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LEARNER ERRORS RELATED TO LINEAR EQUATIONS IN GRADE 10

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A research project submitted in partial fulfilment of the requirements of the degree of Masters of Science (Mathematics Education)

University of the Witwatersrand
Johannesburg
South Africa
10 October 2016

DECLARATION

I declare that this research report is my own work and no part of it has been copied from another source (unless indicated as a quote). All phrases, sentences and paragraphs taken directly from other works have been cited and the reference recorded in full in the reference list.

Stephen Tebeila

10 October 2016

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My family, for always supporting me

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Abstract

This research investigates three different kinds of errors made by learners using a discursive approach when dealing with equations in grade 10. The data for the study is made up of Grade 10 learners' responses to a pre-test and post-test, and is part of a much larger data set that was collected by the Wits Maths Connect Secondary Project (WMCS) in 2013. Questions like substitution and multiplying and simplifying as well as factorization are given attention as they are related to equations.

The data for this study is collected by coding pre-test and post-test scripts. The data is analyzed using codes and using commognition. The role played by the equal sign in equations and operations that are performed on symbols as well as well as errors that creep in executing these operations are given attention in the study. Special attention is given to linear equations in this study. Errors identified by Brodie and Berger (2010) are confirmed. The finding is that extent to which errors made by grade 10 learners when dealing with equations stem from errors in arithmetic is substantial. The other finding is that extent to which grade 10 learners made errors related to basic algebra, when dealing with equations, is substantial.

Chapter 1 Introduction

1.1 Introduction

The study focused on learner errors in basic algebra, arithmetic and linear equations, using a discursive approach. Discursive, according to Sfard (2008, p. 66), means "personal constructs" that originate in public discourses that support only certain versions of such public discourses". The data used in the study consisted of Grade 10 learners' responses to a pre-test and a post-test, and was part of a much larger data set collected by the Wits Maths Connect Secondary Project (WMCS) in 2013. While the learner responses had already been analysed by the WMCS at the level of correct/incorrect, the analysis here focused on the errors themselves. This study analysed in depth the errors of one class (N=45), selected from the larger data set as a typical case.

1.2 Rationale

I conducted a pre-analysis of 15 pre-test and 15 post-test scripts, with attention given to learner solutions in simplifying basic algebra and solving equations. The pre-analysis was used to develop a plan for the error analysis and the kind of errors in the solutions. My analysis shows that prior learner understanding of arithmetic and basic algebra rules seem to impact negatively on how they solved equations and simplified expressions, as reflected by many of the errors.

Equations with one unknown (linear equations) are a key element of elementary mathematics, and were given attention in this study. Examples of quadratic equations which could be factorized to linear equations were included. Factorization provides the link between the two types of equations.

One cannot discuss equations without looking at the equal sign, as that is essential to all equations. According to Andrews and Sayers (2012), the transition from arithmetic to algebra is difficult as a result of factors that include the ambiguous role played by the meaning of mathematics symbols, and the particular importance of the equals sign. Sfard and Linchevski (1994), note that arithmetic is about dealing with numbers and with number calculations

The notion of the difficulty of the transition from arithmetic to algebra is supported by Andrews and Sayers (2012), when they suggest that, for example, the expression $3 + 4$ is arithmetically a command to carry out an operation. Algebraically, the same expression is an object on which other operations may be performed. They further state that a substantial number of early learner experiences of arithmetic impress upon learners the belief that the equals sign is a command to operate rather than having to do with of equality between two expressions. The former is contrary to the successful solution of equations.

This is supported by the notion that, when faced with equations of the form $1 + 2 = x + 5$, learners would apply the additive inverse to any of the three digits in accordance with their conception of the equal sign as an instruction. The interpretation is that an equation does not give a warning to show that the equal sign connects two equivalent expressions, as in examples such as $1 + 4 = 5$, $2 + 3 = 4 + 1$ and $2x + 1 = 3x + 2$. Thus, associating the equal sign with the calculating process is a serious problem to high school learners.

Why is the study important now in South Africa? Warren (2003) notes that the Trends in International Mathematics and Science Study (TIMSS) reports misconceptions many students have with, for example, understanding what the word 'variable' means; solutions of algebraic equations; and in translating word problems into algebraic symbols. She is of the view that these misconceptions need attention, as algebra plays an important role in the school curriculum.

TIMSS was conducted in South Africa in 1995 and 1999, and TIMSS-R in 2003. Both studies revealed that South African learners are not performing well in mathematics - as pointed out by Mji and Makgato (2006). Pournara, Hodgen, Adler and Pillay (2015) supported the notion of poor performance by mentioning that it is well known that most learners in South Africa attain very low marks in mathematics. They cite:

- Trends in International Mathematics and Science Study (TIMSS),
- the Annual National Assessments (ANA), such as the Programme for International Student Assessment (PISA), the National Senior Certificate (NSC) examinations and
- Southern Africa Consortium for Monitoring Education Quality (SACMEQ).

These all provide evidence of lack of impact on learners, mainly attributed to teachers' poor mathematical knowledge. This poor mathematical knowledge impacts negatively on learner achievement. Unsatisfactory performance in mathematics as shown in TMSS has led to study to determine which particular sections of mathematics contribute to this unsatisfactory performance. The central role of linear equations in mathematics has led to looking specifically at linear equations. The aim is to find out why candidates are doing poorly in linear equations.

1.3 Study Problem

The study problem was to identify and categorize the errors in learners' manipulation of linear equations.

1.4 Research questions

Research questions were:

- To what extent do errors made by grade 10 learners when dealing with equations stem from errors in arithmetic?
- To what extent do errors made by grade 10 learners stem from errors in basic algebra?

The test that WMCS used has been found to be appropriate in investigating the changes in the errors made, as it had questions that are representational, transformational or rule-based and global and meta-level, as will be discussed in the third chapter. These types of question link with the move from arithmetic that involves numbers, to algebra that could involve meta-level operations.

1.5 Outline of the research report

This first chapter provided the introduction to the research report, and also gave attention to the rationale for the research. The context of the research was provided and the relevance of this research discussed. The study has been described as focusing on errors made by learners when

answering questions relating to equations. It was emphasized throughout that a discursive approach will be used. Although the learner responses had already been analyzed by the WMCS at the level of correct/incorrect, that analysis has not focused in detail on the actual errors. Two research questions connected with the research problem of learners performing poorly in linear equations were formulated.

Important background that will be kept in mind is that:

- Numerical relations and processes in moving towards generalization will be important in the analysis; and
- The multi-layered recursive structure of discourses can be probed by looking at sub-discourses of equations, in this case substitution questions; multiply and simplify questions, and factorization questions.

The second chapter will give an outline of the literature relating to this research. Arithmetic, algebra and functions are given attention. Special attention is given to linear equations, and the equals sign in particular. The two theoretical frameworks are also explained in this chapter. Commognition put forth by Sfard is presented as a favoured lens employed here. The discursive framework for categorizing errors put forth by Brodie and Berger (2010) is also presented. An emphasis of routine errors, visual mediator errors and signifier errors is made. Mathematical learning is explained.

The third chapter discusses how the research was conducted. Sampling, design of the study, tests and memos to be used, analysis, validity and reliability and ethical considerations are discussed.

The fourth chapter discusses the analysis of the data, giving special attention to quantitative techniques. Categories for doing the analysis are explained. Here it is shown that routine errors are prevalent, and in particular

- errors that involve multiplication and division,
- errors that involve addition and subtraction and
- errors that involve factorization in that order are the most prevalent errors.

Visual mediator errors and signifier errors are shown to be least prevalent.

Differences and similarities of the pre-test and post-test sets of data are also discussed, using commognition.

Special attention to errors related to linear equations is given. Scanned images of interesting errors made by some learners are discussed, using commognition.

The fifth chapter gives a summary of the findings, recommendations for further research, suggestions for teachers, limitations of the study and the conclusions drawn about the findings.

Chapter 2 Literature Review

2.1 Introduction

In this literature review I discuss the theoretical framework that involves commognition and the discursive approach, research on errors in mathematics in general, and show what a discursive approach adds that was lacking in the commognitive approach on errors. Literature of algebra will then be discussed.

2.2 Conceptual framework

Commognition will be a favored lens employed here. Sfard (2007, p. 14), explains this idea as follows "commognitivists view discourse as the very object of learning". The commognitivist view that will be used here refers to thinking as a form of communication, as put forth by Sfard. She also points out that mathematical learning involves "modifying and extending one's discourse".

Discourse, for Sfard (2007), refers to different communication types that bring some people together, but does not accommodate others. The different communication types will be analyzed by determining whether the test is representational, transformational or rule-based; global or meta-level. The meta-level learning of mathematics is, according to Sfard (2008, p. 161) considered as a "multi-layered recursive structure of discourses". This means that the discourse is made up of sub-discourses, which relate to each other in many ways. The Sfardian (2007) framework provides the tools for exploring the mathematics found on the learners' scripts. The exploratory tools include categorizing errors and looking at them through a discourse perspective, as was done by Brodie and Berger (2010).

This framework moves beyond arguing that errors reflect misconceptions. They develop a discursive account of errors, by using Anna Sfard (2008). They see her commognition theory as providing a stronger theoretical scope for describing and accounting for errors. By analysing a multiple-choice test, Brodie and Berger (2010) developed seven categories for errors, which fall into three larger categories:

- errors as routines,
- errors of visual mediators and
- errors of signifiers.

Errors in arithmetic, basic algebra, linear and quadratic equations as observed in the analysis of responses, were found to be prevalent in a preliminary analysis of the pre-test and post-test written by learners in this study.

The word "errors" according to Brodie & Berger (2010, p. 172), is thought of as "narratives that are endorsed by learners". They associate the word with a pattern of mistakes made by learners that are persistent. A narrative, according to Sfard (2008, p. 300), is considered to be "a series of utterances, spoken or written, that is framed as a description of objects, of relations between objects, or processes with or by objects". The written aspect of narrative will be given attention here. A framework for categorizing errors will be developed in line with looking at them through a discourse perspective as was done by Brodie & Berger (2010).

The three categories will be employed in the study. "Routines" are, according to Sfard (2007, p. 574), "well-defined repetitive patterns" shown in actions. Routines in solving problems will be investigated in this study. According to Sfard (2007, p. 571), visual mediators are "means with which participants of discourses identify the objects of their talk and co-ordinate their communication.

Visual mediators will be given little attention in this study, as graphs will not be used to solve equations and word problems do not form part of the test. Signifiers and objects are used interchangeably in discursive theory according to Brodie and Berger (2010). Signifier refers to the main object used in communication, that is, one for which there exist realization procedures according to Sfard (2008). Objects and signifiers will be used interchangeably in this study.

An example of errors as routines (well-defined repetitive patterns) from the preliminary study is: if $a = 2, b = -5, c = 3,$ then

$a + 2c = 2+23$ (Conjoining).

According to Watson (2009, p. 19), conjoining is "an attempt to express an 'answer' by constructing closure, or students may just not know that letters together in this notation mean 'multiply'. So instead of doing $2c = 2 \times 3$ they treat 2c as a two digit number with the unit c and get 23.

As preliminary analysis also pointed to routine errors being prevalent, special subcategories will be created for routine errors to ensure that they are given sufficient exploration. Codes were developed for all categories and subcategories in the method chapter. All these exploratory tools are provided for by commognition, and will be interpreted with that in mind. That will be done by analysing mathematical discourses taken out of learners' scripts.

Errors of visual mediators (i. e., participants of a discourse identify objects of their talk) will be given less attention, as the initial study does not suggest that graphs will be used to solve an equation like $4t - 10 = 2t$, and the test has no word problems.

An example of errors of signifiers of a primary objects used in communication) seen in the preliminary analysis was $x(x - 2) = 8$ becomes $x = 8$ or $(x - 2) = 8$. We could only equate the two terms on the right hand to zero if we had zero on the right hand side and not when we have eight. The quadratic equation $x^2 - 2x - 8 = 0$ is not factorized into its factors $x = 2$ or $x = -4$, so leads to an error of communication. In this example a wrong signifier of eight is used.

