

https://dspace.org/

The laminar two-fluid wake

Pillay, Kiara

2022
https://hdl.handle.net/10539/34596

Downloaded from a DSpace repository. https://dspace.org/

University of the Witwatersrand

School of Computer Science and Applied Mathematics

Masters Dissertation

The laminar two-fluid wake

Author:
Kiara Pillay

Supervisor:
Professor D.P. Mason

A dissertation submitted in fulfillment of the requirements for the degree of Master of Science to the Faculty of Science

May 2022

Declaration of Authorship

I, Kiara Pillay, declare that this dissertation titled, 'The laminar two-fluid wake' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this dissertation has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this dissertation is entirely my own work.
- I have acknowledged all main sources of help.

Kiara Pillay	31 May 2022
Signed	Date

Abstract

The two-fluid flow behind a slender stationary symmetric object, as well as a slender symmetric self-propelled body, is investigated and governing equations are derived. These flows are also known as the classical and momentumless wakes, respectively, where the drag of the momentumless wake is zero. The upper fluid has a lower density so that instabilities do not form. As an initial study, the case where the two fluids have the same mainstream speed is considered. The fluid interface is unknown, but will be derived during the solution. Since the two fluids have the same mainstream speed, a two-fluid boundary layer upstream of the object will not form. A perturbation on the mainstream flow is considered in order to derive the far wake equations downstream of the object. The problem is formulated in terms of stream functions for the upper and lower fluids which reduces the number of unknowns in each fluid to one, while increasing the order of the partial differential equation by one. Conservation laws for the upper and lower fluid for each wake are derived using the multiplier method and conserved vectors are obtained. Conserved quantities for the two-fluid classical and momentumless wakes are calculated with the aid of the conservation laws. Lie point symmetries associated with the conserved vectors are obtained and new invariant solutions are calculated. The new solutions are compared with the existing solutions of Herczynski, Weidman and Burde [1]. Velocity profiles are plotted and analysed for different values of the parameters.

Keywords: Two-Fluid Flow, Classical Wake, Wake Behind a Self-Propelled Body, Similarity Solutions, Conservation Laws, Lie Point Symmetries

Dedication

To my beloved parents who have been my constant source of inspiration and pillar of strength throughout this journey.

Acknowledgements

This dissertation would not have been possible without the unwavering support of my family, friends, and my incredible supervisors.

Special thanks to my supervisor, Prof. David Mason, for his remarkable mentorship, guidance and immense knowledge that he shared with me. I could not have done this without you.

I would also like to extend my thanks to my previous supervisor, Dr. Ashleigh Hutchinson, who was offered a Post-Doctoral Fellowship in Cambridge and was unable to continue with my supervision due to this. Thank you for your guidance and support and for encouraging me to keep going.

Thank you to Mr. Warren Carlson for selflessly giving up his time to assist me with my code and debugging for hours.

I am eternally grateful to my parents, Collin and Kerusha Pillay, for their unconditional love, inspiration and encouragement that gave me the drive to excel.

To my sister, Nirvana Pillay, thank you for your support and adding laughter to my life.

To Oz Pillay who has been my sunshine in the darkest moments.

Above all, I thank God for his guidance and wisdom that He bestowed upon me to tackle this challenge.

I have been honoured and humbled to have each and every one of you walk this journey with me!

Contents

1 Introduction **1**

1.1 Introduction to wake flows 1

1.2 Two-fluid wakes 2

1.3 Conservation Laws 3

1.4 Outline of dissertation 5

2 Mathematical Model **7**

2.1 Derivation of boundary layer equations 7

2.2 Boundary conditions and interface conditions 10

2.2.1 Pressure difference across the interface 13

2.3 Introducing the velocity deficit 14

2.4 Conserved quantity for the two-fluid wake 16

2.4.1 Classical two-fluid wake 16

2.4.2 Two-fluid wake behind a self-propelled body 20

2.5 Introducing the stream function 25

2.6 Conclusions 26

3 Similarity Solutions **27**

3.1 Introduction 27

3.2 Invariant solution for the classical two-fluid wake 28

3.2.1 Visual representation of the solution 35

3.3 Approach of Herczynski, Weidman and Burde on the classical two-fluid wake . . 37

3.3.1 Visual representation of the solution 45

3.4 Invariant solution for the two-fluid momentumless wake 48

3.4.1 The case k = 0 53

3.4.2 Calculation of the constant B when $k = 0$ 54

3.4.3 The case $k \neq 0$ 57

3.4.4 Calculation of the constant B when $k = \pm\sqrt{6}$ 59

3.4.5 Summary of the invariant solutions for a momentumless wake 63

3.4.6 Visual representation of the solution 67

3.5 Approach of Herczynski, Weidman and Burde on the two-fluid self-propelled wake 71

3.5.1 Interface and boundary conditions 75

3.5.2 Solution of the self-propelled two-fluid wake 77

3.5.3 Calculation of constant B 83

3.5.4 Visual representation of the solution 84

3.6 Conclusions 86

4 Derivation of Conservation Laws **88**

4.1 Introduction . . . 88

4.2 Conservation Law Theory . . . 88

4.3 Conserved Vectors . . . 90

4.3.1 Case 1: Momentumless wake ($a_1 \neq 0$) . . . 95

4.3.2 Case 2: Classical wake ($a_3 \neq 0$) . . . 97

4.3.3 Case 3: Combination wake ($a_2 \neq 0$) . . . 98

4.3.4 Case 4: Remaining case ($a_4 \neq 0$) . . . 99

4.3.5 Case 5: Linear combination of conserved vectors . . . 100

4.4 Conserved quantities . . . 102

4.4.1 Conserved quantity for the two-fluid classical wake . . . 103

4.4.2 Conserved quantity for the two-fluid momentumless wake . . . 108

4.5 Conclusions . . . 114

5 Associated Lie point symmetries and invariant solutions for the two-fluid wake 116

5.1 Introduction . . . 116

5.2 Mathematical formulation in terms of the stream function . . . 117

5.2.1 Velocity components formulation . . . 117

5.2.2 Stream function formulation . . . 118

5.3 Associated Lie point symmetries for the classical wake . . . 119

5.3.1 Case 1: Equation (5.3.8) . . . 120

5.3.2 Case 2: Equation (5.3.9) . . . 122

5.4 Invariant solution for classical two-fluid wake 126
5.4.1 The constants c_2 and c_3 127
5.4.2 Conserved quantity 128
5.4.3 Partial differential equation for the velocity deficit 129
5.4.4 Boundary conditions 129
5.4.5 Interface conditions 130
5.4.6 Velocity components 131
5.4.7 Lie point symmetry 132
5.5 Solution of the ODE for the classical two-fluid wake 133
5.5.1 Interface conditions 134
5.5.2 Consequences of the result $k = 0$ 135
5.5.3 Determination of constant B 138
5.5.4 Visual representation of the solution 138
5.6 Summary of results for the classical two-fluid wake 141
5.7 Associated Lie point symmetry for the wake behind a self-propelled body 142
5.7.1 Case 1: Equation (5.7.6) 143
5.7.2 Case 2: Equation (5.7.7) 145
5.7.3 Solution for associated Lie point symmetry 148
5.8 Invariant solution for the momentumless two-fluid wake 151
5.8.1 Conserved quantity 152
5.8.2 Partial differential equation for the velocity deficit 154

5.8.3 Boundary conditions 154

5.8.4 Interface conditions 155

5.8.5 Velocity components 156

5.8.6 Lie point symmetry 156

5.9 Solution of the ODE for the momentumless two-fluid wake 157

5.9.1 Boundary conditions 159

5.9.2 Interface conditions 160

5.9.3 Calculation of constant B when $k = 0$ 163

5.9.4 Calculation of constant B when $k = \pm\sqrt{6}$ 164

5.9.5 Approximation in the conserved quantity for the momentumless wake . . 165

5.9.6 Visual representation of the solutions for the momentumless wakes 166

5.10 Summary of results for the two-fluid wake behind a self-propelled body 171

5.11 Conclusions 172

6 Conclusions **174**

List of Figures

2.1 Two-fluid wake flow. 8

2.2 Pressure difference across the interface 13

3.1 The velocities $u_1(x, y)$ and $u_2(x, y)$ of the classical two-fluid wake plotted against y at $x = 2$ for $D = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{2}$. 36

3.2 The velocities $u_1(x, y)$ and $u_2(x, y)$ of the classical two-fluid wake plotted against y at $x = 2$ for $D = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{5}$. 37

3.3 The velocities $u_1(x, y)$ and $u_2(x, y)$ for the classical two-fluid wake, given by (3.3.78) and (3.3.79) and derived using the method of Herczynski, Weidman and Burde [1], plotted against y at $x = 1$ for $D = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{2}$. 46

3.4 The velocities $u_1(x, y)$ and $u_2(x, y)$ for the classical two-fluid wake, given by (3.3.78) and (3.3.79) and derived using the method of Herczynski, Weidman and Burde [1], plotted against y at $x = 1$ for $D = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{5}$. 47

3.5 The velocities $u_1(x, y)$ and $u_2(x, y)$ for the momentumless two-fluid wake for $k = 0$ plotted against y at $x = 6$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{2}$. 67

3.6 The velocities $u_1(x, y)$ and $u_2(x, y)$ for the momentumless two-fluid wake for $k = 0$ plotted against y at $x = 6$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{5}$. 68

3.7 The velocity $u(x, y)$ for the momentumless two-fluid wake for $k = +\sqrt{6}$ plotted against y at $x = 3$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and S = 0.1458. 69

3.8 The velocity $u(x, y)$ for the momentumless two-fluid wake for $k = -\sqrt{6}$ plotted against y at $x = 3$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and S = 0.1458. 70

3.9 The velocities $u_1(x,y)$ and $u_2(x,y)$ for the momentumless two-fluid wake, given by (3.5.121) and (3.5.122) and derived using the method of Herczynski, Weidman and Burde [1], plotted against y at $x = 0.55$ for $K = \frac{1}{10}$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{2}$. . . . 85

3.10 The velocities $u_1(x,y)$ and $u_2(x,y)$ for the momentumless two-fluid wake, given by (3.5.121) and (3.5.122) and derived using the method of Herczynski, Weidman and Burde [1], plotted against y at $x = 0.55$ for $K = \frac{1}{10}$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{5}$. . . . 86

5.1 The velocities $u_1(x,y)$ and $u_2(x,y)$ of the classical two-fluid wake plotted against y at $x = 2$ for $D = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{20}$ and $\frac{\nu_1}{\nu_2} = 1$. 139

5.2 The velocities $u_1(x,y)$ and $u_2(x,y)$ of the classical two-fluid wake plotted against y at $x = 2$ for $D = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{20}$. 140

5.3 The velocities $u_1(x,y)$ and $u_2(x,y)$ for the momentumless two-fluid wake for $k = 0$ plotted against y at $x = 6$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{20}$ and $\frac{\nu_1}{\nu_2} = 1$. 167

5.4 The velocities $u_1(x,y)$ and $u_2(x,y)$ for the momentumless two-fluid wake for $k = 0$ plotted against y at $x = 6$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{20}$. 168

5.5 The velocity $u(x,y)$ for the momentumless two-fluid wake for $k = +\sqrt{6}$ plotted against y at $x = 3$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{20}$ and S = 0.1458. 169

5.6 The velocity $u(x,y)$ for the momentumless two-fluid wake for $k = -\sqrt{6}$ plotted against y at $x = 3$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{20}$ and S = 0.1458. 170

Chapter 1

Introduction

1.1 Introduction to wake flows

A wake is considered to be a region where the free flowing fluid of a constant speed is disturbed downstream of a solid body aligned with the mainstream flow. Practical examples of wake flows which are of much interest include a free flowing river perturbed by rocks or dams, ships that create wakes while propelling through water and even aircraft that wade through the condensation in the sky or wind turbines. Obstruction to the fluid flow can often result in turbulent downstream flow which is characterised by a large Reynolds number. Symmetric wakes can be categorised as 'classical' or 'momentumless' where the conserved quantity and drag play an important role in the differentiation of these symmetric wakes.

Goldstein [2] originally investigated the two-dimensional steady flow of the laminar classical wake behind a slender symmetric body aligned with the mainstream flow. Goldstein first investigated the velocity of the wake for a distance that was 0.3645 to 0.5 of the length of the slender body which was measured from the downstream end [2]. This was considered the near wake. Further to this, Goldstein then investigated the velocity distribution of the wake a considerable distance downstream behind the slender body which is referred to as the far wake. Birkhoff and Zarantonello [5] expanded on this to study the momentumless laminar wake behind a slender symmetric self-propelled body. Herczynski, Weidman and Burde [1] also later investigated the momentumless wake behind a self-propelled body, however, this was along the interface of a two-fluid stream. The second moment of the axial momentum deficit is conserved for the momentumless wake and the drag is zero, whereas the drag is conserved for the classical wake. The Navier-Stokes equation in the boundary layer approximation is used to

derive the governing equation describing the flow in the wake.

Other types of wake flows include the turbulent planar wakes [20], two-dimensional laminar classical wake of a shear thinning fluid [21] and the laminar axisymmetric classical wake for power law fluids [22]. Tennekes and Lumley [20] studied and discussed the turbulent planar wake where the eddy viscosity closure model was used to formulate equations for the mean velocities. Tennekes and Lumley made use of a constant eddy viscosity and the results obtained were compared against experimental observations. Prandtl's [19] mixing length model was used in order to obtain a finite wake boundary for the turbulent axisymmetric wake investigated by Swain [23]. Prandtl was able to obtain a similarity solution by assuming large Reynolds numbers and neglecting the kinematic viscosity term since it was assumed that the turbulent viscosity was much greater than the kinematic viscosity. Here, it was also assumed that the mixing length was proportional to the width of the boundary layer. Hunt [24] studied another type of wake, called the wall-wake, for laminar and turbulent flows. A wall-wake flow is produced downstream where an obstructing body is situated on the boundary wall. Since the entire bottom half of the plane is inhabited by the wall, the fluid is then restricted to the upper half of the plane. Therefore, this problem is not symmetric about the axis of the wake and the wake is situated within a boundary layer.

1.2 Two-fluid wakes

Herczynski, Weidman and Burde [1] were among the first researchers to investigate the concept of a laminar classical two-fluid wake. In this work, a stream of fluid flows over a parallel stream of a different density and different viscosity, but the same mainstream speed. The two-fluid system comes into contact with a slender object and a wake forms downstream. Herczynski, Weidman and Burde considered the far wake flow in their research. Considering the fact that the two fluids were of the same mainstream speed, the concept of a boundary layer forming upstream of the object may be neglected. A non-standard similarity solution was used to find analytical solutions in the boundary layer approximation for laminar two-fluid jets and wakes to account for interface deflection in the presence of gravity [1]. Herczynski, Weidman and Burde found that the interface of the classical two-fluid wake deflected upwards and the wake flow is described as a sink of momentum upstream of the spatially developing flow. The position of the fluid interface is determined by balancing the tangential stresses across the interface.

Two-fluid boundary layer flows were first studied by Keulegan [6] and Lock [7]. Keulegan considered the flow of a stream of fluid on top of another fluid at rest [6]. He used Prandtl's

boundary layer theory and a method of successive approximations in order to solve for the two-fluid boundary layer flow. Lock expanded upon Keulegan's work. Lock obtained an approximate solution through the use of the momentum equation [7]. He determined that the solutions were dependent on the velocity ratio, $\frac{U_2}{U_1}$, of the two fluids and the product of the density and viscosity ratios, $\rho\mu$. Numerical solutions were obtained for the case where the lower fluid is at rest as well as for one non-zero value of the velocity ratio. Therefore, the study of the laminar two-fluid wake with different mainstream speeds was first studied by Lock. Potter [8] made use of the works of both Keulegan and Lock to find more accurate approximations to the velocity distribution in the laminar boundary layer of two viscous incompressible fluids. Both fluids are considered to be moving co-current but at different velocities. Numerical solutions were derived through the use of a sextic polynomial for the velocity distribution in the boundary layers. These numerical solutions were compared against Keulegan and Lock's solutions.

Recall that Herczynski, Weidman and Burde studied the classical two-fluid wake for the case where the mainstream speeds of the two fluids are the same [1]. There is a significant difference between this case and that of two fluids flowing with different mainstream speeds. When the two streams flow at different speeds, a two-fluid boundary layer forms upstream of the object. The problem of the two-fluid classical and momentumless wakes will be investigated in this dissertation where the two fluids have the same mainstream speeds so that the two-fluid boundary layer upstream of the object may be neglected. A conserved quantity is used to complete the solution for the far wake flow and similarity solutions will be considered.

1.3 Conservation Laws

Conservation laws are an important factor in the study of differential equations. Furthermore, the use of the Lie symmetry approach has been valuable when solving problems in fluid mechanics. There has been a lot written regarding the many applications of symmetry methods to turbulence problems [26] - [28]. The conserved quantity plays an extremely important role in completing the solution process in jet flows where it is used to solve for the unknown exponent in the similarity solution since the homogeneous boundary conditions are not sufficient to aid in this process. Schlichting [40] [41] studied the laminar two-dimensional free jet where the conserved quantity may be determined through physical arguments or through the use of boundary conditions and the continuity equation where Prandtl's momentum boundary layer equation is integrated. Squire [44], Riley [39] and Schwarz [43] employed the method of integrating Prandtl's momentum boundary layer equation across jets to determine the conserved quantity for radial free jets. Similarly, Goldstein [4] applied this method to determine the con-

served quantity for an axisymmetric free jet. The method of using physical aruguments based on the conservation of volume flux in an incompressible fluid was used by Watson [45] for the two-dimensional liquid jet and Riley [39] and Watson [45] for the radial liquid jet to obtain the conserved quantities. Symmetry methods were applied and the group invariant solution was derived through the use of a linear combination of the Lie symmetries of the flow's governing equation in the work by Mason [29] for a laminar two-dimensional jet and Ruscic and Mason [30] for the laminar axisymmetric jet. Here, the arbitrary constants in the linear combination of Lie point symmetries were solved through the use of the conserved quantity and the boundary conditions. It is important to note that the conserved quantity does not only play a key role in the solution process for jet flows but for wake flows as well and there has been significant progress made in deriving conserved quantities for PDEs.

The different methods that exist in order to establish conservation laws for a PDE can be found in [31] - [36]. Naz [9] summarised that there are nine different approaches that may be used to determine conservation laws for PDEs. Well known methods include Noether's theorem [13] which requires a Langrangian function and the direct method by Laplace [38] which is also considered the most elementary approach. Another approach used to determine conservation laws is the multiplier method developed by Steudel [18]. In this method, the conservation law is written in characteristic form. The characteristics are the multipliers of variational equations, but this result is not true in general. The works of Bluman, Cheviakov and Anco [36] as well as Olver [37] demonstrate that all non-trivial conservation laws of a PDE are produced by the multiplier method.

A modified approach which involves adding a symmetry condition to the direct method was developed by Kara and Mahomed [14]. Kara and Mahomed derived a condition which allowed for a Lie point symmetry to be associated with a conserved vector. The Lie point symmetry associated with the conserved vector, which produces the conserved quantity, was employed to generate an invariant solution instead of the linear combination of all Lie point symmetries of the PDE. This approach is considered to be more direct and has been applied in multiple works which can be found in [10] - [12]. There is also an approach that associates multipliers to symmetries developed by Anco and Kara [48]. In this dissertation, the conservation laws for PDEs for the two-fluid classical and momentumless wakes are derived using the multiplier method. It is important to note that the methods around conservation laws referred to thus far relate to turbulent flows but can be used for problems relating to laminar flows as well. Once the conserved quantity is determined, the Lie point symmetries may be derived followed by new invariant solutions. These solutions can be compared against the existing solutions of Herczynski, Weidman and Burde [1].

1.4 Outline of dissertation

This dissertation is outlined as follows:

- Chapter 2: the mathematical model for the laminar classical and momentumless two-fluid wake is provided. The governing equations are derived using the Navier-Stokes equations which are non-dimensionalised. The boundary layer approximation is imposed and the far-wake equations are found. The boundary and interface conditions are presented for the two-fluid problem. The conserved quantities for the classical and momentumless two-fluid wakes are then derived. The boundary layer equations and the boundary and interface conditions are then formulated in terms of the stream function.

- Chapter 3: similarity solutions are used to solve the problem presented in Chapter 2. A scaling transformation is introduced and the invariant solution is derived. The conserved quantity is used to determine the position of the interface for both the classical and momentumless wakes. The method developed by Herczynski, Weidman and Burde [1] is also applied to the boundary layer equations although the conserved quantity derived in this dissertation is used instead of the conserved quantity derived by the authors. The velocity profiles for all solutions are plotted.

- Chapter 4: the theory and systematic process relating to conservation laws and in particular the multiplier approach which, as mentioned, was developed by Steudel [18], is discussed in order to determine the conserved vectors and conserved quantities relating to the governing PDE of the two-fluid far wake flow. The elementary conserved vectors will be derived first in terms of the velocity components. It will be necessary to determine which conserved vector is relevant to the problem discussed in this dissertation as we notice four different conserved vectors arise. Once the correct elementary conserved vector has been determined, the elementary conserved quantity for the two-fluid far wake can be obtained in terms of the velocity components for the classical and momentumless wakes.

- Chapter 5: in this chapter the conserved vector and conserved quantity are written in terms of the stream function for those conservation laws relating to the classical and momentumless two-fluid far wakes. It is important to note that the general form of the conserved vector must be used which accommodates both fluids. The Lie point symmetry associated with the elementary conserved vector is obtained for the classical and momentumless two-fluid far wakes. The invariant solution may then be calculated from the Lie point symmetry.

- Chapter 6: the conclusions drawn are presented in this chapter.

Chapter 2

Mathematical Model

2.1 Derivation of boundary layer equations

In this section, the laminar two-fluid wake of a symmetric body is explored. It is important to note that the governing equations and boundary and interface conditions for the classical wake and the wake behind a self-propelled body are the same and the difference lies in the conserved quantities. Consider a stream of viscous fluid flowing over a parallel stream which has a different density, viscosity, but same mainstream speed. Both fluids are incompressible and the flows are laminar. The two-fluid flow comes into contact with a slender stationary symmetric object or slender symmetric self-propelled body whose leading edge is located at the interface of the two fluids with a zero angle of incidence. A Cartesian coordinate system (x, y) is used where the origin is defined to be at the trailing edge of the object. Let the index $i = 1$ correspond to the top fluid and $i = 2$ correspond to the bottom fluid. The mainstream velocities, densities, and kinematic viscosities are denoted by U_i, ρ_i, and ν_i respectively. The fluid interface is given by $y = \phi(x)$. The flow is shown in Figure 2.1.

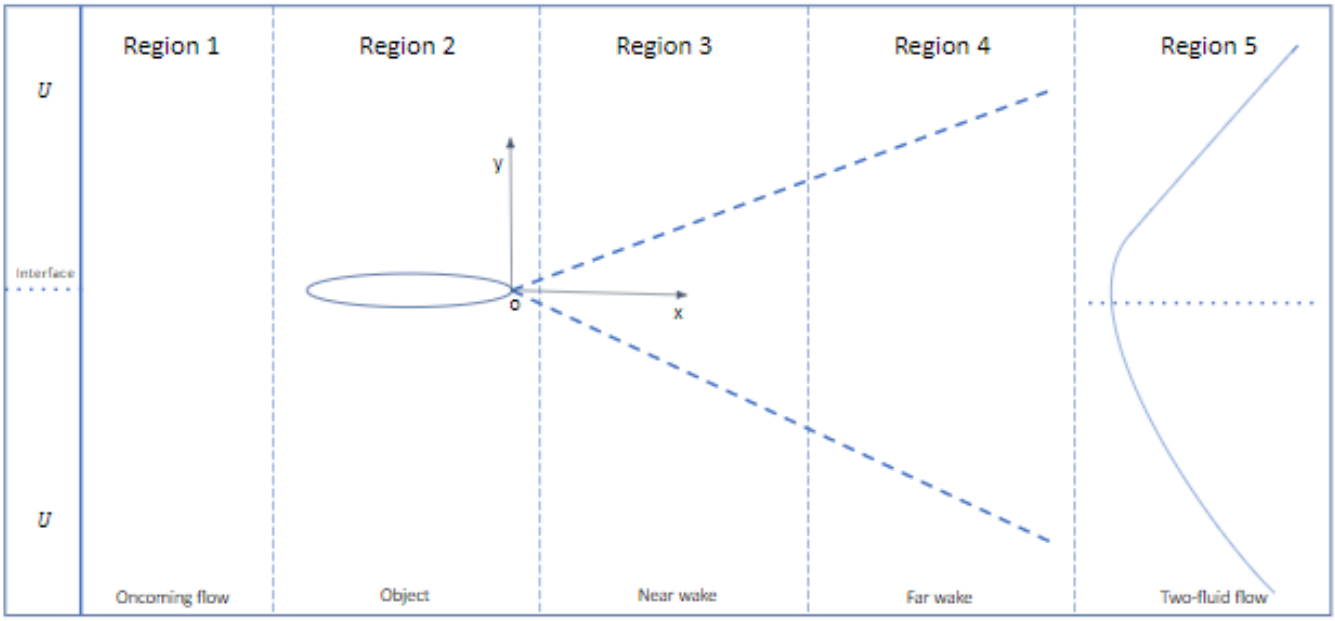

Figure 2.1: Two-fluid wake flow.

Region 1 in Figure 2.1 depicts the upstream flow before coming into contact with the object. In this problem $U_1 = U_2 = U$. Here, U_1 relates to the mainstream speed of the top fluid and U_2 relates to the mainstream speed of the bottom fluid. The two-fluid flow makes contact with the slender object in Region 2 which then leads to the creation of a wake. Region 3 is the wake that is produced behind the object, which is called the near-wake. The wake generated a considerable distance downstream behind the object is shown in Region 4 and is called the far-wake. The work done in this dissertation will focus on the far-wake. Region 5 in Figure 2.1 depicts the two-fluid boundary layer flow that forms. Let the x and y velocity components in the far-wake region be denoted by u_i and v_i and the pressure by P_i. The model is derived from the Navier-Stokes equation which is non-dimensionalised in terms of characteristic quantities defined for the top fluid. The x and y components of the Navier-Stokes equation are given by:

$$u_i\frac{\partial u_i}{\partial x} + v_i\frac{\partial u_i}{\partial y} = -\frac{1}{\rho_i}\frac{\partial P_i}{\partial x} + \nu_i\left(\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2}\right), \tag{2.1.1}$$

$$u_i\frac{\partial v_i}{\partial x} + v_i\frac{\partial v_i}{\partial y} = -\frac{1}{\rho_i}\frac{\partial P_i}{\partial y} + \nu_i\left(\frac{\partial^2 v_i}{\partial x^2} + \frac{\partial^2 v_i}{\partial y^2}\right) - g, \tag{2.1.2}$$

where g is the body force per unit mass in the y-direction.

The continuity equation for the incompressible fluid is:

$$\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} = 0. \tag{2.1.3}$$

The kinematic viscosity, ν_i, is given by:

$$\nu_i = \frac{\mu_i}{\rho_i}. \tag{2.1.4}$$

The process of non-dimensionalisation will now be applied to equations (2.1.1) - (2.1.3) using the following dimensionless variables:

$$\begin{aligned} x^* &= \frac{x}{l}, & y^* &= Re^{\frac{1}{2}}\frac{y}{l}, \\ u_i^* &= \frac{u_i}{U}, & v_i^* &= Re^{\frac{1}{2}}\frac{v_i}{U}, & P_i^* &= \frac{P_i}{\rho_1 U^2}, \end{aligned} \tag{2.1.5}$$

where the Reynolds number, Re, is given by:

$$Re = \frac{Ul}{\nu_1}. \tag{2.1.6}$$

Here, l is the characteristic length of the two-fluid boundary layer.

Substituting the dimensionless variables (2.1.5) into equation (2.1.1) produces:

$$u_i^*\frac{\partial u_i^*}{\partial x^*} + v_i^*\frac{\partial u_i^*}{\partial y^*} = -\frac{\rho_1}{\rho_i}\frac{\partial P_i^*}{\partial x^*} + \frac{\nu_i}{Ul}\frac{\partial^2 u_i^*}{\partial x^{*2}} + \frac{\nu_i Re}{Ul}\frac{\partial^2 u_i^*}{\partial y^{*2}}, \tag{2.1.7}$$

and therefore,

$$u_i^*\frac{\partial u_i^*}{\partial x^*} + v_i^*\frac{\partial u_i^*}{\partial y^*} = -\frac{\rho_1}{\rho_i}\frac{\partial P_i^*}{\partial x^*} + \frac{1}{Re}\frac{\nu_i}{\nu_1}\frac{\partial^2 u_i^*}{\partial x^{*2}} + \frac{\nu_i}{\nu_1}\frac{\partial^2 u_i^*}{\partial y^{*2}}. \tag{2.1.8}$$

Neglecting terms of $\mathcal{O}(Re^{-1})$ reduces equation (2.1.8) to:

$$u_i^*\frac{\partial u_i^*}{\partial x^*} + v_i^*\frac{\partial u_i^*}{\partial y^*} = -\frac{\rho_1}{\rho_i}\frac{\partial P_i^*}{\partial x^*} + \frac{\nu_i}{\nu_1}\frac{\partial^2 u_i^*}{\partial y^{*2}}. \tag{2.1.9}$$

Similarly, the y-component in dimensionless form may be found by substituting the dimensionless variables (2.1.5) into equation (2.1.2):

$$\frac{u_i^*}{Re}\frac{\partial v_i^*}{\partial x^*} + \frac{v_i^*}{Re}\frac{\partial v_i^*}{\partial y^*} = -\frac{\rho_1}{\rho_i}\frac{\partial P_i^*}{\partial y^*} + \frac{1}{Re^2}\frac{\nu_i}{\nu_1}\frac{\partial^2 v_i^*}{\partial x^{*2}} + \frac{1}{Re}\frac{\nu_i}{\nu_1}\frac{\partial^2 v_i^*}{\partial y^{*2}} - \frac{1}{\sqrt{Re}}\frac{lg}{U^2}. \tag{2.1.10}$$

Neglecting terms of $\mathcal{O}(Re^{-1})$ and $\mathcal{O}(Re^{-2})$ in equation (2.1.10) results in:

$$\frac{\rho_1}{\rho_i}\frac{\partial P_i^*}{\partial y^*} = -\frac{gl}{U^2}\frac{1}{\sqrt{Re}}, \tag{2.1.11}$$

that is

$$\frac{\rho_1}{\rho_i}\frac{\partial P_i^*}{\partial y^*} = -\frac{1}{\sqrt{Re}}\frac{1}{F^2}, \tag{2.1.12}$$

where

$$F = \frac{U}{\sqrt{g\,l}} \tag{2.1.13}$$

is the Froude number. The Froude number is the ratio of the mainstream velocity, U, to the speed of small amplitude waves in a fluid of depth l.

Assuming that the Froude number is large so that no stationary lee waves develop and that

$$\frac{1}{\sqrt{Re}}\frac{1}{F^2} \leq \mathcal{O}(\frac{1}{Re}), \tag{2.1.14}$$

equation (2.1.12) reduces to

$$\frac{\partial P_i^*}{\partial y^*} = 0, \tag{2.1.15}$$

which gives that P_i^* is a function of x^* only.

Non-dimensionalising the continuity equation, equation (2.1.3) produces the following result:

$$\frac{\partial u_i^*}{\partial x^*} + \frac{\partial v_i^*}{\partial y^*} = 0. \tag{2.1.16}$$

The boundary and interface conditions are discussed in the next section.

2.2 Boundary conditions and interface conditions

The boundary conditions on u_i^* are deduced based on the following reasoning. As y^* tends to $+\infty$, the velocity component in the x-direction tends to the upper fluid mainstream velocity U_1. In non-dimensionalised form U_1 becomes 1. As y^* tends to $-\infty$, the velocity component in the x-direction tends to the lower fluid mainstream velocity U_1. In non-dimensionalised form U_1 again becomes 1. The first derivative of u_i^* with respect to y^* would then tend to zero.

$$u_1^*(x^*,\infty) = 1, \quad u_2^*(x^*,-\infty) = 1, \quad \frac{\partial u_1^*}{\partial y^*}(x^*,\infty) = 0, \quad \frac{\partial u_2^*}{\partial y^*}(x^*,-\infty) = 0, \quad x^* \geq 0. \tag{2.2.1}$$

The equation of the unknown interface is $y = \phi(x)$. In dimensionless form the interface is

$$y^* = \phi^*(x^*), \qquad \phi^* = \frac{\phi}{\delta}, \qquad \delta = \frac{l}{\sqrt{Re}}. \tag{2.2.2}$$

At the interface the velocity and stress components must balance because the fluids are viscous. Since the two fluids do not mix, a fluid particle on the interface remains on the interface as the fluid evolves.

Consider a fluid particle on the interface that must remain on the interface. This condition is given by

$$\frac{D}{Dt}(y - \phi(x))|_{y=\phi(x)} = 0 \tag{2.2.3}$$

which applies for both fluids. We therefore have in dimensionless form

$$v_i^*(x^*, \phi^*) - u_i^*(x^*, \phi^*)\frac{d\phi^*}{dx^*} = 0. \tag{2.2.4}$$

Now consider the continuity of the components of the stress vector at the fluid interface.

The two-dimensional viscous stress tensor, T^i, in dimensional form is

$$T^i = \begin{bmatrix} \tau_{11}^i & \tau_{12}^i \\ \tau_{21}^i & \tau_{22}^i \end{bmatrix}, \tag{2.2.5}$$

where $\tau_{21}^i = \tau_{12}^i$.

By Cauchy's formula the j-component of the stress vector, $t_j^i(\boldsymbol{n})$, on a surface with unit normal $\boldsymbol{n}$ is

$$t_j^i(\boldsymbol{n}) = n_k \tau_{kj}^i. \tag{2.2.6}$$

The unit normal $\boldsymbol{n}$ in the $+y$ direction is given by

$$\boldsymbol{n} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.2.7}$$

Therefore by Cauchy's formula, since the stress tensor is symmetric,

$$\begin{bmatrix} t_1^i \\ t_2^i \end{bmatrix} = \begin{bmatrix} \tau_{11}^i & \tau_{12}^i \\ \tau_{21}^i & \tau_{22}^i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \tau_{12}^i \\ \tau_{22}^i \end{bmatrix}. \tag{2.2.8}$$

Now, from the Navier-Poisson law for a viscous incompressible fluid, in dimensional form

$$t_1^i = \tau_{12}^i = \mu_i \left(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x} \right),$$
$$t_2^i = \tau_{22}^i = -P_i + 2\mu_i \frac{\partial v_i}{\partial y}. \tag{2.2.9}$$

The characteristic quantity for the stress vector and the stress tensor is the characteristic pressure $\rho_1 U_1^2$. Thus,

$$t_1^{*i} = \tau_{12}^{*i} = \frac{\mu_i}{\mu_1} \left(\frac{1}{\sqrt{Re}} \frac{\partial u_i^*}{\partial y^*} + \frac{1}{Re^{\frac{3}{2}}} \frac{\partial v_i^*}{\partial x^*} \right),$$
$$t_2^{*i} = \tau_{22}^{*i} = -P_i^* + 2\frac{\mu_i}{\mu_1} \frac{1}{Re} \frac{\partial v_i^*}{\partial y^*}. \tag{2.2.10}$$

The tangential and normal stress components must balance at the interface. Therefore,

$$\tau_{12}^{*1}(x^*, \phi^*) = \tau_{12}^{*2}(x^*, \phi^*),$$
$$\tau_{22}^{*1}(x^*, \phi^*) = \tau_{22}^{*2}(x^*, \phi^*). \tag{2.2.11}$$

Finally consider the velocity components. Matching the tangential and normal velocity components at the interface gives

$$u_1^*(x^*, \phi^*) = u_2^*(x^*, \phi^*),$$
$$v_1^*(x^*, \phi^*) = v_2^*(x^*, \phi^*). \tag{2.2.12}$$

In dimensionless form, equations (2.2.4), (2.2.11) and (2.2.12) provide the following interface conditions at $y = \phi(x)$. Terms of $\mathcal{O}(Re^{-\frac{3}{2}})$ are neglected in the tangential matching condition for the stress but terms of $\mathcal{O}(Re^{-1})$ are retained in the normal matching condition for the stress. The normal matching condition for the stress at the interface $y = \phi(x)$ will be considered further later:

$$\mu_1 \frac{\partial u_1^*}{\partial y^*}(x^*, \phi^*) = \mu_2 \frac{\partial u_2^*}{\partial y^*}(x^*, \phi^*), \tag{2.2.13}$$

$$2\frac{\mu_2}{\mu_1} \frac{1}{Re} \frac{\partial v_2^*}{\partial y^*}(x^*, \phi^*) - \frac{2}{Re} \frac{\partial v_1^*}{\partial y^*}(x^*, \phi^*) = P_2^*(x^*, \phi^*) - P_1^*(x^*, \phi^*), \tag{2.2.14}$$

$$u_1^*(x^*, \phi^*) = u_2^*(x^*, \phi^*), \tag{2.2.15}$$

$$v_1^*(x^*, \phi^*) = v_2^*(x^*, \phi^*). \tag{2.2.16}$$

2.2.1 Pressure difference across the interface

Consider Figure 2.2 in aid of the derivation of the pressure difference across the interface.

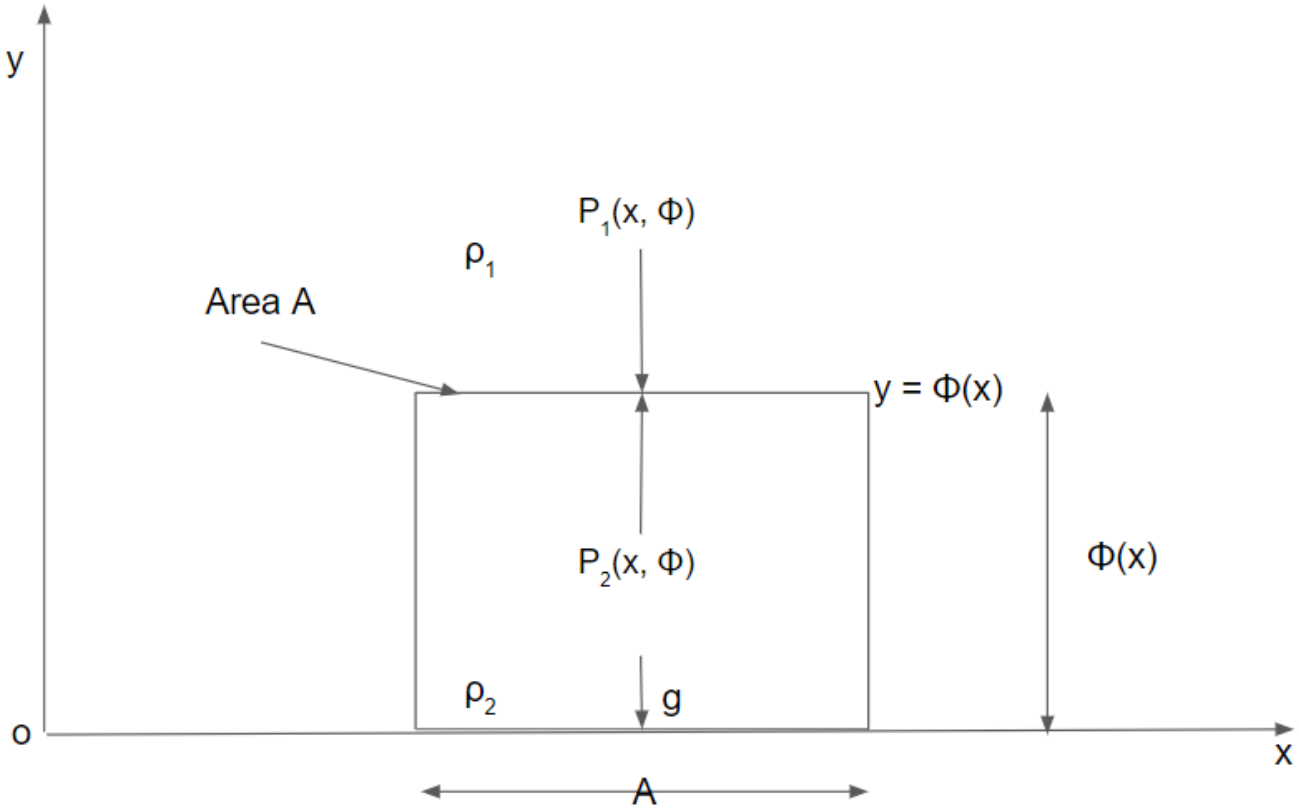

Figure 2.2: Pressure difference across the interface

We know that

$$\text{Upthrust} \;=\; \text{weight of fluid displaced}, \tag{2.2.17}$$

and therefore at the interface

$$\text{Upthrust on } y = \phi(x) = (P_2(x, \phi(x)) - P_1(x, \phi(x)))\; A \tag{2.2.18}$$

and

$$\begin{aligned} \text{Weight of fluid displaced } &= \rho_1 \phi(x) A g - \rho_2 \phi(x)\; A\; g \\ &= (\rho_1 - \rho_2)\phi(x)\; A\; g. \end{aligned} \tag{2.2.19}$$

Therefore

$$(P_2(x, \phi(x)) - P_1(x, \phi(x)))\; A = (\rho_1 - \rho_2)\phi(x)\; A\; g \tag{2.2.20}$$

and

$$P_2(x, \phi(x)) - P_1(x, \phi(x)) = (\rho_1 - \rho_2)\phi(x)\, g. \tag{2.2.21}$$

Now we can non-dimensionalise (2.2.21) using the dimensionless variables (2.1.5) and (2.2.2) which gives

$$P_2^*(x^*, \phi^*(x)) - P_1^*(x^*, \phi^*(x)) = \left(1 - \frac{\rho_2}{\rho_1}\right)\phi^*(x)\frac{l\, g}{U^2}\frac{1}{\sqrt{Re}}. \tag{2.2.22}$$

But using the Froude number (2.1.13), (2.2.22) becomes

$$P_2^*(x^*, \phi^*(x)) - P_1^*(x^*, \phi^*(x)) = \left(1 - \frac{\rho_2}{\rho_1}\right)\phi^*(x)\frac{1}{F^2\sqrt{Re}}. \tag{2.2.23}$$

This condition is used later in Chapter 3 to derive the invariant solution using Herczynski, Weidman and Burde's method.

2.3 Introducing the velocity deficit

In the far wake region, region 4 in Figure 2.1, a perturbation on the oncoming flow is considered.

Consider the following expansions

$$u_i^* = 1 - \bar{w}_i, \tag{2.3.1}$$

$$v_i^* = 0 + \bar{v}_i, \tag{2.3.2}$$

$$P_i^* = 0 + \bar{P}_i, \tag{2.3.3}$$

where $\bar{w}_i, \bar{v}_i$ and $\bar{P}_i$ are small such that their squares are products can be neglected. The dimensionless variable $\bar{w}_i$ is a function of x^* and y^* and is the velocity deficit in the wake in the x-direction.

Substituting equations (2.3.1) and (2.3.2) into equation (2.1.16) results in:

$$-\frac{\partial \bar{w}_i}{\partial x^*} + \frac{\partial \bar{v}_i}{\partial y^*} = 0. \tag{2.3.4}$$

Substituting equations (2.3.1)-(2.3.3) into equation (2.1.9), and neglecting squares are products

of $\bar{w}_i$ and $\bar{v}_i$ results in:

$$\frac{\partial \bar{w}_i}{\partial x^*} = \frac{\rho_1}{\rho_i}\frac{d\bar{P}_i}{dx^*} + \frac{\nu_i}{\nu_1}\frac{\partial^2 \bar{w}_i}{\partial y^{*2}}. \tag{2.3.5}$$

Now in the two inviscid mainstream flows, Euler's equation reduces to

$$\frac{dP_i^*}{dx^*} = 0. \tag{2.3.6}$$

Thus by mainstream matching:

$$\frac{d\bar{P}_i}{dx^*} = 0. \tag{2.3.7}$$

Equation (2.3.5) reduces to:

$$\frac{\partial \bar{w}_i}{\partial x^*} = \frac{\nu_i}{\nu_1}\frac{\partial^2 \bar{w}_i}{\partial y^{*2}}. \tag{2.3.8}$$

Therefore, the boundary layer equations are:

$$-\frac{\partial \bar{w}_i}{\partial x^*} + \frac{\partial \bar{v}_i}{\partial y^*} = 0, \tag{2.3.9}$$

$$\frac{\partial \bar{w}_i}{\partial x^*} = \frac{\nu_i}{\nu_1}\frac{\partial^2 \bar{w}_i}{\partial y^{*2}}. \tag{2.3.10}$$

The boundary conditions in terms of the velocity deficit are given by

$$\bar{w}_1(x^*, \infty) = 0, \quad \bar{w}_2(x^*, -\infty) = 0, \quad \frac{\partial \bar{w}_1}{\partial y^*}(x^*, \infty) = 0, \quad \frac{\partial \bar{w}_2}{\partial y^*}(x^*, -\infty) = 0, \quad x^* \geq 0. \tag{2.3.11}$$

The interface conditions (2.2.13) to (2.2.16) in terms of the velocity deficit are

$$\mu_1 \frac{\partial \bar{w}_1}{\partial y^*}(x^*, \phi^*) = \mu_2 \frac{\partial \bar{w}_2}{\partial y^*}(x^*, \phi^*), \tag{2.3.12}$$

$$2\frac{\mu_2}{\mu_1}\frac{1}{Re}\frac{\partial \bar{v}_2}{\partial y^*}(x^*, \phi^*) - \frac{2}{Re}\frac{\partial \bar{v}_1}{\partial y^*}(x^*, \phi^*) = \bar{P}_2(x^*, \phi^*) - \bar{P}_1(x^*, \phi^*), \tag{2.3.13}$$

$$\bar{w}_1(x^*, \phi^*) = \bar{w}_2(x^*, \phi^*), \tag{2.3.14}$$

$$\bar{v}_1(x^*, \phi^*) = \bar{v}_2(x^*, \phi^*). \tag{2.3.15}$$

2.4 Conserved quantity for the two-fluid wake

In this section, the conserved quantities for the classical two-fluid wake and the two-fluid wake behind a self-propelled body will be derived.

2.4.1 Classical two-fluid wake

In order to determine the conserved quantity, equation (2.3.10) will be integrated across the wake with respect to y^* from $y^* = -\infty$ to $y^* = +\infty$.

Start at the lower fluid:

$$\frac{\partial \bar{w}_2}{\partial x^*} = \frac{\nu_2}{\nu_1}\frac{\partial^2 \bar{w}_2}{\partial y^{*2}}. \tag{2.4.1}$$

Multiplying by $\frac{\rho_2}{\rho_1}$ gives,

$$\frac{\rho_2}{\rho_1}\frac{\partial \bar{w}_2}{\partial x^*} = \frac{\mu_2}{\mu_1}\frac{\partial^2 \bar{w}_2}{\partial y^{*2}}. \tag{2.4.2}$$

Integrate (2.4.2) with respect to y^* at constant x^* from $y^* = -\infty$ to $y^* = \phi(x^*)$:

$$\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x^*)} \frac{\partial \bar{w}_2}{\partial x^*}(x^*, y^*)dy^* = \frac{\mu_2}{\mu_1}\int_{-\infty}^{\phi(x^*)} \frac{\partial^2 \bar{w}_2}{\partial y^{*2}}(x^*, y^*)dy^*. \tag{2.4.3}$$

Now,

$$\int_{-\infty}^{\phi(x^*)} \frac{\partial^2 \bar{w}_2}{\partial y^{*2}}(x^*, y^*)dy^* = \frac{\partial \bar{w}_2}{\partial y^*}\bigg|_{-\infty}^{\phi(x^*)} \tag{2.4.4}$$

$$= \frac{\partial \bar{w}_2}{\partial y^*}(x^*, \phi(x^*)) - \frac{\partial \bar{w}_2}{\partial y^*}(x^*, -\infty). \tag{2.4.5}$$

However, from the boundary conditions (2.3.11),

$$\frac{\partial \bar{w}_2}{\partial y^*}(x^*, -\infty) = 0. \tag{2.4.6}$$

Thus,

$$\int_{-\infty}^{\phi(x^*)} \frac{\partial^2 \bar{w}_2}{\partial y^{*2}}(x^*, y^*)dy^* = \frac{\partial \bar{w}_2}{\partial y^*}(x^*, \phi(x^*)). \tag{2.4.7}$$

The left-hand side of (2.4.3) is rewritten using the Theorem for Differentiation Under the Integral Sign which states in general that [42]

$$\frac{d}{dx}\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy = \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial f}{\partial x}(x,y)dy + f(x,\phi_2(x))\frac{d\phi_2}{dx} - f(x,\phi_1(x))\frac{d\phi_1}{dx}. \tag{2.4.8}$$

The lower limit in (2.4.3) is $y = -\infty$. In order to apply the Theorem replace the lower limit $-\infty$ by $-Bx$ where $B > 0$ is a constant and let $B \to \infty$. Then

$$\frac{d}{dx}\int_{-Bx}^{\phi(x^*)} \bar{w}_2(x^*,y^*)dy^* = \int_{-Bx}^{\phi(x^*)} \frac{\partial \bar{w}_2}{\partial x^*}(x^*,y^*)dy^* + \bar{w}_2(x^*,\phi(x^*))\frac{d\phi(x^*)}{dx^*} - \bar{w}_2(x^*,-Bx)\,B. \tag{2.4.9}$$

Now from the boundary conditions

$$\bar{w}_2(x^*,-\infty) = 0 \tag{2.4.10}$$

Assume the stronger condition that

$$\lim_{B\to\infty} \bar{w}_2(x^*,-Bx)B = 0. \tag{2.4.11}$$

Let $B \to \infty$ in (2.4.9). Then

$$\frac{d}{dx}\int_{-\infty}^{\phi(x^*)} \bar{w}_2(x^*,y^*)dy^* = \int_{-\infty}^{\phi(x^*)} \frac{\partial \bar{w}_2}{\partial x^*}(x^*,y^*)dy^* + \bar{w}_2(x^*,\phi(x^*))\frac{d\phi(x^*)}{dx^*} \tag{2.4.12}$$

and therefore

$$\int_{-\infty}^{\phi(x^*)} \frac{\partial \bar{w}_2}{\partial x^*}(x^*,y^*)dy^* = \frac{d}{dx}\int_{-\infty}^{\phi(x^*)} \bar{w}_2(x^*,y^*)dy^* - \bar{w}_2(x^*,\phi(x^*))\frac{d\phi(x^*)}{dx^*}. \tag{2.4.13}$$

Substituting (2.4.7) and (2.4.13) into (2.4.3) gives

$$\frac{\rho_2}{\rho_1}\left[\frac{d}{dx}\int_{-\infty}^{\phi(x^*)} \bar{w}_2(x^*,y^*)dy^* - \bar{w}_2(x^*,\phi(x^*))\frac{d\phi(x^*)}{dx^*}\right] = \frac{\mu_2}{\mu_1}\frac{\partial \bar{w}_2}{\partial y^*}(x^*,\phi(x^*)). \tag{2.4.14}$$

Next consider the upper fluid:

$$\frac{\partial \bar{w}_1}{\partial x^*} = \frac{\partial^2 \bar{w}_1}{\partial y^{*2}}. \tag{2.4.15}$$

Integrate (2.4.15) with respect to y^* from $y^* = \phi(x^*)$ to $y^* = \infty$ at constant x^*:

$$\int_{\phi(x^*)}^{\infty} \frac{\partial \bar{w}_1}{\partial x^*}(x^*,y^*)dy^* = \int_{\phi(x^*)}^{\infty} \frac{\partial^2 \bar{w}_1}{\partial y^{*2}}(x^*,y^*)dy^*. \tag{2.4.16}$$

Now,

$$\int_{\phi(x^*)}^{\infty} \frac{\partial^2 \bar{w}_1}{\partial y^{*2}}(x^*, y^*)dy^* = \left[\frac{\partial \bar{w}_1}{\partial y^*}(x^*, y^*)\right]_{\phi(x^*)}^{\infty}$$

$$= \frac{\partial \bar{w}_1}{\partial y^*}(x^*, \infty) - \frac{\partial \bar{w}_1}{\partial y^*}(x^*, \phi(x^*))$$

$$= -\frac{\partial \bar{w}_1}{\partial y^*}(x^*, \phi(x^*)), \tag{2.4.17}$$

since boundary condition (2.3.11) gives

$$\frac{\partial \bar{w}_1}{\partial y^*}(x^*, \infty) = 0. \tag{2.4.18}$$

To rewrite the left-hand side of (2.4.16), use again the Theorem for differentiation under an integral sign. Consider

$$\frac{d}{dx^*}\int_{\phi(x^*)}^{Bx} \bar{w}_1(x^*, y^*)dy^* = \int_{\phi(x^*)}^{Bx} \frac{\partial \bar{w}_1}{\partial x^*}(x^*, y^*)dy^* + \bar{w}_1(x^*, Bx)\, B - \bar{w}_1(x^*, \phi(x^*))\frac{d\phi(x^*)}{dx^*}, \tag{2.4.19}$$

where $B > 0$ is a constant. Now from the boundary conditions

$$\bar{w}_1(x^*, \infty) = 0. \tag{2.4.20}$$

Assume the stronger condition

$$\lim_{B\to\infty} \bar{w}_1(x^*, Bx)\, B = 0. \tag{2.4.21}$$

Let $B \to \infty$ in (2.4.19). Then

$$\frac{d}{dx^*}\int_{\phi(x^*)}^{\infty} \bar{w}_1(x^*, y^*)dy^* = \int_{\phi(x^*)}^{\infty} \frac{\partial \bar{w}_1}{\partial x^*}(x^*, y^*)dy^* - \bar{w}_1(x^*, \phi(x^*))\frac{d\phi(x^*)}{dx^*} \tag{2.4.22}$$

and therefore

$$\int_{\phi(x^*)}^{\infty} \frac{\partial \bar{w}_1}{\partial x^*}(x^*, y^*)dy^* = \frac{d}{dx^*}\int_{\phi(x^*)}^{\infty} \bar{w}_1(x^*, y^*)dy^* + \bar{w}_1(x^*, \phi(x^*))\frac{d\phi(x^*)}{dx^*}. \tag{2.4.23}$$

Substituting (2.4.17) and (2.4.23) into (2.4.16) gives

$$\frac{d}{dx^*}\int_{\phi(x^*)}^{\infty} \bar{w}_1(x^*, y^*)dy^* + \bar{w}_1(x^*, \phi(x^*))\frac{d\phi(x^*)}{dx^*} = -\frac{\partial \bar{w}_1}{\partial y^*}(x^*, \phi(x^*)). \tag{2.4.24}$$

Equations (2.4.14) and (2.4.24) separately do not produce a conserved quantity because the right-hand sides are not zero. Therefore, there is not a conserved quantity for the upper fluid nor a conserved quantity for the lower fluid. There is one conserved quantity for the whole wake consisting of the upper and lower fluids.

