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Results in Mathematics



Lower and Upper Bounds for the Generalized Csiszár f-divergence Operator Mapping

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Abstract. Let $\mathbf{A} = \{A_1, ..., A_n\}$ and $\mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators on a Hilbert space $\mathcal{H}$ and f, $h : \mathbb{I} \to \mathbb{R}$ continuous functions with $h > 0$.. We consider the generalized Csiszár f-divergence operator mapping defined by

$$\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) = \sum_{i=1}^{n} P_{f\Delta h}(A_i, B_i),$$

where

$$P_{f\Delta h}(A, B) := h(A)^{1/2} f(h(A)^{-1/2} B h(A)^{-1/2}) h(A)^{1/2}$$

is introduced for every strictly positive operator A and every self-adjoint operator B, where the spectrum of the operators

$$A, A^{-1/2} B A^{-1/2} \text{ and } h(A)^{-1/2} B h(A)^{-1/2}$$

are contained in the closed interval $\mathbb{I}$. In this paper we obtain some lower and upper bounds for $\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B})$ with applications to the geometric operator mean and the relative operator entropy. We verify the information monotonicity for the Csisz ár f-divergence operator mapping and the generalized Csiszár f-divergence operator mapping.

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1. Introduction

The classical perspective function associated to a continuous function $f : [0,\infty) \to \mathbb{R}$ is a function of two variables defined by $P_f(s,t) := sf(\frac{t}{s})$, cf. [19]. For two discrete probability distributions $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$ the f-divergence functional

$$I_f(p,q) = \sum_{i=1}^{n} P_f(p_i, q_i)$$

was introduced by Csiszár [6] as a distance function on the set of discrete probability distributions.

Let f and h be two real valued continuous functions defined on the closed interval $\mathbb{I}$ and $h > 0$. The value $f(A)$ is defined via the functional calculus as usual for a self–adjoint operator A whose spectrum is contained in $\mathbb{I}$. A fully noncommutative perspective of two variables (associated to f), by choosing an appropriate ordering, was introduced in [12] by setting

$$P_f(A,B) := A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$$

and the operator version of a fully noncommutative generalized perspective of two variables (associated to f and h) was also introduced by setting

$$P_{f\Delta h}(A,B) := h(A)^{1/2} f(h(A)^{-1/2} B h(A)^{-1/2}) h(A)^{1/2}$$

for every strictly positive operator A and every self-adjoint operator B on a Hilbert space $\mathcal{H}$, where the spectrum of the operators

$$A, A^{-1/2} B A^{-1/2} \text{ and } h(A)^{-1/2} B h(A)^{-1/2}$$

are contained in the closed interval $\mathbb{I}$. Note that in this situation $P_{f\Delta h}(A,B) = P_f(h(A),B)$. Then, several striking matrix analogues of a classical result for operator convex functions were proved. More precisely, the necessary and sufficient conditions for the joint convexity of a fully noncommutative perspective and generalized perspective function were proved where restricting to the positive commuting matrices ensures Effros' approach announced in [13].

Throughout this section we assume that f and h are continuous real valued functions defined on $[0,\infty)$ and $h > 0$ unless we note otherwise.

Let $\mathbf{A} = \{A_1, ..., A_n\}$ and $\mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators and $f : \mathbb{I} \to \mathbb{R}$ a continuous function. We consider the Csiszár f-divergence operator mapping by setting

$$\mathbf{I}_f(\mathbf{A},\mathbf{B}) = \sum_{i=1}^{n} P_f(A_i, B_i)$$

and the generalized Csiszár f-divergence operator mapping via

$$\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) = \sum_{i=1}^{n} P_{f\Delta h}(A_i, B_i).$$

The joint convexity of the perspective and generalized perspective was proved in [12,25,29].

Theorem 1. *The following statements hold:*

(i) *If f is operator convex, then P_f is jointly convex.*
(ii) *If f is operator convex with $f(0) \leq 0$ and h is operator concave, then $P_{f\Delta h}$ is jointly convex.*
(iii) *If f and h are operator concave with $f(0) \geq 0$, then $P_{f\Delta h}$ is jointly concave.*

These results can be generalized to the Csiszár f-divergence operator mappings [22]. The following corollary is a simple application of the joint convexity of the perspective.

Corollary 1. *The following statements hold:*

(i) *If f is operator convex, then $\mathbf{I}_f$ is jointly convex.*
(ii) *If f is operator convex with $f(0) \leq 0$ and h is operator concave, then $\mathbf{I}_{f\Delta h}$ is jointly convex.*
(iii) *If f and h are operator concave with $f(0) \geq 0$, then $\mathbf{I}_{f\Delta h}$ is jointly concave.*

For a continuous function $g : (0,\infty) \to \mathbb{R}$ the transpose function $\tilde{g}$ of g is defined by

$$\tilde{g}(x) = xg(x^{-1}),\ x > 0.$$

Corollary 2. *Let $g : (0,\infty) \to \mathbb{R}$ be a continuous function. Then,*

$$\mathbf{I}_{\tilde{g}}(\mathbf{A},\mathbf{B}) = \mathbf{I}_g(\mathbf{B},\mathbf{A})$$

for two finite sequences of strictly positive operators $\mathbf{A}$ and $\mathbf{B}$.

2. Upper and Lower Bounds

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty,\infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I. \tag{2.1}$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

For a finite sequence of strictly positive operators $\mathbf{A} = \{A_1, ..., A_n\}$ on a Hilbert space $\mathcal{H}$ and a continuous function f, we set

$$S_{\mathbf{A}} := \sum_{i=1}^{n} A_i \text{ and } S_{f(\mathbf{A})} := \sum_{i=1}^{n} f(A_i),$$

where $f(\mathbf{A}) := \{f(A_1), ..., f(A_n)\}$ is a finite sequence of operators on $\mathcal{H}$.

We have:

Theorem 2. *Let $\mathbf{A} = \{A_1, ..., A_n\}$, $\mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators. If $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a convex function and $h : [a, b] \to (0, \infty)$ is a continuous function such that*

$$mh(A_i) \leq B_i \leq Mh(A_i), \quad i = 1, ..., n \tag{2.2}$$

for some m, $M \in [a, b]$ with $0 < m < M$, then

$$\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) \geq f(t) S_{h(\mathbf{A})} + \varphi(t) \left(S_{\mathbf{B}} - tS_{h(\mathbf{A})}\right) \tag{2.3}$$

for all $t \in (a, b)$ and $\varphi \in \partial f$.

In particular,

$$\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) \geq f\left(\frac{m+M}{2}\right) S_{h(\mathbf{A})} + \varphi\left(\frac{m+M}{2}\right)\left(S_{\mathbf{B}} - \frac{m+M}{2} S_{h(\mathbf{A})}\right).$$

Proof. From (2.1) we have

$$f(x) \geq f(t) + (x - t)\varphi(t) \tag{2.4}$$

for any $x \in [m, M]$ and $t \in (a, b)$.

Using the continuous functional calculus for a selfadjoint operator X with $Sp(X) \subseteq [m, M] \subset (a, b)$ we have from (2.4) in the operator order that

$$f(X) \geq f(t) 1_H + \varphi(t)(X - t1_H) \tag{2.5}$$

for any $t \in (a, b)$.

If the condition (2.2) is valid, then by multiplying both sides by $h(A_i)^{-1/2}$ we get

$$m1_H \leq h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} \leq M1_H.$$

Now, if we take $X = h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}$, $i = 1, ..., n$, in (2.5), then we get

$$f\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) \geq f(t) 1_H + \varphi(t)\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t1_H\right) \tag{2.6}$$

for any $t \in (a, b)$ and $i = 1, ..., n$.

By multiplying both sides of (2.6) with $h(A_i)^{1/2}$ we get

$$\begin{aligned} &h(A_i)^{1/2} f\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) h(A_i)^{1/2} \\ &\geq f(t)\, h(A_i) + \varphi(t)\, h(A_i)^{1/2}\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t 1_H\right) h(A_i)^{1/2} \\ &= f(t)\, h(A_i) + \varphi(t)\,(B_i - t h(A_i)) \end{aligned}$$

for any for any $t \in (a,b)$ and $i = 1, ..., n$, namely

$$P_{f\Delta h}(A_i, B_i) \geq f(t)\, h(A_i) + \varphi(t)\,(B_i - t h(A_i))$$

for all $t \in (a,b)$ and $i = 1, ..., n$.

