A survey of three combinatorial problems.*

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Declaration

I declare that this MSc dissertation is my own, under supervision of Doctor Margaret Archibald, lecturer, Professor Arnold Knopfmacher, research professor, and Professor Charlotte Brennan, associate professor, at the School of Mathematics, University of the Witwatersrand. I have compiled this dissertation and everything included in this dissertation based on research and elaboration on three different papers [11, 37, 47], as have been referenced. It is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg.

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Abstract

This dissertation is based on three different combinatorial papers:

1. The first paper is by Silvia Heubach and Toufik Mansour: Enumeration of 3-Letter Patterns in Compositions. *Combinatorial Number Theory in Celebration of the 70-th Birthday of Ronald Graham.* In: *De Gruyter Proceedings in Mathematics.* 243-264, (2007).

2. The second paper is by Daniel J. Velleman and Gregory S. Warrington: What to expect in a game of memory. *American Mathematical Monthly*, 120:787-805 (2013).

3. The third paper is by Mireille Bousquet-Mélou and Richard Brak: Exactly Solved Models of Polyominoes and Polygons. *Polygons, Polyominoes and Polycubes.* In: *Lecture Notes in Physics*, 775:43-78, (2009).

These three papers have been elaborated on by manner of adding additional proofs, further descriptions of specific combinatorial objects and further explanation of theorems, lemmas and proofs, all contained within the papers.

The first paper deals with the occurrence of specific 3-letter patterns occurring in *compositions* and *words.* A composition of $n \in \mathbb{N}$ is a sequence of parts $\sigma_1\sigma_2\cdots\sigma_k$, where $\sigma_i \in \mathbb{N}$ for $1 \leq i \leq k$ and $\sum_{i=1}^{k}\sigma_i = n$. Words are similar to compositions and also exist as a sequence $\sigma_1\sigma_2\cdots\sigma_k$ of parts, but from an alphabet $[n] = \{1, 2, \cdots, n\}$. Specifically this first paper deals with the process of finding generating functions that enumerate compositions and words that contain specific 3-letter patterns.

In the second paper, the card game of *memory* is discussed. Memory is a game played with two identical decks of cards that are shuffled together. After the two decks have been shuffled together the cards are then laid face down and the game is played through a series of moves. A player performs a move when he flips two cards, one at a time. If the cards match they are removed from play, else they are returned to play face down. The game is played until all the cards have been removed from play. Through the use of a series of theorems and lemmas it is shown that:

1. The average number of moves for a single game of memory is $(3 - 2\ln 2)\, n + 7/8 - 2\ln 2 \approx 1.61n$.

2. The average number of occurrences where the player flips over two identical cards coincidentally is $\ln 2$.

3. The average number of flips until the first match has been found in a game of memory is $2^{2n}/\binom{2n}{n} \sim \sqrt{\pi n}$.

Specific classes of *polyominoes* and *polygons* are enumerated in the third paper. A polyominoe is a plane figure formed by joining a finite number of identical square cells together at their edges. A polygon is a specific type of polyominoe that exists as a polyominoe's border. Through the description of the structure of polyominoes from specific classes, generating functions enumerating these specific classes are found.

The elaboration of the problems, theorems and descriptions of the objects within these three papers forms the basis of this MSc dissertation.

Contents

Part I

Enumeration of 3-letter patterns in compositions [37]

Abstract

A composition of $n \in \mathbb{N}$ is defined as an ordered collection of parts from a set $A \subseteq \mathbb{N}$, whose sum results in n. Generating functions for compositions of n, consisting of m parts in A, that contain specific 3-letter patterns a given number of times are derived. These patterns are made up of two adjacent occurrences of the statistics *levels, rises* and *drops*. Using the generating functions, asymptotics for the pattern-avoiding compositions are derived. Lastly, k-ary words of specific lengths are studied, and the generating functions for such words that contain the aforementioned 3-letter patterns a certain number of times are derived.

1 Introduction

A composition of n, with m parts in $A \subseteq \mathbb{N}$, is defined as a sequence

$$\sigma = \sigma_1 \sigma_2 \ldots \sigma_m$$

where $\sigma_i \in A$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} \sigma_i = n$. The σ_i are referred to as "parts" in the composition σ of n. The objective is to determine the total number of compositions, σ, in which a given pattern τ occurs exactly r times. The patterns are the statistics: rises, levels and drops, and can be illustrated by

$$\sigma = \sigma_1 \sigma_2 \ldots \sigma_{j-1} \sigma_j \ldots \sigma_m$$

where, for $2 \leq j \leq m$,

- a **rise** is given as $\sigma_{j-1}\sigma_j$ such that $\sigma_{j-1} < \sigma_j$;
- a **level** is given as $\sigma_{j-1}\sigma_j$ such that $\sigma_{j-1} = \sigma_j$;
- a **drop** is given as $\sigma_{j-1}\sigma_j$ such that $\sigma_{j-1} > \sigma_j$

in all compositions of n, where $\sigma_i \in A \subseteq \mathbb{N}$, for all $i = 1, 2, \ldots, m$.

Previous authors have studied these statistics in compositions in a variety of different ways. Compositions whose parts were restricted to $\{1, 2\}$ with relation to the Fibonacci sequence were studied by Alladi and Hoggat [2]. Restricting parts to different sets of A, Chinn, Grimaldi and Heubach [21, 22, 23, 33, 34] counted the total number of compositions, rises, levels and drops and looked for correlations with known sequences.

Rises, levels and falls in permutations and compositions have been studied on the set $[n] = \{1, 2, \ldots, n\}$ by Carlitz and co-authors ([16, 17, 18, 19, 20]). (It should be noted that Carlitz et al. added a further rise at the start of each composition and a further fall at the end, but not in [19], dissimilar to that of Alladi and Hoggat in [2].)

Carlitz et al. considered the *specification* of a composition. The specification of a composition is a list consisting of the total number of occurrences of each integer in the compositions, as well as the statistic levels. Weak rises and falls were studied by Rawlings [43]. These statistics occur in a composition where the equality is allowed in the statistic, and Rawlings considered these statistics in the enumeration of compositions related to restricted words by adjacencies. *Ascent variation* is a concept that was introduced by Rawlings. The ascent variation of a composition is the sum of the increments of the statistic rises within the composition. The reason for the concept of ascent variation is a correlation to the perimeter of directed vertically-convex polyominoes.

A polyominoe is a plane figure made up of square cells that are joined together at their edges, it is vertically convex if every vertical line that intersects the polyomino does so only once. To define directedness, consider each cell in a polyomino to be represented as a vertex. If there exists a path from a specific vertex in a polyomino such that it includes every other vertex, and it consists of only East steps or only West steps, and only South steps or only North steps, then that polyominoe is said to be directed. An *upper descent* in such a polyominoe will the index of a column, say k for the column C_k, in such a polyominoe whose uppermost cell is above the uppermost cell of the column to the right of it. A survey of the enumeration of these polyominoes was done by Bousquet-Mélou in [11]. The structure for deriving the generating functions for enumerating rises, levels and falls for an ordered subset $A \subseteq \mathbb{N}$ was developed by Heubach and Mansour [36].

The concept of *waves*, compositions without levels, was introduced by Carlitz et al. *Smirnov sequences* and *Carlitz Compositions* have also been used to refer to waves as can be seen in [32] and [40] respectively. *Up-down* sequences are a specific case of waves. They consist of alternating rises and falls. These up-down sequences were studied in [18] by Carlitz and Scoville, and in [16] by Carlitz. Now, the Simon Newcomb problem [15, 26], can be obtained from the Carlitz problem, of counting compositions by their rises and falls, by neglecting the number of falls. Carlitz and Vaughan [20] studied another version of this problem - an extension of problems for permutations - *viz.* finding generating functions for the number of compositions according to specification, rises, falls and maxima. Pairs of sequences enumerated according to rises, levels and falls were studied by Carlitz and Vaughan in [20]. Taking a probabilistic approach, Gho, Hitzcenko, Louchard and Prodinger studied the distinctness and other attributes of compositions and waves.

Steffen Eger illustrates the connection existing between extended binomial coefficients and weighted restricted integer compositions in [27]. Similar to Alladi and Hoggat, in [31] Gessel and Li also studied Fibonacci numbers, using sums over compositions to obtain formulas for specific sets of Fibonnaci numbers by considering a Fibonacci number as a monoid, and also considering submonoids of such a monoid.

Banderier and Hitczenko [5] studied the enumeration of compositions that are restricted to having parts from a given subset of positive integers, and also having the same number of these parts. In doing so they also found the asymptotics for the probability that two compositions of a positive integer n consist of the same number of parts.

A case in which compositions were studied in the representation of geometric objects was done by A. Knopfmacher and M. E. Mays in [39]. They enumerated various families of graphs by their compositions, and more specifically generalized this enumeration of graph compositions to ordinary compositions of positive integers.

Jean Berstel and Dominique Perrin [8] discuss the origin of the use of words in combinatorics, especially the use of the enumeration of words, where words are used to encode various objects from algebra, geometry, and enumerative combinatorics.

A large field of study for permutations is pattern avoidance, as in [9] and its references. Now using 2-letter patterns to represent rises, levels and drops, we extend the enumeration problems from permutations to compositions. As an example of this consider the pattern represented by 11: two adjacent occurrences of a_i, $a_i a_i$ where $a_i \in A \subseteq \mathbb{N}$. As a result, it can be said that 11 is the same as the occurrence of a level, $a_i a_i$. Now considering rises, it can be seen that the pattern represented by 12 is the same as the occurrence of a a_i adjacent to an a_j, $a_i a_j$ with $a_i < a_j$ and thus corresponds to a rise. In a similar way the pattern represented by 21 can be seen to correspond to a fall.

The objective of of Part I is to study and to generalize results relating to compositions containing specific 3-letter patterns a specific number of times. Now, in the case of 3-letter patterns, 111 can be represented by $a_i a_i a_i$, $a_i \in A \subseteq \mathbb{N}$, which would be the same as two adjacent levels, or simply put: **level+level**. A table for the different cases for 3-letter patterns is given below.

Statistic	Pattern	Statistic	Pattern
level+level	111	rise+rise	123
level+rise	112	rise+drop=peak	121+132+231
level+drop	221	drop+rise=valley	212+213+312

Table 1: Statistics related to 3-letter compositions, and their corresponding patterns.

Using this table, consider the composition of 29 given as 5246822. This composition consists of a drop, 52, a rise, 24, a rise, 46, a rise, 68, a drop, 82, and a level, 22. Now, considering the statistic 3-letter patterns, the makeup of the composition consists, from left to right, of 312 (**drop+rise**), 123 (**rise+rise**), 123 (**rise+rise**), 231 (**rise+drop**), and lastly 211 (**drop+level**). This can simply be expressed as **drop+rise+rise+rise+drop+level**.

Now consider symmetric and non-symmetric compositions: for symmetric compositions each rise is matched by a drop within the composition itself. For non-symmetric compositions there exists a composition whose parts are identical but in reverse order, thus for the asymmetric case the drops and rises are matched between the non-symmetric compositions and its reversed counterpart. As a result it can be seen that the statistic **rise+level**, i.e. 122, occurs just as frequently as the statistic **level+drop**, i.e. 221. From Table 1. it can be seen that only the statistics *peak*, and *valley* are represented by 3 different patterns. This is a result of the size of the adjacent parts present in the two statistics.

The generating functions for the total number of compositions of n with m parts in $A \subseteq \mathbb{N}$, in which the pattern τ, where τ is one of the 6 patterns listed in the table above, occurs r times are derived in Section 2. Section 3 consists of the derivation of the asymptotics for the generating functions for the number of compositions of n that do not contain a specific pattern τ, and are derived through methods relating to complex analysis. In the final section, Section 4, the results obtained in Sections 1 through 3 are applied to k-ary words, elements of $[k]^n$, where the respective $[k]$ is a linearly-ordered alphabet of k letters. Results as in [13, 14] are obtained as well as additional, new results with regard to peaks, and valleys. As a final conclusion it is found that this application illustrates how words form a subclass of a larger combinatorial class of objects that are formed by compositions.

2 Compositions of n with parts in A

Let $A \subseteq \{a_1, a_2, a_3, \ldots\} \subseteq \mathbb{N}$, such that $a_1 < a_2 < \ldots$, so that A is an *ordered subset* of $\mathbb{N}$. The finite and infinite cases of A where possible shall be treated together, and the infinite case shall be specifically dealt with if necessary.

Some basic generating functions are defined:-

Let $C_\tau(n, r)$ ($C_\tau(j; n, r)$ for the finite case respectively) be used to denote the number of compositions of n with parts in $A \subseteq \mathbb{N}$ (with j parts in A for the finite case respectively) in which the pattern τ occurs r times. The generating functions corresponding to these functions will be given by

$$C_\tau(x, y) = \sum_{n,r \geq 0} C_\tau(n, r)\, x^n y^r$$

and,

$$C_\tau(x, y, z) = \sum_{n,r,j \geq 0} C_\tau(j; n, r)\, x^n y^r z^j = \sum_{j \geq 0} C_\tau(j; x, y)\, z^j$$

respectively, where the coefficients of $x^n y^r$ is the total number of compositions of n in which the pattern τ occurs exactly r times.

To generalize this, denote the total number of compositions of n with parts in A, (with j parts in A for the finite case respectively) in which the pattern τ occurs r times and being restricted to having their first l parts as $\sigma_1\sigma_2\ldots\sigma_l$, by $C_\tau(\sigma_1\ldots\sigma_l|n, r)$ ($C_\tau(\sigma_1\ldots\sigma_l|j; n, r)$ for the finite case respectively). Now, the generating functions corresponding to the ones whose first l parts are restricted, are given by

$$C_\tau(\sigma_1\ldots\sigma_l|x, y) = \sum_{n,r \geq 0} C_\tau(\sigma_1\ldots\sigma_l|n, r)\, x^n y^r$$

and,

$$C_\tau(\sigma_1\ldots\sigma_l|x, y, z) = \sum_{n,r,j \geq 0} C_\tau(\sigma_1\ldots\sigma_l|j; n, r)\, x^n y^r z^j = \sum_{j \geq 0} C_\tau(\sigma_1\ldots\sigma_l|j; x, y)\, z^j$$

with initial conditions given by $C_\tau(j; x, y) = 0$, where $j < 0$, $\quad C_\tau(0; x, y) = 1$, and $C_\tau(\sigma_1\ldots\sigma_l|j; x, y) = 0$ when $j \leq l - 1$.

Taking into account the *empty composition* we can deduce that the generating function for the number of compositions of n that contain the pattern τ exactly r times, with j parts in A, is equivalent to the empty composition, 1, added to the sum of all compositions of this type that start with a for all $a \in A$

$$C_\tau(x, y, z) = 1 + \sum_{a \in A} C_\tau(a|x, y, z)\,. \tag{1}$$

Different patterns $\tau = \tau_1\tau_2\tau_3$ in relation to the generating functions $C_\tau(x, y, z)$ will be studied in this section, the aim being to determining the generating functions $C_\tau(x, y, z)$ in their explicit forms. One way this will be done is through the use of recursive equations.

With regard to the pattern represented by 111, a **level + level**, the process is simple, however for the patterns represented by 112 and 221 the process becomes more complex. For the pattern represented by 123, the composition itself is first separated into different pieces. Each of these pieces consists of parts that either come from A or are all larger than a particular part. These steps together with the definition of another generating function form the basis to obtaining an explicit form for $C_{123}(x, y, z)$.

For the patterns peaks and valleys continued fraction expansion is used to find the explicit forms of $C_{peak}(x, y, z)$ and $C_{valley}(x, y, z)$, by again breaking up the compositions into parts based on where the largest or smallest parts occur respectively.

2.1 The pattern 111 (the statistic level+level)

The subsequent theorem introduces a generating function for the number of compositions of n with j parts in A in which the pattern 111 occurs a fixed number of times.

Theorem 2.1. *Let A be any ordered subset of $\mathbb{N}$. Then*

$$C_{111}(x, y, z) = \frac{1}{1 - \sum\limits_{a \in A} \frac{x^a z(1+(1-y)x^a z)}{1+x^a z(1+x^a z)(1-y)}}.$$

Proof. For the pattern 111 to occur in a composition, an integer $a \in A \subseteq \mathbb{N}$ must occur as aaa, three times consecutively. Consider a fixed $a \in A$ and assume that $j \geq 2$, i.e. the composition consists of 2 or more parts.

The number of compositions of n with j parts in A that contain the pattern 111 exactly a specific number of times, and that have a in the first position (i.e start with a), is equivalent to the number of compositions that start with aa and the number of all other compositions that start with ab, where $b \in A$ and $b \neq a$. Hence

$$C_{111}(a|j; x, y) = C_{111}(aa|j; x, y) + \sum_{b \in A,\ b \neq a} C_{111}(ab|j; x, y).$$

Removing a from the first position of the compositions counted by $\sum_{b \in A,\ b \neq a} C_{111}(ab|j; x, y)$, compositions of $n - a$ must be considered.

Thus the resulting generating functions must be multiplied by a factor of x^a to compensate for this difference, and now consist of one less part, thus j must be decremented by 1 to obtain

$$C_{111}(a|j; x, y) = C_{111}(aa|j; x, y) + x^a \sum_{b \in A,\ b \neq a} C_{111}(b|j - 1; x, y).$$

$\sum_{b \in A,\ b \neq a} C_{111}(b|j - 1; x, y)$ is equivalent to all compositions of $n - a$ with $j - 1$ parts in A that contain the pattern represented by 111 exactly a fixed number of times, less the number of compositions of that same type that start with a. Hence

$$C_{111}(a|j; x, y) = C_{111}(aa|j; x, y) + x^a C_{111}(j - 1; x, y) - x^a C_{111}(a|j - 1; x, y). \tag{2}$$

For $C_{111}(aa|j; x, y)$, where $a \in A$ is fixed, assuming $j \geq 3$ the generating function can be split into $|A|$ different cases: the number of compositions that start with aaa and all other compositions that start with aab where $b \in A$ and $b \neq a$. Thus

$$C_{111}\left(aa|j;x,y\right) = C_{111}\left(aaa|j;x,y\right) + \sum_{b\in A,\ b\neq a} C_{111}\left(aab|j;x,y\right).$$

For $C_{111}\left(aaa|j;x,y\right)$ the first a may be removed from the compositions, in doing so one occurrence of the pattern 111 is removed, *viz.* the first aaa is removed thus the generating function is multiplied by y to amend this. The number of parts is less 1, resulting in $j-1$ parts, and compositions of $n-a$ are now considered resulting in a factor of x^a,

$$C_{111}\left(aaa|j;x,y\right) = x^a y C_{111}\left(aa|j-1;x,y\right).$$

Considering $\sum_{b\in A,\ b\neq a} C_{111}\left(aab|j;x,y\right)$, the first two a's may be removed from the compositions. In doing so, the number of parts j is less 2, and n less $2a$, resulting in a factor of x^{2a}. Since aab is not analogous to 111 the number of occurrences of the pattern 111 are not decremented, giving

$$C_{111}\left(aa|j;x,y\right) = x^a y C_{111}\left(aa|j-1;x,y\right) + x^{2a} \sum_{b\in A,\ b\neq a} C_{111}\left(b|j-2;x,y\right).$$

By a similar argument as the one for $\sum_{b\in A,\ b\neq a} C_{111}\left(b|j-1;x,y\right)$, the sum

$$\sum_{b\in A,\ b\neq a} C_{111}\left(b|j-2;x,y\right)$$

is the same as the sum of all the compositions of $n-2a$ with $j-2$ parts in A that contain r occurrences of the pattern 111, less those compositions that start with a, thus

$$C_{111}\left(aa|j;x,y\right) = x^a y C_{111}\left(aa|j-1;x,y\right) + x^{2a} C_{111}\left(j-2;x,y\right) - x^{2a} C_{111}\left(a|j-2;x,y\right). \quad (3)$$

Multiplying (2) by z^j and summing over all $j \geq 1$ gives

$$C_{111}\left(a|x,y,z\right) = C_{111}\left(aa|x,y,z\right) + x^a z C_{111}\left(x,y,z\right) - x^a z C_{111}\left(a|x,y,z\right).$$

Similarly, multiplying (3) by z^j and summing over $j \geq 1$ gives

$$C_{111}\left(aa|x,y,z\right) = x^a y z C_{111}\left(aa|x,y,z\right) + x^{2a} z^2 C_{111}\left(x,y,z\right) - x^{2a} z^2 C_{111}\left(a|x,y,z\right).$$

Thus

$$C_{111}\left(aa|x,y,z\right) = \left(1 + x^a z\right) C_{111}\left(a|x,y,z\right) - x^a z C_{111}\left(x,y,z\right)$$

and,

$$C_{111}\left(aa|x,y,z\right) = \frac{x^{2a} z^2 C_{111}\left(x,y,z\right) - x^{2a} z^2 C_{111}\left(a|x,y,z\right)}{1 - x^a y z}.$$

Equating the right hand sides of these two equations and using algebraic manipulation gives

$$C_{111}\left(a|x,y,z\right) = \frac{x^a z\left(1 + \left(1-y\right) x^a z\right)}{1 + x^a z\left(1 + x^a z\right)\left(1-y\right)} C_{111}\left(x,y,z\right).$$

If this equation is summed for every $a \in A$, then

$$\sum_{a\in A} C_{111}\left(a|x,y,z\right) = C_{111}\left(x,y,z\right) \sum_{a\in A} \frac{x^a z\left(1 + \left(1-y\right) x^a z\right)}{1 + x^a z\left(1 + x^a z\right)\left(1-y\right)}$$

and considering (1) for $\tau = 111$, i.e. $C_{111}(x,y,z) = 1 + \sum_{a\in A} C_{111}(a|x,y,z)$,

$$C_{111}(x,y,z) - 1 = C_{111}(x,y,z) \sum_{a\in A} \frac{x^a z (1 + (1-y) x^a z)}{1 + x^a z (1 + x^a z)(1-y)}.$$

Thus

$$C_{111}(x,y,z) = \frac{1}{1 - \sum\limits_{a\in A} \frac{x^a z(1+(1-y)x^a z)}{1+x^a z(1+x^a z)(1-y)}}.$$

□

By Theorem 2.1, if $A = \mathbb{N}$, $a_i = i$, and $a_i \in A$ for $i \geq 1$ then the number of compositions which avoid the pattern 111 is given by

$$C_{111}(x,0,1) = \frac{1}{1 - \sum\limits_{i\geq 1} \frac{x^i(1+x^i)}{1+x^i(1+x^i)}}.$$

The first coefficients in the generated sequence are: 1, 1, 2, 3, 7, 13, 24, 46, 89, 170, 324, 618, 1183, 2260, 4318, 8249, 15765, 30123, 57556, 109973, 210137,401525, 767216, 1465963, 2801115, 5352275 which correspond to $n = 0, 1, \ldots, 25$ (see Appendix 1.1 for their derivation). It is interesting to note that because these compositions are avoiding the pattern 111 they only contain isolated levels, as two adjacent levels would result in the form $a_i a_i a_i$ for an $a_i \in A$ which is analogous to the pattern 111.

2.2 The patterns 112 and 221 (the statistics level+rise and level+drop)

In this section the generating function for the patterns 112 and 221 will be derived for compositions of n with j parts in A, in which these patterns occur r times.

Theorem 2.2. *Let A be an ordered subset of $\mathbb{N}$, with $|A| = d$. Then*

$$C_{112}(x,y,z) = \frac{1}{1 - \sum\limits_{j=1}^{d} \left(x^{a_j} z \prod\limits_{i=1}^{j-1} \left(1 - (1-y) x^{2a_i} z^2\right) \right)}$$

and,

$$C_{221}(x,y,z) = \frac{1}{1 - \sum\limits_{j=1}^{d} \left(x^{a_j} z \prod\limits_{i=j+1}^{d} \left(1 - (1-y) x^{2a_i} z^2\right) \right)}.$$

2.2.1 Proving the theorem for $C_{112}(x,y,z)$

Proof. The generating function for $C_{112}(x,y,z)$ will be considered first. A variant of equation (2) can be used, with a similar argument as for the proof of Theorem 2.1, to obtain

$$C_{112}(a|x,y,z) = C_{112}(aa|x,y,z) + x^a z C_{112}(x,y,z) - x^a z C_{112}(a|x,y,z),$$

giving

$$C_{112}\left(aa|x,y,z\right) = \left(1+x^a z\right) C_{112}\left(a|x,y,z\right) - x^a z C_{112}\left(x,y,z\right).$$

Similarly, a variant of (3) can be obtained as

$$C_{112}\left(aa|j;x,y\right) = C_{112}\left(aaa|j;x,y\right) + \sum_{b\in A,\ b\neq a} C_{112}\left(aab|j;x,y\right).$$

In this case $\sum_{b\in A,\ b\neq a} C_{112}\left(aab|j;x,y\right)$ is split into two cases, where $b<a$ and where $b>a$. For the latter case, removing the two a's from the start of the compositions reduces the occurrence of the pattern represented by 112 by one, as 112 is analogous to the pattern aab when $b>a$. This results in having to multiply the generating function by a factor of y to correct this.

In the case of the former, removing the two a's from the start of the compositions when $b<a$ does not remove an occurrence of the pattern 112 since when $b<a$, aab is not analogous to the pattern represented by 112. Hence the number of times that pattern occurs in the compositions stays unchanged. Removing the a from the start of the compositions for $C_{112}\left(aaa|j;x,y\right)$ results in $x^a C_{112}\left(aa|j-1;x,y\right)$, by similar arguments as for the proof of Theorem 2.1, as the compositions are of $n-a$ and have one less part. This gives

$$\begin{aligned}
C_{112}\left(aa|j;x,y\right) =& x^a C_{112}\left(aa|j-1;x,y\right) + \sum_{b\in A,\ b<a} C_{112}\left(aab|j;x,y\right) + \sum_{b\in A,\ b>a} C_{112}\left(aab|j;x,y\right)\\
=& x^a C_{112}\left(aa|j-1;x,y\right) + x^{2a} \sum_{b\in A,\ b<a} C_{112}\left(b|j-2;x,y\right)\\
&+ x^{2a} y \sum_{b\in A,\ b>a} C_{112}\left(b|j-2;x,y\right).
\end{aligned}$$

Multiplying this equation by z^j and summing over $j\geq 1$ results in

$$\begin{aligned}
\sum_{j\geq 1} C_{112}\left(aa|j;x,y\right) z^j =& x^a z \sum_{j\geq 1} C_{112}\left(aa|j-1;x,y\right) z^{j-1} + x^{2a} z^2 \sum_{\substack{b\in A\\ b<a}} \sum_{j\geq 1} C_{112}\left(b|j-2;x,y\right) z^{j-2}\\
&+ x^{2a} z^2 y \sum_{\substack{b\in A\\ b>a}} \sum_{j\geq 1} C_{112}\left(b|j-2;x,y\right) z^{j-2}.
\end{aligned}$$

This gives

$$\begin{aligned}
C_{112}\left(aa|x,y,z\right) =& x^a z C_{112}\left(aa|x,y,z\right) + x^{2a} z^2 \sum_{b\in A,\ b<a} C_{112}\left(b|x,y,z\right)\\
&+ x^{2a} z^2 y \sum_{b\in A,\ b>a} C_{112}\left(b|x,y,z\right)
\end{aligned}$$

and,

$$C_{112}\left(aa|x,y,z\right) = \frac{x^{2a} z^2}{1-x^a z} \sum_{b\in A,\ b<a} C_{112}\left(b|x,y,z\right) + \frac{x^{2a} z^2 y}{1-x^a z} \sum_{b\in A,\ b>a} C_{112}\left(b|x,y,z\right).$$

Combining this with

$$C_{112}\left(aa|x,y,z\right) = \left(1+x^a z\right) C_{112}\left(a|x,y,z\right) - x^a z C_{112}\left(x,y,z\right),$$

gives

$$\begin{aligned}C_{112}\left(a|x,y,z\right) =& \frac{x^{2a}z^2}{1-x^{2a}z^2}\sum_{b\in A,\ b<a} C_{112}\left(b|x,y,z\right) + \frac{x^{2a}z^2y}{1-x^{2a}z^2}\sum_{b\in A,\ b>a} C_{112}\left(b|x,y,z\right) \\ &+ \frac{x^a z}{1+x^a z} C_{112}\left(x,y,z\right).\end{aligned} \tag{4}$$

Solving this equation for $C_{112}\left(x,y,z\right)$ explicitly, for the case that A is finite, a system of equations for each $a_i \in A = \{a_1, a_2, \ldots, a_d\}$ is considered. To simplify this system new notation is used:

- $x_0 = C_{112}\left(x,y,z\right)$
- $x_i = C_{112}\left(a_i|x,y,z\right)$
- $\alpha_i = \frac{x^{2a_i}z^2}{1-x^{2a_i}z^2}$
- $\beta_i = \frac{x^{a_i}z}{1+x^{a_i}z}$.

This new notation reduces equation (4) to

$$x_i - \alpha_i \sum_{j<i} x_j - \alpha_i y \sum_{j>i} x_j - \beta_i x_0 = \alpha_i \text{ for } i = 1, \ldots, d.$$

For $i = 1, \ldots, d$, the system of equations is represented in matrix form as

$$\begin{pmatrix} -\beta_1 & 1 & -\alpha_1 y & -\alpha_1 y & \cdots & -\alpha_1 y & -\alpha_1 y \\ -\beta_2 & -\alpha_2 & 1 & -\alpha_2 y & \cdots & -\alpha_2 y & -\alpha_2 y \\ -\beta_3 & -\alpha_3 & -\alpha_3 & 1 & \cdots & -\alpha_3 y & -\alpha_3 y \\ -\beta_4 & -\alpha_4 & -\alpha_4 & -\alpha_4 & \cdots & -\alpha_4 y & -\alpha_4 y \\ \vdots & & & \vdots & & & \vdots \\ -\beta_{d-1} & -\alpha_{d-1} & -\alpha_{d-1} & -\alpha_{d-1} & \cdots & 1 & -\alpha_{d-1} y \\ -\beta_d & -\alpha_d & -\alpha_d & -\alpha_d & \cdots & -\alpha_d & 1 \\ 1 & -1 & -1 & -1 & \cdots & -1 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{d-2} \\ x_{d-1} \\ x_d \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \vdots \\ \alpha_{d-1} \\ \alpha_d \\ 1 \end{pmatrix}.$$

From this system, the explicit value of x_0 is obtained through the use of *Cramer's Rule* where we let M_d be the $(d+1)\times(d+1)$ maxtrix represented in the above system and N_d be the matrix obtained from M_d by replacing the first column with the column on the right hand side of the system giving

$$\begin{aligned} C_{112}\left(x,y,z\right) &= x_0 \\ &= \frac{\det\left(N_d\right)}{\det\left(M_d\right)}. \end{aligned}$$

To simplify the evaluation of $\det(M_d)$, the part of the matrix M_d without the first column is triangularized:

1. For $j = 1, 2, \ldots, d$, subtract α_j times the last row from the j-th row,

$$\begin{pmatrix} -\beta_1-\alpha_1 & 1+\alpha_1 & \alpha_1-\alpha_1 y & \alpha_1-\alpha_1 y & \cdots & \alpha_1-\alpha_1 y & \alpha_1-\alpha_1 y \\ -\beta_2-\alpha_2 & 0 & 1+\alpha_2 & \alpha_2-\alpha_2 y & \cdots & \alpha_2-\alpha_2 y & \alpha_2-\alpha_2 y \\ -\beta_3-\alpha_3 & 0 & 0 & 1+\alpha_3 & \cdots & \alpha_3-\alpha_3 y & \alpha_3-\alpha_3 y \\ -\beta_4-\alpha_4 & 0 & 0 & 0 & \cdots & \alpha_4-\alpha_4 y & \alpha_4-\alpha_4 y \\ \vdots & & & \vdots & & & \vdots \\ -\beta_{d-1}-\alpha_{d-1} & 0 & 0 & 0 & \cdots & 1+\alpha_{d-1} & \alpha_{d-1}-\alpha_{d-1} y \\ -\beta_d-\alpha_d & 0 & 0 & 0 & \cdots & 0 & 1+\alpha_d \\ 1 & -1 & -1 & -1 & \cdots & -1 & -1 \end{pmatrix}.$$

2. Subtract the j-th column from the $(j+1)$-th column, for $j = 1, \ldots, d$, starting from $j = d$, in descending order to obtain

$$\begin{pmatrix} -\beta_1-\alpha_1 & 1+\alpha_1 & -1-\alpha_1 y & 0 & \cdots & 0 & 0 \\ -\beta_2-\alpha_2 & 0 & 1+\alpha_2 & -1-\alpha_2 y & \cdots & 0 & 0 \\ -\beta_3-\alpha_3 & 0 & 0 & 1+\alpha_3 & \cdots & 0 & 0 \\ -\beta_4-\alpha_4 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots & & & \vdots \\ -\beta_{d-1}-\alpha_{d-1} & 0 & 0 & 0 & \cdots & 1+\alpha_{d-1} & -1-\alpha_{d-1} y \\ -\beta_d-\alpha_d & 0 & 0 & 0 & \cdots & 0 & 1+\alpha_d \\ 1 & -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

3. Now M_d is in a form for which its determinant can be easy calculated through cofactor expansion along the last row. Let X_d denote the matrix found by removing the last row and first column of M_d, and let Y_d be the matrix obtained by removing the last row and the second column. Clearly

$$\det(M_d) = (-1)^d \left(\det(X_d) + \det(Y_d)\right).$$

4. Now X_d is a triangular matrix thus

$$\det(X_d) = \prod_{j=1}^{d} (1+\alpha_j).$$

5. For Y_d, we can use cofactor expansion along the last row, to get

$$\det(Y_d) = (1+\alpha_d)\det(Y_{d-1}) - (\alpha_d+\beta_d)\prod_{j=1}^{d-1}(1+\alpha_j y),$$

where Y_{d-1} is obtained from Y_d by deleting the last row and column of Y_d.

By induction on d it will be shown that

$$\det(Y_d) = \sum_{j=1}^{d} (\alpha_j+\beta_j) \prod_{i=j+1}^{d} (1+\alpha_i) \prod_{i=1}^{j-1} (1+\alpha_i y)$$

$$= -\prod_{i=1}^{d}(1+\alpha_i)\sum_{j=1}^{d}\frac{\alpha_j+\beta_j}{1+\alpha_j}\prod_{i=1}^{j-1}\frac{1+\alpha_i y}{1+\alpha_i}.$$

Base case:

$$\begin{aligned}\det(Y_1) &= -(\alpha_1+\beta_1)\\ &= -\sum_{j=1}^{1}(\alpha_1+\beta_1)\prod_{j=2}^{1}(1+\alpha_j)\prod_{i=1}^{0}(1+\alpha_i y)\end{aligned}$$

Thus the base case holds.

Induction Hypothesis: Assume that the hypothesis holds for some $k \geq 1$, i.e.

$$\det(Y_k) = -\sum_{j=1}^{k}(\alpha_j-\beta_j)\prod_{i=j+1}^{k}(1+\alpha_i)\prod_{i=1}^{j-1}(1+\alpha_i y).$$

The Case: $k+1$

$$\begin{aligned}\det(Y_{k+1}) &= (1+\alpha_{k+1})\det(B_k) - (\alpha_{k+1}+\beta_{k+1})\prod_{j=1}^{k}(1+\alpha_j y)\\ &= (1+\alpha_{k+1})\left(-\sum_{j=1}^{k}(\alpha_j+\beta_j)\prod_{i=j+1}^{k}(1+\alpha_i)\prod_{i=1}^{j-1}(1+\alpha_i y)\right)\\ &\quad - (\alpha_{k+1}+\beta k+1)\prod_{j=1}^{k}(1+\alpha_j y)\\ &= -\sum_{j=1}^{k}(\alpha_j+\beta_j)\prod_{i=j+1}^{k+1}(1+\alpha_1)\prod_{i=1}^{j-1}(1+\alpha_i y)\\ &\quad - (\alpha_{k+1}+\beta_{k+1})\prod_{j=1}^{k}(1+\alpha_j y)\\ &= -\sum_{j=1}^{k+1}(\alpha_j+\beta_j)\prod_{i=j+1}^{k+1}(1+\alpha_i)\prod_{i=1}^{j-1}(1+\alpha_i y).\end{aligned}$$

Thus the $k+1$ case holds and hence

$$\det(Y_d) = -\prod_{i=1}^{d}(1+\alpha_i)\sum_{j=1}^{d}\frac{\alpha_j+\beta_j}{1+\alpha_j}\prod_{i=1}^{j-1}\frac{1+\alpha_i y}{1+\alpha_i}\ \text{ for all } d \geq 1.$$

6. Combining these results gives

$$\begin{aligned}\det(M_d) &= (-1)^d\left(\prod_{j=1}^{d}(1+\alpha_j) - \prod_{i=1}^{d}(1+\alpha_i)\sum_{j=1}^{d}\frac{\alpha_j+\beta_j}{1+\alpha_j}\prod_{i=1}^{j-1}\frac{1+\alpha_i y}{1+\alpha_i}\right)\\ &= (-1)^d\prod_{j=1}^{d}(1+\alpha_j)\left(1-\sum_{j=1}^{d}\frac{\alpha_j+\beta_j}{1+\alpha_j}\prod_{i=1}^{j-1}\frac{1+\alpha_i y}{1+\alpha_i}\right).\end{aligned}$$

7. To calculate $\det(Y_d)$, recall that the matrix N_d was obtained from M_d by replacing the first column of M_d by the column on the right hand side of the system, *viz.*

$$\begin{pmatrix} \alpha_1 & 1 & -\alpha_1 y & -\alpha_1 y & \cdots & -\alpha_1 y & -\alpha_1 y \\ \alpha_2 & -\alpha_2 & 1 & -\alpha_2 y & \cdots & -\alpha_2 y & -\alpha_2 y \\ \alpha_3 & -\alpha_3 & -\alpha_3 & 1 & \cdots & -\alpha_3 y & -\alpha_3 y \\ \alpha_4 & -\alpha_4 & -\alpha_4 & -\alpha_4 & \cdots & -\alpha_4 y & -\alpha_4 y \\ \vdots & & & \vdots & & & \vdots \\ \alpha_{d-1} & -\alpha_{d-1} & -\alpha_{d-1} & -\alpha_{d-1} & \cdots & 1 & -\alpha_{d-1} y \\ \alpha_d & -\alpha_d & -\alpha_d & -\alpha_d & \cdots & -\alpha_d & 1 \\ 1 & -1 & -1 & -1 & \cdots & -1 & -1 \end{pmatrix}.$$

To simplify the calculations for $\det(N_d)$ we add the first column of N_d to all the other columns of N_d to obtain a matrix with a triangularized part,

$$\begin{pmatrix} \alpha_1 & 1+\alpha_1 & \alpha_1 - \alpha_1 y & \alpha_1 - \alpha_1 y & \cdots & \alpha_1 - \alpha_1 y & \alpha_1 - \alpha_1 y \\ \alpha_2 & 0 & 1+\alpha_2 & \alpha_2 - \alpha_2 y & \cdots & \alpha_2 - \alpha_2 y & \alpha_2 - \alpha_2 y \\ \alpha_3 & 0 & 0 & 1+\alpha_3 & \cdots & \alpha_3 - \alpha_3 y & \alpha_3 - \alpha_3 y \\ \alpha_4 & 0 & 0 & 0 & \cdots & \alpha_4 - \alpha_4 y & \alpha_4 - \alpha_4 y \\ \vdots & & & \vdots & & & \vdots \\ \alpha_{d-1} & 0 & 0 & 0 & \cdots & 1+\alpha_{d-1} & \alpha_{d-1} - \alpha_{d-1} y \\ \alpha_d & 0 & 0 & 0 & \cdots & 0 & 1+\alpha_d \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

From this $\det(N_d)$ can be found by cofactor expansion along the last row,

$$\det(N_d) = (-1)^d \prod_{i=1}^{d} (1+\alpha_i).$$

8. Putting all this together

$$\begin{aligned} \frac{\det(N_d)}{\det(M_d)} &= \frac{(-1)^d \prod_{i=1}^d (1+\alpha_i)}{(-1)^d \prod_{j=1}^d (1+\alpha_j)\left(1 - \sum_{j=1}^d \frac{\alpha_j+\beta_j}{1+\alpha_j} \prod_{i=1}^{j-1} \frac{1+\alpha_i y}{1+\alpha_i}\right)} \\ &= \frac{1}{1 - \sum_{j=1}^d \left(x^{a_j} z \prod_{i=1}^{j-1} \left(1 - (1-y)\, x^{2a_i} z^2\right)\right)}. \end{aligned}$$

Thus an expression for $C_{112}(x,y,z)$ is obtained for the finite case. In the infinite case when $|A| = \infty$ take the limit as $d \to \infty$ of the above expression.

2.2.2 Proving the theorem for $C_{221}(x, y, z)$

Now considering the generating function for $C_{221}(x,y,z)$, in the case where A is finite, a variant of equation (2) and a similar process to the one for $C_{112}(x,y,z)$ is used to obtain

$$\begin{aligned}C_{221}(a|j;x,y) =& C_{221}(aa|j;x,y) + \sum_{b\in A, b\neq a} C_{221}(ab|j;x,y)\\ =& C_{221}(aa|j;x,y) + x^a \sum_{b\in A, b\neq a} C_{221}(b|j-1;x,y)\\ =& C_{221}(aa|j;x,y) + x^a C_{221}(j-1;x,y) + x^a C_{221}(a|j-1;x,y)\end{aligned}$$

which, after multiplying by z^j and summing over all $j \geq 1$, gives

$$C_{221}(a|x,y,z) = C_{221}(aa|x,y,z) + x^a z C_{221}(x,y,z) + x^a z C_{221}(a|x,y,z).$$

Another variant of equation (3) gives

$$\begin{aligned}C_{221}(aa|j;x,y) =& C_{221}(aaa|j;x,y) + \sum_{b\in A, b\neq a} C_{221}(aab|j;x,y)\\ =& x^a C_{221}(aa|j-1;x,y) + \sum_{b\in A, b<a} C_{221}(aab|j;x,y) + \sum_{b\in A, b>a} C_{221}(aab|j;x,y)\\ =& x^a C_{221}(aa|j-1;x,y) + x^{2a} y \sum_{b\in A, b<a} C_{221}(b|j-1;x,y)\\ &+ x^{2a} \sum_{b\in A, b>a} C_{221}(b|j;x,y).\end{aligned}$$

Where $x^{2a} y \sum_{b\in A, b<a} C_{221}(b|j-1;x,y)$ is obtained from $\sum_{b\in A, b<a} C_{221}(aab|j;x,y)$, by removing the first two a's from the start of the compositions, resulting in a factor of x^{2a}, and as aab is analogous to the pattern represented by 221 when $b < a$, removing these two a's results in one less occurrence of the pattern in the compositions, hence the factor of y.

Multiplying by z^j and summing over all $j \geq 1$ gives

$$\begin{aligned}C_{221}(aa|x,y,z) =& x^a z C_{221}(aa|x,y,z) + x^{2a} z^2 y \sum_{b\in A, b<a} C_{221}(b|x,y,z)\\ &+ x^{2a} z^2 \sum_{b\in A, b>a} C_{221}(b|x,y,z),\end{aligned}$$

which simplifies to

$$C_{221}(aa|x,y,z) = \frac{x^{2a} z^2 y}{1 - x^a y} \sum_{b\in A, b<a} C_{221}(b|x,y,z) + \frac{x^{2a} z^2}{1 - x^a z} \sum_{b\in A, b>a} C_{221}(b|x,y,z).$$

From the first equation derived for $C_{221}(a|x,y,z)$,

$$C_{221}(aa|x,y,z) = (1 - x^a z)\, C_{221}(a|x,y,z) - x^a z C_{221}(x,y,z)$$

and equating this to the second,

$$\begin{aligned}(1-x^az)\,C_{221}(a|x,y,z)-x^azC_{221}(x,y,z)=&\frac{x^{2a}z^2y}{1-x^ay}\sum_{b\in A,b<a}C_{221}(b|x,y,z)\\&+\frac{x^{2a}z^2}{1-x^az}\sum_{b\in A,b>a}C_{221}(b|x,y,z)\,.\end{aligned}$$

This simplifies to

$$\begin{aligned}C_{221}(a|x,y,z)=&\frac{x^{2a}z^2y}{1-x^{2a}z^2}\sum_{b\in A,b<a}C_{221}(b|x,y,z)+\frac{x^{2a}z^2}{1-x^{2a}z^2}\sum_{b\in A,b>a}C_{221}(b|x,y,z)\\&+\frac{x^az}{1-x^az}C_{221}(x,y,z)\,.\end{aligned}$$

To simplify this system let

- $x_0=C_{221}(x,y,z)$,
- $x_i=C_{221}(a_i|x,y,z)$,
- $\alpha_i=\frac{x^{2a_i}z^2}{1-x^{2a_i}z^2}$,
- $\beta_i=\frac{x^{a_i}z}{1+x^{a_i}z}$.

Now

$$x_i-\alpha_iy\sum_{j<i}x_j-\alpha_i\sum_{j>i}x_j-\beta_ix_0=\alpha_i,\ \text{for } i=1,\ldots,d.$$

For $i=1,\ldots,d$, we obtain this system of equations, represented in matrix form as

$$\begin{pmatrix}-\beta_1&1&-\alpha_1&-\alpha_1&\cdots&-\alpha_1&-\alpha_1\\-\beta_2&-\alpha_2y&1&-\alpha_2&\cdots&-\alpha_2&-\alpha_2\\-\beta_3&-\alpha_3y&-\alpha_3y&1&\cdots&-\alpha_3&-\alpha_3\\-\beta_4&-\alpha_4y&-\alpha_4y&-\alpha_4y&\cdots&-\alpha_4&-\alpha_4\\\vdots&&&\vdots&&&\vdots\\-\beta_{d-1}&-\alpha_{d-1}y&-\alpha_{d-1}y&-\alpha_{d-1}y&\cdots&1&-\alpha_{d-1}\\-\beta_d&-\alpha_dy&-\alpha_dy&-\alpha_dy&\cdots&-\alpha_dy&1\\1&-1&-1&-1&\cdots&-1&-1\end{pmatrix}\begin{pmatrix}x_0\\x_1\\x_2\\x_3\\\vdots\\x_{d-2}\\x_{d-1}\\x_d\end{pmatrix}=\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\\\vdots\\\alpha_{d-1}\\\alpha_d\\1\end{pmatrix}.$$

Using Cramer's Rule, and letting M_d be the $(d+1)\times(d+1)$ matrix represented in the above system and N_d be the matrix obtained from M_d by replacing the first column with the column on the right hand side of the system, gives

$$\begin{aligned}C_{221}(x,y,z)=&x_0\\=&\frac{\det(N_d)}{\det(M_d)}.\end{aligned}$$

To evaluate $\det(M_d)$, the part of M_d without the first row is again triangularized:

1. For $j = 1, 2, \ldots, d$, subtract a_j times the last from the j-th row to obtain

$$\begin{pmatrix} -\beta_1-\alpha_1 & 1+\alpha_1 & 0 & 0 & \cdots & 0 & 0 \\ -\beta_2-\alpha_2 & \alpha_2-\alpha_2 y & 1+\alpha_2 & 0 & \cdots & 0 & 0 \\ -\beta_3-\alpha_3 & \alpha_3-\alpha_3 y & \alpha_3-\alpha_3 y & 1+\alpha_3 & \cdots & 0 & 0 \\ -\beta_4-\alpha_4 & \alpha_4-\alpha_4 y & \alpha_4-\alpha_4 y & \alpha_4-\alpha_4 y & \cdots & 0 & 0 \\ \vdots & & & \vdots & & & \vdots \\ -\beta_{d-1}-\alpha_{d-1} & \alpha_{d-1}-\alpha_{d-1} y & \alpha_{d-1}-\alpha_{d-1} y & \alpha_{d-1}-\alpha_{d-1} y & \cdots & 1+\alpha_{d-1} & 0 \\ -\beta_d-\alpha_d & \alpha_d-\alpha_d y & \alpha_d-\alpha_d y & \alpha_d-\alpha_d y & \cdots & \alpha_d-\alpha_d y & 1+\alpha_d \\ 1 & -1 & -1 & -1 & \cdots & -1 & -1 \end{pmatrix}.$$

2. Subtract the $(j+1)$-th column from the j-th column for $j = 2, 3, 4, \ldots, d$, starting from $j = d$, and descending, to get

$$\begin{pmatrix} -\beta_1-\alpha_1 & 1+\alpha_1 & 0 & 0 & \cdots & 0 & 0 \\ -\beta_2-\alpha_2 & -1-\alpha_2 y & 1+\alpha_2 & 0 & \cdots & 0 & 0 \\ -\beta_3-\alpha_3 & 0 & -1-\alpha_3 y & 1+\alpha_3 & \cdots & 0 & 0 \\ -\beta_4-\alpha_4 & 0 & 0 & -1-\alpha_4 y & \cdots & 0 & 0 \\ \vdots & & & \vdots & & & \vdots \\ -\beta_{d-1}-\alpha_{d-1} & 0 & 0 & 0 & \cdots & 1+\alpha_{d-1} & 0 \\ -\beta_d-\alpha_d & 0 & 0 & 0 & \cdots & -1-\alpha_d y & 1+\alpha_d \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

3. Let X_d denote the matrix found by removing the last row and first column of M_d, and let Y_d be the matrix obtained by removing the last row and the last column of M_d. Using cofactor expansion along the last row,

$$\det(M_d) = (-1)^d \left(\det(X_d) + \det(Y_d)\right).$$

4. Now X_d is a triangular matrix, thus

$$\det(X_d) = \prod_{j=1}^{d} (1+\alpha_j).$$

5. For Y_d, cofactor expansion can be used along the last row to get

$$\det(Y_d) = (-1-y\alpha_d)\det(Y_{d-1}) - (\alpha_d+\beta_d)\prod_{j=1}^{d-1}(1-\alpha_j),$$

where Y_{d-1} is obtained from Y_d by deleting the last row and column.

