summary

In this chapter I have first discussed the presentation and categorisation of my data where I begin by describing how my data was collected and it was organised. I also present and discuss themes that emerged from the data which were used as guidance for my analysis. I used a table with categories that have helped me structure the discussion of my analysis per session and per learner. In discussing the findings I have used sessions in which I have discussed use of physical representations and student teacher intervention which have helped me answer my second research question. The findings show that there appears to be a combination of use of strategies by each learner within and across sessions. There is however a dominant reliance to unit counting and more concrete strategies and this concurs with what Ensor et al. (2009) found in the study with South African learners. The difference in our findings is that the learners in Ensor et al. (2009) relied on tally counting, while in my study there was no evidence of tally counting but using objects to count in 1's. This is mostly related to the availability of UC resources because we have seen some evidence of using more sophisticated strategies when they were either not made available or not used by learners (see Figure 4.8.3 above). There seems to be a fairly widespread inability to use number bonds to 10 to avoid UC when in between 'friendly' numbers, even though they have shown some awareness of these bonds when asked specifically about them. So there is quite a strong evidence of localisation of tasks rather than use of bonds of 10 in other task contexts.

Chapter 5-Conclusions

5.1 Conclusions

In this chapter I conclude the research by discussing the findings from my analysis. I discuss the findings by revisiting the questions my study sought to answer. I then discuss implications for teaching and learning and limitations of the study, and lastly I end with recommendations for policy makers.

5.1.1 Findings

I begin by answering the first research question**: 'What addition and subtraction strategies do learners display at the start of the Maths Club?'**

In order to answer this question from the video data I use the first 2 sessions to present strategies learners displayed at the start of the Maths Club.

In session 1 the learners displayed a combination of strategies such as RFIA, PA, SU, UC, GC in 2's, FC. The RFIA appeared to be dominant across all the learners. There was very little evidence of PA since the focus of the session was using groups of 10's and 1's to solve addition and subtraction problems. The ways in which learners used beans reflected lack of awareness of using objects in structured forms – i.e in patterned forms based on groups of 5 or 10. Rather they simply placed beans in unit counted heaps. This pointed to an inability to ***model for representing*** number in structured forms (Gravemeijer & Stephan, 2002). There is a connection between GC and SU, the difference is SU involves GC, but GC might not involve SU because a child might not organise the groups in patterns such that they are able to rapidly enumerate a set without having to count (Anghileri, 2006).

In session 2, the focus of the session was to pair up number bonds of multiples of 10 and number cards were made available to the learners in order to promote immediate enumeration (Anghileri, 2006) of numbers or recalled facts or derived facts. The strategies that were evident are UC while COF and RFIA with UC and COF being dominant. Only NK and KT were able to pair up numbers without counting though the use of counting strategies was dependant on the kind of task and resources made available. There was no evidence of consistency in other tasks either as they reverted to less sophisticated counting strategies. Though using fingers is associated with concrete strategies, in this session fingers were used in a more sophisticated manner as they were used to keep track of counting (Carpenter et al., 1999; Thompson & Smith, 1999)

At the beginning of Grade 3 as mentioned earlier, learners should be exhibiting more sophisticated mental strategies according to the curriculum (DBE, 2011a), but what is reflected in the findings based on sessions 1 and 2 is contrary to what is expected.

Next I answer the second research question**, 'What addition and subtraction strategies and resources does Maths Club suggest to learners?'**

Across the different sessions and in the discussions that took place prior to the MC sessions certain resources and strategies were discussed across the group. In this section and in order to answer this research question from the video data I infer the links between the strategies and resources that were promoted by both the available resources to learners and the 2 student teachers working with the focal group of learners. Whilst the Maths Club offered resources to the learners such as beans, bead strings, maths club money, fingers, the ST's did not always make efficient connections between the resources and the strategies they offered to the learners. For example where ST's suggested a GC in 10's strategy, they did not direct the learners to organise the beans in patterned form so that they were able to rapidly recognise a group of 10. Where the use of beans is concerned the ST's tended to unit count when demonstrating strategies to learners. This transferred to the learners as they unit counted mostly when they used beans to solve problems. When learners organised the beans in heaps the ST's did not model or encourage the learners to organise the beans in patterned form which would promote grouped counting in 10s as was the objective of most sessions. On the contrary, though the use of structured representations of number of beans was not promoted by the ST's effectively, there was some evidence of pointing the learners towards recognising the 10 structure on the bead string as they stopped the learners from counting in one's and promoted using compliments of 10. In session 5 one of the ST's introduces a Part-Part Whole strategy to solve what according to Carpenter et al. (1999) is classified as a

Compare-Change Unknown problem. It took a long time for the learners to grasp the concept. The ST did not repeat this strategy in any other context to promote its continuous use. The ST's offered models such as the 'empty number line' and strategies such as COF, CDF, PA, Sequencing (N10). Some of these strategies differed in their level of sophistication though most of them promoted concrete counting strategies. This finding is related to the work of Ensor and colleagues (2009) which presents evidence of teaching that focuses on concrete counting based strategies rather than promoting shifting over time towards more abstract calculation based strategies. This tendency results in holding learners back.

