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Research Article

Vertex degrees in grid graphs of permutations

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Abstract

A permutation of length n is defined as a finite sequence $a_1, a_2, \dots, a_n$ of distinct positive integers called parts, where for $1 \le i \le n$, the ith part a_i belongs to the set $\{1, 2, \dots, n\}$. In order to study permutations in the context of graph theory, a specific type of graph called a grid graph is defined for each permutation. For each possible fixed degree $i \in \{1, 2, 3, 4\}$, a multivariate generating function is determined to track the total number of vertices of each degree in permutations of length n. A formula is also provided for the total number of horizontal edges in grid graphs corresponding to permutations of length n.

Keywords: permutations; generating functions; grid graphs.

2020 Mathematics Subject Classification: 05A05, 05A15.

1. Introduction

A permutation of length n is defined as a finite sequence $a_1, a_2, \dots, a_n$ of distinct positive integers called parts, where for $1 \le i \le n$, the ith part a_i belongs to $[n] = \{1, 2, \dots, n\}$. There are $n!$ permutations of length n. Let $P(n)$ denote the set of all permutations of length n. For a comprehensive reference to permutation theory, see [5]. Some classical work on permutations can be found in [9, 13]. For some recent work on enumeration of permutations, see, for example, [1, 6–8, 10–12].

We represent each permutation of length n as a graph with vertices and edges, referred to as a grid graph. By a grid graph, we mean that a permutation of length n is represented by n columns of vertices aligned with the x-axis. For $1 \le i \le n$, the i-th part of size k ($1 \le k \le n$) is represented as the i-th vertical column (counted from left to right) with k vertices, each connected by edges to all neighbouring vertices in that column, as well as to vertices at the same height in neighbouring columns. This construction implies that, except for permutations of length 1, any vertex has a graph-theoretic degree belonging to the set $\{1, 2, 3, 4\}$. In Figure 1.1, we provide three examples of grid graphs of length 5.

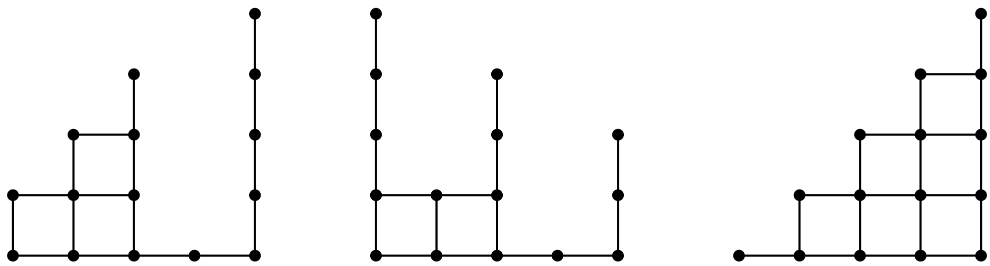

Figure 1.1: Three permutations of length 5, as grid graphs with vertices and edges shown. From left to right, the permutations are 23415, 52413, and 12345.

The number of vertical edges in a grid graph of length n is $1 + 2 + \ldots + (n-1) = \binom{n}{2}$, whereas the number of horizontal edges is given by $\sum_{i=1}^{n-1} \min\{a_i, a_{i+1}\}$. This formula for the total number of horizontal edges across all permutations of length n will be shown in the following section. The number of vertices is given in general by $\binom{n+1}{2}$ for any permutation of n. All vertices have degrees in the set $\{1, 2, 3, 4\}$. (The permutations of 1 are the only exception because the single vertex is of degree 0). Our primary aim is to provide a multivariable generating function for the total number of vertices of each degree in either exterior or interior columns across all permutations of length n.