Adding equations (2.4.14) and (2.4.24) gives

$$\frac{\rho_2}{\rho_1}\left[\frac{d}{dx^*}\int_{-\infty}^{\phi(x^*)}\bar{w}_2(x^*,y^*)dy^* - \bar{w}_2(x^*,\phi(x^*))\frac{d\phi(x^*)}{dx^*}\right] + \frac{d}{dx^*}\int_{\phi(x^*)}^{\infty}\bar{w}_1(x^*,y^*)dy^* + \bar{w}_1(x^*,\phi(x^*))\frac{d\phi(x^*)}{dx^*}$$

$$= \frac{\mu_2}{\mu_1}\frac{\partial\bar{w}_2}{\partial y^*}(x^*,\phi(x^*)) - \frac{\partial\bar{w}_1}{\partial y^*}(x^*,\phi(x^*)). \tag{2.4.25}$$

However, from the interface condition (2.3.12), (2.4.25) reduces to

$$\frac{d}{dx^*}\left[\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x^*)}\bar{w}_2(x^*,y^*)dy^* + \int_{\phi(x^*)}^{\infty}\bar{w}_1(x^*,y^*)dy^*\right] = \left[\frac{\rho_2}{\rho_1}\bar{w}_2(x^*,\phi(x^*)) - \bar{w}_1(x^*,\phi(x^*))\right]\frac{d\phi(x^*)}{dx^*}. \tag{2.4.26}$$

The velocity deficits, $\bar{w}_1$ and $\bar{w}_2$, in (2.4.26) are dimensionless and small such that their squares and products can be neglected. From (2.1.5) and (2.2.2)

$$\bar{w}_1 = \frac{w_1}{U}, \qquad \bar{w}_2 = \frac{w_2}{U}, \qquad \phi^* = \frac{\phi}{\delta}, \qquad \delta = \frac{l}{\sqrt{Re}}, \tag{2.4.27}$$

where δ is an estimate of the width of the wake. For the far wake $\bar{w}_1 << 1$, $\bar{w}_2 << 1$ and $\phi^* << 1$. The right-hand side of (2.4.26) is therefore second order in smallness and can be neglected. Second order terms in smallness have already been neglected in the analysis. The right-hand side of (2.4.26) will be investigated again later when $\phi(x)$ has been determined. It should be noted that the right-hand side of (2.4.25) is first order in smallness and cannot be neglected. It is zero because of the interface condition (2.3.12). The term on the right-hand side of (2.4.26) is therefore neglected. Equation (2.4.26) reduces to

$$\frac{d}{dx^*}\left[\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x^*)}\bar{w}_2(x^*,y^*)dy^* + \int_{\phi(x^*)}^{\infty}\bar{w}_1(x^*,y^*)dy^*\right] = 0. \tag{2.4.28}$$

Thus,

$$\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x^*)}\bar{w}_2(x^*,y^*)dy^* + \int_{\phi(x^*)}^{\infty}\bar{w}_1(x^*,y^*)dy^* = \text{constant independent of } x^*. \tag{2.4.29}$$

Hence,

$$\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x^*)}\bar{w}_2(x^*,y^*)dy^* + \int_{\phi(x^*)}^{\infty}\bar{w}_1(x^*,y^*)dy^* = D^*, \tag{2.4.30}$$

where

$$D^* = \frac{D}{\rho_1 U} \tag{2.4.31}$$

and D is the drag per unit width on the obstacle and D^* is the dimensionless drag. The conserved quantity (2.4.30) agrees with the conserved quantity derived by Herczynski, Weidman and Burde [1].

2.4.2 Two-fluid wake behind a self-propelled body

The wake behind the self-propelled body is referred to as a momentumless wake. For a self-propelled body, the total drag on the body is zero. Hence from (2.4.30), for the two-fluid wake behind a self-propelled body,

$$D^* = \frac{D}{\rho_1 U} = \frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x^*)} \bar{w}_2(x^*, y^*)dy^* + \int_{\phi(x^*)}^{\infty} \bar{w}_1(x^*, y^*)dy^* = 0. \tag{2.4.32}$$

Condition (2.4.32) is first order in $\bar{w}_1$ and $\bar{w}_2$ and is the defining condition for a momentumless wake. It must always be satisfied and determines the form of the conserved quantity for the momentumless wake.

In order to determine the conserved quantity, equation (2.3.10) will be multiplied by y^{*2} and then integrated across the wake with respect to y^* from $y^* = -\infty$ to $y^* = +\infty$.

Start at the lower fluid:

$$\frac{\partial \bar{w}_2}{\partial x^*} = \frac{\nu_2}{\nu_1}\frac{\partial^2 \bar{w}_2}{\partial y^{*2}}. \tag{2.4.33}$$

Multiply by y^{*2} and integrate with respect to y^* at fixed point x^* from $y^* = -\infty$ to $y^* = \phi(x^*)$.

$$\int_{-\infty}^{\phi(x^*)} y^{*2}\frac{\partial \bar{w}_2}{\partial x^*}(x^*, y^*)dy^* = \frac{\nu_2}{\nu_1}\int_{-\infty}^{\phi(x^*)} y^{*2}\frac{\partial^2 \bar{w}_2}{\partial y^{*2}}(x^*, y^*)dy^*. \tag{2.4.34}$$

Consider the left-hand side of (2.4.34) and apply again the Theorem for differentiation under the integral sign. Replace the lower limit $-\infty$ by $-Bx$ where $B > 0$ is a constant. Then

$$\frac{d}{dx^*}\int_{-Bx}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*, y^*)dy^* = \int_{-Bx}^{\phi(x^*)} y^{*2}\frac{\partial \bar{w}_2}{\partial x^*}(x^*, y^*)dy^* + \phi^2(x^*)\,\bar{w}_2(x^*, \phi(x^*))\frac{d\phi(x^*)}{dx^*}$$

$$- B^2\,x^2\,\bar{w}_2(x^*, -Bx)(-B). \tag{2.4.35}$$

From the boundary conditions

$$\bar{w}_2(x^*, -\infty) = 0. \tag{2.4.36}$$

Assume that

$$\lim_{B\to\infty} \bar{w}_2(x^*, -Bx)B^3 = 0. \tag{2.4.37}$$

Let $B \to \infty$ in (2.4.35). Then

$$\frac{d}{dx^*}\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*, y^*)dy^* = \int_{-\infty}^{\phi(x^*)} y^{*2}\frac{\partial \bar{w}_2}{\partial x^*}(x^*, y^*)dy^* + \phi^2(x^*)\,\bar{w}_2(x^*, \phi(x^*))\frac{d\phi(x^*)}{dx^*} \tag{2.4.38}$$

and hence

$$\int_{-\infty}^{\phi(x^*)} y^{*2}\frac{\partial \bar{w}_2}{\partial x^*}(x^*, y^*)dy^* = \frac{d}{dx^*}\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*, y^*)dy^* - \phi^2(x^*)\,\bar{w}_2(x^*, \phi(x^*))\frac{d\phi(x^*)}{dx^*}. \tag{2.4.39}$$

Now consider the right-hand side of (2.4.34). Assume that

$$\lim_{y^*\to-\infty} y^{*2}\frac{\partial \bar{w}_2}{\partial y^*}(x^*, y^*) = 0, \qquad \lim_{y^*\to-\infty} y^*\bar{w}_2(x^*, y^*) = 0. \tag{2.4.40}$$

Then, integrating twice by parts gives

$$\int_{-\infty}^{\phi(x^*)} y^{*2}\frac{\partial^2 \bar{w}_2}{\partial y^{*2}}(x^*, y^*)dy^* = y^{*2}\frac{\partial \bar{w}_2}{\partial y^*}(x^*, y^*)\Big|_{-\infty}^{\phi(x^*)} - \int_{-\infty}^{\phi(x^*)} 2y^*\frac{\partial \bar{w}_2}{\partial y^*}(x^*, y^*)dy^*$$

$$= \phi^2(x^*)\frac{\partial \bar{w}_2}{\partial y^*}(x^*, \phi(x^*)) - 2\phi(x^*)\bar{w}_2(x^*, \phi(x^*)) + 2\int_{-\infty}^{\phi(x^*)} \bar{w}_2(x^*, y^*)dy^*. \tag{2.4.41}$$

Substitute (2.4.39) and (2.4.41) into (2.4.34)

$$\frac{d}{dx^*}\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*, y^*)dy^* - \phi^2(x^*)\bar{w}_2(x^*, \phi(x^*))\frac{d\phi(x^*)}{dx^*} =$$

$$\frac{\nu_2}{\nu_1}\left[2\int_{-\infty}^{\phi(x^*)} \bar{w}_2(x^*, y^*)dy^* + \phi^2(x^*)\frac{\partial \bar{w}_2}{\partial y^*}(x^*, \phi(x^*)) - 2\phi(x^*)\bar{w}_2(x^*, \phi(x^*))\right]. \tag{2.4.42}$$

Hence,

$$\frac{d}{dx^*}\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*, y^*)dy^* = 2\frac{\nu_2}{\nu_1}\int_{-\infty}^{\phi(x^*)} \bar{w}_2(x^*, y^*)dy^* + \text{second order terms}. \tag{2.4.43}$$

Now we need to write the right-hand side of (2.4.43) in the form of the drag, D^*, given by (2.4.32). We therefore multiply (2.4.43) by $\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}$. Hence,

$$\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\frac{d}{dx^*}\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*,y^*)dy^* = 2\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x^*)} \bar{w}_2(x^*,y^*)dy^* + \text{second order terms.} \quad (2.4.44)$$

Now consider the upper fluid:

$$\frac{\partial \bar{w}_1}{\partial x^*} = \frac{\partial^2 \bar{w}_1}{\partial y^{*2}}. \quad (2.4.45)$$

Multiply by y^{*2} and integrate with respect to y^* at fixed point x^* from $y^* = \phi(x^*)$ to $y^* = \infty$.

$$\int_{\phi(x^*)}^{\infty} y^{*2}\frac{\partial \bar{w}_1}{\partial x^*}(x^*,y^*)dy^* = \int_{\phi(x^*)}^{\infty} y^{*2}\frac{\partial^2 \bar{w}_1}{\partial y^{*2}}(x^*,y^*)dy^*. \quad (2.4.46)$$

Consider the left-hand side of (2.4.46). Replace the upper limit ∞ by Bx where $B > 0$ is a constant and apply the Theorem for differentiation under the integral sign. Then

$$\frac{d}{dx^*}\int_{\phi(x^*)}^{Bx} y^{*2}\bar{w}_1(x^*,y^*)dy^* = \int_{\phi(x^*)}^{Bx} y^{*2}\frac{\partial \bar{w}_1}{\partial x^*}(x^*,y^*)dy^* + B^2\, x^2\bar{w}_1(x^*,Bx)\, B$$

$$- \phi^2(x^*)\, \bar{w}_1(x^*,\phi(x^*))\frac{d\phi(x^*)}{dx^*}. \quad (2.4.47)$$

But from the boundary conditions

$$\bar{w}_1(x^*,\infty) = 0. \quad (2.4.48)$$

Assume that

$$\lim_{B\to\infty} \bar{w}_1(x^*,Bx)B^3 = 0. \quad (2.4.49)$$

Let $B \to \infty$ in (2.4.47). Thus

$$\frac{d}{dx^*}\int_{\phi(x^*)}^{\infty} y^{*2}\bar{w}_1(x^*,y^*)dy^* = \int_{\phi(x^*)}^{\infty} y^{*2}\frac{\partial \bar{w}_1}{\partial x^*}(x^*,y^*)dy^* - \phi^2(x^*)\bar{w}_1(x^*,\phi(x^*))\frac{d\phi(x^*)}{dx^*} \quad (2.4.50)$$

and therefore

$$\int_{\phi(x^*)}^{\infty} y^{*2}\frac{\partial \bar{w}_1}{\partial x^*}(x^*,y^*)dy^* = \frac{d}{dx^*}\int_{\phi(x^*)}^{\infty} y^{*2}\bar{w}_1(x^*,y^*)dy^* + \phi^2(x^*)\bar{w}_1(x^*,\phi(x^*))\frac{d\phi(x^*)}{dx^*}. \quad (2.4.51)$$

Now consider the right-hand side of (2.4.46). Assume that

$$\lim_{y^*\to\infty} y^{*2}\frac{\partial \bar{w}_1}{\partial y^*}(x^*,y^*) = 0, \qquad \lim_{y^*\to\infty} y^*\bar{w}_1(x^*,y^*) = 0. \quad (2.4.52)$$

Then, integrating twice by parts gives

$$\int_{\phi(x^*)}^{\infty} y^{*2}\frac{\partial^2\bar{w}_1}{\partial y^{*2}}(x^*,y^*)dy^* = y^{*2}\frac{\partial\bar{w}_1}{\partial y^*}(x^*,y^*)\Big|_{\phi(x^*)}^{\infty} - \int_{\phi(x^*)}^{\infty} 2y^*\frac{\partial\bar{w}_1}{\partial y^*}(x^*,y^*)dy^*$$

$$= -\phi^2(x^*)\frac{\partial\bar{w}_1}{\partial y^*}(x^*,\phi(x^*)) + 2\phi(x^*)\bar{w}_1(x^*,\phi(x^*)) + 2\int_{\phi(x^*)}^{\infty}\bar{w}_1(x^*,y^*)dy^*. \tag{2.4.53}$$

Substitute (2.4.51) and (2.4.53) into (2.4.46)

$$\frac{d}{dx^*}\int_{\phi(x^*)}^{\infty} y^{*2}\bar{w}_1(x^*,y^*)dy^* + \phi^2(x^*)\bar{w}_1(x^*,\phi(x^*))\frac{d\phi(x^*)}{dx^*} =$$

$$2\int_{\phi(x^*)}^{\infty}\bar{w}_1(x^*,y^*)dy^* - \phi^2(x^*)\frac{\partial\bar{w}_1}{\partial y^*}(x^*,\phi(x^*)) + 2\phi(x^*)\bar{w}_1(x^*,y^*). \tag{2.4.54}$$

Hence,

$$\frac{d}{dx^*}\int_{\phi(x^*)}^{\infty} y^{*2}\bar{w}_1(x^*,y^*)dy^* = 2\int_{\phi(x^*)}^{\infty}\bar{w}_1(x^*,y^*)dy^* + \text{second order terms.} \tag{2.4.55}$$

The right-hand side of (2.4.55) has the same form as in the drag, D^*. Now, the upper and lower parts of the wake may be combined by adding (2.4.44) and (2.4.55),

$$\frac{d}{dx^*}\left[\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*,y^*)dy^* + \int_{\phi(x^*)}^{\infty} y^{*2}\bar{w}_1(x^*,y^*)dy^*\right] =$$

$$2\left[\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x^*)}\bar{w}_2(x^*,y^*)dy^* + \int_{\phi(x^*)}^{\infty}\bar{w}_1(x^*,y^*)dy^*\right] + \text{second order terms.} \tag{2.4.56}$$

However, since the total drag, D^*, on the body is zero, it follows from (2.4.32) that

$$\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x^*)}\bar{w}_2(x^*,y^*)dy^* + \int_{\phi(x^*)}^{\infty}\bar{w}_1(x^*,y^*)dy^* = 0. \tag{2.4.57}$$

Therefore, (2.4.56) reduces to

$$\frac{d}{dx^*}\left[\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*,y^*)dy^* + \int_{\phi(x^*)}^{\infty} y^{*2}\bar{w}_1(x^*,y^*)dy^*\right] = \text{second order terms.} \tag{2.4.58}$$

Neglecting second order terms gives,

$$\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*,y^*)dy^* + \int_{\phi(x^*)}^{\infty} y^{*2}\bar{w}_1(x^*,y^*)dy^* = \text{constant independent of } x^*. \tag{2.4.59}$$

Let

$$K = U\left[\frac{\rho_2}{\nu_2}\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*,y^*)dy^* + \frac{\rho_1}{\nu_1}\int_{\phi(x^*)}^{\infty} y^{*2}\bar{w}_1(x^*,y^*)dy^*\right]$$

$$= U\frac{\rho_1}{\nu_1}\left[\frac{\rho_2}{\nu_2}\frac{\nu_1}{\rho_1}\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*,y^*)dy^* + \int_{\phi(x^*)}^{\infty} y^{*2}\bar{w}_1(x^*,y^*)dy^*\right], \tag{2.4.60}$$

and define

$$K^* = \frac{K\nu_1}{U\rho_1}. \tag{2.4.61}$$

Then,

$$K^* = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*,y^*)dy^* + \int_{\phi(x^*)}^{\infty} y^{*2}\bar{w}_1(x^*,y^*)dy^* = \text{constant independent of } x^*. \tag{2.4.62}$$

The quantity K^* is the conserved quantity for the two-fluid momentumless wake.

We now see why we multiplied equations (2.4.33) and (2.4.45) by y^{*2} before integrating. It was necessary to integrate twice by parts to reduce the second derivative $\frac{\partial^2\bar{w}_i}{\partial y^{*2}}$ to $\bar{w}_i(x^*,y^*)$ which occurs in the total drag, D^*.

Herczynski, Weidman and Burde [1] obtain for the conserved quantity for the two-fluid momentumless wake

$$K = U\rho_1\left[\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*,y^*)dy^* + \int_{\phi(x^*)}^{\infty} y^{*2}\bar{w}_1(x^*,y^*)dy^*\right]. \tag{2.4.63}$$

They proved that

$$U\frac{\partial}{\partial x^*}\left[\rho_2\int_{-\infty}^{\phi(x^*)} y^{*2}\bar{w}_2(x^*,y^*)dy^* + \rho_1\int_{\phi(x^*)}^{\infty} y^{*2}\bar{w}_1(x^*,y^*)dy^*\right] =$$

$$2\left[\mu_2\int_{-\infty}^{\phi(x^*)} \bar{w}_2(x^*,y^*)dy^* + \mu_1\int_{\phi(x^*)}^{\infty} \bar{w}_1(x^*,y^*)dy^*\right] \tag{2.4.64}$$

and then stated for a momentumless wake

$$\int_{-\infty}^{\phi(x^*)} \bar{w}_2(x^*,y^*)dy^* = 0 \tag{2.4.65}$$

and

$$\int_{\phi(x^*)}^{\infty} \bar{w}_1(x^*,y^*)dy^* = 0. \tag{2.4.66}$$

However, the integrals (2.4.65) and (2.4.66) are not separately zero. It is the sum of the integrals in the total drag, D^*, given by (2.4.57) which vanishes. The total drag, D^*, given by (2.4.32) for the classical two-fluid wake was derived and used by Herczynski, Weidman and Burde [1] as the conserved quantity for the classical two-fluid wake.

2.5 Introducing the stream function

Stream functions $\psi_i(x, y)$ are introduced, defined by

$$\bar{w}_i(x^*, y^*) = \frac{\partial \psi_i}{\partial y^*}, \quad \bar{v}_i(x^*, y^*) = \frac{\partial \psi_i}{\partial x^*}. \tag{2.5.1}$$

The stream function allows the number of unknowns in each fluid to be reduced from two to one, but the order of the partial differential equation is increased by one. The continuity equation is identically satisfied and equation (2.3.10) becomes

$$\frac{\partial^2 \psi_i}{\partial x^* \partial y^*} = \frac{\nu_i}{\nu_1} \frac{\partial^3 \psi_i}{\partial y^{*3}}. \tag{2.5.2}$$

The boundary conditions (2.3.11) in terms of the stream function are

$$\frac{\partial \psi_1}{\partial y^*}(x^*, \infty) = 0, \quad \frac{\partial \psi_2}{\partial y^*}(x^*, -\infty) = 0, \quad \frac{\partial^2 \psi_1}{\partial y^{*2}}(x^*, \infty) = 0, \quad \frac{\partial^2 \psi_2}{\partial y^{*2}}(x^*, -\infty) = 0, \quad x^* \geq 0. \tag{2.5.3}$$

In terms of the stream function the interface conditions (2.3.12) to (2.3.15) become

$$\mu_1 \frac{\partial^2 \psi_1}{\partial y^{*2}}(x^*, \phi^*) = \mu_2 \frac{\partial^2 \psi_2}{\partial y^{*2}}(x^*, \phi^*), \tag{2.5.4}$$

$$2\frac{\mu_2}{\mu_1} \frac{1}{Re} \frac{\partial^2 \psi_2}{\partial x^* \partial y^*}(x^*, \phi^*) - \frac{2}{Re} \frac{\partial^2 \psi_1}{\partial x^* \partial y^*}(x^*, \phi^*) = \bar{P}_2(x^*, \phi^*) - \bar{P}_1(x^*, \phi^*), \tag{2.5.5}$$

$$\frac{\partial \psi_1}{\partial y^*}(x^*, \phi^*) = \frac{\partial \psi_2}{\partial y^*}(x^*, \phi^*), \tag{2.5.6}$$

$$\frac{\partial \psi_1}{\partial x^*}(x^*, \phi^*) = \frac{\partial \psi_2}{\partial x^*}(x^*, \phi^*). \tag{2.5.7}$$

This concludes the mathematical formulation of the problem.

2.6 Conclusions

In this chapter, the Navier-Stokes equations were used in order to derive the boundary layer equations for the classical and momentumless two-fluid wakes. The governing equations, as well as the boundary and interface conditions, for the classical wake and the momentumless wake are the same. The difference between these two symmetric wakes is the conserved quantities that they satisfy, which will be explored further in the next chapter. The Navier-Stokes equations are non-dimensionalised and a perturbation on the oncoming flow is considered. The velocity deficit is introduced through the perturbation in the x-component of the velocity in order to simply the problem and derive the boundary layer equations. The boundary and interface conditions are derived through balancing tangential and normal stress components and tangential and normal velocity components. These conditions are also expressed in terms of the velocity deficit. The conserved quantities for the classical and momentumless wakes are derived and finally the problem is expressed in terms of the stream function for each fluid.

Chapter 3

Similarity Solutions

3.1 Introduction

Herczynski, Weidman and Burde [1] derived a model based on the two streams of fluid having different densities and viscosities, but the same mainstream speed. This case prevents the need to consider a boundary layer forming upstream of the object. The model is derived from the Navier-Stokes equation which is non-dimensionalised in terms of characteristic quantities defined for the top fluid. The density of the lower fluid is greater than the density of the upper fluid for stability. Herczynski, Weidman and Burde determined that despite the relative magnitude of their absolute fluid kinematic viscosities, the interface of classical two-fluid wakes deflected upward [1]. It was also stated that analytical solutions for the two-fluid classical wake based on the boundary layer approximation characterise asymptotic Goldtein [2] wakes that penetrate differently into each fluid domain. In this chapter the solution will be derived through the use of a scaling transformation as well as the process utilised by Herczynski, Weidman and Burde. It will be applied to the model derived in Chapter 2 and compared against the solutions derived by Herczynski, Weidman and Burde.

In order to keep the notation simple, the bars and stars will be suppressed in the dimensionless quantities.

The governing equations from (2.5.2) are given by:

$$\frac{\partial^2 \psi_i}{\partial x \partial y} = \frac{\nu_i}{\nu_1} \frac{\partial^3 \psi_i}{\partial y^3}, \quad i = 1, 2, \tag{3.1.1}$$

subject to the following boundary and interface conditions where the unknown interface is

defined as

$$y = \phi(x). \tag{3.1.2}$$

The boundary conditions are:

$$\frac{\partial \psi_1}{\partial y}(x,\infty) = 0, \quad \frac{\partial \psi_2}{\partial y}(x,-\infty) = 0, \quad \frac{\partial^2 \psi_1}{\partial y^2}(x,\infty) = 0, \quad \frac{\partial^2 \psi_2}{\partial y^2}(x,-\infty) = 0, \quad x \geq 0. \tag{3.1.3}$$

The interface conditions are:

$$\mu_1 \frac{\partial^2 \psi_1}{\partial y^2}(x,\phi) = \mu_2 \frac{\partial^2 \psi_2}{\partial y^2}(x,\phi), \tag{3.1.4}$$

$$2\frac{\mu_1}{\mu_2}\frac{1}{Re}\frac{\partial^2 \psi_2}{\partial x \partial y}(x,\phi) - \frac{2}{Re}\frac{\partial^2 \psi_1}{\partial x \partial y}(x,\phi) = P_2(x,\phi) - P_1(x,\phi), \tag{3.1.5}$$

$$\frac{\partial \psi_1}{\partial y}(x,\phi) = \frac{\partial \psi_2}{\partial y}(x,\phi), \tag{3.1.6}$$

$$\frac{\partial \psi_1}{\partial x}(x,\phi) = \frac{\partial \psi_2}{\partial x}(x,\phi). \tag{3.1.7}$$

3.2 Invariant solution for the classical two-fluid wake

Consider the following scaling transformation

$$\bar{x} = \lambda^a x, \quad \bar{y} = \lambda^b y, \quad \bar{\psi}_i = \lambda^c \psi_i, \tag{3.2.1}$$

where λ is a constant. Substituting (3.2.1) into equation (3.1.1) and simplifying gives

$$\frac{\partial^2 \bar{\psi}_i}{\partial \bar{x} \partial \bar{y}} = \lambda^{2b-a} \frac{\nu_i}{\nu_1} \frac{\partial^3 \bar{\psi}_i}{\partial \bar{y}^3}. \tag{3.2.2}$$

Equation (3.1.1) is invariant under the scaling transformation provided

$$2b - a = 0, \tag{3.2.3}$$

that is,

$$a = 2b. \tag{3.2.4}$$

Thus, equation (3.1.1) is invariant under the scaling transformation

$$\bar{x} = \lambda^{2b}x, \quad \bar{y} = \lambda^{b}y, \quad \bar{\psi}_i = \lambda^{c}\psi_i. \tag{3.2.5}$$

Suppose that a solution of (3.1.1) is

$$\psi_i = f_i(x, y). \tag{3.2.6}$$

Then, a solution of the invariant transformed partial differential equation is given by

$$\bar{\psi}_i = f_i(\bar{x}, \bar{y}). \tag{3.2.7}$$

Substituting (3.2.5) into (3.2.7) gives

$$\lambda^{c}\psi_i = f_i(\lambda^{2b}x, \lambda^{b}y). \tag{3.2.8}$$

Substitute (3.2.6) into (3.2.8) for ψ_i, we obtain

$$\lambda^{c}f_i(x, y) = f_i(\lambda^{2b}x, \lambda^{b}y). \tag{3.2.9}$$

Differentiating (3.2.9) with respect to λ gives

$$c\lambda^{c-1}f_i(x, y) = 2b\lambda^{2b-1}x\frac{\partial f_i}{\partial \bar{x}} + b\lambda^{b-1}y\frac{\partial f_i}{\partial \bar{y}}. \tag{3.2.10}$$

Let $\lambda = 1$, then $\bar{x} = x$ and $\bar{y} = y$, and $f(x, y)$ satisfies

$$2bx\frac{\partial f_i}{\partial x} + by\frac{\partial f_i}{\partial y} = cf_i. \tag{3.2.11}$$

Equation (3.2.11) is a linear first order partial differential equation. The differential equations of the characteristic curves of (3.2.11) are

$$\frac{dx}{2bx} = \frac{dy}{by} = \frac{df_i}{cf_i}. \tag{3.2.12}$$

The first pair of (3.2.12) gives

$$\frac{y}{x^{\frac{1}{2}}} = t_{i1}, \tag{3.2.13}$$

where t_{i1} is a constant. The first and last terms of (3.2.12) gives

$$\frac{f_i}{x^{\frac{c}{2b}}} = t_{i2}, \tag{3.2.14}$$

where t_{i2} is a constant. The general solution of the first order linear PDE (3.2.11) is

$$t_{i2} = F_i(t_{i1}), \tag{3.2.15}$$

where F_i is an arbitrary function. Therefore, we have

$$\psi_i(x,y) = x^{\omega} F_i\left(\frac{y}{x^{\frac{1}{2}}}\right), \tag{3.2.16}$$

where

$$\omega = \frac{c}{2b}. \tag{3.2.17}$$

The solution which leaves equation (3.1.1) invariant under the scaling transformation (3.2.5) is

$$\psi_i(x,y) = x^{\omega} F_i(\xi), \tag{3.2.18}$$

where

$$\xi = \frac{y}{x^{\frac{1}{2}}}. \tag{3.2.19}$$

Substitute (3.2.18) into equation (3.1.1) where y is expressed in terms of x and ξ. Simplifying gives

$$2\frac{\nu_i}{\nu_1}\frac{d^3F_i}{d\xi^3} + \xi\frac{d^2F_i}{d\xi^2} + (1-2\omega)\frac{dF_i}{d\xi} = 0. \tag{3.2.20}$$

The value of ω cannot be determined from the boundary conditions. It can be obtained, however, from the conserved quantity (2.4.30), expressed in terms of the stream function:

$$D = \frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} \frac{\partial \psi_2}{\partial y}(x,y)dy + \int_{\phi(x)}^{\infty} \frac{\partial \psi_1}{\partial x}(x,y)dy, \tag{3.2.21}$$

where D, the dimensionless drag on the obstacle, is a constant independent of x. Express D in terms of the similarity variables ξ and F_i. Integration in (3.2.21) with respect to y is performed at a given fixed value of x. Hence,

$$dy = x^{\frac{1}{2}}d\xi \tag{3.2.22}$$

and (3.2.21) becomes

$$D = \frac{\rho_2}{\rho_1}x^{\omega}\int_{-\infty}^{\frac{\phi(x)}{x^{\frac{1}{2}}}} \frac{dF_2}{d\xi}d\xi + x^{\omega}\int_{\frac{\phi(x)}{x^{\frac{1}{2}}}}^{\infty} \frac{dF_1}{d\xi}d\xi. \tag{3.2.23}$$

The drag, D, is independent of x if

$$\omega = 0, \qquad \frac{\phi(x)}{x^{\frac{1}{2}}} = k, \tag{3.2.24}$$

where k is a constant. Thus,

$$D = \frac{\rho_2}{\rho_1}\int_{-\infty}^{k}\frac{dF_2}{d\xi}d\xi + \int_{k}^{\infty}\frac{dF_1}{d\xi}d\xi \tag{3.2.25}$$

and the equation of the interface is

$$\phi(x) = kx^{\frac{1}{2}}, \tag{3.2.26}$$

where the constant k is still to be determined. The differential equation (3.2.20) becomes

$$2\frac{\nu_i}{\nu_1}\frac{d^3F_i}{d\xi^3} + \xi\frac{d^2F_i}{d\xi^2} + \frac{dF_i}{d\xi} = 0. \tag{3.2.27}$$

Equation (3.2.27) can be rewritten in the form

$$2\frac{\nu_i}{\nu_1}\frac{d^3F_i}{d\xi^3} + \frac{d}{d\xi}\left(\xi\frac{dF_i}{d\xi}\right) = 0. \tag{3.2.28}$$

This is an example of the Double Reduction Theorem by Sjöberg [46]. If a PDE is reduced to an ODE by a symmetry associated with a conservation law then the ODE can be integrated at least once. In the reduction performed here, the parameter ω in the scaling transformations was determined by the conserved quantity, (3.2.21). The two reductions are to reduce the PDE to an ODE and then to reduce the third order ODE to a second order ODE.

Consider now the boundary and interface conditions given by (2.5.3) to (2.5.7). Now, from (3.2.18), (3.2.19) and (3.2.24),

$$\psi_i(x,y) = F_i(\xi), \qquad \xi = \frac{y}{x^{\frac{1}{2}}}. \tag{3.2.29}$$

Hence,

$$\frac{\partial\psi_i}{\partial x} = -\frac{1}{2x}\xi\frac{dF_i}{d\xi}, \qquad \frac{\partial\psi_i}{\partial y} = \frac{1}{x^{\frac{1}{2}}}\frac{dF_i}{d\xi}, \tag{3.2.30}$$

$$\frac{\partial^2\psi_i}{\partial x\partial y} = -\frac{1}{2x^{\frac{3}{2}}}\left[\frac{dF_i}{d\xi} + \xi\frac{d^2F_i}{d\xi^2}\right], \qquad \frac{\partial^2\psi_i}{\partial y^2} = \frac{1}{x}\frac{d^2F_i}{d\xi^2}. \tag{3.2.31}$$

The boundary conditions (2.5.3) become

$$\frac{dF_1}{d\xi}(\infty) = 0, \qquad \frac{d^2F_1}{d\xi^2}(\infty) = 0, \tag{3.2.32}$$

$$\frac{dF_2}{d\xi}(-\infty) = 0, \qquad \frac{d^2F_2}{d\xi^2}(-\infty) = 0. \tag{3.2.33}$$

The interface conditions (2.5.4) to (2.5.7) at the interface $\xi = k$, expressed in terms of $F_i(\xi)$, become

$$\frac{dF_1}{d\xi}(k) = \frac{dF_2}{d\xi}(k), \tag{3.2.34}$$

$$\frac{d^2F_1}{d\xi^2}(k) = \frac{\mu_2}{\mu_1}\frac{d^2F_2}{d\xi^2}(k), \tag{3.2.35}$$

$$\frac{1}{Rex^{\frac{2}{3}}}\left(1 - \frac{\mu_2}{\mu_1}\right)\frac{dF_1}{d\xi}(k) = P_2(x, \phi(x)) - P_1(x, \phi(x)). \tag{3.2.36}$$

Interface conditions (2.5.6) and (2.5.7) reduce to the same condition in terms of $F_i(\xi)$ given by (3.2.34). Conditions (3.2.34) and (3.2.35) were used to simply interface condition (3.2.36).

Now, integrating (3.2.28) with respect to ξ gives

$$2\frac{\nu_i}{\nu_1}\frac{d^2F_i}{d\xi^2} + \xi\frac{dF_i}{d\xi} = A_i. \tag{3.2.37}$$

Impose boundary conditions (3.2.32) and (3.2.33) for $i = 1$ and $i = 2$ respectively. Assume that $\frac{dF_i}{d\xi}$ tends to zero sufficiently rapidly as $\xi \to \pm\infty$ that

$$\lim_{\xi\to+\infty} \xi\frac{dF_1}{d\xi} = 0, \qquad \lim_{\xi\to-\infty} \xi\frac{dF_2}{d\xi} = 0. \tag{3.2.38}$$

Then

$$A_1 = A_2 = 0. \tag{3.2.39}$$

Thus,

$$2\frac{\nu_i}{\nu_1}\frac{d^2F_i}{d\xi^2} + \xi\frac{dF}{d\xi} = 0. \tag{3.2.40}$$

Now let

$$W_i(\xi) = \frac{dF_i}{d\xi}. \tag{3.2.41}$$

ODE (3.2.40) now becomes

$$2\frac{\nu_i}{\nu_1}\frac{dW_i}{d\xi} + \xi W_i = 0 \tag{3.2.42}$$

and

$$\frac{dW_i}{d\xi} = -\frac{\nu_1}{2\nu_i}\xi W_i. \tag{3.2.43}$$

Solving for W_i gives

$$W_i = \exp\left(B_i^*\right) \exp\left(- \frac{\nu_1}{4\nu_i}\xi^2\right), \tag{3.2.44}$$

where B_i^* is a constant. Therefore,

$$\frac{dF_i}{d\xi} = B_i \exp\left(- \frac{\nu_1}{4\nu_i}\xi^2\right), \tag{3.2.45}$$

where $B_i = \exp\left(B_i^*\right)$. Imposing interface condition (3.2.34) on (3.2.45) gives

$$B_1 \exp\left(- \frac{1}{4}k^2\right) = B_2 \exp\left(- \frac{1}{4}\frac{\nu_1}{\nu_2}k^2\right). \tag{3.2.46}$$

Substituting (3.2.45) into (3.2.43) and imposing interface condition (3.2.35) produces

$$k\left[B_1 \exp\left(- \frac{1}{4}k^2\right) - \frac{\rho_2}{\rho_1}B_2 \exp\left(- \frac{1}{4}\frac{\nu_1}{\nu_2}k^2\right)\right] = 0. \tag{3.2.47}$$

Applying condition (3.2.46) into (3.2.47) gives

$$k\left(1 - \frac{\rho_2}{\rho_1}\right)B_1 \exp\left(- \frac{1}{4}k^2\right) = 0. \tag{3.2.48}$$

However, we consider the case in which

$$\rho_1 \neq \rho_2. \tag{3.2.49}$$

Also,

$$B_1 \neq 0, \tag{3.2.50}$$

because if $B_1 = 0$ then from (3.2.46), $B_2 = 0$ and then

$$\frac{dF_i}{d\xi}(\xi) = 0, \qquad w_i(x, y) = 0. \tag{3.2.51}$$

Thus, from (3.2.48)

$$k = 0. \tag{3.2.52}$$

Then, (3.2.46) becomes

$$B_1 = B_2 = B. \tag{3.2.53}$$

The interface is the x-axis and on the interface it follows from (3.2.45) that

$$\frac{\partial w_i}{\partial y} = \frac{1}{x}\frac{d^2F_i}{d\xi^2}(k) = 0. \tag{3.2.54}$$

The turning point in the velocity profile is therefore on the interface.

Now, the constant B may be determined. The conserved quantity (3.2.25) becomes

$$D = \frac{\rho_2}{\rho_1}\int_{-\infty}^{0}\frac{dF_2}{d\xi}d\xi + \int_{0}^{\infty}\frac{dF_1}{d\xi}d\xi. \tag{3.2.55}$$

However, from (3.2.45)

$$\frac{dF_2}{d\xi} = B\exp\bigg(-\frac{\nu_1}{4\nu_2}\xi^2\bigg), \tag{3.2.56}$$

$$\frac{dF_1}{d\xi} = B\exp\bigg(-\frac{1}{4}\xi^2\bigg). \tag{3.2.57}$$

Thus, replacing ξ by $-\xi$ in the first integral, we obtain

$$D = B\bigg[\frac{\rho_2}{\rho_1}\int_{0}^{\infty}\exp\bigg(-\frac{\nu_1}{4\nu_2}\xi^2\bigg)d\xi + \int_{0}^{\infty}\exp\bigg(-\frac{1}{4}\xi^2\bigg)d\xi\bigg] \tag{3.2.58}$$

$$= B\bigg[\frac{\rho_2}{\rho_1}I_2 + I_1\bigg], \tag{3.2.59}$$

where

$$I_2 = \int_{0}^{\infty}\exp\bigg(-\frac{\nu_1}{4\nu_2}\xi^2\bigg)d\xi, \tag{3.2.60}$$

$$I_1 = \int_{0}^{\infty}\exp\bigg(-\frac{1}{4}\xi^2\bigg)d\xi. \tag{3.2.61}$$

In (3.2.60) let

$$\frac{\nu_1}{4\nu_2}\xi^2 = u, \tag{3.2.62}$$

and therefore

$$I_2 = \bigg(\frac{\nu_2}{\nu_1}\bigg)^{\frac{1}{2}}\int_{0}^{\infty}u^{-\frac{1}{2}}\exp(-u)du. \tag{3.2.63}$$

Letting $\nu_1 = \nu_2$ in (3.2.63) gives

$$I_1 = \int_{0}^{\infty}u^{-\frac{1}{2}}\exp(-u)du. \tag{3.2.64}$$

Substitute (3.2.63) and (3.2.64) into (3.2.59). This gives

$$D = B\left[\frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}} + 1\right]\int_0^\infty u^{-\frac{1}{2}}\exp(-u)du. \tag{3.2.65}$$

However, the Gamma function is [47]

$$\Gamma(n) = \int_0^\infty u^{n-1}\exp(-u)du, \qquad n > 1. \tag{3.2.66}$$

Thus,

$$\int_0^\infty u^{-\frac{1}{2}}\exp(-u)du = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{3.2.67}$$

Therefore, substituting (3.2.67) into (3.2.65) produces

$$D = B\left[\frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}} + 1\right]\sqrt{\pi}. \tag{3.2.68}$$

Now,

$$\frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}} = \frac{\rho_2}{\rho_1}\left(\frac{\mu_2}{\rho_2}\frac{\rho_1}{\mu_1}\right)^{\frac{1}{2}} = \left(\frac{\rho_2\mu_2}{\rho_1\mu_1}\right)^{\frac{1}{2}}. \tag{3.2.69}$$

Therefore

$$B = \frac{D}{\sqrt{\pi}\left[1 + \left(\frac{\rho_2\mu_2}{\rho_1\mu_1}\right)^{\frac{1}{2}}\right]} = \frac{D\,\chi^{\frac{1}{2}}}{\sqrt{\pi}\left[1 + \chi^{\frac{1}{2}}\right]}. \tag{3.2.70}$$

The solution is governed by the parameter

$$\chi = \frac{\rho_1\mu_1}{\rho_2\mu_2}, \tag{3.2.71}$$

as in Herczynski, Weidman and Burde [1] and in Lock [7].

3.2.1 Visual representation of the solution

The invariant solution derived in Section 3.2 is visually represented in Figures 3.1 and 3.2.

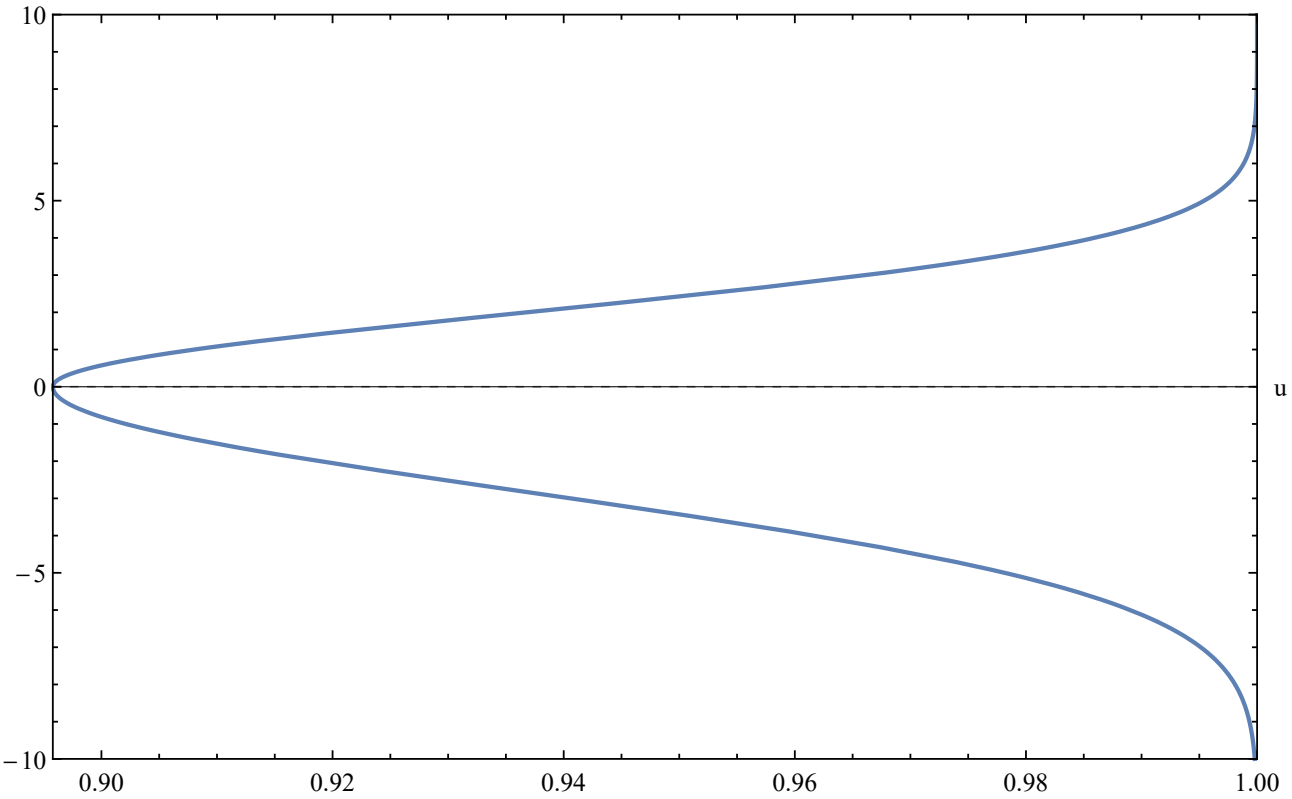

Figure 3.1: The velocities $u_1(x, y)$ and $u_2(x, y)$ of the classical two-fluid wake plotted against y at $x = 2$ for $D = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{2}$.

The velocity profile is given by

$$u_1(x, y) = 1 - w_1(x, y) = 1 - \frac{B}{x^{\frac{1}{2}}} \exp\left(-\frac{1}{4}\frac{y^2}{x} \right), \tag{3.2.72}$$

$$u_2(x, y) = 1 - w_2(x, y) = 1 - \frac{B}{x^{\frac{1}{2}}} \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\frac{y^2}{x} \right) \tag{3.2.73}$$

and

$$B = \frac{D}{\sqrt{\pi}\left[1 + \frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}}\right]}. \tag{3.2.74}$$

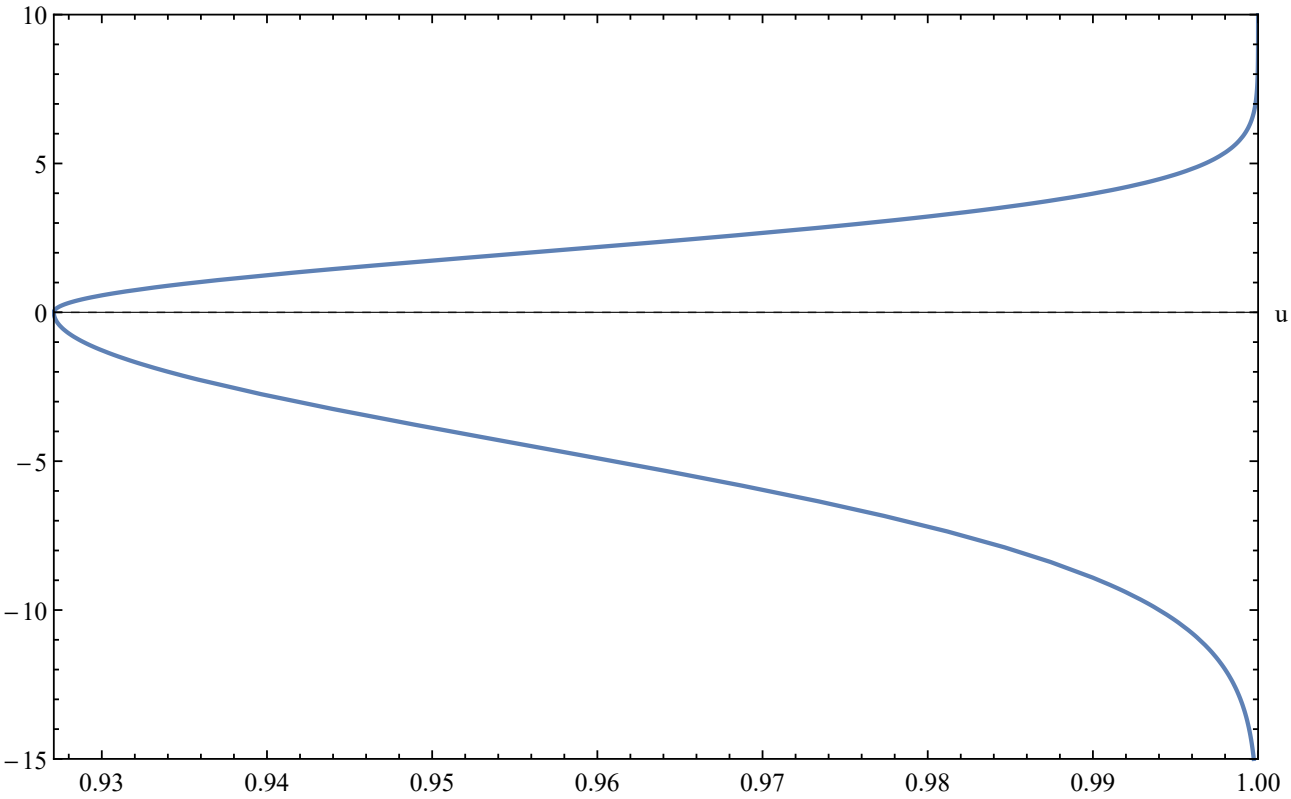

Figure 3.2: The velocities $u_1(x, y)$ and $u_2(x, y)$ of the classical two-fluid wake plotted against y at $x = 2$ for $D = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{5}$.

The effective width of the lower half of the wake in Figure 3.2 is greater than in Figure 3.1. This is because the ratio $\frac{\nu_2}{\nu_1}$ is greater in Figure 3.2 than in Figure 3.1. The diffusion of velocity due to viscosity is therefore stronger in Figure 3.2 resulting in a larger effective width in the lower half of the wake.

3.3 Approach of Herczynski, Weidman and Burde on the classical two-fluid wake

We return to the PDE (3.1.1) in dimensionless form

$$\frac{\partial^2 \psi_i}{\partial x \partial y} = \frac{\nu_i}{\nu_1} \frac{\partial^3 \psi_i}{\partial y^3}, \qquad i = 1, 2. \tag{3.3.1}$$

The conserved quantity is given by (3.2.21):

$$D = \frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} \frac{\partial \psi_2}{\partial y}(x,y)dy + \int_{\phi(x)}^{\infty} \frac{\partial \psi_1}{\partial y}(x,y)dy = \text{constant independent of } x. \tag{3.3.2}$$

The boundary conditions in dimensionless form are from (3.1.3)

$$\frac{\partial \psi_1}{\partial y}(x,\infty) = 0, \qquad \frac{\partial^2 \psi_1}{\partial y^2}(x,\infty) = 0, \tag{3.3.3}$$

$$\frac{\partial \psi_2}{\partial y}(x,-\infty) = 0, \qquad \frac{\partial^2 \psi_2}{\partial y^2}(x,-\infty) = 0. \tag{3.3.4}$$

The interface conditions in dimensionless form are from (3.1.4) to (3.1.7)

$$\frac{\partial \psi_1}{\partial y}(x,\phi) = \frac{\partial \psi_2}{\partial y}(x,\phi), \tag{3.3.5}$$

$$\frac{\partial \psi_1}{\partial x}(x,\phi) = \frac{\partial \psi_2}{\partial x}(x,\phi), \tag{3.3.6}$$

$$\mu_1 \frac{\partial^2 \psi_1}{\partial y^2}(x,\phi) = \mu_2 \frac{\partial^2 \psi_2}{\partial y^2}(x,\phi), \tag{3.3.7}$$

$$2\frac{\mu_1}{\mu_2}\frac{1}{Re}\frac{\partial^2 \psi_2}{\partial x \partial y}(x,\phi) - \frac{2}{Re}\frac{\partial^2 \psi_1}{\partial x \partial y}(x,\phi) = \left(1 - \frac{\rho_2}{\rho_1}\right)\phi(x)\frac{1}{F^2\sqrt{Re}}, \tag{3.3.8}$$

where the Froude number is

$$F = \frac{U}{\sqrt{gl}}. \tag{3.3.9}$$

The coordinate expansion and the expansion of the interface deflection is given by

$$\psi_i(x,y) = f_i^{(0)}(\eta) + \frac{1}{x^{\frac{1}{2}}} f_i^{(1)}(\eta) + \mathcal{O}\left(\frac{1}{x}\right), \tag{3.3.10}$$

$$\phi(x) = \phi^{(0)} + \frac{1}{x^{\frac{1}{2}}}\phi^{(1)} + \frac{1}{x}\phi^{(2)} + \frac{1}{x^{\frac{3}{2}}}\phi^{(3)} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{3.3.11}$$

where

$$\phi^{(0)}, \phi^{(1)}, \phi^{(2)}, \phi^{(3)} \quad \text{are constants} \tag{3.3.12}$$

and

$$\eta = \frac{y - \phi(x)}{x^{\frac{1}{2}}}. \tag{3.3.13}$$

Simplifying (3.3.10), (3.3.11) and (3.3.13) as much as possible before substituting into (3.3.1) is necessary. This is done by considering the conserved quantity and boundary and interface conditions. Consider first the conserved quantity

$$D = \frac{\rho_2}{\rho_1} \int_{-\infty}^{\phi(x)} \frac{\partial \psi_2}{\partial y}(x, y) dy + \int_{\phi(x)}^{\infty} \frac{\partial \psi_1}{\partial y}(x, y) dy. \tag{3.3.14}$$

Express in terms of the variables $f_i^{(n)}$ and η where

$$\eta = \frac{y - \phi(x)}{x^{\frac{1}{2}}}, \qquad \frac{\partial \eta}{\partial y} = \frac{1}{x^{\frac{1}{2}}}. \tag{3.3.15}$$

Also,

$$y = x^{\frac{1}{2}} \eta + \phi(x), \tag{3.3.16}$$

and therefore at constant x,

$$dy = x^{\frac{1}{2}} d\eta. \tag{3.3.17}$$

Now,

$$\frac{\partial \psi_i}{\partial y} = x^{-\frac{1}{2}} \frac{df_i^{(0)}}{d\eta} + x^{-1} \frac{df_i^{(1)}}{d\eta} + \mathcal{O}\left(x^{-\frac{3}{2}}\right). \tag{3.3.18}$$

Thus, substituting (3.3.18) into (3.3.14) gives

$$D = \frac{\rho_2}{\rho_1} \int_{-\infty}^{0} \left(\frac{df_2^{(0)}}{d\eta} + x^{-\frac{1}{2}} \frac{df_2^{(1)}}{d\eta} + \mathcal{O}\left(x^{-1}\right) \right) d\eta + \int_{0}^{\infty} \left(\frac{df_1^{(0)}}{d\eta} + x^{-\frac{1}{2}} \frac{df_1^{(1)}}{d\eta} + \mathcal{O}\left(x^{-1}\right) \right) d\eta. \tag{3.3.19}$$

In (3.3.19), on the interface

$$\eta = \left. \frac{y - \phi(x)}{x^{\frac{1}{2}}} \right|_{y=\phi(x)} = 0. \tag{3.3.20}$$

Now

$$D = \text{constant independent of } x, \tag{3.3.21}$$

provided

$$\frac{df_2^{(1)}}{d\eta} = 0, \qquad \frac{df_2^{(2)}}{d\eta} = 0, ... \tag{3.3.22}$$

$$\frac{df_1^{(1)}}{d\eta} = 0, \qquad \frac{df_1^{(2)}}{d\eta} = 0, ... \tag{3.3.23}$$

The limits are independent of x because of the definition of η. Thus D is independent of x if

$$f_2^{(n)}(\eta) = \text{constant}, \qquad n \geq 1, \tag{3.3.24}$$

$$f_1^{(n)}(\eta) = \text{constant}, \qquad n \geq 1. \tag{3.3.25}$$

The transformation (3.3.10) becomes

$$\psi_i(x, y) = f_i^{(0)}(\eta) + \text{constant}. \tag{3.3.26}$$

But since

$$w_i = \frac{\partial \psi_i}{\partial y}, \qquad v_i = \frac{\partial \psi_i}{\partial x}, \tag{3.3.27}$$

the additive constant in ψ_i does not contribute to w_i and v_i. Therefore, the additive constant can be taken as zero,

$$\psi_i(x, y) = f_i^{(0)}(\eta). \tag{3.3.28}$$

Suppress (0) in $f_i^{(0)}$ and the transformation reduces to

$$\psi_i(x, y) = f_i(\eta), \tag{3.3.29}$$

$$\eta = \frac{y - \phi(x)}{x^{\frac{1}{2}}}, \tag{3.3.30}$$

$$\phi(x) = \phi^{(0)} + x^{-\frac{1}{2}}\phi^{(1)} + x^{-1}\phi^{(2)} + x^{-\frac{3}{2}}\phi^{(3)} + \mathcal{O}\left(x^{-2}\right). \tag{3.3.31}$$

Let us further evaluate

$$\frac{\partial \eta}{\partial y}, \qquad \frac{\partial \eta}{\partial x}. \tag{3.3.32}$$

Now

$$\eta = \frac{y - \phi(x)}{x^{\frac{1}{2}}} = \frac{y}{x^{\frac{1}{2}}} - x^{-\frac{1}{2}}\phi(x), \tag{3.3.33}$$

and

$$y = \phi(x) + \eta x^{\frac{1}{2}}. \tag{3.3.34}$$

From (3.3.33),

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2}\frac{y}{x^{\frac{3}{2}}} + \frac{1}{2}x^{-\frac{3}{2}}\phi(x) - x^{-\frac{1}{2}}\frac{d\phi}{dx}. \tag{3.3.35}$$

Using (3.3.34) to eliminate y and simplifying gives

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2}\frac{\eta}{x} - x^{-\frac{1}{2}}\frac{d\phi}{dx}. \tag{3.3.36}$$

Substituting (3.3.31) into (3.3.36) gives

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2}\frac{\eta}{x} + x^{-2}\phi^{(1)} + x^{-\frac{5}{2}}\phi^{(2)} + \frac{3}{2}x^{-3}\phi^{(3)} + \mathcal{O}\left(x^{-\frac{7}{2}}\right) \tag{3.3.37}$$

and

$$\frac{\partial \eta}{\partial y} = \frac{1}{x^{\frac{1}{2}}}. \tag{3.3.38}$$

The transformed boundary conditions (3.1.3) are

$$\frac{df_1}{d\eta}(\infty) = 0, \qquad \frac{d^2 f_1}{d\eta^2}(\infty) = 0, \tag{3.3.39}$$

$$\frac{df_2}{d\eta}(-\infty) = 0, \qquad \frac{d^2 f_2}{d\eta^2}(-\infty) = 0. \tag{3.3.40}$$

The transformed interface conditions (3.1.4) to (3.1.6) are

$$\frac{df_1}{d\eta}(0) = \frac{df_2}{d\eta}(0), \tag{3.3.41}$$

$$\frac{d^2 f_1}{d\eta^2}(0) = \frac{\mu_2}{\mu_1}\frac{d^2 f_2}{d\eta^2}(0), \tag{3.3.42}$$

$$\frac{1}{Re}\left(1 - \frac{\mu_2}{\mu_1}\right)\frac{df_2}{d\eta}(0)x^{-\frac{3}{2}} = \left(1 - \frac{\rho_2}{\rho_1}\right)\frac{1}{F^2\sqrt{Re}}\phi(x). \tag{3.3.43}$$

The fourth interface condition (3.1.7) is the same as (3.3.41).