Next I answer the third research question, **'What shifts in addition and subtraction strategies can be seen during the course of the Maths Club?'**

The analysis points to very little evidence of shifting from use of concrete strategies to more sophisticated abstract strategies. Findings point to a reliance and dominance of unit counting by learners especially when beans and bead strings were used. It is very interesting that beans are 'unstructured resources' and a bead string is a 'structured resource'-but one that makes tens physically 'separable'. This contrasts with the R10 notes which do not allow for this. Learners work with these resources indicated that they were able to solve addition and subtraction problems using grouped counting in 10s in the context of the R10 notes, but unable to achieve this when working with beans or bead strings. The 'competence' with grouped counting using the base 10 structure was 'localised' within resources that did not allow for any other option This suggests the need for much more careful planning of what resources are available, and when, and in what sequence, and how they are connected. Venkat and Askew (2012) show teachers doing similar unstructured unit counting even when using structured resources such as the abacus and 100 square. Learners in my study tended to use whatever resource was available at that point. When the resources were not made available to them they did not make an effort to look for them, or use the structured counting they illustrate as internatlised tools for calculating. This reliance on concrete strategies is related to what Ensor et al. (2009) found in their study with South African learners.

It was also evident that the PA or TT strategy as described by Thompson (1999b) did not seem to work effectively as the learners tended to revert to unit counting when regrouping the numbers. This observation is in agreement with (Beishuizen et al., 1997; Klein et al., 1998) who found that the sequencing (N10) strategy resulted in fewer errors and enabled making efficient computation choices.

In some research (Carpenter et al., 1999; Thompson, 1999a) as discussed in Chapter 2, it is stipulated that over time modeling strategies are replaced with counting strategies which are then replaced with number fact strategies. I would like to argue that 'over time' in this case would refer to progression across FP grades which suggests that if these authors' hypothesis is true, children at Grade 3 should have replaced at-least modeling strategies with counting strategies. Findings from my study reflect a contrast to this 'replacement' of one strategy with another as children tended to revert to unit counting even whilst executing abstract strategies. In order to help children shift from use of concrete strategies to more abstract strategies Askew and Brown (2003) suggest that a limited number of key representations are to be used rather than a wide range of representations, for example in my study learners were offered beans, bead strings, which did seemed to promote more use of modeling rather than mental reasoning. Use of multiple representations in the form of physical objects in my study did not seem to help the learners shift from use of concrete strategies to more abstract strategies in that when they were presented with either an addition or subtraction sum to solve they did not connect the previous use of a representation to an appropriate strategy to use for the problem. Though some models were used and made available to the learners, when learners were presented with problems to solve, they did not look for appropriate models to use and when they did not use them appropriately but reverted to unit counting. The way the learners used the models in relation to strategies is described as being 'localised' (Venkat & Askew, 2012).

Once learners reflect being comfortable with use of abstract strategies they can shift to using formal written methods. If the learners have not mastered for example, place value, they will use the formal written methods inappropriately. This is especially evident in my study as the use of formal written methods reflected promotion of use of concrete strategies. This finding concurs with Fuson's (1992) findings that U.S. children demonstrated inadequate place-value and multi-digit addition and subtraction knowledge in grade 3. This therefore suggests that learners should first be introduced to mental calculation methods so that mathematical reasoning is not lost (Beishuizen & Anghileri, 1998; Fuson et al., 1997) when using formal written methods to solve addition and subtraction problems.

5.1.2 Implications for teaching and learning

Findings from this study reflect implications for both teaching and learning. Learners at grade 3 level need to be helped to progress from concrete use of strategies to more abstract use since according to Ensor et al. (2009, p.8) 'the inefficiency of progression from concrete methods to more abstract has been offered as a significant contributor to poor mathematics achievement of students in SA schools'. One of the ways to help learners with this progression appears to be the incorporation of appropriate resources into the execution of strategies which implies that teachers need to be much more careful in their planning of what resources are available, when and in what sequence (Venkat & Askew, 2012). In addition to the incorporation of appropriate resources emphasis on structure such as 10 structure on a bead string or beans for example would need to be a focus so that learners may be able to shift from modeling of a situation to modeling for representation of mathematical reasoning.