Two points are important in developing a discursive account of errors as argued by Brodie & Berger (2010). One is that the discursive account suggests that pervasive errors occur because we tend to interact in similar ways, emphasizing social interaction. Social interaction will not be given attention here, as only scripts are available and that is a weakness of the discursive account for this research. The other point is that errors do not disappear. This point will be investigated in the post-test, rather than the pre-test.

Another way that Brodie & Berger (2010) use to account for learner errors is to draw on the work of Sfard (2007, p. 567), according to which mathematical learning involves "modifying and

extending one's discourse". This way relates to dealing with equations by modifying and extending related discourses like arithmetic and basic algebra.

Brodie & Berger (2010) are of the view that errors are not located in the mind of the learner but are located in the interactions between learner, teacher and Mathematics. In this research the interaction between the learner and mathematics as shown in the script will be explored

2.3 Literature of mathematical activities

The representational activities of algebra, as put by Kilpatrick, Swafford and Findel (2001, p. 256), "involve translating verbal information into symbolic expressions and equations".The authors also point out that the transformational - or rule-based - activities are mainly concerned with changing an expression or equation to an equivalent one, using the rules for manipulating algebraic symbols. They further see generalizing and justifying activities as tending to use the representational and transformational elements of algebra. According to Kieran (2004), the global, meta-level, mathematical activities for which algebra is employed as a tool do not belong to algebra exclusively.

This notion of algebra highlights that its discourse involves movement towards generalizing from numbers to symbols. Sfard and Caspi (2012) support this view, mentioning that algebraic thinking is to be found whenever one looks carefully at numerical relations and processes in searching for generalizations. However, that algebra is not only about symbols is again supported by Sfard and Caspi (2012) when they suggest that although symbolization can be important in the development of algebra, it is only a part of the general historical process of algebraic discourse formalization. Other parts include linear equations, which are the focus of this study.

In provoking meta-learning students become active in their own learning. This is supported by Sfard and Linchevski (1994) when they talk of encouraging learners to strive towards understanding as they learn, and.as they work towards making sense of communication geared towards discursive transformation. The meta-level learning of mathematics is, according to Sfard (2008, p. 161), considered as a "multi-layered recursive structure of discourses". This means that the discourse is made up of sub-discourses, which relate to each other in many ways.

2.4 Literature of algebra with special attention given to linear equations

Solution of a linear equation with a single unknown (from here on referred to as a linear equation) is a key component of school mathematics, as seen by Andrews and Sayers (2012). They see the solution of linear equations as a topic located on the border between mathematics as concrete and inductive and mathematics as abstract and deductive. They furthermore see linear equations as offering learners with one of the first authentic opportunities to link their insight of arithmetic and the symbolism of mathematics.

A linear equation like $8 + 5 = x+9$ will be referred to as belonging to elementary algebra. Elementary, or basic algebra is, according to Sfard and Caspi (2012, p. 45), defined as a "meta-discourse of arithmetic", and that means that reflecting on arithmetical processes is crucial in the discourse of elementary algebra. Mathematical learning involves "modifying and extending" one's discourse", as Sfard (2007) also points out. Objects will, as put by Nachlieli and Tabach (2002, p.

10), refer to "those things that are being talked about, do not pre-exist the talk; rather they arise as by-products of the ongoing mathematical conversation". Object-level learning is, in this regard, considered as learning that increases the existing assortment of "routines and endorsed narratives" that currently exist as the latter author further points out.

It can be seen from results of the research done by Norton and Irvin (2007) that much of what the students struggle with could be linked to not understanding arithmetic concepts as evidenced by class discussions, students explanations and examples of the teacher and researcher scaffolding student learning as well as an analysis of work of learners for error patterns, and not understanding equivalence, operations with negative integers, as well as distributive properties. The challenges of not understanding arithmetic concepts, equivalence, basic operations and distributive properties will inform some of the codes used in the discursive approach used to explore the errors made.

That means that there is more to formal mathematics than merely focusing on arithmetic generalizations only, giving attention to a pattern or structure, when setting out to solve certain types of problems. For example, a function like $f(x) = ax + c$ could be looked at in terms of a pattern between *x* and *f(x),* and at the same time be regarded as an object. Using this object, some types of problems that have a linear relationship and that have different values of a and c could then be tackled. The example of function was selected as, according to Schliemann, Carraher, and Brizuela (2007), arithmetical operations could be considered to be functions.

The idea is strongly supported by Sfard and Linchevski (1994) when they suggest that the idea of functions put together, the arithmetical processes that are linked to primary processes, and the formal algebraic processes linked to secondary processes. That is because functions are developed in arithmetic and are useful in formal algebraic processes. The concept of commognition will be used to as a general foundation for understanding how learners think.

Watson (2009, p .8), looks at how learning algebra occurs by focusing on "what learners can do and how their generalizing and use of symbols develop". This research gives information about what learners are doing and thinking, and combines research on mathematics from elementary school to sixteen-year-old learners, and identifies the issues that are important in understanding how children learn mathematics. Of special importance to this research in the article by Watson (2009) is her contribution that algebraic reasoning is about shifts that have to be made between arithmetic and algebra. These contributions are given attention in the next two subsections.

Their importance is a result of the fact that equations are an important part of this research, and are very important in algebra as it helps to consolidate work that would have been done earlier like substitution, addition and subtraction, and also lays a good foundation for further work.

According to Watson (2009, p. 3), algebra has to do with expressing "generalizations about numbers, quantities, relations and functions". So insight into links between numbers, quantities and relations is enabling in terms of good performance in algebra. As basic operations, namely, addition, subtraction, multiplication and division are important when dealing with numbers, the links include these operations. The particular importance of these operations in equations is that they provide inverses for carrying out operations on equations. The notion of the importance of basis operations will inform the choice of questions in the next chapter.

Watson (2009) says that an understanding of algebraic symbols is enhanced by an understanding of operations and being comfortable with using the symbolic shorthand. She is of the view that the learnings mentioned will most likely work out when learners have a knowledge of expressions, and when they are given sufficient time to be fluent at communicating, using the symbolic shorthand. The role played by symbols will inform the choice of questions selected.

According to Watson (2009) a letter can play different roles that include an unknown, a constant, a parameter, or a variable, and students have to decide which role is being assumed (by the letter) in any particular instance.

An example is the different roles played by n in $2n$. $2n$ could be seen as the answer to a question as a result of multiplying 2 by n. Attaching too much importance to substitution disempowers learners in seeing structure. So instances where n appears to be an entity on its own need to be confronted so as to assist learners to be more capable communicators of symbols, and not just concentrate on calculations that involve n. So giving attention to the difference between algebraic expressions as arithmetical structures and as calculations, according to Watson (2009), can enhance students' understanding.

It can be implied from her work that in certain instances an algebraic expression and a calculation or number or another algebraic expression can be equivalent. Now, algebra is often about transformations between these equivalent forms. Failure to understand the role of variable, and inability to deal with substitution will inform the discursive approach used to explore the errors made.

2.5 Differentiating between arithmetical and algebraic equations

Differences between arithmetical and algebraic thinking become clear when one looks at research that points to some conceptual and/or symbolic changes as noted by Filloy and Rojano (1989). These differences are helpful in understanding development of changes from arithmetic to algebra. These changes, according to Filloy and Rojano (1989), can be linked to the ideas and forms of representation of the objects and operations involved in the change from arithmetic to algebra. The ideas and ways in which the objects are represented, as well as operations involved in the change from arithmetic to algebra, inform the formulated hypothesis. It then becomes possible to visualize the changes that are made by learners as they move from arithmetic to algebraic thinking.

For discussion of change that happens or does not happen, the concept of equation will be considered and used in this study. An equation such as

$$Ax + B = C$$

will, in a similar manner to how it was used by Filloy and Rojano (1989), be referred to as being "arithmetical", as its left side can be thought of as a sequence of operations performed on known numbers. The right hand side stands for the result of having performed the left side operations.

An example of an arithmetical equation is

$$2x + 1 = 19.$$

This equation requires only operations on numbers.

Algebraic equations, unlike arithmetic ones, could have unknowns on both sides, and cannot be solved by arithmetic concepts alone. The notion is supported by Filloy and Rojano (1989, p. 19) when they suggest that ways of solving such equations require that the learner be able to operate on the variable as an entity and "understand that the expressions on both sides of the equals sign are of the same nature". An example of an algebraic equation is

$2x + 1 = 10 - x$.

This equation requires operations on algebraic entities.

Transitions from arithmetic to algebra has challenges. The challenges could be the result of different structures in the areas leading to a shift that requires adjustments. Herscovics & Linchevski (1994) support the issue of fundamental difference between the arithmetic and algebraic when speaking of a cognitive demarcation between arithmetic and algebra.

This cognitive gap calls for students having to adapt to how they interpret and solve problems by not merely calculating. Adaptation is geared towards moving towards formalization. Caspi and Sfard (2012), see formalization as a discourse in mathematics with specific meta-rules which regulate it. They furthermore see the rules as being embedded in the algebraic symbolism of mathematics. However, it is possible to think algebraically without letters. Malisani and Spagnolo (2009) support the notion when they say that doing algebra is not only about working with symbols and also when they say that algebraic thinking could be connected with layers of generality that could be factual, contextual and symbolic.

Symbol usage is also connected with some errors creeping in. That is supported by studies such as the Third International Mathematics and Science Study (1998) that have shown the misconceptions that learners have as a result of struggling with not having an understanding of the concept of a variable, solving algebraic equations and in translating word problems into algebraic symbols as pointed out by Warren (2003).

Malisani and Spagnolo (2008) note that some studies maintain that different concepts of the variable present different levels of difficulty to students. They also note serious challenges with the meanings of the generalized number and the variable.

2.6 Past and current research on symbolic interpretation

According to Kieran (2004), as learners move from working with arithmetic ideas to an equation of the form $Ax+ B = Cx + D$, finding a solution involves operations extracted from outside arithmetic, and carrying out operations on unknowns. Key to carrying out operations on unknowns is making meaning of the algebraic equation $Ax+ B = Cx + D$.

Making meaning of algebra includes understanding that an expression on the left hand side of an equation is equal to an expression on the right hand side of the equals sign, and that these expressions are of the same structure. They further say that actions like substituting numbers for the unknowns give meaning to the equality of expressions. That means that students that are working in an arithmetic way will probably not see the relationships between aspects of operations due to focusing their attention on calculations. That means that making a shift will be necessary. This shift includes:

1. Giving attention to relations, and not just concentrating on the calculations and working towards an answer that is a number.
2. Giving attention to carrying out operations, as well as working with their inverses, and on doing and undoing in an equation.
3. Moving away from tackling a problem by only solving it, to representing a problem and also working towards solving it as well.
4. Ensuring that both numbers and letters are given attention, and that it is not numbers alone. That includes:
    (i) working with letters that are not necessarily variables, but could also be unknowns or parameters.
    (ii) accommodating literal expressions that are not closed as responses;
    (iii) Basing a comparison of expressions so as to establish equivalence, on properties and not on an evaluation of numbers;
5. Giving attention to what an equals sign means.

Kieran (2007) noted that recent work suggests factors, including the following, to be impacting on students' interpretation of algebraic notation:

a) The ability of learners to perceive and notice, e.g. $8 + 5$ in

$8 + 5 = x + 9$

can be seen by some learners as a signal to compute, and they will just write down an answer of 13 instead of 4, if attention is not given to x in the equation.

b) Difficulties learners are confronted with when dealing with signs in simplifying expressions, e.g. when some learners are required to simplify $x + 3$ they do not think about the expression itself as being the subject of attention.

2.7 Research that singles out equations

According to Watson (2009, p. 3), "Students often get confused, misapply, or misremember rules for transforming expressions and solving equations", and that means that mathematics is not seen by them as related discourses. Furthermore, mathematically, according to Watson (2009, p.6), '=' means either 'equals to' or 'equivalent to', but is interpreted by learners to mean 'calculate'. An example is an equation like

$1 + 2 = x$

This can be construed arithmetically to be a command to execute 1+2 and not to mean that x is the same as 3. When dealing with

$6 + 3 = x + 5$,

an application of additive operations to any or all of the three digits is executed in line with the concept of the equal sign as an instruction to operate, and not as 6+ 3 is equivalent to x + 5. Watson (2009) also emphasizes that learners need to know what the equation is saying to them, and not just be influenced by how it looks without thinking about what it means. This means that in algebra the emphasis is on relations, and not calculations. For example, the relation 1 + 2 = 3 is often seen as a representation of 3, since 1 + 2 can be calculated, and not as 1+2 being equivalent to 3.

