THE PSYCHOLOGY OF LEARNING AND THE NATURE OF MATHEMATICS

by

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IT is not easy to talk to an unknown audience about two difficult subjects—the Psychology of Learning, about which, in general, very little is known, and the Nature of Mathematics, about which, in general, nothing is known. However, it is important that at this early stage of the course we try to give you some idea of the way people are thinking about mathematics and about learning so that some of the more practical stuff that we shall be dealing with later fits into a framework which at least has been pointed out to you. My thinking at the moment is about ten years old and is dynamic—it is changing all the time—and I am really a student rather than a professor in this particular field. So what I have to say today is a personal point of view culled from a lot of my own work and much of other people's.

You will notice that we have chosen as the title "The Psychology of Learning and the Nature of Mathematics". The reason for this is that we have to understand that the teaching of mathematics is not synonymous with the learning of mathematics. Learning mathematics involves both mathematics (that is, the subject matter) and the living child. It is for this reason that we this morning must look at mathematics, and at the living organism and the way in which it learns.

We can in fact start with either, but I have chosen this morning to start with mathematics. What is mathematics? I find it extraordinarily difficult to answer the question to my own satisfaction. Many people would say that mathematics is the study and classification of pattern. Pattern in this context means any kind of regularity in form or in idea, and I think this is a useful definition, because so many of the things in this world can be described using this particular definition—for example, an electric current, a jet aircraft, the shape of a crystal or the mechanics of the atom. All these have a pattern that can be studied and can be described quite explicitly and clearly using maths.

At a more simple level, let us take the numbers 5, 21 and 3, 047. At first, these seem to be entirely unrelated, but you will see that they can fit in to a number of patterns. The one that is immediately obvious to me is that each of these is an element of the sub-set of numbers which may be described as the odd numbers. This is a simple example. If you take other sub-sets of numbers, say the square numbers, and you write these—

$$(1, 4, 9, 16, \ldots\ldots)$$

and you also write the sub-set of odd numbers as follows—

$$(1, 3, 5, 7, 9, \ldots\ldots)$$

you will see that the pattern of odd numbers is related to the pattern of square numbers. In fact the square numbers are formed by summating the odd numbers. $1 + 3$ gives you the second square number which is 4. If you add 5, you get the third square number, which is 9, and so on.

Another kind of thing which can be said about mathematics is that it is a way of thinking about numbers and about space and about the objects that exist in space and the relationships between these objects. For example, my eye is an object in space and so is the corner of this lectern. We can imagine a line segment between the pupil of my eye and the corner of the lectern and this line segment can be measured by means of mathematics and therefore we can describe their relationship quite explicitly.

There is another kind of definition—mathematics is the discovery of relationships and the expression of these relationships in symbolic form. This is important because mathematics is useless until you have a representational system, a symbol system, so that you can put the mathematics to work.

To me, however, the most useful way of looking at mathematics is that it is the study of an ordered body or structure of related concepts, their expression and use.

A concept is an idea that forms in the mind of a human being. It does not exist without the mind. It is not a thing or an object. A concept happens or is attained as a result of a fusion of certain experience and a human organism. If these things are locked up in your minds, you cannot communicate or use these ideas except idiosyncratically—uniquely, by yourself—unless you have a system of symbols that represent these ideas and forms, if you like, a language with which you can communicate with other human beings. And we should not forget that the nature of mathematics is such that we must, once we have got hold of some of these concepts, have a language available, symbols representing the ideas and relationships which we can then put to use.

It is not often realised that mathematics does not exist and is in itself virtually useless. It has nothing to do with the world; it is not of the world; it is useful in the world, but it is built up in the mind of man and was invented by man. It was not invented by God; it is man-made.

These points are often overlooked, and what is worse, many of us concerned in our early days with teaching mathematics, taught the representational system and not the ideas.

For example:

$$4{,}591 > 437 > 188 < \frac{(144 \times \frac{1}{2})}{\frac{1}{4}}$$

is of no importance as a mathematical statement, but try to explain it in the ordinary language that we use to communicate about other concepts, and see how efficient and elegant this precise language of mathematics is. It saves a lot of time, and this is no more obvious than when you are programming a digital computer.