If we sum over i from 1 to n, we derive the desired result (2.3). □

Corollary 3. *With the assumptions of Theorem 2, we have for any $x \in H \setminus \{0\}$ that*

$$\begin{aligned} \mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) \geq\ & f\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{h(\mathbf{A})} x, x\rangle}\right) S_{h(\mathbf{A})} \\ & + \varphi\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{h(\mathbf{A})} x, x\rangle}\right)\left(S_{\mathbf{B}} - \frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{h(\mathbf{A})} x, x\rangle} S_{h(\mathbf{A})}\right). \end{aligned} \tag{2.7}$$

In particular,

$$\frac{\langle \mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) x, x\rangle}{\langle S_{h(\mathbf{A})} x, x\rangle} \geq f\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{h(\mathbf{A})} x, x\rangle}\right). \tag{2.8}$$

Proof. For $x \in H \setminus \{0\}$ we have

$$\begin{aligned} t_{A,B} = \frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{h(\mathbf{A})} x, x\rangle} &= \frac{\left\langle S_{h(\mathbf{A})}^{1/2}\left(S_{h(\mathbf{A})}^{-1/2} S_{\mathbf{B}} S_{h(\mathbf{A})}^{-1/2}\right) S_{h(\mathbf{A})}^{1/2} x, x\right\rangle}{\left\langle S_{h(\mathbf{A})}^{1/2} x, S_{h(\mathbf{A})}^{1/2} x\right\rangle} \\ &= \frac{\left\langle \left(S_{h(\mathbf{A})}^{-1/2} S_{\mathbf{B}} S_{h(\mathbf{A})}^{-1/2}\right) S_{h(\mathbf{A})}^{1/2} x, S_{h(\mathbf{A})}^{1/2} x\right\rangle}{\left\langle S_{h(\mathbf{A})}^{1/2} x, S_{h(\mathbf{A})}^{1/2} x\right\rangle} \\ &= \frac{\left\langle \left(S_{h(\mathbf{A})}^{-1/2} S_{\mathbf{B}} S_{h(\mathbf{A})}^{-1/2}\right) S_{h(\mathbf{A})}^{1/2} x, S_{h(\mathbf{A})}^{1/2} x\right\rangle}{\left\| S_{h(\mathbf{A})}^{1/2} x\right\|^2}. \end{aligned}$$

If we put

$$u = \frac{S_{h(\mathbf{A})}^{1/2} x}{\left\| S_{h(\mathbf{A})}^{1/2} x\right\|} \neq 0,$$

then $\|u\| = 1$ and

$$t_{\mathbf{A},\mathbf{B}} = \left\langle \left(S_{h(\mathbf{A})}^{-1/2} S_{\mathbf{B}} S_{h(\mathbf{A})}^{-1/2}\right) u, u\right\rangle \in [m, M] \subset (a,b),$$

since, by summing in (2.2), we get

$$mS_{h(\mathbf{A})} \leq S_{\mathbf{B}} \leq MS_{h(\mathbf{A})},$$

which gives

$$mI \leq S_{h(\mathbf{A})}^{-1/2} S_{\mathbf{B}} S_{h(\mathbf{A})}^{-1/2} \leq MI.$$

By taking $t = t_{\mathbf{A},\mathbf{B}}$ in (2.3) we obtain (2.7).

The inequality (2.7) is equivalent to

$$\begin{aligned}
&\langle \mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})y, y\rangle \\
&\geq f\left(\frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{h(\mathbf{A})}x, x\rangle}\right)\langle S_{h(\mathbf{A})}y, y\rangle \\
&\quad + \varphi\left(\frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{h(\mathbf{A})}x, x\rangle}\right)\left(\langle S_{\mathbf{B}}y, y\rangle - \frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{h(\mathbf{A})}x, x\rangle}\langle S_{h(\mathbf{A})}y, y\rangle\right)
\end{aligned} \tag{2.9}$$

for any $y \in H$. This is an inequality of interest in itself.

In particular, if we take in (2.9) $y = x$, then we get the desired result (2.8). □

We also have:

Corollary 4. *With the assumptions of Theorem 2, we have*

$$\begin{aligned}
\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) &\geq 2\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)S_{h(\mathbf{A})} \\
&\quad - \frac{1}{M-m}\left[f(M)\left(MS_{h(\mathbf{A})} - S_{\mathbf{B}}\right) + f(m)\left(S_{\mathbf{B}} - mS_{h(\mathbf{A})}\right)\right].
\end{aligned} \tag{2.10}$$

Proof. If we take the integral mean in the inequality (2.3), then we get

$$\begin{aligned}
&\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) \\
&\geq \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)S_{h(\mathbf{A})} \\
&\quad + \left(\frac{1}{M-m}\int_m^M \varphi(t)\,dt\right)S_{\mathbf{B}} - \left(\frac{1}{M-m}\int_m^M t\varphi(t)\,dt\right)S_{h(\mathbf{A})}.
\end{aligned} \tag{2.11}$$

Observe that, since $\varphi \in \partial f$, hence

$$\frac{1}{M-m}\int_m^M \varphi(t)\,dt = \frac{f(M) - f(m)}{M-m}$$

and

$$\frac{1}{M-m}\int_m^M t\varphi(t)\,dt = \frac{1}{M-m}\left[tf(t)|_m^M - \int_m^M f(t)\,dt\right]$$
$$= \frac{Mf(M)-mf(m)}{M-m} - \frac{1}{M-m}\int_m^M f(t)\,dt$$

and by (2.11) we get

$$\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) \geq \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right) S_{h(\mathbf{A})} + \frac{f(M)-f(m)}{M-m} S_{\mathbf{B}}$$
$$- \left(\frac{Mf(M)-mf(m)}{M-m} - \frac{1}{M-m}\int_m^M f(t)\,dt\right) S_{h(\mathbf{A})}$$
$$= 2\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right) S_{h(\mathbf{A})}$$
$$- \frac{1}{M-m}\left[f(M)(MA-S_{\mathbf{B}}) + f(m)\left(S_{\mathbf{B}} - mS_{h(\mathbf{A})}\right)\right]$$

that proves the desired result (2.10). □

Remark 1. If we take $h(t) = t$, and assume also that f is differentiable on (a,b) while

$$mA_i \leq B_i \leq MA_i, i = 1,...,n \tag{2.12}$$

then

$$\mathbf{I}_f(\mathbf{A},\mathbf{B}) \geq f(t)\,S_{\mathbf{A}} + f'(t)\,(S_{\mathbf{B}} - tS_{\mathbf{A}}) \tag{2.13}$$

for all $t \in (a,b)$.

In particular,

$$\mathbf{I}_f(\mathbf{A},\mathbf{B}) \geq f\left(\frac{m+M}{2}\right) S_{\mathbf{A}} + f'\left(\frac{m+M}{2}\right)\left(S_{\mathbf{B}} - \frac{m+M}{2}S_{\mathbf{A}}\right). \tag{2.14}$$

Moreover, for any $x \in H\setminus\{0\}$ we have that

$$\mathbf{I}_f(\mathbf{A},\mathbf{B}) \geq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right) S_{\mathbf{A}} + f'\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\left(S_{\mathbf{B}} - \frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}S_{\mathbf{A}}\right). \tag{2.15}$$

In particular,

$$\frac{\langle \mathbf{I}_f(\mathbf{A},\mathbf{B})x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle} \geq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right). \tag{2.16}$$

Also, we have

$$\mathbf{I}_f(\mathbf{A},\mathbf{B}) \geq 2\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right) S_{\mathbf{A}} - \frac{1}{M-m}\left[f(M)\left(MS_{\mathbf{A}}-S_{\mathbf{B}}\right)+f(m)\left(S_{\mathbf{B}}-mS_{\mathbf{A}}\right)\right]. \tag{2.17}$$

We observe that for $n=1$, we recapture the results obtained in [10].

