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ON THE FIRST TWO EIGENVALUES OF STURM-LIOUVILLE OPERATORS

JOHANNES NAMO MASEHLA

ABSTRACT. Among the Schrödinger operators with single-well potential defined on $(0, \pi)$ with transition point at $\frac{\pi}{2}$, the gap between the first two eigenvalues of the Dirichlet problem is being investigated. We also show how this extends former results with symmetric potential. Finally we will consider an analogous Dirichlet problem of vibrating strings with single-barrier densities for the ratio of the first two eigenvalues.

1. INTRODUCTION

General bounds on the gaps and ratios between eigenvalues of Sturm-Liouville operators have been the object of considerable attention. The gap between the first two eigenvalues is of particular interest because it represents the first excitation energy. The research was initiated by Singer, Wong, Yau, and Yau in [14] for the multidimensional Schrödiger operator. Their work was followed by improvements for the one-dimensional case with Dirichlet boundary conditions.

In this research report, we analyze the paper written by Micklós Hórvarth in [7]. We consider the Dirichlet problem for Schrödinger operators:

$$-y'' + V(x)y = \lambda y \tag{1.1}$$

$V \in L^1(0, \pi)$ and real-valued acting on $(0, \pi)$,

$$y(0) = y(\pi) = 0. \tag{1.2}$$

Ashbaugh and Benguria in [2] proved that if V is a symmetric single-well potential, then (see [7]) the first two eigenvalues of the Dirichlet problem satisfy:

$$\lambda_2 - \lambda_1 \geq 3 \tag{1.3}$$

with equality if and only if V is constant. Hórvarth then demonstrated a natural way to remove symmetry. The main result is that (1.3) still

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holds for non-symmetric single-well potential if transition point remains the midpoint.

We next consider another Dirichlet problem for vibrating strings:

$$-u'' = \lambda\varrho(x)u \tag{1.4}$$

in $(0, \pi)$

$$u(0) = u(\pi) = 0 \tag{1.5}$$

where $\varrho(x)$ is a positive density function. Huang proved that (see [7]) in the class of concave densities or symmetric single-barrier densities the first two eigenvalues of (1.4)-(1.5) satisfy

$$\frac{\lambda_2}{\lambda_1} \geq 4 \tag{1.6}$$

and in the class of symmetric single-well densities we have,

$$\frac{\lambda_2}{\lambda_1} \leq 4 \tag{1.7}$$

In (1.6) and (1.7) equality holds if and only if ϱ is constant. Hórvarth then showed that (1.6) and (1.7) holds also for non-symmetric densities. The scope of my research report with respect to the manuscript [7] is to provide fully detailed proofs of the theorems, lemmas and corollaries presented in this manuscript. We will begin with the preliminary section to introduce definitions and statements of theorems. Some of the known results will be stated without proofs. The two main theorems will be stated and proved in the last section. In the first part we deal with the Dirichlet problem for Schrödinger operators and the second part with Dirichlet problem for vibrating strings. I will include proofs of results that are not stated in [7] but are relevant for this research. I will show that there exist an optimal single-well potential in some compact set. The existence of an optimal single-barrier density can be proved in a similar manner.

2. Preliminary Definitions and basic results

We let

$$\tau u(x) = \frac{1}{r(x)}\big(- (p(x)u'(x))' + q(x)u(x)\big) \tag{2.1}$$

with $r(x), q(x), \frac{1}{p(x)} \in L^1(a, b)$ and $p(x), r(x) > 0$ almost everywhere in (a, b).

For real λ, every non-trivial solution of u of $(\tau - \lambda)u = 0$ may be written in the polar form

$$u(x) = \rho(x) \sin\theta(x),$$

$$p(x)u'(x) = \rho(x)\cos\theta(x),$$

where $\theta(.)$ is called the Prüfer angle and satisfies

$$\theta(x) = \arctan\frac{u(x)}{p(x)u'(x)}$$

Consider the regular Sturm-Liouville operators $A_{\alpha,\beta}$ on (a,b) with the separated boundary conditions

$$u(a)\cos\alpha - p(a)u'(a)\sin\alpha = 0$$

$$u(b)\cos\beta - p(b)u'(b)\sin\beta = 0,$$

$\alpha, \beta \in \mathbb{R}$. It is well known that $A_{\alpha,\beta}$ is self-adjoint in $L^2(a,b)$ (see [15],p.199).

Let $u(.,\lambda)$ be the solution of $(\tau - \lambda)u = 0$ satisfying the initial conditions,

$$u(a) = \sin\alpha,$$

$$p(a)u'(a) = \cos\alpha$$

and $\theta(.,\lambda)$ be the corresponding Prüfer angle function satisfying $\theta(a,\lambda) = \alpha$. A real number λ is an eigenvalue of A if and if only if

$$\theta(b,\lambda) = \beta \mod \pi$$

As stated in ([15],p.199) we have that

Theorem 2.1. *Let τ be a regular Sturm-Liouville expression on (a,b) with $p(x) > 0$ and $A_{\alpha,\beta}$ a self-adjoint realization of τ with separated boundary condition and $\alpha \in [0,\pi), \beta \in (0,\pi]$. Then the spectrum of A consists of infinitely many simple eigenvalues and is bounded from below. If the eigenvalues are arranged such that*

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n \to \infty,$$

then the eigenfunction u_n corresponding to the eigenvalue λ_n has exactly $n-1$ zeros in (a,b).

The Dirichlet problem (1.1)-(1.2) is a special case of Theorem 2.1 with $p(x) = r(x) = 1, q(x) = V(x)$ whereas the Dirichlet problem (1.4)-(1.5) is a case with $r(x) = \varrho(x), q(x) = 0$ and $p(x) = 1$ when $\alpha = 0, \beta = \pi$ in both cases.

It is well-known (see [5],p.186) that the solutions of the differential equation

$$-y'' = \lambda y$$

satisfying $y(0) = 0$ are of the form

$$y(x) = A_1\sin(\sqrt{\lambda}x)$$

where A_1 is a constant. Hence for the differential equation

$$-y'' + qy = \lambda y$$

$q \in \mathbb{R}$, the solutions are of the form

$$y(x) = B_1 \sin(\sqrt{\lambda - q}x).$$

3. Special Results

Definition 3.1. A function V is called a *single-well* function with transition point at $c \in [0, \pi]$ if V is non-increasing in $[0, c]$ and nondecreasing in $[c, \pi]$.

Definition 3.2. A function V is called a *single-barrier* function with transition point at $c \in [0, \pi]$ if it is nondecreasing in $[0, c]$ and non-increasing in $[c, \pi]$.

Let

$$A_M = \{V | 0 \leq V \leq M, V \text{ is a single-well potential with transition at } \frac{\pi}{2}\},$$

where $M > 0$.

Now consider the Dirichlet problem (1.1)-(1.2) with $V \in A_M$. The eigenvalues $\lambda_1, \lambda_2, ...$ arranged such that $0 < \lambda_1 < \lambda_2 < \dots$ refer to this problem.

Definition 3.3. An optimal single-well potential $V_0 \in A_M$ is when $\lambda_2 - \lambda_1$ is minimal with respect to $V \in A_M$.

Proposition 3.4. *For each $M > 0$, there is an optimal single-well potential $V_0 \in A_M$.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in A_M. We want to show that $(f_n)_n$ has a convergent subsequence in $L^1(\frac{\pi}{2}, \pi)$. Since each f_n is nondecreasing, it has at most countably many discontinuity points. Choose $a_1, a_2, \dots,$ in $(\frac{\pi}{2}, \pi)$ such that $\{a_1, a_2, \dots, \}$ is dense in $(\frac{\pi}{2}, \pi)$ and none of $a_1, a_2, \dots,$ is a discontinuity point of any f_n.

Since $0 \leq f_n \leq M$ for all n, we can define a subsequence $(f_{1,j})_{j=1}^{\infty}$ of $\big(f_n\big)_n$ such that $\big((f_{1,j}(a_1)\big)_{j=1}^{\infty}$ is convergent.

Let $(f_{2,j})_{j=1}^{\infty}$ be a subsequence of $(f_{1,j})_{j=1}^{\infty}$ such that $\big(f_{2,j}(a_2)\big)_{j=1}^{\infty}$ is convergent. Continuing in this manner we have that for each a_k bounded subsequences $\big(f_{k,j}(a_k)\big)_{j=1}^{\infty}$ are convergent.

Consider the diagonal subsequence $\big(f_{n_j}\big)_j = \big(f_{j,j}\big)_{j=1}^{\infty}$. Then $\big(f_{n_j}\big)_j$ is also convergent at a_k since it is a subsequence of a convergent sequence at a_k. .i.e. there is f such that

$$f_{n_j}(a_k) \to f(a_k)$$

as $j \to \infty$. Next we show the limit f is a nondecreasing function.

Since all f_n are nondecreasing, it follows that $f(a_j) \leq f(a_k)$ if $a_j \leq a_k$ which implies f is nondecreasing.

For $x \in (\frac{\pi}{2}, \pi)$ define

$$f(x) = \sup\{f(a_j) \mid a_j \leq x\}$$

If $x = a_k$, then this coincides with the above definition since f is increasing. Let $\epsilon > 0$ be given, then there is k such that $a_k \leq x$ and

$$|f(a_k) - f(x)| < \epsilon$$

by definition of $f(x)$. $f(x)$ is also nondecreasing by definition.

For a continuity point $x \in (\frac{\pi}{2}, \pi)$ of f, choose a_{k_m}, a_{k_l} such that

$$a_{k_m} \to x$$

from below and

$$a_{k_l} \to x$$

from above. This is possible since $\{a_k \mid k \in \mathbb{N}\}$ is dense in $(\frac{\pi}{2}, \pi)$.

Then

$$f(a_{k_m}) = \liminf_j f_{n_j}(a_{k_m}) \leq \liminf_j f_{n_j}(x) \leq \limsup_j f_{n_j}(x)$$

and

$$\limsup_j f_{n_j}(x) \leq \limsup_j f_{n_j}(a_{k_l}) = f(a_{k_l})$$

since the f_{n_j}'s are nondecreasing functions. But

$$f(a_{k_m}) \to f(x)$$

and

$$f(a_{k_l}) \to f(x)$$

since f is continuous at x. Hence

$$f(x) \leq \liminf_j f_{n_j}(x) \leq \limsup_j f_{n_j}(x) \leq f(x)$$

which implies

$$\lim_j f_{n_j}(x) = f(x)$$

Thus we have shown that $f_{n_j}(x) \to f(x) \in A_M$ a.e. in $L^1(\frac{\pi}{2}, \pi)$ and by Lebesgue's Dominated Convergence Theorem,

$$\lim_{j\to\infty} \int_{\frac{\pi}{2}}^{\pi} |f_{n_j}(x) - f(x)|dx = \int_{\frac{\pi}{2}}^{\pi} \lim_{j\to\infty} |f_{n_j}(x) - f(x)|dx = 0$$

Similarly we can construct a convergent subsequence $\big(-f_{n_j}\big)_{j=1}^{\infty}$ of $(-f_n)_{n\in\mathbb{N}}$ on a dense subset of $(0, \frac{\pi}{2})$.