6. Now it is shown by induction on d that

$$\det(Y_d) = -\sum_{j=1}^{d}(\alpha_j+\beta_j)\prod_{i=j+1}^{d}(1+\alpha_i y)\prod_{i=1}^{j-1}(1+\alpha_i)$$

$$= -\prod_{i=1}^{d}(1+\alpha_i)\sum_{j=1}^{d}\frac{\alpha_j+\beta_j}{1+\alpha_j}\prod_{i=j+1}^{d}\left(\frac{1+\alpha_i y}{1+\alpha_i}\right).$$

Base Case:

$$\begin{aligned}\det(Y_1) &= -(\alpha_1+\beta_1)\\ &= -\sum_{j=1}^{1}(\alpha_j+\beta_j)\prod_{i=j+1}^{1}(1+\alpha_i y)\prod_{i=1}^{j-1}(1+\alpha_i)\end{aligned}$$

so the hypothesis holds for the base case.

Induction Hypothesis: Assume that the hypothesis holds for some $k \geq 1$.

Case: k+1

Now

$$\begin{aligned}\det(Y_{k+1}) &= (-1-y\alpha_{k+1})\det(Y_k) - (\alpha_{k+1}+\beta_{k+1})\prod_{j=1}^{k}(1+\alpha_j)\\ &= (-1-y\alpha_{k+1})\left(-\sum_{j=1}^{k}(\alpha_j+\beta_j)\prod_{i=j+1}^{k}(1+\alpha_i y)\prod_{i=1}^{j-1}(1+\alpha_i)\right)\\ &\quad -(\alpha_{k+1}+\beta_{k+1})\prod_{j=1}^{k}(1+\alpha_j)\end{aligned}$$

by the induction hypothesis. Hence

$$\begin{aligned}\det(Y_{k+1}) &= \sum_{j=1}^{k}(\alpha_j+\beta_j)\prod_{i=j+1}^{k+1}(1+\alpha_i y)\prod_{i=1}^{j-1}(1+\alpha_i) - (\alpha_{k+1}+\beta_{k+1})\prod_{j=1}^{k}(1+\alpha_j)\\ &= -\left(\sum_{j=1}^{k}(\alpha_j+\beta_j)\prod_{i=j+1}^{k+1}(1+\alpha_i y)\prod_{i=1}^{j-1}(1+\alpha_i) + (\alpha_{k+1}+\beta_{k+1})\prod_{j=1}^{k}(1+\alpha_j)\right)\\ &= -\sum_{j=1}^{k+1}(\alpha_j+\beta_j)\prod_{i=j+1}^{k+1}(1+\alpha_i y)\prod_{i=1}^{j-1}(1+\alpha_i).\end{aligned}$$

Thus, the hypothesis holds for the $(k+1)$-th case and therefore

$$\begin{aligned}\det(Y_d) &= -\sum_{j=1}^{d}(\alpha_j+\beta_j)\prod_{i=j+1}^{d}(1+\alpha_i y)\prod_{i=1}^{j-1}(1+\alpha_i)\\ &= -\prod_{i=1}^{d}(1+\alpha_i)\sum_{j=1}^{d}\frac{\alpha_j+\beta_j}{1+\alpha_j}\prod_{i=j+1}^{d}\left(\frac{1+\alpha_i y}{1+\alpha_i}\right)\end{aligned}$$

for all $d \geq 1$.

7. Combining these results gives

$$\det\left(M_d\right) = (-1)^d \left(\prod_{j=1}^{d} \left(1+\alpha_j\right) - \prod_{i=1}^{d} \left(1+\alpha_i\right) \sum_{j=1}^{d} \frac{\alpha_j+\beta_j}{1+\alpha_j} \prod_{i=j+1}^{d} \left(\frac{1+\alpha_i y}{1+\alpha_i}\right) \right).$$

8. Now, as before,

$$\det\left(N_d\right) = (-1)^d \prod_{i=1}^{d} \left(1+\alpha_i\right).$$

9. Finally, combining all these results gives

$$\begin{aligned} \frac{\det\left(Y_d\right)}{\det\left(X_d\right)} &= \frac{(-1)^d \prod_{i=1}^{d} \left(1+\alpha_i\right)}{(-1)^d \prod_{j=1}^{d} \left(1+\alpha_j\right) \left(1 - \sum_{j=1}^{d} \frac{\alpha_j+\beta_j}{1+\alpha_j} \prod_{i=j+1}^{d} \left(\frac{1+\alpha_i y}{1+\alpha_i}\right)\right)} \\ &= \frac{1}{1 - \sum_{j=1}^{d} \frac{\alpha_j+\beta_j}{1+\alpha_j} \prod_{i=j+1}^{d} \left(\frac{1+\alpha_i y}{1+\alpha_i}\right)} \\ &= \frac{1}{1 - \sum_{j=1}^{d} \left(x^{a_j} z \prod_{i=j+1}^{d} \left(1 - (1-y)\, x^{2a_i} z^2\right)\right)}. \end{aligned}$$

And thus, for the case where A is finite an explicit expression for $C_{221}(x, y, z)$ is obtained. In the infinite case, when $|A| = \infty$, the limit as $d \to \infty$ of the above expression is taken.

□

From these two results,

$$C_{112}(x, 0, 1) = \frac{1}{1 - \sum_{j \geq 1} x^j \prod_{i=1}^{j-1} \left(1 - x^{2i}\right)}$$

and,

$$C_{221}(x, 0, 1) = \frac{1}{1 - \sum_{i \geq 1} x^i \prod_{j \geq i+1} \left(1 - x^{2j}\right)}$$

which are the two generating functions for the total number of compositions of n, with m parts in the set A, that avoid the patterns represented by 112 and 221 respectively. The following sequence of coefficients is generated by each generating function: $1, 1, 2, 4, 7, 13, 24, 43, 78, 142, 256, 463, 838, 1513, 2735, 4944, 8931, 16139, 29164, 52693, 95213$ and $1, 1, 2, 4, 8, 15, 30, 58, 113, 220, 429, 835, 1627, 3169, 6172, 12023, 23419, 45616, 88853, 173073, 337118$ respectively for $n = 0, 1, \ldots, 20$ (see Appendix 1.2 for their derivation.)

2.2.3 An interesting observation on the differences in the size of the sets of the compositions counted by the generating function C_{112} and C_{221}

Theorem 2.3. *The set of compositions that contain patterns represented by* 112 *is larger than the set of compositions that contain patterns represented by* 221.

Fundamentally this is caused by patterns represented by 112, say *aab* where $a < b$, being smaller when the parts are summed, than patterns represented by 221, say *bba*, i.e. $2a+b < 2b+a$ when $a < b$. Thus patterns represented by 112 form a smaller part of the composition in terms of the sum of their parts than those represented by 221. To show this formally:

Proof. Let C_{112} denote the set of all compositions of n that contain the pattern represented by 112, and let C_{221} denote the set of all compositions of n that contain the pattern represented by 221. Define a map

$$\phi : C_{221} \to C_{112}$$

that maps compositions from C_{221} to C_{112}. It does so by replacing each occurrence of the pattern represented by 221 with the pattern represented by 1121 using the same parts, and each occurrence of the pattern represented by 1121 with 221 using the same parts. For a composition σ from C_{221}, every sequence aab in σ where $a, b \in \mathbb{N}$, and where $a > b$, is replaced with $bbab$, and every sequence $bbab$ in σ is replaced with aab. This results in a new composition of n that now contains the sequences with patterns represented by 112, *viz* 1121, and is thus contained in C_{112}.

ϕ is injective:

$$\text{Let } \sigma, \sigma' \in C_{221} \text{ such that } \phi(\sigma) = \phi(\sigma').$$

Every part of a composition in C_{221} can be part of a sequence representing 221, or 1121 or not part of either of the sequences. For every part α of σ, and every part α' of σ' that is not part of a sequence of parts representing the patterns 1121 or 221

$$\phi(\alpha) = \alpha \quad \text{and} \quad \phi(\alpha') = \alpha'.$$

As $\phi(\sigma) = \phi(\sigma')$ the parts α and α' that are not part of any sequence representing 221 or 1121 are identical for each of the compositions σ and σ'. Consider now a sequence of parts $\alpha\alpha\beta$ in σ representing the pattern 221 , and a similar sequence of parts $\alpha'\alpha'\beta'$ in σ'. By the definition of ϕ

$$\phi(\alpha\alpha\beta) = \beta\beta\alpha\beta \quad \text{and} \quad \phi(\alpha'\alpha'\beta') = \beta'\beta'\alpha'\beta'$$

and as $\phi(\sigma) = \phi(\sigma')$ we have that

$$\beta\beta\alpha\beta = \beta'\beta'\alpha'\beta'$$

so $\alpha = \alpha'$ and $\beta = \beta'$, and thus each occurrence of the sequence of parts representing 221 in σ is identical to the same sequence of parts in σ'.

Consider now a sequence of parts $\delta\delta\gamma\delta$ in σ representing the pattern 1121 , and a similar sequence of parts $\delta'\delta'\gamma'\delta'$ in σ'. By the definition of ϕ,

$$\phi(\delta\delta\gamma\delta) = \gamma\gamma\delta \quad \text{and} \quad \phi(\delta'\delta'\gamma'\delta') = \gamma'\gamma'\delta'$$

and as $\phi(\sigma) = \phi(\sigma')$, we have that

$$\delta\delta\gamma\delta = \delta'\delta'\gamma'\delta'$$

so $\delta = \delta'$ and $\gamma = \gamma'$, and thus each occurrence of the sequence of parts representing 1121 in σ is identical to the same sequence of parts in σ'. Thus $\sigma = \sigma'$ and ϕ is *injective.*

ϕ is not surjective:

Consider a composition $\sigma \in C_{112}$ where every occurrence or the sequence of parts representing the pattern 112 is not followed by a part representing a 1 in that pattern. As ϕ maps every sequence of parts representing the pattern 221 to the sequence representing the pattern 1121 using the same parts, it can never map a composition in C_{221} to the composition $\sigma \in C_{112}$ without there being at least one occurrence of the sequence representing the pattern 1121 in σ. Hence, ϕ is not surjective.

Now as ϕ is injective it implies that $|C_{221}| \leq |C_{112}|$, and as it is not surjective it implies that $|C_{221}| < |C_{112}|$. □

2.3 The pattern 123 (the statistic rise+rise)

Concerning the pattern represented by 123, an explicit form for the generating function for the number of compositions of n with j parts in A in which the pattern represented by 123 occurs a particular number of times will be obtained.

Theorem 2.4. *Let A be any ordered subset of $\mathbb{N}$, with $|A| = d$. Then*

$$C_{123}(x,y,z) = \frac{1}{1 - t^1(A) - \sum_{p=3}^{d} \sum_{j=0}^{p-3} \binom{p-3}{j} t^{p+j}(A)(y-1)^{p-2}}$$

where $t^p(A) = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq d} z^p \prod_{j=1}^{p} x^{a_{i_j}}$, and the coefficients of $x^n y^r z^j$ count the number of compositions of n with j parts in A, in which the pattern 123 occurs exactly r times.

Proof. Consider the case when A is finite, say $A = \{a_1, a_2, \ldots, a_d\}$. Let σ be a composition of n that consists of exactly m parts in A. Let $A_k = A \backslash \{a_1, \ldots, a_k\} = \{a_{k+1}, a_{k+2}, \ldots, a_d\}$. To derive the recursion that will be used in the derivation of an explicit form of the generating function $C_{123}(x,y,z)$ the composition σ will be partitioned into parts: some parts will consist of pieces from A, and others from A_k. These compositions will be partitioned according to the position of the pieces $a_1, a_2, \ldots$. To simplify this process define D^{A_k} to be the generating function for the number of compositions σ of n, with σ having m parts in A_k, and in which the pattern 123 occurs in $a\sigma$, with $a \notin A_k$, exactly r times.

Where $A = \{a_1, a_2, \ldots, a_d\}$, a composition σ of n with m parts in A may either contain the element a_1 or consist entirely of parts from $A \backslash \{a_1\} = A_1$. The generating function for compositions of n with m parts in A_1 that do not contain a_1 is given by $C_{123}^{A_1}(x,y,z)$. If the composition σ contains at least a single occurrence of a_1 then it can be represented as

$$\sigma = \bar{\sigma} a_1 \sigma_{k+1} \ldots \sigma_m$$

where $\bar{\sigma}$ is a composition with parts in A_1, i.e. having generating function $C_{123}^{A_1}(x,y,z)$, and the section following $\bar{\sigma}$ having generating function $C_{123}^{A}(a_1|x,y,z)$. Thus the generating function for this σ, when combining both cases, is given by

$$C_{123}^{A}(x,y,z) = C_{123}^{A_1}(x,y,z) + C_{123}^{A_1}(x,y,z)\, C_{123}^{A}(a_1|x,y,z). \tag{5}$$

For the generating function $C_{123}^{A}(a_1|x,y,z)$, of compositions σ of n with m parts in A starting with a_1 that contain the pattern 123 exactly r times, the inclusion of the element a_1 in the compositions is again considered: if there is only a single occurrence in the composition then $C_{123}^{A}(a_1|x,y,z,)$ may be rewritten as

$$x^{a_1}zD^{A_1}(x,y,z),$$

removing the first a_1 from the start of these compositions, resulting in the factor $x^{a_1}z$. After this a_1 is removed, the new compositions are those generated by $D^{A_1}(x,y,z)$.

In the case that a_1 occurs at least twice, the compositions resemble

$$\sigma = a_1\bar{\sigma}a_1\sigma_{k+1}\dots\sigma_m$$

where $\bar{\sigma}$ is a composition with parts strictly from A_1, i.e. generated by $C_{123}^{A_1}(x,y,z)$. In this case a_1 may be again removed from the start of the compositions, resulting now in the generating function

$$x^{a_1}zD^{A_1}(x,y,z)\,C_{123}^{A}(a_1|x,y,z).$$

The factor $x^{a_1}z$ is a result of removing the first a_1, $D^{A_1}(x,y,z)$ is the generating function of $\bar{\sigma}$, and $C_{123}^{A}(a_1|x,y,z)$ is the generating function for the section of the composition σ that follows $\bar{\sigma}$. Combining these two cases,

$$C_{123}^{A}(a_1|x,y,z) = x^{a_1}zD^{A_1}(x,y,z) + x^{a_1}zD^{A_1}(x,y,z)\,C_{123}^{A}(a_1|x,y,z).$$

Thus

$$C_{123}^{A}(a_1|x,y,z) = \frac{x^{a_1}zD^{A_1}(x,y,z)}{1 - x^{a_1}zD^{A_1}(x,y,z)}.$$

Substitute this into equation (5) to obtain

$$C_{123}^{A}(x,y,z) = C_{123}^{A_1}(x,y,z) + C_{123}^{A_1}(x,y,z)\left(\frac{x^{a_1}zD^{A_1}(x,y,z)}{1 - x^{a_1}zD^{A_1}(x,y,z)}\right).$$

Multiply this by $\left(1 - x^{a_1}zD^{A_1}(x,y,z)\right)$ to get

$$\begin{aligned} C_{123}^{A}(x,y,z)\left(1 - x^{a_1}zD^{A_1}(x,y,z)\right) =& C_{123}^{A_1}(x,y,z)\left(1 - x^{a_1}zD^{A_1}(x,y,z)\right) \\ &+ C_{123}^{A_1}(x,y,z)\,x^{a_1}zD^{A_1}(x,y,z). \end{aligned}$$

This simplifies to

$$C_{123}^{A}(x,y,z) = \frac{C_{123}^{A_1}(x,y,z)}{1 - x^{a_1}zD^{A_1}(x,y,z)}. \tag{6}$$

$D^{A_1}(x,y,z)$ is the generating function for compositions σ of n with m parts in A_1, for which $a\sigma$ contains the pattern represented by 123 exactly r times when $a \notin A_1$, i.e. $a = a_1$. So $a_1\sigma$ contains the pattern represented by 123 exactly r times. Now the inclusion of the element a_2 in the composition σ is considered: if a_2 is not in the composition σ then, by definition, $D^{A_2}(x,y,z)$ is the generating function for the number of occurrences of the pattern 123 in $a\sigma$, where $a \notin A_2$. However if a_2 is in the composition σ then σ can be written as

$$\sigma = \bar{\sigma}^1 a_2 \bar{\sigma}^2 a_2 \dots a_2 \bar{\sigma}^{l+2} \text{ for } l \geq 0,$$

where the $\bar{\sigma}^j$ are compositions with parts in A_2.

A number of sub-cases with regard to this σ now arise:

1. $\bar{\sigma}^1$, $\bar{\sigma}^2$ are both empty compositions: in this case

$$
\begin{aligned}
a_1\sigma =& a_1a_2 \text{ or}\\
a_1\sigma =& a_1a_2a_2\bar{\sigma}^3 \ldots a_2\bar{\sigma}^{l+2} \text{ for } l \geq 1.
\end{aligned}
$$

So $\sigma = a_2$ or $\sigma = a_2a_2\bar{\sigma}^3 \ldots a_2\bar{\sigma}^{l+2}$ for $l \geq 1$. The generating function for $\sigma = a_2$ is $x^{a_2}z$. The generating function for $\sigma = a_2a_2\bar{\sigma}^3 \ldots a_2\bar{\sigma}^{l+2}$ is a little more complex. Removing the first occurrence of a_2 results in a factor of $x^{a_2}z$, and removing the rest of the occurrences of a_2 for each $l \geq 1$, results in a product of $(x^{a_2}z)^l$ as well as the sum of the product of the generating functions, $\left(D^{A_2}(x,y,z)\right)^l$ for each $\bar{\sigma}^l$. All of this combined gives the generating function for the case where $\bar{\sigma}^1$ and $\bar{\sigma}^2$ are empty as

$$
\begin{aligned}
x^{a_2}z + x^{a_2}z\sum_{l\geq 1}\left(x^{a_2}zD^{A_2}(x,y,z)\right)^l =& x^{a_2}z\sum_{l\geq 0}\left(x^{a_2}zD^{A_2}(x,y,z)\right)^l\\
=& \frac{x^{a_2}z}{1-x^{a_2}zD^{A_2}(x,y,z)}.
\end{aligned}
$$

2. Consider now the case when $\bar{\sigma}^2$ is not empty, and $\bar{\sigma}^1$ is still empty. Here:

$$
\begin{aligned}
a_1\sigma =& a_1a_2\bar{\sigma}^2 \text{ or}\\
a_1\sigma =& a_1a_2\bar{\sigma}^2a_2\bar{\sigma}^3 \ldots a_2\sigma^{l+2} \text{ for } l \geq 1.
\end{aligned}
$$

The first thing to note here is that since $\bar{\sigma}^2$ is a composition in A_2, it consists of parts from $\{\sigma_3, \sigma_4, \ldots, \sigma_d\}$. Thus if the first a_2 is removed from $a_1\sigma = a_1a_2\bar{\sigma}^2$ an occurrence of the pattern represented by 123 is also removed, therefore the derived generating function must be multiplied by y to allow for this. Now $\sigma = a_2\bar{\sigma}^2$ or $\sigma = a_2\bar{\sigma}^2a_2\bar{\sigma}^3 \ldots a_2\bar{\sigma}^{l+2}$ for $l \geq 1$. The generating function for $\sigma = a_2\bar{\sigma}^2$ is given by $yx^{a_2}z\left(D^{A_2}(x,y,z)-1\right)$, where the factor $yx^{a_2}z$ is a result of removing the a_2 from the start of the composition, and $D^{A_2}(x,y,z)-1$ is the generating function, less the empty case 1, for $\bar{\sigma}^2$. The generating function for $\sigma = a_2\bar{\sigma}^2a_2\bar{\sigma}^3 \ldots a_2\bar{\sigma}^{l+2}$ can be found in a similar way as the first sub-case. The first part is the same as for $\sigma = a_2\bar{\sigma}^2$, and the part that follows is simply removing all the other occurrences of a_2 as well as the sum of the product for each $l \geq 1$ of the resulting generating functions, and thus the generating function for the case where $\bar{\sigma}^1$ is empty and $\bar{\sigma}^2$ is not empty is given by

$$
yx^{a_2}z\left(D^{A_2}(x,y,z)-1\right)\sum_{l\geq 0}\left(x^{a_2}zD^{A_2}(x,y,z)\right)^l = \frac{yx^{a_2}z\left(D^{A_2}(x,y,z)-1\right)}{1-x^{a_2}zD^{A_2}(x,y,z)}.
$$

3. If $\bar{\sigma}^1$ is not the empty composition, and $\bar{\sigma}^2$ is the empty composition then either

$$
\sigma = \bar{\sigma}^1a_2,
$$

or

$$
\sigma = \bar{\sigma}^1a_2a_2\bar{\sigma}^3 \ldots a_2\sigma^{l+2} \text{ for } l \geq 1.
$$

Now the generating function for $\bar{\sigma}^1a_2$ is similar to the previous $a_2\bar{\sigma}^2$ in the second sub-case, as $\bar{\sigma}^1$ and $\bar{\sigma}^2$ are generated by the same functions. There is however no factor of y, as no occurrence of the pattern represented by 123 is removed by removing the a_2. This means that the generating function for σ in this case is the same as the generating function for the previous case without the factor of y:

$$x^{a_2}z\left(D^{A_2}(x,y,z)-1\right)\sum_{l\geq 0}\left(x^{a_2}zD^{A_2}(x,y,z)\right)^l=\frac{x^{a_2}z\left(D^{A_2}(x,y,z)-1\right)}{1-x^{a_2}zD^{A_2}(x,y,z)}.$$

4. When $\bar{\sigma}^1$ and $\bar{\sigma}^2$ are both non-empty compositions, their generating function is given by $\left(D^{A_2}(x,y,z)-1\right)$. In the case where this is true,

$$\sigma=\bar{\sigma}^1a_1\bar{\sigma}^2$$

or

$$\sigma=\bar{\sigma}^1a_2\bar{\sigma}^2a_2\dots a_2\bar{\sigma}^{l+2}\text{ for }l\geq 1.$$

Now the generating function for $\bar{\sigma}^1a_2\bar{\sigma}^2$ is given as

$$\left(D^{A_2}(x,y,z)-1\right)zx^{a_2}\left(D^{A_2}(x,y,z)-1\right),$$

or

$$zx^{a_2}\left(D^{A_2}(x,y,z)-1\right)^2,$$

where the factor of $x^{a_2}z$ results from removing the a_2, similarly to the previous cases. The generating function for $a_2\bar{\sigma}^3\dots\bar{\sigma}^{l+2}$ is given by the sum of the products

$$\sum_{l\geq 1}\left(x^{a_2}zD^{A_2}(x,y,z)\right)^l$$

as was seen in the previous three cases. Combining these results gives the generating function for the case where $\bar{\sigma}^1$ and $\bar{\sigma}^2$ are not the empty compositions as

$$zx^{a_2}\left(D^{A_2}(x,y,z)-1\right)^2\sum_{l\geq 0}\left(x^{a_2}zD^{A_2}(x,y,z)\right)^l=\frac{x^{a_2}z\left(D^{A_2}(x,y,z)-1\right)^2}{1-x^{a_2}zD^{A_2}(x,y,z)}.$$

Finally, combining all four of the above sub-cases, where σ contains the part a_2, together with the case where it doesn't, and using the notation $D^A:=D^A(x,y,z)$, the following expression for D^{A_1} is obtained:

$$\begin{aligned}D^{A_1}=&D^{A_2}+\frac{x^{a_2}z}{1-x^{a_2}zD^{A_2}}+\frac{x^{a_2}zy\left(D^{A_2}-1\right)}{1-x^{a_2}zD^{A_2}}+\frac{x^{a_2}z\left(D^{A_2}-1\right)}{1-x^{a_2}zD^{A_2}}+\frac{x^{a_2}z\left(D^{A_2}-1\right)^2}{1-x^{a_2}zD^{A_2}}\\=&\frac{\left(1-x^{a_2}z\left(1-y\right)\right)D^{A_2}+x^{a_2}z\left(1-y\right)}{1-x^{a_2}zD^{A_2}}.\end{aligned}\tag{7}$$

An explicit expression for D^A is given in the following lemma:

Lemma 2.5. *For $A=\{a_1,\dots,a_d\}$, and $t^p(A_k)=\sum_{k+1\leq i_1<i_2<\dots<i_p\leq d}z^p\prod_{j=1}^{p}x^{a_{i_j}}$ for all p and $k=0,1,\dots,d$*

$$D^A=\frac{1+\sum_{p=2}^{d}\sum_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}(A)(y-1)^{p-1}}{1-t^1(A)-\sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A)(y-1)^{p-2}}.\tag{8}$$

Proof. $t^p(A_k)$ is a generating function. It counts the total number of partitions with p distinct parts from a specific set. A partition of a set is a particular way in which the set can be divided into smaller sets, where the collection of all the smaller sets is equivalent to the original set. In this case the set is A_k, where $A_k = A\backslash\{a_1,\ldots,a_k\}$ as defined before. Deriving a recursion for $t^p(A_k)$ two cases are considered:

1. In the first case the partitions contain the part a_{k+1}, and their generating function is

$$x^{a_{k+1}}zt^{p-1}\left(A_{k_1}\right).$$

   As the part a_{k+1} forms a part of the partition, it may be eliminated, resulting in one less part, and a partition less a_{k+1}, hence the factor $x^{a_{k+1}}z$.

2. In the second case, a_{k+1} is not contained in the partition and thus the partition consists entirely of parts from A_{k+1}, here the generating function for this case is given by

$$t^p\left(A_{k+1}\right).$$

These two cases combine together to give

$$t^p\left(A_k\right) = t^p\left(A_{k+1}\right) + x^{a_{k+1}}zt^{p-1}\left(A_{k+1}\right). \tag{9}$$

Lemma 2.5 will now be proven by induction on d :-

Base Case: $d = 0, d = 1$

$$\text{For } \ d=0,\ D^{\emptyset} = 1$$

$$\begin{aligned}\text{For } \ d=1,\ D^{\{a_1\}} &= \sum_{n\geq 0} x^{na_1}z^n \\ &= \sum_{n\geq 0}\left(x^{a_1}z\right)^n \\ &= \frac{1}{1-x^{a_1}z}.\end{aligned}$$

Thus the hypothesis holds for the base case.

Induction Hypothesis:

Assume that for some $d-1$, where $d > 2$, the lemma holds (that is $|A| = d-1$.)

Case: $A = \{a_1,\ldots,a_d\}$

Recall equation (7),

$$D^A = \frac{\left(1-x^{a_1}z\left(1-y\right)\right)D^{A_1} + x^{a_1}z\left(1-y\right)}{1-x^{a_1}zD^{A_1}}.$$

Now, by the induction hypothesis for the set A_1 (where $|A_1|= d-1$),

$$D^A = \frac{\left(1-x^{a_1}z\left(1-y\right)\right)\frac{1+\sum\limits_{p=2}^{d-1}\sum\limits_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}(A_1)(y-1)^{p-1}}{1-t^1(A_1)-\sum\limits_{p=3}^{d-1}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A_1)(y-1)^{p-2}}+x^{a_1}z\left(1-y\right)}{1-x^{a_1}z\frac{1+\sum\limits_{p=2}^{d-1}\sum\limits_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}(A_1)(y-1)^{p-1}}{1-t^1(A_1)-\sum\limits_{p=3}^{d-1}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A_1)(y-1)^{p-2}}}$$

$$= \frac{\left(1-x^{a_1}z\left(1-y\right)\right)\left(1+\sum\limits_{p=2}^{d-1}\sum\limits_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1}\right)}{1-t^1\left(A_1\right)-\sum\limits_{p=3}^{d-1}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_1\right)\left(y-1\right)^{p-2}-x^{a_1}z\left(1+\sum\limits_{p=2}^{d-1}\sum\limits_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1}\right)}$$

$$+\frac{x^{a_1}z\left(1-y\right)\left(1-t^1\left(A_1\right)-\sum\limits_{p=3}^{d-1}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_1\right)\left(y-1\right)^{p-2}\right)}{1-t^1\left(A_1\right)-\sum\limits_{p=3}^{d-1}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_1\right)\left(y-1\right)^{p-2}-x^{a_1}z\left(1+\sum\limits_{p=2}^{d-1}\sum\limits_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1}\right)}$$

Let

$$D^A = \frac{s_1}{s_2}.$$

The denominator s_2 will be simplified first:

$$\begin{aligned}
s_2 =& 1-t^1\left(A_1\right)-\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_1\right)\left(y-1\right)^{p-2}-x^{a_1}z\left(1+\sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1}\right)\\
=& 1-t^1\left(A_1\right)-\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_1\right)\left(y-1\right)^{p-2}-x^{a_1}z\\
&+\sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j}x^{a_1}zt^{p+j}\left(A_1\right)\left(y-1\right)^{p-1}.
\end{aligned}$$

The second double sum will be re-indexed as

$$\sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j}x^{a_1}zt^{p+j-1}\left(A_1\right)\left(y-1\right)^{p-2}.$$

Adding this result to the first double sum will result in

$$1-t^1\left(A_1\right)-\sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j}\left(t^{p+j}\left(A_1\right)+x^{a_1}zt^{p+j-1}\left(A_1\right)\right)\left(y-1\right)^{p-2}.$$

Using (9) to simplify the part $\left(t^{p+j}\left(A_1\right)+x^{a_1}zt^{p+j-1}\left(A_1\right)\right)$ to $t^p\left(A\right)$, gives

$$s_2 = 1-t^1\left(A\right)-\sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j}\left(t^{p+j}\left(A\right)\right)\left(y-1\right)^{p-2}.$$

Thus an expression for s_2 is obtained. Now considering the numerator s_1,

$$\begin{aligned}
s_1 = &\left(1 - x^{a_1} z\left(1-y\right)\right)\left(1 + \sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j} t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1}\right) \\
&+ x^{a_1} z\left(1-y\right)\left(1 - t^1\left(A_1\right) - \sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j} t^{p+j}\left(A_1\right)\left(y-1\right)^{p-2}\right) \\
=&1 + \sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j} t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1} - x^{a_1} z\left(1-y\right) \\
&- x^{a_1} z\left(1-y\right)\sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j} t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1} + x^{a_1} z\left(1-y\right) - x^{a_1} z\left(1-y\right) t^1\left(A_1\right) \\
&- x^{a_1} z\left(1-y\right)\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j} t^{p+j}\left(A_1\right)\left(y-1\right)^{p-2} \\
=&1 + x^{a_1} z\left(y-1\right) t^1\left(A_1\right) + \sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j} t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1} \\
&+ x^{a_1} z\sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j} t^{p+j}\left(A_1\right)\left(y-1\right)^{p} + x^{a_1} z\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j} t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1}.
\end{aligned}$$

To simplify the above expression, consider now the coefficients of $\left(y-1\right)^m$ for $m = 0, 1, \ldots, d-1$:

$$\begin{aligned}
&\text{for } \left(y-1\right)^0 \text{ the coefficient is } 1, \\
&\text{for } \left(y-1\right)^1 \text{ the coefficient is } x^{a_1} z t^1\left(A_1\right) + t^2\left(A_1\right) = t^2\left(A\right).
\end{aligned}$$

For powers of $\left(y-1\right)$ greater than 1 consider

$$\begin{aligned}
\sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j} t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1} &+ x^{a_1} z\sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j} t^{p+j}\left(A_1\right)\left(y-1\right)^{p} \\
&+ x^{a_1} z\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j} t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1}.
\end{aligned}$$

The first and third terms may be re-indexed to obtain

$$\begin{aligned}
\sum_{p=1}^{d-2}\sum_{j=0}^{p-1}\binom{p-1}{j} t^{p+j+1}\left(A_1\right)\left(y-1\right)^{p} &+ x^{a_1} z\sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j} t^{p+j}\left(A_1\right)\left(y-1\right)^{p} \\
&+ x^{a_1} z\sum_{p=2}^{d-2}\sum_{j=0}^{p-2}\binom{p-2}{j} t^{p+j+1}\left(A_1\right)\left(y-1\right)^{p}.
\end{aligned}$$

Replacing p with m, and considering each case where the powers of $y-1$ are equivalent in each of the three terms gives the coefficients of $\left(y-1\right)^m$ for $m \geq 2$ as

$$\sum_{j=0}^{m-1}\binom{m-1}{j} t^{m+j+1}\left(A_1\right) + x^{a_1} z\sum_{j=0}^{m-2}\binom{m-2}{j} t^{m+j}\left(A_1\right) + x^{a_1} z\sum_{j=0}^{m-2}\binom{m-2}{j} t^{m+j+1}\left(A_1\right),$$

or re-indexing the sum for the last term as

$$\sum_{j=0}^{m-1}\binom{m-1}{j}t^{m+j+1}\left(A_1\right)+x^{a_1}z\left(\sum_{j=0}^{m-2}\binom{m-2}{j}t^{m+j}\left(A_1\right)+\sum_{j=0}^{m-2}\binom{m-2}{j}t^{m+j+1}\left(A_1\right)\right).$$

This will be simplified using two results:

1.
$$\begin{aligned}\sum_{j=0}^{m-2}\binom{m-2}{j}t^{m+j}\left(A_1\right)+0&=\sum_{j=0}^{m-2}\binom{m-2}{j}t^{m+j}\left(A_1\right)+\binom{m-2}{m-1}t^{2m-1}\left(A_1\right)\\&=\sum_{j=0}^{m-1}\binom{m-2}{j}t^{m+j}\left(A_1\right),\end{aligned}$$

   which is possible since $\binom{m-2}{-1}=\binom{m-2}{m-1}=0$.

2.
$$\begin{aligned}\sum_{j=1}^{m-1}\binom{m-2}{j-1}t^{m+j}\left(A_1\right)+0&=\sum_{j=1}^{m-1}\binom{m-2}{j-1}t^{m+j}\left(A_1\right)+\binom{m-2}{-1}t^{m}\left(A_1\right)\\&=\sum_{j=0}^{m-1}\binom{m-2}{j-1}t^{m+j}\left(A_1\right)\end{aligned}$$

   for the same reason as 1.

Now 1. and 2. give

$$\sum_{j=0}^{m-1}\left(\binom{m-2}{j}+\binom{m-2}{j-1}\right)t^{m+j}\left(A_1\right)=\sum_{j=0}^{m-1}\binom{m-1}{j}t^{m+j}\left(A_1\right),$$

since $\binom{a}{b-1}+\binom{a}{b}=\binom{a+1}{b}$.

So the coefficients of $(y-1)^m$ for $m=2,3,\ldots,d-1$ become

$$\begin{aligned}&\sum_{j=0}^{m-1}\binom{m-1}{j}t^{m+1+j}\left(A_1\right)+\\&x^{a_1}z\sum_{j=0}^{m-1}\binom{m-1}{j}t^{m+j}\left(A_1\right)=\sum_{j=0}^{m-1}\binom{m-1}{j}\left(t^{m+1+j}\left(A_1\right)+x^{a_1}zt^{m+j}\left(A_1\right)\right)\\&\qquad=\sum_{j=0}^{m-1}\binom{m-1}{j}t^{m+j+1}\left(A\right),\end{aligned}$$

where the last part is found by applying the equation in (9). Thus the sum s_1 reduces to

$$s_1=1+t^2\left(A\right)\left(y-1\right)+\sum_{m=1}^{d-1}\sum_{j=0}^{m-1}\binom{m-1}{j}t^{m+j+1}\left(A\right)\left(y-1\right)^m$$

$$=1+\sum_{m=1}^{d-1}\sum_{j=0}^{m-1}\binom{m-1}{j}t^{m+j+1}\left(A\right)\left(y-1\right)^{m}.$$

For s_1 replace the m with p, and re-index the sum to finally obtain

$$s_1=1+\sum_{p=2}^{d}\sum_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}\left(A\right)\left(y-1\right)^{p-1}.$$

Now

$$\begin{aligned}D^{A}=&\frac{s_1}{s_2}\\=&\frac{1+\sum_{p=2}^{d}\sum_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}\left(A\right)\left(y-1\right)^{p-1}}{1-t^{1}\left(A_1\right)-\sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j}\left(t^{p+j}\left(A\right)\right)\left(y-1\right)^{p-2}}\end{aligned}$$

proving Lemma 2.5 (see (8).) □

Moving back to $C_{123}^{A}\left(x,y,z\right)$, using Lemma 2.5 gives

$$\begin{aligned}C_{123}^{A}\left(x,y,z\right)=&\frac{C_{123}^{A}\left(x,y,z\right)}{1-x^{a_1}z\frac{1+\sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1}}{1-t^{1}\left(A_1\right)-\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j}\left(t^{p+j}\left(A_1\right)\right)\left(y-1\right)^{p-2}}}\\=&C_{123}^{A}\left(x,y,z\right)\frac{1-t^{1}\left(A_1\right)-\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j}\left(t^{p+j}\left(A_1\right)\right)\left(y-1\right)^{p-2}}{1-t^{1}\left(A\right)-\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j}\left(t^{p+j}\left(A_1\right)\right)\left(y-1\right)^{p-2}-x^{a_1}z\left(1+\sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}\left(A_1\right)\left(y-1\right)^{p-1}\right)}.\end{aligned}$$

The denominator is just the sum s_2 from the previous case, thus

$$C_{123}^{A}\left(x,y,z\right)=C_{123}^{A}\left(x,y,z\right)\frac{1-t^{1}\left(A_1\right)-\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_1\right)\left(y-1\right)^{p-2}}{1-t^{1}\left(A\right)-\sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A\right)\left(y-1\right)^{p-2}}.$$

This equation can be iterated, using the fact that $C_{123}^{A_d}\left(x,y,z\right)=C_{123}^{\emptyset}\left(x,y,z\right)=1$, to obtain

$$C_{123}^{A}\left(x,y,z\right)=\prod_{k=0}^{d-1}\frac{1-t^{1}\left(A_{k+1}\right)-\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_{k+1}\right)\left(y-1\right)^{p-2}}{1-t^{1}\left(A_k\right)-\sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_k\right)\left(y-1\right)^{p-2}}.$$

Let $0 \leq r \leq d-1$ then

$$\frac{1 - t^1\left(A_{r+1}\right) - \sum\limits_{p=3}^{d-1}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_{r+1}\right)(y-1)^{p-2}}{1 - t^1\left(A_{r}\right) - \sum\limits_{p=3}^{d}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_{r}\right)(y-1)^{p-2}}$$
$$\times\frac{1 - t^1\left(A_{r}\right) - \sum\limits_{p=3}^{d-1}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_{r}\right)(y-1)^{p-2}}{1 - t^1\left(A_{r-1}\right) - \sum\limits_{p=3}^{d}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_{r-1}\right)(y-1)^{p-2}}$$
$$=\frac{1 - t^1\left(A_{r+1}\right) - \sum\limits_{p=3}^{d-1}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_{r+1}\right)(y-1)^{p-2}}{1 - t^1\left(A_{r-1}\right) - \sum\limits_{p=3}^{d}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}\left(A_{r-1}\right)(y-1)^{p-2}}.$$

So, considering the factors in the product, when $r = d-1$, the numerator is 1, as $t^p\left(A_d\right) = 0$, and the fraction's denominator cancels with the numerator of the factor for $r = d$. This cancellation continues until $r = 0$, so each denominator cancels with each numerator except the numerator for $r = d-1$ which has been shown to be 1, and the denominator for $r = 0$. So the generating function $C^A(x,y,z)$ reduces to

$$C_{123}(x,y,z) = \frac{1}{1 - t^1(A) - \sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A)(y-1)^{p-2}}.$$

□

For the infinite case, where $A = \mathbb{N}$, another result must be shown:

$$t^p(\mathbb{N}) = x^{\binom{p+1}{2}}z^p/(x;x)_p\,,$$

where $(a;q)_n = \prod_{j=0}^{n-1}\left(1 - aq^j\right)$.

Proof. Recall that,

$$t^p(A) = \sum_{k+1\leq i_1<i_2<\cdots<i_p\leq d} z^p\prod_{j=1}^{p}x^{a_{i_j}} \text{ for all } p.$$

Now from this definition, extending it to $\mathbb{N}$ where $a_i = i$ for $i \geq 1$,

$$t^p(\mathbb{N}) = z^p\sum_{1\leq i_1<i_2<\cdots<i_{p-1}<i_p} x^{i_1+i_2+\ldots i_p}$$

and summing over the ordered p-tuple $(i_1, i_2, \ldots, i_p)$ the sum may be split into the product of two sums,

$$\begin{aligned}t^p(\mathbb{N}) &= z^p\sum_{1\leq i_1<i_2<\cdots<i_{p-1}} x^{i_1+i_2+\ldots i_{p-1}}\sum_{i>i_{p-1}}x^i.\\ &= z^p\sum_{1\leq i_1<i_2<\cdots<i_{p-1}} x^{i_1+i_2+\ldots i_{p-1}}\frac{x^{i_{p-1}+1}}{1-x}\end{aligned}$$

$$
\begin{aligned}
&= z^p \frac{x}{1-x} \sum_{1\le i_1<i_2<\dots<i_{p-1}} x^{i_1+i_2+\dots 2i_{p-1}} \\
&= z^p \frac{x}{1-x} \sum_{1\le i_1<i_2<\dots<i_{p-2}} x^{i_1+i_2+\dots i_{p-2}} \sum_{i>i_{p-2}} x^{2i} \\
&= z^p \frac{x}{1-x} \sum_{1\le i_1<i_2<\dots<i_{p-2}} x^{i_1+i_2+\dots i_{p-2}} \frac{x^{2i_{p-2}+2}}{1-x^2} \\
&= z^p \frac{x.x^2}{(1-x)(1-x^2)} \sum_{1\le i_1<i_2<\dots<i_{p-2}} x^{i_1+i_2+\dots 3i_{p-2}},
\end{aligned}
$$

continuing in this way

$$
\begin{aligned}
t^p(\mathbb{N}) &= z^p \frac{x^{\sum_{i=1}^{p} i}}{\prod_{j=1}^{p}(1-x^j)} \\
&= z^p \frac{x^{\binom{p+1}{2}}}{(x;x)_p}.
\end{aligned}
$$

□

Using this result,

$$
\begin{aligned}
C_{123}(x,y,z) &= \frac{1}{1-t^1(\mathbb{N}) - \sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j} t^{p+j}(\mathbb{N})(y-1)^{p-2}} \\
&= \frac{1}{1 - z\frac{x}{1-x} - \sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j} z^{p+j}\frac{x^{\binom{p+1+j}{j}}}{(x;x)_{p+j}}(y-1)^{p-2}}.
\end{aligned}
$$

Thus

$$
C_{123}(x,0,1) = \frac{1}{1-\frac{x}{1-x} - \sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j}\frac{x^{\binom{p+1+j}{j}}}{(x;x)_{p+j}}(-1)^{p-2}}
$$

is the generating function for the number of compositions with parts in A, where $|A|= d$, that avoid the pattern represented by 123. For compositions of $n = 0, 1, 2, \dots, 20$ the sequence generated by $C_{123}^{\mathbb{N}}(x, 0, 1)$ is $1, 1, 2, 4, 8, 16, 31, 61, 119, 232, 453, 883, 1721, 3354, 9639, 12735, 24813,$ $48344, 94189, 183506, 357518$ (as derived in Appendix 1.3.). As 123 is a composition of 6, and the smallest representation of the pattern **rise + rise**, the first time this pattern may occur is for a composition of 6 - the term 31 in the generated sequence illustrates this as there are 32 possible compositions of 6, and as shown 31 of these compositions avoid the pattern 123.

2.4 The patterns $\{121, 132, 231\}$ and $\{212, 213, 312\}$ representing the statistics *peak* = rise + drop and *valley* = drop + rise.

The statistics peak and valley will now be considered. Recall that the patterns peak and valley occur in 3 different forms, *viz.* $\{121, 132, 231\}$ and $\{212, 213, 312\}$ respectively. Let $C^A_{peak}(x, y, z)$ and $C^A_{valley}(x, y, z)$ denote the generating functions for the compositions of n with m parts in A that contain the pattern peak or valley respectively exactly r times.

Theorem 2.6. *For $B \subseteq A$ and $s \geq 1$ let*

$$P^s(B) := \{(i_1,\ldots,i_s)\,|a_{i_j} \in B, j=1,2,\ldots,s, \text{ and } i_{2l-1} < i_{2l} \leq i_{2l+1} \text{ for } 1 \leq l \leq \lfloor s/2 \rfloor\}$$
$$Q^s(B) := \{(i_1,\ldots,i_s)\,|a_{i_j} \in B, j=1,2,\ldots,s, \text{ and } i_{2l-1} \leq i_{2l} < i_{2l+1} \text{ for } 1 \leq l \leq \lfloor s/2 \rfloor\}$$

and,

$$M^s(B) = \sum_{(i_1,\ldots,i_s)\in P^s(B)} z^s \prod_{j=1}^{s} x^{a_{i_j}}$$
$$N^s(B) = \sum_{(i_1,\ldots,i_s)\in Q^s(B)} z^s \prod_{j=1}^{s} x^{a_{i_j}}.$$

Then, for $A \subset \{a_1,a_2,a_2,\ldots\} \subset \mathbb{N}$,

$$C^A_{peak}(x,y,z) = \frac{1+\sum_{j\geq 1} M^{2j}(A)(1-y)^j}{1+\sum_{j\geq 1} M^{2j}(A)(1-y)^j - \sum_{j\geq 0} M^{2j+1}(A)(1-y)^j}$$

and,

$$C^A_{valley}(x,y,z) = \frac{1+\sum_{j\geq 1} M^{2j}(A)(1-y)^j}{1+\sum_{j\geq 1} M^{2j}(A)(1-y)^j - \sum_{j\geq 0} N^{2j+1}(A)(1-y)^j}.$$

Proof. For the proof for $C^A_{peak}(x,y,z)$ the notation

$$[c_0,c_1,\ldots,c_{n-1},c_n] = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots + \cfrac{1}{c_{n-1}+1/c_n}}}}$$

will be used. The pattern peak will be considered first: a recursive form for its generating function will be obtained and this will be used to find the explicit form. To derive the recursion for $C^A_{peak}(x,y,z)$, consider the element $a_d \in A = \{a_1,\ldots,a_d\}$, the largest element in the set. A peak will occur around a_d if there is a smaller part on either side of it within the composition. Let $\bar{A}_k = \{a_1,\ldots,a_k\}$ and σ be a composition made up of parts from $A = \{a_1,\ldots,a_d\}$. If a_d is not contained in the composition σ then such a σ is counted in the generating function $C^{\bar{A}_{d-1}}_{peak}(x,y,z)$. Otherwise, if the composition contains a_d, three different cases may occur:

1. The first occurrence of a_d is at the start of the composition σ i.e.

$$\sigma = a_d\sigma'$$

   where σ' is a composition made up of parts in A (and therefore with generating function $C^A_{peak}(x,y,z)$). In this case, as there is no smaller element to the left of the first occurrence of a_d, no peak occurs at the start of the composition σ. Thus a_d may be eliminated, resulting only in a factor of $x^{a_d}z$ in the generating function for this case which is

$$x^{a_d}zC^A_{peak}(x,y,z).$$

2. The second case is when the part a_d is at the end of the composition, i.e. $\sigma = \bar{\sigma} a_d$, and it is the only occurrence of a_d in the composition. $\bar{\sigma}$ is assumed to be a non-empty composition, otherwise a duplicate of case 1 occurs, and since a_d does not occur withing $\bar{\sigma}$, $\bar{\sigma}$ is made up of parts from $\bar{A}_{d-1}$ and its generating function is accounted for in $\left(C_{peak}^{\bar{A}_{d-1}}(x,y,z) - 1\right)$, where the -1 is a result of the removal of the empty case as $\bar{\sigma}$ is non-empty. Similarly to the first case a_d, the largest element, occurs on the outside of the composition σ at the end, and hence no *peak* occurs at this a_d. Removing it results in a factor of $x^{a_d} z$ in the generating function. All of this together gives the generating function for the second case as,

$$x^{a_d} z \left(C_{peak}^{\bar{A}_{d-1}}(x,y,z) - 1\right).$$

3. For the last case, the first a_d occurs in the interior of the composition σ, with a non-empty composition $\bar{\sigma}$ with parts in $\bar{A}_{d-1}$, as before, to the left of a_d, and a non-empty composition σ' with parts from A to the right of a_d. Thus we may represent:

$$\sigma = \bar{\sigma} a_d \sigma'.$$

The generating function for $\bar{\sigma}$ is $\left(C_{peak}^{\bar{A}_{d-1}}(x,y,z) - 1\right)$, and for σ' is $\left(C_{peak}^{A}(x,y,z) - 1\right)$, eliminating the empty composition in each case. Now since σ' is a composition with parts from A it may start with an a_d. If it does then,

$$\sigma = \bar{\sigma} a_d a_d \sigma'$$

and both occurrences of a_d may be removed without removing an occurrence of the pattern peak resulting in a factor of $x^{2a_d} z^2$ in the generating function when this is done. These results combined give the generating function for σ in this case as

$$\left(C_{peak}^{\bar{A}_{d-1}}(x,y,z) - 1\right) x^{2a_d} z^2 C_{peak}^{A}(x,y,z).$$

However, when σ' and $\bar{\sigma}$ are both non-empty compositions, and σ' does not start with an a_d, combined with the fact that $\bar{\sigma}$ is made up of parts from $\bar{A}^{d-1}$, a peak occurs at the first occurrence of a_d in the interior of the composition σ. Thus splitting off the first a_d for the generating function for σ results in a factor of $x^{2a_d} z^2 y$, where the y is a result of removing an occurrence of the pattern *peak*. Now as σ' is a non-empty composition, with parts in A, and it does not start with a_d, then its generating function is given by

$$C_{peak}^{A}(x,y,z) - 1 - x^{a_d} z C_{peak}^{A}(x,y,z).$$

That is, it is the generating function $C_{peak}^{A}(x,y,z)$ less the empty case with generating function 1 and the generating function for compositions of σ' starting with a_d. So for the case where the first a_d occurs in the interior of σ and it has no a_d neighbor, the generating function for σ is

$$\left(C_{peak}^{\bar{A}_{d-1}}(x,y,z) - 1\right) x^{a_d} z y \left(C_{peak}^{A}(x,y,z) - 1 - x^{a_d} z C_{peak}^{A}(x,y,z)\right).$$

Now combining each of these three cases, where $C^A = C_{peak}^{A}(x,y,z)$,

$$C^A = C^{\bar{A}_{d-1}} + x^{a_d} z C^A + x^{a_d} z \left(C^{\bar{A}_{d-1}} - 1\right) + x^{2a_d} z^2 C^A \left(C^{\bar{A}_{d-1}} - 1\right)$$

$$+ x^{a_d} zy \left(C^A - 1 - x^{a_d} z C^A\right) \left(C^{\bar{A}_{d-1}} - 1\right).$$

Which can be rewritten as

$$C^A = \frac{\left(1 + x^{a_d} z \left(1 - y\right)\right) C^{\bar{A}_{d-1}} - x^{a_d} z \left(1 - y\right)}{1 - x^{a_d} z \left(1 - x^{a_d} z \left(1 - y\right)\right) - x^{a_d} z \left(x^{a_d} z \left(1 - y\right) + y\right) C^{\bar{A}_{d-1}}}.$$

Considering the numerator,

$$\begin{aligned}
\left(1 + x^{a_d} z \left(1 - y\right)\right) C^{\bar{A}_{d-1}} - x^{a_d} z \left(1 - y\right) =& C^{\bar{A}_{d-1}} + x^{a_d} z \left(1 - y\right) C^{\bar{A}_{d-1}} - x^{a_d} z \left(1 - y\right) \\
=& x^{a_d} z \left(\left(1 - y\right) C^{\bar{A}_{d-1}} - \left(1 - y\right)\right) + C^{\bar{A}_{d-1}} \\
=& x^{a_d} z \left(1 - y\right) \left(C^{\bar{A}_{d-1}} - 1\right) + C^{\bar{A}_{d-1}},
\end{aligned}$$

and considering the denominator,

$$\begin{aligned}
& 1 - x^{a_d} z \left(1 - x^{a_d} z\right) \left(1 - y\right) - x^{a_d} z \left(x^{a_d} z \left(1 - y\right) + y\right) C^{\bar{A}_{d-1}} \\
=& 1 - x^{a_d} z \left(1 - y\right) + x^{2a_d} z^2 \left(1 - y\right) - x^{2a_d} z^2 \left(1 - y\right) C^{\bar{A}_{d-1}} - x^d zy C^{\bar{A}_{d-1}} \\
=& x^{2a_d} z^2 \left(1 - y\right) \left(1 - C^{\bar{A}_{d-1}}\right) - x^{a_d} z \left(\left(1 - y\right) + y C^{\bar{A}_{d-1}}\right) + 1.
\end{aligned}$$

Dividing the numerator and denominator by $x^{a_d}z\,(1-y)\left(C^{\bar{A}_{d-1}}-1\right)+C^{\bar{A}_{d-1}}$ gives

$$C^A = \frac{1}{1-x^{a_d}z-\frac{C^{\bar{A}_{d-1}}-1}{(1+x^{a_d}z(1-y))C^{\bar{A}_{d-1}}-x^{a_d}z(1-y)}}.$$

Now

$$\begin{aligned}\frac{C^{\bar{A}_{d-1}}-1}{x^{a_d}z\,(1-y)\left(C^{\bar{A}_{d-1}}-1\right)+C^{\bar{A}_{d-1}}} &= \frac{1}{x^{a_d}z\,(1-y)+\frac{C^{\bar{A}_{d-1}}}{C^{\bar{A}_{d-1}}-1}}\\ &= \frac{1}{x^{a_d}z\,(1-y)+\frac{1}{1-\frac{1}{C^{\bar{A}_{d-1}}}}},\end{aligned}$$

hence,

$$C^A = \frac{1}{1-x^{a_d}z-\frac{1}{\left[x^{a_d}z(1-y),1-1/C^{\bar{A}_{d-1}}\right]}}. \tag{10}$$

Lemma 2.7. *Let $A=\{a_1,\ldots,a_d\}$, $b_i=x^{a_i}z$, and $C^A=C^A_{peak}(x,y,z)$, then*

$$C^A = \frac{1}{1-b_d-\frac{1}{[b_d(1-y),b_{d-1},b_{d-1}(1-y),\ldots,b_2,b_2(1-y),b_1]}}.$$

Proof. This will be proven by induction on $|A|=d$.