The following reverse of inequality (2.3) is as follows:

Theorem 3. *Let $\mathbf{A}=\{A_1,...,A_n\}$, $\mathbf{B}=\{B_1,...,B_n\}$ be two finite sequences of strictly positive operators. If $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is a differentiable convex function and $h:[a,b]\to(0,\infty)$ is a continuous function such that the condition (2.2) is valid, then for any $t\in(a,b)$ we have*

$$\begin{aligned}
&\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})\\
&\leq f(t)\,S_{h(\mathbf{A})}+\mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A},\mathbf{B})-t\mathbf{I}_{f'\Delta h}(\mathbf{A},\mathbf{B})\\
&\leq f(t)\,S_{h(\mathbf{A})}+f'(t)\left(S_{\mathbf{B}}-tS_{h(\mathbf{A})}\right)+\left[f'_-(M)-f'_+(m)\right]\mathbf{I}_{(|\cdot|,t)\Delta h}(\mathbf{A},\mathbf{B})\\
&\leq f(t)\,S_{h(\mathbf{A})}+f'(t)\left(S_{\mathbf{B}}-tS_{h(\mathbf{A})}\right)+(M-m)\left[f'_-(M)-f'_+(m)\right]S_{h(\mathbf{A})}
\end{aligned} \tag{2.18}$$

where ℓ is the identity function, i.e. $\ell(t)=t$ and

$$\mathbf{I}_{(|\cdot|,t)\Delta h}(\mathbf{A},\mathbf{B}):=\sum_{i=1}^n h(A_i)^{1/2}\left|h(A_i)^{-1/2}\left(B_i-th(A_i)\right)h(A_i)^{-1/2}\right|h(A_i)^{1/2}.$$

In particular, we have

$$\begin{aligned}
\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) &\leq f\left(\frac{m+M}{2}\right)S_{h(\mathbf{A})}+\mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A},\mathbf{B})-\frac{m+M}{2}\mathbf{I}_{f'\Delta h}(\mathbf{A},\mathbf{B})\\
&\leq f\left(\frac{m+M}{2}\right)S_{h(\mathbf{A})}+f'\left(\frac{m+M}{2}\right)\left(S_{\mathbf{B}}-\frac{m+M}{2}S_{h(\mathbf{A})}\right)\\
&\quad+\left[f'_-(M)-f'_+(m)\right]\mathbf{I}_{(|\cdot|,\frac{m+M}{2})\Delta h}(\mathbf{A},\mathbf{B})\\
&\leq f\left(\frac{m+M}{2}\right)S_{h(\mathbf{A})}+f'\left(\frac{m+M}{2}\right)\left(S_{\mathbf{B}}-\frac{m+M}{2}S_{h(\mathbf{A})}\right)\\
&\quad+\frac{1}{2}(M-m)\left[f'_-(M)-f'_+(m)\right]S_{h(\mathbf{A})}.
\end{aligned} \tag{2.19}$$

Proof. By the gradient inequality we have

$$f'(x)(x-t)+f(t)\geq f(x) \tag{2.20}$$

for any $x\in[m,M]$ and $t\in(a,b)$.

Using the continuous functional calculus for a selfadjoint operator X with $Sp(X)\subseteq[m,M]\subset\mathring{I}$ we have from (2.13) in the operator order that

$$f'(X)(X-t1_H)+f(t)1_H\geq f(X) \tag{2.21}$$

for any $t \in (a,b)$.

Now, if we take $X = h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}$, $i = 1, ..., n$, in (2.21), then we get

$$f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right)\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t1_H\right) + f(t)\, 1_H \geq f\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) \tag{2.22}$$

for any for any $t \in (a,b)$ and $i = 1, ..., n$.

If we multiply both sides of (2.22) by $h(A_i)^{1/2}$, then we obtain

$$\begin{aligned} &h(A_i)^{1/2} f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right)\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t1_H\right) \\ &\quad h(A_i)^{1/2} + f(t)\, h(A_i) \\ &\quad \geq h(A_i)^{1/2} f\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) h(A_i)^{1/2} \end{aligned}$$

for any $t \in (a,b)$ and $i = 1, ..., n$.

If we sum over i from 1 to n, then we get

$$\begin{aligned} &\sum_{i=1}^{n} h(A_i)^{1/2} f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) \\ &\qquad \times \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t1_H\right) h(A_i)^{1/2} + f(t) \sum_{i=1}^{n} h(A_i) \\ &\quad \geq \sum_{i=1}^{n} h(A_i)^{1/2} f\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) h(A_i)^{1/2}. \end{aligned} \tag{2.23}$$

Since

$$\begin{aligned} &\sum_{i=1}^{n} h(A_i)^{1/2} f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) \\ &\qquad \times \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t1_H\right) h(A_i)^{1/2} + f(t) \sum_{i=1}^{n} h(A_i) \\ &\quad = \mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A}, \mathbf{B}) - t\mathbf{I}_{f'\Delta h}(\mathbf{A}, \mathbf{B}) + f(t)\, S_{h(\mathbf{A})} \end{aligned}$$

and

$$\sum_{i=1}^{n} h(A_i)^{1/2} f\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) h(A_i)^{1/2} = \mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}),$$

then by (2.23) we get the first inequality in (2.18).

Now, observe also that

$$\begin{aligned}
&\sum_{i=1}^{n} h(A_i)^{1/2} f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) \\
&\quad \times \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t1_H\right) h(A_i)^{1/2} + f(t) \sum_{i=1}^{n} h(A_i) \\
&= \sum_{i=1}^{n} h(A_i)^{1/2} \left[f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) - f'(t) 1_H\right] \\
&\quad \times \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t1_H\right) h(A_i)^{1/2} \\
&\quad + f'(t) \left(\sum_{i=1}^{n} B_i - t \sum_{i=1}^{n} h(A_i)\right) + f(t) \sum_{i=1}^{n} h(A_i)
\end{aligned}$$

for any $t \in (a, b)$.

Since f' is nondecreasing on (a, b) we have for any $x \in [m, M]$ and $t \in (a, b)$ that

$$\begin{aligned}
0 &\le (f'(x) - f'(t))(x - t) = |(f'(x) - f'(t))(x - t)| \\
&= |f'(x) - f'(t)| |x - t| \le \left[f'_-(M) - f'_+(m)\right] |x - t| \\
&\le (M - m) \left[f'_-(M) - f'_+(m)\right],
\end{aligned}$$

which, as above, implies in the operator order that

$$\begin{aligned}
&h(A_i)^{1/2} \left[f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) - f'(t) 1_H\right] \\
&\times \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t1_H\right) h(A_i)^{1/2} \\
&\le \left[f'_-(M) - f'_+(m)\right] h(A_i)^{1/2} \left|h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t1_H\right| h(A_i)^{1/2} \\
&= \left[f'_-(M) - f'_+(m)\right] h(A_i)^{1/2} \left|h(A_i)^{-1/2} (B_i - th(A_i)) h(A_i)^{-1/2}\right| h(A_i)^{1/2} \\
&\le (M - m) \left[f'_-(M) - f'_+(m)\right] h(A_i).
\end{aligned}$$

If we sum over i from 1 to n, then we get

$$\begin{aligned}
&\sum_{i=1}^{n} h(A_i)^{1/2} \left[f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) - f'(t) 1_H\right] \\
&\quad \times \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t1_H\right) h(A_i)^{1/2} \\
&\le \left[f'_-(M) - f'_+(m)\right] \sum_{i=1}^{n} h(A_i)^{1/2} \left|h(A_i)^{-1/2} (B_i - th(A_i))\right. \\
&\quad \left. h(A_i)^{-1/2}\right| h(A_i)^{1/2}
\end{aligned}$$

$$\leq (M-m)\left[f'_{-}(M)-f'_{+}(m)\right]\sum_{i=1}^{n}h(A_i)$$

This proves the second inequality in (2.18).

We need to prove only the last part of (2.19).

