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Investigating an intervention, informed by variation theory, into the Grade 11 learners' interpretation of algebraic functions

Ramaisa, Mampotse Shirley

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Investigating an intervention, informed by variation theory, into the Grade 11 learners' interpretation of algebraic functions.

Mampotse Shirley Ramaisa

A research report submitted to the Faculty of Science, in partial fulfilment of the requirements for the degree of Master of Science University of the Witwatersrand, Johannesburg.

Johannesburg

18 September 2014

DECLARATION

I declare that this research report is my own work. It is submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

Signature

Date: 18 September 2014

ABSTRACT

This study investigates to what extent and how teaching informed by variation theory could improve the Grade 11 learners' interpretation of algebraic functions. The study adopted a learning study approach, where learner difficulties are elicited in a pre-test, and on the basis of the results of the pre-test, a lesson is planned, informed by variation theory, to make it possible for learners to discern what they found difficult.

In this study, a pre-test on functions was given to three groups of Grade 11 learners (85 learners in all) in the researcher's school that enabled the identification of aspects of functions learners found to be most problematic. The lesson was then taught successively to each group. A post-test at the end of each lesson, together with reflection on the lesson led to refinements for the next lesson to the next group.

The study describes the changes made to the lessons, and the results of the pre and post tests for each of the three groups. The results showed that while each group improved in the post test, the third group outperformed the others, confirming that an intervention, informed by variation theory, did improved learning. All learners were afforded the opportunity to discern the object of learning and their interpretation of functions improved, with the third group improving the most.

Keywords: object of learning, patterns of variation theory, interpretation of functions.

DEDICATION

In memory of my beloved husband, Mokheseng Crosby Ramaisa.

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- I would like to extend my regards and blessings to my mother Sis Martha, my sister Sibongile my daughter Mpho, son Lejakane, and not forgetting my grandson Neo, who continuously motivated, supported and encouraged me throughout the study. I am grateful for their good wishes towards me and desire for me to study.

- I am indebted to the mathematics teachers who willingly gave of their time to be part of this research by forming the panel of observers that observed my lessons, moderated the tests and reflected on the lessons. Without their input and dedication, this study would not have been possible. Their contribution to this study is priceless.

Table of contents

DECLARATION II

ABSTRACT III

DEDICATION IV

ACKNOWLEDGEMENTS V

CHAPTER ONE. 1

1. INTRODUCTION. 1

1.1 AIM. 1

1.2 PERSONAL DILEMMAS. 3

1.3 INTRODUCTION TO THE RESEARCH PROBLEM. 6

1.4 PROBLEM STATEMENT. 12

1.6 CRITICAL QUESTIONS. 13

1.6.1. WHAT ASPECTS OF FUNCTIONS DO LEARNERS FIND TO BE PROBLEMATIC? 13

1.6.2. WHICH OF THESE ASPECTS APPEAR MOST PROBLEMATIC? 13

1.6.3. HOW DOES A VERSION OF A LEARNING STUDY (WHICH IS FRAMED BY VARIATION THEORY) IMPROVE THE GRADE 11 LEARNERS' INTERPRETATION OF ALGEBRAIC FUNCTIONS AND TO WHAT EXTENT? 13

1.7 RATIONALE. 13

1.8 OUTLINE OF THE STUDY BY CHAPTERS. 15

CHAPTER TWO. 18

LITERATURE REVIEW. 18

2.1 INTRODUCTION. 18

2.2 LITERATURE ON FUNCTIONS. 19

2.2.1 WHY MULTIPLE REPRESENTATIONS OF FUNCTIONS ARE SO IMPORTANT? 20

2.3 THE SOUTH AFRICAN CURRICULUM ON FUNCTIONS. 23

2.3.1 FUNCTIONS FOR GRADE 11'S IN THE SOUTH AFRICAN CURRICULUM. 24

2.4 LEARNER MISCONCEPTIONS. 27

2.4.1 MISCONCEPTIONS RELATED TO THE DEFINITION OF A FUNCTION. 29

2.4.2 MISCONCEPTIONS RELATED TO THE CONSTANT FUNCTION. 33

2.4.4 MISCONCEPTIONS ON MULTIPLE REPRESENTATIONS OF FUNCTIONS. 37

2.4.5 MISCONCEPTIONS ON INTERPRETATION OF FUNCTIONS. 46

2.5 A TEACHING PERSPECTIVE. 49

2.5.1 THE DUAL NATURE OF FUNCTIONS. 49

2.6 RELEVANCE OF THE LITERATURE REVIEW TO THE STUDY. 55

2.7 SUMMARY 57

CHAPTER THREE 59

THEORETICAL FRAMEWORK. 59

3.1 INTRODUCTION. 59

3.2 THE FOCUS OF THE STUDY. 59

3.3 THEORETICAL FRAMEWORK 59

3.3.1 OBJECT OF LEARNING. 60
3.3.2 VARIANCE AND INVARIANCE. 61
3.3.3 FEATURES OF VARIATION THEORY. 62

3.4 EXAMPLE SPACE. 65

3.4.1 THE SPACE OF LEARNING. 65
3.4.2 CHOICE OF EXAMPLES. 66

3.5 THE PATTERNS OF VARIATION. 67

3.5.1 SEPARATION. 67
3.5.2 CONTRAST 68
3.5.3 GENERALIZATION. 70
3.5.4 FUSION. 71
3.5.5 SIMULTANEITY. 71

3.6 A LEARNING STUDY. 72

3.7 SUMMARY 73

CHAPTER FOUR 74

METHODOLOGY 74

4 INTRODUCTION. 74

4.1 RESEARCH PARADIGM 74

4.2 RESEARCH DESIGN. 75

4.3 SAMPLE AND SETTING. 77

4.4 DATA COLLECTION STRATEGIES. 78

4.4.1 THE DIAGNOSTIC/PRE-TEST 79
4.4.2 THE INTERVENTION LESSONS. 83
4.4.3 THE POST-TEST. 85

4.5 ETHICAL CONSIDERATIONS. 86

4.7 CHALLENGES DURING DATA COLLECTION. 89

4.8 VALIDITY. 89

4.9 RELIABILITY 90

4.10 LIMITATIONS OF THE STUDY. 91

4.11 SUMMARY 92

CHAPTER FIVE. 93

THE ANALYSIS AND DISCUSSION OF THE INTERVENTION LESSONS. 93

5.1 INTRODUCTION. 93

5.2 THE INTERVENTION LESSONS. 93

5.2.1 INTERVENTION LESSON 1 (GROUP A). 95
5.2.2 A TABLE OF THE INTERVENTION LESSON 1 FOR GROUP A. 98
5.2.3 REFLECTION AND DISCUSSION OF LESSON 1. 100

5.3 INTERVENTION LESSON 2 (GROUP B). 102

5.3.1 INTRODUCTION 102
5.3.2 A TABLE OF THE INTERVENTION LESSON 2 FOR GROUP B. 103
5.3.2 REFLECTION AND DISCUSSION OF LESSON 2. 104

5.4 LESSON 3 (GROUP C). 105

5.4.1 INTRODUCTION 105
5.4.2 A TABLE OF THE INTERVENTION LESSON 3 FOR GROUP C. 108
5.4.3 REFLECTIONS AND DISCUSSION OF LESSON 3. 109

5.5 DISCUSSION OF THE INTERVENTION LESSONS WITH COLLEAGUES. 110

5.6 SUMMARY 111

CHAPTER SIX 114

ANALYSIS AND INTERPRETATION OF TEST DATA. 114

6.1 INTRODUCTION. 114

6.2 MIXED METHODS DESIGN OF DATA ANALYSIS. 115

6.2.1 QUANTITATIVE ANALYSIS OF THE PRE-TEST. 115
6.2.2 RESULTS OF THE DIAGNOSTIC/PRE-TEST. 116

6.3 PERFORMANCE OF THE GROUPS IN EACH PRE-TEST QUESTION. 119

6.3.1 QUESTION 1.1 119

6.4 DISCUSSION OF THE PRE-TEST. 142

6.5 DISCUSSION OF THE POST-TEST. 145

6.5.1 THE POST-TEST. 145

QUESTION 1 145

6.5.2 DISCUSSION OF THEINTERVENTION LESSONS WITH RESPECT TO THE POST-TEST. 147

6.6 RESULTS OF THE POST-TEST. 149

6.7 COMPARISON OF THE PRE-TEST AND THE POST-TEST IN EACH GROUP. 152

6.7.1 COMPARISON OF THE PRE-TEST AND THE POST-TEST FOR LESSON 1 (GROUP A). 153
6.7.2 INTERPRETATION OF THE DATA AFTER LESSON 1 FOR 155
6.7.3 COMPARISON OF THE PRE-TEST AND THE POST-TEST FOR LESSON 2 (GROUP B) 156
6.7.4 INTERPRETATION OF THE DATA AFTER LESSON 2 (GROUP B) 158
6.7.5 COMPARISON OF THE PRE-TEST AND THE POST-TEST FOR LESSON 3 (GROUP C). 159
6.7.6 INTERPRETATION OF THE DATA AFTER LESSON 3 (GROUP C) 162

6.8 DISCUSSION AND INTERPRETATION OF DATA. 163

6.9 SUMMARY 164

CHAPTER SEVEN. 165

CONCLUSION 165

7.1 INTRODUCTION. 165

7.2 REFLECTIONS. 168

7.3 RECOMMENDATIONS. 172

REFERENCES. 173

APPENDIX A 183

INFORMATION SHEET AND CONSENT LETTERS: PARENTS/GUARDIAN 183

APPENDIX B 186

INFORMATION SHEET AND CONSENT LETTERS: LEARNERS 186

APPENDIX C 189

INFORMATION SHEET AND CONSENT LETTERS: EDUCATORS 189

APPENDIX D. 191

PRE-TEST ON FUNCTIONS. 191

APPENDIX E. 200

MEMORANDUM FOR PRE-TEST 200

APPENDIX F 204

POST-TEST ON FUNCTIONS. 204

APPENDIX G. 206

MEMORANDUM TO POST-TEST ON FUNCTIONS. 206

APPENDIX I. 208

INTERVENTION LESSONS. 208

LESSON 1 (GROUP A) 208

LESSON 2 (GROUP B) 211

LESSON 3 (GROUP C) 215

TABLE 1. 219

OBSERVATION SCHEDULE WITH THE SUMMARY OF THE KEY ASPECTS OF THE VIDEOTAPED LESSON. 219

TABLE 2. 220

OBSERVATION SCHEDULE PER LESSON OF 60 MINUTES 220

TABLE 3. 221

TABLE OF SPECIFICATION FOR THE PRE-TEST. 221

TABLE 4. 222

TABLE OF SPECIFICATION FOR THE POST TEST. 222

TABLE 5. 223

FIGURE 2.6: THE QUESTIONNAIRE USED IN THE EXPERIMENTAL AND THE CONTROL GROUPS 223

TABLE 6.....224

FIGURE 2 .7: RESULTS OF THE QUESTIONNAIRE FOR THE CONTROL GROUP AND THE EXPERIMENTAL GROUP.....224

LIST OF FIGURES:

Figure 1: Overall achievement rates in Mathematics from 2008-2011. 7
Figure 2.1: Connecting the definition of a function to the graphical representation. 20
Figure 2.1.2: Mathematical representations and scientific representations. 21
Figure 2.2(a): Visual representation of a physical structure. 21
Figure 2.2(b): Visual representation of a physical structure. 21
Figure 2.2: A quarter circle as a function or non-function. 30
Figure 2.3: Half a circle as a function or non-function. 30
Figures 2.4: Connecting the definition of a function to the graphical representation. 30
Figure 2.5: Representation of the function machine/box as a formula. 38
Figure 2.6: The function machine/ box as a table, a formula and a graph. 38
Figure 4.1: The Convergent parallel design paradigm. 75
Figure 4.2: Action research spiral as identified by Mills (2011) p. 19. 77
Figure 4.3: The 12 directions of equivalence of functional notation as depicted by Van Dyke & Craine (1997). 79
Figure 6.1: Questions 1 to 5 results for the diagnostic test for all groups. 115
Figure 6.1(a): Results of questions 1 to 5 of the pre-test for all groups. 117
Figure 6.2(a): Results of the learner responses for question 1.1 of the pre-test for all the groups. 120
Figure 6.2(b): Results of the pre-test in question 1.2 for all the groups. 123
Figure 6.2(c): Results of the pre-test question 2 for all the groups. 126
Figure 6.3(a): Results of the pre-test for all the groups in question 3. 129
Figure 6.4: Results of the pre-test in question 4 for all the groups. 132
Figure 6.5(a): Results of the pre-test in question 5 for all the groups. 134
Figure 6.5(b): Results of the pre-test in question 5.1 for all the groups. 136
Figure 6.5(c): Results of the pre-test in questions 5.2 to 5.7 for all the groups. 138
Figure 6.6(a): Post-test results in questions 1, 2 and 3 for all the groups. 146
Figure 6.6(b): Comparison of the averages of the results of the pre-test and post-test in all questions for all groups. 146
Figure 6.7(a): Comparison of the results of the pre-test and post-test in all questions for group A. 148
Figure 6.7(b): Comparison of the results of the pre-test and post-test in all the questions for group A. 149

Figure 6.8(a): Comparison of the results of the pre-test and post-test in all questions for group B. 151

Figure 6.8(b): Comparison of the results of the pre-test and post-test in all the questions for group B. 152

Figure 6.9(a): Comparison of the results of the pre-test and post-test in all the questions for group C. 155

Figure 6.9(b): Comparison of the results of the pre-test and post-test in all the questions for group C. 156

LIST OF TABLES:

Table 2.2: An example of the tabular representation of the values of a function. 42
Table 4.1: The sample used in the study. 77
Table 5.2: A detailed table of the first part of the changed intervention lesson for group B adapted from the PowerPoint presentation in appendix I. 103
Table 5.3: A detailed table of the intervention lessons adapted from the Microsoft PowerPoint presentation. 108
Table 6.1: Results of the pre-test in percentages for all groups in all five questions. 116
Table 6.1(a): Results of questions 1 to 5 of the pre-test for all groups. 117
Table 6.2(a): Results of the learner responses for question 1.1 of the pre-test for all the groups. 120
Table 6.2(b): Results of the pre-test in question 1.2 for all the groups. 123
Table 6.2(c): Results of the pre-test question 2 for all the groups. 126
Table 6.3(a): Results of the pre-test for all the groups in question 3. 129
Table 6.4: Results of the pre-test in question 4 for all the groups. 132
Table 6.5(a): Results of the pre-test in question 5 for all the groups. 134
Table 6.5(b): Results of the pre-test in question 5.1 for all the groups. 136
Table 6.5(c): Results of the pre-test in questions 5.2 to 5.7 for all the groups. 138
Table 6.6(a): Post-test results in questions 1, 2 and 3 for all the groups. 146
Table 6.6(b): Comparison of the averages of the results of the pre-test and post-test in all questions for all groups. 146
Table 6.7(a): Comparison of the results of the pre-test and post-test in all questions for group A. 148
Table 6.7(b): Comparison of the results of the pre-test and post-test in all the questions for group A. 149
Table 6.8(a): Comparison of the results of the pre-test and post-test in all questions for group B. 151
Table 6.8(b): Comparison of the results of the pre-test and post-test in all the questions for group B. 152
Table 6.9(a): Comparison of the results of the pre-test and post-test in all the questions for group C. 155
Table 6.9(b): Comparison of the results of the pre-test and post-test in all the questions for group C. 156

ABBREVIATIONS:

AS	Assessment Standards
CK	Curricula Knowledge
DBE	Department of Basic Education
DoE	Department of Education
GDBE	Gauteng Department of Basic Education
GET	General Education and Training
HESA	Higher Education South Africa
FET	Further Education and Training
ICU	Intensive Care Unit
LO	Learning Outcome
NCS	National Curriculum Statement
NSC	National Senior Certificate
PCK	Pedagogic Content Knowledge
SAQA	South Africans Qualifications Authority
SAG	Assessment Guideline
SBA	School Based Assessment
SMK	Subject Matter Knowledge
WMC	Wits Maths Connect

CHAPTER ONE.

1. INTRODUCTION.

1.1 AIM.

> "A student's failure or lack of understanding can be understood in the light of un-discerned aspects. So the discernment of critical aspects is essential for learning. From this theoretical point of departure, in a learning study, the teachers try to find out what the critical aspects are and how they should be brought out in the learning situation in a way that makes discernment possible."
>
> (Runneson, Kullberg & Maunnula, 2006, p. 266).

The above quotation alludes to the idea that because teachers engage in the process of teaching, they should provide opportunities for learners to discern the critical aspects of the concept in a lesson. The failure of learners to discern the critical aspects of the concept in a lesson can result in the learners failing to understand the concept or the learner's lack of understanding of the critical aspects of the concept. The issues of discernment, and affording learners the opportunity to discern the critical aspects of a concept, are explained in Chapters 4 and 5.

The notions of discernment and critical aspects of a concept emerge from variation theory developed in Sweden which is extensively applied by researchers like Marton, Runesson & Tsui (2004) in their research on teaching and learning of concepts. These aspects have been put to use in various Learning Studies[1], which is a focus of this research. These aspects of variation theory are discussed in detail in Chapter 3. The principles that are inherent in variation theory and the challenges of the researcher's own teaching through a version of a Learning Study are very intriguing. I was intrigued by the principles inherent in variation theory and decided to look at my own teaching through a version of a Learning Study.

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[1] A learning study involves a group of teachers coming together to plan and teach a topic that has been identified as problematic to the learners.

The ideas as stated by the quote above clearly emphasise the fact that teaching and learning in general is not a simple and straight forward activity. The ability to teach any subject successfully requires more than the knowledge of the particular subject. Teaching and learning involve the knowledge of how to apply the knowledge of that subject so that the learners can make sense of it, understand it and eventually be able to apply it. What the teachers know and do to make the knowledge of the subject accessible for their learners is of the utmost importance. This study is concerned with mathematics as a subject, and in particular, the teaching of the concept of "function".

"The desire of teachers to achieve meaningful teaching and learning that results in learners' understanding of concepts in any subject, especially in the teaching of mathematical concepts, does not mean that procedural knowledge should be ignored" (Even, 1990, p. 526). Resnick and Ford (1984, p. 52) assert that the "memorisation of certain facts and procedures is important not so much as an end to itself, but as a way to extend the capacity of the working memory".

The memorisation of certain procedures is important in the development of a deep understanding of the concept that is to be learned, particularly in this study, the concept to be learned is the notion of parabolic functions. Parabolic or quadratic functions require the application of certain algebraic procedures to the equation of a parabolic function. These procedures include the factorisation of the equation of the quadratic function to generate the x-intercepts of the parabola. Other procedures involve placing the x-values of the parabolic equation to zero to generate the y-intercept of the parabola as well as completing the square on the quadratic function to generate the turning point and the axis of symmetry of the parabola.

A common issue for mathematics education everywhere in the world is to manage both procedural and conceptual understanding of any concept. Learners need to know what a concept is, why it is that, as well as how to do and carry out procedures in relation to that particular concept. Managing all of these in a classroom can be quite difficult. For all educators that are trying to reform mathematics education, including those in South Africa, this is the key challenge.

Procedural understanding has been emphasised and has been the focus of mathematical teaching and it can be said that relaying procedural understanding is being done successfully. The key issue in the learners' acquisition of mathematical proficiency is by mastering

procedural understanding and conceptual understanding of a concept (Kilpatrick, Swafford & Findell, 2001). Kilpatrick et al. (2001) identified ways in which both procedural and conceptual understanding of any concept could be mastered in order for the learners to become mathematically proficient. One of the ways in which this could be achieved is through strands of mathematical proficiency.

Strands of mathematical proficiency involve learners' ability to "apply knowledge to solve problems, learn new concepts and skills, adapt the knowledge they acquire to different situations, apply mathematical reasoning to different problems and to view mathematics as a useful tool that must constantly be sharpened" (Kilpatrick, et al., 2001, p. 144). These strands are: conceptual understanding; procedural proficiency; strategic competence; adaptive reasoning and productive disposition. As these strands are intertwined, they do not have to be acquired by the learners in any specific order. The importance of the strands of mathematical proficiency for this study are that learners' ability to flexibly convert from one functional representation to the next is one of the main factors that contribute towards learners' ability to interpret functions in all their various representations.

This study explores one possible way in which learners' conceptual understanding could be enhanced. Conceptual understanding allows learners to adapt their knowledge of concepts in order to solve problems in any given situation. Hopefully this study will help in contributing to the approach teachers could apply in the teaching of functions. In doing so, learners could be provided with better opportunities to learn what educators intend for them to learn, which, in this study, is the notion of functions and the parabolic function in particular.

1.2 PERSONAL DILEMMAS.

As the researcher of this study, there is a personal aspect to my role in this study. I am an experienced teacher who has been teaching Mathematics at secondary school level for twenty eight years (28 years). I teach in a school where learner performance is known to be poor. Due to the persistent poor performance of learners, the school has been classified as an under-performing school and has been put in the 'Intensive Care Unit[2]' (ICU) by the Department of

[2] A school that is persistently exhibiting poor performance of learners is classified as an under-performing school and therefore put in ICU because it is "sick".

Basic Education (DBE). A school that has been put under the ICU is one that is constantly visited and monitored by the DBE facilitators to ensure that teaching and learning is constantly taking place and that the teachers adhere to the requirements of the work schedules as designed by the DBE. This monitoring of teaching and learning will continue until the performance of the learners improves to the satisfaction of the DBE.

I also consider myself as a dedicated and hardworking teacher. The learners in my school come from an underprivileged home environment that is based in a community that is riddled with crime and substance abuse by both the learners and some of the members of the community. Some learners come from child-headed households. These factors make conditions in the school where I teach difficult and trying. I began my study in 2012 by administering a diagnostic test to my own class of Grade 11 learners. This test was administered after the section on functions was taught to my Grade 11 learners in 2012 and they performed poorly. The details of the diagnostic test appear in Appendix D.

The aspects of functions that the learners were tested on were based on the sections that I refer to as algebraic functions. By algebraic functions I refer to those functions taught in school that can be represented by an algebraic equation and specifically the line, parabola, hyperbola and the exponential functions.

The Umalusi report of the Grade 12 performances on the algebraic functions from the year 2008 up until the year 2011 included the section on algebraic functions as one of the problematic areas over the years. This report consists of the problematic areas that the Grade 12 learners experienced (from 2008 – 2011) in the National Senior Certificate (NSC) examinations. The contents of the report are elaborated on later in this chapter and the problematic areas in the topic of functions as identified in the report, form the focus of my study.

To my surprise, my learners did much worse than I had expected considering the fact that I had taught them myself. This was very difficult for me to comprehend and accept as these learners were taught by me. I have to say that I was shocked by what I saw and as I could not blame anyone but myself, I was encouraged to improve my teaching more than ever.

This study is based on my experience as a teacher, specifically about focussing on the object of learning (what the lesson intends the learners to be aware of and ultimately discern). In this study, the object of learning is focussed on the interpretation of the key features of a

parabolic function in a lesson. This study focusses on the parabolic function as this happened to be the area where my learners appeared to have the most problems with. I was interested in improving my practice as I could not blame anybody else for the poor performance of my learners after the shocking results of the diagnostic test on functions that was given to my learners.

I structured my teaching, thinking and focussing on a version of a learning study that is mainly used for variation theory. Variation theory concentrates on identifying the object of learning and then with the help of variance and invariance, separation, generalisation, fusion, simultaneity and contrast as the patterns of variation, the learners are provided with opportunities to discern the object of learning. These patterns of variation are elaborated on in Chapter 3, which deals with the theoretical framework that informs this study.

This study is concerned with the aspects of the parabolic function that the learners should discern and the study aims to discover whether this approach will make a difference in the way that my learners performed on the post-test that was written immediately after each intervention lesson, which were planned using the principles of variation theory. This forced me to go back and think more about what I did differently in the lesson(s) that brought about a difference in the learners' performance compared to the lesson(s) prior to the test I my learners underwent in which they performed so poorly.

I did not suddenly change my personality nor did I suddenly become a teacher who uses group work as an approach to teaching. This study is about what it is that I changed in the presentation of my lesson(s), how I changed it, why I changed it and what affect it had on my teaching and my learners' performance. Looking at this study I noticed that the changed approach and structure of my lesson did make a difference but it wasn't immediately very clear to me what exactly the difference was.

I discussed the results of the lessons with my supervisor who pushed me to identify the changes in the lesson(s). I realised that one thing that did change was how I structured my lesson(s) and not so much how I delivered the lesson(s) that brought about the change in the learners' performance. This study is therefore an interesting illustration of how an educator can improve their teaching and learners' learning without uprooting everything that they know.

I taught in the way I normally do and I used the computer as a resource to present the lesson on presentation slides using Microsoft PowerPoint 2010 to prepare the presentation slides. This form of presenting the lesson saved a lot of time brought about by writing on the chalkboard and simultaneously talking to the learners. What really helped in the success of the lessons was that the lessons were well-prepared beforehand with all the critical aspects clearly stated and neatly placed on the slides in presentation form.

The key aspects of the lessons were presented on slides that were introduced one at a time and explained as one sentence at a time giving the learners enough time to concentrate on one key aspect of the lesson. In so doing the learners were afforded the opportunity to discern the variation from one form of the key aspect of the parabolic function to the next one. The lessons were in line with the construct of the theory of variation that is explained in full in Chapter 5 where the intervention lessons are discussed.

As was previously mentioned, the school is an underprivileged school that was recently presented with laptops for being part of a teaching and learning improvement plan initiated by the Wits Maths Connect (WMC) project situated at the Wits University's Department of Mathematics Education, which is aimed at improving the achievement of learners from underprivileged schools in mathematics. The WMC afforded the school a set of tools that enable the practice of teaching and learning of mathematics while using different resources.

In this report, the process that unfolded and how I was provided an opportunity to think carefully and act deliberately in improving my teaching is discussed. I have briefly attempted to introduce the areas that this study focussed on. In the following section, the factors that led to the choice of my research topic and the critical questions that underpin this study are explained explicitly.

1.3 INTRODUCTION TO THE RESEARCH PROBLEM.

The 2012 National Senior Certificate (NSC) report (Umalusi, 2012, p. 98) shows that Mathematics attainment throughout South Africa from the year 2008 to the year 2011 was

poor as is graphically depicted in Figure 1.

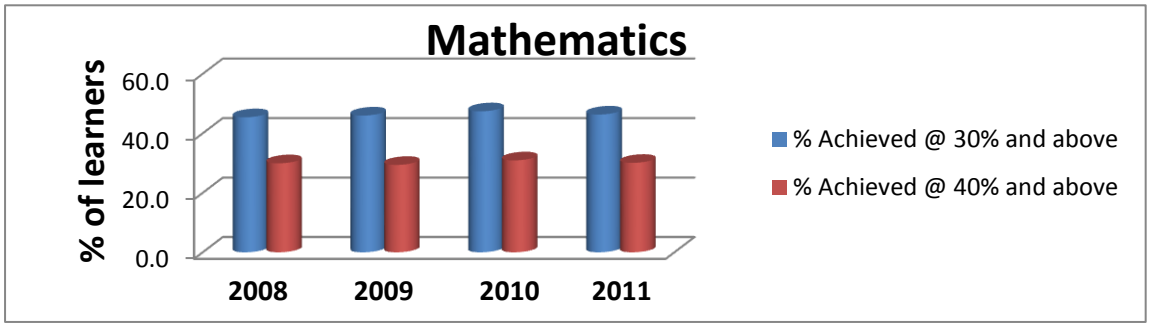


Figure 1: Overall achievement rates in Mathematics from 2008-2011.

The focus of this study is on the section of mathematics concerning 'algebraic' functions and graphs. Algebraic functions constitute about 33% of the overall Mathematics mark for both papers one and two of the Mathematics examination and is therefore an important section of the Mathematics curriculum for learners to be able to obtain a pass percentage in the final matriculation papers. Questions on algebraic functions and their graphs were generally poorly answered by learners in the NSC Grade 12 examination papers in the past years as reported at the beginning of the year road shows.

The road shows are sessions organised by the Gauteng Department of Basic Education (GDBE) at the beginning of each year to inform educators of the learners' performance in the previous year's Grade 12 NSC examinations in selected subjects. The "comprehensive" report on the performance of learners in each question relating to the National Senior Certificate (NSC) Mathematics examination are discussed in relation to the analysis of the results of the NSC and jointly reported by Umalusi and the Higher Education South Africa (HESA) and is published by the Department of Basic Education (DBE).

The Umalusi report is presented for use by teachers, subject advisors, circuit managers and district managers in their preparation of the learners for the current years' NSC examinations. The report contains the overall quantitative analysis of the achievement rates in the subject for the previous years that the curriculum on the subject was written to show an overall trend of learners' performance in the particular subject, the general qualitative overview of learner performance and the qualitative analysis of learner performance in individual questions in all the papers written in that subject.

For each question, the common errors and misconceptions of learners are analysed and suggestions for improvement on those errors and misconceptions are highlighted. As this

study was conducted in the year 2012, for functions and graphs in Mathematics, in the previous years' papers prior to the year 2012, the report found that for the end of 2011 final Mathematics examinations, learners could not deal with inequalities. It was reported that learners could not express the inequality relating to the intervals where the values of x constituted the interval depicting where the given function was positive. Learners also had difficulties when they were required to find the coordinates of the intercepts with both the axes as they wrote the answers to both questions as a unit instead of as separate sections.

The case under discussion was where the learners had to consider the function

$f(x) = \frac{-6}{x-3} - 1$. Learners had to calculate the coordinates of both the x-intercepts and y-intercepts of this hyperbolic function. The learners responded to this question by giving the coordinates of the x-intercepts as one point consisting of the asymptotes of the hyperbola. The response to the coordinates of the x-intercepts was given as $(3; -1)$ rather than specifically answering the individual questions as the coordinates of the one x- intercept to be $(-3; 0)$ where $x = -3$ and $y = 0$ and the coordinates of the y-intercept as $(0; 1)$ representing the coordinates of the other y-intercept where $x = 0$ and $y = 1$, which was required by the questions (Umalusi, 2012). This difficulty is a result of the learners' inability to use the functional notation correctly, their interpretation of the different functional notations as well as the meaning associated with each form of functional notation.

The difficulty with the inequalities arose where learners were required to find the interval where the function $(f(x) > 0)$, meaning the interval where the function was above the x-axis. The answer to this question was that this happened in the interval when

$-3 < x < 3$. Learners who attempted to answer this question failed to notice that x should also be less than 3 (Umalusi, 2012). This error was evident in the pre-test that I gave to my learners in a question that required learners to determine the interval at which the parabolic function was decreasing and for which values of x two functions $f(x) \geq g(x)$, for $x \geq 0$. The learners could not globally interpret the trends where the given functions satisfied the required inequalities.

Questions that required the average gradient between two points on the curve, learners made errors where arbitrary y coordinates were used for the points where x was equal to -2 and 0 into the equation $\frac{f(x_2)-f(x_1)}{x_2-x_1}$. Learners needed to calculate the values of $f(-2)$ and $f(0)$ by

substituting the x-values into the given equation while the $y-$intercept was known instead of substituting $y = -2$ and $y = 0$ and solving for x for both $y-$values (Umalusi, 2012).

This finding is not surprising, based on my experience of dealing with functions, when learners were asked to identify the types of functions when given the functions: $f(x) = x^2$; $f(x) = \frac{2}{x}$; $f(x) = 2^x$; $f(x) = x + 2$. Based on their answers, it can be said that learners are sometimes able to correctly classify the functions as parabola; hyperbola; exponential and straight line respectively but when required to interpret the meaning of $f(2) = \cdots$ in some of the equations above, they could not explain that this was representative of the point on the particular graph where the coordinate of x was 2. Learners are expected to be able to identify and name these functions by the time they reach the Further Education and Training band (FET), which consists of Grades 10, 11 and 12.

According to the Umalusi report, some learners found a question requiring them to draw a sketch graph of the function $y = ax^2 + bx + c$, where $a < 0, b < 0, c < 0$ difficult to answer because the equation consisted of parameters only. This also happens when learners are required to find the solution to the equation $ax^2 + bx + c = 0$ which has two equal roots as the solution. This question expects learners to be able to interpret their knowledge of the nature of the roots and to make use of the discriminant $\Delta = b^2 - 4ac = 0$.

The aims of the questions are to find out whether the learners have the global understanding of the equation of the quadratic function. This presented a problem to the learners as the parameters of the function and the equation were stated in general form and some learners were not used to working with general parameters and therefore found the question unfamiliar. Most learners found the questions unfamiliar and therefore omitted them. Those learners who did attempt the questions managed to score a few marks.

Learners could not link the general function and its equation form to the importance of the nature of the roots, given by the discriminant $\Delta = b^2 - 4ac$, as an approach to such questions. The discriminant stipulates that the roots of the quadratic function/equation will be: real and equal if $\Delta = 0$; real and unequal if $\Delta > 0$ and a perfect square; real, irrational and unequal if $\Delta > 0$ and not a perfect square and non-real if $\Delta < 0$.

Suggestions to improve on these problems were that teachers should be aware that in the Subject Assessment Guideline (SAG) document it is stipulated that there existed a need for learners to know the effect of the various parameters in the general equation of the parabola.

The SAG suggests that learners should first know the basic curve, called the "parent" curve, and then be given ample opportunity to investigate the effects of changing the values of the various parameters as part of an investigative School Based Assessment (SBA) in Grade 11 (Umalusi, 2012, p.101).

The report also emphasises the need for teachers to ensure that the learners are familiar with the standard formulae and defining equations of the various families of functions. These should be taught in such a way that the learners can have a visual picture of the graph from the standard formula of the function and that learners should be exposed to more situations where they are required to "convert flexibly between the various multiple representations of functions (words, symbols, graphs and tables)" (Umalusi, 2012, p. 102).

The report further states that the learners need to pay attention to the notation of the function and that many learners struggled with the derivation of the equation of a cubic function when given a cubic graph and the turning points $T(2;-9)$ and $S(5;18)$. The learners were required to show that $a=21, b=-60$ and $c=4$ for the cubic function: $f(x)=-2x^3+ax^2+bx+c$. Learners did not know how to use the given information in order to arrive at the given values, because they did not know which form(s) of the cubic function they were expected to use (Umalusi, 2012, p. 103).

This problem resurfaced with my learners in the pre-test when they were required to show that the values of a parabola: $f(x)=ax^2+bx+c$ were $a=1$, $b=3$ and $c=2$ for a given sketch of a parabolic function with turning point $P(-\frac{3}{4};-\frac{1}{4})$ and another point $S(2;12)$ on the parabola. My learners did not know which form of the parabolic equation to use in order to show these values, but instead used the given values as if they were originally given without showing how the values were acquired. This is an indication that the learners do not know which form of the equation of an algebraic function should be used given different points of an algebraic graph.

Another problem that was evident for most learners was that when learners were given a function $y=-x^2+3x+2$ and required to complete the square on the right hand side of the function in order to find the turning points of the parabola, they solved the right hand side as they would solve a quadratic equation where the y- value was equal to zero. The learners divided each term on the right hand side by negative one, thereby ignoring the fact that this was not a quadratic equation equal to zero. When they multiply the right hand side by a

negative number they should get the answer of $y = -(x^2 - 3x - 2)$ instead of the answer $y = x^2 - 3x - 2$ as it would be if this was an equation where the value of $y = 0$. This is an indication that the learners do not know the difference between an equation and a function (Umalusi, 2012, p. 104).

Suggestions to improve the way learners understand these variations were that teachers needed to introduce learners to graphical interpretation questions as per the Examination Guideline Document. The importance of understanding the difference between an equation and a function should also be stressed from the General Education and Training (GET) phase (Grades 8 and 9).

This study was conducted before the end of the 2012 National Senior Certificate (NSC) examinations. The 2012 Umalusi report on functions brings out similar problems. According to the 2012 report, these problems persist, which is an indication that the interpretation of functions still remains an area of difficulty beyond the year 2012 which is the year this study was conducted.

In my experience as a Mathematics teacher, I also identified that learners seem to have difficulties in recognising relationships between variables in terms of numerical, graphical, verbal and symbolic representations and to flexibly convert between these representations (tables, graphs, words and formulae). The pre-test that I gave to my learners also confirmed these errors.

The aspects of functions that were tested in the pre-test consisted of:

- Finding the equations of parabolic sketches given three points on the parabola.
- Finding the equations of parabolic sketches given the x-intercepts and another point on the parabola.
- Finding the equations of parabolic sketches given the turning point and another point on the parabola.
- Sketching the graph of the given exponential equation of the graph and explaining what happened to the graph when the horizontal transformation of the exponential function is given.
- Matching different algebraic graphs from one column with their appropriate representations in the next column.

- Interpreting a contextual problem on the exponential function and determining the intervals in which the linear, parabolic and exponential functions were greater or less than the other.
- Finding the distance between two points.

The most pertinent problem that informs this study is the inability of learners to interpret the algebraic functions, specifically the quadratic function from different representations. The functional representations that are the impetus for this study consist of the graphical representation of the parabola and its related equation form, and the interpretation of the different forms of the equation of the parabolic function with regards to the graphical representation.

The pre-test administered to the Grade 11 learners at the beginning of this study revealed that the most problematic area relating to the function concept was that:

(i) Learners tended to substitute the given values of the points on the sketch of the parabola in the wrong general form of the quadratic function.

(ii) When learners were required to find the equation of a parabolic function when they were given the sketch of a parabolic function showing the coordinates of the turning point of the parabola and another point on the parabola, learners tended to substitute the given values of the turning points in the wrong form of the quadratic function.

(iii) This occurrence was also evident when they were required to find the equation of a sketch of a parabolic function where the coordinates of the x- intercepts and another point on the sketch of a parabolic function were given and when the coordinates of any three points on the sketch of a parabolic function were given.

The following section elaborates on the problem statement that underpins this study and the research questions that this study pursues and intends to answer.

1.4 PROBLEM STATEMENT.

With regards to the preamble on the areas of the concern described by the above report as well as my experience in the teaching of mathematics at FET level, I then decided on conducting a study on algebraic functions. The Mathematical problem that this study focusses on is the interpretation of algebraic functions, specifically quadratic functions, by learners

when these functions are presented to them in different representations (graphical and equation form). The study also focussed on the features of algebraic (quadratic) functions, the manipulation of quadratic functions using algebraic processes involving factorisation (resulting in the coordinates of the x- intercepts of the parabola) and completing a square of the quadratic function (resulting in the coordinates of the turning point).

The identification of the different characteristics of the quadratic functions when given the parabola in graphical form and interpretation of the different forms of the parabolic equation and the respective features presented by each form of the parabolic equation are investigated. These aspects of the quadratic function were identified to be problematic for my learners after the learners' dismal performance in the pre-test given to them to find out the areas of functions that they had problems with.

The following section articulates the research questions and the critical questions that inform this study.

1.5 RESEARCH FOCUS.

This study investigates an intervention, informed by variation theory, of Grade 11 learners' interpretation of algebraic functions.

1.6 CRITICAL QUESTIONS.

1.6.1. What aspects of functions do learners find to be problematic?

1.6.2. Which of these aspects appear most problematic?

1.6.3. How does a version of a learning study (which is framed by variation theory) improve the Grade 11 learners' interpretation of algebraic functions and to what extent?

1.7 RATIONALE.

This study was selected because of the discoveries made from the diagnostic test of my Grade 11 learners. From the results of this test on functions, it was discovered that learners did not

know how to find the equation of a parabolic function when they were given the graphical representation of a parabola. The learners also did not know which form of the quadratic function to use in order to find the equation of a parabolic function and also could not interpret the critical aspects and key features related to each form of a quadratic equation. This resulted in the learners' inability to answer most of the questions on the parabolic function or any graphical representation of an algebraic function.

The study is important within the field of Mathematics Education as it might contribute to enhancing the learners' knowledge and interpretation of graphs in Mathematics, Physical Science, Natural Science, Mathematical Literacy and life in general. It could contribute to enabling other educators to reinvestigate their approach to their teaching of algebraic functions. Additionally, the study could inform teachers about the potential of action research informed by variation theory in the teaching of the concept of functions and other Mathematical concepts.

The suggested distribution of marks for Grade 12 in the NSC question papers for Learning Outcome 2 (LO2): Functions, graphs and modelling is $\pm$ 33% of the paper. This has a large influence on the mark in the curriculum and if learners do not attempt this section of the paper, their chances of achieving a promotional percentage in the paper are reduced.

This study also carries some personal weight for the researcher as an educator as it helped to identify the learners' conceptions of a function and the interpretation of functions, including the factors that affect the smooth transition of learner's conceptions of functions from the GET to the FET phase. It helped the researcher to improve the approach to the teaching of algebraic functions, their interpretation and other concepts in Mathematics as well.

The study is also important in South Africa currently, because it could help to improve the teaching of the concept of algebraic functions in the GET and the FET phase. It could contribute to the enrichment of the knowledge base on the topic of functions for education/curriculum planners/ teachers and learners, because a strong understanding of the concept of algebraic function is essential for any student hoping to understand calculus, which is a critical course for the development of future scientists, engineers, and mathematicians.

1.8 OUTLINE OF THE STUDY BY CHAPTERS.

In Chapter 1, the justification and relevance of the study, the aim of the study, the topic of the study and the critical questions that the study aims at answering is discussed.

Chapter 2 describes the literature base that informs the study. The focus is on functions and the research undertaken on functions by other researchers on how and why learners find it difficult to understand, interpret and apply the notion of functions in their daily lives.

Chapter 3 presents at the theoretical framework that informs the study. Research on variation theory is also explored, with a focus on the use of variation theory in learning studies in Mathematics. As the study is informed by variation theory, the patterns of variation that could be employed during the lessons on functions and the dimensions of the function that could be varied are fully discussed.

Chapter 4 looks at the research methodologies applied and all the data collection techniques employed to obtain answers to the critical questions: "what aspects of functions do learners find problematic?" and "which of these aspects appear most problematic"? Challenges during data collection are highlighted and the question of rigor which includes reliability and validity of the data, generalisation and ethical considerations taken are also discussed.

After the identification of the most problematic aspect of functions from the results of the diagnostic test, the setting of the research project is explained in greater detail. This explanation gives insight into how and why the data collection instrument is used and what transpired during the intervention to explore the question "to what extend does a version of a learning study framed by variation theory improve the Grade 11 learners' interpretation of algebraic functions?"

In Chapter 5, the intervention lessons are fully described and it is explained how these lessons were aligned with variation theory. What transpired in the lessons is noted and the reflections of each lesson are also discussed. The transformations that occurred to the lessons after the each consultation with the observers taking part in the study are also described. A summary of the intervention lessons is presented according to the processes followed and presented by Tall & Bakar (1991) without the use of the function machine.

The observation schedules used by the three observers that were involved in the study were also used in the discussion of the intervention lessons and the changes that were to be implemented. The post-test was also used to ascertain where the changes to the intervention lessons could be applied. The limitation of the action research in the form of a version of a learning study is also discussed.

Chapter 6 presents the analysis and explanations of the use of the mixed methods research analysis to analyse the data from the diagnostic/pre- test and the post-test. The features of the diagnostic test are discussed and analysed and the results of the test are analysed. Each question of the diagnostic test and the post-test is analysed and interpreted with reference to the researchers who had done some research on the type of questions being analysed. The sample of learners is divided into three groups and the results of the learners are analysed per group, per question using descriptive statistics.

The three questions in the post-test consists of questions 1.1 and 1.2 and question 5.1 of the pre-test. The first two questions are exactly the same as those in the pre-test; only the numbering had changed from 1.1 and 1.2 to questions 1 and 2 respectively. The third question in the post-test was a modified version of question 5.1 in the pre-test. This question concentrated on testing the ability of learners to flexibly convert between functional representations. The results of the three groups of learners' performance in pre-test and the post-test are compared to ascertain whether the intervention lessons had any impact on the learners' performance.

In Chapter 6, the results of both the pre-test and the post-test are discussed with reference to the literature review. Reference to the analytical and theoretical framework of the study is further elaborated on with regards to the results. The different approaches used in the intervention lessons are compared with the results of the learners' results in both tests. The incorporation of the theoretical framework from the variation theory is fully explained with respect to the learners' results in both tests.

The results of both tests are used to propose answers to the research question, the focus of the study and the critical questions. The intervention lessons, the literature review, the research question and the critical questions of the study are all combined to give results for the main focus of the study. The recommendations as well as the limitations of the study are discussed in relation to the study as a whole.

Chapter 7 concludes the study by reflecting on the findings from the intervention lessons and the comparison of the pre-test and the post-test results. The implications that the findings of this study have for teachers in their teaching and curriculum planners in general are discussed. The difficulties that were experienced by the researcher when undertaking the study and how these difficulties were overcome are also addressed. Recommendations for further research are articulated.

CHAPTER TWO.

LITERATURE REVIEW.

2.1 INTRODUCTION.

This study is concerned with the Grade 11 learners' interpretation of algebraic functions, specifically the version of a learning study (described further below) conducted on a particular sample of the Grade 11 learners that was used in this study. In this chapter, discussions prevalent in the research literature in as far as the notion of functions is concerned are reviewed. In particular, the aspects of algebraic functions that were found to be problematic to the learners from the results of the diagnostic/pre-test and the researchers are discussed.

The literature review is structured along the lines of what the literature informs us about why functions are important and how functions are structured in the South African curriculum. Learners' difficulties in working with functions, as seen by other researchers, are discussed. The learners' misconceptions and what contributes to these misconceptions regarding functions are further explored. Teaching perspectives, what it is that teachers should think about when teaching functions, is explored.

The exploration of literature commences with the definition of a function and will proceed to how these aspects relate to the study. This study is about 'the interpretation of functions' beginning with 'what aspects of functions learners find problematic' and then moving on to 'which of these aspects appear to be most problematic to the learners. All of these aspects are reflected in the results of the diagnostic/pre-test.

The importance of functions and why functions should be taught in school, which aspects of functions are emphasised in the curriculum and misconceptions related to the function concept are discussed. The misconceptions include misconceptions arising as a result of the definition and functional notation. The literature related to multiple representations of functions, in particular the equation and the graphical representation, are discussed.

Literature about learner misconceptions of the constant function brought about by the misinterpretation of the definition of a function is considered. Literature pertaining to the misconceptions of the multiple representations of functions brought about by the prototypes used when functions are first introduced to learners is highlighted. Literature on the misconceptions that learners acquire with regards to the functional notation leading to the problems that learners encounter in their dealings with the function concept also is referred to.

Literature on the nature of functions that leads to learners' inability to interpret functions when they are presented in different ways and the interpretation of functions and the research pertaining to these aspects of functions is explored. The features of functions that emerge as critical for learners to discern the object of learning are discussed. The function that is appropriate to this study is specifically the parabolic function. Additionally, the features of functions that make functions an important topic to be studied are discussed as the focus of this study.

The features of the functions that are referred to are the intercepts with the axes, the shape of the function and the turning point of the function. These features are discussed specifically with regards to the parabolic function in this study. These features are explored in relation to the equation of the parabolic function and the graphical representation of the parabolic function and how the different forms of the equation of the quadratic function can be interpreted. Studies conducted by Mathematics Education researchers on functions with regard to the aforementioned topics are dealt with and their findings are identified and explored.

2.2 LITERATURE ON FUNCTIONS.

A lot of research has been done on the topic of functions and the findings of the research are beneficial to teaching and research in this particular field. Nonetheless, the difficulties associated with the teaching and learning of functions persists as is evident in the Umalusi report that is discussed in Chapter 1. The difficulties are also clear based on the diagnostic/pre-test that the learners wrote at the beginning of this study.

Many reasons have been given for why learners may find it difficult to develop a deep understanding of functions. Some of the reasons given by the researchers are in related to the definition of a function; the multiple representations of functions; the nature of functions; the interpretation of functions; functional notation and the way in which functions are taught.

The aim of this study is however about teaching and students learning of algebraic functions. Therefore, the teaching and learning of algebraic functions should be done in such a way that it enables learners to interpret the parabolic function when it is represented in different ways (specifically from the equation form to the graphical form and from the graphical form to the equation form). The teaching and student learning of functions should afford learners the ability to interpret the different forms of the parabolic equations.

The following section elaborates on the importance of multiple representations of functions and what researchers have found on the importance of functions in school learning and in everyday life.

2.2.1 WHY MULTIPLE REPRESENTATIONS OF FUNCTIONS ARE SO IMPORTANT?

The rationale behind the subject of this study alludes to the fact that a strong understanding of the concept of algebraic function is essential for any student who hopes to pursue Mathematics at tertiary level. The literature also supports the fact that the understanding of functions is important, as expressed in the following quotation: "A strong understanding of the function concept is central to undergraduate mathematics, is a foundation to modern mathematics and essential in related areas of the Sciences. In other words, it is a gateway for any student hoping to understand calculus, a critical course for the development of future scientists, engineers and mathematicians" (Oerhrtman, Carlson & Thompson, 2008, p.151).

According to Leinhardt, Zaslavsky & Stein (1990, p.2) "there are aspects of learning and teaching specific content that are unique and more conspicuous to the particular topic than to the general field of teaching and learning as a whole". The authors focus on the topics of functions and graphs as they discovered that these topics were not dealt with until the

elementary grades or later. Functions and graphs focus on the use of a variety of symbolic systems which can be expanded to other learning areas as well.

Graphing is considered to be important in Mathematics learning as it "enables learners to acquire the transformational notion that is important in regrouping and expanding the number system from counting numbers to rational numbers, adding and subtracting numbers, multiplying and dividing these numbers. These concepts are transferable to other parts of Mathematics as well as other learning areas" (Leinhardt, et al., 1990, p. 5).

Furthermore, Oehrtman, Carlson, & Thompson (1998) state that functions and their graphs should not be construed as isolated concepts as they complement and illuminate each other. The authors surmise that the Mathematical representations and the scientific representations of concepts often follow the processes to flexibly move from one representation to another representation in the fields of Mathematics and Science. This idea is represented in Figure 2.2 below:

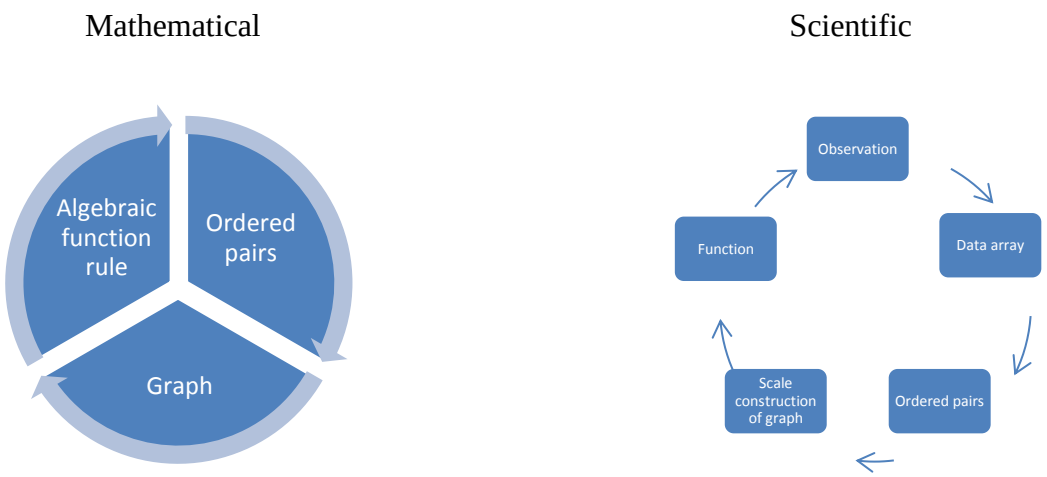


Figure 2.2: Mathematical representations and scientific representations of concepts.

These Mathematical and Scientific representations, when construed as isolated concepts, result in learners who are capable of solving graphical and functional problems in Mathematics being unable to access their knowledge in other learning areas such as Science. Based on this explanation, it becomes important for the teaching of functions to be integrated with other learning areas so as to avoid the isolation of the notion of functions from other related areas. These Mathematical and Scientific representations of functions and their graphs take into consideration the ability of learners to flexibly move from one representation of a

function to another. The approach to integrate learning areas is elaborated on by various researchers.

As far as functions are concerned, Even (1998); Goldin (1987); Janvier (1978); Lesh, Post & Bohr (1987); Marchovits, Eylon, & Bruckheimer (1983); Monk (1988) all address the critical problem of transitioning between and within representations. All these researchers agree that the flexibility in moving from one representation of a concept to another is intertwined with other kinds of knowledge and understanding of problems. Functions are important, because they enable learners to distinguish between visual attributes and physical attributes of physical structures.

This is proved in various research findings. Researchers like (Carlson, 1998; Monk, 1992; Monk & Nemirovsky, 1994) discovered that students have difficulties in distinguishing between visual attributes and graphical representations of physical structures and similar perceptual attributes of a graph of a function modelling a situation.. An example of such a situation is the representation of the side view of a person cycling up a hill, as depicted in Figure 2.3(a).

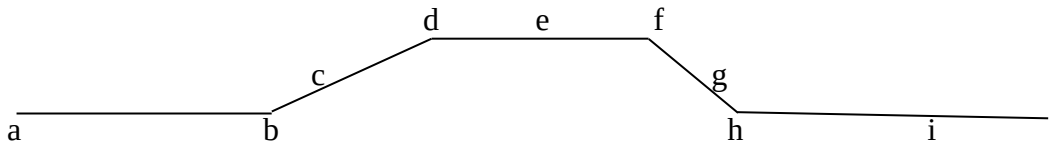


Figure 2.3(a): Visual representation of a physical structure.

The research shows that approximately 88% of A- students make mistakes of representing this scenario as a parabolic function, as shown in Figure 2.3(b) (Monk, 1992):

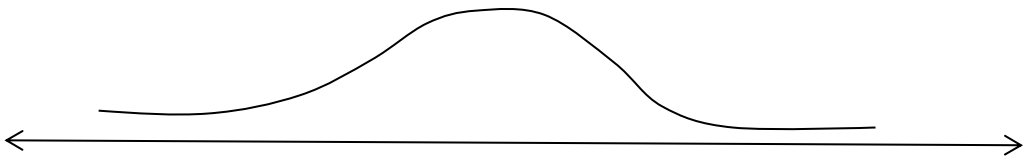


Figure 2.3(b): Visual representation of a physical structure.

The students tend to copy features from the diagram into the graph directly. This is a result of "conflating" the shape of the graph with the visual attributes of a situation under consideration (Monk, 1992; Monk & Nemirovsky, 1994; Carlson, 1998).

It is apparent from the literature, that it is imperative that learners become acquainted with the notion of a function so that they are able to interpret functions in their multiple representations and various attributes. The interpretation of functions forms an important factor that contributes to the importance of the teaching and learning of functions. The interpretation of functions enables learners to interpret many contextual problems that appear and are experienced by learners in their everyday life (Oehrtman et al., 2008).

The literature clearly described why the teaching and learning of functions is so important. Based on this, the South African curriculum is analysed to discover how functions are dealt with in the teaching and learning of specifically algebraic functions.

2.3 THE SOUTH AFRICAN CURRICULUM ON FUNCTIONS.

Before articulating what the South African curriculum suggests in terms of the teaching and learning of functions, it is important to understand what is meant by the word "curriculum".

The curriculum or (curricula) is what the state deems to be important for its citizens to learn and consists of four components. The first component is "Goals", which include the "benchmarks or expectation for teaching and learning and are designed in a form of a scope and sequence of skills to be addressed". The second component is "Methods" which includes specific instructional methods for the teacher that appears in Teacher's Editions of the curriculum. Thirdly, the "Materials" consist of media and tools to be used for teaching and learning. And the fourth component is "Assessment", which aims to measure the student's progress (Eisner, 1994, p. 96).

However, "the South African curriculum takes as its starting point a clear political agenda and the need to transcend the curriculum of the past, which perpetuated race, class, gender and ethnic divisions. It emphasised separateness, rather than common citizenship and nationhood". (Department of Education, 1997, p. 1). The old curriculum included the four components: goals; methods; material and assessment as described above. The South African

curriculum is based on the seven critical outcomes as described by the South Africans Qualifications Authority (SAQA) of 1995 whose primary objective is the promotion of a high quality education and training system in South Africa that embraces the concept of lifelong learning for all.

According to these critical outcomes, learners are required to be able to:

- Identify and solve problems and make decisions using critical and creative thinking
- Work effectively with others as members of a team, group, organisation and community
- Organise and manage themselves and their activities responsibly and effectively
- Collect, analyse, organise and critically evaluate information
- Communicate effectively using visual, symbolic, and/or language skills in various modes
- Use science and technology effectively and critically, showing responsibility towards the environment and the health of others
- Demonstrate an understanding of the world as a set of related systems by recognising that the problem solving contexts do not exist in isolation.

The teaching of the concept of a function satisfies all of these critical outcomes as outlined by the SAQA of 1995. The next section outlines what the South African curriculum expects teachers to concentrate on when teaching functions. It also defines what the learners should know in this regard in the Further Education and Training (FET) phase (Grades 10 to 12).

2.3.1 FUNCTIONS FOR GRADE 11'S IN THE SOUTH AFRICAN CURRICULUM.

The South African curriculum is divided into Learning Outcomes and related Assessment Standards. Functions are part of learning outcome 2 (LO2). Learning outcome 2 and the assessment standards pertaining to it (as prescribed by the National Curriculum Statement (NCS) of South Africa for the learning area of Mathematics in the FET phase for the topic on functions) are detailed and represented in Appendix H.

The aspects of functions that are included in the South African curriculum require learners to understand various types of functions, to find values of the dependent variable by finding

function values and finding the independent variable by solving equations. Learners are also required to be able to describe and use function values, finding a function rule/ formula and be able to transform equations of a function to equivalent expressions through the manipulation of functional equations. In the case of quadratic functions, learners are expected to use the manipulations of factorisation and completing a square to transform a quadratic function into its different forms.

Learners are expected to be able to understand the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and be able to flexibly convert between these representations. They should be able to generate graphs using the point-wise and the global approaches, identify the properties of functions including the domain and range, turning points, intercepts with the axis, maxima and minima, asymptotes, average gradient, intervals on which the functions decrease/ increase and discrete or continuous nature of functions. They should also be able to provide descriptions of situations focussing on trends and features of algebraic functions.

In summary, in Grade 10, the learners should have been introduced to the differences between a relation and a function and be in a position to define these different aspects. The learners also should have been introduced to the sketching and interpretation of the straight line function, the parabolic function, the hyperbolic function and the exponential function. They should also have looked at the vertical movement, the stretch and the intercepts of these functions, but they would not have looked at the horizontal movement of these functions.