It follows from (3.3.43) that

$$\phi(x) = \phi^{(3)}x^{-\frac{3}{2}}. \tag{3.3.44}$$

In order to derive a solution for the classical two-fluid wake, the transformations will need to be substituted into the PDE (3.3.1). Thus, using (3.3.29), (3.3.37), (3.3.38) and (3.3.44)

$$\frac{\partial \psi_i}{\partial y} = \frac{1}{x^{\frac{1}{2}}} \frac{df_i}{d\eta}, \tag{3.3.45}$$

$$\frac{\partial^2 \psi_i}{\partial y^2} = \frac{1}{x} \frac{d^2 f_i}{d\eta^2}, \tag{3.3.46}$$

$$\frac{\partial^3 \psi_i}{\partial y^3} = x^{-\frac{3}{2}} \frac{d^3 f_i}{d\eta^3}, \tag{3.3.47}$$

$$\frac{\partial^2 \psi_i}{\partial x \partial y} = -\frac{1}{2} x^{-\frac{3}{2}} \left(\frac{df_i}{d\eta} + \eta \frac{d^2 f_i}{d\eta^2} \right) + \frac{3}{2} x^{-\frac{7}{2}} \phi^{(3)} \frac{d^2 f_i}{d\eta^2}. \tag{3.3.48}$$

Substitute (3.3.45) to (3.3.48) into PDE (3.3.1) which gives

$$2 \frac{\nu_i}{\nu_1} \frac{d^3 f_i}{d\eta^3} + \frac{d}{d\eta} \left(\eta \frac{df_i}{d\eta} \right) = 3 \phi^{(3)} \frac{d^2 f_i}{d\eta^2} x^{-2}. \tag{3.3.49}$$

For the far wake, $x >> 1$ and terms $\mathcal{O}\left(\frac{1}{x^2} \right)$ are neglected. The ODE (3.3.49) reduces to

$$2 \frac{\nu_i}{\nu_1} \frac{d^3 f_i}{d\eta^3} + \frac{d}{d\eta} \left(\eta \frac{df_i}{d\eta} \right) = 0. \tag{3.3.50}$$

Integrate (3.3.50) with respect to η. This gives

$$2 \frac{\nu_i}{\nu_1} \frac{d^2 f_i}{d\eta^2} + \eta \frac{df_i}{d\eta} = A_i, \tag{3.3.51}$$

where A_i is a constant.

Imposing the boundary conditions (3.3.39) and (3.3.40) at $\eta = \pm\infty$ for $i = 1, 2$ respectively gives

$$A_1 = A_2 = 0. \tag{3.3.52}$$

Hence,

$$2 \frac{\nu_i}{\nu_1} \frac{d^2 f_i}{d\eta^2} + \eta \frac{df_i}{d\eta} = 0. \tag{3.3.53}$$

Let

$$W_i = \frac{df_i}{d\eta}. \tag{3.3.54}$$

Hence

$$2\frac{\nu_i}{\nu_1}\frac{dW_i}{d\eta} + \eta W_i = 0. \tag{3.3.55}$$

Solving (3.3.55) gives

$$W_i = B_i \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\eta^2\right). \tag{3.3.56}$$

Substituting (3.3.54) back into (3.3.56) gives

$$\frac{df_i}{d\eta} = B_i \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\eta^2\right), \tag{3.3.57}$$

where B_i is a constant.

Substituting interface condition (3.3.41) into (3.3.57) gives

$$B_1 = B_2 = B. \tag{3.3.58}$$

Now

$$\frac{d^2 f_i}{d\eta^2} = -\frac{1}{2}\frac{\nu_1}{\nu_i} B\, \eta\, \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\eta^2\right) \tag{3.3.59}$$

and therefore

$$\frac{d^2 f_i}{d\eta^2}(0) = 0. \tag{3.3.60}$$

The interface condition (3.3.42) is identically satisfied.

Since

$$w_i = \frac{\partial \psi_i}{\partial y} = \frac{1}{x^{\frac{1}{2}}}\frac{df_i}{d\eta}, \tag{3.3.61}$$

and

$$v_i = \frac{\partial \psi_i}{\partial x} = \frac{df_i}{d\eta}\frac{\partial \eta}{\partial x}, \tag{3.3.62}$$

in order to calculate w_i and v_i it is sufficient to know $\frac{df_i}{d\eta}$. However, we will continue and solve for $f_i(\eta)$. The streamlines are obtained from ψ_i and therefore from $f_i(\eta)$. Now from (3.3.57)

$$\frac{df_i}{d\eta} = B \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\eta^2\right) \tag{3.3.63}$$

and integrating both sides with respect to η from 0 to η gives

$$f_i(\eta) - f_i(0) = B \int_0^\eta \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}s^2\right) ds. \tag{3.3.64}$$

Now a stream function is determined up to an arbitrary constant. Choose

$$f_i(0) = 0, \qquad i = 1, 2. \tag{3.3.65}$$

Thus

$$f_i(\eta) = B \int_0^\eta \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}s^2\right) ds. \tag{3.3.66}$$

The error function is

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-u^2\right) du. \tag{3.3.67}$$

Transform (3.3.66) to an error function. Let

$$\frac{1}{4}\frac{\nu_1}{\nu_i}s^2 = u^2, \tag{3.3.68}$$

then

$$\eta = 2\left(\frac{\nu_i}{\nu_1}\right)^{\frac{1}{2}} u. \tag{3.3.69}$$

Thus

$$\int_0^\eta \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}s^2\right) ds = \left(\pi\frac{\nu_i}{\nu_1}\right)^{\frac{1}{2}} erf\left(\frac{1}{2}\left(\frac{\nu_1}{\nu_i}\right)^{\frac{1}{2}}\eta\right). \tag{3.3.70}$$

Hence from (3.3.66),

$$f_i(\eta) = B\left(\pi\frac{\nu_i}{\nu_1}\right)^{\frac{1}{2}} erf\left(\frac{1}{2}\left(\frac{\nu_1}{\nu_i}\right)^{\frac{1}{2}}\eta\right).$$

Using the conserved quantity (3.3.19) with (3.3.24) and (3.3.25),

$$D = \frac{\rho_2}{\rho_1}\int_{-\infty}^0 \frac{df_2^{(0)}}{d\eta} d\eta + \int_0^\infty \frac{df_i^{(0)}}{d\eta} d\eta, \tag{3.3.71}$$

the constant B can be determined in the same way as in Section 3.2. Therefore,

$$B = \frac{D}{\sqrt{\pi}\left[1 + \left(\frac{\rho_2\mu_2}{\rho_1\mu_1}\right)^{\frac{1}{2}}\right]}. \tag{3.3.72}$$

From interface condition (3.3.43),

$$\phi^{(3)} = \frac{F^2}{\sqrt{Re}} \frac{\left(1 - \frac{\mu_2}{\mu_1}\right)}{\left(1 - \frac{\rho_2}{\rho_1}\right)} \frac{df_2}{d\eta}(0). \tag{3.3.73}$$

But

$$\frac{df_2}{d\eta}(0) = B. \tag{3.3.74}$$

Hence

$$\phi^{(3)} = B\left(\frac{1 - \frac{\mu_2}{\mu_1}}{1 - \frac{\rho_2}{\rho_1}}\right) \frac{F^2}{\sqrt{Re}}. \tag{3.3.75}$$

where B is given by (3.3.72).

For each side to balance in condition (3.3.43) we require

$$F^2\sqrt{Re} = \mathcal{O}(Re), \tag{3.3.76}$$

that is

$$F^2 = \mathcal{O}(\sqrt{Re}). \tag{3.3.77}$$

This is consistent with the assumptions made on F in (2.1.11) and (2.1.12) when deriving the boundary layer equations.

3.3.1 Visual representation of the solution

The solution derived in Section 3.3 through the use of the Herczynski, Weidman and Burde method is visually represented in Figures 3.3 and 3.4. The same values for D and for the ratios $\frac{\rho_1}{\rho_2}$ and $\frac{\nu_1}{\nu_2}$ as chosen in Figures 3.1 and 3.2 were chosen in Figures 3.3 and 3.4 for comparison. Once $\frac{\rho_1}{\rho_2}$ and $\frac{\nu_1}{\nu_2}$ have been chosen, the ratio $\frac{\mu_1}{\mu_2}$ is determined from $\frac{\mu_1}{\mu_2} = \frac{\nu_1}{\nu_2}\frac{\rho_2}{\rho_1}$.

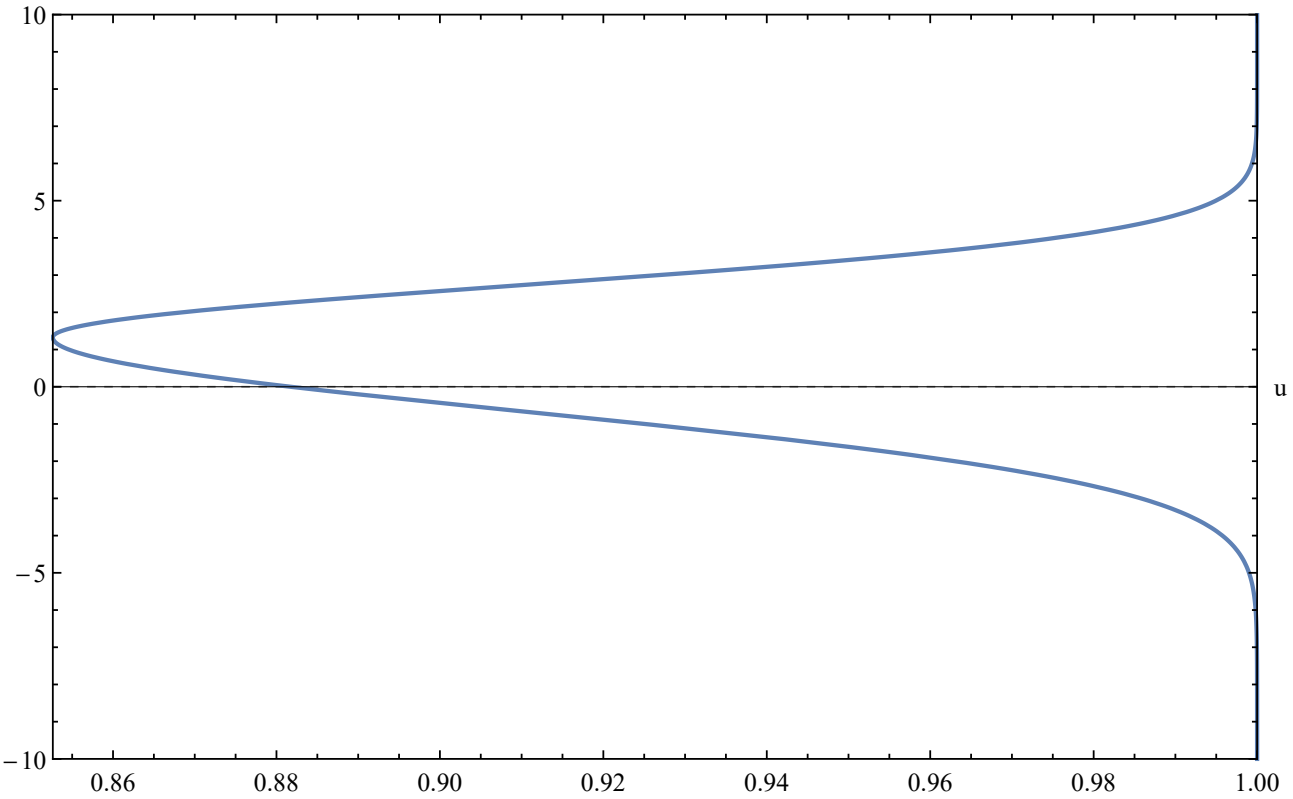

Figure 3.3: The velocities $u_1(x,y)$ and $u_2(x,y)$ for the classical two-fluid wake, given by (3.3.78) and (3.3.79) and derived using the method of Herczynski, Weidman and Burde [1], plotted against y at $x=1$ for $D=1$, $\frac{\rho_1}{\rho_2}=\frac{1}{2}$ and $\frac{\nu_1}{\nu_2}=\frac{1}{2}$.

The velocity profile is given by

$$u_1(x,y) = 1 - w_1(x,y) = 1 - \frac{B}{x^{\frac{1}{2}}}\exp\left(-\frac{1}{4}\frac{(y-\phi(x))^2}{x}\right), \tag{3.3.78}$$

$$u_2(x,y) = 1 - w_2(x,y) = 1 - \frac{B}{x^{\frac{1}{2}}}\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\frac{(y-\phi(x))^2}{x}\right) \tag{3.3.79}$$

where

$$\phi(x) = B\frac{\left(1-\frac{\mu_2}{\mu_1}\right)}{\left(1-\frac{\rho_2}{\rho_1}\right)}\frac{F^2}{\sqrt{Re}}\frac{1}{x^{\frac{3}{2}}} \tag{3.3.80}$$

and

$$B = \frac{D}{\sqrt{\pi}\left[1+\frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}}\right]}. \tag{3.3.81}$$

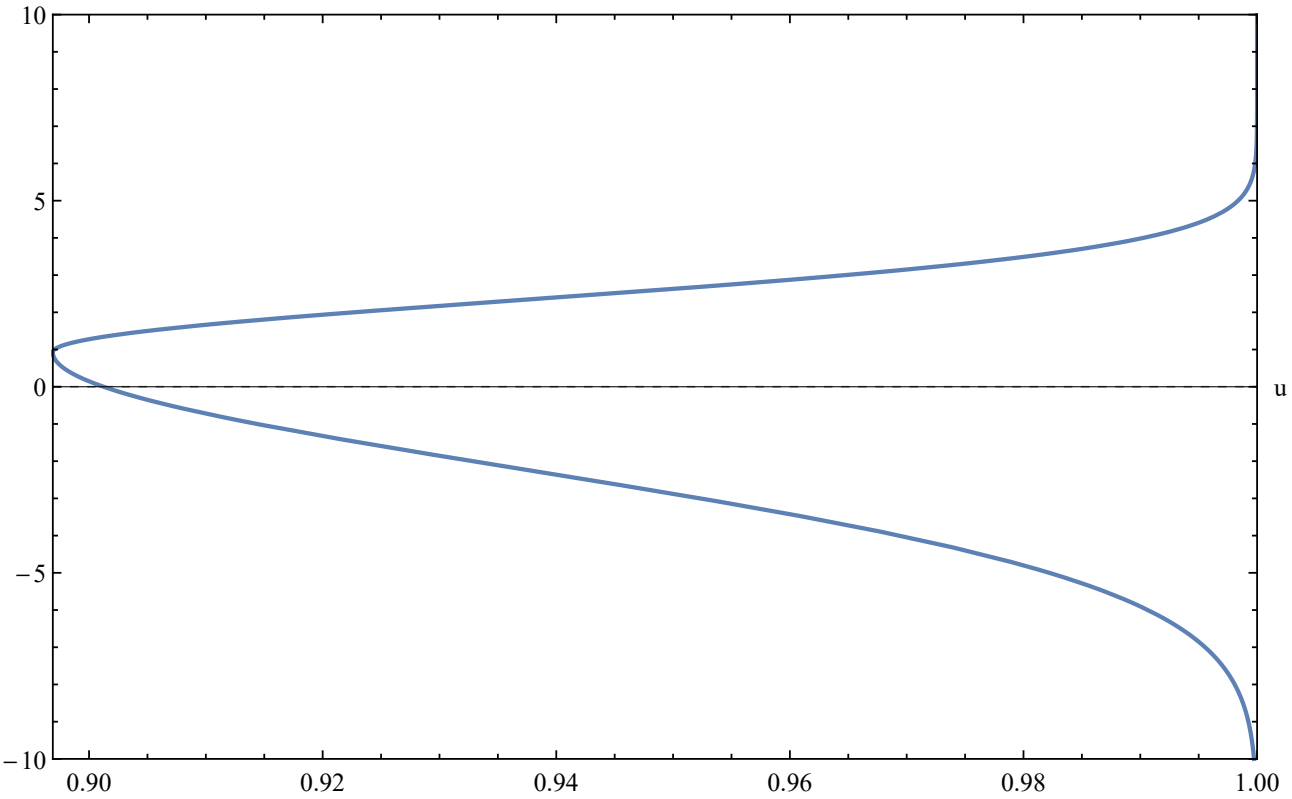

Figure 3.4: The velocities $u_1(x, y)$ and $u_2(x, y)$ for the classical two-fluid wake, given by (3.3.78) and (3.3.79) and derived using the method of Herczynski, Weidman and Burde [1], plotted against y at $x = 1$ for $D = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{5}$.

The interface is displaced by

$$\phi(x) = \phi^{(3)} x^{-\frac{3}{2}}. \tag{3.3.82}$$

The graphs in Figures 3.3 and 3.4 have the same form as in Figures 3.1 and 3.2 but displaced on the y-axis by distance

$$\phi(x) = \phi^{(3)} x^{-\frac{3}{2}}. \tag{3.3.83}$$

The solution of Herczynski, Weidman and Burde [1] tends to the solution derived in Section 3.2 in the limit $x \to \infty$, that is in the limit $\phi(x) \to 0$.

Both solutions make the approximation in the PDE for the far wake that terms of second order in smallness can be neglected. Herczynski, Weidman and Burde [1] make the further approximation that the term of order x^{-2} in the ODE (3.3.49) can be neglected.

3.4 Invariant solution for the two-fluid momentumless wake

The following results from the two-fluid classical wake remain valid

$$\psi_i(x,y) = x^{\omega} F_i(\xi), \tag{3.4.1}$$

$$w_i(x,y) = \frac{\partial \psi_i}{\partial y}(x,y), \tag{3.4.2}$$

$$\xi = \frac{y}{x^{\frac{1}{2}}}. \tag{3.4.3}$$

To obtain ω we use the conserved quantity

$$K = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x)} y^2 w_2(x,y)dy + \int_{\phi(x)}^{\infty} y^2 w_1(x,y)dy. \tag{3.4.4}$$

Now

$$w_i = \frac{\partial \psi_i}{\partial y}(x,y) = x^{\omega-\frac{1}{2}}\frac{dF_i}{d\xi}. \tag{3.4.5}$$

Thus

$$K = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}x^{\omega+1}\int_{-\infty}^{\frac{\phi(x)}{x^{\frac{1}{2}}}} \xi^2\frac{dF_2}{d\xi}d\xi + x^{\omega+1}\int_{\frac{\phi(x)}{x^{\frac{1}{2}}}}^{\infty} \xi^2\frac{dF_1}{d\xi}d\xi = \text{constant independent of } x. \tag{3.4.6}$$

The sufficient conditions for (3.4.6) to be satisfied are

$$\omega + 1 = 0, \qquad \omega = -1, \tag{3.4.7}$$

$$\frac{\phi(x)}{x^{\frac{1}{2}}} = k, \qquad \phi(x) = kx^{\frac{1}{2}}, \qquad k = \text{constant}. \tag{3.4.8}$$

Thus

$$\psi_i(x,y) = \frac{1}{x}F_i(\xi), \tag{3.4.9}$$

$$w_i(x,y) = \frac{1}{x^{\frac{3}{2}}}\frac{df_i}{d\xi} \tag{3.4.10}$$

and the conserved quantity (3.4.6) becomes

$$K = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{k} \xi^2\frac{dF_2}{d\xi}d\xi + \int_{k}^{\infty} \xi^2\frac{dF_1}{d\xi}d\xi. \tag{3.4.11}$$

The PDE in dimensionless form

$$\frac{\partial^2 \psi_i}{\partial x \partial y} = \frac{\nu_i}{\nu_1} \frac{\partial^3 \psi_i}{\partial y^3} \tag{3.4.12}$$

is now expressed in terms of the similarity variables. From (3.4.3)

$$\frac{\partial \xi}{\partial x} = -\frac{1}{2} \frac{\xi}{x}, \tag{3.4.13}$$

$$\frac{\partial \xi}{\partial y} = \frac{1}{x^{\frac{1}{2}}}. \tag{3.4.14}$$

Using (3.4.3), (3.4.9), (3.4.13) and (3.4.14) produces

$$\frac{\partial \psi_i}{\partial x} = -\frac{1}{2x^2} \left[2F_i(\xi) + \xi \frac{dF_i}{d\xi} \right], \tag{3.4.15}$$

$$\frac{\partial^2 \psi_i}{\partial y \partial x} = -\frac{1}{2x^{\frac{5}{2}}} \left[3\frac{dF_i}{d\xi} + \xi \frac{d^2 F_i}{d\xi^2} \right], \tag{3.4.16}$$

$$\frac{\partial \psi_i}{\partial y} = \frac{1}{x^{\frac{3}{2}}} \frac{dF_i}{d\xi}, \tag{3.4.17}$$

$$\frac{\partial^2 \psi_i}{\partial y^2} = \frac{1}{x^2} \frac{d^2 F_i}{d\xi^2}, \tag{3.4.18}$$

$$\frac{\partial^3 \psi_i}{\partial y^3} = \frac{1}{x^{\frac{5}{2}}} \frac{d^3 F_i}{d\xi^3}. \tag{3.4.19}$$

Substitute (3.4.16) and (3.4.19) into (3.4.12) which gives

$$2\frac{\nu_i}{\nu_1} \frac{d^3 F_i}{d\xi^3} + \xi \frac{d^2 F_i}{d\xi^2} + 3\frac{dF_i}{d\xi} = 0. \tag{3.4.20}$$

Now

$$\xi \frac{d^2 F_i}{d\xi^2} = \frac{d}{d\xi} \left(\xi \frac{dF_i}{d\xi} \right) - \frac{dF_i}{d\xi}. \tag{3.4.21}$$

Substituting (3.4.21) into (3.4.20) and simplifying gives

$$2\frac{\nu_i}{\nu_1} \frac{d^3 F_i}{d\xi^3} + \frac{d}{d\xi} \left(\xi \frac{dF_i}{d\xi} \right) + 2\frac{dF_i}{d\xi} = 0. \tag{3.4.22}$$

Equation (3.4.22) illustrates the Double Reduction Theorem [46] that if the form of the similarity solution is obtained from a conserved quantity then the ordinary differential equation can be integrated at least once.

The boundary conditions are the same as for the classical wake:

$$\frac{dF_1}{d\xi}(\infty) = 0, \tag{3.4.23}$$

$$\frac{d^2F_1}{d\xi^2}(\infty) = 0, \tag{3.4.24}$$

$$\frac{dF_2}{d\xi}(-\infty) = 0, \tag{3.4.25}$$

$$\frac{d^2F_2}{d\xi^2}(-\infty) = 0, . \tag{3.4.26}$$

Further boundary conditions may be added later such as no entrainment of the fluid at $y = \pm\infty$.

The interface conditions (2.5.4) to (2.5.7) at the interface $\xi = k$, expressed in terms of $F_i(\xi)$, become

$$\frac{dF_1}{d\xi}(k) = \frac{dF_2}{d\xi}(k), \tag{3.4.27}$$

$$F_1(k) = F_2(k), \tag{3.4.28}$$

$$\frac{d^2F_1}{d\xi^2}(k) = \frac{\mu_2}{\mu_1}\frac{d^2F_2}{d\xi^2}(k), \tag{3.4.29}$$

$$\frac{3}{Re\, x^{\frac{5}{2}}}\left(1 - \frac{\mu_2}{\mu_1}\right)\frac{dF_1}{d\xi}(k) = P_2(x, \phi(x)) - P_1(x, \phi(x)), \tag{3.4.30}$$

Now integrate (3.4.22) with respect to ξ,

$$2\frac{\nu_i}{\nu_1}\frac{d^2F_i}{d\xi^2} + \xi\frac{dF_i}{d\xi} + 2F_i = A_i, \qquad A_i = \text{constant}. \tag{3.4.31}$$

The additional assumption that there is no entrainment of fluid at $y = \pm\infty$ is made

$$v_i(x, y) = -\frac{1}{2x^2}\left[2F_i(\xi) + \xi\frac{dF_i}{d\xi}\right], \tag{3.4.32}$$

and for no entrainment of fluid at $y = +\infty$,

$$v_1(x, \infty) = 0. \tag{3.4.33}$$

Thus

$$2F_1(\infty) + \xi\frac{dF_1}{d\xi}\bigg|_{\xi=\infty} = 0. \tag{3.4.34}$$

Imposing the boundary conditions (3.4.24) and (3.4.34) on (3.4.31) gives

$$A_1 = 0. \tag{3.4.35}$$

Similarly by imposing the boundary conditions at $y = -\infty$ it can be shown that

$$A_2 = 0. \tag{3.4.36}$$

Thus

$$2\frac{\nu_i}{\nu_1}\frac{d^2F_i}{d\xi^2} + \xi\frac{dF_i}{d\xi} + 2F_i = 0. \tag{3.4.37}$$

Multiplying (3.4.37) by ξ and simplifying gives

$$2\frac{\nu_i}{\nu_1}\left[\frac{d}{d\xi}\left(\xi\frac{dF_i}{d\xi}\right) - \frac{dF_i}{d\xi}\right] + \frac{d}{d\xi}(\xi^2F_i) = 0. \tag{3.4.38}$$

Integrating (3.4.38) gives

$$\xi\frac{dF_i}{d\xi} - F_i + \frac{\nu_1}{2\nu_i}\xi^2F_i = E_i, \tag{3.4.39}$$

where E_i = constant. Consider first $i = 1$. From (3.4.23)

$$\frac{dF_1}{d\xi}(\infty) = 0. \tag{3.4.40}$$

Assume that

$$\xi\frac{dF_1}{d\xi}\bigg|_{\xi=\infty} = 0. \tag{3.4.41}$$

Then from no entrainment (3.4.34),

$$F_1(\infty) = 0. \tag{3.4.42}$$

Assume further that

$$\xi^2 F(\xi)|_{\xi=\infty} = 0. \tag{3.4.43}$$

Imposing the boundary conditions at $\xi = +\infty$ on (3.4.39) gives

$$E_1 = 0. \tag{3.4.44}$$

Similarly by imposing the boundary conditions at $\xi = -\infty$ on (3.4.39) it can be shown that

$$E_2 = 0. \tag{3.4.45}$$

Equation (3.4.39) reduces to

$$\xi \frac{dF_i}{d\xi} = \left(1 - \frac{\nu_1}{2\nu_i}\xi^2\right) F_i, \tag{3.4.46}$$

and therefore

$$\frac{dF_i}{F_i} = \left(\frac{1}{\xi} - \frac{\nu_1}{2\nu_i}\xi\right) d\xi. \tag{3.4.47}$$

The ODE (3.4.47) is variable separable and integrating gives

$$\ln F_i = \ln \xi - \frac{\nu_1}{4\nu_i}\xi^2 + B_i^*, \tag{3.4.48}$$

where B_i^* is a constant. Therefore,

$$F_i(\xi) = B_i \xi \exp\left(-\frac{\nu_1}{4\nu_i}\xi^2\right), \tag{3.4.49}$$

where

$$B_i = \exp\left(B_i^*\right). \tag{3.4.50}$$

Now

$$F_1(\xi) = B_1 \xi \exp\left(-\frac{1}{4}\xi^2\right), \tag{3.4.51}$$

$$F_2(\xi) = B_2 \xi \exp\left(-\frac{\nu_1}{4\nu_2}\xi^2\right). \tag{3.4.52}$$

The boundary conditions (3.4.23) to (3.4.26) are all satisfied because of the exponential behaviour of the solution. Now consider interface condition (3.4.28), then

$$B_1\, k\, \exp\left(-\frac{1}{4}k^2\right) = B_2\, k\, \exp\left(-\frac{\nu_1}{4\nu_2}k^2\right). \tag{3.4.53}$$

Substituting (3.4.46) into interface condition (3.4.27) gives

$$B_1\left(1-\frac{1}{2}k^2\right)\exp\left(-\frac{1}{4}k^2\right)=B_2\left(1-\frac{\nu_1}{2\nu_2}k^2\right)\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}k^2\right). \tag{3.4.54}$$

The cases $k=0$ and $k\neq 0$ need to be treated separately. This is because (3.4.53) has k as a factor. It is identically satisfied when $k=0$ and cannot be used when $k\neq 0$.

Now, from (3.4.53), (3.4.54) becomes

$$1-\frac{1}{2}k^2=1-\frac{1}{2}\frac{\nu_1}{\nu_2}k^2. \tag{3.4.55}$$

That is

$$\frac{1}{2}k^2\left(1-\frac{\nu_1}{\nu_2}\right)=0. \tag{3.4.56}$$

Thus

$$\text{either } k\neq 0 \text{ and } \frac{\nu_1}{\nu_2}=1, \tag{3.4.57}$$

$$\text{or } k=0 \text{ and no condition on } \frac{\nu_1}{\nu_2}. \tag{3.4.58}$$

Considering (3.4.53) again, $k=0$ and $k\neq 0$ may be investigated separately.

3.4.1 The case k = 0

When $k=0$ equation (3.4.53) is identically satisfied and (3.4.54) reduces to

$$B_1=B_2=B. \tag{3.4.59}$$

No condition is placed on the ratio $\frac{\nu_1}{\nu_2}$. Now

$$F_1(\xi)=B\xi\exp\left(-\frac{1}{4}\xi^2\right), \tag{3.4.60}$$

$$F_2(\xi)=B\xi\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\xi^2\right) \tag{3.4.61}$$

and

$$F_i(\xi)=B\xi\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2\right). \tag{3.4.62}$$

Thus

$$\frac{dF_i}{d\xi}(\xi) = B\bigg(1 - \frac{1}{2}\frac{\nu_1}{\nu_i}\xi^2\bigg)\exp\bigg(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2\bigg), \tag{3.4.63}$$

$$\frac{d^2F_i}{d\xi^2}(\xi) = -\frac{B}{4}\frac{\nu_1}{\nu_i}\xi\bigg(6 - \frac{\nu_1}{\nu_i}\xi^2\bigg)\exp\bigg(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2\bigg). \tag{3.4.64}$$

Substituting (3.4.64) into the interface condition (3.4.29) gives

$$k\frac{B}{4}\bigg[-(6-k^2)\exp\bigg(-\frac{1}{4}k^2\bigg) + \frac{\rho_2}{\rho_1}\bigg(6 - \frac{\nu_1}{\nu_2}k^2\bigg)\exp\bigg(-\frac{1}{4}\frac{\nu_1}{\nu_2}k^2\bigg)\bigg] = 0. \tag{3.4.65}$$

Since $k = 0$, equation (3.4.65) is identically satisfied and no condition is obtained.

The interface condition (3.4.30) gives the pressure difference across the interface. When $k = 0$, from (3.4.62) and (3.4.63)

$$F_1(\xi) = B\xi\exp\bigg(-\frac{1}{4}\xi^2\bigg), \tag{3.4.66}$$

$$\frac{dF_1}{d\xi}(\xi) = B\bigg(1 - \frac{1}{2}\xi^2\bigg)\exp\bigg(-\frac{1}{4}\xi^2\bigg). \tag{3.4.67}$$

Thus

$$\frac{dF_1}{d\xi}(k) = B\bigg(1 - \frac{1}{2}k^2\bigg)\exp\bigg(-\frac{1}{4}k^2\bigg) = B, \tag{3.4.68}$$

since $k = 0$. Hence (3.4.30) becomes

$$\frac{3}{Re}x^{-\frac{5}{2}}\bigg(1 - \frac{\mu_2}{\mu_1}\bigg)B = P_2(x,0) - P_1(x,0). \tag{3.4.69}$$

3.4.2 Calculation of the constant B when $k = 0$

The conserved quantity for the two-fluid momentumless wake is from (3.4.11)

$$K = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{k}\xi^2\frac{dF_2}{d\xi}d\xi + \int_{k}^{\infty}\xi^2\frac{dF_1}{d\xi}d\xi. \tag{3.4.70}$$

Now

$$k = 0, \qquad \xi = \frac{y}{x^{\frac{1}{2}}}, \tag{3.4.71}$$

and

$$F_1(\xi) = B\xi \exp\left(-\frac{1}{4}\xi^2 \right), \tag{3.4.72}$$

$$F_2(\xi) = B\xi \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\xi^2 \right). \tag{3.4.73}$$

Express (3.4.70) in terms of the invariant solution given by (3.4.72) and (3.4.73). Now

$$\frac{dF_2}{d\xi} = B\left[1 - \frac{1}{2}\frac{\nu_1}{\nu_2}\xi^2\right] \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\xi^2 \right), \tag{3.4.74}$$

$$\frac{dF_1}{d\xi} = B\left[1 - \frac{1}{2}\xi^2\right] \exp\left(-\frac{1}{4}\xi^2 \right). \tag{3.4.75}$$

Thus (3.4.70) becomes

$$K = B\left[\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_0^\infty \xi^2\left(1 - \frac{1}{2}\frac{\nu_1}{\nu_2}\xi^2\right) \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\xi^2 \right)d\xi + \int_0^\infty \xi^2\left(1 - \frac{1}{2}\xi^2\right) \exp\left(-\frac{1}{4}\xi^2 \right)d\xi\right], \tag{3.4.76}$$

which can be written as

$$K = B\left[\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}I_2 + I_1\right], \tag{3.4.77}$$

where

$$I_2 = \int_0^\infty \xi^2\left(1 - \frac{1}{2}\frac{\nu_1}{\nu_2}\xi^2\right) \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\xi^2 \right)d\xi, \tag{3.4.78}$$

$$I_1 = \int_0^\infty \xi^2\left(1 - \frac{1}{2}\xi^2\right) \exp\left(-\frac{1}{4}\xi^2 \right)d\xi. \tag{3.4.79}$$

The following results are used for the Gamma function [47]

$$\Gamma(n) = \int_0^\infty x^{n-1} \exp(-x)dx, \tag{3.4.80}$$

$$\Gamma(n) = (n-1)\Gamma(n-1), \qquad n > 1, \tag{3.4.81}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{3.4.82}$$

Consider first I_2, (3.4.78). Let

$$\frac{1}{4}\frac{\nu_1}{\nu_2}\xi^2 = u. \tag{3.4.83}$$

Then

$$I_2 = 4\left(\frac{\nu_2}{\nu_1}\right)^{\frac{3}{2}}\left[\int_0^\infty u^{\frac{1}{2}}e^{-u}du - 2\int_0^\infty u^{\frac{3}{2}}e^{-u}du\right] \tag{3.4.84}$$

and therefore

$$I_2 = 4\left(\frac{\nu_2}{\nu_1}\right)^{\frac{3}{2}}\left[\Gamma\left(\frac{3}{2}\right) - 2\Gamma\left(\frac{5}{2}\right)\right]. \tag{3.4.85}$$

Now applying (3.4.81) and (3.4.82)

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}, \tag{3.4.86}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}. \tag{3.4.87}$$

Thus

$$I_2 = -4\left(\frac{\nu_2}{\nu_1}\right)^{\frac{3}{2}}\sqrt{\pi}. \tag{3.4.88}$$

The integral I_1 is obtained by setting $\frac{\nu_1}{\nu_2} = 1$ in I_2. Hence

$$I_1 = -4\sqrt{\pi}. \tag{3.4.89}$$

Substituting (3.4.88) and (3.4.89) into (3.4.77) gives

$$K = -4\sqrt{\pi}B\left[\frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}} + 1\right]. \tag{3.4.90}$$

But

$$\frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}} = \frac{\rho_2}{\rho_1}\left(\frac{\mu_2}{\rho_2}\frac{\rho_1}{\mu_1}\right)^{\frac{1}{2}} = \left(\frac{\mu_2\rho_2}{\mu_1\rho_1}\right)^{\frac{1}{2}}. \tag{3.4.91}$$

Hence (3.4.90) takes the form

$$K = -4\sqrt{\pi}B\left[\left(\frac{\mu_2\rho_2}{\mu_1\rho_1}\right)^{\frac{1}{2}} + 1\right], \tag{3.4.92}$$

and therefore

$$B = -\frac{K}{4\sqrt{\pi}\left[1+\left(\frac{\mu_2\rho_2}{\mu_1\rho_1}\right)^{\frac{1}{2}}\right]}. \tag{3.4.93}$$

The solution is governed by the parameter

$$\chi = \frac{\rho_1\mu_1}{\rho_2\mu_2}. \tag{3.4.94}$$

Therefore,

$$B = -\frac{K\,\chi^{\frac{1}{2}}}{4\sqrt{\pi}\left[1+\chi^{\frac{1}{2}}\right]}. \tag{3.4.95}$$

3.4.3 The case $k \neq 0$

Return to equations (3.4.53) and (3.4.54) where $k \neq 0$. Replacing B_1 in (3.4.54) using (3.4.53) gives

$$\left(\frac{\nu_1}{\nu_2} - 1\right)k^2 B_2 \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}k^2\right) = 0 \tag{3.4.96}$$

and since $B_2 \neq 0$ it follows that

$$\frac{\nu_1}{\nu_2} = 1. \tag{3.4.97}$$

Equation (3.4.53) becomes

$$(B_1 - B_2)k \exp\left(-\frac{1}{4}k^2\right) = 0 \tag{3.4.98}$$

and therefore

$$B_1 = B_2. \tag{3.4.99}$$

Hence

$$F_i(\xi) = B \exp\left(-\frac{1}{4}\xi^2\right), \qquad i = 1, 2. \tag{3.4.100}$$

Consider next the interface condition (3.4.29),

$$\frac{d^2F_1}{d\xi^2}(k) = \frac{\mu_2}{\mu_1}\frac{d^2F_2}{d\xi^2}(k). \tag{3.4.101}$$

Now

$$\frac{dF_i}{d\xi}(\xi) = B\left(1 - \frac{1}{2}\xi^2\right)\exp\left(-\frac{1}{4}\xi^2\right), \tag{3.4.102}$$

$$\frac{d^2F_i}{d\xi^2}(\xi) = \frac{B}{4}\xi(\xi^2 - 6)\exp\left(-\frac{1}{4}\xi^2\right) \tag{3.4.103}$$

and (3.4.101) becomes

$$\left(1 - \frac{\mu_2}{\mu_1}\right)k(k^2 - 6)B\exp\left(-\frac{1}{4}k^2\right) = 0. \tag{3.4.104}$$

Now if $\mu_1 = \mu_2$ then since $\nu_1 = \nu_2$ and $\nu = \frac{\mu}{\rho}$ it follows that $\rho_1 = \rho_2$. The upper and lower fluids would then have the same density and the same viscosity and be one fluid. It is therefore assumed that $\mu_1 \neq \mu_2$. Hence (3.4.104) is satisfied provided

$$k^2 = 6, \tag{3.4.105}$$

that is, provided

$$k = +\sqrt{6} \qquad \text{or} \qquad k = -\sqrt{6}. \tag{3.4.106}$$

From (3.4.8) the equation of the interface is

$$y = kx^{\frac{1}{2}} = \pm\sqrt{6}x^{\frac{1}{2}}. \tag{3.4.107}$$

The remaining interface condition (3.4.30) determines the pressure difference across the interface,

$$\frac{3}{Re\, x^{\frac{5}{2}}}\left(1 - \frac{\mu_2}{\mu_1}\right)\frac{dF_1}{d\xi}(k) = P_2(x, \phi(x)) - P_1(x, \phi(x)). \tag{3.4.108}$$

From (3.4.102),

$$\frac{dF_1}{d\xi}(\pm\sqrt{6}) = -2B\exp\left(-\frac{3}{2}\right) \tag{3.4.109}$$

and (3.4.108) becomes

$$-\frac{6}{Re}\left(1 - \frac{\mu_2}{\mu_1}\right)B\exp\left(-\frac{3}{2}\right)x^{-\frac{5}{2}} = P_2(x, \pm\sqrt{6}x^{\frac{1}{2}}) - P_1(x, \pm\sqrt{6}x^{\frac{1}{2}}). \tag{3.4.110}$$

Since $\nu_1 = \nu_2$ it follows that

$$\frac{\mu_2}{\mu_1} = \frac{\rho_2}{\rho_1} > 1 \tag{3.4.111}$$

for stability.

3.4.4 Calculation of the constant B when $k = \pm\sqrt{6}$

The conserved quantity for the two-fluid momentumless wake is from (3.4.11)

$$K = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{k} \xi^2 \frac{dF_2}{d\xi} d\xi + \int_{k}^{\infty} \xi^2 \frac{dF_1}{d\xi} d\xi. \tag{3.4.112}$$

Now

$$\frac{\nu_1}{\nu_2} = 1, \tag{3.4.113}$$

$$F_i(\xi) = B\xi \exp\left(-\frac{1}{4}\xi^2\right), \tag{3.4.114}$$

$$\frac{dF_i}{d\xi} = B\left(1 - \frac{1}{2}\xi^2\right)\exp\left(-\frac{1}{4}\xi^2\right), \tag{3.4.115}$$

$$k = \pm\sqrt{6}. \tag{3.4.116}$$

Substituting the results into the conserved quantity (3.4.112) gives

$$K = B\left[\frac{\rho_2}{\rho_1}\int_{-\infty}^{k}\left(\xi^2 - \frac{1}{2}\xi^4\right)\exp\left(-\frac{1}{4}\xi^2\right)d\xi + \int_{k}^{\infty}\left(\xi^2 - \frac{1}{2}\xi^4\right)\exp\left(-\frac{1}{4}\xi^2\right)d\xi\right]. \tag{3.4.117}$$

Consider first $k = +\sqrt{6}$. Then (3.4.117) becomes

$$K_+ = B\left[\left(\frac{\rho_2}{\rho_1}+1\right)\int_{0}^{\infty}\left(\xi^2 - \frac{1}{2}\xi^4\right)\exp\left(-\frac{1}{4}\xi^2\right)d\xi + \left(\frac{\rho_2}{\rho_1}-1\right)\int_{0}^{\sqrt{6}}\left(\xi^2 - \frac{1}{2}\xi^4\right)\exp\left(-\frac{1}{4}\xi^2\right)d\xi\right]. \tag{3.4.118}$$

Next consider $k = -\sqrt{6}$ then (3.4.117) becomes

$$K_- = B\left[\left(\frac{\rho_2}{\rho_1}+1\right)\int_{0}^{\infty}\left(\xi^2 - \frac{1}{2}\xi^4\right)\exp\left(-\frac{1}{4}\xi^2\right)d\xi - \left(\frac{\rho_2}{\rho_1}-1\right)\int_{0}^{\sqrt{6}}\left(\xi^2 - \frac{1}{2}\xi^4\right)\exp\left(-\frac{1}{4}\xi^2\right)d\xi\right]. \tag{3.4.119}$$

Let

$$J_1 = \int_0^\infty \xi^2 \exp\left(-\frac{1}{4}\xi^2\right) d\xi, \tag{3.4.120}$$

$$J_2 = \int_0^\infty \xi^4 \exp\left(-\frac{1}{4}\xi^2\right) d\xi, \tag{3.4.121}$$

$$J_3 = \int_0^{\sqrt{6}} \xi^2 \exp\left(-\frac{1}{4}\xi^2\right) d\xi, \tag{3.4.122}$$

$$J_4 = \int_0^{\sqrt{6}} \xi^4 \exp\left(-\frac{1}{4}\xi^2\right) d\xi. \tag{3.4.123}$$

First consider (3.4.120). Let

$$\frac{1}{4}\xi^2 = u. \tag{3.4.124}$$

Then

$$I_1 = 4\int_0^\infty u^{\frac{1}{2}} \exp(-u) du = 4\Gamma\left(\frac{3}{2}\right) = 2\Gamma\left(\frac{1}{2}\right) = 2\sqrt{\pi}. \tag{3.4.125}$$

Now consider (3.4.121) and make again the transformation (3.4.124). Then

$$I_2 = 16\int_0^\infty u^{\frac{3}{2}} \exp(-u) du = 16\Gamma\left(\frac{5}{2}\right) = 12\Gamma\left(\frac{1}{2}\right) = 12\sqrt{\pi}. \tag{3.4.126}$$

In order to evaluate the integrals I_3 and I_4 we introduce the Error Function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2) du. \tag{3.4.127}$$

A useful transformation is

$$u^2 = t. \tag{3.4.128}$$

Equation (3.4.127) becomes

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} t^{-\frac{1}{2}} \exp(-t) dt. \tag{3.4.129}$$

This form is useful when integrating by parts. It satisfies the condition

$$\text{erf}(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \exp(-t) dt = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}} = 1. \tag{3.4.130}$$

Consider now I_3 and make the transformation (3.4.124). This gives

$$I_3 = 4 \int_0^{\frac{3}{2}} u^{\frac{1}{2}} \exp(-u) du. \tag{3.4.131}$$

Now, integrating by parts

$$\int_0^{\frac{3}{2}} u^{\frac{1}{2}} \exp(-u) du = \Big[-\exp(-u) u^{\frac{1}{2}} \Big]_0^{\frac{3}{2}} + \frac{1}{2} \int_0^{\frac{3}{2}} u^{-\frac{1}{2}} \exp(-u) du$$

$$= -\left(\frac{3}{2}\right)^{\frac{1}{2}} \exp\left(-\frac{3}{2}\right) + \frac{\sqrt{\pi}}{2} \text{erf}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right). \tag{3.4.132}$$

Thus from (3.4.131)

$$I_3 = 2\sqrt{\pi}\, \text{erf}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right) - 4\left(\frac{3}{2}\right)^{\frac{1}{2}} \exp\left(-\frac{3}{2}\right). \tag{3.4.133}$$

Similarly by making the transformation (3.4.124),

$$I_4 = 16 \int_0^{\frac{3}{2}} u^{\frac{3}{2}} \exp(-u) du. \tag{3.4.134}$$

Integrating by parts gives

$$I_4 = -16\left(\frac{3}{2}\right)^{\frac{3}{2}} \exp\left(-\frac{3}{2}\right) + 24 \int_0^{\frac{3}{2}} u^{\frac{1}{2}} \exp(-u) du. \tag{3.4.135}$$

Using (3.4.132), (3.4.135) becomes

$$I_4 = -48\left(\frac{3}{2}\right)^{\frac{1}{2}} \exp\left(-\frac{3}{2}\right) + 12\sqrt{\pi}\, \text{erf}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right). \tag{3.4.136}$$

Therefore,

$$I_1 = 2\sqrt{\pi}, \tag{3.4.137}$$

$$I_2 = 12\sqrt{\pi}, \tag{3.4.138}$$

$$I_3 = 2\sqrt{\pi}\,\mathrm{erf}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right) - 4\left(\frac{3}{2}\right)^{\frac{1}{2}}\exp\left(-\frac{3}{2}\right), \tag{3.4.139}$$

$$I_4 = 12\sqrt{\pi}\,\mathrm{erf}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right) - 48\left(\frac{3}{2}\right)^{\frac{1}{2}}\exp\left(-\frac{3}{2}\right). \tag{3.4.140}$$

Substituting (3.4.137) to (3.4.140) into (3.4.118) and (3.4.119) gives

$$K_+ = -4\sqrt{\pi}B\left[\frac{\rho_2}{\rho_1} + 1 + \left(\frac{\rho_2}{\rho_1} - 1\right)\left(\mathrm{erf}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right) - \frac{5}{\sqrt{\pi}}\left(\frac{3}{2}\right)^{\frac{1}{2}}\exp\left(-\frac{3}{2}\right)\right)\right], \tag{3.4.141}$$

$$K_- = -4\sqrt{\pi}B\left[\frac{\rho_2}{\rho_1} + 1 - \left(\frac{\rho_2}{\rho_1} - 1\right)\left(\mathrm{erf}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right) - \frac{5}{\sqrt{\pi}}\left(\frac{3}{2}\right)^{\frac{1}{2}}\exp\left(-\frac{3}{2}\right)\right)\right]. \tag{3.4.142}$$

Let

$$S = \mathrm{erf}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right) - \frac{5}{\sqrt{\pi}}\left(\frac{3}{2}\right)^{\frac{1}{2}}\exp\left(-\frac{3}{2}\right), \tag{3.4.143}$$

then

$$K_+ = -4\sqrt{\pi}B\left[\frac{\rho_2}{\rho_1} + 1 + \left(\frac{\rho_2}{\rho_1} - 1\right)S\right], \tag{3.4.144}$$

$$K_- = -4\sqrt{\pi}B\left[\frac{\rho_2}{\rho_1} + 1 - \left(\frac{\rho_2}{\rho_1} - 1\right)S\right]. \tag{3.4.145}$$

We can calculate S as

$$\begin{aligned} S &= \mathrm{erf}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right) - \frac{5}{\sqrt{\pi}}\left(\frac{3}{2}\right)^{\frac{1}{2}}\exp\left(-\frac{3}{2}\right) \\ &= 0.9167355 - 0.7709016 \\ &= 0.1458. \end{aligned} \tag{3.4.146}$$

Solving for the constant B gives

$$k = +\sqrt{6}: \qquad B = B_+ = -\frac{K}{4\sqrt{\pi}\left[\frac{\rho_2}{\rho_1} + 1 + \left(\frac{\rho_2}{\rho_1} - 1\right)S\right]} < 0, \tag{3.4.147}$$

$$k = -\sqrt{6}: \qquad B = B_- = -\frac{K}{4\sqrt{\pi}\left[\frac{\rho_2}{\rho_1} + 1 - \left(\frac{\rho_2}{\rho_1} - 1\right)S\right]} < 0. \tag{3.4.148}$$

3.4.5 Summary of the invariant solutions for a momentumless wake

In this section the three solutions for the wake behind a self-propelled body are summarised. The results are useful for plotting the graphs of the velocity profile.

3.4.5.1 Case 1: $k = 0$

$$F_i(\xi) = B\,\xi\,\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2\right), \qquad \xi = \frac{y}{x^{\frac{1}{2}}}, \tag{3.4.149}$$

$$\frac{dF_i}{d\xi} = \frac{B}{2}\frac{\nu_1}{\nu_i}\left[2\frac{\nu_i}{\nu_1} - \xi^2\right]\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2\right), \tag{3.4.150}$$

$$\frac{d^2F_i}{d\xi^2} = -\frac{B}{4}\left(\frac{\nu_1}{\nu_i}\right)^2\xi\left[6\frac{\nu_i}{\nu_1} - \xi^2\right]\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2\right), \tag{3.4.151}$$

$$B = -\frac{K}{4\sqrt{\pi}\left[1 + \left(\frac{\rho_2\mu_2}{\rho_1\mu_1}\right)^{\frac{1}{2}}\right]} = \frac{K}{4\sqrt{\pi}}\frac{\chi^{\frac{1}{2}}}{\left[1 + \chi^{\frac{1}{2}}\right]}, \tag{3.4.152}$$

$$\chi = \frac{\rho_1\mu_1}{\rho_2\mu_2}, \tag{3.4.153}$$

$$\phi(x) = kx^{\frac{1}{2}} = 0, \tag{3.4.154}$$

$$\frac{3}{Re}\left(1 - \frac{\mu_2}{\mu_1}\right) B\, x^{-\frac{5}{2}} = P_2(x,0) - P_1(x,0), \tag{3.4.155}$$

$$\psi_i(x,y) = \frac{1}{x} F_i(\xi), \tag{3.4.156}$$

$$\begin{aligned} w_i(x,y) &= \frac{1}{x^{\frac{3}{2}}} \frac{dF_i}{d\xi} \\ &= \frac{B}{2x^{\frac{3}{2}}} \frac{\nu_1}{\nu_i} \left[2\frac{\nu_i}{\nu_1} - \xi^2 \right] \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2 \right), \end{aligned} \tag{3.4.157}$$

$$\begin{aligned} \frac{\partial w_i}{\partial y} &= \frac{1}{x^2} \frac{d^2 F_i}{d\xi^2} \\ &= -\frac{B}{4x^2} \left(\frac{\nu_1}{\nu_i} \right)^2 \xi \left[6\frac{\nu_i}{\nu_1} - \xi^2 \right] \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2 \right), \end{aligned} \tag{3.4.158}$$

$$v_i(x,y) = -\frac{1}{2x^2} \left[2F_i(\xi) + \xi \frac{dF_i}{d\xi} \right]. \tag{3.4.159}$$

3.4.5.2 Case 2: $k = \pm\sqrt{6}$

First consider the results satisfied by both $k = +\sqrt{6}$ and $k = -\sqrt{6}$.

$$\frac{\nu_1}{\nu_2} = 1, \qquad \frac{\mu_2}{\mu_1} = \frac{\rho_2}{\rho_1} > 1, \tag{3.4.160}$$

$$\xi = \frac{y}{x^{\frac{1}{2}}}, \tag{3.4.161}$$

$$F_i(\xi) = F(\xi) = B\,\xi\, \exp\left(-\frac{1}{4}\xi^2 \right), \qquad i = 1, 2, \tag{3.4.162}$$

$$\frac{dF_i}{d\xi} = \frac{B}{2}\,(2-\xi^2)\exp\bigg(-\frac{1}{4}\xi^2\bigg), \tag{3.4.163}$$

$$\frac{d^2F_i}{d\xi^2} = -\frac{B}{4}\,\xi\,(6-\xi^2)\exp\bigg(-\frac{1}{4}\xi^2\bigg), \tag{3.4.164}$$

$$-\frac{6}{Re}\bigg(1-\frac{\mu_2}{\mu_1}\bigg)\,B\,\exp\bigg(-\frac{3}{2}\bigg)\,x^{-\frac{5}{2}} = P_2(x,\phi) - P_1(x,\phi), \tag{3.4.165}$$

$$\psi_i(x,y) = \frac{1}{x}F(\xi), \tag{3.4.166}$$

$$\begin{aligned} w_i(x,y) &= \frac{1}{x^{\frac{3}{2}}}\frac{dF}{d\xi} \\ &= \frac{B}{2x^{\frac{3}{2}}}(2-\xi^2)\exp\bigg(-\frac{1}{4}\xi^2\bigg), \end{aligned} \tag{3.4.167}$$

$$\begin{aligned} \frac{\partial w_i}{\partial y} &= \frac{1}{x^2}\frac{d^2F}{d\xi^2} \\ &= -\frac{B}{4x^2}\,\xi\,(6-\xi^2)\exp\bigg(-\frac{1}{4}\xi^2\bigg), \end{aligned} \tag{3.4.168}$$

$$v_i(x,y) = -\frac{1}{2x^2}\bigg[2F + \xi\frac{dF}{d\xi}\bigg], \qquad i = 1,2, \tag{3.4.169}$$

$$\begin{aligned} \mathrm{erf}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-u^2\right)du \\ &= \frac{1}{\sqrt{\pi}}\int_0^{x^2} u^{-\frac{1}{2}}\exp\left(-u\right)du, \end{aligned} \tag{3.4.170}$$

$$S = \mathrm{erf}\bigg(\bigg(\frac{3}{2}\bigg)^{\frac{1}{2}}\bigg) - \frac{5}{\sqrt{\pi}}\bigg(\frac{3}{2}\bigg)^{\frac{1}{2}}\exp\bigg(-\frac{3}{2}\bigg) = 0.1458. \tag{3.4.171}$$

Next consider the results dependent on whether $k = +\sqrt{6}$ or $k = -\sqrt{6}$.