Since $x\in[m,M]$, then $\left|x-\frac{m+M}{2}\right|\leq\frac{1}{2}(M-m)$ that implies in the operator order

$$\left|h(A_i)^{-1/2}B_iAh(A_i)^{-1/2}-\frac{m+M}{2}1_H\right|\leq\frac{1}{2}(M-m)\,1_H,$$

which by multiplication on both sides with $h(A_i)^{1/2}$ and summing over i gives that

$$\mathbf{I}_{(|\cdot|,\frac{m+M}{2})\Delta h}(\mathbf{A},\mathbf{B})\leq\frac{1}{2}(M-m)\,S_{h(\mathbf{A})}.$$

□

Corollary 5. *With the assumptions of Theorem 3, we have for any $x\in H\setminus\{0\}$ that*

$$\begin{aligned}
&\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})\\
&\leq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}\right)S_{h(\mathbf{A})}+\mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A},\mathbf{B})-\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}\mathbf{I}_{f'\Delta h}(\mathbf{A},\mathbf{B})\\
&\leq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}\right)S_{h(\mathbf{A})}\\
&\quad+f'\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}\right)\left(S_{\mathbf{B}}-\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}S_{h(\mathbf{A})}\right)\\
&\quad+\left[f'_{-}(M)-f'_{+}(m)\right]\mathbf{I}_{\left(|\cdot|,\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}\right)\Delta h}(\mathbf{A},\mathbf{B})\\
&\leq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}\right)S_{h(\mathbf{A})}\\
&\quad+f'\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}\right)\left(S_{\mathbf{B}}-\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}S_{h(\mathbf{A})}\right)\\
&\quad+(M-m)\left[f'_{-}(M)-f'_{+}(m)\right]S_{h(\mathbf{A})}. \qquad (2.24)
\end{aligned}$$

In particular,

$$\begin{aligned}
\langle\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})x,x\rangle&\leq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}\right)\langle S_{h(\mathbf{A})}x,x\rangle+\langle\mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A},\mathbf{B})x,x\rangle\\
&\quad-\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}\langle\mathbf{I}_{f'\Delta h}(\mathbf{A},\mathbf{B})x,x\rangle
\end{aligned}$$

$$\leq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}\right)\langle S_{h(\mathbf{A})}x,x\rangle$$
$$+\left[f'_{-}(M)-f'_{+}(m)\right]\left\langle \mathbf{I}_{\left(|\cdot|,\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}\right)\Delta h}(\mathbf{A},\mathbf{B})x,x\right\rangle$$
$$\leq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{h(\mathbf{A})}x,x\rangle}\right)\langle S_{h(\mathbf{A})}x,x\rangle$$
$$+(M-m)\left[f'_{-}(M)-f'_{+}(m)\right]\langle S_{h(\mathbf{A})}x,x\rangle \tag{2.25}$$

for any $x\in H\backslash\{0\}$.

We also have:

Corollary 6. *With the assumptions of Theorem 3, we have*

$$\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})\leq\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)S_{h(\mathbf{A})}$$
$$+\mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A},\mathbf{B})-\frac{m+M}{2}\mathbf{I}_{f'\Delta h}(\mathbf{A},\mathbf{B})$$
$$\leq 2\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)S_{h(\mathbf{A})}$$
$$-\frac{1}{M-m}\left[\Phi(M)\left(MS_{h(\mathbf{A})}-S_{\mathbf{B}}\right)+\Phi(m)\left(S_{\mathbf{B}}-mS_{h(\mathbf{A})}\right)\right]$$
$$+\frac{f'_{-}(M)-f'_{+}(m)}{M-m}\int_m^M \mathbf{I}_{(|\cdot|,t)\Delta h}(\mathbf{A},\mathbf{B})dt. \tag{2.26}$$

Proof. If we take the integral mean in (2.12), then we get

$$\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})$$
$$\leq\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)S_{h(\mathbf{A})}+\mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A},\mathbf{B})-\frac{m+M}{2}\mathbf{I}_{f'\Delta h}(\mathbf{A},\mathbf{B})$$
$$\leq\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)S_{h(\mathbf{A})}+\frac{1}{M-m}\int_m^M f'(t)\left(S_{\mathbf{B}}-tS_{h(\mathbf{A})}\right)dt$$
$$+\frac{f'_{-}(M)-f'_{+}(m)}{M-m}\int_m^M \mathbf{I}_{(|\cdot|,t)\Delta h}(\mathbf{A},\mathbf{B})dt. \tag{2.27}$$

Since, as in the proof of Corollary 4, we have

$$\frac{1}{M-m}\int_m^M f'(t)\left(S_{\mathbf{B}}B-tS_{h(\mathbf{A})}\right)dt$$
$$=\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)S_{h(\mathbf{A})}$$

$$-\frac{1}{M-m}\left[f(M)\left(MS_{h(\mathbf{A})}-S_{\mathbf{B}}\right)+f(m)\left(S_{\mathbf{B}}-mS_{h(\mathbf{A})}\right)\right],$$

then by (2.27) we get the last part of (2.26). □

Remark 2. If we take $h(t)=t$, and assume also that f is differentiable on (a,b) while $mA_i\le B_i\le MA_i$, $i=1,...,n$, then

$$\begin{aligned}\mathbf{I}_f(\mathbf{A},\mathbf{B})&\le f(t)S_{\mathbf{A}}+\mathbf{I}_{f'\ell}(\mathbf{A},\mathbf{B})-t\mathbf{I}_{f'}(\mathbf{A},\mathbf{B})\\&\le f(t)S_{\mathbf{A}}+f'(t)(S_{\mathbf{B}}-tS_{\mathbf{A}})+\left[f'_-(M)-f'_+(m)\right]\mathbf{I}_{(|\cdot|,t)}(\mathbf{A},\mathbf{B})\\&\le f(t)S_{\mathbf{A}}+f'(t)(S_{\mathbf{B}}-tS_{\mathbf{A}})+(M-m)\left[f'_-(M)-f'_+(m)\right]S_{\mathbf{A}}\end{aligned}\tag{2.28}$$

for all $t\in(a,b)$, where ℓ is the identity function, i.e. $\ell(t)=t$ and

$$\mathbf{I}_{(|\cdot|,t)}(\mathbf{A},\mathbf{B}):=\sum_{i=1}^{n}{A_i}^{1/2}\left|{A_i}^{-1/2}(B_i-tA_i){A_i}^{-1/2}\right|{A_i}^{1/2}.$$

In particular, we have

$$\begin{aligned}\mathbf{I}_f(\mathbf{A},\mathbf{B})&\le f\left(\frac{m+M}{2}\right)S_{\mathbf{A}}+\mathbf{I}_{f'\ell}(\mathbf{A},\mathbf{B})-\frac{m+M}{2}\mathbf{I}_{f'}(\mathbf{A},\mathbf{B})\\&\le f\left(\frac{m+M}{2}\right)S_{\mathbf{A}}+f'\left(\frac{m+M}{2}\right)\left(S_{\mathbf{B}}-\frac{m+M}{2}S_{\mathbf{A}}\right)\\&\quad+\left[f'_-(M)-f'_+(m)\right]\mathbf{I}_{\left(|\cdot|,\frac{m+M}{2}\right)}(\mathbf{A},\mathbf{B})\\&\le f\left(\frac{m+M}{2}\right)S_{\mathbf{A}}+f'\left(\frac{m+M}{2}\right)\left(S_{\mathbf{B}}-\frac{m+M}{2}S_{\mathbf{A}}\right)\\&\quad+\frac{1}{2}(M-m)\left[f'_-(M)-f'_+(m)\right]S_{\mathbf{A}}.\end{aligned}\tag{2.29}$$