Evidence of inappropriate use of formal algorithms by learners imply that learners should first understand mental strategies before attempting to apply them in formal written work. In addition, learners need to be able to describe and justify strategies they are using to solve addition and subtraction problems without focusing on producing the correct answer which does not promote thinking and use of strategies.

5.1.3 Limitations of the Study

This study was based on pilot Maths Club that was conducted by inexperienced second year B Ed student teachers and therefore there were a lot of limitations regarding the tasks that were prepared since the Maths Club was linked to a Methodology course and one of the objectives was to apply what they had learnt in the course. In that regard I could only work with what was projected. Though the plan was to use a variety of instruments to collect my data such as video, learner workbooks, field notes, and written work, I later discovered that I could not get any rich data from either their workbooks, written work or field notes because in the learners' workbooks the learners only wrote answers and did not show strategies they used. Field notes were only helpful in generalization of the study across the entire MC group as student teachers did not refer to any specific children while giving feedback of the sessions. My study is therefore based on one data source which is video recording. One of the limitations during the video recording of group of learners in my study was that I could not capture all the learners at the same time so I opted to capture a learner when they were describing their strategies to the student teacher. Lastly, my study is case study based on a sample of 4 learners and 2 student teachers, and therefore cannot be generalized to other settings.

5.1.4 Recommendations

The study has the following recommendations to make for **further research**:

- The introduction of appropriate models in conjunction with addition and subtraction strategies in early foundation phase needs to be investigated.
- 'How children can be mediated effectively to replace over time concrete strategies with more abstract strategies needs to be explored.

Recommendations for Policy

- Policy needs to pay attention into ways that teachers can use resources in ways that help shift learners from unit counting to abstract reasoning.

Recommendations for Practice

- Mathematics educators need to promote use of more sophisticated strategies to the learners right from Grade 1 so that they can be familiar with abstract reasoning as they progress to higher grades. They need to support the appropriation of these strategies into tools for thinking with that extend beyond the physical presence of the resources.

- Foundation Phase educators need to equip themselves regarding knowledge of strategies and links to resources so that they can effectively help their learners exhibit efficient strategies.
- Foundation Phase mathematics educators need to be trained and supported in the area of addition and subtraction strategies and resources.

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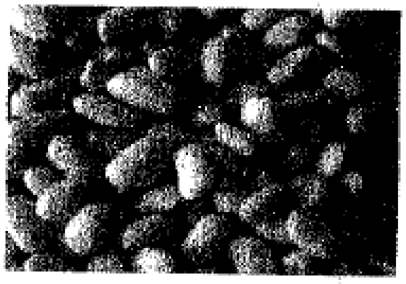
Appendices

Appendix A – Session 1 Tasks

<u>Task 1</u>

Counting beans

You will need:



Make groups of tens and ones. Then do the sums. Use the beans to help you. Follow the steps in the example below.

Example 20 + 31

Count out 25 beans. Then count out another 31 beans.

```
00000      00000
00000      00000
00000      00000
00000      00000
00000      00000
           0
```

Sort each group into tens and those left over.
For 25 you should get:

```
00000 + 00000  + 00000      25 = 10 + 10 +5
00000   00000
```

For 31 you should get:

```
00000 + 00000 + 00000 + 0     31= 10 + 10 + 10+ 1
00000   00000   00000
```

Add all the groups of tens and then add all the groups of ones
10 + 10 +10 + 10 +10 =50 and 5 + 1 = 6 so25 +31 = 56

Now do the following sums on your own:

1. 12 + 14	4. 18 + 62	7. 35 + 24
2. 23 + 24	5. 27 + 12	8. 24 + 23
3. 24 + 35	6. 10 + 19 + 21	9. 14 +12

What do you notice with number 1 and 9?

What do you notice with number 2 and 8?

What do you notice with number 3 and 7?

Task 2

SHOPPING PROBLEMS

Use your bead string to solve the following word sums.

1. Lerato bought 10 sweets on Monday; she then buys 20 sweets on Tuesday. How many sweets has she bought in total?

2. Thandi ate 30 jelly beans during first break; she eats another 10 during second break. How many jelly beans has she eaten altogether?

3. Tammy bought 72 marbles. She lost 24. How many does she have left?

4. Nelly's mom baked 100 biscuits. Nelly and her brother ate a total of 32 biscuits. How many biscuits are left?

5. Brenda has 45 crayons. She gives 33 to her friend. How many crayons does Brenda have left?

Appendix B – Session 3 Tasks

GlenhazelMaths club 12 March 2012

1. **Collect in homework for marking**

Designate 1 of you to mark the homework while the other plays the game with the learners.