Base Case: d=1

$$C^{\bar{A}_1} = \frac{1}{1-x^{a_1}z}$$

therefore the hypothesis holds for the base case.

Induction Hypothesis: Assume that the hypothesis holds for some $k-1\geq 0$. This can be interpreted to be the same as,

$$C^{\bar{A}_{k-1}} = \frac{1}{1-b_{k-1}-\frac{1}{[b_{k-1}(1-y),b_{k-2},b_{k-2}(1-y),\ldots,b_2,b_2(1-y),b_1]}}$$

Case: k

Now, from (10) it is known that

$$\begin{aligned}C^A =& \frac{1}{1-x^{a_k}z-\frac{1}{\left[x^{a_k}z(1-y),1-1/C^{\bar{A}_{k-1}}\right]}}\\ =& \frac{1}{1-b_k-\frac{1}{\left[b_k(1-y),1-1/C^{\bar{A}_{k-1}}\right]}}\end{aligned}$$

and thus, by the induction hypothesis

$$\begin{aligned}C^A =&\frac{1}{1-b_k-\frac{1}{\left[b_k(1-y),1-1/\frac{1}{1-b_{k-1}-\frac{1}{[b_{k-1}(1-y),b_{k-2},b_{k-2}(1-y),\ldots,b_2,b_2(1-y),b_1]}}\right]}}\\ =&\frac{1}{1-b_k-\frac{1}{[b_k(1-y),b_{k-1},b_{k-1}(1-y),\ldots,b_2,b_2(1-y),b_1]}}.\end{aligned}$$

This proves the lemma. □

Now $C^A_{peak}(x,y,z)$ is defined in a recursive form. An explicit version for C^A_{peak} will now be obtained. Where $M^s\left(\bar{A}_d\right) = M^s(A)$, for the cases where s is even and where s is odd two recursion are obtained:

$$\begin{aligned}M^{2s+1}(A) =&b_d M^{2s}(A) + M^{2s+1}\left(\bar{A}_{d-1}\right) \text{ and}\\ M^{2s}(A) =&b_d M^{2s-1}\left(\bar{A}_{d-1}\right) + M^{2s}\left(\bar{A}_{d-1}\right).\end{aligned} \tag{11}$$

Recall that

$$M^s(A) = \sum_{(i_1,\ldots,i_s)\in P^s(A)} z^s \prod_{p=1}^{s} x^{a_{i_j}}$$

and,

$$P^s(A) = \{(i_1,\ldots,i_s)\,|a_{i_j}\in A, j=1,\ldots,s \text{ and } i_{2l-1} < i_{2l} \le i_{2l+1} \text{ for } 1\le l\le \lfloor s/2\rfloor\}.$$

Now for $M^s(A)$ there are two cases for s: either s is odd or s is even. In the case that s is odd, say $2s+1$, then for $(i_1,\ldots,i_{2s+1})\in P^{2s+1}(A)$, $i_{2s}\le i_{2s+1}$, and since $a_d\in A$ there are cases where $i_{2s+1}=d$ and the rest of the cases where $i_{2s+1}\ne d$. If $i_{2s+1}=d$ then it may be that $i_{2s}=d$ as $i_{2s}\le i_{2s+1}$. For these cases the equation for $M^{2s+1}(A)$ becomes

$$\begin{aligned}\sum_{(i_1,\ldots,i_{2s},d)\in P^{2s+1}(A)} z^{2s+1}\prod_{j=1}^{2s+1} x^{a_{i_j}} =& \sum_{(i_1,\ldots,i_{2s})\in P^{2s}(A)} z^{2s}\prod_{j=1}^{2s} x^{a_{i_j}} z x^{a_d}\\ =& b_d M^s(A).\end{aligned}$$

If instead, $i_{2s+1}\ne d$, then for all $i_j\le i_{2s+1}$, $i_j<d$, and the equation for $M^{2s+1}(A)$ becomes,

$$\sum_{(i_1,\ldots,i_{2s},d)\in P^{2s+1}\left(\bar{A}_{d-1}\right)} z^{2s+1}\prod_{j=1}^{2s+1} x^{a_{i_j}} =M^{2s+1}\left(\bar{A}_{d-1}\right).$$

Combining these cases,

$$M^{2s+1}(A) =b_d M^{2s}(A) + M^{2s+1}\left(\bar{A}_{d-1}\right).$$

Considering now the case where s is even, say $2s$, the same parts in the tuple are considered. For $(i_1, \ldots, i_{2s}) \in P^s(A)$, the last part is different to the odd case as, $i_{2s-1} < i_{2s}$. Again, note that $a_d \in A$ so for cases where $i_{2s} = d$ for each $i_j < i_{2s}$, $i_j < d$, and thus for these cases $M^{2s}(A)$ becomes

$$\begin{aligned}\sum_{(i_1,\ldots,i_{2s-1},d)\in P^{2s}(A)} z^{2s}\prod_{j=1}^{2s} x^{a_{i_j}} =& \sum_{(i_1,\ldots,i_{2s-1})\in P^{2s-1}\left(\bar{A}_{d-1}\right)} z^{2s-1}\prod_{j=1}^{2s-1} x^{a_{i_j}}zx^{a_d}\\ =& b_d M^{2s-1}\left(\bar{A}_{d-1}\right).\end{aligned}$$

For the cases where $i_{2s} < d$, then $i_j < d$ for all i_j. The equation for $M^{2s}(A)$ for these cases is given by

$$\sum_{(i_1,\ldots,i_{2s-1},i_{2s})\in P^{2s}\left(\bar{A}_{d-1}\right)} z^{2s}\prod_{j=1}^{2s} x^{a_{i_j}} = M^{2s}\left(\bar{A}_{d-1}\right).$$

Combining results for both cases,

$$M^{2s}(A) = b_d M^{2s-1}\left(\bar{A}_{d-1}\right) + M^{2s}\left(\bar{A}_{d-1}\right).$$

Both these recursions will be used later in the proof. To simplify further calculations let $G_d = \frac{1}{[b_d(1-y), b_{d-1}, b_{d-1}(1-y), \ldots, b_2, b_2(1-y), b_1]}$.

Lemma 2.8. *For all* $d \geq 2$, $G_d = \frac{\sum_{j\geq 0} M^{2j+1}(A)(1-y)^j}{1+\sum_{j\geq 1} M^{2j}(A)(1-y)^j} = \frac{G_d^1}{G_d^2}$.

Proof. This lemma will be proven by induction on $d = |A|$.

Base Case: d=2

$$\begin{aligned}G_2 =& \frac{1}{[b_2(1-y), b_1]}\\ =& \frac{1}{b_2(1-y) + \frac{1}{b_1}}\\ =& \frac{b_1}{b_1 b_1 (1-y) + 1}\end{aligned}$$

therefore $G_2^1 = b_1$ and $G_2^2 = b_1 b_2 (1-y) + 1$, and hence the hypothesis holds for the base case.

Induction Hypothesis: for $d \geq 3$ assume that the hypothesis holds for $d-1$, that is

$$G_{d-1} = \frac{G_{d-1}^1}{G_{d-1}^2}.$$

Case: d

By definition,

$$G_d = \frac{1}{b_d(1-y) + \frac{1}{b_d + G_{d-1}}}.$$

Thus by the induction hypothesis

$$
\begin{aligned}
G_d =& \frac{1}{b_d\left(1-y\right)+\frac{1}{b_d+\frac{G_{d-1}^1}{G_{d-1}^2}}} \\
=& \frac{b_{d-1}+\frac{G_{d-1}^1}{G_{d-1}^2}}{1+b_d\left(1-y\right)\left(b_{d-1}+\frac{G_{d-1}^1}{G_{d-1}^2}\right)} \\
=& \frac{G_{d-1}^2 b_{d-1}+G_{d-1}^1}{G_{d-1}^2+b_d\left(1-y\right)\left(G_{d-1}^2 b_{d-1}+G_{d-1}^1\right)}.
\end{aligned}
$$

Now, simplifying the numerator,

$$
\begin{aligned}
b_{d-1}G_{d-1}^2+G_{d-1}^1 =& b_{d-1}\left(1+\sum_{j\geq 1} M^{2j}\left(\bar{A}_{d-1}\right)\left(1-y\right)^j\right)+\sum_{j\geq 0} M^{2j+1}\left(\bar{A}_{d-1}\right)\left(1-y\right)^j \\
=& b_{d-1}+\sum_{j\geq 1} b_{d-1}M^{2j}\left(\bar{A}_{d-1}\right)\left(1-y\right)^j+\sum_{j\geq 0} M^{2j+1}\left(\bar{A}_{d-1}\right)\left(1-y\right)^j \\
=& \sum_{j\geq 0}\left(M^{2j}\left(\bar{A}_{d-1}\right)b_{d-1}+M^{2j+1}\left(\bar{A}_{d-1}\right)\right)\left(1-y\right)^j \\
=& \sum_{j\geq 0} M^{2j+1}\left(A\right)\left(1-y\right)^j \\
=& G_d^1.
\end{aligned}
$$

Similarly, simplifying the denominator, using the result for the numerator,

$$
\begin{aligned}
G_{d-1}^2+b_d\left(1-y\right)\left(G_{d-1}^2 b_{d-1}+G_{d-1}^1\right) =& G_{d-1}^2+b_d\left(1-y\right)G_d^1 \\
=& 1+\sum_{j\geq 1} M^{2j}\left(\bar{A}_{d-1}\right)\left(1-y\right)^j \\
& +b_d\left(1-y\right)\sum_{j\geq 0} M^{2j+1}\left(A\right)\left(1-y\right)^j \\
=& 1+\sum_{j\geq 1} M^{2j}\left(\bar{A}_{d-1}\right)\left(1-y\right)^j \\
& +\sum_{j\geq 1} b_d M^{2j-1}\left(A\right)\left(1-y\right)^j \\
=& 1+\sum_{j\geq 1}\left(M^{2j}\left(\bar{A}_{d-1}\right)+b_d M^{2j-1}\left(A\right)\right)\left(1-y\right)^j \\
=& 1+\sum_{j\geq 1} M^{2j}\left(A\right)\left(1-y\right)^j \\
=& G_d^2.
\end{aligned}
$$

Combining the results for the numerator and the denominator,

$$
G_d = \frac{G_d^1}{G_d^2},
$$

proving the lemma. $\square$

Recall,

$$C^A = \frac{1}{1 - b_d - G_d},$$

by Lemma 2.7. Using Lemma 2.8,

$$\begin{aligned}
C^A =& \frac{1}{1 - b_d - G_d} \\
=& \frac{1}{1 - b_d - \frac{G_d^1}{G_d^2}} \\
=& \frac{G_d^2}{G_d^2 - b_d G_d^2 - G_d^1} \\
=& \frac{1 + \sum_{j \geq 1} M^{2j}(A)(1-y)^j}{1 + \sum_{j \geq 1} M^{2j}(A)(1-y)^j - b_d \left(1 + \sum_{j \geq 1} M^{2j}(A)(1-y)^j\right) - \sum_{j \geq 0} M^{2j+1}(A)(1-y)^j} \\
=& \frac{1 + \sum_{j \geq 1} M^{2j}(A)(1-y)^j}{1 - b_d - M^1(A) + \sum_{j \geq 1} M^{2j}(A)(1-y)^j - \sum_{j \geq 1} \left(b_d M^{2j}(A) + M^{2j+1}(A)\right)(1-y)^j}
\end{aligned}$$

and thus, using (11),

$$\begin{aligned}
C^A =& \frac{1 + \sum_{j \geq 1} M^{2j}(A)(1-y)^j}{1 - M^1(A) + \sum_{j \geq 1} M^{2j}(A)(1-y)^j - \sum_{j \geq 1} \left(M^{2j+1}(A)\right)(1-y)^j} \\
=& \frac{1 + \sum_{j \geq 1} M^{2j}(A)(1-y)^j}{1 + \sum_{j \geq 1} M^{2j}(A)(1-y)^j - \sum_{j \geq 0} \left(M^{2j+1}(A)\right)(1-y)^j}.
\end{aligned}$$

Hence the proof for the explicit form for the generating function $C^A_{peak}(x, y, z)$ is completed. □

The proof for valley is almost identical, with slight differences: when considering the compositions, the smallest element in A, a_1 is considered instead of a_d, and $A_k = \{a_{k+1}, a_{k+2}, \ldots, a_d\}$ is used instead of $\bar{A}_k$. Lastly, with regard to the recursions,

$$M^s(A_k) = b_{k+1} N^{s-1}(A_{k+1}) + M^s(A_{k+1}) \text{ and } N^s(A_k) = b_{k+1} M^{s-1}(A_k) + N^s(A_{k+1})$$

are used, and they are obtained by separating the elements of the tuples in $P^s(A_k)$: instead of looking at the last element, and whether it is equal to d or not, the first element is considered, and where it is equal to $k+1$ or greater than $k+1$.

Theorem 2.6 will now be applied to $A = \mathbb{N}$ to obtain $\forall s \geq 1$

$$\begin{aligned}
M^{2s}(\mathbb{N}) &= \sum_{1 \leq i_1 < i_2 \leq i_3 \ldots i_{2s-2} \leq i_{2s-1} < i_{2s}} x^{i_1 + \cdots + i_{2s}} z^{2s} = \frac{z^{2s} x^{s(s+2)}}{(x;x)_{2s}} \\
M^{2s+1}(\mathbb{N}) &= \sum_{1 \leq i_1 < i_2 \leq i_3 \ldots i_{2s-1} < i_{2s} \leq i_{2s+1}} x^{i_1 + \cdots + i_{2s+1}} z^{2s+1} = \frac{z^{2s+1} x^{s^2+3s+1}}{(x;x)_{2s+1}} \\
N^{2s+1}(\mathbb{N}) &= \sum_{1 \leq i_1 \leq i_2 < i_3 \ldots i_{2s-1} \leq i_{2s} < i_{2s+1}} x^{i_1 + \cdots + i_{2s+1}} z^{2s+1} = \frac{z^{2s+1} x^{(s+1)^2}}{(x;x)_{2s+1}}.
\end{aligned}$$

These equations are obtained in the following way (similar to $t^p(\mathbb{N})$):

1. $M^{2s}(\mathbb{N})$**:**

   Recall

$$\begin{aligned}
M^{2s}(\mathbb{N}) = & \sum_{1\le i_1<i_2\le i_3\ldots i_{2s-2}\le i_{2s-1}<i_{2s}} x^{i_1+\cdots+i_{2s}} z^{2s}\\
=& z^{2s} \sum_{1\le i_1<i_2\le i_3\ldots i_{2s-2}\le i_{2s-1}} x^{i_1+\cdots+i_{2s-1}} \sum_{i>i_{2s-1}} x^i\\
=& z^{2s} \sum_{1\le i_1<i_2\le i_3\ldots i_{2s-2}\le i_{2s-1}} x^{i_1+\cdots+i_{2s-1}} \frac{x^{i_{2s-1}+1}}{1-x}\\
=& z^{2s} \frac{x}{1-x} \sum_{1\le i_1<i_2\le i_3\ldots i_{2s-2}\le i_{2s-1}} x^{i_1+\cdots+2i_{2s-1}}\\
=& z^{2s} \frac{x}{1-x} \sum_{1\le i_1<i_2\le i_3\ldots i_{2s-3}<i_{2s-2}} x^{i_1+\cdots+i_{2s-2}} \sum_{i\ge i_{2s-2}} x^{2i}\\
=& z^{2s} \frac{x}{1-x} \sum_{1\le i_1<i_2\le i_3\ldots i_{2s-3}<i_{2s-2}} x^{i_1+\cdots+i_{2s-2}} \frac{x^{2i_{2s-2}}}{1-x^2}\\
=& z^{2s} \frac{x}{(1-x)(1-x^2)} \sum_{1\le i_1<i_2\le i_3\ldots i_{2s-3}<i_{2s-2}} x^{i_1+\cdots+3i_{2s-2}}\\
=& z^{2s} \frac{x}{(1-x)(1-x^2)} \sum_{1\le i_1<i_2\le i_3\ldots i_{2s-4}\le i_{2s-3}} x^{i_1+\cdots+i_{2s-3}} \sum_{i>i_{2s-3}} x^{3i}\\
=& z^{2s} \frac{x}{(1-x)(1-x^2)} \sum_{1\le i_1<i_2\le i_3\ldots i_{2s-4}\le i_{2s-3}} x^{i_1+\cdots+i_{2s-3}} \frac{x^{3i_{2s-3}+3}}{1-x^3}\\
=& z^{2s} \frac{x.x^3}{(1-x)(1-x^2)(1-x^3)} \sum_{1\le i_1<i_2\le i_3\ldots i_{2s-4}\le i_{2s-3}} x^{i_1+\cdots+4i_{2s-3}}\\
=& \ldots\\
=& z^{2s} \frac{x.x^3\ldots x^{2s-1}}{(1-x)(1-x^2)\ldots(1-x^{2s-1})} \sum_{1\ge i_1} x^{2si_1}\\
=& z^{2s} \frac{x.x^3\ldots x^{2s-1}}{(1-x)(1-x^2)\ldots(1-x^{2s-1})} \cdot \frac{x^{2s}}{1-x^{2s}}\\
=& z^{2s} \frac{x^3.x^5\ldots x^{2s+1}}{(1-x)(1-x^2)\ldots(1-x^{2s})}.
\end{aligned}$$

All the odd powers of x^i for $i = 3, 5, \ldots, 2s+1$ are multiplied together in the numerator, and the denominator becomes

$$\begin{aligned}\prod_{i=1}^{2s}\left(1-x^i\right) &= \prod_{i=0}^{2s-1}\left(1-x^{i+1}\right)\\ &= (x;x)_{2s}\,.\end{aligned}$$

Thus

$$\begin{aligned}M^{2s}(\mathbb{N}) =& z^{2s}\frac{x^{\sum_{i=1}^{s}(2i+1)}}{(x;x)_{2s}}\\ =& \frac{z^{2s}x^{s(s+2)}}{(x;x)_{2s}}.\end{aligned}$$

2. The process for $M^{2s+1}(\mathbb{N})$ is similar:

$$\begin{aligned}M^{2s+1}(\mathbb{N}) =& \sum_{1\le i_1<i_2\le i_3<\ldots\le i_{2s-1}<i_{2s}\le i_{2s+1}} x^{i_1+\ldots i_{2s+1}}z^{2s+1}\\ =& z^{2s+1}\sum_{1\le i_1<i_2\le i_3<\ldots\le i_{2s-2}\le i_{2s-1}<i_{2s}} x^{i_1+\ldots i_{2s}}\sum_{i\ge i_{2s}} x^i\\ =& z^{2s+1}\sum_{1\le i_1<i_2\le i_3<\ldots\le i_{2s-2}\le i_{2s-1}<i_{2s}} x^{i_1+\ldots i_{2s}}\frac{x^{i_{2s}}}{1-x}\\ =& z^{2s+1}\frac{1}{1-x}\sum_{1\le i_1<i_2\le i_3<\cdots<i_{2s-2}\le i_{2s-1}<i_{2s}} x^{i_1+\ldots 2i_{2s}}\\ =& z^{2s+1}\frac{1}{1-x}\sum_{1\le i_1<i_2\le i_3<\ldots\le i_{2s-3}<i_{2s-2}\le i_{2s-1}} x^{i_1+\ldots i_{2s-1}}\sum_{i>i_{2s-1}} x^{2i}\\ =& z^{2s+1}\frac{1}{1-x}\sum_{1\le i_1<i_2\le i_3<\ldots\le i_{2s-3}<i_{2s-2}\le i_{2s-1}} x^{i_1+\ldots i_{2s-1}}\frac{x^{2i_{2s-1}+2}}{1-x^2}\\ =& z^{2s+1}\frac{x^2}{(1-x)\left(1-x^2\right)}\sum_{1\le i_1<i_2\le i_3<\ldots\le i_{2s-3}<i_{2s-2}\le i_{2s-1}} x^{i_1+\ldots 3i_{2s-1}}\\ =& \ldots\\ =& z^{2s+1}\frac{x^2.x^4\ldots x^{2s}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^{2s}\right)}\sum_{1\le i_1} x^{(2s+1)i_{2s}}\\ =& z^{2s+1}\frac{x^2.x^4\ldots x^{2s}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^{2s}\right)}\cdot\frac{x^{2s+1}}{1-x^{2s+1}}\\ =& z^{2s+1}\frac{x^2.x^4\ldots x^{2s}x^{2s+2}x^{-1}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^{2s+1}\right)}.\end{aligned}$$

Using results similar to the derivation of $M^{2s}(\mathbb{N})$, it can be seen that all the even powers of x^i for $i = 2, 4, \dots, 2s+2$ are multiplied together along with a term of x^{-1} in the numerator and the denominator is obtained in an identical manner. Thus

$$\begin{aligned}M^{2s}(\mathbb{N}) =& z^{2s+1} \frac{x^{\sum_{i=1}^{s+1}(2i)-1}}{(x;x)_{2s+1}} \\ =& z^{2s+1} \frac{x^{s^2+3s+1}}{(x;x)_{2s+1}}.\end{aligned}$$

3. Now, for $N^{2s+1}(\mathbb{N})$, the $<$ and $\leq$ alternate differently, where $\leq$ occurs to the left of an odd indexed i_j, and $<$ is to the right. In this case

$$\begin{aligned}N^{2s+1}(\mathbb{N}) =& \sum_{1\leq i_1\leq i_2<i_3\leq i_4<\cdots<i_{2s-1}\leq i_{2s}<i_{2s+1}} x^{i_1+\cdots+i_{2s+1}} z^{2s+1} \\
=& z^{2s+1} \sum_{1\leq i_1\leq i_2<i_3\leq i_4<\ldots\leq i_{2s-2}<q i_{2s-1}\leq i_{2s}} x^{i_1+\cdots+i_{2s}} \sum_{i>i_{2s}} x^i \\
=& z^{2s+1} \sum_{1\leq i_1\leq i_2<i_3\leq i_4<\ldots\leq i_{2s-2}<q i_{2s-1}\leq i_{2s}} x^{i_1+\cdots+i_{2s}} \frac{x^{i_{2s}+1}}{1-x} \\
=& z^{2s+1} \frac{x}{1-x} \sum_{1\leq i_1\leq i_2<i_3\leq i_4<\ldots\leq i_{2s-2}<q i_{2s-1}\leq i_{2s}} x^{i_1+\cdots+2i_{2s}} \\
=& z^{2s+1} \frac{x}{1-x} \sum_{1\leq i_1\leq i_2<i_3\leq i_4<\cdots<i_{2s-3}\leq i_{2s-2}<i_{2s-1}} x^{i_1+\cdots+i_{2s-1}} \sum_{i\geq i_{2s-1}} x^{2i} \\
=& z^{2s+1} \frac{x}{1-x} \sum_{1\leq i_1\leq i_2<i_3\leq i_4<\cdots<i_{2s-3}\leq i_{2s-2}<i_{2s-1}} x^{i_1+\cdots+i_{2s-1}} \frac{x^{2i_{2s-1}}}{1-x^2} \\
=& z^{2s+1} \frac{x}{(1-x)(1-x^2)} \sum_{1\leq i_1\leq i_2<i_3\leq i_4<\cdots<i_{2s-3}\leq i_{2s-2}<i_{2s-1}} x^{i_1+\cdots+3i_{2s-1}} \\
=& z^{2s+1} \frac{x}{(1-x)(1-x^2)} \sum_{1\leq i_1\leq i_2<i_3\leq i_4<\ldots\leq i_{2s-4}<i_{2s-3}<i_{2s-2}} x^{i_1+\cdots+i_{2s-2}} \sum_{i>i_{2s-2}} x^{3i} \\
=& z^{2s+1} \frac{x}{(1-x)(1-x^2)} \sum_{1\leq i_1\leq i_2<i_3\leq i_4<\ldots\leq i_{2s-4}<i_{2s-3}<i_{2s-2}} x^{i_1+\cdots+i_{2s-2}} \frac{x^{3i_{2s-2}+3}}{1-x^3} \\
=& z^{2s+1} \frac{x.x^3}{(1-x)(1-x^2)(1-x^3)} \sum_{1\leq i_1\leq i_2<i_3\leq i_4<\ldots\leq i_{2s-4}<i_{2s-3}<i_{2s-2}} x^{i_1+\cdots+4i_{2s-2}} \\
=& \dots \\
=& z^{2s+1} \frac{x.x^3\dots x^{2s-1}}{(1-x)(1-x^2)\dots(1-x^{2s})} \sum_{1\leq i_1} x^{(2s+1)i_1} \\
=& z^{2s+1} \frac{x.x^3\dots x^{2s-1}}{(1-x)(1-x^2)\dots(1-x^{2s})} \frac{x^{(2s+1)}}{1-x^{2s+1}} \\
=& z^{2s+1} \frac{x.x^3\dots x^{2s-1}.x^{2s+1}}{(1-x)(1-x^2)\dots(1-x^{2s+1})}.\end{aligned}$$

Thus, in a similar way to $M^{2s}(\mathbb{N})$, all the odd powers of x^i are multiplied together for $i = 1, 3, 5 \ldots, 2s+1$. The denominator is again obtained in an identical way to the first two examples and this gives

$$\begin{aligned} N^{2s+1}(\mathbb{N}) =& z^{2s+1} \frac{x^{\sum_{i=1}^{s+1}(2i-1)}}{(x;x)_{2s+1}} \\ =& z^{2s+1} \frac{x^{(s+1)^2}}{(x;x)_{2s+1}}. \end{aligned}$$

These expressions for $M^{2s}(\mathbb{N})$, $M^{2s+1}(\mathbb{N})$ and $N^{2s+1}(\mathbb{N})$ can now be substituted back into the generating functions in Theorem 2.6 to obtain

$$C^{\mathbb{N}}_{peak}(x,0,1) = \frac{1 + \sum_{j\geq 1} \frac{x^{j(j+2)}}{(x;x)_{2j}}}{1 + \sum_{j\geq 1} \frac{x^{j(j+2)}}{(x;x)_{2j}} - \sum_{j\geq 0} \frac{x^{j^2+3j+1}}{(x;x)_{2j+1}}}$$

and,

$$C^{\mathbb{N}}_{valley}(x,0,1) = \frac{1 + \sum_{j\geq 1} \frac{x^{j(j+2)}}{(x;x)_{2j}}}{1 + \sum_{j\geq 1} \frac{x^{j(j+2)}}{(x;x)_{2j}} - \sum_{j\geq 0} \frac{x^{(j+1)^2}}{(x;x)_{2j+1}}},$$

the generating functions for the total number of compositions of n with parts in $\mathbb{N}$, without peaks and valleys respectively. The generating functions generate the following sequences of coefficients for $n = 0, 1, \ldots, 20$: $1, 1, 2, 4, 7, 13, 22, 38, 64, 107, 177, 293, 481, 789, 1291, 2110, 3445, 5621, 9167, 14947, 24366$ and $1, 1, 2, 4, 8, 15, 28, 52, 96, 177, 326, 600, 1104, 2032, 3740, 6884, 12672, 23327, 42942, 79052, 145528$ respectively (see Appendix 1.4 for their derivation.) The composition with the smallest parts that contain a peak is 121, of 4, and the composition with the smallest parts that contain a valley is 212, of 5.

For reference in Section 4 it is important to note that there are the same number of valleys in compositions of n with m parts as there are peaks in compositions of $m(n+1)-n$ with m parts. This can be shown by considering compositions of n with m parts, say

$$\sigma = \sigma_1\sigma_2 \ldots \sigma_{i-1}\sigma_i\sigma_{i+1} \ldots \sigma_m$$

where $\sigma_{i-1}\sigma_i\sigma_{i+1}$ forms a valley for some $i = 2, \ldots, m-1$. That is $\sigma_{i-1} > \sigma_i$ and $\sigma_i < \sigma_{i+1}$. Now replace each part in the composition σ with $(n+1) - \sigma_i$. Thus, denoting the transformed composition by σ' and the parts of σ' by σ'_i,

$$\begin{aligned} \sigma'_{i-1} &= (n+1) - \sigma_{i-1} \\ \sigma'_i &= (n+1) - \sigma_i \\ \sigma'_{i+1} &= (n+1) - \sigma_{i+1} \end{aligned}$$

and $\sigma'_{i-1} < \sigma'_i$ and $\sigma'_i > \sigma'_{i+1}$. Hence the valley in σ become a peak in σ' and by this transformation, each valley in σ became a peak in σ'. Now, σ' is a composition of $m(n+1) - \sum_{i=1}^m \sigma_i = m(n+1) - n$. Therefore the number of valleys in compositions of n with m parts is the same as the number of peaks in compositions of $m(n+1)-n$ of m parts. This will be used in Section 4.

3 Asymptotics for the number of compositions avoiding τ

The focus of this section will be to determine the asymptotics for the derived generating functions that avoid a specific pattern τ, where τ is any one of the patterns represented by 111, 112, 221, 123, valley or peak. This will be done using tools from complex analysis, namely singularity analysis to find the dominant pole. To simplify calculations let

$$C_{\tau}(z) = C_{\tau}(z, 0, 1) = \sum_{n \geq 0} C_{\tau}(n, 0) z^n.$$

Now, for each τ, $C_{\tau}(n, 0)$ is meromoprhic as it is just a fraction in each case. Theorem 5.2.1 from [51] will be used to obtain the asymptotics for each τ. Now, $C_{\tau}(n, 0)$ counts the number of compositions of n with parts in $\mathbb{N}$ that contain the pattern τ exactly 0 times. Thus it is counting a subset of compositions of n, and there are 2^{n-1} total compositions of n with parts in $\mathbb{N}$, thus

$$C_{\tau}(n, 0) \leq 2^{n-1}.$$

So

$$\begin{aligned} C_{\tau}(z) =& \sum_{n \geq 0} C_{\tau}(n, 0) z^n \\ \leq& \sum_{n \geq 0} 2^{n-1} z^n \\ =& \frac{1}{2} \sum_{n \geq 0} (2z)^n \\ =& \frac{1}{2} \left(\frac{1}{1 - 2z} \right). \end{aligned}$$

Thus the radius of convergence of $C_{\tau}(z)$ is greater than 0.5. Now, for each generating function for τ

$$C_{\tau}(z) = \frac{1}{f(z)}.$$

Thus the dominant pole of $C_{\tau}(z)$ is the least positive value ρ for which $f(\rho) = 0$. And as the radius of convergence for $C_{\tau}(z)$ is greater than 0.5 it must be that $\rho > 0.5$ for each of the patterns τ. Considering the nature of the poles for the generating functions the argument principle will be used (from Theorem 4.10a [35]): for a simple closed curve Γ the number of times that the transformed curve $f(\Gamma)$ winds around the origin, i.e. the winding number of $f(\Gamma)$, is equal to the number of zeros of that function. By the argument principle, let the simple closed curve be the circle $r = |z| = 0.7$ and the transformation be the $f(z)$ where $C_{\tau}(z) = \frac{1}{f(z)}$ for each pattern τ.

Clearly, by the graphs in Appendix 1.7 the winding number is 1 for the generating function for each τ. This means that each $f(z)$ has only one zero for each τ, and hence each ρ obtained is the dominant pole, now for each τ,

$$\begin{aligned}
\tau =& 111 &: \rho = 0.523351,\\
\tau =& 112 &: \rho = 0.55344,\\
\tau =& 221 &: \rho = 0.513387,\\
\tau =& 123 &: \rho = 0.513286,\\
\tau =& peak &: \rho = 0.61359,\\
\tau =& valley &: \rho = 0.543207,
\end{aligned}$$

(as derived in Appendix 1.5.)

By Theorem 5.2.1 in [51], since ρ is a simple pole, and the ρ is the dominant pole, we have

$$C_\tau(n,0) = K \cdot v^n + O\left(\left(\frac{1}{r}\right)^n\right),$$

where $K = \frac{-1}{\rho f'(\rho)}$ and $v = \frac{1}{\rho}$. Thus

$$\begin{aligned}
C_{111}(n,0) =& 0.499301 \cdot 1.91076^n + O\left((10/7)^n\right)\\
C_{112}(n,0) =& 0.692005 \cdot 1.80688^n + O\left((10/7)^n\right)\\
C_{221}(n,0) =& 0.545362 \cdot 1.94785^n + O\left((10/7)^n\right)\\
C_{123}(n,0) =& 0.576096 \cdot 1.94823^n + O\left((10/7)^n\right)\\
C_{peak}(n,0) =& 1.394560 \cdot 1.62975^n + O\left((10/7)^n\right)\\
C_{valley}(n,0) =& 0.728207 \cdot 1.84092^n + O\left((10/7)^n\right),
\end{aligned}$$

(see Appendix 1.6 for the asymptotic derivation.)

4 Counting occurrences of subword patterns

This section will deal with subword patterns in words on k letters, see [13, 14] for examples of this topic. Results obtained in the previous section will now be extended to words on k letters. For this section some definitions are needed:

1. An alphabet $[k] = \{1, 2, \ldots, k\}$ is a totally ordered set of k letters.
2. Words are elements of $[k]^n$.

For a word to contain a pattern τ it must contain a subsequence of consecutive letters that resemble the pattern represented by τ, i.e. that are isomorphic to τ. A word is said to avoid τ if it does not contain τ. Some notation will be defined:

1. The reversal of the pattern τ will be denoted by $r(\tau)$ and will represent the pattern as read from right to left, for example

$$r(221) = 122.$$

2. The complement of the pattern τ will be denoted by $c(\tau)$ and will be the instance where each element τ_i in the pattern τ is replaced by $k+1-\tau_i$, for example, on the alphabet [4],

$$c(113) = 442.$$

3. Now, with regard to the aforementioned notation, the symmetry class for τ is given by

$$\{\tau, r(\tau), c(\tau), c(r(\tau))\}.$$

As there is no restriction on the size of the letters of a word (as there is with the elements of a composition) the patterns represented by each element of the symmetry class occur the exact same number of times in words consisting of k letters. The structure of words is quite similar to that of compositions, and only minor alternations need be done to obtain the generating function for words from the generating functions already obtained for compositions. The generating functions for the number of words, m letters long, from the alphabet A in which the pattern τ occurs r times, can be obtained from the generating functions

$$C_{\tau}^{A}(x, y, z),$$

for the total number of compositions of n consisting of exactly m parts from the set A and that contain the pattern τ exactly r times as

$$C_{\tau}^{A}(1, y, z),$$

since the size restriction on the sum of elements, letters in this case, does not apply to words. Applying this to the following theorems for compositions results in similar theorems for words:

1. Theorem 2.1:

$$C_{111}^{[k]}(1, y, z) = \frac{1 + z(1+z)(1-y)}{1-(k-1+y)z-(k-1)(1-y)z^2}.$$

2. Theorem 2.2:

$$C_{112}^{[k]}(1, y, z) = C_{221}^{[k]}(1, y, z) = \frac{(1-y)z}{(1-y)z - 1 + (1-(1-y)z^2)^k}.$$

From this the generating function for words avoiding the pattern $112 = r(211)$, consisting of m letters, is obtained to be

$$c_{112}^{[k]}(1, 0, z) = C_{221}^{[k]}(1, 0, z) = \frac{z}{z-1+(1-z^2)^k}.$$

3. For Theorem 2.3, consider $t_k^p([k])$. When $x = 1$

$$\begin{aligned} t_k^p([k]) &= \sum_{1 \le i_1 < i_2 < \cdots < i_p \le k} z^p \prod_{j=1}^{p} 1^{a_{i_j}} \\ &= \sum_{1 \le i_1 < i_2 < \cdots < i_p \le k} z^p. \end{aligned}$$

Thus $t_k^p([k])$ sums the number of all the ordered subsets of size p of a set of k elements, i.e. the number of ways of forming a word of length p with letters in the alphabet $[k]$ that is counted as $\binom{k}{p}$ ways, multiplied by z^p, hence

$$t_k^p([k]) = \binom{k}{p} z^p.$$

Therefore

$$C_{123}^{[k]}(1, y, z) = \frac{1}{1 - kz - \sum_{p=3}^{k} \sum_{j=0}^{p-3} \binom{p-3}{j} \binom{k}{p+j} z^{p+j} (y-1)^{p-1}}.$$

4. The results for words containing peaks and valleys will now be derived from the previous results for compositions: for words it is given that $A = [k]$, and $x = 1$. Applying this to Theorem 2.6

$$C_{peak}^{[k]}(1, y, z) = \frac{1 + \sum_{j \ge 1} M^{2j}([k])(1-y)^j}{1 + \sum_{j \ge 1} M^{2j}([k])(1-y)^j - \sum_{j \ge 0} M^{2j+1}([k])(1-y)^j}.$$

To simplify this further, $M^s([k])$ must be determined. Now

$$\begin{aligned} M^s([k]) &= \sum_{(i_1, \ldots, i_s) \in P^s([k])} z^s \prod_{j=1}^{s} 1^{a_{i_j}} \\ &= \sum_{(i_1, \ldots, i_s) \in P^s([k])} z^s \\ &= |P^s([k])| z^s. \end{aligned}$$

Now

$$P^s([k]) = \{(i_1, \ldots, i_s) \,|\, a_{i_j} \in [k], j = 1, 2, \ldots, s, i_{sl-1} < i_{2l-1} \le i_{2s} \text{ for } 1 \le l \le \lfloor s/2 \rfloor\}.$$

To calculate $|P^s([k])|$ from this definition consider strict inequalities where each element in the tuple is strictly less than the element to its right, and let

$$j_n = i_n + \lfloor (n-1)/2 \rfloor,$$

so the strict inequalities all hold.

Then

$$\begin{aligned}\left|P^{2s}\left([k]\right)\right| =& \left|\left\{\left(j_1, j_2, \ldots, j_{2s}\right) \middle| 1 \leq j_1 < j_2 < \cdots < j_{2s} \leq k+s-1\right\}\right| \\ =& \binom{k+s-1}{2s},\end{aligned}$$

the number of subsets of size $2s$ that can be made from $k+s-1$ distinct elements (where for the last inequality, $j_{2s} \leq k + \lfloor (2s-1)/2 \rfloor = k+s-1$). Similarly

$$\begin{aligned}\left|P^{2s+1}\left([k]\right)\right| =& \left|\left\{\left(j_1, j_2, \ldots, j_{2s+1}\right) \middle| 1 \leq j_1 < j_2 < \cdots < j_{2s+1} \leq k+s\right\}\right| \\ =& \binom{k+s}{2s+1},\end{aligned}$$

the number of subsets of size $2s+1$ that can be made from $k+s$ distinct elements (where for the last inequality, $j_{2s+1} \leq k + \lfloor (2s)/2 \rfloor = k+s$). And so

$$\begin{aligned}M^{2l}\left([k]\right) =& z^{2l}\binom{k-1+l}{2l}, \\ M^{2l+1}\left([k]\right) =& z^{2l+1}\binom{k+l}{2l+1}\end{aligned}$$

which gives the final result as

$$C_{peak}^{[k]}\left(1, y, z\right) = \frac{\sum_{j\geq 0} z^{2j}\left(1-y\right)^j \binom{k-1+l}{2j}}{\sum_{j\geq 0} z^{2j}\left(1-y\right)^j \binom{k-1+j}{2j} - \sum_{j\geq 0} z^{2j+1}\left(1-y\right)^j \binom{k+j}{2j+1}}.$$

Due to the nature of the complement of a word,

$$c\left(peak\right) = valley,$$

and therefore it can be seen that

$$C_{valley}^{[k]}\left(1, y, z\right) = C_{peak}^{[k]}\left(1, y, z\right).$$

Now

$$C_{peak}^{[k]}\left(1, 0, z\right) = C_{valley}^{[k]}\left(1, 0, z\right) = \frac{\sum_{j\geq 0} z^{2j}\binom{k-1+l}{2j}}{\sum_{j\geq 0} z^{2j}\binom{k-1+j}{2j} - \sum_{j\geq 0} z^{2j+1}\binom{k+j}{2j+1}},$$

which is the generating function for the number of words of m letters from the alphabet $[k]$ that do not contain valleys or peaks respectively.

Part II

What to expect in a game of memory [47]

Abstract

Memory is a game played with two identical decks of cards. The two identical decks are shuffled together and then their combined cards are laid face down individually. A player plays the game through a series of moves, which involve flipping over two cards, if the two cards are identical then they are removed from play, otherwise they are both flipped back over and the player moves on to the next move. The objective of the game is to remove all the cards from play.

Consider two identical decks of cards, say of n cards each. It will be shown that as $n \to \infty$,

1. The average number of moves for a single game of memory is $(3 - 2\ln 2)\, n + 7/8 - 2\ln 2 \approx 1.61n$.
2. The average number of occurrences where the player flips over two identical cards coincidentally is $\ln 2$.
3. The average number of flips until the first match has been found in a game of memory is $2^{2n}/\binom{2n}{n} \sim \sqrt{\pi n}$.

1 Introduction

There are two versions of memory, a two player version as in [1], and a solitaire version. Both versions consist of two identical decks of cards. Each deck consists of n cards, and the two decks are shuffled together to obtain a single deck of size $2n$ cards. Each of the $2n$ cards are laid face down individually and then the game commences. It is played by a series of moves whereby a player flips two cards, if the two cards match then they are removed from the game and play continues to the next move, if they don't match they are returned, face down, to play. Play ends when all the pairs of cards have been removed and the game is finished.

For the two-player version, both players are competing to see who can remove the most pairs of cards from play. This version has its own winning strategies, a risky and a safe strategy illustrated in [1], the author E. Alfthan describes such a game of memory in terms of the game of Nim and Misère. This version will not be dealt with - only the solitaire version, played by a single player, will be considered.

In dealing with the solitaire case it will be assumed that the player has perfect memory and can remember the position of each of the cards that he has flipped. It will also be assumed that the player will use the optimal strategy when playing the game, that is,

1. If the player knows the positions of two identical cards he will flip both and remove them from play.
2. If the player does not know the position of any two identical cards then he begins a move by flipping a single unknown card. If the player knows the position of its identical partner then the player flips that card as well and both are removed from play.

3. If the player does not know the position of any pair of identical cards, and has already flipped a card during a move he flips another unknown card. If this card matches the first, both cards are removed from play otherwise both cards are flipped face down and returned back into play.

The player continues playing moves in this way until all cards are removed from the game and the game is finished.

Considering the mechanics of the game, every card in the game of memory is turned over at least once: once in the case where it is a coincidental match, and at most twice when it has already been flipped in an earlier move and its match is found after the first flip of a move. This places the total number of card flips required from start to finish for the a game of memory to be between $2n$, when each card of the $2n$ cards are flipped over exactly once before their pair is removed, and $4n$ where each card is flipped over twice before its pair is removed. Hence, as a move in the game of memory consists of two flips, a game of memory consists of at least n moves and at most $2n$ moves.

It should be noted that for any game of memory the last two cards will always match and hence the last card is flipped exactly once. This narrows the range of moves to between n and $2n - 1$ moves.

A term that will be used throughout the paper is that of a *lucky* move. When a player performs a move whereby two unknown cards are flipped and they coincidentally match then the move will be called lucky.

As the cards are shuffled and flipped at random, the cards may be represented as laid out in a single row, and flipped going from left to right, without affecting the problem - the random order of the cards will remain unchanged.

Consider the two following examples relating to the aforementioned format for the problem:

Example 1: Suppose the two identical decks in a game of memory contain 6 cards each, i.e. $n = 6$, and that the shuffled deck of 12 cards are laid out in a single row as,

$$164431262553,$$

then the player proceeds as follows:

1. The first two cards, 1 and 6, are turned over. They don't match and are returned to play face down.

2. The next two cards, first the 4 and then the 4 to the right of it, are turned over resulting in a lucky move as both cards match and they are removed from play.

3. Next, first the 3 and then the 1 cards are flipped over and since they do not match they are returned to play face down.

4. The location of both 1's are now known, and thus, continuing with the optimal algorithm, both 1's are turned over and removed from play.

5. Next, first the 2 card then the 6 card are turned over, they do not match and are returned to play.

6. The second 2 is flipped over and its match has already been seen in the move in the 5. step, thus adhering to the optimal algorithm, the previous 2's are both flipped and removed from the game.

7. Another lucky move occurs and the two adjacent 5's are flipped, the leftmost 5 card first then the one to the right of it, and removed from play.

8. Lastly, the remaining two cards, *viz.* the two 3's, are flipped and the game is ended.

Example 2: For this example, the card order for the shortest and the longest games of memory will be illustrated:

As the shortest possible game of $2n$ cards consists of n moves, each move must be a lucky move and hence the cards may be laid out in some form analogous to

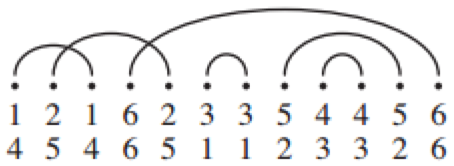
$$112233\ldots(n-1)\,(n-1)\,nn$$

where each card is laid out next to its copy.

Certainly, if the cards are laid out so that no two adjacent cards match, in the specific order illustrated below, then there will never be a lucky move, and due to matches laid out such that the second card in each pair is in an even position from the left each card, except the last, will be flipped twice, resulting in the maximal $2n-1$ moves as laid out in

$$123142\ldots k\,(k-2)\ldots(n-1)\,(n-3)\,n\,(n-2)\,(n-1)\,n.$$

A structure called an interconnection network [30, 41] will be introduced to illustrate these examples:



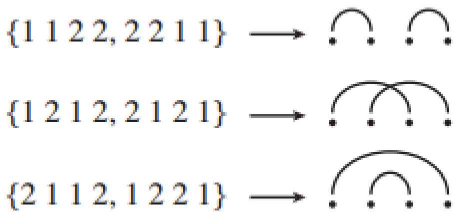
[47]

Figure 1: An example of an interconnection network.

For a deck of $2n$ cards, interconnection networks can be seen to be the $2n$ cards represented as points along with n endpoint-disjoint arcs connecting the identical pairs of cards. For any two deals, if pairs of cards appear in the same place for either deal, where the cards may be different but still form a pair, both deals will have the same interconnection networks, as illustrated by Figure 1.

Hence for $2n$ cards consisting of n identical pairs, for a single interconnection network there are n choices of pairs of cards for the first pair, $n-1$ cards for the second pair, and $n-j+1$ choices of cards for the j-th pair. This results in an $n!$-to-1 map from deals of $2n$ cards to a single interconnection network of $2n$ cards.