Tall, Lima and Healy (2014) have shown that some learners solve linear equations based on their previous experience of arithmetic operations, in which operations are executed to obtain an answer, relying on procedural embodiments as they work towards a solution. They see procedural embodiments as procedures which have to do with embodied actions on the symbols that lead to picking the symbols up and moving them to the other side of an equation. They also see procedural embodiments to be including 'principles' like changing signs or putting symbols in an equation so as to obtain the answer. Procedural embodiments were seen by them to having worked for some learners, but were also not remembered by other learners, leading to a wide range of errors. Procedural embodiments are explained in the following examples:

$2x-4 = 2+x$

$2x-x = 2+4$,

obtained by the operation of shifting x to the left hand side and 4 to the right- hand side and changing sign. and

$x = 6$ obtained by simplifying

Tall et al. (2014), describe these steps as "swop sides, swop signs".

Another example is the equation $3x - 1 = 5 + x$, which is solved by shifting the 1 to the right and the x to the left and changing signs to get: $3x - x = 5+1$

$2x = 6$.

The next step would then be $x = \frac{6}{2} = 3$, and in this step, 2 that is associated with the $2x$ in

$2x = 6$ is moved from the left side to the right and is then placed below 6.

Tall et al. (2014), refer to the procedure to be "swop sides and place underneath". They noted that in an attempt to employ these rules when dealing with $2x = 6$ learners made the following errors:

a) $x = 6 - 2$ b) $x = \frac{6}{-2}$ and $x = \frac{2}{6}$

Tall et al. (2014), refer to the procedure outlined in the last step of problem 2 as 'swop sides and place underneath'. They call the steps in the two examples procedural embodiments.

These researchers observed that with neither type of equation had the learners employed the general principle of doing the same thing on the left side and on the right side of an equation.

They did not see the equation as a balance to 'do the same thing to both sides'; instead, they paid attention to shifting symbols around mainly relying on 'swop sides, swop signs' and 'swop sides and place underneath'. According to Tall et al. (2014), procedural embodiment worked for some learners but not for others. They failed in some cases, as some learners tended to misremember these tactics, and consequently made errors. The issues raised here of procedural embodiments was explored in this study using commognition in the results chapter, that deals specifically with linear equations.

2.8 Conclusion

Literature selection has been informed by the fact that linear equations are the focus of this study. An argument that linear equations offer learners with one of the first authentic opportunities to link an insight of arithmetic and the symbolism of mathematics was advanced.

It turned out that differences that are helpful in determining developmental changes from arithmetic to algebra can be linked with the ideas and the forms of representation of the objects and operations involved. It then became possible to visualize the changes made by learners as they move from arithmetic to algebraic thinking by looking at that change in terms of forms of representation of the objects and operations involved.

Filloy and Rojano (1989) see the change from arithmetic to algebra being clarified by some conceptual and/or symbolic changes .These changes were seen to be linking up with the ideas and the forms of representation of the objects and operations. Representations of the objects and the operations involved in the change help to formulate a hypothesis about the change from arithmetic to algebra. The nature of the change that happens or does not happen was discussed by looking at how representations of objects and operations connect with linear equations. The mere fact that change can happen or not happen means that codes in the next chapter must include that. The importance of change here is that it informs the choice of the methodology used in the next chapter. The one group pre-test-post-test seems to be ideal in this case. The deliberations on linear equations

link up with discourses that involve substitution questions, multiply and simplify questions, as well as factorization, due to the multi-layered recursive structure of algebraic discourses. All the latter will inform the choice of codes in discursive approach used to explore the errors made in the next chapter The role played by basic operations and order of operations will have to be part of the deliberations as questions in the test given make the operations relevant.

Kieran (2004) argues that operations from working with arithmetic ideas to algebraic equations extracted from outside arithmetic. This comes from the fact that the transition involves operations on unknowns. The operations are no longer about calculations only. Relationships between aspects of the operations have also become relevant. Making a shift was seen as necessary in order to accommodate the transition.

Making a shift was seen to be giving attention to relations and to carrying out operations, as well as working with their inverses, and on doing and undoing, giving attention to both numbers and letters, as well as giving attention to what an equal sign means. The equal sign will also be given attention in the codes in the next chapter due to its importance.

Kieran (2007) added to "making a shift" by looking at factors that impact on students' interpretation of algebraic notation and difficulties learners are faced with when dealing with signs in simplifying expressions. This contribution can be construed as linking up with giving attention to working with errors while formulating a hypothesis.

Sfard and Caspi (2012) add to the discussion about the shift and change, emphasizing that reflecting on arithmetical processes is crucial in the discourse of elementary algebra.

Sfard (2007) sees modifying and extending one's discourse to be impacting on the shift and change. Norton and Irvin (2007) see the shift and change being influenced by not understanding arithmetic concepts, and having had on understanding equivalence, operations with negative integers, and distributive properties or not. All these impediments to a smooth shift will inform the codes used in the next chapter.

To enhance a smooth transition Sfard and Linchevski (1994) are suggesting that the idea of function integrates the arithmetical processes linked to primary processes, and the formal algebraic processes linked to secondary processes. This view is shared by Carraher, Schiemann and Brizuela (2006) when they suggested that arithmetical operations could be considered as functions.

Watson (2009) adds to 'shift' contribution by combining research on mathematics from elementary school to sixteen-year-old-learners, and also identifies the issues that are important in understanding how children learn mathematics. Of special importance to this research are her contributions about algebraic reasoning, as well as shifts that have to be made between arithmetic and algebra:

Equations help to consolidate work that would have been done earlier, like substitution, addition and subtraction.

Algebra has to do with expressing generalizations about numbers, quantities, relations and functions. An insight into links between numbers, quantities and relations is enabling in terms of good performance in algebra. Basic operations are an important part of the link.

A smooth shift will be enhanced by learners with knowledge of expressions, and, given sufficient time, fluent at communicating mathematics using the symbolic shorthand.

Understanding the different roles played by a letter, and not associating them only with substitution will enhance a smooth transition.

As this research is about linear equations, attention has been given to equations. The point raised by Watson (2009) that the sign '=' means either 'equal to' or 'equivalent to' but is interpreted by learners to mean 'calculate' was emphasized. That contributes to what affects a smooth shift from arithmetic to algebra by introducing errors.

The work of Tall et al. (2014), showing that some learners solved linear equations based on obtaining an answer as is done in arithmetic, was emphasized. These researchers contribute the notion of some learners tending to pick the symbols up and move them to the other side of an equation, as well as changing signs so as to obtain the answer, being misremembered by some learners and leading to errors. They observed that the general principle of doing the same thing on the left hand side and on the right hand side of an equation was not employed in the two instances mentioned. As a result of that errors were often introduced at the level of grade 10. This notion of procedural embodiments will be probed in the chapter that deals specifically with linear equations.

Chapter 3 Research Methods

3.1 Introduction

The previous two chapters detailed an overview of the research, rationale, background and conceptual framework. A detailed literature review was made and linked to the conceptual framework. This chapter describes the method used in this research.

The study is based on a detailed analysis of scripts collected in 2013 as part of the WMCS learning gains project. The development of models of professional development for secondary mathematics teachers is an essential aspect of the work of the WMCS. The models are aimed at strengthening teachers' relationship to mathematics. The idea is that learning gains at all levels of secondary schooling can be attained. Special attention to conceptualising, designing and implementing a professional development programme and then using research to check its impact on learner achievement, is vitally necessary

This research report focuses on errors made by learners in a pre-test and a post-test. The conclusion about the link between the literature review and the discursive framework impacting on the method inform the choice of the one-group pre-test - post-test research methods and the choice of codes. These codes are useful in categorising the types of learner errors. The notion of shift, raised in the literature review, informs the exploration of what happens in terms of learner performance and errors made in the pre-test and post-test. The learners will not necessarily change from operating arithmetically to operating algebraically at the same time and the change if any will be gradual. It is hoped that the time between February, when the pre-test was written, and the October post-test, will give clues about the errors learners make in the transition from arithmetic to algebra. The discourse used by learners in the pre-test and in the post-test will be explored to determine whether any learning took place.

The study makes use of more than qualitative and quantitative analyses of learners' errors. The next two chapters give attention to these. The study was guided by the following questions:

- To what extent do errors made by grade 10 learners when dealing with equations stem from errors in arithmetic?
- To what extent do errors made by grade 10 learners stem from errors in basic algebra?

3.2 Sampling

The study is based on a detailed analysis of scripts collected in the academic year 2013 as part of the WMCS learning gains project. An analysis of forty-five grade 10 pre-test, and forty-five post-test scripts of 14 - 18 year old learners in one class was done. The 45 scripts were chosen to ensure that there was a sufficient number of learner scripts by the same teacher for consistency in what learners have written. It is important to focus on the same teacher to remove the notion of any improvement on the analysis by the involvement of two or more teachers within the stated period. Any other class with a similar number of learners who wrote the pre-test and post-test could have been chosen for the same teacher.

3.3 Design of study

Attention was given to developing a clear understanding of common errors through learners' scripts. Paying attention to these errors was our major concern. The teachers' role was not considered in this study.

We conducted a one-group pre-test post-test analysis of scripts. The same test was used for the sake of consistency. The analysis of pre-test scripts included classifying the errors and answering the research questions. The analysis of post-test scripts included all that had been done for the pre-test analysis and a comparison was made. Conclusions, informed by the research questions, were then drawn.

A qualitative research method of the pre- and post-test scripts was used to identify errors noticed the responses. The qualitative research method, as defined by Hatch (2002, p. 10) is a process of "looking for patterns of relationships among the specifics". In this study the responses were sorted, into three possible categories of errors: errors as routines, errors of visual mediators and errors of signifiers. This is discussed in more detail in 4.1. Responses of learners were analysed to determine the errors made per sub-question.

The fact that the study gives attention to patterns makes it exploratory, in line with Borg, Gall and Gall (1996, p. 759), define it as "a method for discovering patterns". The fact of patterns arrived at by summarizing findings made the study descriptive, as in Opie (2004, p. 208), who defines it as: "describe or summarize a number of observations". Describing or summarising can take place only when observation of types of errors and learner performance has been made.

According to Leedy & Ormond (2005), one- group pre-test and post-test designs are linked to comparing participant groups and measuring the resultant change. Change may occur due to treatment or interventions. However much more work is needed in order to validly / reliably ascribe changes to an intervention. The work would need a controlled experiment or a pseudo-experiment to investigate the possible impact of interventions. The same test is used as a pre-test and a post-test to make it easier to investigate learning gains.

3.4 The test

The questions used in the analysis chapters were developed by the WMCS project. The selected questions contain a range of difficulty. For example, sample questions 1, 2 and 3 cover grade 9 content, and could therefore be considered easy at Grade 10 level. Questions 3a and 3b are difficult at grade 9 level, so can be considered to be moderate at grade 10 level. Question 5b can be considered difficult at grade 10 level. The researcher's experience in marked matric exams has shown that learners tend to equate each of the terms multiplied on the left side of the equation to the right side, although the right side is not zero. The selected questions assess evaluation, multiplication and simplification, factorisation and equations.

Table 3-4 explains each selected question. They are also discussed in the following chapters.

Table 3.4 Explanation of each question

Question	Description – what was tested
1(a), evaluation	Substitution, multiplication of two positive numbers and addition
1(b), evaluation	Substitution, multiplication of a positive number by a negative number, as well as addition of negative numbers
1(c), evaluation	
2(a), multiplication and simplification	Multiplying a binomial by a monomial, using the distributive law and addition of like terms
2(b), multiplication and simplification	Multiplication of two binomials, using the distributive law; multiplication of a binomial and a monomial using the distributive law, and addition of like terms
3(a), factorization	Factorizing by taking out a common factor.
5(b), factorization	Factorizing a trinomial.
5(a),solving linear equations	Introducing an additive inverse on both sides of an equation, adding like terms and using a multiplicative inverse on both sides of an equation. Learners to check the solution by inspection.
5(b),solving quadratic equations	Application of the distributive law, writing an equation in standard form and solving an equation by inspection and application of a x b = 0 implies a = 0 or b = 0 were tested. The solutions could be checked as in previous.
5(c),solving linear equations	Introducing additive inverse on both sides of an equation, using the multiplicative inverse on both sides of an equation and adding like terms. The solution could be checked as previous.

