

https://dspace.org/

# Method of Lie symmetry for analytical solutions periodicity and attractivity of a family of tenthorder difference equations

**Mensah Folly-Gbetoula; K Anani**

https://hdl.handle.net/10539/47146
WOS

Downloaded from a DSpace repository. https://dspace.org/

Afrika Matematika (2025) 36:122
https://doi.org/10.1007/s13370-025-01342-5

# Method of Lie symmetry for analytical solutions, periodicity and attractivity of a family of tenth-order difference equations

**M. Folly-Gbetoula**[1] · **Kwassi Anani**[2]

Received: 8 October 2024 / Accepted: 1 June 2025
© The Author(s) 2025

**Abstract**
Symmetry is a powerful tool for finding analytical solutions to differential equations, both partial and ordinary, via the similarity variables or via the invariance of the equation under group transformations. It is the largest group of transformations that leaves the differential equation invariant. It is now known that this differential equation method plays the same role when it comes to the study of difference equations. Difference equations can be used to model various phenomena where the changes occur in discrete manner. The use of symmetries on recurrence equations, usually, leads to reductions of order and hence eases the process of finding their solutions. One of the aims of this work is to employ symmetries to generalize some results in the literature. We present new generalized formula solutions of a class of difference equations and we investigate the periodicity and behavior of these solutions.

**Keywords** Difference equation · Symmetry · Reduction · Group invariant solutions

**Mathematics Subject Classification** 39A10 · 39A13 · 39A99

## 1 Introduction

Difference equations have attracted a lot of attention. In practice, they are are used to model phenomena in which the variable is discrete. The idea of symmetries was pioneered by Sophus Lie (1842–1899), where he applied it to differential equations. In 1987, Maeda demonstrated that some modified version of Lie's approach could also be applied to find solutions of ordinary difference equations [12].

✉ M. Folly-Gbetoula
Mensah.Folly-Gbetoula@wits.ac.za

[1] School of Mathematics, University of the Witwatersrand, 2050 Johannesburg, South Africa

[2] Département de Mathématiques, Faculté des Sciences, Université de Lomé, 1515 Lomé, Togo

Published online: 25 June 2025

In this study, we use Lie symmetry analysis to find solutions of a certain class of difference equations. Due to the generality of the proposed symmetry analysis method, it is applicable to a wide range of models governed by high-order difference equations, which naturally arise in diverse fields such as mathematical biology, economics, and physics. Our work is inspired by the work of Sanbo and Elsayed [13], where the authors studied the difference equations

$$x_{n+1} = \frac{x_{n-9}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}, \quad n = 0, 1, \ldots. \tag{1}$$

In their work, Sanbo and Elsayed [13], obtained expressions for solutions of (1) and analyzed the dynamical behavior of these solutions. The solutions were proved by mathematical induction.

Our approach is different from the method Sanbo and Elsayed used to find the solutions. We use Lie symmetry analysis to solve the generalized difference equation:

$$x_{n+1} = \frac{x_{n-9}}{a_n + b_n x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}, \tag{2}$$

where $a_n$ and $b_n$ are real numbers, and $x_i, i = 0, \ldots, 9$ are the initial values. Clearly, this is more general than what was considered by Sanbo and Elsayed. For similar work on difference equations, see [1–4, 10, 11].

### 1.1 Symmetries for difference equations

Consider the tenth-order ordinary recurrence equation

$$x_{n+10} = \Omega(n, x_n, x_{n+2}, x_{n+4}, x_{n+6}, x_{n+8}), \quad \frac{\partial \Omega}{\partial x_n} \neq 0, \tag{3}$$

where $\Omega$ is a smooth function and $n$ an independent variable. The solution of (3) may be expressed as

$$x_n = f(n, c_1, \ldots, c_{10}), \tag{4}$$

where $f$ is an arbitrary function.

**Definition 1.1** The forward shift operator is given by

$$S : n \mapsto n + 1, \qquad S^i x_n = x_{n+i}. \tag{5}$$

The tenth-order ordinary recurrence equation (3) admits a prolonged symmetry generator $X$ given as

$$X = Q\frac{\partial}{\partial x_n} + Q(n+2, x_{n+2})\frac{\partial}{\partial x_{n+2}} + Q(n+4, x_{n+4})\frac{\partial}{\partial x_{n+4}} + Q(n+6, x_{n+6})\frac{\partial}{\partial x_{n+6}} + Q(n+8, x_{n+8})\frac{\partial}{\partial x_{n+8}}, \tag{6}$$

that satisfies the symmetry condition

$$S^{10}Q - X\Omega = 0. \tag{7}$$

The function $Q = Q(n, x_n)$ is called the characteristic of the group of transformations. For more details on this, see [5–9].