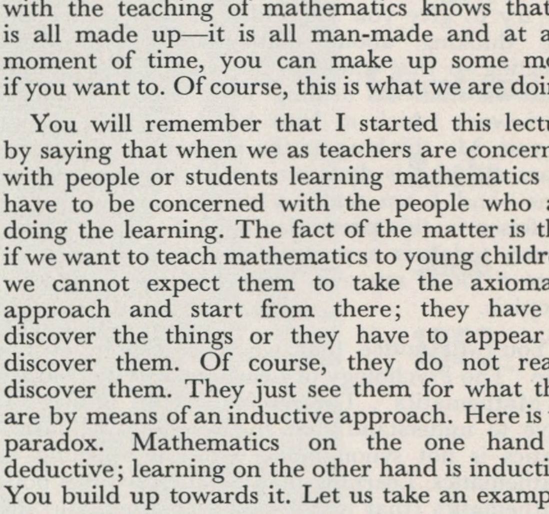
The second point I would like to make about mathematics is that it is essentially abstract and deductive. What do I mean by abstract? I mean that it has no actual form. The ideas have no real form: they do not exist. If you take the simple, natural numbers, these natural numbers do not exist as such. Nobody has ever seen a number. And yet numbers, these highly abstract things, are used quite well by quite young children (if they have had the right experience) with complete understanding.

The number 7, for example, does not exist. It is a property of sets of objects, and represents an infinite class of sets of objects that can vary in every possible way except that there are always 7 discrete elements or objects in the set. That is a bit of a mouthful, but that is what 7 really is.

Furthermore, mathematics is essentially deductive. Numbers, for example, behave in accordance with certain rules or laws, or, if you like, axioms. You can deduce the behaviour of any of the sets of numbers from the laws which man has made up about them. Arithmetic depends upon a number of such laws and is extremely limited by a set of such laws. It is a remarkable fact that most people going to university in Britain cannot write down on a piece of paper the laws that govern the behaviour of numbers in simple arithmetic.

Mathematics, then, is deductive. What we actually do is we have a number of undefined terms as a "point", and from these terms, we set up a number of postulates or we make a number of assumptions. Having made these assumptions, we then proceed by the use of logic to prove the theorems from them. Then we have an axiom, which is a law, and we start making the numbers follow these laws. This is sensible and obvious, but very few people ever seem to realise it. It is amazing to me that hardly anybody concerned with the teaching of mathematics knows that it is all made up—it is all man-made and at any moment of time, you can make up some more if you want to. Of course, this is what we are doing.

You will remember that I started this lecture by saying that when we as teachers are concerned with people or students learning mathematics we have to be concerned with the people who are doing the learning. The fact of the matter is that if we want to teach mathematics to young children, we cannot expect them to take the axiomatic approach and start from there; they have to discover the things or they have to appear to discover them. Of course, they do not really discover them. They just see them for what they are by means of an inductive approach. Here is the paradox. Mathematics on the one hand is deductive; learning on the other hand is inductive. You build up towards it. Let us take an example:



Here are some simple geometric shapes. Suppose we cut these shapes out of paper, using a template, and suppose we divide them up into triangles. As you know, if you put the three internal angles of a triangle together, you will get what some people call a straight angle, or an angle of 180°.

Let us put some order into what we are doing by making a tabulation. "A" is the number of sides and "B" is the number of straight angles:

A	(Sides)	3	4	5	6	7	8
B	(Straight Angles)	1	2	3	4	5	6

I suppose that as you are all mathematicians, you would say that the sum of the internal angles of any straight-sided polygon could be summarised as $S = (n - 2)$ straight angles or angles of 180°. It seems obvious that that is always going to be true and I am sure you could tell me by interpolation or extrapolation how many straight angles a polygon of 16 sides has.

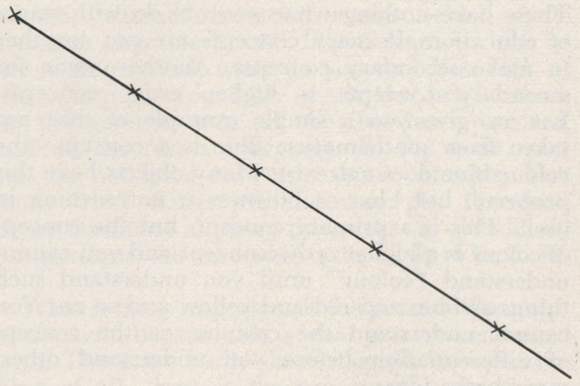
But how do you know that this is always going to be true? This is a hypothetical question. Inductively, you are beginning to get a feel for this; you are beginning to say: "I can classify this". But you can never be quite sure. You think that there will always be another odd number, but you are never really sure. So inductive reasoning is never a hundred per cent certain, but it is good enough for most of us.