Note: In the table above monomial stands for a number, a variable or a product of a number and a variable, where all exponents are whole numbers. Examples are 12, x, $2xy$ and $3x^2$. A binomial is the sum of two monomials.

As the research is about linear equations the questions chosen for analysis were those that dealt with equations and/or application of skills in solving questions involving equations. The questions in the test that were analyzed are as follows:

Question 1

1. If $a = 2, b = -5, c = 3,$ evaluate the following. Show all steps in your work.

a) $a + 2c$

Question 1a) tested whether the learners could translate symbols to numbers and whether they understood whether the expression 2*c* stand for multiplication, although there is no multiplication sign between 2 and c. A possible error, if the multiplication sign is not seen, could involve writing 2 x 3 to yield 23. The question also tested whether the order of operations is understood. A possible error resulting from the wrong order of operations is adding *a* and 2 before multiplying 2 by *c.*

b) $3b - c$

Question 1b) tested whether the learners could translate symbols to numbers and whether they understood that the expression 3*b* stands for multiplication, although there is no multiplication sign between 3 and *b*. A possible error, if the multiplication sign is not seen, could involve writing 3 x (-5) as -35 or as 35. The question also tested whether the order of operations is understood. A possible error resulting from wrong order of operations is adding *b* and –*c* before multiplying 3 and *b.*

c) $a + 4(c - b)$

Question 1c) tested whether the learners could translate symbols to numbers and whether they understood the order of operations. A possible error resulting from wrong order of operations is adding *a* and 4 before multiplying 4 by *c- b.*

These three expressions were included because the study also considered arithmetic and basic algebra. The questions were representational as they involve translating data. The questions are also transformational because of the carrying out of the operations. Letters in the questions were used as symbols that stood for numbers and that are key in generalized arithmetic.

Question 2

Multiply out and simplify.

a) $4(3m - 2) + 5$

b) $(m + 2)(m - 3) + m(m + 3)$

These transformational expressions were included because the study also considers arithmetic and basic algebra. The questions are transformational, as they involve changing the form of an expression to an equivalent one, using rules for manipulating algebraic symbols.

The questions tested the ability to apply the distributive law when simplifying expressions into equivalent ones. They are more cognitively demanding because they do not have numeric answers,

and are meant to test whether learners are making a transition towards working with abstraction in terms of symbols. Possible errors will include adding unlike terms after applying the distributive law. Another source of errors will be applying the distributive law incorrectly.

Question 3

Factorize fully.

a) -3a + 15

b) $b^2 + 2b - 15$

These transformational expressions were included because the study also looks at arithmetic and basic algebra. Like the previous question, they are transformational. They could be simplified using factorization and could be classified as testing procedure. The minus sign in a) makes the problem slightly difficult, as it is expected that learners will encounter difficulties in noting that +15 could be written as -(-15), thus factoring out a factor of -1. Question b) is easy, as it tested only procedure. Possible errors include ignorance of prime factors and the wrong allocation of the minus sign to the factors 3 and 5. The minus sign in 15 makes the question slightly difficult because of the allocation of the minus sign, but nonetheless the question mainly tested procedure.

Question 5

Solve for the unknown.

a) $4t - 10 = 2t$

This equation with unknowns on both sides cannot be solved by arithmetic-based methods, and can be classified as global at the level of grade 10 learners. It involves more than just using algebra as a tool to transform the equation. A possible error will be adding unlike terms.

b) $x(x - 2) = 8$

This transformational question involves changing the form of an expression to an equivalent one using algebraic laws. The steps involved are: the ability to apply the distributive law; have zero on one side of an equation; and factorize to solve a quadratic equation.

c) $3 - \frac{2m}{2m+1} = 7$

The ability to deal with fractions in equations is tested here. In this equation. rules of algebraic operations apply, so the problem is transformational. A possible error could be ignoring the m+1 in the fraction $\frac{2m}{m+1}$ and treating the fraction as 2m. Other errors include inability to find the correct lowest common denominator, ignoring the minus sign and not following the order of operations.

An interesting equation that could be included would be to solve for x in $x + 1 = x + 1$. If students do it arithmetically they get $x = x$. The real algebraic test is interpreting the algebraic entity. Another interesting question would be to solve for x in $x + 1 = x + 2$. The arithmetic process yields $x = x + 1$ and that is yet another algebraic entity. The last two questions show that algebra is not

only about a process, but also showing understanding by interpreting the algebraic entities. This understanding will lead to *x* being any real number in the first case, and not existing in the second.

3.5 Analyzing the test

The analysis of the data was made in two phases. The first involved identifying errors made in all the questions, informed by categories derived from the discursive approach of Brodie and Berger (2010). These categories are routine errors, visual mediator errors and signifier errors. The difference between the categories used here and those of Brodie and Berger (2010) is that in this research routine errors were placed in subcategories. This study uses the already-existing framework of Brodie and Berger (2010). It confirms the presence of errors, and in the conclusion makes recommendations on how those errors can inform planning.

The second phase involves paying special attention to errors in the questions involving equations. The reason for this is that linear equations are the focus of this study and also that most of the errors were in the preliminary work made in the equations. The data was analysed using commognition discourses.

The notion of different types of changes, namely a learner increasing or decreasing the mark from pre-test to post-test was analyzed by investigating different types of errors and discussing them in terms of the commognition discourse

To investigate the mark change (increase or decrease) in more detail errors that a decrease or increase or stay the same from post-test to pre-test will be investigated. In the first 10 scripts marked there were learners who got a solution right in the pre-test but wrong for the same question in the post-test. To ensure that this decrease is given attention coding will include a category of 1.

It is expected that learners will tend to forget, misinterpret situations and misapply rules, as was seen in the analysis of 15 pre- and post-test scripts. Thus: if

$a = 2$, $b = -5$, $c = 3$, then

$a + 2c = 2 + 23$ (Conjoining) and

$a + 4(c - b) = 2 + 4(3 - 5) = 6(3 - 5)$.

Evidence of understanding what an equation means will be provided by working out the following:

$$x + 2 = 0$$

$x = -2$ (Deduce).

This method will be accommodated when developing a memo for marking the scripts.

The linear equation comes from question b), and that becomes clear when the following is considered:

$$x(x - 2) = 8$$

$$x^2 - 2x - 8 = 0$$

$$(x + 2)(x - 4) = 0$$

Or:

$$x + 2 = 0$$

$$x + 2 - 2 = 0 - 2 \text{ (Subtract one both sides)}$$

x = -2 (Simplify)

It is expected that most learners will use this method.

Or $x + 2 = 0$

$x + 2 = -2 + 2$ (Looks the same)

$x = -2$ (Deduce).

It is expected that only exceptional learners will use the last method. It is also expected that many learners will struggle with an equation with a variable on either side, such as:

$$5x + 2 = x + 6.$$

Learners struggling with an equation with a variable on both sides became evident in the preliminary marking of 15 pre- and post-test scripts, in which such equations could not be solved by most learners. It could be solved by the technique shown above:

$$5x + 2 = x + 6$$

$x + 4x + 2 = x + 4(1) + 2$ (Looks the same),

x = 1 (Deduce)

3.6 Validity and reliability

Validity will be mainly limited to the questions that relate to basic algebra and equations. It will be ensured by analyzing scripts of a test which has already been used successfully by WMCS. The validity of a measurement instrument, according to Leedy & Ormond (2005, p. 28), is "the extent to which the instrument measures what it is supposed to measure". In this situation employing the measurement tool involves fitting the errors to the three categories.

The validity was boosted by breaking the three major categories into smaller categories so as to ensure that the subcategories that are embedded in each of the three categories are not missed. Validity in the study will be enhanced by using the pre- and post-test marked scripts of the same learners. Justification of the decision to use a test and to strengthen validity is derived from Leedy & Ormond (2005) when they note that in a one-group pre-test, post-test analysis, it is at least known that a change has taken place - and this is a change study. There is not one possible explanation of the change that might be observed.

Reliability of the research analysis will, as put by Bertram & Christiansen (2014, p. 207), refer to "the extent to which a measure, procedure or instrument yields the same result in repeated trials". In this research reliability will be improved by using the same test as pre-test and post-test. The notion is supported by Charles (1995), who suggests that consistency with which an individual's scores remain relatively the same can be worked out using the test-retest method at two different times. In this instance that will refer to pre-test and post-test scripts, that will be marked and analyzed at consecutive times, though with some overlaps, allowing for investigating the learning changes. Consistency will be enhanced by allowing for some time overlaps to analyze the data, and using the same memo to mark the test. The time frame for completing this study is Aug 2015 to end March 2016.

To enhance reliability 10 pre- and post-tests scripts were given to a teacher to mark and code. The marks obtained were the same, and there were almost no error differences. That indicates that the analysis tools for coding and marking are reliable.

GDE clearance was obtained by WMCS. Ethics clearance was obtained from WSoE (Wits School of Education, protocol number 2015ECE020H). Confidentiality and anonymity were assured by using "school x" as the name of the school, and not mentioning names of learners or the educator.

3.7 Conclusion

This chapter has given details about how the data was collected and analysed. The next two chapters will give an analysis of the data and discuss findings. The discussion was informed by the research questions:

- To what extent do errors made by grade 10 learners when dealing with equations stem from errors in arithmetic?
- To what extent do errors made by grade 10 learners stem from errors in basic algebra?

Sampling and design features of the study were discussed. That involved the reasons for choosing the one group pre-test/ post-test design in keeping with investigating the shift from pre-test to post-

test, and determine whether there are clues about the shift from arithmetic to algebra. The questions selected involved evaluation, multiplication and simplification, factorization and solving equations, as they are linked with research questions about arithmetic and basic algebra.

Coding was informed by literature conclusions, which point out skills that relate to errors in arithmetic and basic algebra, using the discursive framework. Coding will also be informed by progress or its lack in the shift pre-test to post-test, and hopefully also from arithmetic to basic algebra. Consistency in what the learners wrote was enhanced through a sufficient number of scripts and the same teacher.

The collection and analysis of errors as reflected in the scripts was the major concern. A one-group pre-test/post-test analysis of the scripts was done. The same test was used for the sake of consistency. The pre-test analysis of scripts included classifying the errors and answering the research questions while doing a quantitative analysis of performance. The post-test analysis was identical. The post-test and pre-test analyses were compared, and conclusions informed by research questions were drawn.

A qualitative, exploratory and descriptive research method of the pre- and post-test scripts was used to identify errors that were noticed in responses to questions. The responses were sorted into the three categories of errors: errors as routines, errors of visual mediators and errors of signifiers.

The pre-test/post-test analysis was found to be ideal for this research, as it explored the change resulting from treatments/interventions. This provided insights into the shift from arithmetic to algebra through the linear equation and related questions.

The test questions were analyzed. The reason for classifying the questions was to explore whether the questions were shifting from relatively easy levels like calculating, to meta-level within the same discourse. That was done in keeping with algebra being a meta-discourse of arithmetic.

The substitution questions were found to be representational, as they involved translating data. Letters in the questions were used as symbols that stood for numbers, and these letters were key in generalized arithmetic. The 'multiply out and simplify' questions were also found to be transformational, as they involved changing one form of an expression to an equivalent one using rules for manipulating algebraic symbols. The question on factorization was also found to be transformational. The sub-questions on equations showed signs of shifting towards meta-level

The equation with unknowns on both sides, i.e., $4t - 10 = 2t$ was classified as global, as it could not be solved by arithmetic-based methods, and because it involved more than just using algebra as a tool to transform. The equation $x(x - 2) = 8$ was classified as transformational, as it involved changing the form of one expression to an equivalent one, using an algebraic law.