They would have concentrated on the following parameters: $y = ax + q;\ y = ax^2 + q;$ $y = \frac{a}{x} + q; y = ab^x + q; b > 0$, where the a represents the shape of the quadratic/parabolic function. For example, when $a > 0$ the parabola is concave upwards and when $a < 0$ the parabola is concave downwards, the movement of the straight line and the hyperbolic functions from the first quadrant to the third quadrant when $a > 0$ and from the second quadrant to the fourth quadrant when $a < 0$. Their lessons should also have included the exponential function, where the movement from the first quadrant to the second quadrant for $a > 0$ and the movement from the second quadrant to the first quadrant when $a < 0$ as well as the vertical movement of the straight line, the quadratic, the exponential and the hyperbolic functions upwards when $q > 0$ and downwards when $q < 0$.

In Grade 11, the curriculum requires that the teachers and learners look at more of the parameters when dealing with functions and also that they look at the exponential function in more detail. The curriculum for Grade 11 emphasises the multiple representations of algebraic functions and the shifting in both the vertical and the horizontal axes. For instance, the teaching of functions should put more emphasis on algebraic functions that include all parameters represented by the following algebraic functions: $y = mx + c$ for linear functions; $y = a(x + p)^2 + q;\ y = ax^2 + bx + c$ and $y = a(x - x_2)(x - x_2)$ for quadratic functions; $y = \frac{a}{x+p} + q$ for hyperbolic functions and $y = ab^{x+p} + q; b > 0$ for exponential functions. These parameters should be dealt with in more detail as they involve all transformations as required by the Grade 11 curriculum.

The Grade 11 curriculum also emphasises representations of functions, for example assessment standard 11.2.1(b) which states that learners should be able to recognise relationships between variables in terms of numerical, graphical, verbal and symbolic representations and be able to flexibly convert between these representations (tables, graphs, words and formulae), intervals on which the function increases/decreases and the discrete and continuous nature of graphs.

The curriculum also focusses on the characteristics and features of the functions, describing a situation by interpreting graphs, drawing graphs from a description of a situation with special focus on trends and pertinent features of the graphs including the application of the function concept to contextual situations. The formal definition of a function is only dealt with at length in Grade 12. The detail is described in Appendix H which is based on the curriculum across the FET phase.

Based on the Umalusi (2012) report and the results of the diagnostic test given to my learners at the beginning of the study, it is clear that these sections are the sections that the learners have problems with. In particular, learners struggle with the aspects of functions that deal with multiple representations and interpretations of algebraic functions.

In the South African curriculum, as detailed in Appendix H, functions are grouped together with algebra and equations as these are interrelated. The main focus of this study is the intertwining of the quadratic equation with the quadratic function. The correspondence between these concepts is such that in order for one to simplify an equation representing a function, one needs to be conversant with the manipulations involved in algebraic equations

in the form of factorisation of these equations. The algebraic manipulations include, more specifically, factorisation of the quadratic equations and completing the square of the quadratic equation.

In order for learners to find the specific features of a quadratic/parabolic function, the learners have to know that by factorising the quadratic function in the general form: $= ax^2 + bx + c$, they will be in a position to see the features of a parabolic function represented by the factorised form of the quadratic/parabolic equation:

$y = a(x - x_1)(x - x_2)$. The features presented by this general form of the parabolic function are the x- intercepts of the parabola namely: $x = x_1$ and $x = x_2$. By completing the square on the general quadratic/parabolic function: $= ax^2 + bx + c$, the quadratic/parabolic function becomes: $y = a(x - p)^2 + q$, with the features of the parabolic function represented being the turning point at $(p; q)$ and the axis of symmetry represented by the equation: $x = p$. The parameters a and c give the dimensions of the quadratic/parabolic function, namely the shape (concave upwards for $a > 0$ and concave downwards for $a < 0$) and the y-intercept of the parabolic function respectively.

Similarly, the Grade 11 curriculum expects the learners to be able to determine the equation of the parabolic function when they are given a parabolic graph. The learners should be able to substitute the given values of the given variables in the appropriate form of the quadratic/parabolic equation in order to find the equation of the parabola.

This forms the focus of this study and answers the critical question 'which aspect of the function appears to be the most problematic?' for learners.

It's interesting to see what researchers have found about learners' difficulties in working with functions in general. Literature on the aspects of functions that learners find problematic and the misconceptions that learners have on these aspects of functions and what contributes to learners developing these misconceptions are discussed in the next section.

2.4 LEARNER MISCONCEPTIONS.

In this study, all the aspects of functions that learners have difficulties with are discussed. The aspects of functions that are problematic to learners include the definition of a function

that manifests itself in the learners' inability to answer questions that require them to find the domain and range of algebraic functions. Another problem area is the asymptotes of the given algebraic functions and finding the intervals in which the function is increasing or decreasing, the discrete and continuous nature of the graphs between certain intervals and finally moving flexibly between the different representations of functions. All these problematic aspects are described in the analysis of the diagnostic/pre-test, which is discussed in Chapter 6.

The results of the diagnostic/pre-test also revealed that the functional notation was also problematic for learners as they were unable to answer questions that required them to determine the values of the functions by substituting the given values into the functions.

The aspect of functions that appeared to be the most problematic for the learners, concerns the inability of learners to flexibly move between the graphical representation of functions and the equation representation of functions. This is proved by the learners' inability to find the equation of the quadratic/parabolic function when they were given the function in graphical form with the key features of the functions highlighted on the graph. The learners experienced problems with the sketch of the graph of a function when they were given the equation of the function with the vertical movement included in the equation of the parent graph. They also had difficulties with the interpretation of the horizontal movement from the equation of the parent graph of an exponential function.

Misconceptions, according to Nesher (1987, p. 34), denote a "line of thinking that causes a series of errors all resulting from an incorrect underlying premise, rather than sporadic, unconnected and non-systematic errors". These misconceptions are formed as a result of students "incorrectly generalising prior knowledge to grapple with new tasks" (Smith, 2008, p. 135).

While dealing with what the literature says about learner misconceptions and the factors contributing to the development of these misconceptions, the extent to which the definition of a function contributes to the learners' misconceptions of the notion of a function and the misconceptions resulting from how the definition of a function is introduced is discussed. This discussion includes the concept of the function machine and how it results in the misconceptions that learners ultimately possess.

These misconceptions on functions provide a lens through which this study is looked at and the literature on these misconceptions will help frame and focus this study.

2.4.1 MISCONCEPTIONS RELATED TO THE DEFINITION OF A FUNCTION.

The definition of a function is universally accepted and it does not matter which curriculum one follows and which country teaching takes place. Dolciani, Sorgenfrey, Brown & Kane (1986, p. 58) define a function from the perspective of a set as described by: "Let D and R be two sets. A function from D to R is a rule that assigns to each member of D a unique member of R". This trend of defining a function was followed by researchers such as Thompson (1994a); Cuoco (1994); Schwingenhof et al. (1992) & Sierpinska (1988).

Since this study focusses on Grade 11 learners' interpretation of algebraic functions, with particular focus on the question: 'what aspects of functions do learners find problematic?' Based on the definition of a function, the learners should appreciate that a function is a particular or specific relationship between x and y. The learners see the definition rather operationally and they do not see it as this particular or specific relationship that can be represented in a number of ways (as an equation, a table, graphically and verbally). They just have the operational and the formal definition, which makes it quite difficult for learners to understand at this level.

The formal definition of the function is not stressed as much in Grade 11, but the learners are being prepared for the formal definition that is required in Grade 12. The formal definition should be developed while learning about functions and it should not simply be learnt by heart and by regurgitating. In order for learners to understand the function notation, they should know its features and the effects of changing the dimensions of the function represented by its parameters. The definition of a function is one aspect, but there are many other aspects of a function that the learners have to know in order for them to fully understand the notion of a function.

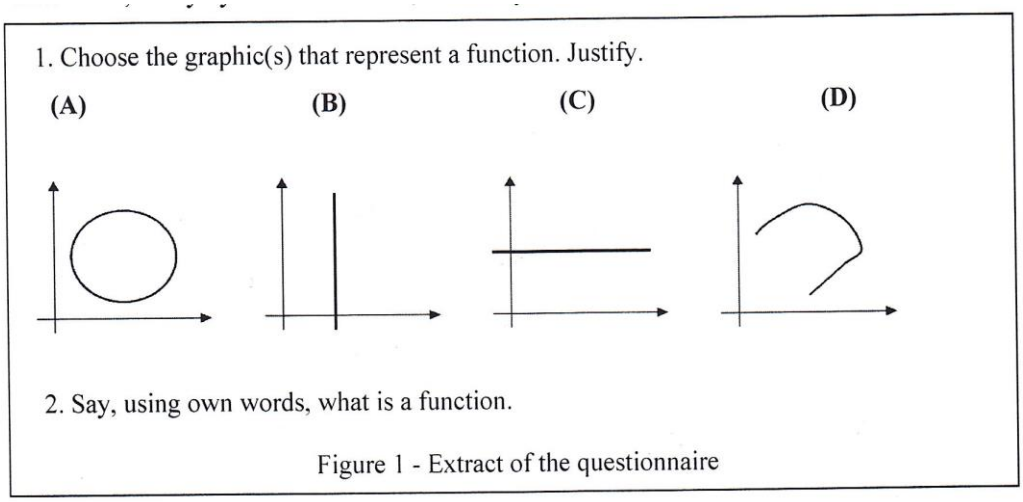
The inability of learners to interpret and apply the definition of a function, together with other aspects of the function to be discussed later, results in learners' inability to answer a question in the pre-test that required them to determine the value of a function:

$g(x^2) + g\left(\frac{1}{x}\right) - 28$ given the function: $g(x) = -x + 14$. The learners' interpretation of the question was that they had to determine the value of: $x^2 + \frac{1}{x} - 28 = 0$.

The literature states that the way in which the definition of a function is interpreted by the learners appeared to lead to learners' developing misconceptions on functions and is therefore very difficult and problematic to the learners. The literature on the definition of a function, as discussed below, shows how the misconceptions relating to the definition of a function are manifested by the learners and explains how these misconceptions can be averted.

Research on the definition of functions, as reviewed by Wagner (1981b) shows that students often believe that functions should be linear, continuous and be defined by an equation (Sajka, 2003). This was confirmed by research conducted by Saraiva (2001) and Texeira (2005). These beliefs were confirmed by the research where learners were required to choose from a series of graphs that represented functions and to justify their answers.

The graphs that were used by these researchers are represented in Figure 2.1 below.


1. Choose the graphic(s) that represent a function. Justify.

(A) (B) (C) (D)

2. Say, using own words, what is a function.

Figure 1 - Extract of the questionnaire

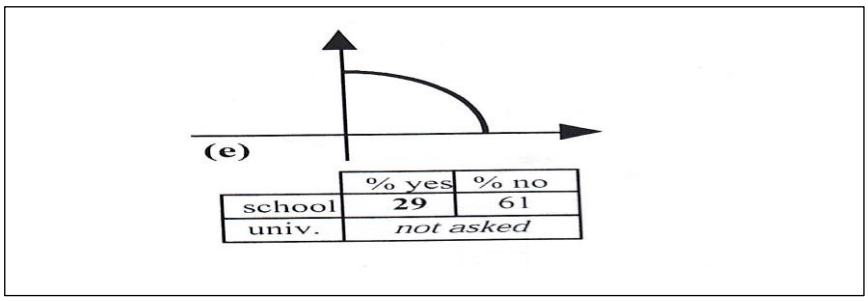
Figure 2.1: Connecting the definition of a function to the graphical representation.

Out of 24 students, only 7 (29%) identified option C correctly. 16 (67%) students chose options B and C only 1 (4%) chose option A. The results confirm that some students did not successfully connect the definition of a function to the graphs that represent functions, for example, some chose closed circles and lines parallel to the y –axis as functions. The reasons given for their choices of these graphs as functions were that the graphs were smooth and uniform. The reason given for the graph in option D as not being a function was that it was not a smooth graph and that it was not uniform.

Other researchers discovered that graphical representations as taught by teachers in the earlier grades also lead to the misconception that a graphical representation of a function should be

in the form of a smooth and uniform graph. This misconception stems from the learners' disassociation of a function from its definition.

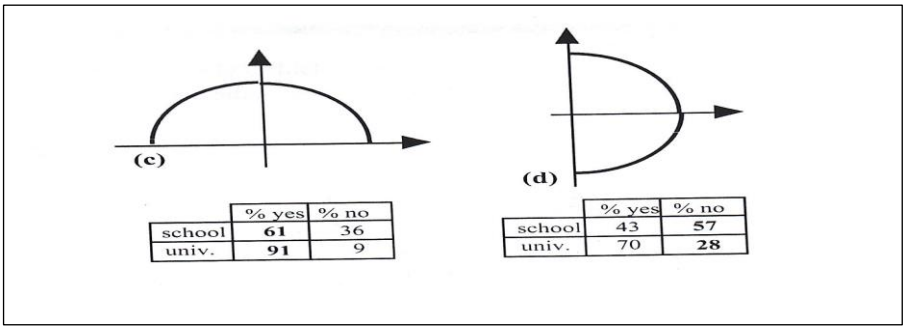
These findings are reiterated by Tall & Bakar (1991) in the study with Mathematics students at school level and university level. The students were required to state whether the graphs in the figures below were functions or not and these were some of the results: the graph of a quarter circle and a graph of half a circle is not a function, because it is not complete and this could be as a result of learners being exposed to the notion of a function to be a "natural totality given by a formula" and that it was essential to have it all and not an unnaturally selected part of a whole in order for it be a function (Tall et al., 1991, p.4). The students disregarded the definition of the function when giving their answers. This is depicted in Figures 2.2 and 2.3.



	% yes	% no
school	**29**	61
univ.	*not asked*	

Figure 2.2: A quarter circle as a function or non-function.

The responses of the students to whether half a circle was a function or non-function is depicted in Figure 2.3.

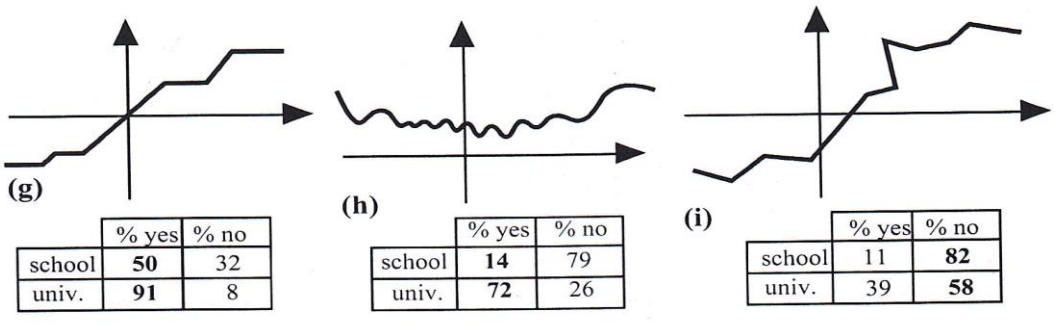


	% yes	% no
school	**61**	36
univ.	**91**	9

	% yes	% no
school	43	**57**
univ.	70	**28**

Figure 2.3: Half a circle as a function or non-function.

In the same study by Tall & Bakar (1991) with Mathematics students at school level and university level who were required to state whether the graphs in Figures 2.4 below were functions or not, the results were as follows:



	% yes	% no
school	**50**	32
univ.	**91**	8

	% yes	% no
school	**14**	79
univ.	**72**	26

	% yes	% no
school	11	**82**
univ.	39	**58**

Figures 2.4: Connecting the definition of a function to the graphical representation.

The general notion of a function for the majority of these students was that a function should be represented by a regular and smooth graph. This is another area where the definition of a function was not properly applied to its graphical representation as depicted in the graphical representations above.

Further misconceptions resulting from the definition of a function were reported after a study performed by Oerhtman et al., (2008). When students were faced with computing $f(x) = 7$ after being given the graph of $f(x) = x^2 + 3x + 9$, 38% of the students who understood the definition of a function obtained correct answers and 62% of those who memorised the definition were unable to give the correct answer. After requesting the students to justify their answers, the authors then came to a conclusion that students who memorised the definition of a function did not possess the process view of a function and that the students with a process view of a concept were able to understand aspects of functions better including the composition of functions and the inverses of functions.

Oerhtman et al. (2008) explain that a student who has the process view of the concept is one who is able to imagine the entire process without having to perform one action at a time and in so doing is able to interpret and apply the definition of a function to a variety of functional representations (graphical, verbal, equation and tabular form). From the literature it is evident that the definition of a function as it is universally accepted, results in the learners having a limited conception of a function which leads to the belief and misconception that a function should be defined by an equation (Sajka, 2003) and be represented by a smooth or regular graph (Tall et al., 1991). This prompted Even (1990) to extend the definition of a function to one that is inclusive of all types of representations of functions and the arbitrary nature of a function.

"The arbitrary nature of functions refers to the relationship between the two sets on which the function is defined and the sets themselves" (Even, 1990, p. 528). By the 'arbitrary nature' of the function the author means that functions do not have to be described by any specific expression, follow some regularity or be described by a graph with any particular shape (Even, 1990, p. 528). The author further elaborates on the definition of a function as to include that: "there is an assignment of a single value to each number", (p.529).

In other words, the misconceptions that learners had as far as the definition of a function is concerned is that a function that is not defined by an equation, which has both the independent and the dependent variable, was not a function. This misconception is elaborated on in the section relating to misconceptions of learners with regards to the constant function in the following section.

The following section deals with the misconceptions relating to the interpretation of the constant function with regards to a function's definition.

2.4.2 MISCONCEPTIONS RELATED TO THE CONSTANT FUNCTION.

In this study, the misconception with regards to the constant function is evidenced in the results of the diagnostic/pre-test when the learners were asked to give the equations of the asymptotes of the parent graph of an exponential function and the vertically transformed version of the same exponential function. The learners gave the equations of the exponential functions as their answers to these questions requiring them to give the equations of the asymptotes. As the equations of the asymptotes are special cases of the equations of the straight line, this sheds light on the fact that the learners do not understand what an asymptote is.

In other words, the equation of the asymptote is problematic to the learners as it is a straight line which does not conform to the general form of a straight line $y = mx + c$. The important issue is that the learners could possibly point to the asymptote, draw the asymptote, but

actually saying what the equation of the asymptote is, interpreting that and writing it in equation form was something they were unable to do. In order to do this they need to know the global view of the function and also need to know the special case of the linear function or equation. From the definition of a function, it appears as if a function should be represented by two variables and since the equations of the asymptotes consist of only one variable, the learners could not give the correct answers.

As already noted, the definition of a function has been reported by researchers to be difficult for learners to comprehend and interpret. The constant function, as it is represented, appears to differ from the general form of this algebraic function representing a linear function that the learners are familiar with in that some of the parameters are missing from the general form of a linear function.

Research by Sfard (Sfard, 1992; Markovitz et al., 1986) revealed that students generally had difficulties with the constant function as they believed that a function involved a change in the independent variable which then influences a change in the dependent variable according to the definition of a function. This was confirmed in a study by Carlson et al. (1998), where about 7% of A- students in college algebra could produce a correct answer to a question requiring them to give examples of functions. In this case, all of its output values were equal to each other as is the case with a constant function like $y = 5$ while the remaining 93% of A- students could only give an example of $y = x$ as an example of such a function as they justified their answers with the fact that $y = 5$ did not represent a function because it does not vary.

This misconception on a constant function prevailed and is reiterated by a majority of students in the research undertaken by Monk (1994). These students viewed a constant function as a non-function as it did not vary. As the definition of a function involves a dependent and independent variable, the constant function was viewed as having only one variable and therefore could not be a function. Similarly, in the study conducted by Tall & Bakar (1991), a constant function like: $y = 0$ was generally considered to be a non-function as the notion of a function was that it should be represented by a formula with two variables. This formula should be compiled in such a way that one variable, usually the x-variable, is the independent variable and the other variable, usually the y- variable, is dependent variable.

These findings lead to the fact that when teaching the definition of a function, one has to be aware of the pitfalls that are associated with the definition. The way in which functions are taught, beginning with the prototypes that are used in order to make the learning of the notion of a function possible, to how the definition of a function should be introduced to the learners are of incredible importance

In the following section, the misconceptions of learners regarding the functional notation will be stated and discussed.

2.4.3 MISCONCEPTIONS WITH FUNCTIONAL NOTATION.

In this study, the misconceptions of functional notation are proven by the learners' inability to answer a related question in the diagnostic/pre-test. This question required the learners to calculate the distance of a boy on skates after three seconds where the distance of the boy at the beginning of the event is defined by the function: $s(t) = 3^{t+1} + 5$. The learners who attempted this question, changed the equation and substituted 3 for $s(t)$ and found the value of : $3 = 3^{t+1} + 5$. This misconception of learners is in line with the findings by Carlson et al. (1998) and shows that the learners could not express distance (s) as a function of time (t).

This misconception is shown by Carlson et al. (1998) in a study that shows that one of the recurring misconceptions and errors among students is that when they are asked to express speed (s) as a function of time (t), many high-performing pre-calculus students could not represent this as $s(t)$. This is not unexpected as this is the state of affairs in South Africa as the Umalusi report has proved functional notation is a problem for learners in general. Additionally, the results of the pre-test (which are discussed in Chapter 6) also confirm that functional notation presents a problem for learner. The Umalusi report of 2011 also refers to functional notation as one of the areas that learners found problematic and is therefore included as part of the FET curriculum in South Africa. These misconceptions are also apparent in this study as the learners could not relate the functional notation to the contextual problem presented to them. This is discussed in detail in Chapter 6.

One of the reasons given by researchers for the misconception of functional notation is that the different notations of functions make the function concept look like different concepts instead of one 'unifying' concept (Maclane, 1986). The author further gives many examples of functions like algebraic operations that give rise to examples of functions of numbers, geometric definitions that produce trigonometric functions, exponential functions and their inverses that result in logarithmic functions that are numeric functions. The author further concedes that in space, distance is a function with real values that function in pairs of points in Boolean algebra, intersections and unions are functions of pairs of sets and in geometry, length is a real valued function of curves.

Coady & Pegg (1993) in their research on tertiary students' difficulties with the function notation, discovered that students responded to the question "If y is increased by t, find an expression for $3y^2 + 2y$" as if they were required to find the expressions for the following:

1. $3y^2 + 2y + t$
2. $t(3y^2 + 2y)$
3. $3(y^2 + 1) + 2(y + 1)$
4. $3(yt)^2 + 2yt$
5. $3(y^2 + 1) + 2(y + 1)$
6. $y + t = 3y^2 + 2y$
7. $t = 3y^2 + 2y.$

Some of these students ignored the phrase 'y is increased by t', some treated the expression $3y^2 + 2y$ as either a single term or two independent terms. Others simply added or multiplied a t as they saw fit. These responses indicate that the students had misconceptions with the function notation.

The companion brief of the pitfalls identified by the Commission of Mathematical Instruction in the international seminar known as the PCMI makes comments on other common notations of functions and elaborates on what these representations draw the attention of the students to (Hazzan & Goldenberg , 1997, p. 263). These are the representations that were in the brief. Firstly, $f(x) = x^2 + 1$ which is a notation that draws attention to the functional nature of the relationship under consideration? Secondly, $y = x^2 + 1$, which is a functional notation that supportsthe graphical representation in the Cartesian plane. Thirdly, $y = f(x) = x^2 + 1$ which puts emphasis on the functional character of the graphical form. Fourthly, $f: x \rightarrow x^2 + 1$ as a notation that is meant to represent the idea of a mapping from one set to another.

Finally, $(x, y) \in \{y = x^2 + 1\}$ as the notation that emphasises the idea of a set of points (Vinner, 1983, p. 301).

These different functional notations are included in the South African curriculum for the FET phase and the students are expected to be familiar with all of them. These different functional notations is one of the problem areas that the Umalusi report covers and is one facet of the notion of a function that the learners generally have difficulties with. This is further proven by the following studies and the results of the pre-test that informs this study.

From engaging with literature that defines mathematical functions, it emerged that functions have different functional notations which include, among others, the function represented by $f_1(n) = n^2 = \sum_{k=1}^{n}(2k - 1)$ for $n \in N$ is viewed by learners as two separate functions: $f_1(n) = n^2$ and $f_2 = \sum_{k=1}^{n}(2k - 1)$, although they produce the same results on the set of natural numbers (Carlson et al., 1998; Sfard, 1992).

To put it more simply, the different functional notations have different and important uses in different situations. It is important for learners to understand how they should be applied in the different situations. As noted above, these notations place different emphasis on the various ideas that are placed on the functions. These ideas can range from those representing the functional nature of the relationship under consideration, the graphical representation in the Cartesian plane, the mapping from one set to the next, the set of points on the function, computing the value of the function at a particular point and the value of the point x – coordinate(s) corresponding to a particular value of the function.

The following section deals with the misconceptions of learners with regards to the multiple representations of functions, which forms the focus of this study.

2.4.4 MISCONCEPTIONS ON MULTIPLE REPRESENTATIONS OF FUNCTIONS.

In order for learners to be able to interpret information given in the different symbolic systems of functions, they need to understand the notion of functions in all its representations. These include the verbal, graphical, tabular and equation form.

One of the reasons why learners develop misconceptions on the notion of functions, is the way in which functions are introduced to them using prototypes (Tall et al., 1991, p.2). The most commonly used method of teaching functions to lower grades is the use of the function machine/box as a representation of a function.

Tall et al. (1991, p.2) assert that the concept of a function is considered as foundational in Mathematics yet it proves to be "elusive and subtle" for students. These authors suggest that the function machine/box is an image of the concept and can act as a "cognitive root" for the students. .A cognitive root is a concept that a learner understands at the beginning of the learning sequence of a particular concept and allows the learners to expand their learning at a later stage while being useful to them as they develop and learn more sophisticated meanings of the concept (Tall et al., 1991, p.2).

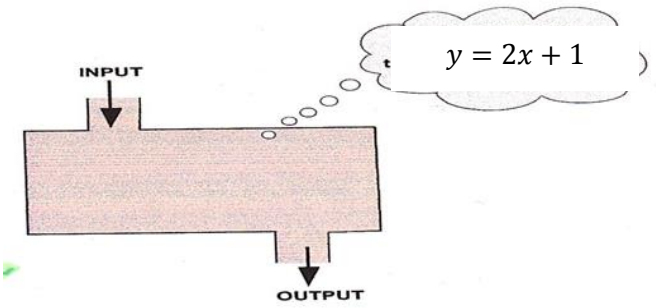
The function machine/box acts as a cognitive root for the learners, because "it can be represented visually as an input-output box and embodies both an object like status and a process aspect from the input to the output mode" (Tall et al. 1991, p. 3). The authors further assert that the function box, as is it is used in the GET phase, is a simplification of the notion of a function and gives rise to the misconception that "all functions are given by a formula". This misconception is carried on by learners to the higher grades and leads to the learners not being able to use the definition of a function in order to separate a function from a relation and to be unable to move between the representations of functions (table, equation, graphical and verbal form).

The misconceptions that result from the use of the function box is the fact that learners require "cognitive reconstruction" in later developmental stages (Tall et al. 1991p.5). Since the function box does not have an explicit range or domain and is usually introduced using natural numbers as "the set of possible inputs", this leads to the understanding that functions have a "natural" range restricted to real numbers.

This representation of a function as function machine/box is known as the cognitive root. Tall (1992, p. 497) defines a cognitive root as "an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built". This is a meaningful example or "prototype" for the learner at the beginning of learning a concept. "This prototype allows the initial development through a strategy of cognitive expansion of the concept and is

robust enough to remain useful as more sophisticated understanding of the concept is developed" (Tall, McGowen & DeMarois, 2001, p.215).

The introduction of the function through the use of a function box that serves as a machine where there is input, process and output is commonly used in the introduction of the function (Figure 2.5).



x	y
1	3
2	5
3	7
4	9
5	11
6	13
7	15

Figure 2.5: Representation of the function machine/box as a formula.

This representation of a function presents learners with obstacles that give rise to them believing that all functions will always have a domain and range of natural numbers. It also gives rise to other beliefs as was presented by the findings of Dreyfuss & Eisenbrg (1982), Arnold (1992) and Barnes (1988). These beliefs include that the majority of students, because of this type of introduction to the concept of function, believe that a function needs to be represented only by a formula while only a few of the students could identify the concept of a function when it was represented graphically.

Tall et al.(2001) suggest that the introduction of a cognitive root as a function machine/box should be represented in all the functional representations and not only as a formula. These representations include the function as a table, a formula, a graph and in its verbal form as this would alleviate the misconception that a function will always be represented by a

formula (Figure 2.6).:

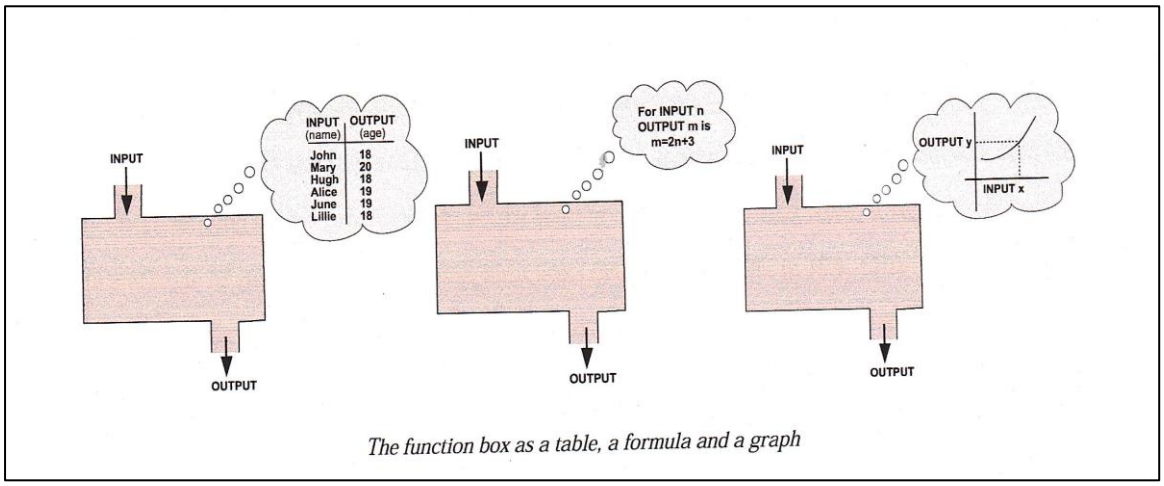


Figure 2.6: The function machine/box as a table, a formula and a graph.

The authors also advise that the function box approach may be used in different ways to engrave in the learners a sense of greater generality through everyday examples by making use of a procedure rather than a simple formula Figure 2. 6). These procedures should be such that the learners are afforded the opportunity to see the function machine as representative of the idea of a function that includes process (input-output) and object, with the various representations of functions (a table, a graph, a formula, a procedure) seen as methods of controlling this input-output cognitive root (Tall et al., 1991, p. 3).

Empirical data that was collected to test the use of the function machine/box as a cognitive root in the introduction of a function in all its representations revealed that 49% of the students began their college course operating at the process level of the function machine/box together with the use of tables. 20% of the students operated on a higher level of algebraic manipulation and 1% made use of graphs when solving college functional problems (De Marois, 1998, p.147).

The use of the function machine/box in the different functional representations, revealed that of the students who were exposed to the introduction of the function machine/box in all its representations, the ones who became successful at college level were those who progressed and operated at the precept level (where they demonstrated flexibility in viewing a function as either a process or an object). The mid-range students were those who acted at an object level (where they could view the function as inputs to higher-order processes). The

unsuccessful students were those who were only able to use the function machine/box in a step-by-step manner and did not make an attempt at the different procedures available and were therefore operating at a process level (De Marois, 1998, pp. 173 – 175).

Furthermore, in research conducted by Davis & McCowen (2001, p. 215), the results shows that the use of the function machine using the different representations of functions as a cognitive root gave students the mental picture "embodying the salient/prominent features of the idea of a function". This is because the use of the input-output and process and object using the various representations of functions (table, formula, verbal and graphical) to describe different processes, enhanced student's ability to flexibly move between the various representations of functions and therefore their understanding was enhanced.

The students in the sample group of this research used the function machine notion to organise their thinking as they worked from one representation of a function to the next. Their ability to use functional notation and differentiate between evaluating, simplifying and solving functions improved. At the end of the researchers' intervention process, the students could differentiate between the representations $f(x) = 3$ and $f(3)$ by using the words input and output correctly. They could differentiate between $f(x) = 3$ as that which required the input value that would give the output value of 3 and $f(3)$ to require the value of the output when the input value was 3.

Additionally, Tall & Marois (1996, p. 3); Cuoco (1994); Schwingendorf et al. (1992) and Sierpinska (1988) describe the different aspects that constitute the function concept. The function is described in terms of facets (representations) and different layers (development via process and object). The facets or representations consist of geometric (graphical), numeric (tabular), word (verbal) and symbolic (equation). The layers consist of the mental development of the concept and includes (process ↔ object). This is similar to the way in which Sfard (1992) begins with a process to an object: : "process" acting on a familiar object which is first "interiorised", then "condensed" in terms of "input/output" without going through the component steps/processes and then "reified" as an "object-like" entity (Sfard, 1992, p.2). Sfard (1991) and Sfard (1992) is elaborated upon in the section on the teachers' perspective, which focusses on what it is that the teacher should think about when teaching functions in a Grade 11 class.

Tall & Marois (1996, p. 3); Cuoco (1994); Schwingendorf et al. (1992) & Sierpinska (1988) further suggest that students who can view functions in all these layers and facets are more able to understand the function concept than those who view the function in only one facet or layer. These authors continue by asserting that the action – process – object layers results in "learners thinking of functions as actions that have to be performed as calculations or manipulations" (Cuoco, 1994, p. 122).

The result of this is that those learners who view functions as processes think that functions are "single valued transformations that are as a result of these transformations" (Cuoco, 1994, p. 122). Some students view functions as "atomic structures that can be inputs and outputs to higher order processes" (Cuoco, 1994, p. 123). Those students who can view the function concept as both processes and objects are in a better position to understand the concept of functions (Cuoco, 1994, p. 123).

Tall & Marois (1996, p. 3); Cuoco (1994); Schwingendorf et al. (1992) & Sierpinska (1988) discovered that if the intermingling of the facets and the layers is not developed at an earlier stage of teaching and learning by being taught these facets and layers simultaneously, the result will be the isolation of these facets and layers from one another. This manner of teaching the concept of a function enables learners to analyse a problem and choose the appropriate way of solving a particular problem.

Other researchers further suggest that it might be "wrongheaded to focus on graphs, expressions, or tables as representations of function but rather a focus should instead be on functions as a representation of *something* that is representable to the students such as aspects of a specific situation" (Thompson, 1994, p.39). The author further asserts that the focus on graphs, expressions, or tables as representations of functions might later be construed by learners as if a function will always have a "natural" domain and range, rather than the domain and range being specifiable in the definition and the problem under discussion.

He advises that it would therefore be an advantage at an early stage to teach the function box as an input-output process taking the elements from a specific domain A into a range B to attempt to move closer to the formal definition of a function at a later stage in the teaching process. This introduction of the concept of a function could alleviate the conflicts that occur

when dealing with constant functions like $y = 3$. Constant functions have been found to persist in the learners as they tend to associate this equation as representing a number 3 and not a function that could be represented using a straight line that is parallel to the x axis (Carlson, Oehrtman & Thompson, 1998).

Even's (1998) research focusses on the importance of learners' ability to flexibly move between the representations of functions. She discovered that there are factors that are involved in linking the representations of functions. These factors depend on the context in which the function is represented. These factors consist of two different approaches, namely the point-wise approach and the global approach. These approaches could be applied when solving functions in the context in which they appeared. Even (1998) differentiates between the point-wise approach which deals with discrete points of a function and the global approach that looks at the functions' extremes, which can be useful, especially when the function is given in symbolic, general form of $y = ax^2 + bx + c$, where no numeric values are given.

The two approaches, the point-wise and the global approach, supplement each other and can be used when flexibly moving between representations. Both of these approaches are found to be critical in the learning and understanding of functions by other researchers such as Breidenback, Dubinsky & Harel (1992); Dubinsky (1991); Bell & Javier (1981); Janvier (1978); Lovell (1971); Marynyanskii (1975); Monk (1988) ; Sfard (1991).

The point-wise approach is mostly applicable where the learner can use the operational (calculation) process to determine discrete values of a function when such operational procedures are required. The global approach is useful when looking at the trends followed by the function in specific intervals as the Grade 11 curriculum on functions requires learners to be able to do.

Representations of functions, according to Thompson (1984), consist of the symbolic or equation representations, which is $g(x) = x^2 + 1$. The graphical representation, $y = x^2 + 1$, mapping, $f: x \to x^2 + 1$ and verbal/descriptive as in – 'One added to the square of certain number; implicit as in a function resulting from the parameters of a solution of equations and tabular representation (Table 2.2) are all representations of a function.

x	-2	-1	0	1	2
$y = x^2 + 1$	5	2	1	2	5

Table 2.2: An example of the tabular representation of the values of a function.

These representations of functions according to Even (1998, p. 105) are important in that "the ability to identify and represent the same thing in different representations and flexibility in moving from one representation to another, allows one to see rich representations, develop a better conceptual understanding and strengthens one's ability to solve problems". In order for learners to fully develop insight into and understandings of the essence of a concept and its many facets, they should be able to connect the different representations of that same concept.

In their research on functions and graphs, Tall & Bakar (1991, p.9) discovered that learners could not "construct the abstract concept of a function in the absence of examples of the concept of the function in prototype or graphical form". The researchers came to the conclusion that this built-in prototype or visualisation of a function by means of a graph often leads to the limitations that are absent or do not apply to the abstract concept of a function (Tall et al., 1991). In other words, learners form mental pictures of functions that are misleading and therefore form misconceptions, which could be averted if the mental pictures are formed together with the correct interpretation that is in line with the definition of a function.

Learners' misconceptions with the interpretations of functions give rise to learners having misleading mental pictures or prototypes of functions. Some of these misleading mental pictures are that all functions should have a 'smooth' or 'regular' pattern and that complete circles are functions as they have a 'smooth' or 'regular' pattern and also that positive or negative semi circles and quarter circles were not functions as they were not 'complete' (Tall & Bakar, 1991, p. 5).

The literature indicates that there are specific and definite problems with the teaching and learning of the concept of functions. The way in which the function is introduced to the learners is critical to the learners' understanding of the concept. Given all the misconceptions that are now well-known, they are further explored in this study. Given the distinction

between global and point-wise thinking, this is also explored. The different approaches to the teaching and learning of functions referred to in the literature have their advantages and disadvantages. Despite these advantages and advantages are essential for teachers to be aware of.

In this study, a different approach to the teaching and learning of the concept of functions is attempted. The aspect of functions that learners find to be most problematic is the use of the appropriate form of the quadratic function in the derivation of the equation from the given graphical representation of a parabola. This approach is informed by the theory of variation which is expanded on in the following chapter. Through the theory of variation, the critical question 'How does a version of a learning study (which is framed by variation theory) improve the Grade 11 learners' interpretation of algebraic functions and to what extent this takes place is attempted to be answered.

The flexibility of learners to move from the graphical representation of a quadratic function to the equation form and from the equation form of the quadratic function to the graphical form informs the main part of this study. This is the aspect of functions that is the most problematic to the learners. For Grade 11 learners to be able to correctly answer questions on algebraic functions, they should be able to flexibly move between the graphical representation and the equation form of these algebraic functions.

From the discussion of why the teaching of functions is important, the misconceptions resulting from the definition of a function, the misconceptions with the constant function and the misconceptions with the multiple representations of functions above, it is evident that multiple representations of functions are a necessary and are a pertinent skill that should be emphasised in the teaching and learning of functions. The ability of learners to flexibly move between the functional representations of functions will enhance their understanding of the notion of a function.

The literature on the misconceptions on the interpretation of functions and their relevance to this study are discussed in the following section. If the learners do not understand the different functional notations and other aspects of functions as previously described, then their chances of performing well is reduced.

2.4.5 MISCONCEPTIONS ON INTERPRETATION OF FUNCTIONS.

The misconceptions concerned with the interpretation of functions as required by the study are all related to the misconceptions described above. They are all a result of, and stem from, the misconceptions of the definition of a function, the misconceptions with the constant function and the misconceptions with the representations and notation of functions above.

The interpretations of functions are included in the expectations of the South African curriculum on functions and as the Umalusi report and the pre-test given to the learners discussed in chapter one, learners have problems with the interpretation of functions. Their problems include the translation of functions (the horizontal and vertical shifts), interpretation of local processes (regarding point-to-point attention), global interpretation processes (detecting trends), global and general (what happens to y as x increases and visa-versa), continuation (interpolation/extrapolation), rate (how fast should a car travel to reach a certain distance in one hour), qualitative interpretation (looking at an entire graph to gain the general meaning of the situation) and quantitative (a collection of isolated points).

The literature of misconceptions of the interpretation of functions serves as a guide to understanding the causes of the learners' misunderstanding of the concept of functions. In this study, it is evident in learners' interpretation of the quadratic function when it is represented in different algebraic forms. These different algebraic forms of a quadratic function are represented in general form by using the parameters and variables, for example: $y = ax^2 + bx + c; f(x) = a(x - p)^2 + q$ and $g(x) = a(x - x_1)(x - x_2)$ which highlight different features of a quadratic function. These different forms of the quadratic function form the basis of this study and are elaborated on in the methodology. The intervention lessons undergone in this study identify this as the most problematic section on functions that the learners in the sample seem to have.

This study focuses on the correct interpretation of these different forms of the quadratic functions as they are key to the learners' ability to determine the equations of quadratic functions when they are given the quadratic functions in graphical form. It also helps the learners to interpret the quadratic functions and to represent the quadratic function in graphical form.

An example of the interpretation of an equation in the form, $f(x) = a(x-p)^2 + q$ in numerical form, $f(x) = -1(x-3)^2 + 2$ immediately informs the learner that the graph is one of a quadratic function that is concave downwards with the axis of symmetry at $x = 3$ and a maximum value at $y = 2$.

As discussed in the Umalusi report and as it happened with the learners in the diagnostic/pre-test, learners cannot interpret quadratic functions given as parameters only. This is evident in a question where the learners, according to the report, were required to draw a sketch graph of the function $y = ax^2 + bx + c$, where $a < 0$, $b < 0$, and $c < 0$. Some learners could not access the appropriate form of a prototype that could deal with a general case regarding the parameters a, b and c of the required sketch as there were no numerals in the quadratic function.

Some misconceptions noted by Oehrtman, Carlson & Thompson (1998) in their research on A-students who have completed college algebra was that when these students attempted to find the value of $f(x + a)$, about 43% of them added a instead of substituting $(x + a)$ into the function. When probed for their reasoning behind this process, they provided some rule or procedure, showing that they were not thinking and interpreting "$x + a$" as a value at which the function is being evaluated but as "a" being added to the function $f(x)$. This form of evaluating also appeared in the Umalusi report and could be due to the students' misconceptions of the interpretation of the function notation.

In other studies by Thompson (1994) and Carlson (1998) it is clear discovered that students cannot distinguish between algebraically-defined functions and equations. Students interpret and view functions as two equations separated by an equal sign. In order to help students distinguish between the use of an equal sign as a means of defining a relationship between two varying quantities and a statement of equality of two expressions, the two authors developed instructional interventions that promoted students' thinking about algebraically-defined functions. They explain that the use of an equal sign as a means of equating the output values of two functions and the act of solving an equation as a means of finding the input value(s) where the output values of these functions are equal makes it clearer (Oehrtman et al. (2008, p.152).

This misconception has been reiterated in the Umalusi report where the learners solved quadratic functions as if they were quadratic equations. When learners were given a function like $y = -x^2 + 3x + 2$ they solved it as they would solve an equation where one side was equal to zero and divided each term by negative one without considering that this was not an equation equal to zero. When they multiplied the right hand side by a negative number they should get the answer of $y = -(x^2 - 3x - 2)$ instead of the answer $y = x^2 - 3x - 2$ as it would be the case if this was an equation with $y = 0$.

Carlson, Oehrtman & Thompson (1998) assert that there are different types of interpretations that can be expected from the learners. Some of the interpretations referred to by the authors are translation (the horizontal and vertical shifts), interpretation of local processes (regarding point-to-point attention), global interpretation processes (detecting trends), global and general (what happens to y as x increases and visa-versa), continuation (interpolation/extrapolation), rate (how fast should a car travel to reach a certain distance in one hour), qualitative interpretation (looking at an entire graph to gain the general meaning of the situation) and quantitative (a collection of isolated points).

In this study, the diagnostic/pre-test required learners to determine the values of x for the given functions $f(x) \geq g(x)$, where $x \geq 0$. Chapter 6 discusses the fact that learners in this study could not interpret the global processes when they were expected to detect trends where one function was greater than the other in a specified interval.

The literature informs about the misconceptions that learners have with regards to the definition of the function and how the definition of a function should be introduced to the learners. It further explores the misconceptions of functional notation and how these misconceptions are derived, how they could be alleviated and what misconceptions emanate from these different functional notations, the representation of functions and how these different representations should be linked together in order to enhance deeper learner understanding of the function concept and its interpretations. Keeping the literature in mind, the following section discusses the suggested teacher's thinking of function while teaching the concept. In doing so, the literature is elaborated on and an indication of what the teacher's focus should be on to help learners reach a level where they can understand functions in all its representations.

2.5 A TEACHING PERSPECTIVE.

Since the study is about teaching and learning of functions, the literature on functions as seen from the previous sections, informs about the misconceptions learners have when dealing with the definition of a function, the representation of functions, the interpretations of functions and the functional notation. The most important aspect of functions that teachers have to be aware of is the nature of functions and how the nature of functions plays an important part in the learners' understanding of the concept.

This dual nature also exists for functions and is a contributing factor to the understanding of the notion of a function. The ability of learners to interpret functions is explained as the dual nature of functions, which is discussed in the following section.

2.5.1 THE DUAL NATURE OF FUNCTIONS.

Many researchers, including Sfard (1991), Tall & DeMarois (1998),; Cuoco (1994), Schwingendorf & Dubinsky (1991), & Sierpinska (1988) state that functions have a dual nature as do most Mathematical concepts do. After analysing several examples, Sfard (1992, p. 61) concludes that "many mathematical notions had been conceived operationally long before their structural definitions and representations were formulated". One example of such a Mathematical notion is rational numbers which emerged as a result of measuring operations such as ratios.

The dual nature of Mathematical concepts is explained by Sfard (1987) and Sfard (1991) as consisting of the operational process and the structural process. The author further elucidates that when operations are performed on existing structures, new objects are formed and when further operations are performed on these new objects, other new objects are the result.

In her earlier work Sfard (1991) suggests that operational concepts/notions consist of "processes, algorithms and actions and they are important as they follow a sequenced and detailed process while structural thinking involves referring to a mathematical entity as an

object (referring to it as if it was a real thing), a static structure existing somewhere in space and time (by recognising an idea at a glance and being able to manipulate it wholly without going into detail)" (Sfard, 1992, p.61). Sfard (1991) asserts that in order for learners to speak about Mathematical objects, they must be able to deal with products of some process without bothering about the processes that produced these products themselves. The author refers to this process as reification. According to Sfard (1991), the understanding of a concept involves the operational conceptions of the concept and should precede the structural conception.. In other words, in order for learners to understand the concept of a function, they have to understand the operational stages of the function and the processes involved in its formation. Teachers should teach learners in such a way that the learners are able to understand how that particular function came about.

Sfard (1991) defines reification as a process that involves substituting talk about actions or processes with talk about objects. She further refers to reification as "an act of replacing sentences about processes and actions with propositions about states and objects" (Sfard, 2008, p.44). She goes on to state that "reification influences the discourse by shaping it in the image of discourses on material objects" (Sfard, 2008, p.44). She also contends that the process of concept formation involves these three stages: interiorisation, condensation and reification. This means that the teachers should be aware of whether the learners have obtained the reification stage or not when teaching the concept of functions.

The question that arises is clear, when does this reification stage occur? According to Sfard (1991), the stage of reification occurs when one is capable of conceiving the notation of a function as a fully-fledged structure and this new structure is detached from the processes that produced it so that other processes could then be performed on this new object/entity which now serves as an input to produce another structure which serves as an output. In order for a learner to be able to interpret the concept of a function, the learner should have successfully undergone the three stages of concept formation (interiorisation, condensation and reification) that leads to reification.

The stage of interiorisation involves learners having to be acquainted and proficient with the process which results in a new concept being formed. In other words, interiorisation involves the learners' ability to go through the processes of finding the intercepts of the graph, the turning point, the asymptotes and then drawing the graphs followed by interpreting whether the graphs are linear, parabolic, hyperbolic or exponential. Learners should also be able to

interpret these functions when they are presented in all the representations of functions (table form, graphical form, equation form and verbal form).

Condensation involves the 'squeezing' of lengthy sequences of operations into more manageable units. In this way a new concept is officially born at this stage of condensation. In other words, the resulting quadratic forms represent the new concepts that can then be interpreted to give the features of a function without the learners having to go through the processes of deriving them (Sfard, 2008).

At the reification stage the learner will be able to solve contextualised problems that he/she can represent as functions. A learner who is at this stage will be able to interpret a contextual problem like calculating the height that could be reached by a golf ball being hit by a golfer from two given points on the ground and at a given speed. This learner will be able to interpret this movement as one that formed a parabolic curve given the starting points represented by the x- intercepts of the parabola and then having to calculate the turning point of this parabola using the form of the quadratic function: $y = a(x - x_1)(x - x_2)$ where x_1 and x_2 are the x- intercepts.

In order to investigate and compare the effects of introducing a new concept on functions to students using the operational approach to the structural approach, Sfard (1992, p,70) gave a questionnaire to students in two groups of 22 to 25 year old students undertaking a course on elementary Mathematics. The one group consisted of students who were introduced to the concept of a function using the operational approach and is referred to as the control group and the second group, referred to as the experimental group, was introduced to the concept of a function using the structural approach.

Two basic principles that were the basis of Sfard's operational method were that new concepts should not be taught in structural terms and that a structural concept should not be introduced as long as the students could understand the concept in the absence of the structural being introduced. The structural approach means that new concepts are taught using the new structures as readymade objects. The questionnaire that was used appears in Table 5 of the appendix and the results of groups, the control group and the experimental group, appear in Table 6 of the appendix.

The results demonstrate that the 'prominent trend' of students learning a new concept (functions in this case) was their ability to associate a function with 'computational

processes' (operationally) rather than the 'object like' (structural) entities. The results also suggest that when students are taught the concept of functions using the structural approach, the learners would eventually learn to also think structurally and thereby reduce the misconception that arise.

Sfard's research on this group of 22 – 25 year old students undertaking a course on elementary Mathematics showed a considerable improvement on their ability to translate one form of representation of functions to another namely (table form, set notation, algebraic formulae, computer programmes, graphs and diagrams of different kinds). The students' gradual improvement in their ability on this difficulty could be attributed to the students being exposed to a rich set of appropriate problems that include introducing new concepts of function from the structural approach. (Sfard, 1992).

The results of the questionnaire also show that the trend was improved from the control group (the group taught new concepts through the operational process) to the experimental group (the group taught new functional concepts structurally) by a significant margin. The author advises that students who needed to pursue Mathematics at higher level institutions were the ones who could be taught the notion of functions both structurally and operationally as it was a crucial skill necessary for higher level knowledge.

Breidenback et al., (1992); Monk, (1987) & Thompson, (1994) distinguish between the action (computation) and process views of functions by noting that a function is not rigidly tied to specific computations or rules that define how to determine the output from a given input. This distinction is synonymous to Sfard's operational (action view) and structural (process view).The process view takes into account the ability of a learner to find "the input value(s) for which the function $f(x) = 6$ (both algebraically and graphically) while the action (computation) view would require learners to solve for an equation $f(x) = 6$ for some specified function f" given in any representation of a function (Oerhtman et al., 2008, p. 161).

Dubinsky & Harel (1992, p. 85) further explain that " the process conception involved a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce that same transformed quantity. In so doing the subject is able to think about the transformation as a complete activity beginning with objects

of some kind, doing something to these objects, obtaining new objects as a result of what was done. When a subject has a process conception, he or she will be able, for example, to combine it with other processes, or even reverse it and thus strengthen the student's process conception".

This is a reflection and confirmation of the fact that the learners who are unable to treat functions as objects (structurally) and not as processes (operationally) can develop this ability after being exposed to treating functions as objects through a series of different examples. These findings confirm observations about processes and objects in Mathematics thinking of by many researchers like Sfard (1992Heinrics, 1991; Dubinsky, 1986; Dubinsky and Schwingendorf, 1990(b); Sfard, 1987, 1988(a), 1991; Harel and Kaput, 1991 cited in Sfard, 1992, p. 71; Breidenback, 1992; Monk, 1987 & Thompson, 1994.

The ability of students to treat functions as both objects and processes leads to the students having fewer to no problems with the interpretation of contextual situations that involve the notion of functions. . With regards to the teaching and learning of functions, finding the appropriate way to enable the learners to see the function as both objects and processes can be difficult. The introduction of functions using both the structural (as an object) approach and the operational (as a computation) approach will help the learners in their understanding of the notion of a function in that they will be in a position to flexibly move between the different representations of functions.

As Sfard (1991) and Tall et al. (1991) have highlighted, multiple representations present difficulties in the teaching and learning of functions due to the abstract notion of a function. Additionally, a function takes many forms in terms of representations and learners tend to separate these representations from each other instead of connecting them together to form a unity. Instead of seeing the table, the equation and the graph as all particular representations of a parabola, learners think that they are not connected. Research proves this, because learners treat graphs as separate and consider some graphs to be functions even though they are not. As a result, learners are not able to flexibly move between the multiple representations of the function.

If some graphs are not represented in a familiar and regular form or are not in standard form (the form they are accustomed to), learners cannot recognise them as functions. This confirms

the fact that the learners cannot link the different representations of a function to the graphical representation.

All of the interpretations of functions, representations of functions, notation of functions referred to by the authors above are included in the South African curriculum on functions for the FET phase specifically for the Grade 11. These areas are reported to pose a problem for the learners, which is evident from the Umalusi report. .

The literature on the nature of functions enables this study to use the representations of functions in both the operational (performing operations on the equation $y = ax^2 + bx + c$) and structural view (identifying the features presented in graphical form) in order to help learners overcome the errors and misconceptions that persist when they have to interpret functions in a variety of ways. This is done by applying different operations on the standard form of the quadratic function $y = ax^2 + bx + c$ to produce the different algebraic forms of the parabolic functions.

The operation of completing the square on the standard quadratic form produces the turning point and the equation of the axis of symmetry of the parabola. The operation of factorising the standard form of the quadratic function produces the x-intercepts of the parabola. The different resulting forms of the parabolic equations enable the learners to identify the features of the parabola in each form of the different quadratic forms.

The two representations of the parabolic function (equation form and graphical form) are the representations that are the most problematic to the learners in this study's sample. It is apparent from the literature that learners definitely have to be able to interpret the function in all its representations and be able to flexibly move between the multiple representations of functions. The learners should be able to operate from both the operational (procedural) process to the structural (object) and be able to use the structural (object) views in order to relate them to the parabolic function. All of these are, in my experience, aspects of algebraic functions that are the most problematic to learners.

This study concentrates on the aspects of functions that are most problematic to the learners. These aspects are found in the results of the diagnostic/pre-test that is discussed in Chapter 6. The aspects of functions that are most problematic to the learners are their ability to completely reach the stage of reification. At this stage, the learners should have interiorised the concept of a function and be able to view the quadratic function globally and point-wise

and be able to operate from both the operational approach and the structural approach. In order for the learners to be able to do this, they should be able to interpret the functional nature of the relationship and know which form of the quadratic equation to use when finding the equation of a parabola, as well as when these different forms of the quadratic equations are applicable.

The functional nature of the quadratic function are in the following forms: $y = ax^2 + bx + c$, which informs the learners of the global appearance of the quadratic function.. Learners should know that this form of the quadratic function should be applied to derive the equation of a parabola when the graphical form is given with any three points on the graph. $f(x) = a(x-p)^2 + q$, is a functional notation that supports the graphical representation in the Cartesian plane. It shows the points of the features of the parabola including the global shape of the parabola, the equation of the axis of symmetry and the turning point of the parabolic function.

2.6 RELEVANCE OF THE LITERATURE REVIEW TO THE STUDY.

The literature on the constant function, functional notation, and multiple representations of functions, interpretations of functions and the dual nature of functions is used in the analysis of the diagnostic/pre-test. The errors and misconceptions that the learners have regarding these aspects of functions are categorised according to the content in the literature.

The literature on multiple representations of functions, interpretations of functions and the dual nature of functions is used in this study to enhance learner understanding of how to interpret the parabolic graphs in the different representations. The appropriate forms of the parabolic equations in order to find the parabolic equations of the given parabola are used to reinforce learner interpretation of the parabolic graphs. The intervention lessons are used in the application of the suggestions from the researchers on the multiple representations of functions, interpretations of functions and the dual nature of functions.

The learners are helped in acquiring the knowledge of when and how each of the equations of the parabolic function should be used to derive the equation when the graphical form is given. They should be able to use the appropriate forms of the quadratic function namely:

$f(x) = a(x-p)^2 + q$ when a graph of the parabola with the turning point and another point on the graph of the parabola is given; $y = f(x) = a(x-x_1)(x-x_2)$ when a graph of the parabola with the x-intercepts and another point on the graph of the parabola is given; $y = ax^2 + bx + c$ when a graph of the parabola with any three points on the graph of the parabola is given. The learners should also be able to draw the graph of a parabolic function when they are given the parabolic equations in the following forms:
$y = f(x) = a(x-p)^2 + q$, which puts emphasis on the functional character of the parabola and informs the learners of the turning point and the axis of symmetry and the shape due to the value of the parameter a; $y = f(x) = a(x-x_1)(x-x_2)$ which informs the learners of the x- intercepts and the shape due to the value of the parameter a; $y = f(x) = ax^2 + bx + c$ which informs the learners that they have to sketch the graph of a parabolic function and should apply the operational processes on the equation to calculate the turning point, the axis of symmetry and the x- intercepts.

The key focus in this study is the interpretation of functions by the Grade 11 learners. In order to explore this focus, it is necessary to ascertain whether the learners have the structural view or not. It is interesting to note whether the problem the learners have relate to this or not. Another difficulty in the teaching and learning of the concept of a function is that it is an abstract idea which can only be given in different representations. Learners have difficulty relating to the idea and rather treat each representation as a thing in itself and are seldom able to flexibly move from the graphical representation to the equation form and from the equation form to the graphical representation of algebraic functions in particular the parabolic function.