In the game of memory, if the cards are dealt in such a way that they are removed in the order 1 to n when the game is played, then the deal is called a *standard deal.* This is the same as when, for each $i=1,2,\ldots,n-1$, the second i in the deal is found to be to the left of the second $i+1$, resulting in the removal of the pair of i's before the pair of $i+1$'s for each $i=1,2,\ldots,n-1$, which is by definition a standard deal.



[47]

Figure 2: The different interconnection networks for the set of cards $\{1, 2\}$.

Using the aforementioned definitions, and considering a single deck of $2n$ cards consisting of two identical decks of size n, three results will be stated and proven:

1. The average number of moves in a game of memory is $(3 - 2\ln 2)\, n + 7/8 - 2\ln 2 \approx 1.61n$.

2. The average number of times that a player unknowingly flips two identical cards, i.e. the expected number of lucky moves is ln2.

3. The expected number of moves in a game of memory until the first match is found is $2^{2n}/\binom{2n}{n} \sim \sqrt{\pi n}$.

Consider now the multiset $\{1, 1, 2, 2, \ldots, n, n\}$; clearly a game of memory is equivalent to some permutation of this set. Denote the set of all of these standard permutations by M_n. Every game of memory is thus some element of this set M_n.

For each game $\sigma \in M_n$, σ is defined as a map $\sigma : [n] \to [n]$. Say, for example, for the deal 1212 from M_2, $\sigma(1) = \sigma(3) = 1$ and $\sigma(2) = \sigma(4) = 2$. Considering the cardinality of M_n leads to the following lemma.

Lemma 1.1. *Let $n \geq 1$, for a deck of $2n$ cards consisting of 2 identical decks of size n, the number of total possible standard games that may occur is given by*

$$(2n-1)\,!! = \frac{(2n)\,!}{2^n n!},$$

where $(2n-1)\,!! = 1 \cdot 3 \cdot 5 \ldots (2n-1)$, the double factorial.

Proof. This will be proven by induction on n.

Base Case: $n = 1$

For $n = 1$, the formula is trivially true.

Induction Hypothesis:

Assume that for some $n \geq 2$ the number of standard games for a game of memory from two identical decks of size $n - 1$ is given by

$$(2\,(n-1) - 1)\,!! = (2n-3)\,!!$$

Case: n

By adding two copies of the card n to a standard deal of size $2(n-1)$, a standard deal of size $2n$ may be formed: putting an n at the end of the deal to ensure that the deal remains standard and an n in any of the $2(n+1)-1$ places between the initial $2(n-1)$ cards. Thus the number of games of memory possible from a deck of $2n$ cards formed in this way is given by

$$\begin{aligned}(2n-3)!!\,(2(n-1)+1) &= (2n-3)!!\,(2n-1) \\ &= (2n-1)!! \\ &= \frac{(2n)!}{2^n n!}.\end{aligned}$$

□

For $\sigma \in M_n$, where $n \geq 1$, the position of the second card in the pair that are matched first in the game σ will be denoted by $f(\sigma)$. As $\sigma \in M_n$ is defined to be a standard deal $f(\sigma)$ is the position of the second 1 card in the deal, as the first matching pair will be a pair of 1's.

Denote the total number of games $\sigma \in M_n$ that have $f(\sigma) = j$ by $a(n,j)$. Say for a game $\sigma \in M_n$ that $f(\sigma) = j$. As it is the second occurrence of the card in the first matching pair there must be at least $n-1$ cards to the right of that card - the second occurrence of each of the $2,3\ldots,n$ cards for each of the pairs for $2,3,\ldots,n$. As the card at position j is the second occurrence of the card in the first match, there must be at least one card to the left of it *viz.* the first card in the pair, hence

$$2 \leq j \leq n+1.$$

So if $j < 2$ or $j > n+1$ then $a(n,j) = 0$. Using this notation, the probability that for $\sigma \in M_n$, $f(\sigma) = j$, is given by

$$\frac{\text{the number of games that have } f(\sigma) = j}{\text{the total number of games}} = \frac{a(n,j)}{(2n)!/(2^n n!)}.$$

From this the expected value of $f(\sigma)$ can be calculated to be

$$\begin{aligned}\sum_{j=2}^{n+1} \text{Prob}(f(\sigma) = j) \cdot j &= \sum_{j=2}^{n+1} j \cdot \frac{a(n,j)}{(2n)!/(2^n n!)} \\ &= \frac{\sum_{j=2}^{n+1} j \cdot a(n,j)}{(2n)!/(2^n n!)}. \end{aligned} \tag{1}$$

Considering the numerator, $\sum_{j=2}^{n+1} j \cdot a(n,j)$, the following lemma is given.

Lemma 1.2. $\forall n \geq 2$

$$\sum_{j=2}^{n+1} j \cdot a(n,j) = 2^n n!\,.$$

Proof. This will be proven by induction on n.

Base Case: $n = 2$

$$\begin{aligned}\sum_{j=2}^{3} j \cdot a(n,j) =& 2 \cdot a(2,2) + 3 \cdot a(2,3) \\ =& 8.\end{aligned}$$

As $a(2,2) = 1$, since 1122 is the only deal for $n = 2$ where $f(\sigma) = 2$; and $a(2,3) = 2$ as 1212 and 2112 are the only standard deals for $n = 2$ where $f(\sigma) = 3$. Now $2^n n! = 2^2 2! = 8$, thus the hypothesis holds for the base case.

Induction Hypothesis:

Assume that the hypothesis holds for some $n - 1$ where $n \geq 3$.

Case: n

There are two ways that $\sigma \in M_n$ with $f(\sigma) = j$ can be made from $\sigma' \in M_{n-1}$: where $f(\sigma') = j$ or $f(\sigma') = j - 1$.

1. If $f(\sigma') = j$, then the first n is inserted into σ' after the position j, preserving $f(\sigma') = j$, in any of $(2n - (1 + j)) = (2n - 1 - j)$ positions and the second n is added at the end of σ' as the result must still be a standard permutation, resulting in $\sigma \in M_n$ with $f(\sigma) = j$. This σ can be created in $(2n - 1 - j) \cdot a(n-1, j)$ ways.
2. If $f(\sigma') = j - 1$, then the first n card is inserted in σ' before the position $j - 1$, resulting in $f(\sigma) = j$, in any of $j - 1$ places, and the second n is added at the end of σ' ensuring that the resulting permutation $\sigma \in M_n$ is still standard. This can be done in $(j - 1) \cdot a(n-1, j-1)$ ways.

The two ways in which $\sigma \in M_n$ can be created from $\sigma' \in M_{n-1}$, where $f(\sigma') = j$ or $f(\sigma') = j - 1$, leads to the recurrence

$$a(n,j) = (2n - 1 - j) \cdot a(n-1, j) + (j - 1) \cdot a(n-1, j-1).$$

Thus

$$\begin{aligned}\sum_{j=2}^{n+1} j \cdot a(n,j) &= \sum_{j=2}^{n+1} j \cdot \left[(2n - 1 - j) \cdot a(n-1, j) + (j - 1) \cdot a(n-1, j-1)\right] \\ &= \sum_{j=2}^{n+1} j \cdot (2n - 1 - j) \cdot a(n-1, j) + \sum_{j=2}^{n+1} j \cdot (j - 1) \cdot a(n-1, j-1) \\ &= \sum_{j=2}^{n+1} j \cdot (2n - 1 - j) \cdot a(n-1, j) + \sum_{j=1}^{n} j \cdot (j + 1) \cdot a(n-1, j),\end{aligned}$$

and as $a\left(n-1, j\right)=0$ if $j>n$,

$$\sum_{j=2}^{n+1} j \cdot (2n-1-j) \cdot a\left(n-1, j\right) = \sum_{j=2}^{n} j \cdot (2n-1, j) \cdot a\left(n-1, j\right).$$

So

$$\begin{aligned}
\sum_{j=2}^{n+1} j \cdot a\left(n, j\right) =& \sum_{j=2}^{n} j \cdot (2n-1-j) \cdot a\left(n-1, j\right) + \sum_{j=1}^{n} j \cdot (j+1) \cdot a\left(n-1, j\right) \\
=& \sum_{j=2}^{n} 2nj \cdot a\left(n-1, j\right) \\
=& 2n \cdot \sum_{j=2}^{n} j \cdot a\left(n-1, j\right) \\
=& 2n \cdot 2^{n-1}\left(n-1\right)! \quad \text{(by the induction hypothesis)} \\
=& 2^n n!.
\end{aligned}$$

$\square$

The results from Lemma 1.2 will now be used in the next theorem, and its corollary.

Theorem 1.3. *Let $n \geq 1$, then the position of the second card in the first matching pair is expected to be at*

$$\frac{2^{2n}}{\binom{2n}{n}}.$$

Proof. By equation (1) the expected value of $f\left(\sigma\right)$, which is the expected position of the first match, is

$$\frac{\sum_{j=2}^{n+1} j \cdot a\left(n, j\right)}{\left(2n\right)!/\left(2^n n!\right)}.$$

By Lemma 1.2.,

$$\begin{aligned}
\frac{\sum_{j=2}^{n+1} j \cdot a\left(n, j\right)}{\left(2n\right)!/\left(2^n n!\right)} =& \frac{2^n n!}{\left(2n\right)!/\left(2^n n!\right)} \\
=& \frac{2^{2n}}{\left(2n\right)!/\left(n!\right)^2} \\
=& \frac{2^{2n}}{\binom{2n}{n}}.
\end{aligned}$$

$\square$

Corollary 1.3.1.

$$E\left(f\left(\sigma\right)\right) = \frac{2^{2n}}{\binom{2n}{n}} \sim \sqrt{\pi n}.$$

Proof. Stirling's approximation [50] states that

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Thus

$$\begin{aligned}
\frac{2^{2n}}{\binom{2n}{n}} &= \frac{2^{2n}}{(2n)!/(n!)^2} \\
&= \frac{2^{2n}(n!)^2}{(2n)!} \\
&\sim \frac{2^{2n}\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2}{\left(\sqrt{2\pi 2n}\left(\frac{2n}{e}\right)^{2n}\right)} \\
&= \frac{2^{2n}(2\pi n)\left(\frac{n}{e}\right)^{2n}}{2\sqrt{\pi n}2^{2n}\left(\frac{n}{e}\right)^{2n}} \\
&= \sqrt{\pi n}.
\end{aligned}$$

□

A main result with regard to the game of memory will be stated and proven using a number of smaller lemmas and theorems in the next theorem.

Theorem 1.4. *The average number of moves, from start to finish, for a game of memory consisting of two identical decks of size n is given by*

$$(3 - 2\ln 2)\,n + 7/8 - 2\ln 2 + \epsilon_n$$

where $lim_{n\to\infty}\epsilon_n = 0$.

Proof. Consider now $f(\sigma)$, the position of the second card in the first matching pair in a standard deal $\sigma \in M_n$. The number of moves required until the first pair is removed from the game depends on $f(\sigma)$. If $f(\sigma)$ is odd then flipping the card at position $f(\sigma)$ will be part of the first step in a move, and as such the first occurrence of 1 will have already been flipped, thus the position of both 1's is known and both are flipped and removed from the game. Doing so will take $f(\sigma)+1$ flips, or $(f(\sigma)+1)/2$ moves.

Now if $f(\sigma)$ is even, then the card at position $f(\sigma)$ will be flipped as the second flip in a move. If the card at position $f(\sigma)-1$ is also a 1 then a lucky move occurs and both 1's are removed from the game. However, if the card at position $f(\sigma)-1$ is not a 1 then an additional move is required whereby the first occurrence of 1 along with the occurrence of 1 at $f(\sigma)$ are both flipped and removed from the game. In the case where $f(\sigma)$ is even and a lucky move occurs, $f(\sigma)/2$ moves are required until the first match. If a lucky move did not occur then $f(\sigma)/2+1$ moves are required until the first match.

After the first match both 1's have been removed from σ, as σ is a standard deal the next matching pair will be of 2's. Once again the player continues playing moves, starting from position $f(\sigma)+1$. Say the next match is found at the card at position $f(\sigma)+k$, the second occurrence of the 2 card, for some $k \geq 1$. Again the number of moves until the second match depends on whether k is odd or even.

If k is odd, then as in the case for the first match, it will take $k+1$ flips from the last match, until the second match, or $(k+1)/2$ moves. If k is even and a lucky move occurs whereby the

2 card was at the position immediately before $f(\sigma) + k$ then it takes $k/2$ moves from the first match to the second match. Otherwise if a lucky move does not occur it takes $k/2 + 1$ moves from the first match to the second match. Each matching pair is removed from the game in this way until the game is finished.

The notion of a *block* in a deal $\sigma \in M_n$ will now be introduced: a block is a part of a partition of a standard deal $\sigma \in M_n$. Each block starts at the end of a previous block and ends when the second occurrence of a card in a matching pair is found. The first block starts at the start of the deal and the last ends at the end of the deal.

Denote the length of each of these blocks for σ as $b_i(\sigma)$ for $i = 1, 2, \ldots, n$ where the blocks are listed in the order they appear in the deal from left to right. Clearly $b_1(\sigma) = f(\sigma)$. And as $\sigma \in M_n$, $\sum_{i=1}^{n} b_i(\sigma) = 2n$, as the blocks form a partition of all the $2n$ cards for $\sigma \in M_n$.

When the last two cards in a block form a matching pair then that block is said to be a *lucky* block. Now by the previous result, discussing the number of moves until each match, if blocks are considered then the number of flips required in the i-th block until the i-th pair are matched is determined by whether $b_i(\sigma)$ is odd or even. The number of moves can be represented by the equation

$$b_i(\sigma)/2 + a_i(\sigma),$$

where

$$a_i(\sigma) = \begin{cases} \frac{1}{2} & \text{if } b_i(\sigma) \text{ odd} \\ 0 & \text{if } b_i(\sigma) \text{ even and } i\text{-th block is lucky} \\ 1 & \text{if } b_i(\sigma) \text{ even and } i\text{-th block is not lucky.} \end{cases}$$

Denote the total number of even length blocks by $e(\sigma)$, and denote the total number of these blocks which are lucky blocks by $L(\sigma)$. Then the total number of moves in a game of memory is given by

$$\begin{aligned}
\sum_{i=1}^{n} \left(\frac{b_i(\sigma)}{2} + a_i(\sigma) \right) =& \frac{\sum_{i=1}^{n} b_i(\sigma)}{2} + \sum_{n=1}^{n} a_i(\sigma) \\
=& \frac{\sum_{i=1}^{n} b_i(\sigma)}{2} + \\
& \frac{1}{2} \cdot (\text{number of odd blocks}) + 1 \cdot (\text{number of even blocks that are not lucky}) \\
=& \frac{\sum_{i=1}^{n} b_i(\sigma)}{2} + \frac{1}{2} \cdot (n - e(\sigma)) + 1 \cdot (e(\sigma) - L(\sigma)) \\
=& \frac{3n}{2} + \frac{e(\sigma)}{2} - L(\sigma).
\end{aligned}$$

Denote the total number of blocks consisting of j cards in every standard deal of $\sigma \in M_n$ by $b(n, j)$ for $j \geq 2$. Similarly denote the total number of lucky blocks contained in these deals consisting of j cards by $L(n, j)$ for $j \geq 2$.

The first block may have the greatest possible length, which occurs if the second occurrence of 1 is at position $n + 1$, followed by the single cards $2, 3, 4, \ldots, n$. Hence, if $j > n + 1$, then $b(n, j) = L(n, j) = 0$. Using the notation $b(n, j)$, the expected value for the total amount of blocks consisting of j cards in all standard deals of $\sigma \in M_n$ is

$$\bar{b}(n, j) = \frac{b(n, j)}{(2n)!/(2^n n!)},$$

and for lucky blocks,

$$\bar{L}(n,j) = \frac{L(n,j)}{(2n)!/(2^n n!)},$$

which leads to the expected values

$$E[e(\sigma)] = \sum_{i=1}^{\infty} \bar{b}(n,2i)$$

for even blocks and,

$$E[L(\sigma)] = \sum_{i=1}^{\infty} \bar{L}(n,2i),$$

for even lucky blocks. Recall that the length of a game is

$$\frac{3n}{2} + \frac{e(\sigma)}{2} - L(\sigma).$$

Thus by the above results the expected length in given by

$$\frac{3n}{2} + \frac{E[e(\sigma)]}{2} - E[L(\sigma)] = \frac{3n}{2} + \frac{\sum_{i=1}^{\infty} \bar{b}(n,2i)}{2} - \sum_{i=1}^{\infty} \bar{L}(n,2i). \tag{2}$$

This equation (2) will be simplified using results that will later be shown. These results are the asymptotic limits of $\bar{b}(n,j)$ and $\bar{L}(n,j)$. For sufficiently large n

$$\bar{b}(n,j) \approx \frac{2(2n+1)}{j(j+1)(j+2)}, \text{ and for } j \geq 2,\ \bar{L}(n,j) \approx \frac{1}{j(j-1)}. \tag{3}$$

These two approximations for $\bar{b}(n,j)$ and $\bar{L}(n,j)$ will be shown to be true by taking the limits of the following equations and showing them to be 0.

$$\dot{b}(n,j) = \bar{b}(n,j) - \frac{2(2n+1)}{j(j+1)(j+2)} \quad \text{and for} \quad j \geq 2,\ \dot{L}(n,j) = \bar{L}(n,j) - \frac{1}{j(j-1)}. \tag{4}$$

This is the same as

$$\bar{b}(n,j) = \dot{b}(n,j) + \frac{2(2n+1)}{j(j+1)(j+2)} \quad \text{and for} \quad j \geq 2,\ \bar{L}(n,j) = \dot{L}(n,j) + \frac{1}{j(j-1)}.$$

By the equations given in (4), equation (2) may now be rewritten as

$$\begin{aligned}\sum_{i=1}^{\infty} \bar{b}(n,2i) &= \sum_{i=1}^{\infty} \left(\frac{2(2n+1)}{2i(2i+1)(2i+2)} + \dot{b}(n,2i) \right) \\ &= \sum_{i=1}^{\infty} \frac{2(2n+1)}{2i(2i+1)(2i+2)} + \sum_{i=1}^{\infty} \dot{b}(n,2i).\end{aligned}$$

Now

$$\frac{2}{j(j+1)(j+2)} = \frac{1}{j} - \frac{2}{j+1} + \frac{1}{j+2}, \tag{5}$$

so

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{2\left(2n+1\right)}{2i\left(2i+1\right)\left(2i+2\right)} &= \left(2n+1\right) \lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{1}{2i} - \frac{2}{2i+1} + \frac{1}{2i+2}\right) \\
&= \left(2n+1\right) \lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{1}{2i} - \frac{2}{2i+1} + \frac{1}{2\left(i+1\right)}\right) \\
&= \left(2n+1\right) \lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{4} - \frac{2}{5} + \frac{1}{6} + \cdots + \frac{1}{2N} - \frac{2}{2N+1} + \frac{1}{2N+2}\right) \\
&= \left(2n+1\right) \lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{1}{2} - \frac{2}{3} + \frac{2}{4} - \frac{2}{5} + \cdots + \frac{2}{2N} - \frac{2}{2N+1} + \frac{1}{2N+2}\right) \\
&= \left(2n+1\right) \lim_{N\to\infty} \sum_{i=1}^{N} \left(\left(\frac{3}{2} - 2\left(1 - \frac{1}{2}\right)\right) - \frac{2}{3} + \frac{2}{4} - \frac{2}{5} + \cdots + \frac{2}{2N} - \frac{2}{2N+1} + \frac{1}{2N+2}\right) \\
&= \left(2n+1\right) \lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{3}{2} + \frac{1}{2N+2} - 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \frac{1}{2N+1}\right)\right).
\end{aligned}$$

Now, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{1}{2N+1}$ is the alternating harmonic series, which converges to ln2, thus

$$\lim_{N\to\infty} 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{1}{2N+1}\right) = 2\ln 2.$$

So, as $\lim_{N\to\infty} \frac{1}{2N+2} = 0$, the first part of the expression for $\sum_{i=1}^{\infty} \bar{b}\left(n, 2i\right)$ simplifies to

$$\sum_{i=1}^{\infty} \frac{2\left(2n+1\right)}{2i\left(2i+1\right)\left(2i+2\right)} = \left(2n+1\right)\left(\frac{3}{2} - 2\ln 2\right).$$

Now for lucky even blocks,

$$\begin{aligned}
\sum_{i=1}^{\infty} \bar{L}\left(n, 2i\right) &= \sum_{i=1}^{\infty} \left(\frac{1}{2i\left(2i-1\right)} + \dot{L}\left(n, 2i\right)\right) \\
&= \sum_{i=1}^{\infty} \frac{1}{2i\left(2i-1\right)} + \sum_{i=1}^{\infty} \dot{L}\left(n, 2i\right).
\end{aligned}$$

Again, the first part of the expression for $\sum_{i=1}^{\infty} \bar{L}\left(n, 2i\right)$ will be calculated. The fraction within the sum will be decomposed as

$$\frac{1}{j\left(j-1\right)} = \frac{1}{j-1} - \frac{1}{j},$$

so

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{1}{2i\left(2i-1\right)} &= \lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{1}{2i-1} - \frac{1}{2i}\right) \\
&= \lim_{N\to\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2N-1} - \frac{1}{2N}\right).
\end{aligned}$$

Now $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2N-1} - \frac{1}{2N}$ is the alternating harmonic sequence again, and its limit is known from the previous result, so

$$\sum_{i=1}^{\infty} \frac{1}{2i\left(2i-1\right)} = \ln 2.$$

Thus far, the following equations are known,

$$\sum_{i=1}^{\infty} \bar{b}\,(n, 2i) = (2n+1)\left(\frac{3}{2} - 2\ln 2\right) + \sum_{i=1}^{\infty} \dot{b}\,(n, 2i) \tag{6}$$

and,

$$\sum_{i=1}^{\infty} \bar{L}\,(n, 2i) = \ln 2 + \sum_{i=1}^{\infty} \dot{L}\,(n, 2i)\,. \tag{7}$$

The results from (6) and (7) may now be substituted back into equation (2) to obtain

$$\frac{3n}{2} + \frac{\sum_{i=1}^{\infty} \bar{b}\,(n, 2i)}{2} - \sum_{i=1}^{\infty} \bar{L}\,(n, 2i) = (3 - 2\ln 2)\,n + \frac{3}{4} - 2\ln 2 + \frac{\sum_{i=1}^{\infty} \dot{b}\,(n, 2i)}{2} - \sum_{i=1}^{\infty} \dot{L}\,(n, 2i)\,.$$

The last two results needed to prove Theorem 1.4 will be to show that

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \dot{b}\,(n, 2i) = \frac{1}{4} \quad \text{and} \quad \lim_{n\to\infty} \sum_{i=1}^{\infty} \dot{L}\,(n, 2i) = 0.$$

1.1 $\lim_{\mathbf{n}\to\infty} \sum_{\mathbf{i=1}}^{\infty} \dot{\mathbf{b}}\,(\mathbf{n}, \mathbf{2i}) = \frac{\mathbf{1}}{\mathbf{4}}$

Consider first $b\,(n, j)$, the total number of blocks of length j in all standard deals $\sigma \in M_n$. Trivially for the deal 11, $b\,(1, 2) = 1$, $b\,(1, j) = 0$ if $j \neq 2$, since the only deal for $n = 1$ is 11. Recall that $\sigma \in M_n$ may be formed from $\sigma' \in M_{n-1}$ by adding an n to the end of σ' and inserting an n somewhere before the added n.

When the n is inserted it may be placed in any of $2n-1$ positions. If it is inserted in a block of length j the length of that block is incremented by 1. The n added to the end of the deal results either in a block of length 1 if the second n is not inserted next to the first, or a block of length 2 if it is, as the second last block ends just before it.

Creating $\sigma \in M_n$ from $\sigma' \in M_{n-1}$ in this way results in the following three recursions:

1. The first recursion is given by,

$$b\,(n, 1) = (2n-2) \cdot b\,(n-1, 1) + (2n-2) \cdot \frac{(2\,(n-1))\,!}{2^{n-1}\,(n-1)\,!}.$$

This recursion is derived in the following way: there are $(2n-2)$ ways to insert an n into a deal $\sigma' \in M_{n-1}$ avoiding a particular block of size 1, and there are exactly $b\,(n-1, 1)$ blocks of size 1. This explains the first term in the recursion. For every game of the $\frac{(2(n-1))!}{2^{n-1}(n-1)!}$ games a block of size 1 is added at the end if only a single n is added to the end of the deal and the second n is not inserted next to it. Avoiding the position next to the n added at the end of the deal, the second n may be inserted into $2n-2$ places. This explains the second term in the recursion.

2. The second recursion is given by,

$$b\,(n, 2) = b\,(n-1, 1) + (2n-3) \cdot b\,(n-1, 2) + \frac{(2\,(n-1))\,!}{2^{n-1}\,(n-1)\,!}.$$

Forming $\sigma \in M_n$ from $\sigma' \in M_{n-1}$ in the same way as for the first recursion, an n is appended to the end of the deal σ', and another n is inserted to the left of that n. Each block of size 1 that the n is inserted into becomes a block of size 2, and there are $b\,(n-1,1)$ blocks of size 1 in all deals $\sigma' \in M_{n-1}$, thus the term $b\,(n-1,1)$. There are $2n-3$ ways that n may be inserted into σ' such that it avoids a particular block of size 2 and there are a total of $b\,(n-1,2)$ blocks of size 2 in all $\sigma' \in M_{n-1}$, hence the term $(2n-3)\cdot b\,(n-1,2)$. Considering the last term, if both n's are put at the end of any of the $\frac{(2(n-1))!}{2^{n-1}(n-1)!}$ deals, then a new block of size 2 is created in σ', resulting in the final term in the second recursion.

3. The third recursion, for $j \geq 3$, is given by

$$b\,(n,j) = (j-1)\cdot b\,(n-1,j-1) + (2n-1-j)\cdot b\,(n-1,j)\,.$$

Consider blocks of size j where $j \geq 3$, when forming $\sigma \in M_n$ from $\sigma' \in M_{n-1}$ by adding an n to the end of σ' and inserting an n into σ' in the same way as in the last two recursions. Inserting n anywhere in a block of size $j-1$, except at the end, results in a block of size j. This n may be inserted in such a block in any of $j-1$ positions, and since there are $b\,(n-1,j-1)$ blocks of size $j-1$ in all deals $\sigma' \in M_{n-1}$ this explains the first term. The second term is a result of inserting an n into $\sigma' \in M_{n-1}$ by avoiding a specific block of size j, of which there are $b\,(n-1,j)$ in all $\sigma' \in M_{n-1}$. This n may be inserted into any of $(2(n-1)-j+1) = (2n-j-1)$ positions, thus resulting in $(2n-1-j)\cdot b\,(n-1,j)$.

Dividing each of the 3 recurrences by $\frac{(2n)!}{2^n n!}$,

$$\bar{b}\,(1,2) = 1\ ,\ \text{and}\ \ \bar{b}\,(1,j) = 0\ \ \text{for}\ \ j \neq 2.$$

For $n \geq 2$, the first recursion becomes

$$\begin{aligned}
\frac{b\,(n,1)}{(2n)\,!\,/(2^n n!\,)} &= \frac{(2n-2)\cdot b\,(n-1,1)}{(2n)\,!\,/2^n n!} + \frac{(2n-2)\,\frac{(2n-2)!}{2^{n-1}(n-1)!}}{(2n)\,!\,/2^n n!} \\
&= \frac{(2n-2)\cdot b\,(n-1,1)}{2n\,(2n-1)\,(2n-2)\,!\,/(2^{n-1}\,(n-1)\,!\,2n)} + \frac{\frac{(2n-2)(2n-2)!2^n n!}{2^{n-1}(n-1)!}}{(2n)\,!} \\
&= \frac{(2n-2)\cdot \bar{b}\,(n-1,1)}{2n-1} + \frac{(2n-2)\,(2n-2)\,!\,2n}{(2n)\,!} \\
&= \frac{(2n-2)\cdot \bar{b}\,(n-1,1)}{2n-1} + \frac{(2n-2)\,2n}{2n\,(2n-1)} \\
&= \frac{(2n-2)\cdot\big(\bar{b}\,(n-1,1)+1\big)}{2n-1}.
\end{aligned}$$

The second recursion becomes

$$\begin{aligned}
\frac{b\,(n,2,)}{(2n)\,!\,/(2^n n!\,)} =&\frac{b\,(n-1,1)}{(2n)\,!\,/(2^n n!\,)} + \frac{(2n-3)\cdot b\,(n-1,2)}{(2n)\,!\,/(2^n n!\,)} + \frac{\frac{(2n-2)!}{2^{n-1}(n-1)!}}{(2n)\,!\,/(2^n n!\,)} \\
=&\frac{b\,(n-1,1)}{(2n-1)\,!\,/(2^{n-1}\,(n-1)\,!\,)} + \frac{(2n-3)\cdot b\,(n-1,2)}{(2n-1)\,!\,/(2^{n-1}\,(n-1)\,!\,)} + \frac{(2n-2)\,!\,2n}{(2n)\,!} \\
=&\frac{\bar{b}\,(n-1,1)}{2n-1} + \frac{(2n-3)\cdot \bar{b}\,(n-1,2)}{2n-1} + \frac{1}{2n-1} \\
=&\frac{\bar{b}\,(n-1,1) + (2n-3)\cdot \bar{b}\,(n-1,2) + 1}{2n-1}.
\end{aligned}$$

And the third recursion gives

$$\begin{aligned}
\frac{b\,(n,j)}{(2n)\,!\,/(2^n n!\,)} =&\frac{(j-1)\cdot b\,(n-1,j-1)}{(2n)\,!\,/(2^n n!\,)} + \frac{(2n-1-j)\cdot b\,(n-1,j)}{(2n)\,!\,/(2^n n!\,)} \\
=&\frac{(j-1)\cdot b\,(n-1,j-1)}{(2n-1)\,!\,/(2^{n-1}\,(n-1)\,!\,)} + \frac{(2n-1-j)\cdot b\,(n-1,j)}{(2n-1)\,!\,/(2^{n-1}\,(n-1)\,!\,)} \\
=&\frac{(j-1)\cdot b\,(n-1,j-1)}{2n-1} + \frac{(2n-1-j)\cdot b\,(n-1,j)}{2n-1} \\
=&\frac{(j-1)\cdot b\,(n-1,j-1) + (2n-1-j)\cdot b\,(n-1,j)}{2n-1}.
\end{aligned}$$

These 5 results for $\bar{b}\,(n,j)$ will now be stated in terms of $\dot{b}\,(n,j)$ using equation (4) in the following lemma.

Lemma 1.5.

$$\dot{b}\,(1,2) = \frac{3}{4},$$

and for $j \neq 2$

$$\dot{b}\,(1,j) = -\frac{6}{j\,(j+1)\,(j+2)}.$$

Considering $n \geq 2$*,*

$$\begin{aligned}
\dot{b}\,(n,1) =&\frac{(2n-2)\cdot \dot{b}\,(n-1,1) + 1}{2n-1}, \\
\dot{b}\,(n,2) =&\frac{\dot{b}\,(n-1,1) + (2n-3)\cdot \dot{b}\,(n-1,2) + 1}{2n-1},
\end{aligned}$$

and for $j \geq 3$*,*

$$\dot{b}\,(n,j) = \frac{(j-1)\cdot \dot{b}\,(n-1,j-1) + (2n-1-j)\cdot \dot{b}\,(n-1,j)}{2n-1}.$$

Proof. By equation (4)

$$\begin{aligned}
\dot{b}(1,2) =& \bar{b}(1,2) - \frac{2(2+1)}{2(2+1)(2+2)} \\
=& \bar{b}(1,2) - \frac{3}{12} \\
=& \frac{b(1,2)}{(2!)/(2^1 \cdot 1!)} - \frac{1}{4} \\
=& 1 - \frac{1}{4} \\
=& \frac{3}{4},
\end{aligned}$$

and for $j \geq 3$,

$$\begin{aligned}
\dot{b}(1,j) =& \bar{b}(1,j) - \frac{2(2\cdot 1+1)}{j(j+1)(j+2)} \\
=& 0 - \frac{6}{j(j+1)(j+2)} \text{ (as } b(i,j)=0 \text{ when } j \neq 2) \\
=& -\frac{6}{j(j+1)(j+2)} \text{ (as } b(i,j)=0 \text{ when } j \neq 2).
\end{aligned}$$

Now, for $n \geq 2$,

$$\begin{aligned}
\dot{b}(n,1) =& \bar{b}(n,1) - \frac{2(2n+1)}{1\cdot 2\cdot 3} \\
=& \frac{(2n-2)\left(\bar{b}(n-1,1)+1\right)}{2n-1} - \frac{2(2n+1)}{6} \\
=& \frac{1}{2n-1}(2n-2)\left(\dot{b}(n-1,1) + \frac{2(2(n-1)+1)}{1.2.3} + 1\right) - \frac{2(2n+1)}{6} \\
=& \frac{2n-2}{2n-1} + \frac{(2n-2)\cdot\dot{b}(n-1,1)}{2n-1} + \frac{2(2n-1)(2n-2) - 2(2n+1)(2n-1)}{6(2n-1)} \\
=& \frac{(2n-2)\cdot\dot{b}(n-1,1)}{2n-1} + \frac{2\left(4n^2-6n+2\right) - 2\left(4n^2-1\right)}{6(2n-1)} + \frac{2n-2}{2n-1} \\
=& \frac{(2n-2)\cdot\dot{b}(n-1,1)}{2n-1} + \frac{-12n+6}{6(2n-1)} + \frac{2n-2}{2n-1} \\
=& \frac{(2n-2)\cdot\dot{b}(n-1,1) - 2n+1+2n-2}{2n-1} \\
=& \frac{(2n-2)\cdot\dot{b}(n-1,1) - 1}{2n-1},
\end{aligned}$$

$$
\begin{aligned}
\dot{b}\left(n,2\right) =&\bar{b}\left(n,2\right)-\frac{2\left(2n+1\right)}{2.3.4}\\
=&\frac{\bar{b}\left(n-1,1\right)+\left(2n-3\right)\cdot\bar{b}\left(n-1,2\right)+1}{2n-1}-\frac{2n+1}{12}\\
=&\frac{1}{2n-1}\left(\frac{2\left(2n-1\right)}{1\cdot2\cdot3}+\dot{b}\left(n-1,1\right)+\left(2n-3\right)\left(\frac{2\left(2n-1\right)}{2.3.4}+\dot{b}\left(n-1,2\right)\right)+1\right)-\frac{2n+1}{12}\\
=&\frac{1}{3}+\frac{\dot{b}\left(n-1,1\right)}{2n-1}+\frac{2n-3}{12}+\frac{\left(2n-3\right)\cdot\dot{b}\left(n-1,2\right)}{2n-1}+\frac{1}{2n-1}-\frac{2n+1}{12}\\
=&\frac{\dot{b}\left(n-1,1\right)+\left(2n-3\right)\cdot\dot{b}\left(n-1,2\right)+1}{2n-1}
\end{aligned}
$$

and,

$$
\begin{aligned}
\dot{b}\left(n,j\right) =&\bar{b}\left(n,j\right)-\frac{2\left(2n+1\right)}{j\left(j+1\right)\left(j+1\right)}\\
=&\frac{\left(j-1\right)\cdot\bar{b}\left(n-1,j-1\right)+\left(2n-1-j\right)\cdot\bar{b}\left(n-1,j\right)}{2n-1}-\frac{2\left(2n+1\right)}{j\left(j+1\right)\left(j+2\right)}\\
=&\frac{j-1}{2n-1}\left(\dot{b}\left(n-1,j-1\right)+\frac{2\left(2\left(n-1\right)+1\right)}{\left(j-1\right)j\left(j+1\right)}\right)+\frac{2n-1-j}{2n-1}\left(\dot{b}\left(n-1,j\right)+\frac{2\left(2\left(n-1\right)+1\right)}{j\left(j+1\right)\left(j+2\right)}\right)\\
&-\frac{2\left(2n+1\right)}{j\left(j+1\right)\left(j+2\right)}\\
=&\frac{\left(j-1\right)\cdot\dot{b}\left(n-1,j-1\right)+\left(2n-1-j\right)\cdot\dot{b}\left(n-1,j\right)}{2n-1}+\frac{\left(j-1\right)2\left(2n-1\right)}{\left(j-1\right)j\left(j+1\right)\left(2n-1\right)}\\
&+\frac{\left(2n-1-j\right)2\left(2n-1\right)}{j\left(j+1\right)\left(j+2\right)\left(2n-1\right)}-\frac{2\left(2n+1\right)}{j\left(j+1\right)\left(j+2\right)}\\
=&\frac{\left(j-1\right)\cdot\dot{b}\left(n-1,j-1\right)+\left(2n-1-j\right)\cdot\dot{b}\left(n-1,j\right)}{2n-1}\\
&+\frac{2\left(j+2\right)}{j\left(j+1\right)\left(j+2\right)}+\frac{\left(2n-1-j\right)2}{j\left(j+1\right)\left(j+2\right)}-\frac{2\left(2n+1\right)}{j\left(j+1\right)\left(j+2\right)}\\
=&\frac{\left(j-1\right)\cdot\dot{b}\left(n-1,j-1\right)+\left(2n-1-j\right)\cdot\dot{b}\left(n-1,j\right)}{2n-1}+\frac{2j+4+4n-2-2j-4n-2}{j\left(j+1\right)\left(j+2\right)}\\
=&\frac{\left(j-1\right)\cdot\dot{b}\left(n-1,j-1\right)+\left(2n-1-j\right)\cdot\dot{b}\left(n-1,j\right)}{2n-1}.
\end{aligned}
$$

□

Using results from Lemma 1.5, the next lemma is obtained.

Lemma 1.6. *1. For $n \geq 1$,*

$$\dot{b}(n,1) = -1.$$

2. For $n \geq 1$,

$$\dot{b}(n,2) = \frac{3}{4(2n-1)}.$$

3. For $n \geq 2$,

$$\dot{b}(n,3) = 2 \cdot \dot{b}(n,2) = \frac{3}{2(2n-1)}.$$

Proof. Each of the three results will be proven by induction on n.

Proving (1):

Base Case: $n = 1$

$$\begin{aligned}\dot{b}(1,1) =& \frac{(2 \cdot 1 - 2) \cdot \dot{b}(0,1) - 1}{2.1 - 1} \\ =& -1.\end{aligned}$$

Therefore (1) holds for the base case.

Induction Hypothesis:

Assume that for some $k \geq 2$,

$$\dot{b}(k,1) = -1.$$

Case: $k+1$

$$\begin{aligned}\dot{b}(k+1,1) =& \frac{(2(k+1) - 2) \cdot \dot{b}(k,1) - 1}{2(k+1) - 1} \\ =& \frac{(2k)(-1) - 1}{2k+1} \\ =& -\frac{2k+1}{2k+1} \\ =& -1.\end{aligned}$$

And thus (1) holds for all $n \geq 1$.

Proving (2):

Base Case: $n = 1$

$$\begin{aligned}\dot{b}(1,2) =& \frac{\dot{b}(0,1) + (2\cdot 1 - 3)\,\dot{b}(0,2) + 1}{2\cdot 1 - 1}\\ =& \frac{\dot{b}(0,1) - \dot{b}(0,2) + 1}{1}.\end{aligned}$$

Now

$$\begin{aligned}\dot{b}(0,1) =& \bar{b}(0,1) - \frac{2(2\cdot 0+1)}{1\cdot 2\cdot 3} = -\frac{2}{6}\\ \dot{b}(0,2) =& \bar{b}(0,2) - \frac{2(2\cdot 0+1)}{2\cdot 3\cdot 4} = -\frac{2}{24},\end{aligned}$$

so

$$\begin{aligned}\dot{b}(1,2) =& -\frac{2}{6} - \left(-\frac{2}{24}\right) + 1\\ =& \frac{3}{4}.\end{aligned}$$

Therefore the base case holds for (2).

Induction Hypothesis:

Assume that for some $k \geq 1$

$$\dot{b}(k,2) = \frac{3}{4(3k-1)}.$$

Case: $k + 1$

Now

$$\begin{aligned}\dot{b}(k+1,2) =& \frac{\dot{b}(k,1) + (2(k+1)-3)\cdot\dot{b}(k,2) + 1}{2(k+1)-1}\\ =& \frac{-1 + (2k-1)\,\frac{3}{4(2k-1)} + 1}{2k+1}\\ =& \frac{3(2k-1)}{4(2k-1)(2k+1)}\\ =& \frac{3}{4(2k+1)}\\ =& \frac{3}{4(2(k+1)-1)}.\end{aligned}$$

Thus (2) holds for all $n \geq 1$.

Proving (3):

Base Case: $n = 2$

$$\begin{aligned}\dot{b}(2,3) =& \frac{2.\dot{b}(1,2) + 0.\dot{b}(1,3)}{3} \\ =& \frac{2.\dot{b}(1,2)}{3} \\ =& \frac{2\left(\frac{3}{4}\right)}{3} \\ =& \frac{1}{2}.\end{aligned}$$

Thus the hypothesis holds for the base case.

Induction Hypothesis:

Let $k \geq 2$ and assume that

$$\dot{b}(k,3) = \frac{3}{2(2k-1)}.$$

Case: $k+1$

By Lemma 1.5,

$$\begin{aligned}\dot{b}(k+1,3) =& \frac{2 \cdot \dot{b}(k,2) + (2k-2) \cdot \dot{b}(k,3)}{2k+1} \\ =& \frac{2\left(\frac{3}{4(2k-1)}\right) + (2k-2)\frac{3}{2(2k-1)}}{2k+1} \\ =& \frac{3 + (2k-2)3}{(2k+1)2(2k-1)} \\ =& \frac{6k-3}{(2k+1)2(2k-1)} \\ =& \frac{3}{2(2k+1)} \\ =& \frac{3}{2(2(k+1)-1)}.\end{aligned}$$

Thus (3) holds for all $n \geq 2$. □

Now, still considering $\dot{b}(n,j)$, the next lemma follows.

Lemma 1.7. $\forall n, j \geq 2$

$$|\dot{b}(n,j)| \leq \frac{3}{4} \cdot \frac{2 \cdot 4 \cdot 6 \ldots (2n-2)}{3 \cdot 5 \cdot 7 \ldots (2n-1)} \tag{8}$$

Proof. This lemma will be proven by induction on n.

Base Case: $n = 1,\ j = 2$

$$\begin{aligned} |\dot{b}\,(1,2)\,| &= \left|\frac{3}{4}\right| \\ &\leq \frac{3}{4}, \end{aligned}$$

as the products in the right fraction become empty products.

If $j \geq 2$, then by Lemma 1.5

$$\begin{aligned} |\dot{b}\,(1,j)\,| &= \left|-\frac{6}{j\,(j+1)\,(j+2)}\right| \\ &\leq \left|-\frac{6}{2\cdot 3\cdot 4}\right| \\ &= \left|-\frac{6}{24}\right| \\ &= \frac{1}{4} \leq \frac{3}{4}. \end{aligned}$$

Thus the base case holds for $n = 1$.

Induction Hypothesis:

Assume that the hypothesis holds for some $n - 1 \geq 1$. That is

$$\begin{aligned} |\dot{b}\,(n-1,j)\,| &\leq \frac{3}{4}\cdot\frac{2\cdot 4\cdot 6\ldots(2\,(n-1)-2)}{3\cdot 5\cdot 7\ldots(2\,(n-1)-1)} \\ &= \frac{3}{4}\frac{2\cdot 4\cdot 6\ldots(2n-4)}{3\cdot 5\cdot 7\ldots(2n-3)}. \end{aligned}$$

Let

$$C = \frac{3}{4}\cdot\frac{2\cdot 4\cdot 6\ldots(2n-4)}{3\cdot 5\cdot 7\ldots(2n-3)}.$$

Case: n

For the case of n, where $|\dot{b}\,(n-1,j)\,| \leq C$, there are three different possibilities to consider:

1. For the first case, consider $j = 2$. Lemma 1.6 gives

$$\begin{aligned} |\dot{b}\,(n,2)\,| &= \frac{3}{4\,(2n-1)} \leq \frac{3}{4}\cdot\frac{2}{2n-1} \\ &\leq \frac{3}{4}\cdot\frac{2}{2n-1}\cdot\frac{4\cdot 6\cdot 8\ldots(2n-2)}{3\cdot 5\cdot 7\ldots(2n-3)} \\ &= \frac{2}{4}\cdot\frac{2\cdot 4\cdot 6\ldots(2n-2)}{3\cdot 5\cdot 7\ldots(2n-1)}. \end{aligned}$$

2. For the second case consider $3 \leq j \leq 2n-1$. Then, by Lemma 1.5

$$\begin{aligned}
\left|\dot{b}(n,j)\right| &= \left|\frac{(j-1)\cdot\dot{b}(n-1,j-1)+(2n-1-j)\cdot\dot{b}(n-1,j)}{2n-1}\right| \\
&\leq \frac{j-1}{2n-1}\cdot\left|\dot{b}(n-1,j-1)\right| + \frac{2n-1-j}{2n-1}\cdot\left|\dot{b}(n-1,j)\right| \\
&\leq \frac{j-1}{2n-1}\cdot C + \frac{2n-1-j}{2n-1}\cdot C \text{ (by the induction hypothesis)} \\
&= \frac{2n-2}{2n-1}\cdot C \\
&= \frac{2n-2}{2n-1}\cdot\frac{3}{4}\cdot\frac{2\cdot 4\cdot 6\ldots(2n-4)}{3\cdot 5\ldots(2n-3)} \\
&= \frac{3}{4}\frac{2\cdot 4\cdot 6\ldots(2n-2)}{3\cdot 5\cdot 7\ldots(2n-1)}.
\end{aligned}$$

3. Lastly, consider $j \geq 2n$. It is assumed that $n \geq 2$ thus $j \geq n+2$, this means that for equation (4), $\bar{b}(n,j) = 0$ by definition. Hence, by equation (4)

$$\dot{b}(n,j) = -\frac{2(2n+1)}{j(j+1)(j+2)}.$$

   Therefore

$$\begin{aligned}
\left|\dot{b}(n,j)\right| &= \frac{2(2n+1)}{j(j+1)(j+2)} \\
&\leq \frac{2(2n+1)}{j(2n+1)(2n+2)} \text{ (as } j \geq 2n) \\
&= \frac{1}{n(2n+2)} \\
&\leq \frac{1}{2n-1} \text{ (as } n(2n+2) \geq 2n-1) \\
&\leq \frac{3}{4}\cdot\frac{2}{2n-1} \\
&\leq \frac{3}{4}\cdot\frac{2}{2n-1}\cdot\frac{4\cdot 6\cdot 8\ldots(2n-2)}{3\cdot 5\cdot 7\ldots(2n-3)} \\
&= \frac{3}{4}\cdot\frac{2\cdot 4\cdot 6\ldots(2n-2)}{3\cdot 5\cdot 7\ldots(2n-1)},
\end{aligned}$$

and thus the hypothesis holds for all $n, j \geq 2$. □

Considering the bound in the last lemma,

$$C = \frac{2\cdot 4\cdot 6\ldots(2n-2)}{3\cdot 5\cdot 7\ldots(2n-1)}$$

from this and infinite product of fractions can be obtained

$$C^2 = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2n-2}{2n-3}\cdot\frac{2n-2}{2n-1}\cdot\frac{1}{2n-1},$$

as $\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2n-2}{2n-3}\cdot\frac{2n-2}{2n-1}$, known as the Wallis product [49]. From [49], it is known that as $n\to\infty$ the product converges to $\frac{\pi}{2}$. Thus, using the result from Lemma 1.6,

$$\begin{aligned}\left|\dot{b}\left(n,j\right)\right| \leq &\frac{3}{4}\sqrt{\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2n-2}{2n-3}\cdot\frac{2n-2}{2n-1}\right)\cdot\frac{1}{2n-1}}\\ \leq &\frac{3}{4}\sqrt{\frac{\pi}{2\left(2n-1\right)}}.\end{aligned} \tag{9}$$

Thus as $n\to\infty$, then $\left|\dot{b}\left(n,j\right)\right|\to 0$, and thus equation (3) holds as $n\to\infty$, that is the asymptotic limit for $\bar{b}\left(n,j\right)$ is found to be (3). Now consider

$$\sum_{j=1}^{\infty}\dot{b}\left(n,j\right)=\sum_{j=1}^{\infty}\bar{b}\left(n,j\right)-\sum_{j=1}^{\infty}\frac{2\left(2n+1\right)}{j\left(j+1\right)\left(j+2\right)}.$$

The expected number of blocks, by definition, is given by $\sum_{j=1}^{\infty}\bar{b}\left(n,j\right)$, and $\sum_{j=1}^{\infty}\bar{b}\left(n,j\right)=n$ as the expected number of blocks will always be n for any deal $\sigma\in M_n$. Now

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{2\left(2n+1\right)}{j\left(j+1\right)\left(j+2\right)} &=\left(2n+1\right)\lim_{N\to\infty}\sum_{j=1}^{N}\left(\frac{1}{j}-\frac{2}{j+1}+\frac{1}{j+2}\right)\\ &=\left(2n+1\right)\lim_{N\to\infty}\left(1-\frac{2}{2}+\frac{1}{3}+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\cdots+\frac{1}{N}-\frac{2}{N+1}+\frac{1}{N+2}\right)\\ &=\left(2n+1\right)\lim_{N\to\infty}\left(1-\frac{2}{2}+\frac{1}{2}+\left(\frac{1}{3}-\frac{2}{3}+\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{2}{4}+\frac{1}{4}\right)+\cdots\right.\\ &\quad\left.+\left(\frac{1}{N}-\frac{2}{N}+\frac{1}{N}\right)+\frac{1}{N+1}-\frac{2}{N+1}+\frac{1}{N+2}\right)\\ &=\left(2n+1\right)\lim_{N\to\infty}\left(1-1+\frac{1}{2}-\frac{2}{N+1}+\frac{1}{N+1}+\frac{1}{N+2}\right)\\ &=\left(2n+1\right)\lim_{N\to\infty}\left(\frac{1}{2}-\frac{1}{N+1}+\frac{1}{N+2}\right)\\ &=\left(2n+1\right)\frac{1}{2}\\ &=n+\frac{1}{2}.\end{aligned}$$

And this gives an explicit formula for $\sum_{j=1}^{\infty}\dot{b}\left(n,j\right)$,

$$\begin{aligned}\sum_{j=1}^{\infty}\dot{b}\left(n,j\right)&=\sum_{j=1}^{\infty}\bar{b}\left(n,j\right)-\sum_{j=1}^{\infty}\frac{2\left(2n+1\right)}{j\left(j+1\right)\left(j+2\right)}\\ &=n-\left(n+\frac{1}{2}\right)=-\frac{1}{2}.\end{aligned}$$

Recall that Lemma 1.6 gave $\dot{b}\left(n,1\right)=-1$, and thus the sum can be shifted one term,

$$\sum_{j=2}^{\infty}\dot{b}\left(n,j\right)=\frac{1}{2}. \tag{10}$$

Considering the terms in (10) for which j is odd and the terms for which it is even leads to the next lemma.