With regard to the question $3 - \frac{2m}{2m+1} = 7$, the rules of algebraic operations are applied in finding the solution, and that means the problem is transformational. So the selected questions contribute to a shift towards meta-level. A suggestion was then made about interesting equations involving interpreting an algebraic entity that could be included to explore the shift towards meta-level. The resulting errors are analyzed in terms of the main framework, namely, commognition.

Validity was ensured by analyzing test scripts which had already been used successfully by WMCS, and enhanced by using the pre-tests and post-test marked scripts of the same learners by the same teacher. Reliability was enhanced by using the same test as pre-test and post-test. The notion is supported by Charles (1995). Consistency was enhanced by allowing for some time interval as data analysis was done, using the same memo to mark the test. GDE clearance and ethics clearance was obtained from WSoE (Wits School of Education). Confidentiality and anonymity was respected by using "school x" as the name of the school, and not mentioning names of learners or the educator.

Chapter 4 Results

4.1 Introduction

In this chapter a memorandum that was developed for marking the pre-test and post-test scripts is discussed. The scripts were marked according to the memorandum in order to aid in the discursive framework in terms of setting up the codes that will be used to investigate the errors made by learners. This was enhanced by assessing the performance of learners in certain questions that were selected on the basis of their relevance to the research questions.

The challenges of misunderstanding arithmetic concepts, equivalence, basic operations and distributive properties as discussed in the literature review also assisted in setting up some of the codes used in the discursive approach used to investigate the errors made. The fact that routine errors, visual mediator errors and signifier errors are not located in the minds but are located in learner responses to the test makes the approach discursive.

The difficulty encountered by learners of misunderstanding the role played by variables in linear equations, and the inability to substitute these variables, also informed the discursive approach used to explore the errors made. The emphasis given to the role played by the equals sign is also included in the choice of codes employed.

The question of coding used was also informed by the progress made by the learners when they got a correct solution, or the lack thereof when they left blanks, in the shift from pre-test to post-test. The latter issue was highlighted in the conclusion of the method chapter.

The learner scripts were coded in such a way that at the most three errors could be accommodated in the case of a solution to a question that has more than one error. The data obtained was then analyzed using codes in a spreadsheet. Extracts from some learner scripts were used to provide evidence and aid analysis.

Attention was given to linear equations to determine whether a new discourse happens without hindrance. This was accomplished by comparing the changes in codes that occur from pre-test to post-test. The notion of some types of errors reducing or not reducing or staying the same was scrutinized by application of the codes. All these were done to aid in answering the research questions:

To what extent do errors made by grade 10 learners when dealing with linear equations stem from errors in arithmetic?

To what extent do errors made by grade 10 learners stem from errors in basic algebra?

4.2 Test

4.2.1 Marking memorandum

Although the Wits Maths Connect Secondary project only focused on coding and did not provide any memorandum, it was felt necessary in this study to mark the learner scripts at the beginning to compare learner performance in the pre-test and in the post-test. The mark allocation was adjusted slightly so that attention could be

given to important skills mentioned in the literature, like basic operations, equivalence, the role of the equals sign, order of operations, the importance of the distributive law and factorization.

Table 4.2.1 Memo

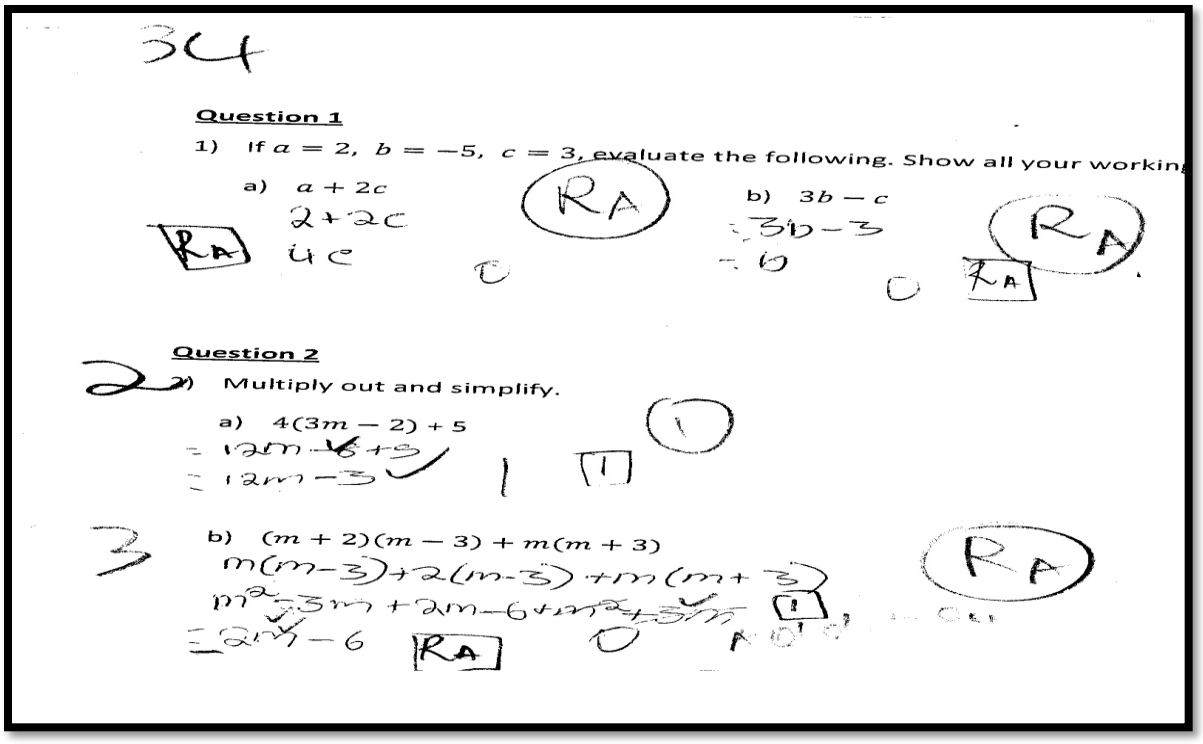
Question	Solution	Mark allocation	Total mark
1. If $a = 2, b = -5, c = 3$ evaluate the following. Show all your working. a) $a + 2c$	= 2 + 2 (3) = 2 + 6 = 8	√ for correct substitution √ for multiplication of 2 by 3 √ for answer	3 marks
b) $3b - c$	= 3(-5) – 3 -15 – 3 = -18	√ for correct substitution √ for multiplication of 3 by -5 √ for answer	3 marks
c) $a + 4(c - b)$	= 2 + 4(3- - 5) = 2 + 4(3+5) = 2 +4(8) = 2+ 32 = 34	√ for correct substitution √ for +5 √ for multiplying 4 by 8 √ for answer	4 marks
2. Multiply and simplify a) $4(3m - 2) + 5$	$=12m - 8 + 5$ $=12m - 3$	√ for $12m - 8$ √ for -3	2 marks

b) $(m+2)(m-3)+m(m+3)$	$=m^2-m-6+m^2+3m$ $=2m^2+2m-6$	√ for m^2-m-6 √ for m^2+3m √ $2m^2$ √ $2m$	4 marks
3. Factorise fully. a) $-3a+15$	$=3(-a+5)$	√ for both factors	1 mark
b) $b^2+2b-15$	$=(b+5)(b-3)$	√ for $(b+5)$ √ for $(b-3)$	2 marks
5.Solve for the unknown a) $4t-10=2t$	$4t-10+10=2t+10$ $4t-2t=2t+10-2t$ $\frac{2t}{2}=\frac{10}{2}$	√ for subtracting 2t both sides $t=5$ √ for answer	2 marks
b) $x^2-2x=8$	$x^2-2x-8=0$ $(x+2)(x-4)=0$ $(x+2)=0$ or $(x-4)=0$ $x=-2$ or $x=4$	√ for standard form √ for both factors √ for equating each factor to zero √ for both answers	4 marks

c) $3 - \frac{2m}{m+1} = 7$	$3(m+1) - 2m = 7(m+1)$ $3m + 3 - 2m = 7m + 7$ $m + 3 - 3 = 7m + 7 - 3$ $m - 7m = 7m + 4 - 7m$ $\frac{-6m}{-6} = \frac{-4}{-6}$ $m = \frac{2}{3}$	√ for multiply throughout by $(m+1)$ √ for subtract 3 both sides √ for subtract 7m both sides √ for answer	4 marks

Notes: Underlining was done where there was an error. Marks for each question were captured in a spreadsheet. Awarding of marks for each question was stopped after three mistakes were picked up, thereafter for the same question only underlining of errors was done. Teacher P was chosen from a secondary school and her particulars are available on request. Both pre-test and post-test were marked. After marking and analysis of pre-test and post-test, coding of the script was then done. Marking was done by using ticks.Ten scripts were given to a teacher to check whether marking and coding would differ much with the researcher's, and the finding has been that there was no difference in terms of the outcome of the process. The teacher was the first to mark and code the selected scripts and so did not see the codes allocated by the researcher. That has boosted reliability of the codes and the memo to some extent. The codes made by the teacher were seen by the researcher when he did his own coding. Five pre-test and 5 post-test scripts and codes were given to a teacher to do the error coding. A code allocated by a teacher was inserted inside a rectangle as opposed to the coding done by the researcher that was put in a circle. Figure 4.2.1 illustrates the codes.

Figure 4.2.1 Learner 34



In the figure for learner 34 the researcher's code is in a circle and the teacher's code is in a rectangle. For Q1 (a) there is agreement that the problem is adding unlike variables 2 and 2c to yield 4c. Although no substitution is done the actual error seen is adding unlike terms.

For Q1 (b) learner 34 seems to have subtracted 3 from 3 and simply retained m. There is also agreement that adding unlike terms was seen here. For Q2 (a) there is agreement that the solution is correct. For Q2 (b) the addition error has been noticed in both cases. However the marker seems to have put a 1 in the rectangle for the mark and not for 1 representing a correct solution in line with the codes. The argument here is the teacher cannot say there is an addition error and give full marks for the question. Mark(s) would have been lost for the addition error.

The script above and other scripts provided evidence that the coding done by the researcher did not differ much.

4.2.2 Explanation of calculations done

The pre-test and post-test marks for all learners were entered in a spreadsheet. The average for each question was calculated. For easier comparison of the pre-test and post-test, the mark for each question was expressed as a percentage of the mark for the question.

The average mark was obtained in the usual way of adding all the marks and dividing by the frequency However individual marks will be considered when deeper analysis of errors is done.

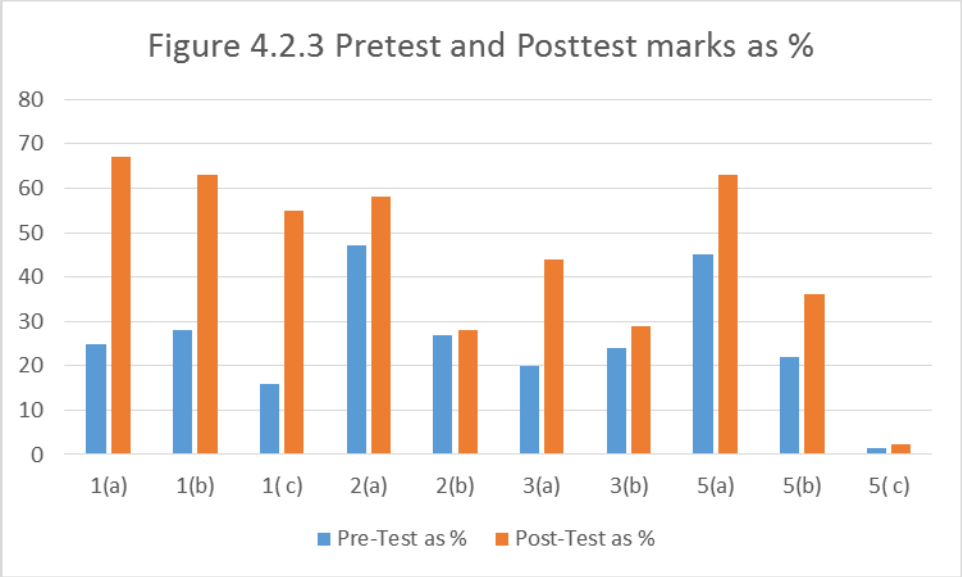
4.2.3 Average Marks for Pre-test and Post-test

In the table below Q1 is a question on substitution, Q2 is a question on multiplication and simplification, Q3 is a question on factorization and Q5 is a question on solving equations.