## 2 Main results

We will use Lie point symmetry in this section to obtain solutions to the tenth-order ordinary recurrence equation (2). We consider the equivalent tenth-order ordinary recurrence equation

$$x_{n+10} = \frac{x_n}{A_n + B_n x_n x_{n+2} x_{n+4} x_{n+6} x_{n+8}}, \tag{8}$$

of (2), where $A_n = a_{n+9}$ and $B_n = b_{n+9}$. Applying the condition (7) to (8), we obtain

$$Q(n+10, \Omega) - Q(n, x_n)\frac{\partial \Omega}{\partial x_n} - Q(n+2, x_{n+2})\frac{\partial \Omega}{\partial x_{n+2}} - Q(n+4, x_{n+4})\frac{\partial \Omega}{\partial x_{n+4}} - Q(n+6, x_{n+6})\frac{\partial \Omega}{\partial x_{n+6}} - Q(n+8, x_{n+8})\frac{\partial \Omega}{\partial x_{n+8}} = 0, \tag{9}$$

where $\Omega$ is the right-hand side expression in (8). At this step, any convenient operator can be applied to (9), provided that it leads to a reduction in the recurrence equation. Applying the operator

$$L = \frac{\partial}{\partial x_n} + \frac{A_n}{B_n x_n^2 x_{n+4} x_{n+6} x_{n+8}}\frac{\partial}{\partial x_{n+2}}, \tag{10}$$

to (9) and simplifying give

$$-\alpha_n Q(n, x_n) + \frac{2\alpha_n}{x_n}Q(n, x_n) + \alpha_n Q(n+2, x_{n+2}) + \frac{\alpha_n}{x_{n+4}}Q(n+4, x_{n+4}) + \frac{\alpha_n}{x_{n+6}}Q(n+6, x_{n+6}) + \frac{\alpha_n}{x_{n+8}}Q(n+8, x_{n+8}) = 0, \tag{11}$$

where $Q'(n, x_n)$ denotes the derivative of $Q$ with respect to $x_n$. To overcome the challenge of dealing with different arguments, we differentiate (11) with respect to $x_n$ and obtain the following:

$$\frac{d}{dx_n}\left(-Q'(n,x_n)+\frac{2}{x_n}Q(n,x_n)\right)=0. \tag{12}$$

Thus, the solution is

$$Q(n,x_n)=\gamma_n x_n^2+\alpha_n x_n+\beta_n, \tag{13}$$

where $\alpha_n$, $\beta_n$ and $\gamma_n$ are some functions of $n$.

By substituting $Q(n,x_n)$ in the symmetry condition (7), we obtain that $\gamma_n=\beta_n=0$, and $\alpha_n$ must satisfy

$$\alpha_n+\alpha_{n+2}+\alpha_{n+4}+\alpha_{n+6}+\alpha_{n+8}=0. \tag{14}$$

Thus, the symmetry generator takes the form

$$X=Q(n,x_n)\frac{\partial}{\partial x_n}=\alpha_n x_n\frac{\partial}{\partial x_n}, \tag{15}$$

where $\alpha_n$ satisfies (14). Note that the same symbol is used for the generator and its prolongation. The solutions of (14) are

$$\alpha_n=e^{\frac{i(2kn\pi)}{10}}, \tag{16}$$

$k=1,2,3,4,6,7,8,9$. It follows from (15) and (16) that

$$X_k=e^{i\frac{2kn\pi}{10}}x_n\frac{\partial}{\partial x_n}, \tag{17}$$

$k=1,2,3,4,6,7,8,9$, are symmetries of (8). We introduce the canonical coordinate defined as

$$S_n=\int\frac{1}{Q(n,x_n)}dx_n=\int\frac{1}{\alpha_n x_n}dx_n, \tag{18}$$

we get

$$S_n\alpha_n=\ln|x_n|. \tag{19}$$

Let $F_n=\exp\{-(S_n\alpha_n+S_{n+2}\alpha_{n+4}+S_{n+6}\alpha_{n+6}+S_{n+8}\alpha_{n+8})\}$. Hence, we have

$$F_n=\frac{1}{x_n x_{n+2}x_{n+4}x_{n+6}x_{n+8}}. \tag{20}$$

We note that

$$\begin{aligned} X(F_n) =& e^{i\frac{2kn\pi}{10}} x_n \frac{\partial F_n}{\partial x_n} + e^{i\frac{2k(n+2)\pi}{10}} x_{n+2} \frac{\partial F_n}{\partial x_{n+2}} + e^{i\frac{2k(n+4)\pi}{10}} x_{n+4} \frac{\partial F_n}{\partial x_{n+4}} \\ &+ e^{i\frac{2k(n+6)\pi}{10}} x_{n+6} \frac{\partial F_n}{\partial x_{n+6}} + e^{i\frac{2k(n+8)\pi}{10}} x_{n+8} \frac{\partial F_n}{\partial x_{n+8}} \\ =& -\left( e^{i\frac{2kn\pi}{10}} + e^{i\frac{2k(n+2)\pi}{10}} + e^{i\frac{2k(n+4)\pi}{10}} + e^{i\frac{2k(n+6)\pi}{10}} \right. \\ &\left. + e^{i\frac{2k(n+8)\pi}{10}} \right) \left( \frac{-1}{x_n x_{n+2} x_{n+4} x_{n+6} x_{n+8}} \right) \\ =& \frac{1}{x_n x_{n+2} x_{n+4} x_{n+6} x_{n+8}} \left( \alpha_n + \alpha_{n+2} + \alpha_{n+4} + \alpha_{n+6} + \alpha_{n+8} \right) \\ =& 0. \end{aligned} \tag{21}$$