Lemma 1.8. *Let $n \geq 1$, then there exists $k_n, l_n \in \mathbb{N}$ such that $2 \leq k_n < l_n \leq n+2$ and*

$$\begin{aligned}\dot{b}(n,2) \leq& \dot{b}(n,3) \leq \cdots \leq \dot{b}(n,k_n),\\ \dot{b}(n,k_n) \geq& \dot{b}(n,k_n+1) \geq \cdots \geq \dot{b}(n,l_n)\\ \dot{b}(n,l_n) \leq& \dot{b}(n,l_n+1) \leq \cdots.\end{aligned}$$

Proof. This lemma will be proven by induction on n.

Base Case: For $n = 1, n = 2$,

$$\dot{b}(1,2) \leq \dot{b}(1,2).$$

Now

$$\begin{aligned}\dot{b}(1,2) =& \frac{3}{4}\\ \geq& -\frac{6}{3(3+1)(3+2)}\\ =& \dot{b}(1,3)\end{aligned}$$

and,

$$\begin{aligned}\dot{b}(1,3) =& -\frac{6}{3(3+1)(3+2)}\\ \leq& -\frac{6}{k(k+1)(k+2)} \quad \forall k > 3.\end{aligned}$$

Thus, the theorem holds for the base case $n = 1$ where $k_1 = 2$ and $l_1 = 3$. Now showing the base case for $n = 2$,

$$\begin{aligned}\dot{b}(2,2) =& \frac{3}{4(2 \cdot 2 - 1)}\\ =& \frac{3}{12}\\ \leq& \frac{3}{2(2 \cdot 3 - 1)}\\ =& \frac{3}{10}\\ =& \dot{b}(2,3).\end{aligned}$$

Now

$$
\begin{aligned}
\dot{b}(2,3) =& \frac{3}{10} \\
\geq& \frac{(4-1)\cdot \dot{b}(2-1,4-1) + (2.2-1-4)\cdot \dot{b}(2-1,4)}{2.2-1} \\
=& \frac{3\cdot \dot{b}(1,3) - \dot{b}(1,4)}{3} \\
=& \frac{3\left(-\frac{6}{3\cdot 4\cdot 5}\right) - \left(-\frac{6}{4\cdot 5\cdot 6}\right)}{3} \\
=& \frac{-15.6}{3\cdot 3\cdot 4\cdot 5\cdot 6} \\
=& \dot{b}(2,4)
\end{aligned}
$$

and,

$$\dot{b}(2,2) \leq 2\cdot \dot{b}(2,2) = \dot{b}(n,3)\,.$$

Thus the base case holds for $n=2$ with $k_2 = 3$ and $l_2 = 4$.

Induction Hypothesis:

Assume that for $n \geq 3$ there exists k_{n-1}, l_{n-1} such that for $2 \leq k_{n-1} < l_{n-1} \leq n+1$,

$$
\begin{aligned}
\dot{b}(n-1,2) \leq& \dot{b}(n-1,3) \leq \cdots \leq \dot{b}(n-1,k_{n-1}) \\
\dot{b}(n-1,k_{n-1}) \geq& \dot{b}(n-1,k_{n-1}+1) \geq \cdots \geq \dot{b}(n-1,l_{n-1}) \\
\dot{b}(n-1,l_{n-1}) \leq& \dot{b}(n-1,l_{n-1}+1) \leq \cdots .
\end{aligned}
$$

Case: n

For the n case, it will be shown that

$$
\begin{aligned}
\dot{b}(n,2) \leq& \dot{b}(n,3) \leq \cdots \leq \dot{b}(n,k_{n-1}) && (11) \\
\dot{b}(n,k_{n-1}+1) \geq& \dot{b}(n,k_{n-1}+2) \geq \cdots \geq \dot{b}(n,l_{n-1}) && (12) \\
\dot{b}(n,l_{n-1}+1) \leq& \dot{b}(n,l_{n-1}+2) \leq \cdots . && (13)
\end{aligned}
$$

Proving (11): If $j=2$, then by Lemma 1.6

$$\dot{b}(n,2) \leq 2\cdot \dot{b}(n,2) = \dot{b}(n,3)$$

so (11) holds for $j=2$.

Now consider $2 \leq j < k_{n-1} < l_{n-1} \leq n+1 \leq 2n-2$ (since $n \geq 3$ it implies that $n+1 \leq 2n-2$). The induction hypothesis gives

$$\dot{b}(n-1, j-1) \leq \dot{b}(n-1, j) \leq \dot{b}(n-1, j+1).$$

j is chosen such that $j < k_{n-1}$, and since $j, k_{n-1} \in \mathbb{N}$ it implies that $j+1 \leq k_{n-1}$, therefore the above inequality holds by the induction hypothesis. Now

$$\begin{aligned}
\dot{b}(n, j) =& \frac{(j-1) \cdot \dot{b}(n-1, j-1) + (2n-1-j) \cdot \dot{b}(n-1, j)}{2n-1} \\
\leq& \frac{(j-1) \cdot \dot{b}(n-1, j) + (2n-1-j)}{2n-1} \text{ (by the above inequality from the induction hypothesis)} \\
=& \frac{(2n-2) \cdot \dot{b}(n-1, j)}{2n-1} \\
=& \frac{(j-j) \cdot \dot{b}(n-1, j) + (2n-2) \cdot \dot{b}(n-1, j)}{2n-1} \\
=& \frac{j \cdot \dot{b}(n-1, j) + (2n-2-j) \cdot \dot{b}(n-1, j)}{2n-1} \\
\leq& \frac{j \cdot \dot{b}(n-1, j) + (2n-2-j) \cdot \dot{b}(n-1, j+1)}{2n-1} \text{ (by the above inequality)} \\
=& \dot{b}(n, j+1),
\end{aligned} \tag{14}$$

thus proving (11). To prove (12) suppose $k_{n-1} + 1 \leq j < l_{n-1}$ then by the induction hypothesis,

$$\dot{b}(n-1, j-1) \geq \dot{b}(n-1, j) \geq \dot{b}(n-1, j+1).$$

Again note that as $j < l_{n-1}$ and $j, l_{n-1} \in \mathbb{N}$ then $j+1 \leq l_{n-1}$. Now

$$\begin{aligned}
\dot{b}(n, j) =& \frac{(j-1) \cdot \dot{b}(n-1, j-1) + (2n-1-j) \cdot \dot{b}(n-1, j)}{2n-1} \\
\geq& \frac{(j-1) \cdot \dot{b}(n-1, j) + (2n-1-j) \cdot \dot{b}(n-1, j)}{2n-1} \text{ (by the above inequality)} \\
=& \frac{(2n-2) \cdot \dot{b}(n-1, j)}{2n-1} \\
=& \frac{(2n-2) \cdot \dot{b}(n-1, j) + (j-j) \cdot \dot{b}(n-1, j)}{2n-1} \\
=& \frac{j \cdot \dot{b}(n-1, j) + (2n-2-j) \cdot \dot{b}(n-1, j)}{2n-1} \\
\geq& \frac{j \cdot \dot{b}(n-1, j) + (2n-2-j) \cdot \dot{b}(n-1, j+1)}{2n-1} \\
=& \dot{b}(n, j+1).
\end{aligned}$$

Thus (12) is proven. In proving (13), consider $j \geq l_{n-1} + 1$. The result from (14) holds if $j \leq 2n - 2$ so it must be shown that $\dot{b}(n, j) \leq \dot{b}(n, j+1)$ for $j \geq 2n - 1$. Since $n \geq 3$ this inequality may be rewritten as $j \geq n + 2$, and by definition $\bar{b}(n, j) = \bar{b}(n, j+1) = 0$. Using Equation (4) and considering this particular j,

$$\begin{aligned}\dot{b}(n, j) &= -\frac{2(2n+1)}{j(j+1)(j+2)} \\ &\leq -\frac{2(2n+1)}{(j+1)(j+2)(j+3)} = \dot{b}(n, j+1),\end{aligned}$$

and hence proving (13) and Lemma 1.8. $\square$

Lemma 1.9. *Consider a monotonic sequence*

$$a_1, a_2, \ldots, a_k$$

where $|a_i| \leq d$ *for all* $i = 1, 2, \ldots, k$. *Let*

$$A = \sum_{\substack{i \text{ odd} \\ i \leq k}} a_i,$$

$$B = \sum_{\substack{i \text{ even} \\ i \leq k}} a_i = \sum_{i=1}^{k} a_i - A,$$

then

$$|B - A| \leq 2d.$$

Proof. For the proof of this lemma two cases must be considered: the case where the sequence is monotonically increasing, and the case where it is monotonically decreasing. If it is monotonically increasing and k is even then

$$A = a_1 + a_3 + \cdots + a_{k-1} \leq a_2 + a_4 + \cdots + a_k = B,$$

thus

$$A - a_1 = a_3 + \cdots + a_{k-1} \geq a_2 + a_4 + \cdots + a_{k-2} = B - a_k,$$

and,

$$\begin{aligned}& A - a_1 \geq B - a_k \\ \iff & 0 \leq B - A \leq a_k - a_1 \\ \iff & |B - A| \leq |a_k - a_1| \\ \iff & |B - A| \leq |a_k| + |a_1| \ \text{(triangle inequality)} \\ \iff & |B - A| \leq 2d \ (\text{as } |a_i| \leq d \text{ for all } i = 1, 2, \ldots, k.)\end{aligned}$$

If k is odd then

$$A = a_1 + a_3 + \cdots + a_k \geq a_2 + \cdots + a_{k-1} = B,$$

so

$$\begin{aligned}
A - a_k &= a_1 + a_3 + \cdots + a_{k-2} \leq a_2 + a_4 + \cdots + a_{k-1} = B \text{ and} \\
A - a_1 &= a_3 + a_5 + \cdots + a_k \geq a_2 + a_4 \cdots + a_{k-1} = B,
\end{aligned}$$

which gives

$$\begin{aligned}
& A - a_k \leq B \\
\iff & A - B \leq a_k,
\end{aligned}$$

and,

$$\begin{aligned}
& A - a_1 \geq B \\
\iff & A - B \geq a_1.
\end{aligned}$$

Hence,

$$a_1 \leq A - B \leq a_k,$$

so

$$|B - A| \leq |a_k| \leq d.$$

Thus proving the lemma for a monotonically increasing sequence. The monotonically decreasing case will now be considered. Suppose the sequence is monotonically decreasing and k is even, then,

$$A = a_1 + a_3 + \cdots + a_{k-1} \geq a_2 + a_4 + \cdots + a_k = B$$

so

$$\begin{aligned}
& A - a_1 \leq B - a_k \\
\iff & 0 \leq A - B \leq a_1 - a_k \\
\iff & |A - B| = |B - A| \leq |a_1| + |a_2| \text{ (by triangle inequality)} \\
\iff & |B - A| \leq 2d.
\end{aligned}$$

If k is odd then

$$A = a_1 + a_3 + \cdots + a_k \geq a_2 + a_4 + \cdots + a_{k-1} = B,$$

which gives

$$\begin{aligned}
A - a_k &\geq a_2 + a_4 + \ldots a_{k-1} = B \text{ and} \\
A - a_1 &\leq a_2 + a_4 + \ldots a_{k-1} = B.
\end{aligned}$$

Thus,

$$\begin{aligned}
A - B &\geq a_k, \\
A - B &\leq a_1,
\end{aligned}$$

which leads to,

$$a_k \leq A - B \leq a_1.$$

Hence,

$$|A - B| = |B - A| \leq d \leq 2d.$$

Proving the case when the sequence is monotonically decreasing, and proving the lemma. $\square$

The first limit in Theorem 1.4 will now be evaluated in the next lemma.

Lemma 1.10.

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \dot{b}\,(n, 2i) = \frac{1}{4}$$

Proof. Recall equation (9), $\left|\dot{b}\,(n, j)\right| \leq \frac{3}{4} \cdot \sqrt{\frac{\pi}{2(2n-1)}}$. Lemma 1.8 states that

$$\dot{b}\,(n, 2)\,, \dot{b}\,(n, 3)\,, \ldots, \dot{b}\,(n, 2N + 1)$$

is made up of no more than 3 monotonic subsequences for any $N \in \mathbb{N}$ and by Lemma 1.9, the absolute value of the difference between the sum of the even and the sum of the odd terms in one such a sequence is bounded by $2d$, where $d = \frac{3}{4} \cdot \sqrt{\frac{\pi}{2(2n-1)}}$ for this case. Consider now

$$\left|\sum_{i=1}^{N} \dot{b}\,(n, 2i) - \sum_{i=1}^{N} \dot{b}\,(n, 2i + 1)\right|.$$

This is the absolute value of the difference between the sum of the odd and the sum of the even terms between at most 3 monotonic sequences. Hence, by Lemma 1.9, it must be no more than $3\,(2d)$, that is,

$$\begin{aligned}
&\left|\sum_{i=1}^{N} \dot{b}\,(n, 2i) - \sum_{i=1}^{N} \dot{b}\,(n, 2i + 1)\right| \leq 6d \\
\iff &-6d \leq \sum_{i=1}^{N} \dot{b}\,(n, 2i) - \sum_{i=1}^{N} \dot{b}\,(n, 2i + 1) \leq 6d \\
\iff &\sum_{i=1}^{N} \dot{b}\,(n, 2i + 1) - 6d \leq \sum_{i=1}^{N} \dot{b}\,(n, 2i) \leq \sum_{i=1}^{N} \dot{b}\,(n, 2i + 1) + 6d.
\end{aligned}$$

$\sum_{i=1}^{N} \dot{b}\,(n, 2i)$ is added to obtain

$$\sum_{i=1}^{2N+1} \dot{b}\,(n, i) - 6d \leq 2\sum_{i=1}^{N} \dot{b}\,(n, 2i) \leq \sum_{i=1}^{2N+1} \dot{b}\,(n, i) + 6d.$$

Recall that equation (10) gives $\sum_{j=2}^{\infty} \dot{b}(n,j) = \frac{1}{2}$ thus taking the limit as $N \to \infty$ of the above inequality gives

$$\frac{1}{2} - 6d \leq 2\sum_{i=1}^{\infty} \dot{b}(n,2i) \leq \frac{1}{2} + 6d.$$

Substituting the expression for d gives

$$\frac{1}{4} - 3\left(\frac{3}{4}\sqrt{\frac{\pi}{2(2n-1)}}\right) \leq \sum_{i=1}^{\infty} \dot{b}(n,2i) \leq \frac{1}{4} + 3\left(\frac{3}{4}\sqrt{\frac{\pi}{2(2n-1)}}\right).$$

Finally taking the limit as $n \to \infty$ of the above inequality results in,

$$\frac{1}{4} \leq \sum_{i=1}^{\infty} \dot{b}(n,2i) \leq \frac{1}{4}$$

which completes the proof. $\square$

1.2 $\lim_{\mathbf{n}\to\infty} \dot{\mathbf{L}}(\mathbf{n}, \mathbf{2i}) = \mathbf{0}$

Consider now $L(n,j)$. Trivially $L(1,2) = 1$ and, $L(1,j) = 0$ if $j \geq 3$, as for $n = 1$ the only deal is 11 and it is lucky. Two recursions will be used. Obtaining these recursions is similar to obtaining the recursions for $b(n,j)$ where a standard deal $\sigma \in M_n$ is formed from a standard deal $\sigma' \in M_{n-1}$ by adding an n card to the end of the deal σ' and inserting a second n card in any of $2n-1$ positions before the n at the end.

1. The first recursion is given by

$$L(n,2) = (2n-3) \cdot L(n-1,2) + \frac{(2n-2)!}{2^{n-1}(n-1)!}.$$

   It is obtained by considering that the n can be inserted in any one of $2n-3$ positions to avoid a specific lucky block, and there are $L(n-1,2)$ lucky blocks of length 2 in all $\sigma' \in M_{n-1}$. This results in the first term. The other n may also be inserted next to the n at the end of the deal σ' creating an additional lucky block of length 2 for each of the deals $\sigma' \in M_{n-1}$, resulting in the last term.

2. The second recursion, for $j \geq 3$, is given by

$$L(n,j) = (j-2) \cdot L(n-1,j-1) + (2n-1-j) \cdot L(n-1,j).$$

   Now the n may be inserted into a lucky block of size $j-1$, to make it a block of size j, in any one of $j-2$ positions so that the lucky part of the block remains lucky, avoiding the matching pair at the end of the block. There are exactly $L(n-1,j-1)$ lucky blocks of size $j-1$ in all deals σ', hence the first term. Considering the second term, it results from avoiding a particular block of size $j-1$ there are exactly $(2n-1-j)$ positions that the n may be inserted to avoid such a lucky block of size j and there are $L(n-1,j)$ blocks of size j in all deals $\sigma' \in M_{n-1}$.

Recall $\bar{L}(n,j) = \frac{L(n,j)}{(2n)!/(2^n n!)}$ so $\bar{L}(1,2) = 1$ and $\bar{L}(1,j) = 0$ for $j \geq 3$. For $n \geq 2$, dividing both recursions by $\frac{(2n)!}{(2^n n!)}$ gives

$$\begin{aligned}\frac{L(n,2)}{(2n)!/(2^n n!)} =& \frac{(2n-3)\cdot L(n-1,2)}{(2n)!/(2^n n!)} + \frac{(2n-2)!/\left(2^{n-1}(n-1)!\right)}{(2n)!/(2^n n!)}\\ =& \frac{2n(2n-3)\cdot L(n-1,2)}{2n(2n-1)(2n-2)!/(2^{n-1}(n-1)!)} + \frac{(2n-2)!\,2n}{(2n)!}\\ =& \frac{2n-3}{2n-1}\cdot\frac{L(n-1,2)}{(2(n-2))!/(2^{n-1}(n-1))!} + \frac{2n}{2n(2n-1)},\end{aligned}$$

therefore

$$\bar{L}(n,2) = \frac{(2n-3)\cdot\bar{L}(n-1,2)+1}{2n-1}.$$

For the second recursion,

$$\begin{aligned}\frac{L(n,j)}{(2n)!/(2^n n!)} =& \frac{(j-2)\cdot L(n-1,j-1)}{(2n)!/(2^n n!)} + \frac{(2n-1-j)\cdot L(n-1,j)}{(2n)!/(2^n n!)}\\ =& \frac{(j-2)\cdot L(n-1,j-1)}{(2n-1)!/(2^{n-1}(n-1)!)} + \frac{(2n-1-j)\cdot L(n-1,j)}{(2n-1)!/(2^{n-1}(n-1)!)}\\ =& \frac{(j-2)\cdot\bar{L}(n-1,j-1)}{2n-1} + \frac{(2n-1-j)\cdot\bar{L}(n-1,j)}{2n-1}.\end{aligned}$$

Therefore

$$\bar{L}(n,j) = \frac{(j-2)\cdot\bar{L}(n-1,j-1) + (2n-1-j)\cdot\bar{L}(n-1,j)}{2n-1}.$$

As $\dot{L}(n,j) = \bar{L}(n,j) - \frac{1}{j(j-1)}$,

$$\begin{aligned}\dot{L}(1,2) =& \bar{L}(1,2) - \frac{1}{2(2-1)}\\ =& 1 - \frac{1}{2}\\ =& \frac{1}{2},\end{aligned}$$

and for $j \geq 3$,

$$\begin{aligned}\dot{L}(1,j) =& \bar{L}(1,j) - \frac{1}{j(j-1)}\\ =& 0 - \frac{1}{j(j-1)}\\ =& -\frac{1}{j(j-1)}.\end{aligned}$$

Now, for $n \geq 2$,

$$\begin{aligned}
\dot{L}(n,2) =& \bar{L}(n,2) - \frac{1}{2(2-1)} \\
=& \frac{(2n-3)\cdot \bar{L}(n-1,2)+1}{2n-1} - \frac{1}{2} \\
=& \frac{(2n-3)\left(\frac{1}{2(2-1)} + \dot{L}(n-1,2)\right)+1}{2n-1} - \frac{1}{2} \\
=& \frac{(2n-3)+2(2n-3)\cdot \dot{L}(n-1,2)+2}{2(2n-1)} - \frac{1}{2} \\
=& \frac{(2n-1)+2(2n-3)\cdot \dot{L}(n-1,2)}{2(2n-1)} - \frac{2n-1}{2(2n-1)} \\
=& \frac{(2n-3)\cdot \dot{L}(n-1,2)}{2n-1}.
\end{aligned}$$

For $j \geq 3$,

$$\begin{aligned}
\dot{L}(n,j) =& \bar{L}(n,j) - \frac{1}{j(j-1)} \\
=& \frac{(j-2)\cdot \bar{L}(n-1,j-1)}{2n-1} + \frac{(2n-1-j)\cdot \bar{L}(n-1,j)}{2n-1} - \frac{1}{j(j-1)} \\
=& \frac{j-2}{2n-1}\left(\frac{1}{(j-1)(j-2)} + \dot{L}(n-1,j-1)\right) + \frac{2n-1-j}{2n-1}\left(\frac{1}{j(j-1)} + \dot{L}(n-1,j)\right) \\
& - \frac{1}{j(j-1)} \\
=& \frac{(j-2)\cdot \dot{L}(n-1,j-1) + (2n-1-j)\cdot \dot{L}(n-1,j)}{2n-1} \\
& + \frac{j-2}{(2n-1)(j-1)(j-2)} + \frac{2n-1-j}{j(j-1)(2n-1)} - \frac{1}{j(j-1)} \\
=& \frac{(j-2)\cdot \dot{L}(n-1,j-1) + (2n-1-j)\cdot \dot{L}(n-1,j)}{2n-1} \\
& + \frac{j(j-2)+(2n-1-j)(j-2)-(j-2)(2n-1)}{j(j-1)(j-2)(2n-1)} \\
=& \frac{(j-2)\cdot \dot{L}(n-1,j-1) + (2n-1-j)\cdot \dot{L}(n-1,j)}{2n-1} + \frac{0}{j(j-1)(j-2)(2n-1)} \\
=& \frac{(j-2)\cdot \dot{L}(n-1,j-1) + (2n-1-j)\cdot \dot{L}(n-1,j)}{2n-1}.
\end{aligned}$$

These results lead to the following lemma.

Lemma 1.11. *Let $j \geq 2$, then $\forall n$,*

$$\left|\dot{L}(n,j)\right| \leq \frac{1}{2(2n-1)},$$

and $\forall n \geq 2$,

$$\dot{L}(n,2) = \dot{L}(n,3) = \frac{1}{2(2n-1)}.$$

Proof. This lemma will be proven by induction on n.

Proving the inequality in the lemma:

Base Case: $n = 1$,

For $n = 1$, and $j = 2$,

$$\begin{aligned}\dot{L}(1,2) =& \bar{L}(1,2) - \frac{1}{2(2-1)} \\ =& 1 - \frac{1}{2} \\ =& \frac{1}{2}.\end{aligned}$$

So

$$\left|\dot{L}(1,2)\right| = \frac{1}{2} \leq \frac{1}{2(2-1)},$$

and for $j > 2$,

$$\begin{aligned}\left|\dot{L}(1,j)\right| =& \left|-\frac{1}{j(j-1)}\right| \\ =& \frac{1}{j(j-1)} \\ \leq& \frac{1}{3(3-1)} \text{ (as } j > 2) \\ =& \frac{1}{6} \leq \frac{1}{2} = \frac{1}{2(2\cdot 1-1)}.\end{aligned}$$

Thus the hypothesis is true for the base case $n = 1$.

Induction Hypothesis:

Assume that the hypothesis holds for some $n > 1$ and $j \geq 2$, that is,

$$\left|\dot{L}(n,j)\right| \leq \frac{1}{2(2n-1)}.$$

Case: $n+1$

Now, for $n+1$ and $3 \leq j \leq 2n-1$,

$$\begin{aligned}
\left|\dot{L}(n+1,j)\right| =& \left|\frac{(j-2)\cdot\dot{L}(n,j-1)+(2n-1-j)\cdot\dot{L}(n,j)}{2n-1}\right| \\
\leq& \left|\frac{(j-2)\cdot\dot{L}(n,j-1)}{2n-1}\right| + \left|\frac{(2n-1-j)\cdot\dot{L}(n,j)}{2n-1}\right| \\
=& \frac{j-2}{2n-1}\cdot\dot{L}(n,j-1)+\frac{2n-1-j}{2n-1}\cdot\dot{L}(n,j) \\
\leq& \frac{j-2}{2n-1}\cdot\frac{1}{2(2n-1)}+\frac{2n-1-j}{2n-1}\cdot\frac{1}{2(2n-1)} \text{ (by the induction hypothesis)} \\
=& \frac{j-2+2n-1-j}{2(2n-1)^2} \text{ (as } n\geq 2 \text{ and } 3\leq j\leq 2n-1) \\
=& \frac{2n-3}{2(2n-1)^2} \\
=& \frac{(2n-3)(2n+1)}{2(2n-1)^2(2n+1)} \\
=& \frac{4n^2-4n-3}{(4n^2-4n+1)\,2(2n+1)} \\
\leq& \frac{4n^2-4n+1}{(4n^2-4n+1)\,2(2n+1)} \\
=& \frac{1}{2(2n+1)} \\
=& \frac{1}{2(2(n+1)-1)}
\end{aligned}$$

Therefore the hypothesis holds $\forall n \geq 1$ proving the inequality in the lemma.

Proving the second part of the lemma:

Base Case: $n=2$

For the base case, when $n=2$,

$$\begin{aligned}
\dot{L}(2,2) =& \frac{(2\cdot 2-3)\cdot\dot{L}(1,2)}{2\cdot 2-1} \\
=& \frac{\dot{L}(1,2)}{3} = \frac{1}{6} = \frac{1}{2(2\cdot 2-1)},
\end{aligned}$$

and,

$$\begin{aligned}\dot{L}(2,3) =& \frac{\dot{L}(1,2) + (2\cdot 2 - 1 - 3)\cdot \dot{L}(1,3)}{2.2-1} \\ =& \frac{\dot{L}(1,2) + 0\cdot \dot{L}(1,3)}{3} \\ =& \frac{\dot{L}(1,2)}{3} = \frac{1}{6}.\end{aligned}$$

Therefore $\dot{L}(2,2) = \dot{L}(2,3) = \frac{1}{2(2\cdot 2-1)}$ and the hypothesis holds for the base case.

Induction Hypothesis:

Assume that the hypothesis is true for some $n-1 > 1$, that is,

$$\dot{L}(n-1,2) = \dot{L}(n-1,3) = \frac{1}{2(2(n-1)-1)}.$$

Case: n

For the case n,

$$\begin{aligned}\dot{L}(n,2) =& \frac{(2n-3)\cdot \dot{L}(n-1,2)}{2n-1} \\ =& \frac{(2n-3)\,\frac{1}{2(2(n-1)-1)}}{2n-1} \text{ (by the induction hypothesis)} \\ =& \frac{1}{2(2n-1)},\end{aligned}$$

and,

$$\begin{aligned}\dot{L}(n,3) =& \frac{(3-2)\cdot \dot{L}(n-1,3-1) + (2n-1-3)\cdot \dot{L}(n-1,3)}{2n-1} \\ =& \frac{\dot{L}(n-1,2) + (2n-4)\cdot \dot{L}(n-1,3)}{2n-1} \\ =& \frac{\dot{L}(n-1,2) + (2n-4)\cdot \dot{L}(n-1,2)}{2n-1} \text{ (by the induction hypothesis)} \\ =& \frac{(2n-3)\cdot \dot{L}(n-1,2)}{2n-1} \\ =& \frac{(2n-3)\left(\frac{1}{2(2(n-1)-1)}\right)}{2n-1} \\ =& \frac{1}{2(2n-1)},\end{aligned}$$

proving the lemma. $\square$

Lemma 1.12. $\forall n \geq 1$

$$\sum_{j=2}^{\infty} \bar{L}(n,j) = 1, \quad and$$

$$\sum_{j=2}^{\infty} \dot{L}(n,j) = 0.$$

Proof. The first equation will be proven by induction on n.

Base Case: $n = 1$

$$\begin{aligned}\sum_{j=2}^{\infty} \bar{L}(1,j) =& \bar{L}(1,2) + 0 + 0 + \ldots \\ =& 1.\end{aligned}$$

Thus the hypothesis for the first equation holds for the base case.

Induction Hypothesis

Assume that the hypothesis holds for some $n-1 > 1$, that is

$$\sum_{j=2}^{\infty} \bar{L}(n-1,j) = 1.$$

Case: n

For the case n,

$$\begin{aligned}\sum_{j=2}^{\infty} \bar{L}(n,j) =& \bar{L}(n,2) + \sum_{j=3}^{\infty} \bar{L}(n,j) \\
=& \bar{L}(n,2) + \sum_{j=3}^{\infty} \frac{(j-2)\cdot\bar{L}(n-1,j-1) + (2n-1-j)\cdot\bar{L}(n-1,j)}{2n-1} \\
=& \frac{(2n-3)\cdot\bar{L}(n-1,2)+1}{2n-1} + \sum_{j=3}^{\infty} \frac{(j-2)\cdot\bar{L}(n-1,j-1) + (2n-1-j)\cdot\bar{L}(n-1,j)}{2n-1} \\
=& \frac{(2n-3)\cdot\bar{L}(n-1,2)+1}{2n-1} + \frac{1}{2n-1}\sum_{j=3}^{\infty}\big(j\cdot\bar{L}(n-1,j-1) \\
&-2\cdot\bar{L}(n-1,j-1) + 2n\cdot\bar{L}(n-1,j) - \bar{L}(n-1,j) - j\cdot\bar{L}(n-1,j)\big) \\
=& \frac{(2n-3)\cdot\bar{L}(n-1,2)+1}{2n-1} + \frac{1}{2n-1}\left(\sum_{j=3}^{\infty} j\cdot\bar{L}(n-1,j-1) - 2\sum_{j=3}^{\infty}\bar{L}(n-1,j-1)\right. \\
&\left.+2n\sum_{j=3}^{\infty}\bar{L}(n-1,j) - \sum_{j=3}^{\infty}\bar{L}(n-1,j) - \sum_{j=3}^{\infty} j\cdot\bar{L}(n-1,j)\right)\end{aligned}$$

$$
\begin{aligned}
=&\frac{(2n-3)\cdot\bar{L}(n-1,2)+1}{2n-1}+\frac{1}{2n-1}\left(\sum_{j=2}^{\infty}(j+1)\cdot\bar{L}(n-1,j)-2\sum_{j=2}^{\infty}\bar{L}(n-1,j)\right.\\
&+2n\sum_{j=2}^{\infty}\bar{L}(n-1,j)-2n\cdot\bar{L}(n-1,2)-\sum_{j=2}^{\infty}\bar{L}(n-1,j)+\bar{L}(n-1,2)\\
&\left.-\sum_{j=2}^{\infty}j\cdot\bar{L}(n-1,j)+2\cdot\bar{L}(n-1,2)\right)\\
=&\frac{(2n-3)\cdot\bar{L}(n-1,2)+1}{2n-1}+\frac{1}{2n-1}\left(\sum_{j=2}^{\infty}j\cdot\bar{L}(n-1,j)+\sum_{j=2}^{\infty}\bar{L}(n-1,j)\right.\\
&-2\cdot 1+2n\cdot 1-2n\cdot\bar{L}(n-1,2)-1+\bar{L}(n-1,2)\\
&\left.-\sum_{j=2}^{\infty}j\cdot\bar{L}(n-1,j)+2\cdot\bar{L}(n-1,2)\right)\ \text{(by the induction hypothesis)}\\
=&\frac{(2n-3)\cdot\bar{L}(n-1,2)+1}{2n-1}\\
&+\frac{1}{2n-1}\left(1-2+2n-2n\cdot\bar{L}(n-1,2)-1+\bar{L}(n-1,2)+2\cdot\bar{L}(n-1,2)\right)\\
=&\frac{2n-2+1}{2n-1}\\
=&1
\end{aligned}
$$

proving the first equation in this lemma.

For the second equation, $\sum_{j=2}^{\infty}\dot{L}(n,j)$, recall the definition of $\dot{L}(n,j)$,

$$
\begin{aligned}
\sum_{j=2}^{\infty}\dot{L}(n,j)=&\sum_{j=2}^{\infty}\bar{L}(n,j)-\sum_{j=2}^{\infty}\frac{1}{j(j-1)}\\
=&1-\sum_{j=2}^{\infty}\frac{1}{j(j-1)}\ \text{(by the first equation in this lemma.)}
\end{aligned}
$$

Consider the second term in the above equation,

$$
\begin{aligned}
\sum_{j=2}^{\infty}\frac{1}{j(j-1)}&=\lim_{N\to\infty}\sum_{j=2}^{N}\left(\frac{1}{j-1}-\frac{1}{j}\right)\\
&=\lim_{N\to\infty}\left(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{N-1}-\frac{1}{N}\right)\\
&=\lim_{N\to\infty}\left(1-\frac{1}{N}\right)=1.
\end{aligned}
$$

Clearly the sum of the partial fraction decomposition forms a telescoping series. This gives,

$$\sum_{j=2}^{\infty} \dot{L}\left(n, j\right) = 1 - 1 = 0,$$

thus proving the second equation and the lemma. □

Lemma 1.13. *Let $n \geq 1$, then $\exists p_n \in \mathbb{N}$ such that for $2 \leq p_n \leq n+2$,*

$$\begin{aligned}
&\dot{L}\left(n, 2\right) \geq \dot{L}\left(n, 3\right) \geq \ldots \geq \dot{L}\left(n, p_n\right), \\
&\dot{L}\left(n, p_n\right) \leq \dot{L}\left(n, p_n + 1\right).
\end{aligned}$$

Proof. This lemma will be proven by induction on n.

Base Case: For $n = 1$ it will be shown that $p_1 = 3$,

$$\begin{aligned}
\dot{L}\left(1, 2\right) =& \frac{1}{2} \\
\dot{L}\left(1, 3\right) =& -\frac{1}{3\left(3-1\right)} \\
=& -\frac{1}{6} \\
\implies \dot{L}\left(1, 2\right) \geq& \dot{L}\left(1, 3\right)
\end{aligned}$$

and,

$$\begin{aligned}
\dot{L}\left(1, 3\right) &= -\frac{1}{6} \\
\dot{L}\left(1, 3+k\right) &= -\frac{1}{\left(3+k\right)\left(3+k-1\right)} \\
&= -\frac{1}{\left(3+k\right)\left(2+k\right)} \leq -\frac{1}{6} \text{ (for all } k \geq 1.)
\end{aligned}$$

Now,

$$\begin{aligned}
\dot{L}\left(1, 3+\left(k+1\right)\right) =& -\frac{1}{\left(3+\left(k+1\right)\right)\left(2+\left(k+1\right)\right)} \\
=& -\frac{1}{\left(4+k\right)\left(3+k\right)} \\
\geq& -\frac{1}{\left(3+k\right)\left(2+k\right)} \\
=& \dot{L}\left(1, 3+k\right)
\end{aligned}$$

proving the base case for $n = 1$.

Now for $n = 2$ it will be shown that $p_2 = 4$. We have

$$\dot{L}\left(2, 2\right) = \frac{1}{6} = \dot{L}\left(2, 3\right)$$

and,

$$\begin{aligned}
\dot{L}(2,4) =& \frac{(4-2)\cdot \dot{L}(2-1,4-1) + (2\cdot 2-1-4)\cdot \dot{L}(2-1,4)}{2.2-1} \\
=& \frac{2\cdot \dot{L}(1,3) - \dot{L}(1,4)}{3} \\
=& \frac{2\cdot \left(-\frac{1}{6}\right) - \left(-\frac{1}{12}\right)}{3} \\
=& -\frac{1}{12} \leq \frac{1}{6}.
\end{aligned}$$

For the second part,

$$\dot{L}(2,4) = -\frac{1}{12}$$

and $\forall k \geq 1$,

$$\begin{aligned}
\dot{L}(2,4+k) =& \frac{(4+k-2)\cdot \dot{L}(2-1,4+k-1) + (2\cdot 2-1-(4+k))\cdot \dot{L}(2-1,4+k)}{2\cdot 2-1} \\
=& \frac{(2+k)\cdot \dot{L}(1,3+k) - (1+k)\cdot \dot{L}(1,4+k)}{3} \\
=& \frac{(2+k)\left(-\frac{1}{(3+k)(2+k)}\right) - (1+k)\left(-\frac{1}{(4+k)(3+k)}\right)}{3} \\
=& -\frac{1}{(4+k)(3+k)} \geq -\frac{1}{2},
\end{aligned}$$

and,

$$\begin{aligned}
\dot{L}(2,4+(k+1)) =& \frac{(4+(k+1)-2)\cdot \dot{L}(2-1,4+(k+1)-1) + (2\cdot 2-1-(4+(k+1)))\cdot \dot{L}(2-1,4+(k+1))}{2\cdot 2-1} \\
=& \frac{(3+k)\cdot \dot{L}(1,4+k) - (2+k)\cdot \dot{L}(1,5+k)}{3} \\
=& \frac{(3+k)\left(-\frac{1}{(4+k)(3+k)}\right) - (2+k)\left(-\frac{1}{(5+k)(4+k)}\right)}{3} \\
=& -\frac{1}{(5+k)(4+k)} \geq -\frac{1}{(4+k)(3+k)} = \dot{L}(2,4+k),
\end{aligned}$$

proving the base case for $n = 2$.

Induction Hypothesis:

Suppose that the hypothesis holds for some $n - 1 \geq 2$.

Case: n

It will be shown that,

$$\dot{L}(n,2) \geq \dot{L}(n,3) \geq \ldots \geq \dot{L}(n,p_{n-1}) \text{ and}$$
$$\dot{L}(n,p_{n-1}+1) \leq \dot{L}(n,p_{n-1}+2) \leq \ldots.$$

Consider $2 \leq j \leq p_{n-1}$. For $j=2$,

$$\dot{L}(n,j) \geq \dot{L}(n,j+1),$$

by Lemma 1.11. Suppose $j \geq 3$ and note that $2 \leq j \leq p_{n-1} \leq (n-1)+2 = n+1 \leq 2n-2$ for $n \geq 3$. Thus by the induction hypothesis,

$$\dot{L}(n-1,j-1) \geq \dot{L}(n-1,j) \geq \dot{L}(n-1,j+1).$$

Therefore

$$\begin{aligned}
\dot{L}(n,j) =& \frac{(j-2)\cdot\dot{L}(n-1,j-1)+(2n-1-j)\cdot\dot{L}(n-1,j)}{2n-1} \\
\geq& \frac{(j-2)\cdot\dot{L}(n-1,j)+(2n-1-j)\cdot\dot{L}(n-1,j)}{2n-1} \text{ (by the induction hypothesis)} \\
=& \frac{(2n-3)\cdot\dot{L}(n-1,j)}{2n-1} \\
=& \frac{(2n-3)\cdot\dot{L}(n-1,j)+(j-j)\cdot\dot{L}(n-1,j)}{2n-1} \\
=& \frac{j\cdot\dot{L}(n-1,j)+(2n-3-j)\cdot\dot{L}(n-1,j)}{2n-1} \\
\geq& \frac{j\cdot\dot{L}(n-1,j)+(2n-3-j)\cdot\dot{L}(n-1,j+1)}{2n-1} \text{ (by the induction hypothesis)} \\
=& \dot{L}(n,j+1).
\end{aligned}$$

Suppose $j \geq p_{n-1}+1$. Consider two cases:

1. $j \leq 2n-2$:

   Then, by the induction hypothesis,

   $$\dot{L}(n-1,j-1) \leq \dot{L}(n-1,j) \leq \dot{L}(n-1,j+1).$$

2. $j \geq 2n-1$:

   In this case, as $n \geq 3$, it implies that $j \geq n+2$. So,

   $$\bar{L}(n,j) = \bar{L}(n,j+1) = 0.$$

Therefore,

$$\begin{aligned}\dot{L}(n,j) =& \bar{L}(n,j) - \frac{1}{j(j-1)}\\ =& -\frac{1}{j(j-1)}\\ \leq& -\frac{1}{(j+1)j} = \dot{L}(n,j+1)\end{aligned}$$

proving the lemma.

□

Lemma 1.14.

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}\dot{L}(n,2i) = 0.$$

Proof. Using Lemma 1.13 it can be seen that the sequence of $\dot{L}(n,2), \dot{L}(n,3), \ldots, \dot{L}(n,2N+1)$ contains no more than 2 monotonic subsequences. By Lemma 1.11, $\left|\dot{L}(n,j)\right| \leq \frac{1}{2(2n-1)}$, thus by Lemma 1.9 the absolute value of the difference of the sum of the even and the sum of the odd terms in the sequence is the difference of at most 2 monotonic subsequences. Thus it is bounded by $2(2d)$ where $d = \frac{1}{2(2n-1)}$. That is,

$$\left|\sum_{i=1}^{N}\dot{L}(n,2i) - \sum_{i=1}^{N}\dot{L}(n,2i+1)\right| \leq 4d,$$

so,

$$\begin{aligned}&-4d \leq \sum_{i=1}^{N}\dot{L}(n,2i) - \sum_{i=1}^{N}\dot{L}(n,2i+1) \leq 4d\\ \iff& \sum_{i=1}^{N}\dot{L}(n,2i+1) - 4d \leq \sum_{i=1}^{N}\dot{L}(n,2i) \leq \sum_{i=1}^{N}\dot{L}(n,2i+1) + 4d.\end{aligned}$$

Add $\sum_{i=1}^{N}\dot{L}(n,2i)$ to both sides to obtain,

$$\sum_{i=2}^{2N+1}\dot{L}(n,i) - 4d \leq 2\sum_{i=1}^{N}\dot{L}(n,2i) \leq \sum_{i=2}^{2N+1}\dot{L}(n,i) + 4d,$$

taking the limit as $N\to\infty$, and $n\to\infty$, gives,

$$\lim_{n\to\infty}\left(\sum_{i=1}^{\infty}\dot{L}(n,i) - 4d\right) \leq 2\lim_{n\to\infty}\sum_{i=1}^{\infty}\dot{L}(n,2i) \leq \lim_{n\to\infty}\left(\sum_{i=1}^{\infty}\dot{L}(n,i) + 4d\right),$$

and by Lemma 1.12,

$$
\begin{aligned}
& 0 - 4 \lim_{n\to\infty} d \le 2 \lim_{n\to\infty} \sum_{i=1}^{\infty} \dot{L}\left(n, 2i\right) \le 0 + 4 \lim_{n\to\infty} d \\
\iff & -2 \lim_{n\to\infty} \frac{1}{2\left(2n-1\right)} \le \lim_{n\to\infty} \sum_{i=1}^{\infty} \dot{L}\left(n, 2i\right) \le 0 + 2 \lim_{n\to\infty} \frac{1}{2\left(2n-1\right)} \\
\iff & 0 \le \lim_{n\to\infty} \sum_{i=1}^{\infty} \dot{L}\left(n, 2i\right) \le 0,
\end{aligned}
$$

thus the lemma follows. □

And this proves Theorem 1.4. By Lemma 1.10 and 1.14 the ϵ_n in Theorem 1.4 is of

$$
\begin{aligned}
O\left(d_1 + d_2\right) =& O\left(\frac{1}{\sqrt{n}} + \frac{1}{n}\right) \\
=& O\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

□

As a corollary to this result the expected number of lucky moves in a game of memory can also be obtained.

Corollary 1.14.1. *For a game of memory consisting of 2 identical decks of size n, the expected number of lucky moves in a game tends to* $\ln 2$ *as $n \to \infty$.*

Proof. As a move consists of two flips, the cards of a single block are flipped two at a time, one after the other. Therefore there is a lucky move for each even block as the last two cards will only be flipped in a single move if the block length is even. Thus the expected number of lucky moves is

$$
\sum_{i=1}^{\infty} \bar{L}\left(n, 2i\right) = \ln 2 + \sum_{i=1}^{\infty} \dot{L}\left(n, 2i\right),
$$

and hence $\sum_{i=1}^{\infty} \bar{L}\left(n, 2i\right) \to \ln 2$ as $n \to \infty$. □

Part III
Exactly solved models of polyominoes and polygons [11]

Abstract

A polyomino is the interior of a polygon on the square lattice. It is a plane geometric figure formed from square cells of equal width that are joined together at their edges. As a polyomino may form the interior of a polygon, a polygon may be thought to be a special type of polyomino. Generating functions for the enumeration of specific classes of polyominoes and polygons will be derived using first a recursive description for the structure of the polyomino and then using one of three different methods to obtain the generating function for the enumeration for that class of polyominoes from this structure.

1. Polyominoes whose structure is linear and recursive will be counted by the first method, and the resulting generating function will be rational.
2. Those polyominoes having an algebraic structure will be considered next, and the method used in their enumeration will result in an algebraic generating function.
3. The class of convex polyominoes will be enumerated last, using the Temperley method, whereby a new column is added to the polyominoes when defining their recursive construction.

1 Introduction

Polyominoes are plane geometric figures; thus the parameters considered when enumerating polyominoes are their area and perimeter. As each cell is a square of area 1 the area of a polyomino may be used to count the number of blocks of that polyomino. The edges of the square cells shared by other cells don't contribute to the perimeter of the polyomino, so the perimeter measures the length of the border around the polyomino, as it should.

As the perimeter along the north border of a polyomino is equivalent to the perimeter along the south border, and the perimeter along the east border is equivalent to the perimeter along the west border, the perimeter of a polyomino is always even.

Now the perimeter of a polyomino consists of a vertical perimeter and a horizontal perimeter, the perimeter may thus be partitioned into these two parts which may be considered individually when enumerating polyominoes.

Using this information, let C denote a class of polyominoes. The complete generating function for C will be given by,

$$C\left(x,y,q\right)=\sum_{P\in C}x^{\mathrm{hp}(P)/2}y^{\mathrm{vp}(P)/2}q^{a(P)}$$

where $\mathrm{hp}\left(P\right)$ denotes the horizontal perimeter of a polyomino P; $\mathrm{vp}\left(P\right)$ denotes the vertical perimeter of P; and $a\left(P\right)$ denotes the area of P. Therefore the coefficient of $x^r y^s q^t$ enumerates all those polyominoes from the class C that have horizontal semi-perimeter r; vertical semi-perimeter s; and area t.

Let $C\left(x,y,z\right)$ denote such a generating function. Thus $C\left(t,t,1\right)$ is the perimeter generating function, where the coefficients of t^n enumerate all those polyominoes having half perimeter n. Similarly $C\left(x,y,1\right)$ is the anisotropic version, the perimeter generating function distinguishing between horizontal and vertical perimeters. Lastly, it can be seen that $C\left(1,1,q\right)=\sum_{P\in C}q^{a(P)}$

is the area generating function for the class C, where the coefficient of q^n counts all polyominoes from the class C which have area n.

This generating function $C(x, y, q)$ can be used to obtain many different results relating to the class of polyominoes C. Some of the results obtainable, such as the asymptotics relating to such a generating function, are found through the use of results from Complex Analysis as illustrated by Sedgewick and Flajolet in [30].

Due to the nature of polyominoes, the various different types and classes, the task of enumerating all polyominoes is tremendously complex. Sub-classes of polyominoes are thus studied instead to simplify this task. The enumeration of certain classes of polyominoes is applicable to fields outside of combinatorics:

1. In number theory partitions themselves are enumerated as well as in relation to representations of symmetric groups.

2. Andrews too considered the enumeration of partitions exclusively as described in [4].

3. In considering directed percolation, directed polyominoes are used, these directed polyominoes also appear as binary search networks in the field of theoretical computer science as shown in [52]

The majority of classes of polyominoes are formed by considering convexity and directedness conditions relating to polyominoes. These classes will be defined and their conditions explained in the following sections.

1.1 Three different counting methods

The following three methods may be used to enumerate a large number of classes of polyominoes. Each method starts by recursively describing the structure of the polyominoes relating to a specific class. This structure is then translated to a generating function, $C(x, y, q)$. The methods differ in the way in which the recursive construction of the polyominoes is described. Each of these methods is applicable to specific classes of polyominoes.

The first method that will be discussed is applicable to all classes of polyominoes that have a linear recursive structure. The resulting generating function is rational. The second method is applicable to those polyominoes whose structure is algebraic and gives rise to an algebraic generating function. The last method is applicable to the class of all convex polyominoes, and is sometimes referred to as the Temperley approach or the Temperley method. It consists of adding an extra column to the polyomino when describing its recursive structure and was used by Temperley in 1956 as illustrated in [46]. It can result in either an algebraic or a rational generating function.

Other methods also exist which utilize less generalized techniques to enumerate specific classes of polygons. Among the tools used in those techniques two occur most frequently: the first appears in a paper by Polya [42] and it concerns staircase polyominoes. This tool may also be used in the case of polygons that may be self-intersecting and works by eliminating those polygons which self-intersect through an inclusion-exclusion principle. More general polygons have been enumerated in [28, 29] using an extension of this tool.

Bijections are the second tool that are used most frequently in the enumeration of specific classes of polyonimoes - a bijection is found between the class being counted and one with a simpler recursive structure, thus the original class may be enumerated by proxy of the simpler class. Examples of the use of this technique can be seen in [12, 25].

1.2 Different classes of polyominoes and their animals

Consider a polyomino P. It is made up of square cells. An *animal* is defined to be the set of all centers of each square cell in P. Let A be the animal that corresponds to the polyomino P. As all the cells of P are connected, so each of the centers that form A are also connected. These centers will be called *vertices*.