2. Game to start:

a) Play spill the beans with 10 beans in the cup.

Let each learner have a chance to be the "spiller". Let the others get the number as quickly as possible.

Play it one round with them all shouting out loud.

Play another round where each of them must write their answer on a piece of paper and put that piece of paper upside down in front of them. When everyone has their piece of paper in front of them, look at the answers and see who was right. Get the ones who have it right to explain how they worked it out.

The second version of the game you should keep a written record for the table.

On the table	Under the cup
3	7
4	6
Etc	

b) Play spill the beans with 20 beans in the cup.

Do the same again but this time with 20 beans in the cup.

Let everyone look at the written records. Try to get learners to see the pattern i.e. 3 goes with 7 in version (a) and 3 goes with 17 in version (b) etc.

3. Give back homework

If there are any things which many learners have done wrong or misunderstood in the homework then explain that to them briefly.

4. **Working with the bead strings**

Give each learner a bead string.

Ask them the following questions

a) How many blue beads are there in each group. How many orange/ red beads in each group.

b) How many beads are there on the string (make sure they count up in 10s not in 1s for this)

c) i) On your bead string make the number 12. Ask them "what number am I showing you?"

ii) On your bead string make the number 23. Ask them "what number am I showing you".

iii) Do the same for a few other numbers.

d) Ask them to show you 14, 25, 63 on their bead strings.

e) Get them to use the bead string to count up in 10s

i) start at 0 ii) starting at 2 iii) starting at 24

f) Get them to start at a number (e.g.) 32 and count back 15 steps using the bead string as a help. Do this for a few numbers. (The idea here is just for them to practice counting backwards because you noticed quite a few students couldn't do this in the first session).

g) Get them to use the bead string to do the following additions:

i) 26 + 8

ii) 26 + 18 (here try to encourage them to do 26 + 10 + 8 or something similar. Not to count up in 1s)

iii) 26 + 28

iv) 26 + 38

Make them record their answers on a piece of paper (they must write in full i.e. 26 + 8 = 34)

h) If there is time do some more "strings" of sums

By a string of sums I mean something like 34 + 7; 34 + 17; 34 + 27 etc.

Appendix C – Session 5 Tasks

Glenhazel Maths Club: Second half of the year

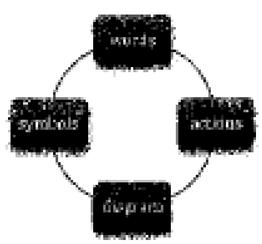
We will be focusing on making sense of addition and subtraction using join and separate problems. We will also look at emphasizing the structure of numbers (tens and units) and seeing how this works in doing subtraction (and addition).
In doing this we want to make sure that we

- Link subtraction and addition
- Give the learners are good understanding of what subtraction and addition mean – linking it to story problems.
- Emphasize both the idea of subtraction as take away and subtraction as difference
- Provide a good understanding of place value so that the standard column algorithm can make sense.

First few weeks: JOIN AND SEPARATE PROBLEMS

In the first week we will focus on "join" and "separate" problems. (see the readings for a discussion of these). We are doing this to build a connection between addition and subtraction, help learners see the meaning of addition and subtraction and give both an understanding of subtraction as take away and subtraction as difference.

We are going to be using story problems. We want the learners to **make sense** of the stories – so if you just tell them "OK so what you obviously need to do here is subtract" you are taking away all the thinking we want them to do. We want them to be making connections (remember the reading about the connectionist teacher from the 1st week?) and one set of connections we want them to make are those between the following



WORKSHEET FOR MATHS CLUB 23 July 2012

1. Anna has 6 sweets. Babalwa gives her 9 more. How many sweets does Anna have now?
2. Patrica has 15 sweets. Puleng has 6 sweets. How many sweets do they have altogether.
3. Dudu has 7 sweets. How many more does she need to have 13 sweets?
4. Kelebogile has 4 sweets. How many more does she need so she has 18 sweets?
5. Gogo has 19 sweets. She eats 6 of them. How many does she have left?
6. Jabu has 18 sweets. He eats some of them. There are 10 left. How many did He eat.
7. Lulu has 5 sweets. Maria has 9 sweets. How many more sweets does Maria have than Lulu.
8. Maru has 8 sweets. Nandipha has 13 sweets. How many more sweets does Nandipha have than Maru?
9. Thato has 12 sweets. Thaba has 18 sweets. How many more sweets must Thato get to have as many as Thaba?
10. Vuyo has 23 sweets. Wandile has 16 sweets. How many fewer sweets does Vuyo have than Wandile?