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The reader may be wondering about the motivation for setting permutation theory in the field of graph theory. There are some precedents for this unification in the current authors' own recent work, which explored the contrast between bargraphs viewed either as a column of vertices or as a column of interconnected square cells each with area one. For these precedents, in the context of *partitions* of integers, see [2], in *compositions* of integers, see [3] and in Catalan words, see [4], where the contrast between the two underlying theoretical frameworks of grid graphs and polyomino model are explored. In these three papers, the name "c-graphs" has been used as a synonym (i.e., equivalent in meaning) for what are here called "grid graphs".

The primary method used in this paper is constructive in nature and was recently used in the same three papers, [2], [3], and [4]. By the constructive method, we mean a column-by-column approach where a particular column (say the ith, where $1 \leq i \leq n$) is chosen and all permutations of n are "constructed" around this column. The details of this approach follow in the remaining sections of the paper.

2. Counting the number of horizontal edges

We consider all permutations of length n. Our general approach here is constructive in nature. Initially, we fix interior column m (i.e., neither the first nor the last) of size b, and count all horizontal edges that have one vertex in this column. Eventually, we will sum over possible m in the range $2 \leq m \leq n-1$ and all b in the range $1 \leq b \leq n$.

Let $m \geq 2$ and let a, b, c be the respective heights of the $(m-1)$th, mth, and $(m+1)$th columns of an otherwise arbitrary grid graph. Each such configuration has $(n-3)!$ different occurrences, due to the number of permutations of all other columns. We illustrate this situation in Figure 2.1 below.

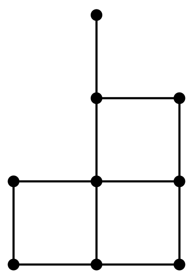

Figure 2.1: Part of a grid graph. Only columns $m-1$, m and $m+1$ are shown. These three columns are from left-to-right of heights a, b, and c, respectively. Here $b > \max\{a, c\}$.

To summarise, we are considering 3 successive columns with parts a, b, and c respectively. In Figure 2.1, we consider the case $b > \max\{a, c\}$. We will separately consider other cases later.

So the total number of edges with one vertex in column b, i.e., in the middle column, is given by

$$(n-3)!(a+c). \tag{1}$$

If a, b, and c meet the conditions stated above, by using symmetry, we obtain altogether that the total number of horizontal edges having one vertex in column b where position m varies as shown, is given by

$$2\sum_{m=2}^{n-1}\sum_{b=3}^{n}\sum_{c=2}^{b-1}\sum_{a=1}^{c-1}(n-3)!(a+c) = \frac{1}{4}(n-2)(n+1)!. \tag{2}$$

The factor 2 in the above equation arises because of the symmetry of the cases $a < c$ and $c < a$.

We next consider the case in which $b < \min\{a, c\}$. For this case, we obtain the total number of horizontal edges with one vertex in the middle column b as

$$2\sum_{m=2}^{n-1}\sum_{c=3}^{n}\sum_{a=2}^{c-1}\sum_{b=1}^{a-1}(n-3)!(2b) = \frac{1}{6}(n-2)(n+1)!. \tag{3}$$

Finally, if b is between a and c in size, we obtain the total number of horizontal edges with one vertex in column b as

$$2\sum_{m=2}^{n-1}\sum_{c=3}^{n}\sum_{b=2}^{c-1}\sum_{a=1}^{b-1}(n-3)!(a+b) = \frac{1}{4}(n-2)(n+1)!. \tag{4}$$

What is left out of the previous discussion are the cases where a horizontal edge has a vertex in the first or last column. By symmetry, the total number of these are the same. So, we consider the case where the 1st column is a and the second is b and double for the symmetry. The number of such horizontal edges across all permutations of n is

$$2\sum_{a=2}^{n}\sum_{b=1}^{a-1}(n-2)!b+2\sum_{b=2}^{n}\sum_{a=1}^{b-1}(n-2)!a=\frac{2}{3}(n+1)!. \tag{5}$$

Adding together Equations (2), (3), (4), and (5) and dividing by 2 (since each horizontal edge has 2 vertices) we have proved the following proposition.