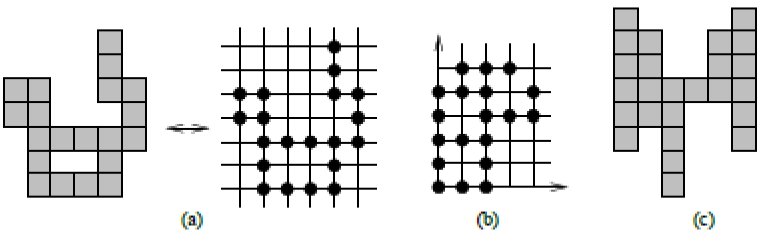
An animal is said to be directed if there exists a vertex v_0 such that there exists a path between v_0 and every other point A consisting entirely of only east or only west steps, and only north or only south steps. Polyominoes may thus be North-West, South-West, North-East and South-East directed.

Considering now the convexity property of polyominoes:

1. If every vertical line that intersects a polyomino intersects it exactly once then that polyomino is column-convex,

2. similarly, if the same holds for horizontal lines, then the polyomino is row-convex.

Distinct classes may be formed by considering each of the four direction properties of polyominoes as well as each of the four convexity properties. There are 31 classes of polyominoes that adhere to at least 1 convexity property, and 4 classes of polyominoes that have at least one directional property. The next figure illustrates some of these polyominoes, as well some animals from the different polyominoes.



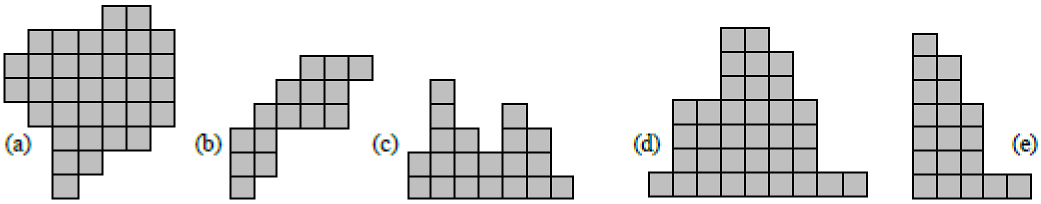
[11]

Figure 1: (a) shows a polyomino on the left, and that polyomino's animal on the right. The animal shown in (b) is North-East Directed, as every other vertex in the animal can be reached starting from the bottom leftmost vertex, using only North and East steps. The polygon shown by (c) can be intersected only once by a vertical line and is thus column-convex.

The five classes of polyominoes that will be dealt with in the next sections are:

1. Convex polyominoes, which are those polyominoes that are column- and row-convex;
2. staircase polyominoes, which are North-East (or South-West) directed polyominoes;
3. bargraphs, which are the special type of polyominoes known as polygons, and are North-East and North-West directed;
4. stacks is the fourth class of polyominoes that will be considered, and are a subclass of bargraphs, that are row-convex;
5. lastly, the polyominoes known as partitions, or Ferrers diagrams will be considered which are polygons that are convex and North-East, North-West and South-East directed.

The next figure illustrates some examples of these polygons.



[11]

Figure 2: The figure in (a) represents a convex polygon. A staircase polygon is shown in (b). (c) gives an example of a bargraph. (d) is a stack polygon. The diagram in (e) represents a Ferrers diagram.

Concerning the generating functions and recursions for the different classes of polyominoes, two definitions will be needed:

1. **A rational function**: a function is said to be rational if it can be defined by a rational fraction, that is expressed as a fraction with a polynomial numerator and denominator.
2. **An algebraic function**: a function is algebraic if it is the root of a non-trivial polynomial.

Now, starting with the enumeration of classes for the first of the three methods, the enumeration of bargraphs will be considered.

2 Constructing linear models and obtaining rational series

2.1 The class of bargraphs enumerated by area

In this subsection bargraphs will be enumerated with regard to their area. Let the number of bargraphs that have area n be denoted by b_n. Trivially it is true that $b_1 = 1$. Bargraphs with an area greater than 2 can be partitioned into two different subsets:

- bargraphs with a rightmost column consisting only of a single block, which will be referred to as bargraphs of type 1,
- or bargraphs for which the rightmost column consists of two blocks or more, which will be referred to as bargraphs of type 2.

Bargraphs of area $n \geq 2$ may be formed from bargraphs of area $n-1$ by the addition of a single block. To obtain bargraphs of the first subset add the block at the end on the right of any bargraph of area $n-1$. This results in a bargraph of area n whose rightmost column consists of a single block. To obtain bargraphs of the second subset add the block onto the rightmost column of any bargraph or area $n-1$. The result is a bargraph of area n with a rightmost column of height at least 2.

Since the two different types of bargraphs come from mutually exclusive sets the following recursion is obtained

$$\begin{aligned} b_1 &= 1 \\ \text{if } n \geq 2 \text{ then } b_n &= b_{n-1} + b_{n-1} \\ &= 2b_{n-1}. \end{aligned}$$

Thus

$$b_n = 2^{n-1},$$

and the area generating function of bargraphs may be given by

$$\begin{aligned} B(q) :=& \sum_{n\geq 1} b_n q^n \\ =& \sum_{n\geq 1} 2^{n-1} q^n \\ =& q \sum_{n\geq 0} (2q)^n \\ =& \frac{q}{1-2q}, \end{aligned}$$

where the coefficients of q^n enumerate the number of bargraphs with area n.

2.2 Constructing polyominoes as linear objects

Constructing bargraphs in the way described in the previous subsection uses two steps:

1. Partitioning a set into two disjoint sets and using the union of these disjoint smaller sets instead of the entire larger set to represent the larger set,
2. and forming a larger object from a smaller one by adding a single cell to it.

When considering the area generating function $B(q)$ of bargraphs, the two aforementioned steps give an expression for the generating function as

$$B(q) = q + q(B(q) + B(q)).$$

The single q represents the area of the trivial case of a single cell bargraph, and the $q(B(q) + B(q))$ is a result of concatenating an extra cell onto either one of two different bargraphs, where the factor q increments the area of those bargraphs by 1.

If any object in a particular class, with a given size, can be formed exactly once

- from a finite number of starting objects,
- using a disjoint union to represent a larger set,
- and adding a cell to a smaller object to obtain a larger object in a different set,

then that class is defined to be *linear*.

In continuing with deriving an exact functional expression for $B(q)$ we denote the area generating function for bargraphs of the first type, that is bargraphs of whose rightmost column consists of a single block only, by $\tilde{B}(q)$. These bargraphs have only a single cell in their rightmost column and thus are formed by adding a single cell to the right of any bargraph, or exist as the single cell only, thus

$$\tilde{B}(q) = q + qB(q),$$

where q represents the area of the single cell bargraph and $qB(q)$ represents forming a bargraph of type 1 by adding a cell to the right of any bargraph.

Returning to the method used in obtaining this generating function, generally the generating function for such a linear class, consisting of k different disjoint subclasses, is obtained from the system

$$B_i(q) = P_i(q) + q\sum_{j=1}^{k} a_{i,j} B_j(q) \text{ where } 1 \le i \le k$$

as the first component, where $a_{i,j} \in \mathbb{N}$, $P_i(q) = x^i b_i + \cdots + x_1 b_1 + b_0$ and $b_i \in \mathbb{N}\ \forall i$.

As can be seen from the area generating function of bargraphs, $P_i(q)$ represents the number of all the initial objects, all of type i, while the $a_{i,j}$ represent the total number of ways that a cell may be added to the objects of type j creating objects of type i. $\mathbb{N}$-rational series are the series obtained from the above equation, and it can be seen in [45] that there is a strong correlation between $\mathbb{N}$-rational series and the theory of regular languages.

2.3 Further linear constructions

Three more problems will be considered and solved by the first method, through the use of a linear recursive description. Two additional tools will be used for these three problems:

1. To utilize the linear recursive description of a class, bijections will first be found between polyominoes and objects that have a linear recursive structure. The first method can then be applied to this class of objects.
2. Directed graphs will be used to describe more complex linear constructions.

2.3.1 Enumerating Ferrers diagrams

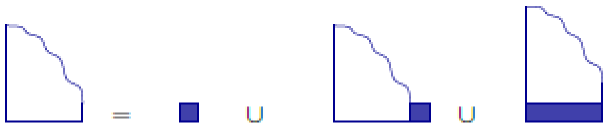
The first of the three problems is the enumeration of Ferrers diagrams by perimeter. A Ferrers diagram is a polygon that is convex and directed, as such the class of Ferrers diagrams may be divided into three disjoint subsets:

1. The first set consists of a single object - the unique diagram of half-perimeter two, that is the single cell.

2. The second set is all diagrams whose interior is at least two blocks wide, having rightmost column height 1.

3. The third set is all diagrams having all columns with height 2 or greater.

This may be illustrated by the following figure.



[11]

Figure 3: The linear structure of Ferrers diagrams.

The diagram in the first set is obtained trivially. The second diagram in the set can be constructed by simply adding a column of height one to any Ferrers diagram of width at least 1. Diagrams obtained by duplicating the bottom row of any non-empty diagram illustrate the third set.

Using these three disjoint sets it will be shown that the North-East border of Ferrers diagrams may be described as a path starting from the North-West corner of the diagram and ending at the South-East corner.

The set of these words may be used to give a linear construction of Ferrers diagrams by perimeter. The North-East border of a Ferrers diagram consists entirely of South and East steps if traversed going from the North-West corner to the South-East corner. Thus such a border may be represented as a word over the alphabet $\{e, s\}$ where e represents East steps and s represents South steps.

Traversing the border of a Ferrers diagram in this way the first step is always East, starting at the top left corner, and the last step is always South, ending at the bottom right corner. Hence, all words starting with an e and ending with an s over the alphabet $\{e, s\}$ encode Ferrers diagrams along the North-East border.

Denote the class of these words by F and the set of all prefixes of length at least 1 to these words by L. From this the following linear description is obtained

$$\begin{aligned}\mathsf{F} =& \mathsf{L}s\\ \mathsf{L} =& \{e\} \cup \mathsf{L}e \cup \mathsf{L}s \quad \text{where} \quad \mathsf{L}s = \{us \mid u \in \mathsf{L}\}.\end{aligned}$$

Every word encoding a Ferrers diagram in this way can be seen to consist of a prefix appended to the letter s, thus $\mathsf{F} = \mathsf{L}s$. This prefix must start with an e as all words encoding Ferrers diagrams over this alphabet start with an e. It may be only the letter e, or have length greater than 1, consisting of another similar prefix appended to the letter s as it is not a complete word, thus

$$\mathsf{L} = \{e\} \cup \mathsf{L}e \cup \mathsf{L}s,$$

a union of disjoint sets.

The generating function enumerating these words by length will thus be obtained using the following system

$$\begin{aligned} F(t) =& tL(t) \\ L(t) =& t + tL(t) + tL(t) \\ =& t + 2tL(t) . \end{aligned}$$

Since a word encoding the North-East boundary is made up of all the steps along that boundary its length is equivalent to the half-perimeter of the diagram. The above system therefore gives,

$$\begin{aligned} F(t) =& \frac{t^2}{1-2t} \\ =& \sum_{n\geq 2} 2^{n-2} t^n , \end{aligned}$$

which is the length generating function of Ferrers diagrams, where the coefficients of t^n enumerate Ferrers diagrams with perimeter $2n$.

Recall that the coefficients of y^n enumerate the polyominoes with vertical perimeter $2n$ and the coefficients of x^m enumerate polyominoes with horizontal perimeter $2m$. Considering the border as consisting of East and South steps separately, and using the linear description given, where $F(x, y)$ is the generating function for Ferrers Diagrams, we obtain

$$\begin{aligned} F(x,y) &= xL(x,y) \\ \text{where } L(x,y) &= x + xL(x,y) + yL(x,y) . \end{aligned}$$

The anisotropic perimeter generating function of Ferrers diagrams can be obtained from this, where the coefficients of x^n counts the East steps and y^n counts the South steps encoding the border, as

$$\begin{aligned} F(x,y) =& \frac{xy}{1-x-y} \\ =& xy \sum_{n,m\geq 0} \binom{n+m}{m} x^m y^n \quad \text{(the bivariate generating function)} \\ =& \sum_{n,m\geq 0} \binom{n+m}{m} x^{m+1} y^{n+1} \\ =& \sum_{n,m\geq 1} \binom{n+m-2}{m-1} x^m y^n . \end{aligned}$$

2.3.2 Enumerating stack polygons

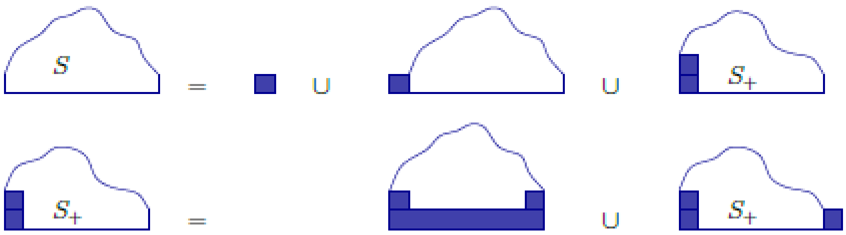
The perimeter generating function of stack polygons will be obtained next, using an identical argument, and still by the first method. Recall that stack polygons are row-convex bargraphs, this means they are row- and column-convex.

First consider that a stack may be only the unique polygon of half-perimeter 2, that is the single cell. A stack polygon of half-perimeter greater than 2 may occur as one of two types:

1. As stacks whose leftmost column has height 1,

2. or stacks whose leftmost column has height greater than 1.

This divides the stacks into two disjoint subsets with criterion identical to that for the case of bargraphs. The first type of stack can be formed from the class of all stacks by simply adding a single block to the left of any stack, increasing the horizontal perimeter by 2, or may exist simply as the single cell. Stacks of the second kind can be formed in two ways. They can be obtained from any stack by duplicating the bottom row, increasing the vertical perimeter by 2, resulting in at least two blocks in the leftmost column, or they can be formed from stacks already having leftmost column height 1 by adding a block to the right of such stacks. This gives the following recursive diagram for stack polygons,



[11]

Figure 4: The decomposition of the set of stack polygons into a recursive structure.

The leads to the following system, where the coefficients of x^n enumerate stacks with horizontal perimeter $2n$ and the coefficients of y^m enumerate stacks with vertical perimeter $2m$, given by

$$\begin{aligned} S(x,y) =& xy + xS(x,y,) + S_+(x,y) \\ S_+(x,y) =& yS(x,y) + xS_+(x,y). \end{aligned}$$

Therefore the perimeter generating function for stack polygons is

$$S(x,y) = \frac{xy(1-x)}{(1-x)^2 - y}.$$

2.3.3 Enumerating polygons that are column-convex by their area

Column-convex polygons will now be counted by area. Let P be a polygon from the class of column-convex polygons, consisting of n cells. Starting from the bottom, leftmost cell label the cells from 1 to n in the following order:

1. The leftmost column is labeled from bottom to top starting at 1.
2. The cell in the right bordering column that is closest to the cell in the first column is labeled next.
3. The cells below this cell are then labeled from top to bottom, then the cells above this cell are labeled from bottom to top.
4. This process is repeated from step 2 for each column, going from left to right, until all the cells have been labeled.

This process is illustrated by the following diagram.

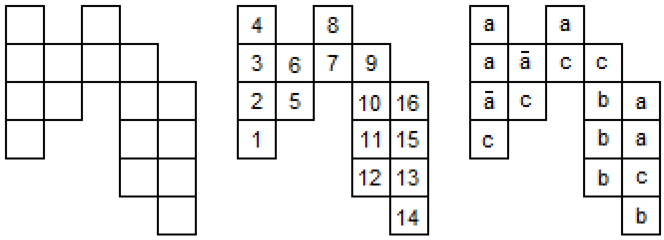


Figure 5: An example of the labeling process described, encoding the column-convex polyomino to the word $caaac\bar{a}cacbbbcbaa$.

The labeling will now be used to encode a column-convex polygon of n cells to a word of n letters over the alphabet $\{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$. The word will be formed going from the labels 1 to n. Let $w = w_1 w_2 \dots w_n$ be a word encoding a column-convex polygon. Then for $i = 1, 2, \dots, n$

1. $w_i = c$ if the cell labeled i has the smallest label in its column,
2. $w_i = b$ if the i^{th} cell is below the cell labeled c,
3. and $w_i = a$ of the i^{th} cell is above the cell labeled c.

A bar will be used to denote specific cells: if there is a cell to the East and a cell to the South of the cell i, and if there exists no cell to the South-East of the particular cell i, then the letter encoding it, w_i, will have a bar above it. Note that only one cell per column may be encoded with a bar, as only one cell per column can have an East neighbor but no South-East neighbor in a column-convex polyomino. It should also be noted that the last column of a column-convex polyomino can not be encoded with any barred letters, as none of the cells in the rightmost column have an east neighbor.

This encoding allows any column-convex polygon to be encoded as a unique word w over the alphabet $\{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$. Each of these coding words encodes a column-convex polygon uniquely, and hence the class of these coding words is the same size as the class of all column-convex polygons.

We will thus focus on the enumeration of the class of these coding words instead. The set of prefixes, including the empty prefix ϵ, of these coding words will be denoted by L, and every word may be constructed from different prefixes. Each prefix may only end on a certain combination of letters, encoding the rightmost column of a column-convex polyomino. As a result of this L can be partitioned into 5 disjoint subsets, *viz.* $\mathsf{L}_1, \mathsf{L}_2, \mathsf{L}_3, \mathsf{L}_4$, and L_5, each of which shall be described as follows:

$$\begin{aligned}
\mathsf{L}_1 =& \{\epsilon\},\\
\mathsf{L}_2 =& \mathsf{L}_1 c \cup \mathsf{L}_2 a \cup \mathsf{L}_3 a \cup \mathsf{L}_4 c,\\
\mathsf{L}_3 =& \mathsf{L}_2 c \cup \mathsf{L}_3 b \cup \mathsf{L}_3 c,\\
\mathsf{L}_4 =& \mathsf{L}_2 \bar{a} \cup \mathsf{L}_3 \bar{a} \cup \mathsf{L}_4 a \cup \mathsf{L}_5 b,\\
\mathsf{L}_5 =& \mathsf{L}_2 \bar{c} \cup \mathsf{L}_3 \bar{b} \cup \mathsf{L}_3 \bar{c} \cup \mathsf{L}_5 b.
\end{aligned}$$

Together, all 5 sets $\mathsf{L}_1, \mathsf{L}_2, \ldots, \mathsf{L}_5$ form a partition of L and make up every possible prefix to the words encoding column-convex polygons, as will be explained using the following two diagrams:

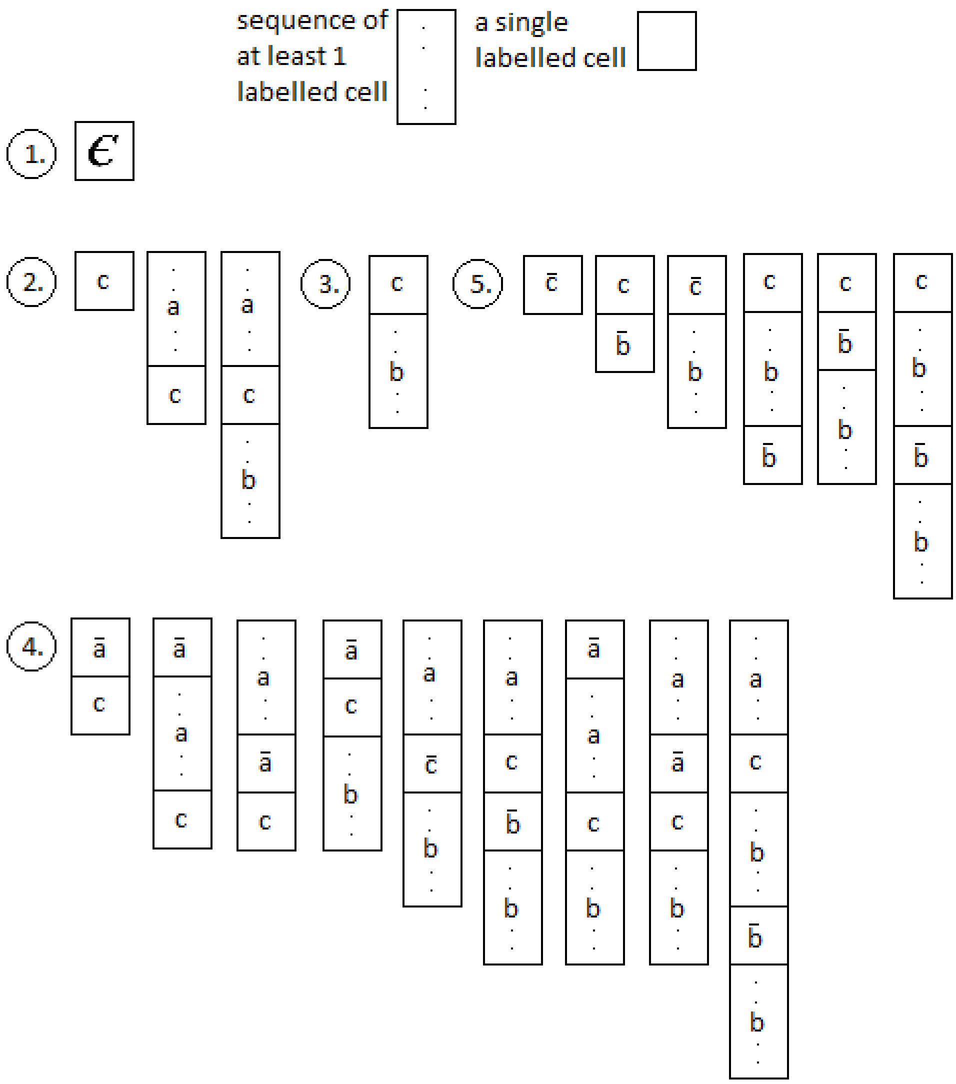
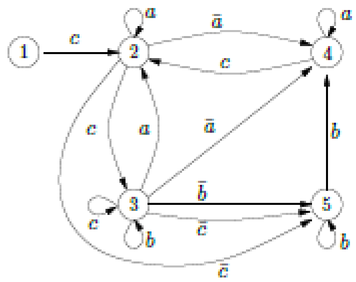


Figure 6: This diagram illustrates all possible rightmost columns which each of the prefixes may be encoding, and in which set the prefixes belong depending on the rightmost column of the column-convex polyomino which they encode. Some prefixes do not encode a completed column-convex polyomino, as is the case in set 4, and 5. Each of the numbered sets i corresponds to a subset L_i.



[11]

Figure 7: An illustration, demonstrating the encoding of column-convex polygons to words, starting at the vertex 1. A prefix in one of the sets L_i starts at 1 and ends at i.

The subsets are classified according the letters on which the prefixes in L end. The first diagram shows that the five subsets L_1, L_2, L_3, L_4 and L_5, form a partition of L based on the last column of the polyomino which the prefixes in L encode, as each of the possible rightmost columns allow a prefix to be in one subset only, and each possible rightmost column is considered.

As the prefixes in L_4 and L_5 have barred letters, $\bar{a}, \bar{b}$ and $\bar{c}$, the words following their prefixes must still have extra columns added since the bar above the letter signifies a new column.

The prefixes contained in both L_2 and L_3 can be seen as complete words, encoding certain column-convex polygons completely. Translating the sets into a series representation that enumerates these words by length gives

$$\begin{aligned}
L_1 =& 1, \\
L_2 =& q\left(L_1 + L_2 + L_3 + L_4\right), \\
L_3 =& q\left(L_2 + L_3 + L_3\right), \\
L_4 =& q\left(L_2 + L_3 + L_4 + L_5\right), \\
L_5 =& q\left(L_2 + L_3 + L_3 + L_5\right).
\end{aligned}$$

As L_2 and L_3 are seen as encoding polyominoes completely i.e. there are no more letters needed so that their prefixes fully encode a polyomino, the area generating function for column-convex polygons is given by

$$C(q) = \mathsf{L}_2(q) + \mathsf{L}_3(q).$$

The above system of equations may be solved to obtain

$$C(q) = \frac{q(1-q)^3}{1 - 5q + 7q^2 - 4q^3}.$$

This is the area generating function for column-convex polyominoes, where the coefficient of q^n enumerates the total number of column-convex polyominoes having area n (see Appendix 3.1 for the solution.)

A column-convex polygon may be North-East directed and is North-East directed if the word encoding it does not contain the letter b, as b will lead to the requirement of a South step in reaching all the vertices in the polygon from a given vertex. Adjusting the area generating function $C(q)$ for the case of directed column-convex polygons, all terms $\mathsf{L}_i b$ must then be removed from the recursive linear description. Hence the linear recursive description becomes

$$L_1 = 1,$$

$$
\begin{aligned}
L_2 =& q\left(L_1+L_2+L_3+L_4\right), \\
L_3 =& q\left(L_2+L_3\right), \\
L_4 =& q\left(L_2+L_3+L_4\right).
\end{aligned}
$$

The set of prefixes L_5 becomes irrelevant. Solving this system gives the area generating function for directed column-convex polygons as

$$
DC(q)=\frac{q(1-q)}{1-3q+q^2}.
$$

(For the solution please see Appendix 3.2.)

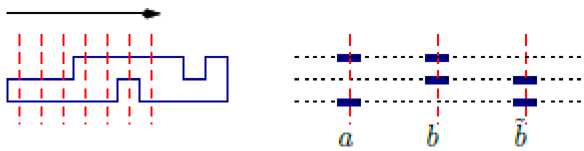
2.3.4 Enumerating polygons whose height is restricted to a strip

The next set of polyominoes that will be enumerated are polygons whose height is confined to a strip of a given height. These polygons thus have a linear structure. Confining polyominoes in this way to fit in a strip aids in the enumeration of polyominoes, self-avoiding-walks as well as self-avoiding polygons as can be seen in [3, 6, 44, 53, 54].

Enumerating polyominoes in this way gives an approximation to the enumeration of the same type of polygons but in a class independent of height. Thus, increasing the height restriction of the strip leads to more polyominoes being counted and a better approximation.

Consider now self-avoiding polygons restricted to lie in a strip of a fixed height k. The horizontal edges of a self-avoiding polygon lying in such a strip determine the polygon completely, as shown in the next diagram.

This polygon may be intersected by a finite number of vertical lines spaced one cell length from each other. The points at which the edges of the polygon intersect a particular line will be referred to as a *state*. As the height of a strip is fixed, the number of edges at specific positions that one of these vertical lines may intersect is finite, hence there are a finite number of states.



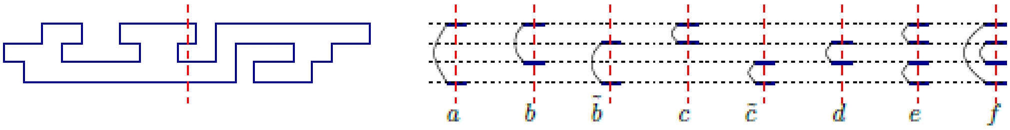
[11]

Figure 8: The illustration of states on a 2-strip, and their mapping to letters over an alphabet of 3 letters. The polygon on the left is encoded to the word $\tilde{b}\tilde{b}\tilde{b}aabaa\tilde{b}a$.

Those states may be used to encode a self-avoiding polygon of a given height k to a word, listing the states as they occur from left to right.

The above figure illustrates a self-avoiding polygon restricted to height 2 encoded to a word over the alphabet $\{a, b, \tilde{b}\}$. Therefore the set made up of these words for any given height k has a structure that is linear.

Another example of encoding self-avoiding polygons is the way illustrated by the next figure over the alphabet $\{a, b, \tilde{b}, c, \tilde{c}, d, e, f\}$.



[11]

Figure 9: The 8 states used to encode self-avoiding polygons restricted to a 3-strip. The polygon on the left is encoded to the word $d\tilde{b}af\tilde{c}\tilde{c}eaaf\tilde{c}aceeabcc$.

It should be noted that the position of the intersections of the horizontal boundaries of the self-avoiding polygon intersecting the vertical lines is important and used to encode the self-avoiding polygon to a word.

Let the height k of the strip be fixed. Denote by S the set of all the words that encode a self-avoiding polygon restricted to having height no more than k. Let L denote the set of all prefixes of the words in S. Now L has a linear structure, containing the words encoding incomplete self-avoiding polygons. L is also Markovian with memory 1, as for each $w \in \mathsf{L}$, the set a of letters for which $wa \in \mathsf{L}$ is dependent only on the letter at the end of the prefix w. Thus denote by L_a the set of all prefixes that end in a distinct letter a for every distinct letter a in the alphabet encoding the self-avoiding polygon to a strip of height k. A graph can be used to encode the linear structure, and from this graph the equations for the different sets L_a can be obtained for each distinct a.

Now, considering self-avoiding polygons restricted to a strip of height 2, from Figure 7, it can clearly be seen that in the words encoding these polygons b cannot follow $\tilde{b}$, and $\tilde{b}$ cannot follow b. Thus the linear structure of the prefixes is given as follows

$$\mathsf{L}_a = \left(\epsilon \cup \mathsf{L}_a \cup \mathsf{L}_b \cup \mathsf{L}_{\tilde{b}}\right) a,$$

as a may follow a, b and $\tilde{b}$,

$$\mathsf{L}_b = \left(\epsilon \cup \mathsf{L}_a \cup \mathsf{L}_b\right) b,$$

as b may only follow a or b, and

$$\mathsf{L}_{\tilde{b}} = \left(\epsilon \cup \mathsf{L}_a \cup \mathsf{L}_{\tilde{b}}\right) \tilde{b},$$

as $\tilde{b}$ may only follow a or $\tilde{b}$.

This structure can now be translated to linear equations, counting the horizontal edges, where the coefficients of x^n count the total number of incomplete self-avoiding polygons, in a strip of height 2, having horizontal perimeter $2n$, and the coefficients of y^m counts those having vertical perimeter $2m$.

Adding a horizontal edge is thus equivalent to multiplying by $\sqrt{x}$, and adding a vertical edge is equivalent to multiplying by $\sqrt{y} = z$ in the generating function. To get a self-avoiding polygon consisting of just a from the initial vertex you need two vertical and two horizontal edges, i.e. it has generating function $\sqrt{x}\sqrt{x}zz = xz^2$. To add a to the end of the prefixes in L_a you only need two horizontal edges, i.e. multiply the generating function by $\sqrt{x}\sqrt{x}$ resulting in

$$\mathsf{L}_a x.$$

To add a to the end of the prefixes in L_b you need a single vertical edge, and two horizontal edges, thus the factor of $z\sqrt{x}\sqrt{x}$ in

$$\mathsf{L}_b z\sqrt{x}\sqrt{x} = \mathsf{L}_b zx.$$

Adding a to $\mathsf{L}_{\tilde{b}}$ is the same as adding a to L_b hence it has generating function

$$\mathsf{L}_{\tilde{b}} zx.$$

All of this combines to give

$$\mathsf{L}_a = \left(\epsilon + \mathsf{L}_a + \mathsf{L}_b + \mathsf{L}_{\tilde{b}}\right) a.$$

Consider now the prefixes L_b: to have b from the initial vertex takes one vertical edge and two horizontal edges, that has generating function zx. Having a b after an a takes a vertical step and two horizontal steps, giving

$$\mathsf{L}_a z\sqrt{x}\sqrt{x} = \mathsf{L}_a zx.$$

Finally, to have a b state follow a b state takes only two horizontal edges, and this has generating function

$$\mathsf{L}_b\sqrt{x}\sqrt{x} - \mathsf{L}_b x.$$

Combining each case gives

$$\mathsf{L}_b = \left(z + z\mathsf{L}_a + \mathsf{L}_b\right) x.$$

The set of prefixes $\mathsf{L}_{\tilde{b}}$ have a similar linear equation to L_b,

$$\mathsf{L}_{\tilde{b}} = \left(z + z\mathsf{L}_a + \mathsf{L}_{\tilde{b}}\right) x,$$

as only $\tilde{b}$ states can be followed by a $\tilde{b}$. Adding a letter to any word in the set L represented by one of the above equations is illustrated by the second graph in the middle of Figure 9.

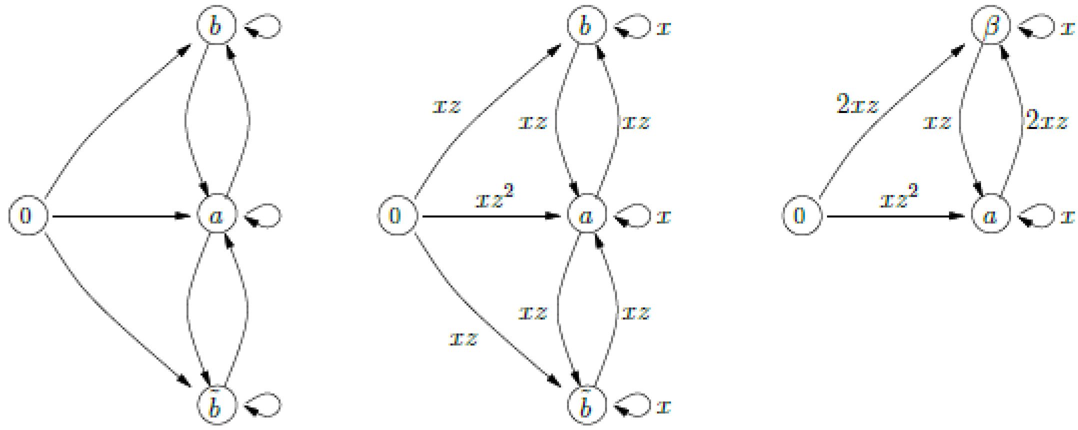
The generating function of self-avoiding polygons restricted to a strip having height 2 can be obtained from the classes of prefixes corresponding to incomplete polygons by adding the required number of vertical edges to complete the polygons. The polygons ending in an a state need two vertical edges to be complete, resulting in a factor of z^2 when considering the generating function. The prefixes ending in a b and $\tilde{b}$ need only 1 vertical edge and thus a factor of z in the generating function. This translates to the generating function for all self-avoiding polygons restricted to a strip of height 2, given by

$$S_2\left(x, y\right) = z^2 L_a + zL_b + zL_{\tilde{b}}.$$

Consider the states b and $\tilde{b}$, from Figure 9 it can be seen that they contribute exactly the same to the linear structure of the self-avoiding polygon in a 2-strip. They can thus be replaced by a unique state β representing both states. The incomplete self-avoiding polygons which end in a β state, that is not in an a state, is equivalent to the sum of the generating function enumerating the number of self-avoiding polygons ending in a b and a $\tilde{b}$ state. This gives the equation

$$L_\beta = L_b + L_{\tilde{b}}.$$

The next figure illustrates a directed graph for encoding polygons restricted to a specific height, for $k = 2$. Every word illustrated by the graph is represented as a path starting from a vertex 0.



[11]

Figure 10: The graph illustrating the linear structure of these polyominoes.

Now, the above system may be simplified to

$$\begin{aligned} L_a =& x\left(z^2 + L_a + zL_\beta\right), \\ L_\beta =& x\left(2z + 2zL_a + L_\beta\right), \\ S_2\left(x, y\right) =& z^2 L_a + zL_\beta. \end{aligned}$$

This can be solved to obtain

$$S_2\left(x, y\right) = \frac{xy\left(2 - 2x + y + 3xy\right)}{\left(1 - x\right)^2 - 2x^2 y}.$$

(See Appendix 3.3 for the solution.)

Polygons existing entirely of $\tilde{b}$ states are identical to those existing entirely of the same number of b states. Thus the series for $S_2\left(x, y\right)$ counts polygons of height 1 twice. To correct this the generating function for self-avoiding polygons of height 1 given by

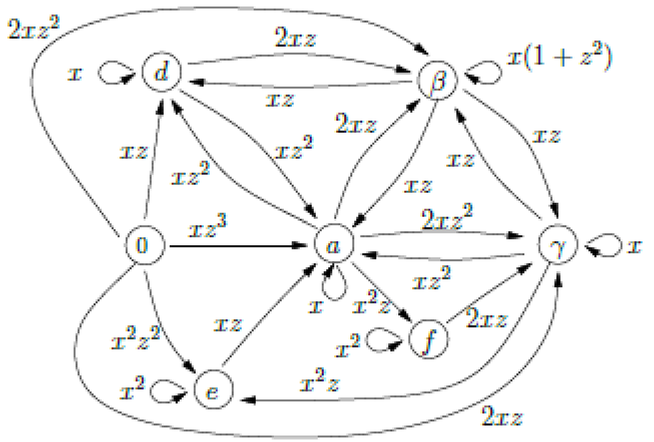
$$\begin{aligned} S_1\left(x, y\right) =& xy \sum_{i \geq 0} x^i \\ =& \frac{xy}{1 - x}, \end{aligned}$$

should be subtracted from $S_2\left(x, y\right)$ to get the generating function for self-avoiding polygons in a 2-strip as

$$\frac{xy\left(2 - 2x + y + 3xy\right)}{\left(1 - x\right)^2 - 2x^2 y} - \frac{xy}{1 - x}.$$

2.3.5 Counting self-avoiding polygons of height 3

Consider now self-avoiding polygons in a 3-strip. Using the same alphabet as in Figure 8, the following figure illustrates their linear structure.



[11]

Figure 11: The graph which illustrates the linear structure of self-avoiding polygons, encoded by words formed as a path from the root vertex 0.

In the same way that the b and the $\tilde{b}$ states in self-avoiding polygons in a 2-strip were combined in a β state, they are again combined in a β state, however here now too the c and the $\tilde{c}$ states play symmetric roles and can be combined in a γ state similarly.

Having the x and z weights the same as for a 2-strip, Figure 10 can be used to obtain the following linear recursive system of equations for self-avoiding polygons in a 3-strip, as

$$\begin{aligned}
L_a =& x\left(z^3+L_a+zL_\beta+z^2L_\gamma+z^2L_d+zL_e\right),\\
L_\beta =& x\left(2z^2+2zL_a+\left(1+z^2\right)L_\beta+zL_\gamma+2zL_d\right),\\
L_\gamma =& x\left(2z+2z^2L_a+zL_\beta+L_\gamma+2zL_f\right),\\
L_d =& x\left(z+z^2L_a+zL_\beta+L_d\right),\\
L_e =& x^2\left(z^2+zL_\gamma+L_e\right),\\
L_f =& x^2\left(zL_a+L_f\right).
\end{aligned}$$

This system may be solved to obtain the generating function for self-avoiding polygons in a 3-strip by first completing the polygons ending in the various states by adding the appropriate number of vertical edges to get (see Appendix 3.6 for solution.)

$$\begin{aligned}
S_3\left(x,y\right) =& z^3L_a+z^2L_\beta+zL_\gamma+zL_d+z^2L_f\\
=& \frac{xyN\left(x,y\right)}{D\left(x,y\right)},
\end{aligned}$$

where

$$N(x,y) = 3\left(x+1\right)^2\left(1-x\right)^5+\left(5x+2\right)\left(2x-1\right)\left(x+1\right)^2\left(x-1\right)^3y$$

$$
\begin{aligned}
&- (x-1)\left(6x^6+4x^5-18x^4-6x^3+11x^2+8x+1\right)y^2\\
&- x(x+1)\left(2x^4+6x^3-8x^2+4x+1\right)y^3,
\end{aligned}
$$

and

$$
\begin{aligned}
D(x,y) = &(x+1)^2(x-1)^6 - x(1+4x)(x+1)^2(x-1)^4 y\\
&+ x^2\left(3x^4+4x^3-6x^2-8x-3\right)(x-1)^2 y^2\\
&+ x^3(x+1)\left(x^3+3x^2-5x+3\right)y^3.
\end{aligned}
$$

$S_3(x,y)$ is the anisotropic perimeter generating function of self-avoiding polygons of height at most 3. The half-perimeter generating function of the same polygons can be obtained by letting $x=y=t$, that is

$$
S_3(t) = \frac{t\left(-8t^9+4t^8+10t^7-20t^6-t^5-t^4+7t^3+3t^2-7t+3\right)}{4t^{10}-2t^9-5t^8+8t^7-t^6+2t^5-4t^4+2t^3+3t^2-4t+1}.
$$

By considering a cylinder instead of a strip, this method may be improved as in [6].

2.3.6 Obtaining q-analogues of the already obtained generating functions

Obtaining now the complete generating function for Ferrers diagrams the area must also be considered. Let the coefficients of q^n count Ferrers diagrams with area n.

Referring back to Figure 3, the unique single cell of half-perimeter 2 has area 1, which translated to a factor of q. Adding a cell to the right of any Ferrers diagram increases the area of the diagram by 1, resulting in a factor of q. Duplicating the bottom row increases the area of the Ferrers diagram by the length of the bottom row, as the length of a single cell is 1 which is the same as the area of the cell. Thus the length of a Ferrers diagram is the same as the half-horizontal perimeter. The coefficients of x^n in the generating function for Ferrers diagrams counts Ferrers diagrams having horizontal half-perimeter n, or having length n. Thus increasing the area by the length of the diagram results in a factor of q for each x. That translates to the generating function

$$
yF(xq,y,q).
$$

All this combined gives an expression for the complete generating function for Ferrers diagrams as

$$
F(x,y,q) = xyq + xqF(x,y,q) + yF(xq,y,q)
$$

a q-analogue of $F(x,y,1)$. The term $yF(xq,y,q)$ makes the expression non-linear. An explicit form of this equation will be determined through iteration,

$$
\begin{aligned}
F(x,y,q) =& \frac{xyq}{1-xq} + \frac{y}{1-xq}F(xq,y,q)\\
=& \frac{xyq}{1-xq} + \frac{y}{1-xq}\frac{xyq^2}{1-xq^2} + \frac{y}{1-xq}\frac{y}{1-xq^2}\left(\frac{xyq^3}{1-xq^3} + \frac{y}{1-xq^3}F\left(xq^3,y,q\right)\right)\\
=& \frac{xyq}{1-xq} + \frac{xy^2q^2}{(1-xq)(1-xq^2)} + \frac{xy^3q^3}{(1-xq)(1-xq^2)(1-xq^3)}\\
&+ \frac{y^3}{(1-xq)(1-xq^2)(1-xq^3)}F\left(xq^3,y,q\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \ldots \\
&= \sum_{n \geq 1} \frac{x y^n q^n}{(xq)_n},
\end{aligned}
$$

where

$$(xq)_0 = 1, \text{ and } (xq)_n = (1 - xq)\left(1 - xq^2\right) \ldots (1 - xq^n).$$

Finding now the complete generating function enumerating stack polygons, the same factor q is introduced as for Ferrers diagrams, and Figure 4 will now be used again.

Identically the area of the single cell is 1, resulting in a factor of q. Adding a single cell to a stack increases its area by 1, which results in multiplying the stack generating function by a factor of q. Now those stacks whose leftmost column has height at least 2 are made from two sets, as described previously. Duplicating the bottom row of any stack increases its area by the length of that stack, in the same way as for Ferrers diagrams, which gives the term

$$yS(xq, y, q),$$

Adding a single cell to the right of stacks whose left most column has height at least 2 increases the area of those stacks by 1, hence the term $xqS_+(x, y, q)$. This all together gives

$$S(x, y, q) = \frac{xyq}{1 - xq} + \frac{y}{(1 - xq)^2} S(xq, y, q).$$

which again is non-linear as a result of the $S(xq, y, q)$ term.

Solving this by iteration gives,

$$
\begin{aligned}
S(x, y, q) =& \frac{xyq}{1 - xq} + \frac{y}{(1 - xq)^2} S(xq, y, q) \\
=& \frac{xyq}{1 - xq} + \frac{y}{(1 - xq)^2} \frac{xyq^2}{1 - xq^2} + \frac{y}{(1 - xq)^2} \frac{y}{(1 - xq^2)^2} S\left(xq^2, y, q\right) \\
=& \frac{xyq}{1 - xq} + \frac{y}{(1 - xq)^2} \frac{xyq^2}{1 - xq^2} + \frac{y}{(1 - xq)^2} \frac{y}{(1 - xq^2)^2} \frac{xyq^3}{1 - xq^3} \\
&+ \frac{y}{(1 - xq)^2} \frac{y}{(1 - xq^2)^2} \frac{y}{(1 - xq^3)^2} S\left(xq^3, y, q\right) \\
=& \frac{xyq}{1 - xq} + \frac{xy^2q^2}{(1 - xq)(1 - xq)(1 - xq^2)} + \frac{xy^3q^3}{(1 - xq)(1 - xq^2)(1 - xq)(1 - xq^2)(1 - xq^3)} \\
&+ \frac{y^3}{(1 - xq)(1 - xq^2)(1 - xq^3)(1 - xq)(1 - xq^2)(1 - xq^3)} S\left(xq^3, y, q\right) \\
=& \ldots \\
=& \sum_{n \geq 1} \frac{xy^n q^n}{(xq)_{n-1} (xq)_n}.
\end{aligned}
$$

which finally gives the complete generating function for stacks as,

$$S(x, y, q) = \sum_{n \geq 1} \frac{xy^n q^n}{(xq)_{n-1} (xq)_n}.$$

2.4 Constructing algebraic models and obtaining algebraic series

The second method will now be described and illustrated through a number of examples. It consists of using algebraic models and obtaining algebraic series.

2.4.1 Counting the class of bargraphs by perimeter

The first example will be counting bargraphs by perimeter. Considering the perimeter, the set of all bargraphs will be partitioned into subsets based on the criterion of the position of the first column of height 1 from left. The description of these classes of bargraphs will be illustrated by the following figure.

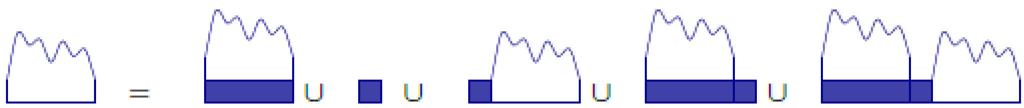

[11]

Figure 12: Constructing bargraphs recursively, this time to obtain an algebraic structure.

Each representation of the set of bargraphs in the above figure, shown as a general bargraph having its bottom row duplicated in the first set; which is added to the left of the single cell in the third set; whose bottom row is again duplicated in the fourth set, with a single cell added the to right of the duplicated row; and whose bottom row is duplicated in the left part of the fourth set, and who is added to the right of the single cell added to the right of the duplicated bottom row in the fifth set, represents a non-empty set of bargraphs.

Using this decomposition it can be seen that there are bargraphs having no column of height 1, shown by the first object in the figure. A single cell is a bargraph of one column of height 1, and is its own set; next, the class of bargraphs whose first column has height 1; second to the last is the class of bargraphs whose last column is the only one having height 1; and lastly is the set of all bargraphs that have some internal column having height 1, that is not the first or the last column.

So the partition of bargraphs according to this criterion has been explained. These classes of bargraphs may be encoded by words that describe their entire top border, i.e. the part of the boundary not on the base of the bargraph. Denote by B the set of these words over the alphabet $\{n, s, e\}$ where n, s and e encode North, South and East steps respectively. Let L represent the part of the words from B without the first or last step. It may be noted that the first step will always be North and the last South. Thus

$$\mathsf{B} = n\mathsf{L}s,$$

and using Figure 11

$$\mathsf{L} = n\mathsf{L}s \cup \{e\} \cup e\mathsf{L} \cup n\mathsf{L}se \cup n\mathsf{L}se\mathsf{L}.$$

As the bargraphs from the first set have no column of height 1, there will be at least two North starting steps and two South ending steps, hence the term $n\mathsf{L}s$ when encoding their border. The set $\{e\}$ is trivial as the top boundary for a single cell can be encoded as nes. The second step in the border of the bargraphs from the third set will always be East since the bargraphs from it have their first column height 1, this gives the $e\mathsf{L}$ term. The second last set's bargraphs only have their last column height 1, so the first two steps will be North and the last two steps will be encoded by ses, giving the term $n\mathsf{L}se$.

Finally the last set of bargraphs have two parts. The first part has no column of height 1, so the first two steps on the border will be nn. That part is followed by a column of height 1, resulting in a guaranteed two steps of the border encoded by se. This is followed by the last part which may or may not have a column of height 1. All of these given terms combine to give, $n\mathsf{L}se\mathsf{L}$.

Using the description of the partition of bargraphs into disjoint sets based on the occurrence of a column of height 1, the following system for the anisotropic generating function of bargraphs is obtained where the coefficients of $x^n y^m$ enumerate the number of bargraphs having horizontal perimeter $2n$ and vertical perimeter $2m$,

$$\begin{aligned} B(x,y) =& yL(x,y) \\ L(x,y) =& yL(x,y) + x + xL(x,y) + xyL(x,y) + xyL(x,y)^2 . \end{aligned}$$

Solving these equations gives,

$$B(x,y) = \frac{1-x-y-xy-\sqrt{(1-y)\left((1-x)^2 - y(1+x)^2\right)}}{2x},$$

the perimeter generating function of bargraphs (see Appendix 3.4 for the solution.) This generating function is clearly algebraic. The equation for this generating function may be determined differently, without the use of words.

Bargraphs may still be partitioned into sets according to the occurrence of a column of height 1, as was done earlier. Firstly there are two disjoint classes into which the set of bargraphs may be partitioned - those having no column of height 1, which will be referred to as *thick* bargraphs, and those that have at least a single column of height 1.

The set of thick bargraphs can be obtained from the set of all bargraphs by simply duplicating the bottom row of any bargraph. This set thus has generating function $yB(x,y)$. The set of bargraphs having at least 1 column of height 1 may be partition into 4 disjoint subsets. The first set contains only the single cell bargraph which has the perimeter generating function xy. The rest of the sets all have at least two columns, one of which has height 1. The first of these contains bargraphs whose first column has height 1. These bargraphs may be formed from the set of all bargraphs by simply adding a column of height 1 to the left of any bargraph, and thus have perimeter generating function

$$xB(x,y).$$

The second of these sets is made up of bargraphs whose last column has height 1, and it is the only column of height 1 in the bargraph. These may be formed by adding a column of height 1 to the right of any thick bargraph and thus have perimeter generating function

$$xyB(x,y) = xyB(x,y).$$

The third set has first column height 1 internally, that is not the first or last column of the bargraph. This may be created by adding a column of height 1 to the right of any thick bargraph and adding any bargraph to the right of this column. The perimeter generating function for this set is thus given by

$$yB(x,y)\,xy^{-1}B(x,y) = xB(x,y)^2.$$

Adding all these perimeter generating functions gives the expression for the anisotropic perimeter generating function of bargraphs as

$$B(x,y) = yB(x,y) + xy + xB(x,y) + xyB(x,y) + xB(x,y)^2.$$

2.5 Enumerating sets of algebraic objects

Outlining the second method, it consists of two steps:

1. Partitioning a set into disjoint sets having simpler structure,

2. and then taking the Cartesian product of these sets.