- $\underline{k = +\sqrt{6}}$

$$\phi(x) = kx^{\frac{1}{2}} = +\sqrt{6}x^{\frac{1}{2}}, \tag{3.4.172}$$

$$B = B_+ = -\frac{K}{4\sqrt{\pi}\left[\frac{\rho_2}{\rho_1} + 1 + \left(\frac{\rho_2}{\rho_1} - 1\right)S\right]} < 0. \tag{3.4.173}$$

- $\underline{k = -\sqrt{6}}$

$$\phi(x) = kx^{\frac{1}{2}} = -\sqrt{6}x^{\frac{1}{2}}, \tag{3.4.174}$$

$$B = B_- = -\frac{K}{4\sqrt{\pi}\left[\frac{\rho_2}{\rho_1} + 1 - \left(\frac{\rho_2}{\rho_1} - 1\right)S\right]} < 0. \tag{3.4.175}$$

Finally we note an important property of the interface of the momentumless wake with $k = \pm\sqrt{6}$. From (3.4.168)

$$\frac{\partial w_i}{\partial y} = -\frac{B}{4x^2}\xi\,(6 - \xi^2)\exp\left(-\frac{1}{4}\xi^2\right). \tag{3.4.176}$$

Hence

$$\frac{\partial w_i}{\partial y} = 0 \tag{3.4.177}$$

when

$$\xi = \pm\sqrt{6}, \tag{3.4.178}$$

that is, when

$$y = \pm\sqrt{6}x^{\frac{1}{2}}. \tag{3.4.179}$$

But the equation of the interface is

$$y = \pm\sqrt{6}x. \tag{3.4.180}$$

Hence the interface passes through the turning points of the velocity deficit $w_i(x, y)$.

3.4.6 Visual representation of the solution

The invariant solution for the momentumless wake for $k = 0$ derived in Section 3.4 is visually represented for $\frac{\nu_1}{\nu_2} = \frac{1}{2}$ in Figure 3.5 and for $\frac{\nu_1}{\nu_2} = \frac{1}{5}$ in Figure 3.6.

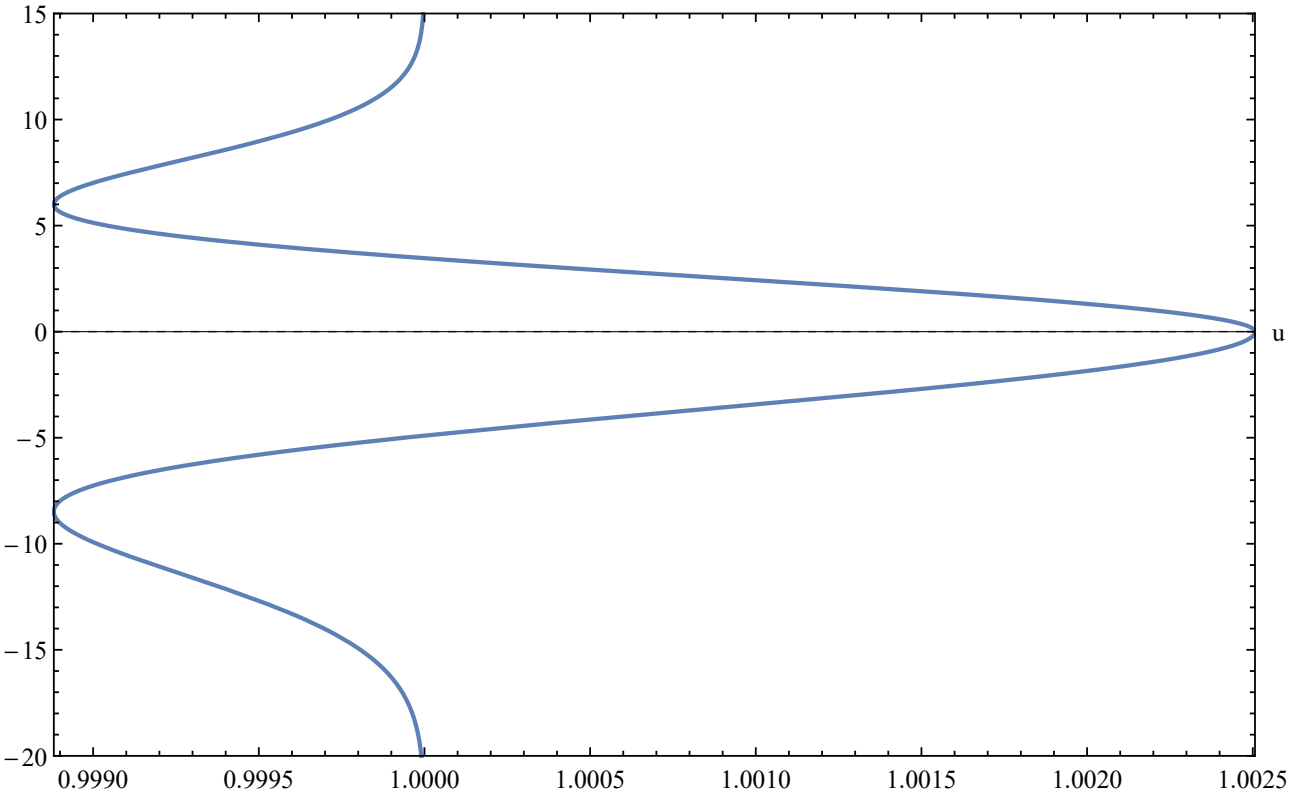

Figure 3.5: The velocities $u_1(x, y)$ and $u_2(x, y)$ for the momentumless two-fluid wake for $k = 0$ plotted against y at $x = 6$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{2}$.

The velocity profile is given by

$$u_1(x, y) = 1 - w_1(x, y) = 1 - \frac{1}{2x^{\frac{3}{2}}} B \left[2 - \frac{y^2}{x} \right] \exp\left(-\frac{1}{4}\frac{y^2}{x} \right), \tag{3.4.181}$$

$$u_2(x, y) = 1 - w_2(x, y) = 1 - \frac{1}{2x^{\frac{3}{2}}} \frac{\nu_1}{\nu_2} B \left[2\frac{\nu_2}{\nu_1} - \frac{y^2}{x} \right] \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\frac{y^2}{x} \right) \tag{3.4.182}$$

and

$$B = -\frac{K}{4\sqrt{\pi}\left[1 + \frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1} \right)^{\frac{1}{2}} \right]}. \tag{3.4.183}$$

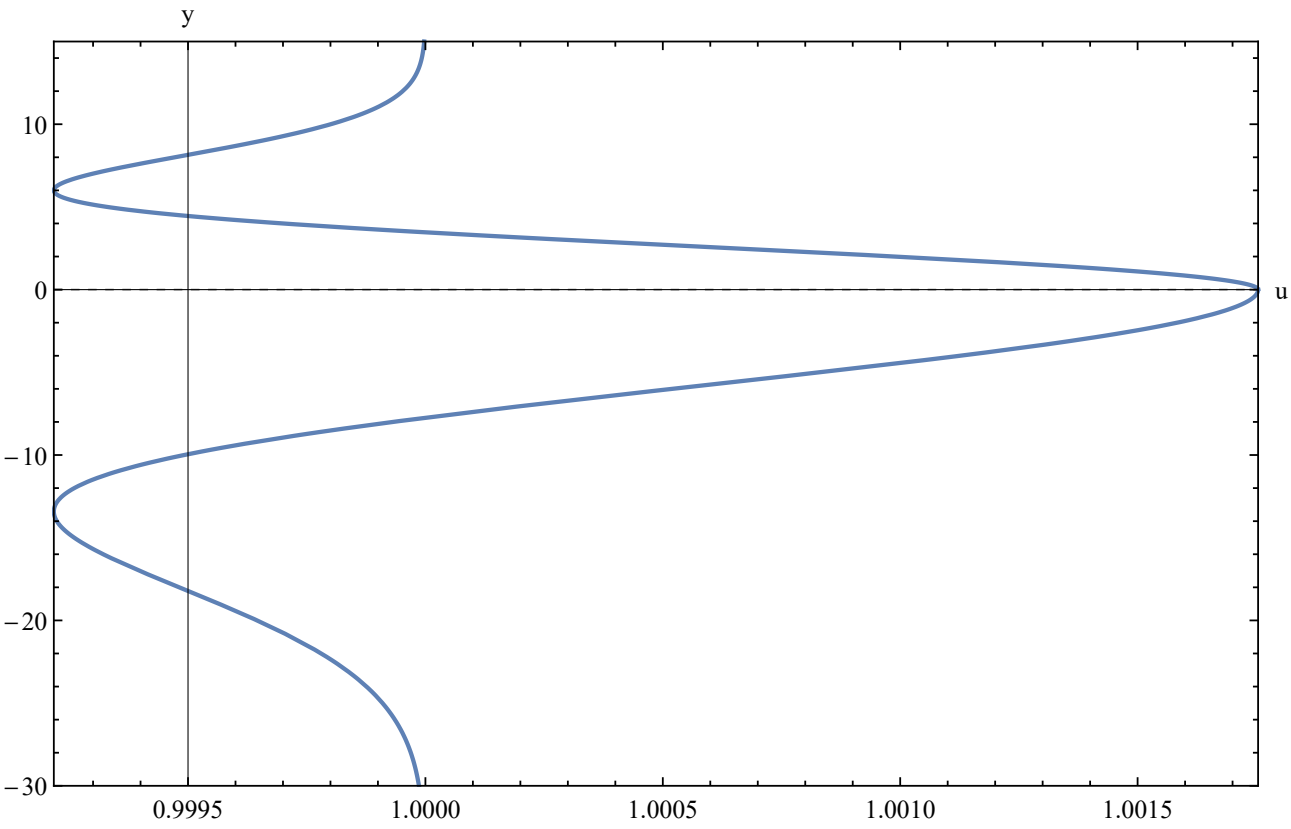

Figure 3.6: The velocities $u_1(x, y)$ and $u_2(x, y)$ for the momentumless two-fluid wake for $k = 0$ plotted against y at $x = 6$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{5}$.

We see that the effective width of the lower half of the wake in Figure 3.6 is greater than that in Figure 3.5. This is due to the stronger diffusion of velocity caused by the larger viscosity ratio $\frac{\nu_2}{\nu_1}$ in Figure 3.6.

The invariant solution for the momentumless wake for $k = \pm\sqrt{6}$ derived in Section 3.4 is visually represented for $+\sqrt{6}$ in Figure 3.7 and for $-\sqrt{6}$ in Figure 3.8.

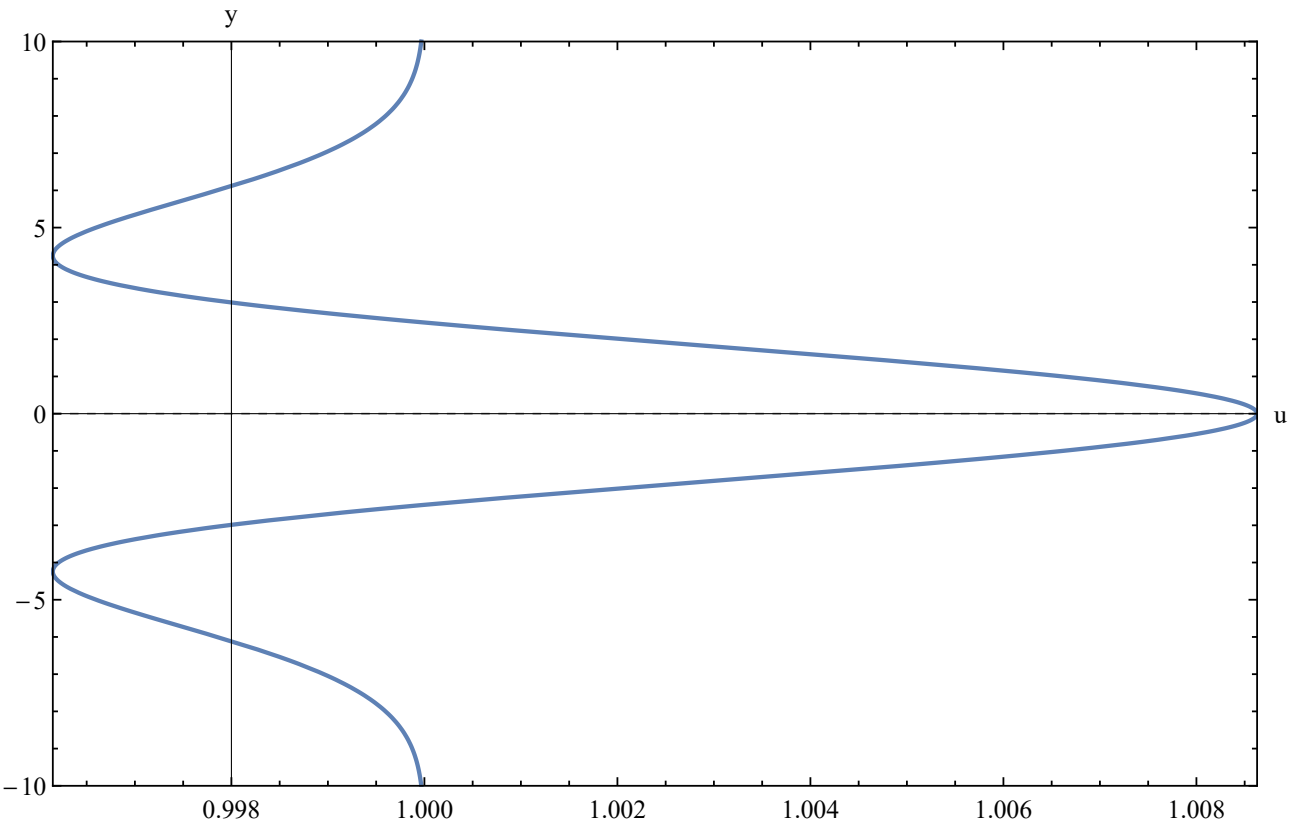

Figure 3.7: The velocity $u(x, y)$ for the momentumless two-fluid wake for $k = +\sqrt{6}$ plotted against y at $x = 3$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and S $= 0.1458$.

The velocity profile is given by

$$u(x,y) = 1 - w_1(x,y) = 1 - \frac{B_+}{x^{\frac{3}{2}}}\left(1 - \frac{y^2}{2x}\right)\exp\left(-\frac{1}{4}\frac{y^2}{x}\right) \tag{3.4.184}$$

and

$$B_+ = -\frac{K}{4\sqrt{\pi}\left[\frac{\rho_2}{\rho_1} + 1 + \left(\frac{\rho_2}{\rho_1} - 1\right)S\right]}. \tag{3.4.185}$$

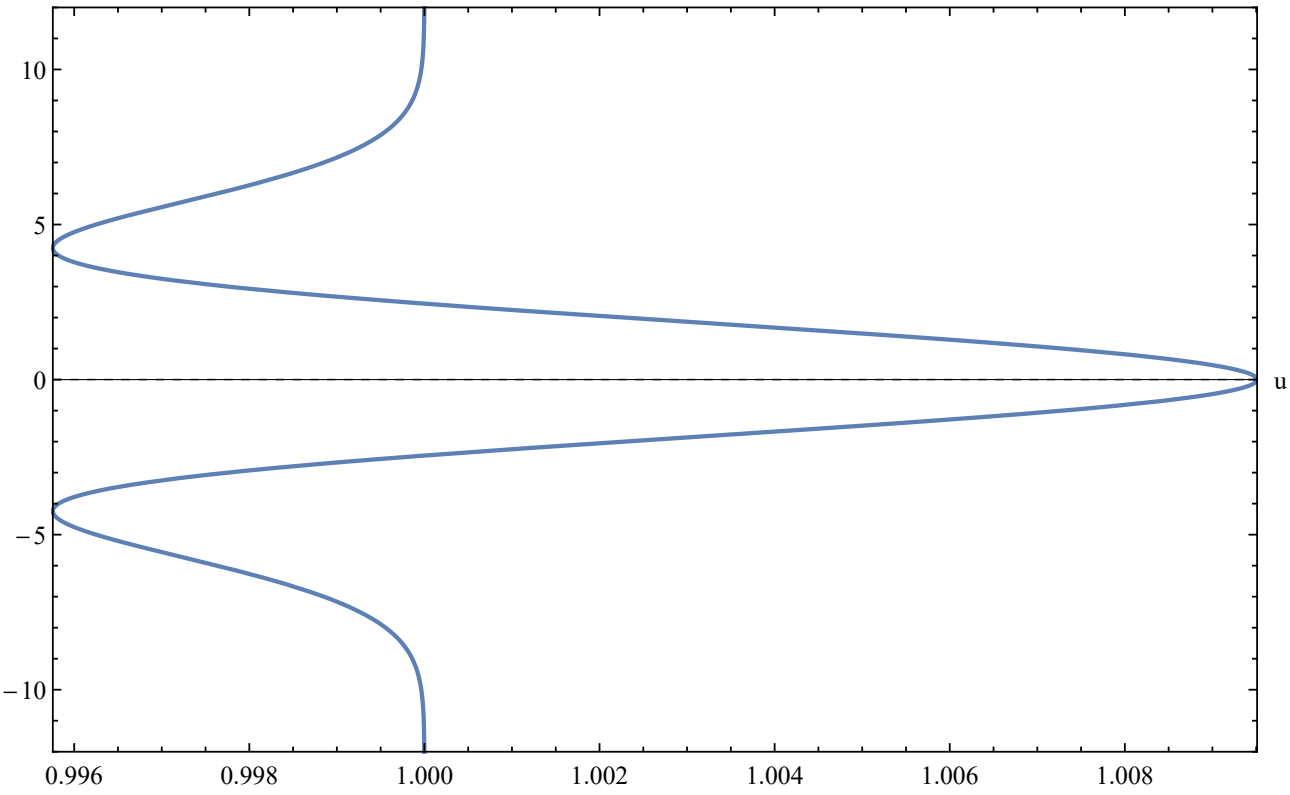

Figure 3.8: The velocity $u(x, y)$ for the momentumless two-fluid wake for $k = -\sqrt{6}$ plotted against y at $x = 3$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and S $= 0.1458$.

The velocity profile is given by

$$u(x, y) = 1 - w_1(x, y) = 1 - \frac{B_-}{x^{\frac{3}{2}}}\left(1 - \frac{y^2}{2x}\right)\exp\left(-\frac{1}{4}\frac{y^2}{x}\right) \tag{3.4.186}$$

and

$$B_- = -\frac{K}{4\sqrt{\pi}\left[\frac{\rho_2}{\rho_1} + 1 - \left(\frac{\rho_2}{\rho_1} - 1\right)S\right]}. \tag{3.4.187}$$

Figures 3.5, 3.6, 3.7 and 3.8 all have the same form, however, the point of the interface on the y-axis differs. The interface is given by

$$y = kx^{\frac{1}{2}}. \tag{3.4.188}$$

When $k = 0$, the interface is at $y = 0$ as for Figures 3.5 and 3.6. The interface lies on $y = +\sqrt{6}x^{\frac{1}{2}}$ in Figure 3.7 and $y = -\sqrt{6}x^{\frac{1}{2}}$ in Figure 3.8 which are the local turning points for the velocity profile.

3.5 Approach of Herczynski, Weidman and Burde on the two-fluid self-propelled wake

An outline will now be given of the derivation by Herczynski, Weidman and Burde [1] of the solution for the two-fluid momentumless wake but using the conserved quantity, (2.4.62), which is based on the requirement that the total drag on the body is zero and not the conserved quantity, (2.4.63), used by Herczynski, Weidmand and Burde.

The PDE for the stream function in dimensionless form is given by

$$\frac{\partial^2 \psi_i}{\partial x \partial y} = \frac{\nu_i}{\nu_1} \frac{\partial^3 \psi_i}{\partial y^3}. \tag{3.5.1}$$

The conserved quantity for the two-fluid momentumless wake which we will use is

$$K^* = \frac{\rho_2}{\rho_1} \frac{\nu_1}{\nu_2} \int_{-\infty}^{\phi(x)} y^2 w_2(x,y) dy + \int_{\phi(x)}^{\infty} y^2 w_1(x,y) dy. \tag{3.5.2}$$

The boundary conditions are the same as for a classical two-fluid wake:

$$\frac{\partial \psi_1}{\partial y}(x, \infty) = 0, \tag{3.5.3}$$

$$\frac{\partial^2 \psi_1}{\partial y^2}(x, \infty) = 0, \tag{3.5.4}$$

$$\frac{\partial \psi_2}{\partial y}(x, -\infty) = 0, \tag{3.5.5}$$

$$\frac{\partial^2 \psi_2}{\partial y^2}(x, -\infty) = 0. \tag{3.5.6}$$

The interface conditions are

$$\frac{\partial \psi_1}{\partial y}(x,\phi(x)) = \frac{\partial \psi_2}{\partial y}(x,\phi(x)), \tag{3.5.7}$$

$$\frac{\partial \psi_1}{\partial x}(x,\phi(x)) = \frac{\partial \psi_2}{\partial x}(x,\phi(x)), \tag{3.5.8}$$

$$\frac{\partial^2 \psi_1}{\partial y^2}(x,\phi(x)) = \frac{\mu_2}{\mu_1}\frac{\partial^2 \psi_2}{\partial y^2}(x,\phi(x)), \tag{3.5.9}$$

$$\frac{2}{Re}\left[\frac{\mu_2}{\mu_1}\frac{\partial^2 \psi_2}{\partial y}(x,\phi(x)) - \frac{\partial^2 \psi_1}{\partial x \partial y}(x,\phi(x))\right] = \left(1-\frac{\rho_2}{\rho_1}\right)\phi(x)\frac{1}{F^2\sqrt{Re}}, \tag{3.5.10}$$

where the Froude number is

$$F = \frac{U}{\sqrt{gl}}. \tag{3.5.11}$$

Assume that

$$F = Re^{\frac{1}{4}}. \tag{3.5.12}$$

The terms on each side of (3.5.10) therefore balance.

The coordinate expansion and expansion of the interface deflection used by Herczynski, Weidman and Burde [1] are given by

$$\psi_i(x,y) = f_i^{(0)}(\eta) + x^{-\frac{1}{2}} f_i^{(1)}(\eta) + x^{-1} f_i^{(2)}(\eta) + x^{-\frac{3}{2}} f_i^{(3)}(\eta) + \mathcal{O}(x^{-2}), \tag{3.5.13}$$

$$\phi(x) = \phi^{(0)} + x^{-\frac{1}{2}}\phi^{(1)} + x^{-1}\phi^{(2)} + x^{-\frac{3}{2}}\phi^{(3)} + x^{-2}\phi^{(4)} + \mathcal{O}(x^{-\frac{5}{2}}), \tag{3.5.14}$$

where

$$\phi^{(0)}, \phi^{(1)}, \phi^{(2)}, \phi^{(3)}, \phi^{(4)} \text{ are constants,} \tag{3.5.15}$$

and

$$\eta = \frac{y-\phi(x)}{x^{\frac{1}{2}}}. \tag{3.5.16}$$

Equations (3.5.13) and (3.5.14) are expansions in powers of $x^{-\frac{1}{2}}$ for the far wake. As for the classical wake, this extends the coordinate expansion of Goldstein [3] to an expansion of $\phi(x)$.

These equations will be simplified as much as possible before substituting into the PDE (4.3.3). This is done by using the conserved quantity and the interface conditions. Express the integrals of the conserved quantity in terms of $f_i^{(n)}(\eta)$ and η. From (3.5.16)

$$\frac{\partial \eta}{\partial y} = \frac{1}{x^{\frac{1}{2}}} \tag{3.5.17}$$

and

$$y = x^{\frac{1}{2}}\eta + \phi(x). \tag{3.5.18}$$

At constant x,

$$dy = x^{\frac{1}{2}} d\eta. \tag{3.5.19}$$

Now

$$\frac{\partial \psi_i}{\partial y} dy = \frac{\partial \psi_i}{\partial \eta} d\eta$$

$$= \left[\frac{df_i^{(0)}}{d\eta} + x^{-\frac{1}{2}} \frac{df_i^{(1)}}{d\eta} + x^{-1} \frac{df_i^{(2)}}{d\eta} + x^{-\frac{3}{2}} \frac{df_i^{(3)}}{d\eta} + \mathcal{O}(x^{-2}) \right] d\eta. \tag{3.5.20}$$

Also,

$$y^2 = x\eta^2 + 2x^{\frac{1}{2}}\phi(x)\eta + \phi(x)^2$$

$$= x\eta^2 + x^{\frac{1}{2}}[2\eta\phi^{(0)}] + x^0[2\eta\phi^{(1)} + \phi^{(0)2}] + x^{-\frac{1}{2}}[2\eta\phi^{(2)} + 2\phi^{(0)}\phi^{(1)}]$$

$$+ x^{-1}[2\eta\phi^{(3)} + 2\phi^{(0)}\phi^{(2)} + \phi^{(1)2}] + x^{-\frac{3}{2}}[2\eta\phi^{(4)} + 2\phi^{(0)}\phi^{(3)} + 2\phi^{(1)}\phi^{(2)}] + \mathcal{O}(x^{-2}). \tag{3.5.21}$$

Thus, using (3.5.20) and (3.5.21) and simplifying

$$y^2 \frac{\partial \psi_i}{\partial y} dy = x \left[\eta^2 \frac{df_i^{(0)}}{d\eta} \right] d\eta + x^{\frac{1}{2}} \left[\eta^2 \frac{df_i^{(1)}}{d\eta} + 2\phi^{(0)}\eta \frac{df_i^{(0)}}{d\eta} \right] d\eta$$

$$+ x^0 \left[\eta^2 \frac{df_i^{(2)}}{d\eta} + 2\phi^{(0)}\eta \frac{df_i^{(1)}}{d\eta} + (2\eta\phi^{(1)} + \phi^{(0)2}) \frac{df_i^{(0)}}{d\eta} \right] d\eta$$

$$+ x^{-\frac{1}{2}} \left[\eta^2 \frac{df_i^{(3)}}{d\eta} + 2\phi^{(0)}\eta \frac{df_i^{(2)}}{d\eta} + (2\eta\phi^{(1)} + \phi^{(0)2}) \frac{df_i^{(1)}}{d\eta} + (2\eta\phi^{(0)} + 2\phi^{(0)}\phi^{(1)}) \frac{df_i^{(0)}}{d\eta} \right] d\eta$$

$$+ \mathcal{O}(x^{-1}). \tag{3.5.22}$$

Now

$$K^* = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x)} y^2\frac{\partial\psi_2}{\partial y}dy + \int_{\phi(x)}^{\infty} y^2\frac{\partial\psi_1}{\partial y}dy = \text{constant independent of } x. \tag{3.5.23}$$

Substitute the expansions (3.5.13), (3.5.14) and (3.5.16) into (3.5.23). The limits of integration with respect to η become

$$\int_{-\infty}^{0} d\eta, \qquad \int_{0}^{\infty} d\eta, \tag{3.5.24}$$

which are independent of x. The integrands in (3.5.23) are given by (3.5.22) for $i = 2$ and $i = 1$. Separating by powers of x produces the following.

$$x: \qquad \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{0} \eta^2\frac{df_2^{(0)}}{d\eta}d\eta + \int_{0}^{\infty} \eta^2\frac{df_1^{(0)}}{d\eta}d\eta. \tag{3.5.25}$$

Since K^* is independent of x, this coefficient must vanish. Hence

$$\frac{df_2^{(0)}}{d\eta} = 0, \qquad \frac{df_1^{(0)}}{d\eta} = 0. \tag{3.5.26}$$

$x^{\frac{1}{2}}$: Making use of (3.5.26) gives

$$\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{0} \eta^2\frac{df_2^{(1)}}{d\eta}d\eta + \int_{0}^{\infty} \eta^2\frac{df_1^{(1)}}{d\eta}d\eta. \tag{3.5.27}$$

Since K^* is independent of x, this coefficient must vanish. Hence

$$\frac{df_2^{(1)}}{d\eta} = 0, \qquad \frac{df_1^{(1)}}{d\eta} = 0. \tag{3.5.28}$$

x^0: Making use of (3.5.26) and (3.5.28) gives

$$\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{0} \eta^2\frac{df_2^{(2)}}{d\eta}d\eta + \int_{0}^{\infty} \eta^2\frac{df_1^{(2)}}{d\eta}d\eta. \tag{3.5.29}$$

This coefficient is independent of x. We take the leading term in the expansion (3.5.13) in powers of $x^{-\frac{1}{2}}$ which gives K^* independent of x. Hence

$$\psi_i(x,y) = \frac{1}{x}f_i^{(2)}(\eta) + \mathcal{O}\left(\frac{1}{x^{\frac{3}{2}}}\right). \tag{3.5.30}$$

We can stop here because we are making an expansion in powers of $x^{-\frac{1}{2}}$ and neglect terms of lower order than x^0. The conserved quantity is

$$K^* = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{0} \eta^2\frac{df_2^{(2)}}{d\eta}d\eta + \int_{0}^{\infty} \eta^2\frac{df_1^{(2)}}{d\eta}d\eta. \tag{3.5.31}$$

To keep the notation simple, we will write $f_i^{(2)}$ as f_i.

3.5.1 Interface and boundary conditions

Consider first (3.5.9),

$$\frac{\partial^2\psi_1}{\partial y^2}(x,\phi(x)) = \frac{\mu_2}{\mu_1}\frac{\partial^2\psi_2}{\partial y^2}(x,\phi(x)). \tag{3.5.32}$$

Now

$$\psi_i(x,y) = \frac{1}{x}f_i(\eta), \tag{3.5.33}$$

$$\frac{\partial\psi_i}{\partial y} = x^{-\frac{3}{2}}\frac{df_i}{d\eta}, \tag{3.5.34}$$

$$\frac{\partial^2\psi_i}{\partial y^2} = x^{-2}\frac{d^2f_i}{d\eta^2}. \tag{3.5.35}$$

Also, when $y = \phi(x)$ then $\eta = 0$. Thus

$$\frac{\partial^2\psi_i}{\partial y^2}(x,\phi(x)) = x^{-2}\frac{d^2f_i}{d\eta^2}(0). \tag{3.5.36}$$

Therefore, interface condition (3.5.9) becomes

$$\frac{d^2f_1}{d\eta^2}(0) = \frac{\mu_2}{\mu_1}\frac{d^2f_2}{d\eta^2}(0). \tag{3.5.37}$$

Similarly interface condition (3.5.7) becomes

$$\frac{df_1}{d\eta}(0) = \frac{df_2}{d\eta}(0). \tag{3.5.38}$$

Consider next interface condition (3.5.10). Then from (3.5.34)

$$\frac{\partial^2\psi_i}{\partial x\partial y} = -\frac{3}{2}x^{-\frac{5}{2}}\frac{df_i}{d\eta} + x^{-\frac{3}{2}}\frac{d^2f_i}{d\eta^2}\frac{\partial\eta}{\partial x} \tag{3.5.39}$$

and

$$\frac{\partial^2\psi_i}{\partial x\partial y}(x,\phi(x)) = -\frac{3}{2}x^{-\frac{5}{2}}\frac{df_i}{d\eta}(0) + x^{-\frac{3}{2}}\frac{d^2f_i}{d\eta^2}(0)\frac{\partial\eta}{\partial x}\bigg|_{\eta=0}. \tag{3.5.40}$$

Then equation (3.5.10) becomes

$$\frac{2}{Re}\left[\frac{3}{2}x^{-\frac{5}{2}}\left(\frac{df_1}{d\eta}(0) - \frac{\mu_2}{\mu_1}\frac{df_2}{d\eta}(0)\right) + x^{-\frac{3}{2}}\left(\frac{\mu_2}{\mu_1}\frac{d^2f_2}{d\eta^2}(0) - \frac{d^2f_1}{d\eta^2}(0)\right)\frac{\partial\eta}{\partial x}\bigg|_{\eta=0}\right] = \frac{1}{F^2\sqrt{Re}}\left(1-\frac{\rho_2}{\rho_1}\right)\phi(x). \tag{3.5.41}$$

Using (3.5.37) and (3.5.38), equation (3.5.41) reduces to

$$\frac{3}{Re}\left(1-\frac{\mu_2}{\mu_1}\right)\frac{df_2}{d\eta}(0)x^{-\frac{5}{2}} = \frac{1}{F^2\sqrt{Re}}\left(1-\frac{\rho_2}{\rho_1}\right)\phi(x) \tag{3.5.42}$$

and therefore

$$\phi(x) = 3\frac{F^2}{\sqrt{Re}}\frac{\left(1-\frac{\mu_2}{\mu_1}\right)}{\left(1-\frac{\rho_2}{\rho_1}\right)}\frac{df_2}{d\eta}(0)x^{-\frac{5}{2}}. \tag{3.5.43}$$

Thus

$$\phi(x) = \phi^{(5)}x^{-\frac{5}{2}}, \tag{3.5.44}$$

where

$$\phi^{(5)} = 3\frac{F^2}{\sqrt{Re}}\frac{\left(1-\frac{\mu_2}{\mu_1}\right)}{\left(1-\frac{\rho_2}{\rho_1}\right)}\frac{df_2}{d\eta}(0). \tag{3.5.45}$$

Express the remaining interface condition (3.5.8) in terms of the transformation. Now

$$\psi_i(x,y) = \frac{1}{x}f_i(\eta) \tag{3.5.46}$$

and

$$\frac{\partial\psi_i}{\partial x} = -\frac{1}{x^2}f_i(\eta) + \frac{1}{x}\frac{df_i}{d\eta}\frac{\partial\eta}{\partial x}. \tag{3.5.47}$$

But

$$\eta = \frac{y-\phi(x)}{x^{\frac{1}{2}}}$$

$$= \frac{y}{x^{\frac{1}{2}}} - \phi^{(5)}x^{-3}. \tag{3.5.48}$$

Hence

$$\frac{\partial\eta}{\partial x} = -\frac{1}{2}\frac{\eta}{x} + \frac{5}{2}\phi^{(5)}x^{-4}. \tag{3.5.49}$$

Now (3.5.47) becomes

$$\frac{\partial\psi_i}{\partial x} = -\frac{1}{x^2}f_i(\eta) + \frac{1}{x}\frac{df_i}{d\eta}\left(-\frac{1}{2}\frac{\eta}{x} + \frac{5}{2}\phi^{(5)}x^{-4}\right) \tag{3.5.50}$$

and

$$\frac{\partial \psi_i}{\partial x}(x, \phi(x)) = -\frac{1}{x^2} f_i(0) + \frac{5}{2}\phi^{(5)} \frac{df_i}{d\eta}(0) x^{-5}. \tag{3.5.51}$$

The interface condition (3.5.8) becomes

$$-\frac{1}{x^2} f_1(0) + \frac{5}{2}\phi^{(5)} \frac{df_1}{d\eta}(0) x^{-5} = -\frac{1}{x^2} f_2(0) + \frac{5}{2}\phi^{(5)} \frac{df_2}{d\eta}(0) x^{-5}. \tag{3.5.52}$$

But from (3.5.38), (3.5.52) reduces to

$$f_1(0) = f_2(0). \tag{3.5.53}$$

This is a new interface condition that does not apply to the classical wake.

The boundary conditions (3.5.3) to (3.5.6) become:

$$\frac{df_1}{d\eta}(\infty) = 0, \tag{3.5.54}$$

$$\frac{d^2 f_1}{d\eta^2}(\infty) = 0, \tag{3.5.55}$$

$$\frac{df_2}{d\eta}(-\infty) = 0, \tag{3.5.56}$$

$$\frac{d^2 f_2}{d\eta^2}(-\infty) = 0. \tag{3.5.57}$$

This completes the simplification of the transformation (3.5.13) to (3.5.15).

3.5.2 Solution of the self-propelled two-fluid wake

We now derive the solution for the two-fluid momentumless wake. Return to the PDE

$$\frac{\partial^2 \psi_i}{\partial x \partial y} = \frac{\nu_i}{\nu_1} \frac{\partial^3 \psi_i}{\partial y^3}. \tag{3.5.58}$$

Make the following transformations:

$$\psi_i(x,y) = \frac{1}{x} f_i(\eta), \tag{3.5.59}$$

$$\eta = \frac{y - \phi(x)}{x^{\frac{1}{2}}}, \tag{3.5.60}$$

$$\phi(x) = \phi^{(5)} x^{-\frac{5}{2}}, \tag{3.5.61}$$

$$\frac{\partial \eta}{\partial x} = -\frac{\eta}{2x} + \frac{5}{2}\phi^{(5)} x^{-4}, \tag{3.5.62}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{x^{\frac{1}{2}}}. \tag{3.5.63}$$

Thus

$$\frac{\partial \psi_i}{\partial y} = x^{-\frac{3}{2}} \frac{df_i}{d\eta}, \tag{3.5.64}$$

$$\frac{\partial^2 \psi_i}{\partial y^2} = x^{-2} \frac{d^2 f_i}{d\eta^2}, \tag{3.5.65}$$

$$\frac{\partial^3 \psi_i}{\partial y^3} = x^{-\frac{5}{2}} \frac{d^3 f_i}{d\eta^3}, \tag{3.5.66}$$

$$\frac{\partial^2 \psi_i}{\partial x \partial y} = -\frac{1}{2} x^{-\frac{5}{2}} \left[3\frac{df_i}{d\eta} + \eta \frac{d^2 f_i}{d\eta^2} \right] + \frac{5}{2}\phi^{(5)} x^{-\frac{11}{2}} \frac{d^2 f_i}{d\eta^2}. \tag{3.5.67}$$

Substitute (3.5.66) and (3.5.67) into the PDE (3.5.58) and simplifying gives

$$2\frac{\nu_i}{\nu_1}\frac{d^3 f_i}{d\eta^3} + \eta \frac{d^2 f_i}{d\eta^2} + 3\frac{df_i}{d\eta} = 5\phi^{(5)} x^{-3} \frac{d^2 f_i}{d\eta^2}. \tag{3.5.68}$$

For the far wake $x >> 1$ and terms $\mathcal{O}\left(\frac{1}{x^3}\right)$ are neglected. The solution is more accurate than that for the classical wake in which larger terms $\mathcal{O}\left(\frac{1}{x^2}\right)$ were neglected. The ODE (3.5.68) reduces to

$$2\frac{\nu_i}{\nu_1}\frac{d^3 f_i}{d\eta^3} + \eta \frac{d^2 f_i}{d\eta^2} + 3\frac{df_i}{d\eta} = 0. \tag{3.5.69}$$

Before integrating (3.5.69) we will calculate the velocity components which will be required in the boundary conditions:

$$w_i = \frac{\partial \psi_i}{\partial y} = \frac{1}{x^{\frac{3}{2}}} \frac{df_i}{d\eta}, \tag{3.5.70}$$

$$v_i = \frac{\partial \psi_i}{\partial x} = -\frac{1}{x^2} f_i(\eta) - \frac{1}{2x^2} \eta \frac{df_i}{d\eta} + \frac{5}{2} \phi^{(5)} \frac{1}{x^5} \frac{df_i}{d\eta}, \tag{3.5.71}$$

$$\frac{\partial w_i}{\partial y} = \frac{\partial}{\partial y} \left(\frac{1}{x^{\frac{3}{2}}} \frac{df_i}{d\eta} \right) = \frac{1}{x^2} \frac{d^2 f_i}{d\eta^2}. \tag{3.5.72}$$

Now the ODE (3.5.69) can be rewritten as

$$2 \frac{\nu_i}{\nu_1} \frac{d^3 f_i}{d\eta^3} + \frac{d}{d\eta} \left(\eta \frac{df_i}{d\eta} \right) + 2 \frac{df_i}{d\eta} = 0. \tag{3.5.73}$$

Integrating (3.5.73) with respect to η gives

$$2 \frac{\nu_i}{\nu_1} \frac{d^2 f_i}{d\eta^2} + \eta \frac{df_i}{d\eta} + 2 f_i = A_i, \qquad A_i = \text{ constant}. \tag{3.5.74}$$

The boundary conditions at $\eta = \pm\infty$ are given by (3.5.54) to (3.5.57). We make the additional assumption that there is no entrainment of fluid at $y = \pm\infty$, that is

$$v_1(x, +\infty) = 0 \qquad \text{and} \qquad v_2(x, -\infty) = 0. \tag{3.5.75}$$

Then from (3.5.71) for $i = 1$,

$$-\frac{1}{x^2} f_1(\infty) - \frac{1}{2x^2} \eta \frac{df_1}{d\eta}\bigg|_{\eta=\infty} + \frac{5}{2} \phi^{(5)} \frac{1}{x^5} \frac{df_1}{d\eta}(\infty) = 0. \tag{3.5.76}$$

From (3.5.54)

$$\frac{df_1}{d\eta}(\infty) = 0. \tag{3.5.77}$$

Assume that

$$\lim_{\eta \to \infty} \eta \frac{df_1}{d\eta} = 0. \tag{3.5.78}$$

We will find later from the solution that $\frac{df_i}{d\eta} \to 0$ exponentially as $\eta \to \infty$ and therefore (3.5.78) is satisfied.

Then (3.5.76) becomes

$$\frac{1}{x^2} f_1(\infty) = 0. \tag{3.5.79}$$

In equation (3.5.68), terms of $\mathcal{O}\left(\frac{1}{x^3}\right)$ were neglected. Terms of order $\frac{1}{x^2}$ are retained. Hence from (3.5.79)

$$f_1(\infty) = 0. \tag{3.5.80}$$

Substituting (3.5.55), (3.5.78) and (3.5.80) into (3.5.74) produces

$$A_1 = 0. \tag{3.5.81}$$

Similarly by imposing the boundary conditions at $\eta = -\infty$ we find that

$$f_2(-\infty) = 0 \qquad \text{and} \qquad A_2 = 0. \tag{3.5.82}$$

Equation (3.5.74) becomes

$$2\frac{\nu_i}{\nu_1}\frac{d^2 f_i}{d\eta^2} + \eta\frac{df_i}{d\eta} + 2f_i = 0. \tag{3.5.83}$$

Multiply (3.5.83) by η which gives

$$2\frac{\nu_i}{\nu_1}\eta\frac{d^2 f_i}{d\eta^2} + \eta^2\frac{df_i}{d\eta} + 2\eta f_i = 0, \tag{3.5.84}$$

and simplifying to integrate gives

$$2\frac{\nu_i}{\nu_1}\left[\frac{d}{d\eta}\left(\eta\frac{df_i}{d\eta}\right) - \frac{df_i}{d\eta}\right] + \frac{d}{d\eta}(\eta^2 f_i) = 0. \tag{3.5.85}$$

Integrate (3.5.85) with respect to η,

$$2\frac{\nu_i}{\nu_1}\eta\frac{df_i}{d\eta} - 2\frac{\nu_i}{\nu_1}f_i + \eta^2 f_i = C_i^*, \tag{3.5.86}$$

where C_i^* is a constant. Thus

$$\eta\frac{df_i}{d\eta} - f_i + \frac{\nu_1}{2\nu_i}\eta^2 f_i = C_i, \tag{3.5.87}$$

where C_i is a constant.

From the boundary conditions (3.5.54) and (3.5.80)

$$\eta = +\infty : \qquad \frac{df_1}{d\eta}(\infty) = 0, \qquad f_1(\infty) = 0 \ \text{(no entrainment)}. \tag{3.5.88}$$

Assume the stronger conditions

$$\lim_{\eta\to\infty} \eta\frac{df_1}{d\eta} = 0, \qquad \lim_{\eta\to\infty} \eta^2 f_1(\eta) = 0. \tag{3.5.89}$$

Thus

$$C_1 = 0. \tag{3.5.90}$$

Similarly, from the boundary conditions (3.5.56) and (3.5.82)

$$\eta = -\infty : \qquad \frac{df_2}{d\eta}(-\infty) = 0, \qquad f_2(-\infty) = 0, \tag{3.5.91}$$

and assume the stronger conditions

$$\lim_{\eta\to-\infty} \eta\frac{df_2}{d\eta} = 0, \qquad \lim_{\eta\to\infty} \eta^2 f_2(\eta) = 0. \tag{3.5.92}$$

Thus

$$C_2 = 0. \tag{3.5.93}$$

Equation (3.5.87) reduces to

$$\frac{df_i}{f_i} = \left(\frac{1}{\eta} - \frac{\nu_1}{2\nu_i}\eta\right)d\eta. \tag{3.5.94}$$

The ODE (3.5.94) is variable separable and the solution is

$$f_i(\eta) = B_i\,\eta\,\exp\left(-\frac{\nu_1}{4\nu_i}\eta^2\right), \tag{3.5.95}$$

where

$$B_i = \text{ constant}. \tag{3.5.96}$$

The boundary conditions (3.5.89) and (3.5.92) are indeed satisfied because of the exponential behaviour of the solution.

Consider now the interface conditions. The interface condition (3.5.38) is

$$\frac{df_1}{d\eta}(0) = \frac{df_2}{d\eta}(0). \tag{3.5.97}$$

But $f_i(\eta)$ is given by (3.5.95) and therefore

$$\frac{df_i}{d\eta} = B_i\left[1 - \frac{\nu_1}{2\nu_i}\eta^2\right]\exp\left(-\frac{\nu_1}{4\nu_i}\eta^2\right). \tag{3.5.98}$$

Hence

$$\frac{df_i}{d\eta}(0) = B_i, \tag{3.5.99}$$

and the interface condition (3.5.97) becomes

$$B_1 = B_2 = B. \tag{3.5.100}$$

Consider next the interface condition (3.5.53)

$$f_1(0) = f_2(0). \tag{3.5.101}$$

From (3.5.95) for $f_i(\eta)$ we see that the condition is identically satisfied.

Consider next the interface condition (3.5.37)

$$\frac{d^2 f_1}{d\eta^2}(0) = \frac{\mu_2}{\mu_1}\frac{d^2 f_2}{d\eta^2}. \tag{3.5.102}$$

From (3.5.98), it can be derived that

$$\frac{d^2 f_i}{d\eta^2} = B_i\bigg[-\frac{3}{2}\frac{\nu_1}{\nu_i}\eta + \frac{\nu_1^2}{4\nu_i^2}\eta^3\bigg] \exp\bigg(-\frac{\nu_1}{4\nu_i}\eta^2\bigg). \tag{3.5.103}$$

Thus

$$\frac{d^2 f_i}{d\eta^2}(0) = 0 \tag{3.5.104}$$

and interface condition (3.5.102) is identically satisfied.

Finally consider the interface condition (3.5.42)

$$\frac{3}{Re}\bigg(1 - \frac{\mu_2}{\mu_1}\bigg)\frac{df_2}{d\eta}(0) = \frac{1}{F^2\sqrt{Re}}\bigg(1 - \frac{\rho_2}{\rho_1}\bigg)\phi^{(5)}. \tag{3.5.105}$$

Using (3.5.99) and (3.5.100), (3.5.105) becomes

$$\frac{3}{Re}\bigg(1 - \frac{\mu_2}{\mu_1}\bigg)B = \frac{1}{F^2\sqrt{Re}}\bigg(1 - \frac{\rho_2}{\rho_1}\bigg)\phi^{(5)}. \tag{3.5.106}$$

Hence

$$\phi^{(5)} = \frac{3F^2}{\sqrt{Re}}\frac{\bigg(1 - \frac{\mu_2}{\mu_1}\bigg)}{\bigg(1 - \frac{\rho_2}{\rho_1}\bigg)}B. \tag{3.5.107}$$

3.5.3 Calculation of constant B

It remains to calculate the constant B which is obtained from the conserved quantity. We know that

$$K^* = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{0}\eta^2\frac{df_2}{d\eta}d\eta + \int_0^{\infty}\eta^2\frac{df_1}{d\eta}d\eta, \tag{3.5.108}$$

$$\frac{df_2}{d\eta} = B\left[1-\frac{\nu_1}{2\nu_2}\eta^2\right]\exp\left(-\frac{\nu_1}{4\nu_2}\eta^2\right), \tag{3.5.109}$$

$$\frac{df_1}{d\eta} = B\left[1-\frac{1}{2}\eta^2\right]\exp\left(-\frac{1}{4}\eta^2\right). \tag{3.5.110}$$

Substituting (5.2.2) and (5.2.3) into (5.2.1) and simplifying gives

$$K^* = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}B\int_0^{\infty}\eta^2\left(1-\frac{\nu_1}{2\nu_2}\eta^2\right)\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\eta^2\right)d\eta + B\int_0^{\infty}\eta^2\left(1-\frac{1}{2}\eta^2\right)\exp\left(-\frac{1}{4}\eta^2\right)d\eta$$

$$= B\left[\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}I_2 + I_1\right], \tag{3.5.111}$$

where

$$I_1 = \int_0^{\infty}\eta^2\left(1-\frac{1}{2}\eta^2\right)\exp\left(-\frac{1}{4}\eta^2\right)d\eta, \tag{3.5.112}$$

$$I_2 = \int_0^{\infty}\eta^2\left(1-\frac{\nu_1}{2\nu_2}\eta^2\right)\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\eta^2\right)d\eta \tag{3.5.113}$$

Equation (5.2.4) for K^* is exactly the same as (3.4.77) for K derived in Section 3.4.2. The integrals I_1 and I_2 were also evaluated in Section 3.4.2. It was found from equation (3.4.90) that

$$K^* = -4\sqrt{\pi}B\left[1+\frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}}\right]. \tag{3.5.114}$$

Thus

$$B = -\frac{K^*}{4\sqrt{\pi}\left[1+\frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}}\right]}. \tag{3.5.115}$$

But

$$\frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}} = \frac{\rho_2}{\rho_1}\left(\frac{\mu_2}{\rho_2}\frac{\rho_1}{\mu_1}\right)^{\frac{1}{2}} = \left(\frac{\rho_2\mu_2}{\rho_1\mu_1}\right)^{\frac{1}{2}} = \frac{1}{\chi^{\frac{1}{2}}}, \tag{3.5.116}$$

where

$$\chi = \frac{\rho_1\mu_1}{\rho_2\mu_2}. \tag{3.5.117}$$

The solution is therefore governed by the parameter χ and

$$B = -\frac{K^*}{4\sqrt{\pi}}\frac{\chi^{\frac{1}{2}}}{\left[1+\chi^{\frac{1}{2}}\right]}. \tag{3.5.118}$$

We have found that the solution for the momentumless wake for $k = 0$ and the solution for the momentumless wake using the method of Herczynski, Weidman and Burde but the conserved quantity, (3.4.4), depends on the parameter χ. Herczynski, Weidman and Burde used the conserved quantity

$$\frac{K}{\rho_1 U} = \frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} y^2 w_2(x,y)dy + \int_{\phi(x)}^{\infty} y^2 w_1(x,y)dy \tag{3.5.119}$$

which is not consistent with zero total drag on the body and found that the solution depends on a new parameter

$$\Omega = \frac{\rho_1\mu_2^3}{\rho_2\mu_1^3}. \tag{3.5.120}$$

3.5.4 Visual representation of the solution

The solution derived in Section 3.5 through the use of the Herczynski, Weidman and Burde method is visually represented in Figures 3.9 and 3.10. The same values for the ratios $\frac{\rho_1}{\rho_2}$ and $\frac{\nu_1}{\nu_2}$ as chosen in Figures 3.5 and 3.6 were chosen in Figures 3.9 and 3.10 for comparison. Once $\frac{\rho_1}{\rho_2}$ and $\frac{\nu_1}{\nu_2}$ have been chosen, the ratio $\frac{\mu_1}{\mu_2}$ is determined from $\frac{\mu_1}{\mu_2} = \frac{\nu_1}{\nu_2}\frac{\rho_2}{\rho_1}$.