Table 4.2.3 Pre-test Average and Post-test Average

	Pre-test Average (A_P)	A_{Pr} %	Post-test Average	A_P%	Comments: 30% is reasonable at FET.
Q1 (a) 3 marks	0,76	25%	2	67%	All the Q1 average marks have shown an increase from pre-test to post-test. The pre-test average in each case is less than half the mark for the question. The post-test average is more than half the mark for the question
Q1 (b) 3 marks	0,85	28%	1,89	63%	
Q1 (c) 4 marks	0,65	16%	2,21	55%	
Q2 (a) 2 marks	0,93 1,07	47% 27%	1,15	58%	All the Q2 average marks have shown an increase from pre-test to post-test. The pre-test average in each case is less than half the mark for the question. The post-test average is more than 50% the mark for 2a but not for 2b.
Q2 (b) 4 marks			1,10	28%	
Q3 (a) 1 mark	0,19	20%	0,44	44%	All the Q3 average marks have increased from pre-test to post-test. The pre-test average in each case is less than 50 % the mark for the question. The post-test average is more than half the mark for the question
Q3 (b) 2 marks	0,47	24%	0,57	29%	
Q5 (a) 2 marks	0,90	45%	1,25	63%	All the Q5 average marks are increasing from pre-test to post-test. The pre-test average in each Case is less than 50% the mark for the question. The post-test average is more than 50% mark for the question. Based on 30% s 1a, 1b and 1c have significant changes
Q5 (b) 4 marks	0,88	22%	1,43	36%	
Q5 (c) 4 marks	0,05	1,3%	0,09	2,3%	

Note: In the table above A_{Pr} stands for the average mark of the pre-test question expressed as a percentage of the total mark for the question and A_P stands for the average mark of the post-test question expressed as a percentage of the total mark for the question. A significant increase is 30%.



Note: The question number is on the horizontal axis and the average mark is on the on the vertical axis

4.2.4 Analysis of performance

The low averages obtained in the pre-test justifies the fact that all these questions that are related to equations required to be given attention in the analysis. The low percentage obtained in the equation questions, contributes substantially to the poor performance. The latter justifies the importance of having to investigate what could have led to poor performance in this very important question on equations.

Solving equations is capable of singly indicating mastery of substitution when checking, multiplication when applying the distributive law to terms that are inside brackets and also factorizing if dealing with quadratic equations. This equation question also indicates whether further work like trigonometric equations, financial mathematics and analytical geometry will not be mastered because of inability to master equations.

The post-test results are also indicative of the fact that all these chosen questions are still not well mastered after the February to October intervention. It is only in the case of substitution where a substantial improvement is noticed. The marks show a substantial percentage point increase from pre-test to post-test in Q1 (a), Q1 (b) and Q1 (c) where the increases are

The least increases were in Q2 and Q3. Q5 (a), Q5 (b) and Q5 (c) had respective increases of 18%, 22% and 1%. So Q5a and 5b seem to show some change.That Q5 (c) shows almost no increase was indicative of the fact that fractions and dealing with inverses is almost out of reach for the learners. So codes in the error analysis must include skills in the solution of equations. This includes knowing the importance of the equal sign, substitution as that helps in checking the answers obtained, factorization, inverses, division and subtraction.

Increases in Q3 (a) and Q3 (b) of respectively18% and 14% show a insignificant improvement from pre-test to post-test. Although the changes are insignificant, codes in the analysis include factorisation for a more detailed analysis"

Increases in Q2 (a) and Q2 (b) of respectively 11% and 1% show that performance in arithmetic and basic algebra skills shows no increase from pre-test to post-test. The question involves multiplication, and addition of terms that have variables and numbers. Although the changes are insignificant, codes in the analysis nonetheless include arithmetic and basic skills in arithmetic for a more detailed analysis.

Coding schemes in the next section must also be informed by literature conclusions which point out skills that relate to errors in arithmetic and basic algebra, using the discursive framework. Coding itself will be informed by data related to progress or lack of that in the shift from pre-test to post-test and hopefully also from arithmetic to basic algebra.

4.3 Analysis of the pre-test and post-test by giving special attention to errors

4.3.1 Three major categories of errors

Three categories, namely,

- errors as routines R,
- errors of visual mediators V_m and
- errors of signifiers S

put forth by Brodie & Berger (2010), form a significant part of the pre-test and post-test analysis. Furthermore, extra categories will be introduced to accommodate findings that cannot be matched with the mentioned three categories. Examples here are $3b - c = (3b - c)(c - 3)$ which was made by learner 29 and $a + 2c = 2c + 2c$ was made by learner 32. These errors could not be matched with routine errors, visual mediator errors and signifier errors. Findings include instances where learners got the question correct, instances where a learner rewrote the question or left a blank as well as instances where the learner's work could not be explained. An example of the latter is factorizing $-3a + 15$ to yield 5^{-2}. More coding information is given on 4.3.5 to 4.3.7.

4.3.2 Routine errors

According to Sfard (2007, p. 574), routines are "well-defined repetitive patterns" shown in actions. An explanation used by Brodie & Berger (2010) to describe a well-defined repetitive pattern is learners selecting an answer which is an answer to an intermediate step of a procedure, and this response is called a 'halting signal'. In the test an intermediate step could involve getting one of several terms of an answer, distributing correctly for only one term and not for all the terms. For example to multiply a sum of two numbers by a third number one multiplies each addend in the sum by the third number using the distributive law and then add the products. The numbers could be replaced by variables to yield a (b + c) = ab + ac. Routines categories without adhering to any order will be as follows:

R_F = routine error related to factorization whether the factorization is the difference of two squares, taking out a common factor or factorizing a trinomial. An example is

$2xy - 4xz = 2x(y+2z)$

where only one term in the bracket is correct and the other term in the bracket is incorrect when factorizing by taking out a common factor**.** Another example that illustrates an error in factorization is $-3a + 15 = -3(a +15)$ instead of $-3a +15 = -3(a-5)$. The choice of this error code is informed by poor performance in this question and by literature review. These were called routine factorization common error.

R_E = wrong application of equals sign. Certain times this error will impact negatively in getting the right answer and, at certain times, the right answer is obtained although there are three equal sign in an equation. E.g. = $4t - 2t = 10$ resulting in $t = 5$. Another example is $4t-10=2t$ becomes 4t-10-2t which is an expression and not equation. Another example is $a+2c$ became $a=-2c$. In this study attention will be given to instances in which not understanding an equals led to a wrong answer. This code was informed by marking the scripts where some expression were seen to being changed to equations and the other way round. These were called routine equation errors.

R_V = errors related to dropping numbers or variables to make an expression easy to deal with. An example here is reducing m+1 to m e.g. $\frac{2m}{m+1}=10$ resulting in $\frac{2\cancel{m}}{\cancel{m}}=10$. In the example $m+1$ is written as m by dropping 1 so as to make it easy to cancel. These were called routine variable errors.

R_S = errors related to substitution, such as putting $4(c-b)=4(3-5)$ when c = 3 and b = -5. These were called routine subtraction errors.

R_M = errors related to multiplication or division as well as using the distributive law. An example is 4(3m-2) +5=12m-2+5 instead of 4(3m-2) +5 = 12m-8 +5. Another example is $a+4(c-b)$ becomes a + 4b - c. Another example is 2+2x3c=4x3c=12c. In the last example the order of operations has not been mastered, leading to a multiplication error. This code

is informed by common question response in the 'simplify and multiply' question and also by literature review. They were called routine multiplication errors.

R_A = routine error that pertains to adding or subtracting unlike terms. Inability to work with additive inverse in an equation will also fall here. An example is 4t-10 becomes 6t. Another example is $3-\frac{2m}{m+1}=7$ becomes $\frac{2m}{m+1}=7+3$. In this example -3 was supposed to be added both sides and this means that the additive inverse is not well understood. These were called routine addition errors.

N = errors related to not having mastered basic arithmetic like addition, multiplication and simplification. Possible errors here will be: wrong addition, wrong subtraction, wrong division, wrong multiplication, wrong simplification, misunderstanding of product, can't simplify. Examples include -2-3 = 5, 2-3 = 1, -2.-3 = -6, -6/-3 = -2. This code was informed by poor performance in the multiply and simplify question where a lot of arithmetic errors were seen to be impacting negatively. Literature review also informs the choice. These were called the routine arithmetic errors.

4.3.3 Visual mediator errors

V_M = errors related to visual mediators and these are "means with which participants of discourses identify the objects of their talk and co-ordinate their communication". Examples of visual mediators include written symbols, such as numerals, tables, algebraic formulae, and lines. Errors relating to visual mediators were limited to conjoining in this study as errors around substitution of symbols for numbers lead to conjoining. An example of errors as visual mediators from the preliminary study is if $a = 2$, $b = -5$, $c = 3$, then $a + 2c = 2 + 23$ (Conjoining). Here conjoining is "an attempt to express an 'answer' by constructing closure, or students may just not know that letters together in this notation mean 'multiply'" Errors of visual mediators) will be limited to conjoining as initial study does not suggest that graphs will be used to solve for example $4t - 10 = 2t$ and the test has no word problems. These errors were called visual mediator errors.

4.3.4 Signifiers errors

S = signifier errors. Signifiers are "words or symbols that function as nouns in utterances of discourse participants". Signifiers and objects are used interchangeably in discursive theory. Errors of signifiers involve a learner employing techniques that have been working in colloquial discourse or in a previous mathematical discourse. An example used by Brodie & Berger (2010) involves a learner employing techniques that have been working in colloquial discourse or in a previous mathematical discourse. Here a learner may use $ab = 0$ implies $a = 0$ or $b = 0$ to be the same as $ab = 1$ implies $a = 1$ or $b = 1$, inserting a new signifier into a discursive template that has worked. Another example is $15 = 3^5$ instead of 15= 3 x 5. An example of signifiers (primary objects used in communication) seen in the preliminary analysis was $x(x - 2) = 8$ becomes $x = 8$ or $(x - 2) = 8$. In this example a learner does not understand that $ab = 0$ implies that $a = 0$ or $b = 0$.

4.3.5 No response

0 = there is no response. This question code was informed by lack of progress in the shift from pre-test to post-test. No response errors were called blanks.

4.3.6 Correct answer

1 = correct answer. It has to be mentioned that 1 is not a mark but is a code. This category does not directly deal with errors but is useful in enabling the analysis to include the fact that errors were not made in some questions as learners got everything correct. The code was informed by progress or lack of that in the shift from pre-test to post-test and also hopefully, from arithmetic to basic algebra as pointed out in the conclusion of the method chapter.

4.3.7 Errors that cannot be explained with codes used in the study

X = will be errors that cannot be explained using categories and framework used here; e.g a +2c = 2c (4+a). Another example is $2c + a = \frac{2c}{a}$. This code is informed by errors I could not explain using codes developed in this study. An error from such a category cannot be explained in terms of mathematical laws that are relevant at the level of grade 10 and could not easily be matched with developed codes. Errors in this category were called can't explain errors.

4.4 Coding responses and populating tables

The coded responses are populated and counted in the pre-test attachment called Appendix 1 and in the post-test attachment called Appendix 2

A response to a question that had many errors was coded to accommodate different kinds of errors made. At most three different kinds were coded for each question. This was done to get a richer understanding of errors and to accommodate the diversified nature of responses made and to accommodate the fact that some learners made many errors in one question. Three errors were accommodated as usually after three errors the interpretation of errors became much more uncertain. The other reason for coding a solution using several codes is so that useful data is not sacrificed. However it is more difficult to analyze when multiple coding is employed as many codes have to be kept in mind when coding.