Issues pertinent to this study are around the different operations that can be performed to the different quadratic forms of the quadratic equations of a function and how these different forms of the quadratic functions can be linked to the graphical representation in order to generate the equations of the functions represented graphically. This study also looks at how learners interpret these different forms of the quadratic functions in order for them to generate the graphical representations of the quadratic function from these different forms.

This study does not focus so much on domain and range, on the definition of the function or the functional notation. The knowledge that the learners' problems concerning functions could be due to their misconceptions emanating from these aspects of functions is worth

contemplating in order to answer the critical question ‘What aspects of functions do learners find problematic?’

It is evident for this study that if learners do not fully understand the difference between a relation and a function in the early years of dealing with functions, they will not be in a position to interpret the different representations and notations of a function. It is therefore important to make learners aware of the link between the definition of a function and the functions they deal with in class.

The use of the definition and the teaching of learners in the different approaches are useful in this study. This study focusses on the three algebraic forms of the quadratic function:

$f(x) = ax^2 + bx + c;\ g(x) = \ a(x-p)^2 + q; h(x) = \ \ a(x-x_1)(x-x_2)$. These different forms of the quadratic functions are used in the different representations of the parabolic function. These are the graphical form and the equation form. The graphical form is used where learners are expected to interpret the graphical form given particular features of the parabolic graph, including the x-intercepts and another point on the graph, the y-intercept and another point on the graph and the turning point and another point on the graph. Learners are also expected to interpret the graph given three points on the parabola and the choice of the appropriate form of the equation of the parabola to be used in finding the equation.

In addition to these approaches, the study investigates how the structural approach can be used using the theory of variation. The theory of variation is discussed in the following chapter.

2.7 SUMMARY

This chapter focuses on reviewing literature pertaining to why functions are important. The literature on aspects of functions that the learners find problematic is alluded to. The literature on the misconceptions learners encounter when dealing with the notion of a function is discussed. These misconceptions as encountered by learners involving the definition of a function, the interpretation of a function, the multiple representation of a function, and the notation of a function and the nature of functions are also described.

How the curriculum across the General Education and Training (GET) and the FET phases expects teachers to teach the concept of functions is highlighted. The literature on the

different ways in which learners could be helped to overcome these misconceptions is discussed as well as the manner in which the functions are introduced in the early stages of teaching was explored and how the use of prototypes and function machine contributed to the understanding and also the misconceptions that learners struggle with.

What the teachers have to think about when teaching the notion of a function and the way in which the teachers should approach the introduction of a function to the learners is linked to the dual nature of the function.. The relevance of the literature review to the study was discussed and how the literature helps the analysis of the diagnostic/pre-test and the how the intervention lessons should be framed using the concepts on the multiple representations of functions, interpretation of functions and the dual nature of functions.

The theoretical framework that informs the study is discussed in the following chapter. The patterns of variation theory that are applied in the study and how they are applied is explained and elaborated on.

CHAPTER THREE

THEORETICAL FRAMEWORK.

3.1 INTRODUCTION.

As this study is informed by the theory of variation, it is imperative that the theory of variation is defined and the features of the theory of variation discussed. This includes exploring the literature pertaining to the use of this theory in the teaching and learning of functions.

This chapter describes what variation theory entails, its features and why it was chosen as the theoretical framework that frames this study. The features of variation theory includes an object of learning which consists of the intended object of learning, the enacted object of learning and the lived object of learning. Secondly, the patterns of variation which include contrast, separation, generalisation, fusion variance and invariance, simultaneity as well as the application of these features in a version of learning study.

3.2 THE FOCUS OF THE STUDY.

This study focusses on the use the patterns of variation, as alluded to by Marton, Runesson and Tsui (2004), to try and answer the research question, 'to what extent does a version of a learning study framed by variation theory improve the Grade 11 learners' interpretation of algebraic functions?'

In order to answer this research question, it is necessary to identify which aspects of functions learners have problems with and which of these aspects of functions are most problematic for learners.

3.3 THEORETICAL FRAMEWORK.

"A theoretical framework provides a particular perspective or lens through which to examine a topic" (Smith E, 2008, p. 136). Variation theory is selected as the theoretical framework for this study, because this theory provides different approaches to the teaching and learning of any concept. According to Runneson (2006), the key concepts in variation theory include –

object of learning, features, variance and invariance, example space, patterns of variation including contrast, generalisation, separation, fusion and simultaneity. These key concepts of variation form the theoretical framework that is used as a tool to frame the study. How these concepts are applied in a version of a learning study is explained in the following sections.

3.3.1 OBJECT OF LEARNING.

Variation theory involves having an object of learning which implies that opportunities to learn are made available in various ways. Variation theory is a theory accounting for differences in learning and it provides a way to describe the conditions that are necessary for learning. Runesson (2005) refers to the object of learning as what is possible for learners to learn. Furthermore in variation theory, to learn implies to experience, understand, be aware, perceive or see something in a different way (Runesson, 2006).

Runesson (2006) reports that variation theory is not a theory of the mechanisms of learning but a theory of the relation between the object of learning and the learner. It is a way in which a learner is afforded the opportunity to arrive at the discernment (awareness) of the object of learning. Variation theory's aim involves the ultimate discernment by the learners of the different variables and their roles in different learning situations (Rowland, 2008).

Marton (1997, p. 157) proposes that "variation constitutes what is learned" and therefore variation needs to be practiced in any learning situation. Marton (2005) believes that learning only happens when there is some variation present and discerned and therefore sees learning as the discernment of variation. The author also states that it is important to consider invariance as well as variation and work out how much or how little variation is necessary for learners to notice what it is hoped they will see and be aware of.

The object of learning is the particular capability that the teacher expects the learners to acquire and learn after the learning process is complete. In other words, it is "a specific insight, skill or capability that the students are expected to develop during a lesson or during a limited sequence of lessons" (Marton & Pang, 2006, p.194). The object of learning has three categories, the intended object of learning, the enacted object of learning and the lived object of learning (Marton & Pang, 2006, p.194; Runesson, 2006).

The intended object of learning is what the teacher intends the learners to discern and therefore learn. In other words, it is what the teacher expects learners to have learned at the

end of the lesson or a series of lessons concerning a specific concept of the learning area/subject.

The enacted object of learning is what is done during the process of learning and creates a situation that makes it possible for the students to learn, experience or discern. The enacted object of learning d depends on how the lesson is structured and the conditions of learning that will make it possible for the object of learning to be discerned by the students. The use of questions by the teacher among other teaching approaches/methods in a learning situation constitutes the enacted object of learning. Runesson (2005, p.70) states that the enacted object of learning is "co-constituted by the interaction between the teacher and the learner or between the learners themselves.

The lived object of learning is what the learners actually learn, discern or encounter since learning is the process of being capable of doing 'something' and is as 'a result of having a certain experience of that something' (Marton et al., 2004; Runesson, 2005, p.70)..

In this study, the object of learning, which is the particular capability that the research expects the learners to acquire and learn during the intervention lessons, is the interpretation of functions from the graphical form to the equation form and from the equation form to the graphical form. The object of learning includes the application of the different forms of the parabolic equation to the equations of the given graphical representations of the parabola.

The theory of variation involves varying one thing and keeping others constant. This process is called variance and invariance, which is one of the key aspects of variation theory and is discussed in the following section.

3.3.2 VARIANCE AND INVARIANCE.

Marton and Pang (2006) state that invariance involves explaining the same concept in different ways while variance involves contrasting what is to what is not. The authors further state that "even though the experience of a certain pattern of variation and invariance is necessary for a given instance of learning, it might not be a necessary nor sufficient condition of learning for the given instance of learning" (Marton and Pang, 2006, p200). In other words, "an educator can create the necessary conditions that are necessary for the learner to experience a certain pattern of variation and invariance to be able to discern a quality, a dimension of variation, or a relation" in order to be afforded the opportunity to learn a concept (Marton and Pang, 2006, p200).

This study applies the key feature of variance and invariance by making use of the patterns of variation. Through these patterns of variation, the critical features of a parabola are brought to the surface of the learning to be discerned. These patterns of variation are separation, contrast, generalisation, fusion and simultaneity and are discussed in the following sections.

3.3.3 FEATURES OF VARIATION THEORY.

According to Marton et al. (2004), when one experiences an object on the table, for example, a blue cylindrical, ceramic mug with a handle, this means that the "blue" and "cylindrical" are values in the dimension of variation as they are seen in relation to the experiences that can be varied.

Features of the mug such as the colour, shape and material are simultaneously discerned as a dimension of the mug that could be varied. This concept represents a pattern of variation, which could be applied to functions as one of the patterns of variation. "These aspects are necessary for defining the object of learning in question and they are referred to as the features of the mug" (Marton et al., 2004, p. 15).

Marton et al. (2004) further state that in Mathematics, different strategies for solving a problem constitute different values in the patterns of variation. Therefore, a strategy used in solving a problem is an instance of a variation strategy used. "Variation enables learners to experience the features that are critical for particular learning and the development of certain capabilities". (Marton et al., 2004, p. 15).

Experiencing something entails discerning an aspect or feature of that object. Values will then be experienced in the corresponding dimension of variation. The particular experienced aspect is discerned as a value in the dimension of that variation. The patterns of variation theory emanate from the theory of variation, because variation theory provides the manner in which variation can be applied in the teaching and learning of functions. Therefore the patterns of variation frame the study.

In this study, functions have features that are necessary for defining and differentiating them from other concepts in Mathematics. These features of functions are the focus of the study and these features will be the values that are varied using variation theory in order to afford the learners an opportunity to learn. This approach is what constitutes variation theory.

Before the theoretical framework that is used in this study is elaborated on, an explanation of how this framework can be a success is needed. According to the theory of reification as seen by Sfard (1991) and elaborated in a cognitive root by Tall et al. (1991), the students have to be expedient in the internalisation and condensation stages for them to have reached the reification stage. A learner who has reached the reification stage is then able to use structures as objects in order to derive other structures.

This study uses the theory of variation as a tool to afford learners the opportunity to discern the object of learning. This is the purpose of the intervention lessons that are further discussed in Chapter 5. Hopefully, after the intervention lessons and the use of variation theory as a theoretical framework, the learners are afforded the opportunity to reach the reification stage.

Marton et al. (2004) argues that most studies place their focus on what is taught and relayed as content and concepts and do not focus on how the teaching of the concept is constituted. This study is concerned with finding out how teaching informed by variation theory can bring about the discernment of the object of learning and therefore improve the learners' understanding of the concept of functions or not. In other words, the dimension of variation theory as seen and explained by the aforementioned authors is used as a lens through which the teaching of the notion of functions in the learning study is described.

In the classroom, methodological tools consisting of the patterns of variation can be used to afford learners the opportunities to discern the object of learning. This is described by researchers like Marton, Runesson & Tsui (2004); Marton et. al., (2006); Pang, (2006); Runesson, (2005); Marton & Pang (2006); Pang & Pang (2004); Pang (2002); Runesson, Holmqvist & Marton (2009) and elaborated by Watson & Mason, (2005) and Rowland, (2008); (Marton (1997) and Chik & Lo (2004). These tools consist of the patterns of variation, namely: contrast, generalisation, separation, fusion and simultaneity. All of these patterns are discussed in the following sections.

According to Marton et al. (2004) when one experiences an object on the table, for example, as a blue cylindrical, ceramic mug with a handle(an instance), this means that the "blue" and "cylindrical" are values in the dimension of variation as they are seen in relation to the experiences that they can vary.

Features of the mug such as the colour, shape and material are simultaneously discerned as a dimension of the mug that could be varied. This concept represents a pattern of variation and thus could be applied to functions as one of the patterns of variation. "These aspects are necessary for defining the object of learning in question and they are referred to as the features of the mug" (Marton et al., 2004, p. 15).

Marton et al. (2004) further posit that in mathematics, different strategies for solving a problem constitute different values in the patterns of variation thus a strategy as used in solving a problem is an instance of a strategy used. "Variation enables learners to experience the features that are critical for a particular learning and the development of certain capabilities". (Marton et al., 2004, p. 15).

Experiencing something entails discerning an aspect or feature of that object and values will then be experienced in the corresponding dimension of variation. The particular experienced aspect is discerned as a value in the dimension of variation. The patterns of variation theory emanating from the theory of variation because variation theory provides the manner in which variation can be applied in the teaching and learning of functions and therefore the patterns of variation frame the study.

In my study, functions have features that are necessary for defining and differentiating them from other concepts in mathematics. These features of functions will be the focus of the study and these features of functions will be the values that will be varied using variation theory in order to afford the learners an opportunity to learn. This is thus what constitutes variation theory.

Before I elaborate on the theoretical framework that is used in my study, I will explain how this framework can be a success. According to the theory of reification as seen by Sfard (1991) and elaborated in a cognitive root by Tall et al. (1991), the students have to be expedient in the internalisation and condensation stages for them to have reached the reification stage. A learner who has reached the reification stage is then able to use structures as objects in order to derive other structures.

The study will use the theory of variation as a tool to afford learners the opportunity to discern the object of learning that will be the purpose of the intervention lessons that will be discussed in chapter five. Hopefully, after the intervention lessons and the use of variation

theory as a theoretical framework, the learners will be afforded the opportunity to reach the reification stage.

The tools that can be used to afford learners the opportunities to discern the object of learning as alluded to by Marton et al. (2004) and consisting of the patterns of variation namely: contrast, generalisation, separation, fusion and simultaneity will be discussed in the following sections.

3.4 EXAMPLE SPACE.

3.4.1 THE SPACE OF LEARNING.

Another aspect of variation theory that is salient to the teaching and learning of functions is the space of learning. A space of learning is the space created by the teacher in the classroom or a learning environment by using different approaches to the teaching of a concept. This space consists of examples that are given to the students in a learning environment. In order for learners to experience variation in mathematics, a space for such experiences to occur has to be created and is called the space of learning. The space of learning involves the use of examples that can help the learners to experience the patterns of variation as referred to by Marton et al. (2004). These are contrast, generalisation, separation and fusion.

A space of learning is formed when a critical aspect of the topic is brought to the foreground of the learners' awareness. This is similar to when a noise is discerned against a background of silence or a tall person is contrasted with a short person, then the critical aspect of tallness and shortness is discerned. Learning according to variation theory involves the ability to discern differences in the critical features of what is to be learned. What is learned depends on taking other conditions into consideration as well (Runesson, 2006, p.402).

Rowland (2008, p.150) contends that there are two types of example spaces that can be used when dealing with variation theory. The first type of example space is inductive in nature and can be created by motivating students to give examples of "something" Rowland (2008, p.150) .The examples are usually "particular" instances of the general Rowland (2008, p.150). The second type of examples can be through exercises. The aim of exercises is to facilitate retention of the procedure by repetition. The teacher's choice of examples can

facilitate the learners' awareness of the nature of the concept through variation of examples within the chosen category.

Skemp (1979) posits that the role of examples helps learners in their formulation and comprehension of the concept as well as the assimilation of the concept through subsequent examples may occur in future. The space of learning involves the meticulous planning of the lessons and the proper use of appropriate examples that facilitate the discernment of the object of learning by the learners.

The examples used in the study at the beginning of the intervention lessons (which is explored in Chapter 5) constitute the space of learning for this study. The choice of examples and how they are introduced to the learners at the beginning of each intervention lesson provides us with sufficient information on whether the particular choice of examples was meaningful for the study or not. The choice of examples is discussed in the following section.

3.4.2 CHOICE OF EXAMPLES.

Research conducted by Rowland (2008) shows how the choice of examples by 12 elementary trainee teachers during their final school placement influenced the discernment of the object of attention by the learners who were aged between 4 and 7 years. The research emphasises the diversity of knowledge required from teachers in order for variation theory to be effective.

Schulman (1987) as cited in Rowland (2008, p. 15) emphasises three categories of teacher content knowledge that are important to the application of variation theory. These categories are subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curricular knowledge (CK). In order for variation theory to be effective in a learning situation, the teacher should be well-vested in these three types of knowledge. The knowledgeable teacher should be able provide learners with a robust (meaningful, relevant) choice of examples that will enhance learning and discernment of the object of learning.

A learning study was used by Runesson (2006) whereby she asserts that variation theory accounts for the differences in learning. She provides ways to describe the conditions necessary for learning and involved the experiencing and the understanding of perceiving and seeing something in a different way, which could lead to learners' ability to interpret functions presented in different forms such as graphically, verbally, in tabular form and in

equation form. A version of a learning study is used in this study as a catalyst to enable the learners' discernment of the object of learning. This is discussed further in Chapter 5.

The author further states that variation is a necessary condition for the discernment of the object of learning, the critical features of what is to be learned, seeing the differences in the object of learning and essentially forms a space of learning. In order to provide learners with the opportunities to discern the critical features of the object of learning variation could be used.

When using variation theory, there must be something that varies while another one remains invariant. In choosing examples, one should be cautious to use systematic variation. The chosen examples should follow a carefully-planned variation that enables the learners to discern the object of learning. Carefully-chosen examples and their structure promote mathematically significant variation (Watson & Mason, 2005).

For learners to be able to discern the chosen object of learning, examples given to the learners should involve the variance and the invariance aspect of variation. There are different patterns of variation theory that create the space of learning. These patterns of variation involve the variance and the invariance aspect of the theory of variation which is discussed in the following section.

3.5 THE PATTERNS OF VARIATION.

Patterns of variation, referred to by Marton et al. (2004) and the other researchers, consist of the different approaches that could be applied during the process of teaching and learning to ensure that the object of learning is discerned by the learners. These patterns of variation and the manner in which they are applied in this study are discussed in the following sections.

3.5.1 SEPARATION.

Marton et al. (2004) and Runesson (2006) refer to separation as the process where a certain aspect of something is varied while other aspects are unvaried. Research on learners' ability to discern the critical aspects of a cuboid was conducted by Jaworski (1994) and analysed by Runesson (2005). In this research, the pattern of a "separation" is identified as important as the shape of the cuboid was kept the same and the different ratios, forms and sizes were

varied to enhance the object of learning. The object of learning was the shape and critical aspects of the cuboid.

Separation allows learners to experience certain aspects of a function by separating them from other aspects of the function and then varying it separately while keeping other aspects of the same function the same. Examples of separation are seen in the following functions:

$$f(x) = x^2 + 2x + 1; f(x) = x^2 + 2x - 3; f = x^2 + 2x - 4 \; and \; f(x) = x^2 + 2x - 6.$$

In these functions, only the constant term is varied and the other terms are kept constant. The resulting critical feature of separation in this case is the different points of the parabola touching the y – axis (the y- intercepts), the x- intercepts and the turning points, but the shape of the parabola is kept the same. In this study, the dimension of separation is used by keeping the graphical form of the quadratic function the same and varying the features of the function on the graph. Learners are then required to find the equation of the quadratic function by using the appropriate form of the general quadratic function.

3.5.2 CONTRAST.

Contrast involves a contrasting experience that must be present in order for the learners to discern the object of learning through variation. In order to experience something, a person must experience something else to compare it to (Marton et al. 2004, p.15). These authors explain contrast as the ability to experience something that is different from the one that has been experienced in order to understand the contrasting effect of the two things. It involves experiencing a certain aspect of something in order to differentiate it from other aspects.

Functions could be presented with the same constant values that represent different functions such as a linear function, a parabolic function, a hyperbolic function and an exponential function. The differentiation in the equations could lie in the position of the numerals and the parameters, for example:

$$y = \frac{1}{2}x + 1; y = \frac{x}{2} + 1; y = 2x + 1; y = x^2 + 1; y = \frac{2}{x} + 1; y = 2^x + 1.$$

The application of this dimension of variation theory in the teaching and learning of functions might generate some improvement in the learner's understanding of the notion of a function.

This dimension of variation is implemented in this study by teaching the learners the different forms of the quadratic function, specifically which form of the quadratic function should be used to generate the equation of a quadratic function when they are given specific features of the function in the different representations (graphical and equation).

The forms of the quadratic function that are in contrast to each other are:

$f(x) = ax^2 + bx + c;\ g(x) = \ a(x-p)^2 + q; h(x) = \ \ a(x-x_1)(x-x_2).$

The contrast is presented in the equation and the graphical form of the quadratic function.. In the test, the quadratic function was given in equation form as above and the learners were required to graphically showing the features given by each form. Thereafter, the graphical form of the same quadratic function was shown with the features shown in the graphs and the learners were required to give the form of the equation appropriate to the sketch of the graph drawn.

The first form of the quadratic function informs the learners about the shape of the quadratic function depicted by the parameter "a" and the y- intercept given by the parameter "c". For $a > 0$ the shape of the quadratic function will be concave up whereas for $a < 0$ the shape of the quadratic function will be concave down.

The form of the quadratic function represented by $g(x) = \ a(x-p)^2 + q$ informs the learners of the features concerning the axis of symmetry and the turning point of the quadratic function. The learners should be in a position to conclude that the axis of symmetry is given by $x = p$ and the turning point given by the parameters "p" and "q" by applying the functional notation $(p; q)$ that is included in the South African curriculum for the FET phase.

The form of the quadratic function represented by $h(x) = \ \ a(x-x_1)(x-x_2)$ informs the learners of the features concerning the x-intercepts of the quadratic function. According to this dimension of variation the learners should be in a position to conclude that the x-intercepts are represented by x_1 and x_2. This is included in the South African curriculum for the FET phase, which expects learners to be able to show the intercepts of functions with the axes.

The form of the parabolic equation represented by $f(x) = ax^2 + bx + c$ informs the learners of the features concerning the shape of the parabola and the y-intercept of the quadratic function. The other features of the parabola can be attained through the algebraic

manipulation of the equation. By using factorisation the x- intercepts can be found and by completing a square the axis of symmetry and the turning point can be found. The turning point and the axis of symmetry can also be found through substitution into the appropriate equations.

3.5.3 GENERALIZATION.

Generalisation takes place when one is able to find a general formula or appearance of a pattern after a series of carefully chosen examples. According to Marton & Tsui (2004), in order for learners to understand the meaning of the number like say "three", they have to experience different appearances of the number "three". For example, three bananas, three people, three houses or three dogs. The learners' ability to identify the varying general appearances of the following equations uses generalisation.

$f(x) = ax^2 + bx + c;\ g(x) = a(x-p)^2 + q; h(x) = a(x-x_1)(x-x_2)\ and\ f: x \rightarrow ax^2 + bx + c$.

Generalisation of the standard form of the quadratic function is due to the different operations that are performed on the function. The operations that are performed on the standard form of the quadratic function can be demonstrated by using the following specific example:

The standard form of a specific quadratic function would be $f(x) = x^2 - 3x + 2$, which is generally written as $f(x) = ax^2 + bx + c$. After the operation of completing the square, the standard form becomes $f(x) = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$ generally written as $f(x) = a(x-p)^2 + q$. After factorising, the standard form, $f(x) = (x-2)(x-1)$,is the result generally written as $f(x) = a(x-x_1)(x-x_2)$.

The use of these generalised forms of the quadratic function are important for this study as the learners are expected to choose form these forms of the quadratic function to find the equations of quadratic functions when they are given the critical points of the functions in graphical form. This study is concerned with the ability of learners to use the appropriate form of the quadratic function and to substitute the given features appearing in graphical form.

3.5.4 FUSION.

Fusion involves varying several aspects of an object that are critical for the discernment of the object of learning at the same time (Marton et al., 2004). Fusion is the ability to discern several critical aspects of something simultaneously in order to separate them, as it often happens in everyday life that it is seldom only one aspect of something that varies at a time (Marton et al., 2004, p. 16).

These varying critical aspects of the object (function in this case) have to be experienced by the learners simultaneously as in the following examples of linear functions where more than one parameter is varied at the same time:

$y = x; y = x + 1; y = 2x; y = 2x + 2$ This leads to the conclusion that well-designed lessons with systematic variation result in effective learning taking place (Marton et. al., 2006).

In this study, fusion is applied to the different forms of the quadratic function by varying more than one parameter at a time. This is done to enable to identify the effect the parameters have on the graphical form of the quadratic function.

3.5.5 SIMULTANEITY.

Learning, according to Marton et al. (2004), can be viewed as the development of capabilities and values. The kinds of capabilities that are focussed on in learning are those that empower learners to deal with situations in ways that allow them to simultaneously focus on features that are critical for a particular aim. Since learners can simultaneously experience only that which they can discern, learners can only discern that which they experiences to vary. Learners can only experience variation if they experience different aspects of something at the same time. In order to discern a feature of something, learners should experience variation in a dimension corresponding to that feature of something.

Simultaneity in variation theory emphasises that in order "to experience variation amounts to experiencing different instances at the same time" (Marton et. al., 2004, p.17). Aspects discerned simultaneously are discerned as a pattern of variation called simultaneity and are a result of varying different features of a concept at the same. Looking at different forms of a quadratic function at the same time and deciding which form of the quadratic function should be used when certain features are given is another instance of simultaneity.

The different forms of a quadratic function under scrutiny are:

$y = ax^2 + bx + c$; $y = a(x - x_1)(x - x_2)$; and $y = a(x - p)^2 + q$;

$y = ax^2 + bx + c$ can be used when a graph of a parabolic function is given with any three points on the graph other than the x- intercepts and another point on the graph or the turning point and another point on the graph.

$y = a(x - x_1)(x - x_2)$ can be used when the x-intercept and another point on the graph is given.

$y = a(x - p)^2 + q$ can be used when the turning point of the parabola and another point on the parabola are given.

These forms of the quadratic function are given to the learners at the beginning of the lesson and the graphical form presented to them so that they could identify which form of the quadratic equation could help them generate the equation of the given parabolic function.

Simultaneously exposing leaners to the different forms of the quadratic function might enable them to make the relevant connections when faced with choosing the appropriate form of the quadratic equation to use and under the different circumstances in which it could be used.

In the following section a learning study is explained and the way in which it will be used in this study is elaborated on.

3.6 A LEARNING STUDY.

According to Lo, Marton, Pang & Pang (2004), Pang (2002), Runesson et al. (2009) (2009) variation theory has been used in the design of learning situations. One of the learning situations designed using variation theory is a model called “the learning study” which is used to capture what is important for learning in the form of an object of learning. Therefore, in order to apply the patterns of variation as stated above, a learning study can be used.

A learning study involves a group of teachers coming together to plan and teach a topic that has been identified as problematic to learners. The process involved in a learning study is testing the learners to ascertain their previous and existing knowledge on a particular topic and the areas that are problematic to them.

The results of the test then form the basis of the object of learning or topic to be taught. The planning of the lesson takes into account issues pertaining to what it is that the teachers expect the learners to be capable of developing and what aspects of the lesson are necessary for learners to discern these capabilities.

The first lesson is then planned with these questions in mind and one teacher teaching the lesson is videotaped. After the first lesson, the learners are then tested on the same test that was written before the learning study with some new questions added. The results of the test are then analysed by the group of teachers together with the videotaped lesson. A revised lesson is planned according to the results of the test, the interviews on the lessons and the videotaped lesson. The revised lesson is based on whether the objectives of the lesson were achieved or not and at what degree these objectives were achieved.

The second teacher uses the revised second lesson plan to teach another class while being videotaped and observed by the other teachers. The second class is then tested using the same test written by the first class. The process is repeated until about four teachers have been afforded the opportunity to teach four different classes using the revised lesson plan. Each revised lesson is planned after the analysis of the results of the previous class' test results, after interviews are conducted and the videotaped lesson reviewed to see what can be improved or changed in the planning of the next lesson (if necessary).

This study uses a simplified version of a learning study as only one teacher does the teaching of three different classes. The procedure is the same except for the one teacher conducting the three lessons. The version of a learning study that is used in this study is similar to the one used in the action research approach. The complete process is discussed in the following chapter.

3.7 SUMMARY.

In this chapter, the theoretical framework that underpins the study is outlined, the tools that are used in the framework are discussed and the ways in which they are used in the study are elaborated on. In the following chapter, the research approach and methods used in the study are discussed in detail.

CHAPTER FOUR

METHODOLOGY

4 INTRODUCTION.

This chapter covers the research methodologies that are employed to answer the research question 'which aspects of functions do learners find most problematic?' After identifying the most problematic aspect of functions from the results of the diagnostic test, the object of learning that informs the planning of the intervention lessons is identified. The setting of the research project is explained in greater detail in order to give insight as to how data collection instruments are used in order to explore the main question of 'to what extent does a version of a learning study framed by variation theory improve the Grade 11 learners' interpretation of algebraic functions?'

Challenges during data collection are highlighted and the question of rigor which includes the reliability and validity of the data, generalisation, limitations of the study and ethical considerations taken are also discussed.

4.1 RESEARCH PARADIGM.

A paradigm is a way of looking at the world. It is composed of certain philosophical assumptions that guide and direct thinking and action. "Within each paradigm there exists ontological beliefs (about the nature of reality), epistemological beliefs (about the nature of knowledge and the most appropriate ways of producing that knowledge when investigating the world) (Johnson & Onwuegbuzie, 2004, p. 112).

Lincoln and Guba (1983, p.200) identify three questions that help define a paradigm – the ontological question, "What is the nature of reality?" the epistemological question, "What are the nature and knowledge and the relationship between the knower and the would-be known?" and the methodological question, "How can the knower go about obtaining the desired knowledge and understandings?"

The researcher's paradigm is a transformative paradigm, which includes both practical and participatory research. The research approach is action research and the ontology for this research is about the nature of reality. The ontological question for the study is "how do we

think about mathematical objects? How do the learners think about functions as functions are things that exist in reality?' Objects do not exist simply as we perceive. Objects are perceived in social settings. Objects do not speak for themselves and so when one presents a function to learners, it is not a function until someone like a teacher has made it a function. For the purposes of this study,

Mathematical objects that appear as representations in various forms in classroom practices, do not speak for themselves. They take on meaning in social settings and that is why learners have to discern functions as they do not speak for themselves. These questions that define a paradigm highlight the nature and knowledge of the notion of a function that the learners possess and how they interpret this knowledge when answering the questions in the pre-test. In order to answer this question, the learners were given a pre-test which sheds light on how they interpret and view the nature of reality, which is the notion of a function.

The epistemological questions for the study are: 'What is a function?' and 'What aspects of functions do learners have the most problems with?' These questions highlight what the nature of functions is and the knowledge that the learners have concerning functions. The epistemology of this study is what the learners need to know about functions. A function is an abstract notion, a uniform notion, but it can never be presented once, it can only be presented piece-by-piece. It can only be partial and the way we present it to learners is through piecewise methods which are mainly by presenting it through its points. In order to answer these questions, the learners were given a pre-test which will shed light to how the learners interpret and view functions.

The methodological question for this study is 'To what extent does a version of a learning study framed by variation theory improve the Grade 11 learners' interpretation of algebraic functions?' The intervention lessons attempt to answer this methodological question which is further discussed in Chapter 6. The study examines ways in which the research could benefit the learners and teachers who are interested in improving their teaching practice.

4.2 RESEARCH DESIGN.

This study applies the transformative paradigm. The transformative paradigm includes both practical and participatory action researchers whose ultimate aim is to bring about change. This paradigm uses a transformative theory to develop the program theory and the research approach (Fowler, 2009). Fowler (2009) states that the most appropriate research approach

for the transformative paradigm is participatory action research as it involves the people who are the research participants in the planning of the lessons, conducting analysis of the data collected, interpreting the data and the use of the research. The transformative paradigm could follow the convergent parallel design, which includes both qualitative and quantitative data collection and analysis methods, as depicted in Figure 4.1.

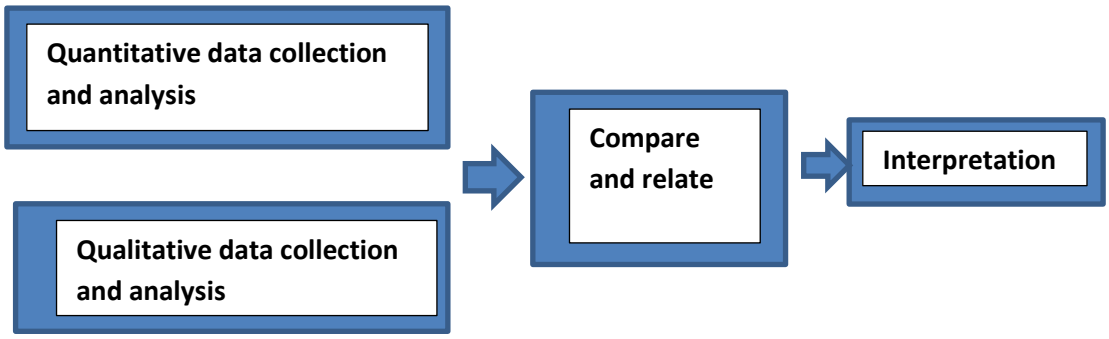


Figure 4.1: The convergent parallel design paradigm.

Convergent parallel design involves simultaneously collecting both qualitative and quantitative data, merging the data and using the results to understand and interpret the research problem (Creswell, 2012).

This study makes use of the transformative paradigm as its research approach is in the form of practical action research. A version of a learning study was used as the study involves only one teacher doing the teaching of the three different classes.

According to Creswell (2012, p. 577) “action research designs are systematic procedures used in educational settings when there is a problem to solve”. The author further elaborates on two types of action research, - practical action research and participatory action research. He further explains that in practical action research “a teacher or teachers are involved in research concerning problems in their classrooms in order to improve their students’ learning and also their own professional performance” (Creswell, 2012, p. 579). Both qualitative and quantitative data collection methods can be used in practical action research.

Participatory action research, on the other hand, is mostly used outside of education. Rather than focussing on individual teachers in solving immediate problems, it has a social and community orientation and focusses on research aimed at “emancipating” or “changing” society. It espouses many of the ideas used in school-based practical research and is mostly quantitative in data collection and analysis (Creswell, 2012, p. 579).

This study is an action research study informed by a learning study on the variation theory to done in a Grade 11 Mathematics classes at a school situated in one township. It was conducted according to the procedures of practical action research using a version of a learning study.

The purpose of practical action research is to research a specific school/classroom situation with the aim of improving practice (Schmuch, 1997). In practical action research, a small-scale research project which focusses on a specific issue is undertaken by an individual teacher with the help of his/her team of educators at a particular school or a group of teachers whose aim is to "research problems in their own classrooms so that they can improve their learners' learning and their own professional performance" (Creswell, 2012. p.579).

The aim of this study is to improve the performance of learners on the topic of functions and particularly to improve the researcher's teaching and other educators' teaching of this topic.

The action research spiral identified by Mills (2011, p.19) was followed where an area of focus was identified. This area of focus then forms the object of learning from the results of the diagnostic/pre-test. A version of learning study using the action research approach follows a research spiral. This action research spiral is seen in Figure 4.2.

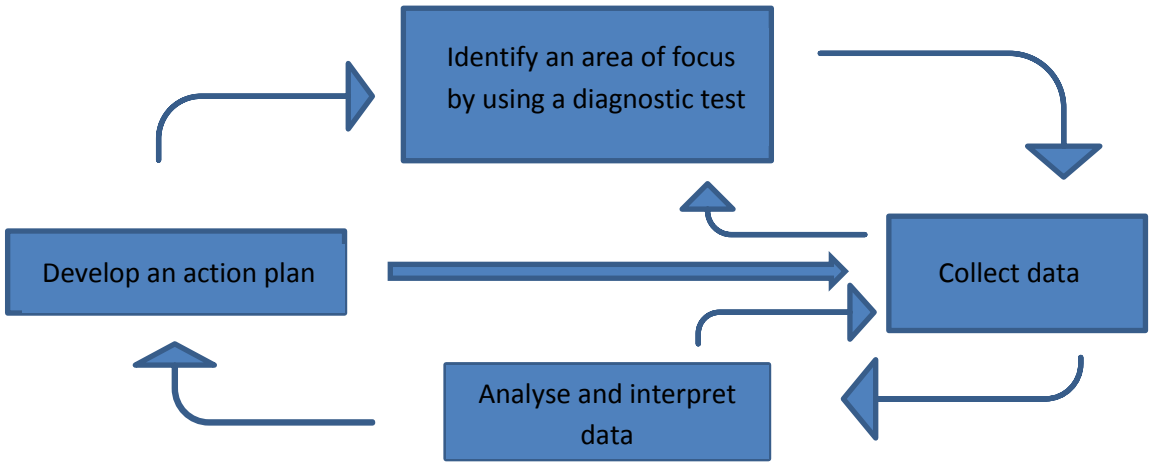


Figure 4.2: Action research spiral as identified by Mills (2011 p.19).

4.3 SAMPLE AND SETTING.

This study is conducted at a school situated in a township in an urban area with Grade 11 Mathematics classes. "A sample is a subgroup of the target population that the researcher

plans to study for the purpose of making a generalisation about the target population" (Creswell, 2012, p.627). Convenience sampling where the participants are easily accessible as they are learners who the researcher is teaching was applied in this study. The principal of the school is aware that this research on a topic on functions is conducted and the school is part of the Wits Maths Connect project. In practical action research, a small-scale research project which focusses on a specific issue is undertaken by an individual teacher with the help of his/her team of educators at a particular school. This small-scale practical action research was undertaken by the researcher in three Grade 11 Mathematics classes, supported by three of my colleagues. The three classes of learners consist of 85 learners, 35 of whom are boys and 50 are girls. The number of boys and girls in the three classes is shown in Table 4.1.

Class	Boys	Girls	Total
A	13	18	**31**
B	13	19	**32**
C	9	13	**22**
Total	**35**	**50**	**85**

Table 4.1: The sample of students in the study.

4.4 DATA COLLECTION STRATEGIES.

The data collection was undertaken in three parts. The first part consists of the use of the diagnostic/pre-test as a data collection tool given to the learners to answer the critical question 'what aspects of functions do learners find problematic?' This data collection tool lead to the selection of the object of learning that was used in the planning of the intervention lessons.

The second part of the study involves the intervention lessons as a data collection tool employed in the three groups to answer the research question 'to what extent does a version of a learning study framed by variation theory improve the Grade 11 learners' interpretation of algebraic functions?' The third part of the study comprises of the post-test that was given to the learners after each intervention lesson to find out whether the intervention lessons were successful or not.

Mixed methods design research is a procedure used in collecting and "mixing" both qualitative and quantitative methods of analysing data in a single study or a series of studies to understand a research problem (Creswell & Clark, 2011, p.179). In this study, the mixed methods research design is used to analyse the data quantitatively in order to best highlight and compare the different achievement levels of the learners in both the pre-test and post-test. The qualitative analysis of the data is used to ascertain the object of learning that informed the intervention process. Both the quantitative and qualitative data provide a better understanding of the research problem (Miles & Huberman, 1994).

4.4.1 THE DIAGNOSTIC/PRE-TEST.

Diagnostic tests are a way of collecting data about individuals and allow data to be analysed qualitatively and quantitatively (Creswell, 2012). The type of diagnostic test that was used is criterion referenced testing and "involves the interpretation of the individual's performance scores by measuring and interpreting the individual's score on a narrow domain of knowledge or skill" (Borg & Gall, 2007, p. 210).

This study focussed on the domains/features of the interpretation of functions and the learners' ability to flexibly move from one functional representation to another. This is explained by the 12 directions of equivalence of functional notation as depicted by Van Dyke & Craine (1997) in Figure 4.3.

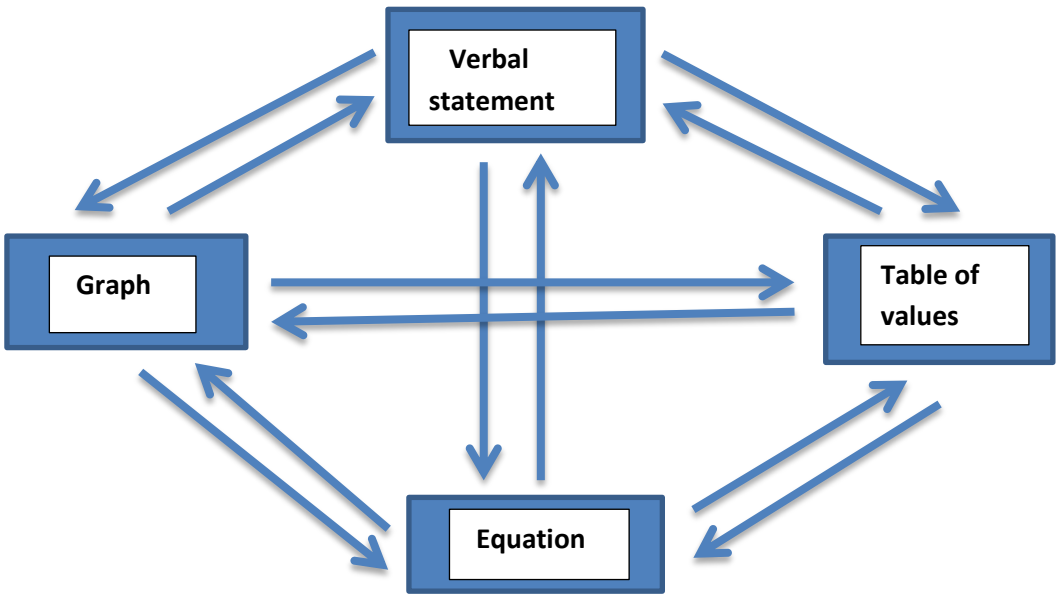


Figure 4.3: The 12 directions of equivalence of functional notation as depicted by Van Dyke & Craine (1997).

Diagnostic tests enable teachers to identify learners who are in need of remedial intervention and the skills or areas of the subject matter content on which the remediation/intervention should/could be effectively applied (Borg & Gall, 2007). Borg and Gall (2007) state that a diagnostic test can be used quantitatively and qualitatively and that when they are used quantitatively, a score of say 50 is assigned to the entire test and each test item is assigned a numeric score. When the tests are used qualitatively, each test item is assigned a domain or critical feature that the test concentrates on. Numeric scores were used to calculate the percentage improvement of the learners in each group between the pre-test and the post-test on each test item and the effects, if any, of the intervention programme that was undertaken in the three classes.

The four criteria that are used in a test to prove or show that a test is of sufficient quality to be used in educational research, is the test's objectivity, standard conditions of administration and scoring, standards for interpretation and fairness (Gall & Borg, 2007). By objectivity the authors refer to the administration and scoring of the tests in such a way that it is not to be distorted by the biases of the individual who administers the tests. The diagnostic test that was used is a closed-form test and is objective, because all the scores apply a scoring key which is universally used. This scoring is in accordance with the assessment standards as set out in the NCS assessment policy for the Grade 11 Mathematics curriculum. Assessment was in the form of the scoring of the test and the allocation of scores per test item in accordance with the memorandum, depicting the possible steps that the learners could follow in order to arrive at a particular/required answer (Appendix E).

Standard conditions of administration and scoring increases the objectivity of the test and includes the memorandum that specifies the procedures that should be followed to order to arrive at the desired answer. The test also specifies the amount of time that the individuals should take to complete the test. The memorandum was designed in such a way that it includes the alternative procedures that could possibly be taken for a specific test item if more than one such procedure is possible.

These procedures include consistency in administration and scoring as they stipulate the scores for each test item which is included in the features of the test itself and what the objectives of the test are. Standards for interpretation imply that the tests are scored according to a set of norms and criterion set for the test as the criterion of the diagnostic test set out above (Borg & Gall, 2007). By fairness, it is implied that different individuals of equal ability

and from different subgroups, for instance males and females, should earn the same scores on each item of the test irrespective of their sexual orientation and subgroup (Borg & Gall, 2007).

As a starting point, the learners wrote a pre-test that was prepared as a diagnostic test and was approved by my supervisor (Appendix D). This pre-test was designed according to the requirements of the National Curriculum Statement for the teaching of concepts on functions and complied with the objectivity, standard conditions of administration and scoring, standards for interpretation and fairness, as stated by Gall and Borg (2007).

The first part of the study consists of data that was collected from this pre-test which was analysed both qualitatively and quantitatively. An object of learning was then derived from the results of the diagnostic/pre-test and a process of intervention undertaken through the designing of a lesson plan that was focussed at addressing this object of learning.

The objective of the pre-test is to serve as a diagnostic process in order to ascertain which aspects of functions learners found most problematic. In developing the diagnostic assessment instrument/pre-test, learners' prescribed textbooks, the National Curriculum Statement and past examination question papers were used as a guide to the type of questions that the learners were faced with in the examination. This diagnostic pre-test consisted of five questions.

Question 1 consisted of two parts and required learners to use the appropriate form of the equation of the quadratic function to find the equation of the parabola where the parabola was given in graphical form. This question required learners to use the dual nature of functions to interpret the structural view by using the operational processes on the structural view.

In part one of the first question, the parabolic graph was given with the turning point and the $y-$intercept. The question required the learners to use the form of the quadratic function $y = a(x-p)^2 + q$. In part two of the first question, the parabolic graph was given with the x-intercepts and another point and the learners were expected to use the form of the quadratic function $y = a(x-x_1)(x-x_2)$ in order to find the equation of the parabola.

Question 2 required learners to apply their knowledge of multiple representations by moving from the equation form of the function to the graphical form. This question consisted of three parts. In part one, the learners were expected to sketch the graphs of the exponential function using transformation processes. Part two required learners to use the interpretation of

functions by stating what the domain and range of each function were. Part three of the question required learners to interpret the transformation that took place by giving an explanation of how the given transformations had an effect on the first exponential function drawn.

In question 3 the learners had to show their knowledge of the multiple representations of algebraic functions by matching the algebraic graphs in one column to the appropriate representation of the graphs in another column. The first column consisted of the graphical representations of the algebraic functions which were to be matched to either the verbal form, table form or equation form that appeared in the second column.

Question 4 consisted of three parts and required learners to apply their knowledge of functions.. Part one required learners to interpret algebraic functions given in contextualised situations using substitution to answer the question. Part two required learners to use the appropriate substitution to show that they could interpret the situation and use the appropriate substitutions to show their understanding of the functional notation of algebraic functions. Part three required learners to interpret the function using substitution and their ability to apply the notion of a function in everyday life.

Question 5 was a typical examination question and consisted of seven parts. Part one required learners to use their knowledge of functions by using the dual nature of functions to determine the equation of a parabolic function from the given graph. Part two required learners to interpret the function. The interpretation of the function was done by calculating the distance between the x- intercepts of the parabolic function. In part three, learners were expected to interpret the parabolic function and determine the vertical distance between two points. Part four required learners to determine the trend at which the function was decreasing by finding the interval for which values of x the given algebraic functions decreased. Part five required learners to interpret the trend where one function was greater than the other one. Part six required learners to interpret the function by calculating the vertical distance between two given point. The last part of the questions required learners to interpret the functional notation of the given algebraic graphs.

Three groups of learners were also given the test immediately after the intervention lesson was undergone. The answer sheets together with the question papers were collected from the learners after each group had written the pre-test. The answer sheets were marked according

to the memorandum as it appears in Appendix E and moderated by the teachers observing the intervention lessons.

Briefly, the analysis of the pre-test revealed that there were a lot of errors including the errors that correspond to the errors found by previous research. These errors and some of the learner responses to these questions are discussed in more detail in Chapter 6 after the analysis of the pre-test and the post-test is done. These errors include learners' inability to interpret the transformation of functions, plot graphs of functions, flexibly move between functional representations (verbal, equation, tabular and equation), correctly apply the functional notation, use the dual nature of functions and use the appropriate form of the quadratic function to generate the equation of the function. Given the extent of errors, it became important to focus the study and select only some things for deliberate attention.

Focus was place on the errors that stood out the most and errors that resulted in the learners' inability to correctly answer most questions. The errors that were common were that the learners had difficulties in identifying the appropriate form of the quadratic equation to find the equation of the parabolic/quadratic function. The details of the test analysis appear in Chapter 6.

The results from the diagnostic test enabled the choice of the object of learning and thus informed the route that should be taken when planning for the intervention lessons. The planning of the intervention lessons is discussed in the next section and the details are discussed in Chapter 5. The comprehensive PowerPoint lessons appear in Appendix I.

4.4.2 THE INTERVENTION LESSONS.

The learners were divided into three groups and are referred to as Groups A, B and C. These groups form the sample of this study. All intervention lessons took place in the computer laboratories using PowerPoint on the computer and the chalkboard for further explanation of the concepts. The intervention lessons were video recorded and audio recorded so as to provide a full record of the lessons. The intervention lessons are discussed in the following chapter.

The videotapes were used to ensure that there is an accurate record of the lessons for reflection purposes in preparation for the planning of the next lesson(s). The videotapes together with the notes the observers made when they observed were used to analyse the lessons using the observation schedule as a guide and to recommend the changes that could

be made to the lessons. The details of the intervention lessons are discussed in the next chapter where it is shown how the observation schedules were used to analyse the intervention lessons that tried to address the object of learning and how the theory of variation was implemented.

The intervention lessons were planned based on the results of the pre-test that appears in Appendix D according. The pre-test was assessed according to the memorandum in Appendix E. The analysis of the pre-test which is explored in Chapter 6 shows whether the learners were able to discern and interpret the features of a quadratic function or not. The identification and interpretation of the features of a quadratic function involve the use of the appropriate form of the equation to find the equation of the parabolic function. This was explored in the intervention lessons with the help of the analysis of the post-test.

The difficulties that the learners experienced with the identification of the appropriate form of the quadratic function to use to find the equation of the given parabolic function, is discussed in Chapter 6. The intervention lessons concentrated on providing leaners with opportunities to discern the object of learning which included what form of the quadratic equation should to be used and under what conditions it could be applied. The full versions of the intervention lessons are discussed in the following chapter.

The intervention process consists of seven parts. The first part of the intervention process involves the planning of the first lesson to be conducted with the first group. The team of educators who were involved in the research formed part of the planning group. The second part of the intervention process involved the presentation of the lesson to the first group. The third part of the intervention process involved giving the first group a test based on the lesson. This test constituted the post-test as it appears in Appendix F.

The fourth part of the intervention process involved the marking of the post-test according to the memorandum in Appendix G. The fifth part of the intervention process involved moderation of the marked post-test by the panel of teachers involved in the research. The sixth part of the intervention process involved the analysis of the lessons using the summary of the key aspects of the videotaped lessons that appear in Table 1 together with the observation schedule as it appears in Table 2 of the appendix. The analysis was done by all the members of the panel with the researcher also being a part of the panel.. The seventh part of the intervention process involved the reflecting on the lesson and refining it in preparation for the second group B.

The cycle is then repeated from the second part of the intervention process for the second group B and the third group C.

4.4.3 THE POST-TEST.

The post-test as the third data collection strategy consists of the learners writing a post-test to ascertain whether the intervention lessons were effective or not. The results of the post-test gave answers to the critical question 'to what extent does a version of a learning study framed by variation theory improve the Grade 11 learners' interpretation of algebraic functions?'

As it was discovered from the results of the pre-test that the learners had difficulties with the identification of the appropriate form of the quadratic function to be used in order to find the equation of the given parabolic function, the post-test consisted of the same questions from the pre-test that required learners to use this knowledge. Questions 1.1 and 1.2 were unchanged and an additional question which was similar to question 5.1 of the pre-test was added. The questions were renamed question 1, 2 and 3 to prevent learners from recognising that they were the same questions as in the pre-test.

Question 1 required learners to use the appropriate form of the equation of the quadratic function to find the equation of the parabola when the parabola was given in graphical form. This question required learners to use the dual nature of functions to interpret the structural view of the function by using the operational processes on the structural view.

In the first question, the parabolic graph was given with the turning point and the $y-$intercept and required the learners to use the form of the quadratic function $y = a(x-p)^2 + q$.

In the second question, the parabolic graph was given with the x-intercepts and another point and the learners were expected to use the form of the quadratic function defined by: $y = a(x-x_1)(x-x_2)$ in order to find the equation of the parabola.

Question 3 was given in verbal form and required learners to interpret the parabola as having three distinct points that were neither the x- intercepts of the parabola nor the turning point. In order to find the equation of the parabola, the learners were expected to use the equation of the parabola: $y = ax^2 + bx + c$ by substituting the three given points to end up with three quadratic equations in three unknowns. In order for the learners to find the values of the

parameters a, b and c, they had to use algebraic manipulations to solve the three equations simultaneously.

Before any data could be collected from the learners, some ethical considerations had to be met.

4.5 ETHICAL CONSIDERATIONS.

Since the sample consisted of learners whose ages range from 16 years to 20 years, consent letters were given to the parents of the learners and as well as consent letters given to the learners. These are available in appendices A and B. The parents were given the liberty to choose whether they allowed their children to participate in the research or not. It was explained to the learners and their parents that the research results would not affect the learners' academic marks. Learners' anonymity would be respected. Learners' confidentiality would also be guaranteed by using coding of learners in order to guarantee learners' anonymity and confidentiality (Creswell, 2012; Opie, 2004).

The consent letters contained the explanation that the aim of the research was to find ways and means of improving the teaching and learning of functions in Mathematics in Grade 11 in the school and particularly to enhance learner participation and performance regarding this topic. The consent letters also gave the learners and their parents' information about exactly what the process of the research would involve. This includes that the researcher, who is their teacher and registered with the University of the Witwatersrand, would require the learners to take a Mathematics test on functions, conduct the lesson in their classes and audio record the lessons and to video record a lesson on functions in the classes. The process also includes an interview the learner if selected and an audio recording of the interview if the learner was chosen for the interview.

The letters also gave the parents and the learners the information that the results from the learners' tests and the videotape will be used to study the teacher's teaching and to study and improve on the teacher's teaching ability. The researcher, her supervisor and the teachers involved with the observation of the lessons and the videotapes would see the learners' tests and the videotaped lessons. The teachers who would be involved with the planning and analysis of the lessons and tests would be given letters to pledge their willingness to keep the

anonymity and confidentiality of this information as seen in Appendix C. The researcher was would write a report which would be discussed at conferences and in journal articles. The learners' results, videotapes and, if they were selected, the learners' interview information, would be used for the duration of the project and stored for a further five years. Thereafter the content would be destroyed.

The letters explained that the videotape would be used to provide a full record of the lessons. The researcher would use the videotapes to ensure that she had an accurate record of the classes she teaches and thereafter observes. The researcher and observers were going to use the videotapes together with the notes they made during their observation of the lessons to analyse the lessons and to recommend the changes that could be made to the lessons for the classes that followed each lesson. The videotape would focus on the teacher who is also the researcher in this case. The camera person would, however possible, focus the video away from learners' faces, to help secure confidentiality. The videotape would only be used for the purposes of the research project.

The consent letters informed the parents and the learners that the names of the learners would not appear in any reports or articles. Pseudonyms were to be used instead and also that the research was completely separate from the learner's school work. All information obtained for research purposes would not affect the learners' assessment in school. There would also be no problem if the learners do not want to take part in the research. If they chose not to participate, this would not affect them in any way and if they decided that they no longer wanted to continue participating in the study, they were free to withdraw at any time. They learners could simply inform the researcher who would then make the necessary changes to her research process. Learners who decided not to be part of the research would not be victimised or excluded from the process, but the video camera would be directed away from them as they would be put in a position in the class where this could be done.

4.6 METHODS USED IN THE STUDY.

The study uses the mixed methods research design. Mixed methods design research is a procedure used in collecting and "mixing" both qualitative and quantitative methods of analysing data in a single study or a series of studies to understand a research problem (Creswell & Clark, 2011, p.179). In this study, the mixed methods research design is used to analyse the data quantitatively in order to best highlight and compare the different

achievement levels of the learners in both the pre-test and post-test. The qualitative analysis of the data is used to ascertain the object of learning that informed the intervention process. Both the quantitative and qualitative data provide a better understanding of the research problem (Miles & Huberman, 1994).

Quantitative and qualitative methods of data analysis are used in this study. Quantitative methods of data analysis are used to see the bigger picture of the aspects of functions that the learners find most problematic. Quantitative analysis of data involves "summed scores of individuals added over several questions that measure the same variable where individual items are added to compute an overall score for a variable" (Creswell, 2012, p.178). Qualitative methods of data collection are used to provide a more detailed and much richer description of the aspects of functions that the learners encountered and found most problematic (Dayer, 1995).

A "table of specifications" of the learning area objectives are drawn, "identifying" the different levels of "understanding, application, analysis and synthesis" (Popham, 1991, p. 13). The table of specifications for the pre- test appear in Table 3 after the appendices. The learners' skills in manipulating the given information in order to arrive at the required responses are also the focus of the diagnostic tests. Short answer items are included in order to give the researcher information about learners' thinking processes in producing the answers (Greeno, 1978).

The table of specification for the observation schedule includes the types of examples used by the teacher and the representations used at the start of the lesson. This is used for further explanation by the teacher and for further work by the learners. It also includes the kind of words that were used in explaining the key concepts of the lesson and the aids used in explaining these key concepts. The types of activities the learners were engaged in during the lesson include whether they were listening, asking questions, answering questions, copying from the board, solving the problems and writing their solutions, discussing their thinking with other learners in groups and explaining their thinking to the class or doing exercises. The time spent on each of these activities was also recorded.

4.7 CHALLENGES DURING DATA COLLECTION.

Out of a total of 120 learners who were present at the beginning of the research when writing the pre-test and comprise of all the learners of my Grade 11 Mathematics class, the data of only 85 of these learners was analysed. The reason for this is that some of the learners either did not write the pre-test or the post-test due to absenteeism and some not consenting to be part of the study. As the ethical considerations state that no learner should be penalised nor prevented from being part of the study, all learners were allowed to write the tests and attend the intervention lessons but only the data analysed for the study was of the learners who consented to take part in the study and also wrote both the pre-test and the post-test.

4.8 VALIDITY.

Test validity is defined as the "degree to which evidence and theory support the interpretation of test scores and observation schedules entailed by proposed uses of tests and observation schedules (Sandoval, J., Frisby, C. L., Geisinger, R. F., Ramos-Grenier, J. & Scheuneman, J.D.1998, p.143). The validity of a test is measured through its ability to help individuals administering it to be able to measure particular characteristics or attributes that it was set to measure (Sandoval et.al., 1998, p.143). La Marca (2001) states that there are five types of evidence required for demonstrating the validity of a test and the observation schedule. The types of evidence are the tests/observation schedules content, response processes, internal structure, relationship to other variables and consequences of testing and observation schedules. Evidence from the test/observation schedules refers to the relationship between tests'/observation schedules' content and the criterions/ features or domains that it is supposed to measure (La Marca, 2001). The content-related validity as it will be used in the diagnostic test/observation schedule will be a representative of the Mathematics learning area content as it appears in the work schedule and curriculum for the Grade 11 Mathematic syllabus on functions (the use of the appropriate form of the equation of the quadratic function to find the equation of the parabola).. This leads to the identification of the validity of the test items.