Shifting equation (20) twice and substituting the expression of $x_{n+10}$ given in (8), we obtain

$$F_{n+2} = A_n F_n + B_n. \tag{22}$$

By iterating (22), we get

$$F_{2n+j} = F_j \left( \prod_{t=0}^{n-1} A_{2t+j} \right) + \sum_{i=0}^{n-1} \left( B_{2i+j} \prod_{k_2=i+1}^{n-1} A_{2k_2+j} \right), \quad j = 0, 1. \tag{23}$$

Also,

$$x_{n+10} = \frac{F_n}{F_{n+2}} x_n, \tag{24}$$

whose iteration yields

$$\begin{aligned} x_{10n+k} =& x_k \left( \prod_{s=0}^{n-1} \frac{F_{10s+k}}{F_{10s+k+2}} \right), \quad k = 0, 1, 2, \ldots, 9 \\ =& x_k \left( \prod_{s=0}^{n-1} \frac{F_{10s+2\lfloor \frac{k}{2} \rfloor + \tau(k)}}{F_{10s+2\lfloor \frac{k}{2} \rfloor + \tau(k)+2}} \right), \\ =& x_k \left( \prod_{s=0}^{n-1} \frac{F_{2(5s+\lfloor \frac{k}{2} \rfloor) + \tau(k)}}{F_{2(5s+1+\lfloor \frac{k}{2} \rfloor) + \tau(k)}} \right), \end{aligned} \tag{25}$$

because we can always write any integer $i$ in the form $i = 2\lfloor \frac{i}{2} \rfloor + \tau(i)$, where $\tau(i)$ is the remainder when $a$ is divided by 2 and $\lfloor \cdot \rfloor$ is the floor function.

Substituting equation (23) into equation (25), we get

$$
\begin{aligned}
x_{10n+k} =& x_k \prod_{s=0}^{n-1} \frac{F_{\tau(k)} \left( \prod_{t=0}^{5s+\lfloor \frac{k}{2} \rfloor -1} A_{2t+\tau(k)} \right) + \sum_{i=0}^{5s+\lfloor \frac{k}{2} \rfloor -1} \left( B_{2i+\tau(k)} \prod_{k_2=i+1}^{5s+\lfloor \frac{k}{2} \rfloor -1} A_{2k_2+\tau(k)} \right)}{F_{\tau(k)} \left( \prod_{t=0}^{5s+\lfloor \frac{k}{2} \rfloor} A_{2t+\tau(k)} \right) + \sum_{i=0}^{5s+\lfloor \frac{k}{2} \rfloor} \left( B_{2i+\tau(k)} \prod_{k_2=i+1}^{5s+\lfloor \frac{k}{2} \rfloor} A_{2k_2+\tau(k)} \right)} \\
=& x_k \prod_{s=0}^{n-1} \frac{\left( \prod_{t=0}^{5s+\lfloor \frac{k}{2} \rfloor -1} A_{2t+\tau(k)} \right) + \sum_{i=0}^{5s+\lfloor \frac{k}{2} \rfloor -1} \left( \frac{B_{2i+\tau(k)}}{F_{\tau(k)}} \prod_{k_2=i+1}^{5s+\lfloor \frac{k}{2} \rfloor -1} A_{2k_2+\tau(k)} \right)}{\left( \prod_{t=0}^{5s+\lfloor \frac{k}{2} \rfloor} A_{2t+\tau(k)} \right) + \sum_{i=0}^{5s+\lfloor \frac{k}{2} \rfloor} \left( \frac{B_{2i+\tau(k)}}{F_{\tau(k)}} \prod_{k_2=i+1}^{5s+\lfloor \frac{k}{2} \rfloor} A_{2k_2+\tau(k)} \right)},
\end{aligned}
\tag{26}
$$

where $1/F_{\tau(k)} = x_{\tau(k)} x_{\tau(k)+2} x_{\tau(k)+4} x_{\tau(k)+6} x_{\tau(k)+8}$.

## 3 Case where $A_n = A$ and $B_n = B$

Let $A_n = A$ and $B_n = B$ be fixed in equation (26). We obtain

$$
x_{10n+k} = x_k \left( \prod_{s=0}^{n-1} \frac{A^{5s+\lfloor \frac{k}{2} \rfloor} + \frac{B}{F_{\tau(k)}} \sum_{s=0}^{5s+\lfloor \frac{k}{2} \rfloor -1} A^s}{A^{5s+1+\lfloor \frac{k}{2} \rfloor} + \frac{B}{F_{\tau(k)}} \sum_{s=0}^{5s+\lfloor \frac{k}{2} \rfloor} A^s} \right) \tag{27}
$$

$$
= x_k \left( \prod_{s=0}^{n-1} \frac{A^{5s+\lfloor \frac{k}{2} \rfloor} + B x_{\tau(k)} x_{\tau(k)+2} x_{\tau(k)+4} x_{\tau(k)+6} x_{\tau(k)+8} \sum_{s=0}^{5s+\lfloor \frac{k}{2} \rfloor -1} A^s}{A^{5s+1+\lfloor \frac{k}{2} \rfloor} + B x_{\tau(k)} x_{\tau(k)+2} x_{\tau(k)+4} x_{\tau(k)+6} x_{\tau(k)+8} \sum_{s=0}^{5s+\lfloor \frac{k}{2} \rfloor} A^s} \right) \tag{28}
$$