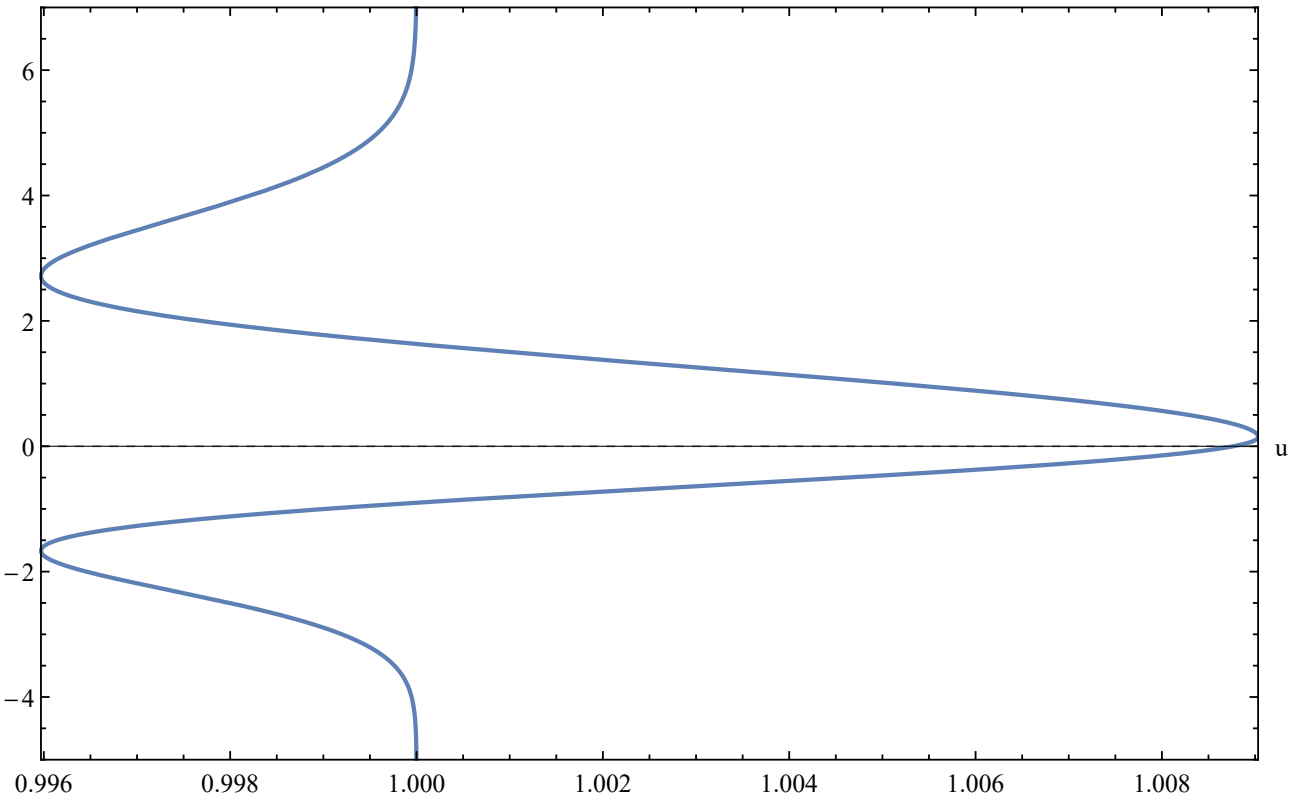

Figure 3.9: The velocities $u_1(x,y)$ and $u_2(x,y)$ for the momentumless two-fluid wake, given by (3.5.121) and (3.5.122) and derived using the method of Herczynski, Weidman and Burde [1], plotted against y at $x = 0.55$ for $K = \frac{1}{10}$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{2}$.

The velocity profile is given by

$$u_1(x,y) = 1 - w_1(x,y) = 1 - \frac{B}{x^{\frac{3}{2}}}\left[1 - \frac{1}{2}\frac{(y-\phi(x))^2}{x}\right]\exp\left(-\frac{1}{4}\frac{(y-\phi(x))^2}{x}\right), \tag{3.5.121}$$

$$u_2(x,y) = 1 - w_2(x,y) = 1 - \frac{B}{x^{\frac{3}{2}}}\left[1 - \frac{1}{2}\frac{\nu_1}{\nu_2}\frac{(y-\phi(x))^2}{x}\right]\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\frac{(y-\phi(x))^2}{x}\right) \tag{3.5.122}$$

and

$$B = -\frac{K^*}{4\sqrt{\pi}\left[1 + \frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}}\right]}. \tag{3.5.123}$$

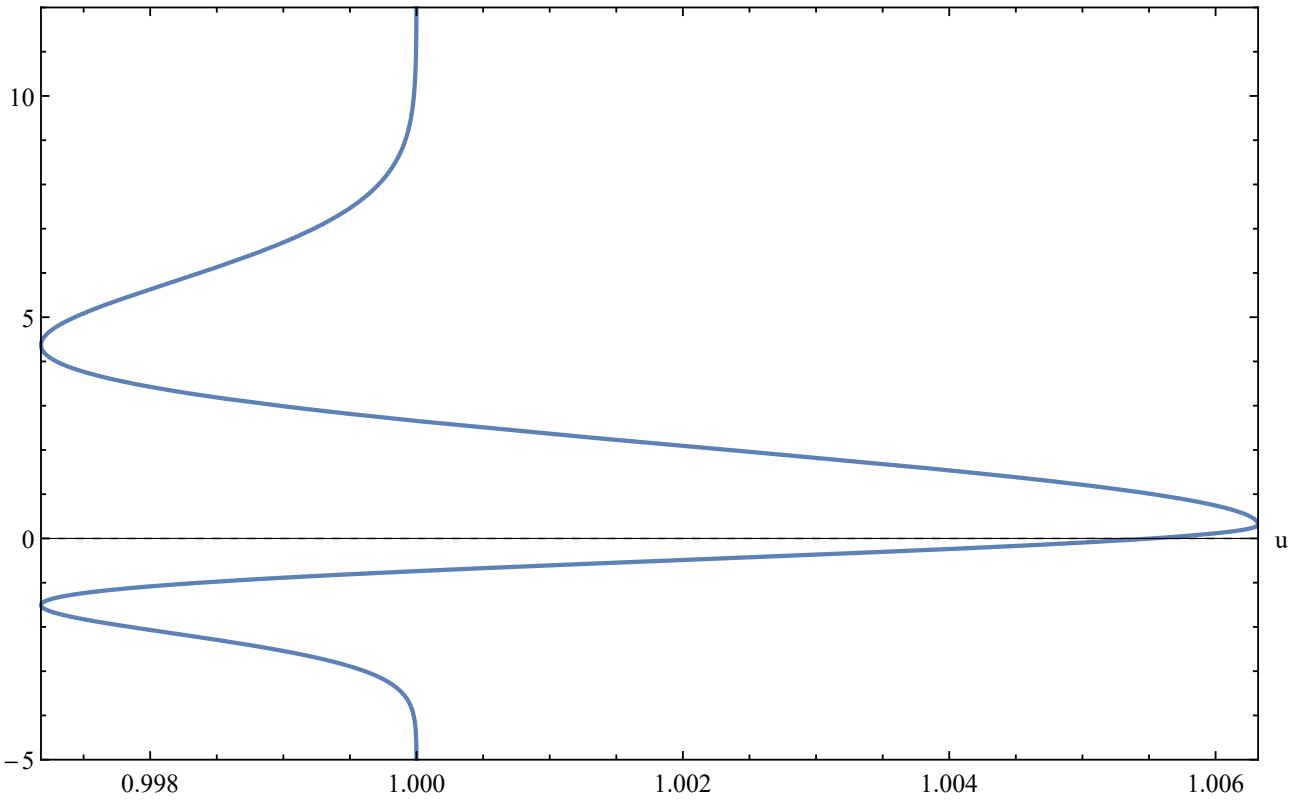

Figure 3.10: The velocities $u_1(x, y)$ and $u_2(x, y)$ for the momentumless two-fluid wake, given by (3.5.121) and (3.5.122) and derived using the method of Herczynski, Weidman and Burde [1], plotted against y at $x = 0.55$ for $K = \frac{1}{10}$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{5}$.

The graphs in Figures 3.9 and 3.10 have the same form as in Figures 3.5 and 3.6 but displaced on the y-axis by distance

$$\phi(x) = \phi^{(5)} x^{-\frac{5}{2}}. \tag{3.5.124}$$

The solution of Herczynski, Weidman and Burde [1] tends to the solution derived in Section 3.4 in the limit $x \to \infty$, that is in the limit $\phi(x) \to 0$.

3.6 Conclusions

Two solutions for the classical and momentumless two-fluid wake were derived in this chapter. A scaling transformation was introduced and an invariant solution was found for the classical and momentumless wakes through imposing of the boundary and interface conditions to reduce the resulting PDE. The results were also visually represented and it was found that the classical

wake has an interface at the y-axis while the momentumless wake has three different potential interfaces.

Subsequent to this, the method by Herczynski, Weidman and Burde was studied and replicated. The conserved quantity derived in this dissertation differed to that of Herczynski, Weidman and Burde. It was decided to use the conserved quantity derived in Chapter 2 because it satisfied the condition that the total drag on the body is zero. The coordinate expansion and expansion of the interface deflection that was used by Herczynski et al. was also incorporated into the derivation of the solution and a new corrected solution was found for the momentumless wake. The solutions for the classical and momentumless wakes were also visually presented and are similar to the solutions using the scaling transformation, however, the interface is deflected.

Chapter 4

Derivation of Conservation Laws

4.1 Introduction

In this chapter the conservation laws for a single fluid wake and a two-fluid wake will be explored. In Section 4.2 a brief review of the relevant theory regarding conservation laws and the different approaches that exist will be discussed [9]. In Section 4.3 the boundary layer equations for the two-fluid wake are considered. The conserved vectors are determined by elementary manipulations through the use of multipliers. Once the conserved vectors are found they will be used to determine the conserved quantity. This is done in Section 4.4 for the classical two-fluid wake flow and for the two-fluid wake flow behind a self-propelled body. Section 4.5 concludes this chapter by summarising the conclusions.

4.2 Conservation Law Theory

The formulation of physical laws like the balance laws for energy and momentum play a role in the mathematical idea of a conservation law [9]. Conserved quantities may be used in solving differential equations where a similarity solution exists. Boundary conditions are also used in the solution process.

There are many different methods that may be applied in finding conservation laws and conserved vectors. One such method is Noether's theorem [13]. This method relies on the use of a suitable Lagrangian for Euler-Lagrange differential equations. Conserved vectors are de-

termined by a formula based on the Noether symmetry associated with a specific Lagrangian. There are methods that do not rely on the use of a Lagrangian including the direct method which was first used by Laplace, however, in this dissertation we make use of the multiplier approach developed by Steudel [18] which can be used to generate conservation laws for a system of partial differential equations (PDEs).

Consider a k-th order system of PDEs of n independent variables and m dependent variables given by:

$$G_j(x, \psi, \psi_{(1)}, ..., \psi_{(k)}) = 0, \qquad j = 1, 2, ..., m. \tag{4.2.1}$$

The independent and dependent variables are denoted by $x = (x^1, x^2, ..., x^n)$ and $\psi = (\psi^1, \psi^2, ..., \psi^m)$ respectively.The collection of i-th order partial derivatives of the dependent variables is denoted by $\psi_{(i)}$. The system given by (4.2.1) is assumed to be of maximal rank and locally solvable.

The total derivative of ψ^j with respect to x^i is given by:

$$\psi^j_i = D_i(\psi^j), \quad \psi^j_{ip} = D_p D_i(\psi^j), \quad ...., \tag{4.2.2}$$

where the total derivative operator with respect to x^i is presented as:

$$D_i = \frac{\partial}{\partial x^i} + \psi^j_i \frac{\partial}{\partial \psi^j} + \psi^j_{ip} \frac{\partial}{\partial \psi^j_p} + ..., \qquad i = 1, 2, ..., n. \tag{4.2.3}$$

The Euler operator can annihilate a differential function if and only if the differential function is a total divergence. The Euler operator is defined as:

$$E^*_{\psi^j} = \frac{\partial}{\partial \psi^j} + \sum_{s \geq 1} (-1)^s D_{i_s} \frac{\partial}{\partial \psi^j_{i_1 ... i_s}}, \qquad j = 1, 2, ..., m, \quad i_1 \leq i_2 \leq ... \leq i_s. \tag{4.2.4}$$

The symmetry operator is given by:

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^j \frac{\partial}{\partial \psi^j} + \sum_{s \leq 1} \zeta^j_{i_1 ... i_s} \frac{\partial}{\partial \psi^j_{i_1 ... i_s}}, \qquad i_1 \leq i_2 \leq ... \leq i_s, \tag{4.2.5}$$

where $\zeta^j_{i_1 ... i_s}$ are given by:

$$\zeta^j_i = D_i(\eta^j) - \psi^j_p D_i(\xi^p), \tag{4.2.6}$$

$$\zeta^j_{i_1 ... i_s} = D_{i_s}(\zeta^j_{i_1 ... i_{s-1}}) - \psi^j_{p i_1 ... i_{s-1}} D_{i_s}(\xi^p), \quad s > 1. \tag{4.2.7}$$

A set of functions $T^1, T^2, ..., T^n$ consists of the components of a local conserved vector if and only if they satisfy

$$D_i T^i = 0, \tag{4.2.8}$$

for all solutions of system (4.2.1).

The multiplier Λ has components $\Lambda^1, \Lambda^2, ..., \Lambda^n$. The multiplier can depend on the independent and dependent variables and the partial derivatives up to the k-th order of the dependent variables. The multiplier for system (4.2.1) has the following property:

$$\Lambda^j G_j(x, \psi, \psi_{(1)}, ..., \psi_{(k)}) = D_j T^j, \tag{4.2.9}$$

for all functions ψ.

The determining equations may be obtained by applying the Euler operator to equation (4.2.9) giving:

$$E^*_{\psi^j}\left[\Lambda^p G_p\right] = 0, \quad j = 1, 2, ..., m. \tag{4.2.10}$$

The determining equations (4.2.10) are not only satisfied for all solutions of system (4.2.1) but for all functions $\psi = (\psi^1, \psi^2, ..., \psi^m)$. The determining equations (4.2.10) can be solved by separating powers and products of the partial derivatives of ψ^j since the partial derivatives of ψ^j are independent.

After determining the multiplier Λ, solutions for ψ may be considered, followed by calculating the components of the conserved vector T. The conserved vector $T = (T^1, T^2, ..., T^n)$ is invariant under the action of the symmetry operator (4.2.5) [16] provided

$$X(T^i) + T^i D_p(\xi^p) - T^p D_p(\xi^i) = 0, \quad i = 1, 2, ..., n, \tag{4.2.11}$$

where X is prolongated to as high an order in the derivatives as required. A generator (4.2.5) that satisfies (4.2.11) is called a Lie point symmetry associated with the conserved vector T of the system (4.2.1). The invariant solution generated by X can be determined once the Lie point symmetry X has been found.

4.3 Conserved Vectors

In this section the elementary conserved vectors are derived for the boundary layer equations

$$\frac{\partial w_i}{\partial x} = \frac{\nu_i}{\nu_1}\frac{\partial^2 w_i}{\partial y^2}, \tag{4.3.1}$$

$$-\frac{\partial w_i}{\partial x} + \frac{\partial v_i}{\partial y} = 0. \tag{4.3.2}$$

When x, y, w_i, v_i and the partial derivatives of w_i and v_i are considered as independent variables, the notation $w_{ix}, w_{iy}, w_{ixy}, v_{ix}, v_{iy}, v_{ixy}$ will be used.

Multipliers $\Lambda_1 = \Lambda_1(w_i, v_i, x, y)$ and $\Lambda_2 = \Lambda_2(w_i, v_i, x, y)$ for the system of equations (4.3.1) and (4.3.2) have the property that

$$\Lambda_1\left(w_{ix} - \frac{\nu_i}{\nu_1}w_{iyy}\right) + \Lambda_2(-w_{ix} + v_{iy}) = D_x T^1_{(i)} + D_y T^2_{(i)} \tag{4.3.3}$$

for all functions $w_i(x, y)$ and $v_i(x, y)$.

The total derivative operators are defined as:

$$D_x = \frac{\partial}{\partial x} + w_{ix}\frac{\partial}{\partial w_i} + v_{ix}\frac{\partial}{\partial v_i} + w_{ixx}\frac{\partial}{\partial w_{ix}} + v_{ixx}\frac{\partial}{\partial v_{ix}} + w_{ixy}\frac{\partial}{\partial w_{iy}} + v_{ixy}\frac{\partial}{\partial v_{iy}} + ... \tag{4.3.4}$$

$$D_y = \frac{\partial}{\partial y} + w_{iy}\frac{\partial}{\partial w_i} + v_{iy}\frac{\partial}{\partial v_i} + w_{iyy}\frac{\partial}{\partial w_{iy}} + v_{iyy}\frac{\partial}{\partial v_{iy}} + w_{iyx}\frac{\partial}{\partial w_{ix}} + v_{iyx}\frac{\partial}{\partial v_{ix}} + ... \tag{4.3.5}$$

The determining equations for multipliers Λ_1 and Λ_2 are:

$$E_{w_i}\left[\Lambda_1\left(w_{ix} - \frac{\nu_i}{\nu_1}w_{iyy}\right) + \Lambda_2(-w_{ix} + v_{iy})\right] = 0, \tag{4.3.6}$$

$$E_{v_i}\left[\Lambda_1\left(w_{ix} - \frac{\nu_i}{\nu_1}w_{iyy}\right) + \Lambda_2(-w_{ix} + v_{iy})\right] = 0, \tag{4.3.7}$$

where E_{w_i} and E_{v_i} are the standard Euler operators and are given by:

$$E_{w_i} = \frac{\partial}{\partial w_i} - D_x\frac{\partial}{\partial w_{ix}} - D_y\frac{\partial}{\partial w_{iy}} + D_x^2\frac{\partial}{\partial w_{ixx}} + D_xD_y\frac{\partial}{\partial w_{ixy}} + D_y^2\frac{\partial}{\partial w_{iyy}} - ... \tag{4.3.8}$$

$$E_{v_i} = \frac{\partial}{\partial v_i} - D_x\frac{\partial}{\partial v_{ix}} - D_y\frac{\partial}{\partial v_{iy}} + D_x^2\frac{\partial}{\partial v_{ixx}} + D_xD_y\frac{\partial}{\partial v_{ixy}} + D_y^2\frac{\partial}{\partial v_{iyy}} - ... \tag{4.3.9}$$

The expansion of equations (4.3.6) and (4.3.7) gives:

$$\frac{\partial\Lambda_1}{\partial w_i}\left(w_{ix} - \frac{\nu_i}{\nu_1}w_{iyy}\right) + \frac{\partial\Lambda_2}{\partial w_i}(-w_{ix} + v_{iy}) + D_x(\Lambda_2 - \Lambda_1) - \frac{\nu_i}{\nu_1}D_y^2(\Lambda_1) = 0, \tag{4.3.10}$$

$$\frac{\partial\Lambda_1}{\partial v_i}\left(w_{ix} - \frac{\nu_i}{\nu_1}w_{iyy}\right) + \frac{\partial\Lambda_2}{\partial v_i}(-w_{ix} + v_{iy}) - D_y(\Lambda_2) = 0. \tag{4.3.11}$$

When expanded, equations (4.3.10) and (4.3.11) become

$$w_{ix}\frac{\partial\Lambda_1}{\partial w_i}-\frac{\nu_i}{\nu_1}w_{iyy}\frac{\partial\Lambda_1}{\partial w_i}-w_{ix}\frac{\partial\Lambda_2}{\partial w_i}+v_{iy}\frac{\partial\Lambda_2}{\partial w_i}-\frac{\partial\Lambda_1}{\partial x}-w_{ix}\frac{\partial\Lambda_1}{\partial w_i}-v_{ix}\frac{\partial\Lambda_1}{\partial v_i}+\frac{\partial\Lambda_2}{\partial x}+w_{ix}\frac{\partial\Lambda_2}{\partial w_i}+$$

$$v_{ix}\frac{\partial\Lambda_2}{\partial v_i}-\frac{\nu_i}{\nu_1}\frac{\partial^2\Lambda_1}{\partial y^2}-\frac{\nu_i}{\nu_1}w_{iy}\frac{\partial^2\Lambda_1}{\partial y\partial w_i}-\frac{\nu_i}{\nu_1}v_{iy}\frac{\partial^2\Lambda_1}{\partial y\partial v_i}-\frac{\nu_i}{\nu_1}w_{iy}\frac{\partial^2\Lambda_1}{\partial w_i\partial y}-\frac{\nu_i}{\nu_1}w_{iyy}\frac{\partial\Lambda_1}{\partial w_i}-\frac{\nu_i}{\nu_1}w_{iy}w_{iy}\frac{\partial^2\Lambda_1}{\partial w_i^2}-$$

$$\frac{\nu_i}{\nu_1}v_{iy}w_{iy}\frac{\partial^2\Lambda_1}{\partial w_i\partial v_i}-\frac{\nu_i}{\nu_1}v_{iy}\frac{\partial^2\Lambda_1}{\partial v_i\partial y}-\frac{\nu_i}{\nu_1}v_{iyy}\frac{\partial\Lambda_1}{\partial v_i}-\frac{\nu_i}{\nu_1}w_{iy}v_{iy}\frac{\partial^2\Lambda_1}{\partial w_i\partial v_i}-\frac{\nu_i}{\nu_1}v_{iy}v_{iy}\frac{\partial^2\Lambda_1}{\partial v_i^2}=0, \tag{4.3.12}$$

$$w_{ix}\frac{\partial\Lambda}{\partial v_i}-\frac{\nu_i}{\nu_1}w_{iyy}\frac{\partial\Lambda_1}{\partial v_i}-w_{ix}\frac{\partial\Lambda_2}{\partial v_i}+v_{iy}\frac{\partial\Lambda_2}{\partial v_i}-\frac{\partial\Lambda_2}{\partial y}-w_{iy}\frac{\partial\Lambda_2}{\partial w_i}-v_{iy}\frac{\partial\Lambda_2}{\partial v_i}=0. \tag{4.3.13}$$

Now the partial derivatives of $w_i(x,y)$ and $v_i(x,y)$ in (4.3.12) and (4.3.13) are independent because (4.3.3) holds for all functions $w_i(x,y)$ and $v_i(x,y)$ and not only for solutions of the two partial differential equations (4.3.1) and (4.3.2). We can therefore separate equations (4.3.12) and (4.3.13) by equating the coefficients of powers and products of the partial derivatives of w_i and v_i.

Separating equation (4.3.12) gives the following results.

$$w_{ix}: \qquad 0 = 0, \tag{4.3.14}$$

$$w_{iy}: \qquad \frac{\partial^2 \Lambda_1}{\partial y \partial w_i} = 0, \tag{4.3.15}$$

$$w_{iy}^2: \qquad \frac{\partial^2 \Lambda_1}{\partial w_i^2} = 0, \tag{4.3.16}$$

$$w_{iyy}: \qquad \frac{\partial \Lambda_1}{\partial w_i} = 0, \tag{4.3.17}$$

$$v_{ix}: \qquad \frac{\partial \Lambda_2}{\partial v_i} - \frac{\partial \Lambda_1}{\partial v_i} = 0, \tag{4.3.18}$$

$$v_{iy}: \qquad \frac{\partial \Lambda_2}{\partial w_i} - 2\frac{\nu_i}{\nu_1}\frac{\partial^2 \Lambda_1}{\partial y \partial v_i} = 0, \tag{4.3.19}$$

$$v_{iy}^2: \qquad \frac{\partial^2 \Lambda_1}{\partial v_i^2} = 0, \tag{4.3.20}$$

$$v_{iyy}: \qquad \frac{\partial \Lambda_1}{\partial v_i} = 0, \tag{4.3.21}$$

$$v_{iy}w_{iy}: \qquad \frac{\partial^2 \Lambda_1}{\partial w_i \partial v_i} = 0. \tag{4.3.22}$$

The remainder is

$$\frac{\partial \Lambda_2}{\partial x} - \frac{\partial \Lambda_1}{\partial x} - \frac{\nu_i}{\nu_1}\frac{\partial^2 \Lambda_1}{\partial y^2} = 0. \tag{4.3.23}$$

Separating equation (4.3.13) gives the following results

$$w_{ix}: \qquad \frac{\partial \Lambda_1}{\partial v_i} - \frac{\partial \Lambda_2}{\partial v_i} = 0, \tag{4.3.24}$$

$$w_{iy}: \qquad \frac{\partial \Lambda_2}{\partial w_i} = 0, \tag{4.3.25}$$

$$w_{iyy}: \qquad \frac{\partial \Lambda_1}{\partial v_i} = 0, \tag{4.3.26}$$

$$v_{iy}: \qquad 0 = 0. \tag{4.3.27}$$

The remainder is

$$\frac{\partial \Lambda_2}{\partial y} = 0. \tag{4.3.28}$$

Now

$$\Lambda_1 = \Lambda_1(w_i, v_i, x, y), \qquad \Lambda_2 = \Lambda_2(w_i, v_i, x, y). \tag{4.3.29}$$

From equations (4.3.17), (4.3.18), (4.3.21), (4.3.25) and (4.3.28)

$$\Lambda_1 = \Lambda_1(x, y), \qquad \Lambda_2 = \Lambda_2(x). \tag{4.3.30}$$

With this form for Λ_1 and Λ_2 there is only one other equation not identically zero, the remainder (4.3.23)

$$\frac{d\Lambda_2}{dx}(x) - \frac{\partial \Lambda_1}{\partial x}(x, y) - \frac{\nu_i}{\nu_1}\frac{\partial^2 \Lambda_1}{\partial y^2}(x, y) = 0. \tag{4.3.31}$$

To make progress we look for a multiplier Λ_1 that is independent of x. Instead of looking for multipliers of the form (4.3.29) we should have looked for multipliers of the form

$$\Lambda_1 = \Lambda_1(w_i, v_i, y), \qquad \Lambda_2 = \Lambda_2(w_i, v_i, x, y). \tag{4.3.32}$$

With

$$\Lambda_1 = \Lambda_1(y) \tag{4.3.33}$$

equation (4.3.31) becomes

$$\frac{d\Lambda_2}{dx}(x) = \frac{\nu_i}{\nu_1}\frac{d^2 \Lambda_1}{dy^2}(y). \tag{4.3.34}$$

By the technique of separation of variables, each side of (4.3.34) must be a constant, $2a_1$. Thus

$$\frac{d\Lambda_2}{dx}(x) = 2a_1, \tag{4.3.35}$$

$$\frac{\nu_i}{\nu_1}\frac{d^2\Lambda_1}{dy^2}(y) = 2a_1. \tag{4.3.36}$$

Hence, integrating

$$\Lambda_1(y) = a_1\frac{\nu_1}{\nu_i}y^2 + a_2 y + a_3, \tag{4.3.37}$$

$$\Lambda_2(x) = 2a_1 x + a_4, \tag{4.3.38}$$

where a_2, a_3 and a_4 are constants. The factor $\frac{\nu_1}{\nu_i}$ in (4.3.37) cannot be transformed away by letting

$$b_1 = \frac{\nu_1}{\nu_i}a_1 \tag{4.3.39}$$

because a_1 also occurs in (4.3.38). Therefore,

$$\Lambda_1(y)\left[w_{ix} - \frac{\nu_i}{\nu_1}w_{iyy}\right] + \Lambda_2(x)[-w_{ix} + v_{iy}] = D_x T^1_{(i)} + D_y T^2_{(i)} \tag{4.3.40}$$

for all functions $w_i(x, y)$ and $v_i(x, y)$.

4.3.1 Case 1: Momentumless wake ($a_1 \neq 0$)

Consider

$$a_1 = 1, \qquad a_2 = 0, \qquad a_3 = 0, \qquad a_4 = 0. \tag{4.3.41}$$

Then

$$\Lambda_1(y) = \frac{\nu_1}{\nu_i}y^2, \qquad \Lambda_2(x) = 2x \tag{4.3.42}$$

and

$$\frac{\nu_1}{\nu_i}y^2\left[w_{ix} - \frac{\nu_i}{\nu_1}w_{iyy}\right] + 2x[-w_{ix} + v_{iy}] = D_x T^1_{(i)} + D_y T^2_{(i)} \tag{4.3.43}$$

for all functions $w_i(x, y)$ and $v_i(x, y)$.

We have to find the conserved vector $[T^1_{(i)}, T^2_{(i)}]$. Now (4.3.43) is

$$y^2\left[\frac{\nu_1}{\nu_i}w_{ix} - w_{iyy}\right] + 2x[-w_{ix} + v_{iy}] = D_x T^1_{(i)} + D_y T^2_{(i)}. \tag{4.3.44}$$

Consider

$$D_x\left[\frac{\nu_1}{\nu_i}y^2 w_i - 2xw_i\right] + D_y[-y^2 w_{iy} + 2yw_i + 2xv_i]$$

$$= \frac{\nu_1}{\nu_i}y^2 w_{ix} - 2w_i - 2xw_{ix} - 2yw_{iy} - y^2 w_{iyy} + 2w_i + 2yw_{iy} + 2xv_{iy}$$

$$= y^2\left[\frac{\nu_1}{\nu_i}w_{ix} - w_{iyy}\right] + 2x[-w_{ix} + v_{iy}]. \tag{4.3.45}$$

Hence

$$\frac{\nu_1}{\nu_i}y^2\left[w_{ix} - \frac{\nu_i}{\nu_1}w_{iyy}\right] + 2x[-w_{ix} + v_{iy}] = D_x\left[\frac{\nu_1}{\nu_i}y^2 w_i - 2xw_i\right] + D_y[-y^2 w_{iy} + 2yw_i + 2xv_i] \tag{4.3.46}$$

for arbitrary functions $w_i(x, y)$ and $v_i(x, y)$. When $w_i(x, y)$ and $v_i(x, y)$ are solutions of the system of PDEs

$$\frac{\partial w_i}{\partial x} = \frac{\nu_i}{\nu_1}\frac{\partial^2 w_i}{\partial y^2}, \tag{4.3.47}$$

$$-\frac{\partial w_i}{\partial x} + \frac{\partial v_i}{\partial y} = 0, \tag{4.3.48}$$

then (4.3.46) reduces to

$$D_x\left[\frac{\nu_1}{\nu_i}y^2 w_i - 2xw_i\right] + D_y\left[-y^2 w_{iy} + 2yw_i + 2xv_i\right]\Big|_{\text{PDE (4.3.47) and (4.3.48)}} = 0. \tag{4.3.49}$$

Thus a conserved vector of the system of PDEs (4.3.47) and (4.3.48) is

$$T^1_{(i)} = \frac{\nu_1}{\nu_i}y^2 w_i - 2xw_i, \tag{4.3.50}$$

$$T^2_{(i)} = -y^2 w_{iy} + 2yw_i + 2xv_i. \tag{4.3.51}$$

4.3.2 Case 2: Classical wake ($a_3 \neq 0$)

Consider

$$a_1 = 0, \qquad a_2 = 0, \qquad a_3 = 1, \qquad a_4 = 0. \tag{4.3.52}$$

Then

$$\Lambda_1(y) = 1, \qquad \Lambda_2(x) = 0 \tag{4.3.53}$$

and

$$w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy} = D_x T^1_{(i)} + D_y T^2_{(i)}. \tag{4.3.54}$$

We have to find the conserved vector $[T^1_{(i)}, T^2_{(i)}]$. Consider

$$D_x[w_i] + D_y\left[-\frac{\nu_i}{\nu_1} w_{iy} \right] = w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy}. \tag{4.3.55}$$

Hence

$$w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy} = D_x(w_i) + D_y\left[-\frac{\nu_i}{\nu_1} w_{iy} \right] \tag{4.3.56}$$

for an arbitrary function $w_i(x, y)$. When $w_i(x, y)$ is a solution of the PDE

$$\frac{\partial w_i}{\partial x} = \frac{\nu_i}{\nu_1} \frac{\partial^2 w_i}{\partial y^2}, \tag{4.3.57}$$

then (4.3.56) reduces to

$$D_x\left[w_i \right] + D_y\left[-\frac{\nu_i}{\nu_1} w_{iy} \right]\Big|_{\text{PDE (4.3.57)}} = 0. \tag{4.3.58}$$

Thus a conserved vector of the PDE (4.3.57) is

$$T^1_{(i)} = w_i, \tag{4.3.59}$$

$$T^2_{(i)} = -\frac{\nu_i}{\nu_1} w_{iy}. \tag{4.3.60}$$

4.3.3 Case 3: Combination wake ($a_2 \neq 0$)

Consider

$$a_1 = 0, \qquad a_2 = 1, \qquad a_3 = 0, \qquad a_4 = 0. \tag{4.3.61}$$

Then

$$\Lambda_1(y) = y, \qquad \Lambda_2(x) = 0. \tag{4.3.62}$$

Now

$$\Lambda_1(y)\left[w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy}\right] + \Lambda_2(x)[-w_{ix} + v_{iy}] = D_x T^1_{(i)} + D_y T^2_{(i)}. \tag{4.3.63}$$

Thus

$$y\left[w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy}\right] = D_x T^1_{(i)} + D_y T^2_{(i)}. \tag{4.3.64}$$

Consider

$$D_x\left[yw_i\right] + D_y\left[-\frac{\nu_i}{\nu_1} y w_{iy} + \frac{\nu_i}{\nu_1} w_i\right] = y w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy} - \frac{\nu_i}{\nu_1} y w_{iyy} + \frac{\nu_i}{\nu_1} w_{iy}$$

$$= y\left[w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy}\right]. \tag{4.3.65}$$

Hence

$$y\left[w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy}\right] = D_x\left[yw_i\right] + D_y\left[-\frac{\nu_i}{\nu_1} y w_{iy} + \frac{\nu_i}{\nu_1} w_i\right] \tag{4.3.66}$$

for an arbitrary function $w_i(x, y)$. When $w_i(x, y)$ is a solution of the PDE

$$\frac{\partial w_i}{\partial x} = \frac{\nu_i}{\nu_1} \frac{\partial^2 w_i}{\partial y^2}, \tag{4.3.67}$$

then (4.3.66) reduces to

$$D_x\left[yw_i\right] + D_y\left[-\frac{\nu_i}{\nu_1} y w_{iy} + \frac{\nu_i}{\nu_1} w_i\right]\Big|_{\text{PDE (4.3.67)}} = 0. \tag{4.3.68}$$

Thus a conserved vector of the PDE (4.3.67) is

$$T^1_{(i)} = y w_i, \tag{4.3.69}$$

$$T^2_{(i)} = \frac{\nu_i}{\nu_1}(w_i - y w_{iy}). \tag{4.3.70}$$

4.3.4 Case 4: Remaining case ($a_4 \neq 0$)

Consider

$$a_1 = 0, \qquad a_2 = 0, \qquad a_3 = 0, \qquad a_4 = 1. \tag{4.3.71}$$

Then

$$\Lambda_1(y) = 0, \qquad \Lambda_2(x) = 1. \tag{4.3.72}$$

Now

$$\Lambda_1(y)\left[w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy}\right] + \Lambda_2(x)[-w_{ix} + v_{iy}] = D_x T^1_{(i)} + D_y T^2_{(i)}. \tag{4.3.73}$$

Thus

$$-w_{ix} + v_{iy} = D_x T^1_{(i)} + D_y T^2_{(i)}. \tag{4.3.74}$$

Consider

$$D_x[-w_i] + D_y[v_i] = -w_{ix} + v_{iy}. \tag{4.3.75}$$

Hence

$$-w_{ix} + v_{iy} = D_x[-w_i] + D_y[v_i] \tag{4.3.76}$$

for arbitrary functions $w_i(x, y)$ and $v_i(x, y)$. When $w_i(x, y)$ and $v_i(x, y)$ are a solution of the PDE

$$-\frac{\partial w_i}{\partial x} + \frac{\partial v_i}{\partial y} = 0, \tag{4.3.77}$$

then (4.3.76) reduces to

$$D_x[-w_i] + D_y[v_i]|_{\text{PDE (4.3.77)}} = 0. \tag{4.3.78}$$

Thus a conserved vector of the PDE (4.3.77) is

$$T^1_{(i)} = -w_i, \tag{4.3.79}$$

$$T^2_{(i)} = v_i. \tag{4.3.80}$$

4.3.5 Case 5: Linear combination of conserved vectors

Consider

$$a_1 \neq 0, \qquad a_2 \neq 0, \qquad a_3 \neq 0, \qquad a_4 \neq 0. \tag{4.3.81}$$

Then

$$\Lambda_1(y) = a_1 \frac{\nu_1}{\nu_i} y^2 + a_2 y + a_3, \qquad \Lambda_2(x) = 2a_1 x + a_4 \tag{4.3.82}$$

and

$$\left[a_1 \frac{\nu_1}{\nu_i} y^2 + a_2 y + a_3\right]\left[w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy}\right] + [2a_1 x + a_4][-w_{ix} + v_{iy}] = D_x T^1_{(i)} + D_y T^2_{(i)} \tag{4.3.83}$$

for all functions $w_i(x, y)$ and $v_i(x, y)$.

We have to find the conserved vector $[T^1_{(i)}, T^2_{(i)}]$. Based on our experience with the four conserved vectors separately, we consider

$$D_x\left[a_1\left(\frac{\nu_1}{\nu_i} y^2 w_i - 2x w_i\right) + a_2(y w_i) + a_3(w_i) + a_4(-w_i)\right] +$$

$$D_y\left[a_1(-y^2 w_{iy} + 2y w_i + 2x v_i) + a_2\left(-\frac{\nu_i}{\nu_1} y w_{iy} + \frac{\nu_i}{\nu_1} w_i\right) + a_3\left(-\frac{\nu_i}{\nu_1} w_{iy}\right) + a_4(v_i)\right]$$

$$= a_1\left[\frac{\nu_1}{\nu_i} y^2\left(w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy}\right) + 2x(-w_{ix} + v_{iy})\right] + a_2\left[y\left(w_{ix}\frac{\nu_i}{\nu_1} w_{iyy}\right)\right] + a_3\left[w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy}\right] + a_4[-w_{ix} + v_{iy}]$$

$$= \left(a_1 \frac{\nu_1}{\nu_i} y^2 + a_2 y + a_3\right)\left(w_{ix} - \frac{\nu_i}{\nu_1} w_{iyy}\right) + (2x a_1 + a_4)(-w_{ix} + v_{iy}). \tag{4.3.84}$$

Equation (4.3.84) is satisfied for arbitrary functions $w_i(x, y)$ and $v_i(x, y)$. When $w_i(x, y)$ and $v_i(x, y)$ are solutions of the system of PDEs

$$\frac{\partial w_i}{\partial x} = \frac{\nu_i}{\nu_1} \frac{\partial^2 w_i}{\partial y^2}, \tag{4.3.85}$$

$$-\frac{\partial w_i}{\partial x} + \frac{\partial v_i}{\partial y} = 0, \tag{4.3.86}$$

then (4.3.84) reduces to

$$D_x\left[a_1\left(\frac{\nu_1}{\nu_i}y^2w_i - 2xw_i\right) + a_2(yw_i) + a_3(w_i) + a_4(-w_i)\right] +$$

$$D_y\left[a_1(-y^2w_{iy} + 2yw_i + 2xv_i) + a_2\left(-\frac{\nu_i}{\nu_1}yw_{iy} + \frac{\nu_i}{\nu_1}w_i\right) + a_3\left(-\frac{\nu_i}{\nu_1}w_{iy}\right) + a_4(v_i)\right]\Big|_{\text{PDE (4.3.85) and (4.3.86)}}$$

$$= 0. \tag{4.3.87}$$

Thus the general conserved vector of the system of PDEs (4.3.85) and (4.3.86) with multipliers $\Lambda_1(y)$ and $\Lambda_2(x)$ is of the form

$$T^1_{(i)} = a_1\left(\frac{\nu_1}{\nu_i}y^2w_i - 2xw_i\right) + a_2(yw_i) + a_3(w_i) + a_4(-w_i), \tag{4.3.88}$$

$$T^2_{(i)} = a_1(-y^2w_{iy} + 2yw_i + 2xv_i) + a_2\left(-\frac{\nu_i}{\nu_1}yw_{iy} + \frac{\nu_i}{\nu_1}w_i\right) + a_3\left(-\frac{\nu_i}{\nu_1}w_{iy}\right) + a_4(v_i). \tag{4.3.89}$$

The conserved vector with components (4.3.88) and (4.3.89) is a linear combination of four conserved vectors obtained by setting all $a_i = 0$ except one, in turn. The four conserved vectors are

$$a_1 = 1, \quad a_i = 0, \quad i \neq 1:$$

$$T^1 = \frac{\nu_1}{\nu_i}y^2w_i - 2xw_i, \tag{4.3.90}$$

$$T^2 = -y^2w_{iy} + 2yw_i + 2xv_i. \tag{4.3.91}$$

$$a_2 = 1, \quad a_i = 0, \quad i \neq 2:$$

$$T^1 = yw_i, \tag{4.3.92}$$

$$T^2 = \frac{\nu_i}{\nu_1}(w_i - yw_{iy}). \tag{4.3.93}$$

$$a_3 = 1, \quad a_i = 0, \quad i \neq 3 :$$

$$T^1 = w_i, \tag{4.3.94}$$

$$T^2 = -\frac{\nu_i}{\nu_1} w_{iy}. \tag{4.3.95}$$

$$a_4 = 1, \quad a_i = 0, \quad i \neq 4 :$$

$$T^1 = -w_i, \tag{4.3.96}$$

$$T^2 = v_i. \tag{4.3.97}$$

The general conserved vector $\boldsymbol{T}$ is the linear combination

$$\boldsymbol{T} = a_1 \boldsymbol{T^{(1)}} + a_2 \boldsymbol{T^{(2)}} + a_3 \boldsymbol{T^{(3)}} + a_4 \boldsymbol{T^{(4)}}. \tag{4.3.98}$$

We can think of

$$\boldsymbol{T^{(1)}}, \boldsymbol{T^{(2)}}, \boldsymbol{T^{(3)}}, \boldsymbol{T^{(4)}} \tag{4.3.99}$$

as forming a basis of conserved vectors of the PDEs (4.3.85) and (4.3.86) for multipliers $\Lambda_1(y)$ and $\Lambda_2(x)$.

4.4 Conserved quantities

We now treat T^1 and T^2 as functions of the independent variables x and y:

$$T^1 = T^1(x, y), \tag{4.4.1}$$

$$T^1 = T^1(x, y, w_i(x, y), v_i(x, y), w_{ix}(x, y), w_{iy}(x, y), v_{ix}(x, y), v_{iy}(x, y), ...), \tag{4.4.2}$$

$$\frac{\partial T^1}{\partial x} = D_x T^1, \tag{4.4.3}$$

and

$$T^2 = T^2(x, y), \tag{4.4.4}$$

$$T^2 = T^2(x, y, w_i(x, y), v_i(x, y), w_{ix}(x, y), w_{iy}(x, y), v_{ix}(x, y), v_{iy}(x, y), ...), \tag{4.4.5}$$

$$\frac{\partial T^2}{\partial y} = D_y T^2. \tag{4.4.6}$$

Thus

$$D_x T^1 + D_y T^2 = \frac{\partial T^1}{\partial x}(x, y) + \frac{\partial T^2}{\partial y}(x, y). \tag{4.4.7}$$

The conserved vectors with x and y regarded as independent variables therefore satisfy

$$\frac{\partial T^1}{\partial x}(x, y) + \frac{\partial T^2}{\partial y}(x, y) = 0. \tag{4.4.8}$$

4.4.1 Conserved quantity for the two-fluid classical wake

The conserved quantity for the classical wake is obtained from the conserved vector

$$T^1_{(i)} = w_i, \qquad T^2_{(i)} = -\frac{\nu_i}{\nu_1} w_{iy} \tag{4.4.9}$$

and the interface condition (shear stress at the interface)

$$\frac{\partial w_1}{\partial y}(x, \phi(x)) = \frac{\mu_2}{\mu_1} \frac{\partial w_2}{\partial y}(x, \phi(x)). \tag{4.4.10}$$

We derive expressions for $\frac{\partial w_1}{\partial y}(x, \phi(x))$ and $\frac{\partial w_2}{\partial y}(x, \phi(x))$.

4.4.1.1 Lower fluid

$$T^1 = w_2, \qquad T^2 = -\frac{\nu_2}{\nu_1} \frac{\partial w_2}{\partial y}. \tag{4.4.11}$$

Now

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y} = 0, \tag{4.4.12}$$

and therefore

$$\frac{\partial}{\partial x}(w_2) + \frac{\partial}{\partial y}\left(- \frac{\nu_2}{\nu_1}\frac{\partial w_2}{\partial y}\right) = 0. \tag{4.4.13}$$

Since (4.4.10) has the factor $\frac{\mu_2}{\mu_1}$ we multiply (4.4.13) by $\frac{\rho_2}{\rho_1}$. Hence

$$\frac{\rho_2}{\rho_1}\frac{\partial w_2}{\partial x} = \frac{\mu_2}{\mu_1}\frac{\partial^2 w_2}{\partial y^2}. \tag{4.4.14}$$

Integrate both sides of (4.4.14) with respect to y from $y = -\infty$ to $y = \phi(x)$.

$$\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} \frac{\partial w_2}{\partial x}(x,y)dy = \frac{\mu_2}{\mu_1}\int_{-\infty}^{\phi(x)} \frac{\partial^2 w_2}{\partial y^2}(x,y)dy. \tag{4.4.15}$$

Consider first the right-hand side of (4.4.15):

$$\int_{-\infty}^{\phi(x)} \frac{\partial^2 w_2}{\partial y^2}(x,y)dy = \left[\frac{\partial w_2}{\partial y}(x,y)\right]_{-\infty}^{\phi(x)} \tag{4.4.16}$$

$$= \frac{\partial w_2}{\partial y}(x,\phi(x)) - \frac{\partial w_2}{\partial y}(x,-\infty). \tag{4.4.17}$$

However, from the boundary condition at $y = -\infty$,

$$\frac{\partial w_2}{\partial y}(x,-\infty) = 0. \tag{4.4.18}$$

Hence

$$\int_{-\infty}^{\phi(x)} \frac{\partial^2 w_2}{\partial y^2}(x,y)dy = \frac{\partial w_2}{\partial y}(x,\phi(x)). \tag{4.4.19}$$

Consider next the left-hand side of (4.4.15). To evaluate the left-hand side of (4.4.15) replace the lower limit by $-Bx$ where $B > 0$ is a constant. Using the formula for differentiation under the integral sign [42]

$$\frac{d}{dx}\int_{-Bx}^{\phi(x)} w_2(x,y)dy = \int_{-Bx}^{\phi(x)} \frac{\partial w_2}{\partial x}(x,y)dy + w_2(x,\phi(x))\frac{d\phi(x)}{dx} - w_2(x,-Bx)(-B). \tag{4.4.20}$$

Now from the boundary conditions

$$w_2(x,-\infty) = 0. \tag{4.4.21}$$

Assume the stronger condition that

$$\lim_{B\to\infty} w_2(x,-Bx)B = 0. \tag{4.4.22}$$

Let $B \to \infty$ in (4.4.20). Then

$$\frac{d}{dx}\int_{-\infty}^{\phi(x)} w_2(x,y)dy = \int_{-\infty}^{\phi(x)} \frac{\partial w_2}{\partial x}(x,y)dy + w_2(x,\phi(x))\frac{d\phi(x)}{dx} \tag{4.4.23}$$

and therefore

$$\int_{-\infty}^{\phi(x)} \frac{\partial w_2}{\partial x}(x,y)dy = \frac{d}{dx}\int_{-\infty}^{\phi(x)} w_2(x,y)dy - w_2(x,\phi(x))\frac{d\phi(x)}{dx}. \tag{4.4.24}$$

Substitute (4.4.19) and (4.4.24) into (4.4.15),

$$\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} \frac{\partial w_2}{\partial x}(x,y)dy = \frac{\mu_2}{\mu_1}\int_{-\infty}^{\phi(x)} \frac{\partial^2 w_2}{\partial y^2}(x,y)dy, \tag{4.4.25}$$

which gives

$$\frac{\rho_2}{\rho_1}\left[\frac{d}{dx}\int_{-\infty}^{\phi(x)} w_2(x,y)dy - w_2(x,\phi(x))\frac{d\phi(x)}{dx}\right] = \frac{\mu_2}{\mu_1}\frac{\partial w_2}{\partial y}(x,\phi(x)). \tag{4.4.26}$$

Thus

$$\frac{\mu_2}{\mu_1}\frac{\partial w_2}{\partial y}(x,\phi(x)) = \frac{d}{dx}\left(\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} w_2(x,y)dy\right) - \frac{\rho_2}{\rho_1}w_2(x,\phi(x))\frac{d\phi(x)}{dx}. \tag{4.4.27}$$

4.4.1.2 Upper fluid

For the upper fluid,

$$T^1 = w_1, \qquad T^2 = -\frac{\partial w_1}{\partial y}. \tag{4.4.28}$$

Now

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y} = 0, \tag{4.4.29}$$

therefore

$$\frac{\partial w_1}{\partial x} = \frac{\partial^2 w_1}{\partial y^2}. \tag{4.4.30}$$

Integrate both sides of (4.4.30) from $y = \phi(x)$ to $y = \infty$ with respect to y:

$$\int_{\phi(x)}^{\infty} \frac{\partial w_1}{\partial x}(x,y)dy = \int_{\phi(x)}^{\infty} \frac{\partial^2 w_1}{\partial y^2}(x,y)dy. \tag{4.4.31}$$

Consider first the right-hand side of (4.4.31).

$$\int_{\phi(x)}^{\infty} \frac{\partial w_1}{\partial x}(x,y)dy = \left[\frac{\partial w_1}{\partial y}(x,y)\right]_{\phi(x)}^{\infty} \tag{4.4.32}$$

$$= \frac{\partial w_1}{\partial y}(x,\infty) - \frac{\partial w_1}{\partial y}(x,\phi(x)). \tag{4.4.33}$$

However, from the boundary condition at $y = +\infty$

$$\frac{\partial w_1}{\partial y}(x,\infty) = 0. \tag{4.4.34}$$

Hence

$$\int_{\phi(x)}^{\infty} \frac{\partial w_1}{\partial x}(x,y)dy = -\frac{\partial w_1}{\partial y}(x,\phi(x)). \tag{4.4.35}$$

Consider next the left-hand side of (4.4.31). Apply again the Theorem for differentiation under the integral sign. Replace the upper limit as in the integral by Bx where $B > 0$ is a constant. Then

$$\frac{d}{dx}\int_{\phi(x)}^{Bx} w_1(x,y)dy = \int_{\phi(x)}^{Bx} \frac{\partial w_1}{\partial x}(x,y)dy + w_1(x,Bx)B - w_1(x,\phi(x))\frac{d\phi(x)}{dx}. \tag{4.4.36}$$

But from the boundary conditions

$$w_1(x,\infty) = 0. \tag{4.4.37}$$

Assume the stronger condition

$$\lim_{B\to\infty} w_1(x,Bx)B = 0. \tag{4.4.38}$$

Let $B \to \infty$ in (4.4.36). Then

$$\frac{d}{dx}\int_{\phi(x)}^{\infty} w_1(x,y)dy = \int_{\phi(x)}^{\infty} \frac{\partial w_1}{\partial x}(x,y)dy - w_1(x,\phi(x))\frac{d\phi(x)}{dx} \tag{4.4.39}$$

and

$$\int_{\phi(x)}^{\infty} \frac{\partial w_1}{\partial x}(x,y)dy = \frac{d}{dx}\int_{\phi(x)}^{\infty} w_1(x,y)dy + w_1(x,\phi(x))\frac{d\phi(x)}{dx}. \tag{4.4.40}$$

Substitute (4.4.35) and (4.4.40) into (4.4.31) which gives

$$\frac{\partial w_1}{\partial y}(x,\phi(x)) = -\frac{d}{dx}\int_{\phi(x)}^{\infty} w_1(x,y)dy - w_1(x,\phi(x))\frac{d\phi(x)}{dx}. \tag{4.4.41}$$

4.4.1.3 Two fluids

Equations (4.4.27) and (4.4.41) do not separately generate conserved quantities. We do not have a conserved quantity for the upper fluid and a conserved quantity for the lower fluid. There is one conserved quantity for the whole wake consisting of the upper fluid and the lower fluid. Substitute (4.4.27) and (4.4.41) into the interface condition (4.4.10):

$$\frac{\partial w_1}{\partial y}(x,\phi(x)) = \frac{\mu_2}{\mu_1}\frac{\partial w_2}{\partial y}(x,\phi(x)). \tag{4.4.42}$$

Then

$$-\frac{d}{dx}\int_{\phi(x)}^{\infty} w_1(x,y)dy - w_1(x,\phi(x))\frac{d\phi(x)}{dx} = \frac{d}{dx}\left(\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} w_2(x,y)dy\right) - \frac{\rho_2}{\rho_1}w_2(x,\phi(x))\frac{d\phi(x)}{dx}, \tag{4.4.43}$$

and therefore

$$\frac{d}{dx}\left[\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} w_2(x,y)dy + \int_{\phi(x)}^{\infty} w_1(x,y)dy\right] = \left[\frac{\rho_2}{\rho_1}w_2(x,\phi(x)) - w_1(x,\phi(x))\right]\frac{d\phi(x)}{dx}. \tag{4.4.44}$$

But from the interface condition

$$w_2(x,\phi(x)) = w_1(x,\phi(x)). \tag{4.4.45}$$

Thus

$$\frac{d}{dx}\left[\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} w_2(x,y)dy + \int_{\phi(x)}^{\infty} w_1(x,y)dy\right] = \left(\frac{\rho_2}{\rho_1} - 1\right)w_1(x,\phi(x))\frac{d\phi(x)}{dx}. \tag{4.4.46}$$

The term

$$w_1(x,\phi(x))\frac{d\phi(x)}{dx} \tag{4.4.47}$$

is second order in smallness and is therefore neglected. Second order small terms have already been neglected in the derivation of the diffusion equation for the wake. Note that the terms in (4.4.42) are first order in smallness and cannot be neglected. If these terms could be neglected then we would have conserved quantities in the upper and lower wakes separately. By neglecting (4.4.47), equation (4.4.46) reduces to

$$\frac{d}{dx}\left[\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} w_2(x,y)dy + \int_{\phi(x)}^{\infty} w_1(x,y)dy\right] = 0. \tag{4.4.48}$$

Hence

$$\frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} w_2(x,y)dy + \int_{\phi(x)}^{\infty} w_1(x,y)dy = \text{constant independent of } x \tag{4.4.49}$$

and

$$D^* = \frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} w_2(x,y)dy + \int_{\phi(x)}^{\infty} w_1(x,y)dy, \tag{4.4.50}$$

where D^* is the total dimensionless drag on the obstacle. The constant D^* is the conserved quantity for the two-fluid classical wake. In the solution which we derive for the two-fluid classical wake, $\phi(x) = 0$ and the term (4.4.47) is zero. Equation (4.4.48) is then satisfied without the assumption that the term (4.4.47) can be neglected.