Proposition 2.1. *The total number of horizontal edges $H(n)$ in grid graphs of permutations of length n is given by*

$$H(n)=\frac{1}{3}(n-1)(n+1)!. \tag{6}$$

From this, we see that the average number of horizontal edges in grid graphs of permutations of length n is given by $\frac{1}{3}(n-1)(n+1)$.

3. A multivariate generating function in permutations of length n tracking the total number of vertices q_i of degree i where $i \in \{1,2,3,4\}$

Before we define the generating functions $F_{int}(x;q_1,q_2,q_3,q_4)$ for the *total* degree product of all interior columns (excluding the first and last column) across all permutations of n, we give some explanation. Jumping ahead, see Example 3.1, where the bold term in the generating function, namely $4x^4q_1q_3^2$, means that across all permutations of 4 there are 4 interior columns of size 3 (i.e., the total q power), each having 1 degree one vertex and 2 degree three vertices. We also need to explain the column-by-column constructive procedure that follows. As per Figure 2.1 we focus on the mth column, i.e., the middle column b ($1 \le b \le n$) of a set of three successive arbitrary columns $\{a,b,c\}$. We then allow $2 \le m \le n-1$ to vary over all interior column positions. We also note that any particular m value yields the same result as any other. Finally, we have the following definition:

$$F_{int}(x;q_1,q_2,q_3,q_4):=\sum_{n\ge 1}x^n(n-3)!\sum_{m=2}^{n-1}\sum_{b=1}^{n}\sum_{AT(b)}\sum_{s=1}^{b}\prod_{i=1}^{4}q_i^{t(s)}, \tag{7}$$

where $AT(b)$ is the set of all triples $\{a,b,c\}$ as described just above this definition; where q_i tracks the number of vertices $t(s)$ of degree i and where $t(s) \in \{0,1,\dots,b\}$ depending on the number of occurrences of degree i vertices as s ranges through each vertex of b. The $(n-3)!$ factor in the above definition allows for the permutation of all columns other than those in the triple $\{a,b,c\}$. For an explicit review of the terms of $[x^n]F_{int}(x;q_1,q_2,q_3,q_4)$ in the case where $n=4$, see Example 3.1 again.

Also, by a slight modification of the explanation above, we note that external columns (i.e., the first or last column) can only have vertices of degree one to three. Since the degree of each vertex in the first or last column depends only on the single adjacent column, we focus here on pairs $\{a,b\}$ instead of triplets $\{a,b,c\}$, where a is in the first column and b is in the second. We now have the following definition.

For $n \ge 2$,

$$F_{ext}(x;q_1,q_2,q_3):=2\sum_{n\ge 2}x^n(n-2)!\sum_{a=1}^{n}\sum_{AD(a)}\sum_{s=1}^{a}\prod_{i=1}^{3}q_i^{t(s)}, \tag{8}$$

where $AD(a)$ is the set of all doublets $\{a,b\}$ as described just above this definition; where $t(s) \in \{0,1,\dots,a\}$ depending on the number of occurrences of degree i vertices as s ranges through each vertex of a. The $(n-2)!$ factor in the above definition allows for the permutation of all columns other than those in the doublet $\{a,b\}$. There is a symmetrical process for the last column. This accounts for the initial 2 in the above generating function.

As already stated, we use the same constructive method as in the previous section to simultaneously count these four statistics, but this time, we obtain a generating function (as opposed to a counting formula).

We refer the reader once more to Figure 2.1 where we are considering 3 successive columns with parts a, b and c respectively. As illustrated in Figure 2.1, we first consider the case $b > \max\{a,c\}$. Without loss of generality, we will set $a < c$. Using symmetry, the case $c < a$ has an identical result, which we capture by doubling. Cases other than $b > \max\{a,c\}$ will be considered later.

