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OPTIMIZATION

An inertial-type method for solving image restoration problems

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Abstract
We first establish weak convergence results regarding an inertial Krasnosel'skiĭ-Mann iterative method for approximating common fixed points of countable families of nonexpansive mappings in real Hilbert spaces with no extra assumptions on the considered countable families of nonexpansive mappings. The method of proof and the imposed conditions on the iterative parameters are different from those already available in the literature. We then present some applications to the Douglas–Rachford splitting method and image restoration problems, and compare the performance of our method with that of other popular inertial Krasnosel'skiĭ-Mann methods which can be found in the literature.

Keywords Image Restoration · Douglas–Rachford splitting · Hilbert space · Inertial Krasnosel'skiĭ-Mann method · Weak convergence

1 Introduction

Many results regarding the approximation of fixed points of nonexpansive mappings have been established in the literature (see, for example, Bauschke and Combettes 2011; Bauschke et al. 2011; Berinde 2007; Cegielski 2012; Krasnoselskii 1955; Mann 1953; Chang et al. 2002 and references therein). Recently, several iterative methods for constructing common fixed points of families of nonexpansive mappings have been proposed and studied. For instance, the results in Bauschke (1996), Kimura et al. (2005), O'Hara et al. (2003), Shimizu and Takahashi (1997), Shioji and Takahashi (1997) concern approximating common fixed points of finite families of nonexpansive mappings, while those in Aoyama et al. (2007), Cho et al. (2008), Klin-eam and Suantai (2010), Nilsrakoo and Saejung (2009a), Nilsrakoo and Saejung (2009b), Sahu et al. (2013) aim at finding common fixed points of countable families of such mappings.

Inertial Krasnosel'skiĭ-Mann iterations have been proposed in order to improve the convergence speed of the already available methods for finding fixed points of nonexpansive mappings. Some of these results can be found, for example, in Bot et al. (2015), Dong (2020), Dong et al. (2018), Dong et al. (2019), Dong et al. (2019), Maingé (2008), Vong and Liu (2018). Numerical examples concerning image reconstruction and signal processing have shown that the inertial Krasnosel'skiĭ-Mann iterative methods outperform the classical Krasnosel'skiĭ-Mann iterative methods (see, for instance, Bot et al. 2015; Dong 2020) through optimum choice of the inertia and iterative parameters. One of the major contributions of these results is that they indicate how to weaken and relax the conditions imposed on the inertial term. More recent results in this direction can be found in Dong (2020).

One of the limitations of the inertial Krasnosel'skiĭ-Mann iterative methods in Bot et al. (2015), Dong (2020), Dong et al. (2018), Dong et al. (2019), Dong et al. (2019), Maingé (2008), Vong and Liu (2018) is that some strong assumptions are imposed on the inertial factor of the proposed methods. In addition, these inertial Krasnosel'skiĭ-Mann methods are only considered for a single nonexpansive mapping. As far as we know, no inertial-type Krasnosel'skiĭ-Mann iterative methods have been proposed and studied for countable families of nonexpansive mappings so that results pertaining to

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such methods imply the results concerning known inertial Krasnosel'skiĭ-Mann methods as corollaries.

This leads us to ask the following question:

Question: Can we propose a new and useful inertial-type Krasnosel'skiĭ-Mann iterative method for countable families of nonexpansive mappings in Hilbert spaces?

Our aim in this paper is to give an affirmative answer to this question. We now present a brief summary of our answer:

- We propose an inertial-type Krasnosel'skiĭ-Mann iterative method for approximating common fixed points of countable families of nonexpansive mappings in Hilbert spaces with new conditions on the iterative parameters.
- We establish a weak convergence theorem for our proposed method under weak conditions on the iterative parameters.
- We give some applications of our results to the inertial Douglas–Rachford splitting method.
- We present numerical illustrations of our proposed method and compare our method with some important and recent inertial Krasnosel'skiĭ-Mann iterations which are available in the literature. Our numerical experiments show that our method has a faster convergence speed than these recent inertial Krasnosel'skiĭ-Mann iterations.

The organization of the paper is as follows. In Sect. 2, we present the relationship of our results with existing ones in the literature and highlight the contribution of this paper to existing results. In Sect. 3, we present several results which will be needed in our convergence analysis. The convergence analysis of our inertial-type Krasnosel'skiĭ-Mann iterative method for countable families of nonexpansive mappings is presented in Sect. 4. We provide some applications of our results to solving monotone inclusion problems and the Douglas–Rachford operator splitting method in Sect. 5. Section 6 contains numerical implementations of our theoretical analysis. Concluding remarks are given in Sect. 7.

2 Relation with existing results

Let H be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. A mapping $T : H \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in H.$$

One of the most important methods for finding a fixed point of a nonexpansive mapping T is the Krasnosel'skiĭ-Mann iteration (Krasnoselskii 1955; Mann 1953; Reich 1979): $x_1 \in H$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \quad \forall n = 1, 2, \ldots, \tag{2.1}$$

where $\alpha_n \in (0, 1)$. It is well-known that the Krasnosel'skiĭ-Mann iteration (2.1) converges weakly to a fixed point of T under some appropriate conditions on $\{\alpha_n\}$. For more details regarding the convergence of (2.1), see, for example, Bauschke et al. (2004), Chidume (2009), Cho et al. (2008), Genel and Lindenstrauss (1975), Goebel and Reich (1984), Opial (1967), Reich (1979), Yao and Liou (2008) and references therein.

Recently, convergence results for a combination of the Krasnosel'skiĭ-Mann iterative method (2.1) and an inertial extrapolation step (that is, inertial Krasnosel'skiĭ-Mann iterations) have been considered in the literature. It has been shown that the addition of an inertial extrapolation step to existing optimization methods does increase the convergence speed of the optimization methods (see, for instance, Alvarez and Attouch 2001, Attouch et al. 2000, 2014, Attouch and Czarnecki 2002, Attouch and Peypouquet 2016, Beck and Teboulle 2009, Bot and Csetnek 2016a, 2016b, Chen et al. 2015, Lorenz and Pock 2015, Maingé 2008, Ochs et al. 2015, Polyak 1964). In particular, the inertial Krasnosel'skiĭ-Mann iteration is of the form: $x_0, x_1 \in H$,

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (1 - \alpha_n)w_n + \alpha_n T w_n, \end{cases} \tag{2.2}$$

where $\theta_n \in [0, 1)$ is the inertial factor and $\alpha_n \in (0, 1)$. Weak convergence results for (2.2) have been established in Bot et al. (2015), Dong (2020), Dong et al. (2018), Dong et al. (2019), Dong et al. (2019), Maingé (2008), Vong and Liu (2018). Several conditions have been imposed on $\{\theta_n\}$ in order to obtain weak convergence of $\{x_n\}$ to a fixed point of T. For instance, the following conditions were imposed on $\{\theta_n\}$ in several papers in order to obtain the convergence of (2.2), which follows the style of (Alvarez and Attouch 2001, Theorem 2.1)

$$0 \leq \theta_n \leq \theta < 1, \quad \sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty. \tag{2.3}$$

Although the summability condition (2.3) can be verified in practice by using a suitable on-line rule, it involves the iterates $\{x_n\}$ and $\{x_{n-1}\}$ that are a priori unknown. Thus, it is desirable to give alternative conditions in which the knowledge of the iterates is not required.

To overcome this setback, the authors in Bot et al. (2015) provided the following conditions: the inertial sequence $\{\theta_n\}$ and the coefficient $\{\alpha_n\}$ satisfy

$$\begin{aligned} 0 = \theta_1 \leq \theta_n \leq \theta_{n+1} \leq \theta < 1 \ \forall n \geq 1, \quad 0 < \liminf \alpha_n \leq \alpha_n \\ \leq \frac{\delta - \theta[\theta(1+\theta) + \theta\delta + \sigma]}{\delta[1 + \theta(1+\theta) + \theta\delta + \sigma]}, \end{aligned} \tag{2.4}$$

with

$$\delta > \frac{\theta^2(1+\theta)+\theta\sigma}{1-\theta^2}, \tag{2.5}$$

where $\sigma, \delta > 0$. Similar conditions were assumed in Dong et al. (2018), Dong et al. (2019), Dong et al. (2019), Maingé (2008), Vong and Liu (2018).
Summing up, we highlight our contributions in this paper as follows.

- Clearly, conditions (2.4) and (2.5) are very complicated and restrictive. In particular, it is difficult to determine the upper bound θ of the inertial sequence $\{\theta_n\}$. To circumvent these difficulties, we provide new conditions on the inertial sequence $\{\theta_n\}$ and the coefficients $\{\alpha_n\}$ in (4.1). In contrast to the above conditions, our proposed conditions in (4.1) are much more relaxed. For instance, in Propositions 4.13–4.15, we have provided several simple ways of choosing the inertial sequence $\{\theta_n\}$ and the coefficients $\{\alpha_n\}$ so that our proposed conditions in (4.1) are satisfied.
- Our proposed algorithm does not involve the on-line rule $\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\|^2 < \infty$ assumed in (2.3). Instead, we propose another way of choosing the inertial parameters which is different from the one used in (Alvarez and Attouch 2001, Theorem 2.1).
- Furthermore, in the case where $\theta_n = 0$ for all $n \geq 1$, the authors in Bot and Meier (2021) imposed some strong assumptions, which are difficult to satisfy in practice, on the countable families of nonexpansive mappings (see condition (iii) of (Bot and Meier 2021, Theorem 2.1)). Our method (even with θ_n not necessarily equal to 0) does not require such strong and restrictive conditions on the countable families of nonexpansive mappings. Moreover, to the best of our knowledge, no inertial-type Krasnosel'skiĭ-Mann iterative method has so far been proposed and studied for countable families of nonexpansive mappings. Thus, the techniques employed in this paper are different from the ones which have up to now been used in the literature.
- Our results are applied to solving image restoration problems.

3 Preliminaries

In this section, we recall and present several results which will be needed in our convergence analysis.

Lemma 3.1 (Opial 1967). *Let C be a nonempty subset of H and let $\{w_n\}$ be a sequence contained in H that satisfies the following two conditions:*

(i) $\lim_{n\to\infty}\|w_n - y\|$ *exists for any $y \in C$;*
(ii) *every subsequential weak limit point of $\{w_n\}$ belongs to C.*

Then, $\{w_n\}$ converges weakly to a point in C.

Definition 3.2 A mapping T is said to be demiclosed if whenever a sequence $\{w_n\}$ weakly converges to y and the sequence $\{Tw_n\}$ strongly converges to z, it follows that $T(y) = z$.