The set $\{\lambda_2(V) - \lambda_1(V) : V \in A_M\}$ is bounded below. So there is a sequence $(V_n)_{n\in\mathbb{N}} \subset A_M$ such that

$$\lim_{n\to\infty} \big(\lambda_2(V_n) - \lambda_1(V_n)\big) = \inf\{\lambda_2(V) - \lambda_1(V) : V \in A_M\}$$

Since we have shown that every sequence in A_M has a convergent subsequence, we may assume that there is $V_0 \in A_M$ such that

$$\lim_{n\to\infty} V_n = V_0$$

in $L^1(0,\pi)$.

For the differential equations

$$-y'' + V_0 y = \lambda y$$

and

$$-y'' + V_n y = \lambda y$$

the complete knowledge of the set of solutions can be obtained if we know the fundamental matrix of the associated first order system (see [5]p.69,[13]p.69). The solutions depend on the parameters of these differential equations. The solutions $y(x,\lambda)$ of

$$-y'' + V(x)y = \lambda y$$

with boundary conditions $y(0,\lambda) = 0$ and $y'(0,\lambda) = 1$ depend analytically on λ and continuously on V. The eigenvalue equation is $y(\lambda,\pi) = 0$. Hence by Rouché's theorem the eigenvalues depend continuously on V. Then, for the first and second eigenvalues

$$\lim_{n\to\infty} \big(\lambda_2(V_n) - \lambda_1(V_n)\big) = \inf\{\lambda_2(V) - \lambda_1(V) : V \in A_M\} = \lambda_2(V_0) - \lambda_1(V_0)$$

□

Definition 3.5. We normalize the eigenfunctions y_n corresponding to eigenvalues λ_n such that $y_n > 0$ for small $x > 0$ and

$$\int_0^\pi y_n^2 = 1 \tag{3.1}$$

Proposition 3.6. *Consider the Dirichlet problem (1.1)-(1.2) with $V \in L^1(0,\pi)$. If y_1, y_2 are respectively the first and second eigenfunction then $\frac{y_2}{y_1}$ is decreasing on $(0,\pi)$.*

Proof. We only need to show that $\left(\frac{y_2}{y_1}\right)' < 0$. y_1 has no zeros and is positive in $(0,\pi)$ by Theorem 2.1. y_2 has an inner zero $x_0, 0 < x_0 < \pi$, and $y_2 > 0$ in $(0,x_0)$ and $y_2 < 0$ in (x_0,π).

$$\big(\frac{y_2(x)}{y_1(x)}\big)' = y_1(x)^{-2}\big(y_2'(x)y_1(x) - y_2(x)y_1'(x)\big)$$

then

$$\begin{aligned}\big(\frac{y_2(x)}{y_1(x)}\big)' =& y_1(x)^{-2}\int_0^x (y_1(t)y_2''(t) + y_2'(t)y_1'(t))dt \\ &- y_1(x)^{-2}\int_0^x (y_2(t)y_1''(t) + y_1'(t)y_2'(t))dt\end{aligned}$$

by fundamental theorem of calculus and the fact that $y_n(0) = 0$ for all n. Substituting $y_n'' = (V - \lambda_n)y_n$ in the above equation for $n = 1, 2$ and simplifying we get

$$\begin{aligned}\int_0^x y_1(t)y_2''(t)dt &= \int_0^x y_1(t)(V(t) - \lambda_2)y_2(t)dt \\ &= \int_0^x V(t)y_1(t)y_2(t)dt - \int_0^x \lambda_2 y_1(t)y_2(t)dt\end{aligned}$$

$$\begin{aligned}\int_0^x y_2(t)y_1''(t)dt &= \int_0^x y_2(t)(V(t) - \lambda_1)y_1(t)dt \\ &= \int_0^x V(t)y_1(t)y_2(t)dt - \int_0^x \lambda_1 y_1(t)y_2(t)dt\end{aligned}$$

Then

$$\big(\frac{y_2(x)}{y_1(x)}\big)' = \frac{1}{y_1^2(x)}\int_0^x (\lambda_1 - \lambda_2)y_1(t)y_2(t)dt < 0$$

since $\lambda_1 < \lambda_2$ and $y_1, y_2 > 0$ on $(0, x_0)$.

Similarly on (x_0, π),

$$\begin{aligned}\big(\frac{y_2(x)}{y_1(x)}\big)' &= -y_1^{-2}(x)\big(\int_x^\pi (y_1(t)y_2''(t)dt + y_2'(t)y_1'(t))dt - \int_x^\pi (y_2(t)y_1''(t) + y_1'(t)y_2'(t))dt\big) \\ &= -\frac{1}{y_1^2(x)}\big(\int_x^\pi y_1(t)(V(t) - \lambda_2)y_2(t)dt - \int_x^\pi y_2(t)(V(t) - \lambda_1)y_1(t)dt\big) \\ &= -\frac{1}{y_1^2(x)}\int_x^\pi (\lambda_1 - \lambda_2)y_1(t)y_2(t)dt < 0\end{aligned}$$

since $\lambda_1 < \lambda_2$ and $y_1 y_2 < 0$ on (x_0, π). □

Following from the above proposition we have points $x_\pm$ with

$$0 \le x_- < x_0 < x_+ \le \pi$$

such that

$$\big|\frac{y_2}{y_1}\big| < 1$$

for $x \in (x_-, x_+)$ and

$$|\frac{y_2}{y_1}| > 1$$

for $x \in (0, x_-) \cup (x_+, \pi)$. Note that we take $x_- = 0$ if $\frac{y_2}{y_1}$ does not attain the value 1 on $(0, x_0)$ and $x_+ = \pi$ if $\frac{y_2}{y_1}$ does not attain the value -1 on (x_0, π).

From above

$$|\frac{y_2}{y_1}|^2 < 1$$

for

$$x \in (x_-, x_+)$$

which implies

$$\frac{y_2^2}{y_1^2} < 1$$

i.e.

$$y_2^2 - y_1^2 < 0 \tag{3.2}$$

for $x \in (x_-, x_+)$ and similarly

$$y_2^2 - y_1^2 > 0 \tag{3.3}$$

for $x \in (0, x_-) \cup (x_+, \pi)$.

Proposition 3.7. *Consider the Dirichlet problem:*

$$-y''(x,t) + V(x,t)y(x,t) = \lambda(t)y(x,t)$$

with boundary conditions

$$y_n(0,t) = y_n(\pi,t) = 0.$$

The derivative of the n-th eigenvalue $\lambda_n(t)$ with respect to t, $\dot{\lambda}_n$ is given by

$$\dot{\lambda}_n = \int_0^\pi \dot{V} y_n^2 dx$$

where the eigenfunctions y_n are normalized as in (3.1) and $V(x,t) \in L^1(0,\pi)$ is a potential function that depends integrably on x and differentiably on t.

Proof.

$$-y_n''(x,t) + V(x,t)y_n(x,t) = \lambda_n(t)y_n(x,t)$$

for each eigenfunction $y_n(x,t)$ and its corresponding eigenvalue $\lambda_n(t)$. Differentiating the above equation with respect to t we get

$$-\dot{y}_n'' + \dot{V}y_n + V\dot{y}_n = \lambda_n\dot{y}_n + \dot{\lambda}_n y_n$$

multiplying by y_n we get

$$-\dot{y}_n'' y_n + \dot{V} y_n^2 + V \dot{y}_n y_n = \lambda_n \dot{y}_n y_n + \dot{\lambda}_n y_n^2 \tag{3.4}$$

Integrating with respect to x from 0 to π,

$$\int_0^\pi -\dot{y}_n'' y_n dx + \int_0^\pi \dot{V} y_n^2 dx + \int_0^\pi V \dot{y}_n y_n dx = \int_0^\pi \lambda_n \dot{y}_n y_n dx + \int_0^\pi \dot{\lambda}_n y_n^2 dx. \tag{3.5}$$

The first term becomes

$$\begin{aligned}
\int_0^\pi -\dot{y}_n'' y_n dx &= -\dot{y}_n' y_n \big|_0^\pi + \int_0^\pi \dot{y}_n' y_n' dx \\
&= \int_0^\pi \dot{y}_n' y_n' dx \\
&= \dot{y}_n y_n' \big|_0^\pi - \int_0^\pi \dot{y}_n y_n'' dx \\
&= -\int_0^\pi \dot{y} y_n'' dx \\
&= \int_0^\pi (\lambda_n - V) y_n \dot{y}_n dx
\end{aligned}$$

after integrating by parts twice and using the boundary conditions and the fact that

$$\dot{y}_n(0,t) = \dot{y}_n(\pi,t) = 0.$$

Hence

$$\dot{\lambda}_n = \int_0^\pi \dot{V} y_n^2 dx$$

□

Proposition 3.8. *For $M \geq 4$, an optimal $V_0 \in A_M$ is a step function with at most one jump on $(0,\pi)$.*

Proof. Let $V_0, V_1 \in A_M$ be single-well potentials. Define

$$V(x,t) = tV_1(x) + (1-t)V_0(x). \tag{3.6}$$

then

$$\dot{V}(x,t) = V_1(x) - V_0(x) \tag{3.7}$$

The derivative of λ_n with respect to t, $\dot{\lambda}_n$, is given by Proposition 3.7 which implies

$$(\lambda_2 - \lambda_1)\dot{} = \int_0^\pi (V_1(x) - V_0(x))(y_2^2(x,t) - y_1^2(x,t))dt. \tag{3.8}$$

It is shown in Proposition 3.4 that there is an optimal potential which we now take it to be $V_0 \in A_M$.