Hence, the generating function for the total number of vertices of each possible degree in column b, i.e., the middle column in position m (allowing this column to take any internal position in the permutation) is given by

$$\begin{aligned}
&2\sum_{m=2}^{n-1}\sum_{b=3}^{n}\sum_{c=2}^{b-1}\sum_{a=1}^{c-1}(n-3)!q_1q_2^{b-1-c}q_3^{c-a+1}q_4^{a-1}\\
&=\frac{1}{(-1+q_2)(q_2-q_3)(-1+q_3)(q_2-q_4)(q_3-q_4)(-1+q_4)}\times\\
&\quad\times 2(n-2)!q_1q_3^2\Big(q_2^n(-1+q_3)(q_3-q_4)(-1+q_4)-q_3^n(-1+q_4)q_4+q_3\left(q_4^2-q_4^n\right)\\
&\quad+q_3^2(-q_4+q_4^n)+q_2^2(-q_3^n(-1+q_4)+q_4-q_4^n+q_3(-1+q_4^n))\\
&\quad+q_2\left(-q_4^2+q_4^n+q_3^n\left(-1+q_4^2\right)-q_3^2(-1+q_4^n)\right)\Big).
\end{aligned}\tag{9}$$

Now, we consider the case in which $b<\min\{a,c\}$ and again, before using symmetry, assume that $a<c$. Also, the case $b=1$ is exceptional and therefore these cases are included separately as the first term in the generating function below. If $n\geq 2$, the generating function for the total number of vertices of each possible degree in column b is given by

$$\begin{aligned}
&2(n-3)!\left(q_2\sum_{m=2}^{n-1}\sum_{c=3}^{n}\sum_{a=2}^{c-1}1\right)+2(n-3)!\sum_{m=2}^{n-1}\sum_{c=4}^{n}\sum_{a=3}^{c-1}\sum_{b=2}^{a-1}q_3^2q_4^{b-2}\\
&=(n-2)!\left(\frac{q_3^2\left(-\left(n^2-3n+2\right)q_4^3+2\left(n^2-4n+3\right)q_4^2-\left(n^2-5n+6\right)q_4+2q_4^n\right)}{(q_4-1)^3q_4}+\left(n^2-3n+2\right)q_2\right).
\end{aligned}\tag{10}$$

The last case for the interior column generating function is where b is between a and c in size. We again use symmetry and assume $a<b<c$. The generating function for the degree of vertices in the middle column b is

$$\begin{aligned}
&2\sum_{m=2}^{n-1}\sum_{c=3}^{n}\sum_{b=2}^{c-1}\sum_{a=1}^{b-1}(n-3)!q_2q_3^{b-a}q_4^{a-1}\\
&=\frac{2q_2q_3(n-2)!\left((q_4-1)^2q_3^n-q_3^2\left(q_4^n-nq_4+n-1\right)+q_3\left(2q_4^n-nq_4^2+n-2\right)-q_4^n+(n-1)q_4^2-(n-2)q_4\right)}{(1-q_3)^2(q_3-q_4)(1-q_4)^2}.
\end{aligned}\tag{11}$$

Finally, what is left out of the above three cases is the generating function for the degree count in the first or last column. We again count this only in the first column and, using the symmetry between the first and last columns, double the answer. For this, we let the first column be of size a and the second column be of size b. The cases $b=1$ and $a=1$ are different from the other cases and are therefore separated out in the generating function below as the first and third terms, respectively. For $n\geq 2$, we obtain as per definition 8 the following theorem.