We also have

$$\begin{aligned}\mathbf{I}_f(\mathbf{A},\mathbf{B})&\le f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)S_{\mathbf{A}}+\mathbf{I}_{f'\ell}(\mathbf{A},\mathbf{B})-\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\mathbf{I}_{f'}(\mathbf{A},\mathbf{B})\\&\le f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)S_{\mathbf{A}}+f'\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\left(S_{\mathbf{B}}-\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}S_{\mathbf{A}}\right)\\&\quad+\left[f'_-(M)-f'_+(m)\right]\mathbf{I}_{\left(|\cdot|,\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)}(\mathbf{A},\mathbf{B})\\&\le f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)S_{\mathbf{A}}+f'\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\left(S_{\mathbf{B}}-\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}S_{\mathbf{A}}\right)\\&\quad+(M-m)\left[f'_-(M)-f'_+(m)\right]S_{\mathbf{A}}\end{aligned}\tag{2.30}$$

and

$$\begin{aligned}&\langle\mathbf{I}_f(\mathbf{A},\mathbf{B})x,x\rangle\\&\le f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\langle S_{\mathbf{A}}x,x\rangle+\langle\mathbf{I}_{f'\ell}(\mathbf{A},\mathbf{B})x,x\rangle-\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\langle\mathbf{I}_{f'}(\mathbf{A},\mathbf{B})x,x\rangle\end{aligned}$$

$$\leq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\langle S_{\mathbf{A}}x,x\rangle+\left[f'_{-}(M)-f'_{+}(m)\right]\left\langle \mathbf{I}_{\left(|\cdot|,\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)}(\mathbf{A},\mathbf{B})x,x\right\rangle$$

$$\leq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\langle S_{\mathbf{A}}x,x\rangle+(M-m)\left[f'_{-}(M)-f'_{+}(m)\right]\langle S_{\mathbf{A}}x,x\rangle \quad (2.31)$$

for $x\in H\setminus\{0\}$.

Also, we have

$$\begin{aligned}\mathbf{I}_f(\mathbf{A},\mathbf{B}) &\leq \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)S_{\mathbf{A}}\\ &\quad+\mathbf{I}_{f'\ell}(\mathbf{A},\mathbf{B})-\frac{m+M}{2}\mathbf{I}_{f'}(\mathbf{A},\mathbf{B})\\ &\leq 2\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)S_{\mathbf{A}}\\ &\quad-\frac{1}{M-m}\left[\Phi(M)(MS_{\mathbf{A}}-S_{\mathbf{B}})+\Phi(m)(S_{\mathbf{B}}-mS_{\mathbf{A}})\right]\\ &\quad+\frac{f'_{-}(M)-f'_{+}(m)}{M-m}\int_m^M \mathbf{I}_{(|\cdot|,t)}(\mathbf{A},\mathbf{B})dt. \end{aligned}\quad (2.32)$$

We observe that for $n=1$, we recapture the results obtained in [10].

3. Information Monotonicity

We now prove the information monotonicity of the generalized Csiszár f -divergence operator as follows. In our main results we find the complementary inequalities and some other applications. Note that for a positive linear map Φ and a finite sequence of strictly positive operators $\mathbf{A}=\{A_1,...,A_n\}$ by $\Phi(\mathbf{A})$ we mean the finite sequence of strictly positive operators $\{\Phi(A_1),...,\Phi(A_n)\}$.

Theorem 4. *If $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is a positive linear map, f an operator convex function on an interval $[a,b]$ and $h:[a,b]\to(0,\infty)$ a continuous function such that the condition (2.2) is valid for some m, $M\in[a,b]$ with $0<m<M$, then*

$$\mathbf{I}_{f\Delta h}(\Phi(\mathbf{A}),\Phi(\mathbf{B}))\leq\Phi(\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}))$$

for any strictly positive operators A_i, any self-adjoint operator B_i with

$$Sp(h(A_i)^{-1/2}B_ih(A_i)^{-1/2}),Sp(h(\Phi(A_i))^{-1/2}\Phi(B_i)h(\Phi(A_i))^{-1/2})\subseteq[m,M]$$

and any Hilbert spaces H, K.

Proof. Following an idea of Ando [2] to a fixed strictly positive operator $A_i\in\mathcal{B}(H)$ we set

$$\Psi(X)=h(\Phi(A_i))^{-1/2}\Phi(h(A_i)^{1/2}Xh(A_i)^{1/2})h(\Phi(A_i))^{-1/2}$$

and notice that $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a unital linear map. By operator convexity of f and Choi-Davis-Jensen inequality, we realize that

$$f(\Psi(X)) \leq \Psi(f(X))$$

for all positive invertible operator X. Therefore,

$$f\Big(h(\Phi(A_i))^{-1/2}\Phi(h(A_i)^{1/2}Xh(A_i)^{1/2})h(\Phi(A_i))^{-1/2}\Big)$$
$$\leq h(\Phi(A_i))^{-1/2}\Phi(h(A_i)^{1/2}f(X)h(A_i)^{1/2})h(\Phi(A_i))^{-1/2}$$

For $X = h(A_i)^{-1/2}B_iA_i^{-1/2}$ this entails

$$P_{f\Delta h}(\Phi(A_i), \Phi(B_i)) \leq \Phi(P_{f\Delta h}(A_i, B_i).$$

If we sum over i from 1 to n, then we get

$$\sum_{i=i}^{n} P_{f\Delta h}(\Phi(A_i), \Phi(B_i)) \leq \sum_{i=i}^{n} \Phi(P_{f\Delta h}(A_i, B_i)$$
$$= \Phi\left(\sum_{i=i}^{n} P_{f\Delta h}(A_i, B_i)\right),$$

which entails the result. □

We now prove a complementary inequality for the information monotonicity of the operator perspective.

Corollary 7. *Let f be a differentiable function on $[a, b]$ and $h : [a, b] \to (0, \infty)$ a continuous function such that the condition (2.2) is valid for some m, $M \in [a, b]$ with $0 < m < M$, any strictly positive operators A_i and any self-adjoint operator B_i.*

(i) *If f' is decreasing, then*

$$f(m)S_{h(\mathbf{A})} + f'(b)(S_{\mathbf{B}} - mS_{h(\mathbf{A})}) \leq \mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B})$$
$$\leq f(m)S_{h(\mathbf{A})} + f'(a)(S_{\mathbf{B}} - mS_{h(\mathbf{A})}).$$

(ii) *If f' is increasing, then the reverse inequalities in part (i) hold.*

Proof. (i) Regarding the assumption f' is decreasing and so $f'(b) \leq f'(t) \leq f'(a)$ for any $t \in [a, b]$. Define $g(t) := f(t) - f'(b)t$ and $k(t) := f'(a)t - f(t)$ for any $t \in [a, b]$. Since g is increasing, $g(m) \leq g(t)$ for any $t \in [m, M]$ and so

$$f(m) - f'(b)m \leq f(t) - f'(b)t,$$

which entails

$$f(m) + f'(b)(t - m) \leq f(t) \tag{3.1}$$

for any $t \in [m, M]$. Since k is increasing, $k(m) \leq k(t)$ for any $t \in [m, M]$ and hence

$$f'(a)m - f(m) \leq f'(a)t - f(t).$$

This inequality signifies

$$f(t) \leq f(m) + f'(a)(t-m) \tag{3.2}$$

for any $t \in [m, M]$. Combining the inequalities (3.1) and (3.2), one can yield

$$f(m) + f'(b)(t-m) \leq f(t) \leq f(m) + f'(a)(t-m) \tag{3.3}$$

for any $t \in [m, M]$.

Using the continuous functional calculus for a selfadjoint operator X with $Sp(X) \subseteq [m, M] \subset (a, b)$ we have from (3.3) in the operator order that

$$f(m)1_H + f'(b)(X - m1_H) \leq f(X) \leq f(m)1_H + f'(a)(X - m1_H) \tag{3.4}$$

for any $t \in [m, M]$. If we take $X = h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}$, $i = 1, ..., n$, in (3.4) and multiplying both sides by $h(A_i)^{-1/2}$ we get

$$\begin{aligned} f(m)h(A_i) + f'(b)(B_i - mh(A_i)) &\leq h(A_i) f(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}) h(A_i) \\ &\leq f(m)h(A_i) + f'(a)(B_i - mh(A_i)) \end{aligned}$$

If we sum over i from 1 to n, then we get the result.