4.4.2 Conserved quantity for the two-fluid momentumless wake

The conserved quantity for the momentumless wake behind a self-propelled body is obtained from the conserved vector

$$T^1_{(i)} = \frac{\nu_1}{\nu_i}y^2 w_i - 2xw_i, \qquad T^2_{(i)} = -y^2 w_{iy} + 2yw_i + 2xv_i \tag{4.4.51}$$

and the condition that the total drag on the obstacle is zero:

$$D^* = \frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} w_2(x,y)dy + \int_{\phi(x)}^{\infty} w_1(x,y)dy. \tag{4.4.52}$$

We derive expressions for

$$\int_{-\infty}^{\phi(x)} w_2(x,y)dy \qquad \text{and} \qquad \int_{\phi(x)}^{\infty} w_1(x,y)dy. \tag{4.4.53}$$

4.4.2.1 Lower fluid

The components of the conserved vector are

$$T^1 = \frac{\nu_1}{\nu_2}y^2 w_2 - 2xw_2, \qquad T^2 = -y^2\frac{\partial w_2}{\partial y} + 2yw_2 + 2xv_2. \tag{4.4.54}$$

Now the conservation law can be expressed as

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y} = 0, \tag{4.4.55}$$

therefore

$$\frac{\partial}{\partial x}\left(\frac{\nu_1}{\nu_2}y^2 w_2 - 2xw_2\right) + \frac{\partial}{\partial y}\left(-y^2\frac{\partial w_2}{\partial y} + 2yw_2 + 2xv_2\right) = 0. \tag{4.4.56}$$

However, consider

$$\frac{\partial}{\partial x}(-2xw_2) + \frac{\partial}{\partial y}(2xv_2) = -2w_2 + 2x\bigg(- \frac{\partial w_2}{\partial x} + \frac{\partial v_2}{\partial y}\bigg) = -2w_2 \tag{4.4.57}$$

since from conservation of mass

$$-\frac{\partial w_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0. \tag{4.4.58}$$

Equation (4.4.56) becomes

$$\frac{\partial}{\partial x}\bigg(\frac{\nu_1}{\nu_2}y^2 w_2\bigg) - 2w_2 + \frac{\partial}{\partial y}\big(- y^2\frac{\partial w_2}{\partial y} + 2yw_2\bigg) = 0. \tag{4.4.59}$$

Integrate (4.4.59) with respect to y from $y = -\infty$ to $y = \phi(x)$.

$$\int_{-\infty}^{\phi(x)} \frac{\partial}{\partial x}\bigg(\frac{\nu_1}{\nu_2}y^2 w_2\bigg)dy - 2\int_{-\infty}^{\phi(x)} w_2(x,y)dy + \int_{-\infty}^{\phi(x)} \frac{\partial}{\partial y}\big(- y^2\frac{\partial w_2}{\partial y} + 2yw_2\bigg)dy = 0. \tag{4.4.60}$$

Now

$$\begin{aligned}\int_{-\infty}^{\phi(x)} \frac{\partial}{\partial y}\big(- y^2\frac{\partial w_2}{\partial y} + 2yw_2\bigg)dy &= \bigg[- y^2\frac{\partial w_2}{\partial y} + 2yw_2\bigg]_{-\infty}^{\phi(x)} \\ &= -\phi^2(x)\frac{\partial w_2}{\partial y}(x,\phi(x)) + 2\phi(x)w_2(x,\phi(x)) \\ &\quad + \bigg(y^2\frac{\partial w_2}{\partial y}\bigg)\bigg|_{y=-\infty} + (2yw_2)|_{y=-\infty}.\end{aligned} \tag{4.4.61}$$

But, from the boundary conditions at $y = -\infty$

$$w_2(x,-\infty) = 0, \qquad \frac{\partial w_2}{\partial y}(x,-\infty) = 0. \tag{4.4.62}$$

We assume the stronger conditions

$$yw_2(x,y)|_{y=-\infty} = 0, \qquad y^2\frac{\partial w_2}{\partial y}(x,y)\bigg|_{y=-\infty} = 0 \tag{4.4.63}$$

Then (4.4.61) becomes

$$\int_{-\infty}^{\phi(x)} \frac{\partial}{\partial y}\big(- y^2\frac{\partial w_2}{\partial y} + 2yw_2\bigg)dy = -\phi^2(x)\frac{\partial w_2}{\partial y}(x,\phi(x)) + 2\phi(x)w_2(x,\phi(x)). \tag{4.4.64}$$

Also to rewrite the first term in (4.4.60) use the Theorem of differentiation under the integral sign. Replace the lower limit $-\infty$ by $-Bx$ where $B > 0$ is a constant. Then

$$\frac{d}{dx}\int_{-Bx}^{\phi(x)} y^2 w_2(x,y)dy = \int_{-Bx}^{\phi(x)} y^2 \frac{\partial w_2}{\partial x}(x,y)dy + \phi^2(x)\, w_2(x,\phi(x))\frac{d\phi(x)}{dx}$$

$$- B^2\, x^2\, w_2(x,-Bx)(-B). \tag{4.4.65}$$

But from the boundary conditions

$$w_2(x,-\infty) = 0. \tag{4.4.66}$$

Assume the stronger condition

$$\lim_{B\to\infty} B^3\, w_2(x,-Bx) = 0. \tag{4.4.67}$$

Let $B \to \infty$ in (4.4.65). Then

$$\frac{d}{dx}\int_{-\infty}^{\phi(x)} y^2 w_2(x,y)dy = \int_{-\infty}^{\phi(x)} y^2 \frac{\partial w_2}{\partial x}(x,y)dy + \phi^2(x)\, w_2(x,\phi(x))\frac{d\phi(x)}{dx} \tag{4.4.68}$$

and therefore

$$\int_{-\infty}^{\phi(x)} y^2 \frac{\partial w_2}{\partial x}(x,y)dy = \frac{d}{dx}\int_{-\infty}^{\phi(x)} y^2 w_2(x,y)dy - \phi^2(x)\, w_2(x,\phi(x))\frac{d\phi(x)}{dx}. \tag{4.4.69}$$

Using (4.4.64) and (4.4.69), equation (4.4.60) becomes

$$2\int_{-\infty}^{\phi(x)} w_2(x,y)dy = \frac{\nu_1}{\nu_2}\frac{d}{dx}\int_{-\infty}^{\phi(x)} y^2 w_2(x,y)dy - \frac{\nu_1}{\nu_2}\phi^2(x) w_2(x,\phi(x))\frac{d\phi(x)}{dx}$$

$$- \phi^2(x)\frac{\partial w_2}{\partial y}(x,\phi(x)) + 2\phi(x) w_2(x,\phi(x)). \tag{4.4.70}$$

4.4.2.2 Upper fluid

The components of the conserved vector are

$$T^1 = y^2 w_1 - 2x w_1, \qquad T^2 = -y^2\frac{\partial w_1}{\partial y} + 2y w_1 + 2x v_1. \tag{4.4.71}$$

Now the conservation law can be expressed as

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y} = 0, \tag{4.4.72}$$

and therefore

$$\frac{\partial}{\partial x}(y^2 w_1 - 2x w_1) + \frac{\partial}{\partial y}\bigg(- y^2 \frac{\partial w_1}{\partial y} + 2y w_1 + 2x v_1\bigg) = 0. \tag{4.4.73}$$

Now,

$$\frac{\partial}{\partial x}(-2x w_1) + \frac{\partial}{\partial y}(2x v_1) = -2w_1 + 2x\bigg(- \frac{\partial w_1}{\partial x} + \frac{\partial v_1}{\partial y}\bigg) = -2w_1 \tag{4.4.74}$$

since from conservation of mass

$$-\frac{\partial w_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0. \tag{4.4.75}$$

Therefore, (4.4.73) becomes

$$\frac{\partial}{\partial x}(y^2 w_1) - 2w_1 + \frac{\partial}{\partial y}\bigg(- y^2 \frac{\partial w_1}{\partial y} + 2y w_1\bigg) = 0. \tag{4.4.76}$$

Integrate (4.4.76) with respect to y from $y = \phi(x)$ to $y = +\infty$:

$$\int_{\phi(x)}^{\infty} \frac{\partial}{\partial x}(y^2 w_1)dy - 2\int_{\phi(x)}^{\infty} w_1(x,y)dy + \int_{\phi(x)}^{\infty} \frac{\partial}{\partial y}\bigg(- y^2 \frac{\partial w_1}{\partial y} + 2y w_1\bigg)dy = 0. \tag{4.4.77}$$

Now

$$\begin{aligned}\int_{\phi(x)}^{\infty} \frac{\partial}{\partial y}\bigg(- y^2 \frac{\partial w_1}{\partial y} + 2y w_1\bigg)dy &= \bigg[- y^2 \frac{\partial w_1}{\partial y} + 2y w_1\bigg]_{\phi(x)}^{\infty} \\ &= -y^2 \frac{\partial w_1}{\partial y}(x,y)\bigg|_{y=\infty} + 2y w_1(x,y)|_{y=\infty} \\ &\quad + \phi^2(x)\frac{\partial w_1}{\partial y}(x,\phi(x)) - 2\phi(x) w_1(x,\phi(x)).\end{aligned} \tag{4.4.78}$$

But from the boundary conditions at $y = +\infty$,

$$w_1(x,\infty) = 0, \qquad \frac{\partial w_1}{\partial y}(x,\infty) = 0. \tag{4.4.79}$$

Assume the stronger conditions

$$y\, w_1(x,y)|_{y=\infty} = 0, \qquad y^2 \frac{\partial w_1}{\partial y}(x,y)\bigg|_{y=\infty} = 0. \tag{4.4.80}$$

Then (4.4.78) becomes

$$\int_{\phi(x)}^{\infty} \frac{\partial}{\partial y}\bigg(- y^2 \frac{\partial w_1}{\partial y} + 2y w_1\bigg)dy = \phi^2(x)\frac{\partial w_1}{\partial y}(x,\phi(x)) - 2\phi(x) w_1(x,\phi(x)). \tag{4.4.81}$$

In order to rewrite the first term in (4.4.77) we apply again the Theorem for differentiation under the integral sign. Replace the upper limit ∞ in the integral by Bx where $B > 0$ is a constant. Then

$$\frac{d}{dx}\int_{\phi(x)}^{Bx} y^2 w_1(x,y)dy = \int_{\phi(x)}^{Bx} y^2 \frac{\partial w_1}{\partial x}(x,y)dy + B^2\, x^2\, w_1(x,Bx)\, B - \phi^2(x) w_1(x,\phi(x))\frac{d\phi(x)}{dx}. \tag{4.4.82}$$

But from the boundary conditions at $y = +\infty$

$$w_1(x,\infty) = 0. \tag{4.4.83}$$

We assume the stronger condition

$$\lim_{B\to\infty} B^3\, w_1(x,Bx) = 0. \tag{4.4.84}$$

Let $B \to \infty$ in (4.4.82). Then

$$\frac{d}{dx}\int_{\phi(x)}^{\infty} y^2 w_1(x,y)dy = \int_{\phi(x)}^{\infty} y^2 \frac{\partial w_1}{\partial x}(x,y)dy - \phi^2(x) w_1(x,\phi(x))\frac{d\phi(x)}{dx} \tag{4.4.85}$$

and therefore

$$\int_{\phi(x)}^{\infty} y^2 \frac{\partial w_1}{\partial x}(x,y)dy = \frac{d}{dx}\int_{\phi(x)}^{\infty} y^2 w_1(x,y)dy + \phi^2(x) w_1(x,\phi(x))\frac{d\phi(x)}{dx}. \tag{4.4.86}$$

Using (4.4.81) and (4.4.86), equation (4.4.77) becomes

$$2\int_{\phi(x)}^{\infty} w_1(x,y)dy = \frac{d}{dx}\int_{\phi(x)}^{\infty} y^2 w_1(x,y)dy + \phi^2(x) w_1(x,\phi(x))\frac{d\phi(x)}{dx}$$

$$+ \phi^2(x)\frac{\partial w_1}{\partial y}(x,\phi(x)) - 2\phi(x) w_1(x,\phi(x)). \tag{4.4.87}$$

4.4.2.3 Two fluids

Equations (4.4.70) and (4.4.87) do not separately generate conserved quantities and there is not a conserved quantity for the upper wake and a conserved quantity for the lower wake. This is because the terms

$$\int_{-\infty}^{\phi(x)} w_2(x,y)dy, \qquad \int_{\phi(x)}^{\infty} w_1(x,y)dy, \tag{4.4.88}$$

are only first order in smallness and cannot be neglected. They are eliminated using the property of a self-propelled wake that the total drag is zero:

$$D^* = \frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} w_2(x,y)dy + \int_{\phi(x)}^{\infty} w_1(x,y)dy = 0. \tag{4.4.89}$$

Substitute (4.4.70) and (4.4.87) into (4.4.89)

$$\frac{\rho_2}{\rho_1}\left[\frac{d}{dx}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x)} y^2 w_2(x,y)dy - \frac{\nu_1}{\nu_2}\phi^2(x)w_2(x,\phi(x))\frac{d\phi(x)}{dx} - \phi^2(x)\frac{\partial w_2}{\partial y}(x,\phi(x)) + 2\phi(x)w_2(x,\phi(x))\right]$$

$$+\frac{d}{dx}\int_{\phi(x)}^{\infty} y^2 w_1(x,y)dy + \phi^2(x)w_1(x,\phi(x))\frac{d\phi(x)}{dx} + \phi^2(x)\frac{\partial w_1}{\partial y}(x,\phi(x)) - 2\phi(x)w_1(x,\phi(x)) = 0. \tag{4.4.90}$$

Simplifying (4.4.90) gives

$$\frac{d}{dx}\left[\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x)} y^2 w_2(x,y)dy + \int_{\phi(x)}^{\infty} y^2 w_1(x,y)dy\right] = \frac{\rho_2}{\rho_1}w_2(x,\phi(x))\phi^2(x)\frac{d\phi(x)}{dx}$$

$$+\frac{\rho_2}{\rho_1}\frac{\partial w_2}{\partial y}(x,\phi(x))\phi^2(x) - 2\frac{\rho_2}{\rho_1}w_2(x,\phi(x))\phi(x) - w_1(x,\phi(x))\phi^2(x)\frac{d\phi(x)}{dx}$$

$$-\frac{\partial w_1}{\partial y}(x,\phi(x))\phi^2(x) + 2w_1(x,\phi(x))\phi(x). \tag{4.4.91}$$

Let

$$R(\phi(x),w_1,w_2) = \frac{\rho_2}{\rho_1}w_2(x,\phi(x))\phi^2(x)\frac{d\phi(x)}{dx} + \frac{\rho_2}{\rho_1}\frac{\partial w_2}{\partial y}(x,\phi(x))\phi^2(x) - 2\frac{\rho_2}{\rho_1}w_2(x,\phi(x))\phi(x)$$

$$- w_1(x,\phi(x))\phi^2(x)\frac{d\phi(x)}{dx} - \frac{\partial w_1}{\partial y}(x,\phi(x))\phi^2(x) + 2w_1(x,\phi(x))\phi(x). \tag{4.4.92}$$

The function $R(\phi(x),w_1,w_2)$ depends only on second order terms in smallness. Equation (4.4.91) becomes

$$\frac{d}{dx}\left[\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x)} y^2 w_2(x,y)dy + \int_{\phi(x)}^{\infty} y^2 w_1(x,y)dy\right] = R(\phi(x),w_1,w_2). \tag{4.4.93}$$

If we neglect $R(\phi(x),w_1,w_2)$ then

$$\frac{d}{dx}\left[\frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x)} y^2 w_2(x,y)dy + \int_{\phi(x)}^{\infty} y^2 w_1(x,y)dy\right] = 0. \tag{4.4.94}$$

Hence

$$K^* = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x)} y^2 w_2(x,y)dy + \int_{\phi(x)}^{\infty} y^2 w_1(x,y)dy = \text{constant independent of } x. \quad (4.4.95)$$

The constant K^* is the conserved quantity for the two-fluid momentumless wake. As discussed in Chapter 2 it differs from the conserved quantity derived by Herczynski, Weidman and Burde [1].

4.4.2.4 Simplification of $R(\phi(x), w_1, w_2)$

Consider interface conditions

$$w_1(x,\phi(x)) = w_2(x,\phi(x)), \quad (4.4.96)$$

$$\frac{\partial w_1}{\partial y}(x,\phi(x)) = \frac{\mu_2}{\mu_1}\frac{\partial w_2}{\partial y}(x,\phi(x)). \quad (4.4.97)$$

By using (4.4.96) and (4.4.97), equation (4.4.92) for R can be simplified to

$$R(\phi(x), w_1, w_2) = \left(\frac{\rho_2}{\rho_1} - 1\right) w_1(x,\phi(x))\phi(x)\left(\phi(x)\frac{d\phi(x)}{dx} - 2\right)$$

$$+ \left(\frac{\nu_1}{\nu_2} - 1\right)\frac{\partial w_1}{\partial y}(x,\phi(x))\phi^2(x). \quad (4.4.98)$$

The quantity R will be investigated further in Chapter 5.

4.5 Conclusions

Conservation laws were discussed in this chapter. The multiplier approach developed by Steudel [18] was applied in order to generate the conservation laws for the system of PDEs governing the classical and momentumless two-fluid wakes. The relevant theory was reviewed and a systematic method in which to apply the multiplier approach was outlined. Four conserved vectors in terms of the velocity components were generated. The linear combination of conserved vectors was also derived. The first conserved vector related to the wake of a self-propelled body. The second conserved vector belonged to a new type of wake which can be referred to as the combination

wake. It will not be considered further here. It had been found earlier for a one-fluid wake [17]. The third conserved vector described the classical wake. The final conserved vector was related to the conservation of mass equation. The conserved vectors will be formulated in terms of the stream function in Chapter 5.

Using the conserved vectors the conserved quantities for the classical two-fluid wake and the momentumless two-fluid wake were derived in a systematic way. They agree with the conserved quantities derived in Chapter 2. Although the conserved vectors were derived for the upper wake and the lower wake separately the conserved quantities depend on both the upper and lower wakes. There is not a conserved quantity for the upper wake and a conserved quantity for the lower wake.

It may be cumbersome to find higher order multipliers by hand and could be investigated using software programs. Unfortunately this will not be useful in this dissertation.

Chapter 5

Associated Lie point symmetries and invariant solutions for the two-fluid wake

5.1 Introduction

We consider the classical wake behind a fixed body and the wake behind a self-propelled body. The wake behind a self-propelled body is sometimes referred to as the momentumless wake.

We first consider the classical wake and calculate the Lie point symmetry associated with the conserved vector for the upper wake and the Lie point symmetry associated with the conserved vector for the lower wake and then use the two associated Lie point symmetries to generate the invariant solution for the two-fluid classical wake.

We next consider the momentumless wake and calculate the Lie point symmetry associated with the conserved vector for the upper wake and the Lie point symmetry associated with the conserved vector for the lower wake and the use the two associated Lie point symmetries to generate the invariant solution for the two-fluid momentumless wake.

The problem will be formulated in terms of the stream function for each wake.

5.2 Mathematical formulation in terms of the stream function

We summarise the formulation in terms of the velocity components in Section 2.3 and the stream function in Section 2.5.

5.2.1 Velocity components formulation

PDE:

$$\frac{\partial w_i}{\partial x} = \frac{\nu_i}{\nu_1}\frac{\partial^2 w_i}{\partial y^2}. \tag{5.2.1}$$

Boundary conditions:

$$w_1(x,\infty) = 0, \tag{5.2.2}$$

$$\frac{\partial w_1}{\partial y}(x,\infty) = 0, \tag{5.2.3}$$

$$w_2(x,-\infty) = 0, \tag{5.2.4}$$

$$\frac{\partial w_2}{\partial y}(x,-\infty) = 0. \tag{5.2.5}$$

Interface conditions where the interface is $y = \phi(x)$:

$$w_1(x,\phi(x)) = w_2(x,\phi(x)), \tag{5.2.6}$$

$$v_1(x,\phi(x)) = v_2(x,\phi(x)), \tag{5.2.7}$$

$$\mu_1\frac{\partial w_1}{\partial y}(x,\phi(x)) = \mu_2\frac{\partial w_2}{\partial y}(x,\phi(x)), \tag{5.2.8}$$

$$2\frac{\mu_2}{\mu_1}\frac{1}{Re}\frac{\partial v_2}{\partial y}(x,\phi(x)) - \frac{2}{Re}\frac{\partial v_1}{\partial y}(x,\phi(x)) = P_2(x,\phi(x)) - P_1(x,\phi(x)). \tag{5.2.9}$$

5.2.2 Stream function formulation

PDE:

$$\frac{\partial^2 \psi_i}{\partial x \partial y} = \frac{\nu_i}{\nu_1} \frac{\partial^3 \psi_i}{\partial y^3}. \tag{5.2.10}$$

Boundary conditions:

$$\frac{\partial \psi_1}{\partial y}(x, \infty) = 0, \tag{5.2.11}$$

$$\frac{\partial^2 \psi_1}{\partial y^2}(x, \infty) = 0, \tag{5.2.12}$$

$$\frac{\partial \psi_2}{\partial y}(x, -\infty) = 0, \tag{5.2.13}$$

$$\frac{\partial^2 \psi_2}{\partial y^2}(x, -\infty) = 0. \tag{5.2.14}$$

Interface conditions where the interface is $y = \phi(x)$:

$$\frac{\partial \psi_1}{\partial y}(x, \phi(x)) = \frac{\partial \psi_2}{\partial y}(x, \phi(x)), \tag{5.2.15}$$

$$\frac{\partial \psi_1}{\partial x}(x, \phi(x)) = \frac{\partial \psi_2}{\partial x}(x, \phi(x)), \tag{5.2.16}$$

$$\frac{\partial^2 \psi_1}{\partial y^2}(x, \phi(x)) = \frac{\mu_2}{\mu_1} \frac{\partial^2 \psi_2}{\partial y^2}(x, \phi(x)), \tag{5.2.17}$$

$$2\frac{\mu_2}{\mu_1} \frac{1}{Re} \frac{\partial^2 \psi_2}{\partial x \partial y}(x, \phi(x)) - \frac{2}{Re} \frac{\partial^2 \psi_1}{\partial x \partial y}(x, \phi(x)) = P_2(x, \phi(x)) - P_1(x, \phi(x)), \tag{5.2.18}$$

where

$$w_i(x, y) = \frac{\partial \psi_i}{\partial y}(x, y), \qquad v_i(x, y) = \frac{\partial \psi_i}{\partial x}. \tag{5.2.19}$$

5.3 Associated Lie point symmetries for the classical wake

The conserved vectors for the upper wake and lower wake of the classical two-fluid wake are

$$T^1_{(i)} = w^i, \qquad T^2_{(i)} = -\frac{\nu_i}{\nu_1} w^i_y, \qquad i = 1, 2. \tag{5.3.1}$$

Now,

$$w^i = \frac{\partial \psi^i}{\partial y}. \tag{5.3.2}$$

Hence the conserved vectors are

$$T^1_{(i)} = \frac{\partial \psi^i}{\partial y}, \qquad T^2_{(i)} = -\frac{\nu_i}{\nu_1} \psi^i_{yy}, \qquad i = 1, 2. \tag{5.3.3}$$

The Lie point symmetry

$$X_{(i)} = \xi^1_i(x, y, \psi^i)\frac{\partial}{\partial x} + \xi^2_i(x, y, \psi^i)\frac{\partial}{\partial y} + \eta_i(x, y, \psi^i)\frac{\partial}{\partial \psi^i} \tag{5.3.4}$$

is associated with the conserved vector (5.3.3) provided [14] [15]

$$X_{(i)}(T^s_{(i)}) + T^s_{(i)} D_k(\xi^k_i) - T^k_{(i)} D_k(\xi^s_i) = 0, \qquad s = 1, 2, \tag{5.3.5}$$

where

$$D_1 = D_x = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx}\frac{\partial}{\partial \psi_x} + \psi_{yx}\frac{\partial}{\partial \psi_y} + ..., \tag{5.3.6}$$

$$D_2 = D_y = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{xy}\frac{\partial}{\partial \psi_x} + \psi_{yy}\frac{\partial}{\partial \psi_y} + ... \tag{5.3.7}$$

By expanding (5.3.5) we obtain the two components

$$s = 1: \qquad X_{(i)}(T^1_{(i)}) + T^1_{(i)} D_2(\xi^2_i) - T^2_{(i)} D_2(\xi^1_i) = 0, \tag{5.3.8}$$

$$s = 2: \qquad X_{(i)}(T^2_{(i)}) + T^2_{(i)} D_1(\xi^1_i) - T^1_{(i)} D_1(\xi^2_i) = 0. \tag{5.3.9}$$

In order to keep the notation simple, in the following calculations the index (i) will be suppressed in all quantities except in the ratio $\frac{\nu_i}{\nu_1}$ and in the interface conditions. The results apply to both the upper and lower wakes and the index (i) or 1 and 2 can be inserted when necessary.

5.3.1 Case 1: Equation (5.3.8)

$$X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0. \tag{5.3.10}$$

Now

$$T^1 = \psi_y \tag{5.3.11}$$

and

$$\begin{aligned} X(T^1) &= X(\psi_y) \\ &= \left(\xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial \psi} + \zeta_2 \frac{\partial}{\partial \psi_y}\right)\psi_y \\ &= \zeta_2. \end{aligned} \tag{5.3.12}$$

But,

$$\begin{aligned} \zeta_s &= D_s(\eta) - \psi_k D_s(\xi^k) \\ &= D_s(\eta) - \psi_1 D_s(\xi^1) - \psi_2 D_s(\xi^2). \end{aligned} \tag{5.3.13}$$

Hence

$$\begin{aligned} \zeta_2 &= D_2(\eta) - \psi_1 D_2(\xi^1) - \psi_2 D_2(\xi^2) \\ &= D_y\eta(x,y,\psi) - \psi_x D_y(\xi^1(x,y,\psi)) - \psi_y D_y(\xi^2(x,y,\psi)) \\ &= \frac{\partial \eta}{\partial y} + \psi_y \frac{\partial \eta}{\partial \psi} - \psi_x \frac{\partial \xi^1}{\partial y} - \psi_x \psi_y \frac{\partial \xi^1}{\partial \psi} - \psi_y \frac{\partial \xi^2}{\partial y} - (\psi_y)^2 \frac{\partial \xi^2}{\partial \psi} \end{aligned} \tag{5.3.14}$$

and therefore

$$\begin{aligned} X(T^1) &= \zeta_2 \\ &= \frac{\partial \eta}{\partial y} + \psi_y \frac{\partial \eta}{\partial \psi} - \psi_x \frac{\partial \xi^1}{\partial y} - \psi_x \psi_y \frac{\partial \xi^1}{\partial \psi} - \psi_y \frac{\partial \xi^2}{\partial y} - (\psi_y)^2 \frac{\partial \xi^2}{\partial \psi}. \end{aligned} \tag{5.3.15}$$

Also using (5.3.3)

$$T^1 D_2(\xi^2) = \psi_y \left(\frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} \right) \xi^2(x, y, \psi)$$

$$= \psi_y \frac{\partial \xi^2}{\partial y} + (\psi_y)^2 \frac{\partial \xi^2}{\partial \psi} \tag{5.3.16}$$

and

$$T^2 D_2(\xi^1) = -\frac{\nu_i}{\nu_1} \psi_{yy} \left(\frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} \right) \xi^1(x, y, \psi)$$

$$= -\frac{\nu_i}{\nu_1} \frac{\partial \xi^1}{\partial y} \psi_{yy} - \frac{\nu_i}{\nu_1} \psi_y \psi_{yy} \frac{\partial \xi^1}{\partial \psi}. \tag{5.3.17}$$

Hence (5.3.10) becomes

$$\frac{\partial \eta}{\partial y} + \psi_y \frac{\partial \eta}{\partial \psi} - \psi_x \frac{\partial \xi^1}{\partial y} - \psi_x \psi_y \frac{\partial \xi^1}{\partial \psi} - \psi_y \frac{\partial \xi^2}{\partial y} - (\psi_y)^2 \frac{\partial \xi^2}{\partial \psi} +$$

$$\psi_y \frac{\partial \xi^2}{\partial y} + (\psi_y)^2 \frac{\partial \xi^2}{\partial \psi} - \frac{\nu_i}{\nu_1} \frac{\partial \xi^1}{\partial y} \psi_{yy} - \frac{\nu_i}{\nu_1} \psi_y \psi_{yy} \frac{\partial \xi^1}{\partial \psi} = 0. \tag{5.3.18}$$

Separate according to powers and products of the independent partial derivatives of ψ.

$$\psi_y \psi_{yy} : \qquad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{5.3.19}$$

$$\psi_{yy} : \qquad \frac{\partial \xi^1}{\partial y} = 0, \tag{5.3.20}$$

$$\psi_x \psi_y : \qquad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{5.3.21}$$

$$\psi_y : \qquad \frac{\partial \eta}{\partial \psi} = 0, \tag{5.3.22}$$

$$\psi_x : \qquad \frac{\partial \xi^1}{\partial y} = 0. \tag{5.3.23}$$

The remainder is

$$\frac{\partial \eta}{\partial y} = 0. \tag{5.3.24}$$

Hence

$$\xi^1 = \xi^1(x), \qquad \xi^2 = \xi^2(x, y, \psi), \qquad \eta = \eta(x). \tag{5.3.25}$$

5.3.2 Case 2: Equation (5.3.9)

Consider next the second component, (5.3.9):

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0. \tag{5.3.26}$$

Now

$$T^1 = \psi_y, \qquad T^2 = -\frac{\nu_i}{\nu_1}\psi_{yy} \tag{5.3.27}$$

and

$$X(T^2) = -\frac{\nu_i}{\nu_1}X(\psi_{yy}) = -\frac{\nu_i}{\nu_1}\zeta_{22}. \tag{5.3.28}$$

But

$$\zeta_{ij} = D_i(\zeta_j) - \psi_{ik}D_j(\xi^k) \tag{5.3.29}$$

and therefore

$$\zeta_{22} = D_2(\zeta_2) - \psi_{2k}D_2(\xi^k) = D_2(\zeta_2) - \psi_{21}D_2(\xi^1) - \psi_{22}D_2(\xi^2). \tag{5.3.30}$$

Now from (5.3.14)

$$\zeta_2 = \frac{\partial \eta}{\partial y} + \psi_y\frac{\partial \eta}{\partial \psi} - \psi_x\frac{\partial \xi^1}{\partial y} - \psi_x\psi_y\frac{\partial \xi^1}{\partial \psi} - \psi_y\frac{\partial \xi^2}{\partial y} - \psi_y^2\frac{\partial \xi^2}{\partial \psi} \tag{5.3.31}$$

and using the results (5.3.25), ζ_2 simplifies to

$$\zeta_2 = -\frac{\partial \xi^2}{\partial y}\psi_y - \frac{\partial \xi^2}{\partial \psi}\psi_y^2. \tag{5.3.32}$$

Thus

$$D_2(\zeta_2) = \left(\frac{\partial}{\partial y} + \psi_y\frac{\partial}{\partial \psi} + \psi_{yy}\frac{\partial}{\partial \psi_y}\right)\left[-\frac{\partial \xi^2}{\partial y}\psi_y - \frac{\partial \xi^2}{\partial \psi}\psi_y^2\right]$$

$$= -\frac{\partial^2 \xi^2}{\partial y^2}\psi_y - 2\frac{\partial^2 \xi^2}{\partial y \partial \psi}\psi_y^2 - \frac{\partial^2 \xi^2}{\partial \psi^2}\psi_y^3 - \frac{\partial \xi^2}{\partial y}\psi_{yy} - 2\frac{\partial \xi^2}{\partial \psi}\psi_y\psi_{yy}. \tag{5.3.33}$$

Also

$$\psi_{21}D_2(\xi^1) = \psi_{yx}D_y(\xi^1(x)) = 0, \tag{5.3.34}$$

and

$$\psi_{22}D_2(\xi^2) = \psi_{yy}D_y(\xi^2(x,y,\psi)) = \frac{\partial\xi^2}{\partial y}\psi_{yy} + \frac{\partial\xi^2}{\partial\psi}\psi_y\psi_{yy}. \tag{5.3.35}$$

Substitute (5.3.33), (5.3.34) and (5.3.35) into (5.3.30):

$$\zeta_{22} = -\frac{\partial^2\xi^2}{\partial y^2}\psi_y - 2\frac{\partial^2\xi^2}{\partial y\partial\psi}\psi_y^2 - \frac{\partial^2\xi^2}{\partial\psi^2}\psi_y^3 - 2\frac{\partial\xi^2}{\partial y}\psi_{yy} - 3\frac{\partial\xi^2}{\partial\psi}\psi_y\psi_{yy}. \tag{5.3.36}$$

Hence from (5.3.28)

$$X(T^2) = \frac{\nu_i}{\nu_1}\left[\frac{\partial^2\xi^2}{\partial y^2}\psi_y + 2\frac{\partial^2\xi^2}{\partial y\partial\psi}\psi_y^2 + \frac{\partial^2\xi^2}{\partial\psi^2}\psi_y^3 + 2\frac{\partial\xi^2}{\partial y}\psi_{yy} + 3\frac{\partial\xi^2}{\partial\psi}\psi_y\psi_{yy}\right]. \tag{5.3.37}$$

Also

$$T^2D_1(\xi^1) = -\frac{\nu_i}{\nu_1}\frac{d\xi^1}{dx}\psi_{yy}, \tag{5.3.38}$$

and

$$T^1D_1(\xi^2) = \psi_y\left(\frac{\partial}{\partial x} + \psi_x\frac{\partial}{\partial\psi}\right)\xi^2(x,y,\psi) = \frac{\partial\xi^2}{\partial x}\psi_y + \frac{\partial\xi^2}{\partial\psi}\psi_x\psi_y. \tag{5.3.39}$$

Substitute (5.3.37), (5.3.38) and (5.3.39) into (5.3.26). This gives

$$\frac{\nu_i}{\nu_1}\left[\frac{\partial^2\xi^2}{\partial y^2}\psi_y + 2\frac{\partial^2\xi^2}{\partial y\partial\psi}\psi_y^2 + \frac{\partial^2\xi^2}{\partial\psi^2}\psi_y^3 + 2\frac{\partial\xi^2}{\partial y}\psi_{yy} + 3\frac{\partial\xi^2}{\partial\psi}\psi_y\psi_{yy}\right] - \frac{\nu_i}{\nu_1}\frac{d\xi^1}{dx}\psi_{yy} - \frac{\partial\xi^2}{\partial x}\psi_y$$

$$- \frac{\partial\xi^2}{\partial\psi}\psi_x\psi_y = 0. \tag{5.3.40}$$

Separate according to the powers and products of the independent partial derivatives of ψ.

$$\psi_y\psi_{yy}: \qquad 3\frac{\nu_i}{\nu_1}\frac{\partial\xi^2}{\partial\psi} = 0, \tag{5.3.41}$$

$$\psi_{yy}: \qquad \frac{\nu_i}{\nu_1}\left[2\frac{\partial\xi^2}{\partial y} - \frac{d\xi^1}{dx}\right] = 0, \tag{5.3.42}$$

$$\psi_x\psi_y: \qquad \frac{\partial\xi^2}{\partial\psi} = 0, \tag{5.3.43}$$

$$\psi_y^3: \qquad \frac{\nu_i}{\nu_1}\frac{\partial^2\xi^2}{\partial\psi^2} = 0, \tag{5.3.44}$$

$$\psi_y^2: \qquad 2\frac{\nu_i}{\nu_1}\frac{\partial^2\xi^2}{\partial y\partial\psi} = 0, \tag{5.3.45}$$

$$\psi_y: \qquad \frac{\nu_i}{\nu_1}\frac{\partial^2\xi^2}{\partial y^2} - \frac{\partial\xi^2}{\partial x} = 0. \tag{5.3.46}$$

Now $\nu_i \neq 0$. From (5.3.41) and (5.3.25)

$$\xi^2 = \xi^2(x, y). \tag{5.3.47}$$

Equations (5.3.43), (5.3.44) and (5.3.45) are identically satisfied. The system (5.3.41) to (5.3.46) reduces to

$$2\frac{\partial\xi^2}{\partial y}(x, y) = \frac{d\xi^1}{dx}(x), \tag{5.3.48}$$

$$\frac{\nu_i}{\nu_1}\frac{\partial^2\xi^2}{\partial y^2} = \frac{\partial\xi^2}{\partial x}(x, y). \tag{5.3.49}$$

Differentiate (5.3.48) with respect to y,

$$\frac{\partial^2\xi^2}{\partial y^2}(x, y) = 0. \tag{5.3.50}$$

Using (5.3.50), equation (5.3.49) becomes

$$\frac{\partial\xi^2}{\partial x}(x, y) = 0 \tag{5.3.51}$$

and therefore

$$\xi^2 = \xi^2(y). \tag{5.3.52}$$

Equation (5.3.48) becomes

$$2\frac{d\xi^2}{dy}(y) = \frac{d\xi^1}{dx}(x). \tag{5.3.53}$$

By separation of variables, each side must be a constant, $2c_1$. Thus

$$\frac{d\xi^1}{dx}(x) = 2c_1, \qquad \frac{d\xi^2}{dy}(y) = c_1 \tag{5.3.54}$$

and

$$\xi^1(x) = 2c_1x + c_2, \qquad \xi^2(y) = c_1y + c_3. \tag{5.3.55}$$

Since from (5.3.25),

$$\eta = \eta(x) \tag{5.3.56}$$

it follows that the Lie point symmetry (5.3.4) is

$$X = (2c_1x + c_2)\frac{\partial}{\partial x} + (c_1y + c_3)\frac{\partial}{\partial y} + \eta(x)\frac{\partial}{\partial \psi}. \tag{5.3.57}$$

We consider the general case for which $c_1 \neq 0$. Divide (5.3.57) by c_1 and define

$$X^* = \frac{1}{c_1}X, \qquad c_2^* = \frac{c_2}{c_1}, \qquad c_3^* = \frac{c_3}{c_1}, \qquad \eta^*(x) = \frac{1}{c_1}\eta(x). \tag{5.3.58}$$

The associated Lie point symmetry (5.3.57) becomes

$$X^* = (2x + c_2^*)\frac{\partial}{\partial x} + (y + c_3^*)\frac{\partial}{\partial y} + \eta^*(x)\frac{\partial}{\partial \psi}. \tag{5.3.59}$$

Suppressing the star gives

$$X = (2x + c_2)\frac{\partial}{\partial x} + (y + c_3)\frac{\partial}{\partial y} + \eta(x)\frac{\partial}{\partial \psi}, \tag{5.3.60}$$

where $\eta(x)$ is an arbitrary function.

5.4 Invariant solution for classical two-fluid wake

Now $\psi = \Psi(x, y)$ is an invariant solution of the PDE

$$\frac{\partial^2 \psi_i}{\partial x \partial y} = \frac{\nu_i}{\nu_1} \frac{\partial^3 \psi_i}{\partial y^3} \tag{5.4.1}$$

generated by the Lie point symmetry X provided

$$X(\psi - \Psi(x, y))|_{\psi=\Psi(x,y)} = 0. \tag{5.4.2}$$

Thus

$$(2x + c_2)\frac{\partial \Psi}{\partial x} + (y + c_3)\frac{\partial \Psi}{\partial y} = \eta(x). \tag{5.4.3}$$

The differential equations of the characteristic curves are

$$\frac{dx}{(2x + c_2)} = \frac{dy}{(y + c_3)} = \frac{d\Psi}{\eta(x)}. \tag{5.4.4}$$

The first pair of terms give

$$\frac{dx}{2\left(x + \frac{c_2}{2}\right)} = \frac{dy}{y + c_3} \tag{5.4.5}$$

and integrating we obtain

$$\frac{1}{2} \ln \left(x + \frac{c_2}{2}\right) = \ln (y + c_3) + \text{constant}. \tag{5.4.6}$$

Hence

$$\frac{y + c_3}{\left(x + \frac{c_2}{2}\right)^{\frac{1}{2}}} = k_1, \tag{5.4.7}$$

where k_1 is a constant. The first and last terms in (5.4.4) give

$$\frac{dx}{2x + c_2} = \frac{d\Psi}{\eta(x)} \tag{5.4.8}$$

and therefore

$$\Psi(x, y) - \int^x \frac{\eta(x)dx}{(2x + c_2)} = k_2, \tag{5.4.9}$$

where k_2 is a constant. The general solution of the PDE (5.4.3) is

$$k_2 = F(k_1), \tag{5.4.10}$$

where F is an arbitrary function. Hence

$$\Psi(x, y) = F(k_1) + \int^x \frac{\eta(x)dx}{(2x + c_2)}. \tag{5.4.11}$$

But $\psi = \Psi(x, y)$. Hence

$$\psi(x, y) = F(\xi) + G(x), \tag{5.4.12}$$

where

$$\xi = \frac{y + c_3}{\left(x + \frac{c_2}{2}\right)^{\frac{1}{2}}}, \tag{5.4.13}$$

$$G(x) = \int^x \frac{\eta(x)dx}{(2x + c_2)}, \tag{5.4.14}$$

and $\eta(x)$ is an arbitrary function. Hence the function $G(x)$ is also arbitrary.

5.4.1 The constants c_2 and c_3

The coordinate in the invariant solution is

$$\xi_1 = \frac{y + c_3}{\left(x + \frac{1}{2}c_2\right)^{\frac{1}{2}}}. \tag{5.4.15}$$

We choose the origin of the coordinates in the upper and lower wakes to be at $y = 0$. Hence $c_3 = 0$. Also there is a singularity in ξ at

$$x = -\frac{1}{2}c_2. \tag{5.4.16}$$

But a singularity, if it exists, will be at the obstacle, $x = 0$. Hence $c_2 = 0$ and

$$\xi = \frac{y}{x^{\frac{1}{2}}}. \tag{5.4.17}$$

The similarity coordinate is the same in the upper and lower wakes. The Lie point symmetry (5.3.60) reduces to

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \eta(x)\frac{\partial}{\partial \psi}, \tag{5.4.18}$$

where $\eta(x)$ is an arbitrary function.

5.4.2 Conserved quantity

The conserved quantity for the two-fluid classical wake is

$$D^* = \frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} w_2(x,y)dy + \int_{\phi(x)}^{\infty} w_1(x,y)dy, \tag{5.4.19}$$

where D^* is the total drag on the obstacle. In Chapter 3 the conserved quantity was used to determine the unknown parameter ω_i in the invariant solution

$$\psi_i(x,y) = x^{\omega_i}F_i(\xi), \qquad \xi = \frac{y}{x^{\frac{1}{2}}}. \tag{5.4.20}$$

Since the Lie point symmetry was associated with the conserved vector, the unknown parameter has been determined and the invariant solution does not contain an unknown parameter. The conserved quantity is used to obtain the equation of the interface.

The conserved quantity is first expressed in terms of the invariant solution. Now

$$\psi_i(x,y) = F_i(\xi) + G_i(x), \qquad \xi = \frac{y}{x^{\frac{1}{2}}} \tag{5.4.21}$$

and

$$w_i = \frac{\partial\psi_i}{\partial y} = \frac{dF_i}{d\xi}\frac{\partial\xi}{\partial y} = \frac{1}{x^{\frac{1}{2}}}\frac{dF_i}{d\xi}, \qquad dy = x^{\frac{1}{2}}d\xi. \tag{5.4.22}$$

Hence from (5.4.19)

$$D^* = \frac{\rho_2}{\rho_1}\int_{-\infty}^{\frac{\phi(x)}{x^{\frac{1}{2}}}} \frac{dF_2}{d\xi}d\xi + \int_{\frac{\phi(x)}{x^{\frac{1}{2}}}}^{\infty} \frac{dF_1}{d\xi}d\xi = \text{constant independent of } x. \tag{5.4.23}$$

For D^* to be a constant independent of x,

$$\frac{\phi(x)}{x^{\frac{1}{2}}} = k, \qquad k = \text{constant}. \tag{5.4.24}$$

Then the equation of the interface is

$$\phi(x) = kx^{\frac{1}{2}}. \tag{5.4.25}$$

The conserved quantity becomes

$$D^* = \frac{\rho_2}{\rho_1}\int_{-\infty}^{k} \frac{dF_2}{d\xi}d\xi + \int_{k}^{\infty} \frac{dF_1}{d\xi}d\xi. \tag{5.4.26}$$

5.4.3 Partial differential equation for the velocity deficit

We next express the partial differential equation for the velocity deficit in terms of the invariant solution:

$$\frac{\partial^2\psi_i}{\partial x\partial y}=\frac{\nu_i}{\nu_1}\frac{\partial^3\psi_i}{\partial y^3}. \tag{5.4.27}$$

From (5.4.21)

$$\frac{\partial\xi}{\partial x}=-\frac{1}{2}\frac{\xi}{x},\qquad \frac{\partial\xi}{\partial y}=\frac{1}{x^{\frac{1}{2}}}, \tag{5.4.28}$$

$$\frac{\partial\psi}{\partial x}=-\frac{1}{2x}\xi\frac{dF}{d\xi}+\frac{dG}{dx},\qquad \frac{\partial^2\psi}{\partial y\partial x}=-\frac{1}{2x^{\frac{3}{2}}}\left(\frac{dF}{d\xi}+\xi\frac{d^2F}{d\xi^2}\right), \tag{5.4.29}$$

$$\frac{\partial\psi}{\partial y}=\frac{1}{x^{\frac{1}{2}}}\frac{dF}{d\xi},\qquad \frac{\partial^2\psi}{\partial y^2}=\frac{1}{x}\frac{d^2F}{d\xi^2},\qquad \frac{\partial^3\psi}{\partial y^3}=\frac{1}{x^{\frac{3}{2}}}\frac{d^3F}{d\xi^3}. \tag{5.4.30}$$

Therefore (5.4.27) reduces to the third order ordinary differential equation

$$2\frac{\nu_i}{\nu_1}\frac{d^3F}{d\xi^3}+\frac{d}{d\xi}\left(\xi\frac{dF}{d\xi}\right)=0. \tag{5.4.31}$$

Equation (5.4.31) is an example of the double reduction theorem [46]. Since the general form of the invariant solution was determined by the conserved quantity the ODE (5.4.31) can be integrated at least once.

The ODE (5.4.31) is independent of the arbitrary function $G(x)$ in the stream function (5.4.21).

5.4.4 Boundary conditions

Consider not the boundary conditions expressed in terms of the invariant solution (5.4.21). We have

$$\frac{\partial\psi}{\partial y}=\frac{1}{x^{\frac{1}{2}}}\frac{dF}{d\xi},\qquad \frac{\partial^2\psi}{\partial y^2}=\frac{1}{x}\frac{d^2F}{d\xi^2}. \tag{5.4.32}$$

Then the boundary conditions (5.2.11) to (5.2.14) become

$$\frac{dF_1}{d\xi}(\infty) = 0, \tag{5.4.33}$$

$$\frac{d^2F_1}{d\xi^2}(\infty) = 0, \tag{5.4.34}$$

$$\frac{dF_2}{d\xi}(-\infty) = 0, \tag{5.4.35}$$

$$\frac{d^2F_2}{d\xi^2}(-\infty) = 0. \tag{5.4.36}$$

5.4.5 Interface conditions

We next express the interface conditions (5.2.15) to (5.2.18) in terms of the invariant solution. The equation of the interface is

$$y = \phi(x) = kx^{\frac{1}{2}} \tag{5.4.37}$$

and on the interface

$$\xi = \frac{y}{x^{\frac{1}{2}}} = \frac{\phi(x)}{x^{\frac{1}{2}}} = \frac{kx^{\frac{1}{2}}}{x^{\frac{1}{2}}} = k. \tag{5.4.38}$$

The results (5.4.21) and (5.4.28) to (5.4.30) are used to rewrite the interface conditions in terms of the invariant solution.

Tangential velocity continuous (5.2.15):

$$\frac{\partial\psi_1}{\partial y}(x,\phi(x)) = \frac{\partial\psi_2}{\partial y}(x,\phi(x)), \tag{5.4.39}$$

$$\frac{dF_1}{d\xi}(k) = \frac{dF_2}{d\xi}(k). \tag{5.4.40}$$

Normal velocity continuous (5.2.16):

$$\frac{\partial\psi_1}{\partial x}(x,\phi(x)) = \frac{\partial\psi_2}{\partial x}(x,\phi(x)),$$

$$-\frac{1}{2x}k\frac{dF_1}{d\xi}(k) + \frac{dG_1}{dx} = -\frac{1}{2x}k\frac{dF_2}{d\xi}(k) + \frac{dG_2}{dx}. \tag{5.4.41}$$

Using (5.4.40), equation (5.4.41) becomes

$$\frac{dG_1}{dx} = \frac{dG_2}{dx}. \tag{5.4.42}$$

Tangential stress continuous (5.2.17):

$$\frac{\partial^2\psi_1}{\partial y^2}(x,\phi(x)) = \frac{\mu_2}{\mu_1}\frac{\partial^2\psi_2}{\partial y^2}(x,\phi(x)), \tag{5.4.43}$$

$$\frac{d^2F_1}{d\xi^2}(k) = \frac{\mu_2}{\mu_1}\frac{d^2F_2}{d\xi^2}(k). \tag{5.4.44}$$

Normal stress continuous (5.2.18):

$$2\frac{\mu_2}{\mu_1}\frac{1}{Re}\frac{\partial^2\psi_2}{\partial x\partial y}(x,\phi(x)) - \frac{2}{Re}\frac{\partial^2\psi_1}{\partial x\partial y}(x,\phi(x)) = P_2(x,\phi(x)) - P_1(x,\phi(x)), \tag{5.4.45}$$

$$\frac{1}{Re}\left[-\frac{\mu_2}{\mu_1}\frac{dF_2}{d\xi}(k) - k\frac{\mu_2}{\mu_1}\frac{d^2F_2}{d\xi^2}(k)\right]x^{-\frac{3}{2}} + \frac{1}{Re}\left[\frac{dF_1}{d\xi}(k) + k\frac{d^2F_1}{d\xi^2}(k)\right]x^{-\frac{3}{2}} = P_2(x,\phi(x)) - P_1(x,\phi(x)). \tag{5.4.46}$$

However, using (5.4.40) and (5.4.44), equation (5.4.46) becomes

$$\frac{1}{Re}\left[-\frac{\mu_2}{\mu_1}\frac{dF_1}{d\xi}(k) - k\frac{d^2F_1}{d\xi^2}(k)\right]x^{-\frac{3}{2}} + \frac{1}{Re}\left[\frac{dF_1}{d\xi}(k) + k\frac{d^2F_1}{d\xi^2}(k)\right]x^{-\frac{3}{2}} = P_2(x,\phi(x)) - P_1(x,\phi(x)). \tag{5.4.47}$$

Hence

$$\frac{1}{Re}\left(1-\frac{\mu_2}{\mu_1}\right)\frac{dF_1}{d\xi}(k)x^{-\frac{3}{2}} = P_2(x,\phi(x)) - P_1(x,\phi(x)). \tag{5.4.48}$$

5.4.6 Velocity components

The velocity components become:

$$w_i(x,y) = \frac{\partial\psi_i}{\partial y} = x^{-\frac{1}{2}}\frac{dF_i}{d\xi}, \tag{5.4.49}$$

$$v_i(x,y) = \frac{\partial\psi_i}{\partial x} = -\frac{1}{2x}\xi\frac{dF_i}{d\xi} + \frac{dG_i}{dx}, \tag{5.4.50}$$

where from (5.4.42)

$$\frac{dG_1}{dx} = \frac{dG_2}{dx}, \tag{5.4.51}$$

and

$$\frac{\partial w_i}{\partial y}(x,y) = \frac{\partial^2\psi_i}{\partial y^2} = \frac{1}{x}\frac{d^2F_i}{d\xi^2}. \tag{5.4.52}$$

5.4.7 Lie point symmetry

The Lie point symmetry (5.4.18) is

$$X_i = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \eta_i(x)\frac{\partial}{\partial \psi} \tag{5.4.53}$$

where from (5.4.14) with $c_2 = 0$,

$$G_i(x) = \frac{1}{2}\int^x \frac{1}{x}\eta_i(x)dx. \tag{5.4.54}$$

Thus

$$\frac{dG_1}{dx} = \frac{1}{2x}\eta_1(x), \qquad \frac{dG_2}{dx} = \frac{1}{2x}\eta_2(x), \tag{5.4.55}$$

and it follows from (5.4.51) that

$$\eta_1(x) = \eta_2(x) = \eta(x). \tag{5.4.56}$$

The Lie point symmetry is the same in both wakes:

$$X_1 = X_2 = X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \eta(x)\frac{\partial}{\partial \psi}. \tag{5.4.57}$$

Also from (5.4.51) and integrating

$$G_1(x) = G_2(x) + G_0, \qquad G_0 = \text{constant}. \tag{5.4.58}$$

But

$$\psi_i(x, y) = F_i(\xi) + G_i(x). \tag{5.4.59}$$

Since a stream function is determined up to an arbitrary additive constant we can take $G_0 = 0$ and then

$$G_1(x) = G_2(x) = G(x) \tag{5.4.60}$$

and

$$\psi_i(x, y) = F_i(\xi) + G(x). \tag{5.4.61}$$

5.5 Solution of the ODE for the classical two-fluid wake

The solution will be outlined for comparison with the similarity solution presented in Chapter 3, Section 3.2.

Return to the ODE

$$2\frac{\nu_i}{\nu_1}\frac{d^3F_i}{d\xi^3} + \frac{d}{d\xi}\left(\xi\frac{dF_i}{d\xi}\right) = 0. \tag{5.5.1}$$

Integrating with respect to ξ gives

$$2\frac{\nu_i}{\nu_1}\frac{d^2F_i}{d\xi^2} + \xi\frac{dF_i}{d\xi} = A_i, \tag{5.5.2}$$

where $A_i, i = 1, 2$, are constants. At $i = 1$, the following boundary conditions must be satisfied,

$$\frac{d^2F_1}{d\xi^2}(\infty) = 0, \qquad \frac{dF_1}{d\xi}(\infty) = 0 \tag{5.5.3}$$

and therefore

$$A_1 = 0. \tag{5.5.4}$$

Similarly at $i = 2$, the following boundary conditions are imposed,

$$\frac{d^2F_2}{d\xi^2}(-\infty) = 0, \qquad \frac{dF_2}{d\xi}(-\infty) = 0 \tag{5.5.5}$$

and therefore

$$A_2 = 0. \tag{5.5.6}$$

Thus

$$2\frac{\nu_i}{\nu_1}\frac{d^2F_i}{d\xi^2} + \xi\frac{dF_i}{d\xi} = 0. \tag{5.5.7}$$

Let

$$W_i = \frac{dF_i}{d\xi}, \tag{5.5.8}$$

then the ODE becomes

$$2\frac{\nu_i}{\nu_1}\frac{dW_i}{d\xi} + \xi W_i = 0. \tag{5.5.9}$$

Separation of variables gives

$$\frac{dW_i}{W_i} = -\frac{\nu_1}{2\nu_i}\xi d\xi \tag{5.5.10}$$

and solving (5.5.10), we obtain

$$W_i(\xi) = B_i \exp\bigg(-\frac{\nu_1}{4\nu_i}\xi^2\bigg), \tag{5.5.11}$$

where B_i, $i = 1, 2$, are constants. Hence

$$\frac{dF_i}{d\xi} = B_i \exp\bigg(-\frac{\nu_1}{4\nu_i}\xi^2\bigg). \tag{5.5.12}$$

Since $w_i(x, y)$ and $v_i(x, y)$ given by (5.4.49) and (5.4.50) depend on $\frac{dF_i}{d\xi}$ and are independent of $F_i(\xi)$, it is not necessary to integrate further and obtain $F_i(\xi)$.