The next result is well-known (Browder 1968).

Lemma 3.3 *Let $C \subset H$ be nonempty, closed and convex, and let $T : C \to C$ be a nonexpansive mapping. Then, $I - T$ is demiclosed at 0.*

We now specialize a result in Bruck (1973) (proved for all strictly convex Banach spaces) to real Hilbert spaces.

Lemma 3.4 *Let C be a nonempty, closed and convex subset of H. Let $\{S_k\}$ be a sequence of nonexpansive mappings of C into H and let $\{\beta_k\}$ be a sequence of positive real numbers such that $\sum_{k=1}^{\infty}\beta_k = 1$. If the intersection $\cap_{k=1}^{\infty}F(S_k)$ of the fixed point sets of $\{S_k\}$ is nonempty, then the well-defined mapping $T = \sum_{k=1}^{\infty}\beta_k S_k$ has a fixed point and $F(T) = \cap_{k=1}^{\infty}F(S_k)$.*

Let $x_{n+1} = x_n + \theta_n(x_n - x_{n-1})$. Then, we have for all $n \geq 1$ (Attouch and Cabot 2020),

$$x_{n+1} - x_n = \left(\prod_{j=1}^{n}\theta_j\right)(x_1 - x_0),$$

which implies that

$$x_n = x_1 + \left(\sum_{j=1}^{n-1}\prod_{j=1}^{l}\theta_j\right)(x_1 - x_0).$$

Thus, the sequence $\{x_n\}$ converges if and only if $x_1 = x_0$ or if $\sum_{l=1}^{\infty}\prod_{j=1}^{l}\theta_j < \infty$.
Therefore, throughout this paper, we assume that

$$\sum_{l=i}^{\infty}\left(\prod_{j=i}^{l}\theta_j\right) < \infty \ \ \forall i \geq 1. \tag{3.1}$$

Next, we define the sequence $\{t_i\}$ in $\mathbb{R}$ by

$$t_i := \sum_{l=i-1}^{\infty}\left(\prod_{j=i}^{l}\theta_j\right) = 1 + \sum_{l=i}^{\infty}\left(\prod_{j=i}^{l}\theta_j\right) \tag{3.2}$$

with the convention that $\prod_{j=i}^{i-1}\theta_j = 1 \ \forall i \geq 1$.

Remark 3.5 (See also Attouch and Cabot 2020). In view of assumption (3.1), the sequence $\{t_i\}$ given by (3.2) is well-defined and

$$t_i = 1 + \theta_i t_{i+1} \ \ \forall i \geq 1. \tag{3.3}$$

In the following proposition, we provide a criterion for ensuring assumption (3.1).

Proposition 3.6 (Attouch and Cabot 2020, Proposition 3.1). *Let $\{\theta_n\}$ be a sequence such that $\theta_n \in [0, 1)$ $\forall n \geq 1$. Assume that*

$$\lim_{n\to\infty} \left(\frac{1}{1-\theta_{n+1}} - \frac{1}{1-\theta_n} \right) = c,$$

for some $c \in [0, 1)$. Then,

(i) *Assumption (3.1) holds, and $t_{n+1} \sim \frac{1}{(1-c)(1-\theta_n)}$ as $n \to \infty$,*
(ii) *The equivalence $1-\theta_n \sim 1-\theta_{n+1}$ holds true as $n \to \infty$. Hence, $t_{n+1} \sim t_{n+2}$ as $n \to \infty$.*

Remark 3.7 An example of a sequence satisfying assumption (3.1) is $\theta_n = 1 - \frac{\hat{\theta}}{n}$, $\hat{\theta} > 1$.
Indeed, we have

$$\lim_{n\to\infty} \left(\frac{1}{1-\theta_{n+1}} - \frac{1}{1-\theta_n} \right) = \lim_{n\to\infty} \left(\frac{1}{\hat{\theta}}(n+1) - \frac{1}{\hat{\theta}} n \right) = \frac{1}{\hat{\theta}} \in [0, 1),$$

which by Proposition 3.6 implies that assumption (3.1) holds. Note that this example ($\theta_n = 1 - \frac{\hat{\theta}}{n}$, $\hat{\theta} > 1$) falls within the setting of Nesterov's extrapolation methods, and many practical choices for θ_n satisfy assumption (3.1) (see Attouch and Cabot 2020, Beck and Teboulle 2009, Chambolle and Dossal 2015, Nesterov 1983).

The corresponding finite sum expression of $\{t_i\}$ is defined for $i, n \geq 1$ by

$$t_{i,n} := \begin{cases} \sum_{l=i-1}^{n-1} \left(\prod_{j=i}^{l} \theta_j \right) = 1 + \sum_{l=i}^{n-1} \left(\prod_{j=i}^{l} \theta_j \right), & i \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Similarly to Remark 3.5, we see that $\{t_{i,n}\}$ is well-defined and

$$t_{i,n} = 1 + \theta_i t_{i+1,n} \ \forall i \geq 1, \ \ n \geq i+1. \tag{3.5}$$

Lemma 3.8 (Attouch and Cabot 2020, Lemma B.1). *Let $\{a_n\}$, $\{\theta_n\}$ and $\{b_n\}$ be sequences of real numbers satisfying*

$$a_{n+1} \leq \theta_n a_n + b_n \quad \text{for every} \ \ n \geq 1.$$

Assume that $\theta_n \geq 0$ for every $n \geq 1$.

(a) For every $n \geq 1$, we have

$$\sum_{i=1}^{n} a_i \leq t_{1,n} a_1 + \sum_{i=1}^{n-1} t_{i+1,n} b_i,$$

where $\{t_{i,n}\}$ is as defined in (3.4).

(b) Under (3.1), assume that the sequence $\{t_i\}$ defined by (3.2) satisfies $\sum_{i=1}^{\infty} t_{i+1}[b_i]_+ < \infty$. Then, the series $\sum_{i\geq 1} [a_i]_+$ is convergent and

$$\sum_{i=1}^{\infty} [a_i]_+ \leq t_1 [a_1]_+ + \sum_{i=1}^{\infty} t_{i+1} [b_i]_+ ,$$

where $[t]_+ := \max\{t, 0\}$ for any $t \in \mathbb{R}$.

Lemma 3.9 (Attouch and Cabot 2020, Lemma 2.1). *Let $\{x_n\}$ be a sequence in H and let $\{\theta_n\}$ be a sequence of real numbers. Given a point $z \in H$, define the sequence $\{\Gamma_n\}$ by $\Gamma_n := \frac{1}{2}\|x_n - z\|^2$. Then,*

$$\begin{aligned} &\Gamma_{n+1} - \Gamma_n - \theta_n(\Gamma_n - \Gamma_{n-1}) \\ &\quad = \frac{1}{2}(\theta_n + \theta_n^2)\|x_n - x_{n-1}\|^2 + \langle x_{n+1} - w_n, w_n - z \rangle \\ &\quad\quad + \frac{1}{2}\|x_{n+1} - w_n\|^2, \end{aligned} \tag{3.6}$$

where $w_n = x_n + \theta_n(x_n - x_{n-1})$.

Lemma 3.10 *We have*

$$\begin{aligned} 2\langle x, y \rangle &= \|x\|^2 + \|y\|^2 - \|x - y\|^2 \\ &= \|x + y\|^2 - \|x\|^2 - \|y\|^2 \ \forall x, y \in H. \end{aligned}$$

4 Main results

We begin by listing the assumptions that we use in our convergence analysis.

Assumption 4.1 (a) $\{\beta_n^k\}$ is a family of non-negative numbers with indices $n, k \in \mathbb{N}$, where $k \leq n$ and the following conditions are satisfied:

(i) $\sum_{k=1}^{n} \beta_n^k = 1 \ \ \forall n \geq 1$;
(ii) $\lim_{n\to\infty} \beta_n^k > 0 \ \ \forall k \geq 1$;

(iii) $\sum_{n=1}^{\infty}\sum_{k=1}^{n}|\beta_{n+1}^k-\beta_n^k|<\infty$;

(b) $\{S_k\}$ is a countable family of nonexpansive mappings such that $\cap_{k=1}^{\infty}F(S_k)\neq\emptyset$;

(c) $\theta_n\in[0,1)$ and $\alpha_n\in(0,1)$ for all $n\geq 1$ such that $\liminf_{n\to\infty}\alpha_n>0$. Assume further that there exists $\epsilon\in(0,1)$ such that for n large enough, we have

$$
\begin{aligned}
&(1-\epsilon)\frac{(1-\alpha_{n-1})}{\alpha_{n-1}}(1-\theta_{n-1})\\
&\quad\geq\theta_n t_{n+1}\Big(1+\theta_n+\Big[\frac{(1-\alpha_n)}{\alpha_n}(1-\theta_n)\\
&\quad-\frac{(1-\alpha_{n-1})}{\alpha_{n-1}}(1-\theta_{n-1})\Big]_+\Big).
\end{aligned}
\tag{4.1}
$$

Remark 4.2 Condition (a) in Assumption 4.1 is satisfied when (see Aoyama et al. 2007)

$$
\beta_n^k=\begin{cases}2^{-k}, & k<n\\ 2^{1-k}, & k=n\end{cases}
\tag{4.2}
$$

for $n,k\geq 1$ with $k\leq n$.

Remark 4.3 Condition (c) in Assumption 4.1 is different from the conditions imposed on $\{\theta_n\}$ in Bot et al. (2015), Dong et al. (2019), Dong et al. (2019), Dong et al. (2018), Dong (2020), Maingé (2008), Vong and Liu (2018).

We now introduce our new inertial-type method.

Algorithm 4.4 *Choose sequences $\{\theta_n\}$ and $\{\alpha_n\}$ such that the conditions from Assumption 4.1 hold. For arbitrary $x_0,x_1\in H$, let the sequence $\{x_n\}$ be generated by*

$$
\begin{cases}w_n=x_n+\theta_n(x_n-x_{n-1}),\\ x_{n+1}=(1-\alpha_n)w_n+\alpha_n\sum_{k=1}^{n}\beta_n^kS_kw_n,\ \forall n\geq 1.\end{cases}
\tag{4.3}
$$

We begin our analysis with the following lemmas.

Lemma 4.5 *Let Assumption 4.1 hold and let $\{x_n\}$ be a sequence generated by Algorithm 4.4. Then, the following inequality holds:*

$$
\begin{aligned}
&\Gamma_{n+1}-\Gamma_n\leq\theta_n(\Gamma_n-\Gamma_{n-1})\\
&\quad+\frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2-\frac{1}{2}\alpha_n(1-\alpha_n)\|w_n\\
&\quad-\sum_{k=1}^{n}\beta_n^kS_kw_n\|^2,
\end{aligned}
$$

where $\Gamma_n:=\frac{1}{2}\|x_n-z\|^2\ \forall z\in\cap_{k=1}^{\infty}F(S_k)$.