(ii) Applying a similar way as in the proof of part (i) we reach the result. □

Corollary 8. *If $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a positive linear map, f an operator convex function on $[a, b]$ and $h : [a, b] \to (0, \infty)$ a continuous function such that the condition (2.2) is valid for some m, $M \in [a, b]$ with $0 < m < M$, any strictly positive operators A_i and any self-adjoint operator B_i, then*

$$\begin{aligned} \Phi(\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B})) \leq \mathbf{I}_{f\Delta h}(\Phi(\mathbf{A}), \Phi(\mathbf{B})) &+ f(m)(S_{\Phi(h(\mathbf{A}))} - S_{h(\Phi(\mathbf{A}))}) \\ &+ (f'(b) - f'(a)) S_{\Phi(\mathbf{B})} \\ &+ m(f'(a) S_{h(\Phi(\mathbf{A}))} - f'(b) S_{\Phi(h(\mathbf{A}))}). \end{aligned}$$

In particular, if $\Phi \circ h = h \circ \Phi$, then

$$\begin{aligned} \Phi(\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B})) \leq \mathbf{I}_{f\Delta h}(\Phi(\mathbf{A}), \Phi(\mathbf{B})) &\\ + (f'(b) - f'(a))(S_{\Phi(\mathbf{B})} - m S_{\Phi(h(\mathbf{A}))}). & \end{aligned}$$

Moreover, if $h(t) = t$, then

$$\begin{aligned} \Phi(\mathbf{I}_f(\mathbf{A}, \mathbf{B})) \leq \mathbf{I}_f(\Phi(\mathbf{A}), \Phi(\mathbf{B})) &\\ + (f'(b) - f'(a))(S_{\Phi(\mathbf{B})} - m S_{\Phi(\mathbf{A})}). & \end{aligned}$$

Proof. Since f is operator convex, so that f is differentiable and f' is increasing. Using Corollary 7 (ii) we find that

$$\begin{aligned} f(m) S_{h(\mathbf{A})} + f'(b)(S_{\mathbf{B}} - m S_{h(\mathbf{A})}) &\geq \mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) \\ &\geq f(m) S_{h(\mathbf{A})} + f'(a)(S_{\mathbf{B}} - m S_{h(\mathbf{A})}). \end{aligned} \tag{3.5}$$

Substituting $\Phi(\mathbf{A}), \Phi(\mathbf{B})$ by $\mathbf{A}, \mathbf{B}$ in the second inequality one can deduce

$$\mathbf{I}_{f\Delta h}(\Phi(\mathbf{A}), \Phi(\mathbf{B})) \geq f(m) S_{h(\Phi(\mathbf{A}))} + f'(a)(S_{\Phi(\mathbf{B})} - m S_{h(\Phi(\mathbf{A}))}). \tag{3.6}$$

It follows from (3.6) that

$$\mathbf{I}_{f\Delta h}(\Phi(\mathbf{A}),\Phi(\mathbf{B}))-f(m)S_{h(\Phi(\mathbf{A}))}-f'(a)(S_{\Phi(\mathbf{B})}-mS_{h(\Phi(\mathbf{A}))})\geq 0. \tag{3.7}$$

By linearity and positiveness of Φ, the second inequality in (3.5) leads to the following inequality

$$f(m)\Phi(S_{h(\mathbf{A})})+f'(b)(\Phi(S_{\mathbf{B}})-m\Phi(S_{h(\mathbf{A})}))\geq \Phi(\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})). \tag{3.8}$$

Since $\Phi(S_{h(\mathbf{A})})=S_{\Phi(h(\mathbf{A}))}$ and $\Phi(S_{\mathbf{B}})=S_{\Phi(\mathbf{B})}$, from (3.8) we have

$$f(m)S_{\Phi(h(\mathbf{A}))}+f'(b)(S_{\Phi(\mathbf{B})}-mS_{\Phi(h(\mathbf{A}))})\geq \Phi(\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})). \tag{3.9}$$

Summing the inequalities (3.7) and (3.9), we reach the desired result.

Note that if $\Phi\circ h=h\circ\Phi$, then $S_{\Phi(h(\mathbf{A}))}=S_{h(\Phi(\mathbf{A}))}$ and the result follows. □

4. Applications for Operator Geometric Mean

Assume that A, B are positive invertible operators on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. We use the following notations for operators

$$A\nabla_{\nu}B:=(1-\nu)A+\nu B,$$

the *weighted operator arithmetic mean* and

$$A\sharp_{\nu}B:=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2},$$

the *weighted operator geometric mean,* where $\nu\in[0,1]$. When $\nu=\frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

The definition $A\sharp_{\nu}B$ can be extended accordingly for any real number ν.

The following inequality is well as the operator *Young inequality* or operator *ν-weighted arithmetic–geometric mean inequality*: [16]

$$A\sharp_{\nu}B\leq A\nabla_{\nu}B \text{ for all } \nu\in[0,1]. \tag{4.1}$$

We consider the continuous function $f_{\nu}:[0,\infty)\to[0,\infty)$, $f_{\nu}(x)=x^{\nu}$ for $\nu\in[0,1]$. Consider the convex function $f=-f_{\nu}$. We consider the Csisz ár f-divergence operator mapping by setting

$$\mathbf{I}_{f_{\nu}}(\mathbf{A},\mathbf{B})=-\sum_{i=1}^{n}A_i\sharp_{\nu}B_i$$

and the generalized Csiszár f-divergence operator mapping via

$$\mathbf{I}_{f_{\nu}\Delta h}(\mathbf{A},\mathbf{B})=-\sum_{i=1}^{n}h(A_i)\sharp_{\nu}B_i.$$

Assume that $\mathbf{A}=\{A_1,...,A_n\}$, $\mathbf{B}=\{B_1,...,B_n\}$ satisfy the condition

$$mA_i\leq B_i\leq MA_i,\quad i=1,...,n$$

then by (2.13) we derive

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \leq t^\nu (1-\nu) S_{\mathbf{A}} + \nu t^{\nu-1} S_{\mathbf{B}} \tag{4.2}$$

for all $t \in (a,b)$ which contains the interval $[m,M] \subset (0,\infty)$.

In particular,

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \leq (1-\nu) \left(\frac{m+M}{2}\right)^\nu S_{\mathbf{A}} + \nu \left(\frac{m+M}{2}\right)^{\nu-1} S_{\mathbf{B}}. \tag{4.3}$$

Moreover, for any $x \in Hc\{0\}$ we have that

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \leq (1-\nu) \left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{\mathbf{A}} x, x\rangle}\right)^\nu S_{\mathbf{A}} + \nu \left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{\mathbf{A}} x, x\rangle}\right)^{\nu-1} S_{\mathbf{B}}. \tag{4.4}$$

In particular,

$$\left\langle \sum_{i=1}^{n} A_i \sharp_\nu B_i x, x \right\rangle \leq \langle S_{\mathbf{B}} x, x\rangle^\nu \langle S_{\mathbf{A}} x, x\rangle^{1-\nu}, \tag{4.5}$$

which holds for any $A_i,\ B_i \geq 0$. Moreover, if we take the supremum over $\|x\| = 1$, we deduce the norm inequality

$$\left\| \sum_{i=1}^{n} A_i \sharp_\nu B_i \right\| \leq \|S_{\mathbf{A}}\|^{1-\nu} \|S_{\mathbf{B}}\|^\nu$$

for $\nu \in [0,1]$.

We observe that, if we write the inequality (4.1) for A_i, B_i an sum, then we get

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \leq (1-\nu) S_{\mathbf{A}} + \nu S_{\mathbf{B}},$$

which is equivalent to

$$\sum_{i=1}^{n} \langle A_i \sharp_\nu B_i x, x\rangle \leq (1-\nu) \langle S_{\mathbf{A}} x, x\rangle + \nu \langle S_{\mathbf{B}} x, x\rangle \tag{4.6}$$

for any $x \in H \setminus \{0\}$.

Since by the scalar *A-G-inequality* we have

$$\langle S_{\mathbf{B}} x, x\rangle^\nu \langle S_{\mathbf{A}} x, x\rangle^{1-\nu} \leq (1-\nu) \langle S_{\mathbf{A}} x, x\rangle + \nu \langle S_{\mathbf{B}} x, x\rangle,$$

for any $x \in H \setminus \{0\}$, then we can conclude that the inequality (4.5) is better than (4.6).