Homework

1. Darius has 9 sweets. Ephraim has 8 sweets. How many do they have altogether?
2. Patience has 24 sweets. She gives 7 to her friend. How many does she have now?
3. Linda has 17 sweets. She gives some to her friend. Now she has 12 sweets left. How many did she give to her friend?
4. Kelebogile has some sweets. She gives 4 to her friend. Now she has 11 sweets. How many did she have to start with?
5. Patricia has 15 sweets. Puleng has 6 sweets. How many sweets must Patricia eat so she has the same number of sweets as Puleng?
6. Joshua has 12 sweets. Phetiwe has 15 sweets. How many more sweets does Phetiwe have than Joshua?
7. Sizwe has 23 sweets. Thabo has 5 less sweets than Sizwe. How many sweets does Thabo have?

Appendix D – Session 6 Tasks

Maths Club 6 August

In this class we will try and help them with strategies for doing calculations.

One method you can use is the Maths Club money. Review the handout from 16 July for how to use the money. (I've reprinted some of it below so you can read it here).

The Maths Club money is meant to help learners who are still drawing individual dots and counting them or are counting on on their fingers. It is also intended to help the learners who are using column addition or subtraction but doing it incorrectly. If a learner is doing column addition or subtraction efficiently you do NOT need to force them to do it using money. Also if a learner is using a number line or bead string well (i.e. not counting one by one) then you do NOT need to force them to do it using money.

Try out the sums for yourself using the Maths Club money. How can you make the connections with the way the learners have been doing the work? How can you make connections with where learners need to get to?

One idea that we will play with over a few weeks is using "Maths club money" to help them get an idea of place value and its use in calculations. The "Maths club money" we start with has two types of notes a R10 note and R1 note. You play the role of banker and have a packet with ten R1 notes in it and you also have a R10 note. On the table there are other R10 notes and R1 notes. You first need to establish with learners that ten R1 is the same as one R10 and that they can happily swop one R10 note at the bank for ten R1 notes and vice versa. Then working with two learners (I'll call them Anna and Buhle) you give them questions to act out as follows:

Example 1:
Anna has R57. (Let her take R57 from the table).
She must give Buhle R20. (Let her give Buhle R20)
How much does Anna have left?

Example 2:
Anna has R46 (Let her take R46 from the table).
She must give Buhle R23 (Let her give Buhle R23)
How much does Anna have left?

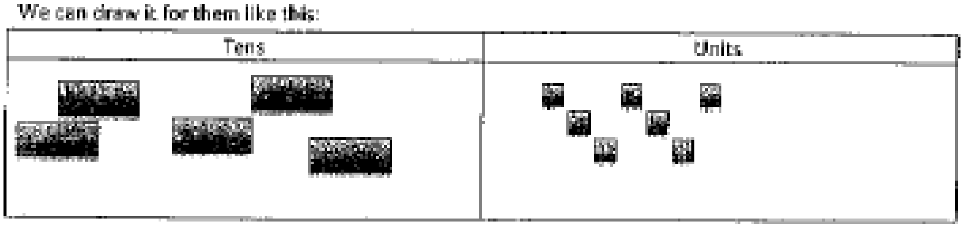
Example 3:
Anna has R62 (Let her take R62 from the table)
She must give Buhle R7 (Here you need to let them think about what to do – hopefully they'll come up with the idea that they give the bank R10 and get ten R1 notes in exchange).
How much does Anna have left?

Now again we can work on helping our learners make connections.
First of all we can draw similar pictures to model the word sums e.g. for example 1:

57	
20	?

And we can write 57 – 20 = ?

But the main point we want them to see here is what they've done in the calculation
We can draw it for them like this:



Maths Club 6 August

1. I have R23 and you have R45. How much money do we have altogether?

2. I have R34 and you have R52. How much money do we have altogether?

3. I have R36 and you have R25. How much money do we have altogether?

4. I have R27 and you have R18. How much money do we have altogether?

5. You have R67 and you give me R23. How much money do you have left?

6. You have R43 and you give me R11. How much money do you have left?

7. You have R42 and you give me R28. How much money do you have left?

8. You have R56 and you give me R39. How much money do you have left?

9. I have R46 and you have R54. How much more money do you have than me?

10. I have R35 and you have R52. How much more money do you have than me?

Homework

1. I have R24 and you have R45. How much do we have altogether?

2. I have R46 and I give you R13. How much do I have left?

3. I have R57 and I earn another R64. How much do I have now?

4. I have R53 and I buy a book for R24. How much do I have now?

5. I have R37 and you have R61. How much more money do you have than me?

Maths Club 13 August

1. I have R133 and you have R245. How much money do we have altogether?

2. I have R236 and you have R357. How much money do we have altogether?

3. You have R267 and you give me R123. How much money do you have left?

4. You have R267 and you give me R138. How much money do you have left?

5. You have R267 and you give me R173. How much money do you have left?

6. You have R267 and you give me R178. How much money do you have left?

7. You have R235 and I have R230. How much more money do you have than me?

8. You have R235 and I have R228. How much more money do you have than me?

9. You have R235 and I have R123. How much more money do you have than me?

10. You have R235 and I have R157. How much more money do you have than me?

Homework

1. I have R224 and you have R145. How much do we have altogether?

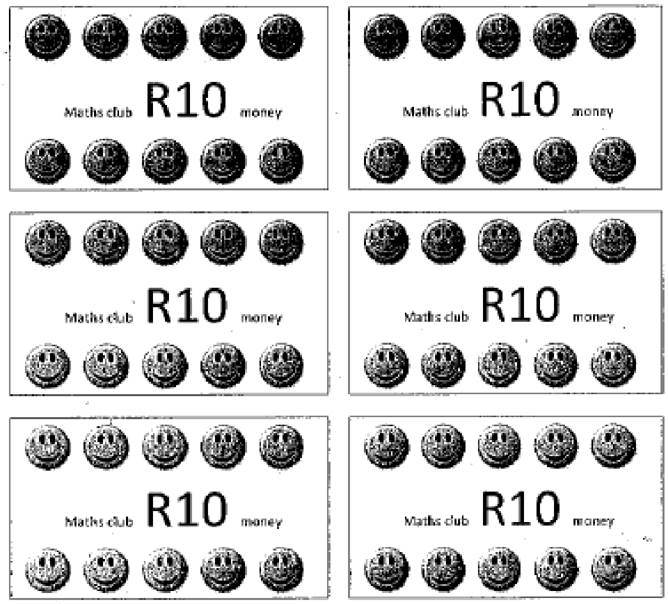
2. I have R246 and I give you R113. How much do I have left?

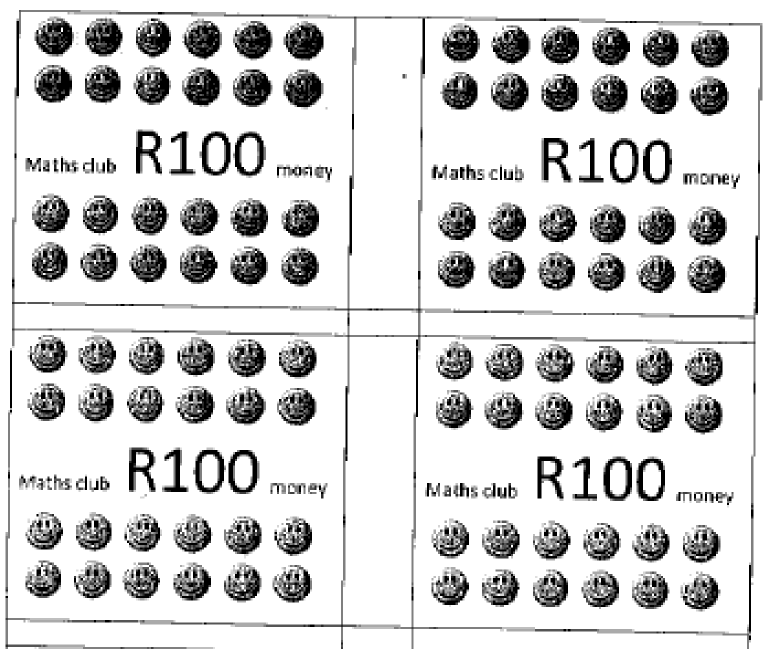
3. I have R257 and I give you R168. How much do I have left

4. I have R253 and you have R247. How much more money do I have than you?

5. I have R347 and you have R168. How much more money do I have than you?

Appendix E – Maths Club Money





Appendix F – Ethics Clearance Letter

Wits School of Education

27 St Andrews Road, Parktown, Johannesburg, 2193 Private Bag 3, Wits 2050, South Africa
Tel: +27 11 717-3064 Fax: +27 11 717-3100 E-mail: enquiries@educ.wits.ac.za Website: www.wits.ac.za

Student Number:
696188

Protocol Number:
2012ECE120

Date: 19-Aug-2012

Dear Thulelah Takane

Application for Ethics Clearance:

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate has considered your application for ethics clearance for your proposal entitled:

Investigating addition and subtraction strategies of grade 3 learners within the context of a Maths Club

The committee recently met and I am pleased to inform you that clearance was granted. The committee was delighted about the ways in which you have taken care of and given consideration to the ethical dimensions of your research project. Congratulations to you and your supervisor!