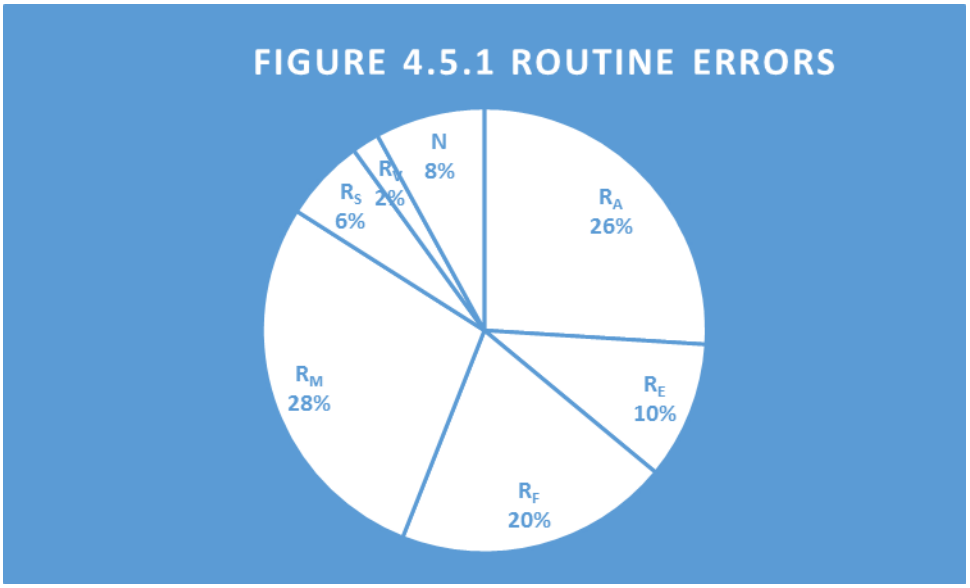
The pre-test was coded and then populated in a table. Totals of codes for each category were calculated for all 45 learners. The same process was repeated for the post-test. Results of the tests are attached. To enhance comparing post-test errors the results were then put side by side.

4.5 Pre-test Tables and comments

4.5.1 Routine errors

Table 4.5.1 Routine errors in pre-test

	Question1	Question 2	Question 3	Question 5	Total
R_A	21	41	5	31	98
R_E	10	1	1	28	40
R_F	4	5	56	10	75
R_M	25	42	3	37	107
R_S	15	5	0	2	22
R_V	0	5	0	4	9
N	21	10	0	0	31
Total	96	109	65	112	382



From the total column of table 4.5.1 above it can be seen that the highest number of errors (107) is to be found in the subcategory of routine multiplication errors. Figure 4.5.1 helps to show the finding. That means the routine multiplication errors are more prevalent amongst the routine errors. Question 5 contributed a lot to these errors, and that does not come as a surprise since question 5 deals with multiplication and routine multiplication errors deals with multiplication errors. Question 3 contributes 3 routine multiplication errors as it deals with factorization.

The next higher total in the totals column is routine addition with 98 errors. The greatest contributor to the total is question 2. This does not come as a surprise as that question involves a lot of simplification after multiplication, and that simplification involves addition. Question 3 contributes the least number of errors to routine addition errors since it deals with factorization.

An important observation in the totals column is that the highest number of errors in this column is to be found in the first four rows. Each question contributes substantially to at least one of these 4 routine errors.

The totals row shows that question 5 has the most number of errors, and that is in line with the implication that question 5 contributes most errors to the routine errors. Question 5 contributes substantially to routine errors mainly because of the prevalence of routine multiplication errors and routine addition errors in this question. That means that multiplication, division distributive law, addition, subtraction contribute substantially to question 5 errors.

The contributions of routine factorization errors and routine substitution errors to question 5 are small due to the nature of the equation questions used in the test. Only one of these questions 5 (b) required factorization and so there could not be many factorization errors as the two other equations, namely 5 (a) and 5 (c) did not require factorization. None of the learners verified their answers by checking, and so there could not be errors introduced due to substitution.

The fact that N is zero for question five is simply a result of the equations chosen being algebraic.

4.5.2 Signifier, visual mediator, can't explain errors and blanks

Table 4.5.2 Signifier, visual mediator, can't explain errors and blanks

	Question 1	Question 2	Question 3	Question 5	Total
S	2	0	1	0	3
V_M	19	0	1	2	22
X	12	1	22	14	49
Blanks	13	6	5	20	44
Total	46	7	29	36	118

From the totals column of table 4.5.2 it can be seen that the greatest number of errors are to be found in the categories of blanks and can't explain errors. That means that blanks and can't explain errors are more prevalent as compared to signifier errors and visual mediator errors. From the totals row we see that question 1 has the most number of errors in the categories considered here, with visual mediator errors contributing substantially. That was to be expected as conjoining errors are related to substitution and this is a substitution question. Question 5 has almost the least number of signifier errors and visual mediator errors because of the nature of the questions selected. The questions did not require tables, graphs, flowcharts hence low visual mediator errors, and there was only one quadratic equation hence low signifier errors. However the presence of visual mediator errors cannot be ignored. The presence of signifier errors in two of the question means that those errors can't be ignored.

Considering the last column, we find that the category of blanks to be highest for question 3. That means that it would be expected that when solving equations that involve factorization, learners would most likely encounter more serious problems when it comes to the stage of factorizing, as compared to errors introduced by other categories considered in table 4.5.2.

The number of errors in multiplication and routine addition (respectively107 and 98) is in each case bigger than the total column in the table 4.5.2 and that means routine errors are relatively more prevalent than the categories considered here.

The equations questions had the most number of routine errors mainly routine addition errors and routine multiplication errors but the lowest number of visual mediator errors and signifier errors.

4.5.3 Comparison between routine errors and other errors

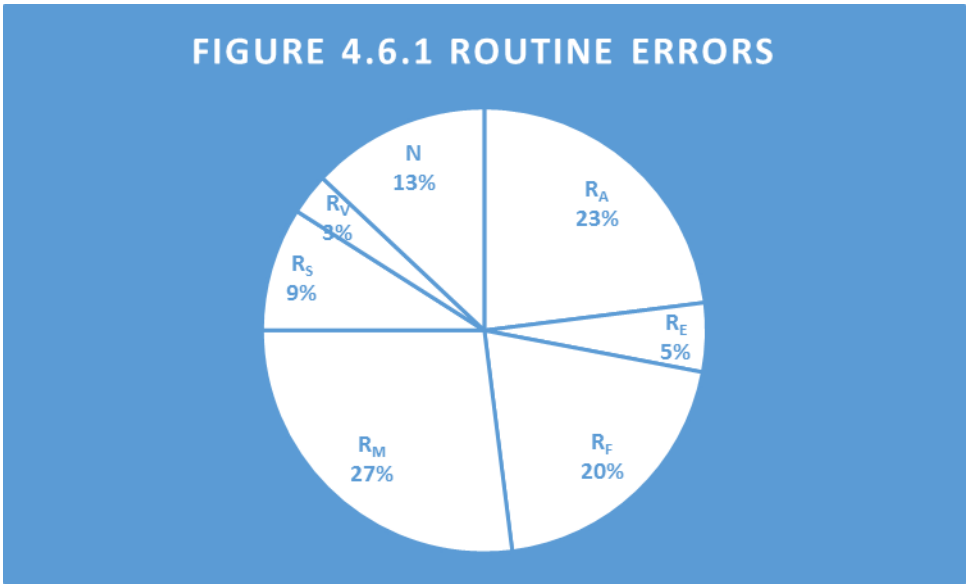
The fact that routine errors had a total of 258 and the group of all other errors had a total of 107 means that routine errors contributed more than twice the number of other categories combined meaning that routine errors are more prevalent. That means that errors in the routine categories impact on inefficient working out in all the questions investigated.

4.6 Post-test Tables and comments

4.6.1 Routine errors

Table 4.6.1 Routine errors in post-test

	Question1	Question 2	Question 3	Question 5	Total
R_A	9	30	4	22	65
R_E	3	1	1	10	15
R_F	0	3	46	8	57
R_M	6	34	1	36	77
R_S	18	3	0	3	24
R_V	1	3	0	4	8
N	32	4	0	0	36
Total	69	78	52	83	282



From the total column of table 4.6.1 above it can be seen that routine multiplication provides the highest number of errors in the total column. So routine multiplication errors are more prevalent amongst the routine errors. As in the case of the pre-test Question 5 contributes substantially to these errors, and that does not come as a surprise since question 5 deals with multiplication and routine multiplication is linked multiplication errors. Question 3 contributes no errors to routine multiplication errors as it deals with factorization.

Same as in the pre-test the next higher total in the totals column is routine addition that has 65 errors. Question 2 contributes substantially to these errors as it makes 46% of 65 which is almost equals to the sum of all errors made in other questions. That is not surprising since Question 2 involves a lot of simplification after multiplication would have been done. This simplification involves addition. Question 3 contributes the least number of errors to routine addition errors since it deals with factorization.

Biggest numbers of errors are located in the top four totals that involve routine addition errors, routine multiplication errors and routine factorization errors in the total column. These rows are high because of the contributions made by question 5 errors. The implication is that question 5 contributes substantially to routine errors and that errors linked with routine addition errors, routine multiplication errors, routine factorization errors, routine equation errors are more prevalent in question 5 since this question is about equations.

Like in the pre-test the totals row shows that question 5 has the most number of errors, and that is in line with the implication that question 5 contributes most errors to the routine errors. Question 5 contributes most to routine errors mainly because of the prevalence of routine multiplication errors and routine addition errors in this question. Thus

multiplication, division distributive law, addition, subtraction contribute substantially to question 5 errors.

The contributions of routine factorization errors and routine substitution errors to question 5 are small due to the nature of the equation used in the test. Only one of these questions 5 (b) required factorization and factorization errors could only come in question 5(b). None of the learners checked their answers, and so there could not be errors introduced due to substitution.

What is common about the pre-test and post-test is that Question 5 has a prevalence of routine addition errors and routine multiplication errors.

4.6.2 Signifier, visual mediator, can't explain errors and blanks

Table 4.6.2 Signifier, visual mediator, can't explain errors and blanks

	Question 1	Question 2	Question 3	Question 5	Total
S	1	0	1	0	2
V_M	1	0	0	0	1
X	6	3	12	6	27
Blank	3	2	6	32	43
Total	11	5	19	38	73

From the total column of the table above it can be seen that the greatest number of errors are to be found in the category of blanks. That means that blanks errors are more prevalent as compared to signifier errors and visual mediator errors. Visual mediator errors are almost nonexistent. From the totals row of table 4.6.2 we see that question 5 has the most number of blanks in the categories considered here and that question 5 is the last in terms of signifier errors and visual mediator errors. The questions did not require diagrams, tables, graphs, flowcharts hence low visual mediator errors, and there was only one quadratic equation hence low signifier errors. But the presence of signifier errors and visual mediator errors in other questions means that those errors are important and need attention too.

The number of errors in routine multiplication errors, 107 in pre-test and 77 in post-test are bigger than numbers in the other categories. The same argument holds for routine addition errors. So the conclusion is that routine multiplication and routine addition are the most prevalent errors. And these errors are mainly made in question 5. The equations questions had the most number of routine errors, mainly routine addition errors and routine multiplication errors but almost the least number of visual mediator errors and signifier errors.

From the last column of the table 4.6.2 it can be seen that the greatest number of errors are to be found in the category of blanks, using my categories. This means that blanks are more prevalent as compared to can't explain errors, signifier errors and visual mediator errors. Question 3 which dealt with factorization contributes a lot to these errors.

Question 2 contributed the least number of errors in this category and as the question involves multiplication, the implication is that at least most learners do not leave blanks or rewrite an equation when dealing with multiplication of numbers that do not form part of equations.

The category of can't explain is the next question where many errors were found when the last column is considered. Question 3 contributes mostly to this category and the implication is that equations question will have more challenges as compared to other categories in table 4.6.2 namely, signifier errors and visual mediator errors.

The last row shows that Question 5 which was based on equations as looked at in this research had the most number of errors due to blanks and can't explain errors. This does not come as a surprise as errors related to routine factorization, routine addition and routine multiplication collectively contribute to Question 5 being found to be challenging. Question 2 which was based on multiplication had the least number of the sum of errors in this category group.