Let us take another simple example. Let us try to get an approximate value for π. You know that π is a special sort of number called an irrational number. Nobody has yet discovered its exact value, and it is highly probable that they never will, because in fact it is not an exact number. You cannot work it out to a finite number of decimal places. You cannot do this with any irrational number. Despite the fact that we have deductively said this, somebody used a computer several years ago and worked it out to about 14,000 decimal places. Of course, it just goes on and on. However, you can inductively build up a fairly good notion of π by wrapping pieces of paper round cylinders or taking circular objects and drawing round them and measuring across the diameters and seeing the apparent relationship between the diameter and the circumference.

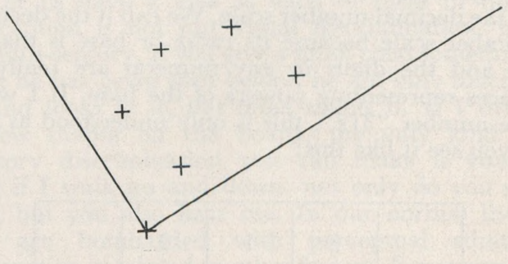
Sometimes you cannot believe your eyes. Here is something else for you to do. Suppose you had a piece of string and you twined it round the equator, tightly, and then you increased the length of this piece of string by approximately six inches. How far off the equator would the piece of string stand if it was the same distance away from the earth all the way round the circumference? This is a simple question, but the answer is almost unbelievable.

So the inductive approach is a very exciting and important approach. It is the discovery approach that so many of us feel to be absolutely essential to the learning of mathematics. But it is not essentially in the nature of maths, because maths is deductive, and sooner or later you will have to take an axiomatic approach. But remember my warning at the very beginning, that sometimes you have to turn away from mathematics towards the child and this is something that you should consider very carefully.

Now let us come back to this question of the ordered structure of concepts. Here is a very difficult question. What are these concepts that form the structure? Let us make some drawings to describe this structure.



Now is it a linear structure like this? No, it is not like this at all; it is much more complex; it is multi-dimensional. It is rather like this:



one solitary concept down here and a sort of stellation of concepts, not just on one plane, but in many planes, many of them inter-related in a complex and fascinating fashion.

So in studying the structure involved, you cannot think of these concepts any longer as belonging to arithmetic or one of the algebras, or one of the geometries, or the calculus, because in fact these things that we call algebra and geometry and calculus are on the outside. These are all things that are made possible by the structure that lies underneath, and this again is a completely new notion. Hardly anybody understands it. Yet it is basic to any thinking about mathematics and makes sense of much that we are doing. We are not trying to fuse the things on the outside. There is no need to fuse them. They are each dependent upon ideas and, of course, with these ideas, we can make up some new mathematics, perhaps: the so-called modern mathematics is not really modern or new. It is simply a new arrangement of some of these ideas, with perhaps some more thrown in. This is of fundamental importance.

The best way to differentiate sharply in the structure is to differentiate between what are called primary concepts and secondary concepts. These have nothing whatsoever to do with stages of education. Primary concepts are put together to make secondary concepts. Another name for secondary concepts is higher order concepts. Let me give you a simple example of this, not taken from mathematics. Blue is a concept—the colour blue does not exist. Many objects have this property, but blue or blueness is not a thing in itself. This is a primary concept, but the concept of colour is a higher order concept and you cannot understand "colour" until you understand such things as blue and red and yellow and so on. You cannot understand the calculus or the concept of differentiation before you understand other, more basic, ideas.

Perhaps a simple example of primary concepts, higher order concepts and the way all these things are related, is to consider the notion of place value. You know that we have systems of numeration to express the numbers. Let us take the system of numeration which we could describe as the decimal number scale. We call it the decimal number scale because its radix or base is that of 10 and the digits in any numeral are really in places representing powers of the base. If I write the number "312", this is only understood as 312 if you see it like this:

	3	1	2	
10^3	10^2	10^1	10^0	10^{-1}

the digits in this numeral being in the places of which the 1 is in the base position, the base being 10 and the power being 1. The 3 therefore, is the base to the second power. Notice that the exponent or the index is increasing by 1. Therefore if we go the other way, the units position will be the base to the zeroth power and if you want to put a decimal point here, then the next position to the right will be the power to the minus 1. Any number which is between the second power and the third power will have to, or may be expressed, not by writing it like this, but by writing it as a fractional exponent. Another word for the fractional exponent is the logarithm, and logarithms need not be to the base of 10, but can be to the base of any number. If you want to understand logarithms, you ought to work in other bases.