For $x \neq y$ and $p \in \mathbb{R}\setminus\{-1,0\}$, we define the *p- logarithmic mean* (*generalized logarithmic mean*) $L_p(x,y)$ by

$$L_p(x,y) := \left[\frac{y^{p+1} - x^{p+1}}{(p+1)(y-x)}\right]^{1/p}.$$

From (2.17) we derive

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \leq 2L_\nu^\nu(m,M)S_{\mathbf{A}} - \frac{1}{M-m}\left[M^\nu\left(MS_{\mathbf{A}} - S_{\mathbf{B}}\right) + m^\nu\left(S_{\mathbf{B}} - mS_{\mathbf{A}}\right)\right]. \tag{4.7}$$

From the inequality (2.28) we derive

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \geq t^\nu S_{\mathbf{A}} + \nu t^{\nu-1}\left(S_{\mathbf{B}} - tS_{\mathbf{A}}\right) - \nu\left(M-m\right)\left(m^{\nu-1} - M^{\nu-1}\right)S_{\mathbf{A}}$$

for all $t \in (a,b)$ which contains the interval $[m,M] \subset (0,\infty)$.

From (2.29) we get

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \geq \left(\frac{m+M}{2}\right)^\nu S_{\mathbf{A}} + \nu\left(\frac{m+M}{2}\right)^{\nu-1}\left(S_{\mathbf{B}} - \frac{m+M}{2}S_{\mathbf{A}}\right) - \frac{1}{2}\nu\left(M-m\right)\left(m^{\nu-1} - M^{\nu-1}\right)S_{\mathbf{A}}. \tag{4.8}$$

From (2.31) we derive

$$\left\langle \sum_{i=1}^{n} A_i \sharp_\nu B_i x, x \right\rangle \geq \left\langle S_{\mathbf{B}}x, x\right\rangle^\nu \left\langle S_{\mathbf{A}}x, x\right\rangle^{1-\nu} - \nu\left(M-m\right)\left(m^{\nu-1} - M^{\nu-1}\right)\left\langle S_{\mathbf{A}}x, x\right\rangle \tag{4.9}$$

for any $x \in H \setminus \{0\}$.

5. Applications for Relative Operator Entropy

Kamei and Fujii [14], defined the *relative operator entropy* $S\left(A|B\right)$, for positive invertible operators A and B, by

$$S\left(A|B\right) := A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}, \tag{5.1}$$

which is a relative version of the operator entropy [3,15,21,32].

Consider the logarithmic function $\ln$. Then the relative operator entropy can be interpreted as the perspective of $-\ln$, namely

$$P_{-\ln}\left(A,B\right) = -S\left(A|B\right).$$

Consider the convex function $f = -\ln$. We consider the Csiszár f -divergence operator mapping by setting

$$\mathbf{I}_{-\ln}(\mathbf{A},\mathbf{B}) = -\sum_{i=1}^{n} S\left(A_i|B_i\right) = -\sum_{i=1}^{n} A_i^{1/2}\left(\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)A_i^{\frac{1}{2}}$$

and the generalized Csiszár f-divergence operator mapping via

$$\mathbf{I}_{(-\ln)\Delta h}(\mathbf{A},\mathbf{B}) = -\sum_{i=1}^{n} S\left(h\left(A_i\right)|B_i\right)$$
$$= -\sum_{i=1}^{n} h\left(A_i\right)^{1/2}\left(\ln\left(h\left(A_i\right)^{-1/2} B_i h\left(A_i\right)^{-1/2}\right)\right) h\left(A_i\right)^{1/2}.$$

From Remark 1 we get

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \leq \left(\ln t\right) S_{\mathbf{A}} + t^{-1}\left(S_{\mathbf{B}} - tS_{\mathbf{A}}\right) \tag{5.2}$$

for all $t > 0$.

In particular, if $\mathbf{A} = \{A_1, ..., A_n\}$, $\mathbf{B} = \{B_1, ..., B_n\}$ satisfy the condition (2.12), then

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \leq \ln\left(\frac{m+M}{2}\right) S_{\mathbf{A}} + \left(\frac{m+M}{2}\right)^{-1}\left(S_{\mathbf{B}} - \frac{m+M}{2} S_{\mathbf{A}}\right).$$

Moreover, for any $x \in H\backslash\{0\}$ we have that

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \leq \ln\left(\frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{\mathbf{A}}x, x\rangle}\right) S_{\mathbf{A}} + \left(\frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{\mathbf{A}}x, x\rangle}\right)^{-1}\left(S_{\mathbf{B}} - \frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{\mathbf{A}}x, x\rangle} S_{\mathbf{A}}\right).$$

In particular,

$$\left\langle \sum_{i=1}^{n} S\left(A_i|B_i\right)x, x\right\rangle \leq \langle S_{\mathbf{A}}x, x\rangle \ln\left(\frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{\mathbf{A}}x, x\rangle}\right). \tag{5.3}$$

The following inequality for the relative operator entropy is well known

$$S\left(A|B\right) \leq B - A$$

for any A, B positive invertible operators. This inequality is equivalent to

$$\langle S\left(A|B\right)x, x\rangle \leq \langle Bx, x\rangle - \langle Ax, x\rangle \tag{5.4}$$

for any $x \in H$.

By summing over i in (5.4) we get

$$\left\langle \sum_{i=1}^{n} S\left(A_i|B_i\right)x, x\right\rangle \leq \langle S_{\mathbf{B}}x, x\rangle - \langle S_{\mathbf{A}}x, x\rangle. \tag{5.5}$$

We know the following elementary inequality that holds for the logarithm

$$\ln t \leq t - 1 \text{ for any} t > 0.$$

If we take in this inequality $t = \frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{\mathbf{A}}x, x\rangle} > 0$, $x \in H$, $x \neq 0$ and multiply with $\langle S_{\mathbf{A}}x, x\rangle > 0$, then we get

$$\langle S_{\mathbf{A}}x, x\rangle \ln\left(\frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{\mathbf{A}}x, x\rangle}\right) \leq \langle S_{\mathbf{B}}x, x\rangle - \langle S_{\mathbf{A}}x, x\rangle \tag{5.6}$$

for any $x \in H$, $x \neq 0$.

Therefore, by (5.3) and (5.6) we have

$$\left\langle \sum_{i=1}^{n} S\left(A_i|B_i\right) x, x \right\rangle \leq \langle S_{\mathbf{A}} x, x\rangle \ln\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{\mathbf{A}} x, x\rangle}\right) \leq \langle S_{\mathbf{B}} x, x\rangle - \langle S_{\mathbf{A}} x, x\rangle$$

for any $x \in H$, $x \neq 0$.

This shows that the inequality (5.3) is better than (5.5).

Also, we have

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \leq 2 \ln I\left(m, M\right) S_{\mathbf{A}} - \frac{1}{M-m}\left[\ln\left(M\right)\left(M S_{\mathbf{A}} - S_{\mathbf{B}}\right) + \ln\left(m\right)\left(S_{\mathbf{B}} - m S_{\mathbf{A}}\right)\right], \quad (5.7)$$

where $I\left(m, M\right)$ is the *identric mean* defined for x, $y > 0$ by

$$I(x, y) := \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)} \quad (5.8)$$

and satisfies the equality

$$\frac{1}{M-m}\int_m^M \ln t dt = \ln I\left(m, M\right).$$

From Remark 2 we obtain

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \geq \ln\left(t\right) S_{\mathbf{A}} + t^{-1}\left(S_{\mathbf{B}} - t S_{\mathbf{A}}\right) - \left(M-m\right)\left(m^{-1} - M^{-1}\right) S_{\mathbf{A}}$$

for $t \in [m, M] \subset (0, \infty)$.