Evidence from the responses refer to the evidence that the processes actually engaged by the test and that the observation schedule are consistent with a particular construct. In this study,

evidence refers to the learners' responses to questions on functions and their ability to use the appropriate form of the quadratic equation in finding the equation of the parabola. Evidence from internal structures refers to the ability of the learners to answer similar test items in the same way.

4.9 RELIABILITY.

The reliability of the test/observation schedule means the ability of these instruments to yield the same results for a respondent across repeated administrations (Borg & Gall, 1989). The reliability of the instruments sheds light on the question "does the test/observation schedule always yield the same score for an individual when it is administered on several occasions?" (Borg & Gall, 1989, p.170). The several occasions referred to in this study are the diagnostic test in the form of a pre-test, the observation from the intervention that took place after the results of the diagnostic test, and the post-test that was administered after the intervention.

The results of the pre-test and those of the post-test are not the same, because the post-test was written after the intervention process was performed.

Credibility involves trustworthiness, dependability, confirmability and transferability. Opie (2004, pp.72-72) explains credibility as something that can be obtained through the explanations of the data gathering procedures, data to be represented in such a way that anyone could be able to re-analyse it, data that does not fit in with the intentions of the researcher is included, biases acknowledged, relationships between claims and supporting evidence are clearly expressed, primary data is distinguished from secondary data, interpretation is clearly distinguished from description, information on what transpired during the study is fully related and reliable procedures are used to check the quality of the data.

Credibility can be obtained by means of the process of triangulation where data gathering procedures are separately or independently applied to the same subjects in such a way that statements can be made about the subjects that are separately warranted. Different researchers obtaining the same data set of the said subjects will ensure "triangulation by researchers" while the same data set obtained from the same subject, but using different procedures will ensure "triangulation by procedures" through the tests and the observation schedules (Opie, 2004, p.73).

In this study, different people on the panel that was involved in the study moderated the marking of the pre-test and the post-test and obtained more-or-less similar data results. Therefore, thus triangulation by procedures obtained though the tests and observation schedules were used to obtain the credibility of the study.

4.10 LIMITATIONS OF THE STUDY.

It is important to take cognisance of the limitations of using an action research approach and one such limitation is the issue of generalizability. Action research produces results which are not generalizable, but the ideas or conclusions derived from another person's action research process can always be tried out by other persons in their own practice. This can be done to see if the results work for them as one person's improvement can be another person's deterioration (Hamilton, 1981).

Borg and Gall (2007) assert that the correlation between the pre-test and the post-test could be positive and this does not necessarily mean that the remediation/intervention programme was a success as opposed to the experimental research where there is a control group, which does not go through the intervention/remedial programme and the experimental group that goes through the remedial/intervention programme for cause and effect relationships to be easily measurable. Since the sample size is small, the correlation is not calculated, but the results of the two tests are compared using percentages of improvement from the pre-test to the post-test.

The reliability of a test and the findings of the research depend on the sample from which the test scores and the findings were derived (Baker, 2002). This means that if the researcher uses a different population from the one used in the studies' development, its reliability and the characteristics of the test items might be different in that the test items might be too easy or too difficult for some individuals as individuals are different in their comprehension and abilities of retention of information. The level of ability of the learners might impact negatively or positively to the measurement error or % improvement.

4.11 SUMMARY.

This chapter discusses the research paradigm employed in the study, the sample and setting of the study is discussed, the research methodology is outlined, the tests' objectivity, and standard conditions of administration and scoring, standards for interpretation and fairness and the dilemmas that the study encountered are highlighted. The data collection instruments, data collection techniques and methods of analysis are outlined. The sample and setting of the study are described and the ethical issues taken into consideration are discussed. The questions of rigor which include validity and reliability are considered. The next chapter further elaborates on the data analysis and the results of the study.

CHAPTER FIVE.

THE ANALYSIS AND DISCUSSION OF THE INTERVENTION LESSONS.

5.1 INTRODUCTION.

This chapter focuses on the intervention lessons, how and why they were constructed as they were, what ensued during the three intervention lessons both from the researcher and the learners' perspective and actions. How and why the lessons were aligned with the variation theory is explained. The discussions with the observers of the intervention lessons, the reflections made by the research team after each lesson and how and why these reflections influenced the changes made to the lessons are discussed.

5.2 THE INTERVENTION LESSONS.

The lessons were designed in the pursuit of the discernment of the object of learning by the learners. The object of learning which the intervention lessons were addressing is 'discerning the features of a quadratic function and the identification and use of the appropriate form of the equation to generate the equation of the quadratic function from the graph(s) or any information about the quadratic function'. Audio-visual materials were used in the collection of data during the intervention lessons. One of the advantages of audio-visual materials is that the video data can always be re-examined for accurate recollection of what transpired during the intervention lessons (Opie, 2004). The videotapes show all the written information that was presented in the lessons including the different approaches and changes that were made to the lessons. This information can be documented and accurately reported on.

The function machine acts as a cognitive root which includes both the object status and the process aspect leading to an output mode. In order to dispel the notion that all functions are given by a formula, the ideas emanating from the use of the input (object) followed by→process (action) thereafter → output (outcome) as alluded to by Tall, et al. (1991) and Sfard (1991) was used as a tool to structure the lesson plans. This tool is used by presenting

the input (object), process (action) and output (outcome) in different representations during the intervention lessons as presented in the tables of the intervention lessons (Tables 5.1, 5.2 and 5.3).

Variation theory was applied using the patterns of variation theory, namely contrast, separation, generalisation, fusion and simultaneity. The object of learning that all the intervention lessons focussed on was 'discerning the features of a quadratic function and identification and use of the appropriate form of the equation to generate the equation of the quadratic function from the graph(s) or any information about the quadratic function'**.**

The input (object), process (operation) and output (outcome) were all attended to. The detailed hard copy of the intervention lessons are presented in Appendix I. The PowerPoint presentation was presented to learners using animation where each sentence of the PowerPoint was presented separately from another by pressing the enter button. Each sentence was revealed on its own. The PowerPoint presentation and the videotapes of the intervention lessons are saved and available on a USB flash stick. During the intervention lessons, three observers recorded and documented the lessons according to the observation schedule that is represented in Table 1 Appendix I.

The modifications of the lessons were notably at the beginning of the lessons after consultation and reflection with the three observers after each lesson. The feedback from the observers from the review of the video recordings of the participation of the learners during each lesson, the questions that the learners asked during the lessons together with the results of the post-test played an important part in the changes made for the intervention lessons.

From the review of the video and audio recordings it was discovered that the learners were constantly requesting to view the slide that showed the different forms of the quadratic equation again. Especially when they were asked which equation should be used when they had to give the equation appropriate for a particular question asked by the teacher (the researcher).

This line of questioning by the learners prompted the review of the lessons so as to avoid similar questions arising from the other groups. The importance of simultaneity was decided upon for the first lesson in order for learners to fully discern what was kept the same and what was varied from the different forms of the three equations of the quadratic function. As the lessons progressed, the changes made to the respective lessons were based on the learner

participation in the lessons and the results of the post-test after each lesson. The intervention lessons are fully described in the following section.

5.2.1 INTERVENTION LESSON 1 (GROUP A).

With the object of learning being 'discerning the features of a quadratic function and identification and use of the appropriate form of the equation to generate the equation of the quadratic function from the graph(s) or any information about the quadratic function' as the focus of the lessons, the first intervention lesson was performed with Group A and named Intervention Lesson 1.

The introduction of the lesson comprises of the general form of the quadratic function $y = ax^2 + bx + c$ in equation form and the learners were required to identify the key features of the function emanating from this algebraic representation. At first, the learners did not understand what was meant by the key features until they were given an example of the importance of the parameter 'a' in this general form of the quadratic function.

In order to focus attention on the importance of the parameter 'a', separation was used as the pattern of variation where the learners were required to give the shape of the parabola where examples of graphs of different signs of the value of the parameter 'a' were shown on the chalkboard and concluded in the slide. This enabled learners to discern the effect of the sign of the parameter 'a' on the shape of the parabola. The learners were then able to conclude that if '$a > 0$', the parabola was concave upwards and that if '$a < 0$', the parabola was concave downwards. The parameters 'b' and 'c' were kept constant when this variance of the parameter 'a' was done.

After these examples, where only the parameter 'a' was discussed, the learners were asked what the parameter 'c' represented and they answered in unison by giving the other critical feature as represented by the parameter 'c' which is the y-intercept of the parabola. After the general form of the quadratic function was discussed with the learners in this form, they were given a different form of the general quadratic function. They were presented with $y = a(x - x_1)(x - x_2)$ from which they were required to extract the key features of the parabola. The key features of a parabola that the learners extracted were the x-intercepts of the parabola. These features from this form of the parabolic function were then revealed from the slide on the computer.

The next form of the general equation of the quadratic function was then presented to the learners on the next slide. The general form of the quadratic function was revealed from the slide as $y = a(x - p)^2 + q$. The learners were required to extract the key features from this form of the equation of the parabola which they identified as the turning point and the axis of symmetry. The learners' response was then confirmed by revealing the correct answers from this form of the parabolic function on the slide on the computer.

The pattern of variation that was applied was that of separation by keeping the equations of the parabola in their general forms and varying the forms of the general equations.

From the general forms of the equations of the parabola, the learners were then given a specific equation of a parabolic function in the form: $y = x^2 - 4x + 3$ which was in line with the form of the general equation $y = ax^2 + bx + c$. Learners were asked to extract the key features of the specific equation of a parabola as they did with the general form. The specific key features were revealed in the slide on the computer.

The learners were then asked to factorise the specific equation of the parabolic function and they found the answer of: $y = (x - 1)(x - 3)$ which was the form of the second general equation of the parabola that the learners were exposed to at the beginning of the lesson. The learners could extract the specific key features of the parabola from be the specific equation of the parabola. The key features that were revealed from the slide were the x-intercepts of $x = 1$ and $x = 3$.

The learners were then required to complete the square on the standard form of the same specific equation of the quadratic function which resulted in the following form of the equation: $y = (x - 2)^2 - 1$ which was the form of the third general equation of the parabola that the learners were exposed to at the beginning of the lesson. They were required to interpret the key features emanating from this form of the quadratic function which are the turning point and the axis of symmetry of the parabola.

The learners were asked to plot the points they had discovered from this specific equation of a parabola after completing the algebraic manipulations that they were required to perform to arrive at the different forms of the same parabolic function. They were asked to join the points they had plotted in their books and compare their sketches with the sketch that was revealed from the slide. Two more examples were revealed from the slides and the same process that was discussed above was repeated.

The learners were then given the graphical representation of the same specific function and they were required to find the equation of the parabolic functions. The first graphical representation of the function had the points of the x-intercepts and another point labelled on the quadratic function. The second graphical representation of the same parabolic graph was presented to the learners with the turning point and another point on the parabola and the third graphical representation of the same function had three points labelled on it. The learners were required to find the equations of these graphical representations.

The learners requested the teacher to go back to the slides that contained the three forms of the quadratic function before they could find the equations of each one of the given graphical representations. After the learners found the equations of the first three graphical representations of the first graph, one learner asked why the equations of the three different graphical representations were the same and yet they used three different general forms of the equation of the parabola. The presenter then asked the learners to look at the three graphical representations and tell her whether the graphs were the same or not.

The learners discovered that the three graphs were the same and only the labelled points were different.

Another two different examples of graphical representation of the graphs were given to the learners. In each example the same parabolic graph was presented with the different points labelled on the graphs as discussed above. In each case the learners requested the presenter to go back to the slides where the general forms were given. From each graphical representation with the three different points labelled on the graphs, the learners found the same equation. The learners' results were confirmed by the appearance of the same equations from the PowerPoint slides.

The notion of variance using the dimension of variation prominent in Lesson 1 was separation. The specific equation of the quadratic function was kept the same and the different forms from the algebraic manipulations of the specific equation of the parabola were the result. The algebraic manipulations of factorisation and completing a square were performed on the same specific equation of the parabola to produce the different forms of the same specific equation, which forms the pattern of variation called separation as the equation was kept the same and the forms of the equation were kept the same.

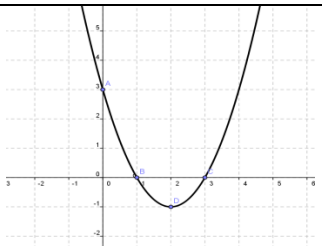
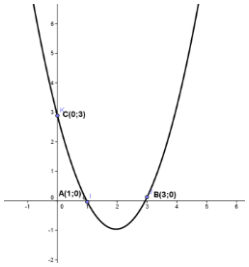
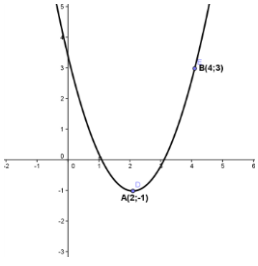
With regards to the graphical representations of the function, the pattern of variation that was applied in Lesson 1 was that of separation. The same function was presented to the learners, but the points labelled on the graphical representation were varied. This variation of the points given in the graphical representation of the same parabola required learners to be able to use the appropriate form of the equation of the quadratic function in order to find the equation of the same parabola.

A detailed table of the intervention Lesson 1 for Group A is presented in the following section.

5.2.2 A TABLE OF THE INTERVENTION LESSON 1 FOR GROUP A.

The detailed table of the intervention lesson for Group A adapted from the PowerPoint presentation in Appendix I is presented in Table 5.1 below.

Object of learning. **Discerning the key features of a quadratic function, specifically identification and use of the appropriate form of the equation to generate the equation of the quadratic function from the graph(s) or other information about the quadratic function.**		
LESSON 1 (GROUP A)	**DURATION ≈ 60 MINUTES.**	
Input (Object)	**Process (Action)**	**Output**
General form of the quadratic function: $y = ax^2 + bx + c$	Identify the key features of the function related to the shape of the quadratic function from this general form of the quadratic function; The shape of the parabola and the y-intercept thereof.	Shape of the parabola: If $a > 0$ ∪ concave upwards. If $a < 0$, the shape of the parabola is concave downwards. c Is the y-intercept of the parabolic function.
$y = a(x - x_1)(x - x_2)$	Identify the key features of the function from this form of the equation of a parabola.	The x −intercepts of the parabola are $x = x_1$ and $x = x_2$
$y = a(x - p)^2 + q$	Identify the key feature of the function from this form of the equation of the function.	The turning point of the parabola = $(p; q)$ and the axis of symmetry is $x = p$.
Specific equation: $y = x^2 - 4x + 3$	Identify the key features of the function.	Shape of the parabola: $a > 0; a = 1$ ∴ ∪ concave upwards , 3 Is the y-intercept of the parabolic function.

	Factorising the quadratic function: $y = (x-1)(x-3)$. For the x −intercept put $y = 0$: $0 = (x-1)(x-3)$ $\therefore x = 1 \; or \; x = 3$.	The x −intercepts of the parabola are: $x = 1$ and $x = 3$.
	Completing the square: $y = (x-2)^2 - 1$	The turning point of the parabola is: $(p;q) = (2;-1)$ and the axis of symmetry is $x = 2$.
	Sketching the graph of $y = x^2 - 4x + 3$ From the above information.	
Two more specific equations are given.	The same process as above was followed for each of the specific equations of functions dealt with.	The graphs of the equations were sketched.
. 	Identify the key features of the parabolic function: y −intercept = 3 and the x − intercepts are: $x = 1 \; and \; x = 3$. The form of the quadratic function to be used: $y = a(x-x_1)(x-x_2)$ Substituting the critical values: $3 = a(x-1)(x-3)$ $3 = 3a$ $1 = a$	Equation of the parabolic function is: $y = x^2 - 4x + 3$
	Identify the key features of the parabolic function: $(2;-1)$ is the turning point and $(4;3)$ another point on the parabola. The form of the quadratic function to be used: $y = a(x-p)^2 + q$ Substituting the critical values: $3 = a(x-2)^2 - 1$ $4 = 4a$ $1 = a$	Equation of the parabolic function is: $y = (x-2)^2 - 1$ $y = x^2 - 4x + 4 - 1$ $y = x^2 - 4x + 3$
Given any three points on the parabola which are not classified as any of the features of the parabola.	**Identifying which form of the quadratic function is applicable in this case by solving three functions simultaneously.**	**The final equation of the parabolic function.**

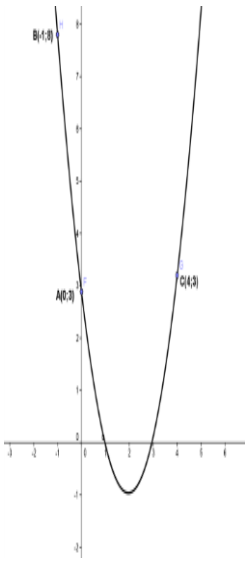
	Identify the key features of the parabolic function: Three points are given: $(0;3), (4;3)$ and $(-1;8)$. The form of the quadratic function to be used: $y = ax^2 + bx + c$ Solving the three functions simultaneous by substituting the given points on the graph one at a time: For point $(0;3)$: $3 = a(0)^2 + b(0) + c$ $3 = c$ … (1) For point (4;3): $3 = a(4)^2 + b(4) + 3$ $3 = a16 + 4b + 3$ $0 = 4a + b$ … (2) For point $(-1;8)$: $8 = a(-1)^2 + b(-1) + 3$ $5 = a - b$ … (3) (2) + (3): $5 = 5a$ $1 = a$ … (4) (4) in (3): $5 = 1 - b$ $-4 = b$	Equation of the parabolic function is: $y = x^2 - 4x + 3$
Two more parabolic graphs were presented.	The same process ensued for the two parabolic functions as above.	The equations of the parabolic graphs were generated by the learners and confirmed in the slide as fully presented in the appendix I.

Table 5.1: A detailed table of the intervention lesson for Group A adapted from the PowerPoint presentation in Appendix I.

After the lesson, the learners were immediately given a post-test (Appendix F) to find out whether the intervention lesson was a success or not. The pre-test consisted of three questions. The first two were the same questions as the ones given for the pre-test and a third question was given where three points on a parabolic graph were given and the learners were required to find the equation of that parabola. The answer sheets and question papers were collected from the learners. The answer sheets were marked according to the memorandum, as it appears in Appendix G, by myself and moderated by the panel of Mathematics educators taking part in the research. The results of the post-test are discussed in the next chapter.

5.2.3 REFLECTION AND DISCUSSION OF LESSON 1.

Three colleagues from the department of Mathematics who were observing the lesson and making notes according to the observation schedule in Table 1 of Appendix I joined me in the computer room to analyse the results of the tests and to discuss and reflect on the lesson. The

videotape was also used to analyse the lesson and comments were made by the observers while referring to their observation sheets. From the videotaped lesson it was noticed that the learners frequently asked the presenter to go back to the previous slides which contained the different forms of the general quadratic functions. The learners seemed to forget the different forms of the parabolic functions.

The learners constantly needed to refer back to the different forms of the equations of the parabola when they had to answer questions requiring them to find the equation of a parabola when they were given a sketch of the parabola with certain key features reflected. From the feedback from the panel and my reflection on the lesson for Group A as well as the line of questioning that the learners embarked on, importance of simultaneity to be built into the lessons became apparent.

It was suggested that the lesson could be improved for the next group by showing the learners all the three different forms of the general quadratic function simultaneously at the beginning of the lesson. The three forms could also be written on the chalkboard for faster and convenient accessibility. The observations made by the team was that the separation of the forms of the equations of the general quadratic equations confused the learners into thinking that the different forms of the quadratic equation of a parabola should always give different equations. The other observation made by the panel was that the learners could not remember the different forms of the quadratic function when they needed to apply them as they were on separate slides of the presentation.

Since these equations were discussed separately, the learners seemed to view the forms of the quadratic function as separate entities and therefore found it disturbing that the equations of the represented parabolic graphs were the same. This confusion of the learners necessitated the application of the pattern of variation called simultaneity within the theory of variation. The incorporation of this pattern of variation in the changing of the next lesson is discussed in the next section.

The results of the post-test, which are fully discussed in the next chapter, revealed that some of the learners still used the incorrect form of the quadratic equation while answering the questions. Some of the learners could not associate the points labelled on the graphs with the appropriate forms of the general equations of the parabola.

The next lesson was then planned together with the observers incorporating their suggestions. The next group was then taught the following day with the alterations to the intervention Lesson 1 been made according to the reflections made by the panel and the researcher.

5.3 INTERVENTION LESSON 2 (GROUP B).

5.3.1 INTRODUCTION.

With the object of learning being ’discerning the features of a quadratic function and identification and use of the appropriate form of the equation to generate the equation of the quadratic function from the graph(s) or any information about the quadratic function’ as the focus of the lessons, the second intervention lesson was performed with Group B and named Intervention Lesson 2 for Group B.

Due to the difficulties that were experienced by the learners in the first lesson, the changes were made at the beginning of the lesson. In order to avoid going back to the slides that contained the forms of the general equations of the parabolic functions, these general forms were all written on the chalkboard and on the same slide at the beginning of the lesson.

The different forms of the general equations of the parabola were written on the first slide. The learners were then requested to mention the key features of the parabolic function represented by each of the forms of the equation of the quadratic function. From the forms of the general equations of the parabola, the learners were given the specific equation of a parabola. The learners seemed to have a problem with where the different forms of the equations of the parabola came from as one learner asked where the general forms came from and which points on the graphs should be substituted into which equation? This confusion was echoed by the learners’ agreement with these questions.

From these questions, it was evident that the learners needed to be taken through the derivation of the forms of the general equations of the parabola. Hopefully, the derivations of the general forms of the equations of the parabola would also answer the learner’s question of which points on the graph were appropriate for which form of the general equation of the parabola. This was then incorporated in the next lesson for Group C as will be discussed in the next section.

In this Group B as in Intervention Lesson 2, simultaneity was the pattern of variation that was added to the already existing pattern of variation that was prominent in lesson 1, namely separation. Simultaneity was applied at the beginning of the lesson where the three general forms of the quadratic function were simultaneously placed on the first slide and written on the chalkboard for the learners to experience the different forms of the quadratic function at the same time. The learners were asked to give the features of a parabola represented by each form of the quadratic function at the same time. The specific equations were given to the learners and the rest of the lesson progressed similarly to the Intervention Lesson 1 for Group A (Appendix I).

5.3.2 A TABLE OF THE INTERVENTION LESSON 2 FOR GROUP B.

The detailed table of the beginning of the intervention lesson for Group B adapted from the PowerPoint presentation in Appendix I is presented in table 5.2 below.

LESSON 2 (GROUP B) DURATION ≈ 60 MINUTES.		
Object of learning. **Discerning the key features of a quadratic function, specifically identification and use of the appropriate form of the equation to generate the equation of the quadratic function from the graph(s) or other information about the quadratic function.**		
Input (Object)	**Process (Action)**	**Output**
The three different forms of the quadratic equations were simultaneously presented to the learners. $y = ax^2 + bx + c$ $y = a(x - x_1)(x - x_2)$ $y = a(x - p)^2 + q$	The key features of each of the different forms were individually discussed with the learners. For $y = ax^2 + bx + c$ For $y = a(x - x_1)(x - x_2)$ For $y = a(x - p)^2 + q$	The key features of the parabola are: Shape of the parabola: If $a > 0$ ∪ concave upwards. If $a < 0$ ∩ concave downwards. c Is the y −intercept of the parabola. The key features are: The x −intercepts of the parabola namely $x = x_1$ and $x = x_2$. The critical features are: turning point of the parabola = $(p; q)$ and the axis of symmetry is $x = p$.
Specific equation: $y = x^2 - 4x + 3$	Identify the key features of the function.	Shape of the parabola: $a > 0; a = 1$ ∴ ∪ concave upwards , 3 Is the y −intercept of the parabola.

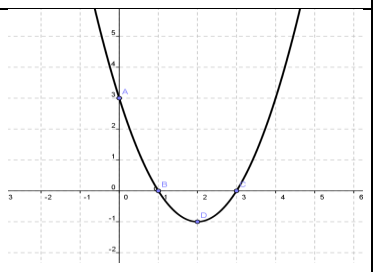
	Factorising the quadratic function: $y = (x-1)(x-3)$ For the x −intercept put $y = 0$: $0 = (x-1)(x-3)$ $\therefore x = 1 \; or \; x = 3$	The x −intercepts of the parabola are: $x = 1$ and $x = 3$
	Completing the square: $y = (x-2)^2 - 1$	The turning point of the parabola = $(2; -1)$ and the axis of symmetry is $x = 2$.
	Sketching the graph of $y = x^2 - 4x + 3$ From the above information.	
From this point onwards, the lesson 2 is the same as lesson 1.		

Table 5.2: A detailed table of the first part of the changed intervention lesson for Group B adapted from the PowerPoint presentation in Appendix I.

The same post-test was given to the learners immediately after the lesson. The answer sheets and question papers were collected from the learners. The answer sheets were marked using the memorandum as it appears in Appendix G by myself and moderated by the panel of Mathematics educators taking part in the research. The results of the post-test were discussed together with the review of the video and audio recordings of the lesson. The reflection of the second lesson resulted in the planning of the third lesson.

5.3.2 REFLECTION AND DISCUSSION OF LESSON 2.

As was the case after Lesson 1, three colleagues from the Department of Mathematics, who were observing the lessons and making notes according to the observation schedule in Table 5.2, joined me in the computer room to analyse the results of the tests of all the groups and to discuss and reflect on the two lessons. The learners' persistence in asking the question where

does this general form of the equation come from and how does one know which points on the graphs should be substituted into which equation inspired the team to reassess the introduction of the lesson.

This observation from the video recording and the audio recording of the lesson prompted the suggestion that the lesson be improved for the next group by beginning by showing the learners the three different forms of the specific quadratic function simultaneously and showing the learners how the forms of the general quadratic functions were derived. The results of the post-test which are discussed in the next chapter revealed that some of the learners even went as far as substituting the incorrect points of the given graph into the general forms of the equations of the parabola.

This was attributed to the fact that the learners could not associate the general forms of the equations with the particular points as they had no idea of how the general forms of the quadratic equations were derived. From the feedback of my colleagues and my reflection on the lesson and the results of the post-test, the importance of beginning the next lesson with the specific quadratic functions followed by the general forms of the quadratic functions became clear.

With these learner difficulties and questions in mind, the last lesson was planned and took on the form as described in the following section.

5.4 LESSON 3 (GROUP C).

5.4.1 INTRODUCTION.

With the object of learning being ’discerning the features of a quadratic function and identification and use of the appropriate form of the equation to generate the equation of the quadratic function from the graph(s) or any information about the quadratic function’ as the focus of the lessons, the third intervention lesson was performed with Group C and named Intervention Lesson 3.

Intervention Lesson 3 followed the same format as the other two intervention lessons except for the introduction. For Group C in Intervention Lesson 3, the lesson began with the specific quadratic function which appeared in the three forms of the equation. The three forms of the

specific quadratic function were given to the learners on the same slide. This same specific quadratic function used in its different forms was

$y = x^2 - 4x + 3$, which represents the standard form of a quadratic function. Followed by $y = (x - 1)(x - 3)$, which represents the factorised form of the equation and $y = (x - 2)^2 - 1$ which represents the form of the equation of a quadratic function resulting from completing the square. This constitutes the pattern of variation called simultaneity.

From the standard form of the equation of a parabola, the learners were required to discuss the key features of this specific standard form of the parabola. They were reminded that in order to find the y-intercept of the function, they had to put $x = 0$ in the parabola's equation. This specific equation was then associated with its general form. The parameters 'a', 'b' and 'c' of the equation of the general form of the parabola were equated to the numerical values of the coefficients of the different variables 'x^2', 'x' and the constant value 'c'of the specific equation.

The generalisation pattern of variation theory was introduced from the specific equations. The standard form of the specific quadratic function $y = x^2 - 4x + 3$ was analysed and its features highlighted. Thereafter the general form was given for the standard form of this specific quadratic function as $y = ax^2 + bx + c$.

The learners were then required to factorise the specific equation of the parabola. They were then reminded that in order for them to find the x-intercepts of the specific equation, they had to put $y = 0$ in the specific equation of the parabola and solve for x. The resulting form of the specific equation was then compared with the general form of the equation of the parabola. The parameters 'x_1' and 'x_2' of the general form of the quadratic equation were compared to the values of the specific equation, $x = 1$ and $x = 3$, which represented the x-intercepts of the specific equation.

The use of the same specific equation of the parabola in its different forms represented the pattern of separation as discussed in Lesson 1.. The specific quadratic equation was factorised to get the factors as follows:

$y = x^2 - 4x + 3 = (x - 1)(x - 3)$. The key features of the specific equation were highlighted as the x- intercepts where $x = 1$ and $x = 3$. This was followed by the presentation of the general form of $y = a(x - x_1)(x - x_2)$.

The completion of the square was then performed on the specific equation and the resulting form of the specific equation compared with the general form of the equation of the parabola to show the turning point $(p; q)$ represented by the values $(2; -1)$ in the specific equation. The learners were reminded that in order for them to find the equation of the axis of symmetry for the parabola, they had to solve the equation $x - p = 0$ as represented in the specific equation as $x - 2 = 0$. to yield the constant equation $x = p$ from the general equation of the parabola and $x = 2$ from the specific equation of the parabola.

A step-by-step approach was used with these learners in order for them to see how the equation of the general form of the parabola was derived from the use of the specific equation of a parabola.

Finally, the completion of the square was performed on the same specific quadratic function which resulted in the form of the quadratic function $y = x^2 - 4x + 3 = (x - 2)^2 - 1$. This form of the equation highlighted the axis of symmetry as $x = 2$ and the turning point $(2; -1)$ followed by the appropriate general form of $y = a(x - p)^2 + q$. The details of Lesson 2 are shown in Appendix I. From this point onwards, the lesson was similar to Lessons 1 and 2.

As this was not the first time that the learners saw the general forms of the quadratic function, they could associate the different specific forms of the quadratic function with their appropriate general forms as follows:

- $y = x^2 - 4x + 3$ as corresponding to the general form $y = ax^2 + bx + c$;
- $y = (x - 1)(x - 3)$ as corresponding to the general form $y = a(x - x_1)(x - x_2)$
- $y = (x - 2)^2 - 1$ as corresponding to the general form $y = a(x - p)^2 + q$

This constituted the pattern of variation called generalisation. By using the processes of factorisation and completing the square on the specific quadratic function as discussed above, the learners were afforded the opportunity to connect the general forms of the different quadratic functions on their own before being given the general form of the quadratic function. After the learners had undergone the processes of factorisation and completing the square, the learners were then shown how the general form of the quadratic function was derived and written as this was the weakest group of the three groups.

The lesson was planned in such a way that the specific quadratic functions in their different forms should first be given to the learners representing simultaneity as the pattern of variation. From this point onwards, the learners were given an opportunity to derive the different forms themselves by applying the processes of factorisation and completing the

square. From these results, the learners were afforded the opportunity to generate a form of the quadratic functions that they could associate with the generalised forms of the different forms of the quadratic functions by themselves. This process represented generalisation as a pattern of variation.

For this group, more than one pattern of variation was applied to the lesson as is discussed above. The table of the first part of Intervention Lesson 3 for Group C is represented in Table 5.3 in the following section.

5.4.2 A TABLE OF THE INTERVENTION LESSON 3 FOR GROUP C.

The detailed table of the intervention lesson for Group C adapted from the PowerPoint presentation in Appendix I is presented in Table 5.3 below.

<table>
<tr><th colspan="3">LESSON 3 (GROUP C) DURATION ≈ 60 MINUTES.</th></tr>
<tr><td colspan="3"><u>Object of learning.</u>
Discerning the features of a quadratic function, specifically identification and use of the appropriate form of the equation to generate the equation of the quadratic function from the graph(s) or other information about the quadratic function.</td></tr>
<tr><th>Input (Object)</th><th>Process (Action)</th><th>Output</th></tr>
<tr><td rowspan="2">The three general forms of the quadratic equation were presented to the learners: $y = ax^2 + bx + c$
$y = a(x - x_1)(x - x_2)$
$y = a(x - p)^2 + q$

The specific equation was thereafter presented to the learners.
$y = x^2 - 4x + 3$</td><td>Specific equation is given as:
$y = x^2 - 4x + 3$
Discerning the general form of the equation to be:
$y = ax^2 + bx + c$.
For the y-intercept, put $x = 0$ in the specific equation to get: $y = 3 = c$</td><td>The key features are:
The shape of the parabola: $a > 0; a = 1$ ∴ the graph is ∪ concave upwards , and the y −intercept of the parabola is at $y = 3$.</td></tr>
<tr><td>Factorising the specific equation we get:
$y = (x - 1)(x - 3)$.
which can be written in the general form of :
$y = a(x - x_1)(x - x_2)$.
Putting $y = o$ and solving the equation we get:
$0 = (x - 1)(x - 3)$
∴ $x = 1$ or $x = 3$, which represent the x-intercepts of the parabola.</td><td>The key features are:
The x −intercepts of the parabola are $x = x_1$ and $x = x_2$ which are $x = 1$ and $x = 3$ for the specific equation.</td></tr>
</table>

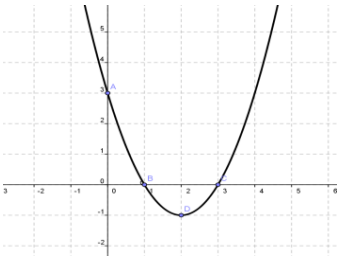
	Completing the square we get: $y = (x-2)^2 - 1$ Discerning the general form of the equation to be: $y = a(x-p)^2 + q$. For the equation of the axis of symmetry, put $x - 2 = 0$. We get $x = 2$ as the equation of the axis of symmetry.	The key features are: The turning point of the parabola = $(2; -1)$ and the axis of symmetry is $x = 2$. In the general form, the turning point is $(p; q)$ and the equation of the axis of symmetry is $x = p$.
	Sketching the parabolic graph with the help of the key features.	
From this point onwards, the lesson is similar to lessons 1 and lesson 2.		

Table 5.3: A detailed table of Intervention Lessons 3 adapted from the Microsoft PowerPoint presentation.

After the lesson, the learners were immediately given the same post-test as was given to the previous groups. The learners' answer sheets together with the question papers were collected from the learners. I marked the scripts as per memorandum in Appendix G and then the panel members moderated the answer sheets. The results of the test were analysed by the entire team and deliberation on these results, the lesson, the video and the audio recording of the lesson ensued. The results of the post-test are discussed in the next chapter.

5.4.3 REFLECTIONS AND DISCUSSION OF LESSON 3.

As for lesson 2, the three colleagues from the Department of Mathematics gave their input on the lessons, the results of the test, the video and audio recordings of the third lesson were reviewed. The results of the pre-test for all three groups were compared with the results of the post-test in all three groups and the of the tests of all the groups in order to discuss and reflect on all three lessons. From the feedback of my colleagues and my reflection on the lessons, the

importance of starting the lesson with the specific quadratic functions followed by the general forms of the quadratic functions became evident.

The incorporation of more than one pattern of variation, as discussed above, was seen as the only way in which all the problems that the learners were experiencing across the lessons, could be dealt with all at once.

Planning for separation was important in that with the specific parabolic function, the learners are able to clearly see the varying forms of the parabolic equation while the parabolic equation itself remains unchanged. The importance of planning for simultaneity is that the forms of the parabolic equation can be seen at the same time and the learners are able to choose the appropriate form of the quadratic equation to be used in the relevant context. Planning for generalisation was important so as to enable the learners to choose the appropriate form of the general quadratic function when required to find the equation of a parabola given certain features of the parabola.

This approach made the learners aware of how the general formulae were derived and thus made the application of these general forms of the equations easy for them to use in appropriate situations. Lesson 3 for Group C incorporated the pattern of variation theory called fusion. Many patterns of variation were used, which constitutes a fusion of the patterns of variation.

The results of all the groups are discussed in detail in the next chapter.

5.5 DISCUSSION OF THE INTERVENTION LESSONS WITH COLLEAGUES.

The discussions with colleagues focussed on what the learners seemed to not understand, why they seemed not to understand it and how best we could make them understand. We used what we experienced while sitting in on the lessons and conducting the lessons and what they wrote as their comments in their observation schedules. The discussion included the results of the pre-test and the comparison of the results with the results of the post-test. The judgements were made and decisions on how to change the lessons were based on these factors.

The results of the observers' notes, with the help of the observation schedule, the pre-test and the post-test together with the three approaches to the lessons were compared by the panel and myself. Although the post-test did not form a great part of the decisions to change the lessons, it played an important role in giving answers to the research question 'to what extent does a version of a learning study framed by variation theory improve the Grade 11 learners' interpretation of algebraic functions?' It was not a matter of whether the learners had improved or not, but to what extent they did improve.

The conclusion resulting from these discussions was that it was more effective for the teaching and learning approach to follow the route taken in Lesson 3. It was also concluded that the learners understood better when the teaching approach was from the specific to the general as the learners seemed to be able to operate in the operational stage rather than the structural stage. This is in line with the findings of researchers like Dubinsky (1986); Sfard (1987, 1988(a), 1991); Dubinsky and Schwingendorf (1990(b)); Heinrics (1991); Harel and Kaput (1991); Sfard (1992); Breidenback (1992); Monk (1987) & Thompson (1994) who found that learners tend to understand better from the specific to the general as they become better equipped to understand and appreciate where the general originates from.

The discussion with colleagues revealed that all three lessons did, in varying degrees, result in the discernment of the object of learning. The results are detailed in the following chapter.

5.6 SUMMARY.

In this chapter, the intervention lessons that took place within the three groups of learners in Grade 11 are discussed. The reflections by the observers and the researcher were noted and the reasons for the changes made to each lesson plan highlighted.

The pattern of variation theory which was brought to the lessons through the use of the same specific quadratic functions presented in different forms. These forms resulted from the algebraic operations of factorisation and completing the square to get a different form of the same specific quadratic function, which constitutes the pattern of separation. This is in line with the aspect of variance with the help of the pattern of variation called separation as discussed by Marton & Tsui (2004) where the authors assert that by keeping the equation of the function the same and varying one parameter, learners will experience separation.

In the intervention lessons, separation, as a pattern of variation was used in all the lessons by having the same graphical representation of a parabolic function and varying the labelling of the points on the parabola. Another use of separation as a pattern of variation was evident in all the lessons when a specific equation of a parabola was kept the same and the different forms of the parabolic equation varied.

Similarly, in this study the parabolic function and its graph was kept the same and the given critical points on the parabola were varied, showing the pattern of separation in all the lessons. Learners were given the opportunity to flexibly move between the two different forms of the quadratic function/parabola, namely from the graphical form to the equation form and vice versa.

The reasons for the inclusion of simultaneity as the patterns of variation were as a result of the learners' inability to remember the different forms of the general equations of the parabola in intervention Lesson 1 for group A. The learners' inability to substitute the correct points that appeared on the graphical representations of the parabola necessitated the inclusion of simultaneity as a pattern of variation to the lessons. This was done in Lesson 2 and Lesson 3.

The graphical representation and equation representations of functions were used to showcase the element of contrast as a dimension of variation theory (Marton et al., 2004). The last lesson applied the pattern of variation called fusion, as many patterns of variation were used to present the lesson.

The learners' questions on how the general forms of the quadratic function were derived necessitated the inclusion of the generalisation pattern of variation to the Intervention Lesson 3 for Group C. The generalisation pattern of variation theory is another pattern of variation theory which was brought to the lessons through the use of the same specific quadratic functions presented in different forms. These forms resulted from the algebraic operations of factorisation and completing the square to get a different form of the same specific quadratic function.

The generalisation pattern of variation was brought into the last lesson, because of the problems that the learners had regarding the derivations of the general forms of the quadratic function. This was brought to the last lesson by applying the algebraic manipulations to the specific standard form of the specific equations to produce the different forms of the

parabolic equations. These different forms of the specific equations were then written in the general forms of the respective equations of the parabola, which formed the generalisation pattern of variation. These algebraic operations performed on the specific equation of a parabola are factorisation of the specific equation and completing the square on the specific equation. This is in line with the pattern of variation called generalisation as proposed by Marton et al. (2004).

A comparison of the results of the three groups from the pre-test and the post-test is discussed extensively in the data analysis in the following chapter.

CHAPTER SIX.

ANALYSIS AND INTERPRETATION OF TEST DATA.

6.1 INTRODUCTION.

In this chapter I analyse the diagnostic test for the whole sample which gave answers to the critical questions 'What aspects of functions do learners find to be problematic? And 'Which of these aspects appear most problematic?' The results of the post-test for the whole sample will then be analysed and will give answers to the research question 'To what extent does a version of a learning study framed by variation theory improve the Grade 11 learners' interpretation of algebraic functions?'

The analysis of the pre-test will be detailed for each question and a discussion on the performance of the learners in each question discussed in detail. Some of the transcripts from the learners' responses in the pre-test will be shown to highlight the types of errors that the majority of learners were displaying when answering the pre-test. A discussion of the results of the pre-test and the post-test will follow the analysis of the results for both tests. The analysis of the results of the pre-test and the post-test will be discussed for each individual group to find out the impact that each intervention lesson had on each group. The extent to which the changes to the intervention lessons had an impact on the learners' performance in the post-tests will be discussed for each group. Reference to the literature on functions will be linked to the findings of the study.

The mixed methods research method will be discussed and used to analyse the data. This method includes both the quantitative and qualitative analysis of the results. The quantitative analysis of the results of the diagnostic/Pre-test and the Post-test are considered and any improvements from the pre-test as observed in the post-test are investigated. The quantitative analysis of the five questions of the pre-test and specifically the first three questions leading to the choice of the intervention process will be examined.

The results of both tests will be presented as a horizontal or vertical bar graph and in table form. The bar graphs are used for their visual impact and the table form is used to showcase the exact numeric results in percentages or in average percentages. Any overlaps between the

two tests will be noted. The outcomes of the pre-test and post-test will be analysed and compared for each group. These results will be discussed in relation to each intervention lesson that took place with each group. The patterns of variations that were used in each group will be discussed together with the results of learners in both the tests. The main question of the study 'To what extent does a version of a learning study framed by variation theory improve the Grade 11 learners' interpretation of functions?' will be answered.

6.2 MIXED METHODS DESIGN OF DATA ANALYSIS.

As explained in chapter four, the mixed methods design research which is a procedure used in collecting and "mixing" both qualitative and quantitative methods of analysing data in a single study or a series of studies to understand a research problem, was used in this study. The quantitative and qualitative analysis of the data is used to ascertain the object of learning that informed the intervention process as they together provide a better understanding of the research problem (Miles & Huberman, 1994).

6.2.1 QUANTITATIVE ANALYSIS OF THE PRE-TEST.

In this study summed scores are used to sum up the scores of learners for each question in each group and for the whole sample of learners. These scores are then grouped according to particular intervals representing the levels of the scores as required by the National Curriculum Statement (2012). The spread sheet is used to generate the graphical representation of the data resulting from both the diagnostic and the post-test. I have not done extensive qualitative study of the nature of errors and their proliferation but some evidence from learner scripts is provided and so some qualitative data is present in the process of showing the problems learners were having with respect to the notion of the interpretation of functions.

Initially it was thought that the diagnostic and post-test data could be treated statistically in an attempt to determine whether any differences were statistically significant or not. The small sizes of the intervals render this approach not to be viable. As such, the data are represented graphically for visual inspection and analysis and thereafter represented as a

summary in table form to elaborate on the graphical representation of the results of the pre-test.

6.2.2 RESULTS OF THE DIAGNOSTIC/PRE-TEST.

A sample of 85 learners forming three groups A, B and C completed the diagnostic test consisting of five questions as in appendix D and the results of the performance of each group in each question are shown in their respective percentage intervals. Figure 6.1(a) shows the percentage scores that the learners achieved in the diagnostic test according to the individual questions in each group and table 6.1 shows the table thereof. The mean percentage of the number of learners obtaining a particular interval of percentage is shown both graphically and through a table for both the diagnostic test and the post-test data for greater clarity. The learners' diagnostic test generated the data that is presented across the levels/intervals as in figure 6.1 below.

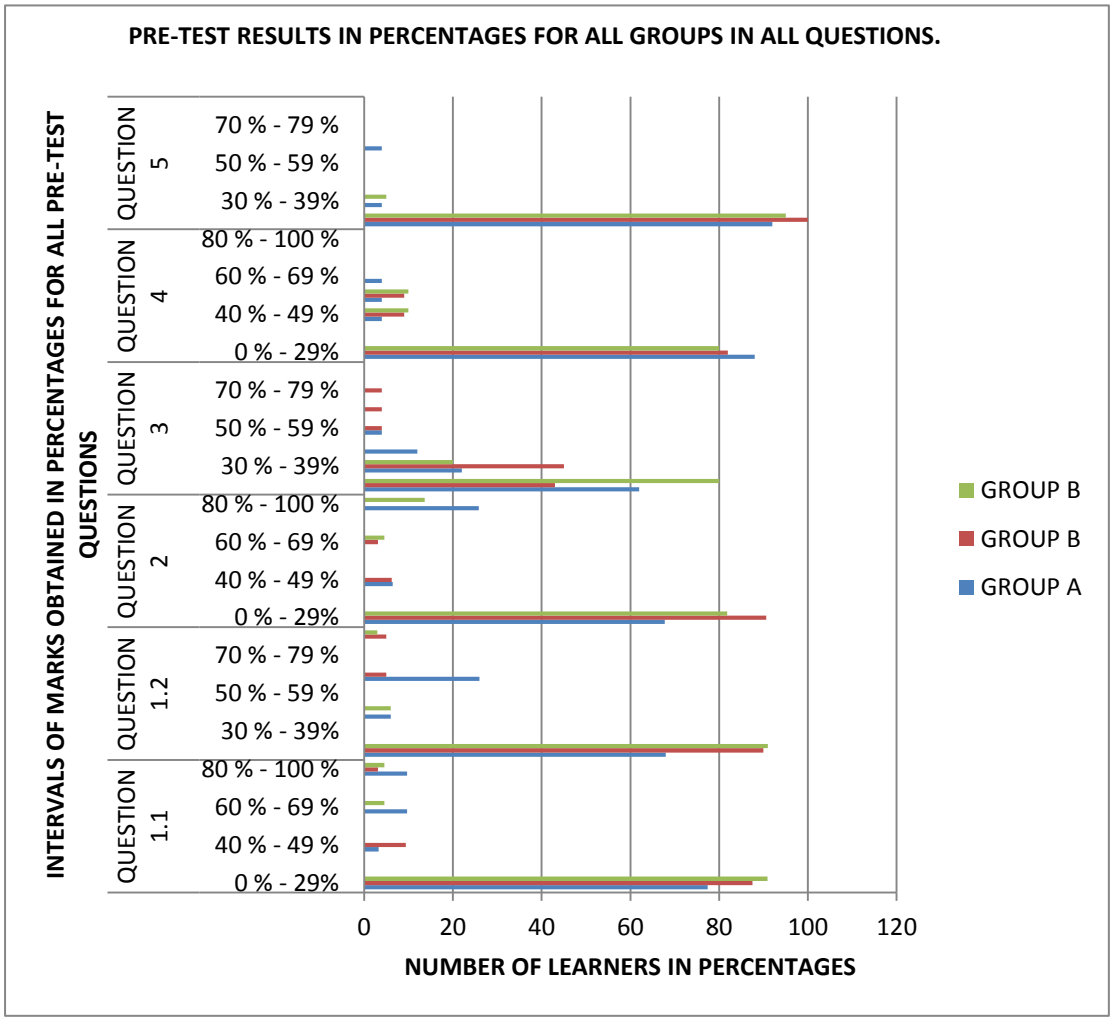


Figure 6.1: Questions 1 to 5 results for the diagnostic test for all groups.

Table 6.1 below shows the attainment of learners in all questions of the pre-test for all groups in percentages.

RESULTS OF PRE-TEST IN PERCENTAGES FOR ALL GROUPS PER TEST QUESTION				
INTERVALS IN PERCENTAGES FOR ALL GROUPS	**INTERVAL IN PERCENTAGES**	**GROUP A**	**GROUP B**	**GROUP C**
QUESTION 1.1	**0 % - 29%**	**77**	**88**	**91**
	30 % - 39%	**0**	**0**	**0**
	40 % - 49 %	**3**	**9**	**0**
	50 % - 59 %	**0**	**0**	**0**
	60 % - 69 %	**10**	**0**	**5**
	70 % - 79 %	**0**	**0**	**0**
	80 % - 100 %	**10**	**3**	**5**
QUESTION 1.2	**0 % - 29%**	**68**	**90**	**91**
	30 % - 39%	**0**	**0**	**0**
	40 % - 49 %	**6**	**0**	**6**
	50 % - 59 %	**0**	**0**	**0**
	60 % - 69 %	**26**	**5**	**0**
	70 % - 79 %	**0**	**0**	**0**
	80 % - 100 %	**0**	**5**	**3**
QUESTION 2	**0 % - 29%**	**68**	**91**	**82**
	30 % - 39%	**0**	**0**	**0**
	40 % - 49 %	**6**	**6**	**0**
	50 % - 59 %	**0**	**0**	**0**
	60 % - 69 %	**0**	**3**	**5**
	70 % - 79 %	**0**	**0**	**0**
	80 % - 100 %	**26**	**0**	**14**
QUESTION 3	**0 % - 29%**	**62**	**43**	**80**
	30 % - 39%	**22**	**45**	**20**
	40 % - 49 %	**12**	**0**	**0**
	50 % - 59 %	**4**	**4**	**0**
	60 % - 69 %	**0**	**4**	**0**
	70 % - 79 %	**0**	**4**	**0**
	80 % - 100 %	**0**	**0**	**0**
QUESTION 4	**0 % - 29%**	**88**	**82**	**80**
	30 % - 39%	**0**	**0**	**0**
	40 % - 49 %	**4**	**9**	**10**
	50 % - 59 %	**4**	**9**	**10**
	60 % - 69 %	**4**	**0**	**0**
	70 % - 79 %	**0**	**0**	**0**
	80 % - 100 %	**0**	**0**	**0**
QUESTION 5	**0 % - 29%**	**92**	**100**	**95**
	30 % - 39%	**4**	**0**	**5**
	40 % - 49 %	**0**	**0**	**0**
	50 % - 59 %	**0**	**0**	**0**
	60 % - 69 %	**4**	**0**	**0**
	70 % - 79 %	**0**	**0**	**0**
	80 % - 100 %	**0**	**0**	**0**

Table 6.1: Results of the pre-test in percentages for all groups in all five questions.

The graphical representations shown in figure 6.1(a) and table 6.1(a) below show a clear picture of the performance of all the learners in the entire sample in all the pre-test questions.

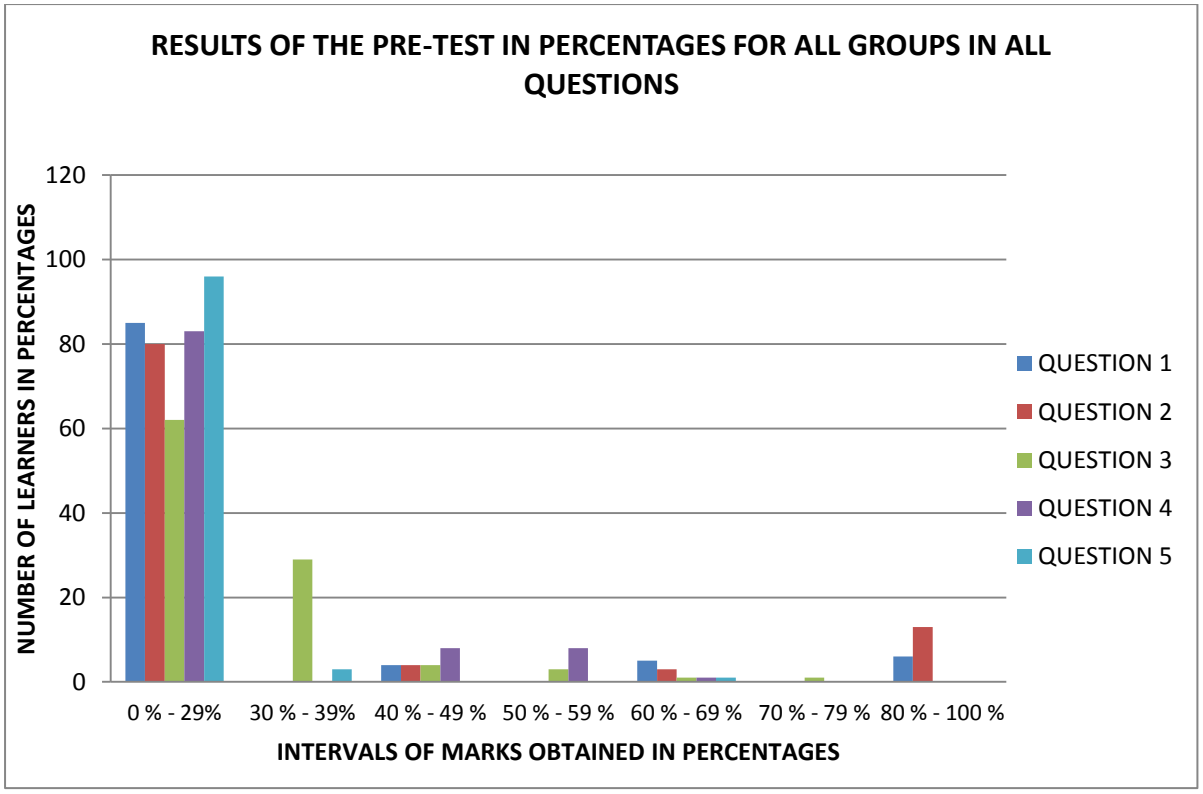


Figure 6.1(a): Results of questions 1 to 5 of the pre-test for all groups.

ANALYSIS OF THE RESULTS OF THE PRE-TEST					
	NUMBER OF LEARNERS IN PERCENTAGES				AVERAGE NUMBER OF LEARNERS IN PERCENTAGES
	INTERVALS IN PERCENTAGES	GROUP A	GROUP B	GROUP C	
QUESTION 5	**0% - 29%**	86	98	98	**94**
	30% - 39%	0	0	0	**0**
	40% - 49%	3	0	0	**1**
	50% - 59%	0	0	0	**0**
	60% - 69%	4	0	0	**1**
	70% - 79%	0	0	0	**0**
	80% - 100%	8	3	2	**4**
	TOTAL	**100**	**100**	**100**	**100**

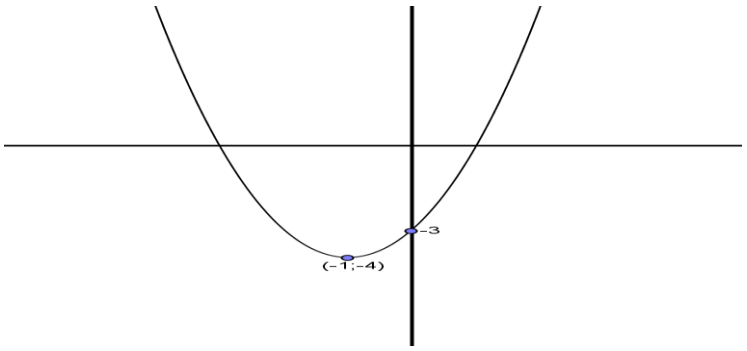
Table 6.1(a): Results of questions 1 to 5 of the pre-test for all groups.

As the graph clearly shows, the performance of all the learners in the sample was very poor. In the next section, the performance of the learners will be analysed for each question in each group. The areas and functional concepts that were particularly problematic to the learners will be highlighted. The nature of the errors that were prevalent for the learners and where the learners' performance was particularly weak will be evident from the analysis of the results of the learner's performance in each question. This analysis led to identifying the object of learning that was discussed in chapter five.

6.3 PERFORMANCE OF THE GROUPS IN EACH PRE-TEST QUESTION.

6.3.1 QUESTION 1.1

In question 1.1, the learners were required to find the equation of the parabolic graph where the turning point of the parabola and another point on the graph was given.



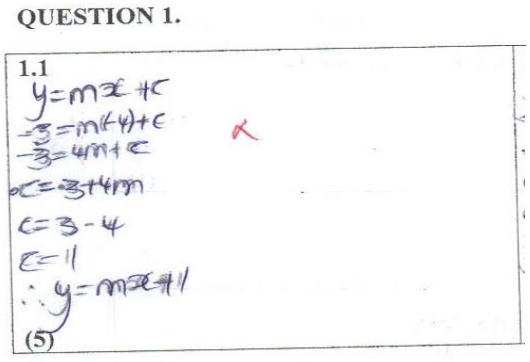
(5)

In this question learners were required to interpret the graphical representation of a parabola and be able to move from the graphical representation of a function to the equation form of the parabola. In order for the learners to be able to do this, they had to be able to recognise the given key features of the parabolic function and substitute the given points on the parabola in the appropriate form of the parabolic equation.

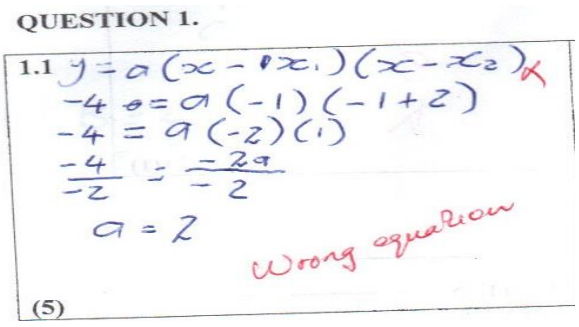
The majority of learners used the incorrect form of the parabolic equation as is evidenced by the transcripts that are shown below.

Some of the extracts from the learner responses for question 1.1 of the pre-test from group A:

Response 1.



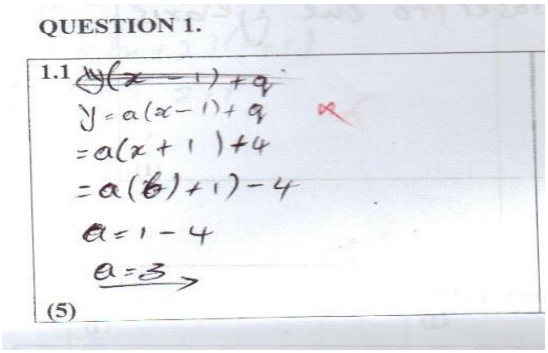
Response 2.



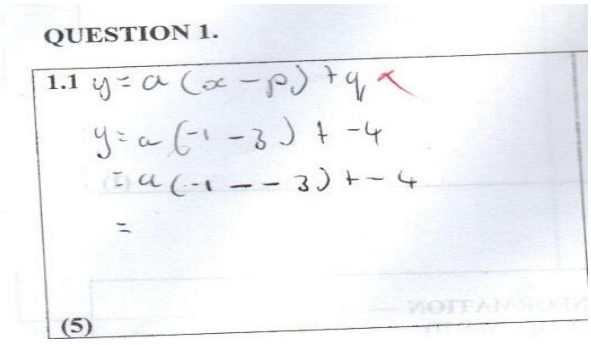
Extracts 1.1(a): Extracts from some of the learner responses for question 1.1 from group A.