We distinguish 3 cases:

(1) $x_- \leq \frac{\pi}{2} < x_+$

Let

$$V_1(x) = \begin{cases} V_0(x_-) & \text{on } (0, \frac{\pi}{2}) \\ V_0(x_+) & \text{on } (\frac{\pi}{2}, \pi) \end{cases}$$

On $(0, x_-) \cup (x_+, \pi)$

$$V_1(x) - V_0(x) = \begin{cases} V_0(x_-) - V_0(x) \leq 0 & \text{for } x \in (0, x_-) \\ V_0(x_+) - V_0(x) \leq 0 & \text{for } x \in (x_+, \pi) \end{cases}$$

since $V_0(x_\mp)$ is a minimum of V_0 respectively on $(0, x_-]$ and $[x_+, \pi)$.

On (x_-, x_+) for $x \in (x_-, \frac{\pi}{2})$,

$$V_1(x) - V_0(x) = V_0(x_-) - V_0(x) \geq 0$$

since $\frac{\pi}{2}$ is the transition point of a single-well. For $x \in (\frac{\pi}{2}, x_+)$

$$V_1(x) - V_0(x) = V_0(x_+) - V_0(x) \geq 0$$

also by the fact that $V_1(x_+)$ is maximum of V_0 on $(\frac{\pi}{2}, x_+)$.

By the optimality of V_0 the derivative $(\lambda_2 - \lambda_1)\dot{}$ must be nonnegative at $t = 0$:

$$0 \leq (\lambda_2 - \lambda_1)\dot{} = \int_0^\pi (V_1(x) - V_0(x))(y_2^2(x,0) - y_1^2(x,0))dx$$

Since the product $(V_1(x) - V_0(x))(y_2^2(x,0) - y_1^2(x,0))$ is non-positive by above and inequalities (3.2), (3.3), this is possible only when $V_1 - V_0 = 0$ i.e. when

$$V_1 = V_0$$

almost everywhere. i.e. the optimal potential V_0 must be a step-function with the only jump at $\frac{\pi}{2}$. This proves the proposition in this case.

(2) $\frac{\pi}{2} < x_-$.

Let

$$V_1(x) = \begin{cases} V_0(\frac{\pi}{2}) & \text{on } (0, x_-) \\ V_0(x_+) & \text{on } (x_-, \pi) \end{cases}$$

then $V_1 \in A_M$.

$$V_1(x) - V_0(x) = V_0(\frac{\pi}{2}) - V_0(x) \leq 0$$

on $(0, x_-)$,

$$V_1(x) - V_0(x) = V_0(x_+) - V_0(x) \leq 0$$

on (x_+, π) and

$$V_1(x) - V_0(x) = V_0(x_+) - V_0(x) \geq 0$$

on (x_-, x_+) since V_0 is non-decreasing on $(\frac{\pi}{2}, x_+) \supset (x_-, x_+)$. Define V as in (3.6). Then

$$\dot{V}(x,t) = V_1(x) - V_0(x)$$

$V \in A_M$ if $0 \leq t \leq 1$. Then by the optimality of V_0 the derivative $\dot{(\lambda_2 - \lambda_1)}$ must be non-negative at $t = 0$:

$$0 \leq \dot{(\lambda_2 - \lambda_1)} = \int_0^\pi (V_1(x) - V_0(x))(y_2^2(x,0) - y_1^2(x,0))dx$$

But the product $(V_1(x) - V_0(x))(y_2^2(x,0) - y_1^2(x,0))$ is non-positive on $(0, \pi)$. This is only possible when

$$V_1 = V_0$$

i.e. if V_0 is a step function with the jump at x_-. We want to show that the transition point must be at $\frac{\pi}{2}$.

Let

$$V_2(x) = \begin{cases} 0 & \text{on } (0, x_-), \\ M & \text{on } (x_-, \pi). \end{cases} \tag{3.9}$$

Recall that

$$\int_0^\pi y_2^2(x,0) = 1$$

and

$$\int_0^\pi y_1^2(x,0) = 1$$

by normalization condition. Then

$$\begin{aligned} 0 &= \int_0^\pi (y_2^2(x,0) - y_1^2(x,0))dx \\ &= \int_0^{x_-} (y_2^2(x,0) - y_1^2(x,0))dx + \int_{x_-}^\pi (y_2^2(x,0) - y_1^2(x,0))dx \end{aligned}$$

the first term is positive by equation (3.3) which implies the second term is negative. i.e.

$$\int_0^{x_-} (y_2^2(x,0) - y_1^2(x,0))dx > 0,$$

$$\int_{x_-}^\pi (y_2^2(x,0) - y_1^2(x,0))dx < 0.$$

With V defined as

$$V(x,t) = tV_2(x) + (1-t)V_0(x)$$

by the optimality of $V_0 = V_1$

$$\begin{aligned}
0 \leq \dot{(\lambda_2 - \lambda_1)} &= \int_0^{\pi} (V_2(x) - V_0(x))(y_2^2(x,0) - y_1^2(x,0))dx \\
&= \int_0^{x_-} (V_2(x) - V_0(x))(y_2^2(x,0) - y_1^2(x,0))dx \\
&+ \int_{x_-}^{\pi} (V_2(x) - V_0(x))(y_2^2(x,0) - y_1^2(x,0))dx \\
&= -V_0(\frac{\pi}{2}) \int_0^{x_-} (y_2^2(x,0) - y_1^2(x,0))dx \\
&+ (M - V_0(x_+)) \int_{x_-}^{\pi} (y_2^2(x,0) - y_1^2(x,0))dx
\end{aligned}$$

the first term is non-positive since the integral is positive by above, the second term must be non-negative. i.e. $M - V_0(x_+) \leq 0$ since the integral is negative. i.e.

$$M \leq V_0(x_+)$$

But M is the upper bound which implies

$$V_0(x_+) = M$$

and

$$V_0(\frac{\pi}{2}) = 0$$

i.e. the optimal potential V_0 must be of the form V_2. We will show this is impossible.

The second eigenfunction of the potential V_2 (Equation 3.9) can be expressed by

$$y_2(x) = \begin{cases} A\sin(\sqrt{\lambda_2}x) & \text{on } (0, x_-), \\ B\sin(\sqrt{\lambda_2 - M}(\pi - x)) & \text{on } (x_-, \pi). \end{cases}$$

The constants $(A, B) \neq (0,0)$ must be chosen such that y_2 is continuous at x_-. The only inner zero x_0 is on (x_-, x_+), so on $(0, \frac{\pi}{2})$ $y_2 \neq 0$ since $\frac{\pi}{2} < x_- < x_0$. This is only possible when

$$\sqrt{\lambda_2}\frac{\pi}{2} < \pi$$

i.e.

$$\lambda_2 < 4$$

For $M \geq 4, \lambda_2 - M < 0$ which implies

$$0 = y_2(x_0) = B\sin(\sqrt{\lambda_2 - M}(\pi - x_0))$$

is impossible.

(3) $x_+ < \frac{\pi}{2}$ (similar to the above case)

□

Let

$$B_M = \{\varrho \mid \frac{1}{M} \leq \varrho \leq M, \varrho \text{ is a single-barrier density with transition point at} \frac{\pi}{2}\},$$

where $M > 1$.

We now consider the Dirichlet problem (1.4)-(1.5). The eigenvalues $\lambda_1, \lambda_2, \ldots$ arranged such that $0 < \lambda_1 < \lambda_2 < ...$ refer to this problem.

Definition 3.9. An optimal single-barrier density $\varrho_0 \in B_M$ is when $\frac{\lambda_2}{\lambda_1}$ is minimal with respect to $\varrho \in B_M$.

The existence of an optimal single-barrier density can be proved in similar way as in Proposition 3.4 for the single-well potential.

Definition 3.10. We normalize the eigenfunctions u_n such that $u_n > 0$ for small $x > 0$ and

$$\int_0^\pi u_n^2 \varrho dx = 1 \tag{3.10}$$

Proposition 3.11. *Consider the Dirichlet problem (1.4)-(1.5) with $0 < \varrho \in L^1(0,\pi)$. If u_1, u_2 are respectively first and second eigenfunction then $\frac{u_2}{u_1}$ is decreasing on $(0,\pi)$.*

Proof. We only need to show $\left(\frac{u_2}{u_1}\right)' < 0$. On $(0, x_0)$

$$\begin{aligned}
\left(\frac{u_2(x)}{u_1(x)}\right)' &= \frac{1}{u^2(x)}(u_2'(x)u_1(x) - u_2(x)u_1'(x)) \\
&= \frac{1}{u^2(x)}\Big(\int_0^x (u_1(t)u_2''(t) + u_2'(t)u_1'(t))dt \\
&- \int_0^x (u_2(t)u_1''(t) + u_1'(t)u_2'(t))dt\Big) \\
&= \frac{1}{u^2(x)}\int_0^x (u_1(t)u_2''(t) - u_2(t)u_1''(t))dt \\
&= \frac{1}{u^2(x)}\int_0^x (\lambda_1 - \lambda_2)\varrho(t)u_1(t)u_2(t)dt \\
&< 0
\end{aligned}$$

by the fundamental theorem of calculus and the fact that $u_n(0) = 0$ for all n. We used the fact that $u_1 > 0$ and $u_2 > 0$ on $(0, x_0)$ in the last step. Similarly on (x_0, π)

$$\begin{aligned}
\big(\frac{u_2(x)}{u_1(x)}\big)' &= \frac{1}{u^2(x)}(u_2'(x)u_1(x) - u_2(x)u_1'(x)) \\
&= -\frac{1}{u^2(x)}\int_x^\pi (u_1(t)u_2''(t) + u_2'(t)u_1'(t))dt \\
&+ \frac{1}{u^2(x)}\int_x^\pi (u_2(t)u_1''(t) + u_1'(t)u_2'(t))dt \\
&= -\frac{1}{u^2(x)}\int_x^\pi (u_1(t)u_2''(t) - u_2(t)u_1''(t))dt \\
&= -\frac{1}{u^2(x)}\int_x^\pi (\lambda_1 - \lambda_2)\varrho(t)u_1(t)u_2(t)dt \\
&< 0
\end{aligned}$$

since $u_1 > 0$ and $u_2 < 0$ on (x_0, π). □

Then it follows that there exist $x_\pm$, $0 \le x_- < x_0 < x_+ \le \pi$ such that

$$u_1^2 - u_2^2 < 0 \tag{3.11}$$

on $(0, x_-) \cup (x_+, \pi)$ and

$$u_1^2 - u_2^2 > 0 \tag{3.12}$$

on (x_-, x_+).