5.5.1 Interface conditions

Impose on (5.5.12) the interface condition (5.4.40):

$$\frac{dF_1}{d\xi}(k) = \frac{dF_2}{d\xi}(k), \tag{5.5.13}$$

Condition (5.5.13) becomes

$$B_1 \exp\bigg(-\frac{1}{4}k^2\bigg) = B_2 \exp\bigg(-\frac{\nu_1}{4\nu_2}k^2\bigg). \tag{5.5.14}$$

Impose next the interface condition (5.4.44):

$$\frac{d^2F_1}{d\xi^2}(k) = \frac{\mu_2}{\mu_1}\frac{d^2F_2}{d\xi^2}(k). \tag{5.5.15}$$

Now, from (5.5.12)

$$\frac{d^2F_i}{d\xi^2} = -\frac{1}{2}\frac{\nu_1}{\nu_i}B_i\xi \exp\bigg(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2\bigg). \tag{5.5.16}$$

Then (5.5.15) becomes

$$B_1 k \exp\bigg(-\frac{1}{4}k^2\bigg) = \frac{\rho_2}{\rho_1}B_2 k \exp\bigg(-\frac{1}{4}\frac{\nu_1}{\nu_2}k^2\bigg). \tag{5.5.17}$$

Replace B_2 in terms of B_1 using interface condition (5.5.14). Simplifying gives

$$k\bigg(1 - \frac{\rho_2}{\rho_1}\bigg)B_1 \exp\bigg(-\frac{1}{4}k^2\bigg) = 0. \tag{5.5.18}$$

Now we are considering the two fluids with

$$\rho_1 \neq \rho_2. \tag{5.5.19}$$

Also

$$B_1 \neq 0 \tag{5.5.20}$$

because if $B_1 = 0$ then from (5.5.14), $B_2 = 0$ and therefore

$$w_i(x, y) = x^{-\frac{1}{2}} \frac{dF_i}{d\xi} = 0. \tag{5.5.21}$$

Thus

$$k = 0. \tag{5.5.22}$$

5.5.2 Consequences of the result $k = 0$

5.5.2.1 Interface is the x-axis

Since

$$\phi(x) = kx^{\frac{1}{2}} = 0 \tag{5.5.23}$$

it follows that the interface is the x-axis. Also the turning point of the velocity deficit is on the interface. For

$$\frac{\partial w_i}{\partial y}(x, y) = \frac{1}{x} \frac{d^2 F_i}{d\xi^2}. \tag{5.5.24}$$

But from (5.5.12)

$$\frac{d^2 F_i}{d\xi^2} = -\frac{1}{2} \frac{\nu_1}{\nu_2} \xi B \exp\left(-\frac{1}{4} \frac{\nu_1}{\nu_i} \xi^2 \right), \tag{5.5.25}$$

and therefore

$$\frac{d^2 F_i}{d\xi^2}(0) = 0. \tag{5.5.26}$$

Hence

$$\frac{\partial w_i}{\partial y}(x, 0) = \frac{1}{x} \frac{d^2 F_i}{d\xi^2}(0) = 0 \tag{5.5.27}$$

and the turning point is $(x, 0)$ on the x-axis.

Also, since $k = 0$, equation (5.5.14) reduces to

$$B_1 = B_2. \tag{5.5.28}$$

Hence from (5.5.12),

$$\frac{dF_i}{d\xi} = B \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2 \right). \tag{5.5.29}$$

5.5.2.2 Conserved quantity

In Chapter 4, Section 4.4.1, equation (4.4.50) it was found that

$$D^* = \frac{\rho_2}{\rho_1}\int_{-\infty}^{\phi(x)} w_2(x, y)dy + \int_{\phi(x)}^{\infty} w_1(x, y)dy = \text{constant independent of } x \tag{5.5.30}$$

provided

$$\left(\frac{\rho_2}{\rho_1} - 1\right) w_1(x, \phi(x))\frac{d\phi(x)}{dx} = 0. \tag{5.5.31}$$

The term in (5.5.31) is second order in smallness because w_i and $\frac{d\phi}{dx}$ are both small and was neglected because terms of second order in smallness had already been neglected in the derivation of the diffusion equation. We see that since $k = 0$ the condition (5.5.31) is identically satisfied.

5.5.2.3 y-component of velocity on interface

Now

$$v_{iy}(x, y) = \frac{\partial \psi_i}{\partial x} = -\frac{1}{2x}\xi\frac{dF_i}{d\xi} + \frac{dG}{dx}, \tag{5.5.32}$$

and

$$v_{iy}(x, 0) = \frac{dG}{dx}. \tag{5.5.33}$$

But on the interface between two fluids the normal component of velocity vanishes. Hence

$$v_{iy}(x, 0) = \frac{dG}{dx} = 0 \tag{5.5.34}$$

and therefore

$$v_{iy}(x, y) = -\frac{1}{2x}\xi\frac{dF_i}{d\xi}. \tag{5.5.35}$$

5.5.2.4 Stream function

The stream function is

$$\psi_i(x,y) = F_i(\xi) + G(x). \tag{5.5.36}$$

But since

$$\frac{dG}{dx} = 0, \tag{5.5.37}$$

it follows that

$$G(x) = G_0 = \text{constant}. \tag{5.5.38}$$

Hence

$$\psi_i(x,y) = F_i(\xi) + G_0. \tag{5.5.39}$$

Since a stream function is determined up to an arbitrary additive constant we can take

$$G_0 = 0 \tag{5.5.40}$$

and therefore

$$\psi_i(x,y) = F_i(\xi). \tag{5.5.41}$$

5.5.2.5 Lie point symmetry

From (5.4.57)

$$X_1 = X_2 = X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \eta(x)\frac{\partial}{\partial \psi}, \tag{5.5.42}$$

where from (5.4.54)

$$G(x) = \frac{1}{2}\int^x \frac{1}{x}\eta(x)dx. \tag{5.5.43}$$

But

$$\frac{dG}{dx} = \frac{1}{2x}\eta(x) = 0 \tag{5.5.44}$$

using (5.5.37). The Lie point symmetry therefore reduced to

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{5.5.45}$$

5.5.3 Determination of constant B

The constant B in (5.5.29) is obtained from the conserved quantity. Since $\phi(x) = 0$ the conserved quantity (5.5.30) is

$$D^* = \frac{\rho_2}{\rho_1}\int_{-\infty}^{0} w_2(x,y)dy + \int_0^{\infty} w_1(x,y)dy. \tag{5.5.46}$$

But

$$w_i(x,y) = x^{-\frac{1}{2}}\frac{dF_i}{d\xi}, \qquad \xi = \frac{y}{x^{\frac{1}{2}}}, \qquad dy = x^{\frac{1}{2}}d\xi, \qquad \frac{dF_i}{d\xi} = B\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2\right). \tag{5.5.47}$$

Therefore, the conserved quantity can be rewritten as

$$D^* = B\left[\frac{\rho_2}{\rho_1}\int_{-\infty}^{0}\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\xi^2\right)d\xi + \int_0^{\infty}\exp\left(-\frac{1}{4}\xi^2\right)d\xi\right]. \tag{5.5.48}$$

Equation (5.5.48) for B is exactly the same as (3.2.58) for B in the similarity solution. It was found that

$$B = \frac{D}{\sqrt{\pi}\left[1 + \frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}}\right]} \tag{5.5.49}$$

which can be rewritten as

$$B = \frac{D}{\sqrt{\pi}\left[1 + \left(\frac{\rho_2\mu_2}{\rho_1\mu_1}\right)^{\frac{1}{2}}\right]} = \frac{D\,\chi^{\frac{1}{2}}}{\sqrt{\pi}\left[1 + \chi^{\frac{1}{2}}\right]} \tag{5.5.50}$$

where

$$\chi = \frac{\rho_1\mu_1}{\rho_2\mu_2}. \tag{5.5.51}$$

The parameter χ is used in the paper by Herczynski, Weidman and Burde [1].

5.5.4 Visual representation of the solution

The invariant solution derived in Section 5.5 is visually represented in Figures 5.1 and 5.2.

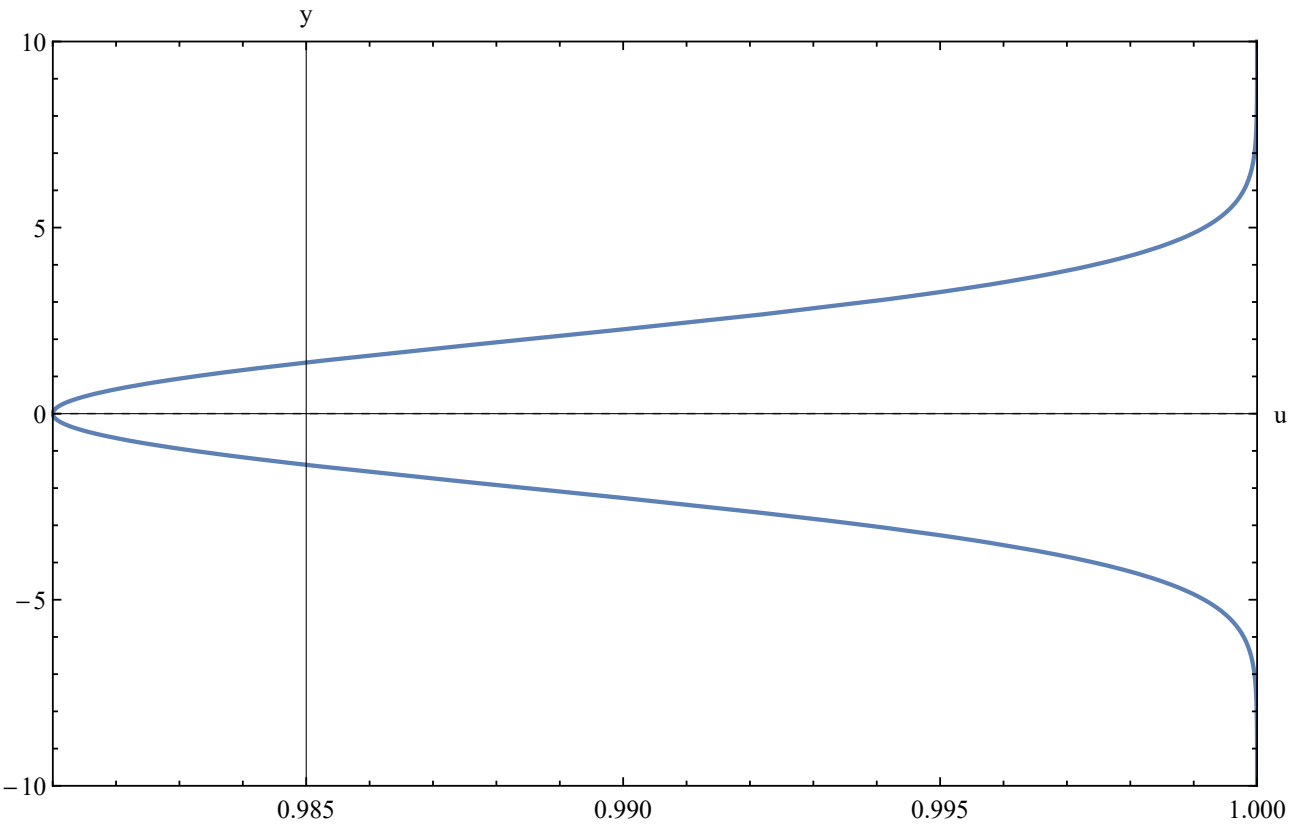

Figure 5.1: The velocities $u_1(x,y)$ and $u_2(x,y)$ of the classical two-fluid wake plotted against y at $x=2$ for $D=1$, $\frac{\rho_1}{\rho_2}=\frac{1}{20}$ and $\frac{\nu_1}{\nu_2}=1$.

The velocity profile is given by

$$u_1(x,y)=1-w_1(x,y)=1-\frac{B}{x^{\frac{1}{2}}}\exp\left(-\frac{1}{4}\frac{y^2}{x}\right), \tag{5.5.52}$$

$$u_2(x,y)=1-w_2(x,y)=1-\frac{B}{x^{\frac{1}{2}}}\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\frac{y^2}{x}\right) \tag{5.5.53}$$

and

$$B=\frac{D}{\sqrt{\pi}\left[1+\frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}}\right]}. \tag{5.5.54}$$

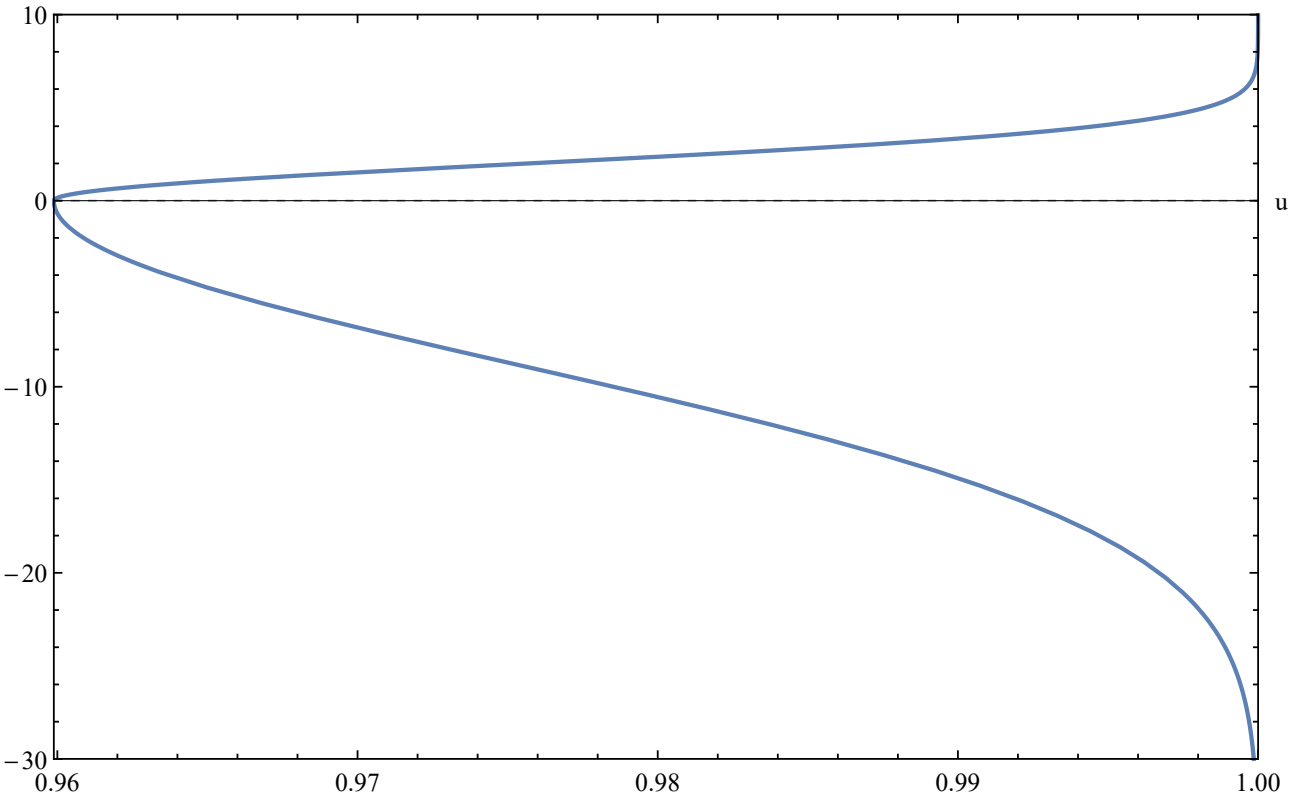

Figure 5.2: The velocities $u_1(x, y)$ and $u_2(x, y)$ of the classical two-fluid wake plotted against y at $x = 2$ for $D = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{20}$.

The velocity profile (5.5.52) and (5.5.53) is exactly the same as (3.2.72) and (3.2.73) for the similarity solution derived in Chapter 3. We therefore choose different values of the parameters in the graphs from those used in Chapter 3. In Figure 5.1 the effect of the density ratio is investigated by taking $\frac{\rho_1}{\rho_2} = \frac{1}{20}$ and $\frac{\nu_1}{\nu_2} = 1$. In Figure 5.2 the effect of the viscosity ratio is investigated by taking $\frac{\nu_1}{\nu_2} = \frac{1}{20}$ and $\frac{\rho_1}{\rho_2} = \frac{1}{2}$.

From Figure 5.1 it is seen that the higher value of the density ratio $\frac{\rho_2}{\rho_1}$ increases the width of both the upper and lower wakes. From Figure 5.2 it is seen that the higher viscosity ratio $\frac{\nu_2}{\nu_1}$ increases the effective width of the lower wake.

5.6 Summary of results for the classical two-fluid wake

The results derived for the classical two-fluid wake are summarised below.

$$k = 0, \qquad \phi(x) = kx^{\frac{1}{2}} = 0, \qquad \xi = \frac{y}{x^{\frac{1}{2}}}, \tag{5.6.1}$$

$$\frac{dF_i}{d\xi} = B \exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2 \right), \tag{5.6.2}$$

$$B = \frac{D^*}{\sqrt{\pi}\left[1 + \left(\frac{\rho_2\mu_2}{\rho_1\mu_1}\right)^{\frac{1}{2}}\right]}, \tag{5.6.3}$$

$$\psi_i(x, y) = F_i(\xi), \tag{5.6.4}$$

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{5.6.5}$$

$$w_i(x, y) = \frac{\partial \psi_i}{\partial y} = x^{-\frac{1}{2}}\frac{dF_i}{d\xi}, \tag{5.6.6}$$

$$v_i(x, y) = \frac{\partial \psi_i}{\partial x} = -\frac{1}{2x}\xi\frac{dF_i}{d\xi}, \tag{5.6.7}$$

$$\frac{\partial w_i}{\partial y}(x, y) = \frac{1}{x}\frac{d^2F_i}{d\xi^2}. \tag{5.6.8}$$

The remaining interface condition (5.4.48) gives the pressure difference across the interface:

$$P_2(x, \phi(x)) - P_1(x, \phi(x)) = \frac{1}{Re}\left(1 - \frac{\mu_2}{\mu_1}\right)\frac{dF_1}{d\xi}(k)x^{-\frac{3}{2}}. \tag{5.6.9}$$

Using (5.6.2) and (5.6.3), (5.6.9) becomes

$$P_2(x, \phi(x)) - P_1(x, \phi(x)) = \frac{1}{Re}\left(1 - \frac{\mu_2}{\mu_1}\right)\frac{D^*\, x^{-\frac{3}{2}}}{\sqrt{\pi}\left[1 + \left(\frac{\rho_2\mu_2}{\rho_1\mu_1}\right)^{\frac{1}{2}}\right]}. \tag{5.6.10}$$

If terms of order $\frac{1}{Re}$ are neglected then

$$P_2(x, \phi(x)) = P_1(x, \phi(x)) \tag{5.6.11}$$

and the pressure is continuous across the interface.

5.7 Associated Lie point symmetry for the wake behind a self-propelled body

We suppress the symbol i on ψ, w, v and Lie point symmetry X to keep the notation simple.The conserved vectors for the upper wake and lower wake of the momentumless two-fluid wake are

$$T^1_{(i)} = \frac{\nu_1}{\nu_i} y^2 w - 2xw, \qquad T^2_{(i)} = -y^2 \frac{\partial w}{\partial y} + 2yw + 2xv, \qquad i = 1, 2. \tag{5.7.1}$$

Now,

$$w = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}. \tag{5.7.2}$$

The conserved vectors expressed in terms of the stream function are

$$T^1_{(i)} = \frac{\nu_1}{\nu_i} y^2 \frac{\partial \psi}{\partial y} - 2x \frac{\partial \psi}{\partial y}, \qquad T^2_{(i)} = -y^2 \frac{\partial^2 \psi}{\partial y^2} + 2y \frac{\partial \psi}{\partial y} + 2x \frac{\partial \psi}{\partial x}, i = 1, 2. \tag{5.7.3}$$

The Lie point symmetry

$$X = \xi^1(x, y, \psi) \frac{\partial}{\partial x} + \xi^1(x, y, \psi) \frac{\partial}{\partial y} + \eta(x, y, \psi) \frac{\partial}{\partial \psi} \tag{5.7.4}$$

is associated with the conserved vector (T^1, T^2) provided

$$X(T^s) + T^s D_k(\xi^k) - T^k D_k(\xi^s) = 0, \qquad s = 1, 2. \tag{5.7.5}$$

By expanding (5.7.5) we obtain the two components

$$s = 1 : \qquad X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0, \tag{5.7.6}$$

$$s = 2 : \qquad X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0, \tag{5.7.7}$$

where

$$D_1 = D_x = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx} \frac{\partial}{\partial \psi_x} + \psi_{yx} \frac{\partial}{\partial \psi_y} + ..., \tag{5.7.8}$$

$$D_2 = D_y = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \psi_{yy} \frac{\partial}{\partial \psi_y} + ... \tag{5.7.9}$$

5.7.1 Case 1: Equation (5.7.6)

From (5.7.6)

$$X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0. \tag{5.7.10}$$

Now

$$T^1 = \frac{\nu_1}{\nu_i} y^2 \psi_y - 2x\psi_y \tag{5.7.11}$$

and therefore

$$X(T^1) = \left(\xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial \psi} + \zeta_2 \frac{\partial}{\partial \psi_y}\right)\left[\frac{\nu_1}{\nu_i} y^2 \psi_y - 2x\psi_y\right]$$

$$= -2\xi^1 \psi_y + 2\frac{\nu_1}{\nu_i} y \xi^2 \psi_y + \frac{\nu_1}{\nu_i} y^2 \zeta_2 - 2x\zeta_2. \tag{5.7.12}$$

But

$$\zeta_s = D_s(\eta) - \psi_k D_s(\eta^k). \tag{5.7.13}$$

Hence

$$\zeta_2 = D_y(\eta) - \psi_k D_y(\eta^k)$$

$$= D_y(\eta) - \psi_x D_y(\eta^1) - \psi_y D_y(\eta^2), \tag{5.7.14}$$

and expanding gives

$$\zeta_2 = \frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial \psi}\psi_y - \frac{\partial \xi^1}{\partial y}\psi_x - \frac{\partial \xi^1}{\partial \psi}\psi_x\psi_y - \frac{\partial \xi^2}{\partial y}\psi_y - \frac{\partial \xi^2}{\partial \psi}\psi_y^2. \tag{5.7.15}$$

Thus, from (5.7.12)

$$X(T^1) = -2\xi^1\psi_y + 2\frac{\nu_1}{\nu_i} y\xi^2\psi_y + \frac{\nu_1}{\nu_i} y^2 \left[\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial \psi}\psi_y - \frac{\partial \xi^1}{\partial y}\psi_x - \frac{\partial \xi^1}{\partial \psi}\psi_x\psi_y - \frac{\partial \xi^2}{\partial y}\psi_y - \frac{\partial \xi^2}{\partial \psi}\psi_y^2\right]$$

$$- 2x\left[\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial \psi}\psi_y - \frac{\partial \xi^1}{\partial y}\psi_x - \frac{\partial \xi^1}{\partial \psi}\psi_x\psi_y - \frac{\partial \xi^2}{\partial y}\psi_y - \frac{\partial \xi^2}{\partial \psi}\psi_y^2\right]. \tag{5.7.16}$$

Also

$$T^1 D_2(\xi^2) = \frac{\nu_1}{\nu_i} y^2 \frac{\partial \xi^2}{\partial y}\psi_y + \frac{\nu_1}{\nu_i} y^2 \frac{\partial \xi^2}{\partial \psi}\psi_y^2 - 2x\frac{\partial \xi^2}{\partial y}\psi_y - 2x\frac{\partial \xi^2}{\partial \psi}\psi_y^2, \tag{5.7.17}$$

$$T^2 D_2(\xi^1) = -y^2 \frac{\partial \xi^1}{\partial y}\psi_{yy} - y^2 \frac{\partial \xi^1}{\partial \psi}\psi_y\psi_{yy} + 2y\frac{\partial \xi^1}{\partial y}\psi_y + 2y\frac{\partial \xi^1}{\partial \psi}\psi_y^2 + 2x\frac{\partial \xi^1}{\partial y}\psi_x + 2x\frac{\partial \xi^1}{\partial \psi}\psi_x\psi_y. \tag{5.7.18}$$

Using (5.7.16), (5.7.17) and (5.7.18), equation (5.7.10) becomes

$$
\begin{aligned}
&-2\xi^1\psi_y+2\frac{\nu_1}{\nu_i}y\xi^2\psi_y+\frac{\nu_1}{\nu_i}y^2\left[\frac{\partial\eta}{\partial y}+\frac{\partial\eta}{\partial\psi}\psi_y-\frac{\partial\xi^1}{\partial y}\psi_x-\frac{\partial\xi^1}{\partial\psi}\psi_x\psi_y-\frac{\partial\xi^2}{\partial y}\psi_y-\frac{\partial\xi^2}{\partial\psi}\psi_y^2\right]\\
&-2x\left[\frac{\partial\eta}{\partial y}+\frac{\partial\eta}{\partial\psi}\psi_y-\frac{\partial\xi^1}{\partial y}\psi_x-\frac{\partial\xi^1}{\partial\psi}\psi_x\psi_y-\frac{\partial\xi^2}{\partial y}\psi_y-\frac{\partial\xi^2}{\partial\psi}\psi_y^2\right]\\
&+\frac{\nu_1}{\nu_i}y^2\frac{\partial\xi^2}{\partial y}\psi_y+\frac{\nu_1}{\nu_i}y^2\frac{\partial\xi^2}{\partial\psi}\psi_y^2-2x\frac{\partial\xi^2}{\partial y}\psi_y-2x\frac{\partial\xi^2}{\partial\psi}\psi_y^2\\
&+y^2\frac{\partial\xi^1}{\partial y}\psi_{yy}+y^2\frac{\partial\xi^1}{\partial\psi}\psi_y\psi_{yy}-2y\frac{\partial\xi^1}{\partial y}\psi_y-2y\frac{\partial\xi^1}{\partial\psi}\psi_y^2-2x\frac{\partial\xi^1}{\partial y}\psi_x-2x\frac{\partial\xi^1}{\partial\psi}\psi_x\psi_y=0 \qquad (5.7.19)
\end{aligned}
$$

Separate (5.7.19) by the powers and products of the partial derivatives of ψ which are independent.

$$\psi_y\psi_{yy}: \qquad \frac{\partial\xi^1}{\partial\psi}=0, \qquad (5.7.20)$$

$$\psi_{yy}: \qquad \frac{\partial\xi^1}{\partial y}=0, \qquad (5.7.21)$$

$$\psi_x\psi_y: \qquad \frac{\partial\xi^1}{\partial\psi}=0, \qquad (5.7.22)$$

(5.7.23)

$$\psi_y^2: \qquad \frac{\partial\xi^1}{\partial\psi}=0, \qquad (5.7.24)$$

$$\psi_y: \qquad -2\xi^1+2\frac{\nu_1}{\nu_i}y\xi^2+\left(\frac{\nu_1}{\nu_i}y^2-2x\right)\frac{\partial\eta}{\partial\psi}-2y\frac{\partial\xi^1}{\partial y}=0, \qquad (5.7.25)$$

$$\psi_x: \qquad \frac{\partial\xi^1}{\partial y}=0. \qquad (5.7.26)$$

The remainder is

$$\frac{\nu_1}{\nu_i}y^2\frac{\partial\eta}{\partial y}-2x\frac{\partial\eta}{\partial y}=0. \qquad (5.7.27)$$

From (5.7.20) and (5.7.21),

$$\xi^1=\xi^1(x). \qquad (5.7.28)$$

The remaining equations reduce to

$$-2\xi^1(x) + 2\frac{\nu_1}{\nu_i}y\xi^2 + \left(\frac{\nu_1}{\nu_i}y^2 - 2x\right)\frac{\partial\eta}{\partial\psi} = 0, \tag{5.7.29}$$

$$\left(\frac{\nu_1}{\nu_i}y^2 - 2x\right)\frac{\partial\eta}{\partial y} = 0. \tag{5.7.30}$$

Therefore

$$\xi^1 = \xi^1(x), \qquad \xi^2 = \xi^2(x, y, \psi), \qquad \eta = \eta(x, y, \psi), \tag{5.7.31}$$

provided (5.7.29) and (5.7.30) are satisfied.

5.7.2 Case 2: Equation (5.7.7)

From (5.7.7)

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0. \tag{5.7.32}$$

Now

$$T^2 = -y^2\psi_{yy} + 2y\psi_y + 2x\psi_x. \tag{5.7.33}$$

Now

$$X(T^2) = \left[\xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial\psi} + \zeta_1\frac{\partial}{\partial\psi_x} + \zeta_2\frac{\partial}{\partial\psi_y} + \zeta_{22}\frac{\partial}{\partial\psi_{yy}}\right]$$

$$= 2\xi^1\psi_x - 2y\xi^2\psi_{yy} + 2\xi^2\psi_y + 2x\zeta_1 + 2y\zeta_2 - y^2\zeta_{22}. \tag{5.7.34}$$

We need to calculate ζ_1, ζ_2 and ζ_{22} where

$$\zeta_n = D_n(\eta) - \psi_k D_n(\xi^k), \tag{5.7.35}$$

$$\zeta_{mn} = D_m(\zeta_n) - \psi_{mk} D_n(\xi^k). \tag{5.7.36}$$

Now

$$\zeta_1 = D_1(\eta) - \psi_1 D_1(\xi^1) - \psi_2 D_1(\xi^2)$$

$$= \frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial\psi}\psi_x - \frac{d\xi^1}{dx}\psi_x - \frac{\partial\xi^2}{\partial x}\psi_y - \frac{\partial\xi^2}{\partial\psi}\psi_x\psi_y, \tag{5.7.37}$$

$$\zeta_2 = D_2(\eta) - \psi_1 D_2(\xi^1) - \psi_2 D_2(\xi^2)$$

$$= \frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial \psi}\psi_y - \frac{\partial \xi^2}{\partial y}\psi_y - \frac{\partial \xi^2}{\partial \psi}\psi_y^2. \tag{5.7.38}$$

Conside next ζ_{22}:

$$\zeta_{22} = D_2(\zeta_2) - \psi_{2k} D_2(\xi^k)$$

$$= D_y(\zeta_2) - \psi_{xy} D_y(\xi^1(x)) - \psi_{yy} D_y(\xi^2)$$

$$= D_y(\zeta_2) - \psi_{yy} D_y(\xi^2). \tag{5.7.39}$$

Now

$$D_y(\zeta_2) = D_y\left[\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial \psi}\psi_y - \frac{\partial \xi^2}{\partial y}\psi_y - \frac{\partial \xi^2}{\partial \psi}\psi_y^2\right]. \tag{5.7.40}$$

But

$$D_y\left(\frac{\partial \eta}{\partial y}\right) = \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial \psi \partial y}\psi_y, \tag{5.7.41}$$

$$D_y\left(\frac{\partial \eta}{\partial \psi}\psi_y\right) = \frac{\partial^2 \eta}{\partial y \partial \psi}\psi_y + \frac{\partial^2 \eta}{\partial \psi^2}\psi_y^2 + \frac{\partial \eta}{\partial \psi}\psi_{yy}, \tag{5.7.42}$$

$$D_y\left(-\frac{\partial \xi^2}{\partial y}\psi_y\right) = -\frac{\partial^2 \xi^2}{\partial y^2}\psi_y - \frac{\partial^2 \xi^2}{\partial y \partial \psi}\psi_y^2 - \frac{\partial \xi^2}{\partial y}\psi_{yy}, \tag{5.7.43}$$

$$D_y\left(-\frac{\partial \xi^2}{\partial \psi}\psi_y^2\right) = -\frac{\partial^2 \xi^2}{\partial y \partial \psi}\psi_y^2 - \frac{\partial^2 \xi^2}{\partial \psi^2}\psi_y^3 - 2\frac{\partial \xi^2}{\partial \psi}\psi_y\psi_{yy} \tag{5.7.44}$$

and therefore

$$D_y(\zeta_2) = \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial \psi \partial y}\psi_y + \frac{\partial^2 \eta}{\partial y \partial \psi}\psi_y + \frac{\partial^2 \eta}{\partial \psi^2}\psi_y^2 + \frac{\partial \eta}{\partial \psi}\psi_{yy} - \frac{\partial^2 \xi^2}{\partial y^2}\psi_y - \frac{\partial^2 \xi^2}{\partial y \partial \psi}\psi_y^2 - \frac{\partial \xi^2}{\partial y}\psi_{yy}$$

$$- \frac{\partial^2 \xi^2}{\partial y \partial \psi}\psi_y^2 - \frac{\partial^2 \xi^2}{\partial \psi^2}\psi_y^3 - 2\frac{\partial \xi^2}{\partial \psi}\psi_y\psi_{yy}. \tag{5.7.45}$$

Also

$$-\psi_{yy} D_y(\xi^2) = -\frac{\partial \xi^2}{\partial y}\psi_{yy} - \frac{\partial \xi^2}{\partial \psi}\psi_y\psi_{yy}. \tag{5.7.46}$$

We then obtain

$$\begin{aligned}\zeta_{22} &= D_y(\zeta_2) - \psi_{yy} D_y(\xi^2) \\ &= \frac{\partial^2\eta}{\partial y^2} + \frac{\partial^2\eta}{\partial\psi\partial y}\psi_y + \frac{\partial^2\eta}{\partial y\partial\psi}\psi_y + \frac{\partial^2\eta}{\partial\psi^2}\psi_y^2 + \frac{\partial\eta}{\partial\psi}\psi_{yy} - \frac{\partial^2\xi^2}{\partial y^2}\psi_y - \frac{\partial^2\xi^2}{\partial y\partial\psi}\psi_y^2 - \frac{\partial\xi^2}{\partial y}\psi_{yy} \\ &\quad - \frac{\partial^2\xi^2}{\partial y\partial\psi}\psi_y^2 - \frac{\partial^2\xi^2}{\partial\psi^2}\psi_y^3 - 2\frac{\partial\xi^2}{\partial\psi}\psi_y\psi_{yy} - \frac{\partial\xi^2}{\partial y}\psi_{yy} - \frac{\partial\xi^2}{\partial\psi}\psi_y\psi_{yy}. \end{aligned} \tag{5.7.47}$$

We next evaluate the following two terms

$$T^2 D_1(\xi^1) = -y^2\frac{d\xi^1}{dx}\psi_{yy} + 2y\frac{d\xi^1}{dx}\psi_y + 2x\frac{d\xi^1}{dx}\psi_x, \tag{5.7.48}$$

$$-T^1 D_1(\xi^2) = -\frac{\nu_1}{\nu_i}y^2\frac{\partial\xi^2}{\partial x}\psi_y - \frac{\nu_1}{\nu_i}y^2\frac{\partial\xi^2}{\partial\psi}\psi_x\psi_y + 2x\frac{\partial\xi^2}{\partial x}\psi_y + 2x\frac{\partial\xi^2}{\partial\psi}\psi_x\psi_y. \tag{5.7.49}$$

Return now to (5.7.32)

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0. \tag{5.7.50}$$

Substituting in (5.7.34) for $X(T^2)$ gives

$$2\xi^1\psi_x - 2y\xi^2\psi_{yy} + 2\xi^2\psi_y + 2x\zeta_1 + 2y\zeta_2 - y^2\zeta_{22} + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0. \tag{5.7.51}$$

Substituting (5.7.37), (5.7.38), (5.7.47), (5.7.48) and (5.7.49) into (5.7.51) produces

$$\begin{aligned}&2\xi^1\psi_x - 2y\xi^2\psi_{yy} + 2\xi^2\psi_y + 2x\frac{\partial\eta}{\partial x} + 2x\frac{\partial\eta}{\partial\psi}\psi_x - 2x\frac{d\xi^1}{dx}\psi_x - 2x\frac{\partial\xi^2}{\partial x}\psi_y - 2x\frac{\partial\xi^2}{\partial\psi}\psi_x\psi_y + 2y\frac{\partial\eta}{\partial y} + \\ &2y\frac{\partial\eta}{\partial\psi}\psi_y - 2y\frac{\partial\xi^2}{\partial y}\psi_y - 2y\frac{\partial\xi^2}{\partial\psi}\psi_y^2 - y^2\frac{\partial^2\eta}{\partial y^2} - y^2\frac{\partial^2\eta}{\partial y\partial\psi}\psi_y - y^2\frac{\partial^2\eta}{\partial y\partial\psi}\psi_y - y^2\frac{\partial^2\eta}{\partial\psi^2}\psi_y^2 - y^2\frac{\partial\eta}{\partial\psi}\psi_{yy} + \\ &y^2\frac{\partial^2\xi^2}{\partial y^2}\psi_y + y^2\frac{\partial^2\xi^2}{\partial y\partial\psi}\psi_y^2 + y^2\frac{\partial\xi^2}{\partial y}\psi_{yy} + y^2\frac{\partial\xi^2}{\partial y\partial\psi}\psi_y^2 + y^2\frac{\partial^2\xi^2}{\partial\psi^2}\psi_y^3 + 2y^2\frac{\partial\xi^2}{\partial\psi}\psi_y\psi_{yy} + y^2\frac{\partial\xi^2}{\partial y}\psi_{yy} + \\ &y^2\frac{\partial\xi^2}{\partial\psi}\psi_y\psi_{yy} - y^2\frac{d\xi^1}{dx}\psi_{yy} + 2y\frac{d\xi^1}{dx}\psi_y + 2x\frac{d\xi^1}{dx}\psi_x - \frac{\nu_1}{\nu_i}y^2\frac{\partial\xi^2}{\partial x}\psi_y - \frac{\nu_1}{\nu_i}y^2\frac{\partial\xi^2}{\partial\psi}\psi_x\psi_y + 2x\frac{\partial\xi^2}{\partial x}\psi_y + \\ &2x\frac{\partial\xi^2}{\partial\psi}\psi_x\psi_y = 0. \end{aligned} \tag{5.7.52}$$

Separate (5.7.52) according to the powers and products of the partial derivatives of ψ.

$$\psi_y\psi_{yy}: \qquad 3y^2\frac{\partial\xi^2}{\partial\psi}=0, \tag{5.7.53}$$

$$\psi_{yy}: \qquad -2y\xi^2+2y^2\frac{\partial\xi^2}{\partial y}-y^2\frac{\partial\eta}{\partial\psi}-y^2\frac{d\xi^1}{dx}=0, \tag{5.7.54}$$

$$\psi_x\psi_y: \qquad \frac{\nu_1}{\nu_i}y^2\frac{\partial\xi^2}{\partial\psi}=0, \tag{5.7.55}$$

$$\psi_y^3: \qquad y^2\frac{\partial^2\xi^2}{\partial\psi^2}=0, \tag{5.7.56}$$

$$\psi_y^2: \qquad -2y\frac{\partial\xi^2}{\partial\psi}-y^2\frac{\partial^2\eta}{\partial\psi^2}+2y^2\frac{\partial^2\xi^2}{\partial y\partial\psi}=0, \tag{5.7.57}$$

$$\psi_y: \qquad 2\xi^2-2y\frac{\partial\xi^2}{\partial y}+y^2\frac{\partial^2\xi^2}{\partial y^2}-\frac{\nu_1}{\nu_i}y^2\frac{\partial\xi^2}{\partial x}+2y\frac{d\xi^1}{dx}+2y\frac{\partial\eta}{\partial\psi}-2y^2\frac{\partial^2\eta}{\partial y\partial\psi}=0, \tag{5.7.58}$$

$$\psi_x: \qquad \xi^1(x)+x\frac{\partial\eta}{\partial\psi}=0. \tag{5.7.59}$$

The remainder is

$$2x\frac{\partial\eta}{\partial x}+2y\frac{\partial\eta}{\partial y}-y^2\frac{\partial^2\eta}{\partial y^2}=0. \tag{5.7.60}$$

5.7.3 Solution for associated Lie point symmetry

So far it has been shown that ξ^1, ξ^2 and η are of the form (5.7.31) and that they satisfy the conditions (5.7.29), (5.7.30) and (5.7.53) to (5.7.59).

From (5.7.53) we have

$$\frac{\partial\xi^2}{\partial\psi}(x,y,\psi)=0 \tag{5.7.61}$$

and therefore

$$\xi^2=\xi^2(x,y). \tag{5.7.62}$$

Now from (5.7.30)

$$\left(\frac{\nu_1}{\nu_i}y^2 - 2x\right)\frac{\partial \eta}{\partial y} = 0. \tag{5.7.63}$$

If

$$\frac{\nu_1}{\nu_i}y^2 - 2x \neq 0, \tag{5.7.64}$$

then

$$\frac{\partial \eta}{\partial y} = 0. \tag{5.7.65}$$

Therefore

$$\eta = \eta(x, \psi). \tag{5.7.66}$$

The remainder (5.7.60) reduces to

$$\frac{\partial \eta}{\partial x} = 0. \tag{5.7.67}$$

Hence

$$\eta = \eta(\psi). \tag{5.7.68}$$

Since $\xi^2 = \xi^2(x, y)$, equation (5.7.57) reduces to

$$\frac{d^2\eta}{d\psi^2} = 0 \tag{5.7.69}$$

and therefore

$$\eta(\psi) = a_1\psi + a_2, \tag{5.7.70}$$

where a_1 and a_2 are constants. From (5.7.59)

$$\xi^1(x) = -x\frac{\partial \eta}{\partial \psi} = -a_1 x. \tag{5.7.71}$$

Now from (5.7.29)

$$-2\xi^1(x) + 2\frac{\nu_1}{\nu_i}y\xi^2 + \left(\frac{\nu_1}{\nu_i}y^2 - 2x\right)\frac{\partial \eta}{\partial \psi} = 0. \tag{5.7.72}$$

Since,

$$\xi^1(x) = -a_1 x, \qquad \xi^2 = \xi^1(x, y), \qquad \eta(\psi) = a_1\psi + a_2 \tag{5.7.73}$$

equation (5.7.29) reduces to

$$\xi^2 = -\frac{1}{2}a_1 y. \tag{5.7.74}$$

Therefore

$$\xi^1(x) = -a_1 x, \qquad \xi^2(y) = -\frac{1}{2}a_1 y, \qquad \eta(\psi) = a_1\psi + a_2. \tag{5.7.75}$$

We now verify that the remaining equations (5.7.54) and (5.7.58) are identically satisfied.

Consider the left-hand side of (5.7.54)

$$-2y\xi^2 + 2y^2\frac{\partial \xi^2}{\partial y} - y^2\frac{\partial \eta}{\partial \psi} - y^2\frac{d\xi^1}{dx} = a_1y^2 - a_1y^2 - a_1y^2 + a_1y^2 = 0. \tag{5.7.76}$$

Equation (5.7.54) is identically satisfied.

Consider the left-hand side of (5.7.58)

$$2\xi^2 - 2y\frac{\partial \xi^2}{\partial y} + y^2\frac{\partial^2 \xi^2}{\partial y^2} - \frac{\nu_1}{\nu_i}y^2\frac{\partial \xi^2}{\partial x} + 2y\frac{d\xi^1}{dx} + 2y\frac{\partial \eta}{\partial \psi} - 2y^2\frac{\partial^2 \eta}{\partial y \partial \psi} =$$

$$- a_1 y + a_1 y - 2a_1 y + 2a_1 y = 0. \tag{5.7.77}$$

Equation (5.7.58) is identically satisfied. Therefore all conditions are satisfied.

The associated Lie point symmetry is

$$X = -a_1 x\frac{\partial}{\partial x} - \frac{1}{2}a_1 y\frac{\partial}{\partial y} + (a_1\psi + a_2)\frac{\partial}{\partial \psi}. \tag{5.7.78}$$

Consider the general case in which $a_1 \neq 0$. Multiply (5.7.78) by $-\frac{2}{a_1}$ which gives

$$-\frac{2}{a_1}X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\left(\psi + \frac{a_2}{a_1}\right)\frac{\partial}{\partial \psi}. \tag{5.7.79}$$

Let

$$X^* = -\frac{2}{a_1}X, \qquad c^* = \frac{a_2}{a_1}. \tag{5.7.80}$$

Then

$$X^* = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2(\psi + c^*)\frac{\partial}{\partial \psi}. \tag{5.7.81}$$

We suppress the star. The associated Lie point symmetry for the upper wake and the lower wake are of the form

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2(\psi + c)\frac{\partial}{\partial \psi}. \tag{5.7.82}$$

Unlike the associated Lie point symmetry (5.3.60) for the classical wake, the associated Lie point symmetry (5.7.82) for the momentumless wake does not contain an arbitrary function.

5.8 Invariant solution for the momentumless two-fluid wake

Now $\psi = \Psi(x, y)$ is an invariant solution of the PDE

$$\frac{\partial^2 \psi_i}{\partial x \partial y} = \frac{\nu_i}{\nu_1} \frac{\partial^3 \psi_i}{\partial y^3} \tag{5.8.1}$$

generated by the Lie point symmetry X provided

$$X(\psi - \Psi(x, y))|_{\psi=\Psi(x,y)} = 0. \tag{5.8.2}$$

Using the Lie point symmetry (5.7.82), condition (5.8.2) becomes

$$\left[2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2(\psi + c)\frac{\partial}{\partial \psi}\right](\psi - \Psi(x, y))|_{\psi=\Psi(x,y)} = 0, \tag{5.8.3}$$

that is, provided

$$2x\frac{\partial \Psi}{\partial x} + y\frac{\partial \Psi}{\partial y} = -2(\Psi + c). \tag{5.8.4}$$

The differential equations of the characteristic curves of the PDE (5.8.4) are

$$\frac{dx}{2x} = \frac{dy}{y} = -\frac{d\Psi}{2(\Psi + c)}. \tag{5.8.5}$$

The first pair of terms are

$$\frac{dx}{2x} = \frac{dy}{y} \tag{5.8.6}$$

and integrating gives

$$\frac{1}{2}\ln(x) = \ln(y) + \text{constant}. \tag{5.8.7}$$

Thus

$$\frac{y}{x^{\frac{1}{2}}} = k_1, \tag{5.8.8}$$

where k_1 is a constant. The first and last term in (5.8.5) give

$$\frac{dx}{2x} = -\frac{d\Psi}{2(\Psi + c)} \tag{5.8.9}$$

and integrating we obtain

$$\ln(x) = -\ln(\Psi + c) + \text{constant}. \tag{5.8.10}$$

Hence

$$x(\Psi + c) = k_2, \tag{5.8.11}$$

where k_2 is a constant. The general solution is

$$k_2 = F(k_1), \tag{5.8.12}$$

where F is an arbitrary function. Hence

$$x(\Psi + c) = F\left(\frac{y}{x^{\frac{1}{2}}}\right). \tag{5.8.13}$$

However, $\Psi(x, y) = \psi$ and therefore

$$\psi = \frac{1}{x}F\left(\frac{y}{x^{\frac{1}{2}}}\right) - c. \tag{5.8.14}$$

Let

$$\xi = \frac{y}{x^{\frac{1}{2}}} \tag{5.8.15}$$

and then

$$\psi(x, y) = \frac{1}{x}F(\xi) - c. \tag{5.8.16}$$

5.8.1 Conserved quantity

In the approach in Chapter 3 for the momentumeless two-fluid wake, the conserved quantity was used to determine the unknown parameter ω_i in the invariant solution

$$\psi_i(x, y) = x^{\omega_i}F_i(\xi), \qquad \xi = \frac{y}{x^{\frac{1}{2}}}. \tag{5.8.17}$$

Since the Lie point symmetry was associated with the conserved vector for the momentumless two-fluid wake, the unknown parameter has been determined and the invariant solution does not contain an unknown parameter:

$$\psi_i(x, y) = \frac{1}{x}F_i(\xi), \qquad \xi = \frac{y}{x^{\frac{1}{2}}}. \tag{5.8.18}$$

The conserved quantity gives the equation of the interface

$$y = \phi(x). \tag{5.8.19}$$

The conserved quantity for the momentumless two-fluid wake is given by (4.4.95)

$$K^* = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x)} y^2 w_2(x,y)dy + \int_{\phi(x)}^{\infty} y^2 w_1(x,y)dy = \text{constant independent of } x. \tag{5.8.20}$$

We express the conserved quantity in terms of the invariant solution. Now

$$w_i = \frac{\partial \psi_i}{\partial y}. \tag{5.8.21}$$

Hence

$$K^* = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\phi(x)} y^2 \frac{\partial \psi_2}{\partial y}dy + \int_{\phi(x)}^{\infty} y^2 \frac{\partial \psi_1}{\partial y}dy. \tag{5.8.22}$$

Now

$$\psi_i(x,y) = \frac{1}{x}F_i(\xi), \qquad \xi = \frac{y}{x^{\frac{1}{2}}}, \qquad \frac{\partial \xi}{\partial x} = -\frac{\xi}{2x}, \qquad \frac{\partial \xi}{\partial y} = \frac{1}{x^{\frac{1}{2}}}, \qquad dy = x^{\frac{1}{2}}d\xi. \tag{5.8.23}$$

Hence

$$\frac{\partial \psi_i}{\partial y} = \frac{1}{x}\frac{dF_i}{d\xi}\frac{\partial \xi}{\partial y} = x^{-\frac{3}{2}}\frac{dF_i}{d\xi}. \tag{5.8.24}$$

Therefore the conserved quantity (5.8.22) becomes

$$K^* = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{\frac{\phi(x)}{x^{\frac{1}{2}}}} \xi^2 \frac{dF_2}{d\xi}d\xi + \int_{\frac{\phi(x)}{x^{\frac{1}{2}}}}^{\infty} \xi^2 \frac{dF_1}{d\xi}d\xi. \tag{5.8.25}$$

Since K^* is a constant independent of x,

$$\frac{\phi(x)}{x^{\frac{1}{2}}} = k, \qquad k = \text{constant}, \tag{5.8.26}$$

and

$$\phi(x) = kx^{\frac{1}{2}}. \tag{5.8.27}$$

The interface therefore is

$$y = \phi(x) = kx^{\frac{1}{2}}. \tag{5.8.28}$$

The conserved quantity becomes

$$K^* = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{k} \xi^2 \frac{dF_2}{d\xi}d\xi + \int_{k}^{\infty} \xi^2 \frac{dF_1}{d\xi}d\xi. \tag{5.8.29}$$

5.8.2 Partial differential equation for the velocity deficit

We express the partial differential equation

$$\frac{\partial^2 \psi_i}{\partial x \partial y} = \frac{\nu_i}{\nu_1} \frac{\partial^3 \psi_i}{\partial y^3} \tag{5.8.30}$$

in terms of the invariant solution

$$\psi_i(x, y) = \frac{1}{x} F_i(\xi), \qquad \xi = \frac{y}{x^{\frac{1}{2}}}. \tag{5.8.31}$$

Now

$$\frac{\partial \psi}{\partial x} = -\frac{1}{2x^2}\left[2F(\xi) + \xi \frac{dF}{d\xi}\right], \tag{5.8.32}$$

$$\frac{\partial^2 \psi}{\partial y \partial x} = -\frac{1}{2} x^{-\frac{5}{2}}\left[3\frac{dF}{d\xi} + \xi \frac{d^2F}{d\xi^2}\right], \tag{5.8.33}$$

$$\frac{\partial \psi}{\partial y} = x^{-\frac{3}{2}} \frac{dF}{d\xi}, \qquad \frac{\partial^2 \psi}{\partial y^2} = x^{-2} \frac{d^2F}{d\xi^2}, \qquad \frac{\partial^3 \psi}{\partial y^3} = x^{-\frac{5}{2}} \frac{d^3F}{d\xi^3}. \tag{5.8.34}$$

Substituting (5.8.31) to (5.8.34) into (5.8.30) and simplifying gives

$$2\frac{\nu_i}{\nu_1}\frac{d^3F_i}{d\xi^3} + \frac{d}{d\xi}\left(\xi \frac{dF_i}{d\xi}\right) + 2\frac{dF_i}{d\xi} = 0. \tag{5.8.35}$$

5.8.3 Boundary conditions

Equations (5.8.34) are used to express the boundary conditions (5.2.11) to (5.2.14) in terms of the invariant solution. We obtain for the boundary conditions

$$\frac{dF_1}{d\xi}(\infty) = 0. \tag{5.8.36}$$

$$\frac{d^2F_1}{d\xi^2}(\infty) = 0. \tag{5.8.37}$$

$$\frac{dF_2}{d\xi}(-\infty) = 0. \tag{5.8.38}$$

$$\frac{d^2F_2}{d\xi^2}(-\infty) = 0. \tag{5.8.39}$$

5.8.4 Interface conditions

We express the interface conditions (5.2.15) to (5.2.18) in terms of the invariant solution

$$\psi_i(x,y) = \frac{1}{x}F_i(\xi), \qquad \xi = \frac{y}{x^{\frac{1}{2}}}. \tag{5.8.40}$$

The interface is

$$y = \phi(x) = kx^{\frac{1}{2}} \tag{5.8.41}$$

and on the interface

$$\xi = \frac{\phi(x)}{x^{\frac{1}{2}}} = k. \tag{5.8.42}$$

The partial derivatives of ψ with respect to x and y are given by (5.8.32) to (5.8.34).

Tangential velocity continuous (5.2.15):

$$\frac{\partial\psi_1}{\partial y}(x,\phi(x)) = \frac{\partial\psi_2}{\partial y}(x,\phi(x)), \tag{5.8.43}$$

$$\frac{dF_1}{d\xi}(k) = \frac{dF_2}{d\xi}(k). \tag{5.8.44}$$

Normal velocity continuous (5.2.16):

$$\frac{\partial\psi_1}{\partial x}(x,\phi(x)) = \frac{\partial\psi_2}{\partial x}(x,\phi(x)), \tag{5.8.45}$$

$$2F_1(k) + k\frac{dF_1}{d\xi}(k) = 2F_2(k) + k\frac{dF_2}{d\xi}(k). \tag{5.8.46}$$

Using (5.8.44), equation (5.8.46) reduces to

$$F_1(k) = F_2(k). \tag{5.8.47}$$

Tangential stress continuous (5.2.17):

$$\frac{\partial^2\psi_1}{\partial y^2}(x,\phi(x)) = \frac{\mu_2}{\mu_1}\frac{\partial^2\psi_2}{\partial y^2}(x,\phi(x)), \tag{5.8.48}$$

$$\frac{d^2F_1}{d\xi^2}(k) = \frac{\mu_2}{\mu_1}\frac{d^2F_2}{d\xi^2}(k). \tag{5.8.49}$$

Normal stress continuous (5.2.18):

$$2\frac{\mu_2}{\mu_1}\frac{1}{Re}\frac{\partial^2\psi_2}{\partial x\partial y}(x,\phi(x)) - \frac{2}{Re}\frac{\partial^2\psi_1}{\partial x\partial y}(x,\phi(x)) = P_2(x,\phi(x)) - P_1(x,\phi(x)), \tag{5.8.50}$$

$$\frac{1}{Re}\bigg[-3\frac{\mu_2}{\mu_1}\frac{dF_2}{d\xi}(k) - k\frac{\mu_2}{\mu_1}\frac{d^2F_2}{d\xi^2}(k)\bigg]x^{\frac{5}{2}} + \frac{1}{Re}\bigg[3\frac{\mu_2}{\mu_1}\frac{dF_1}{d\xi}(k) + k\frac{\mu_2}{\mu_1}\frac{d^2F_1}{d\xi^2}(k)\bigg]x^{\frac{5}{2}}$$

$$= P_2(x,\phi(x)) - P_1(x,\phi(x)). \tag{5.8.51}$$

Substituting (5.8.44) and (5.8.49) reduces (5.8.51) to

$$\frac{3}{Re}\bigg(1-\frac{\mu_2}{\mu_1}\bigg)\frac{dF_1}{d\xi}(k)x^{\frac{5}{2}} = P_2(x,\phi(x)) - P_1(x,\phi(x)). \tag{5.8.52}$$

5.8.5 Velocity components

The velocity components expressed in terms of the invariant solution are

$$w_i(x,y) = \frac{\partial\psi_i}{\partial y} = x^{-\frac{3}{2}}\frac{dF_i}{d\xi}, \tag{5.8.53}$$

$$v_i(x,y) = \frac{\partial\psi_i}{\partial x} = -\frac{1}{2x^2}\bigg[2F_i(\xi) + \xi\frac{dF_i}{d\xi}\bigg], \tag{5.8.54}$$

$$\frac{\partial w_i}{\partial y}(x,y) = \frac{1}{x^2}\frac{d^2F_i}{d\xi^2}. \tag{5.8.55}$$

5.8.6 Lie point symmetry

We can take the additive constant c in the stream function

$$\psi(x,y) = \frac{1}{x}F(\xi) - c \tag{5.8.56}$$

to be zero because it does not contribute to the velocity components. Therefore from (5.7.82)

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\psi\frac{\partial}{\partial\psi}. \tag{5.8.57}$$

5.9 Solution of the ODE for the momentumless two-fluid wake

We will outline the solution for comparison with the similarity solution given in Chapter 3, Section 3.4.