Theorem 3.1. *For the generating function defined in Equation (8) the coefficient of x^n is given by*

$$\begin{aligned}
[x^n]F_{ext}(x;q_1,q_2,q_3)&=2q_1(n-2)!\Big(\sum_{a=2}^{n}q_2^{a-1}(n-2)!+q_1\sum_{a=3}^{n}\sum_{b=2}^{a-1}q_2^{a-b}q_3^{b-1}+q_1\sum_{b=2}^{n}1+\sum_{b=3}^{n}\sum_{a=2}^{b-1}q_2^2q_3^{a-2}\Big)\\
&=2(n-2)!\left(\frac{q_2^2\left(q_3^n-(n-1)q_3^2+(n-2)q_3\right)}{(1-q_3)^2q_3}+(n-1)q_1+\frac{q_1\left(q_2-q_2^n\right)}{1-q_2}\right.\\
&\qquad\left.+\frac{q_1\left((q_3-1)q_3q_2^n+q_2^2\left(q_3-q_3^n\right)+q_2\left(q_3^n-q_3^2\right)\right)}{(1-q_2)(q_2-q_3)(1-q_3)}\right).
\end{aligned}\tag{12}$$

In Equations (13)–(15), we introduce the following generating functions and notation. Firstly,

$$DF_{ext}(x,q_1)=\frac{\partial}{\partial q_1}F_{ext}(x;q_1,q_2,q_3)|_{\{q_1,q_2,q_3\}=\{1,1,1\}}.\tag{13}$$

Secondly,

$$DF_{ext}(x,q_2)=\frac{\partial}{\partial q_2}F_{ext}(x;q_1,q_2,q_3)|_{\{q_1,q_2,q_3\}=\{1,1,1\}}.\tag{14}$$

Finally,

$$DF_{ext}(x,q_3) = \frac{\partial}{\partial q_3} F_{ext}(x;q_1,q_2,q_3)|_{\{q_1,q_2,q_3\}=\{1,1,1\}}. \tag{15}$$

By doing the differentiation in Equations (13) to (15), we have proved the following corollary.

Corollary 3.1. *The total number of vertices of degrees one to three in the first and last columns of all permutations of length n is given, respectively, by*

$$[x^n]DF_{ext}(x,q_1) = (n+2)(n-1)!, \tag{16}$$

$$[x^n]DF_{ext}(x,q_2) = \frac{1}{3}(n^2+7n-12)(n-1)!, \tag{17}$$

and

$$[x^n]DF_{ext}(x,q_3) = \frac{1}{3}(n-2)(2n-3)(n-1)!. \tag{18}$$

The exponential generating functions associated with Equations (13) to (15) are given respectively by

$$\sum_{n\geq 0}(n+2)(n-1)!\frac{x^n}{n!} = \frac{1+2x-2x^2}{(1-x)^2}, \tag{19}$$

$$\frac{1}{3}\sum_{n\geq 0}(n^2+7n-12)(n-1)!\frac{x^n}{n!} = \frac{x(12-15x+4x^2)}{3(1-x)^2} + 4\log(1-x) \tag{20}$$

and

$$\frac{1}{3}\sum_{n\geq 0}(n-2)(2n-3)(n-1)!\frac{x^n}{n!} = \frac{x(12-15x+4x^2)}{3(1-x)^2} + 4\log(1-x). \tag{21}$$

Adding together Equations (9), (10), and (11), in accordance with definition (7), we have proved the following theorem.

Theorem 3.2. *For $n \geq 2$, the coefficient of x^n in the generating function $F_{int}(x;q_1,q_2,q_3,q_4)$ as defined in Equation (7) is given by*