Proposition 3.12. *For the Dirichlet problem:*

$$-u''(x,t) = \lambda(t)\varrho(x,t)u(x,t)$$

with boundary conditions

$$u_n(0,t) = u_n(\pi,t) = 0$$

for all t and n. The derivative of the n-th eigenvalue $\lambda_n(t)$ with respect to t, $\dot\lambda_n$ is given by

$$\dot\lambda_n = -\lambda_n \int_0^\pi \dot\varrho u_n^2 dx \tag{3.13}$$

where the eigenfunctions u_n are normalized as in (3.10) and $\varrho(x,t) \in L^1(0,\pi)$ is a density function which depends integrably on x and differentiably on t.

Proof.

$$-u_n''(x,t) = \lambda_n(t)\varrho(x,t)u_n(x,t)$$

for each eigenfunction $u_n(x,t)$ and its corresponding eigenvalue $\lambda_n(t)$. Differentiating the above equation with respect to t we get

$$\dot{u}_n'' + \dot{\lambda}_n\varrho u_n + \lambda_n\dot{\varrho}u_n + \lambda_n\varrho\dot{u}_n = 0$$

Multiplying by u_n and integrating with respect to x from 0 to π we get

$$\begin{aligned}\int_0^\pi \big(\dot{u}_n'' u_n + \dot{\lambda}_n\varrho u_n^2 + \lambda_n\dot{\varrho}u_n^2 + \lambda_n\varrho\dot{u}_n u_n\big)dx &= u_n\dot{u}_n'\big|_0^\pi - \int_0^\pi \dot{u}_n' u_n' dx \\ &+ \dot{\lambda}_n\int_0^\pi \varrho u_n^2 dx + \int_0^\pi \lambda_n\dot{\varrho}u_n^2 dx + \int_0^\pi \lambda_n\varrho\dot{u}_n u_n dx \\ &= -\dot{u}_n u_n'\big|_0^\pi + \int_0^\pi u_n''\dot{u}_n dx + \dot{\lambda}_n + \int_0^\pi \lambda_n\big(\dot{\varrho}u_n^2 + \varrho u_n\dot{u}_n\big)dx \\ &= \int_0^\pi \big(u_n'' + \lambda_n\varrho u_n\big)\dot{u}_n dx + \dot{\lambda}_n + \int_0^\pi \lambda_n\dot{\varrho}u_n^2 dx\end{aligned}$$

which gives

$$\dot{\lambda}_n = -\lambda_n\int_0^\pi \dot{\varrho}u_n^2 dx.$$

after integrating twice by parts and using boundary conditions $u_n(0,t) = u_n(\pi,t) = \dot{u}_n(0,t) = \dot{u}_n(\pi,t) = 0$.

□

Proposition 3.13. *For $M > 1$, an optimal density $\varrho_0 \in B_M$ is a step function with at most one jump on $(0,\pi)$.*

Proof. Let ϱ_0, $\varrho_1 \in B_M$ be single-barrier densities. Define

$$\varrho(x,t) = t\varrho_1(x) + (1-t)\varrho_0(x) \tag{3.14}$$

Then

$$\dot{\varrho}(x,t) = \varrho_1(x) - \varrho_0(x) \tag{3.15}$$

The derivative of λ_n with respect to t, $\dot{\lambda}_n$ is given in Proposition 3.12. Then

$$\begin{aligned}\Big(\frac{\lambda_2}{\lambda_1}\Big)^{\cdot} &= \frac{\dot{\lambda}_2}{\lambda_1} - \frac{\lambda_2}{\lambda_1^2}\dot{\lambda}_1 \\ &= -\frac{\lambda_2}{\lambda_1}\int_0^\pi \dot{\varrho}u_2^2(x,t)dx + \frac{\lambda_2}{\lambda_1}\int_0^\pi \dot{\varrho}u_1^2(x,t)dx \\ &= \frac{\lambda_2}{\lambda_1}\int_0^\pi \big(\varrho_1(x) - \varrho_0(x)\big)(u_1^2(x,t) - u_2^2(x,t))dx\end{aligned}$$

There exist an optimal density giving the minimal ratio $\frac{\lambda_2}{\lambda_1}$ which we now take it to be $\varrho_0 \in B_M$. We distinguish 3 cases:

(1) $x_- \leq \frac{\pi}{2} < x_+$

Let

$$\varrho_1(x) = \begin{cases} \varrho_0(x_-) & \text{on } (0, \frac{\pi}{2}) \\ \varrho_0(x_+) & \text{on } (\frac{\pi}{2}, \pi). \end{cases}$$

On $(0, x_-) \cup (x_+, \pi)$,

$$\dot{\varrho} = \varrho_1(x) - \varrho_0(x) = \begin{cases} \varrho_0(x_-) - \varrho_0(x) \geq 0 & \text{for } x \in (0, x_-) \\ \varrho_0(x_+) - \varrho_0(x) \geq 0 & \text{for } x \in (x_+, \pi) \end{cases}$$

since $\varrho_0(x_-)$ and $\varrho_0(x_+)$ are maxima of ϱ_0 respectively on $(0, x_-)$ and (x_+, π). On $(x_-, \frac{\pi}{2})$,

$$\varrho_1(x) - \varrho_0(x) = \varrho_0(x_-) - \varrho_0(x) \leq 0$$

since $\varrho_0(x_-)$ is the minimum of ϱ_0 on $(x_-, \frac{\pi}{2})$. On $(\frac{\pi}{2}, x_+)$,

$$\varrho_1(x) - \varrho_0(x) = \varrho_0(x_+) - \varrho_0(x) \leq 0$$

since $\varrho_0(x_+)$ is the minimum of ϱ_0 on $(\frac{\pi}{2}, x_+)$. So

$$\varrho_1(x) - \varrho_0(x) \leq 0$$

on (x_-, x_+). $\varrho \in B_M$ if $0 \leq t \leq 1$. By the optimality of ϱ_0, $\dot{\left(\frac{\lambda_2}{\lambda_1}\right)}$ must be nonnegative at $t = 0$:

$$0 \leq \dot{\left(\frac{\lambda_2}{\lambda_1}\right)} = \frac{\lambda_2}{\lambda_1} \int_0^\pi \big(\varrho_1(x) - \varrho_0(x)\big)\big(u_1^2(x,0) - u_2^2(x,0)\big)dx$$

But the product $\big(\varrho_1(x) - \varrho_0(x)\big)\big(u_1^2(x,0) - u_2^2(x,0)\big)$ is non-positive by above and inequalities (3.11) and (3.12). This is only possible when $\varrho_1 - \varrho_0 = 0$. i.e. when

$$\varrho_1 = \varrho_0$$

hence the optimal density ϱ_0 must be a step function with the only jump at $\frac{\pi}{2}$. This proves the result in this case.

(2) $\frac{\pi}{2} < x_-$

Let

$$\varrho_1(x) = \begin{cases} \varrho_0(\frac{\pi}{2}) & \text{on } (0, x_-) \\ \varrho_0(x_+) & \text{on } (x_-, \pi). \end{cases}$$

Then $\varrho_1 \in B_M$. On $(0, x_-)$,

$$\varrho_1(x) - \varrho_0(x) = \varrho_0(\frac{\pi}{2}) - \varrho_0(x) \geq 0$$

since $\varrho_0(\frac{\pi}{2})$ is the maximum of ϱ_0. On (x_+, π),

$$\varrho_1(x) - \varrho_0(x) = \varrho_0(x_+) - \varrho_0(x) \geq 0$$

since $\varrho_0(x_+)$ is the maximum of ϱ_0 on (x_+, π). So

$$\varrho_1(x) - \varrho_0(x) \geq 0$$

on $(0, x_-) \cup [x_+, \pi)$. On (x_-, x_+),

$$\varrho_1(x) - \varrho_0(x) = \varrho_0(x_+) - \varrho_0(x) \leq 0$$

since $\varrho_0(x)$ is a non-increasing function on $(\frac{\pi}{2}, x_+) \supset (x_-, x_+)$.

$\varrho \in B_M$ if $0 \leq t \leq 1$. Then by the optimality of ϱ_0, $\dot{\left(\frac{\lambda_2}{\lambda_1}\right)}$ must be nonnegative at $t = 0$:

$$0 \leq \dot{\left(\frac{\lambda_2}{\lambda_1}\right)} = \frac{\lambda_2}{\lambda_1} \int_0^\pi \big(\varrho_1(x) - \varrho_0(x)\big)\big(u_1^2(x,0) - u_2^2(x,0)\big)dx$$

but the product $\big(\varrho_1(x) - \varrho_0(x)\big)\big(u_1^2(x,0) - u_2^2(x,0)\big)$ is non-positive by above and inequalities (3.11) and (3.12). This is only possible only when $\varrho_1 - \varrho_0 = 0$. i.e. when

$$\varrho_1 = \varrho_0$$

hence the optimal density ϱ_0 must be a step function with the only jump at x_-. Now with

$$\varrho(x,t) = t\varrho_2(x) + (1-t)\varrho_1(x),$$

where

$$\varrho_2(x) = \begin{cases} M & \text{on } (0, x_-) \\ \frac{1}{M} & \text{on } (x_-, \pi). \end{cases}$$

Recall that

$$\int_0^\pi u_1^2(x,0)\varrho dx = 1$$

and

$$\int_0^\pi u_2^2(x,0)\varrho dx = 1$$

by normalization condition in (3.10). Then

$$\begin{aligned} 0 &= \int_0^\pi \big(u_1^2(x,0) - u_2^2(x,0)\big)\varrho dx \\ &= \int_0^{x_-} \big(u_1^2(x,0) - u_2^2(x,0)\big)\varrho dx + \int_{x_-}^\pi \big(u_1^2(x,0) - u_2^2(x,0)\big)\varrho dx \end{aligned}$$

But $u_1^2 - u_2^2 < 0$ on $(0, x_-)$ by inequality (3.11) and $\varrho > 0$. Hence

$$\int_0^{x_-} \big(u_1^2(x,0) - u_2^2(x,0)\big)dx < 0$$

$$\int_{x_-}^{\pi} \big(u_1^2(x,0) - u_2^2(x,0)\big)\varrho(x,0)dx = \int_{x_-}^{\pi} \big(u_1^2(x,0) - u_2^2(x,0)\big)\varrho_0(x_+)dx$$
$$> 0$$

which implies

$$\int_{x_-}^{\pi} \big(u_1^2(x,0) - u_2^2(x,0)\big)dx > 0$$

since $\varrho_0(x_+) > 0$. By the optimality of ϱ_0, $\big(\frac{\lambda_2}{\lambda_1}\big)^{\cdot}$ must be nonnegative at $t = 0$:

$$\begin{aligned}
0 \leq \frac{\lambda_1}{\lambda_2}\big(\frac{\lambda_2}{\lambda_1}\big)^{\cdot} &= \int_0^{\pi} \big(\varrho_2(x) - \varrho_0(x)\big)\big(u_1^2(x,0) - u_2^2(x,0)\big)dx \\
&= \int_0^{x_-} \big(\varrho_2(x) - \varrho_1(x)\big)\big(u_1^2(x,0) - u_2^2(x,0)\big)dx \\
&+ \int_{x_-}^{\pi} \big(\varrho_2(x) - \varrho_1(x)\big)\big(u_1^2(x,0) - u_2^2(x,0)\big)dx \\
&= \big(M - \varrho_0(\frac{\pi}{2})\big) \int_0^{x_-} \big(u_1^2(x,0) - u_2^2(x,0)\big)dx \\
&+ \big(\frac{1}{M} - \varrho_0(x_+)\big) \int_{x_-}^{\pi} \big(u_1^2(x,0) - u_2^2(x,0)\big)dx
\end{aligned}$$