We return to the PDE (5.8.35),

$$2\frac{\nu_i}{\nu_1}\frac{d^3F_i}{d\xi^3} + \frac{d}{d\xi}\left(\xi\frac{dF_i}{d\xi}\right) + 2\frac{dF_i}{d\xi} = 0. \tag{5.9.1}$$

Equation (5.9.1) can be integrated at least once. This is another example of the Double Reduction Theorem of Sjöberg [46]. The invariant form of (5.9.1) was obtained by associating the Lie point symmetry with a conserved vector. Integrate (5.9.1) with respect to ξ:

$$2\frac{\nu_i}{\nu_1}\frac{d^2F_i}{d\xi^2} + \xi\frac{dF_i}{d\xi} + 2F_i = A_i, \tag{5.9.2}$$

where A_i = constant.

Impose first the boundary conditions at $\xi = +\infty$. The boundary conditions at $\xi = +\infty$ are

$$\frac{dF_1}{d\xi}(\infty) = 0, \qquad \frac{d^2F_1}{d\xi^2}(\infty) = 0. \tag{5.9.3}$$

We assume the stronger condition

$$\lim_{\xi\to\infty}\xi\frac{dF_1}{d\xi}(\infty) = 0. \tag{5.9.4}$$

We will check that the solution derived satisfies this condition. We make the additional assumption that there is no entrainment of fluid at $y = \pm\infty$. We will also check that the solution derived satisfies this condition. From (5.8.54)

$$v_i(x,y) = -\frac{1}{2x^2}\left[2F_i(\xi) + \xi\frac{dF_i}{d\xi}\right] \tag{5.9.5}$$

and the condition for no entrainment implies

$$v_i(x,\pm\infty) = 0. \tag{5.9.6}$$

Thus at $\xi = +\infty$,

$$v_1(x,\infty) = 0: \qquad 2F_1(\infty) + \xi\frac{dF_1}{d\xi}\bigg|_{\xi=\infty} = 0. \tag{5.9.7}$$

Since it is assumed that

$$\lim_{\xi\to\infty}\xi\frac{dF_1}{d\xi}(\xi)=0 \tag{5.9.8}$$

it follows that

$$F_1(\infty)=0. \tag{5.9.9}$$

Therefore imposing the boundary condition at $\xi=+\infty$ on (5.9.2) we obtain

$$A_1=0. \tag{5.9.10}$$

Similarly, by imposing the boundary conditions at $\xi=-\infty$ on (5.9.2) we find that

$$A_2=0. \tag{5.9.11}$$

Equation (5.9.2) becomes

$$2\frac{\nu_i}{\nu_1}\frac{d^2F_i}{d\xi^2}+\xi\frac{dF_i}{d\xi}+2F_i=0. \tag{5.9.12}$$

In order to integrate (5.9.12) multiply by ξ. This gives

$$2\frac{\nu_i}{\nu_1}\xi\frac{d^2F_i}{d\xi^2}+\xi^2\frac{dF_i}{d\xi}+2\xi F_i=0, \tag{5.9.13}$$

which can be rewritten as

$$2\frac{\nu_i}{\nu_1}\left[\frac{d}{d\xi}\left(\xi\frac{dF_i}{d\xi}\right)-\frac{dF_i}{d\xi}\right]+\frac{d}{d\xi}(\xi^2F_i)=0. \tag{5.9.14}$$

Integrating with respect to ξ gives

$$\xi\frac{dF_i}{d\xi}-F_i+\frac{\nu_1}{2\nu_i}\xi^2F_i=E_i, \tag{5.9.15}$$

where E_i is a constant. Consider first the boundary conditions at $\xi=+\infty$. We have

$$\lim_{\xi\to\infty}\xi\,\frac{dF_1}{d\xi}=0,\qquad F_1(\infty)=0. \tag{5.9.16}$$

Assume also the stronger condition

$$\lim_{\xi\to\infty}\xi^2\,F(\xi)=0. \tag{5.9.17}$$

We will check that the solution derived also satisfies this condition. Imposing the boundary conditions at $\xi=+\infty$ on (5.9.15) gives

$$E_1=0. \tag{5.9.18}$$

Similarly by imposing the boundary condition at $\xi = -\infty$ we obtain

$$E_2 = 0. \tag{5.9.19}$$

Therefore equation (5.9.15) becomes

$$\xi \frac{dF_i}{d\xi} = \left(1 - \frac{\nu_1}{2\nu_i}\xi^2\right) F_i. \tag{5.9.20}$$

By separation of variables we obtain

$$\frac{dF_i}{F_i} = \left[\frac{1}{\xi} - \frac{\nu_1}{2\nu_i}\xi\right] d\xi \tag{5.9.21}$$

and integration gives

$$\ln(F_i) = \ln(\xi) - \frac{\nu 1}{4\nu_i}\xi^2 + B_i^*, \tag{5.9.22}$$

where B_i^* is a constant. Hence

$$F_i = B_i\, \xi \exp\left(- \frac{\nu_1}{4\nu_i}\xi^2\right), \tag{5.9.23}$$

where

$$B_i = \exp\,(B_i^*). \tag{5.9.24}$$

For the momentumless wake the velocity component v_i given by (5.8.54) depends on $F_i(\xi)$ as well as $\frac{dF_i}{d\xi}$ and $F_i(\xi)$ was therefore derived. For the classical wake the velocity components w_i and v_i given by (5.6.6) and (5.6.7) depend only on $\frac{dF_i}{d\xi}$ and it was not necessary to calculate $F_i(\xi)$.

5.9.1 Boundary conditions

The boundary conditions (5.8.36) to (5.8.39) are all identically satisfied because of the exponential behaviour of the solution. The assumptions

$$\lim_{\xi\to\infty} \xi \frac{dF_1}{d\xi} = 0, \qquad \lim_{\xi\to\infty} \xi^2 F_1(\infty) = 0, \tag{5.9.25}$$

$$\lim_{\xi\to-\infty} \xi \frac{dF_2}{d\xi} = 0, \qquad \lim_{\xi\to-\infty} \xi^2 F_2(-\infty) = 0 \tag{5.9.26}$$

are also identically satisfied because of the exponential behaviour of the solution.

5.9.2 Interface conditions

We express the interface conditions in terms of the solution (**??**) for $F_i(\xi)$ and its derivatives:

$$F_i = B_i\,\xi \exp\bigg(-\frac{\nu_1}{4\nu_i}\xi^2\bigg), \tag{5.9.27}$$

$$\frac{dF_i}{d\xi} = B_i\bigg(1-\frac{\nu_1}{2\nu_i}\xi^2\bigg)\exp\bigg(-\frac{\nu_1}{4\nu_i}\xi^2\bigg), \tag{5.9.28}$$

$$\frac{d^2F_i}{d\xi^2} = -\frac{1}{4}B_i\,\frac{\nu_1}{\nu_i}\,\xi\bigg(6-\frac{\nu_1}{\nu_i}\xi^2\bigg)\exp\bigg(-\frac{\nu_1}{4\nu_i}\xi^2\bigg). \tag{5.9.29}$$

Normal velocity (5.8.47):

$$F_1(k) = F_2(k), \tag{5.9.30}$$

$$B_1\,k\,\exp\bigg(-\frac{1}{4}k^2\bigg) = B_2\,k\,\exp\bigg(-\frac{1}{4}\frac{\nu_1}{\nu_2}k^2\bigg). \tag{5.9.31}$$

Tangential velocity (5.8.44):

$$\frac{dF_1}{d\xi}(k) = \frac{dF_2}{d\xi}(k), \tag{5.9.32}$$

$$B_1\bigg(1-\frac{1}{2}k^2\bigg)\exp\bigg(-\frac{1}{4}k^2\bigg) = B_2\bigg(1-\frac{\nu_1}{2\nu_2}k^2\bigg)\exp\bigg(-\frac{1}{4}\frac{\nu_1}{\nu_2}k^2\bigg). \tag{5.9.33}$$

Tangential stress (5.8.49):

$$\frac{d^2F_1}{d\xi^2}(k) = \frac{\mu_2}{\mu_1}\frac{d^2F_2}{d\xi^2}(k), \tag{5.9.34}$$

$$B_1\,k\,(6-k^2)\exp\bigg(-\frac{1}{4}k^2\bigg) = \frac{\rho_2}{\rho_1}B_2\,k\,\bigg(6-\frac{\nu_1}{\nu_2}k^2\bigg)\exp\bigg(-\frac{1}{4}\frac{\nu_1}{\nu_2}k^2\bigg), \tag{5.9.35}$$

where we used

$$\frac{\mu_2}{\mu_1}\frac{\nu_1}{\nu_2} = \frac{\rho_2}{\rho_1} \tag{5.9.36}$$

Multiply the interface conditions (5.9.31), (5.9.33) and (5.9.35) by

$$\exp\bigg(\frac{1}{4}k^2\bigg) \tag{5.9.37}$$

the following interface conditions are obtained

$$k\left[B_1 - B_2 \exp\left(\frac{1}{4}\left(1 - \frac{\nu_1}{\nu_2}\right)k^2\right)\right] = 0, \tag{5.9.38}$$

$$B_1\left(1 - \frac{1}{2}k^2\right) - B_2\left(1 - \frac{\nu_1}{2\nu_2}k^2\right)\exp\left(\frac{1}{4}\left(1 - \frac{\nu_1}{\nu_2}\right)k^2\right) = 0, \tag{5.9.39}$$

$$k\left[B_1(6 - k^2) - B_2\frac{\rho_2}{\rho_1}\left(6 - \frac{\nu_1}{\nu_2}k^2\right)\exp\left(\frac{1}{4}\left(1 - \frac{\nu_1}{\nu_2}\right)k^2\right)\right] = 0. \tag{5.9.40}$$

We consider two cases, $k = 0$ and $k \neq 0$.

5.9.2.1 Case 1: $k = 0$

Equations (5.9.38) and (5.9.40) are identically satisfied. Equation (5.9.39) becomes

$$B_1 = B_2 = B. \tag{5.9.41}$$

The solution (5.9.23) becomes

$$F_i = B\,\xi\,\exp\left(-\frac{\nu_1}{4\nu_i}\xi^2\right). \tag{5.9.42}$$

5.9.2.2 Case 2: $k \neq 0$

Equations (5.9.38) to (5.9.40) become

$$B_1 - B_2 \exp\left(\frac{1}{4}\left(1 - \frac{\nu_1}{\nu_2}\right)k^2\right) = 0, \tag{5.9.43}$$

$$B_1\left(1 - \frac{1}{2}k^2\right) - B_2\left(1 - \frac{\nu_1}{2\nu_2}k^2\right)\exp\left(\frac{1}{4}\left(1 - \frac{\nu_1}{\nu_2}\right)k^2\right) = 0, \tag{5.9.44}$$

$$B_1(6 - k^2) - B_2\frac{\rho_2}{\rho_1}\left(6 - \frac{\nu_1}{\nu_2}k^2\right)\exp\left(\frac{1}{4}\left(1 - \frac{\nu_1}{\nu_2}\right)k^2\right) = 0. \tag{5.9.45}$$

From (5.9.43)

$$B_1 = B_2 \exp\left(\frac{1}{4}\left(1 - \frac{\nu_1}{\nu_2}\right)k^2\right). \tag{5.9.46}$$

Substitute (5.9.46) into (5.9.44). Thus

$$B_2\left(1-\frac{\nu_1}{\nu_2}\right)\exp\left(\frac{1}{4}\left(1-\frac{\nu_1}{\nu_2}\right)k^2\right)=0. \tag{5.9.47}$$

For a non-trivial solution ($B_2 \neq 0$) it is necessary that

$$\frac{\nu_1}{\nu_2}=1, \qquad \nu_1=\nu_2. \tag{5.9.48}$$

From (5.9.46), since $\nu_1=\nu_2$

$$B_1=B_2=B. \tag{5.9.49}$$

With $B_1=B_2=B$ and $\nu_1=\nu_2$, equation (5.9.45) reduces to

$$B\,(6-k^2)\left(1-\frac{\rho_2}{\rho_1}\right)=0. \tag{5.9.50}$$

If $\rho_1=\rho_2$ then since $\nu_1=\nu_2$ it follows that $\mu_1=\mu_2$. There would be only one fluid. We therefore assume

$$\rho_1 \neq \rho_2. \tag{5.9.51}$$

Hence from (5.9.50), for $B \neq 0$

$$6-k^2=0 \tag{5.9.52}$$

and therefore

$$k=\pm\sqrt{6}. \tag{5.9.53}$$

The solution (5.9.23) becomes

$$F_i(\xi)=B\xi\exp\left(-\frac{1}{4}\xi^2\right). \tag{5.9.54}$$

Finally, the fourth interface condition (5.8.52) is

$$\frac{3}{Re}\left(1-\frac{\mu_2}{\mu_1}\right)\frac{dF_1}{d\xi}(k)x^{-\frac{5}{2}}=P_2(x,\phi(x))-P_1(x,\phi(x)). \tag{5.9.55}$$

This interface condition gives the pressure difference across the interface.

When $k=0$ we have from (5.9.28)

$$\frac{dF_1}{d\xi}(k)=B\left(1-\frac{1}{2}k^2\right)\exp\left(-\frac{1}{4}k^2\right)\Big|_{k=0}=B. \tag{5.9.56}$$

Therefore (5.9.55) becomes

$$\frac{3}{Re}\left(1-\frac{\mu_2}{\mu_1}\right)B\,x^{-\frac{5}{2}} = P_2(x,0) - P_1(x,0). \tag{5.9.57}$$

When $k = \pm\sqrt{6}$ we have from (5.9.28)

$$\frac{dF_1}{d\xi}(\pm\sqrt{6}) = B(1-\frac{1}{2}k^2)\exp\left(-\frac{1}{4}k^2\right)\Big|_{k=\pm\sqrt{6}}$$

$$= -2B\exp\left(-\frac{3}{2}\right). \tag{5.9.58}$$

Therefore (5.9.55) becomes

$$-\frac{6}{Re}B\left(1-\frac{\mu_2}{\mu_1}\right)\exp\left(-\frac{3}{2}\right)x^{-\frac{5}{2}} = P_2(x,\pm\sqrt{6}) - P_1(x,\pm\sqrt{6}). \tag{5.9.59}$$

5.9.3 Calculation of constant B when $k = 0$

The constant B is obtained from the conserved quantity which is when $k = 0$,

$$K = \frac{\rho_2}{\rho_1}\frac{\nu_1}{\nu_2}\int_{-\infty}^{0}\xi^2\frac{dF_2}{d\xi}d\xi + \int_0^{\infty}\xi^2\frac{dF_1}{d\xi}d\xi, \tag{5.9.60}$$

where $F_i(\xi)$ is given by (5.9.42) and

$$\frac{dF_2}{d\xi} = B\left(1-\frac{1}{2}\frac{\nu_1}{\nu_2}\xi^2\right)\exp\left(-\frac{\nu_1}{4\nu_2}\xi^2\right), \tag{5.9.61}$$

$$\frac{dF_1}{d\xi} = B\left(1-\frac{1}{2}\xi^2\right)\exp\left(-\frac{1}{4}\xi^2\right). \tag{5.9.62}$$

Equations (5.9.60), (5.9.61) and (5.9.62) are exactly the same as equations (3.4.70), (3.4.74) and (3.4.75) used to calculate the constant B in Section 3.4.2 for the similarity solution for the momentumless wake. It was found that

$$B = -\frac{K}{4\sqrt{\pi}\left[1+\frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}}\right]} \tag{5.9.63}$$

which can be rewritten in terms of the parameter

$$\chi = \frac{\rho_1\mu_1}{\rho_2\mu_2} \tag{5.9.64}$$

as

$$B = -\frac{K\,\chi^{\frac{1}{2}}}{4\sqrt{\pi}\left[1+\chi^{\frac{1}{2}}\right]}. \tag{5.9.65}$$

5.9.4 Calculation of constant B when $k = \pm\sqrt{6}$

The constant B is obtained from the conserved quantity (5.8.29) with $\frac{\nu_1}{\nu_2} = 1$,

$$K = \frac{\rho_2}{\rho_1}\int_{-\infty}^{k} \xi^2 \frac{dF_2}{d\xi}d\xi + \int_{k}^{\infty} \xi^2 \frac{dF_1}{d\xi}d\xi, \tag{5.9.66}$$

where K is prescribed and k takes the values $+\sqrt{6}$ and $-\sqrt{6}$. From (5.9.54)

$$F_i(\xi) = B\,\xi \exp\left(-\frac{1}{4}\xi^2\right) \tag{5.9.67}$$

and

$$\frac{dF_1}{d\xi} = \frac{dF_2}{d\xi} = B\left(1-\frac{1}{2}\xi^2\right)\exp\left(-\frac{1}{4}\xi^2\right). \tag{5.9.68}$$

The value of the constant B will depend on whether $k = +\sqrt{6}$ or $k = -\sqrt{6}$. Equations (5.9.66) and (5.9.68) are the same as (3.4.112) and (3.4.115) used in Section 3.4.4 to calculate the constant B for the similarity solution for the momentumless wake. The results (3.4.147) and (3.4.148) therefore apply here:

$$k = +\sqrt{6}: \qquad B = B_+ = -\frac{K}{4\sqrt{\pi}\left[\frac{\rho_2}{\rho_1}+1+\left(\frac{\rho_2}{\rho_1}-1\right)S\right]} < 0, \tag{5.9.69}$$

$$k = -\sqrt{6}: \qquad B = B_- = -\frac{K}{4\sqrt{\pi}\left[\frac{\rho_2}{\rho_1}+1-\left(\frac{\rho_2}{\rho_1}-1\right)S\right]} < 0, \tag{5.9.70}$$

where

$$S = \mathrm{erf}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right) - \frac{5}{\sqrt{\pi}}\left(\frac{3}{2}\right)^{\frac{1}{2}}\exp\left(-\frac{3}{2}\right) = 0.1458 \tag{5.9.71}$$

and the Error Function is defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-u^2\right)du. \tag{5.9.72}$$

5.9.5 Approximation in the conserved quantity for the momentumless wake

In Section 4.4.2.4 the term neglected in the derivation of the conserved quantity for the momentumless wake, $R(\phi(x), w_1, w_2)$, was simplified. It was shown that

$$R(\phi(x), w_1, w_2) = \left(\frac{\rho_2}{\rho_1} - 1\right) w_1(x, \phi(x))\phi(x)\left(\phi(x)\frac{d\phi(x)}{dx} - 2\right)$$

$$+ \left(\frac{\nu_1}{\nu_2} - 1\right)\frac{\partial w_1}{\partial y}(x, \phi(x))\phi^2(x). \tag{5.9.73}$$

There are three momentumless wakes.

5.9.5.1 Case 1: $k = 0$

Then

$$R(\phi(x), w_1, w_2) = 0 \tag{5.9.74}$$

and there is no approximation made in the derivation of the conserved quantity.

5.9.5.2 Case 2: $k = \pm\sqrt{6}$

For this wake

$$\frac{\nu_1}{\nu_2} = 1. \tag{5.9.75}$$

Also

$$\phi(x) = k\, x^{\frac{1}{2}}, \qquad \frac{d\phi}{dx} = \frac{1}{2}\, k\, x^{-\frac{1}{2}} \tag{5.9.76}$$

and therefore

$$\phi(x)\frac{d\phi}{dx} = \frac{1}{2}\, k^2 = 3. \tag{5.9.77}$$

Equation 5.9.73 reduces to

$$R(\phi(x), w_1, w_2) = \left(\frac{\rho_2}{\rho_1} - 1\right) w(x, \phi(x))\phi(x). \tag{5.9.78}$$

Now

$$w_1(x,y) = \frac{B}{x^{\frac{3}{2}}}\left(1 - \frac{1}{2}\xi^2\right)\exp\left(-\frac{1}{4}\xi^2\right) \tag{5.9.79}$$

and therefore

$$w_1(x,y) = \frac{B}{x^{\frac{3}{2}}}\left(1 - \frac{1}{2}k^2\right)\exp\left(-\frac{1}{4}k^2\right) \tag{5.9.80}$$

$$= -\frac{2B}{x^{\frac{3}{2}}}\exp\left(-\frac{3}{2}\right) \tag{5.9.81}$$

$$= \mathcal{O}\left(x^{-\frac{3}{2}}\right) \qquad \text{as } x \to \infty. \tag{5.9.82}$$

Also

$$\phi(x) = k\,x^{\frac{1}{2}} = \mathcal{O}\left(x^{\frac{1}{2}}\right) \qquad \text{as } x \to \infty. \tag{5.9.83}$$

Hence

$$R(\phi(x), w_1, w_2) = \mathcal{O}\left(\frac{1}{x}\right) \qquad \text{as } x \to \infty. \tag{5.9.84}$$

For the three momentumless wakes there is no approximation for $k = 0$ and for $k = \pm\sqrt{6}$ since the neglected term is $\mathcal{O}\left(\frac{1}{x}\right)$ which is small for the far wake, $x >> 1$.

5.9.6 Visual representation of the solutions for the momentumless wakes

The invariant solution for the momentumless wake for $k = 0$ is visually represented in Figures 5.3 and 5.4.

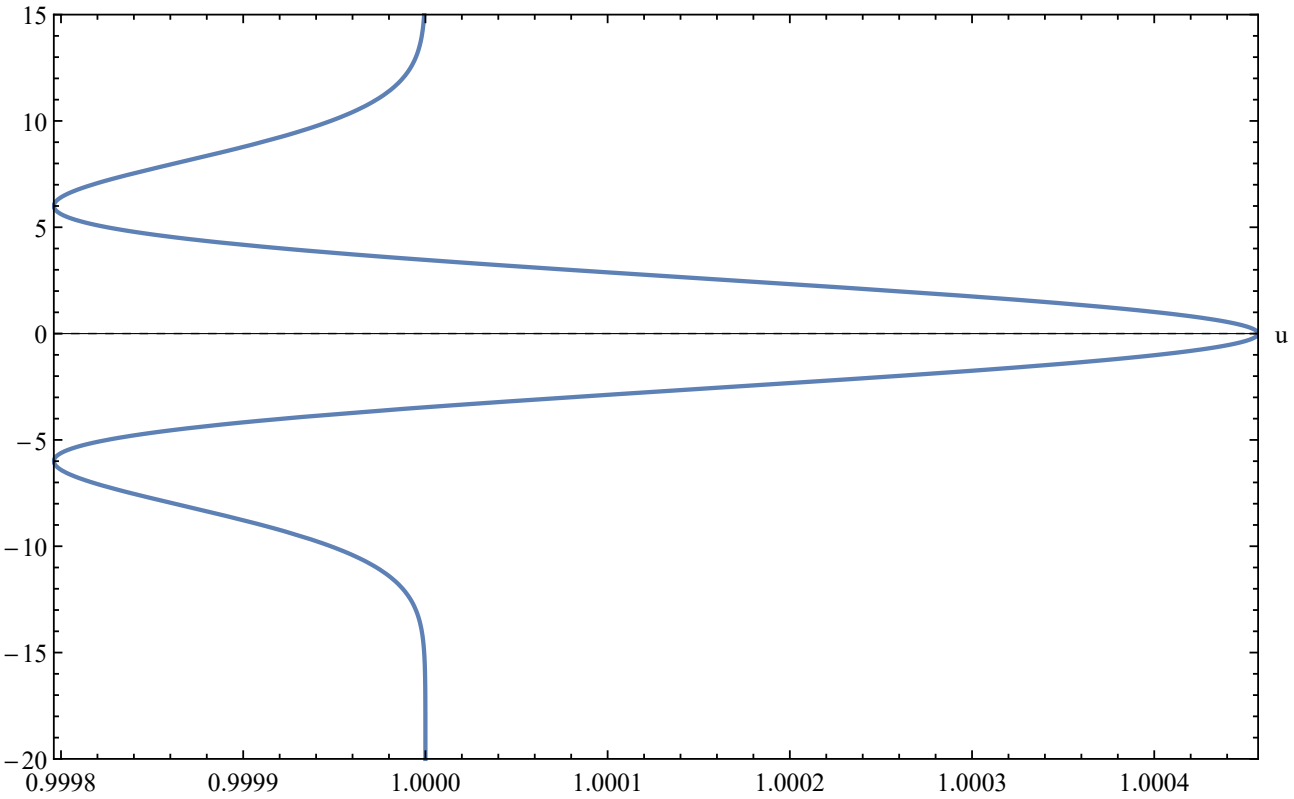

Figure 5.3: The velocities $u_1(x,y)$ and $u_2(x,y)$ for the momentumless two-fluid wake for $k=0$ plotted against y at $x=6$ for $K=1$, $\frac{\rho_1}{\rho_2}=\frac{1}{20}$ and $\frac{\nu_1}{\nu_2}=1$.

The velocity profile is given by

$$u_1(x,y)=1-w_1(x,y)=1-\frac{1}{2x^{\frac{3}{2}}}B\left[2-\frac{y^2}{x}\right]\exp\left(-\frac{1}{4}\frac{y^2}{x}\right), \tag{5.9.85}$$

$$u_2(x,y)=1-w_2(x,y)=1-\frac{1}{2x^{\frac{3}{2}}}\frac{\nu_1}{\nu_2}B\left[2\frac{\nu_2}{\nu_1}-\frac{y^2}{x}\right]\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_2}\frac{y^2}{x}\right) \tag{5.9.86}$$

and

$$B=-\frac{K}{4\sqrt{\pi}\left[1+\frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}}\right]}. \tag{5.9.87}$$

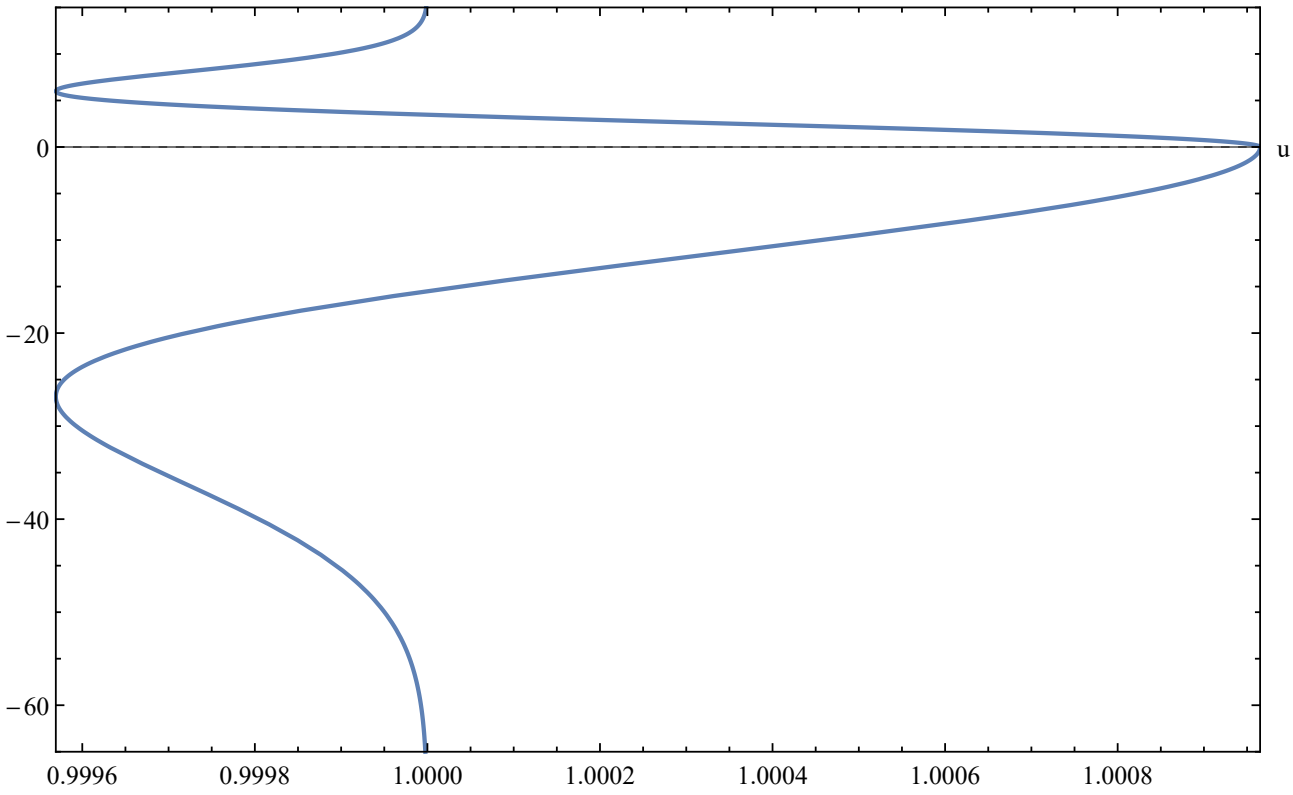

Figure 5.4: The velocities $u_1(x, y)$ and $u_2(x, y)$ for the momentumless two-fluid wake for $k = 0$ plotted against y at $x = 6$ for $K = 1$, $\frac{\rho_1}{\rho_2} = \frac{1}{2}$ and $\frac{\nu_1}{\nu_2} = \frac{1}{20}$.

The velocity profile (5.9.85) and (5.9.86) is the same as (3.4.181) and (3.4.182) for the similarity solution derived in Chapter 3. We draw graphs of the velocity profiles again but with different values for the parameters from Chapter 3. In Figure 5.3 we choose $\frac{\rho_1}{\rho_2} = \frac{1}{20}$ and $\frac{\nu_1}{\nu_2} = 1$ to examine the effect of the density ratio. In Figure 5.4 the effect of the viscosity ratio is examined by choosing $\frac{\nu_1}{\nu_2} = \frac{1}{20}$ and $\frac{\rho_1}{\rho_2} = \frac{1}{2}$.

From Figure 5.3 we see that the effect of the larger density ratio $\frac{\rho_1}{\rho_2}$ is to reduce the magnitude of the velocity difference at the turning points in the wake. From Figure 5.4 we see that the effect of increasing the viscosity ratio $\frac{\nu_2}{\nu_1}$ is to increase the effective width of the lower wake. The effective width of the upper wake is not altered.

The invariant solution for the momentumless wake for $k = \pm\sqrt{6}$ is visually represented for $+\sqrt{6}$ in Figure 5.5 and for $-\sqrt{6}$ in Figure 5.6.

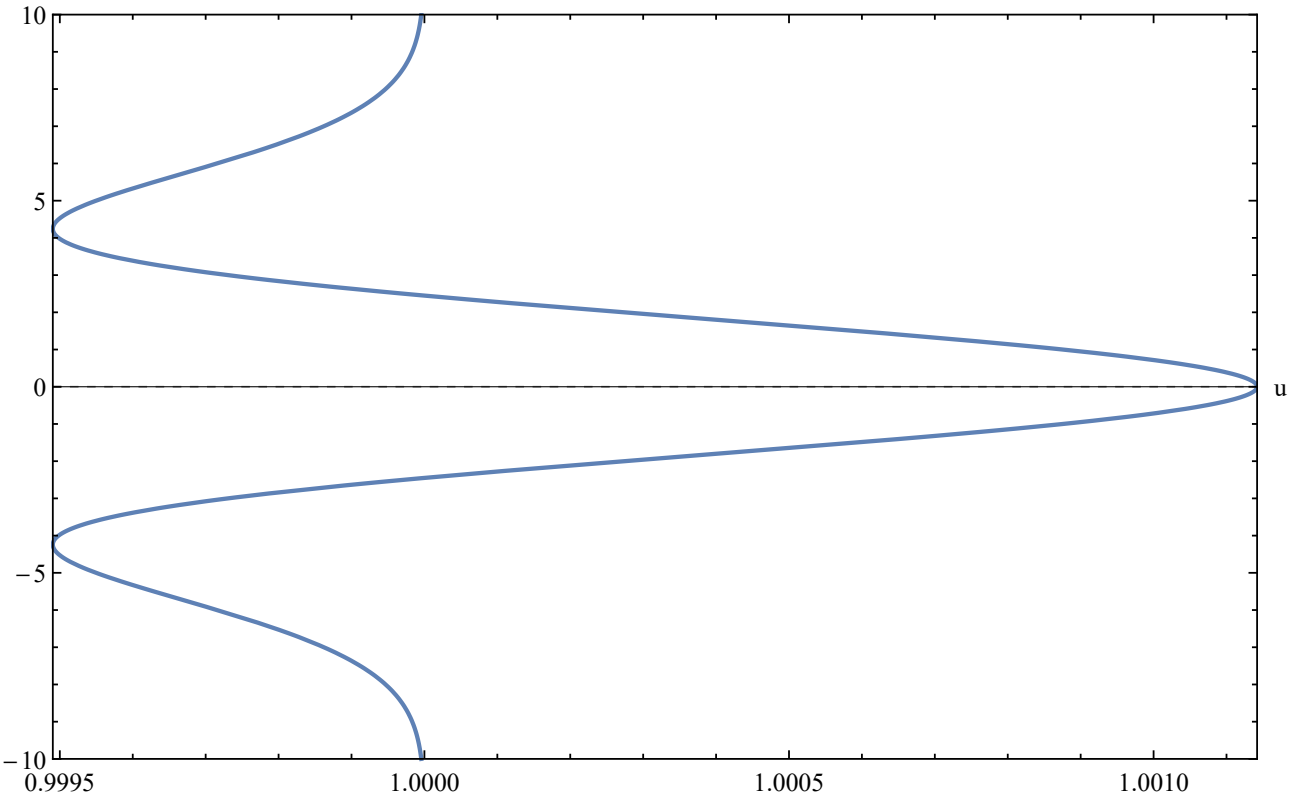

Figure 5.5: The velocity $u(x,y)$ for the momentumless two-fluid wake for $k=+\sqrt{6}$ plotted against y at $x=3$ for $K=1$, $\frac{\rho_1}{\rho_2}=\frac{1}{20}$ and S $=0.1458$.

The velocity profile is given by

$$u_1(x,y)=u_2(x,y)=1-w_1(x,y)=1-\frac{B_+}{x^{\frac{3}{2}}}\left(1-\frac{y^2}{2x}\right)\exp\left(-\frac{1}{4}\frac{y^2}{x}\right) \tag{5.9.88}$$

and

$$B_+=-\frac{K}{4\sqrt{\pi}\left[\frac{\rho_2}{\rho_1}+1+\left(\frac{\rho_2}{\rho_1}-1\right)S\right]}. \tag{5.9.89}$$

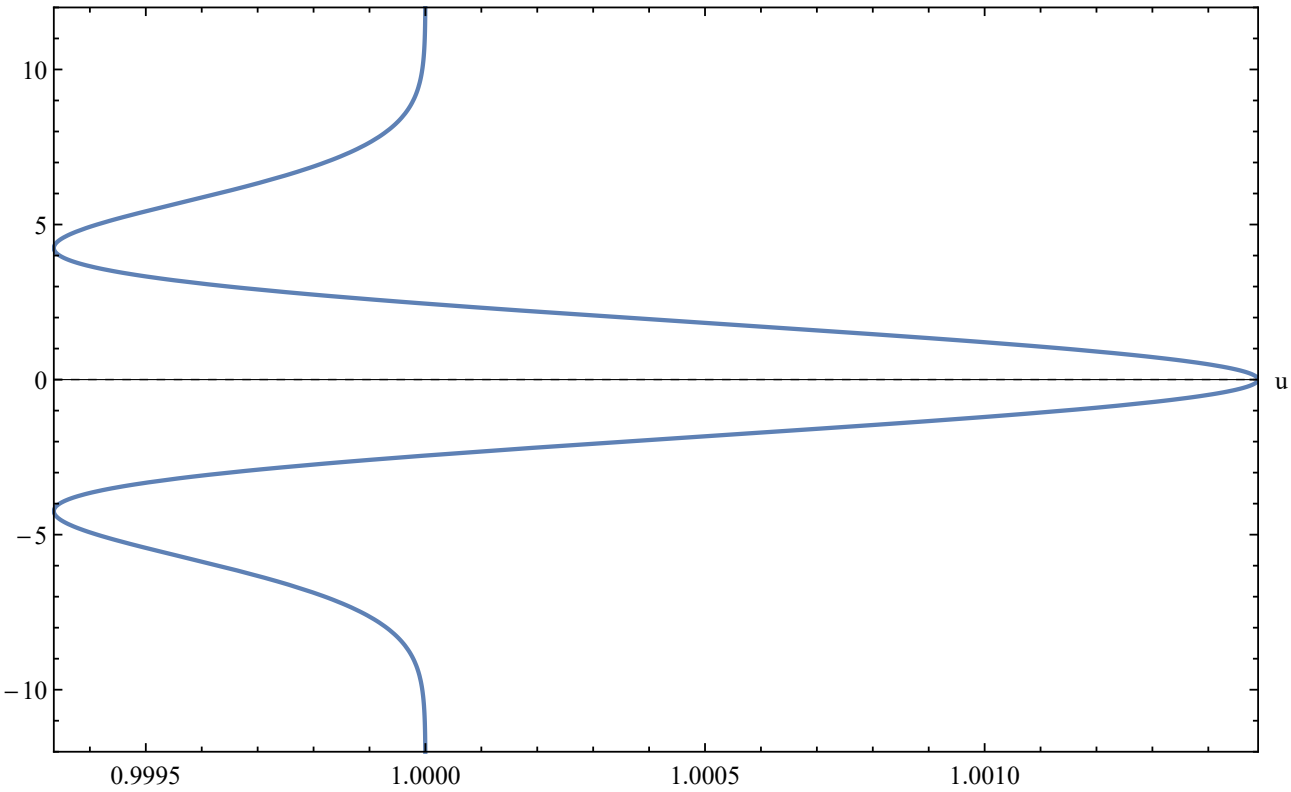

Figure 5.6: The velocity $u(x,y)$ for the momentumless two-fluid wake for $k=-\sqrt{6}$ plotted against y at $x=3$ for $K=1$, $\frac{\rho_1}{\rho_2}=\frac{1}{20}$ and S $=0.1458$.

The velocity profile is given by

$$u_1(x,y)=u_2(x,y)=1-w_1(x,y)=1-\frac{B_-}{x^{\frac{3}{2}}}\left(1-\frac{y^2}{2x}\right)\exp\left(-\frac{1}{4}\frac{y^2}{x}\right) \tag{5.9.90}$$

where

$$B_-=-\frac{K}{4\sqrt{\pi}\left[\frac{\rho_2}{\rho_1}+1-\left(\frac{\rho_2}{\rho_1}-1\right)S\right]} \tag{5.9.91}$$

and

$$S=\operatorname{erf}\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right)-\frac{5}{\sqrt{\pi}}\left(\frac{3}{2}\right)^{\frac{1}{2}}\exp\left(-\frac{3}{2}\right)=0.1458 \tag{5.9.92}$$

Since the velocity profiles are the same as in Chapter 3 for the similarity solution, a different value for the density ratio, $\frac{\rho_1}{\rho_2}=\frac{1}{20}$, is chosen in Figures 5.5 and 5.6.

It is seen in Figures 5.5 and 5.6 that the increase in the density ratio, $\frac{\rho_2}{\rho_1}=20$, produces a decrease in the magnitude of the velocity deficit.

The interface is

$$y = kx^{\frac{1}{2}}. \tag{5.9.93}$$

In Figures 5.3 and 5.4, $k = 0$ and the interface is at $y = 0$. In Figure 5.5, $k = +\sqrt{6}$ and the interface is at $y = +\sqrt{6}x^{\frac{1}{2}}$ while in Figure 5.6, $k = -\sqrt{6}$ and the interface is at $y = -\sqrt{6}x^{\frac{1}{2}}$.

5.10 Summary of results for the two-fluid wake behind a self-propelled body

The results derived thus far for the two-fluid wake behind a self-propelled body are summarised below.

$$\psi_i(x,y) = \frac{1}{x}F_i(\xi), \qquad \xi = \frac{y}{x^{\frac{1}{2}}}, \tag{5.10.1}$$

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\psi\frac{\partial}{\partial \psi}, \tag{5.10.2}$$

$$w_i(x,y) = x^{-\frac{3}{2}}\frac{dF_i}{d\xi}, \tag{5.10.3}$$

$$v_i(x,y) = -\frac{1}{2x^2}\left[2F_i(\xi) + \xi\frac{dF_i}{d\xi}\right], \tag{5.10.4}$$

$$\frac{\partial w_i}{\partial y} = x^{-2}\frac{d^2F_i}{d\xi^2}. \tag{5.10.5}$$

When $k = 0$,

$$\frac{\nu_1}{\nu_2} \neq 1, \tag{5.10.6}$$

$$F_i(\xi) = B\xi\exp\left(-\frac{1}{4}\frac{\nu_1}{\nu_i}\xi^2\right), \tag{5.10.7}$$

$$B = -\frac{K}{4\sqrt{\pi}\left[1 + \left(\frac{\mu_2\rho_2}{\mu_1\rho_1}\right)^{\frac{1}{2}}\right]}. \tag{5.10.8}$$

When $k = \pm\sqrt{6}$,

$$\frac{\nu_1}{\nu_2} = 1, \tag{5.10.9}$$

$$F_i(\xi) = B\xi \exp\left(-\frac{1}{4}\xi^2\right), \tag{5.10.10}$$

$$B_+ = -\frac{K}{4\sqrt{\pi}\left[\frac{\rho_2}{\rho_1} + 1 + \left(\frac{\rho_2}{\rho_1} - 1\right)S\right]}, \tag{5.10.11}$$

$$B_- = -\frac{K}{4\sqrt{\pi}\left[\frac{\rho_2}{\rho_1} + 1 - \left(\frac{\rho_2}{\rho_1} - 1\right)S\right]}, \tag{5.10.12}$$

$$S = erf\left(\left(\frac{3}{2}\right)^{\frac{1}{2}}\right) - \frac{5}{\sqrt{\pi}}\left(\frac{3}{2}\right)^{\frac{1}{2}} \exp\left(-\frac{3}{2}\right) = 0.1458. \tag{5.10.13}$$

5.11 Conclusions

We started Chapter 5 by summarising the formulation of the problem in terms of the stream function which is the same for both the classical wake and the momentumless wake. The boundary and interface conditions in terms of the stream function are also provided. This was derived in Chapter 2.

The associated Lie point symmetry for the classical two-fluid wake was derived using the conserved vector. This conserved vector was determined in Chapter 4. The Lie point symmetry was then used to derive the invariant solution. There is one exact solution for the two-fluid classical wake. For this solution, the interface is at $\phi(x) = 0$. The results obtained by deriving the invariant solution and solving the ODE are the same results as derived in Section 3.2. We found that we could derive the solution for the classical two-fluid wake by a similarity solution and also by the use of conservation laws and Lie point symmetries.

Similarly, we derived the Lie point symmetry associated with the conserved vector for the momentumless wake and then used the Lie point symmetry to derive the invariant solution. The results are the same as the results derived in Section 3.4 which means that we have

developed two methods of deriving the solution for the two-fluid wake behind a self-propelled body.

Chapter 6

Conclusions

In Chapter 2, the governing equations were derived for the two-fluid far wake. This was done through non-dimensionalising the Navier-Stokes equations and then imposing the boundary layer approximation. The equations have been expressed in terms of the velocity components where the y-component of the mean velocity is given by $\bar{v}$ and the velocity deficit in the x-direction is given by $\bar{w}$. The situation was expressed diagrammatically and the problem of the two-fluid wake was described. The boundary conditions were first derived in terms of the velocity components and the dimensionless form given. The interface was first defined by $y = \phi(x)$, and the interface conditions were defined based on the fact that the velocity and stress components should balance at the interface and a fluid particle on the interface should remain on the interface. The conserved quantities for the classical and momentumless wake were derived. A stream function, ψ, was introduced. The governing equations as well as the boundary and interface conditions were defined in terms of the newly introduced stream function.

In Chapter 3, a scaling transformation was introduced and an invariant solution was derived for both the classical and momentumless wakes through the use of the boundary and interface conditions, as well as the conserved quantity which aided in the determination of the equation of the interface. For the classical two-fluid wake, the interface was determined to be at $y = 0$, whereas there were three potential interface equations determined for the momentumless wake. Further to this, the method presented by Herczynski. Weidman and Burde [1] was also replicated. For the momentumless wake a new conserved quantity was used which was derived based on the condition that the total drag on the body is zero. The solution was derived through the use of a coordinate expansion and expansion of the interface deflection. The velocity profiles for all solutions were plotted and it is clear from the profiles that the

general shape of the velocity profiles were the same. The interface is deflected upwards in the solution derived through the Herczynski. Weidman and Burde approach.

In Chapter 4, conservation laws were discussed and the multiplier method was explored. The multiplier method was applied to the governing equations for the two-fluid far wake flow. The conserved vectors were determined in which four cases were discovered. The first belonged to the wake of a self-propelled body, the second to the combination wake, the third to the classical wake and the fourth was related to the conservation of mass equation. An alternative method using a linear combination of conserved vectors was also used and this method easily demonstrates how to generate the general form of the conserved vector and further break down the general form into the four cases. All of the conserved vectors were obtained in terms of the velocity components. The conserved quantities for the combined upper and lower wakes were then obtained from the conserved vectors and boundary and interface conditions. For the classical wake the matching of the tangential stress at the interface was used to derive the conserved quantity while for the momentumless wake the requirement that the total drag on the body is zero was used to derive the conserved quantity. The conserved quantities generated in Chapter 4 are the same as the conserved quantities originally derived in Chapter 2.

In Chapter 5, the conserved vectors and the conserved quantities for the classical and momentumless two-fluid far wakes were rewritten in terms of the stream function which was introduced in Chapter 2. Lie point symmetries associated with the conserved vectors derived in Chapter 4 were determined for the classical and momentumless wakes. The Lie point symmetries were then used to generate the invariant solutions. The solutions derived in this chapter are the same as those derived in Chapter 3 using a scaling transformation. Hence, two methods have been used to generate the invariant solution for the classical two-fluid wake and the two-fluid wake behind a self-propelled body. The velocity profiles were also plotted. Since these are the same as those in Chapter 3 different values of the parameters were used. One exact analytical solution was derived for the classical wake where the interface is $y = 0$ and three exact solutions were derived for the momentumless wake corresponding to the three possible interface positions.

Bibliography

[1] A. Herczynski, P. D. Weidman, G.I. Burde. Two-fluid jets and wakes. Physics of Fluids. AIP., (2004). Vol. 16, No. 4, pp. 1037-1048.

[2] S. Goldstein. Concerning some solutions of the boundary layer equations in hydrodynamics. Proc. Cambridge Phil. Soc., (1929). Vol. 26, Pt. 1, 1-30.

[3] S. Goldstein. On the two-dimensional steady flow of a viscous fluid behind a solid body. Proc. R. Soc. Lond.Math. Phys. Eng. Sci. 142 (1933) 545-561.

[4] S. Goldstein. Modern Developments in Fluid Dynamics, Volume 1, Clarendon Press, Oxford, 1938, pp. 148 and 149.

[5] G. Birkhoff, E. H. Zarantonello. Jets, Wakes, and Cavities (Series in Applied Mathematics and Mechanics, vol. 2). (1957).

[6] G.H. Keulegan. Laminar flow at the interface of two liquids. US Bureau of Standards. Journal of Research., (1944). 32, pp. 303.

[7] R.C. Lock. The velocity distribution in the laminar boundary layer between parallel streams. The Quarterly Journal of Mechanics and Applied Mathematics., (1951). 4(1), pp. 42-63.

[8] O.E. Potter. Laminar boundary layers at the interface of co-current parallel streams. The Quarterly Journal of Mechanics and Applied Mathematics., (1957). 10(3), pp. 302-311.

[9] R. Naz. Symmetry Solutions and Conservation Laws for some Partial Differential Equations in Fluid Mechanics. PhD Thesis, University of the Witwatersrand, Johannesburg, South Africa, (2008).

[10] R. Naz, I. Naeem. Group invariant solutions for two-dimensional free, wall and liquid jets having finite fluid velocity at orifice, in: World Academy of Science, Engineering and Technology, vol 43, 2008, pp 275-281.

[11] D.P. Mason, D.L. Hill. Invariant solution for an axisymmetric turbulent free jet using a conserved vector. Commun. Nonlinear Sci. Numer. Simulat. 18 (2013) 1607-1622.

[12] M. Anthonyrajah, D.P. Mason. Conservation laws and invariant solution in the Fanno model for turbulent compressible flow. Math. Comput. Appl. 15 (2010) 529-542.

[13] E. Noether. Invariant variation problems. Transport Theory and Statistical Physics. 1971 Jan 1;1(3):186-207.

[14] A.H. Kara, F.M. Mahomed. Relationship between symmetries and conservation laws. International Journal of Theoretical Physics. 2000 Jan;39(1):23-40.

[15] A.H. Kara, F.M. Mahomed. A basis of conservation laws for partial differential equations. J. Nonlinear Math. Phys. 9 (Supplement 2) (2002) 60-72.

[16] A.J. Hutchinson. Turbulent wake flows: Lie group analysis and conservation laws, PhD Thesis, University of the Witwatersrand, Johannesburg, South Africa, (2018).

[17] A.J. Hutchinson, N. Hale, K. Born, and D.P. Mason, 2021. Prandtl's extended mixing length model applied to the two-dimensional turbulent classical far wake. Proceedings of the Royal Society A, 477(2246), p.20200875.

[18] H. Steudel, Über die zuordnung zwischen invarianzeigenschaften und erhaltungssatzen, Z. Naturforsch A, 1962, 17(2), 129-132.

[19] L. Prandtl, in: Proceedings of the Second International Congress for Applied Mechanics, Zurich 1926, p 62.

[20] H. Tennekes, J.L. Lumley. A First Course in Turbulence, MIT Press, Cambridge, Massachusetts and London 1971, pp. 113-124.

[21] Z. Rotem. A note on boundary-layer solutions for pseudoplastic fluids. Chem. Engng. Sci. 21 (1966) 618-620.

[22] P.D. Weidman, C.W. Van Atta. The laminar axisymmetric wake for power-law fluids. Acta Mech. 146 (2001) 239-245.

[23] L.M. Swain. On the turbulent wake behind a body of revolution. Proc. R. Soc. Lond. Ser. A 125 (1929) 647-659.

[24] J.C.R. Hunt. A theory for the laminar wake of a two-dimensional body in a boundary layer. J. Fluid Mech. 49 (1971) 159-178.

[25] J.C.R. Hunt, P.S. Jackson. Wakes behind two-dimensional surface obstacles in turbulent boundary layers. J. Fluid Mech. 64 (1974) 529-563.

[26] G. Unal. Application of equivalence transformations to inertial subrange of turbulence. Lie Groups Appl. 1 (1994) 232-240.

[27] N.H. Ibragimov, G. Unal. Lie groups in turbulence. Lie Groups Appl. 1 (1994) 98-103.

[28] M. Oberlack. A unified approach for symmetries in plane parallel turbulent shear flows. J. Fluid Mech. 42 (2001) 299-328.

[29] D.P.Mason. Group invariant solution and conservation law for a free laminar two-dimensional jet, J. Nonlinear Math. Phys. 9 (Supplement 2) (2002) 92-101.

[30] I. Ruscic, D.P. Mason. Group invariant solution and conservation law for a steady laminar axisymmetric free jet. Quaest.Math. 27 (2004) 171-183.

[31] R. Naz, F.M. Mahomed, D.P. Mason. Comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics. Appl. Math. Comput. 205 (2008) 212-230.

[32] S.C. Anco, G.W. Bluman. Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications. Euro. J. Appl.Math. 13 (2002) 545-566.

[33] S.C. Anco, G.W. Bluman. Direct construction method for conservation laws of partial differential equations Part II: General treatment. Euro. J. Appl.Math. 13 (2002) 567-585.

[34] S.C. Anco, G. Bluman. Direct Construction of Conservation Laws from Field Equations. Phys. Rev. Lett. 78 (1997), 2869-2873.

[35] T. Wolf. A comparison of four approaches to the calculation of conservation laws. Euro. J. Appl.Math. 13 (2002), 129-152.

[36] G.W. Bluman, A.F. Cheviakov, S.C. Anco. Applications of Symmetry Methods to Partial Differential Equations. Springer- Verlag New York, Applied Mathematical Sciences 168, 2010.

[37] P.J. Olver. Applications of Lie Groups to Differential Equations. Springer- Verlag New York, Graduate Texts in Mathematics 107 (1986).

[38] P.S. Laplace. Traite de Mecanique Celeste, Vol 1, Paris, 1798.[English translation, Celestrial Mechanics, New York 1966].

[39] N. Riley. Radial jets with swirl, Part I. Incompressible Flow, Quart.J.Mech. and Applied Math. 1962b. 15 435-458.

[40] H. Schlichting. Laminare Strahlausbreitung, Z. Angew. Math. Mech. 1933. 13 260-263.

[41] H. Schlichting. Boundary Layer Theory, Sixth Edition, McGraw-Hill, New York, 1968, pp. 170-174.

[42] R.P. Gillespie. Integration, Oliver and Boyd, Edinburgh, 1959, pp. 113-116.

[43] W.H.Schwarz. The radial free jet, Chem. Eng, Sci. 1963. 18 779-786.

[44] H.B. Squire. 50 fahre Grenzschichtforschung, edited by Gorter H and Tollmien W, Braunschweig: Vieweg, 47-54. 1955.

[45] E.J. Watson. The radial spread of a liquid jet over a horizontal plane. J. Fluid Mech. 1964. 20 481-499.

[46] A. Sjöberg. Double reduction of PDEs from the association of symmetries with conservation laws with applications, Appl. Math. Comput. 184, (2007), 608 - 616.

[47] R.P. Gillespie. Integration, Oliver and Boyd, Edinburgh, 1959, pp. 90-95.

[48] S.C. Anco, A.H. Kara. Symmetry-invariant conservation laws of partial differential equations. European Journal of Applied Mathematics. 2018 ,78-117.