$$
= \begin{cases}
x_k \prod_{s=0}^{n-1} \frac{1+B(5s+\lfloor \frac{k}{2} \rfloor) x_{\tau(k)} x_{\tau(k)+2} x_{\tau(k)+4} x_{\tau(k)+6} x_{\tau(k)+8}}{1+B(5s+\lfloor \frac{k}{2} \rfloor +1) x_{\tau(k)} x_{\tau(k)+2} x_{\tau(k)+4} x_{\tau(k)+6} x_{\tau(k)+8}}, \text{when } A = 1; \\
x_k \prod_{s=0}^{n-1} \frac{A^{5s+\lfloor \frac{k}{2} \rfloor} + \frac{B(1-A^{5s+\lfloor \frac{k}{2} \rfloor})}{1-A} x_{\tau(k)} x_{\tau(k)+2} x_{\tau(k)+4} x_{\tau(k)+6} x_{\tau(k)+8}}{A^{5s+1+\lfloor \frac{k}{2} \rfloor} + \frac{B(1-A^{5s+1+\lfloor \frac{k}{2} \rfloor})}{1-A} x_{\tau(k)} x_{\tau(k)+2} x_{\tau(k)+4} x_{\tau(k)+6} x_{\tau(k)+8}}, \text{when } A \neq 1;
\end{cases}
\tag{29}
$$

for $k = 0, 1, 2 \cdots, 9$.

### 3.1 Case where $A = -1$

For the case $A = -1$, we have

$$
x_{10n+k} = x_k \left( \prod_{s=0}^{n-1} \frac{(-1)^s + \frac{B((-1)^{\lfloor \frac{k}{2} \rfloor} - (-1)^s)}{2} x_{\tau(k)} x_{\tau(k)+2} x_{\tau(k)+4} x_{\tau(k)+6} x_{\tau(k)+8}}{-(-1)^s + \frac{B((-1)^{\lfloor \frac{k}{2} \rfloor} + (-1)^s)}{2} x_{\tau(k)} x_{\tau(k)+2} x_{\tau(k)+4} x_{\tau(k)+6} x_{\tau(k)+8}} \right). \tag{30}
$$

The above implies that:

- For $k = 0, 1$;

$$x_{10n+k} = \begin{cases} x_k & \text{if } n \text{ even} \\ \frac{x_k}{-1+Bx_kx_{k+2}x_{k+4}x_{k+6}x_{k+8}} & \text{if } n \text{ odd} \end{cases}, \tag{31a}$$

- For $k = 2, 3$;

$$x_{10n+k} = \begin{cases} x_k & \text{if } n \text{ even} \\ x_k(-1 + Bx_{k-2}x_kx_{k+2}x_{k+4}x_{k+6}) & \text{if } n \text{ odd} \end{cases}, \tag{31b}$$

- For $k = 4, 5$;

$$x_{10n+k} = \begin{cases} x_k & \text{if } n \text{ even} \\ \frac{x_k}{-1+Bx_{k-4}x_{k-2}x_kx_{k+2}x_{k+4}} & \text{if } n \text{ odd} \end{cases}, \tag{31c}$$

- For 6, 7;

$$x_{10n+k} = \begin{cases} x_k & \text{if } n \text{ even} \\ x_k(-1 + Bx_{k-6}x_{k-4}x_{k-2}x_kx_{k+2}) & \text{if } n \text{ odd} \end{cases}, \tag{31d}$$

- For 8, 9;

$$x_{10n+k} = \begin{cases} x_k & \text{if } n \text{ even} \\ \frac{x_k}{(-1+Bx_{k-8}x_{k-6}x_{k-4}x_{k-2}x_k)} & \text{if } n \text{ odd} \end{cases}. \tag{31e}$$

### 3.2 Known results

Recall that we shifted equation (2) forward 9 times to obtain (8), the solution of which is provided by (29). We reverse the equations in (29) 9 times to obtain the solution of the difference equation (2), which is given as

$$x_{10n+k-9} = x_{k-9}\prod_{s=0}^{n-1} \frac{1 + B(5s + \lfloor \frac{k}{2} \rfloor)x_{\tau(k)-9}x_{\tau(k)-7}x_{\tau(k)-5}x_{\tau(k)-3}x_{\tau(k)-1}}{1 + B(5s + \lfloor \frac{k}{2} \rfloor + 1)x_{\tau(k)-9}x_{\tau(k)-7}x_{\tau(k)-5}x_{\tau(k)-3}x_{\tau(k)-1}}, \tag{32}$$

if $A = 1$; and

$$x_{10n+k-9} = x_{k-9}\prod_{s=0}^{n-1} \frac{A^{5s+\lfloor \frac{k}{2} \rfloor} + \frac{B(1-A^{5s+\lfloor \frac{k}{2} \rfloor})}{1-A}x_{\tau(k)-9}x_{\tau(k)-7}x_{\tau(k)-5}x_{\tau(k)-3}x_{\tau(k)-1}}{A^{5s+1+\lfloor \frac{k}{2} \rfloor} + \frac{B(1-A^{5s+1+\lfloor \frac{k}{2} \rfloor})}{1-A}x_{\tau(k)-9}x_{\tau(k)-7}x_{\tau(k)-5}x_{\tau(k)-3}x_{\tau(k)-1}}, \tag{33}$$