The first term is non-positive since the integral is negative and M is the upper bound. Hence the second term must be nonnegative i.e.

$$\varrho_0(x_+) \leq \frac{1}{M}$$

but $\frac{1}{M}$ is the lower bound which implies

$$\varrho_0(x_+) = \frac{1}{M}$$

and

$$\varrho_0(\frac{\pi}{2}) = M$$

So the optimal density ϱ_0 must of the form ϱ_2. We will show that this is impossible. The second eigenfunction of ϱ_2 is given by

$$u_2(x) = \begin{cases} a\sin(\sqrt{\lambda_2 M}x) & \text{on } (0, x_-), \\ b\sin(\sqrt{\lambda_2 \frac{1}{M}}(\pi - x)) & \text{on } (x_-, \pi). \end{cases}$$

Constants $(a, b) \neq (0, 0)$ must be chosen such that u_2 is continuous at x_-. Since the zero of u_2, x_0 is in (x_-, x_+),

$$\sqrt{\lambda_2 M}x_- < \pi$$

and

$$\sqrt{\lambda_2 \frac{1}{M}}(\pi - x_-) > \pi$$

which implies

$$M < \frac{1}{\lambda_2}\frac{\pi^2}{x_-^2}$$

and

$$M < \lambda_2 \frac{(\pi - x_-)^2}{\pi^2}.$$

Multiplying these inequalities we get

$$M^2 < \frac{(\pi - x_-)^2}{x_-^2} < \frac{\pi^2}{4x_-^2} < 1$$

since $x_- > \frac{\pi}{2}$ which contradicts the fact that $M > 1$. Hence the case $\frac{\pi}{2} < x_-$ is impossible.

(3) $x_+ < \frac{\pi}{2}$ (similar)

□

4. MAIN RESULTS

Lemma 4.1. *Define $f(t) = \sqrt{t}\cot(\sqrt{t}\frac{\pi}{2})$ for real t and let $m > 0$. Then the first two solutions of the equation $f(t) = -f(t-m)$ satisfy*

$$t_2 - t_1 > 3.$$

Proof. For $t > 0$,

$$f(t) = \sqrt{t}\cot(\sqrt{t}\frac{\pi}{2}) = \sqrt{t}\frac{\cos(\sqrt{t}\frac{\pi}{2})}{\sin(\sqrt{t}\frac{\pi}{2})}$$

is real since $\sin\theta$ and $\cos\theta$ are real for real θ. Hence f is real.

We now show that $f(t)$ is strictly decreasing on the intervals $(-\infty, 4), (4, 16)$, and generally on $(4n^2, 4(n+1)^2), n \geq 1$. We have

$$\begin{aligned} f'(t) &= \frac{1}{2\sqrt{t}}\cot(\sqrt{t}\frac{\pi}{2}) - \frac{\pi}{4\sqrt{t}}\sqrt{t}\csc^2(\sqrt{t}\frac{\pi}{2}) \\ &= \frac{2\cos(\sqrt{t}\frac{\pi}{2})\sin(\sqrt{t}\frac{\pi}{2})}{4\sqrt{t}\sin^2(\sqrt{t}\frac{\pi}{2})} - \frac{\sqrt{t}\pi}{4\sqrt{t}\sin^2(\sqrt{t}\frac{\pi}{2})} \\ &= \frac{\sin(\sqrt{t}\pi) - \sqrt{t}\pi}{4\sqrt{t}\sin^2(\sqrt{t}\frac{\pi}{2})} \end{aligned}$$

for $t > 0, t \neq 4n^2$,

Let

$$g(z) = \sin z - z$$

$g(0) = 0$ and $g'(z) = \cos z - 1 \leq 0$, then $g(z) < 0$ is a non-increasing function for $z > 0$ which implies $f'(t) < 0$. i.e. $f(t)$ is decreasing for $t \in (4n^2, 4(n+1)^2), n \in \mathbb{N} \cup \{0\}$.

For $t < 0, \sqrt{t} = \sqrt{-(-t)} = i\sqrt{-t}$.

$$\begin{aligned} f(t) &= \sqrt{-(-t)}\cot(\sqrt{-(-t)}\frac{\pi}{2}) \\ &= i\sqrt{-t}\cot(i\sqrt{-t}\frac{\pi}{2}) \\ &= \sqrt{-t}\coth(\sqrt{-t}\frac{\pi}{2}) \in \mathbb{R} \end{aligned}$$

since $\sqrt{-t}$ and $\coth(\sqrt{-t}\frac{\pi}{2})$ are real.

Then

$$\begin{aligned} f'(t) &= \frac{-\cosh(\sqrt{-t}\frac{\pi}{2})}{2\sqrt{-t}\sinh(\sqrt{-t}\frac{\pi}{2})} + \big(\frac{-\pi}{4\sqrt{-t}}\big)\frac{\sqrt{-t}(\sinh^2(\sqrt{-t}\frac{\pi}{2}) - \cosh^2(\sqrt{-t}\frac{\pi}{2}))}{\sinh^2(\sqrt{-t}\frac{\pi}{2})} \\ &= -\frac{2\cosh(\sqrt{-t}\frac{\pi}{2})\sinh(\sqrt{-t}\frac{\pi}{2})}{4\sqrt{-t}\sinh^2(\sqrt{-t}\frac{\pi}{2})} + \frac{-\pi\sqrt{-t}(-1)}{4\sqrt{-t}\sinh^2(\sqrt{-t}\frac{\pi}{2})} \\ &= \frac{\sqrt{-t}\pi - \sinh(\sqrt{-t}\pi)}{4\sqrt{-t}\sinh^2(\sqrt{-t}\frac{\pi}{2})} \end{aligned}$$

Let

$$d(z) = z - \sinh z.$$

Then $d(0) = 0$, $d'(z) = 1 - \cosh z < 0$ for $z \neq 0$, and hence $d(z)$ is a decreasing function which is negative for $z \neq 0$. It follows that $f'(t) < 0$ for $t < 0$, and hence $f(t)$ is a decreasing function of t for $t < 0$.

Consider $h(t) = f(t) + f(t-m)$ for $m \geq 0$. Let t_1 and t_2 be the first two zeros of $h(t)$.

Claim 1 : $t_1(m) \in (1, 4)$ and $t_2(m) \in (4, \min\{m+4, 16\})$ for all $m > 0$.

$f(t)$ and $f(t-m)$ are both decreasing so $h(t) = f(t) + f(t-m)$ is also decreasing for all $m \geq 0$. $f(t)$ has poles at $t = 4n^2$ and $f(t-m)$ has poles at $t = m + 4n^2$ for integers $n \geq 1$. For $n = 1$, $f(t)$ has a pole at $t = 4$. As

$$t \to 4^-, f(t) \to -\infty$$

and

$$t \to 4^+, f(t) \to \infty$$

Then $h(t) = f(t) + f(t-m) \to \pm\infty$ as $t \to 4^{\pm}$. So $h(t)$ has a pole at $t = 4$.

Let t_0 be the first zero of $f(t)$. Then $h(t_0) = f(t_0) + f(t_0 - m) > 0$ and

$h(t) < 0$ as $t \to 4^-$. So there is $t_1(m) \in (t_0, 4)$ such that $h(t_1(m)) = 0$. Hence $t_1(m) \in (1, 4)$ for all $m \geq 0$ since $t_0 = 1$.

We want to show that $t_2(m) \in (4, \min\{m+4, 16\})$ for all $m > 0$. $f(t)$ is decreasing and has exactly one zero between any two poles. So $h(t)$ is also decreasing and has exactly one zero between any two poles. Hence the second zero $t_2(m) \in (4, \min\{m+4, 16\})$ for all $m > 0$.

Claim 2: $f(t_1(m)) < 0$ and $f(t_2(m)) > 0$ for $0 < m < 8$.

$$0 = h(t_1(m)) = f(t_1(m)) + f(t_1(m) - m))$$

But $f(t_1(m)) < f(t_1(m) - m)$ since $f(t)$ is decreasing on $(-\infty, 4)$. Hence we must have $f(t_1(m)) < 0$ and $f(t_1(m) - m) > 0$.

$t_2(m) \in (4, 7)$ if $m \leq 3$ from the claim above. So $f(t_2(m)) > 0$ for $m \leq 3$ since $f(t)$ is decreasing on $(4, 16)$ and $f(t) = 0$ at $t = 9$.

For $3 < m < 8$ at $t = m + 1$, $f(t - m) = 0$. So

$$h(m+1) = f(m+1) > 0$$

since $m + 1 < 9$. $f(9) = 0$ so

$$h(9) = f(9 - m) < 0$$

since $1 < (9 - m) < 4$ if $m > 5$. Hence $t_2(m) \in (m + 1, 9)$ which implies $f(t_2(m)) > 0$ if $m > 5$. $h(m+1) = f(m+1) > 0$ if $3 < m \leq 5$. But there is a pole at $t = m + 4 \leq 9$. So $t_2(m) \in (m + 1, m + 4)$ if $3 < m \leq 5$. Hence $f(t_2(m)) > 0$ for $0 < m < 8$.