Proof Let $z\in\cap_{k=1}^{\infty}F(S_k)$. Then, from (4.3) and Lemma 3.10, we obtain

$$
\begin{aligned}
&\langle x_{n+1}-w_n,w_n-z\rangle\\
&\quad=-\alpha_n\langle w_n-\sum_{k=1}^{n}\beta_n^kS_kw_n,w_n-z\rangle\\
&\quad=-\alpha_n\langle w_n-\sum_{k=1}^{n}\beta_n^kS_kw_n,w_n-\sum_{k=1}^{n}\beta_n^kS_kw_n\rangle-\alpha_n\langle w_n\\
&\qquad-\sum_{k=1}^{n}\beta_n^kS_kw_n,\sum_{k=1}^{n}\beta_n^kS_kw_n-z\rangle\\
&\quad=-\alpha_n\|w_n-\sum_{k=1}^{n}\beta_n^kS_kw_n\|^2-\frac{1}{2}\alpha_n\Big[\|w_n-z\|^2-\|w_n\\
&\qquad-\sum_{k=1}^{n}\beta_n^kS_kw_n\|^2-\|\sum_{k=1}^{n}\beta_n^kS_kw_n-z\|^2\Big]\\
&\quad=-\frac{\alpha_n}{2}\|w_n-\sum_{k=1}^{n}\beta_n^kS_kw_n\|^2-\frac{\alpha_n}{2}\|w_n-z\|^2\\
&\qquad+\frac{\alpha_n}{2}\|\sum_{k=1}^{n}\beta_n^k(S_kw_n-z)\|^2\\
&\quad\leq-\frac{\alpha_n}{2}\|w_n-\sum_{k=1}^{n}\beta_n^kS_kw_n\|^2-\frac{\alpha_n}{2}\|w_n-z\|^2\\
&\qquad+\frac{\alpha_n}{2}\Big(\sum_{k=1}^{n}\beta_n^k\Big)^2\|w_n-z\|^2\\
&\quad=-\frac{\alpha_n}{2}\|w_n-\sum_{k=1}^{n}\beta_n^kS_kw_n\|^2.
\end{aligned}
\tag{4.4}
$$

Using (4.4) in Lemma 3.9 and noting that $x_{n+1}-w_n=\alpha_n(\sum_{k=1}^{n}\beta_n^kS_kw_n-w_n)$, we obtain

$$
\begin{aligned}
&\Gamma_{n+1}-\Gamma_n\leq\theta_n(\Gamma_n-\Gamma_{n-1})+\frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2\\
&\qquad-\frac{1}{2}\alpha_n\|w_n-\sum_{k=1}^{n}\beta_n^kS_kw_n\|^2\\
&\qquad+\frac{1}{2}\|x_{n+1}-w_n\|^2\\
&\quad=\theta_n(\Gamma_n-\Gamma_{n-1})\\
&\qquad+\frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2\\
&\qquad-\frac{1}{2}\alpha_n(1-\alpha_n)\|w_n-\sum_{k=1}^{n}\beta_n^kS_kw_n\|^2.
\end{aligned}
$$

$\square$

Lemma 4.6 *Let assumption (3.1) and Assumption 4.1 hold, and let $\{x_n\}$ be a sequence generated by Algorithm 4.4. Then,* *the following inequality holds:*

$$\sum_{i=1}^{n-1}\frac{(1-\alpha_{i-1})}{\alpha_{i-1}}(1-\theta_{i-1}) - \theta_i t_{i+1}\left(1+\theta_i+\left[\frac{(1-\alpha_i)}{\alpha_i}(1-\theta_i) - \frac{(1-\alpha_{i-1})}{\alpha_{i-1}}(1-\theta_{i-1})\right]_+\right)\|x_i-x_{i-1}\|^2$$
$$\le 2t_1|\Gamma_1-\Gamma_0| + 2\Gamma_0 + t_1\frac{(1-\alpha_0)}{\alpha_0}(1-\theta_0)\|x_1-x_0\|^2,$$

where t_i is as defined in (3.2).

Proof From (4.3) and Lemma 3.10, we obtain

$$\begin{aligned}
\|\sum_{k=1}^{n}\beta_n^k S_k w_n - w_n\|^2 &= \alpha_n^{-2}\|x_{n+1}-x_n-(x_n-x_{n-1}) +(1-\theta_n)(x_n-x_{n-1})\|^2\\
&= \alpha_n^{-2}\|x_{n+1}-2x_n+x_{n-1}\|^2 +\alpha_n^{-2}(1-\theta_n)^2\|x_n-x_{n-1}\|^2\\
&\quad +2\alpha_n^{-2}(1-\theta_n)\langle x_{n+1}-2x_n+x_{n-1}, x_n-x_{n-1}\rangle\\
&= \alpha_n^{-2}\|x_{n+1}-2x_n+x_{n-1}\|^2 +\alpha_n^{-2}(1-\theta_n)^2\|x_n-x_{n-1}\|^2\\
&\quad +\alpha_n^{-2}(1-\theta_n)\left[\|x_{n+1}-x_n\|^2 -\|x_n-x_{n-1}\|^2-\|x_{n+1}-2x_n+x_{n-1}\|^2\right]\\
&= \alpha_n^{-2}\theta_n\|x_{n+1}-2x_n+x_{n-1}\|^2 +\alpha_n^{-2}(1-\theta_n)^2\|x_n-x_{n-1}\|^2\\
&\quad +\alpha_n^{-2}(1-\theta_n)\left[\|x_{n+1}-x_n\|^2-\|x_n-x_{n-1}\|^2\right]\\
&\ge \alpha_n^{-2}(1-\theta_n)^2\|x_n-x_{n-1}\|^2 +\alpha_n^{-2}(1-\theta_n)\left[\|x_{n+1}-x_n\|^2 -\|x_n-x_{n-1}\|^2\right].
\end{aligned}\tag{4.5}$$

Using (4.5) in Lemma 4.5, we get

$$\begin{aligned}
\Gamma_{n+1}-\Gamma_n &\le \theta_n(\Gamma_n-\Gamma_{n-1}) + \frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2\\
&\quad -\frac{1}{2}\alpha_n^{-1}(1-\alpha_n)(1-\theta_n)^2\|x_n-x_{n-1}\|^2\\
&\quad -\frac{1}{2}\alpha_n^{-1}(1-\alpha_n)(1-\theta_n)\left[\|x_{n+1}-x_n\|^2-\|x_n-x_{n-1}\|^2\right].
\end{aligned}\tag{4.6}$$

It follows from Lemma 3.8(a) and (4.6) that

$$\begin{aligned}
\Gamma_n-\Gamma_0 = \sum_{i=1}^{n}(\Gamma_i-\Gamma_{i-1}) &\le t_{1,n}(\Gamma_1-\Gamma_0) + \sum_{i=1}^{n-1} t_{i+1,n}\Big[\Big(\frac{1}{2}(\theta_i+\theta_i^2)\\
&\quad -\frac{1}{2}\frac{(1-\alpha_i)}{\alpha_i}(1-\theta_i)^2\Big)\|x_i-x_{i-1}\|^2\\
&\quad -\frac{1}{2}\frac{(1-\alpha_i)}{\alpha_i}(1-\theta_i)\Big(\|x_{i+1}-x_i\|^2-\|x_i-x_{i-1}\|^2\Big)\Big].
\end{aligned}$$

Let $s_i := \frac{1-\alpha_i}{\alpha_i}$. Noting that $t_{i,n}\le t_i$ $\forall i$ and $\Gamma_n\ge 1$ $\forall n\ge 1$, we get

$$\begin{aligned}
&\sum_{i=1}^{n-1} t_{i+1,n}\Big[\Big(s_i(1-\theta_i)^2-(\theta_i+\theta_i^2)\Big)\|x_i-x_{i-1}\|^2\\
&\quad +s_i(1-\theta_i)\Big(\|x_{i+1}-x_i\|^2-\|x_i-x_{i-1}\|^2\Big)\Big]\\
&\le 2t_{1,n}(\Gamma_1-\Gamma_0)+2(\Gamma_0-\Gamma_n)\\
&\le 2t_1|\Gamma_1-\Gamma_0|+2\Gamma_0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&2t_1|\Gamma_1-\Gamma_0|+2\Gamma_0\\
&\ge \sum_{i=1}^{n-1} t_{i+1,n}\Big[\Big(s_i(1-\theta_i)^2-(\theta_i+\theta_i^2)\Big)\|x_i-x_{i-1}\|^2 + s_i(1-\theta_i)\Big(\|x_{i+1}-x_i\|^2-\|x_i-x_{i-1}\|^2\Big)\Big]\\
&= \sum_{i=1}^{n-1} t_{i+1,n}\Big(s_i(1-\theta_i)^2-(\theta_i+\theta_i^2)\Big)\|x_i-x_{i-1}\|^2\\
&\quad +\sum_{i=1}^{n-1}\Big(t_{i,n}s_{i-1}(1-\theta_{i-1})-t_{i+1,n}s_i(1-\theta_i)\Big)\|x_i-x_{i-1}\|^2\\
&\quad +t_{1,n}s_{i-1}(1-\theta_{n-1})\|x_n-x_{n-1}\|^2-t_{1,n}s_0(1-\theta_0)\|x_1-x_0\|^2\\
&\ge \sum_{i=1}^{n-1} t_{i+1,n}\Big(s_i(1-\theta_i)^2-(\theta_i+\theta_i^2)\Big)\|x_i-x_{i-1}\|^2\\
&\quad +\sum_{i=1}^{n-1}\Big(t_{i,n}s_{i-1}(1-\theta_{i-1})-t_{i+1,n}s_i(1-\theta_i)\Big)\|x_i-x_{i-1}\|^2\\
&\quad - t_{1,n}s_0(1-\theta_0)\|x_1-x_0\|^2\\
&\ge \sum_{i=1}^{n-1} t_{i+1,n}\Big(s_i(1-\theta_i)^2-(\theta_i+\theta_i^2)\Big)\|x_i-x_{i-1}\|^2\\
&\quad +\sum_{i=1}^{n-1}\Big(t_{i,n}s_{i-1}(1-\theta_{i-1})-t_{i+1,n}s_i(1-\theta_i)\Big)\|x_i-x_{i-1}\|^2\\
&\quad - t_1s_0(1-\theta_0)\|x_1-x_0\|^2\\
&= \sum_{i=1}^{n-1}\Big[t_{i+1,n}\Big(s_i(1-\theta_i)^2-(\theta_i+\theta_i^2)\Big)+t_{i,n}s_{i-1}(1-\theta_{i-1})\\
&\quad -t_{i+1,n}s_i(1-\theta_i)\Big]\|x_i-x_{i-1}\|^2\\
&\quad - t_1s_0(1-\theta_0)\|x_1-x_0\|^2.
\end{aligned}\tag{4.7}$$