If a class of objects can be defined through these two steps, that is through the use of disjoint unions and the concatenation of elements from different sets through Cartesian products, and be described recursively in a non-ambiguous manner, then that class is said to be algebraic. The size of the elements in an algebraic class will be assumed to be additive when two such sets are concatenated.

The use of algebraic objects now allows for the concatenation of objects from different classes, and thus generalizes linear constructions which previously only allowed the concatenation of an object and a cell. The concatenation of objects from two classes results in the set obtained by taking the Cartesian product of the two classes. Translated to generating functions, the generating function for the set of concatenated objects is simply the product of the generating functions of the two concatenated sets. As the generating function of an algebraic class is itself algebraic, by the definition of an algebraic function it will be the first term in the solution of the system of polynomials,

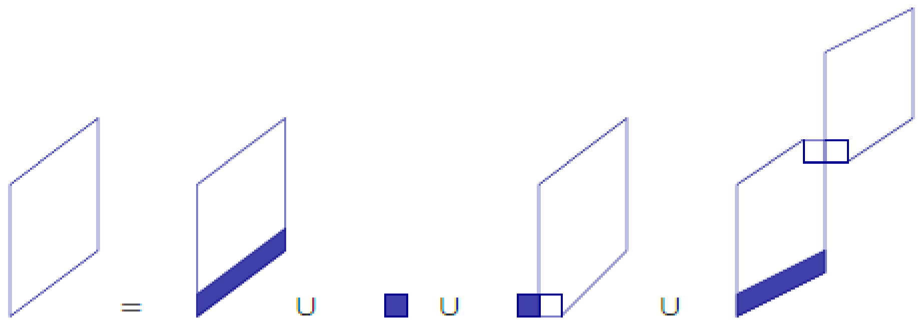
$$A_i = P_i\left(t, A_1, \ldots, A_k\right) \quad \text{for} \quad 1 \leq i \leq k$$

for polynomials P_i whose coefficients are natural numbers.

More examples of the second method will be shown, obtaining algebraic models and generating functions for each.

2.5.1 Counting staircase polygons by their perimeter

The enumeration of staircase polygons by perimeter will be considered next. Recall that a staircase polyomino is North-East directed and is also convex. With these properties in mind staircase polyominoes may be constructed recursively as illustrated by the following figure.



[11]

Figure 13: Constructing staircase polygons recursively.

The set of staircase polyominoes can be partitioned into 2 sets. The one set contains staircase polyominoes whose width remains the same when their bottom most cells are deleted, and will be called thick staircase polygons. The second set contains all staircase polygons that are not thick.

Keeping with the algebraic construction of the generating functions, let $S(x,y)$ be the anisotropic perimeter generating function enumerating staircase polygons, where the coefficient of $x^n y^m$ enumerate staircase polygons with horizontal perimeter $2n$ and vertical perimeter $2m$.

Thick staircase polygons may be formed from the set of all staircase polygons by simply duplicating each of the cells at the bottom of every column. The resulting generating function is given by

$$yS(x,y).$$

The vertical perimeter is increased by 1 as a result of the left most duplicate cell and increased by 1 as a result of the right most duplicated cell, hence the factor of y.

Consider now the second set containing staircase polygons that are not thick. The single cell is contained in this set and has generating function xy. The rest of this set consists of staircase polygons having at least two columns, and can be split into two types:

1. Staircase polygons whose first column is a single cell, which may be formed from any staircase polygon by adding a single cell to their left, and thus the set has generating function

$$xS(x,y),$$

2. and staircase polygons that consist of a thick polygon concatenated to a non-thick polygon by two columns overlapping at one cell only, and thus the set has generating function

$$yS(x,y)\,y^{-1}S(x,y) = S(x,y)^2,$$

where the factor of y^{-1} is a result of removing a vertical edge of length 1 from the perimeter of the thick polgyon and the non-thick polygon at the overlap.

The generating functions for these sets are added together to obtain the expression for the anisotropic perimeter generating function of staircase polygons as

$$S(x,y) = yS(x,y) + xy + xS(x,y) + S(x,y)^2.$$

Solving for $S(x,y)$ gives

$$\begin{aligned} S(x,y) =& \frac{1}{2}\left(1 - x - y - \sqrt{1 - 2x - 2y - 2xy + x^2 + y^2}\right) \\ =& \sum_{p,q\geq 1} \frac{1}{p+q-1}\binom{p+q-1}{p}\binom{p+q-1}{q} x^p y^q. \end{aligned}$$

This expression is found by the Lagrange Inversion Formula, and its full solution is shown in [10]. To obtain the isotropic semi-perimeter generating function let $x = y = t$ as in previous examples to obtain

$$S(t,t) = \frac{1}{2}\left(1 - 2t - \sqrt{1-4t}\right) = \sum_{n\geq 1} C_n t^{n+1},$$

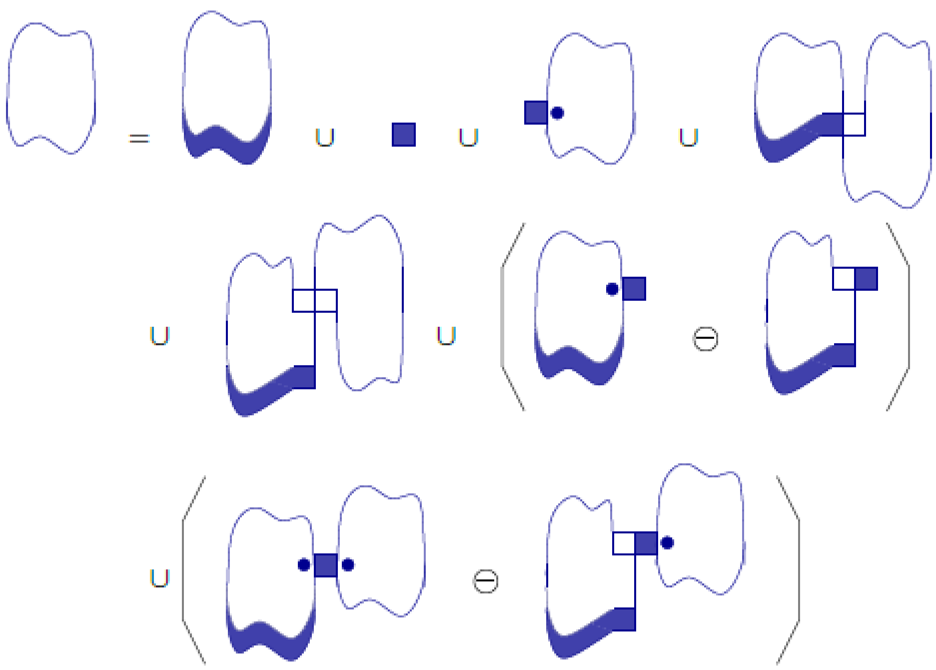
where $C_n = \binom{2n}{n}/(n+1)$ represents the n^{th} Catalan number and the coefficients of t^n enumerate staircase polygons having semi-perimeter n. In [10, 24] the same method is demonstrated for the directed-convex polygons, and general convex polygons.

2.5.2 Enumerating the set of column-convex polygons by perimeter

For the next example column-convex polygons will be enumerated by perimeter. Denote the set of column-convex polygons by C, and let $C(x,y)$ be the anisotropic perimeter generating function enumerating these polygons.

The partition of C into two subsets will be used in the recursive construction of C. Let C_1 denote the set of all column-convex polygons that have a cell marked in the last column, and have generating function $C_1(x,y)$. By the symmetry of column-convex polygons the generating function for the set C_1 counts those column-convex polygons which have a cell in the first column that is marked.

Let C_2 be the set of all column-convex polygons having a cell marked in the first, and in the last column uniquely, and having generating function $C_2(x,y)$. Using C_1 and C_2 the recursive construction of C is shown in the next figure.



[11]

Figure 14: The symbolic decomposition illustrating the way in which column-convex polgyons are constructed recursively.

The concept of thickness will again be introduced for column-convex polygons where a column-convex polygon is said to be thick if its width does not change when the cells at the bottom row of each column are eliminated.

Thick column-convex polygons may be constructed from any column-convex polygons by simply adding cells at the bottom of each column, duplicating the bottom row, and thus have generating function

$$\sqrt{y}\sqrt{y}C(x,y) = yC(x,y),$$

where the factor of y is a result of increasing the polygon's vertical perimeter by 1 on either side of the duplicated row.

Finding the perimeter generating function of column-convex polygons that are not-thick, we must first consider the disjoint subsets forming the set of these polygons. The single cell is a subset of this set and has generating function xy. Column-convex polygons that are not-thick but have at least two columns compromise the rest of this set. The subset of these column-convex polygons of at least two columns may again be divided into disjoint subsets, based on the criterion of the position of the first two columns from left that overlap by only a single edge. The first such set of these column-convex polygons are those consisting of column-convex polygons whose first column has height 1, and may be formed from any column-convex polygon having a marked cell in its first column, by simply adding a single cell to the left of the column, adjacent to the cell which is marked. It thus has perimeter generating function $xC_1(x,y)$. The factor of x is a result of adding a cell to the left which increases the horizontal perimeter by 2.

The second subset comprises of column-convex polygons that consist of a thick column-convex polygon appended to a general column-convex polygon so that the two polygons are joined by a single cell only.

Again this set will be separated into smaller disjoint subsets. The polygons in this set may be constructed in a number of different ways, depending on where the thick and general column-convex polygons are joined. The bottom rightmost cell of the thick column-convex polygon may be joined to the top leftmost cell of the general column-convex polygon. The set of these polygons will thus have generating function

$$yC(x,y)\,y^{-1}C(x,y) = C(x,y)^2,$$

as two edges are removed from the vertical perimeter where the polygons overlap we have a factor of y^{-1}.

The polygons may overlap at the top rightmost cell of the thick column-convex polygon and the bottom leftmost cell of the general column-convex polygon giving a set with perimeter generating function

$$C(x,y)^2,$$

by identical reasoning as for the previous set.

If the leftmost column of the general column-convex polygon which overlaps with the rightmost column of the thick column-convex polygon has height 1 then two cases may occur:

1. The general polygon only consists of this column of height 1 and the generating function for the polygons constructed from the thick polygon and the general polygon which is obtained by adding a single cell to the right of a marked cell in the thick column-convex polygon is

   $$xyC_1(x,y).$$

   It should be noted that this generating function also enumerates the cases where the top of the leftmost cell of the thick polygon is added to the bottom of the rightmost cell of a polygon consisting of a cell which is already counted by the generating function $C^2(x,y)$. Thus when considering the perimeter generating function for all column-convex polygons, the case must be subtracted, as

   $$xyC_1(x,y) - xyC(x,y).$$

2. The second set contains the polygons where the column of the general polygon which joins to the right of the thick polygon has height 1 and where the general polygon has at least two columns. It may be obtained by joining thick polygons at the marked cell on their left

to a general polygon at a marked cell on its right, by a single cell only. This set therefore has generating function

$$yC_1(x,y)\,xy^{-1}C_1(x,y)$$

where the xy^{-1} term is a result of the joining cell that adds two horizontal edges and removes two vertical edges. This generating function, similar to the previous one, counts those column-convex polygons for which the top cell of the rightmost column of a thick column-convex polygon overlaps with the bottom cell of the left most column, a column of height 1, of a general polygon which is also counted by the generating function $C^2(x,y)$ thus considering the perimeter generating function for all column-convex polygons this case must be subtracted giving the generating function

$$yC_1(x,y)\,xy^{-1}C_1(x,y) - yC(x,y)\,xy^{-1}C_1(x,y) = x\left(C_1(x,y)^2 - C(x,y)\,C_1(x,y)\right).$$

The generating function for each of these subsets may now be added together, with the particular double counted parts subtracted, to obtain an expression for the perimeter generating function of column-convex polygons given by

$$C = yC + xy + xC_1 + 2C^2 + xy(C_1 - C) + x\left(C_1^2 - CC_1\right),$$

where $C := C(x,y)$, $C_1 := C_1(x,y)$ and $C_2 := C_2(x,y)$. C_1 may now be constructed in a similar way.

C_1 may now again be partitioned into two sets, one containing thick polygons, and the other polygons that are not thick. Thick polygons of the set C_1 may be formed in a number of ways:

1. They may be formed by simply duplicating the bottom row of a polygon already from C_1,
2. or they may be formed from a general polygon by duplicating the bottom row and marking the rightmost cell in that row.

This set thus has generating function,

$$yC + yC_1 = y(C + C_1).$$

The set of non-thick polygons of type C_1 is more complex to decompose. The polygons in this set may be formed as follows:

1. They may occur as the single cell that is marked, with generating function

$$xy.$$

2. They may be formed by attaching a single cell to the left of a polygon that has a marked cell in its leftmost column, next to the marked cell, and then marking a cell in the rightmost column so that the polygon to which the cell is attached has generating function C_2. This created polygon thus has generating function,

$$xC_2.$$

3. They can be formed by joining a thick polygon to the left of a general polygon so that the bottom rightmost cell of the left polygon is joined to the top leftmost cell of the general polygon by a single edge, or such that the top rightmost cell of the thick polygon is joined to the bottom leftmost cell of the general polygon by a single edge. For both cases a cell in the last column is marked. These polygons thus have generating function

$$CC_1 + CC_1 = 2CC_1$$

4. These polygons may also be formed by attaching a single cell to the right of a thick polygon of type C_1, next to the marked cell. Their generating function is given by

$$xyC_1.$$

However this generating function counts those polygons formed by joining a thick polygon to the left of a single cell polygon such that the thick polygon is joined at its top rightmost cell to the single cell, which is already counted by the generating function $2CC_1$. So when considering the generating function for all column-convex polygons the generating function for this case must be subtracted, which is xyC, giving

$$xyC_1 - xyC = xy\left(C_1 - C\right).$$

5. Lastly, non-thick column-convex polygons of type C_1 may be formed by joining a thick polygon of type C_1 at its marked cell to the left of a single cell and joining the single cell to the left of the marked cell of a polygon of type C_2, which itself has a marked cell in its rightmost column. Polygons formed in this way thus have generating function

$$xC_1C_2.$$

This generating function counts those polygons formed by joining a thick polygon of type C_1 to the left of a polygon whose leftmost column is a single cell, which is again also counted by the generating function $2C_1C$ thus when considering the generating function for all column-convex polygons the generating function for this case must be subtracted, that is xCC_2, giving

$$xC_1C_2 - xCC_2 = x\left(C_1C_2 - CC_2\right).$$

Two equations are now obtained, for C and for C_1. An equation for C_2 will now be obtained using Figure 13. Again, polygons of type C_2 will be partitioned into two sets based on whether they are thick or not. Thick polygons of type C_2 may be formed in 4 ways:

1. They may be formed from a polygon whose leftmost row has a marked cell, counted by C_1, by duplicating the bottom row and marking the rightmost cell in this row, having generating function

$$yC_1.$$

2. Similarly, they may be formed from polygons of type C_1 by duplicating the bottom row and marking the leftmost cell in the row, having generating function

$$yC_1.$$

3. They can be formed from general polygons by duplicating the bottom row and marking the leftmost and the rightmost cells of the duplicated row giving generating function

$$yC.$$

4. Or they may simply be formed by duplicating the bottom row of polygon of type C_2, giving generating function

$$yC_2.$$

Altogether thick column-convex polygons of type C_2 have generating function

$$y\left(C + 2C_1 + C_2\right).$$

Column-convex polygons that are not thick and are of type C_2 will now be broken into a number of different sets:

1. They may occur trivially as the single cell with generating function

$$xy.$$

2. These polygons may be obtained by adding a single cell to the left of the cell marked in the left column of polygons of type C_2, thus having generating function

$$xC_2.$$

3. These polygons may be formed by joining a thick polygon that has a marked cell in its leftmost column to the left of a polygon of type C_2, such that the bottom rightmost cell of the thick polygon is joined to the top leftmost cell of the polygon to which it is joined. Now the thick polygon may be formed in two ways, by duplicating the bottom row of a polygon that has a marked cell in its leftmost column, or by duplicating the bottom row of a column-convex polygon and marking the leftmost cell in the duplicated row, thus giving the generating function

$$CC_1 + C_1C_1.$$

4. Using the same steps as in 3. but joining the top rightmost cell of the thick polygon to the bottom leftmost cell of the general polygon instead gives column-convex polygons which are not thick and are of type C_2 with identical generating function,

$$\left(C + C_1\right)C_1.$$

5. A polygon of type C_2 that is not thick may be formed by adding a single cell to the right of a marked cell in the rightmost column of a thick polygon of type C_2. Now this thick polygon may be obtained in two ways: by duplicating the bottom row of a polygon of type C_2 or by duplicating the bottom row of a polygon of type C_1 and marking the leftmost cell in the duplicated row. Polygons created in this way thus have generating function

$$xyC_1 + xyC_2.$$

The generating function xyC_1 counts polygons formed by joining a cell to the right of the top rightmost column of a thick column-convex polygon and this is also counted by

$2\left(C+C_1\right)C_1$. Thus considering the generating function for all column-convex polygons the generating function for this case xyC must be subtracted. Similarly xyC_2 counts the polygons created when a single cell is added to the right of the top rightmost polygon of a thick polygon of type C_1, which is counted by $2\left(C+C_1\right)C_1$, then by identical reasoning the generating function for this case xyC_1 must also be subtracted. This altogether gives,

$$xy\left(\left(C_1-C\right)+\left(C_2-C_1\right)\right).$$

6. Lastly, column-convex polygons that are not thick and are of type C_2 may be formed by joining a single cell to the right of the marked cell in the rightmost column of a thick column-convex polygon of type C_2, and joining to the right of the single cell a column-convex polygon of type C_2 at the marked cell in its leftmost column. As with the case in 5., the thick column-convex polygon may be formed in 2 ways, and in the same two ways, giving generating function

$$xyC_2y^{-1}C_2+xyC_1y^{-1}C_2=x\left(C_2^2+C_1C_2\right).$$

However xC_2^2 counts those polygons formed by joining the top rightmost cell of a thick column-convex polygon of the type C_2 to the bottom leftmost cell of a polygon of type C_2, which is already counted by $2\left(C+C_1\right)C_1$, and xC_1C_2 counts those polygons formed by joining the top rightmost cell of a thick column-convex polygon of type C_1 to the bottom leftmost cell of a polygon of type C_2, which is also counted by $2\left(C+C_1\right)C_1$. Thus considering the generating function for all column-convex polygons, these two cases must be subtracted as $-\left(xCC_2+xC_1C_2\right)$, giving

$$x\left(\left(C_1C_2-CC_2\right)+\left(C_2^2-C_1C_2\right)\right).$$

An expression for the generating function C_2 may now be obtained as the sum of all these generating functions, for the thick and non-thick cases, as

$$\begin{aligned}C_2=&y\left(C+2C_1+C_2\right)+xy+xC_2+2\left(C+C_1\right)C_1+xy\left(\left(C_1-C\right)+\left(C_2-C_1\right)\right)\\&+x\left(\left(C_1C_2-CC_2\right)+\left(C_2^2-C_1C_2\right)\right).\end{aligned}$$

This gives 3 equations in 3 unknown generating functions that can be solved to give

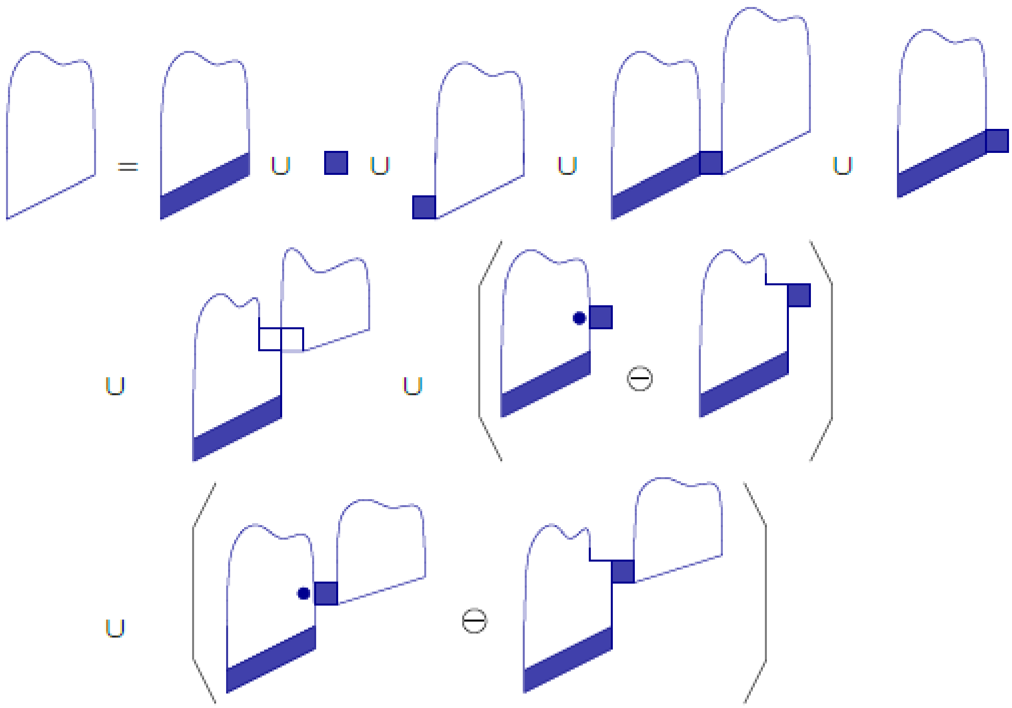
$$\begin{aligned}&\left(-5xy-18+2xy^2-18y^2+36y+2x\right)C^4+(y-1)\left(5xy^2-21y^2+42y-14xy+5x-21\right)C^3\\&+2\left(y-1\right)^2\left(-4y^2+2xy^2+8y-7xy-4+2x\right)C^2+\left(y-1\right)^3\left(xy^2-y^2+2y-6xy+x-1\right)C\\&\qquad -xy\left(y-1\right)^4=0\end{aligned}$$

which is a quartic equation in terms of C. Solving for C thus gives 4 roots, one of which is the generating function for column-convex polygons. This is found to be,

$$C\left(x,y\right)=\left(1-y\right)\left(1-\frac{2\sqrt{2}}{3\sqrt{2}-\sqrt{1+x+\sqrt{\left(1-x\right)^2-16\frac{xy}{(1-y)^2}}}}\right).$$

2.5.3 Enumerating directed column-convex polygons by their perimeter

A construction very similar to that shown in Figure 13 will be used to obtain the construction of the set of directed column-convex polygons. As the set of directed column-convex polygons is a subset of the set of column-convex polygons the construction of the set of directed column-convex polygons is almost identical, with one added restriction - for every pair of adjacent columns, the right column may not be lower then the left, preserving directedness. The construction of the set of directed column-convex polygons may be illustrated by the following figure.



[11]

Figure 15: The symbolic decomposition of directed column-convex polygons, illustrating the way in which their set is constructed, recursively.

Note that in each case, when two directed column-convex polygons are joined, at the joining columns the right column's bottommost cell is at the same level or higher than the left column's bottommost cell. This figure then translates to the generating function,

$$D = yD + xy + xD + xD^2 + xyD + D^2 + xy\,(D_1 - D) + x\,(D_1 - D)\,D.$$

As in the previous example an expression for the generating function D_1, counting all the directed column-convex polygons with a marked cell in the last column, will be obtained. The only objects that change in the construction of this set are the first, and third to sixth. The single cell with generating function xy stays the same, and the last two sets already contain only directed column-convex polygons of type D_1.

The sets that change will now be described. Thick directed column-convex polygons of type D_1 may be formed in two ways:

1. By duplicating the bottom row of a directed column-convex polygon of type D_1, and marking the rightmost cell in the duplicated row,
2. or by duplicating the bottom row of a directed column-convex polygon that is already of type D_1.

The set of thick directed column-convex polygons thus has generating function

$$yD + yD_1 = y\left(D + D_1\right).$$

Considering now the remaining four sets,

1. Counting directed column-convex polygons of type D_1, the set of polygons counted in the third set represented in Figure 14 are formed by joining the single cell to the bottom left of a directed column-convex polygons of type D_1, now having generating function

$$xD_1.$$

2. Considering the set represented by the fourth term in Figure 14, the general polygon added on the right becomes a polygon of type D_1, so the generating function for the set is now

$$xDD_1.$$

3. Using the fifth term in Figure 14 to count directed column-convex polygons of type D_1, the cell added to the right is marked, so the generating function is essentially unchanged, and remains

$$xyD.$$

4. The sixth term can be used to count directed column-convex polygons of type D_1, by restricting the general directed column-convex polygon on the right to be of type D_1, it changes the generating function for the set to be

$$DD_1.$$

Combining all the cases, together with those that don't change, where C is simply replaced by D, gives an expression for the generating function of directed column-convex polygons of type D_1, as

$$D_1 = y\left(D + D_1\right) + xy + xD_1 + xDD_1 + xyD + DD_1 + xy\left(D_1 - D\right) + x\left(D_1 - D\right)D_1.$$

We now have two equations in terms of two unknown generating functions. Solving for D gives,

$$D^3 + 2\left(y-1\right)D^2 + \left(y-1\right)\left(x+y-1\right)D + xy\left(y-1\right) = 0$$

a cubic equation with 3 roots. The correct root is found to be the only root that yields positive coefficients,

$$D = -\frac{2\left(y-1\right)}{3} + \frac{\left(1+i\sqrt{3}\right)\left(2y+3xy-y^2-3x-1\right)}{3\sqrt[3]{4\left(M+N\right)}} - \frac{\left(1-i\sqrt{3}\right)\sqrt[3]{M+N}}{6\sqrt[3]{2}}$$

where

$$\begin{aligned} M =& 18x + 6y - 9xy - 6y^2 - 9xy^2 + 2y^3 - 2, \text{ and} \\ N =& \sqrt{4\left(2y+3xy-y^2-3x-1\right)^3 + \left(18x+6y-9xy-6y^2-9xy^2+2y^3-2\right)^2}. \end{aligned}$$

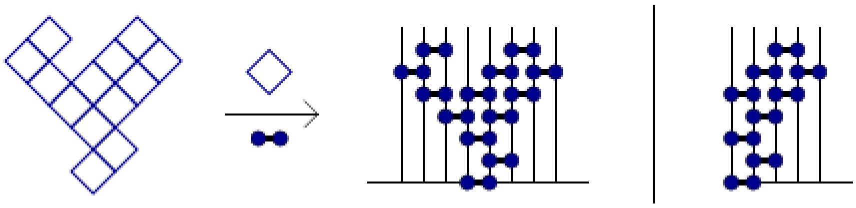
(See Appendix 3.5 for the solution.)

2.5.4 Enumerating directed polyominoes by their area

The next set of polyominoes that will be counted are directed polyominoes, and they will be counted by their area. Their structure is algebraic, and was discovered to be so by Viennot in [48] as he was developing the concept of a *heap*.

A heap may be defined as a number of objects that are placed vertically in a 'heap', where each object is at least partially on top of another. A minimal object in a heap is the object that lies the lowest in the heap.

Now considering the set being enumerated, directed polyominoes may be transformed into heaps. To do this, rotate any directed polyomino 45 degrees, and replace each cell by an object called a *dimer*, which is simply the left and right vertex of the cell, after rotation, joined by a line. This is illustrated by the next figure.

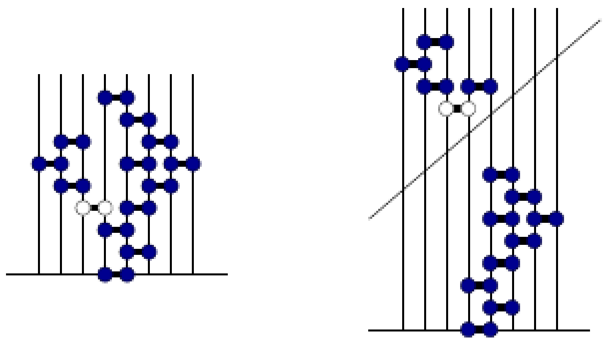


[11]

Figure 16: Applying the aforementioned process to a directed polygon to obtain a stack is shown on the left of the figure. A half-pyramid is shown on the right.

These heaps each have a minimal piece which is unique. Every heap that is obtained applying the aforementioned process to directed polyominoes will be referred to as a *pyramid*. A half-pyramid will be such a heap where there are no dimers to the left of the minimal piece.

The tool which will allow for the algebraic description of these heaps is the *product* of heaps. The product of heaps is defined to be the operation whereby one heap is aligned so that it's rightmost dimer is over another heap's leftmost dimer and is then 'dropped' onto it, and its dimers are allowed to 'fall' into place on top of the heap onto which it is dropped. The next figure illustrates this. Note that the rightmost dimer of the left heap 'falls' onto the top dimer of the right heap, and thereby becomes the top dimer of the product of the two heaps.



[11]

Figure 17: The pyramid on the left is a product of a pyramid and a half-pyramid, as illustrated by the diagram on the right.

The set of these pyramids may be partitioned into two sets: the set of half-pyramids, and its complement whose elements are formed as products of a pyramid and a half pyramid, as shown in Figure 17. Denote by $D(q)$ the generating function for the set of pyramids where the coefficients of q^n enumerate the pyramids having n dimers.

Denote similarly by $H(q)$ the generating function of the set of half-pyramids. By the partition of the set of pyramids as explained, the following expression is obtained,

$$\begin{aligned} D(q) =& H(q) + H(q) D(q) \\ =& H(q)(1 + D(q)). \end{aligned}$$

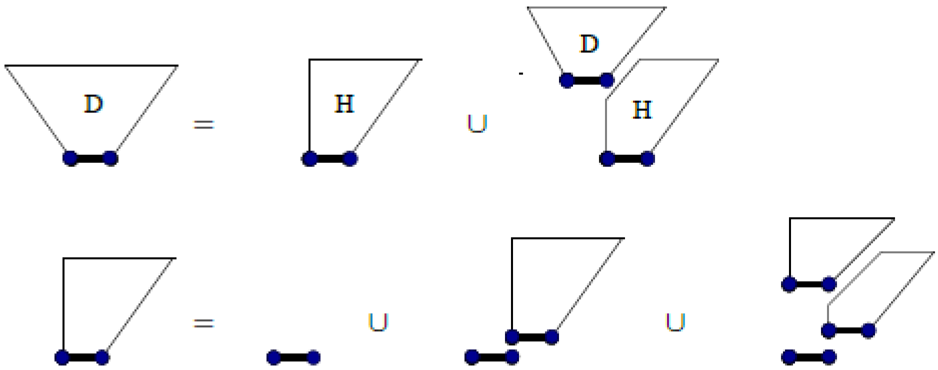
Decomposing the set of half-pyramids, a half-pyramid may be only the single dimer, with generating function q, the product of one half-pyramid and a single dimer, with generating function

$$qH(q)$$

or the product of two half-pyramids and a single dimer with generating function

$$qH(q) H(q) = qH(q)^2.$$

All three cases are illustrated in the following figure.



[11]

Figure 18: Partitioning pyramids into 3 sets.

These three cases combine to give

$$H(q) = q + qH(q) + qH^2(q).$$

As a dimer represents a cell, it represents a part of a polyomino having area 1, thus counting pyramids by dimers is identical to enumerating directed polyominoes by area, and thus the generating functions are the same, so $D(q)$ is also the area generating function of directed polyominoes. Using these two equations to solve for $D(q)$ gives

$$D(q) = \frac{1}{2}\left(\sqrt{\frac{1+q}{1-3q}} - 1\right).$$

2.5.5 Obtaining q-analogues from the already obtained generating functions

A q-analogue will now be found for bargraphs to obtain their complete generating function. It has already been shown that the perimeter generating function of bargraphs is algebraic. Figure 11 will be used to find the q-analogue.

Joining one bargraph to another results in a bargraph whose area is the sum of the areas of the two bargraphs. Duplicating the bottom row of a bargraph increases its area by the length of the bottom row, which is the same as its horizontal half-perimeter which is enumerated by the coefficients of x^n in the generating function. Thus the complete generating function of bargraphs may be obtained from Figure 11 as

$$B(x,y,q) = yB(xq,y,q) + xyq + xqB(x,y,q) + xyqB(xq,y,q) + xqB(xq,y,q)\,B(x,y,q)\,,$$

a q-analogue of $B(x,y,1)$.

This expression for $B(x,y,q)$ is not algebraic because of the $B(xq,y,q)$ terms. The next and final section will illustrate a technique to solve such an equation, and the expression will be solved using it.

3 Temperley's approach: adding an extra layer

The third method, known as Temperley's Approach, will be explained and illustrated through three examples. It was used by Temperley in [46] to obtain a functional equation for generating functions enumerating column-convex polygons by perimeter, and thus became known as Temperley's Approach. It had however been used in a more complex way, as can be seen in [7, 38], before Temperley.

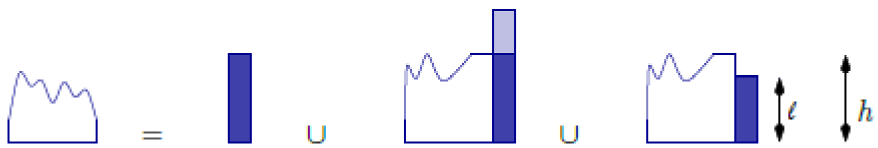
3.1 Enumerating bargraphs by their perimeter and area

The first example illustrating this method will be to find a functional equation for the complete generating function of bargraphs, $B(x,y,q)$, which was left for this section at the end of the last.

Along with area and perimeter, the height of the rightmost column of a bargraph will now also be considered when enumerating bargraphs. Considering this new parameter the generating function for bargraphs now becomes

$$B(x,y,q,s) = \sum_{h\geq 1} B_h(x,y,q)\,s^h,$$

where $B_h(x,y,q)$ is the complete generating function of bargraphs whose rightmost column has height h, and therefore the coefficients of s^h enumerate all bargraphs whose rightmost column has height h. Constructing the set of bargraphs in this way, with the rightmost column height parameter, is illustrated by the next figure.



[11]

Figure 19: Constructing bargraphs recursively, this time by the Temperley approach of adding a slice.

Denote the class of bargraphs by B. This class may now be partitioned into 3 disjoint subsets with regard to this new parameter:

1. The first set contains bargraphs consisting only of a single column, and thus has generating function

$$\sum_{n\geq 1} x\left(yqs\right)^n = xysq \sum_{n\geq 0}\left(ysq\right)^n$$

the factor of x is a result of the width being constant, and as it is a single column the column height is equal to the vertical half-perimeter and the column's area, thus the $\left(yqs\right)^h$ term.

2. The second set contains bargraphs where the rightmost column is not lower than the column to the left of it. These bargraphs are formed by duplicating the last row of a bargraph, and possibly adding a column on top of this duplicated column. The generating function for bargraphs with the duplicated rightmost column is thus given by

$$xB\left(x, y, q, sq\right)$$

as the area is increased by the height of the duplicated column, and the horizontal perimeter is increased by 1. The generating function for a column attached to the polygon whose height may possibly be 0 is given by

$$\sum_{n\geq 0}\left(ysq\right)^n = \frac{1}{1-ysq}.$$

Combining these 2 generating function gives the series

$$\frac{x}{1-ysq} B\left(x, y, q, sq\right),$$

which is the generating function for this set.

3. The third set contains bargraphs whose second last column is taller than its last column. The generating function enumerating bargraphs whose second last column has height h, and whose last column has height less than h is given by

$$xB_h\left(x, y, q\right)\sum_{l=1}^{h-1}\left(sq\right)^l,$$

that is, the bargraph whose rightmost column has height h, attached to the left of a column whose height is less than h. Now the generating function enumerating bargraphs for all heights h is given by

$$x\sum_{h\geq 1} B_h\left(x, y, q\right)\sum_{l=1}^{h-1}\left(sq\right)^l,$$

which is the generating function for the third set.

Using the notation $B\left(s\right) := B\left(x, y, q, s\right)$ the generating functions for the three sets into which the set of bargraphs is partitioned may now be added to obtain the complete generating function for the set of bargraphs

$$B\left(s\right) = \frac{xysq}{1-ysq} + \frac{xsq}{1-sq}B\left(1\right) + \frac{xsq\left(y-1\right)}{\left(1-sq\right)\left(1-ysq\right)}B\left(sq\right).$$

Iterating this equation gives,

$$\begin{aligned}
B(s) =& \frac{xysq}{1-ysq} + \frac{xsq}{1-sq}B(1) + \frac{xsq(y-1)}{(1-sq)(1-ysq)}\frac{xysq^2}{1-ysq^2} + \frac{xsq(y-1)}{(1-sq)(1-ysq)}\frac{xsq^2}{1-sq^2}B(1) \\
&+ \frac{xsq(y-1)}{(1-sq)(1-ysq)}\frac{xsq^2(y-1)}{(1-sq^2)(1-ysq^2)}B\left(sq^2\right) \\
=& \frac{xysq}{1-ysq} + \frac{xsq}{1-sq}B(1) + \frac{xsq(y-1)}{(1-sq)(1-ysq)}\frac{xysq^2}{1-ysq^2} \\
&+ \frac{xsq(y-1)}{(1-sq)(1-ysq)}\frac{xsq^2}{1-sq^2}B(1) + \frac{xsq(y-1)}{(1-sq)(1-ysq)}\frac{xsq^2(y-1)}{(1-sq^2)(1-ysq^2)}\frac{xysq^3}{1-ysq^3} \\
&+ \frac{xsq(y-1)}{(1-sq)(1-ysq)}\frac{xsq^2(y-1)}{(1-sq^2)(1-ysq^2)}\frac{xsq^3}{1-sq^3}B(1) \\
&+ \frac{xsq(y-1)}{(1-sq)(1-ysq)}\frac{xsq^2(y-1)}{(1-sq^2)(1-ysq^2)}\frac{xsq^3(y-1)}{(1-sq^3)(1-ysq^3)}B\left(sq^3\right) \\
=& \frac{xysq}{1-syq} + \frac{x^2s^2q^{1+2}(y-1)}{(1-sq)(1-ysq)(1-ysq^2)} + \frac{x^3s^3yq^{1+2+3}(y-1)^2}{(1-sq)(1-sq^2)(1-ysq)(1-ysq^2)(1-ysq^3)} \\
&+ \frac{xyq}{1-sq}B(1) + \frac{x^2s^2q^{1+2}(y-1)}{(1-sq)(1-sq^2)(1-ysq)}B(1) \\
&+ \frac{x^3s^3q^{1+2+3}(y-1)^2}{(1-sq)(1-sq^2)(1-sq^3)(1-ysq)(1-ysq^2)}B(1) + \dots \\
=& \sum_{n\geq 1} \frac{(xs(y-1))^{n-1}q^{\binom{n}{2}}}{(sq)_{n-1}(ysq)_{n-1}}\left(\frac{xysq^n}{1-ysq^n} + \frac{xsq^n}{1-sq^n}B(1)\right).
\end{aligned}$$

Now let $s=1$, note that $B(1) = B(x,y,q,1)$ which is the complete generating function of bargraphs. The above equation may be solved to give

$$B(x,y,q) = \frac{I_+}{1-I_-},$$

where

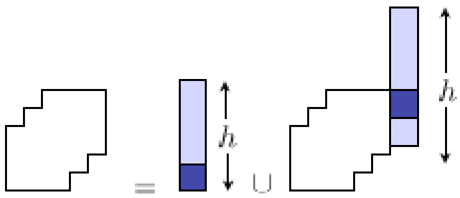
$$I_+ = \sum_{n\geq 1} \frac{x^n (y-1)^{n-1} q^{\binom{n+1}{2}}}{(q)_{n-1}(yq)_n} \text{ and, } I_- = \sum_{n\geq 1} \frac{x^n (y-1)^{n-1} q^{\binom{n+1}{2}}}{(q)_n (yq)_{n-1}}.$$

3.2 Enumerating staircase polygons

The next set of polyominoes that will be enumerated by the Temperley Approach will be that of staircase polygons. Similarly to the previous example the height of the rightmost column will again be considered with regard to the generating function for staircase polygons,

$$S(x,y,q,s) = \sum_{h\geq 1} S_h(x,y,q)\, s^h,$$

being analogous to $H(x,y,q,s)$. The set of staircase polygons may be partitioned into 2 disjoint subsets as illustrated by the following figure.



[11]

Figure 20: the set of staircase polygons partitioned into the 2 described subsets.

1. The first subset will contain those staircase polygons consisting of a single column only, and thus has generating function

$$\frac{xyqs}{1-yqs}.$$

2. The next set contains staircase polygons which consist of two or more columns. The polygons in this set may be decomposed as follows: they can be formed by joining a staircase polygon to the left of a column. The generating function for the subset of these staircase polygons is given by

$$S\left(x,y,q,1\right)$$

as the height of the rightmost column does not play a role yet, when considering the whole polygon. The added column however may not be lower than the rightmost column of the staircase polygon to which it is attached, as the resulting polygon must still be a staircase polygon. This column consists of a single cell, ensuring it is not empty, with generating function $xyqs$, which is attached to the right of the cell at the top of the rightmost column of the staircase-polygon. At the top of this cell is a possibly empty column with generating function

$$\sum_{n\geq 0}\left(yqs\right)^{n}=\frac{1}{1-yqs},$$

and at the bottom is another possibly empty column with generating function

$$\sum_{n\geq 0}\left(qs\right)^{n}=\frac{1}{1-qs},$$

with no y^n term as it does not increase the vertical half-perimeter where it is attached.

Combining these generating functions gives

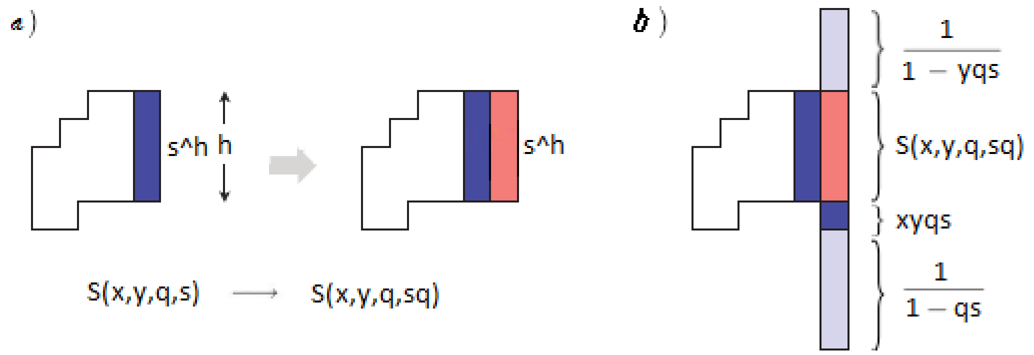
$$\frac{xyqs}{1-yqs}+S\left(x,y,q,1\right)\cdot xyqs\cdot\frac{1}{1-qs}\cdot\frac{1}{1-yqs}.$$

This series counts those cases where the bottom column attached to the cell in the last column of the polygon extends below the second last column of the polygon. The case where this happens must thus be subtracted, as these are not staircase polygons.

The polygons for which this happens can be formed in the following way:

1. Consider a staircase polygon, it has generating function
$$S\left(x,y,q,s\right).$$
Duplicating the last column, the area of the new staircase polygon is increased by the length of the duplicated last column, and the horizontal perimeter is increased by 1. These polygons therefore have generating function
$$xS\left(x,y,q,sq\right).$$
2. Attach to the top of the duplicated column a possibly empty column having generating function
$$\sum_{n\geq 0}\left(yqs\right)^{n}=\frac{1}{1-yqs}.$$
3. Attach to the bottom of the duplicated row a single cell, with generating function yqs, to ensure that the last column extends below the second last column.
4. Finally, attach below the cell added in 3. a possibly empty column having generating function
$$\sum_{n\geq 0}\left(qs\right)^{n}=\frac{1}{1-qs}$$
where the absence of a y^n term is a result of the column not adding to the vertical perimeter as a result of where it is attached to the staircase polygon.

Thus the generating function for the case where the last column extends below the second last column is given by
$$S\left(x,y,q,sq\right)\cdot xyqs\cdot\frac{1}{1-qs}\cdot\frac{1}{1-yqs},$$
and its construction is illustrated by the following figure.



[11]

Figure 21: The process of duplicating the last column is shown in (a). Case (b) shows the construction of the generating function for the unwanted cases.

This generating function may now be subtracted from the first generating function to obtain the series

$$S(x,y,q,s) = \frac{xyqs}{1-yqs} + S(x,y,q,1)\cdot xyqs\cdot\frac{1}{1-qs}\cdot\frac{1}{1-yqs} - S(x,y,q,sq)\cdot xyqs\cdot\frac{1}{1-qs}\cdot\frac{1}{1-yqs}$$

which is the generating function for staircase polygons. Let $S(1) \equiv S(x,y,q,1)$. By iteration, where $S(x,y,q,s) \equiv S(s)$, it can be found that,

$$\begin{aligned}
S(s) =& \frac{xyqs}{1-yqs} + S(1)\frac{xyqs}{(1-qs)(1-yqs)} \\
& - \left(\frac{xyq^2s}{1-yq^2s} + \left(S(1) - S\left(sq^2\right)\right)\frac{xyq^2s}{(1-q^2s)(1-y^2qs)}\right)\frac{xyqs}{(1-qs)(1-yqs)} \\
=& \frac{xyqs}{1-yqs} - \frac{x^2y^2q^3s^2}{(1-qs)(1-yqs)(1-yq^2s)} + S(1)\frac{x^2y^2q^3s^2}{(1-qs)(1-q^2s)(1-yqs)(1-y^2qs)} \\
& + S\left(sq^2\right)\frac{xyq^2s}{(1-q^2s)(1+yq^2s)} \\
& = \dots \\
& = \sum_{n\geq 1}(-1)^{n+1}\frac{x^ny^nq^{\binom{n+1}{2}}s^n}{(q)_{n-1}((ys)q)_n} - S(1)\sum_{n\geq 1}(-1)^n\frac{x^ny^nq^{\binom{n+1}{2}}s^n}{(q)_n((ys)q)_n}
\end{aligned}$$

and setting $s = 1$, the expression may be solved for $S(1)$, to obtain the complete generating function for staircase polygons,

$$S(x,y,q,1) = y\frac{J_1}{J_0}$$

where J_1 and J_0 are two q-Bessel functions,

$$J_1(x,y,q) = \sum_{k\geq 1}(-1)^{k+1}\frac{x^kq^{\binom{k+1}{2}}}{(q)_{k-1}(yq)_k} \text{ and } J_0(x,y,q) = \sum_{k\geq 0}(-1)^k\frac{x^kq^{\binom{k+1}{2}}}{(q)_k(yq)_k}.$$

3.3 Enumerating column-convex polygons

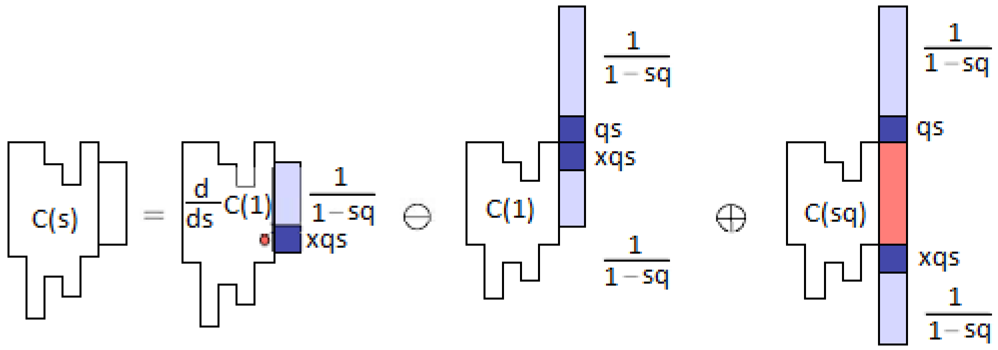
The complete generating function for column-convex polygons will now be obtained. To do so the height of the last column of a column-convex polygon will be considered. Based on the criterion of the height of the last column, as in previous examples, the set of column-convex polygons may be partitioned into a number of subsets. One such subset is the subset containing column-convex polygons whose last column is neither higher nor lower than the second last column. This set may be called the *contained* set of column-convex polygons. The generating function for this set is given by,

$$\frac{xsq}{1-sq}\frac{\partial}{\partial d}C(1) - \frac{xs^2q^2}{(1-sq)^2}(C(1) - C(sq)).$$

For a polygon where the rightmost column has height h, there are h cells in the rightmost column that may be marked, thus

$$\frac{\partial C}{\partial s}(1) = h \cdot C(x, y, q, 1)$$

counts column-convex polygons that have a cell marked in their rightmost column. The generating function for the contained set will now be explained, and its structure will be illustrated by the following figure.



[11]

Figure 22: Illustrating symbolically the construction of the contained set: the subtraction of certain incorrect objects and the addition of unnecessarily subtracted parts.

Let C be a column-convex polygon. C may be formed by joining a column-convex polygon to the left of a column. Mark the cell in the rightmost column of the column-convex polygon that is next to the bottommost cell of the joined column. The joined column is attached and formed as follows:

1. A single cell is added to the right of the marked cell of the column-convex polygon, having generating function xqs, ensuring the column is not empty.
2. Above this added cell a possibly empty column is added having generating function

$$\sum_{n \geq 0} (sq)^n = \frac{1}{1 - sq}.$$

Combining these two cases gives

$$xqs \cdot \frac{1}{1 - sq} \cdot \frac{\partial}{\partial s} C(1).$$

This series counts those polygons whose last column exceeds the height of the second last column, which do not belong to the contained set. Therefore the generating function for these polygons must be subtracted from this generating function when finding the generating function for the contained set. Polygons from this unwanted case where the last column exceeds the height of the second last column are formed in the following way:

1. A single cell is attached to the right of the cell at the top of the rightmost column of a column-convex polygon. This cell has generating function xqs, and the column-convex polygon has generating function $C(1)$.