Claim 3: Fix m, so that $t_i(m) = t_i$ for $i = 1, 2$ then $f'(t_2 - m) < f'(t_2)$ for $0 < m < 8$.

$$\begin{aligned} f'(t) &= \frac{1}{2\sqrt{t}} \cot(\sqrt{t}\frac{\pi}{2}) - \sqrt{t}(1 + \cot^2(\sqrt{t}\frac{\pi}{2}))\frac{\pi}{4\sqrt{t}} \\ &= \frac{1}{2t}\sqrt{t}\cot(\sqrt{t}\frac{\pi}{2}) - \frac{\pi}{4}\big(1 + \frac{1}{t}[\sqrt{t}\cot(\sqrt{t}\frac{\pi}{2})]^2\big) \end{aligned}$$

then

$$f'(t) = \frac{1}{2t}f(t) - \frac{\pi}{4}\big(1 + \frac{1}{t}f^2(t)\big) \tag{4.1}$$

where $t \neq 0$. So we have expressed $f'(t)$ as a function of $f(t)$.

$$\begin{aligned} f'(t_2-m)-f'(t_2) &= \frac{1}{2(t_2-m)}f(t_2-m)-\frac{\pi}{4(t_2-m)}f^2(t_2-m) \\ &\quad -\frac{1}{2t_2}f(t_2)+\frac{\pi}{4t_2}f^2(t_2) \\ &= -\frac{f(t_2)}{2(t_2-m)}-\frac{\pi}{4(t_2-m)}f^2(t_2)-\frac{1}{2t_2}f(t_2) \\ &\quad +\frac{\pi}{4t_2}f^2(t_2) \\ &= \frac{1}{(t_2-m)t_2}f(t_2)\big(-\frac{t_2}{2}-\frac{\pi t_2}{4}f(t_2)-\frac{t_2-m}{2} \\ &\quad +\frac{\pi}{4}(t_2-m)f(t_2)\big) \\ &= \frac{1}{(t_2-m)t_2}f(t_2)\big(\frac{m-2t_2}{2}-\frac{\pi}{4}mf(t_2)\big) \end{aligned}$$

since $f(t_2) = -f(t_2-m)$. $t_2 > m+1$ since $t_2(m) \in (m+1, 9)$. So $t_2 - m > 1 > 0, f(t_2) > 0, m - 2t_2 < 0$ by Claim 2 which implies

$$f'(t_2-m) < f'(t_2) \tag{4.2}$$

Claim 4: $f'(t_1-m) > f'(t_1)$ for $0 < m < 8$.

We show that

$$f'(t_1-m) > -\frac{\pi}{4}$$

and

$$f'(t_1) < -\frac{\pi}{4}.$$

$$\begin{aligned} f'(t_1)+\frac{\pi}{4} &= \frac{f(t_1)}{2t_1}-\frac{\pi}{4t_1}f^2(t_1) \\ &< 0 \end{aligned}$$

since $f(t_1) < 0$ so $f'(t_1) < -\frac{\pi}{4}$.

$$f'(t)+\frac{\pi}{4} = \frac{f(t)}{2t}\big(1-\frac{\pi f(t)}{2}\big)$$

Since $f'(0)$ must be finite, $1-\frac{\pi f(t)}{2} = 0$ when $t = 0$. $1-\frac{\pi f(t)}{2}$ is increasing with t since $f(t)$ is decreasing. So $1-\frac{\pi f(t)}{2}$ and t have the same sign which implies $1-\frac{\pi f(t_1-m)}{2}$ and $t_1 - m$ have the same sign. Then

$$f'(t_1-m)+\frac{\pi}{4} = \frac{f(t_1-m)}{2(t_1-m)}\big(1-\frac{\pi f(t_1-m)}{2}\big) > 0$$

since $f(t_1(m)-m) > 0$. Hence $f'(t_1-m) > -\frac{\pi}{4}$.

Claim 5: $t_1(m)$ and $t_2(m)$ are increasing functions in m. For $i = 1, 2$ differentiating

$$f(t_i(m)) = -f(t_i(m) - m)$$

in m we get

$$f'(t_i(m))t_i'(m) = f'(t_i(m) - m)(1 - t_i'(m)) \tag{4.3}$$

From (4.3) $f'(t_i(m) - m)$ and $f'(t_i(m))$ are always negative so $(1 - t_i'(m))$ and $t_i'(m)$ must have the same sign. If $t_i'(m) < 0$, $1 - t_i'(m) > 0$ which contradicts the fact that $(1 - t_i'(m))$ and $t_i'(m)$ must have the same sign. Hence $t_i'(m) > 0$ and $1 - t_i'(m) > 0$.

From (4.3) for $i = 1$ we get

$$\begin{aligned} f'(t_1(m))t_1'(m) &= f'(t_1(m) - m)(1 - t_1'(m)) \\ &> f'(t_1(m))(1 - t_1'(m)) \end{aligned}$$

from the Claim 4 above. Hence

$$t_1'(m) < \frac{1}{2}$$

for $0 < m < 8$. From (4.3) for $i = 2$ we get

$$\begin{aligned} f'(t_2(m))t_2'(m) &= f'(t_2(m) - m)(1 - t_2'(m)) \\ &< f'(t_2(m))(1 - t_2'(m)) \end{aligned}$$

from Claim 3. Hence

$$t_2'(m) > \frac{1}{2}$$

for $0 < m < 8$. So

$$t_1'(m) < \frac{1}{2} < t_2'(m). \tag{4.4}$$

then

$$(t_2 - t_1)'(m) > 0$$

i.e the function $t_2 - t_1$ increases with m. For $0 < m < 3$,

$$h(1) = f(1) + f(1 - m) > 0$$

and

$$h(m + 1) = f(m + 1) < 0$$

since $f(t)$ is decreasing with t. Hence $t_1(m) \in (1, m + 1)$ since $h(t)$ is also decreasing. So

$$\lim_{m \to 0} t_1(m) = 1$$

and

$$\lim_{m \to 0} t_2(m) = 4$$

by Claim 1. So

$$\lim_{m\to 0}(t_2(m)-t_1(m))=3.$$

Hence $t_2-t_1>3$ for $0<m<8$. Note that t_2 jumps at $m=0$. For $m>8$, $t_2(m)\in(9,\min\{m+4,16\})$ and $t_1(m)\in(1,4)$. So $t_2-t_1>5>3$. Hence $t_2-t_1>3$ for all $m>0$.

□

Theorem 4.2. *Let V be a (not necessarily symmetric) single-well potential on $[0,\pi]$ with a transition point at $a=\pi/2$. Then the first two eigenvalues of the Dirichlet problem (1.1)-(1.2) satisfy*

$$\lambda_2-\lambda_1\geq 3$$

with equality if and only if V is constant. If $a\neq\pi/2$, there are single-well potentials V with $\lambda_2-\lambda_1<3$.

Proof. It was shown in Proposition 3.8 that for $M\geq 4$ the optimal V_0 must be of the form

$$V_0=\begin{cases}0 & \text{on } (0,\frac{\pi}{2}),\\ m & \text{on } (\frac{\pi}{2},\pi).\end{cases}$$

or

$$V_0=\begin{cases}m & \text{on } (0,\frac{\pi}{2}),\\ 0 & \text{on } (\frac{\pi}{2},\pi).\end{cases}$$

for some $m\geq 0$ since the potential can be shifted and the difference in eigenvalues is unaffected. Since the differential equations of the two potentials yield the same eigenvalue we deal only with the first form of V_0. The eigenfunction corresponding to the eigenvalue λ can be expressed as

$$y(x)=\begin{cases}C\sin(\sqrt{\lambda}x) & \text{on } (0,\frac{\pi}{2}),\\ D\sin(\sqrt{\lambda-m}(\pi-x)) & \text{on } (\frac{\pi}{2},\pi).\end{cases}$$

The nonzero constants C,D have to be chosen such that $y(x)$ is C^1-smooth at $\frac{\pi}{2}$. This is possible if and only if the ratios $\frac{y'}{y}$ are the same from both sides i.e.

$$\sqrt{\lambda}\cot(\sqrt{\lambda}\frac{\pi}{2})=-\sqrt{\lambda-m}\cot(\sqrt{\lambda-m}\frac{\pi}{2}) \tag{4.5}$$

The eigenvalues are real solution λ of (4.5). We allow cases when both sides of the equation are infinite. But by Lemma 4.1 $\lambda_2-\lambda_1>3$ if $m>0$. Hence the optimal V_0 (where $\lambda_2-\lambda_1=3$) is when $m=0$ i.e. the optimal potential V_0 must be constant. This proves the Theorem in the case when the transition point a is at $\frac{\pi}{2}$.

Suppose $\frac{\pi}{2} < a < \pi$. Let

$$V(x,t) = \begin{cases} t & \text{if } x \in (0,a), \\ 0 & \text{if } x \in (a,\pi). \end{cases}$$

Then for $t = 0$, the eigenfunctions are given by

$$y_n(x,0) = A_n \sin nx.$$

Using the normalization condition we get

$$\begin{aligned} 1 = A_n^2 \int_0^\pi \sin^2(nx)dx &= \frac{A_n^2}{2}\int_0^\pi (1-\cos 2nx)dx \\ &= \frac{A_n^2}{2}x\Big|_0^\pi - \frac{\sin 2nx}{2n}\Big|_0^\pi \\ &= \frac{A_n^2}{2}\pi \end{aligned}$$

i.e.

$$A_n = \sqrt{\frac{2}{\pi}}.$$

Then $y_1(x,0) = \sqrt{\frac{2}{\pi}}\sin x$ and $y_2(x,0) = \sqrt{\frac{2}{\pi}}\sin 2x$.

$$\begin{aligned} \int_0^a (y_2^2(x,0) - y_1^2(x,0))dx &= \frac{2}{\pi}\int_0^a (\sin^2 2x - \sin^2 x)dx \\ &= \frac{1}{\pi}\int_0^a \big((1-\cos 4x) - (1-\cos 2x)\big)dx \\ &= \frac{1}{\pi}\int_0^a (\cos 2x - \cos 4x)dx \\ &= \frac{1}{\pi}\big(\frac{\sin 2a}{2} - \frac{\sin 4a}{4}\big) \\ &= \frac{\sin 2a}{2\pi}(1-\cos 2a) < 0 \end{aligned}$$

since $\sin 2a < 0$ and $1 - \cos 2a > 0$ for $\frac{\pi}{2} < a < \pi$. Then by (3.8)

$$\dot{(\lambda_2 - \lambda_1)} = \int_0^a (y_2^2(x,0) - y_1^2(x,0)dx < 0$$

so for small $t > 0$ the single-well potentials $V(x,t)$ gives an eigenvalues gap $\lambda_2 - \lambda_1 < 3$.

For $a = \pi$, we can choose

$$V(x,t) = \begin{cases} t & \text{if } x \in (0, \frac{2\pi}{3}), \\ 0 & \text{if } x \in (\frac{2\pi}{3}, \pi). \end{cases}$$

to get $\lambda_2 - \lambda_1 < 3$. By symmetry we can find a single-well potential that gives $\lambda_2 - \lambda_1 < 3$ for $a < \frac{\pi}{2}$.