Some extracts from the learner responses for question 1.1 of the pre-test from group B:

Response 3.



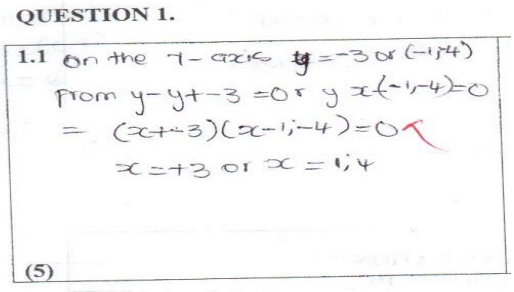
Response 4.



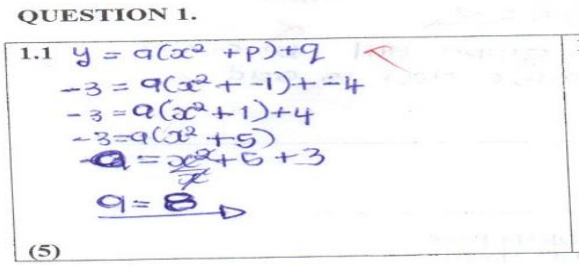
Extracts 1.1(b): Extracts from some of the learner responses for question 1.1 from group B.

Some of the extracts from the learner responses for question 1.1 of the pre-test from group C:

Response 5.



Response 6.



Extracts 1.1(c): Extracts from some of the learner responses for question 1.1 from group C.

In response 1, the learner used the equation of a straight line and also substituted values haphazardly which indication that the learner did not know which function the learner was dealing with. In response 2 the learner used the wrong form of the parabolic function and also substituted incorrectly into the equation of the quadratic function.

In responses 3 and 4, the learners incorrectly wrote the required form of the parabolic equation and also incorrectly substituted the points given in the sketch of the parabola. In responses 5 and 6 the learners used the incorrectly written form of the parabolic equation and also incorrectly substituted the values of the given points.

The results of the learner responses for question 1.1 of the pre-test are shown in figure 6.2(a) and Table 6.2(a) below.

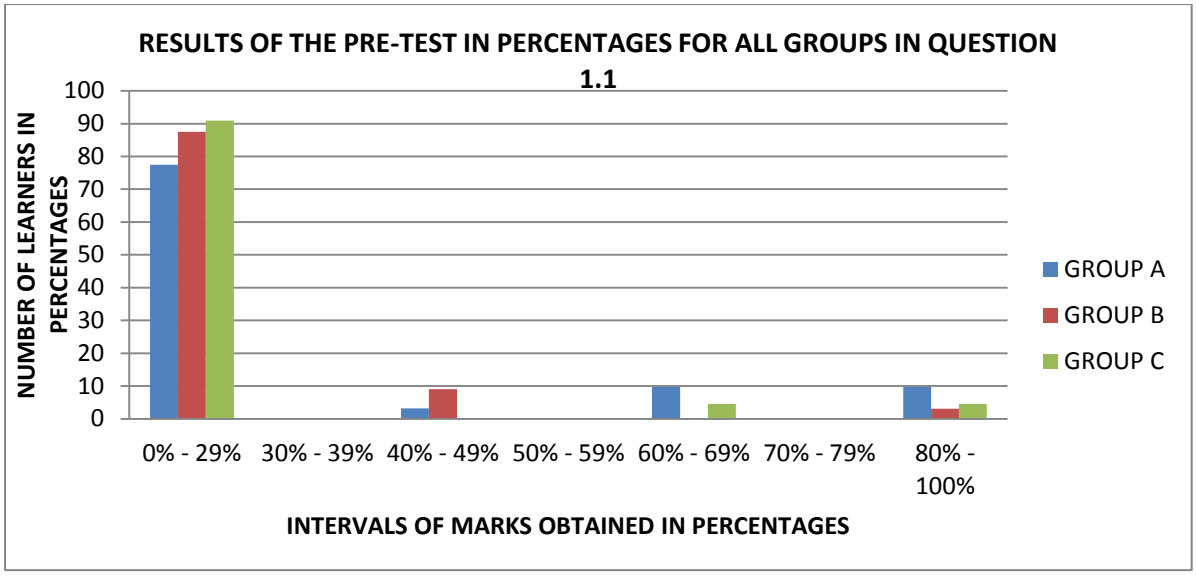


Figure 6.2(a): Results of the learner responses for question 1.1 of the pre-test for all the groups.

Table 6.2(a) below shows the performance of the learners in question 1.1 of the pre-test.

ANALYSIS OF THE RESULTS OF THE PRE-TEST					
	NUMBER OF LEARNERS IN PERCENTAGES				AVERAGE NUMBER OF LEARNERS IN PERCENTAGES
	INTERVALS IN PERCENTAGES	GROUP A	GROUP B	GROUP C	
QUESTION 1.1	**0% - 29%**	77	88	91	**85**
	30% - 39%	0	0	0	**0**
	40% - 49%	3	9	0	**4**
	50% - 59%	0	0	0	**0**
	60% - 69%	10	0	5	**5**
	70% - 79%	0	0	0	**0**
	80% - 100%	10	3	5	**6**
		100	**100**	**100**	**100**

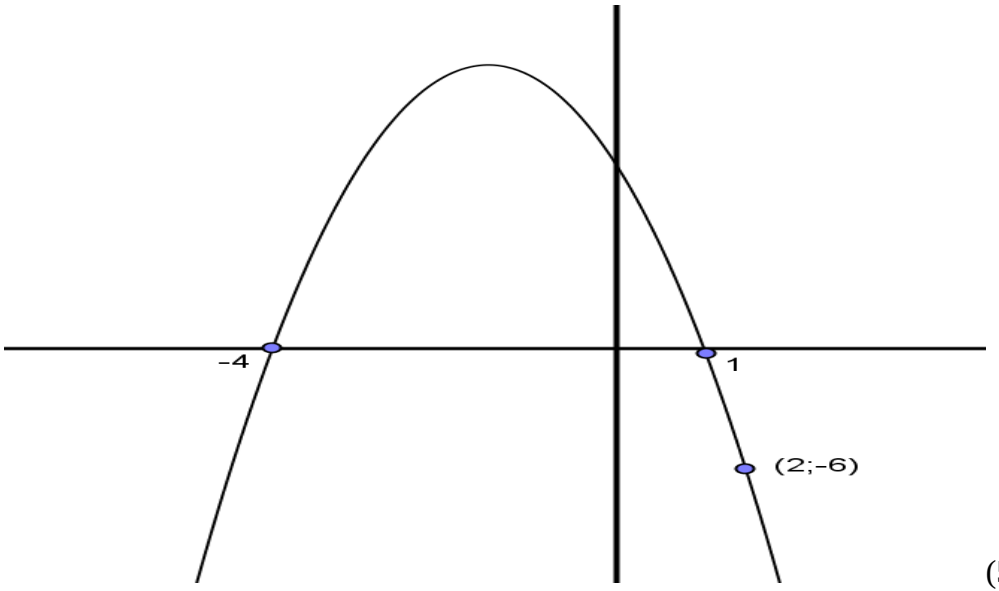
Table 6.2(a): Results of the pre-test in questions 1.1 for all groups.

The majority of the learners used the incorrect form of the parabolic equation and some of the learners that used the appropriate form of the parabolic function substituted the given points incorrectly. Some of the learners used equations of functions that were not parabolic in nature. The learners' overall average performance in question 1.1 for the interval 0% to 29% was 85% as it appears in the table 6.2(a) above. The average number of learners who obtained between 80% and 100% for question 1.1 for all the groups was 6%.

The majority of learners performed poorly in this question. The errors that were prevalent were the choice of the incorrect equation of a parabolic function. The substitution of the incorrect points chosen equation of a parabolic function, together with the incorrect algebraic manipulations of the chosen equations also contributed to the learners' poor performance in this question.

Question 1.2

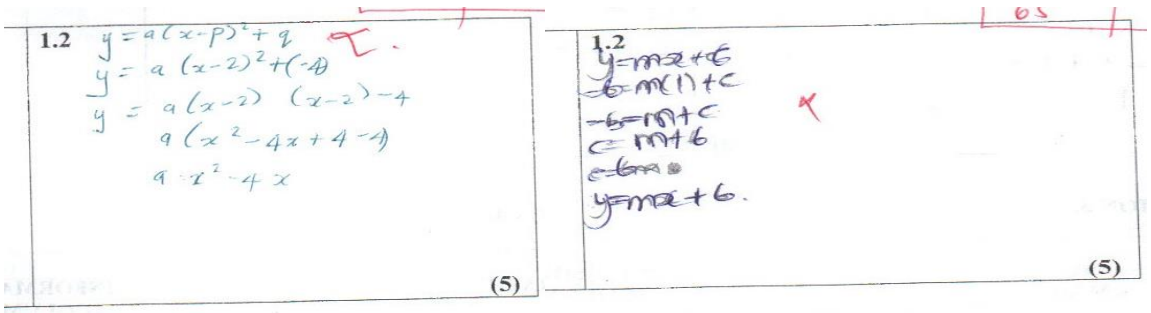
In question 1.2, the learners were required to find the equation of the parabolic graph where the x-intercepts of the parabola and another point on the graph was given.



(5)

Similar to question 1.1 above, this question required learners to interpret the graphical representation of a parabola and be able to move from the graphical representation of a function to the equation form of the parabola. In order for the learners to be able to do this, they had to be able to recognise the given key features of the parabolic function and substitute the given points on the parabola in the appropriate form of the parabolic equation. Some of the extracts from the learner responses for question 1.2 of the pre-test from group A are shown in extracts 1.2(a) below.

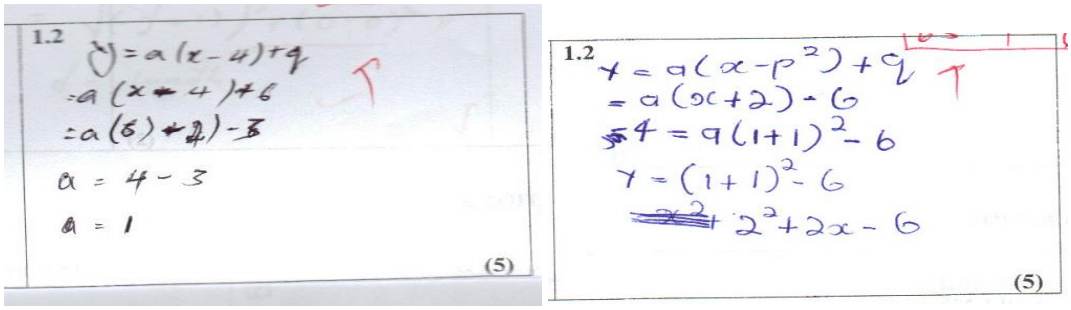
Response7. Response 8.



Extracts 1.2(a): Some of the extracts from learner responses for question 1.2 from group A.

Some of the extracts from the learner responses for question 1.2 of the pre-test from group B are shown in extract 1.2(b) below.

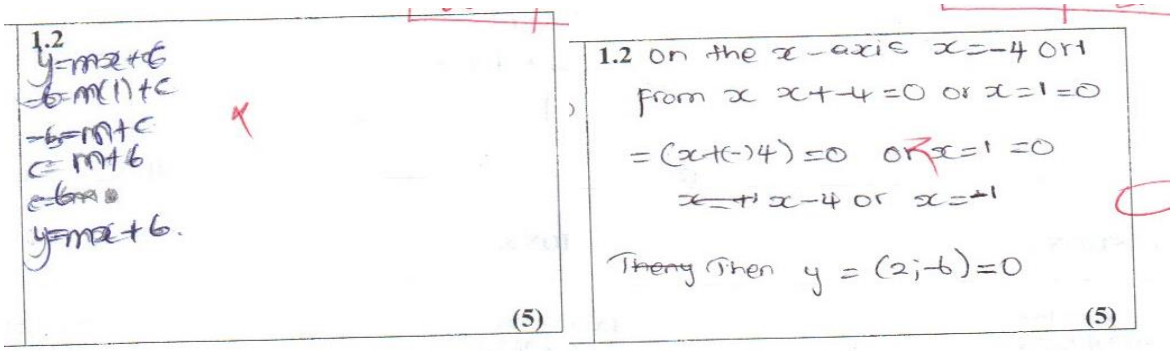
Response 9. Response 10.



Extracts 1.2(b): Some of the extracts from learner responses for question 1.2 from group B.

Some of the extracts from the learner responses for question 1.2 of the pre-test from group C:

Response11. Response12.



Extracts 1.2(c): Some of the extracts from learner responses for question 1.2 from group C.

As was the case with question 1.1 above, the majority of learners used the incorrect form of the parabolic equation as is evidenced by the transcripts that are shown from learners from all groups for question 1.2. In response 7, the learner 7 used the incorrect form of the parabolic equation and also substituted the wrong points into the wrong equation of a parabola. The learner in response 8 used the equation of a straight line instead and then went on to substitute values of points that appear in the sketch. The learners in responses 9 and 10 used the incorrect form of the equation of a parabolic function and also substituted the wrong points given on the parabolic graph.

The majority of learners made the errors that these extracts show and this could be attributed to the fact that the majority of learners could not interpret the given information from the graphical form of the parabolic function because they used the incorrect form of the parabolic function whilst other learner used any equation of a function that they remembered.

The results of the learner responses for question 1.2 of the pre-test are shown in figure 6.2(b) and table 6.2(b) below.

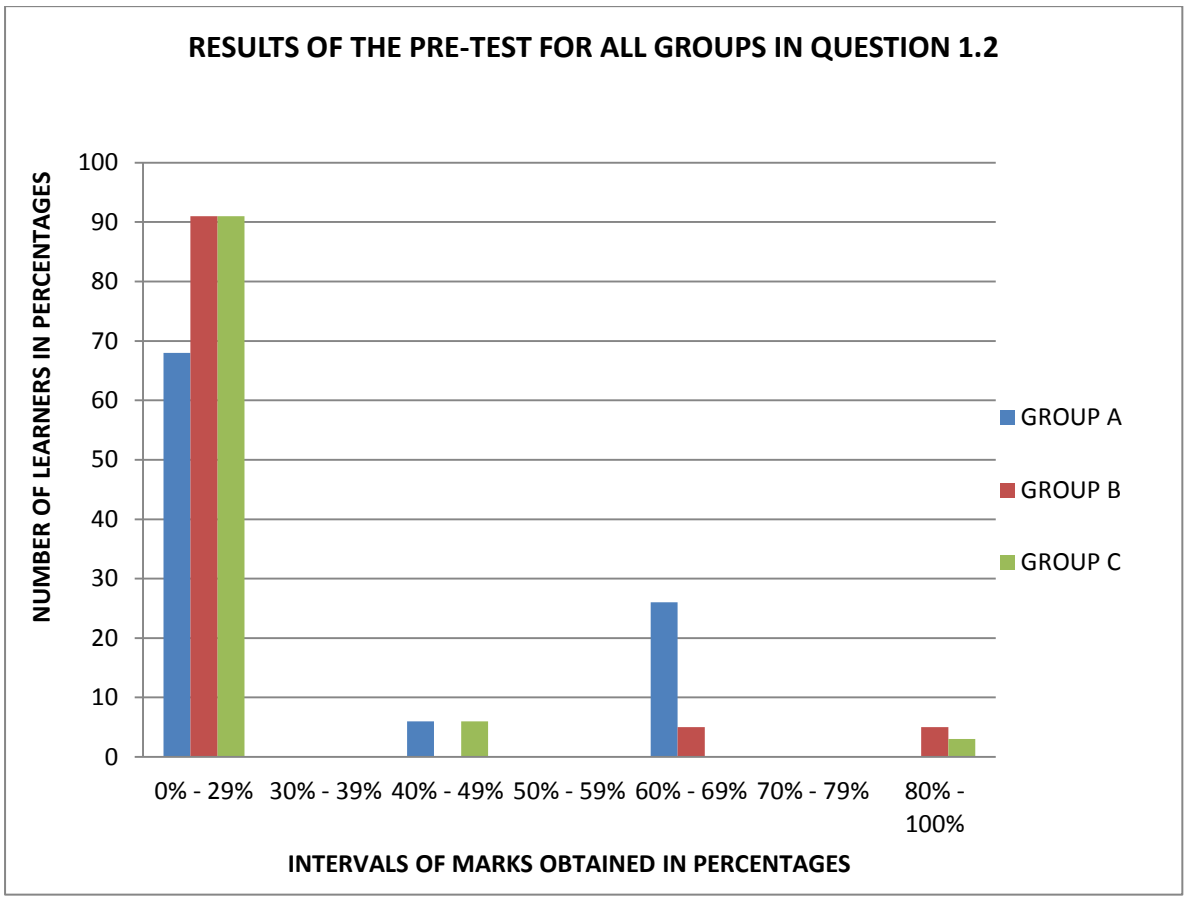


Figure 6.2(b): Results of the pre-test in question 1.2 for all the groups.

ANALYSIS OF THE RESULTS OF THE PRE-TEST					
	NUMBER OF LEARNERS IN PERCENTAGES				AVERAGE NUMBER OF LEARNERS IN PERCENTAGES
	INTERVALS IN PERCENTAGES	GROUP A	GROUP B	GROUP C	
QUESTION 1.2	**0% - 29%**	68	91	91	**83**
	30% - 39%	0	0	0	**0**
	40% - 49%	6	0	6	**4**
	50% - 59%	0	0	0	**0**
	60% - 69%	26	4	0	**10**
	70% - 79%	0	0	0	**0**
	80% - 100%	0	5	3	**3**
	TOTAL	**100**	**100**	**100**	**100**

Table 6.2(b): Results of the pre-test in question 1.2 for all the groups.

Of the 85 learners in the sample, the performance of the groups in the 0% to 29% interval ranged from 68% to 90% for all the groups. This question was also poorly answered because the majority of learners used the incorrect form of the parabolic equation. Some of the errors that were made by the learners were concerned with the incorrect applications of the algebraic manipulations.

The majority of those that used the appropriate form of the parabolic equation substituted the incorrect values of the given points of the parabola. Some of the errors that the learners made were purely computational in that instead of writing the product of a negative value and a positive value as being negative, some of the learners gave an answer that was positive. The learners' overall average performance in question 1.2 for the interval 0% to 29% was 83% as it appears in the table 6.2(b) above. The average number of learners who obtained between 80% and 100% for question 1.2 for all the groups was 3%.

As Sfard (1991) and Tall et al. (1991) have highlighted, multiple representations present difficulties in the teaching and learning of functions. This, the researchers attribute to the abstract notion of a function that takes many forms in terms of representations and learners tend to think that these representations are separated from each other instead of connecting them together to form a unity. Instead of seeing the equation and the graph as all particular representations of a parabola, they think that they are not connected. This could be the reason for the learners' inability to flexibly move from the graphical representation of a parabola to the equation of the same parabola.

Question 2.

2.1 On the **DIAGRAM SHEET,** on the same system of axes, draw the graphs of

$g(x) = 2^x$ *and* $f(x) = 2^x - 4$. Clearly show the coordinates of the intercepts on the y-axes as well as the asymptote of g and equation of the asymptote of f. (5)

2.2 Write down the domain of g and the range of f. (3)

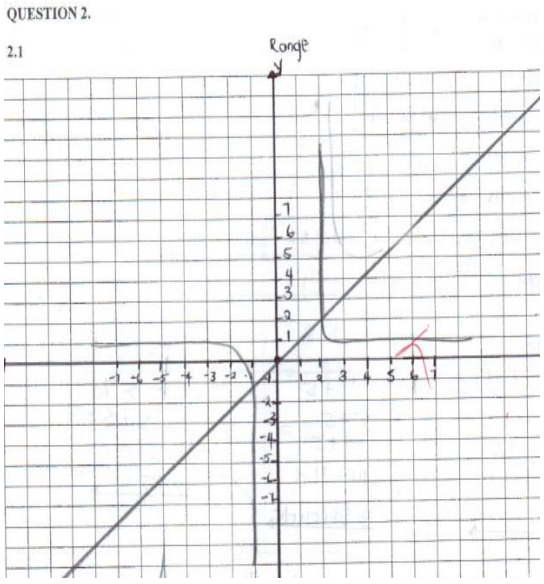
2.3 Explain what will happen to the graph of *f*, if the equation of *f* changes to

$f(x) = 2^{x-4} - 4$? (2) **[10]**

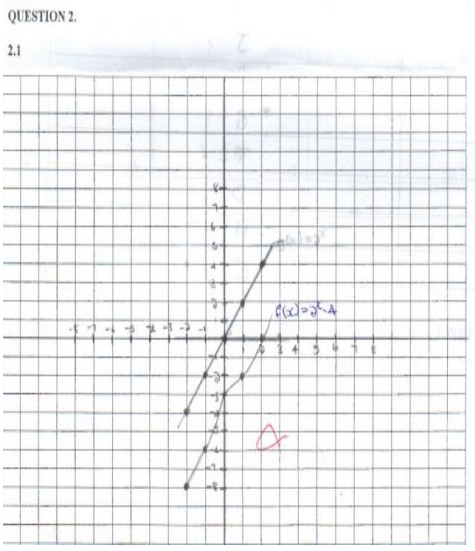
Question 2.1 required learners to be able to produce sketches of exponential functions. The learners were given the algebraic representation of the function and thereafter clearly showing that they know what is meant by the intercepts and the asymptotes. In question 2.2, the learners were expected to find the trend of the functions by giving the domain and range of one of the functions. In question 2.3 the learners were required to interpret what the horizontal shift of the function would do to the vertically shifted parent graph.

Some of the extracts from the learner responses for question 2.1 of the pre-test from the sample are shown in extracts 2.1 below.

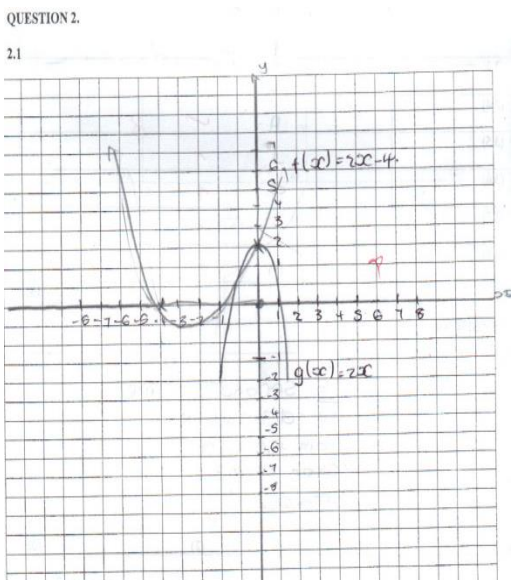
Response 1.



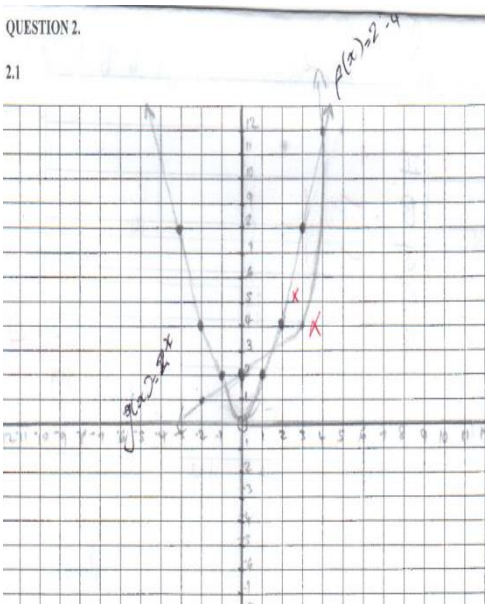
Response 2.



Response 3.



Response 4



Extracts 2.1: Some of the extracts from learner responses for question 2.1 from the sample.

Most of the learners sketched the graphs of one of the functions that appear in exrtacts 2.1 above. This implied that the learners were unable to identify the shape of the function from the equation of the function. The results of the learner responses for question 1.2 of the pre-test are shown in figure 6.2(c) and 6.2(c) below.

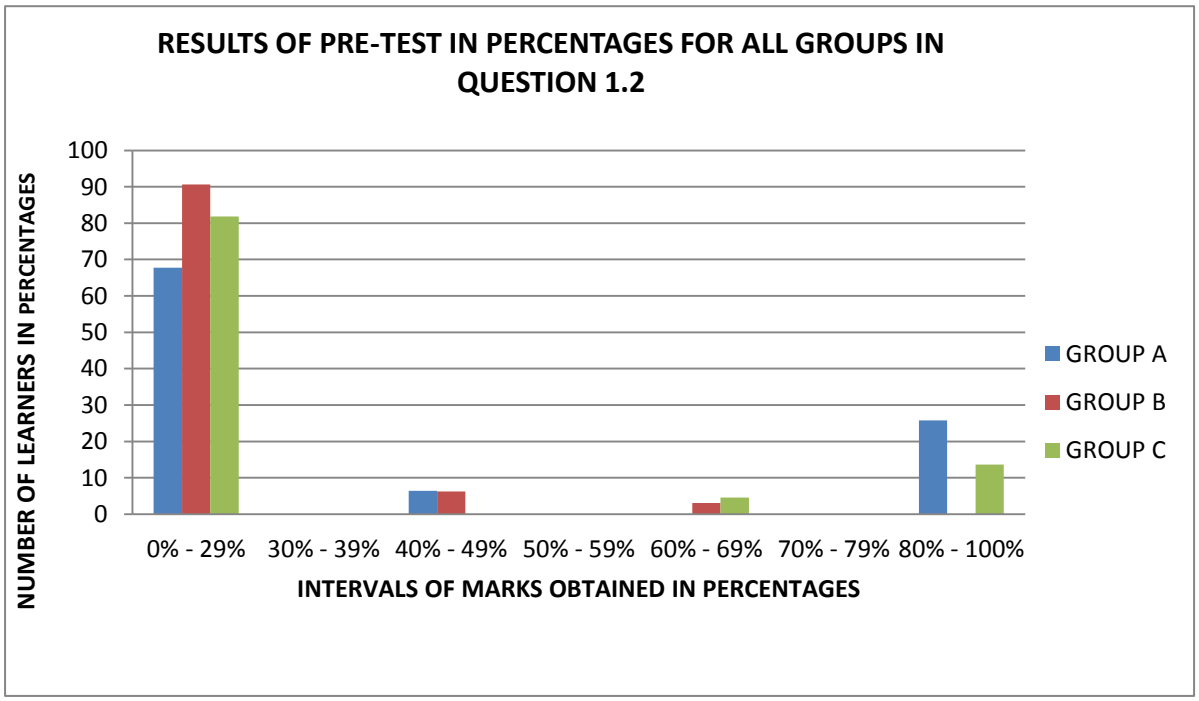


Figure 6.2(c): Results of the pre-test question 2 for all the groups.

ANALYSIS OF THE RESULTS OF THE PRE-TEST					
	NUMBER OF LEARNERS IN PERCENTAGES				AVERAGE NUMBER OF LEARNERS IN PERCENTAGES
	INTERVALS IN PERCENTAGES	GROUP A	GROUP B	GROUP C	
QUESTION 2	**0% - 29%**	68	91	82	**80**
	30% - 39%	0	0	0	**0**
	40% - 49%	6	6	0	**4**
	50% - 59%	0	0	0	**0**
	60% - 69%	0	3	5	**3**
	70% - 79%	0	0	0	**0**
	80% - 100%	26	0	14	**13**
	TOTAL	**100**	**100**	**100**	**100**

Table 6.2(c): Results of the pre-test in question 2 of the diagnostic test for all groups.

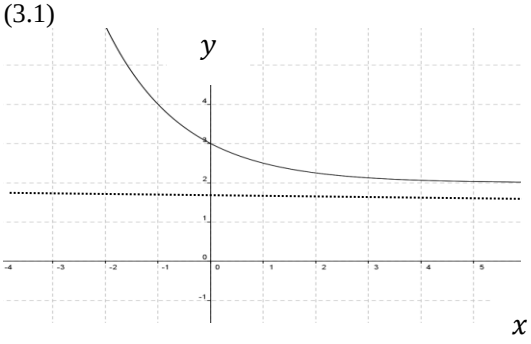
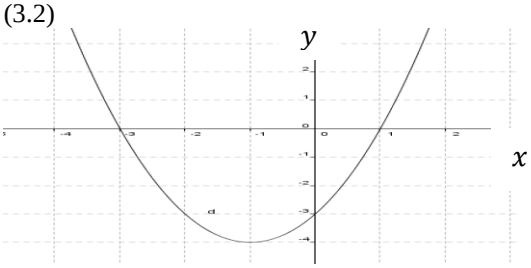
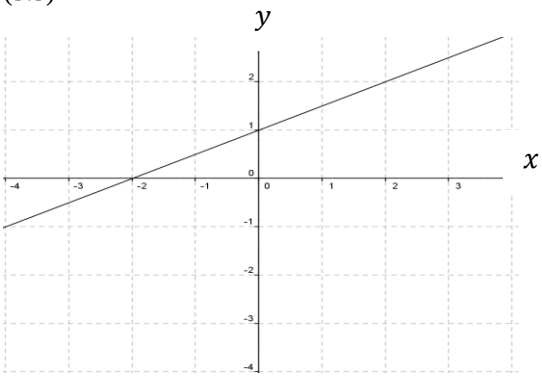
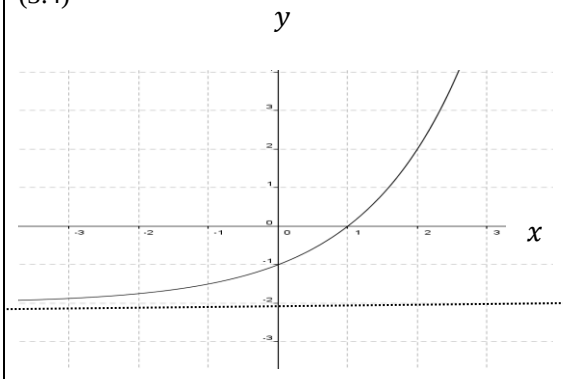
The most prominent error that the learners made was the identification of the particular function to be sketched. Some learners sketched graphs of linear functions and others sketched graphs of a parabola. The learners did not make use of their knowledge of the shape associated with this particular type of an equation of a function. The learners did not use the point-to-point process of plotting a graph in question 2.1. The learners' overall average performance in question 1.2 for the interval 0% to 29% was 80% as it appears in the table 6.2(b) above. The average number of learners who obtained between 80% and 100% for question 1.2 for all the groups was 13%.

According to Leinhardt, Zaslavsky & Stein (1990), the learners should be able to apply some of the interpretations of graphical representations such as translation, interpretation of local processes (regarding point-to-point attention), global interpretation processes (detecting trends), global and general (what happens to y as x increases), continuation (interpolation/extrapolation) when interpreting functions.

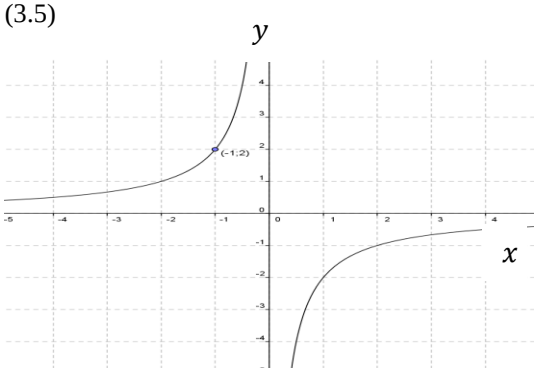
In question 2.1, the interpretation of transformation was not properly applied by the learners in the sketching of the given exponential function. The application of transformation was not applied by the learners when explaining what the effects of changing the function $f(x)$ in question 2.3 would do to the parent graph of the exponential function.

Question 3.

Use the table provided in the answer sheet to match a graph in column (I) to its appropriate representation in column (II).

Column (I)	Column (II)
(3.1) 	(A) x | 0 | 1 | 2 y | -1 | 0 | 2 (2)
(3.2) 	(B) $h(x) = -\frac{2}{x}$ (2)
	(C) $y = x + 1$ (2)
(3.3) 	(D) A certain number subtracted from four. (2)
(3.4) 	(E) x | y -1 | 4 1 | 0 0 | 2 2 | 6 (2)

(3.5)



(F)

x	-1	0	1	2
y	-4	-3	0	9

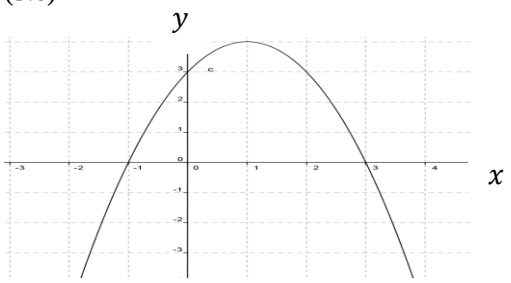
(2)

(G)

$h(x) = -\frac{2}{x} + 1$

(2)

(3.6)



(H)

One added to two divided by a certain number.

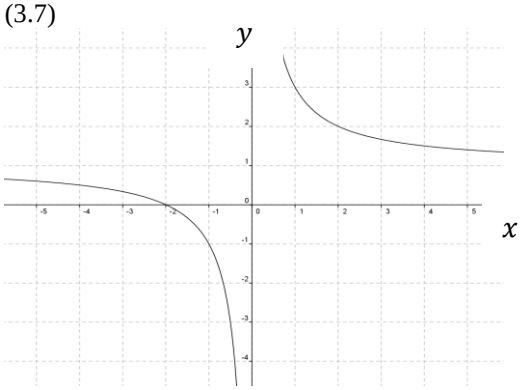
(2)

(I)

$h(x) = 2^{-x} + 2$

(2)

(3.7)



(J)

$h(x) = -x^2 + 2x + 3$

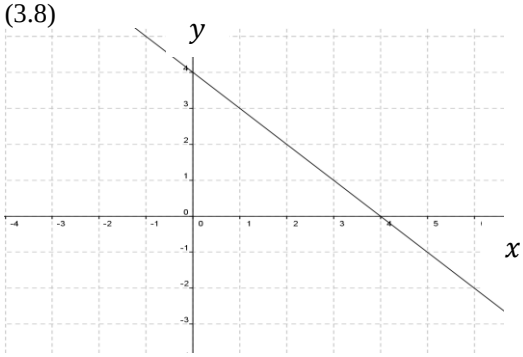
(2)

(K)

$y = 2^x + 3$

(2)

(3.8)



(L)

One added to half of a certain number.

(2)

(M)

$xy = x + 1$

(2)

[18]

The results of the learner responses for question 3 of the pre-test are shown in figure 6.3(a) and table 6.3(a) below.

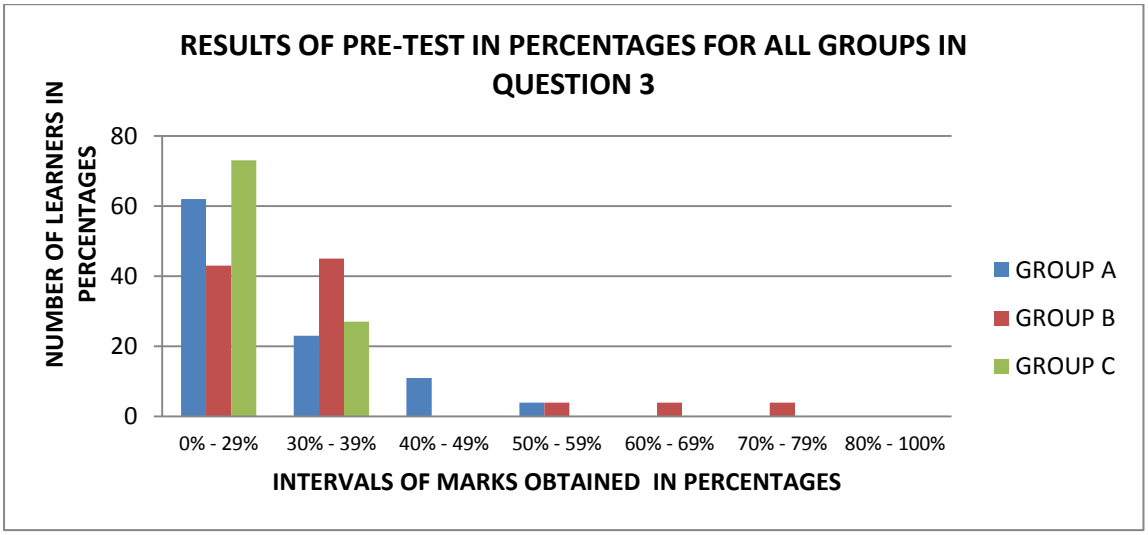


Figure 6.3(a): Results of the pre-test for all the groups in question 3.

ANALYSIS OF THE RESULTS OF THE PRE-TEST					
	NUMBER OF LEARNERS IN PERCENTAGES				AVERAGE NUMBER OF LEARNERS IN PERCENTAGES
	INTERVALS IN PERCENTAGES	GROUP A	GROUP B	GROUP C	
QUESTION 3	**0% - 29%**	62	43	73	**59**
	30% - 39%	23	45	27	**32**
	40% - 49%	11	0	0	**4**
	50% - 59%	4	4	0	**3**
	60% - 69%	0	4	0	**1**
	70% - 79%	0	4	0	**1**
	80% - 100%	0	0	0	**0**
	TOTAL	**100**	**100**	**100**	**100**

Table 6.3(a): Results of the pre-test in questions 3 for all groups.

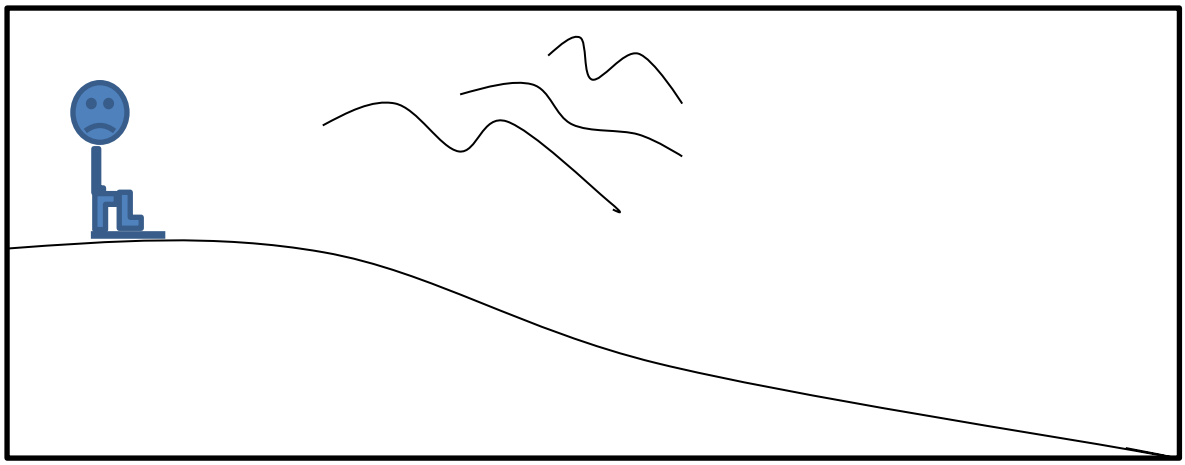
Although this question was fairly answered by the learners, some of the prominent errors that were evident involved the learners' inability to flexibly move from one graphical representation to the other. In this case, the learners had to be able to move from the graphical representation to either the (i) equation form or (ii) table form or (iii) verbal form. The errors that the learners made could be due to the learners' inability to treat functions as both objects and processes.

The learners' overall average performance in question 1.2 for the interval 0% to 29% was 59% as it appears in the table 6.2(b) above. The average performance of the learners in this question was 32% for the interval 30% to 39% which was better that in the previous

questions that have been discussed above. The average number of learners who obtained between 80% and 100% for question 1.2 for all the groups was 0%.

According to Sfard (1991), the ability of students to treat functions as both objects and processes leads to the students having fewer to no problems with moving from one functional representation to the next. The learners were unable to find the appropriate way to enable them to match the given graphical representation of functions to their appropriate verbal, equation and tabular form. The functions were given in the structural (as an object) approach and the learners had to apply the operational (as a computation) approach in order to help them in their matching of the respective forms of the graphical representations of the functions. This approach could have helped the learners in their understanding of the notion of a function in that they would have been in a position to flexibly move between the different representations of functions.

Question 4.



A boy takes part in a skate board competition. His distance from the tuck-shop at any stage of the race is given by the equation: $s(t) = 3^{t+1} + 5$ where $s(t)$ is the distance in *metres* and *t* is time in *seconds.*

4.1 Prove that before the race has started, he was 8*m* away from the tuck-shop. (2)

4.2 Calculate the distance from the tuck-shop after 3 seconds. (2)

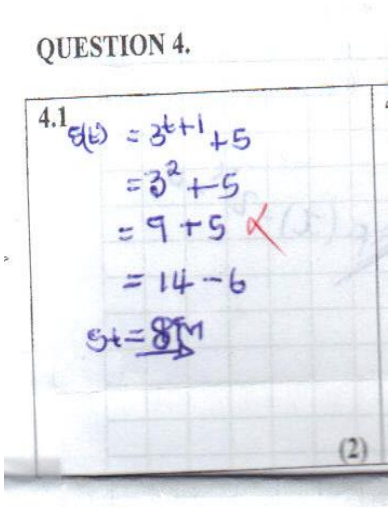
4.3 The formula $s(t) = 3^{t+1} + 5$ is not reliable enough to measure his distance after *t* seconds. Use any value of *t*, $t \in N$, and $6 \leq t \leq 9$ to prove that this statement is true. (3)

[7]

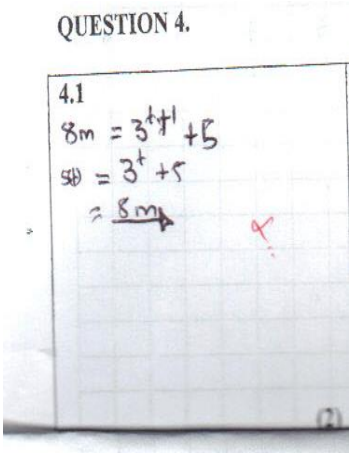
Question 4 required learners to be aware that before the start of the race, the t was zero seconds. The learners had to interpret that the value to be substituted for t was $t = 0$ and then solve the equation with this value substituted into the equation.

Some of the extracts from learner responses for question 4 from the sample are shown in extracts 3.1 below.

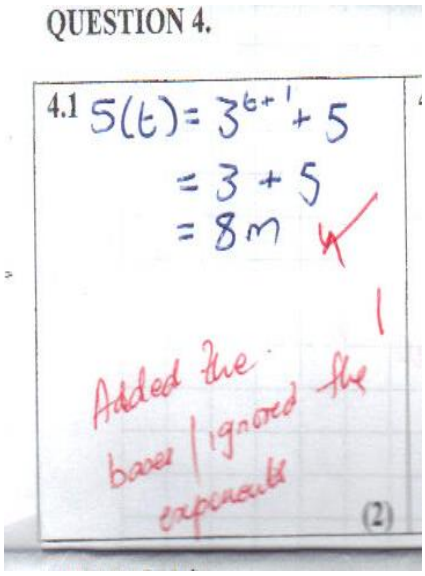
Response 1.



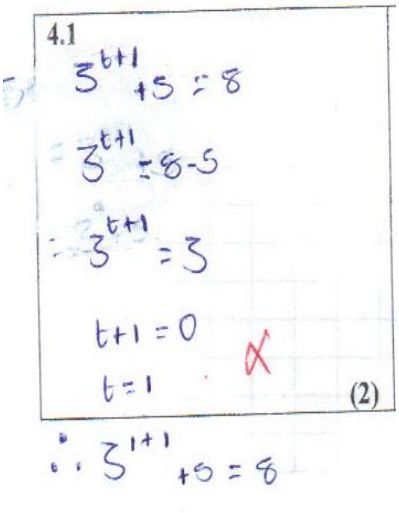
Response 2.



Response 3.



Response 4.



Extracts 3.1: Some of the extracts from learner responses for question 4.1 from the sample.

The learners' errors were computational in nature as the majority of learners could not interpret the time before the beginning of the race as being equal to zero. Most of the learners ignored the exponent and treated the bases of the exponent as natural numbers which could be added together and give the required answer of 8.

The results of the learner responses for question 4 of the pre-test are shown in figure 6.4 and table 6.4 below.

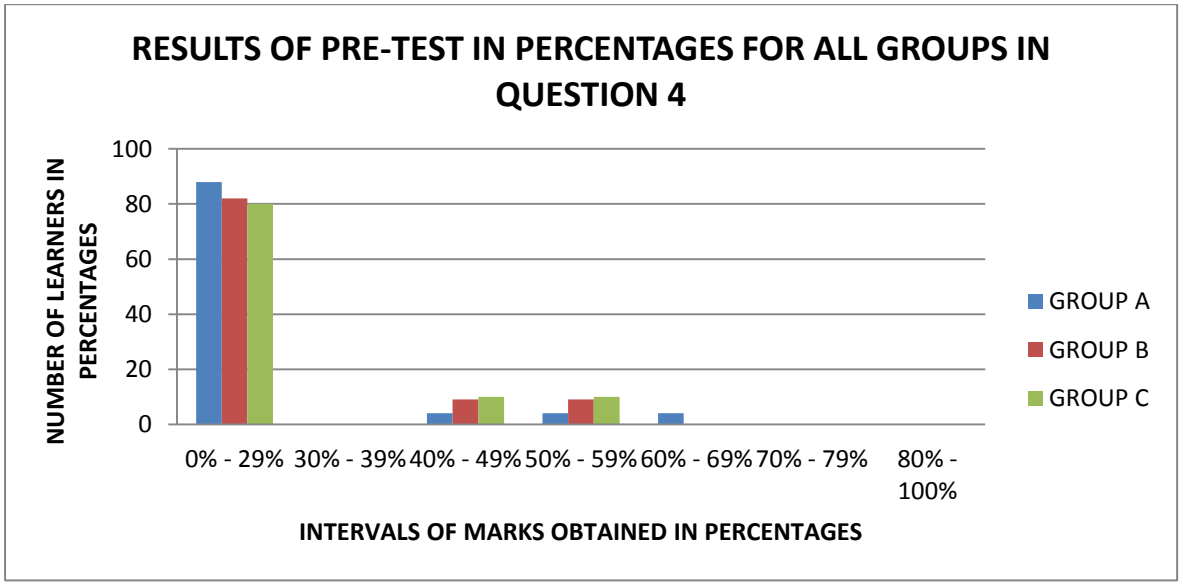


Figure 6.4: Results of the pre-test in question 4 for all the groups.

ANALYSIS OF THE RESULTS OF THE PRE-TEST					
	NUMBER OF LEARNERS IN PERCENTAGES				AVERAGE NUMBER OF LEARNERS IN PERCENTAGES
	INTERVALS IN PERCENTAGES	GROUP A	GROUP B	GROUP C	
QUESTION 4	**0% - 29%**	88	82	80	**83**
	30% - 39%	0	0	0	**0**
	40% - 49%	4	9	10	**8**
	50% - 59%	4	9	10	**8**
	60% - 69%	4	0	0	**1**
	70% - 79%	0	0	0	**0**
	80% - 100%	0	0	0	**0**
	TOTAL	**100**	**100**	**100**	**100**

Table 6.4: Results of the pre-test in question 4 for all groups.

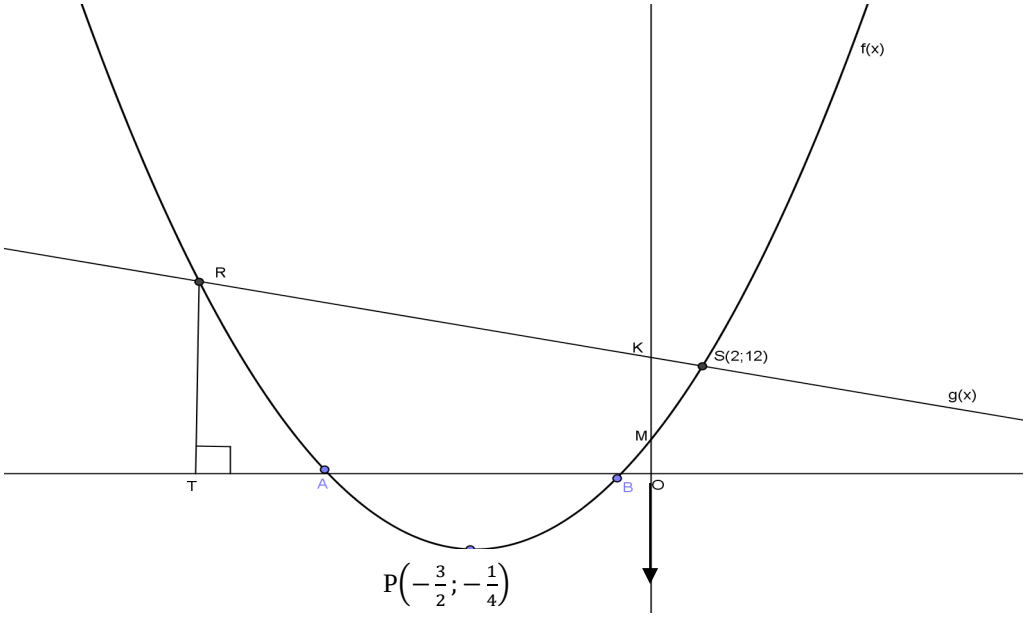
Between 80% and 90% of learners obtained marks in the interval 0% and 29%. The majority of learners could not interpret the contextualised question 4. According to Sfard (1991), the ability of students to treat functions as both objects and processes leads to the students having fewer to no problems with the interpretation of contextual situations that involve the notion of functions as has been elaborated on in chapter two of the literature review.

This question involved the interpretation of functions from a contextualised approach and the inability of learners to interpret the contextualised situation, resulted in most of the learners

not being able to answer all of question 4 correctly. The learners' overall average performance in question 1.2 for the interval 0% to 29% was 83% as it appears in the table 6.2(b) above. The average number of learners who obtained between 80% and 100% for question 1.2 for all the groups was 0%.

Question 5.

A parabola $f(x) = ax^2 + bx + c$ with turning point P$\left(-\frac{3}{2}; -\frac{1}{4}\right)$ and a straight line $g(x) = -x + 14$ intersect at the point $S(2\,;12)$. The two graphs, not drawn to scale, are drawn below. A and B are the x intercepts of the parabola. K is the y-intercept of f. RT is a straight line parallel to the y-axis.



5.1 Show that $a = 1,\ b = 3\ and\ c = 2$ (5)

5.2 Calculate the distance between A and B. (4)

5.3 Determine the length of KM, the distance between the two y-intercepts. (3)

5.4 For which values of x is f a decreasing function? (2)

5.5 Determine for which values of x is $f(x) \geq g(x)$, where $x \geq 0$. (2)

5.6 Determine the length of RT. (5)

5.7 Determine the value of $g(x^2) + g\left(\frac{1}{x}\right) - 28$. (5)

[26]

For a more precise and visual analysis of question 5 of the pre-test the mixed methods analysis is used for better understanding of the concepts of functions where learners had

serious problems. The performance of the learners in question 5 was the poorest of the other questions. For this reason, the analysis of the responses of the learners for question 5 will be separated. Question 5.1 will be analysed on its own as it forms part of the analysis that compares the pre-test and the post-test.

Question 5.2 up to question 5.7 will be analysed as one unit. These questions were not tested in the post-test but were used to answer the research question 'What aspects of functions did learners have problems with?' The learners' overall average performance in question 5 for the interval 0% to 29% was 94% as it appears in the table 6.2(a) below. The average number of learners who obtained between 80% and 100% for question 5 for all the groups was 4% as the table below attests. The learners' overall average performance in question 5 for the interval 0% to 29% was 94% as it appears in the table 6.5(a) below. The average number of learners who obtained between 80% and 100% for question 5 for all the groups was 4%.

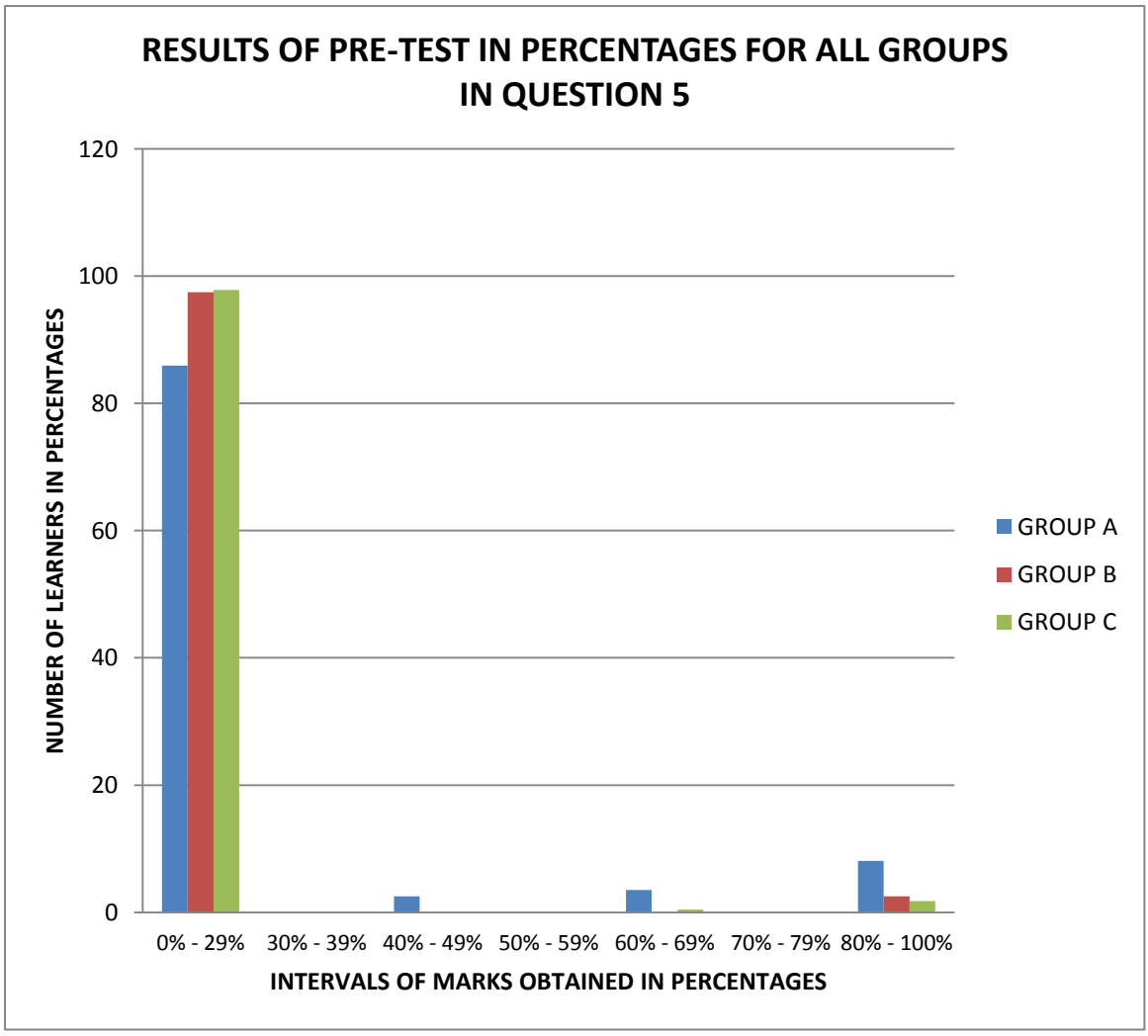


Figure 6.5(a): Results of the pre-test in question 5 for all the groups.

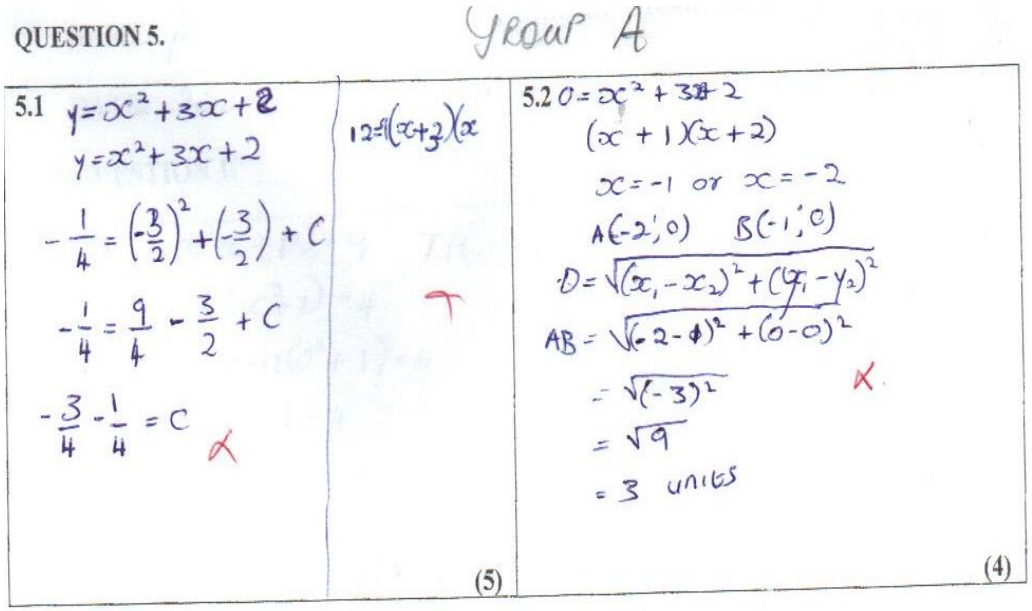
ANALYSIS OF THE RESULTS OF THE PRE-TEST					
	NUMBER OF LEARNERS IN PERCENTAGES				AVERAGE NUMBER OF LEARNERS IN PERCENTAGES
	INTERVALS IN PERCENTAGES	GROUP A	GROUP B	GROUP C	
QUESTION 5	**0% - 29%**	86	98	98	**94**
	30% - 39%	0	0	0	**0**
	40% - 49%	3	0	0	**1**
	50% - 59%	0	0	0	**0**
	60% - 69%	4	0	0	**1**
	70% - 79%	0	0	0	**0**
	80% - 100%	8	3	2	**4**
	TOTAL	**100**	**100**	**100**	**100**

Table 6.5(a): Results of the pre-test in question 5 for all the groups.