if $A \neq 1$. If we set

$$j = 9 - k, \tag{34a}$$

then,

$$\lfloor \frac{j}{2} \rfloor = 4 - \lfloor \frac{k}{2} \rfloor, \tag{34b}$$

and

$$\tau(j) = 1 - \tau(k), \tag{34c}$$

for $k = 0, 1, \ldots, 9$. Therefore, from (32), (33) and (34), we have

$$x_{10n-j} = x_{-j} \prod_{s=0}^{n-1} \frac{1 + B(5s + 4 - \lfloor \frac{j}{2} \rfloor) x_{-\tau(j)-8} x_{-\tau(j)-6} x_{-\tau(j)-4} x_{-\tau(j)-2} x_{-\tau(j)}}{1 + B(5s + 5 - \lfloor \frac{j}{2} \rfloor) x_{-\tau(j)-8} x_{-\tau(j)-6} x_{-\tau(j)-4} x_{-\tau(j)-2} x_{-\tau(j)}}, \tag{35}$$

if $A = 1$; and

$$x_{10n-j} = x_{-j} \prod_{s=0}^{n-1} \frac{A^{5s+4-\lfloor \frac{j}{2} \rfloor} + \frac{B(1-A^{5s+4-\lfloor \frac{j}{2} \rfloor})}{1-A} x_{-\tau(j)-8} x_{-\tau(j)-6} x_{-\tau(j)-4} x_{-\tau(j)-2} x_{-\tau(j)}}{A^{5s+5-\lfloor \frac{j}{2} \rfloor} + \frac{B(1-A^{5s+5-\lfloor \frac{j}{2} \rfloor})}{1-A} x_{-\tau(j)-8} x_{-\tau(j)-6} x_{-\tau(j)-4} x_{-\tau(j)-2} x_{-\tau(j)}}, \tag{36}$$

if $A \neq 1$.

### 3.2.1 The case: $A = 1$ and $B = 1$

Set $M_j = 5 - \lfloor \frac{j}{2} \rfloor$, $P_j = \prod_{k=0}^{4} a_{\tau(j)+2k}$ and $x_{-j} = a_j$. We have

$$P_j = \prod_{k=0}^{4} a_{\tau(j)+2k} = x_{-\tau(j)-8} x_{-\tau(j)-6} x_{-\tau(j)-4} x_{-\tau(j)-2} x_{-\tau(j)}. \tag{37}$$

Then, equation (35) becomes

$$x_{10n-j} = a_j \prod_{s=0}^{n-1} \left( \frac{1 + (5s + M_j - 1) P_j}{1 + (5s + M_j) P_j} \right), \tag{38}$$

which is the same as Theorem 2 in [13].

### 3.2.2 Case: $A = 1$ and $B = -1$

In this case, equation (35) is given as

$$x_{10n-j} = a_j \prod_{s=0}^{n-1} \left( \frac{1-(5s+M_j-1)P_j}{1-(5s+M_j)P_j} \right), \tag{39}$$

which is the same as Theorem 5 in [13].

#### 3.2.3 Case: $A = -1$ and $B = 1$

In this case, equation (36) becomes

$$x_{10n-j} = a_j \prod_{s=0}^{n-1} \frac{(-1)^{5s+M_j-1} + \frac{(1-(-1)^{5s+M_j-1})}{2} P_j}{(-1)^{5s+M_j} + \frac{(1-(-1)^{5s+M_j})}{2} P_j}, \tag{40}$$

which was obtained in Theorem 7 of [13].

#### 3.2.4 Case: $A = -1$ and $B = -1$

As above, in this case, equation (36) yields

$$x_{10n-j} = a_j \prod_{s=0}^{n-1} \frac{(-1)^{5s+M_j-1} + \frac{(-1-(-1)^{5s+M_j-2})}{2} P_j}{(-1)^{5s+M_j} + \frac{(-1-(-1)^{5s+M_j-1})}{2} P_j}, \tag{41}$$

which was obtained in Theorem 10 of [13].

## 4 Periodic property and stability

In this section, we provide sufficient conditions for existence of periodic solutions with periods 1, 5, 10 and 20. We further prove that the equilibrium points are non-hyperbolic.

**Theorem 1** *Let $x_n$ be a solution of*

$$x_{n+10} = \frac{x_n}{-1 + B x_n x_{n+2} x_{n+4} x_{n+6} x_{n+8}} \tag{42}$$

with initial conditions $x_i,\ i = 0, \ldots, 9$; and $B$ is a non-zero constant. Then equation (42) has a 20-periodic solution.

***Proof*** Clearly, $A = -1$. It follows from (31) that the 20-periodic solutions are given by

$$\begin{aligned}&\ldots, x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, \frac{x_0}{-1+Bx_0x_2x_4x_6x_8},\\ &\frac{x_1}{-1+Bx_1x_3x_5x_7x_9}, x_2(-1+Bx_0x_2x_4x_6x_8), x_3(-1+Bx_1x_3x_5x_7x_9),\\ &\frac{x_4}{-1+Bx_0x_2x_4x_6x_8}, \frac{x_5}{-1+Bx_1x_3x_5x_7x_9}, x_6(-1+Bx_0x_2x_4x_6x_8),\\ &x_7(-1+Bx_1x_3x_5x_7x_9), \frac{x_8}{-1+Bx_0x_2x_4x_6x_8}, \frac{x_9}{-1+Bx_1x_3x_5x_7x_9}, \cdots.\end{aligned} \tag{43}$$

□

In Fig. 1, we use the initial conditions $x_0 = 1; x_1 = -1; x_2 = 2;$. $x_3 = -2; x_4 = 1/4; x_5 = 1; x_6 = -1; x_7 = 2; x_8 = -2; x_9 = 1/4$From the graph, we can verify that the solution is indeed periodic with period 20.