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely

MMabeta

Matsie Mabeta
Wits School of Education

011 717 3416

Appendix G – Ethics Information Letters and Consent Forms

LETTER TO THE PRINCIPAL

DATE:

Dear Principal

My name is Thulelah Blessing Takane and I am a Master Student in the School of Education at the University of the Witwatersrand.

I am doing research on addition and subtraction of grade 3 learners within the context of the Maths Club.

My research involves video-taping and audio taping the children in your school. I will also be interviewing your child as an individual as well as keeping field-notes and analysing samples of his/her work. This investigation will be done through-out the course of the Maths Club which consists of 13 lessons altogether. I intend focusing my research on a group of 6 learners.

The reason why I have chosen your school is because of the Maths Club already being conducted in your school within which my study will be based.
I was wondering whether you would mind if I invited the children in your child to be a participants in my research and use the information mentioned in the paragraph above for the purposes of my research.

The research participants will not be advantaged or disadvantaged in any way. They will be reassured that they can withdraw their permission at any time during this project without any penalty. There are no foreseeable risks in participating in this study. The participants will not be paid for this study.

The names of the research participants and identity of the school will be kept confidential at all times and in all academic writing about the study. Your individual privacy will be maintained in all published and written data resulting from the study.

All research data will be destroyed between 3-5 years after completion of the project.

Please let me know if you require any further information.
Thank you very much for your help.

Yours sincerely,

SIGNATURE:

NAME : Thulelah Blessing Takane
ADDRESS : 24-313 Wits Junction, Cnr Rich and Boundary Road, Parktown, 2193
EMAIL : thulelahblessing@gmail.com
TELEPHONE NUMBERS : 011- 717 3319 (Office) or 0769183569

INFORMATION SHEET PARENTS

DATE

Dear Parent

My name is Thulelah Blessing Takane and I am a Master Student in the School of Education at the University of the Witwatersrand.

I am doing research on addition and subtraction of grade 3 learners within the context of the Maths Club.

My research involves video-taping and audio taping your child together with others. I will also be interviewing your child as an individual as well as keeping field-notes and analysing samples of his/her work. This investigation will be done through-out the course of the Maths Club which consists of 13 lessons altogether. Your child is amongst a group of 6 learners on which I intend focusing my research.

The reason why I have chosen your child's class is because there are two student teachers who are leading the group who have agreed for me to work with the group.
I was wondering whether you would mind if I invited your child to be a participant in my research and use the information mentioned in the paragraph above for the purposes of my research.

Your child will not be advantaged or disadvantaged in any way. S/he will be reassured that s/he can withdraw her/his permission at any time during this project without any penalty. There are no foreseeable risks in participating and your child will not be paid for this study.

Your child's name and identity will be kept confidential at all times and in all academic writing about the study. His/her individual privacy will be maintained in all published and written data resulting from the study.

All research data will be destroyed between 3-5 years after completion of the project.

Please let me know if you require any further information.
Thank you very much for your help.

Yours sincerely,

SIGNATURE:

NAME : Thulelah Blessing Takane
ADDRESS : 24-313 Wits Junction, Cnr Rich and Boundary Road, Parktown, 2193
EMAIL : thulelahblessing@gmail.com
TELEPHONE NUMBERS : 011- 717 3319 (Office) or 0769183569

Consent Form Parents Videotaping

Please fill and return the reply slips below and indicate your willingness to allow your child to take part in my voluntary research project called:

Investigating grade 3 addition and subtraction strategies within the context of a Maths Club

Permission to be videotaped

I, ______________________ the parent of ____________________

give/do not give* my consent to let my child be video recorded in class.

[ ] I know that I may withdraw from the study at any time and that my child will not be advantaged or disadvantaged in any way.

[ ] I know that the videotapes will be used for this study only.

[ ] I know that the tapes will be destroyed between 3-5 years after completion of this project.

Parent Signature: ______________________ Date: ___________________

Contact person:

NAME: Thulelah Blessing Takane

ADDRESS: 24-313 Wits Junction, Cnr Rich and Boundary Road, Parktown, 2193

TEL NUMBER: 011- 717 3319 (Office) or 0769183569

INFORMATION SHEET STUDENT TEACHERS

DATE

Dear Student Teacher

My name is Thulelah Blessing Takane and I am a Master Student in the School of Education at the University of the Witwatersrand.