2. A possibly empty column is attached below the cell, and has generating function

$$\sum_{n\geq 0} (sq)^n = \frac{1}{1-sq}.$$

3. Another cell, having generating function qs, is attached above this cell to ensure that these polygons have the last column height greater than the second last column.
4. Above this second cell a possibly empty column is joined having generating function

$$\sum_{n\geq 0} (sq)^n = \frac{1}{1-sq}.$$

Altogether these generating functions may be combined to give

$$xq^2s^2C(1) \cdot \frac{1}{1-sq}\frac{1}{1-sq}.$$

However subtracting this from the series already obtained for the contained set of column-convex polygons, also subtracts those cases where the last column of the polygon extends above and below the second last column, which are not counted by the first obtained series. Thus these cases must be added again to correct this. The polygons to be added are formed as follows:

1. The last column of a column-convex polygon is duplicated, thus having generating function

$$C(sq).$$

2. A cell is added to the top of the duplicated column, with generating function qs, ensuring that the height of the last column of the polygon exceeds that of the second last column.
3. A possibly empty column is added to the top of the added cell, having generating function

$$\sum_{n\geq 0} (sq)^n = \frac{1}{1-sq}.$$

4. At the bottom of the duplicated column, another cell is added, with generating function xqs, ensuring that the last column extends below the second last column.
5. And below the cell added in 4. a possibly empty column is added with generating function

$$\sum_{n\geq 0} (sq)^n = \frac{1}{1-sq}.$$

These 5 parts are combined to get the expression

$$xq^2s^2 \cdot C(sq) \cdot \frac{1}{1-sq} \cdot \frac{1}{1-sq},$$

and the complete generating function for the contained set of column-convex polygons is thus obtained to be

$$\frac{xsq}{1-sq}\frac{\partial}{\partial s}C(1) - \frac{xs^2q^2}{(1-sq)^2}(C(1) - C(sq)).$$

Obtaining the generating functions for the other cases where the last column is above or below the second last column are obtained in ways similar to those used for staircase polygons. All these cases give the expression,

$$C(s) = \frac{xsyq}{1-syq} + \frac{xsq}{1-sq}\frac{\partial C}{\partial s}(1) + \frac{xs^2q^2(2y-syq-1)}{(1-sq)^2(1-syq)}C(1) + \frac{xs^2q^2(1-y)^2}{(1-sq)^2(1-syq)^2}C(sq).$$

Solving this expression requires a number of steps. First this equation is iterated to obtain an expression in terms of $C(1)$, and $C'(1)$, for $C(s)$.

$$\begin{aligned}
C(s) =& \frac{xsyq}{1-syq} + \frac{xsq}{1-sq}C'(s) + \frac{xs^2q^2(2y-syq-1)}{(1-sq)^2(1-syq)}C(1) + \frac{xs^2q^2(1-y)^2}{(1-sq)^2(1-syq)^2}C(sq) \\
=& \frac{xsyq}{1-syq} + \frac{xsq}{1-sq}C'(1) + \frac{xs^2q^2(2y-syq-1)}{(1-sq)^2(1-syq)}C(1) \\
&+ \frac{xs^2q^2(1-y)^2}{(1-sq)^2(1-syq)^2}\left(\frac{xsyq^2}{1-syq^2} + \frac{xsq^2}{1-sq^2}C'(1) + \frac{xs^2q^3\left(2y-syq^2-1\right)}{(1-sq^2)^2(1-syq^2)}C(1)\right. \\
&\left.+\frac{xs^2q^3(1-y)^2}{(1-sq^2)^2(1-syq^2)^2}C\left(sq^2\right)\right) \\
=& \frac{xsyq}{1-syq} + \frac{x^2s^3q^{1+2}q^1(1-y)^2y}{(1-sq)(1-sq)(1-syq)(1-syq)(1-syq^2)} + \frac{xsq}{1-sq}C'(1) \\
&+ \frac{x^2s^3q^{1+2}q^1(1-y)^2}{(1-sq)(1-sq)(1-sq^2)(1-syq)(1-syq)}C'(1) + \frac{xs^2q^2(2y-syq-1)}{(1-sq)(1-sq)(1-syq)}C(1) \\
&+ \frac{x^2s^4q^{2+3}(1-y)^2\left(2y-syq^2-1\right)}{(1-sq)(1-sq^2)(1-sq)(1-sq^2)(1-syq)(1-syq)(1-syq^2)}C(1) \\
&+ \frac{x^2s^4q^{2+3}(1-y)^4}{(1-sq)(1-sq^2)(1-sq)(1-sq^2)(1-syq)(1-syq^2)(1-syq)(1-syq^2)}C\left(sq^2\right) \\
=& \ldots \\
=& \sum_{n\geq 1}\frac{x^ns^{2n-1}yq^{\binom{n+1}{2}}q^{n-1}(1-y)^{2n-2}}{(sq)_{n-1}(sq)_{n-1}(syq)_{n-1}(syq)_n} + \sum_{n\geq 1}\frac{x^ns^{2n-1}q^{\binom{n+1}{2}}q^{n-1}(1-y)^{2n-2}}{(sq)_{n-1}(sq)_n(syq)_{n-1}(syq)_{n-1}}C'(1) \\
&+ \sum_{n\geq 1}\frac{x^ns^{2n}(2y-syq^n-1)(1-y)^{2n-2}q^nq^{\binom{n+1}{2}}}{(sq)_n(sq)_n(syq)_{n-1}(syq)_n}C(1)
\end{aligned}$$

Next, let $s=1$ in this expression, to obtain a linear equation in terms of $C(1)$ and $C'(1)$. Also, differentiate this expression, and let $s=1$, to obtain a second equation in terms of $C(1)$ and $C'(1)$ (It should be noted that $\frac{d}{dx}(xq)_n = \sum_{m=1}^{n}\frac{(xq)_n(-q^m)}{1-xq^m}$.) This gives a system of two equations in two unknowns, which may be solved to give

$$C(x,y,q,1) = y\frac{(1-y)X}{1+W+yX},$$

where

$$X = \frac{xq}{(1-y)(1-yq)} + \sum_{n\geq 2}\frac{(-1)^{n+1}x^n(1-y)^{2n-4}q^{\binom{n+1}{2}}\left(y^2q\right)_{2n-2}}{(q)_{n-1}(yq)_{n-2}(yq)_{n-1}^2(yq)_n\left(y^2q\right)_{n-1}},$$

and

$$W=\sum_{n\geq 1}\frac{(-1)^n\,x^n\,(1-y)^{2n-3}\,q^{\binom{n+1}{2}}\,\left(y^2q\right)_{2n-1}}{(q)_n\,(yq)_{n-1}^3\,(yq)_n\,(y^2q)_{n-1}}.$$

This concludes the demonstration of three different methods which may be used to enumerate a large number of different classes of polyominoes.

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Appendices

Appendix 1.1

Defining the generating function $c_{111}(x,0,1)$

$$\text{C111}[x\_] = \frac{1}{1 - \sum_{i=1}^{100} \frac{x^i \left(1+x^i\right)}{1+x^i \left(1+x^i\right)}};$$

Extracting the coefficients of x^n and displaying them from n=0,1,...,25 in a table.

```
Table[SeriesCoefficient[Series[%, {x, 0, 25}], n], {n, 0, 25}]
```

```
{1, 1, 2, 3, 7, 13, 24, 46, 89, 170, 324, 618, 1183, 2260, 4318, 8249, 15765,
 30123, 57556, 109973, 210137, 401525, 767216, 1465963, 2801115, 5352275}
```

Appendix 1.2

Defining the generating function $c_{112}(x,0,1)$

```
C112[x_] = 1 / (1 - Sum_{i=1}^{100} x^i Prod_{j=1}^{i-1} (1 - x^(2 j)));
```

$$C112[x\_] = \frac{1}{1 - \sum_{i=1}^{100} x^{i} \prod_{j=1}^{i-1} \left(1 - x^{2j}\right)};$$

Extracting the coefficients of x^n and displaying them from n=0,1,...,25 in a table.

```
Table[SeriesCoefficient[Series[%, {x, 0, 25}], n], {n, 0, 25}]
{1, 1, 2, 4, 7, 13, 24, 43, 78, 142, 256, 463, 838, 1513, 2735, 4944, 8931,
 16139, 29164, 52693, 95213, 172042, 310855, 561682, 1014898, 1833794}
```

Defining the generating function $c_{221}(x,0,1)$

$$C221[x\_] = \frac{1}{1 - \sum_{i=1}^{100} \left(x^{i} \prod_{j=i+1}^{100} \left(1 - x^{2j}\right)\right)};$$

Extracting the coefficients of x^n and displaying them from n=0,1,...,25 in a table.

```
Table[SeriesCoefficient[Series[%, {x, 0, 25}], n], {n, 0, 25}]
{1, 1, 2, 4, 8, 15, 30, 58, 113, 220, 429, 835, 1627, 3169, 6172, 12023, 23419,
 45616, 88853, 173073, 337118, 656656, 1279065, 2491423, 4852911, 9452731}
```

Appendix 1.3

Defining the generating function $c_{123}(x,0,1)$

$$\texttt{C123[x\_]} = \frac{1}{1 - \frac{x}{1-x} - \sum_{p=3}^{100} \sum_{j=0}^{p-3} \texttt{Binomial}[p-3,\, j] \frac{x^{\texttt{Binomial}[p+1+j,2]}}{\prod_{i=0}^{p+j-1}\left(1-x^{i+1}\right)} (-1)^{p-2}};$$

Extracting the coefficients of x^n and displaying them from n=0,1,...,25 in a table.

```
Table[SeriesCoefficient[Series[%, {x, 0, 25}], n], {n, 0, 25}]
```

```
{1, 1, 2, 4, 8, 16, 31, 61, 119, 232, 453, 883, 1721, 3354, 6536, 12 735, 24 813, 48 344,
 94 189, 183 506, 357 518, 696 534, 1 357 019, 2 643 798, 5 150 746, 10 034 865}
```

Appendix 1.4

Defining the generating function $C^N_{\text{peak}}(x,0,1)$

$$\texttt{Cpeak[x\_]} = \frac{1+\sum_{j=1}^{50}\frac{x^{j(j+2)}}{\prod_{i=0}^{2j-1}(1-x^{i+1})}}{1+\sum_{j=1}^{50}\frac{x^{j(j+2)}}{\prod_{i=0}^{2j-1}(1-x^{i+1})}-\sum_{j=0}^{50}\frac{x^{j^2+3j+1}}{\prod_{i=0}^{2j}(1-x^{i+1})}};$$

Extracting the coefficients of x^n and displaying them from n=0,1,...,25 in a table.

```
Table[SeriesCoefficient[Series[%, {x, 0, 25}], n], {n, 0, 25}]
{1, 1, 2, 4, 7, 13, 22, 38, 64, 107, 177, 293, 481, 789, 1291, 2110,
 3445, 5621, 9167, 14 947, 24 366, 39 716, 64 732, 105 500, 171 941, 280 220}
```

Defining the generating function $C^N_{\text{valley}}(x,0,1)$

$$\texttt{Cpeak[x\_]} = \frac{1+\sum_{j=1}^{50}\frac{x^{j(j+2)}}{\prod_{i=0}^{2j-1}(1-x^{i+1})}}{1+\sum_{j=1}^{50}\frac{x^{j(j+2)}}{\prod_{i=0}^{2j-1}(1-x^{i+1})}-\sum_{j=0}^{50}\frac{x^{(j+1)^2}}{\prod_{i=0}^{2j}(1-x^{i+1})}};$$

Extracting the coefficients of x^n and displaying them from n=0,1,...,25 in a table.

```
Table[SeriesCoefficient[Series[%, {x, 0, 25}], n], {n, 0, 25}]
{1, 1, 2, 4, 8, 15, 28, 52, 96, 177, 326, 600, 1104, 2032, 3740, 6884, 12 672,
 23 327, 42 942, 79 052, 145 528, 267 905, 493 192, 907 928, 1 671 424, 3 076 959}
```

Appendix 1.5

Finding ρ for each pattern τ

τ = 111

$$\texttt{FindRoot}\left[1-\sum_{i=1}^{100}\frac{x^i\,(1+x^i)}{1+x^i\,(1+x^i)},\ \{x,\ 0.7\}\right]$$

```
{x → 0.523351}
```

τ = 112

$$\texttt{FindRoot}\left[1-\sum_{i=1}^{100}x^i\prod_{j=1}^{i-1}\left(1-x^{2\,j}\right),\ \{x,\ 0.7\}\right]$$

```
{x → 0.55344}
```

τ = 221

$$\texttt{FindRoot}\left[1-\sum_{i=1}^{100}\left(x^i\prod_{j=i+1}^{100}\left(1-x^{2\,j}\right)\right),\ \{x,\ 0.7\}\right]$$

```
{x → 0.513387}
```

τ = 123

$$\texttt{FindRoot}\left[1-\frac{x}{1-x}-\sum_{p=3}^{50}\sum_{j=0}^{p-3}\texttt{Binomial}[p-3,\ j]\ \frac{x^{\texttt{Binomial}[p+1+j,2]}}{\prod_{i=0}^{p+j-1}\left(1-x^{i+1}\right)}\ (-1)^{p-2},\ \{x,\ 0.5\}\right]$$

```
{x → 0.513286}
```

τ = peak

$$\text{FindRoot}\left[1+\sum_{j=1}^{50}\frac{x^{j\,(j+2)}}{\prod_{i=0}^{2\,j-1}\left(1-x^{i+1}\right)}-\sum_{j=0}^{50}\frac{x^{j^2+3\,j+1}}{\prod_{i=0}^{2\,j}\left(1-x^{i+1}\right)},\ \{x,\ 0.7\}\right]$$

{x → 0.61359}

τ = valley

$$\text{FindRoot}\left[1+\sum_{j=1}^{50}\frac{x^{j\,(j+2)}}{\prod_{i=0}^{2\,j-1}\left(1-x^{i+1}\right)}-\sum_{j=0}^{50}\frac{x^{(j+1)^2}}{\prod_{i=0}^{2\,j}\left(1-x^{i+1}\right)},\ \{x,\ 0.7\}\right]$$

{x → 0.543207}

Appendix 1.6

Deriving the asymptotics for the generating functions for each pattern τ

$\tau = 111$

$$\text{DF111}[x\_] = D\left[1 - \sum_{i=1}^{100} \frac{x^i (1 + x^i)}{1 + x^i (1 + x^i)}, x\right];$$

Finding K

$$\frac{-1}{0.5233508892675827\grave{} * \text{DF111}[0.5233508892675827\grave{}]}$$

0.499301

Finding v

$$v = \frac{1}{0.5233508892675827\grave{}}$$

1.91076

$\tau = 112$

$$\text{DF112}[x\_] = D\left[1 - \sum_{i=1}^{100} x^i \prod_{j=1}^{i-1} \left(1 - x^{2j}\right), x\right];$$

Finding K

$$\frac{-1}{0.553439701710276\grave{} * \text{DF112}[0.553439701710276\grave{}]}$$

0.692005

Finding v

$$v = \frac{1}{0.553439701710276\grave{}}$$

1.80688

τ = 221

$$\text{DF221}[x\_] = D\left[1 - \sum_{i=1}^{100}\left(x^{i}\prod_{j=i+1}^{100}\left(1 - x^{2j}\right)\right), x\right];$$

Finding K

$$\frac{-1}{0.5133872861580691\grave{} * \text{DF221}[0.5133872861580691\grave{}]}$$

0.545362

Finding v

$$v = \frac{1}{0.5133872861580691\grave{}}$$

1.94785

τ = 123

$$\text{DF123}[x\_] = D\left[1 - \frac{x}{1-x} - \sum_{p=3}^{50}\sum_{j=0}^{p-3}\text{Binomial}[p-3, j]\,\frac{x^{\text{Binomial}[p+1+j,2]}}{\prod_{i=0}^{p+j-1}\left(1 - x^{i+1}\right)}\,(-1)^{p-2}, x\right];$$

Finding K

$$\frac{-1}{0.513285839366507\grave{} * \text{DF123}[0.513285839366507\grave{}]}$$

0.576096

Finding v

$$v = \frac{1}{0.513285839366507\grave{}}$$

1.94823

τ = peak

In[5]:=
$$\text{DFpeak}[x\_] = D\left[\frac{1 + \sum_{j=1}^{50}\frac{x^{j\,(j+2)}}{\prod_{i=0}^{2j-1}\left(1-x^{i+1}\right)} - \sum_{j=0}^{50}\frac{x^{j^2+3j+1}}{\prod_{i=0}^{2j}\left(1-x^{i+1}\right)}}{1 + \sum_{j=1}^{50}\frac{x^{j\,(j+2)}}{\prod_{i=0}^{2j-1}\left(1-x^{i+1}\right)}^{2}}, x\right];$$

Finding K

In[6]:=
$$\frac{-1}{0.6135900248916283\text{`} * \text{DFpeak}[0.6135900248916283\text{`}]}$$

Out[6]= 1.39456

Finding v

$$v = \frac{1}{0.6135900248916283\text{`}}$$

1.62975

τ = valley

In[7]:=
$$\text{DFvalley}[x\_] = D\left[\frac{1 + \sum_{j=1}^{50} \frac{x^{j\,(j+2)}}{\prod_{i=0}^{2\,j-1} (1-x^{i+1})} - \sum_{j=0}^{50} \frac{x^{(j+1)^2}}{\prod_{i=0}^{2\,j} (1-x^{i+1})}}{1 + \sum_{j=1}^{50} \frac{x^{j\,(j+2)}}{\prod_{i=0}^{2\,j-1} (1-x^{i+1})}}, x\right];$$

Finding K

In[8]:=
$$\frac{-1}{0.5432069327550826\text{`} * \text{DFvalley}[0.5432069327550826\text{`}]}$$

Out[8]= 0.728207

Finding v

$$v = \frac{1}{0.5432069327550826\text{`}}$$

1.84092

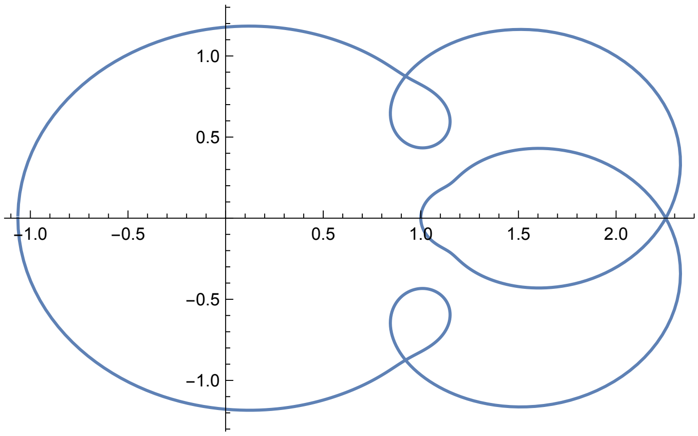
Appendix 1.7

The graphs of f(Γ) for each statistic τ, where |Γ|<0.7

τ=111

ListLinePlot[

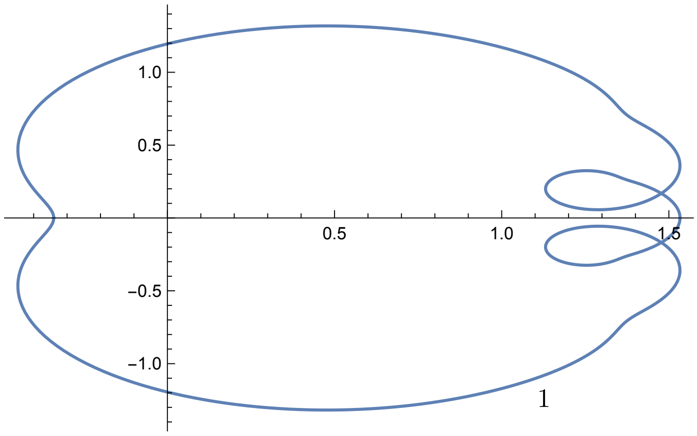
$$\{Re[\#], Im[\#]\} \ \& \ /@ Table\left[1 - \sum_{i=1}^{25} \frac{(0.7\, e^{i\theta})^{i}\left(1 + (0.7\, e^{i\theta})^{i}\right)}{1 + (0.7\, e^{i\theta})^{i}\left(1 + (0.7\, e^{i\theta})^{i}\right)}, \{\theta, 0, 2\pi, 0.01\}\right]\Big]$$



τ=112

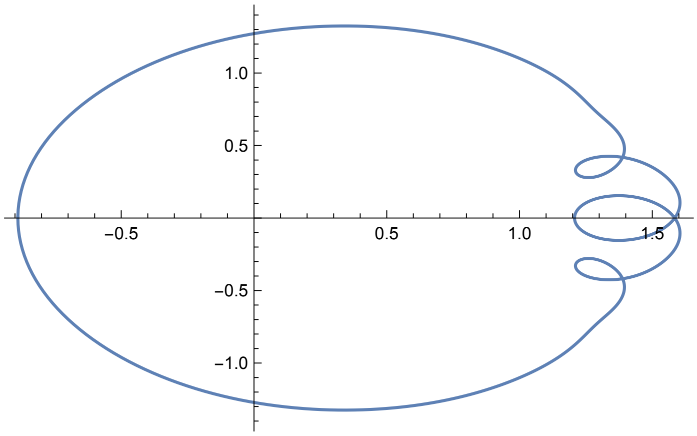
ListLinePlot[

$$\{Re[\#], Im[\#]\} \ \& \ /@ Table\left[1 - \sum_{i=1}^{25} (0.7\, e^{i\theta})^{i} \prod_{j=1}^{i-1}\left(1 - (0.7\, e^{i\theta})^{2j}\right), \{\theta, 0, 2\pi, 0.01\}\right]\Big]$$



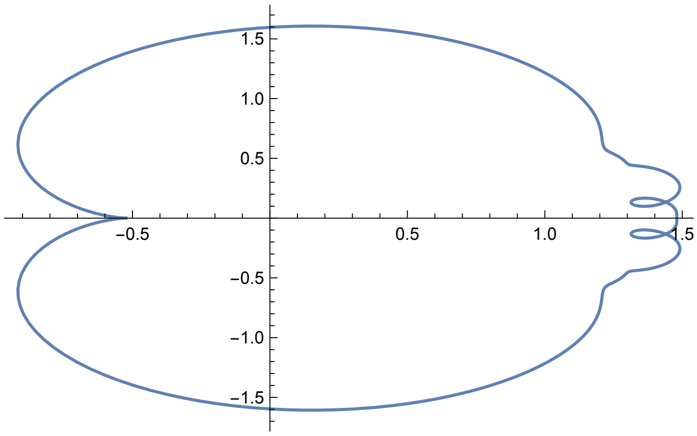
τ=221

$$\texttt{ListLinePlot}\left[\{\texttt{Re}[\#],\ \texttt{Im}[\#]\}\ \&\ /@\ \texttt{Table}\left[1-\sum_{i=1}^{25}\left(\left(0.7\,e^{i\,\theta}\right)^{i}\prod_{j=i+1}^{25}\left(1-\left(0.7\,e^{i\,\theta}\right)^{2\,j}\right)\right),\ \{\theta,\ 0,\ 2\,\pi,\ 0.01\}\right]\right]$$



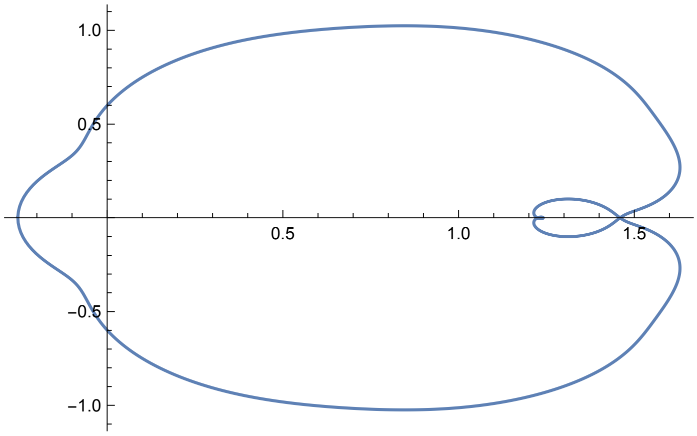
τ=123

$$\texttt{ListLinePlot}\left[\{\texttt{Re}[\#],\ \texttt{Im}[\#]\}\ \&\ /@\ \texttt{Table}\left[1-\frac{\left(0.7\,e^{i\,\theta}\right)}{1-\left(0.7\,e^{i\,\theta}\right)}-\sum_{p=3}^{25}\sum_{j=0}^{p-3}\texttt{Binomial}[p-3,\ j]\ \frac{\left(0.7\,e^{i\,\theta}\right)^{\texttt{Binomial}[p+1+j,2]}}{\prod_{i=0}^{p+j-1}\left(1-\left(0.7\,e^{i\,\theta}\right)^{i+1}\right)}\ (-1)^{p-2},\ \{\theta,\ 0,\ 2\,\pi,\ 0.01\}\right]\right]$$



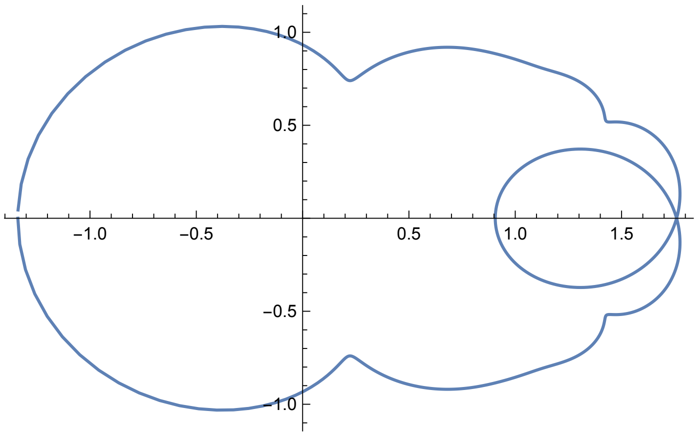
τ=peak

$$\texttt{ListLinePlot}\left[\{\texttt{Re}[\#], \texttt{Im}[\#]\} \ \& \ /@ \ \texttt{Table}\left[1 + \sum_{j=1}^{25} \frac{\left(0.7\, e^{i\theta}\right)^{j(j+2)}}{\prod_{i=0}^{2j-1}\left(1 - \left(0.7\, e^{i\theta}\right)^{i+1}\right)} - \sum_{j=0}^{25} \frac{\left(0.7\, e^{i\theta}\right)^{j^2+3j+1}}{\prod_{i=0}^{2j}\left(1 - \left(0.7\, e^{i\theta}\right)^{i+1}\right)}, \{\theta, 0, 2\pi, 0.01\}\right]\right]$$



τ=valley

$$\texttt{ListLinePlot}\left[\{\texttt{Re}[\#], \texttt{Im}[\#]\} \ \& \ /@ \ \texttt{Table}\left[1 + \sum_{j=1}^{25} \frac{\left(0.7\, e^{i\theta}\right)^{j(j+2)}}{\prod_{i=0}^{2j-1}\left(1 - \left(0.7\, e^{i\theta}\right)^{i+1}\right)} - \sum_{j=0}^{25} \frac{\left(0.7\, e^{i\theta}\right)^{(j+1)^2}}{\prod_{i=0}^{2j}\left(1 - \left(0.7\, e^{i\theta}\right)^{i+1}\right)}, \{\theta, 0, 2\pi, 0.01\}\right]\right]$$



Appendix 3.1

Solving the system of equations

```
Solve[{lh1 == 1, lh2 == q (lh1 + lh2 + lh3 + lh4),
  lh3 == q (lh2 + lh3 + lh3), lh4 == q (lh2 + lh3 + lh4 + lh5), cQ == lh2 + lh3,
  lh5 == q (lh2 + lh3 + lh3 + lh5)}, {cQ, lh1, lh2, lh3, lh4, lh5}]
```

Obtaining the desired solution as cQ

$$\left\{\left\{\mathrm{cQ} \to \frac{(-1+q)^3\, q}{-1+5q-7q^2+4q^3},\ \mathrm{lh1} \to 1,\ \mathrm{lh2} \to -\frac{q-4q^2+5q^3-2q^4}{-1+5q-7q^2+4q^3},\right.\right.$$
$$\left.\left.\mathrm{lh3} \to -\frac{q^2\left(1-2q+q^2\right)}{-1+5q-7q^2+4q^3},\ \mathrm{lh4} \to -\frac{q^2-q^3+q^4}{-1+5q-7q^2+4q^3},\ \mathrm{lh5} \to \frac{(-1+q)\, q^2}{-1+5q-7q^2+4q^3}\right\}\right\}$$

Appendix 3.2

Solving the system of equations

```
Solve[{lh1 == 1, lh2 == q (lh1 + lh2 + lh3 + lh4), lh3 == q (lh2 + lh3),
  lh4 == q (lh2 + lh3 + lh4), dQ == lh2 + lh3}, {dQ, lh1, lh2, lh3, lh4}]
```

Obtaining the desired solution as dQ

$$\left\{\left\{\mathrm{dQ} \to -\frac{(-1+q)\, q}{1-3q+q^2},\ \mathrm{lh1} \to 1,\ \mathrm{lh2} \to -\frac{-q+2q^2-q^3}{1-3q+q^2},\ \mathrm{lh3} \to -\frac{-q^2+q^3}{1-3q+q^2},\ \mathrm{lh4} \to \frac{q^2}{1-3q+q^2}\right\}\right\}$$

Appendix 3.3

Solving the system of equations

```
Solve[{la == x (z^2 + la + z * lb), lb == x (2 z + 2 z * la + lb), sTwo == z^2 * la + z * lb},
  {sTwo, la, lb}] /. z → (√y)
```

Obtaining the desired solution as sTwo

$$\left\{\left\{\text{sTwo} \to -\frac{2\,x\,y - 2\,x^2\,y + x\,y^2 + 3\,x^2\,y^2}{-1 + 2\,x - x^2 + 2\,x^2\,y},\right.\right.$$

$$\left.\left.\text{la} \to -\frac{x\,y + x^2\,y}{-1 + 2\,x - x^2 + 2\,x^2\,y},\ \text{lb} \to -\frac{2\,\sqrt{y}\,\left(x - x^2 + x^2\,y\right)}{-1 + 2\,x - x^2 + 2\,x^2\,y}\right\}\right\}$$

Appendix 3.4

Solving the system of equations

```
Solve[{bXY == y * lXY, lXY == y * lXY + x + x * lXY + x * y * lXY + x * y * lXY * lXY}, {bXY, lXY}]
```

Obtaining two different solutions bXY

$$\left\{\left\{bXY \to \frac{1}{2}\left(-1+\frac{1}{x}-y-\frac{y}{x}-\frac{\sqrt{-4x^2y+(-1+x+y+xy)^2}}{x}\right),\right.\right.$$

$$\left.lXY \to \frac{1-x-y-xy-\sqrt{-4x^2y+(-1+x+y+xy)^2}}{2xy}\right\},$$

$$\left\{bXY \to \frac{1}{2}\left(-1+\frac{1}{x}-y-\frac{y}{x}+\frac{\sqrt{-4x^2y+(-1+x+y+xy)^2}}{x}\right),\right.$$

$$\left.\left.lXY \to \frac{1-x-y-xy+\sqrt{-4x^2y+(-1+x+y+xy)^2}}{2xy}\right\}\right\}$$

Testing the coefficients of the second solution for bXY

$$\text{Series}\left[\frac{1}{2}\left(-1+\frac{1}{x}-y-\frac{y}{x}+\frac{\sqrt{-4x^2y+(-1+x+y+xy)^2}}{x}\right),\{x,0,5\},\{y,0,5\}\right]$$

$$\frac{1-y+O[y]^6}{x}+\left(-1-y+O[y]^6\right)+\left(-y-y^2-y^3-y^4-y^5+O[y]^6\right)x+$$
$$\left(-y-3y^2-5y^3-7y^4-9y^5+O[y]^6\right)x^2+\left(-y-6y^2-16y^3-31y^4-51y^5+O[y]^6\right)x^3+$$
$$\left(-y-10y^2-40y^3-105y^4-219y^5+O[y]^6\right)x^4+$$
$$\left(-y-15y^2-85y^3-295y^4-771y^5+O[y]^6\right)x^5+O[x]^6$$

```
SeriesCoefficient[%, {3, 3}]
```

-16

The coefficient is negative, and hence this is not the desired solution. The first solution,

$\frac{1}{2}\left(-1+\frac{1}{x}-y-\frac{y}{x}-\frac{\sqrt{-4x^2y+(-1+x+y+xy)^2}}{x}\right)$, is the desired solution

Appendix 3.5

Finding the roots of the expression for D

```
Roots[d^3 + 2 (y - 1) d^2 + (y - 1) (x + y - 1) d + x * y (y - 1) == 0, d]
```

$$d == -\frac{2}{3}(-1+y) - \left(2^{1/3}\left(-1-3x+2y+3xy-y^2\right)\right) \Big/ \left(3\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3+\sqrt{4\left(-1-3x+2y+3xy-y^2\right)^3+\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3\right)^2}\right)^{1/3}\right) + \frac{1}{3\times 2^{1/3}}\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3+\sqrt{4\left(-1-3x+2y+3xy-y^2\right)^3+\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3\right)^2}\right)^{1/3} \;||$$

$$d == -\frac{2}{3}(-1+y) + \left(\left(1+i\sqrt{3}\right)\left(-1-3x+2y+3xy-y^2\right)\right) \Big/ \left(3\times 2^{2/3}\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3+\sqrt{4\left(-1-3x+2y+3xy-y^2\right)^3+\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3\right)^2}\right)^{1/3}\right) - \frac{1}{6\times 2^{1/3}}\left(1-i\sqrt{3}\right)\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3+\sqrt{4\left(-1-3x+2y+3xy-y^2\right)^3+\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3\right)^2}\right)^{1/3} \;||$$

$$d == -\frac{2}{3}(-1+y) + \left(\left(1-i\sqrt{3}\right)\left(-1-3x+2y+3xy-y^2\right)\right) \Big/ \left(3\times 2^{2/3}\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3+\sqrt{4\left(-1-3x+2y+3xy-y^2\right)^3+\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3\right)^2}\right)^{1/3}\right) - \frac{1}{6\times 2^{1/3}}\left(1+i\sqrt{3}\right)\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3+\sqrt{4\left(-1-3x+2y+3xy-y^2\right)^3+\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3\right)^2}\right)^{1/3}$$

Testing the roots manually to check which one of the three roots are the correct root

$$\text{Series}\Big[-\frac{2}{3}(-1+y) - \left(2^{1/3}\left(-1-3x+2y+3xy-y^2\right)\right)\Big/\left(3\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3+\sqrt{4\left(-1-3x+2y+3xy-y^2\right)^3+\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3\right)^2}\right)^{1/3}\right) + \frac{1}{3\times 2^{1/3}}\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3+\sqrt{4\left(-1-3x+2y+3xy-y^2\right)^3+\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3\right)^2}\right)^{1/3},\ \{x,0,5\},\{y,0,5\}\Big];$$

`FullSimplify[SeriesCoefficient[%, {3, 3}]]`

$$-\frac{37}{2}$$

Clearly this is not the right root, as the coefficient is negative.

$$\text{Series}\Big[-\frac{2}{3}(-1+y) + \left(\left(1-i\sqrt{3}\right)\left(-1-3x+2y+3xy-y^2\right)\right)\Big/\left(3\times 2^{2/3}\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3+\sqrt{\left(4\left(-1-3x+2y+3xy-y^2\right)^3+\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3\right)^2\right)}\right)^{1/3}\right) - \frac{1}{6\times 2^{1/3}}\left(1+i\sqrt{3}\right)\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3+\sqrt{\left(4\left(-1-3x+2y+3xy-y^2\right)^3+\left(-2+18x+6y-9xy-6y^2-9xy^2+2y^3\right)^2\right)}\right)^{1/3},\ \{x,0,5\},\{y,0,5\}\Big];$$

`FullSimplify[SeriesCoefficient[%, {3, 3}]]`

$$-\frac{37}{2}$$

Clearly this is not the right root, as the coefficient is negative.

$$\text{Series}\Big[-\frac{2}{3}(-1+y)+\Big(\big(1+i\sqrt{3}\big)\big(-1-3x+2y+3xy-y^2\big)\Big)\Big/\Big(3\times 2^{2/3}\Big(-2+18x+6y-9xy-6y^2-9xy^2+2y^3+\sqrt{\big(4\big(-1-3x+2y+3xy-y^2\big)^3+\big(-2+18x+6y-9xy-6y^2-9xy^2+2y^3\big)^2\big)}\Big)^{1/3}\Big)-\frac{1}{6\times 2^{1/3}}\big(1-i\sqrt{3}\big)\Big(-2+18x+6y-9xy-6y^2-9xy^2+2y^3+\sqrt{\big(4\big(-1-3x+2y+3xy-y^2\big)^3+\big(-2+18x+6y-9xy-6y^2-9xy^2+2y^3\big)^2\big)}\Big)^{1/3},\ \{x,0,5\},\ \{y,0,5\}\Big];$$

```
FullSimplify[SeriesCoefficient[%, {3, 3}]]
37
```

This is the correct root, as the coefficient of $x^3 y^3$ should be 37 and the other two roots are incorrect.

Appendix 3.6

Solving the system of equations

```
Solve[{x*z^3 + (x - 1)*a + x*z*b + x*z^2*g + z^2*x*d + z*x*e == 0,
   2*z^2*x + 2*z*x*a + (x*z^2 + x - 1)*b + x*z*g + 2*x*z*d == 0,
   2*x*z + 2*z^2*x*a + z*x*b + (x - 1)*g + 2*x*z*f == 0, x*z + x*z^2*a + x*z*b + (x - 1) d == 0,
   x^2*z^2 + x^2*z*g + (x^2 - 1) e == 0, x^2*z*a + (x^2 - 1) f == 0,
   s3 == z^3*a + z^2*b + z*g + z*d + z^2*f}, {a, b, g, d, e, f, s3}] /. z → (√y)
```

$$\Big\{\Big\{a \to -\Big(\Big(\big(-x + x^2\big)\, y^{3/2}\, \big(1 + 3x - x^2 - 6x^3 - x^4 + 3x^5 + x^6 - xy - x^2y - x^4y - x^5y\big)\Big) \Big/ \big(1 - 4x + 4x^2 + 4x^3 - 10x^4 + 4x^5 + 4x^6 - 4x^7 + x^8 - xy - 2x^2y + 9x^3y - 15x^5y + 6x^6y + 7x^7y - 4x^8y - 3x^2y^2 - 2x^3y^2 + 7x^4y^2 + 8x^5y^2 - 11x^6y^2 - 2x^7y^2 + 3x^8y^2 + 3x^3y^3 - 2x^4y^3 - 2x^5y^3 + 4x^6y^3 + x^7y^3\big)\Big),$$

$$b \to -\Big(\Big(2\,\big(-xy + x^2y + 3x^3y - 3x^4y - 3x^5y + 3x^6y + x^7y - x^8y - x^2y^2 - x^3y^2 + 3x^4y^2 + 2x^5y^2 - 3x^6y^2 - x^7y^2 + x^8y^2 + x^3y^3 - x^4y^3 - x^5y^3 + 2x^6y^3 + x^7y^3\big)\Big) \Big/ \big(1 - 4x + 4x^2 + 4x^3 - 10x^4 + 4x^5 + 4x^6 - 4x^7 + x^8 - xy - 2x^2y + 9x^3y - 15x^5y + 6x^6y + 7x^7y - 4x^8y - 3x^2y^2 - 2x^3y^2 + 7x^4y^2 + 8x^5y^2 - 11x^6y^2 - 2x^7y^2 + 3x^8y^2 + 3x^3y^3 - 2x^4y^3 - 2x^5y^3 + 4x^6y^3 + x^7y^3\big)\Big),$$

$$g \to -\Big(\Big(2\sqrt{y}\,\big(-x + 3x^2 - x^3 - 5x^4 + 5x^5 + x^6 - 3x^7 + x^8 + 3x^3y - 3x^4y - 6x^5y + 6x^6y + 3x^7y - 3x^8y - x^2y^2 - x^3y^2 + 3x^4y^2 + 2x^5y^2 - 6x^6y^2 + x^7y^2 + 2x^8y^2 + x^3y^3 - x^4y^3 + 2x^6y^3\big)\Big) \Big/ \big(1 - 4x + 4x^2 + 4x^3 - 10x^4 + 4x^5 + 4x^6 - 4x^7 + x^8 - xy - 2x^2y + 9x^3y - 15x^5y + 6x^6y + 7x^7y - 4x^8y - 3x^2y^2 - 2x^3y^2 + 7x^4y^2 + 8x^5y^2 - 11x^6y^2 - 2x^7y^2 + 3x^8y^2 + 3x^3y^3 - 2x^4y^3 - 2x^5y^3 + 4x^6y^3 + x^7y^3\big)\Big),$$

$$d \to \Big((-1 + x)\, x\, \big(-\sqrt{y} + 2x\sqrt{y} + x^2\sqrt{y} - 4x^3\sqrt{y} + x^4\sqrt{y} + 2x^5\sqrt{y} - x^6\sqrt{y} - xy^{3/2} + 2x^2y^{3/2} + 2x^3y^{3/2} - 4x^4y^{3/2} - x^5y^{3/2} + 2x^6y^{3/2} - xy^{5/2} - 3x^2y^{5/2} - x^3y^{5/2} + 5x^4y^{5/2} + 2x^5y^{5/2} + x^2y^{7/2} + x^3y^{7/2}\big)\Big) \Big/ \big(1 - 4x + 4x^2 + 4x^3 - 10x^4 + 4x^5 + 4x^6 - 4x^7 + x^8 - xy - 2x^2y + 9x^3y - 15x^5y + 6x^6y + 7x^7y - 4x^8y - 3x^2y^2 - 2x^3y^2 + 7x^4y^2 + 8x^5y^2 - 11x^6y^2 - 2x^7y^2 + 3x^8y^2 + 3x^3y^3 - 2x^4y^3 - 2x^5y^3 + 4x^6y^3 + x^7y^3\big),$$

$$e \to \Big(y\,\big(x^2 - 2x^3 - x^4 + 4x^5 - x^6 - 2x^7 + x^8 - x^3y - 2x^4y + 2x^5y + 4x^6y - x^7y - 2x^8y - x^4y^2 + 4x^7y^2 + x^8y^2 + x^5y^3 - x^7y^3\big)\Big) \Big/ \big(1 - 4x + 4x^2 + 4x^3 - 10x^4 + 4x^5 + 4x^6 - 4x^7 + x^8 - xy - 2x^2y + 9x^3y - 15x^5y + 6x^6y + 7x^7y - 4x^8y - 3x^2y^2 - 2x^3y^2 + 7x^4y^2 + 8x^5y^2 - 11x^6y^2 - 2x^7y^2 + 3x^8y^2 + 3x^3y^3 - 2x^4y^3 - 2x^5y^3 + 4x^6y^3 + x^7y^3\big),$$

$$f \to \Big(x^3\,\big(y^2 + 2xy^2 - 3x^2y^2 - 3x^3y^2 + 2x^4y^2 + x^5y^2 - xy^3 - x^4y^3\big)\Big) \Big/ \big(1 - 4x + 4x^2 + 4x^3 - 10x^4 + 4x^5 + 4x^6 - 4x^7 + x^8 - xy - 2x^2y + 9x^3y - 15x^5y + 6x^6y + 7x^7y - 4x^8y - 3x^2y^2 - 2x^3y^2 + 7x^4y^2 + 8x^5y^2 - 11x^6y^2 - 2x^7y^2 + 3x^8y^2 + 3x^3y^3 - 2x^4y^3 - 2x^5y^3 + 4x^6y^3 + x^7y^3\big),$$

$$s3 \to -\Big(\Big(x\,\big(-3y + 9xy - 3x^2y - 15x^3y + 15x^4y + 3x^5y - 9x^6y + 3x^7y - 2y^2 + xy^2 + 15x^2y^2 - 12x^3y^2 - 24x^4y^2 + 21x^5y^2 + 11x^6y^2 - 10x^7y^2 - y^3 - 7xy^3 - 3x^2y^3 + 17x^3y^3 + 12x^4y^3 - 22x^5y^3 - 2x^6y^3 + 6x^7y^3 + xy^4 + 5x^2y^4 - 4x^3y^4 - 2x^4y^4 + 8x^5y^4 + 2x^6y^4\big)\Big) \Big/ \big(1 - 4x + 4x^2 + 4x^3 - 10x^4 + 4x^5 + 4x^6 - 4x^7 + x^8 - xy - 2x^2y + 9x^3y - 15x^5y + 6x^6y + 7x^7y - 4x^8y - 3x^2y^2 - 2x^3y^2 + 7x^4y^2 + 8x^5y^2 - 11x^6y^2 - 2x^7y^2 + 3x^8y^2 + 3x^3y^3 - 2x^4y^3 - 2x^5y^3 + 4x^6y^3 + x^7y^3\big)\Big)\Big\}\Big\}$$

Simplifying the required equation

$$
\begin{aligned}
&\mathbf{Simplify}\Big[\\
&-\Big(\big(x\,(-3\,y+9\,x\,y-3\,x^2\,y-15\,x^3\,y+15\,x^4\,y+3\,x^5\,y-9\,x^6\,y+3\,x^7\,y-2\,y^2+x\,y^2+15\,x^2\,y^2-12\\
&\quad x^3\,y^2-24\,x^4\,y^2+21\,x^5\,y^2+11\,x^6\,y^2-10\,x^7\,y^2-y^3-7\,x\,y^3-3\,x^2\,y^3+17\,x^3\,y^3+12\,x^4\\
&\quad y^3-22\,x^5\,y^3-2\,x^6\,y^3+6\,x^7\,y^3+x\,y^4+5\,x^2\,y^4-4\,x^3\,y^4-2\,x^4\,y^4+8\,x^5\,y^4+2\,x^6\,y^4)\big)\Big/\\
&\quad\big(1-4\,x+4\,x^2+4\,x^3-10\,x^4+4\,x^5+4\,x^6-4\,x^7+x^8-x\,y-2\,x^2\,y+9\,x^3\,y-\\
&\quad 15\,x^5\,y+6\,x^6\,y+7\,x^7\,y-4\,x^8\,y-3\,x^2\,y^2-2\,x^3\,y^2+7\,x^4\,y^2+8\,x^5\,y^2-\\
&\quad 11\,x^6\,y^2-2\,x^7\,y^2+3\,x^8\,y^2+3\,x^3\,y^3-2\,x^4\,y^3-2\,x^5\,y^3+4\,x^6\,y^3+x^7\,y^3\big)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&-\Big(\big(x\,y\,\big(-3-2\,y-y^2+x^7\,(3-10\,y+6\,y^2)+x^4\,(15-24\,y+12\,y^2-2\,y^3)+\\
&\quad x\,(9+y-7\,y^2+y^3)+x^6\,(-9+11\,y-2\,y^2+2\,y^3)-x^3\,(15+12\,y-17\,y^2+4\,y^3)+\\
&\quad x^2\,(-3+15\,y-3\,y^2+5\,y^3)+x^5\,(3+21\,y-22\,y^2+8\,y^3)\big)\big)\Big/\\
&\quad\big(1-x\,(4+y)+x^2\,(4-2\,y-3\,y^2)+x^8\,(1-4\,y+3\,y^2)+x^4\,(-10+7\,y^2-2\,y^3)+\\
&\quad x^5\,(4-15\,y+8\,y^2-2\,y^3)+x^7\,(-4+7\,y-2\,y^2+y^3)+\\
&\quad x^3\,(4+9\,y-2\,y^2+3\,y^3)+x^6\,(4+6\,y-11\,y^2+4\,y^3)\big)\Big)
\end{aligned}
$$

Showing N(x,y)=3(x+1)^2(1-x)^5+(5x+2)(2x-1)(x+1)^2(x-1)^3y-(x-1)(6x^6+4x^5-18x^4-6x^3+11x^2+8x+1)y^2-x(x+1)(2x^4+6x^3-8x^2+4x+1)y^3

In[16]:=

$$
\begin{aligned}
&\mathbf{Expand}\Big[-\big(-3-2\,y-y^2+x^7\,(3-10\,y+6\,y^2)+x^4\,(15-24\,y+12\,y^2-2\,y^3)+\\
&\quad x\,(9+y-7\,y^2+y^3)+x^6\,(-9+11\,y-2\,y^2+2\,y^3)-x^3\,(15+12\,y-17\,y^2+4\,y^3)+\\
&\quad x^2\,(-3+15\,y-3\,y^2+5\,y^3)+x^5\,(3+21\,y-22\,y^2+8\,y^3)\big)-
\end{aligned}
$$

```
  (3 (x + 1) ^2 (1 - x) ^5 + (5 x + 2) (2 x - 1) (x + 1) ^2 (x - 1) ^3 y -
    (x - 1) (6 x^6 + 4 x^5 - 18 x^4 - 6 x^3 + 11 x^2 + 8 x + 1) y^2 -
    x (x + 1) (2 x^4 + 6 x^3 - 8 x^2 + 4 x + 1) y^3)]
```

Out[16]= 0

Showing D(x,y)=(x+1)^2(x-1)^6-x(1+4x)(x+1)^2(x-1)^4y+x^2(3x^4+4x^3-6x^2-8x-3)(x-1)^2y^2+x^3(3x^4+4x^3-6x^2-8x-3)(x-1)^2y^2+x^3(x+1)(x^3+3x^2-5x+3)y^3

In[18]:=

$$
\begin{aligned}
&\mathbf{Expand}\Big[\big(1-x\,(4+y)+x^2\,(4-2\,y-3\,y^2)+\\
&\quad x^8\,(1-4\,y+3\,y^2)+x^4\,(-10+7\,y^2-2\,y^3)+x^5\,(4-15\,y+8\,y^2-2\,y^3)+\\
&\quad x^7\,(-4+7\,y-2\,y^2+y^3)+x^3\,(4+9\,y-2\,y^2+3\,y^3)+x^6\,(4+6\,y-11\,y^2+4\,y^3)\big)-
\end{aligned}
$$

```
  ((x + 1) ^2 (x - 1) ^6 - x (1 + 4 x) (x + 1) ^2 (x - 1) ^4 y +
    x^2 (3 x^4 + 4 x^3 - 6 x^2 - 8 x - 3) (x - 1) ^2 y^2 +
    x^3 (x + 1) (x^3 + 3 x^2 - 5 x + 3) y^3)]
```

Out[18]= 0