**Theorem 2** *Let $x_n$ be a solution of*

$$x_{n+10} = \frac{x_n}{A + Bx_nx_{n+2}x_{n+4}x_{n+6}x_{n+8}} \tag{44}$$

with initial conditions $x_i,\ i = 0, \ldots, 9$; and $A \neq 1$ and $B$ some non-zero constants. If the initial conditions satisfy

$$x_ix_{i+2}x_{i+4}x_{i+6}x_{i+8} = \frac{1-A}{B} \tag{45}$$

and

$$x_i \neq x_{i+2}, x_i \neq x_{i+5}, \tag{46}$$

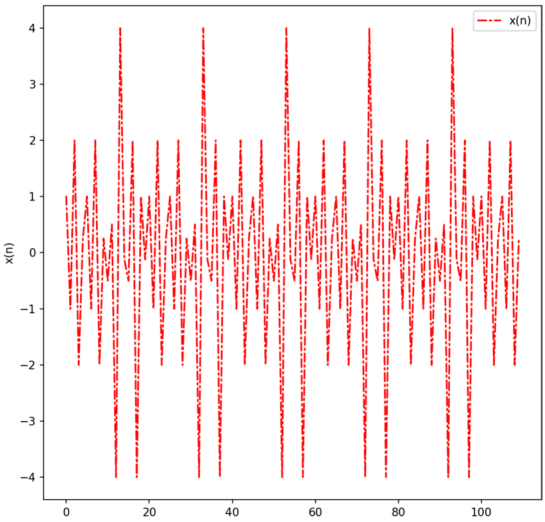



**Fig. 1** Graph of $x_{n+10} = \frac{x_n}{-1-x_nx_{n+2}x_{n+4}x_{n+6}x_{n+8}}$

then equation (44) has a 10-periodic solution.

***Proof*** Assuming $x_i x_{i+2} x_{i+4} x_{i+6} x_{i+8} = \frac{1-A}{B}$ as stated in Theorem 2, equation (29) reduces to

$$x_{10n+j} = x_j \prod_{t=0}^{n-1} \frac{A^{5t+\lfloor \frac{j}{2} \rfloor} + \left( \frac{1-A^{5t+\lfloor \frac{j}{2} \rfloor}}{1-A} \right) B \left( \frac{1-A}{B} \right)}{A^{5t+\lfloor \frac{j}{2} \rfloor+1} + \left( \frac{1-A^{5t+\lfloor \frac{j}{2} \rfloor+1}}{1-A} \right) B \left( \frac{1-A}{B} \right)} \tag{47}$$

$$= x_j. \tag{48}$$

Imposing the condition $x_i \neq x_{i+2}, x_i \neq x_{i+5}$, we note that the solution is periodic with period 10. □

In Fig. 2, we use the initial conditions $x_0 = 1; x_1 = -1; x_2 = 2; x_3 = -2; x_4 = 1/4; x_5 = -1/4;\ x_6 = 3; x_7 = -3; x_8 = 2/3; x_9 = 2/3$ for $A = 2$ and $B = -1$. They satisfy

$$x_0 x_2 x_4 x_6 u_8 = x_1 x_3 x_5 x_7 x_9 = \frac{1-A}{B} \quad \text{and} \quad x_i \neq x_{i+2}, x_i \neq x_{i+5}. \tag{49}$$

From the graph, we can verify that the solution is indeed periodic with period 10.

**Theorem 3** *Let $x_n$ be a solution of*

$$x_{n+10} = \frac{x_n}{A + B x_n x_{n+2} x_{n+4} x_{n+6} x_{n+8}} \tag{50}$$

with initial conditions $x_i,\ i = 0, \ldots, 9$; and $A \neq 1$ and $B$ some non-zero constants. If the initial conditions satisfy

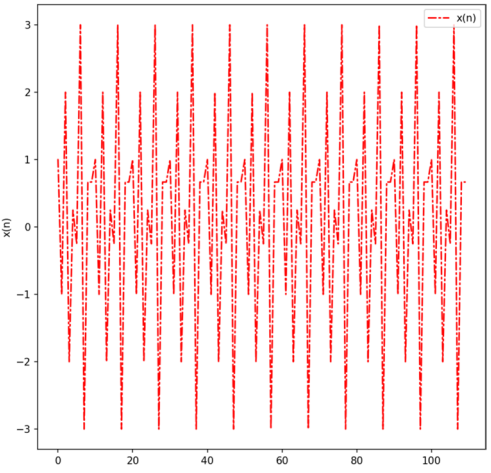



**Fig. 2** Graph of $x_{n+10} = \frac{x_n}{2 - x_n x_{n+2} x_{n+4} x_{n+6} x_{n+8}}$

$$x_0x_1x_2x_3x_4 = \frac{1-A}{B} \tag{51}$$

and

$$x_i \neq x_{i+2}, x_i = x_{i+5}, \tag{52}$$

then equation (50) has a 5-periodic solution.