I am doing research on investigating grade 3 addition and subtraction learner strategies within the context of a Maths Club.

My investigation involves video-taping and audio taping you together with others. I will also be interviewing you as an individual as well as keeping field-notes and analysing samples of your work. This investigation will be done through-out the course of the Maths Club which consists 13 lessons altogether.

The reason why I have chosen your group is because in my observation both in your Methodology course and the Maths Club I have found you to be committed and willing to participate when required to do so. I was wondering whether you would mind if I invite you to participate in my study.

Your name and identity will be kept confidential at all times and in all academic writing about the study. Your individual privacy will be maintained in all published and written data resulting from the study. Your work will not be analysed, but the learners work will be analysed in relation to your work.

All research data will be destroyed between 3-5 years after completion of the project.

You will not be advantaged or disadvantaged in any way. Your participation is voluntary, so you can withdraw your permission at any time during this project without any penalty. There are no foreseeable risks in participating and you will not be paid for this study.

Please let me know if you require any further information.
Thank you very much for your help.

Yours sincerely,

SIGNATURE

NAME : Thulelah Blessing Takane
ADDRESS : 24-313 Wits Junction, Cnr Rich and Boundary Road, Parktown, 2193
EMAIL : thulelahblessing@gmail.com
TELEPHONE NUMBERS : 011-717 3319 (Office) or 0769183569

Consent Form Student Teachers Videotaping

Please fill and return the reply slips below and indicate your willingness to be videotaped in my voluntary research project called:

Investigating addition and subtraction strategies of grade 3 learners within the context of a maths club

Permission to be videotaped

I, ________________________

give/do not give* my consent to be videotaped for this project.

[ ] I know that I may withdraw from the study at any time and that I will not be advantaged or disadvantaged in any way.

[ ] I know that the videotapes will be used for this study only.

[ ] I know that the tapes will be destroyed between 3-5 years after completion of this project.

Student Teacher Signature: ________________________ Date: _____________

Contact person:
NAME: Thulelah Blessing Takane
ADDRESS: 24-313 Wits Junction, Cnr Rich and Boundary Road, Parktown, 2193
TEL NUMBER: 011- 717 3319 (Office) or 0769183569
*please delete where applicable

INFORMATION SHEET LEARNERS

DATE

Dear Learner

My name is Thulelah Blessing Takane and I am a Master Student in the School of Education at the University of the Witwatersrand.

I am doing research on shifts that can be seen on learners' addition and subtraction strategies within the context of the Maths Club.

My investigation involves video-taping and audio taping you together with others. I will also be interviewing you as an individual as well as keeping field-notes and analysing samples of your work. This investigation will be done through-out the course of the Maths Club which consists 13 lessons altogether.

I was wondering whether you would mind if I asked for your permission to allow me to go ahead with my investigation as describe in the paragraph above. I need your help with participating in my study.

Remember, this is not a test, it is not for marks and it is voluntary, which means that you don't have to do it. Also, if you decide halfway through that you prefer to stop, this is completely your choice and will not affect you negatively in any way.

I will not be using your own name but I will make one up. So no one can identify you, and all information about you will be kept confidential in all my writing about the study. Also, all collected information I will store safely and destroy between 3-5 years after I have completed.

Your parents have also been given an information sheet and consent form, but at the end of the day it is your decision to join us in the study.

I look forward to working with you!

Please feel free to contact me if you have any questions.

Thank you

SIGNATURE:

NAME : Thulelah Blessing Takane
ADDRESS : 24-313 Wits Junction, Cnr Rich and Boundary Road, Parktown, 2193
EMAIL : thulelahblessing@gmail.com
TELEPHONE NUMBERS : 011-717 3319 (Office) or 0769183569

Consent Form Learners Videotaping

Please fill in the reply slip below if you agree to be videotaped. I will use these videotapes for my study called:

Investigating addition and subtraction strategies of grade 3 learners within the context of a maths club.

Permission to be videotaped

My name is: ________________________

I agree to be video in class	YES/NO
I know that I can stop the videotaping at any time	YES/NO
I know that the videotapes will be used for this project only	YES/NO
I know that the videotapes will be destroyed between 3-5 years after completion of this study.	YES/NO

Sign____________________________ Date__________________________

Contact person:
NAME: Thulelah Blessing Takane
ADDRESS: 24-313 Wits Junction, Cnr Rich and Boundary Road, Parktown, 2193
TEL NUMBER: 011- 717 3319 (Office) or 0769183569
*please delete where applicable