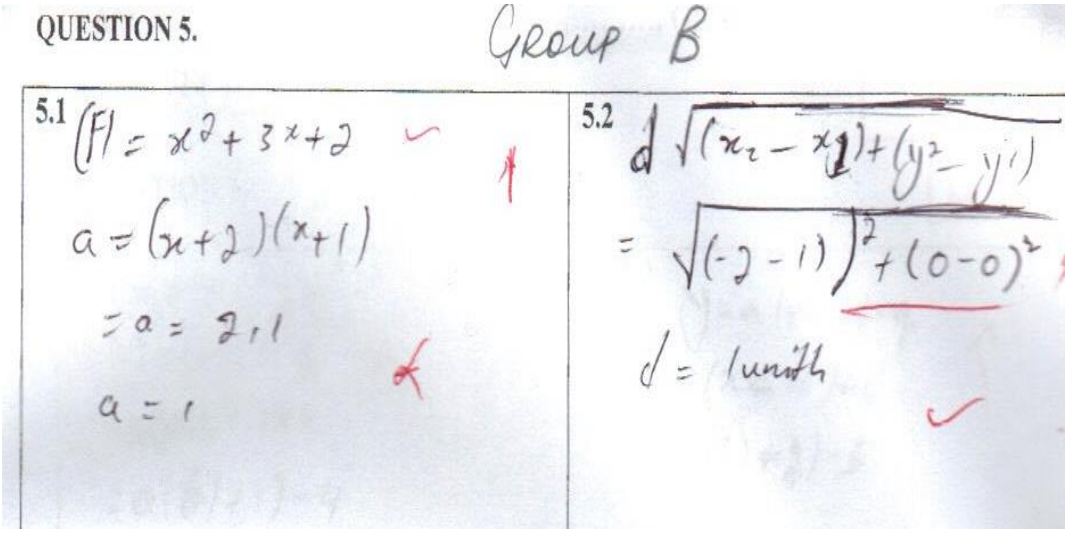
The analysis of question 5.1 is important in that it will be used in the analysis and comparison of the post-test. Question 3 of the post-test was similar to question 5.1. Some of the learner responses from the three groups appear in the following learners' extracts from the learners' answer sheets of the pre-test:

Learner responses for question 5.1 and 5.2 of the pre-test from group A are shown below:

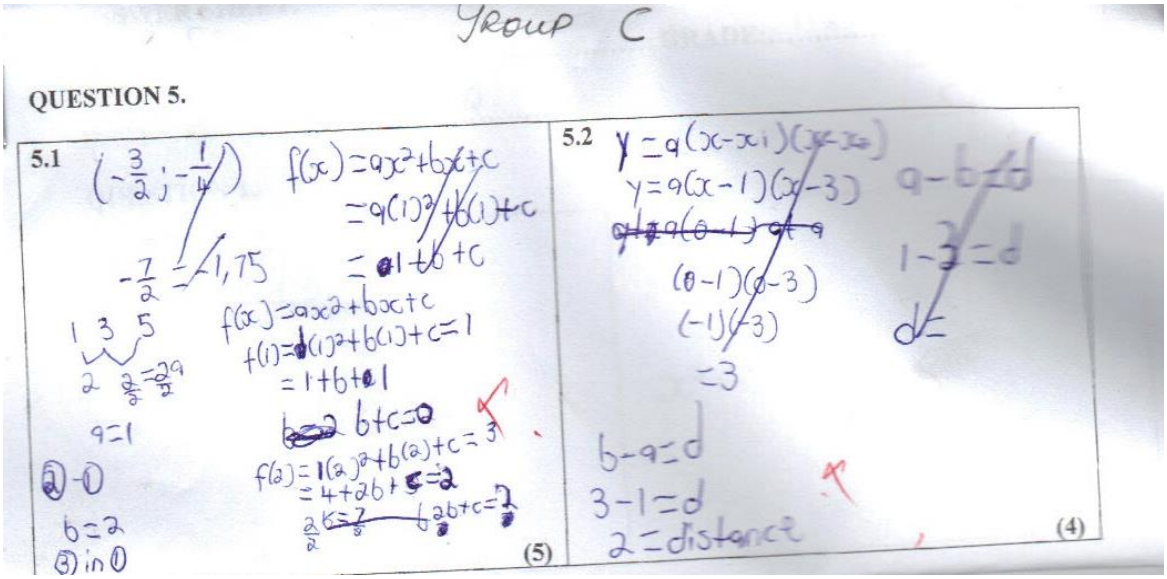
Response 1. Response 2.



Learner responses for question 5.1 and 5.2 of the pre-test from group B are shown below:



Learner responses for question 5.1 and 5.2 of the pre-test from group C are shown below:



Extracts 4.1: Some of the extracts from learner responses for questions 5.1 and 5.2 from groups A, B and C.

Once again, the learners' interpretation of the graphical form of the parabola had an impact on the way in which learners responded to question 5.1. The majority of learners used the incorrect form of the parabolic equation while some of the learners used the given values of the parameters "a", "b" and "c" in the parabolic equation $y = ax^2 + bx + c$. The majority of learners did not attempt questions 5.2 to 5.7 as they could not answer question 5.1 correctly. This inability of learners to answer the remaining questions in question 5, together with the learners' persistent use of the incorrect form of the parabolic equation in questions 1.1 and 1.2 discussed above, led to the choice of the object of learning as was discussed in chapter 5.

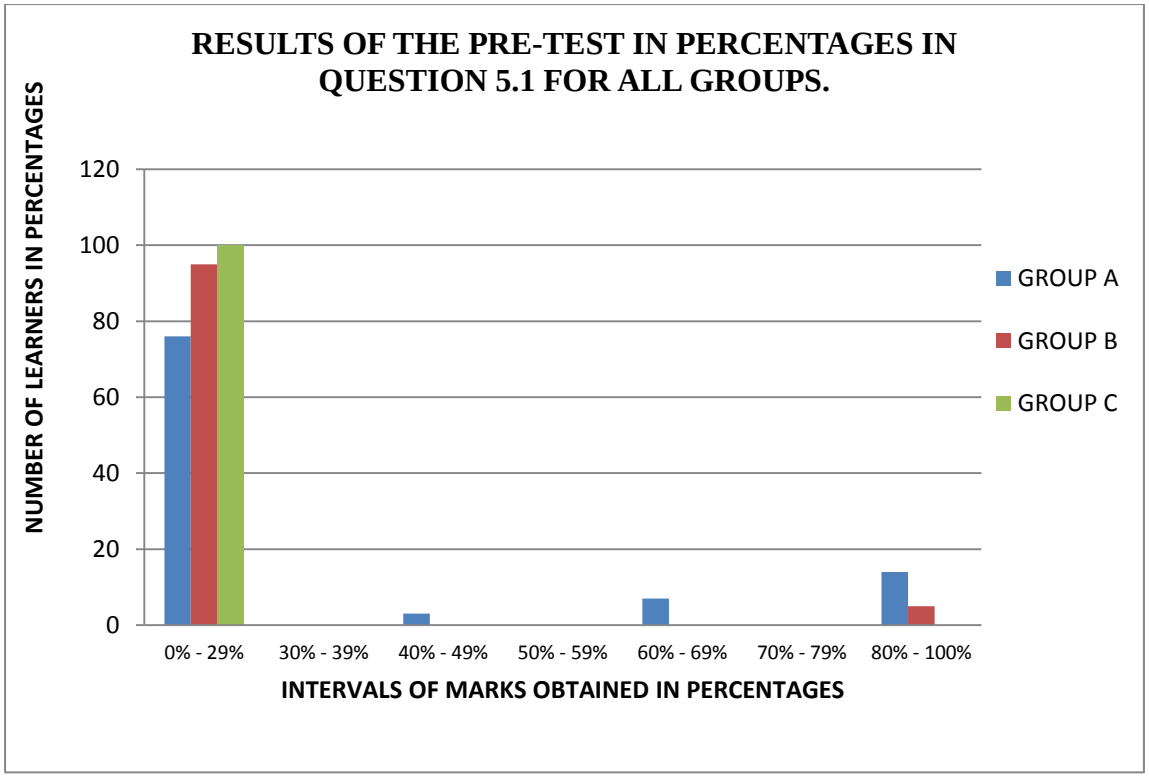


Figure 6.5(b): Results of the pre-test in question 5.1 for all the groups.

ANALYSIS OF THE RESULTS OF THE PRE-TEST					
	NUMBER OF LEARNERS IN PERCENTAGES				AVERAGE NUMBER OF LEARNERS IN PERCENTAGES
	INTERVALS IN PERCENTAGES	GROUP A	GROUP B	GROUP C	
QUESTION 5.1	**0% - 29%**	76	95	100	**90**
	30% - 39%	0	0	0	**0**
	40% - 49%	3	0	0	**1**
	50% - 59%	0	0	0	**0**
	60% - 69%	7	0	0	**2**
	70% - 79%	0	0	0	**0**
	80% - 100%	14	5	0	**6**
	TOTAL	**100**	**100**	**100**	**100**

Table 6.5(b): Results of the pre-test in question 5.1 for all the groups.

The results of the learner responses for question 5.1 of the pre-test as shown in figure 6.5(b) and table 6.5(b) above reveal that learners' overall average performance in question 5.1 for the interval 0% to 29% was 90%. The average number of learners who obtained between 80% and 100% for question 5.1 for all the groups was 6%.

The learner's performance in question 5 was the worst of the learners' performances in all the questions in the pre-test. The inability of learners to answer question 5.1 correctly could be due to the learners' inability to flexibly move between the representations as discussed in questions 1.1 and 1.2 above.

Even (1998) discovered that the factors that were involved in linking representations of functions together as a unit depended on the context in which the function was represented. Even (1998) also further elaborated that these factors consisted of two different approaches namely the point-wise approach and the global approach.

In differentiating between these approaches, she alluded to the fact that the point-wise approach mainly dealt with discrete points of a function. In question 5.1, the learners could use the point-wise approach in order to show that the values of the parameters "a", "b" and "c" were the values that they were required to show. The learners failed to use the given points on the parabola to show this as the points given in the parabola could be used in all forms of the parabolic equation namely: $y = ax^2 + bx + c$; $y = a(x - p)^2 + q$ and $y = a(x - x_1)(x - x_2)$. Even (1998) further asserts that these extremes could be useful especially when the function was given in symbolic form like in the general form as in $y = ax^2 + bx + c$ where no numeric values were given as in question 5.1.

The results of the learner responses for the rest of question 5 of the pre-test are shown in figure 6.5(c) and table 6.5(c) below. These results were combined together as they were

similar across the groups.

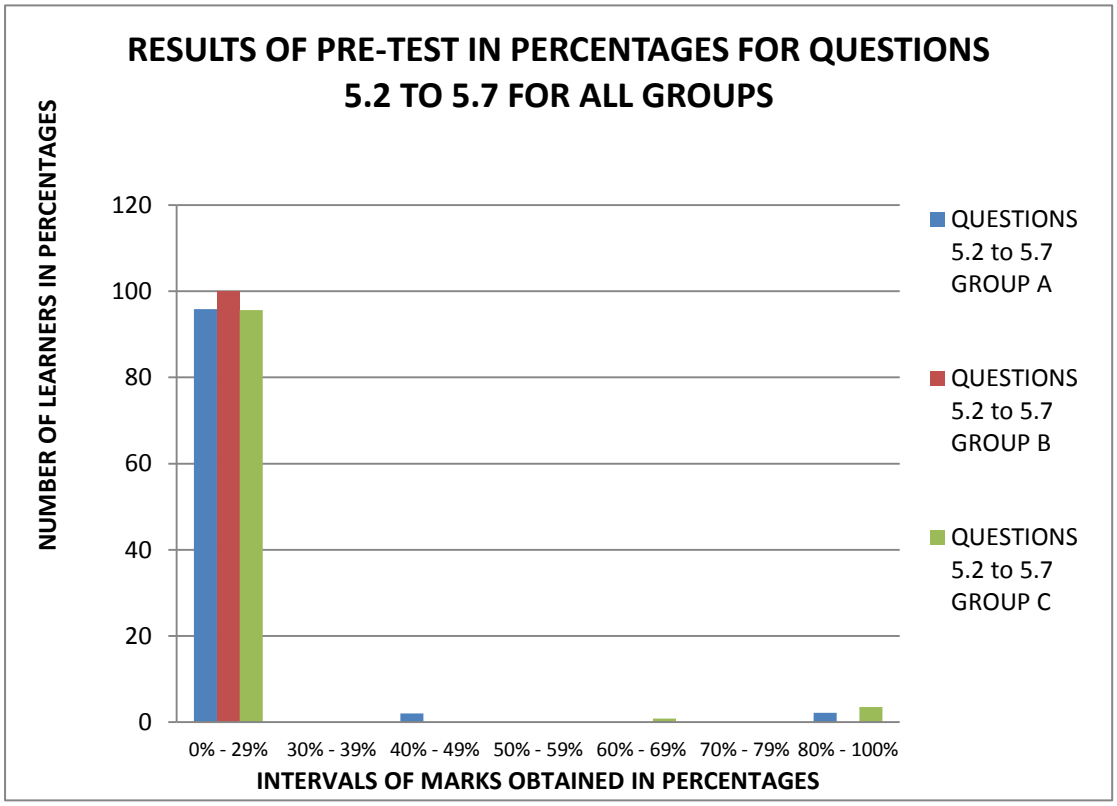


Figure 6.5(c): Results of the pre-test in questions 5.2 to 5.7 for all the groups.

ANALYSIS OF THE RESULTS OF THE PRE-TEST					
	NUMBER OF LEARNERS IN PERCENTAGES				AVERAGE NUMBER OF LEARNERS IN PERCENTAGES
	INTERVALS IN PERCENTAGES	GROUP A	GROUP B	GROUP C	
QUESTIONS 5.2 - 5.7	0% - 29%	96	100	96	97
	30% - 39%	0	0	0	0
	40% - 49%	2	0	0	1
	50% - 59%	0	0	0	0
	60% - 69%	0	0	0	0
	70% - 79%	0	0	0	0
	80% - 100%	2	0	4	2
	TOTAL	100	100	99	100

Table 6.5(c): Results of the pre-test in question 5.1 for all groups in percentages.

The global approach which dealt with looking at the extremes that the functions could be stretched could be used to answer questions 5.4 and 5.5. In dealing with these extremes, the learners could be in a position to find the trends in this Carlson, Oehrtman & Thompson (1998) assert that the different types of interpretations that can be expected from the learners included the global interpretation processes where detection of trends could be applied. These trends include the ability to detect where one graph was greater or smaller than the other one and by looking at an entire graph the learners could detect the general meaning of the situation which would give the learners an idea of the general trend of each function.

6.4 DISCUSSION OF THE PRE-TEST.

The overall results in all groups for questions 1.1, 1.2 and 5 were15%, 17% and 4% respectively refer to the interpretation of quadratic functions. The learners' ability to use the appropriate generalized form of the parabolic equation by identifying the features that were given from the parabolic graphs was of utmost importance for the learners to be able to correctly answer these questions. The analysis shows that for question 1.1, an average of only 15% of the learners could use the appropriate form of the quadratic equation to find the equation of the parabola; for question 1.2, an average of only 17% could use the appropriate form of the quadratic equation to find the equation of the parabola and for question 5, only an average of 4% of the learners could use the appropriate form of the quadratic equation to find the equation of the parabola.

The learners' responses to questions 1.1, 1.2 and 5.1 led to the identification of the object of learning. In these questions learners were required to be able to see the parabolic function both as a process and as an object according to the dual nature of a function as referred to by Sfard (1991). The dual nature of a function consists of the structural nature – as an object and the operational nature – as a process. In these questions the object is given as a graph of a parabola and represented as a structure and the operational nature is a computational process or a well-defined method for getting from one system to another.

In these questions the well-defined method is identifying the features represented in the structure (the graphical form), choosing the appropriate form of the quadratic equation into which these critical features (points, values) can be substituted, perform algorithms and solve the equation and end up with the equation of the parabola. The poor performance of learners

is evidenced by the low percentages scored by the learners in these questions. This is proof that that the dual nature of the function as alluded to by Sfard (1991) was a problematic concept for the learners as the researchers had envisaged.

In question 3.1 to 3.8 learners were required to flexibly move between the different representations of functions and give answers to the question of "What aspects of functions do learners find to be problematic? " This question was fairly attempted and an average of 41% of the learners could match the graphs to its correct representation namely: verbal form, equation form and table form. As mentioned in the literature review Even (1998, p. 526) is of the opinion that "the ability to identify and represent the same thing in different representations and flexibility in moving from one representation to another, allows one to see rich representations, develop a better conceptual understanding and strengthens one's ability to solve problems".

In order for one to fully develop insight into and understandings of the essence of a concept and its many facets, one should be able to connect the different representations of that same concept in different contexts as it is echoed by some researchers in this field who addressed the critical problem of transition between and within representations and they all are in agreement to the conclusion that the flexibility in moving from one representation of a concept to another is intertwined with other kinds of knowledge and understanding of problems in different situations.

Questions 2.1 and 2.3 expected the learners have to interpret the effect of the vertical shift of the graph as it is transformed from:

$g(x) = 2^x \; to \; f(x) = 2^x - 4 \; and \; to \; f(x) = 2^{x-4} - 4,$ and be able to interpret the functions by applying the concepts of translation of a function. This was done with the help of the concept of transformation in the form of translation where the parameters were kept the same and a four was added to the exponent. The average of 20% of the learners could correctly interpret the transformation. According to Leinhardt, Zaslavsky & Stein (1990) functions and graphs focus on the use of one symbolic system to expand and understand another system. Graphs are important in mathematics learning as they enabled learners to acquire the transformational notion that is important in regrouping and expanding the number system from counting numbers to rational numbers, adding and subtracting numbers, multiplying and dividing these numbers and are transferable to other parts of mathematics and learning areas as well. This results in learners who are capable of solving graphical and

functional problems in mathematics being unable to access their knowledge in other learning areas including science and in transferring their functional knowledge to contextual situations as is evident in these questions.

The authors assert that there are different types of interpretations that can be expected from the learners. Some of the interpretations referred to by the authors are translation, interpretation of local processes (regarding point-to-point attention), global interpretation processes (detecting trends), global and general (what happens to $y\ as\ x\ increases$, continuation (interpolation/extrapolation), rate (how fast should a car travel to reach a certain distance in one hour), qualitative interpretation (looking at an entire graph to gain the general meaning of the situation) and quantitative (a collection of isolated points).

Questions 1, 2, 3 and 5 involve the contrasting pattern of variation. It allows learners to experience the different aspects of different functions and be able to identify the contrasting aspects of each type of function: linear functions with the general appearance of $f(x) = mx + c$ as in question 5, quadratic function with its general form of $g(x) = ax^2 + bx + c$ as in questions 1 and 5 and exponential function with its general form of $h(x) = ab^x$ +q as in questions 2 and 3.

Question 4 involves the interpretation of the functions in contextual situations. An average of 17% of the learners managed to interpret the contextualised form of the problem and could calculate the distance travelled and make realistic conclusions as in appendix D.

The analysis of the diagnostic test gives an answer to the critical question ‘What aspects of functions do learners find to be most problematic?’ The results revealed that the aspect of functions that learners found to be most problematic was the ability to interpret the features of a parabolic function from the graphical form of the quadratic function to the equation form of the quadratic function. In the absence of sufficient knowledge on the form of parabolic equation to be used in different situations, the learners are running the risk of not performing well in questions relating to this notion of functions in the examination.

The results of the pre-test then informed the choice of the intervention lessons that were described in the previous chapter. The intervention lessons led to the post-test and will be analysed and interpreted quantitatively in the next sections.

The analysis of the pre-test gives an answer to the critical question ‘What aspects of functions were problematic to the learners’? The answer that resulted from the analysis of the pre-test

was that the learners had problems with most of the aspects of functions that were tested in the pre-test. From the aspects of functions that learners found to be problematic, the aspects of functions that were most problematic were chosen as the identification of the appropriate form of the parabolic equation to be used in the event that a parabolic graph was given in order to find the equation of a parabola. This became the aspect of functions that was the most problematic for the learners as the learners could not answer most of the questions as the majority of questions depended on the learner's knowledge of the equation of the parabolic function as in questions 1.1, 1.2 and 5.1.

6.5 DISCUSSION OF THE POST-TEST.

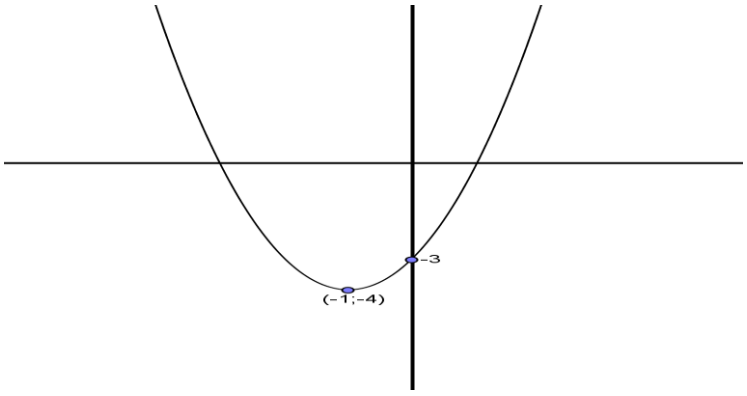
The object of learning that informed the planning of the intervention lessons was chosen from the result of the pre-test. The intervention lessons were divided into three lessons as discussed in chapter five. The first lesson was planned with the panel of observers with the object of learning as the focal point. The learners were immediately given a post-test after each lesson and the results of the post-test after each lesson are discussed below.

6.5.1 THE POST-TEST.

The post-test consisted of three questions. The first two questions of the pos-test are the same two questions of the pre-test. The third question of the post-test consists of a question similar to question 5.1 of the pre-test.

Question 1

In question 1 the learners were required to find the equation of the parabolic graph where the turning point of the parabola and another point on the graph was given.

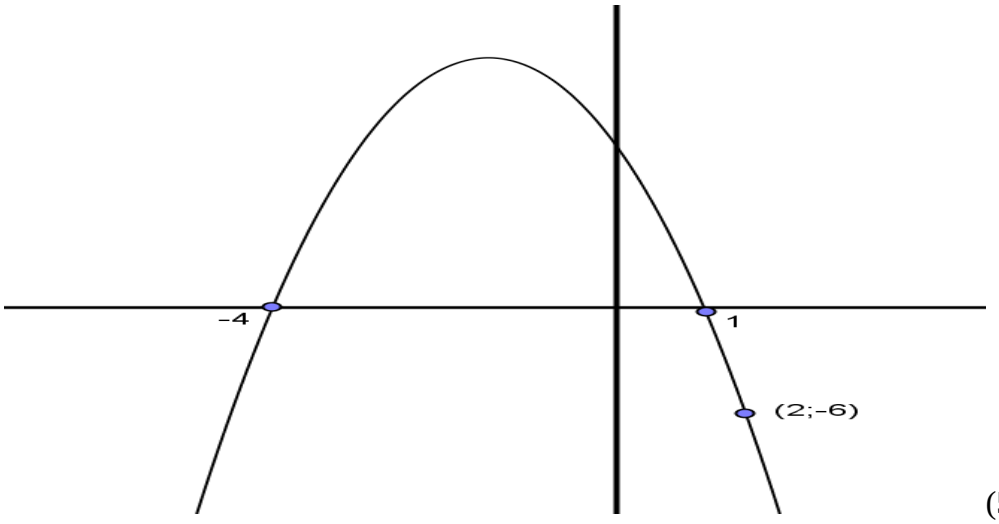


(5)

In this question learners were required to interpret the graphical representation of a parabola and be able to move from the graphical representation of a function to the appropriate form of the equation of a parabola. In order for the learners to be able to do this, they had to be able to recognise the given key features of the parabolic function and substitute the given points on the parabola in the appropriate form of the parabolic equation.

Question 2

In question 2, the learners were required to find the equation of the parabolic graph where the x-intercepts of the parabola and another point on the graph was given.



(5)

Similar to question 1 above, this question required learners to interpret the graphical representation of a parabola and be able to move from the graphical representation of a function to the equation form of the parabola. In order for the learners to be able to do this, they had to be able to recognise the given key features of the parabolic function and substitute the given points on the parabola in the appropriate form of the parabolic equation.

Question 3.

Generate the equation of a parabola passing through the given points.

A(0;-1), B(1;2) and C(-2;5) (10)

In question 3, the learners were given verbal information concerning the problem that they had to solve. This question required learners to convert from the verbal representation of a

function to equation form of the function. In so doing, the learners were afforded the opportunity to interpret the verbal information and convert it to the equation form of functional representation.

6.5.2 DISCUSSION OF THE INTERVENTION LESSONS WITH RESPECT TO THE POST-TEST.

The first intervention lesson was performed on group A. The lesson focussed on the introduction of the lesson using separation as the pattern of variation. The three different forms of the parabolic equation were introduced in their general form on separate slides. The learners were expected to identify the critical features of the parabolic function from each of the general forms of the equations of the parabola separately. The learners had difficulties in identifying the critical features of the parabola from the general forms.

From the identification of the critical features of the parabola from the general forms of the parabolic equations, the learners were given the graphical representations of the same parabola with the critical features shown in separate parabolic graphs. From these graphical forms of the parabola, the learners had to choose the appropriate form of the parabolic equation to be used in order to find the equation of the parabola. The critical features of the parabola were shown as (i) the x- intercepts of the parabola, (ii) the turning point of the parabola and (iii) any three points on the parabola.

After the graphical representations, the learners were given the same specific equation of the parabola in the different forms so that the leaners could identify the critical features that each of the different forms of the parabola represented. The pattern of variation prevalent in this lesson was that of separation in that the graphical representation of the parabola was kept the same while the critical features on the graphical representations were varied. Later on in the same lesson, the same equation of the parabola was presented to the learners while the forms of the parabolic equation were varied.

The same process repeated with three more graphical representations of the same parabola followed by the same equation of the parabola written in the different forms of the parabolic equation. The results of the post-test written by this group of learners will be discussed in the next section.

The difficulties experienced by the learners from group A necessitated the redirection of the manner in which the next lesson should be introduced. The three forms of the equation of the parabola were written on different slides and the learners frequently requested that the previous slides of the different forms of the parabolic equation be shown in order for them to be able to identify the key features of the parabolic function from each the general forms of the equations of the parabola that appeared on other slides.

Some of the responses of the learners in the post-test showed that some learners were still unsure of the form of the parabolic equation to be used. This led to the team of observers and me to come to a conclusion that the general forms of the quadratic functions were taught as separate entities thus the inability of learners to associate the given information with the appropriate general form of the equation of a quadratic function.

Lesson 2 was introduced to the learners with simultaneity as the prevalent pattern of variation. In applying this pattern of variation, all the three forms of the equation of the parabola were written on the same slide and also on the chalkboard for easy accessibility. The learners had immediate access to all the different forms of the general parabolic equations. The learners were required to identify the key features of the parabola from these different forms of the parabolic equations. Thereafter the different forms of the same specific equation of the parabola were shown to the learners in order for them to be able to identify the key features of the parabolic function from each the general forms of the equations of the parabola that appeared on the same slide.

This appeared to pacify the learners and their responses were more spontaneous than with the learners in lesson 1. The responses of the learners were more fluent than in lesson 1. The learners could immediately see the different forms of the parabolic function and were able to identify the key features represented by each form of the parabolic function.

The difficulties experienced by the learners from this group B were different from those in group A. Their difficulties were more concerned with how the general forms of the quadratic equations were derived. This necessitated the redirection of the manner in which the next lesson should be introduced. Some of the responses of the learners in the post-test showed that some learners were not sure of the points to be substituted into the appropriate form of the parabolic equation that was used.

The results of the post-test after lesson 2 will be discussed in the next section. The difficulties that the learners experienced were concerned with how the general forms were derived. This prompted the changes to the lesson that involved how the general forms of the parabolic functions came to be derived.

Lesson 3 was planned a way that would resolve the problem that the learners had with connecting the key features of a graph of a quadratic function to its respective form of the equation of a parabolic function. The manner in which this was done was through the derivation of the different forms of the equations of a quadratic function from the specific equation of a function.

Lesson 3 began with the specific equation of a parabola. From this specific equation, the learners were required to perform operations of factorisation of the specific equation, the completion of the square of the specific equation and the identification of the key features of the original specific equation of a parabola in the form of $y = ax^2 + bx + c$. The algebraic manipulation of the specific equation of a parabola through the processes of factorisation resulted in the derivation of the general form of the parabolic function as

$y = a(x - x_1)(x - x_2)$ and the algebraic manipulations involving completion of the square on the specific equation of a parabola resulted in the derivation of the general form of the parabolic function: $y = a(x - p)^2 + q$. The introduction of the lessons with simultaneity, fusion and generalisation as the patterns of variation resulted in the learners' discernment of the object of learning. The results of the post-test after lesson 3 will be discussed in the next section.

6.6 RESULTS OF THE POST-TEST.

The post-test consisted of three questions as in appendix F and it generated the data as represented in figures 6.4(a) and table 6.4(a) below. The average percentages of both the pre-test and the post-test for all the groups in all the questions are summarised in figure 6.4(b) and table 6.4(b) below. The post-test consisted of questions 1.1 and 1.2 of the pre-test numbered as question 1 and 2. A question 3 was added as a result of the learners' inability to answer question 5.1 of the pre-test. This came about from the analysis of the pre-test question 5.1. The post-test therefore consisted of questions 1.1 and 1.2 of the pre-test and a third

question. These questions were numbered questions 1, 2 and 3. Question 3 required learners to find the equation of a graph of a parabola with three points on the parabola labelled on the graphical representations.

For a more precise and visual analysis of the post-test the mixed methods analysis is used for better understanding of the concepts of functions where learners had serious problems. These results are visually shown in the figures and tables that follow.

The results of the post-test are represented in a horizontal bar graph that appears as Figure 6.7(a) and numerically in table 6.7(a) below.

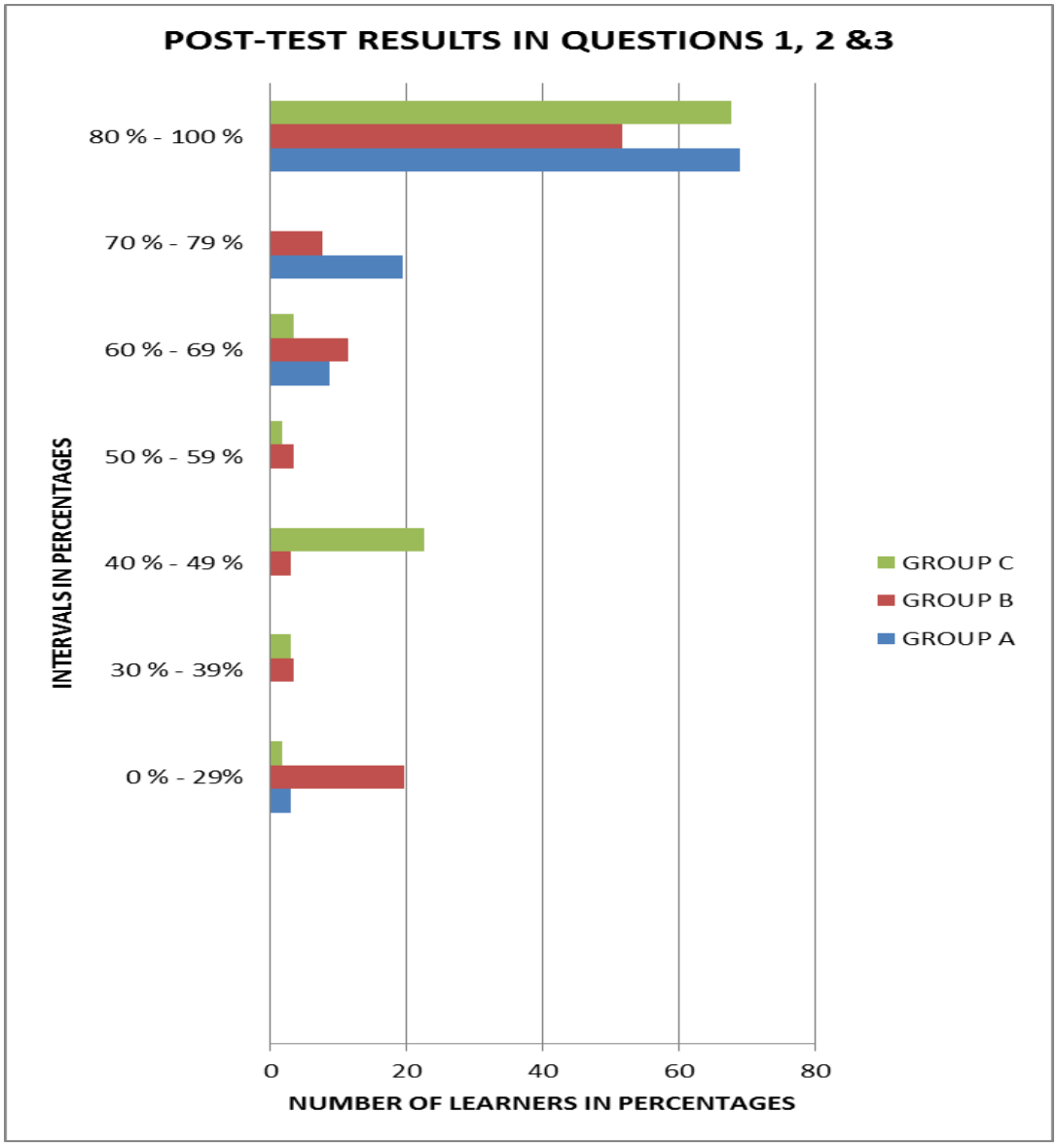


Figure 6.6: Post-test results in questions 1, 2 and 3 for all the groups.

NUMBER OF LEARNERS IN PERCENTAGES FOR ALL GROUPS IN ALL QUESTIONS						
INTERVALS OF MARKS OBTAINED IN PERCENTAGES	GROUP A		GROUP B		GROUP C	
	PRE-TEST	POST-TEST	PRE-TEST	POST-TEST	PRE-TEST	POST-TEST
0% - 29%	74	1	91	20	94	2
30% - 39%	0	0	0	3	0	3
40% - 49%	4	2	3	3	2	23
50% - 59%	0	0	0	3	0	2
60% - 69%	14	6	2	11	2	3
70% - 79%	0	0	0	8	0	0
80% - 100%	8	91	4	52	3	68
TOTAL	100	100	100	100	100	100

Table 6.6: Comparison of the results of the pre-test and post-test in all questions for all groups.

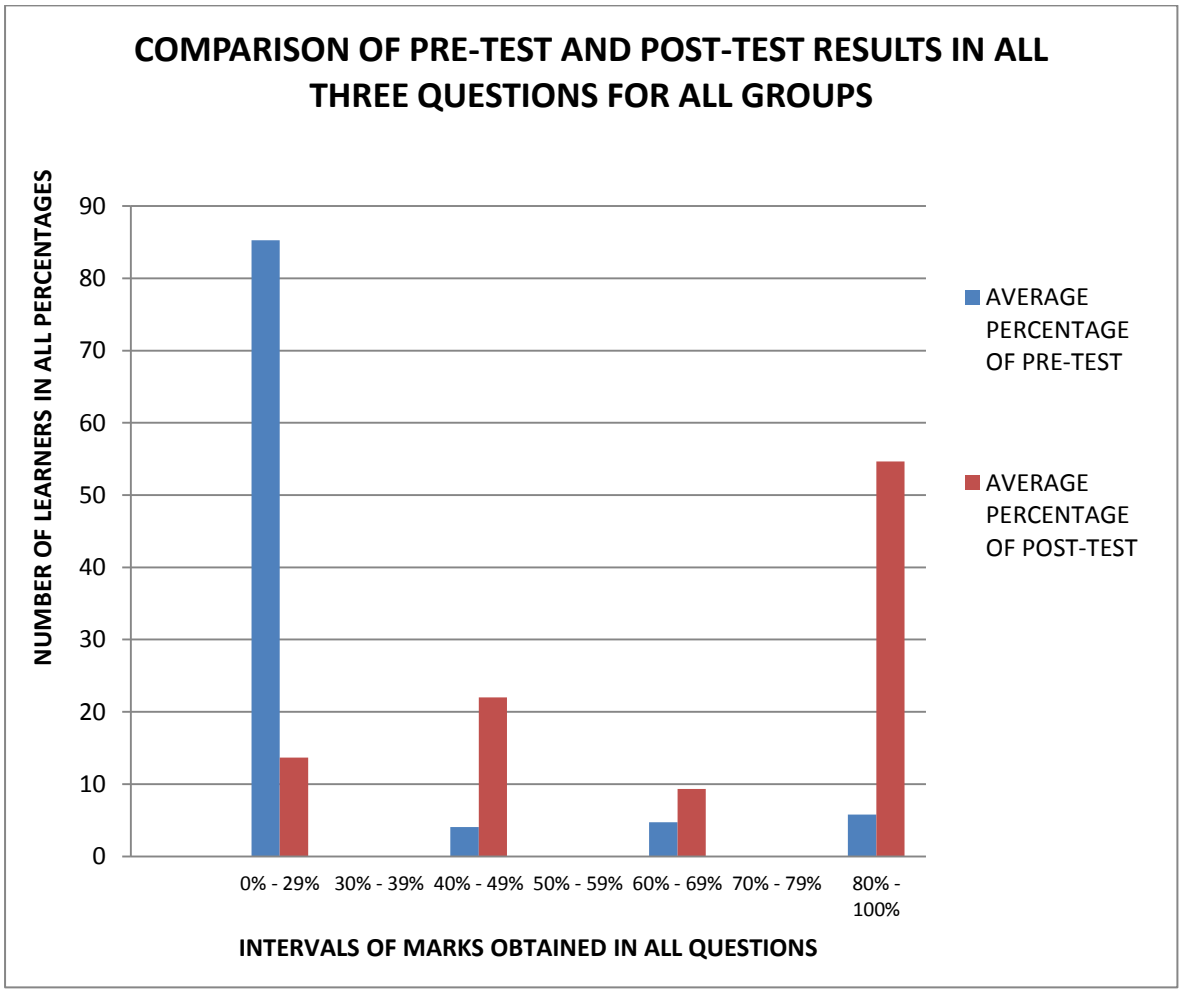


Figure 6.6(a): Comparison of the averages of the results of the pre-test and post-test in all questions for all groups.

AVERAGE PERCENTAGES OF THE RESULTS OF THE PRE- AND POST-TEST IN ALL QUESTIONS FOR ALL GROUPS			
INTERVALS OF MARKS OBTAINED IN PERCENTAGES	**AVERAGE PERCENTAGE OF PRE-TEST**	**AVERAGE PERCENTAGE OF POST-TEST**	**PERCENTAGE INCREASE(+) OR DECREASE(-)**
	GROUPS A, B & C		
0% - 29%	85	14	**-72**
30% - 39%	0	0	**0**
40% - 49%	4	22	**18**
50% - 59%	0	0	**0**
60% - 69%	5	9	**5**
70% - 79%	0	0	**0**
80% - 100%	6	55	**49**
TOTAL	**100**	**100**	

Table 6.6(a): Comparison of the averages of the results of the pre-test and post-test in all questions for all groups.

The learners in the sample's overall performance improved after the intervention lessons. Their performance in the interval 80% to 100% increased from 6% to 55% in all test questions. Their performance in the 0% to 29% decreased from 85% overall to 14%.

In the following section, each group's performance in the pre-test will be compared with their performance in the post-test.

6.7 COMPARISON OF THE PRE-TEST AND THE POST-TEST IN EACH GROUP.

As noted earlier it was thought that the same two questions in both the diagnostic test and the post-test could be treated statistically in an attempt to determine whether the differences were of any statistical significance or not. Figures 6.6(a) and table 6.6(a) shows the comparison of both tests in the first and second questions for both groups respectively:

The questions 1.1 and 1.2 in the diagnostic test are the same questions that were tested in the post-test as questions 1 and 2. The data for the two questions for the post-test is represented in figure 6.6(a). Question 5.1 in the diagnostic test is similar but not the same as question 2 given in the post-test as question 3.

The comparison of the pre-test and the post-test in the three questions as discussed above appear in the next section. These comparisons of the individual groups will try to answer the main question of the study 'To what extent does a version of a learning study framed by variation theory improve the Grade 11's interpretation of functions?'

In order to answer this main question, the results of the learners' performance in both the pre-test and the post-test will be compared for each individual group. In so doing, the effects of each intervention lesson will be measured to ascertain which lesson had the most effect on the performance of the learners.

6.7.1 COMPARISON OF THE PRE-TEST AND THE POST-TEST FOR LESSON 1 (GROUP A).

The comparison of the results of the performance of the learners of group A in both the pre-test and the post-test are shown in figure 6.5.1(a) and table 6.5.1(a) below.

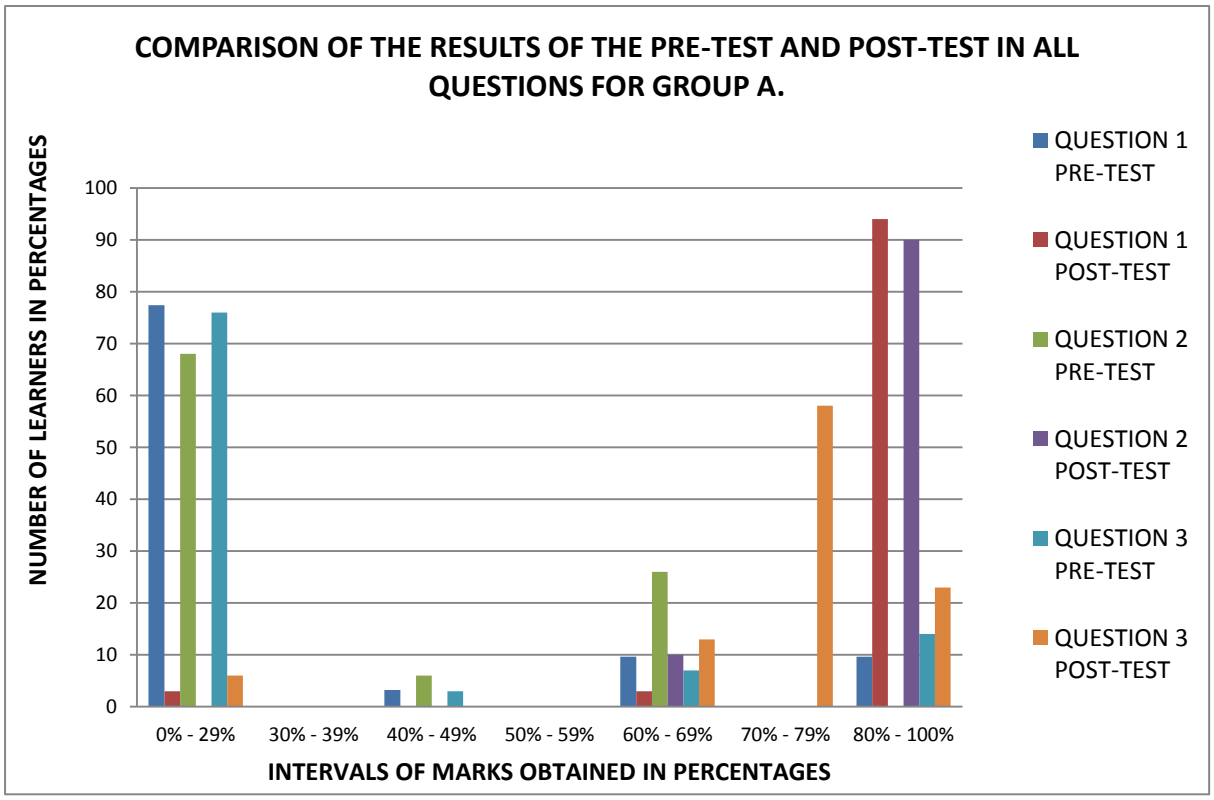


Figure 6.7(a): Comparison of the results of the pre-test and post-test in all questions for group A.

The table 6.7(a) below shows the actual numbers of learners in percentages that were obtained by the learners of group A in both the pre-test and post-test.

NUMBER OF LEARNERS IN PERCENTAGES						
GROUP A						
INTERVALS OF MARKS OBTAINED IN PERCENTAGES	QUESTION 1		QUESTION 2		QUESTION 3	
	PRE-TEST	POST-TEST	PRE-TEST	POST-TEST	PRE-TEST	POST-TEST
0% - 29%	77	3	68		76	6
30% - 39%	0		0			
40% - 49%	3		6		3	
50% - 59%	0		0			
60% - 69%	10	3	26	10	7	13
70% - 79%	0		0			58
80% - 100%	10	94	0	90	14	23
TOTAL	100	100	100	100	100	100

Table 6.7(a): Comparison of the results of the pre-test and post-test in all questions for group A.

The comparison of the average percentage attainment of learners of group A in both the pre-test and the post-test are shown in figure 6.7(b) and table 6.7(b) below.

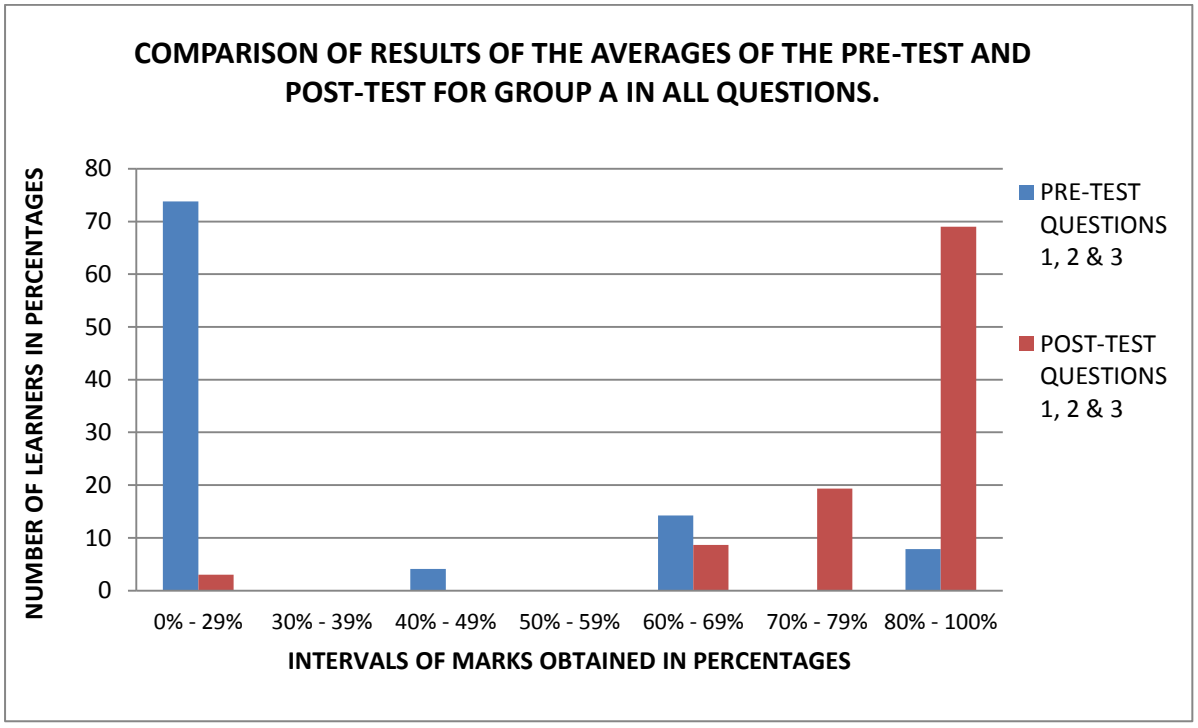


Figure 6.7(b): Comparison of the results of the pre-test and post-test in all the questions for group A.

NUMBER OF LEARNERS IN PERCENTAGES			
GROUP A			
INTERVALS OF MARKS OBTAINED IN PERCENTAGES	**PRE-TEST QUESTIONS 1, 2 & 3**	**POST-TEST QUESTIONS 1, 2 & 3**	**INCREASE(+)/ DECREASE(-)**
0% - 29%	74	3	**-71**
30% - 39%	0	0	**0**
40% - 49%	4	0	**-4**
50% - 59%	0	0	**0**
60% - 69%	14	9	**-6**
70% - 79%	0	19	**19**
80% - 100%	8	69	**61**

Table 6.7(b): Comparison of the results of the pre-test and post-test in all the questions for group A.

6.7.2 INTERPRETATION OF THE DATA AFTER LESSON 1 (GROUP A).

The object of learning that was the focus of the study was "Discerning the features of a quadratic function and identification and use of the appropriate form of the equation of the quadratic function to generate the equation of the quadratic function from the graph(s) of the quadratic function or any information about the quadratic function". The percentage number of learners in the intervals 80% to 100% increased from 8% to 69% and the average percentage number of learners in the interval 0% to 29% decreased from 74% to 3%.for group A.

This improvement could be attributed to the intervention lesson which began with the general forms of each form of the equation of the parabolic function. These general forms of the equation of the parabola were introduced separately at different times. From the general form of the equation of a parabola the lessons continued to the specific equation with each form of the parabolic equation dealt with separately. The pattern of variation that was prominent in lesson 1 was that of separation. In this pattern of variation, the form of the general equation of a parabola was varied but the equation and the graph of the parabola was kept the same.

At the end of the lesson, the learners were ultimately able to find the equation of a parabola by using the appropriate form of the equation of the parabola. They were able to interpret the different forms of the parabolic equations. They were also able to discern the key features of

a parabolic graph. They were able to flexibly move between the equation representation to the graphical representation of the parabola and vice versa.

6.7.3 COMPARISON OF THE PRE-TEST AND THE POST-TEST FOR LESSON 2 (GROUP B).

The second lesson was planned according to the results of the reflections made after lesson 1 for group A. The results after lesson 2 for group B are shown in figure 6.5.2(a) and table 6.5.2(a) below.

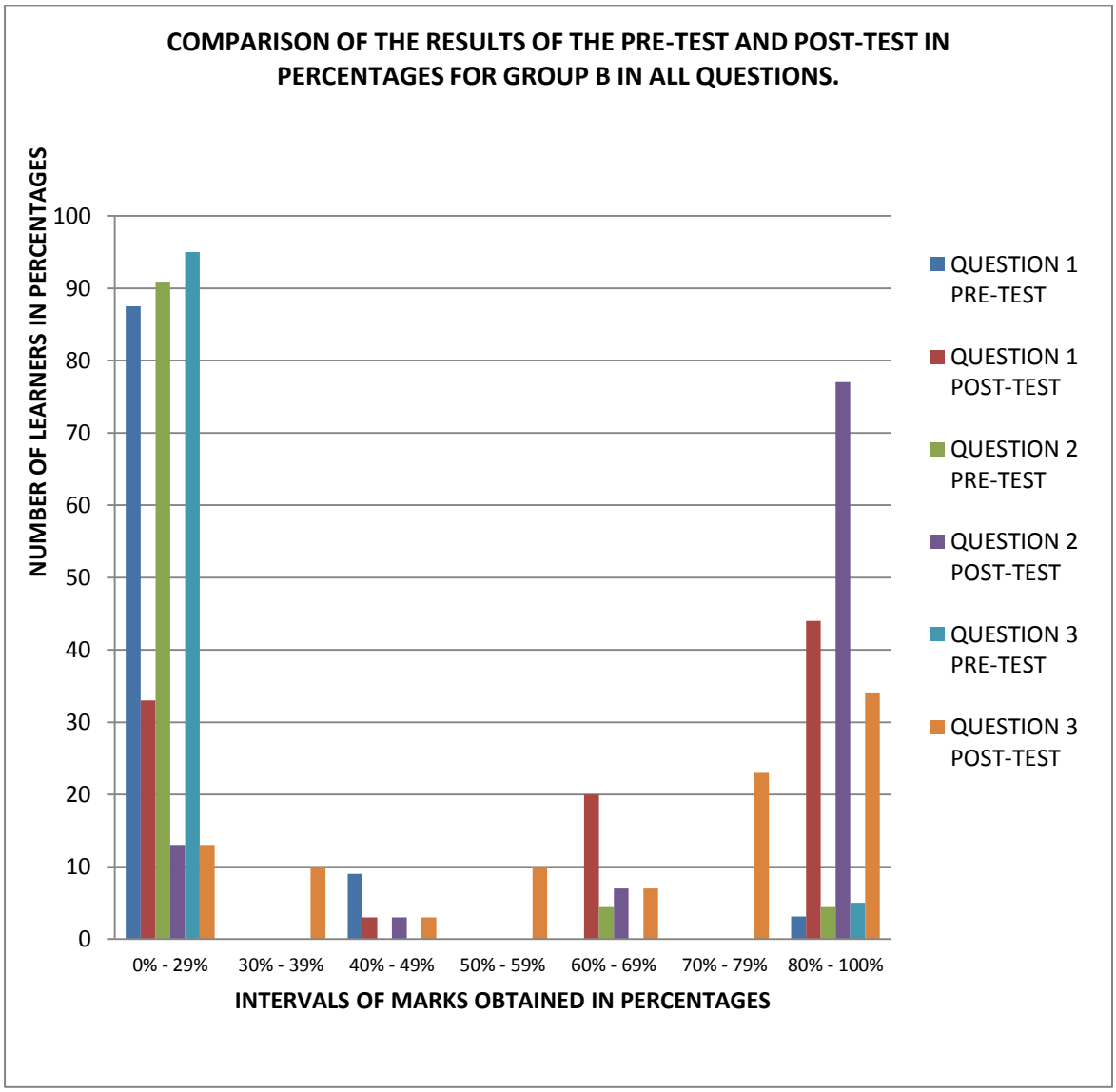


Figure 6.8(a): Comparison of the results of the pre-test and post-test in all questions for group B.

NUMBER OF LEARNERS IN PERCENTAGES						
GROUP B						
INTERVALS OF MARKS OBTAINED IN PERCENTAGES	QUESTION 1		QUESTION 2		QUESTION 3	
	PRE-TEST	POST-TEST	PRE-TEST	POST-TEST	PRE-TEST	POST-TEST
0% - 29%	88	33	91	13	95	13
30% - 39%	0		0			10
40% - 49%	9	3	0	3		3
50% - 59%	0		0			10
60% - 69%	0	20	5	7		7
70% - 79%	0		0			23
80% - 100%	3	44	5	77	5	34
TOTAL	100	100	100	100	100	100

Table 6.8(a): Comparison of the results of the pre-test and post-test in all questions for group B.

The comparison of the average percentage attainment of learners of group B in both the pre-test and the post-test are shown in figure 6.8(b) and table 6.8(b) below.

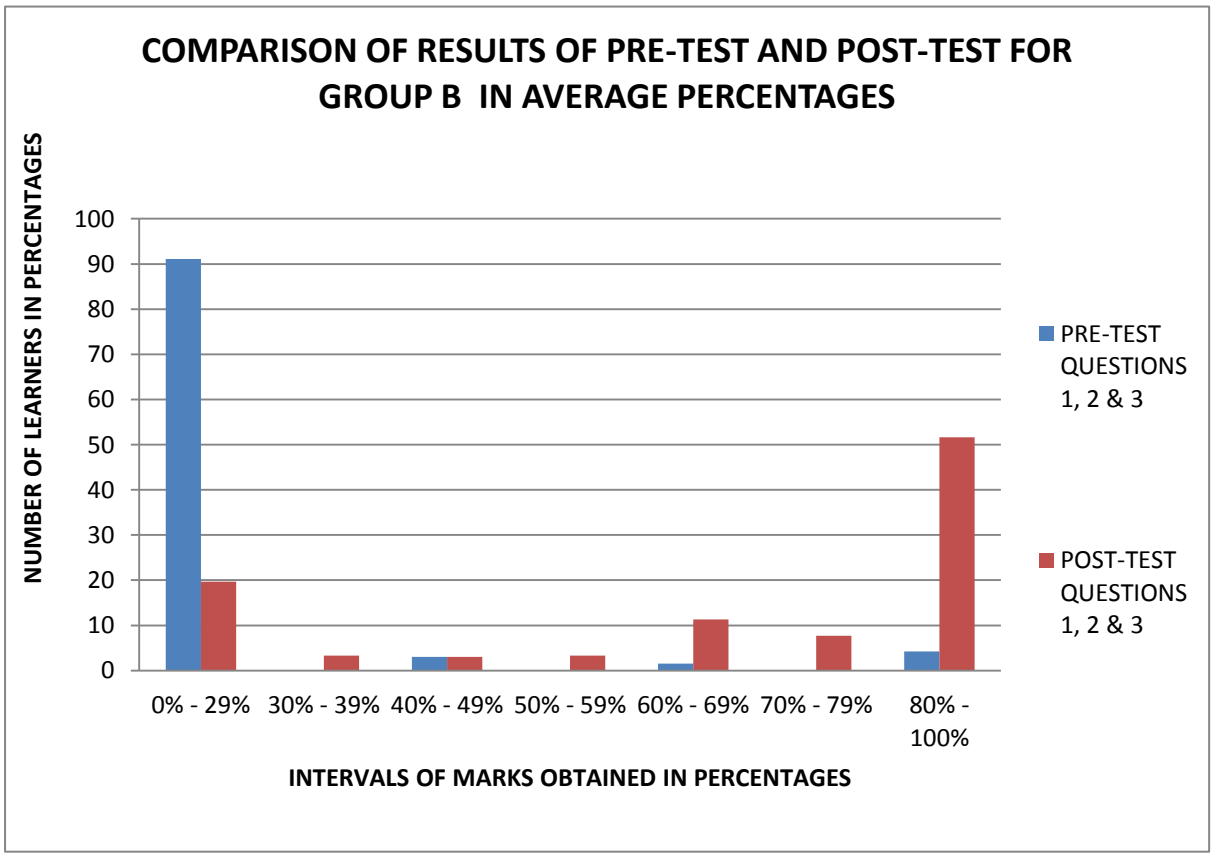


Figure 6.8(b): Comparison of the results of the pre-test and post-test in all the questions for group B.

NUMBER OF LEARNERS IN PERCENTAGES			
GROUP B			
INTERVALS OF MARKS OBTAINED IN PERCENTAGES	**PRE-TEST QUESTIONS 1, 2 & 3**	**POST-TEST QUESTIONS 1, 2 & 3**	**INCREASE(+)/ DECREASE(-)**
0% - 29%	91	20	-71
30% - 39%	0	3	3
40% - 59%	3	3	0
50% - 59%	0	3	3
60% - 69%	2	11	10
70% - 79%	0	8	8
80% - 100%	4	52	47
TOTAL	**100**	**100**	

Table 6.8(b): Comparison of the results of the pre-test and post-test in all the questions for group B.

6.7.4 INTERPRETATION OF THE DATA AFTER LESSON 2 (GROUP B).

The object of learning that was the focus of the study was "Discerning the features of a quadratic function and identification and use of the appropriate form of the equation of the quadratic function to generate the equation of the quadratic function from the graph(s) of the quadratic function or any information about the quadratic function".