***Proof*** The proof is similar to the proof of Theorem 2 and is thus omitted. □

In Fig. 3, we use the initial conditions $x_0 = 1; x_1 = -1; x_2 = 2; x_3 = -2; x_4 = 1/4; x_5 = 1; x_6 = -1; x_7 = 2; x_8 = -2; x_9 = 1/4$ for $A = 2$ and $B = -1$. They satisfy

$$x_0x_2x_4x_6x_8 = \frac{1-A}{B} \quad \text{and} \quad x_i \neq x_{i+2}, x_i = x_{i+5}. \tag{53}$$

From the graph, we can verify that the solution is indeed periodic with period 5.

**Theorem 4** *Let $x_n$ be a solution of*

$$x_{n+10} = \frac{x_n}{A + Bx_nx_{n+2}x_{n+4}x_{n+6}x_{n+8}} \tag{54}$$

with initial conditions $x_i,\ i = 0, \ldots, 9$; and $A \neq 1$ and $B$ some non-zero constants. If the initial conditions satisfy

$$x_i^5 = \frac{1-A}{B} \tag{55}$$

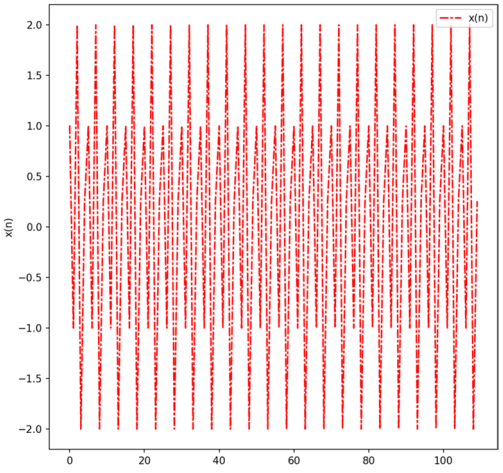



**Fig. 3** Graph of $x_{n+10} = \frac{x_n}{2-x_nx_{n+2}x_{n+4}x_{n+6}x_{n+8}}$

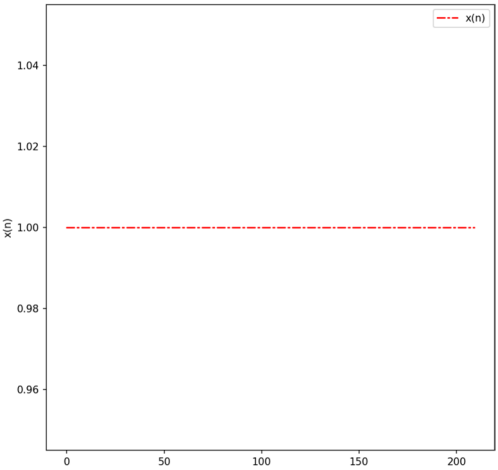



**Fig. 4** Graph of $x_{n+10} = \frac{x_n}{2-x_n x_{n+2} x_{n+4} x_{n+6} x_{n+8}}$

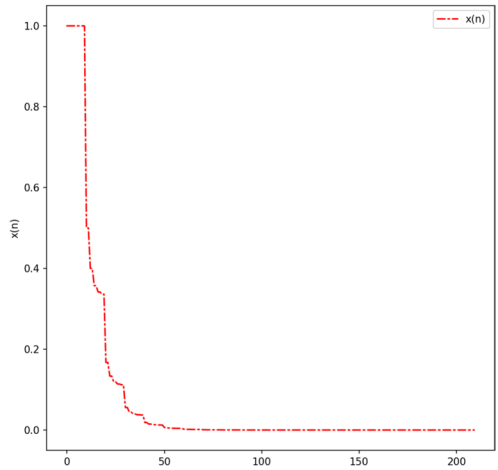



**Fig. 5** Graph of $x_{n+10} = \frac{x_n}{3-x_n x_{n+2} x_{n+4} x_{n+6} x_{n+8}}$

then equation (54) has a 1-periodic solution.

***Proof*** The proof is similar to the proof of Theorem 2 and is thus omitted. □

In Fig. 4, we use the initial conditions $x_i = 1$ for $A = 2$ and $B = -1$. They satisfy

$$x_i^5 = \frac{1-A}{B}. \tag{56}$$

From Fig. 4, we can verify that the solution is indeed periodic with period 1. Despite the fact that all the initial conditions are the same in Fig. 5, where for $A = 3$ and $B = -1$, we do not have a $1-$periodic solution because $x_i \neq \frac{1-A}{B}$.

**Theorem 5** *Given the equation*

$$x_{n+10} = \frac{x_n}{1 + Bx_n x_{n+2} x_{n+4} x_{n+6} x_{n+8}}, \tag{57}$$

the only equilibrium point $\bar{x} = 0$ is non-hyperbolic.