From (2.29) we get the operator inequality

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \geq \ln\left(\frac{m+M}{2}\right) S_{\mathbf{A}} + \left(\frac{m+M}{2}\right)^{-1}\left(S_{\mathbf{B}} - \frac{m+M}{2} S_{\mathbf{A}}\right) - \frac{1}{2}\left(M-m\right)\left(m^{-1} - M^{-1}\right) S_{\mathbf{A}}.$$

From (2.30) we obtain

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \geq \ln\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{\mathbf{A}} x, x\rangle}\right) S_{\mathbf{A}} + \left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{\mathbf{A}} x, x\rangle}\right)^{-1}\left(S_{\mathbf{B}} - \frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{\mathbf{A}} x, x\rangle} S_{\mathbf{A}}\right) - \left(M-m\right)\left(m^{-1} - M^{-1}\right) S_{\mathbf{A}}$$

while from (2.31)

$$\left\langle \sum_{i=1}^{n} S\left(A_i|B_i\right)x,x\right\rangle \geq \ln\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\langle S_{\mathbf{A}}x,x\rangle - (M-m)\left(m^{-1}-M^{-1}\right)\langle S_{\mathbf{A}}x,x\rangle$$

for $x \in H$, $x \neq 0$.

6. Some Applications of the Information Monotonicity

Assume that $\mathbf{A} = \{A_1, ..., A_n\}$, $\mathbf{B} = \{B_1, ..., B_n\}$ satisfy the condition

$$mA_i \leq B_i \leq MA_i, \quad i = 1, ..., n.$$

Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a positive linear map and $h : [a,b] \to (0,\infty)$ a continuous function such that the condition (2.2) is valid for some $m, M \in [a,b]$ with $0 < m < M$. Let $f_\nu : [0,\infty) \to [0,\infty)$, $f_\nu(x) = x^\nu$ for $\nu \in [0,1]$ be a continuous function. Consider the convex function $f = -f_\nu$. We consider the information monotonicity for the Csisz ár f-divergence operator mapping and the generalized Csiszár f-divergence operator mapping. By Theorem 4 we derive

$$-\sum_{i=1}^{n} h(\Phi(A_i))\sharp_\nu \Phi(B_i) \leq \Phi\left(-\sum_{i=1}^{n} h(A_i)\sharp_\nu B_i\right)$$

and hence

$$\sum_{i=1}^{n} h(\Phi(A_i))\sharp_\nu \Phi(B_i) \geq \Phi\left(\sum_{i=1}^{n} h(A_i)\sharp_\nu B_i\right). \tag{6.1}$$

Moreover, by Corollary 8, we get a complementary for this inequality as follows:

$$\begin{aligned}\Phi\left(-\sum_{i=1}^{n} h(A_i)\sharp_\nu B_i\right) \leq & -\sum_{i=1}^{n} h(\Phi(A_i))\sharp_\nu \Phi(B_i) \\ & - m^\nu(S_{\Phi(h(\mathbf{A}))} - S_{h(\Phi(\mathbf{A}))}) \\ & + (-\nu b^{\nu-1} + \nu a^{\nu-1})S_{\Phi(\mathbf{B})} \\ & + m(-\nu a^{\nu-1}S_{h(\Phi(\mathbf{A}))} + \nu b^{\nu-1}S_{\Phi(h(\mathbf{A}))})\end{aligned}$$

and so

$$\begin{aligned}\sum_{i=1}^{n} \Phi\left(h(A_i)\sharp_\nu B_i\right) \geq & \sum_{i=1}^{n} h(\Phi(A_i))\sharp_\nu \Phi(B_i) \\ & + m^\nu(S_{\Phi(h(\mathbf{A}))} - S_{h(\Phi(\mathbf{A}))}) \\ & + \nu(b^{\nu-1} - a^{\nu-1})S_{\Phi(\mathbf{B})} \\ & + m\nu(a^{\nu-1}S_{h(\Phi(\mathbf{A}))} - b^{\nu-1}S_{\Phi(h(\mathbf{A}))}).\end{aligned} \tag{6.2}$$

Combining (6.1) and (6.2) one can reach the information monotonicity and its complementary for the generalized Csiszár f-divergence operator mapping. In particular, when $h(t) = t$, we obtain the information monotonicity for the Csiszár f-divergence operator mapping as follows

$$\begin{aligned}\sum_{i=1}^{n}\Phi(A_i)\sharp_\nu\Phi(B_i) &\geq \sum_{i=1}^{n}\Phi\Big(A_i\sharp_\nu B_i\Big)\\ &\geq \sum_{i=1}^{n}\Phi(A_i)\sharp_\nu\Phi(B_i)\\ &\quad+\nu(b^{\nu-1}-a^{\nu-1})(S_{\Phi(\mathbf{B})}-mS_{\Phi(\mathbf{A})}).\end{aligned}$$

W know that the relative operator entropy can be interpreted as the perspective of $-\ln$, namely

$$P_{-\ln}(A,B) = -S(A|B).$$

Consider the convex function $f = -\ln$. We consider the information monotonicity for the Csisz ár f-divergence operator mapping and the generalized Csiszár f-divergence operator mapping.

By Theorem 4 we derive

$$\begin{aligned}-\sum_{i=1}^{n}S\left(h\left(\Phi(A_i)\right)|\Phi(B_i)\right) &= \mathbf{I}_{(-\ln)\Delta h}(\Phi(\mathbf{A}),\Phi(\mathbf{B}))\\ &\leq \Phi(\mathbf{I}_{(-\ln)\Delta h}(\mathbf{A},\mathbf{B}))\\ &= \Phi\Big(-\sum_{i=1}^{n}S\left(h\left(A_i\right)|B_i\right)\Big)\end{aligned}$$

and hence

$$\sum_{i=1}^{n}S\left(h\left(\Phi(A_i)\right)|\Phi(B_i)\right) \geq \sum_{i=1}^{n}\Phi\Big(S\left(h\left(A_i\right)|B_i\right)\Big). \tag{6.3}$$

On the other hand, by Corollary 8, we deduce

$$\begin{aligned}\Phi\Big(-\sum_{i=1}^{n}S\left(h\left(A_i\right)|B_i\right)\Big) &= \Phi(\mathbf{I}_{(-\ln)\Delta h}(\mathbf{A},\mathbf{B}))\\ &\leq \mathbf{I}_{(-\ln)\Delta h}(\Phi(\mathbf{A}),\Phi(\mathbf{B}))\\ &\quad(-\ln m)(S_{\Phi(h(\mathbf{A}))}-S_{h(\Phi(\mathbf{A}))})+\left(-\frac{1}{b}+\frac{1}{a}\right)S_{\Phi(\mathbf{B})}\\ &\quad+m\left(-\frac{1}{a}S_{h(\Phi(\mathbf{A}))}+\frac{1}{b}S_{\Phi(h(\mathbf{A}))}\right)\\ &= -\sum_{i=1}^{n}S\left(h\left(\Phi(A_i)\right)|\Phi(B_i)\right)\end{aligned}$$

$$(-\ln m)(S_{\Phi(h(\mathbf{A}))} - S_{h(\Phi(\mathbf{A}))}) + \left(\frac{1}{a} - \frac{1}{b}\right) S_{\Phi(\mathbf{B})} + m\left(\frac{1}{b} S_{\Phi(h(\mathbf{A}))} - \frac{1}{a} S_{h(\Phi(\mathbf{A}))}\right).$$

It follows that

$$\sum_{i=1}^{n} \Phi\Big(S\left(h\left(A_i\right)|B_i\right)\Big) \geq \sum_{i=1}^{n} S\left(h\left(\Phi(A_i)\right)|\Phi(B_i)\right) (-\ln m)(S_{\Phi(h(\mathbf{A}))} - S_{h(\Phi(\mathbf{A}))}) + \left(\frac{1}{a} - \frac{1}{b}\right) S_{\Phi(\mathbf{B})} + m\left(\frac{1}{b} S_{\Phi(h(\mathbf{A}))} - \frac{1}{a} S_{h(\Phi(\mathbf{A}))}\right). \tag{6.4}$$

Combining (6.3) and (6.4) we reach the information monotonicity and its complementary for the generalized Csiszár f-divergence operator mapping. In particular, when $h(t) = t$, we have

$$\sum_{i=1}^{n} S\left(\Phi(A_i)|\Phi(B_i)\right) \geq \sum_{i=1}^{n} \Phi\Big(S\left(A_i|B_i\right)\Big) \geq \sum_{i=1}^{n} S\left(\Phi(A_i)|\Phi(B_i)\right) + \left(\frac{1}{a} - \frac{1}{b}\right)(S_{\Phi(\mathbf{B})} - mS_{\Phi(\mathbf{A})}).$$

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