You will now see how what seems to be a very simple thing leads on to something else, how understanding to write numbers down in the decimal number system leads you on to a study of decimals and, if you want to, on to a study of fractions, but it can also lead you to a study of powers or to a study of roots and certainly to a study of logarithms.

There are two kinds of concepts in mathematics. There are class concepts of which number is an example, and operational concepts, such as addition. There is always a danger of confusing the concepts with the techniques.

There was a time in Britain when six and seven-year olds were asked to perform this calculation:

$$\begin{array}{r} 39 \\ \times\ 3 \\ \hline 117 \\ \hline \end{array}$$

I would just like to point out to you the concepts of which that is an expression. First of all there is the concept of place value which enables you to write the number 39 in this way. Then there is the law called the law of closure for multiplication that says that for every a and b there exists a p such that a $\times$ b is the same number as p. In other words, the system that we are using is closed in respect of multiplication and if you multiply two numbers together there will always be a number in the system that will be the product. This is also true of addition. If you take any two natural numbers, you can always find a third number called the sum. This, however, is not true for subtraction. This is an example of the fact that we have made a series of laws, and numbers can only do certain things in accordance with the laws.

If you do not know three nines, you can say nine threes, and you can say this because there is a commutative law for multiplication. Incidentally, you are not multiplying 39 by 3, but you are distributing the 39 into 9 plus 30 and you are using the complicated distributive law.

Ideally, children should not be asked to do this kind of operation until these ideas are beginning to grow in their minds.

I should like to point out to you that this stellation shown in the diagram may have new stars, new concepts, in it. The body of concepts may well change. As you know, new systems of numbers were made up when the other numbers were not good enough. When the natural numbers were not good enough, we had a set of integers; when the integers were not good enough, we had some fractions; when the fractions were not good enough we had some rational numbers; when the rational numbers were not good enough, we added in some irrational numbers to get the set of real numbers and when the real numbers were not good enough for electronic circuits, then we made up some numbers which were imaginary like the square root of minus 1. Then we had the set of complex numbers and when we start landing on the moon, we might need some more. We can make some up, we can make some assumptions and prove them to be true, establish some axioms and away we go.

That is the exciting thing about maths; it can expand like the cosmos. I hope it does not do so very fast because most of us are about five hundred years behind the times already. I should like to quote you what Marshall Stone said in that fascinating book "New Thinking in School Mathematics" which was the report of a conference held in 1959 under the auspices of the O.E.E.C. Stone said: "The educated man whom we envisage as the end product of our elaborate educational process should not be left some two hundred years behind the times in mathematics merely because he is not a specialist in maths or science." (You see, only those people reading for degrees in maths or science were coming across any of the new ideas until very recently. Certainly this was true in 1959.) Stone continued: "Least of all should this be so in an era when maths is growing so vigorously and penetrating so profoundly into so many different domains of thought."

I think that all of us have to realise that when we consider the nature of mathematics, we should leave our thoughts open-ended. We must be prepared to accept new ideas, and it is for us as teachers to decide how these ideas should be learned. That is not necessarily the job of the mathematician who thinks of them.

Well, that is a very brief view of maths. I hope that it has made you realise that it is a very difficult and complex subject.

Let us now have a look at the psychology of learning and particularly the aspect of cognition. There are many kinds of learning. If you want to learn to knock a nail into a piece of wood, then learning is involved. You have got to learn how to hold the hammer; there is a co-ordination of muscles and eyes; it is a very complex kind of learning, but it is essentially different from mathematical learning.

At the beginning of this paper, we said: "What is mathematics?" So let us start off with a similar question: "What is a concept?" It is extraordinarily difficult to describe things and people with whom you are really familiar, and the same is true of concepts. Concepts are fundamental to human thinking—we are using them all the time. This is the best definition of a concept that I know and it is not a very good one: "A concept is a classification or systematic organisation of stimuli, characteristics or events which have common characteristics." If you want to look at the average child, what do you do? You look at the extremes. You look at all the bright ones, all the geniuses and all the slow-learning ones, and all the rest are average. In the same way, you do not have to worry about concepts. You know what things are not concepts, therefore, if you eliminate all those things that are obviously not concepts, the others are quite likely to be them.