Now, from (3.5), we obtain

$$\begin{aligned}
&t_{i+1,n}\Big(s_i(1-\theta_i)^2-(\theta_i+\theta_i^2)\Big)\\
&\quad+t_{i,n}s_{i-1}(1-\theta_{i-1})-t_{i+1,n}s_i(1-\theta_i)\\
&=t_{i+1,n}\big(s_i(1-\theta_i)^2-(\theta_i+\theta_i^2)\big)\\
&\quad+s_{i-1}(1-\theta_{i-1})+\theta_i t_{i+1,n}s_{i-1}(1-\theta_{i-1})\\
&\quad-t_{i+1,n}s_i(1-\theta_i)\\
&=s_{i-1}(1-\theta_{i-1})-\theta_i t_{i+1,n}\Big(s_i(1-\theta_i)\\
&\quad+1+\theta_i-s_{i-1}(1-\theta_{i-1})\Big)\\
&\geq s_{i-1}(1-\theta_{i-1})-\theta_i t_{i+1}\Big(1+\theta_i+\big[s_i(1-\theta_i)\\
&\quad-s_{i-1}(1-\theta_{i-1})\big]_+\Big). \qquad (4.8)
\end{aligned}$$

Combining (4.7) and (4.8), and noting that $s_i=\frac{1-\alpha_i}{\alpha_i}$ $\forall i$, we arrive at the desired conclusion. ◻

Lemma 4.7 *Let assumption (3.1) and Assumption 4.1 hold, and let $\{x_n\}$ be a sequence generated by Algorithm 4.4. Then, $\sum\limits_{n=1}^{\infty} t_{n+1}\theta_n\|x_n-x_{n-1}\|^2<\infty$.*

Proof We may assume without any loss of generality that inequality (4.1) holds true for all $n\geq 1$. That is,

$$\begin{aligned}
&(1-\epsilon)s_{n-1}(1-\theta_{n-1})\\
&\quad\geq\theta_n t_{n+1}\left(1+\theta_n+\big[s_n(1-\theta_n)-s_{n-1}(1-\theta_{n-1})\big]_+\right)\ \forall n\\
&\quad\geq 1, \qquad (4.9)
\end{aligned}$$

where $s_n=\frac{(1-\alpha_n)}{\alpha_n}$. This implies that

$$\begin{aligned}
&\epsilon s_{n-1}(1-\theta_{n-1})\leq s_{n-1}(1-\theta_{n-1})\\
&\quad-\theta_n t_{n+1}\Big(1+\theta_n+\big[s_n(1-\theta_n)-s_{n-1}(1-\theta_{n-1})\big]_+\Big)\ \forall n\\
&\quad\geq 1. \qquad (4.10)
\end{aligned}$$

Also, using (4.9), we obtain

$$\begin{aligned}
&\theta_n t_{n+1}\leq s_{n-1}(1-\theta_{n-1})\\
&\quad-\theta_n t_{n+1}\Big(\theta_n+\big[s_n(1-\theta_n)-s_{n-1}(1-\theta_{n-1})\big]_+\Big)\\
&\quad-\epsilon s_{n-1}(1-\theta_{n-1})\\
&\quad\leq s_{n-1}(1-\theta_{n-1}). \qquad (4.11)
\end{aligned}$$

Combining (4.10) and (4.11), we find that

$$\begin{aligned}
&\epsilon\theta_n t_{n+1}\leq s_{n-1}(1-\theta_{n-1})\\
&\quad-\theta_n t_{n+1}\Big(1+\theta_n+\big[s_n(1-\theta_n)-s_{n-1}(1-\theta_{n-1})\big]_+\Big)\ \forall n\\
&\quad\geq 1.
\end{aligned}$$

Now, replacing n with i and applying Lemma 4.6, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n-1}\epsilon\theta_i t_{i+1}\|x_i-x_{i-1}\|^2\leq 2t_1|\Gamma_1-\Gamma_0|\\
&\quad+2\Gamma_0+t_1s_0(1-\theta_0)\|x_1-x_0\|^2. \qquad (4.12)
\end{aligned}$$

Taking the limit as $n\to\infty$ in (4.12), we see that

$$\sum_{i=1}^{\infty}\theta_i t_{i+1}\|x_i-x_{i-1}\|^2<\infty,$$

which yields the desired conclusion. ◻

Lemma 4.8 *Let assumption (3.1) and Assumption 4.1 hold, and let $\{x_n\}$ be a sequence generated by Algorithm 4.4. Then, we have the following assertions:*

(a) $\lim\limits_{n\to\infty}\|x_n-z\|$ exists for all $z\in\cap_{k=1}^{\infty}F(S_k)$ and consequently, $\{x_n\}$ is bounded.
(b) $\lim\limits_{n\to\infty}\|x_n-w_n\|=0$.
(c) $\lim\limits_{n\to\infty}\|\sum_{k=1}^{n}\beta_n^k S_k w_n-w_n\|=0$.

Proof (a) From Lemma 4.5, it follows that

$$\begin{aligned}
\Gamma_{n+1}-\Gamma_n&\leq\theta_n\Big(\Gamma_n-\Gamma_{n-1}\Big)\\
&\quad+\frac{1}{2}\Big(\theta_n+\theta_n^2\Big)\|x_n-x_{n-1}\|^2\\
&\quad-\frac{1}{2}\alpha_n(1-\alpha_n)\|w_n-\sum_{k=1}^{n}\beta_n^k S_k w_n\|^2\\
&\leq\theta_n\Big(\Gamma_n-\Gamma_{n-1}\Big)+\theta_n\|x_n-x_{n-1}\|^2\\
&\quad-\frac{1}{2}\alpha_n(1-\alpha_n)\|w_n-\sum_{k=1}^{n}\beta_n^k S_k w_n\|^2 \qquad (4.13)\\
&\leq\theta_n\Big(\Gamma_n-\Gamma_{n-1}\Big)+\theta_n\|x_n-x_{n-1}\|^2. \qquad (4.14)
\end{aligned}$$

Applying Lemma 3.8(b) and Lemma 4.7 to (4.14), we get that $\sum\limits_{n=1}^{\infty}\Big[\Gamma_n-\Gamma_{n-1}\Big]_+<\infty$. Since $\Gamma_n=\frac{1}{2}\|x_n-z\|^2$, we conclude that $\lim\limits_{n\to\infty}\|x_n-z\|$ exists for all $z\in\cap_{k=1}^{\infty}F(S_k)$. Hence, $\{x_n\}$ is bounded, as asserted.

(b) From (4.3) and Lemma 4.7, we obtain

$$\sum_{n=1}^{\infty}t_{n+1}\|w_n-x_n\|^2\leq\sum_{n=1}^{\infty}t_{n+1}\theta_n\|x_n-x_{n-1}\|^2<\infty.$$

Since $t_n\geq 1\ \forall n\geq 1$, we conclude that $\lim\limits_{n\to\infty}\|w_n-x_n\|=0$.

(c) Applying Lemma 3.8(a) to (4.13), we obtain

$$\begin{aligned}\Gamma_n - \Gamma_0 &= \sum_{i=1}^{n}\left(\Gamma_i - \Gamma_{i-1}\right)\\ &\le t_{1,n}\Big(\Gamma_1 - \Gamma_0\Big)\\ &\quad + \sum_{i=1}^{n-1} t_{i+1,n}\Big[\theta_i\|x_i - x_{i-1}\|^2\\ &\quad - \frac{1}{2}\alpha_i(1-\alpha_i)\|w_i - \sum_{k=1}^{n}\beta_n^k S_k w_i\|^2\Big]\\ &\le t_1\big(\Gamma_1 - \Gamma_0\big)\\ &\quad + \sum_{i=1}^{n-1} t_{i+1}\theta_i\|x_i - x_{i-1}\|^2\\ &\quad - \frac{1}{2}\sum_{i=1}^{n-1} t_{i+1,n}\alpha_i(1-\alpha_i)\|w_i - \sum_{k=1}^{n}\beta_n^k S_k w_i\|^2,\end{aligned}$$

which implies by Lemma 4.7 that

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{n-1} t_{i+1,n}\alpha_i(1-\alpha_i)\|w_i - \sum_{k=1}^{n}\beta_n^k S_k w_i\|^2\\ &\quad \le \Gamma_0 + t_1(\Gamma_1 - \Gamma_0) + \sum_{i=1}^{n-1} t_{i+1}\theta_i\|x_i - x_{i-1}\|^2 < \infty.\end{aligned}$$

Since $t_{i+1,n} = 0$ for $i \ge n$, letting n tend to ∞, and applying the monotone convergence theorem, we get

$$\frac{1}{2}\sum_{i=1}^{\infty} t_{i+1}\alpha_i(1-\alpha_i)\|w_i - \sum_{k=1}^{n}\beta_n^k S_k w_i\|^2 < \infty. \quad (4.15)$$

Replacing i with n in (4.15), noting that $t_n \ge 1\ \forall n \ge 1$, and using the condition imposed on $\{\alpha_n\}$, we obtain from (4.15) that $\lim\limits_{n\to\infty}\|w_n - \sum_{k=1}^{n}\beta_n^k S_k w_n\| = 0$, as asserted. □

Now, we give the weak convergence result below.