The average percentage number of learners in the intervals 80% to 100% increased from 4% to 52% and the average percentage in the interval 0% to 29% decreased from 91% to 20% for group B.

These general forms of the equation of the parabola were introduced to the learners simultaneously. From these general forms of the equation of a parabola the learners were then required to find the key features of each of the forms of the quadratic functions. The patterns of variation that were prominent in lesson 2 were those of separation and simultaneity. These patterns of variation enabled learners to see the differences and the key features of a parabola from these forms of the parabolic equations. The different forms of the general equation of a parabola were simultaneously varied for learners to be able to see the key features of each form of the quadratic function at the same time.

Through the simultaneous introduction of these patterns of variation, the learners were afforded the opportunity to discern the features of the parabola that were appropriate from each form of the equation of a parabolic function. At the end of the lesson, the learners were ultimately able to find the equation of a parabola by using the appropriate form of the equation of the parabola. They were able to interpret the different forms of the parabolic equations. They were also able to discern the key features of a parabolic graph. They were able to flexibly move between the equation representation to the graphical representation of the parabola and vice versa.

The learners were ultimately able to find the equation of a parabola by using the appropriate form of the equation of the parabola. They were able to interpret the different forms of the parabolic equations. They were also able to discern the key features of a parabolic graph. They were able to flexibly move between the equation representation to the graphical representation of the parabola and vice versa.

6.7.5 COMPARISON OF THE PRE-TEST AND THE POST-TEST FOR LESSON 3 (GROUP C).

The third lesson was planned according to the results of the reflections on lesson 2. It was decided that the next lesson should begin with the specific equation. The change was decided upon due to the questions that the learners of group B posed regarding the derivation of the general forms of the quadratic function. In this lesson, the leaners were afforded the opportunity to derive the general forms of the equations of the parabolic function. In so doing, the learners were exposed to the in-depth knowledge of how to associate the general forms of the equations of the quadratic function with the different forms of the specific equations after having performed algebraic manipulations on the specific equations of the quadratic function. The results of the post-test after lesson 3 for group C are shown in figure 6.9(a) and table 6.9(a) below.

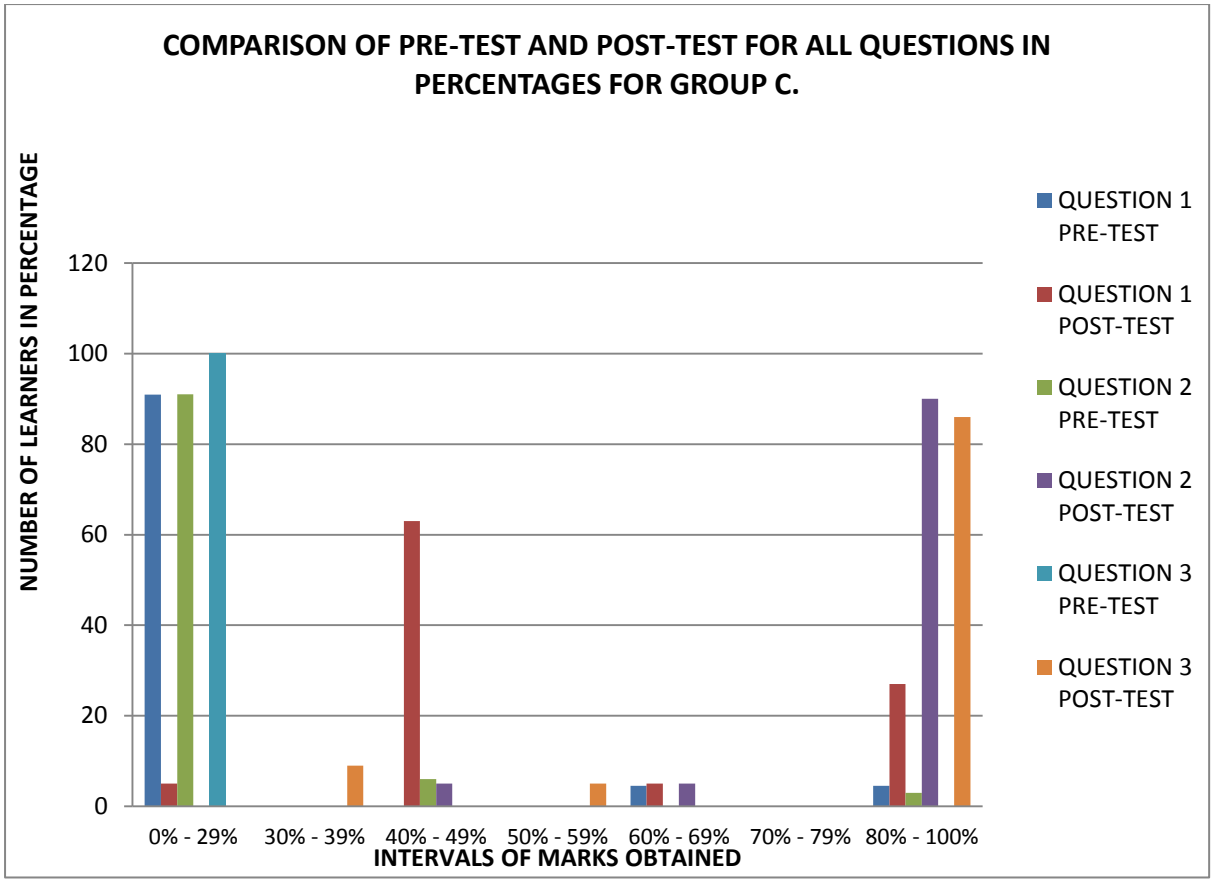


Figure 6.9(a): Comparison of the results of the pre-test and post-test in all the questions for group C.

NUMBER OF LEARNERS IN PERCENTAGES						
GROUP C						
	QUESTION 1		QUESTION 2		QUESTION 3	
	PRE-TEST	POST-TEST	PRE-TEST	POST-TEST	PRE-TEST	POST-TEST
0% - 29%	91	5	91		100	
30% - 39%	0		0			9
40% - 49%	0	63	6	5		
50% - 59%	0		0			5
60% - 69%	5	5	0	5		
70% - 79%	0		0			
80% - 100%	5	27	3	90		86
TOTAL	100	100	100	100	100	100

Table 6.9(a): Comparison of the results of the pre-test and post-test in all the questions for group C.

The comparison of the average percentage attainment of learners of group C in both the pre-test and the post-test are shown in figure 6.9(b) and table 6.9(b) below.

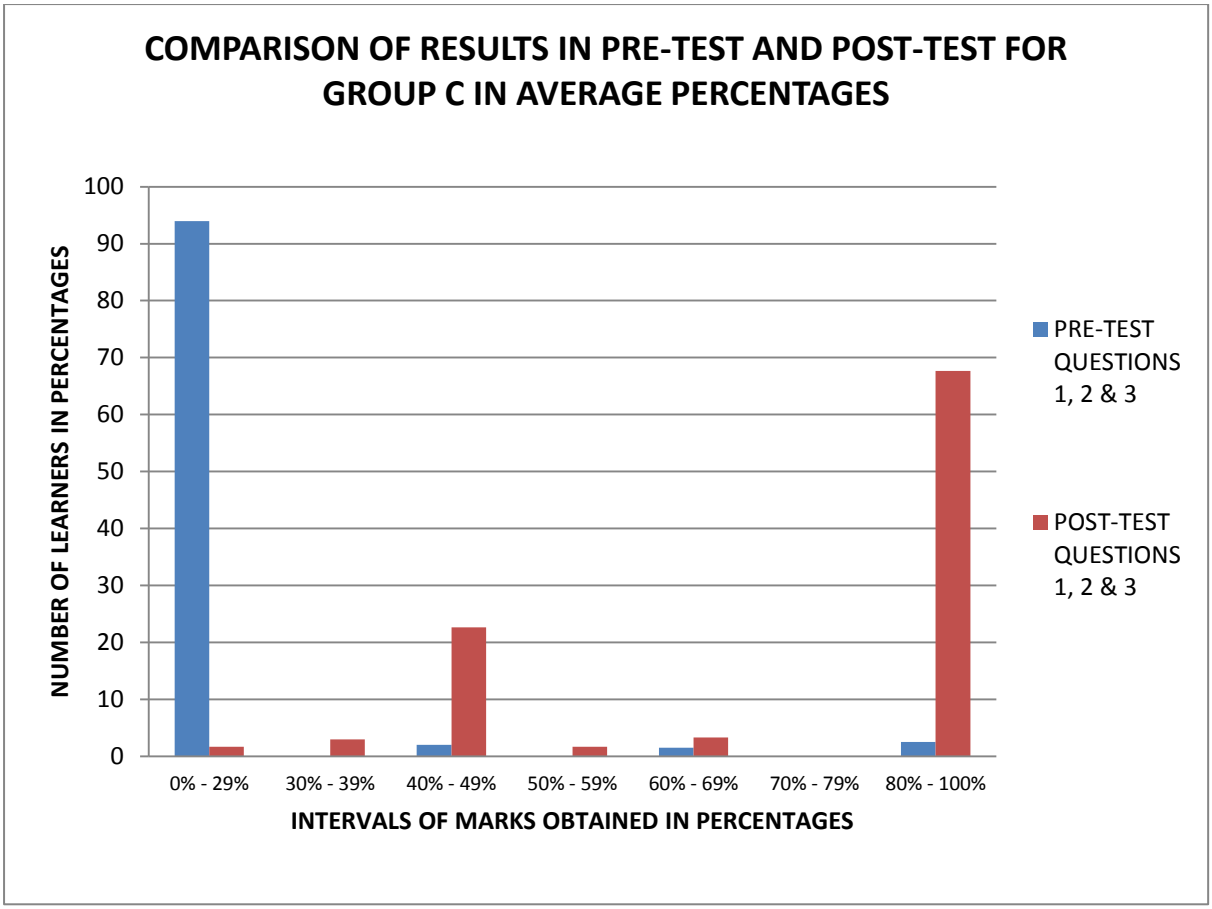


Figure 6.9(b): Comparison of the results of the pre-test and post-test in all the questions for group C.

NUMBER OF LEARNERS IN PERCENTAGES			
GROUP C			
INTERVALS OF MARKS OBTAINED IN PERCENTAGES	**PRE-TEST QUESTIONS 1, 2 & 3**	**POST-TEST QUESTIONS 1, 2 & 3**	**INCREASE(+)/ DECREASE(-)**
0% - 29%	94	2	**-92**
30% - 39%	0	3	**3**
40% - 49%	2	23	**21**
50% - 59%	0	2	**2**
60% - 69%	2	3	**2**
70% - 79%	0	0	**0**
80% - 100%	3	68	**65**
TOTAL	**100**	**100**	

Table 6.9(b): Comparison of the results of the pre-test and post-test in all the questions for group C.

6.7.6 INTERPRETATION OF THE DATA AFTER LESSON 3 (GROUP C).

The object of learning that was the focus of the study was “Discerning the features of a quadratic function and identification and use of the appropriateform of the equation of the quadratic function to generate the equation of the quadratic function from the graph(s) of the quadratic function or any information about the quadratic function”.

The average percentage number of learners in the intervals 80% to 100% increased from 3% to 68% and the average percentage in the interval 0% to 29% decreased from 94% to 2% for group C. The improvement of group C learners was remarkable considering that this group was the poorest of all the groups in the sample.

This improvement could be attributed to the intervention lesson which began the lesson with the introduction of the specific equations of the parabola. The general forms of the equations of the parabola were derived from the specific equation of the parabola. This group improved the most which is indicative of the fact that the lived object of learning for this group was much stronger. This could be due to the manner in which the examples were presented, from the specific to the general using the pattern of separation, simultaneity and generalisation all at once.

The turning point for the learners in this group was that the beginning example consisted of a specific equation of the parabola presented in different forms of the equation of a parabolic function. From this specific equation the various forms of the specific equations of a parabola were derived using algebraic manipulations of factorisation, completing a square and substitution.

The learners began to see the relationship between the function, the equation of the function and the graph of the function. This could be attributed fact that the general equation of a parabola has too many parameters that could be varied. The learners could not see what was varied in the general form of the quadratic function but they were able see what was varied with the specific equation of the quadratic function.

The learners were ultimately able to find the equation of a parabola by using the appropriate form of the equation of the parabola. They were able to interpret the different forms of the

parabolic equations. They were also able to discern the key features of a parabolic graph. They were able to flexibly move between the equation representation to the graphical representation of the parabola and vice versa.

6.8 DISCUSSION AND INTERPRETATION OF DATA.

In this study purposeful sampling is used because the sample consists of learners that I, the researcher, taught and thus the learners were readily available. Ethical considerations were adhered to as explained in the previous chapter.

Figures and tables shown above give a clear picture of the results of the learners' performances in both the pre-test and the post-test. The comparison of the number of learners in percentages obtained in both the diagnostic and the post-test give the answer to the question 'To what extent does a version of a learning study framed by variation theory improve the Grade 11 learners' interpretation of functions?'

The comparison of the pre-test and the post-test results for all groups show that the percentages of the learners who improved in the interpretation of functions was experienced in each group. The use of separation as a pattern of variation after the first intervention lesson performed with group A was noticeable. This is proof that the use of separation as a pattern of variation with this group did provide the learners with the opportunity to discern the object of learning.

The percentage number of learners that increased (+) or decreased (-) in the attainment of the learners in each group in the three questions that were compared from both the pre-test and the post-test shows that the learners' performance improved in all the intervals of marks obtained. The success of the intervention lesson is viewed as such in lieu of the comparison of the results of the diagnostic test and the post-test which consisted of the three questions that were compared from both the tests. These three questions are compared for consistency.

The following section will explore ways in which the intervention lessons were altered and the reasons for these lessons to be changed.

6.9 SUMMARY.

In this chapter the mixed methods research analysis was explained and used to analyse the data from the diagnostic test and the post-test. The important features of the diagnostic test were discussed and analysed and the results of both the pre-test and the post- test analysed and interpreted. Each question of the diagnostic test was analysed with reference to the literature that was reviewed to support the study. The sample of learners was divided into three groups and results of the learners analysed per group per question using intervals as levels of attainment required by the National Curriculum Statement of 2003.

The first two questions of the diagnostic test which are the same two questions appearing in the post-test were compared according to groups to ascertain whether the intervention had any impact on the learners' performance or not. Learner extracts from the pre-test for questions 1.1, 1.2, 5.1 and 5.2 were presented in the analysis of the results of this test. The results of the two questions in the pre- and post-test were compared, analysed and the percentage improvement for each group discussed with reference to the way in which the intervention lessons were presented in each group.

The limitation of the action research was discussed. The next chapter will focus on conclusions that were reached from conducting the study. The discussion of the study as a whole will be embarked on.

CHAPTER SEVEN.

CONCLUSION.

7.1 INTRODUCTION.

This study was mainly concerned with investigating an intervention, informed by variation theory, into the Grade 11 learners' interpretation of algebraic functions. The investigation led to finding an answer to the question "To what extent does a version of a learning study framed by variation theory improve the Grade 11 learners' interpretation of algebraic functions?"

The critical questions that guided this study were the following:

7.1.1 What aspects of functions do learners find to be problematic?

7.1.2 Which of these aspects appear most problematic?

7.1.3 To what extent and how does a version of a learning study (which is framed by variation theory) improve the Grade 11 learners' interpretation of algebraic functions?

The use of a diagnostic test for the whole sample helped in providing answers to the critical questions as stated in 7.1.1, 7.1.2 and 7.1.3 above. These critical questions are: 'What aspects of functions do learners find to be problematic? ; 'Which of these aspects appear most problematic?'; 'To what extent and how does a version of a learning study (which is framed by variation theory) improve the Grade 11 learners' interpretation of algebraic functions?'

The analysis of the pre-test led to the identification of the object of learning which was as a result of the critical question: 'Which of these aspects appear to be the most problematic?'

The categories of learner misconceptions with the notion of functions that were identified from the Umalusi report (2011), the literature review and the pre-test were used to analyse the pre-test. These difficulties that learners had were with: The dual nature of functions as seen by Sfard (1991), the multiple representations of functions as researched by ;Even (1998); Goldin (1987); Janvier (1978); Lesh, Post & Bohr (1987); Moschovich, Schroenfeld &

Arcavi (1993) and Monk (1988). The interpretation of functions as stated by: Tall & Bakar (1991); Thompson (1994); Carlson (1998) and Oehrtman et al. (2008).

The results of the analysis of the pre-test led to the identification of the object of learning which was: 'Discerning the features of a quadratic function and identification and use of the appropriate form of the equation of the quadratic function to generate the equation of the quadratic function from the graph(s) of the quadratic function or any information about the quadratic function'.

The results of the post-test for the whole sample was then analysed and these results gave answers to the research question 'To what extend does a version of a learning study framed by variation theory improve the Grade 11 learners' interpretation of algebraic functions?' This answer became apparent from the intervention lessons conducted in the three groups where different approaches to the lessons were applied.

This topic on the notion of a function was chosen because it is problematic for learners and teachers. Despite the substantial amount of research done on this topic, the difficulties associated with the teaching and learning of the notion of functions still persists. This is evidenced by the reports received from the Grade 12 external examinations board of examiners (Umalusi) every year. Since the mark allocation for this section is $\pm$33% of the entire mathematics paper in the FET phase, learners who manage to obtain a 30% or more on this section stand a better chance of obtaining a pass mark in the mathematics examination.

The theoretical lens that informed the study was drawn from the theory of variation as applied by Marton, et al., (2004). The features of variation theory include amongst others, an object of learning consisting of the intended object of learning, the enacted object of learning and the lived object of learning; the patterns of variation which include contrast, separation, generalisation, fusion: variance, invariance; simultaneity and the application of these features of variation theory in a version of a learning study.

This framework helped me in providing learners opportunities to learn by applying the patterns of variation in the different intervention lessons. These opportunities to learn resulted in the learners actually discerning the object of learning as is evident in their performance in the post-test. The lived object of learning was lived by the learners in varying degrees as was reflected by the results of the post-test in all groups.

This study was influenced by the work of Sfard's (1991) dual nature of functions and how the learners can be enabled to view the function as an object and a structure. The patterns of variation as described by Marton, et al., (2004) are not prescriptive and through the processes of a version of a learning study that was applied in the presentation of the intervention lessons, the most appropriate starting examples in a lesson were identified.

The processes of reflection and discussions after each intervention lesson provided the study with the most appropriate starting examples. The most effective starting example was from the specific example of a quadratic function to the general form and not vice versa. After the application of this strategy, the learners from group C were able to see where the general forms of the quadratic function came from and thereafter were the most improved of the three groups.

The findings of the post-test were that all groups improved in their performance in the post-test compared to the pre-test and the most improved group was the third group where the starting examples were from the specific example to the general form of the specific form of the quadratic function.

This form of introducing the lesson from the specific equation of parabola enabled learners to experience the derivation of the general forms of the quadratic functions. The learners were involved in the algebraic manipulations of the general equation of the parabola using algebraic algorithms for factorisation, completing a square and simultaneously solving three equations.

The application of the algorithms for factorisation to the specific equation of the parabola, enabled learners to see the resulting form of the specific equation of the parabola as one that represents the process used when finding the x-intercepts of the parabola, This conclusion was arrived at by the learners when they substituted the value of $y = 0$ to the factorised form of the specific equation and solving the equation for 'x'. This process enabled learners to see that the resulting form of the specific equation of the parabola resembled the general form of the parabolic equation: $y = a(x - x_1)(x - x_2)$.

The application of the algebraic manipulations of completing a square to the specific equation of the parabola, enabled learners to see the resulting form of the specific equation of the parabola as one that represents the form of the quadratic function which represented the turning point and the axis of symmetry of the parabola. The learners were then in a position

to conclude that the form resulting from completing a square on the specific equation of a parabola resembled the general form of the parabolic equation: $y = a(x - p)^2 + q$.

These findings are in agreement with the findings of many researchers like Sfard (1991); Tall & Marois (1996, p. 3); Dubinsky (1986); Cuoco (1994); Schwingendorf & Dubinsky (1990); Harel & Kaput (1991) & Sierpinska (1988) who agree that functions and many other mathematical concepts have a dual nature. As Sfard (1992, p. 61) came to the conclusion that "many mathematical notions had been conceived operationally long before their structural definitions and representations were formulated". This conclusion was as a result of analysing several examples.

7.2 REFLECTIONS.

I began the study by giving my class of learners a test on functions. This test was given to the learners after I had taught them functions. I have been teaching functions to learners for twenty years and yet the learners still had difficulties with the interpretation of functions. I was of the opinion that procedural understanding was the key to the learners eventually understanding a concept. Procedural understanding of a concept has been emphasised and has been the centre of the focus when teaching concepts to learners and we have been able to do procedural understanding successfully.

The key issue in the learners' acquisition of mathematical proficiency is through the mastery of procedural understanding and conceptual understanding (Kilpatrick, Swafford and Findell, 2001). Kilpatrick, Swafford and Findell (2001) identified ways in which both procedural and conceptual understanding of any concept could be mastered in order that the learners become mathematically proficient.

The authors identified that the proficiency of both the conceptual and procedural should have been achieved by the learner in order to enhance the learners' understanding of the mathematical concepts (functional concepts in my study). One of the ways in which the attainment of both the procedural and conceptual understanding could be achieved was through the acquisition of the strands of mathematical proficiency (Kilpatrick et al., 2001),

These strands of mathematical proficiency involve learners' ability to "apply knowledge to solve problems, learn new concepts and skills, adapt the knowledge they acquire to different

situations, apply mathematical reasoning to different problems and to view mathematics as a useful tool that must constantly be sharpened" (Kilpatrick, et al., 2001, p. 144). These strands are namely: conceptual understanding; procedural proficiency; strategic competence; adaptive reasoning; productive disposition. As these strands are intertwined, they should be acquired by the learners in no specific order.

Instead of using the strands of mathematical proficiency as seen by Kilpatrick et al. (2001), this study explored one possible way in which this could be attained. My study used the patterns of variation as proposed by Marton et al. (2004) in order to afford learners the opportunity to discern the object of learning. In discerning the object of learning, the learners are afforded the opportunity to be proficient in their procedural understanding of a concept and thus enhance their conceptual understanding.

Hopefully this study will help in contributing to the approach teachers could apply in the teaching of the notion of functions. In so doing, learners could be provided better opportunities to learn what was intended for them to learn. In this study the tools provided by variation theory were applied in order to enable learners to acquire mathematical proficiency so as to be able to interpret functions.

After engaging with the literature on functions I was specifically intrigued by the misconceptions on functions brought about by using prototypes as one of the causes of learners developing misconceptions on the notion of functions (Tall, McGowen & Marois, 1991). I was one of the teachers who used the function machine to introduce the concept of a function to the lower grades. I used it in the following order: input (equation)→process (algebraic manipulations) → output (result). The learners in these lower grades have not developed the abstract notion of a function and thus the use of the function machine as a cognitive root suffices in the lower grades.

This study has made me change my approach to the use of the function box to be such that it is used to enable learners to discern the object of learning. flexibly move between the representations of functions namely: table, equation, graphical and verbal form.

The dual nature of functions, though it seems complicated for the teachers as well as the learners to understand, Sfard (1991) asserts that in order for learners to speak about mathematical objects, they must be able to deal with products of some process without bothering about the processes that produced these products themselves. This, the author refers

to as reification. According to Sfard (1991), the understanding of a concept involves the operational conceptions of the concept and should precede the structural conception thereof.

In other words in order for learners to understand the concept of a function, they have to understand the operational stages of the concept of a function and the processes involved in the formation of the function. The teacher(s) should teach learners in such a way that the learners are able to understand how the general formula for any mathematical concept was derived. The method that is mostly effective is one that more-often-than-not follows the process from the specific to the general. The intervention lesson 3 for group C is an example of this statement was applicable. The learners were made aware of the conception of the general form of the quadratic functions through the algebraic processes of factorisation, completing a square and substitution on the specific quadratic function.

When the learners know how the general form of the parabolic function was derived, they will then be in a position to apply the appropriate form of the general form of the parabolic equation to the structure of a parabola in order to derive the equation of that particular structure of a parabola. This was the aim of the study and through the intervention lessons, the learners were able to use the appropriate form of the parabolic equation to find the equation of the given parabola.

The use of the patterns of variation was influential in the choice of examples that were used. These patterns of variation informed me that the choice of examples should be made in a manner that will provide learners the opportunity to discern the object of learning. The theory of variation is one such theory that informs the educator of the manner in which the choice of examples could be made.

The examples that were used in the study were chosen such that they could to enable the learners to discern the key features of the parabola that became prominent when:
(i) The specific equation of a function was kept the same and the forms of the specific equation of the function varied (separation as a pattern of variation applied;
(ii) The graph of the parabola was kept the same and the points on the parabola varied (separation as a pattern of variation used):
(iii) The general forms of the equation of the parabolic function were presented to the learners simultaneously and the key features of the parabolic function from each form of the

general equation of a parabolic function identified (simultaneity as a pattern of variation applied);
(iv) Algebraic manipulations performed on the specific equation of a parabola to enable the learners to associate the result of the specific equation of the parabola to its general form of the quadratic equation (generalisation as a pattern variation applied).

The examples that were chosen enabled the learners to differentiate between the forms of the parabola that could be used when finding the equation of the parabola when the graphical representation of the function was kept the same and only the points on the parabola highlighted.

The use of a version of a learning study helped me in changing the way I planned my lessons and presented them to the learners. A version of a learning study facilitated my awareness of the 'taken-for-granted' use of the non-mathematical language when teaching mathematical concepts. The positive input given by the panel of observers enabled me to develop new ways of as a model for developing my teaching practice. This version of action research facilitated my growth in the ways in which I taught the mathematical concepts to the learners.

The use of a version of action research in the form of a learning study had its challenges. Some of the challenges that I experienced when conducting the study were the learners' commitment. Since the study was conducted outside of the prescribed contact time scheduled by the school policy, not all of the learners who participated at the beginning of the study managed to complete the study. To overcome these challenges with some of the learners, I had to prepare sandwiches and cool drinks to the learners after the intervention lessons.

Some of the teachers in the panel were not fully committed to the study because they did not give feedback during the reflection discussions. These challenges were overcome by probing the teachers by asking them questions related to the lesson and the learners' reactions and questions. These pre-planned questions together with the impromptu questions asked by the researcher/myself helped in breaking the ice and thus giving the inexperienced teachers an opportunity to fruitfully contribute in the discussion sessions.

7.3 RECOMMENDATIONS.

Since the examples used in the interventions lessons of this study on the forms of the quadratic functions: $y = ax^2 + bx + c; y = a(x - p)^2 + q$ and $y = a(x - x_1)(x - x_2)$ were given to the learners with the value of '$a = 1$' and 'a $= -1$', it would be interesting to investigate the starting examples with a different value of 'a' say '$a = 2$' or 'a $= -2$'instead of '$a = 1$' and 'a $= -1$'. The problems become more complex but certainly a follow up would be interesting to see.

It would be interesting to see what aspects of functions become clear as separation, fusion, generalisation, simultaneity and contrast are investigated as patterns of variation in this context. I chose an example that was easier for learners to work within and to work with because of the time within which the study had to be completed.

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APPENDIX A

ETHICAL CONSIDERATIONS.

PROTOCOL NUMBER: 2012ECE056

INFORMATION SHEET AND CONSENT LETTERS:

PARENTS/GUARDIAN

Please complete pages 1, 3 and 4, and return to the school

Dear parent/guardian

Your child …… is invited to be part of a RESEARCH PROJECT:

'Improving the Teaching and Learning of functions in Grade 11'

Undertaken by the Mrs. Ramaisa M. S. for her M Sc degree and as part of the Wits Mathematics Education Project and directed by Professor Jill Adler, University of the Witwatersrand.

The research is being conducted in Realogile High School in Gauteng. The project has the support of the Gauteng Department of Education, and the Johannesburg East District Office.

If you are happy for your child to take part in the research, please sign below.

I am happy for my child to be videotaped and observed in class, write tests and be interviewed if selected as part of the research.

Name of Parent/Guardian:

……..

Name of Learner:

……

Signed: ……

Date: ……

The research is funded by the GDE for the year 2012 and managed by the Wits School of Education. The project has the support of the Gauteng Department of Education and the Johannesburg East District Office.

You may keep the information sheet on the next page.

For more information speak to your child's mathematics teacher and researcher Mrs. Ramaisa.

WHAT WILL THE RESEARCHER DO?

The researcher Mrs. Ramaisa M. S. wants to find ways and means of improving the teaching and learning of functions in mathematics in Grade 11 in your child's school and particularly to enhance learner participation and performance regarding this topic.

The researcher Mrs. Ramaisa M. S. from the Wits School of Education and Wits Maths Connect Project at the University of the Witwatersrand will:

- Require your child to take a mathematics test on functions.
- Conduct classes and make notes in your child's school.
- Videotape and audio record a lesson on functions in your child's class.
- Interview your child if selected.

HOW WILL THE INFORMATION BE USED

The researcher Mrs. Ramaisa M. S. will use the information from your child's tests, videotapes and audio recordings to study and improve on her teaching. Only Mrs. Ramaisa and her supervisor will see your child's test and the videotape. Other teachers in your child's school will not see your child's individual tests, only the overall results for your child's school.

The researchers will write a report which will be discussed at conferences and in journal articles. The results, videotape and audio recording and, if your child is selected, the interview information, will be used for the duration of the project and stored for a further five years. Thereafter they will be destroyed.

YOUR RIGHTS AND THE RIGHTS OF YOUR CHILD

The researcher will not use your child's name in any reports or articles.

The research is completely separate from your child's school work. All information obtained for research purposes will not affect your child's assessment in school.

There will also be no problem if you do not want your child to take part in the research. If you choose that your child should not participate, this will not affect your child in any way.

If you decide that your child should no longer continue participating in the study, you are free to withdraw this consent at any time. You should then inform your child's mathematics teacher Mrs. Ramaisa who will inform the project leaders at Wits Mathematics Education Project.

INFORMATION AND CONSENT TO VIDEO AND AUDIORECORDING

Dear Parent/Guardian

The researcher Mrs. Ramaisa will also videotape and audio record a lesson on functions with your child's class.

WHAT WILL THE RESEARCHER DO WITH THE VIDEOTAPE AND AUDIO RECORDING OF THE LESSON

The videotape will be backed up with an audio recorder to assist with the review of the lesson and transcription where necessary. The videotape provides a full record of the lesson(s). The researcher will use the videotape together with the audio recording of the lesson to ensure that she has an accurate record of the classes she taught and observed. The researcher will use the videotape together with the notes she takes when she teaches the concept of functions.

The videotape will be focused on your child's teacher/the researcher. The camera person will, wherever possible, focus the video away from your child's face and the learners' faces, to help secure confidentiality.

The videotape and audio recording of the lesson will only be used for the purposes of the Research project conducted by the researcher.

The videotape and the audio recording of the lesson will be kept securely for five years after the end of the project and then destroyed.

If you are happy for your child to be videotaped, please sign below.

I am happy for my child to be videotaped during lesson observations as part of the research.

Name of parent/guardian:

……

Name of learner:

……

Signed: ……

Date: ……

YOUR RIGHTS AND THE RIGHTS OF YOUR CHILD

If you decide later that you no longer want your child to be video recorded you can tell your child's mathematics teacher Mrs. Ramaisa M. S, who is also the researcher in this case. You may at any time withdraw your child from being interviewed with no consequence to your child's school performance.

PARENT CONSENT FORM FOR HIS/HER CHILD TO BEING INTERVIEWED

WHAT WILL THE RESEARCHER DO?

If your child is selected for an interview, the interview will be conducted after the post-test has been written, marked and compared with the pre-test. The interview will be conducted in the researcher's office and will last for approximately half an hour. The results of the interview will be kept securely for five years after the end of the project and then destroyed.

If you are happy for your child to be interviewed, please sign below.

I am happy to allow my child to be interviewed if chosen in this research.

Name of parent/guardian: ……

Name of learner: ……

Signed: ……

Date: ……

YOUR RIGHTS AND THE RIGHTS OF YOUR CHILD

You can ask to see the transcript of the interview of your child made by the researcher. If you do decide later that you do not want your child to be audio recorded you can tell your child's mathematics teacher Mrs. Ramaisa M. S. who is the researcher in this case. Your child may at any time withdraw from being interviewed with no consequence to his/her school performance.

APPENDIX B

PROTOCOL NUMBER: 2012ECE056

INFORMATION SHEET AND CONSENT LETTERS: LEARNERS

Dear Learner

You are invited to be part of a RESEARCH PROJECT

'Improving the Teaching and Learning of functions in Grade 11 Mathematics'

Undertaken by Mrs. Ramaisa M. S. for her M Sc Ed degree and as part of the Wits Mathematics Education Project and directed by Professor Jill Adler, University of the Witwatersrand

If you are happy to take part in the research, please sign below.

I am happy to be videotaped and observed in class, write tests, be interviewed if selected as part of the research.

Name: ……

Signed: ……

Date: ……

The research is funded by the GDE for the year 2012 and managed by the Wits School of Education;

The project has the support of the Gauteng Department of Education, and the Johannesburg East District Office.

You may keep the information sheet on the next page.

For more information speak to your teacher Mrs. Ramaisa who is the researcher.

WHAT WILL THE RESEARCHER DO?

The researcher Mrs. Ramaisa M. S. wants to find ways and means of improving the teaching and learning of functions in mathematics in Grade 11 in her school and particularly to enhance learner participation and performance regarding this topic.

The researcher Mrs. Ramaisa M. S. from the Wits School of Education and Wits Math Connect project at the University of the Witwatersrand will:

- Require you to take a mathematics test on functions.
- Conduct your classes and make notes.
- Videotape and audio record a lesson on functions in your class.
- Interview you if selected.

HOW WILL THE INFORMATION BE USED

The researcher Mrs. Ramaisa M. S. will use the information from your tests, the videotape and audio recording to study her teaching and to study and improve on her teaching. Only Mrs Ramaisa and her supervisor will see your tests and the videotape. Other teachers in your school will not have access to this information. The researcher will write a report which will be discussed at conferences and in journal articles. Your results, videotape that is audio recorded and, if you were selected, your interview information, will be used for the duration of the project and stored for a further five years. Thereafter they will be destroyed.

YOUR RIGHTS

The researcher will not use your name in any reports or articles. The research is completely separate from your school work. All information obtained for research purposes will not affect your assessment in school.

There will also be no problem if you do not want to take part in the research. If you choose not to participate, this will not affect you in any way.

If you decide that you no longer want to continue participating in the study, you are free to withdraw at any time with no consequence to your school performance. You should then inform your teacher Mrs. Ramaisa who will inform the project leaders at Wits mathematics education project.

INFORMATION AND CONSENT TO VIDEO AND AUDIO-RECORDING:

Dear learner

The researcher Mrs. Ramaisa M. S. will also videotape and audio record a lesson on functions with your class.

WHAT WILL THE RESEARCHER DO WITH THE VIDEOTAPE AND AUDIO RECORDING OF THE LESSONS

The videotape will be backed up with an audio recorder to assist with the review of the lesson and transcription where necessary.

The videotape and audio recording provides a full record of the lesson.

The researcher will use the videotape and audio recording to ensure that she has an accurate record of the classes she teaches.

The videotape will be focused on your teacher. The camera person will, wherever possible, focus the video away from learners' faces, to help secure confidentiality.

The videotape will only be used for the purposes of the research project. The videotape and audio recording will be kept securely for five years after the end of the project and then destroyed.

If you are happy to be videotaped, please sign below.

I am happy to be videotaped during lesson observations as part of the research.

Name: ……

Signed: ……

Date: ……

LEARNER CONSENT FORM FOR BEING INTERVIEWED

If you are happy to be interviewed, please sign below.

I am happy to be interviewed in this research.

Name: ……

Signed: ……

Date: ……

WHAT WILL THE RESEARCHER DO?

If you are selected for an interview, the researchers would like to have a transcript of the interview. This will help her to have a full and accurate record of the interview. The transcript will be kept securely for five years after the end of the project and then destroyed. The interview will be conducted after the post-test has been written, marked and compared with the pre-test. The interview will be conducted in the researcher's office and will last for approximately half an hour.

YOUR RIGHTS

You can ask to see the transcript of the interview and to look at the video recording of the lesson made by the researcher.

If you do decide later that you do not want to be interviewed you can tell your mathematics teacher Mrs. Ramaisa M. S. who is the researcher in this case. You may at any time withdraw from being interviewed with no consequence to your school performance.

APPENDIX C

PROTOCOL NUMBER: 2012ECE056

INFORMATION SHEET AND CONSENT LETTERS: EDUCATORS

Dear Educator.

You are invited to assist in a RESEARCH PROJECT

'Improving the Teaching and Learning of functions in Grade 11 Mathematics'

Undertaken by Mrs. Ramaisa M. S. for her M Sc Ed degree and as part of the Wits Mathematics Education Project and directed by Professor Jill Adler, University of the Witwatersrand

The research is funded by the GDE for the year 2012 and managed by the Wits School of Education. The project has the support of the Gauteng Department of Education, and the Johannesburg East District Office.

For more information speak to your colleague Mrs. Ramaisa who is the researcher in this case.

WHAT WILL THE RESEARCHER DO?

The researcher Mrs. Ramaisa M. S. wants to find ways and means of improving the teaching and learning of functions in mathematics in Grade 11 in your school and particularly to enhance learner participation and performance regarding this topic.

The researcher Mrs. Ramaisa M. S. from the Wits School of Education and Wits Maths Connect Project at the University of the Witwatersrand will videotape and audio record her teaching of the lessons on functions and require you to:

- Assist her in the research.
- Observe the lesson(s) she will be conducting on functions and fill in an observation schedule she has prepared for the analysis of the observed lesson(s).
- Discuss the videotapes and audiotapes of the lesson(s) on functions in her three Grade 11 classes.

HOW WILL THE INFORMATION BE USED

The researcher Mrs. Ramaisa M. S. will use the information from the tests, videotapes and audio taped lesson(s) to study and improve on her teaching. The information from the pre-test, the video and audiotapes, post-test and the interviews will be kept confidential and the learners' anonymity guaranteed. The videotape will be backed up with an audio recorder to assist with the review of the lesson and transcription where necessary.

The researchers will write a report which will be discussed at conferences and in journal articles. The results, videotape and the interview information, will be used for the duration of the project and stored for a further five years. Thereafter the information will be destroyed.

YOUR RIGHTS

The researcher will not use your name in any reports or articles as you might appear in the videotapes.

The research is completely separate from the learners' school work. All information obtained for research purposes will not affect the learners' assessment in the school.

There will also be no problem if you do not want to assist in the research. If you choose that you do not want to assist in the research, this will not affect you in any way.

If you decide that you no longer want to continue participating in the study, you are free to withdraw this consent at any time with no consequence. You should then inform the mathematics teacher Mrs. Ramaisa who is the researcher in this case.

PLEASE READ THE INFORMATION AND CONSENT TO ASSIST IN THE RESEARCH PROJECT.Dear Educator(s)

The researcher Mrs. Ramaisa will videotape and audio record a lesson on functions with the Grade 11 classes that she teaches. The videotape will be backed up with an audio recorder to assist with the review of the lesson(s) and transcription where necessary.

WHAT WILL THE RESEARCHER DO WITH THE VIDEOTAPE AND THE AUDIO RECORDING OF THE LESSONS

The videotape will be backed up with an audio recorder to assist with the review of the lesson(s) and transcription where necessary and will provide a full record of the lesson(s). The researcher will use the videotape together with the audio recording to ensure that she has an accurate record of the classes she taught and observed by you. The researcher will use the videotape together with the notes you take when she teaches the concept of functions for research purposes only.

The videotape will be focused on the teacher/the researcher. The camera person will, wherever possible, focus the video away from the observers and the learners' faces to help secure confidentiality.

The videotape will only be used for the purposes of the Research project conducted by the researcher. The videotape will be kept securely for five years after the end of the project and then destroyed.

If you are happy to assist in the research and guarantee the confidentiality and anonymity of the learners in the research and pledge not to divulge the information from the research to anyone, please sign below.

I am happy assist in the research and guarantee the anonymity of the learners and confidentiality of the research information:

Name Educator: ……

Signed: ……

Date: ……

APPENDIX D.

PRE-TEST ON FUNCTIONS. DATE: MAY 2012.

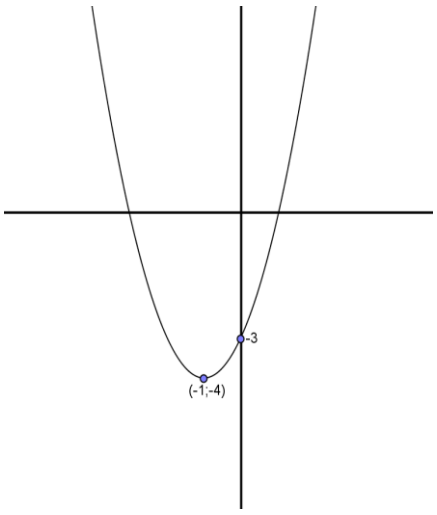
TOTAL = 65 **TIME: $1\frac{1}{2}$ HOURS**

Write your answers on the answer sheet provided.

QUESTION 1.

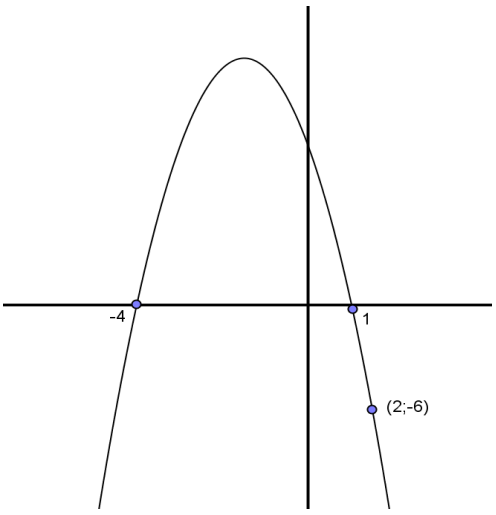
Find the equations of the given curves:

1.1



(5)

1.2



(5) **[10]**

QUESTION 2.

2.1 On the **DIAGRAM SHEET,** on the same system of axes, draw the graphs of

$g(x) = 2^x$ *and* $f(x) = 2^x - 4$. Clearly show the coordinates of the intercepts on the *y*-axes as well as the asymptote of f and equation of the asymptote of *f.* (5)

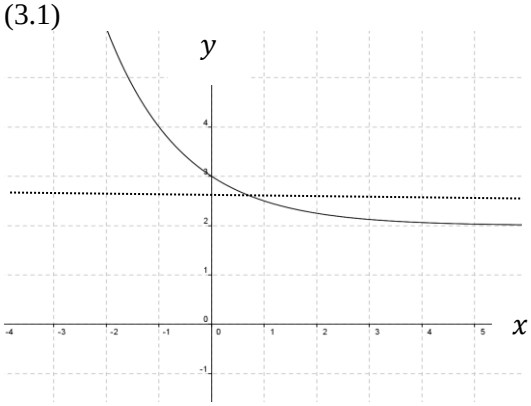
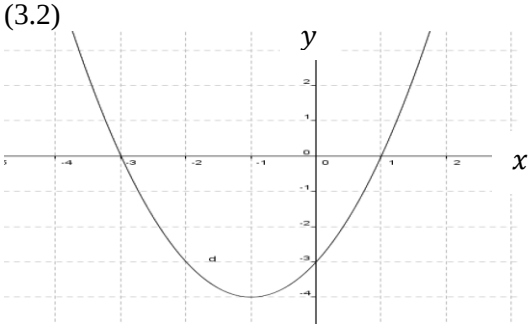
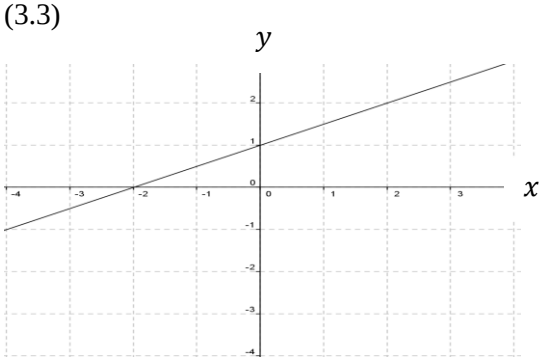
2.2 Write down the domain of g and the range of f. (3)

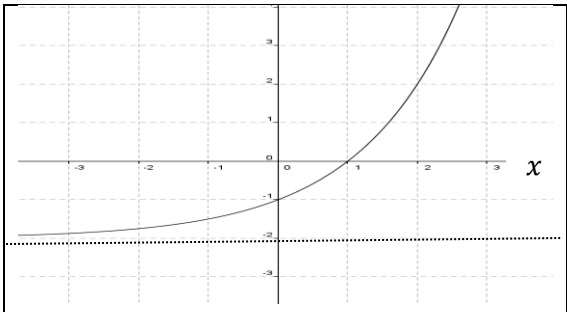
2.3 Explain what will happen to the graph of *f*, if the equation of *f* changes to $f(x) = 2^{x-4} - 4$? (2)

[10]

QUESTION 3.

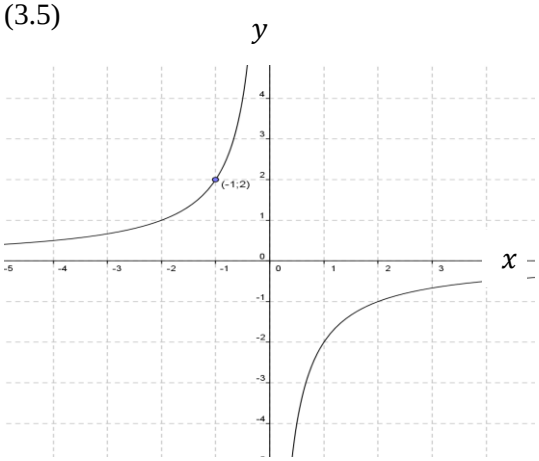
Use the table provided in the answer sheet to match a graph in column (I) to its appropriate representation in column (II).

<table>
<tr><th>Column (I)</th><th>Column (II)</th></tr>
<tr><td>(3.1)

</td><td>(A)
<table><tr><td>x</td><td>0</td><td>1</td><td>2</td></tr><tr><td>y</td><td>-1</td><td>0</td><td>2</td></tr></table>
(2)</td></tr>
<tr><td rowspan="2">(3.2)

</td><td>(B) $h(x) = -\frac{2}{x}$
(2)</td></tr>
<tr><td>(C) $y = x + 1$
(2)</td></tr>
<tr><td>(3.3)

</td><td>(D)
A certain number subtracted from four.
(2)</td></tr>
<tr><td>(3.4)
y</td><td>(E)
<table><tr><td>x</td><td>y</td></tr><tr><td>-1</td><td>4</td></tr><tr><td>1</td><td>0</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>6</td></tr></table></td></tr>
</table>



(2)

(3.5)



(F)

x	-1	0	1	2
y	-4	-3	0	9

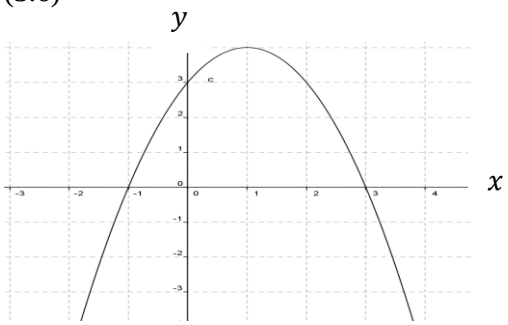
(2)

(G)

$h(x) = -\frac{2}{x} + 1$

(2)

(3.6)



(H)

One added to two divided by a certain number.

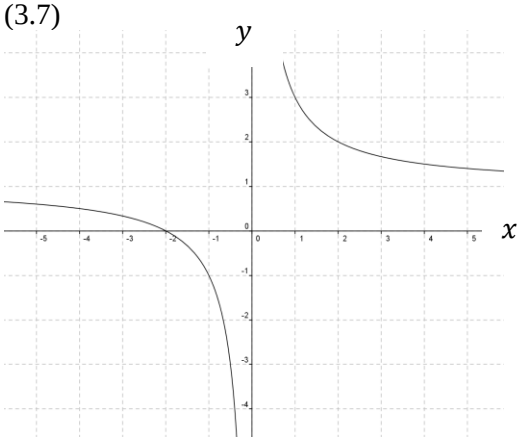
(2)

(I)

$h(x) = 2^{-x} + 2$

(2)

(3.7)



(J)

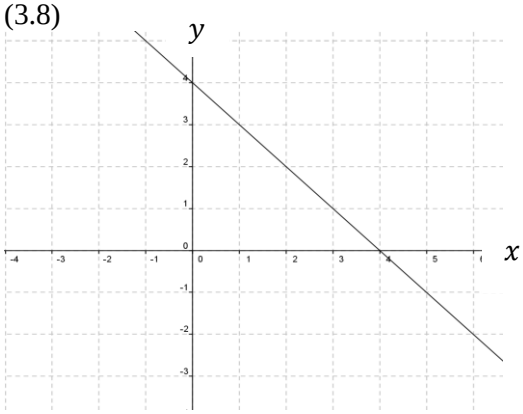
$h(x) = -x^2 + 2x + 3$

(2)

(K)

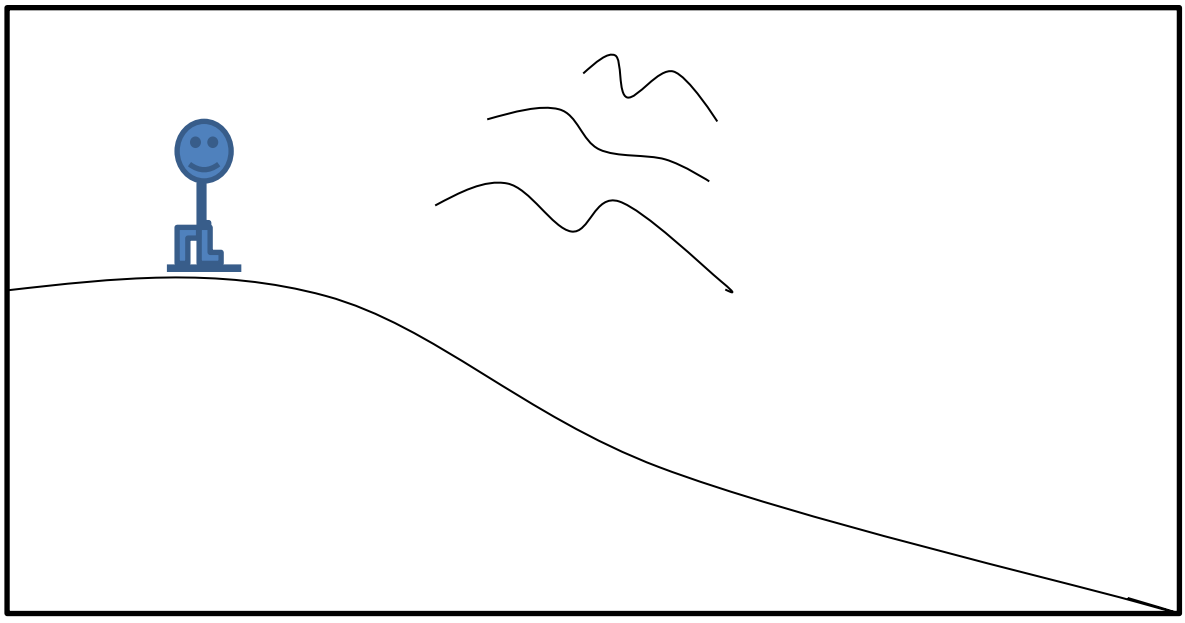
$y = 2^x + 3$

(2)

(3.8)  	(L) One added to half of a certain number. (2)
	(M) $xy = x + 1$ (2)

[18]

QUESTION 4.



A boy takes part in a skate board competition. His distance from the tuck-shop at any stage of the race is given by the equation: $s(t) = 3^{t+1} + 5$ where $s(t)$ is the distance in *metres* and *t* is time in *seconds.*

4.1 Prove that before the race has started, he was $8m$ away from the tuck-shop. (2)

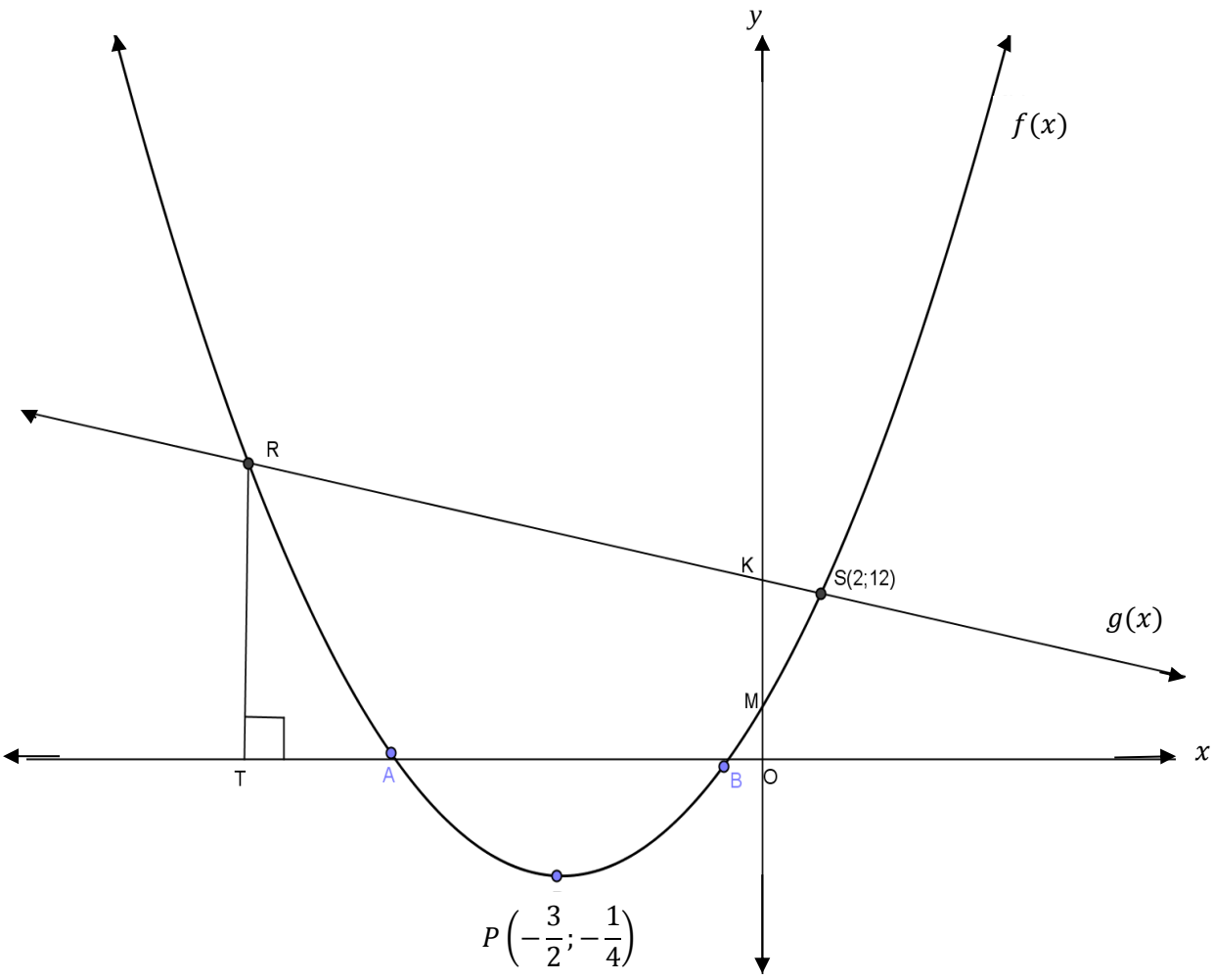
4.2 Calculate the distance from the tuck-shop after 3 seconds. (2)

4.3 The formula $s(t) = 3^{t+1} + 5$ is not reliable enough to measure his distance after *t* seconds. Use any value of *t*, $t \in N$, and $6 \leq t \leq 9$ to prove that this statement is true. (3)

[7]

QUESTION 5.

A parabola $f(x) = ax^2 + bx + c$ with turning point P$\left(-\frac{3}{2}; -\frac{1}{4}\right)$ and a straight line $g(x) = -x + 14$ intersect at the point $S(2\,;12)$. The two graphs, not drawn to scale, are drawn below. A and B are the x intercepts of the parabola. K is the y-intercept of f. RT is a straight line parallel to the y-axis.



5.1 Show that $a = 1,\ b = 3\ and\ c = 2$ (5)

5.2 Calculate the distance between A and B. (4)

5.3 Determine the length of KM, the distance between the two y-intercepts. (3)

5.4 For which values of x is f a decreasing function? (2)

5.5 Determine for which values of x is $f(x) \geq g(x)$, where $x \geq 0$. (2)

5.6 Determine the length of RT. (5)

5.7 Determine the value of $g(x^2) + g\left(\frac{1}{x}\right) - 28$. (5)

[26]

ANSWER SHEET.

NAME: ……. GRADE:…….

DATE: …….

QUESTION 1.

1.1 **(5)**	**1.2** **(5)**

QUESTION 2.

2.1

2.2 **(2)**	**2.3** **(1)**

QUESTION 3.

GRAPH (COLUMN I)	**INFORMATION (COLUMN II)**
(3.1)	
(3.2)	
(3.3)	
(3.4)	
(3.5)	
(3.6)	
(3.7)	
(3.8)	

[18]

QUESTION 4.

4.1 **(2)**	**4.2** **(2)**	**4.3** **(3)**

QUESTION 5.

5.1 **(5)**	**5.2** **(4)**

5.3 **(3)**	**5.4** **(2)**	**5.5** **(2)**

5.6 **(5)**	**5.7** **(5)**

APPENDIX E.