What is more important is the process of concept attainment, because this is what you are concerned with. There are really three stages in this process. First of all, there is the process of perception—taking note of something in some way. Then there is the process of abstraction, meaning drawing away from, and finally the process of generalisation; so it is the cycle of perception, abstraction, generalisation.

Let us have a look at each of these in turn. Perception is not just visual discrimination, but is multi-sensory. If you give a small child something that he has never seen before, he not only touches it and looks at it, but often smells it. If I draw a few shapes on the board, the only kind of sensory discrimination you can make is visual. But if I walk up and down, not only do you see me, but you also hear me. In our normal lives, we are bombarded with perceptual stimuli. Perception is related to other factors. For example, it is related to motivational and emotive states. We have paid little attention to the emotive state in education and absolutely none to motivation. Teaching goes along on the assumption that as far as motivation and emotion are concerned, everything is in order and the children are bound to be listening.

Various psychologists, notably Professor Vernon, have looked at the perceptual modes. Some people use one mode of perception more than others. Most of us are conditioned, because we are supposed to be civilised, to using visual discrimination. If you walk into somebody's house and they say: "Don't you think that fruit looks fresh?" and you walk over and smell it, they will be very cross. So we have conditioned ourselves to use this visual sense. But this is not true of children. Why should we make the assumption that we have got to limit them to one kind of thinking, and how do we know that they discriminate as we do?

Vernon saw that as far as perception was concerned, there are really two kinds of ways in which people perceive things. There is what is called an analytic mode and a synthetic mode. Those of us who are synthetic people look at things as a whole — a kind of Gestalt mode. The analysers look at little bits, so, as I am an analyser, if you invite me to your home, I shall know that the curtain is faded at the edge and that you have got a very nice wood carving on the wall, but it is cracking owing to the temperature changes, but I shall not be able to tell you at all what the general effect of your room is because I shall not have noticed it. Others who might come into the room would have an impression of well-being and would say: "This is a nice house," and you would say: "Did you like the curtains?" and they would say: "Well, I didn't notice them." Adults can be classified, but unfortunately, you cannot do this with children. They are not set in their ways. So whenever we start teaching, we are making use of our own particular perceptual mode and we impose this on children. This is why half the children cannot follow what we are doing, not because they are intellectually incapable, but simply because they do not happen to look at things as we do.

Jean Piaget has been working in the Jean-Jacques Rousseau Institut in Genève for twenty-five years and it is only comparatively recently that his work has become widely known. Piaget has looked at perception very carefully and has discovered that perceptual judgment, visual discrimination, deteriorates in children after the age of six and seven up to the age of nine to eleven. This is just the time when we start teaching them, using this visual sense.

Let us now have a look at the process of abstraction. Visual perception involves the senses and naturally some thinking. Abstraction involves a thought process. You have first of all to perceive certain attributes or qualities; then you have to draw them away from the objects and get them into your mind and put them together in order to define the concept. So abstraction is a mental process and again is affected by emotional and other factors, but the thinking that goes on in the process of abstraction is probably one of the greatest unanswered questions of our time. We know little about it. We know the kinds of things that hold it up, but the actual process is an open question, except in a number of small ways.

The process of abstraction is related to the ways in which people think. I believe we shall eventually be able to construct a sort of model representing the way in which every individual thinks. This model will look something like the models that we have of structures of atoms. I can imagine that we shall have a number of dimensions, not in one plane, but in "n" planes — an n-dimensional model — with a number of these modalities or modes and each one of us will have on a modality a region in which we operate most usually. Thinking is not just a single kind of act, but has a number of aspects which together build up the pattern of us as thinking persons.