$$\begin{aligned}
[x^n]F_{int}(x;q_1,q_2,q_3,q_4) = (n-2)!\Bigg(&(n-2)(n-1)q_2 \\
&+ \frac{2q_3q_2\Big((q_4-1)^2q_3^n - q_3^2\left(q_4^n - nq_4 + n - 1\right) + q_3\left(2q_4^n - nq_4^2 + n - 2\right) - q_4^n\Big)}{(1-q_3)^2(q_3-q_4)(1-q_4)^2} \\
&+ \frac{q_4\left((n-1)q_4 - n + 2\right)}{(1-q_3)^2(q_3-q_4)(1-q_4)^2} \\
&+ \frac{q_3^2\left(2q_4^n + q_4\left((n-1)q_4\left(2(n-3)-(n-2)q_4\right) - (n-3)(n-2)\right)\right)}{(q_4-1)^3q_4} \\
&+ \frac{2q_1q_3^2\left((q_3-1)(q_3-q_4)(q_4-1)q_2^n + q_2^2\left(-\left((q_4-1)q_3^n\right) + q_3\left(q_4^n-1\right) - q_4^n + q_4\right)\right)}{(q_2-1)(q_2-q_3)(q_3-1)(q_2-q_4)(q_3-q_4)(q_4-1)} \\
&+ \frac{q_2\left(\left(q_4^2-1\right)q_3^n - q_3^2\left(q_4^n-1\right) + q_4^n - q_4^2\right) + (q_3-1)q_3q_4^n + q_4\left(q_3(q_4-q_3) - (q_4-1)q_3^n\right)}{(q_2-1)(q_2-q_3)(q_3-1)(q_2-q_4)(q_3-q_4)(q_4-1)}\Bigg).
\end{aligned} \tag{22}$$

Example 3.1. *By setting $n=4$, we obtain*

$$[x^4]F_{int}(x;q_1,q_2,q_3,q_4) = 12q_2 + 8q_2q_3 + 4q_3^2 + \mathbf{4q_1q_3^2} + 4q_2q_3^2 + 4q_1q_2q_3^2 + 4q_1q_3^3 + 4q_2q_3q_4 + 4q_1q_3^2q_4.$$

Here, the total exponent of each term is meaningful. So, the term $12q_2$ has total exponent 1, which indicates that the interior column of the permutation is 1. The degree of each vertex in an internal 1 is two (indicated by q_2) and there are 12 such cases. The terms with a total exponent 3 are $4q_1q_3^2$, $4q_2q_3^2$, $4q_2q_3q_4$. The total exponent of 3 in $4q_1q_3^2$ indicates that 3 is occurring 4 times, where 1 vertex has degree one and 2 vertices have degree three. Furthermore, $4q_2q_3q_4$ indicates that 3 is occurring 4 times, where one vertex has degree two, one vertex has degree three, and one vertex has degree four, and so on.

Thereafter, setting $\{q_1, q_2, q_4\} = \{1, 1, 1\}$, we obtain $[x^4]F_{int}(x; 1, 1, q_3, 1) = 12 + 12q_3 + 20q_3^2 + \mathbf{4q_3^3}$. The bold term means that for the internal columns of permutations of 4, there are four columns that contain exactly 3 vertices of degree three. These are $\{1\mathbf{43}2, 3\mathbf{41}2, 2\mathbf{34}1, 2\mathbf{14}3\}$ where the 4 columns are shown in bold.

By analogy with the previous discussion, we introduce the following notation and generating functions. Firstly

$$DF_{int}(x, q_1) := \frac{\partial}{\partial q_1} F_{int}(x; q_1, q_2, q_3, q_4)|_{\{q_1,q_2,q_3,q_4\}=\{1,1,1,1\}}. \tag{23}$$

Secondly,

$$DF_{int}(x, q_2) := \frac{\partial}{\partial q_2} F_{int}(x; q_1, q_2, q_3, q_4)|_{\{q_1,q_2,q_3,q_4\}=\{1,1,1,1\}}. \tag{24}$$

Thirdly,

$$DF_{int}(x, q_3) := \frac{\partial}{\partial q_3} F_{int}(x; q_1, q_2, q_3, q_4)|_{\{q_1,q_2,q_3,q_4\}=\{1,1,1,1\}} \tag{25}$$

and finally,

$$DF_{int}(x, q_4) := \frac{\partial}{\partial q_4} F_{int}(x; q_1, q_2, q_3, q_4)|_{\{q_1,q_2,q_3,q_4\}=\{1,1,1,1\}}. \tag{26}$$

By evaluating Equations (23) to (26), we have proved the following corollary.