Theorem 4.9 *Let assumption (3.1) and Assumption 4.1 hold, and let $\{x_n\}$ be a sequence generated by Algorithm 4.4. Then, $\{x_n\}$ converges weakly to a common fixed point of $\{S_k\}$.*

Proof Let $z \in \cap_{k=1}^{\infty} F(S_k)$. Since S_k is nonexpansive for each k, we have

$$\|S_k x - z\| \le \|x - z\| \ \forall x \in H, z \in \cap_{k=1}^{\infty} F(S_k).$$

Therefore,

$$\sum_{k=1}^{n}\|\beta_n^k S_k x\| \le \|x - z\| + \|z\|,$$

because $\sum_{k=1}^{n}\beta_n^k = 1$. Now, define

$$T_n x := \sum_{k=1}^{n}\beta_n^k S_k x, \ \ x \in H.$$

It follows from condition (a)(i) in Assumption 4.1 that T_n is nonexpansive for each n. Clearly, $\cap_{k=1}^{\infty} F(S_k) \subset \cap_{n=1}^{\infty} F(T_n)$. Using condition (a)(ii), we see that for each $k \ge 1$, there exists $n_0 \ge 1$ such that $\beta_{n_0}^k > 0$. Hence, $F(T_{n_0}) \subset F(S_k)\ \forall k \ge 1$ by Lemma 3.4, and so $\cap_{n=1}^{\infty} F(T_n) \subset F(S_k)\ \forall k \ge 1$. Therefore,

$$\cap_{n=1}^{\infty} F(T_n) = \cap_{k=1}^{\infty} F(S_k) \ne \emptyset.$$

Let B be a bounded subset of H. Since $\cap_{k=1}^{\infty} F(S_k) \ne \emptyset$, we know that $\{S_k x : x \in B, k \ge 1\}$ is bounded. Define $M := \sup\{\|S_k x\| : x \in B, k \ge 1\}$. Then for each $x \in B$ and $n \ge 1$, we have

$$\begin{aligned}\|T_{n+1}x - T_n x\| &= \|\sum_{k=1}^{n+1}\beta_{n+1}^k S_k x - \sum_{k=1}^{n}\beta_n^k S_k x\|\\ &\le \sum_{k=1}^{n}|\beta_{n+1}^k - \beta_n^k|\|S_k x\| + \beta_{n+1}^{n+1}\|S_{k+1}x\|\\ &\le \sum_{k=1}^{n}|\beta_{n+1}^k - \beta_n^k|M + \Big(1 - \sum_{k=1}^{n}\beta_{n+1}^k\Big)M\\ &= \sum_{k=1}^{n}|\beta_{n+1}^k - \beta_n^k|M + \Big(\sum_{k=1}^{n}\beta_n^k - \sum_{k=1}^{n}\beta_{n+1}^k\Big)M\\ &\le 2M\sum_{k=1}^{n}|\beta_{n+1}^k - \beta_n^k|. \quad (4.16)\end{aligned}$$

Therefore,

$$\sup\{\|T_{n+1}x - T_n x\| : x \in B\} \le 2M\sum_{k=1}^{n}|\beta_{n+1}^k - \beta_n^k|.$$

Using condition (a)(iii), we obtain that

$$\begin{aligned}&\sum_{n=1}^{\infty}\sup\{\|T_{n+1}x - T_n x\| : x \in B\}\\ &\quad \le 2M\sum_{n=1}^{\infty}\sum_{k=1}^{n}|\beta_{n+1}^k - \beta_n^k| < \infty. \quad (4.17)\end{aligned}$$

Now, for $l, m \ge 1$ with $l > m$, we get

$$|\beta_l^k - \beta_m^k| \le \sum_{n=m}^{l-1}|\beta_{n+1}^k - \beta_n^k|$$

for all $k \geq 1$ and $k \leq m$. Hence,

$$\sum_{k=1}^{m} |\beta_l^k - \beta_m^k| \leq \sum_{k=1}^{m}\sum_{n=m}^{l-1} |\beta_{n+1}^k - \beta_n^k|$$
$$= \sum_{n=m}^{l-1}\sum_{k=1}^{m} |\beta_{n+1}^k - \beta_n^k|$$
$$\leq \sum_{n=m}^{l-1}\sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k|.$$

Taking the limit, we get

$$\sum_{k=1}^{m} |\beta^k - \beta_m^k| \leq \sum_{n=m}^{\infty}\sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty,$$

where $\beta^k := \lim_{n\to\infty} \beta_n^k,\ \ k \geq 1$. Thus,

$$\lim_{m\to\infty} \sum_{k=1}^{m} |\beta^k - \beta_m^k| = 0.$$

Therefore,

$$0 \leq |\sum_{k=1}^{\infty} \beta^k - 1| = \lim_{m\to\infty} |\sum_{k=1}^{m} \beta^k - 1|$$
$$= \lim_{m\to\infty} |\sum_{k=1}^{m} \beta^k - \sum_{k=1}^{m} \beta_m^k|$$
$$\leq \lim_{m\to\infty} \sum_{k=1}^{m} |\beta^k - \beta_m^k| = 0,$$

and so $\sum_{k=1}^{\infty} \beta^k = 1$.

Next, define a mapping T by

$$Tx := \sum_{k=1}^{\infty} \beta^k S_k x,\ \ x \in H.$$

It follows from Lemma 3.4 that T is a nonexpansive self-mapping of H such that

$$F(T) = \cap_{k=1}^{\infty} F(S_k) = \cap_{n=1}^{\infty} F(T_n)$$

since $\beta^k > 0\ \ \forall k \geq 1$. Furthermore,

$$\|T_m x - Tx\| = \|\sum_{k=1}^{m} \beta_m^k S_k x - \sum_{k=1}^{\infty} \beta^k S_k x\|$$
$$\leq \|\sum_{k=1}^{m} (\beta_m^k - \beta^k) S_k x\| + \sum_{k=m+1}^{\infty} \|\beta^k S_k x\|$$
$$\leq \sum_{k=1}^{m} |\beta_m^k - \beta^k| M + \sum_{k=m+1}^{\infty} \beta^k M.$$

Passing to the limit, we see that

$$\lim_{m\to\infty} T_m x = Tx,\ \ x \in H. \tag{4.18}$$

As a matter of fact, it also follows directly from (4.17) that the sequence $\{T_n x\}$ is Cauchy for each $x \in H$. Indeed, let $k, l \geq 1$ be such that $k > l$. Then,

$$\|T_k x - T_l x\| \leq \sup\{\|T_k u - T_l u\| : u \in B\}$$
$$\leq \sup\{\|T_k u - T_{k-1} u\| : u \in B\}$$
$$+ \sup\{\|T_{k-1} u - T_l u\| : u \in B\}$$
$$\vdots$$
$$\leq \sum_{n=l}^{k-1} \sup\{\|T_{n+1} u - T_n u\| : u \in B\}$$
$$\leq \sum_{n=l}^{\infty} \sup\{\|T_{n+1} u - T_n u\| : u \in B\}.$$

Hence, the sequence $\{T_n x\}$ is Cauchy and converges strongly to a point Tx in H. Furthermore,

$$\|Tx - T_l x\| = \lim_{k\to\infty} \|T_k x - T_l x\|$$
$$\leq \sum_{n=l}^{\infty} \sup\{\|T_{n+1} u - T_n u\| : u \in B\}$$

for all $x \in H$ and $l \geq 1$. So,

$$\sup\{\|Tx - T_l x\| : x \in B\}$$
$$\leq \sum_{n=l}^{\infty} \sup\{\|T_{n+1} u - T_n u\| : u \in B\}.$$

Thus,

$$\lim_{l\to\infty} \sup\{\|Tx - T_l x\| : x \in B\} = 0. \tag{4.19}$$

Now,

$$\|Tw_n - w_n\| \leq \|Tw_n - T_n w_n\|$$
$$+ \|x_{n+1} - x_n\| + \|x_n - T_n w_n\|$$
$$+ \|x_n - x_{n+1}\| + \|x_n - w_n\|$$
$$= \|Tw_n - T_n w_n\| + 2\|x_{n+1} - x_n\|$$
$$+ \|x_n - w_n\| + \|x_n - T_n w_n\|$$
$$\leq \sup\{\|Tx - T_n x\| : x \in B\} + 2\|x_{n+1} - x_n\|$$
$$+ \|x_n - w_n\| + \|x_n - T_n w_n\|. \tag{4.20}$$

Using Lemma 4.8 (b) and (c), we get

$$\begin{aligned}\|x_n - T_n w_n\| &= \|x_n - \sum_{k=1}^{n} \beta_n^k S_k w_n\| \\ &\le \|\sum_{k=1}^{n} \beta_n^k S_k w_n - w_n\| \\ &\quad + \|x_n - w_n\| \to 0 \ \text{ as } \ n \to \infty. \end{aligned} \tag{4.21}$$

Also, note that

$$\|\sum_{k=1}^{n} \beta_n^k S_k w_n - w_n\| = \frac{1}{\alpha_n}\|x_{n+1} - x_n\|.$$

Using Lemma 4.8 (c) and $\liminf_{n\to\infty} \alpha_n > 0$, we see that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.22}$$

Next, using Lemma 4.8 (b), (4.19), (4.21) and (4.22), we infer from (4.20) that

$$\lim_{n\to\infty} \|T w_n - w_n\| = 0. \tag{4.23}$$

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup p \in H$. Since $(x_n - w_n) \to 0$ as $n \to \infty$, we obtain that $w_{n_j} \rightharpoonup p \in H$. Now (4.23) and Lemma 3.3 imply that $p \in F(T) = \cap_{k=1}^{\infty} F(S_k)$. Summing up, we have established that

(i) $\lim_{n\to\infty} \|x_n - z\|$ exists for each $z \in \cap_{k=1}^{\infty} F(S_k)$;
(ii) any weak subsequential limit point of $\{x_n\}$ belongs to $\cap_{k=1}^{\infty} F(S_k)$. It now follows from Lemma 3.1 that the whole sequence $\{x_n\}$ converges weakly to a point in $\cap_{k=1}^{\infty} F(S_k)$, as asserted. $\square$

For a single nonexpansive mapping, we have the following corollary from Theorem 4.9.

Corollary 4.10 *Let $S : H \to H$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\theta_n\}$ satisfy Assumption 4.1 (c). Suppose $\{x_n\}$ is generated as follows: $x_0, x_1 \in H$,*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (1-\alpha_n)w_n + \alpha_n S w_n, \ \forall n \ge 1. \end{cases}$$

Then, $\{x_n\}$ converges weakly to a fixed point of S.

We give some comments about our contributions in this paper as follows.

Remark 4.11 (a) In Condition (iii) of (Bot and Meier 2021, Theorem 2.1), some strong assumptions, which might be difficult to check in applications, are imposed on the countable families of nonexpansive mappings in order to obtain convergence. In our convergence analysis of Theorem 4.9, we do not impose any such extra conditions on the countable families of nonexpansive mappings in order to obtain weak convergence.
(b) Our method of proofs in Lemmas 4.5,4.6, 4.7, 4.8, Theorem 4.9 and the imposed conditions on the iterative parameters in Assumption 4.1 are different from the methods of proofs and the conditions imposed in Bot et al. (2015), Dong (2020), Dong et al. (2018), Dong et al. (2019), Dong et al. (2019), Maingé (2008), Vong and Liu (2018).