MATHEMATICS RESEARCH PRE-TEST GRADE 11 **MARKS= 65**

MEMORANDUM FOR PRE-TEST

Question	Solutions	Mark allocation	Comments related to the features of functions for the research.
Question 1			**Given a parabola, are the learners able to interpret and use appropriate equations?**
1.1	The Turning point = (-1;-4) Substituting (-1;-4) in $y = a(x-p)^2 + q$ yields $y = a(x+1)^2 - 4$ Substituting point (0;-3) yields $-3 = a(0+1)^2 - 4$ $1 = a$ $\therefore y = (x+1)^2 - 4$ $y = x^2 + 2x - 3$	√ √ √ √ √ **(5)**	Knowledge of the turning point. Appropriate formula used. Correct substitution. Substitution of the correct values. Appropriate equation.
1.2	The x-intercepts are (-4;0) and (1;0) Substituting these in the equation $y = a(x-x_1)(x-x_2)$ $y = a(x+4)(x-1)$ Substituting the point (0;4) yields: $4 = a(0+4)(0-1)$ $4 = -4a$ $\therefore -1 = a$ $y = -1(x+4)(x-1)$ $y = -x^2 - 3x + 4$	√ √ √ √ √ **(5)**	Knowledge of the intercepts. Use of appropriate formula. Correct substitution. Correct substitution of the other point. Appropriate equation.
Question 2			**Given equations are the learners able to draw graphs thereof?**
2.1	Diagram sheet.	√√ √√ √ **(5)**	y-intercepts of $g(x) and\ f(x)$ Asymptotes of $g(x) and\ f(x)$ Shapes of $g(x) and\ f(x)$
2.2	$g(x) = 2^x\ and\ f(x)$ $= 2^x - 4$ The domain of g is $x \in R$. The range of f is $y > -4$	 √ √ **(3)**	Interpretation of the domain and range from the graphs.
2.3	If the equation of $f(x) = 2^x - 4$ *changes to* $f(x) = 2^{x-4} - 4$, the graph of $f(x)$ will move horizontally by 4 units to the right.	√ **(2)**	Interpretation of the horizontal shifts.
Question 3			**Are the learners able to move between different functional**

			representations?
3.1	I	√√ **(2)**	Graphical to equation form.
3.2	F	√√ **(2)**	Graphical to equation form.
3.3	L	√√ **(2)**	Graphical to verbal form.
3.4	A	√√ **(2)**	Graphical to table form.
3.5	B	√√ **(2)**	Graphical to equation form.
3.6	J	√√ **(2)**	Graphical to equation form.
3.7	H	√√ **(2)**	Graphical to verbal form.
3.8	D	√√ **(2)**	Graphical to verbal form.
Question 4			**Are the learners able to apply their knowledge of functions to everyday situations?**
4.1	Before the race started, t=0 $s(0) = 3^{0+1} + 5$ $s(0) = 3 + 5 = 8$ ∴ He was 8m away from the tuck-shop.	√ √ **(2)**	t = 0 Substitution, answer and conclusion.
4.2	After t =3 seconds, the distance is: $s(3) = 3^{3+1} + 5$ $s(3) = 3^4 + 5$ $s(3) = 86m$	√ √ **(2)**	Correct substitution of t = 3 Correct answer in metres.
4.3	After t= 7 seconds the distance from the tuck-shop will be $s(7) = 3^{7+1} + 5$ = 6 566m from the tuck-shop. This is quite a large distance for one to travel in 7 seconds on a skate board.	√ √ √ **(3)**	Choice of any value $6 \leq t \leq 9$ Answer in metres Interpretation of the meaning
Question 5			**Given the general quadratic equation, are the learners able to interpret the information and use their algebraic knowledge in answering the questions?**
5.1	For turning point substitute values $p = -\frac{3}{2}$ and $q = -\frac{1}{4}$ in the equation $f(x) = a(x-p)^2 + q$ $f(x) = a[x - (-\frac{3}{2})]^2 - \frac{1}{4}$ $f(x) = a(x + \frac{3}{2})^2 - \frac{1}{4}$	√ √	Correct interpretation of the turning point and equation to be used. Correct substitution. Substituting the other point on the

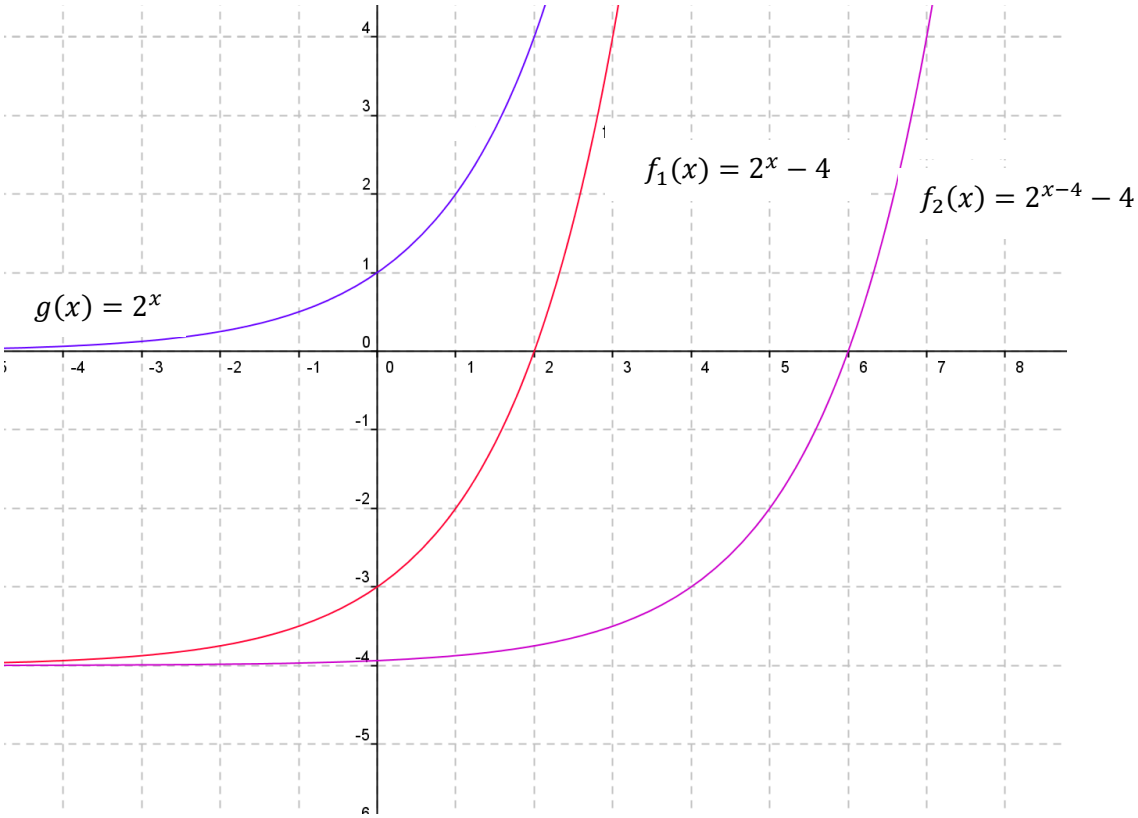
	$f(x)$ passes through the point S(2;12) $12 = a(2+\frac{3}{2})^2 - \frac{1}{4}$ $\frac{49}{4} = a\left(\frac{49}{4}\right)$ $\therefore a = 1$ $f(x) = (x+\frac{3}{2})^2 - \frac{1}{4}$ $f(x) = x^2 + 3x + \frac{9}{4} - \frac{1}{4}$ $f(x) = x^2 + 3x + 2$ $a = 1; b = 3; c = 2$	√ √ √ **(5)**	parabola. Using the other point in order to find the value of a. Appropriate equation and finding the values of a, b and c.
5.2	*For* x-intercepts, put $f(x) = 0$ in the equation $f(x) = (x+\frac{3}{2})^2 - \frac{1}{4}$ $0 = (x+\frac{3}{2})^2 - \frac{1}{4}$ $\frac{1}{4} = (x+\frac{3}{2})^2$ $\pm\frac{1}{2} = x + \frac{3}{2}$ $x = -\frac{3}{2} \pm \frac{1}{2}$ $x = \frac{-3 \pm 1}{2}$ $x = -1 \; or \; x = -2$ $A(-2;0) and \; B(-1;0)$ The distance between A and B is 1 unit	√ √ √ √ **(4)**	Realising that A and B are the x – intercepts of the parabola and are required in order to calculate the distance between A and B. Correct substitution and solutions. Correct coordinates of A and B. Correct distance between A and B.
5.3	K(0;14) is the y-intercept of $g(x)$ and M(0;2) is the y-intercept of $f(x)$. The length of KM is 12 units.	√√ √ **(3)**	Realising that K and M are the$y$ – intercepts of the straight line and the parabola respectively. Correct length of KM.
5.4	$f(x)$ is a decreasing function for $x < -\frac{3}{2}; x \in R$.	√√ **(2)**	Correct Interpretation of the decreasing function.
5.5	$f(x) \geq g(x)$when $x \geq 2$	√√ **(2)**	Interpretation of a function being greater than the other function.
5.6	At R , $f(x) = g(x)$ $x^2 + 3x + 2 = -x + 14$ $x^2 + 4x - 12 = 0$ $(x+6)(x-2) = 0$ $x = -6 \; or \; x = 2$ At R, $x = -6$ $g(-6) = -(-6) + 14 = 20$	√ √ √ √	Interpretation of the point of intersection of the two graphs and the use of simultaneous equations as a means of finding the points R and T. Simplifying. Realising that at R, the x value is

	$\therefore R(-6;20)$ The length of RT is 20 units.	√ **(5)**	the negative one. Correct calculation of the length of RT.
5.7	$g(x^2)+g\left(\frac{1}{x}\right)-28$ $=-x^2+14-\frac{1}{x}+14-28$ $=-x^2-\frac{1}{x}$ $=\frac{-x^3-1}{x}$ $=\frac{-(x^3+1)}{x}$ $=\frac{-(x+1)(x^2-x+1)}{x}$	√ √ √ √ √ **(5)**	Correct substitution of the functions $g(x^2)\ and\ g\left(\frac{1}{x}\right)$ Simplifying. Factorising the sum of two cubes.

NAME: .....

GRADE: .....

DIAGRAM SHEET FOR QUESTION 2.1



APPENDIX F

PROTOCOL NUMBER: 2012ECE056

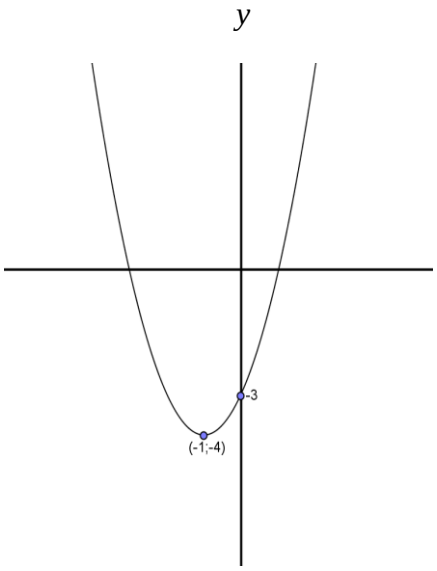
POST-TEST ON FUNCTIONS. DATE: NOVEMBER 2012.

TOTAL = 20 **TIME: 30 MINUTES**

Write your answers on the answer sheet provided.

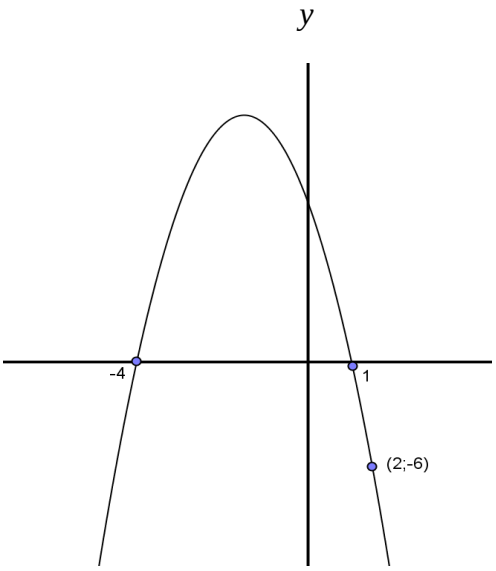
Find the equations of the following curves:

QUESTION 1.



(5)

QUESTION 2



(5)

[10]

QUESTION 3.

Generate the equation of a parabola passing through the given points.

A(0;-1), B(1;2) and C(-2;5) (10)

[10]

PROTOCOL NUMBER: 2012ECE056

ANSWER SHEET.

NAME:…….. GRADE: …….

DATE: ……..

QUESTION 1.	**QUESTION 2**
(5)	**(5)**
QUESTION 3.	
(10)	

APPENDIX G.

PROTOCOL NUMBER: 2012ECE056

MEMORANDUM TO POST-TEST ON FUNCTIONS.

DATE: NOVEMBER 2012. **TOTAL = 20** **TIME: 30 MINUTES**

Question	Solutions	Mark allocation	Comments related to the features of functions for the research.
Question			**Given a parabola, are the learners able to interpret and use appropriate equations?**
1	The Turning point = (-1;-4) Substituting (-1;-4) in $y = a(x-p)^2 + q$ yields $y = a(x+1)^2 - 4$ Substituting point (0;-3) yields $-3 = a(0+1)^2 - 4$ $1 = a$ $\therefore y = (x+1)^2 - 4$ $y = x^2 + 2x - 3$	√ √ √ √ √ **(5)**	Knowledge of the turning point. Appropriate formula used. Correct substitution. Substitution of the correct values. Appropriate equation.
2	The x-intercepts are (-4;0) and (1;0) Substituting these in the equation $y = a(x-x_1)(x-x_2)$ $y = a(x+4)(x-1)$ Substituting the point (0;4) yields: $4 = a(0+4)(0-1)$ $4 = -4a$ $\therefore -1 = a$ $y = -1(x+4)(x-1)$ $y = -x^2 - 3x + 4$	√ √ √ √ √ **(5)**	Knowledge of the intercepts. Use of appropriate formula. Correct substitution. Correct substitution of the other point. Appropriate equation.

3	Equation to be used is: $y = ax^2 + bx + c$ Substituting point A(0;-1) yields: $-1 = a(0)^2 + b(0) + c$ $\therefore -1 = c$ … (1) Substituting point B(1;2) yields: $2 = a(1)^2 + b(1) - 1$ $3 = a + b$ … (2) Substituting point C(-2;5) yields: $5 = a(-2)^2 + b(-2) - 1$ $6 = 4a - 2b$ $\div 2$: $3 = 2a - b$ … (3) (2) + (3): $6 = 3a$ $\therefore 2 = a$ … (4) (4) in (2): $3 = 2 + b$ $1 = b$ Substituting the values of a, b and c into the general equation of the parabola we get: $y = 2x^2 + x - 1$	√ √ √ √ √ √ √ √ √ √ **(10)**	

APPENDIX I.

PROTOCOL NUMBER: 2012ECE056

INTERVENTION LESSONS.

LESSON 1 (GROUP A)

PROTOCOL NUMBER: 2012ECE056
INTERVENTION LESSON .

LESSON 1
GROUP A

Lesson 1.
Group A.

Object of learning.

Discerning the critical features of the parabolic function from the equation form of the function and from the graphical form of the function.

The parabola.

Different forms of the parabola.

$y = ax^2 + bx + c$

- Critical features of the parabola from this form of the equation of the parabola are:
  - the shape of the parabola-
  - for $a > 0 \rightarrow$ concave upwards ∪
  - for $a < 0 \rightarrow$ concave downwards ∩

$f(x) = a(x - x_1)(x - x_2)$

- Critical features of the parabola from this form of the equation are:
  - the x-intercepts namely:
  - $x = x_1$ *and* $x = x_2$

$g(x) = a(x - p)^2 + q$

- Critical features of the parabola from this form of the equation are:
  - the turning point namely $(p; q)$ and
  - the axis of symmetry namely: $x = p$

$y = x^2 - 4x + 3$

- Critical features of the parabola from this form of the equation of the parabola are:
  - the shape of the parabola- concave upwards ∪
  - the y-intercept namely: $y = 3$

Factorising the equation we get:

$y = (x - 1)(x - 3)$

- Critical features of the parabola from this form of the equation are:
  - the x-intercepts namely:
  - $x = 1$ *and* $x = 3$

Completing the square we get:

$y = (x - 2)^2 - 1$

- Critical features of the parabola from this form of the equation are:
  - the turning point namely $(p; q) = (2; -1)$
  - the axis of symmetry namely: $x = 2$

Factorising the equation we get:

$f(x) = (x + 4)(x - 2)$

- Critical features of the parabola from this form of the equation of the parabola are:
  - the x-intercepts namely:
  - $x = -4$ *and* $x = 2$

Completing the square we get:

$f(x) = (x + 1)^2 - 9$

- Critical features of the parabola from this form of the equation of the parabola are:
- the turning point namely $(p; q) = (-1; -9)$
- the axis of symmetry namely: $x = -1$

Factorising the equation we get:

$g(x) = -(x + 2)(x - 4)$

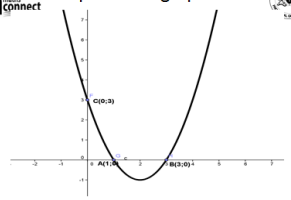
- Critical features of the parabola from this form of the equation of the parabola are:
  - the x-intercepts namely:
  - $x = -2$ *and* $x = 4$

Completing the square we get:

$g(x) = (x - 1)^2 + 9$

- Critical features of the parabola from this form of the equation are:
  - the turning point namely $(p; q) = (1; 9)$
  - the axis of symmetry namely: $x = 1$

Generate the equation of the following parabolic graph.



Solution.

Critical points:

The $x-intercepts$: $x=1$ *and* $x=3$.

The $y-intercept$: $y=3$.

Equation to be used: $y=a(x-x_1)(x-x_2)$

Substituting these critical points in the equation:

Solution continued ...

$3=a(0-1)(0-3)$

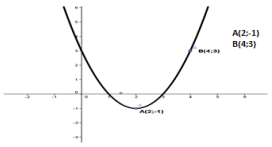
$3=3a$

$\therefore a=1$

Equation is: $y=(x-1)(x-3)$

i.e. $y=x^2-4x+3$

Generate the equation of the following parabolic graph.



Generate the equation of a parabola passing through the given points.

A(0;3), B(4;3) and C(-1;8)

Solution:

Equation to be used: $y=ax^2+bx+c$

Substituting point A(0;3) in the equation:

$3=a(0^2)+b(0)+c$

$3=c$ ... (1)

Solution continued ...

Substituting point B(4;3) in the equation:

$3=a(4)^2+b(4)+3$

$0=16a+4b$

$0=4a+b$ .... (2)

Substituting point C(-1;8) in the equation:

$8=a(-1)^2+b(-1)+3$

$5=a-b$ ... (3)

Solution continued ...

(2) – (3): $2=2a$ ...

$1=a$ ... (4)

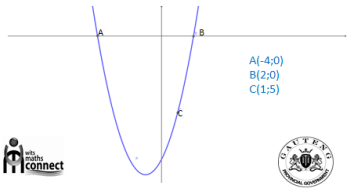
(4) In (3): $-2=2(1)+b$

$-4=b$

∴ Equation of the parabola is:

$y=x^2-4x+3$

Generate the equation of the following parabolic graph.



Solution.

Critical points:

The $x-intercepts$: $x=-4$ *and* $x=2$.

The point (1;5).

Equation to be used:

$y=a(x-x_1)(x-x_2)$

Substituting these critical points in the equation:

Solution continued...

$5=a(1-(-4))(1-2)$

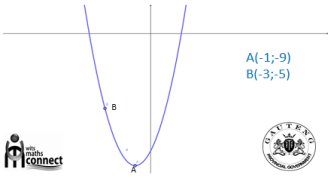
$5=5a$

$\therefore a=1$

Equation is: $y=(x+4)(x-2)$

i.e. $y=x^2+2x-8$

Find the equation of the following parabolic graph.



Solution.

Critical points:

The turning point $(-1;-9)$

The axis of symmetry: $x=1$

and another point $B(-3;-5)$.

Equation to be used:

$y=a(x-p)^2+q$

Solution continued ...

Substituting these critical points into the equation:

$-5=a(-3+1)^2-9$

$\therefore 4=4a$

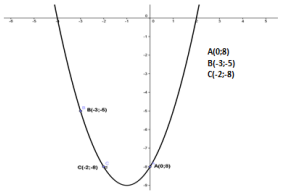
$1=a$

Equation is: $y=(x+1)^2-9$

$y=x^2+2x+1-9$

i.e. $y=x^2+2x-8$

Generate the equation of the following parabolic graph.



Find the equation of the following:

A parabola passing through the points:

A(0;-8), B(-3;-5) and C(-2;-8).

Solution:

Equation to be used: $y=ax^2+bx+c$

Substituting point A(0;-8) in the equation:

$-8=a(0^2)+b(0)+c$

$-8=c$ ... (1)

Solution continued ...

Substituting point B(-3;-5) into the equation:

$-5=a(-3)^2+b(-3)-8$

$3=9a-3b$

$1=3a-b$ .... (2)

Substituting point C(-2;-8) in the equation:

$-8=a(-2)^2+b(-2)-8$

$0=4a-2b$ ... (3)

Solution continued ...

(2) x 2: $2=6a-2b$ ... (4)

(4) – (3): $2=2a$ ... (3)

$1=a$ ... (5)

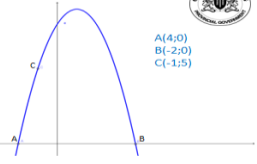
(5) In (2): $1=3(1)-b$

$2=b$

∴ Equation of the parabola is:

$y=x^2+2x-8$

Find the equation of the following parabola.



Solution.

Critical points:

The $x-intercepts$: $x = 4$ *and* $x = -2$.

The point (-1;5).

Equation to be used:

$y = a(x - x_1)(x - x_2)$

Substituting these critical points in the equation:

Solution continued ...

$5 = a(-1 - 4)(-1 - (-2))$

$5 = -5a$

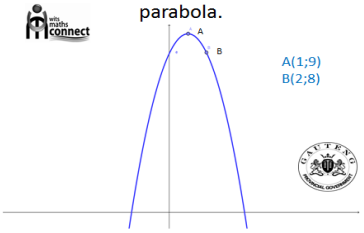
$\therefore -1 = a$

Equation is: $y = -(x - 4)(x + 2)$

$y = -(x^2 - 2x - 8)$

i.e. $y = -x^2 + 2x + 8$

Generate the equation of the following parabola.



Solution.

Critical points:

The turning point $(1; 9)$

The axis of symmetry: $x = 1$

and another point $B(2; 8)$.

Equation to be used: $y = a(x - p)^2 + q$

Solution continued ....

Substituting these critical points into the equation:

$8 = a(2 - 1)^2 + 9$

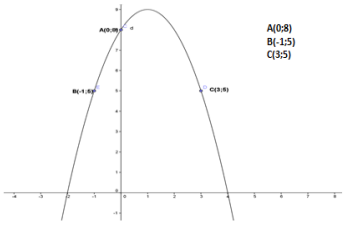
$\therefore -1 = a$

Equation is: $y = -(x - 1)^2 + 9$

$y = -(x^2 - 2x + 1) + 9$

i.e. $y = -x^2 + 2x + 8$

Generate the equation of the following parabolic graph.



Find the equation of the following:

A parabola passing through the following points: A(0;8), B(-1;5) and C(3;5).

Solution:

Equation to be used: $y = ax^2 + bx + c$

Substituting point A(0;8) into the equation:

$8 = a(0)^2 + b(0) + c$

$8 = c$ ... (1)

Solution continued ...

Substituting point B(-1;5) into the equation:

$5 = a(-1)^2 + b(-1) + 8$

$-3 = a - b$ .... (2)

Substituting point C(3;5) into the equation:

$5 = a(3)^2 + b(3) + 8$

$-3 = 9a + 3b$

$-1 = 3a + b$ ... (3)

Solution continued ...

(2) + (3) : $-4 = 4a$

$-1 = a$ ... (4)

(4) In (2): $-3 = -1 - b$

$-2 = -b$

$2 = b$

$\therefore$ Equation of the parabola is:

$y = -x^2 + 2x + 8$

Post-test.

APPENDIX I.

PROTOCOL NUMBER: 2012ECE056

INTERVENTION LESSONS.

LESSON 2 (GROUP B)

PROTOCOL NUMBER: 2012ECE056
INTERVENTION LESSON .

LESSON 2

GROUP B.

Lesson 2.
Group B.

Object of learning.

Discerning the critical features of the parabolic function from the equation form of the function and from the graphical form of the function.

The parabola.

Different forms of the equation of a parabola.

- $y = ax^2 + bx + c$
- $y = a(x - x_1)(x - x_2)$
- $y = a(x - p)^2 + q$

The critical features of each form of equation.

- $y = ax^2 + bx + c$
- Critical features of the parabola from this form of the equation of the parabola are:
  - the shape of the parabola-
  - for $a > 0 \rightarrow$ concave upwards $\cup$
  - for $a < 0 \rightarrow$ concave downwards $\cap$

$f(x) = a(x - x_1)(x - x_2)$

- Critical features of the parabola from this form of the equation are:
  - the x-intercepts namely:
  - $x = x_1 \; and \; x = x_2$

$g(x) = a(x - p)^2 + q$

- Critical features of the parabola from this form of the equation are:
  - the turning point namely $(p; q)$ and
  - the axis of symmetry namely: $x = p$

$y = x^2 - 4x + 3$

- Critical features of the parabola from this form of the equation of the parabola are:
  - the shape of the parabola- concave upwards $\cup$
  - the y-intercept namely: $y = 3$

Factorising the equation we get:

$y = (x - 1)(x - 3)$

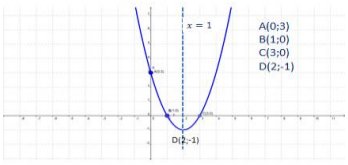
- Critical features of the parabola from this form of the equation are:
  - the x-intercepts namely:
  - $x = 1 \; and \; x = 3$

Completing the square we get:

$y = (x - 2)^2 - 1$

- Critical features of the parabola from this form of the equation are:
  - the turning point namely ;
  - $(p; q) = (2; -1)$
  - the axis of symmetry namely: $x = 2$

Sketching the graph of y:



$f(x) = -x^2 + 2x + 8$

- Critical features of the parabola from this form of the equation of the parabola are:
  - the shape of the parabola- concave downwards $\cap$
  - the y-intercept namely: $y = 8$

Factorising the equation we get:

$f(x) = (x + 4)(x - 2)$

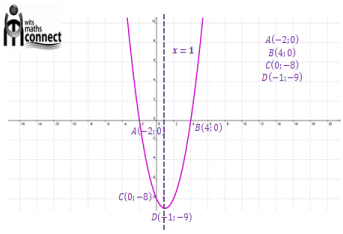
- Critical features of the parabola from this form of the equation of the parabola are:
  - the x-intercepts namely:
  - $x = -4 \; and \; x = 2$

Completing the square we get:

$f(x) = (x+1)^2 - 9$

- Critical features of the parabola from this form of the equation of the parabola are:
- the turning point namely:
- $(p;q) = (-1;-9)$
- the axis of symmetry namely: $x = -1$

Sketching the graph of f(x):



$g(x) = -x^2 + 2x + 8$

- Critical features of the parabola from this form of the equation of the parabola are:
  - the shape of the parabola- concave downwards ∩
  - the y-intercept namely: $y = 8$

Factorising the equation we get:

$g(x) = -(x+2)(x-4)$

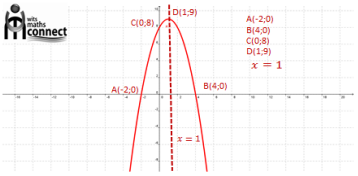
- Critical features of the parabola from this form of the equation of the parabola are:
  - the x-intercepts namely:
  - $x = -2 \text{ and } x = 4$

Completing the square we get:

$g(x) = (x-1)^2 + 9$

- Critical features of the parabola from this form of the equation are:
  - the turning point namely:
  - $(p;q) = (1;9)$
  - the axis of symmetry namely: $x = 1$

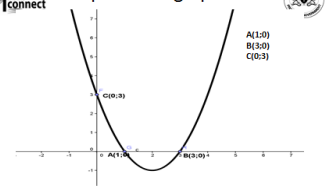
Sketching the graph of g(x):



Questions to ask yourself.

- Which critical points am I given?
- Which general form of the equation is most appropriate to use?
- Which process am I going to follow?

Generate the equation of the following parabolic graph.



Solution.

Critical points:

The $x-intercepts$: $x = 1 \text{ and } x = 3$.

The $y-intercept$: $y = 3$.

Equation to be used:

$y = a(x - x_1)(x - x_2)$

Substituting these critical points in the equation:

Solution continued ...

$3 = a(0-1)(0-3)$

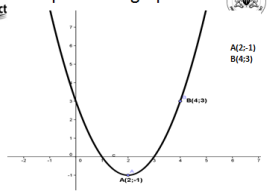
$3 = 3a$

$\therefore a = 1$

Equation is: $y = (x-1)(x-3)$

i.e. $y = x^2 - 4x + 3$

Generate the equation of the following parabolic graph.



Solution.

Critical points:

The turning point $(2;-1)$

The axis of symmetry:

$x = 2 \text{ and another point } B(4;3)$.

Equation to be used:

$y = a(x-p)^2 + q$

Solution continued...

Substituting these critical points into the equation:

$3 = a(4-2)^2 - 1$

$\therefore 4 = 4a$

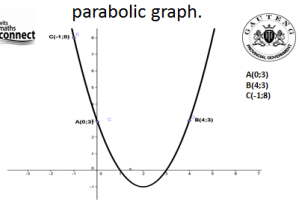
$1 = a$

Equation is: $y = (x-2)^2 - 1$

$y = x^2 - 4x + 4 - 1$

i.e. $y = x^2 - 4x + 3$

Generate the equation of the following parabolic graph.



Generate the equation of a parabola passing through the given points.

A(0;3), B(4;3) and C(-1;8)

Solution:

Equation to be used: $y = ax^2 + bx + c$

Substituting point A(0;3) in the equation:

$3 = a(0^2) + b(0) + c$

$3 = c \quad \ldots \quad (1)$

Solution continued ...

Substituting point B(4;3) in the equation:

$3 = a(4)^2 + b(4) + 3$

$0 = 16a + 4b$

$0 = 4a + b \quad \ldots. \quad (2)$

Substituting point C(-1;8) in the equation:

$8 = a(-1)^2 + b(-1) + 3$

$5 = a - b \quad \ldots \quad (3)$

Solution continued ...

(2) + (3): $5 = 5a \quad \ldots$

$1 = a \quad \ldots \quad (4)$

(4) In (3): $5 = (1) - b$

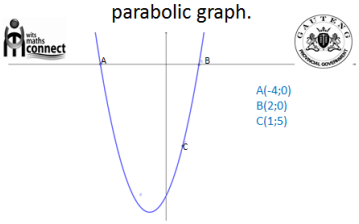
$4 = -b$

$-4 = b$

$\therefore$ Equation of the parabola is:

$y = x^2 - 4x + 3$

Generate the equation of the following parabolic graph.



Solution.

Critical points:

The $x-intercepts$: $x=-4$ *and* $x=2$.

The point (1;5).

Equation to be used:

$y=a(x-x_1)(x-x_2)$

Substituting these critical points in the equation:

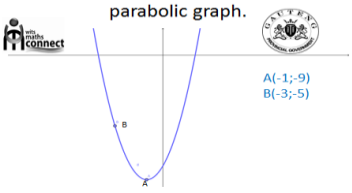
Solution continued...

$$5=a(1-(-4))(1-2)$$
$$5=5a$$
$$\therefore a=1$$

Equation is: $y=(x+4)(x-2)$

i.e. $y=x^2+2x-8$

Find the equation of the following parabolic graph.



Solution.

Critical points:

The turning point $(-1;-9)$

The axis of symmetry: $x=1$

and another point $B(-3;-5)$.

Equation to be used:

$y=a(x-p)^2+q$

Solution continued ...

Substituting these critical points into the equation:

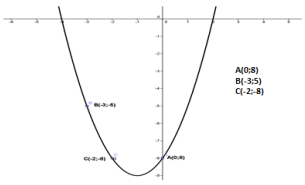
$$-5=a(-3+1)^2-9$$
$$\therefore 4=4a$$
$$1=a$$

Equation is: $y=(x+1)^2-9$

$$y=x^2+2x+1-9$$

i.e. $y=x^2+2x-8$

Generate the equation of the following parabolic graph.



Find the equation of the following:

A parabola passing through the points:

A(0;-8), B(-3;-5) and C(-2;-8).

Solution:

Equation to be used: $y=ax^2+bx+c$

Substituting point A(0;-8) in the equation:

$$-8=a(0^2)+b(0)+c$$
$$-8=c \quad \ldots \quad (1)$$

Solution continued ...

Substituting point B(-3;-5) into the equation:

$$-5=a(-3)^2+b(-3)-8$$
$$3=9a-3b$$
$$1=3a-b \quad \ldots. \quad (2)$$

Substituting point C(-2;-8) in the equation:

$$-8=a(-2)^2+b(-2)-8$$
$$0=4a-2b \quad \ldots \quad (3)$$

Solution continued ...

(2) x 2: $2=6a-2b$ ... (4)

(4) – (3): $2=2a$ ... (3)

$1=a$ ... (5)

(5) In (2): $1=3(1)-b$

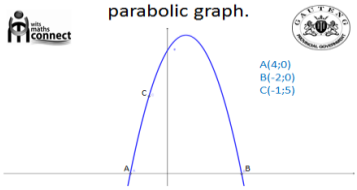
$-2=-b$

$2=b$

∴ Equation of the parabola is:

$y=x^2+2x-8$

Find the equation of the following parabolic graph.



Solution.

Critical points:

The $x-intercepts$: $x=4$ *and* $x=-2$.

The point (-1;5).

Equation to be used:

$y=a(x-x_1)(x-x_2)$

Substituting these critical points in the equation:

Solution continued ...

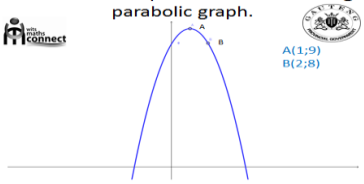
$$5=a(-1-4)(-1-(-2))$$
$$5=-5a$$
$$\therefore -1=a$$

Equation is: $y=-(x-4)(x+2)$

$$y=-(x^2-2x-8)$$

i.e. $y=-x^2+2x+8$

Generate the equation of the following parabolic graph.



Solution.

Critical points:

The turning point $(1;9)$

The axis of symmetry: $x=1$

and another point $B(2;8)$.

Equation to be used: $y=a(x-p)^2+q$

Solution continued ....

Substituting these critical points into the equation:

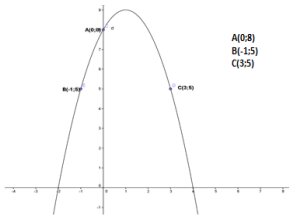
$$8=a(2-1)^2+9$$
$$\therefore -1=a$$

Equation is: $y=-(x-1)^2+9$

$$y=-(x^2-2x+1)+9$$

i.e. $y=-x^2+2x+8$

Generate the equation of the following parabolic graph.



Find the equation of the following:

A parabola passing through the following points:

A(0;8), B(-1;5) and C(3;5).

Solution:

Equation to be used: $y=ax^2+bx+c$

Substituting point A(0;8) into the equation:

$$8=a(0)^2+b(0)+c$$
$$8=c \quad \ldots \quad (1)$$

Solution continued ...

Substituting point B(-1;5) into the equation:

$$5=a(-1)^2+b(-1)+8$$
$$-3=a-b \quad \ldots. \quad (2)$$

Substituting point C(3;5) into the equation:

$$5=a(3)^2+b(3)+8$$
$$-3=9a+3b$$
$$-1=3a+b \quad \ldots \quad (3)$$

Solution continued ...

(2) + (3) : $-4 = 4a$

$-1 = a$ ... (4)

(4) In (2): $-3 = -1 - b$

$-2 = -b$

$2 = b$

∴ Equation of the parabola is:

$$y = -x^2 + 2x + 8$$

Post-test

Post-test.

APPENDIX I.

PROTOCOL NUMBER: 2012ECE056

INTERVENTION LESSONS.

LESSON 3 (GROUP C)

PROTOCOL NUMBER: 2012ECE056
INTERVENTION LESSON .

LESSON 3
GROUP C.

Lesson 3.
Group C.

Object of learning.

Discerning the critical features of the parabolic function from the equation form of the function and from the graphical form of the function.

The parabola.

Different forms of the equation of a parabola.

- $y = x^2 - 4x + 3$
- $y = (x-1)(x-3)$
- $y = a(x-2)^2 - 1$

$y = x^2 - 4x + 3$

- Critical features of the parabola from this form of the equation of the parabola are:
  - the shape of the parabola- concave upwards $\cup$
  - the y-intercept namely: $y = 3$

$y = ax^2 + bx + c$

- $y = x^2 - 4x + 3$ is in the form
- $y = ax^2 + bx + c$ where
- $a = 1; b = -4 \text{ and } c = 3$
- The critical features are
- The $y - intercept \text{ which is } (0;3)$
- The shape of the parabola which is concave upwards.

Factorising the equation we get:

$y = (x-1)(x-3)$

- Critical features of the parabola from this form of the equation are:
  - the x-intercepts namely:
  - $x = 1 \text{ and } x = 3$

$y = a(x - x_1)(x - x_2)$

- $y = (x-1)(x-3)$
- the general form of this parabolic function is
- $y = a(x - x_1)(x - x_2)$
- The critical features are:_
- The $x - intercepts\ x = 1 \text{ and } x = 3$
- In co-ordinate form $(1;0) and\ (3;0)$

Completing the square we get:

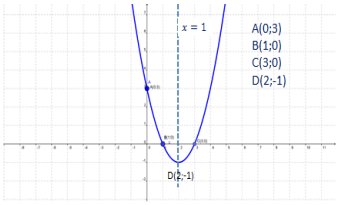
$y = (x-2)^2 - 1$

- Critical features of the parabola from this form of the equation are:
  - the turning point namely ;
  - $(p;q) = (2;-1)$
  - the axis of symmetry namely: $x = 2$

$y = a(x-p)^2 + q$

- $y = (x-2)^2 - 1$ is in the form
- $y = a(x-p)^2 + q$
- The critical features are:
- $The\ turning\ point\ (2;-1) and$
- The axis of symmetry $x = 2$

Sketching the graph of y:



$f(x) = -x^2 + 2x + 8$

- Critical features of the parabola from this form of the equation of the parabola are:
  - the shape of the parabola- concave downwards $\cap$
  - the y-intercept namely: $y = 8$

Factorising the equation we get:

$f(x) = (x + 4)(x - 2)$

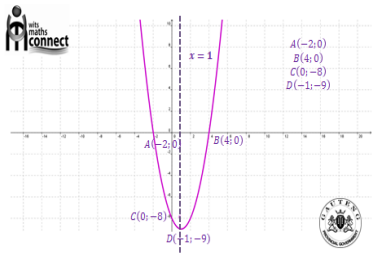
- Critical features of the parabola from this form of the equation of the parabola are:
  - the x-intercepts namely:
  - $x = -4 \; and \; x = 2$

Completing the square we get:

$f(x) = (x + 1)^2 - 9$

- Critical features of the parabola from this form of the equation of the parabola are:
- the turning point namely:
- $(p; q) = (-1; -9)$
- the axis of symmetry namely: $x = -1$

Sketching the graph of f(x):



$g(x) = -x^2 + 2x + 8$

- Critical features of the parabola from this form of the equation of the parabola are:
  - the shape of the parabola- concave downwards $\cap$
  - the y-intercept namely: $y = 8$

Factorising the equation we get:

$g(x) = -(x + 2)(x - 4)$

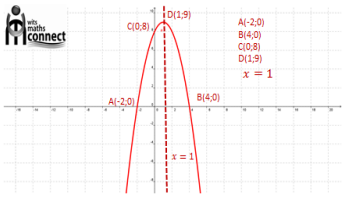
- Critical features of the parabola from this form of the equation of the parabola are:
  - the x-intercepts namely:
  - $x = -2 \; and \; x = 4$

Completing the square we get:

$g(x) = (x - 1)^2 + 9$

- Critical features of the parabola from this form of the equation are:
  - the turning point namely:
  - $(p; q) = (1; 9)$
  - the axis of symmetry namely: $x = 1$

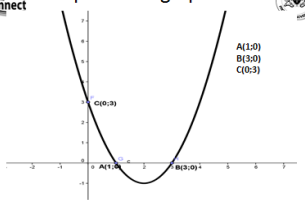
Sketching the graph of g(x):



Questions to ask yourself.

- Which critical points am I given?
- Which general form of the equation is most appropriate to use?
- Which process am I going to follow?

Generate the equation of the following parabolic graph.



Solution.

Critical points:

The $x - intercepts$: $x = 1 \; and \; x = 3$.

The $y - intercept$: $y = 3$.

Equation to be used:

$y = a(x - x_1)(x - x_2)$

Substituting these critical points in the equation:

Solution continued ...

$3 = a(0 - 1)(0 - 3)$

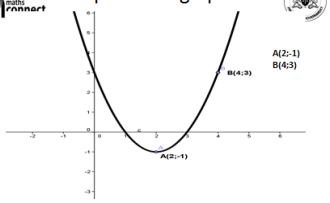
$3 = 3a$

$\therefore a = 1$

Equation is: $y = (x - 1)(x - 3)$

i.e. $y = x^2 - 4x + 3$

Generate the equation of the following parabolic graph.



Solution.

Critical points:

The turning point $(2; -1)$

The axis of symmetry:

$x = 2 \; and \; another \; point \; B(4; 3)$.

Equation to be used:

$y = a(x - p)^2 + q$

Solution continued...

Substituting these critical points into the equation:

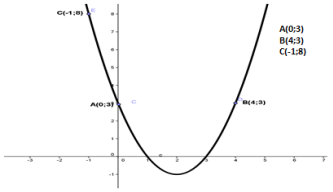
$$3 = a(4-2)^2 - 1$$
$$\therefore 4 = 4a$$
$$1 = a$$

Equation is: $y = (x-2)^2 - 1$

$$y = x^2 - 4x + 4 - 1$$

i.e. $y = x^2 - 4x + 3$

Generate the equation of the following parabolic graph.



Generate the equation of a parabola passing through the given points.

A(0;3), B(4;3) and C(-1;8)

Solution:

Equation to be used: $y = ax^2 + bx + c$

Substituting point A(0;3) in the equation:

$$3 = a(0^2) + b(0) + c$$
$$3 = c \quad \ldots \quad (1)$$

Solution continued ...

Substituting point B(4;3) in the equation:

$$3 = a(4)^2 + b(4) + 3$$
$$0 = 16a + 4b$$
$$0 = 4a + b \quad \ldots. \quad (2)$$

Substituting point C(-1;8) in the equation:

$$8 = a(-1)^2 + b(-1) + 3$$
$$5 = a - b \quad \ldots \quad (3)$$

Solution continued ...

(2) + (3): $5 = 5a$ ...

$1 = a$ ... (4)

(4) In (3): $5 = (1) - b$

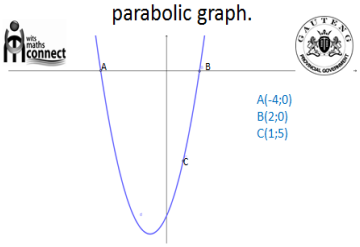
$4 = -b$

$-4 = b$

∴ Equation of the parabola is:

$$y = x^2 - 4x + 3$$

Generate the equation of the following parabolic graph.



Solution.

Critical points:

The $x-intercepts$: $x = -4$ *and* $x = 2$.

The point (1;5).

Equation to be used:

$y = a(x - x_1)(x - x_2)$

Substituting these critical points in the equation:

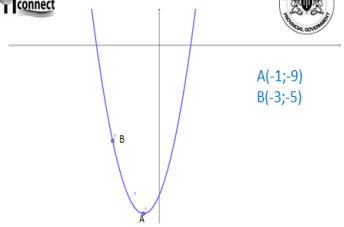
Solution continued...

$$5 = a(1-(-4))(1-2)$$
$$5 = 5a$$
$$\therefore a = 1$$

Equation is: $y = (x+4)(x-2)$

i.e. $y = x^2 + 2x - 8$

Find the equation of the following parabolic graph.



Solution.

Critical points:

The turning point $(-1;-9)$

The axis of symmetry: $x = 1$

and another point $B(-3;-5)$.

Equation to be used:

$y = a(x-p)^2 + q$

Solution continued ...

Substituting these critical points into the equation:

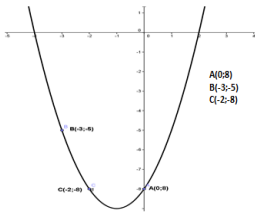
$$-5 = a(-3+1)^2 - 9$$
$$\therefore 4 = 4a$$
$$1 = a$$

Equation is: $y = (x+1)^2 - 9$

$$y = x^2 + 2x + 1 - 9$$

i.e. $y = x^2 + 2x - 8$

Generate the equation of the following parabolic graph.



Find the equation of the following:

A parabola passing through the points:

A(0;-8), B(-3;-5) and C(-2;-8).

Solution:

Equation to be used: $y = ax^2 + bx + c$

Substituting point A(0;-8) in the equation:

$$-8 = a(0^2) + b(0) + c$$
$$-8 = c \quad \ldots \quad (1)$$

Solution continued ...

Substituting point B(-3;-5) into the equation:

$$-5 = a(-3)^2 + b(-3) - 8$$
$$3 = 9a - 3b$$
$$1 = 3a - b \quad \ldots. \quad (2)$$

Substituting point C(-2;-8) in the equation:

$$-8 = a(-2)^2 + b(-2) - 8$$
$$0 = 4a - 2b \quad \ldots \quad (3)$$

Solution continued ...

(2) x 2: $2 = 6a - 2b$ ... (4)

(4) − (3): $2 = 2a$ ... (3)

$1 = a$ ... (5)

(5) In (2): $1 = 3(1) - b$

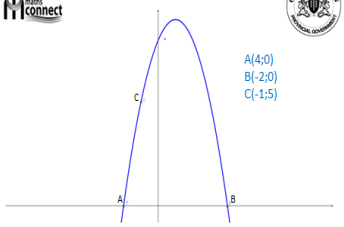
$-2 = -b$

$2 = b$

∴ Equation of the parabola is:

$$y = x^2 - 2x - 8$$

Find the equation of the following parabola.



Solution.

Critical points:

The $x-intercepts$: $x = 4$ *and* $x = -2$.

The point (-1;5).

Equation to be used:

$y = a(x - x_1)(x - x_2)$

Substituting these critical points in the equation:

Solution continued ...

$5 = a(-1 - 4)(-1 - (-2))$

$5 = -5a$

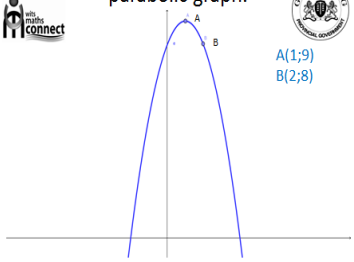
$\therefore -1 = a$

Equation is: $y = -(x - 4)(x + 2)$

$y = -(x^2 - 2x - 8)$

i.e. $y = -x^2 + 2x + 8$

Generate the equation of the following parabolic graph.



Solution.

Critical points:

The turning point $(1; 9)$

The axis of symmetry: $x = 1$

and another point $B(2; 8)$.

Equation to be used: $y = a(x - p)^2 + q$

Solution continued ....

Substituting these critical points into the equation:

$8 = a(2 - 1)^2 + 9$

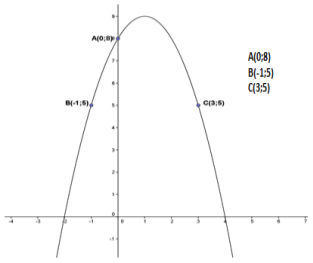
$\therefore -1 = a$

Equation is: $y = -(x - 1)^2 + 9$

$y = -(x^2 - 2x + 1) + 9$

i.e. $y = -x^2 + 2x + 8$

Generate the equation of the following parabolic graph.



Find the equation of the following:

A parabola passing through the following points:

A(0;8), B(-1;5) and C(3;5).

Solution:

Equation to be used: $y = ax^2 + bx + c$

Substituting point A(0;8) into the equation:

$8 = a(0)^2 + b(0) + c$

$8 = c$ ... (1)

Solution continued ...

Substituting point B(-1;5) into the equation:

$5 = a(-1)^2 + b(-1) + 8$

$-3 = a - b$ .... (2)

Substituting point C(3;5) into the equation:

$5 = a(3)^2 + b(3) + 8$

$-3 = 9a + 3b$

$-1 = 3a + b$ ... (3)

Solution continued ...

(2) + (3) : $-4 = 4a$

$-1 = a$ ... (4)

(4) In (2): $-3 = -1 - b$

$-2 = -b$

$2 = b$

$\therefore$ Equation of the parabola is:

$y = -x^2 + 2x + 8$

Post-test.

TABLE 1.

PROTOCOL NUMBER: 2012ECE056

OBSERVATION SCHEDULE WITH THE SUMMARY OF THE KEY ASPECTS OF THE VIDEOTAPED LESSON.

<table>
<tr><td colspan="4">Duration of the lesson is 60 minutes.
Object of learning is ……..</td></tr>
<tr><td colspan="4">What is the relationship between the examples given by the teacher, representations used, questions asked and explanations by the teacher and learners and the activities of the learners in relation to the object of learning?</td></tr>
<tr><td>What examples were used?</td><td>What representations were used?</td><td>What are the questions posed and asked and the explanations given?</td><td>What work did the learners do?</td></tr>
<tr><td>At the start of the lesson:
For further explanation by the teacher.
For further work by the learners.</td><td>Verbal?
Graphical?
Table?
Equation?</td><td>What words were used?

What other aids were used?</td><td>Listening?
Asking questions?
Answering questions?
Copying from the board?
Solving a problem?
Writing their solutions?
Discussing their thinking in groups?
Explaining their thinking in groups?
Doing exercises?</td></tr>
<tr><td colspan="4">Comments:</td></tr>
</table>

TABLE 2.

PROTOCOL NUMBER: 2012ECE056

OBSERVATION SCHEDULE PER LESSON OF 60 MINUTES.

School:…… **Topic:……..**

Teacher: …….... **Observer:……**

Date: …….. **Time: ……..**

Group:…….

	Activity	Duration in minutes																			
		5					10					15					20				
Teacher activity	Examples used at the start of the lesson.																				
	Examples used for further explanation.																				
	Examples used for further work by the learners.																				
Learner activity	Listening?																				
	Asking questions?																				
	Answering questions?																				
	Copying from the board?																				
	Solving a problem?																				
	Writing their solutions?																				
	Discussing their thinking in groups?																				
	Explaining their thinking in groups?																				
	Doing exercises?																				

TABLE 3.

PROTOCOL NUMBER: 2012ECE056

TABLE OF SPECIFICATION FOR THE PRE-TEST.

Question	Category/ features of functions
1.1	Interpretation of the parabolic graph and finding the equation of a graph given the turning point of the parabola and a point by using the equation: $y = a(x - p)^2 + q$.
1.2	Interpretation of the parabolic graph and finding the equation of a graph given the *x*-intercepts of the parabola and a point by using the equation $y = a(x - x_1)(x - x_2)$.
2.1	Using transformation procedures to sketch the graphs of the given exponential function and its horizontal shift.
2.2	Finding the domain and the range of the exponential graphs.
2.3	Interpreting the vertical shift of the exponential graph.
3.1	Application of multiple representations by matching the graph to the equation form.
3.2	Application of multiple representations by matching the graph to the equation form.
3.3	Application of multiple representations by matching the graph to the verbal form.
3.4	Application of multiple representations by matching the graph to the table form.
3.5	Application of multiple representations by matching the graph to the equation form.
3.6	Application of multiple representations by matching the graph to the equation form.
3.7	Application of multiple representations by matching the graph to the verbal form.
3.8	Application of multiple representations by matching the graph to the verbal form.
4.1	Interpretation of functions in real life.
4.2	Calculating the distance travelled from one point to another point.
4.3	Interpretation of functions and making realistic conclusions.
5.1	Interpretation of the parabolic graph and finding the values of the parameters a, b and c given the turning point of the parabola and another point by using the equation: $y = a(x - p)^2 + q$.
5.2	Finding the distance between the two *x*-intercepts.
5.3	Finding the distance between the *y*-intercepts of the two graphs namely the straight line and the parabola.
5.4	Interpreting decreasing and increasing trends of functions.
5.5	Interpretation of the intervals at which one function is greater than another function.
5.6	Finding the length of the perpendicular distance between two points.
5.7	Substituting values into functions using the correct functional notations and interpretation.

Table 6.1(a): Interpretation of the questions for the diagnostic test.

TABLE 4.

PROTOCOL NUMBER: 2012ECE056

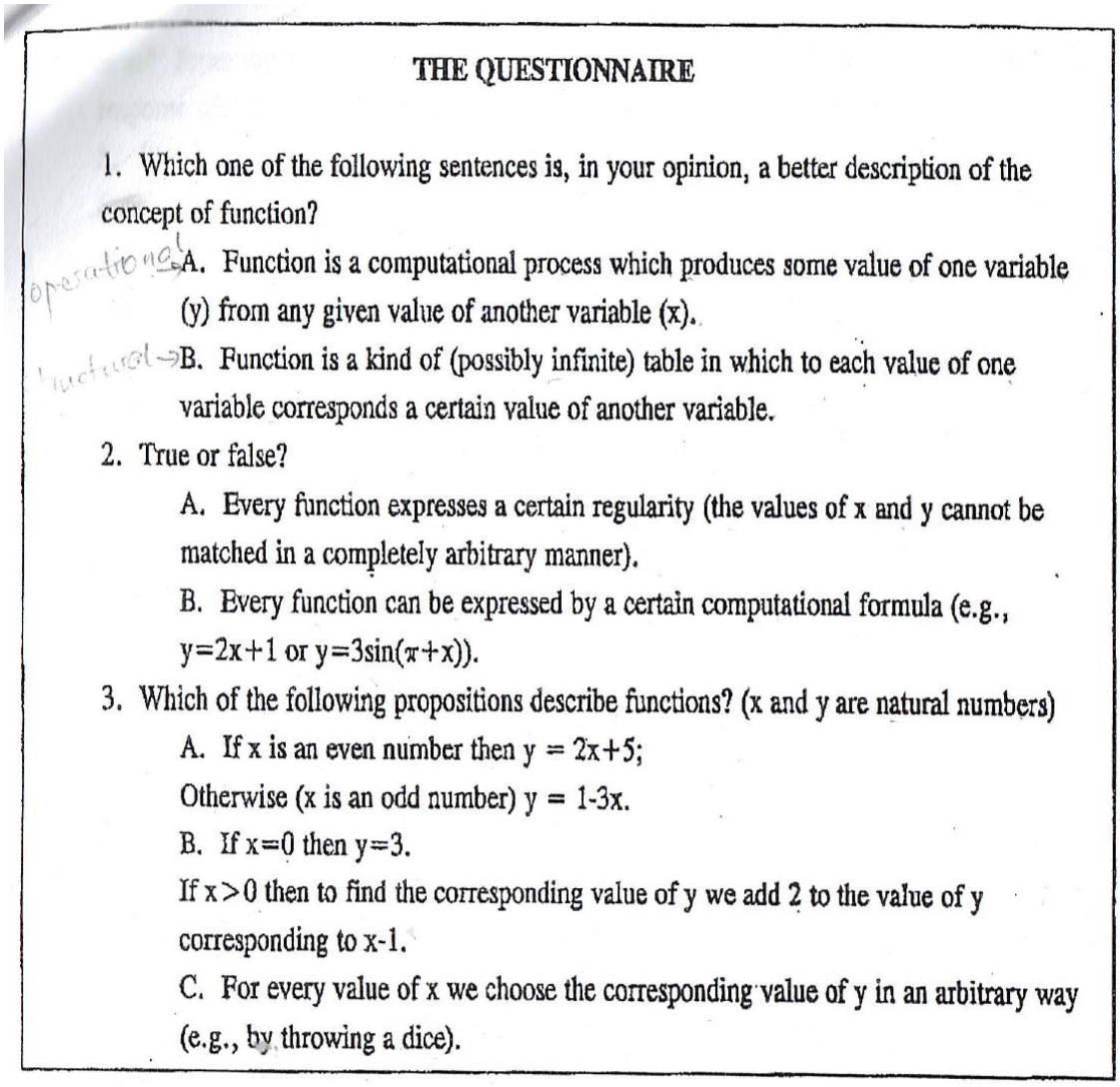
TABLE OF SPECIFICATION FOR THE POST TEST.

Question	Category/ features of functions
1	Interpretation of the parabolic graph and finding the equation of a graph given the turning point of the parabola and a point by using the equation: $y = a(x - p)^2 + q$.
2	Interpretation of the parabolic graph and finding the equation of a graph given the *x*-intercepts of the parabola and a point by using the equation $y = a(x - x_1)(x - x_2)$.
3	Interpretation of the verbal representation of a parabolic graph and finding the equation of a graph given any three points of the parabola by using the equation: $y = ax^2 + bx + c$. Three quadratic equations are to be solved simultaneously in order to find the values of the parameters a, b and c. The conversion from the verbal form of the function to the equation form of the function was the aim of the question.

TABLE 5.

PROTOCOL NUMBER: 2012ECE056

Figure 2.6: The questionnaire used in the experimental and the control groups.



THE QUESTIONNAIRE

1. Which one of the following sentences is, in your opinion, a better description of the concept of function?

   A. Function is a computational process which produces some value of one variable (y) from any given value of another variable (x).

   B. Function is a kind of (possibly infinite) table in which to each value of one variable corresponds a certain value of another variable.

2. True or false?

   A. Every function expresses a certain regularity (the values of x and y cannot be matched in a completely arbitrary manner).

   B. Every function can be expressed by a certain computational formula (e.g., $y=2x+1$ or $y=3\sin(\pi+x)$).

3. Which of the following propositions describe functions? (x and y are natural numbers)

   A. If x is an even number then $y = 2x+5$;
   Otherwise (x is an odd number) $y = 1-3x$.

   B. If $x=0$ then $y=3$.
   If $x>0$ then to find the corresponding value of y we add 2 to the value of y corresponding to x-1.

   C. For every value of x we choose the corresponding value of y in an arbitrary way (e.g., by throwing a dice).

TABLE 6.

PROTOCOL NUMBER: 2012ECE056



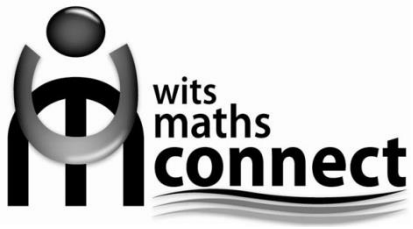




Figure 2 .7: Results of the questionnaire for the control group and the experimental group.

RESULTS	control	exper.
	N=48	N=28
1. percent of students who chose item A	81	50
percent of students who chose item B	19	50
2. percent of students whose answer to A & B were		
yes, yes	46	36
yes, no	26	18
no, yes	21	3
no, no	6	43
3. percent of students who said it was function		
item A	50	93
item B	73	93
item C	17	50