Corollary 3.2. *The total number of vertices of degrees one to four in all interior columns of all permutations of length n are given, respectively, by*

$$[x^n]DF_{int}(x, q_1) = \frac{1}{3}(n-2)n!, \tag{27}$$

$$[x^n]DF_{int}(x, q_2) = \frac{1}{12}(n-2)(n^2+n+12)(n-1)!, \tag{28}$$

$$[x^n]DF_{int}(x, q_3) = \frac{1}{6}(n-2)(-12+7n+n^2)(n-1)! \tag{29}$$

and

$$[x^n]DF_{int}(x, q_4) = \frac{1}{12}(n-3)(n-2)(3n-4)(n-1)!. \tag{30}$$

In a similar way to Equations (19), (20), and (21), the exponential generating functions associated with Equations (23) to (26) are given respectively by

$$\frac{x^3}{3(1-x)^2}, \tag{31}$$

$$\frac{x(12-30x+26x^2-7x^3)}{6(1-x)^3} + 2\log(1-x), \tag{32}$$

$$\frac{x(-12+30x-19x^2+2x^3)}{3(1-x)^3} - 4\log(1-x) \tag{33}$$

and

$$\frac{x(12-30x+22x^2-x^3)}{6(1-x)^3} + 2\log(1-x). \tag{34}$$

By using Corollaries 3.1 and 3.2, we obtain our next theorem.

Theorem 3.3. *For $n \geq 2$ and $i \in \{1, 2, 3, 4\}$, the total number of vertices $Q_i(n)$ of degree i across all permutations of length n is given by*

$$Q_1(n) = \frac{1}{3}(n^2 + n + 6)(n-1)!, \tag{35}$$

$$Q_2(n) = \frac{1}{12}(n^3 + 3n^2 + 38n72)(n-1)!, \tag{36}$$

$$Q_3(n) = \frac{1}{6}(n^3 + 9n^2 - 40n + 36)(n-1)! \tag{37}$$

and

$$Q_4(n) = \frac{1}{12}(n-3)(n-2)(3n-4)(n-1)!. \tag{38}$$

Example 3.2. *By setting $n = 3$ in Equation (37), we obtain $Q_3(3) = 8$. The permutations of 3 are $\{$123,132,213,231,312,321$\}$. The permutations 123 and 132 each contain two vertices of degree 3, as do their reverse-order permutations. The remaining permutations contain none.*

4. The asymptotic proportions of vertices of degrees $1, 2, 3,$ and 4

Since the total number of vertices in permutations of length n is equal to $\binom{n+1}{2}n!$, we can deduce the asymptotic proportions of vertices of different degrees from Equations (35), (36), (37) and (38). The proportions as $n \to \infty$ are respectively $\{r_1, r_2, r_3, r_4\}$ with values given as $\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}\}$. The comparisons which follow are found in [2–4].

By comparison in grid graphs of Catalan words, this same set of ratios is $\{0, 0, 0, 1\}$. The vertices of degree 1, 2, and 3 occurred asymptotically in the ratio $\{\frac{1}{18}, \frac{2}{9}, \frac{13}{18}\}$.

In the case of integer partitions for large $n = 150$, the proportions $\{r_1, r_2, r_3, r_4\}$ were approximately $\{0.012 : 0.171 : 0.377 : 0.44\}$. In the case of degree 4, this proportion seemed to be growing; for the other degrees, the proportion seems to be falling as n increases. It is possible that asymptotically, as $n \to \infty$, integer partition ratios are the same as permutation ones. We leave this as an open problem for an interested researcher.

In the case of integer compositions, the asymptotic proportions $\{r_1, r_2, r_3, r_4\}$ were $\{\frac{105}{882}, \frac{13}{28}, \frac{336}{882}, \frac{7}{196}\}$ or approximately $\{0.119, 0.464, 0.381, 0.035\}$.

We also suggest the following directions for future research to interested readers:

1. Investigating grid graphs of permutations avoiding various patterns of length three.

2. Conducting a similar study for other relevant parameters.

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