Remark 4.12 It is worth noting that many practical choices for the inertial sequence $\{\theta_n\}$ and the coefficients $\{\alpha_n\}$ satisfy (4.1). In fact, in the following propositions, we present several ways of choosing $\{\theta_n\}$ and $\{\alpha_n\}$ so that (4.1) is satisfied.

Proposition 4.13 *Assume that $\{\theta_n\}$ is an increasing real sequence that satisfies $\theta_n \in [0, 1)$ $\forall n \ge 1$ with $\lim_{n\to\infty} \theta_n = \theta$ such that the following condition holds:*

$$1 - 3\theta > 0.$$

Then, (4.1) is satisfied.

Proposition 4.14 *Suppose that $\theta_n \in [0, 1)$ $\forall n \ge 1$ and that there exists $c \in [0, \frac{1}{2})$ such that*

$$\lim_{n\to\infty}\left(\frac{1}{1-\theta_{n+1}} - \frac{1}{1-\theta_n}\right) = c$$

and

$$\liminf_{n\to\infty}(1-\theta_n)^2 > \limsup_{n\to\infty} \frac{\theta_n(1+\theta_n)}{1-2c}.$$

Then, (4.1) is satisfied.

Proposition 4.15 *Suppose that $\theta_n \in [0, 1)$ for all $n \ge 1$. Assume that there exist $c \in [0, 1)$ and $\bar{c} \in [c, 1)$ such that*

$$\lim_{n\to\infty}\left(\frac{1}{1-\theta_{n+1}} - \frac{1}{1-\theta_n}\right) = c, \\ \lim_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\gamma_n(1-\theta_n)} = \bar{c}$$

and

$$\liminf_{n\to\infty} \gamma_n(1-\theta_n)^2 > \limsup_{n\to\infty} \frac{\theta_n(1+\theta_n)}{1-\bar{c}},$$

where $\gamma_n = \frac{1-\alpha_n}{2\alpha_n}$. Then, (4.1) is satisfied.

The proofs of Propositions 4.13–4.15 can be derived by slightly modifying the proofs in Attouch and Cabot (2020).

5 Applications

In this section, we use Theorem 4.9 in order to solve monotone inclusion problems.

Recall that an operator $A : D(A) \subset H \to 2^H$ is said to be monotone if

$$\langle u - v, x - y\rangle \geq 0 \;\; \forall x, y \in D(A), \;\; u \in Ax, v \in Ay,$$

and maximal monotone if it is monotone and its graph

$$G(A) := \{(x, y) : x \in D(A), y \in Ax\}$$

is not properly contained in the graph of any other monotone operator.

The resolvent operator $J_{\gamma A}$ associated with A and $\gamma > 0$ is the mapping $J_{\gamma A} : H \to 2^H$ defined by

$$J_{\gamma A}(x) := (I + \gamma A)^{-1}x, \;\; x \in H, \gamma > 0. \tag{5.1}$$

The resolvent operator $J_{\gamma A}$ is single-valued, nonexpansive and everywhere defined on H whenever A is maximal monotone (see, for example, Bauschke and Combettes 2011). Moreover, $x \in A^{-1}(0)$ (where $A^{-1}(0) := \{x \in H : 0 \in A(x)\}$) if and only if $x = J_{\gamma A}(x) \;\; \forall \gamma > 0$.

Let us consider the following monotone inclusion problem:

$$\text{Find } x \in H \text{ such that } 0 \in Ax, \tag{5.2}$$

where A is a maximal monotone operator defined on H.

Corollary 5.1 *Let $A : H \to 2^H$ be a maximal monotone operator such that $A^{-1}(0) \neq \emptyset$. Suppose $\{x_n\}$ is generated by: $x_0, x_1 \in H$,*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (1-\alpha_n)w_n + \alpha_n J_{\gamma A} w_n \;\; \forall n \geq 1, \end{cases} \tag{5.3}$$

where $\theta_n \in [0, 1)$, $\alpha_n \in (0, 2)$ for all $n \geq 1$ and $\liminf_{n\to\infty} \alpha_n > 0$. Assume that there exists $\epsilon \in (0, 1)$ such that for n large enough, we have

$$\begin{aligned}(1-\epsilon)&\frac{(2-\alpha_{n-1})}{\alpha_{n-1}}(1-\theta_{n-1}) \\ &\geq \theta_n t_{n+1}\left(1 + \theta_n + \left[\frac{(2-\alpha_n)}{\alpha_n}(1-\theta_n) \right.\right.\\ &\quad \left.\left. -\frac{(2-\alpha_{n-1})}{\alpha_{n-1}}(1-\theta_{n-1})\right]_+\right).\end{aligned}$$

Then, $\{x_n\}$ converges weakly to a point in $A^{-1}(0)$.

Proof Let $x^* \in A^{-1}(0)$. Observe that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(w_n - x^*) + \alpha_n(J_{\gamma A}w_n - w_n)\|^2 \\ &= \|w_n - x^*\|^2 + 2\alpha_n\langle w_n - x^*, J_{\gamma A}w_n - w_n\rangle \\ &\quad + \alpha_n^2\|J_{\gamma A}w_n - w_n\|^2 \\ &\leq \|w_n - x^*\|^2 - \alpha_n(2-\alpha_n)\|J_{\gamma A}w_n - w_n\|^2.\end{aligned} \tag{5.4}$$

Following similar arguments to those used above (from Lemma 4.5 to Theorem 4.9) and noting that $A^{-1}(0) = F(J_{\gamma A})$, we obtain the desired conclusion. □

Next, we apply our results to the Douglas–Rachford splitting method for solving the following monotone inclusion problem involving two operators:

$$\text{Find } x \in H \text{ such that } 0 \in (A + B)x, \tag{5.5}$$

where $A, B : H \to 2^H$ are maximal monotone operators on a Hilbert space H (see Douglas and Rachford 1956, Lions and Mercier 1979).

Define

$$R_{\gamma A} := 2J_{\gamma A} - I \quad \text{and} \quad R_{\gamma B} := 2J_{\gamma B} - I$$

for the corresponding *reflections* (also called *Cayley operators*), and observe that these reflections are nonexpansive mappings.

One can show that $0 \in Ax + Bx$ if and only if $x = J_{\gamma B}(y)$, where y is a fixed point of the nonexpansive mapping $R_{\gamma A}R_{\gamma B}$. The Douglas–Rachford splitting method for solving Problem (5.5) is as follows:

$$x_{n+1} := (1-\alpha_n)x_n + \alpha_n R_{\gamma A}R_{\gamma B}x_n, \quad n \geq 1. \tag{5.6}$$

Modifying (5.6), we obtain the following weak convergence result.

Theorem 5.2 *Let $\gamma \in (0, \infty)$ and assume that $0 \in \operatorname{ran}(A + B)$. Assume that $\{\alpha_n\}$ and $\{\theta_n\}$ satisfy Assumption 4.1 (c). Suppose that the sequence $\{x_n\}$ is generated by: $x_0, x_1 \in H$,*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (1-\alpha_n)w_n + \alpha_n(2J_{\gamma_n A} - I)(2J_{\gamma_n B} - I)w_n \;\; \forall n \geq 1. \end{cases} \tag{5.7}$$

Then, $\{x_n\}$ converges weakly to some point $y \in H$ such that $J_{\gamma B}y \in (A + B)^{-1}(0)$, that is, $x := J_{\gamma B}y$ is a solution to the monotone inclusion problem (5.5).

Proof Define $T := R_{\gamma A}R_{\gamma B}$. Then, T is nonexpansive. Applying Theorem 4.9, we reach the desired conclusion. □

6 Numerical experiments

In this section, our focus is on the numerical implementation of our proposed algorithm regarding some of the applications which are presented in Sect. 5. This also involves comparing its performance with that of some algorithms in the literature. The codes are written in MATLAB 2016 (b) running on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30GHz and 8.00 Gb-RAM.

Example 6.1 In this example, we consider Problem (5.5) and use Algorithm (5.7) for the numerical experiments.

As for γ_n, we adopt the method in Dong and Fischer (2010), Dong (2020) for updating γ_n in the following manner. We assume, in addition, that A is continuous. At the n-th iterative step, with a known γ_n, we first calculate

$$\rho_n := \frac{\gamma_n \|A(x_n) - A(x_{n-1})\|}{\|x_n - x_{n-1}\|},$$

and then, we update γ_n by

$$\gamma_{n+1} = \begin{cases} 1.5\gamma_n, & \text{if } \rho_n \leq 0.5, \\ \gamma_n, & \text{if } 0.5 < \rho_n < 2, \\ 0.5\gamma_n, & \text{if } \rho_n \geq 2. \end{cases} \tag{6.1}$$

Let

$$Q := \{x \ x_i \geq 0, \ i = 1, \cdots, N\} \text{ andlet}$$

$$A_1 = \begin{pmatrix} D & -I & & & & \\ -I & D & -I & & & \\ & \cdots & \ldots & & & \\ & & \ldots & \ldots & & \\ & & -I & D & -I & \\ & & & -I & D & \end{pmatrix},$$

$$A_2 = \frac{hc}{2} \begin{pmatrix} O & I & & & & \\ -I & O & I & & & \\ & \cdots & \ldots & & & \\ & & \ldots & \ldots & & \\ & & -I & O & I & \\ & & & -I & O & \end{pmatrix}$$

be two $N \times N$ block tridiagonal matrices, where $N = m^2$, $h = \dfrac{1}{m+1}$, $c = 100$, I is the $m \times m$ identity matrix, and D is the $m \times m$ matrix $4I + M + M^T$ with $M_{i,j} = 1$ if $j = i + 1, \ i = 1, \cdots, m-1$, and $M_{i,j} = 0$ otherwise.
Now, let $A = A_1 + A_2$ and consider the following complementarity problem which is also discussed in Dong (2020, Section 5);

$$0 \in Ax - b + \partial\delta_Q(x)$$

where $\partial\delta_Q$ is the sub-differential of the indicator function defined by

$$\delta_Q(x) = \begin{cases} 0, & \text{if } x \in Q, \\ +\infty, & \text{if } x \notin Q. \end{cases}$$

Setting $b = Ae_1$, where e_1 is the first column of the identity matrix of order $N \times 1$ corresponding to the matrix A, we see that $x^* = e_1$ is the unique solution to the complementarity problem. Thus, we set

$$b := Ae_1, \qquad A(x) := Ax - b, \qquad B(x) := \partial\delta_Q(x).$$

Then, $J_{\gamma_n B}(w_n) = P_Q(w_n) = \max\{0, w_n\}$.
The results of the numerical experiments are presented in Figs. 1 and 2, and in Tables 1 and 2.