□

Lemma 4.3. *Define $f(\lambda) = \sqrt{\lambda}\cot(\sqrt{\lambda}\frac{\pi}{2})$ for real λ and let $m > 1$. Then the first two positive solutions of the equation $f(\lambda) = -f(\lambda m)$ satisfy*

$$\frac{\lambda_2}{\lambda_1} > 4$$

As solutions we allow values of λ for which $f(\lambda)$ and $f(\lambda m)$ are infinite.

Proof. Let $\sqrt{\lambda}\frac{\pi}{2} = t$ and $\sqrt{m} = d$. Then $f(\lambda) = \frac{2t}{\pi}\cot t$ and $f(\lambda m) = \frac{2td}{\pi}\cot td$. Hence we need to show that the first two positive zeros t_1 and t_2 of the function

$$h(t) := \frac{2t}{\pi}\cot t + \frac{2t}{\pi}d\cot(td) \tag{4.6}$$

satisfy the inequality

$$t_2 > 2t_1.$$

$\cot t$ is a decreasing functions $(0,\pi),(\pi,2\pi)$,etc and $\cot(td)$ is a decreasing functions $(0,\frac{\pi}{d}),(\frac{\pi}{d},\frac{2\pi}{d})$,etc for $d > 1$. There is exactly one zero of h between any two poles (i.e. $0, \frac{\pi}{d}, \frac{2\pi}{d}, \pi$, etc.). Hence $h(t)$ is also a decreasing function and has exactly one zero on $(0,\frac{\pi}{d})$. At $t = \frac{\pi}{2d}$,

$$h(t) = \frac{2t}{\pi}(\cot t + d\cot td) = \frac{2t}{\pi}\cot t > 0$$

So there is $t_1 \in (\frac{\pi}{2d}, \min\{\frac{\pi}{d},\frac{\pi}{2}\})$ such that $h(t_1) = 0$ since $h(t)$ is a decreasing function. The next pole of h is at $\min\{\frac{2\pi}{d},\pi\}$. Hence there is $t_2 \in (\frac{\pi}{d}, \min\{\frac{2\pi}{d},\pi\})$ such that $h(t_2) = 0$. Since we are only interested in the first two solutions, we will not consider other poles.

t_1 and t_2 satisfy

$$\frac{2t_i}{\pi}\cot t_i = -\frac{2t_i}{\pi}d\cot t_i d$$

which implies

$$\tan t_i = -\frac{1}{d}\tan(t_i d). \tag{4.7}$$

Claim 1: If $d > 1$ is increasing then t_1 decreases and $t_1 d$ increases with d.

$$\begin{aligned}\frac{\partial}{\partial s}\big(-\frac{1}{s}\tan(ts)\big) &= -\big(\frac{1}{s}\sec^2(ts)t + (-\frac{1}{s^2})\tan(ts)\big)\\ &= -\big(\frac{ts}{s^2\cos^2(ts)} - \frac{\sin(ts)\cos(ts)}{s^2\cos^2(ts)}\big)\\ &= -\frac{2ts - \sin(2ts)}{2s^2\cos^2(ts)} < 0.\end{aligned}$$

So $-\frac{1}{s}\tan(t_1 s)$ decreases as s increases. Choose $\epsilon > 0$ such that for all $d' \in (d, d+\epsilon)$, $t_1 d' \in (\frac{\pi}{2}, \pi)$. Hence

$$\tan t_1 = -\frac{1}{d}\tan(t_1 d) > -\frac{1}{d'}\tan(t_1 d'),$$

$$\tan t_1' = -\frac{1}{d'}\tan(t_1' d').$$

If $t_1 \le t_1'$, then

$$\tan t_1 = -\frac{1}{d}\tan(t_1 d) > -\frac{1}{d'}\tan(t_1 d') \ge -\frac{1}{d'}\tan(t_1' d') = \tan t_1'$$

So $\tan t_1 > \tan t_1'$ which leads to contradiction since $\tan t$ is increasing on $(0, \frac{\pi}{2})$. Hence $t_1 > t_1'$ and t_1 decreases as $d > 1$ is increasing.

$\tau = t_1 d$ is the first solution of

$$\tan\tau = -d\tan\frac{\tau}{d} \tag{4.8}$$

and t_1 is the first solution of

$$\tan t = -\frac{1}{d}\tan(td).$$

We showed that t_1 decreases as $d > 1$ is increasing. The above equations are similar except that in (4.8) d is replaced by $\frac{1}{d}$. $\frac{1}{d}$ decreases as d increases. So the first solution τ_1 of (4.8) increases as d increases. Hence $t_1 d < t_1' d'$ which shows $t_1 d$ increases as $d > 1$ is increasing. Note that it can be shown that for $0 < d < 1$, $t_1 \in (\frac{\pi}{2}, \pi)$ and $t_1 d \in (0, \frac{\pi}{2})$.

Claim 2: For $1 < d \le 3$, $t_1 < \frac{3\pi}{4d}$.

Since $t_1 d$ increases with d, it is sufficient to only check it for $d = 3$. $t_1 \in (\frac{\pi}{2d}, \min\{\frac{\pi}{d}, \frac{\pi}{2}\}) = (\frac{\pi}{6}, \frac{\pi}{3})$.

$$\begin{aligned}\tan t_1 &= -\frac{1}{3}\tan(3t_1)\\ &= -\frac{1}{3}\frac{\tan(2t_1)+\tan t_1}{1-\tan(2t_1)\tan t_1}\\ &= -\frac{1}{3}\frac{\frac{2\tan t_1}{1-\tan^2 t_1}+\tan t_1}{1-\frac{2\tan^2 t_1}{1-\tan^2 t_1}}\\ &= \frac{\frac{\tan^3 t_1}{3}-\tan t_1}{1-3\tan^2 t_1}.\end{aligned}$$

So

$$2\tan t_1(1-\frac{5}{3}\tan^2 t_1)=0,$$

which implies

$$\tan t_1=\sqrt{\frac{3}{5}}<1$$

so $t_1<\frac{\pi}{4}=\frac{3\pi}{4d}$. Hence $t_1<\frac{3\pi}{4d}$ for $1<d\leq 3$.

Claim 3: For $1<d\leq 3, t_2>2t_1$.

For $d=3$, $t_2=\frac{\pi}{2}$ since $h(\frac{\pi}{2})=0$. So it is true for $d=3$ since $t_1<\frac{\pi}{4}$. For $1<d<3$, $t_1d<\frac{3\pi}{4}$ by Claim 2. So $2t_1d\in(\pi,\frac{3\pi}{2})$ for all $1<d<3$. Consider cases: (i) $1<d\leq\frac{3}{2}$

$t_2d\in(\pi,\min\{2\pi,\pi d\})=(\pi,\pi d)$. Hence $t_2d\in(\pi,\frac{3\pi}{2})$.

(ii) $\frac{3}{2}<d\leq 2$

$t_2\in(\frac{\pi}{d},\min\{\frac{2\pi}{d},\pi\})=(\frac{\pi}{d},\pi)$.

$$h\big(\frac{3\pi}{2d}\big)=\cot(\frac{3\pi}{2d})<0.$$

So $t_2\in(\frac{\pi}{d},\frac{3\pi}{2d})$. Hence $t_2d\in(\pi,\frac{3\pi}{2})$.

(iii) $2<d<3$

$t_2\in(\frac{\pi}{d},\frac{2\pi}{d})$. $\cot(\frac{\pi}{2})=0$ so

$$h(\frac{\pi}{2})=d\cot(\frac{\pi}{2}d)>0$$

and

$$h\big(\frac{3\pi}{2d}\big)=\cot\big(\frac{3\pi}{2d}\big)<0$$

since $\cot(\frac{3\pi}{2d}d)=0$. Then $t_2\in(\frac{\pi}{2},\frac{3\pi}{2d})$ which yield $t_2d\in(\frac{\pi d}{2},\frac{3\pi}{2})$. Hence $t_2d\in(\pi,\frac{3\pi}{2})$.

For all the above cases if $t_1 \leq \frac{\pi}{4}$, then $t_2 > 2t_1$ since $t_2 > \frac{\pi}{2}$. In each case, t_2 is a unique zero of h in the intervals $(\frac{\pi}{d}, \frac{3\pi}{2d})$ and $(\frac{\pi}{2}, \frac{3\pi}{2d})$.

When $t_1 > \frac{\pi}{4}$,

$$2t_1 d, t_2 d \in \big(\pi, \frac{3\pi}{2}\big).$$

On $(\frac{\pi}{2}, \frac{3\pi}{2d})$, $\tan t$ is increasing and $-\frac{1}{d}\tan(td)$ is decreasing with t where $td \in (\pi, \frac{3\pi}{2})$. For $d > 1$

$$\tan^2 t_1 < d^2 \tan^2 t_1 \Rightarrow 1 - \tan^2 t_1 > 1 - d^2 \tan^2 t_1$$

When $\frac{\pi}{4} < t_1 < \frac{\pi}{2}$, $1 - \tan^2 t_1 < 0$ and $1 - d^2 \tan^2 t_1 < 0$. Then

$$\frac{1}{1 - \tan^2 t_1} < \frac{1}{1 - d^2 \tan^2 t_1}. \tag{4.9}$$

Then by (4.7),

$$\begin{aligned} -\frac{1}{d}\tan(2t_1 d) &= -\frac{1}{d}\frac{2\tan(t_1 d)}{1 - \tan^2(t_1 d)} \\ &= -\frac{1}{d}\frac{-2d\tan t_1}{1 - d^2\tan^2 t_1} \\ &= \frac{2\tan t_1}{1 - d^2 \tan^2 t_1}. \end{aligned}$$

From inequality (4.9)

$$\tan 2t_1 = \frac{2\tan t_1}{1 - \tan^2 t_1} < \frac{2\tan t_1}{1 - d^2\tan^2 t_1},$$

so

$$-\frac{1}{d}\tan(2t_1 d) > \tan 2t_1$$

Assume $t_2 \leq 2t_1$. Then

$$\begin{aligned} \tan t_2 &= -\frac{1}{d}\tan(t_2 d) \\ &\geq -\frac{1}{d}\tan(2t_1 d) \\ &> \tan 2t_1 \end{aligned}$$

which implies $\tan t_2 > \tan 2t_1$ contradicting the fact that $\tan t$ is increasing on $(\frac{\pi}{2}, \pi)$. Hence $t_2 > 2t_1$.

Claim 4: For $d > 3$ we also have $t_2 > 2t_1$.