In mathematics, the important modality is one which has at one end the word "constructive" and at the other end the word "analytic". I am not talking about perception. I am talking about thinking. Some people tend to think constructively and others tend to think analytically, not just to perceive in this way, but actually to think constructively or analytically. Recent research has shown that children between the ages of birth and eleven tend to think constructively more often than they do analytically, but not always, because this is a modality and you can be at any point on it at any moment of time. In general, children, particularly girls, tend to think constructively. Here is a very interesting point. Most adults who pass examinations in mathematics and science must be able to think analytically or they do not pass the examinations. Therefore it is reasonable to suppose that most teachers of mathematics are analytic thinkers; in fact most teachers are analytic thinkers. They do not have great intuitive leaps into things. They take the component parts very carefully. They look at things in an analytic way and think analytically, but most children do not. This is a very important point because it means that if you are going to teach children and you are going to persuade them to think in your way, you have got an impossible task. With the younger children anyway, you are really making their learning almost impossible because you are interpreting things in an analytic way and they need to interpret them in a constructive way. Therefore I would say categorically that if you want young children to learn mathematics, you would be well advised to give up teaching and devise situations and experiences from which they themselves can learn, using their own thinking mode.

The fact that children are like this does not necessarily mean that they need be so. I have been doing some work with a psychologist from Harvard with young children in which we are beginning to suspect that if you design the right kinds of experiences, very young children can flit from analytic to constructive thinking very well, better than adults, at a very tender age; in fact our work has shown that in complicated situations, five to seven-year olds are able to work much more effectively than adults. This leads me to suggest that education is not only a waste of time but a positively harmful thing. I believe that within the next decade, much that we are doing in schools will be completely and utterly condemned from this point of view alone, not just in relation to the teaching of mathematics, but remember that the teaching of mathematics is above all a thinking activity and therefore we have a greater responsibility than most other teachers.

When you are carrying out the process of abstraction you are really drawing out the various attributes and you have to disregard all the other things. Therefore you are going to make it very difficult if you add to what is called in psychological terms, "the noise." You have got to disregard the psychological noise and draw out the relevant attributes. So if you want children to learn about numbers, it is not necessarily a good thing to make the objects that they are looking at into jam tarts, because the appeal of the jam tarts is greater than the appeal of the mathematics and so, instead of decreasing the noise, you are increasing it and many of the so-called practical approaches to mathematics failed on this score, that you added to the psychological noise and consequently you got less learning. This is something that we could not possibly have discovered for ourselves. We could not understand why this happened. We needed a psychologist to take an objective view and to tell us this.

Generalisation, which is the final stage, is the inclusion in the concept or the class of new things. May I point out that what I have said about the process of concept development must obviously draw your attention to the absolute importance of the nature of the experience. The experience allows perception to take place and abstraction to follow and, we hope, the concept to be attained and to be generalised. Therefore if the experiences that we present to children are not right for one or other reason, either there is too much noise, or they are emotionally wrong, or they are not related to the concept, learning will not take place. I cannot stress too highly to you the importance of the proper experience at the appropriate time. This means that all these notions of teachers teaching whole classes at the same time are completely and utterly useless. This is not to say that you do not have to do it. I am talking about what is apparently so; whether you do it is another matter, but from the mathematical and psychological viewpoint, class teaching is a complete and utter waste of time and very little learning will take place.

That is just the beginning to the psychological part. I have not talked about the stages of development through which children pass because you will know of the developmental psychology of Piaget. I have not talked about motivation, the kinds of motivation. I have not talked about emotional factors. I have not told you of some research that I have been doing which shows the high correlation between anxiety and poor mathematical attainment. I have not told you that concepts take time to grow and can be operational partially at various stages. However it seems to me that mathematics has a unique function to play in education because it is above all the time when children are learning to think and to build up concepts. The attainment of concepts is fundamental to education, and mathematics is a study of such a kind that concept learning should be taking place for much of the time. One cannot ignore what the psychologists are telling us and one cannot ignore the problem of maintaining the right balance at any moment of time for any individual, the right fusion of experience which is mathematically right and psychologically right. This imposes an enormous burden on teachers.

Finally, I should like to leave you with this thought, that if you can give a little more time to looking at your children as they are learning, I think that you will begin to see that everything that I have been saying is in fact basically so, and I do commend to you that if we do nothing at all during this course, we do this — we give you some experiences which will enable you to do less teaching and think more about the nature of learning and the nature of experiences that will enable your children to learn mathematics, remembering that what you do with the five-year olds must be right because it will be used by the seven-year olds, the nine-year olds, the eleven-year olds and the fifteen-year olds.

I think it is a great privilege to accept this challenge of trying to match up the nature of the way children learn with the nature of mathematics and to do our best to see that this kind of learning is as good as it can be.