Example 6.2 (**Image Restoration Problem**). Consider the following problem:

$$\min_{x \in \mathbb{R}^N} \{||Dx - b||_2^2 + \gamma ||x||_1\}, \tag{6.2}$$

where $\gamma > 0$, $x \in \mathbb{R}^N$ is the original image to be recovered, $b \in \mathbb{R}^M$ is the observed image and $D : \mathbb{R}^N \to \mathbb{R}^M$ is the blurring operator. In order to solve the above problem via numerical computations, we choose the regularization parameter $\gamma = 5e^{-2}$. Furthermore, we consider the 256×256 Cameraman image and the 128×128 medical resonance imaging (MRI) from the MATLAB Image Processing Toolbox. Moreover, we use the Gaussian blur of size 9×9 and standard deviation $\sigma = 4$ to create the blurred and noisy image (observed image). Also, we measure the quality of the restored image using the signal-to-noise ratio defined by

$$\text{SNR} = 20 \times \log_{10}\left(\frac{\|x\|_2}{\|x - x^*\|_2}\right),$$

where x is the original image and x^* is the restored image. Note that the larger the SNR, the better the quality of the restored image. We also choose the initial values as $x_0 = \mathbf{0} \in \mathbb{R}^{N \times N}$ and $x_1 = \mathbf{1} \in \mathbb{R}^{N \times M}$.
Our results are reported in Figs. 3 and 4, and in Tables 3 and 4. Figures 3 and 4 show the original, blurred and restored images, while Tables 3 and 4 show the CPU time and the SNR values for each algorithm.

For the numerical comparisons, we employ different choices of θ_n and α_n as chosen in Bot et al. (2015), Dong et al. (2018), Dong (2020), Maingé (2008), Vong and Liu (2018). We conduct the following experiments.

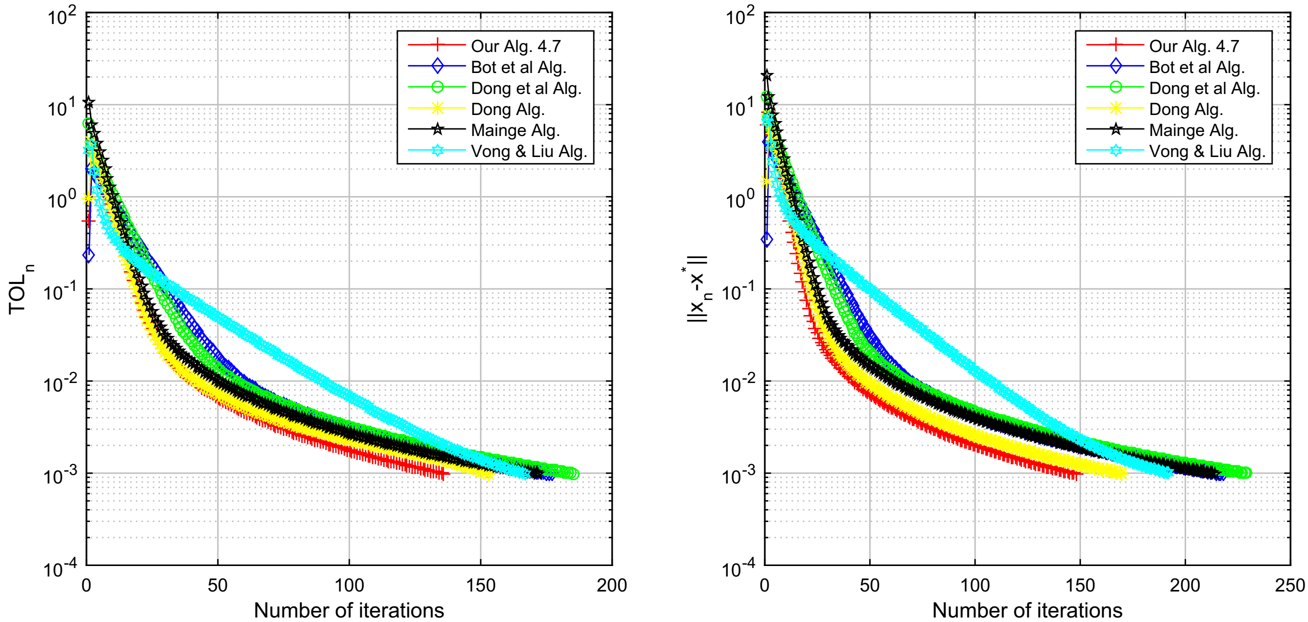

Fig. 1 Comparison for Example 6.1 (Experiment 1): Left: $N = 400$; Right: $N = 1600$

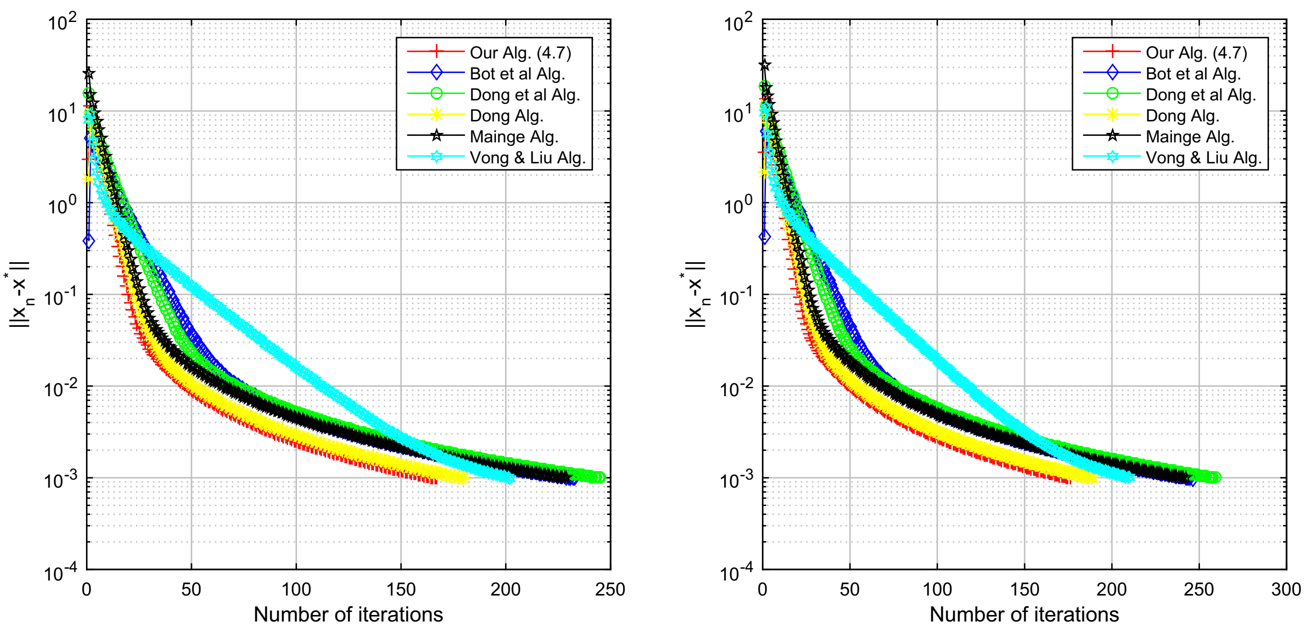

Fig. 2 Comparison for Example 6.1 (Experiment 2): Left: $N = 2500$; Right: $N = 3600$

Table 1 Numerical results for Example 6.1 (Experiment 1)

N		Alg. (5.7)	Bot et al Alg	Dong et al Alg	Dong Alg	Maingé Alg	Vong & Liu Alg
400	CPU Iter	1.0433	1.9486 177	2.2572	1.6639	1.9411	1.9003
		136	177	185	153	172	167
1600	CPU Iter	21.0291	30.3693	32.1375	23.5890	29.9312	26.9191
		156	218	229	170	214	192

Table 2 Numerical results for Example 6.1 (Experiment 2)

N		Alg. (5.7)	Bot et al Alg	Dong et al Alg	Dong Alg	Maingé Alg	Vong & Liu Alg
2500	CPU Iter	57.0660	78.6224	82.3426	60.4150	77.6415	68.2888
		167	233	245	180	229	202
3600	CPU Iter	134.1133	171.8811	184.0907	133.1440	191.7407	150.7633
		176	246	259	189	242	210