$t_1 < \frac{\pi}{4}$ for $d = 3$, hence $t_1 < \frac{\pi}{4}$ for $d > 3$ since t_1 decreases as d increases. Hence $t_2 > 2t_1$ if $t_2 \geq \frac{\pi}{2}$. So we will consider the cases for which $t_2 < \frac{\pi}{2}$.

$t_1 \in \left(\frac{\pi}{2d}, \min\{\frac{\pi}{d}, \frac{\pi}{2}\}\right) = \left(\frac{\pi}{2d}, \frac{\pi}{d}\right)$ and $t_2 \in \left(\frac{\pi}{d}, \min\{\frac{2\pi}{d}, \frac{\pi}{2}\}\right)$. At $t = \frac{3\pi}{2d}$,

$$h(t) = \frac{2t}{\pi}(\cot t + \cot(td)) = \frac{2t}{\pi}\cot t > 0.$$

Hence $t_2 \in (\frac{3\pi}{2d}, \min\{\frac{2\pi}{d}, \frac{\pi}{2}\})$ since h is a monotone decreasing function $(\frac{\pi}{d}, \frac{2\pi}{d})$. If $2t_1 \leq \frac{3\pi}{2d}$ then $t_2 > 2t_1$ since $t_2 > \frac{3\pi}{2d}$. We consider the case when $2t_1 > \frac{3\pi}{2d}$.

Consequently $2t_1 \in \left(\frac{3\pi}{2d}, \frac{\pi}{2}\right)$ and $t_2 \in \left(\frac{3\pi}{2d}, \frac{\pi}{2}\right)$ for $3 < d \leq 4$. So $2t_1 d, t_2 d \in (\frac{3\pi}{2}, 2\pi)$ if $3 < d \leq 4$. $\tan t$ is strictly increasing on $(\frac{3\pi}{2d}, \frac{\pi}{2})$ and $-\frac{1}{d}\tan(td)$ is strictly decreasing with t where $td \in (\frac{3\pi}{2}, 2\pi)$. Similarly we get $2t_1 d, t_2 d \in (\frac{3\pi}{2}, 2\pi)$ for $d > 4$.

$$\tan^2 t_1 < d^2 \tan^2 t_1$$

which implies

$$1 - \tan^2 t_1 > 1 - d^2 \tan^2 t_1.$$

$t_1 < \frac{\pi}{4}$, so $1 - \tan^2 t_1 > 0$. By (4.7)

$$d^2 \tan^2 t_1 = \tan^2(t_1 d) < 1$$

since $\frac{3\pi}{4} < t_1 d < \pi$. Hence $1 - d^2 \tan^2 t_1 > 0$. Then

$$\frac{1}{1 - \tan^2 t_1} < \frac{1}{1 - d^2 \tan^2 t_1}.$$

By (4.7),

$$\begin{aligned} -\frac{1}{d}\tan(2t_1 d) &= -\frac{1}{d}\frac{2\tan(t_1 d)}{1 - \tan^2(t_1 d)} \\ &= -\frac{1}{d}\frac{-2d\tan t_1}{1 - d^2\tan^2 t_1} \\ &= \frac{2\tan t_1}{1 - d^2 \tan^2 t_1}. \end{aligned}$$

By the above inequality

$$\tan 2t_1 = \frac{2\tan t_1}{1 - \tan^2 t_1} < \frac{2\tan t_1}{1 - d^2\tan^2 t_1},$$

so

$$-\frac{1}{d}\tan(2t_1 d) > \tan 2t_1$$

Assume $t_2 \leq 2t_1$, then

$$\begin{aligned}\tan t_2 &= -\frac{1}{d}\tan(t_2 d)\\ &\geq -\frac{1}{d}\tan(2t_1 d)\\ &> \tan 2t_1\end{aligned}$$

so $\tan t_2 > \tan 2t_1$ contradicting the fact that $\tan t$ is increasing on $(\frac{3\pi}{2d}, \frac{\pi}{2})$. Hence $t_2 > 2t_1$. This proves the entire Lemma. □

Remark 4.4.

$$\lim_{d\to 1} t_2(d) = \pi$$

so

$$\lim_{d\to 1} t_2(d)d = \pi.$$

but there is a pole at π. Hence

$$0 = \lim_{d\to 1} h(t_2(d)) = \lim_{d\to 1}\big(\frac{2t_2(d)}{\pi}\cot t_2(d) + \frac{2t_2(d)}{\pi}d\cot(t_2(d)d)\big)$$

so there are solutions for which $f(\lambda)$ and $f(\lambda m)$ are infinite.

$$\lim_{d\to 1} t_1(d) = \frac{\pi}{2}$$

then

$$\lim_{d\to 1} t_1(d)d = \frac{\pi}{2}.$$

$$\lim_{d\to 1} h(t_1(d)) = \lim_{d\to 1}\big(\frac{2t_1(d)}{\pi}\cot t_1(d) + \frac{2t_1(d)}{\pi}d\cot(t_1(d)d)\big)$$

but there is a zero at $\frac{\pi}{2}$ which shows that there are solutions for which $f(\lambda)$ and $f(\lambda m)$ are zero. This shows that $\frac{t_2}{t_1} = 2$ if $d = 1$. i.e. $\frac{\lambda_2}{\lambda_1} = 4$ for $m = 1$.

Theorem 4.5. *Let ϱ be a (not necessarily symmetric) single-barrier density function on $[0,\pi]$ with transition point $a = \frac{\pi}{2}$. Then the first two eigenvalues of the Dirichlet problem (1.4)-(1.5) satisfy*

$$\frac{\lambda_2}{\lambda_1} \geq 4$$

with equality if and only if ϱ is constant. If the transition point $a \neq \frac{\pi}{2}$, there are single-barrier densities ϱ for which $\frac{\lambda_2}{\lambda_1} < 4$.

Proof. It was shown in Proposition 3.13 that the optimal density ϱ_0 must of the form

$$\varrho_0 = \begin{cases} 1 & \text{on } (0, \frac{\pi}{2}),\\ m & \text{on } (\frac{\pi}{2}, \pi).\end{cases}$$

or

$$\varrho_0 = \begin{cases} m & \text{on } (0, \frac{\pi}{2}), \\ 1 & \text{on } (\frac{\pi}{2}, \pi). \end{cases}$$

for some $m \geq 1$ since when multiplying density by an appropriate constant the ratio of eigenvalues in unaffected. The differential equations of the above densities yield the same eigenvalues so we deal with the first form. The eigenfunctions are given by

$$u(x) = \begin{cases} c \sin(\sqrt{\lambda} x) & \text{on } (0, \frac{\pi}{2}), \\ d \sin(\sqrt{\lambda m}(\pi - x)) & \text{on } (\frac{\pi}{2}, \pi). \end{cases}$$

Non-zero constants c and d must chosen such that u is C^1-smooth at $\frac{\pi}{2}$. This is only possible when the ratio $\frac{u'}{u}$ coincides from both sides at $\frac{\pi}{2}$. So

$$\sqrt{\lambda} \cot(\sqrt{\lambda}\frac{\pi}{2}) = -\sqrt{\lambda m} \cot(\sqrt{\lambda m}\frac{\pi}{2}) \tag{4.10}$$

The eigenvalues are real positive solutions of the above equation. We allow cases when both sides are infinite. But by Lemma 4.3 the first two eigenvalues satisfy $\frac{\lambda_2}{\lambda_1} > 4$ if $m > 1$. Hence the optimal density ϱ_0 is obtained when $m = 1$ (see Remark 4.4) i.e. ϱ_0 must be a constant. This proves the Theorem in the case when the transition point a is at $\frac{\pi}{2}$.

Suppose $0 < a < \frac{\pi}{2}$. Let

$$\varrho(x, t) = \begin{cases} t & \text{if } x \in (0, a), \\ 1 & \text{if } (a, \pi). \end{cases}$$

As in the case of the single-well potential the eigenfunctions are given by $u_n(x, 1) = k_n \sin nx$ where $k_n > 0$ for $t = 1$. By normalization condition

$$\int_0^\pi \varrho(x, 1) u_n^2(x, 1) dx = 1,$$

we get

$$k_n^2 = \frac{2}{\pi}.$$

Then for $t = 1$:

$$\begin{aligned}
\dot{\big(\frac{\lambda_2}{\lambda_1}\big)} &= \frac{\lambda_2}{\lambda_1}\int_0^\pi \dot{\varrho}\big(u_1^2(x,1) - u_2^2(x,1)\big)dx \\
&= \frac{2\lambda_2}{\pi\lambda_1}\int_0^a \big(\sin^2 x - \sin^2 2x\big)dx \\
&= \frac{2\lambda_2}{\pi\lambda_1}\big(\int_0^a (1-\cos 2x)dx - \int_0^a (1-\cos 4x)dx\big) \\
&= \frac{2\lambda_2}{\pi\lambda_1}\big((a - \frac{\sin 2a}{2}) - (a - \frac{\sin 4a}{4})\big) \\
&= \frac{2\lambda_2}{\pi\lambda_1}\big(\frac{\cos 2a \sin 2a}{2} - \frac{\sin 2a}{2}\big) \\
&= \frac{\lambda_2}{\pi\lambda_1}\sin 2a(\cos 2a - 1) \\
&< 0
\end{aligned}$$

since $\sin 2a > 0$ and $\cos 2a < 1$ for $0 < a < \frac{\pi}{2}$. So for small $t > 1$ there are densities $\varrho(x,t)$ for which the eigenvalue ratio $\frac{\lambda_2}{\lambda_1} < 4$. Similarly for $\frac{\pi}{2} < a < \pi$ the single-barrier density function

$$\varrho(x,t) = \begin{cases} 1 & \text{if } x \in (0,a), \\ t & \text{if } x \in (a,\pi). \end{cases}$$

gives $\frac{\lambda_2}{\lambda_1} < 4$. For $a = 0$, the single-barrier density function

$$\varrho(x,t) = \begin{cases} 1 & \text{if } x \in (0,\frac{2\pi}{3}), \\ t & \text{if } x \in (\frac{2\pi}{3},\pi). \end{cases}$$

gives $\frac{\lambda_2}{\lambda_1} < 4$. By symmetry for $a = \pi$, the single-barrier density function

$$\varrho(x,t) = \begin{cases} t & \text{if } x \in (0,\frac{2\pi}{3}), \\ 1 & \text{if } x \in (\frac{2\pi}{3},\pi). \end{cases}$$

gives $\frac{\lambda_2}{\lambda_1} < 4$. □

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