***Proof*** The condition $x = x/(1 + Bx^5)$ yields the fixed point $x = 0$ and the characteristic equation of (57) near 0 is $\lambda^{10} - 1 = 0$ whose roots are the tenth roots of unity. The moduli of the roots are all equal to 1. Thus, $x = 0$ is non-hyperbolic. □

**Theorem 6** *The fixed point $\bar{x} = 0$ of (44) is (locally) asymptotically stable for $|A| > 1$. Furthermore, the non-zero fixed points of (44) are non-hyperbolic for all $A \neq 1$.*

***Proof*** Using the fixed point condition on (44) leads to $\bar{x}(A + B\bar{x}^5 - 1) = 0$. For the first part of the proof, observe that the characteristic equation of (44) near $\bar{x} = 0$ is given by $\lambda^{10} - \frac{1}{A} = 0$. Hence, the roots $\lambda_i$ satisfy $|\lambda_i| < 1$ for $|A| > 1$. Thus, $x = 0$ is locally asymptotically stable when $|A| > 1$. For the second part of the proof, we find the non-zero fixed points from the equation $A + Bx^5 - 1 = 0$. In this case, the characteristic equation of (44) near a non-zero equilibrium point is given by

$$0 = \lambda^{10} - (A-1)\lambda^8 - (A-1)\lambda^6 - (A-1)\lambda^4 - (A-1)\lambda^2 - A \tag{58}$$

$$= \frac{1 - \lambda^{10}}{1 - \lambda^2}(\lambda^2 - A), \quad \text{with} \quad \lambda^2 \neq 1. \tag{59}$$

There is a root of the characteristic equation (for example, $\lambda = e^{i2\pi/5}$) whose modulus is equal to 1. □

## 5 Conclusion

In this paper, we have confirmed and extended the findings of Sando and Elsayed [13] by employing Lie symmetry analysis as a foundational tool. Specifically, we demonstrated that the results obtained in our study are consistent with those presented in Sando's work, thereby validating their conclusions within a broader and more generalized framework. Our approach not only reinforces the structural integrity of the equations previously analyzed but also highlights the importance of symmetry methods in studying nonlinear difference equations. Furthermore, we advanced the discussion by examining the periodicity and stability properties of solutions to a more general class of high-order difference equations than those originally considered by Sando and Elsayed. We have proved the existence of non-hyperbolic equilibrium points. This generalization broadens the applicability of their initial results and provides a deeper understanding of the dynamic behavior of solutions of such equations under more flexible assumptions. These contributions demonstrate the power of symmetry techniques in revealing qualitative features of complex discrete systems and suggest further avenues for exploration in related mathematical and applied contexts.

**Author Contributions** The authors contributed equally to this work.

**Funding** Open access funding provided by University of the Witwatersrand.

**Data Availability** Not applicable.

**Materials availability** Not applicable.

**Code availability** Not applicable.

## Declarations

**Conflict of interest** The authors declare that they have no Conflict of interest.

**Ethics approval and consent to participate** Not applicable.

**Consent for publication** Not applicable.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

## References

1. Bukhary, N.A., Elsayed, E.M.: An investigation of the solutions for rational systems of difference equations of order two. Eur. J. Math. Appl. **4**, 1 (2024)
2. Elabbasy, E.M., El-Metwally, H., Elsayed, E.M.: On the difference equation $x_{n+1} = \alpha x_{n-k}/(\beta + \gamma \prod_{i=0}^{k} x_{n-i}$. J. Concr. Appl. Math. **5**(2), 101–113 (2007)
3. Elsayed, E.M.: On the solution of some difference equation. Eur. J. Pure Appl. Math. **4**(3), 287–303 (2011)
4. Elsayed, E.M., Din, Q., Al-Rakhami, F.A., Seyam, N.M.: Dynamics and expressions of solutions of fourth-order rational systems of difference equations. Int. J. Anal. Appl. **22**, 163 (2024)
5. Folly-Gbetoula, M.: Symmetry, reductions and exact solutions of the difference equation $u_{n+2} = (au_n)/(1 + bu_n u_{n+1})$. J. Differ. Equ. Appl. **23**(6), 1017–1024 (2017)
6. Folly-Gbetoula, M., Kara, A.H.: Symmetries, conservation laws, and integrability of difference equations. Adv. Differ. Equ. **2014** (2014)
7. Folly-Gbetoula, M., Mkhwanazi, K., Nyirenda, D.: On a study of a family of higher order recurrence relations. Math. Probl. Eng. **2022**, 11 (2022)
8. Hydon, P.E.: Symmetries and first integrals of ordinary difference equations. Proc. Roy. Soc. Lond. A **456**, 2835–2855 (2000)
9. Hydon, P.E.: Difference Equations by Differential Equation Methods. Cambridge University Press, Cambridge (2014)
10. Imane, D., Touafek, N., Yalçinkaya, I., Almatrafi, M.B.: Explicit formulas for the solutions of a system of difference equations of fourth order in the complex domain. Math. Eng. Sci. Aerosp. **14**(3), 931–959 (2023)
11. Khaliq, A., Sadiq, S., Ahmed, H.M.E., Mahmoud, B.A.A., Al-Sinan, B.R., Ibrahim, T.F.: The dynamical behavior of a three-dimensional system of exponential difference equations. Mathematics **11**(8), 1808 (2023)

12. Maeda, S.: The similarity method for difference equations. IMA J. Appl. Math. **38**, 129–134 (1987)
13. Sanbo, A., Elsayed, E.M.: Some results on a rational recursive sequences. Pan-Am. J. Math. **2**, 7 (2023)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.