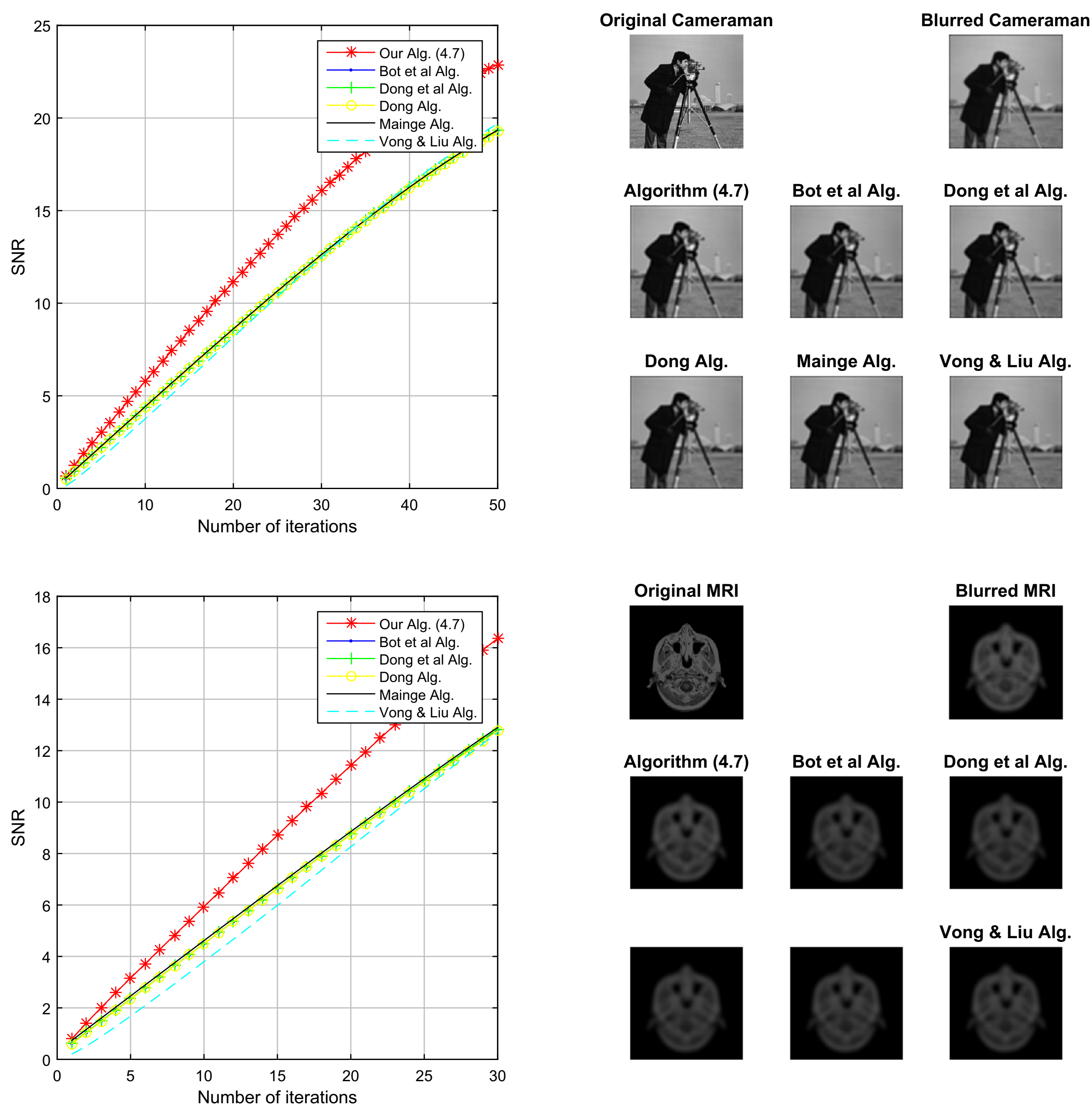

Fig. 3 Comparison for Example 6.2 (Experiment 1): Top: Cameraman; Bottom: MRI

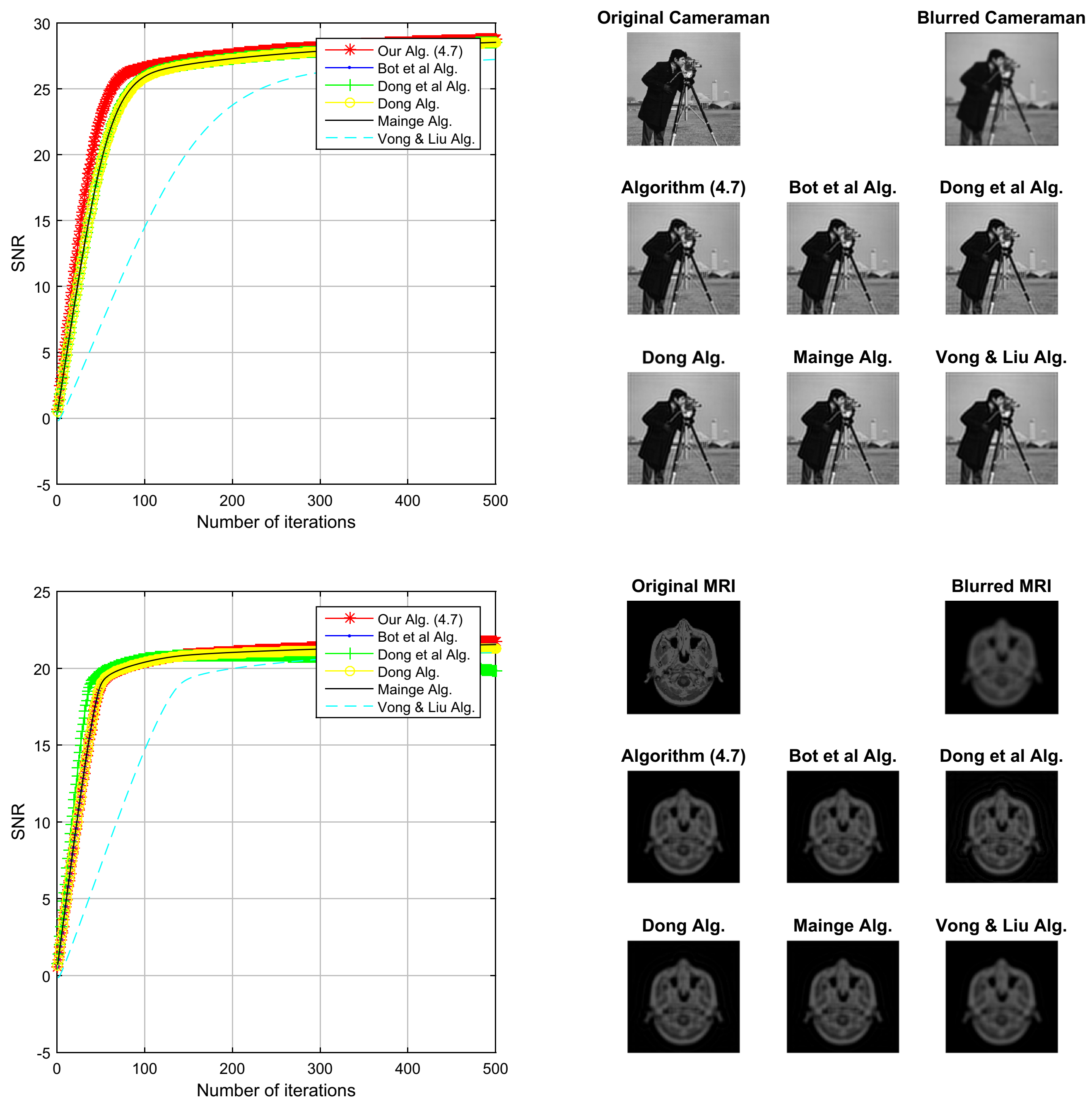

Fig. 4 Comparison for Example 6.2 (Experiment 2): Top: Cameraman; Bottom: MRI

Experiment 1

Algorithm (5.7): $\theta_n = \frac{19}{20} - \frac{1}{(n+1)^{\frac{1}{2}}}$ and $\alpha_n = \frac{1}{20} + \frac{1}{(n+1)^2}$.

(Bot et al. 2015, Algorithm 5): $\theta_n = 0.11$ and $\alpha_n = 0.8$.

(Dong et al. 2018, Algorithm 3.1): $\theta_n = 0.5$, $\mu = 0.4$, $\lambda = 0.5$, $\gamma_n = 0.9$ and $\beta_n = \frac{1}{(n+1)^2}$.

(Dong 2020, Algorithm 1 in Section 4): $\theta_n = 0.21$ and $\alpha_n = 0.7$.

(Maingé 2008, Algorithm 1.2):

$\theta_n = \min\{0.9, \frac{1}{n^{0.01}(n+1)||x_n - x_{n-1}||^2}\}$ and $\alpha_n = \frac{9n}{10n+1}$.

(Vong and Liu 2018, (rmiMann)): $\theta_n = 0.5$, $\mu = 0.4$, $\lambda = 0.5$, $\gamma_n = 0.9$, $\beta_n = \frac{1}{(n+1)^2}$ and $\xi_n = \frac{2}{(n+1)^{1+0.01}}$.

For Example 6.1, we choose in this experiment, $x_1 = ones(N, 1)$ and $x_0 = zeros(N, 1)$ with $N \in \{400, \quad 1600\}$. We use the stopping criterion $\|x_n - x^*\| < 10^{-3}$. The numerical results for this experiment are given in Table 1, Table 3, and Figs. 1 and 3.

Table 3 Numerical results for Example 6.2 (Experiment 1)

Algorithms	Cameraman		MRI	
	CPU	SNR	CPU	SNR
Alg. (5.7)	0.5115	23.0193	0.1063	16.3613
Bot et al Alg	1.1789	19.2980	0.3273	12.7975
Dong et al Alg	1.2531	19.3447	0.4366	12.8332
Dong Alg	1.2559	19.2528	0.3302	12.7639
Maingé Alg	1.2091	19.3465	0.3335	12.9002
Vong & Liu Alg	1.1615	20.5500	0.3327	12.7093

Table 4 Numerical results for Example 6.2 (Experiment 2)

Algorithms	Cameraman		MRI	
	CPU	SNR	CPU	SNR
Alg. (5.7)	10.0794	29.3923	1.3977	22.6102
Bot et al Alg	14.6114	28.5300	3.3212	21.5432
Dong et al Alg	17.9912	28.5362	3.3575	19.8321
Dong Alg	17.9181	28.5185	3.6683	21.3077
Maingé Alg	16.9046	28.5375	3.4686	21.5423
Vong & Liu Alg	15.7208	27.2374	3.9504	21.0194

Experiment 2

Algorithm (5.7): $\theta_n = \frac{9.9}{10} - \frac{1}{(n+1)^{\frac{1}{3}}}$ and $\alpha_n = \frac{1}{11} + \frac{1}{(n+1)}$.
(Bot et al. (2015), Algorithm 5): $\theta_n = 0.04$ and $\alpha_n = 0.9$.
(Dong et al. 2018, Algorithm 3.1): $\theta_n = \frac{n}{2n+1}$, $\mu = 0.8$, $\lambda = 1$, $\gamma_n = 0.9$ and $\beta_n = \frac{1}{(n+1)^{1+1.2}}$.
(Dong 2020, Algorithm 1 in Section 4): $\theta_n = 0.14$ and $\alpha_n = 0.8$.
(Maingé 2008, Algorithm 1.2):
$\theta_n = \min\{0.9, \frac{1}{n^{0.01}(n+1)||x_n - x_{n-1}||}\}$ and $\alpha_n = \frac{9n}{10n+1}$.
(Vong and Liu 2018, rmiMann): $\theta_n = \frac{n}{2n+1}$, $\mu = 0.8$, $\lambda = 1$, $\gamma_n = 0.9$, $\beta_n = \frac{1}{(n+1)^{1+1.2}}$ and $\xi_n = \frac{1}{e^{2n}}$.
For Example 6.1, we choose in this experiment, $x_1 = ones(N, 1)$ and $x_0 = zeros(N, 1)$ with $N \in \{2500, \ 3600\}$.
We use the stopping criterion $\|x_n - x^*\| < 10^{-3}$.
The numerical results for this experiment are given in Table 2, Table 4, Figs. 2 and 4.

7 Final remarks

In this paper, we have presented a modification of the inertial Krasnosel'skiĭ-Mann iterative method for approximating a common fixed point of a countable family of nonexpansive mappings with new conditions on the iterative parameters. Using this method, we have obtained weak convergence results in Hilbert spaces. We have applied our results to monotone inclusion problems and to the Douglas–Rachford splitting method for finding a zero of the sum of two maximal monotone operators. We have also presented several numerical implementations which illustrate our theoretical analysis.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval All the authors approved the final version of this manuscript.

Informed consent All the authors understand the purpose of the research and their individual roles.

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