4.6.3 Routine errors

The fact that routine errors had a total of 369 and the group of all other errors had a total of 146 means that routine errors contributed mostly to unsatisfactory performance as routine errors are more prevalent.

Routine multiplication errors yielded more errors than any other category, and that shows that errors related to multiplication and division contribute mainly to errors that are made when answering all four questions. As these questions are all related to solving equations it can be concluded that routine multiplication errors contribute substantially to errors that are made when solving equations.

4.7 Comparison between pre-test and post-test

What is common in the pre-test analysis and post-test analysis is that routine errors and in particular routine multiplication stands out to be contributing substantially to most errors in both the pre-test and the post-test. It is also clear that visual mediator errors and signifier errors in both post-test and pre-test are minimal.

The total routine errors from pre-test to post-test decreases from 382 to 282 The number of routine multiplication errors from pre-test to post-test goes down from 107 to 77. The number of routine addition errors from pre-test decreases from 98 to 65. The total number of signifier, visual mediator and can't explain and blank errors decreases from 118 to 73. The number of blanks from pre-test to post-test reduces from 44 to 43. So the trend is clear, the number of errors from pre-test to post-test tends to go down. However the errors do not disappear although the number of errors from pre-test to post-test goes down as learners modify and extend their own discourse from arithmetic to algebra.

Increasing substitution errors could be linked with the fact that there is no single learner who was seen to be checking answers by substitution and that means that the discourse that is related to working with substitution is not enhanced. The mere fact that arithmetic errors are increasing from pre-test to post-test means that extent to which errors made by grade 10 learners when dealing with linear equations stem from errors in arithmetic is substantial.

Table 4.7 Comparison between pre-test and post-test

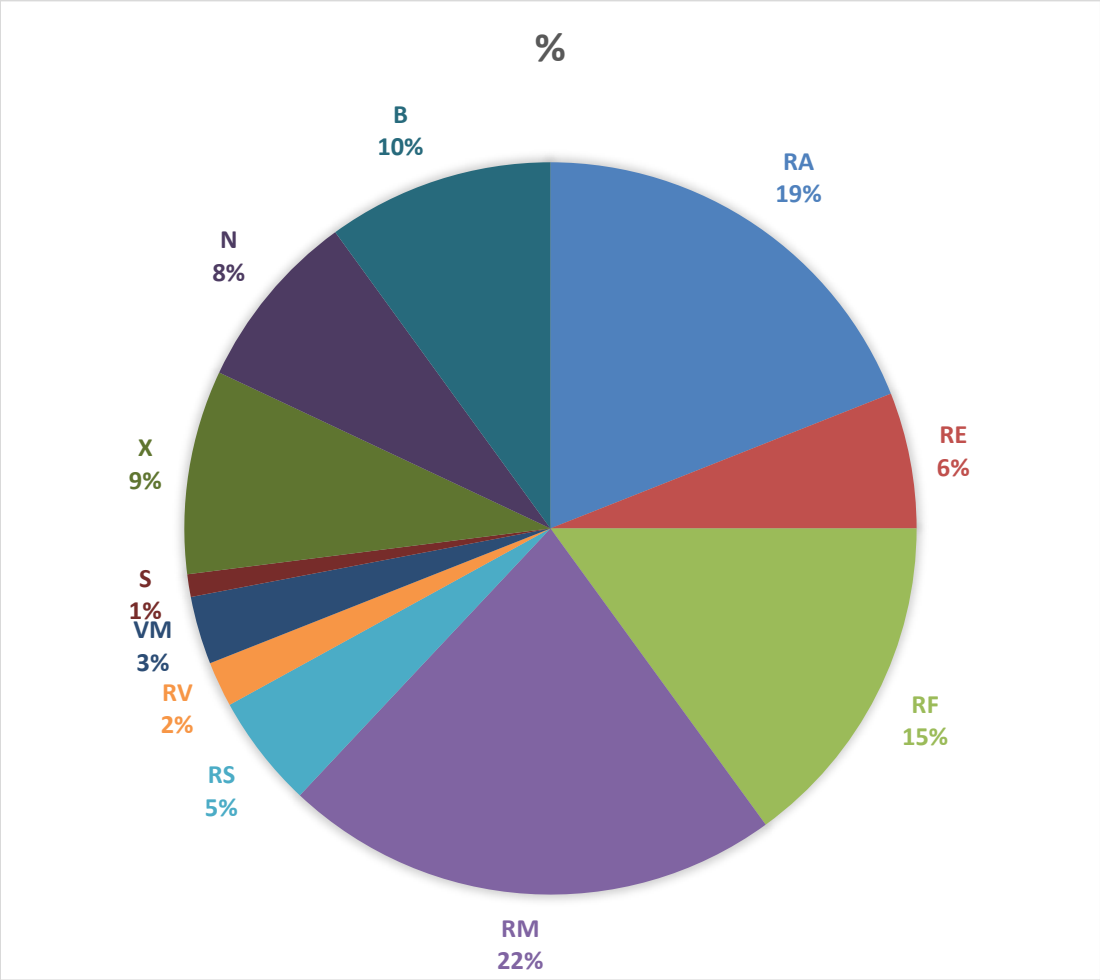
	R	R_A	R_E	R_F	R_M	R_S	R_V	V_M	S	X	N	B
Pre-test	382	98	40	75	107	22	9	22	3	49	31	44
Post-test	282	65	15	57	77	24	8	1	2	27	36	43
Difference	100	33	25	18	30	-2	1	21	1	22	-5	1

4.8 Pre-test and Post-test combined

Table 4.8 Pre-test and post- test errors as percentage of total errors made

	R_A	R_E	R_F	R_M	R_S	R_V	V_M	S	X	N	B	Total
Pre-test	98	40	75	107	22	9	22	3	49	31	44	500
Post-test	65	15	57	77	24	8	1	2	27	36	43	355
Sum	163	55	132	184	46	17	23	5	76	67	87	855
% of total sum	19%	6%	15%	22%	5%	2%	3%	1%	9%	8%	10	100

Figure 4.8 Pre-test and Post-test errors



From table 4.8 it is evident that routine multiplication errors, routine addition errors and routine factorisation errors contribute substantially to the total errors made in both the pre-test and the post- test scripts. The other errors do not disappear from pre-test to post-test.

From the table above we see that errors are mostly caused by inability to handle multiplication and division, denoted routine multiplication; then comes addition and subtraction – routine addition; and finally factorisation, that is routine factorisation. To a lesser extent, substitution rules and reducing an expression with two terms to one term, as endorsed by learners, show errors related to deviations from the real mathematical discourse. Discourse refers to as talked about in the mathematics community.

These enacted rules as endorsed by learners are not necessarily the same for the 45 learners. The rules endorsed by the learners are not the same as they do not make exactly the same mistakes. The example is others would write $2c$ to be $2 + c$ and others to be $2c$.

That there are different rules endorsed by the learners is supported by the fact that some routine multiplication errors revolve around multiplication, some around the distributive law and some around division. To some learners it would seem that multiplication to them must always yield a positive number as demonstrated by 4 by -2 to give 8. The learners also seem to have created their own meanings of what addition means as evidenced by some adding 3 and -5 to yield -2 and others writing 3 -2m to yield m.

As coding was done some scripts that seemed to be differing with others were kept aside for further analysis. The following section pays special attention to why these scripts are different.

4.9 Detailed analysis of some scripts with unexpected changes in errors

4.9.1 Unexpected changes

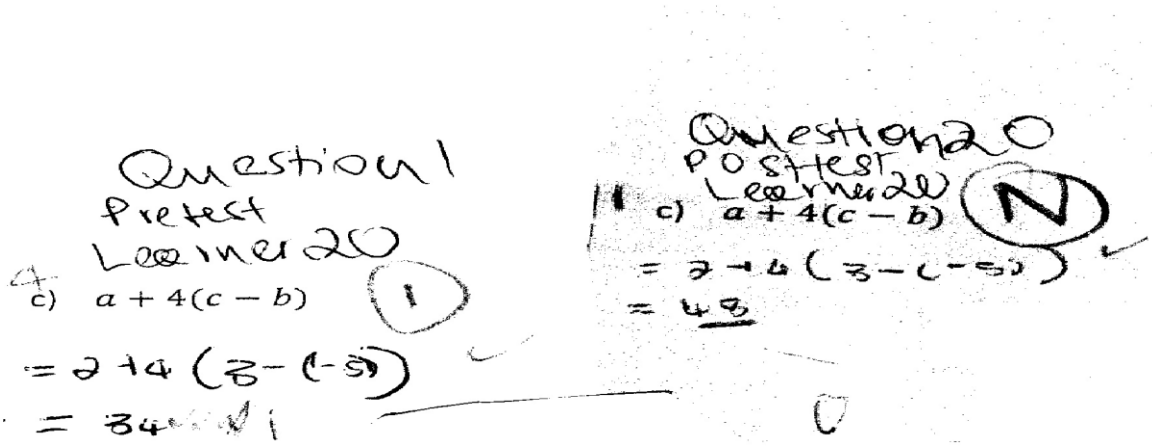
The scripts were coded such that a code is inserted in a circle alongside the question being coded. The mark on the right hand corner is the mark obtained after marking. The other marks on the right of the write up which will often be **0** or **1** or **M** represents the codes allocated by Wits Maths Connect Secondary**.**

A detailed analysis of some interesting differences between post-test and pre-test was done by looking at errors made by each of the 45 learners in the pre-test and in the post-test and that analysis is to be found in appendix 1.

In some cases a comparison between pre-test errors and post-test errors yielded more or less the same results although the number of correct answers tended to increase from pre-test analysis to post-test analysis. There were instances where there were marked differences between pre-test errors and post-test errors. There are also an instance that involved a right pre-test step which was not given a mark in the memo but was a blank in the post-test. Attention will be given to all these differences in this analysis.

Learner 20 made five routine errors in the pre-test as compared to eight in the post-test. It would be expected that the number of errors would decrease from pre-test to post-test. That the decrease did not happen makes this case to be interesting. There were three correct answers in the post-test compared to the pre-test which has two correct answers. The number of errors is bigger in the post-test. What is interesting about one of the correct pre-test answers is that the learners got the answer wrong in the post-test but right in the pre-test. This case will be discussed. Lastly in this section evidence of visual mediator and signifier errors will be discussed. The figure below shows the observation.

Figure 4.9.1 (a) Learner 20



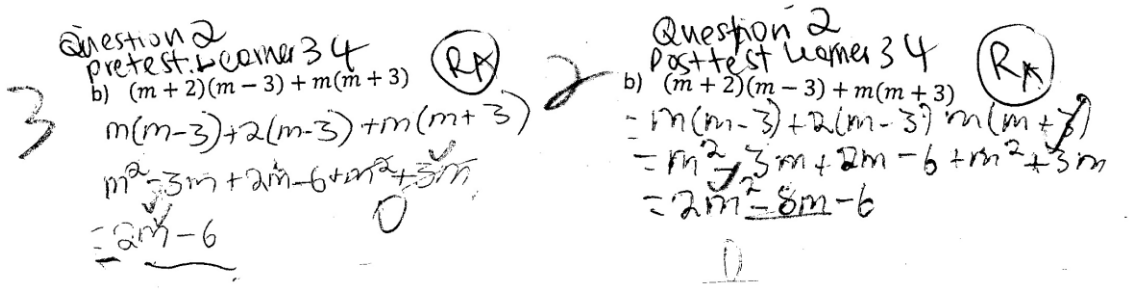
In the pre-test the learner wrote $a+4(c-b)=2+4(3-(-5))$.The expression was then simplified to 34 which is correct. In the post-test the learner wrote

$$a+4(c-b)=2+4(3-(-5)).$$

That is exactly the same as what was done in the pre-test. The difference is that the expression then became 48. So the order of operations was wrong, in this case to get 48 one would have to add 2 and 4 to yield 6. The result would then be multiplied by 3 + 5=8 to yield 48. What makes the solutions different is that a question that was done without errors in the pre- but became a challenge in the post-test. It became a challenge as the solution in the post-test is not correct. This may suggest that the order of operations is unstable for this learner.

Learner 34's marks also show an anomaly in moving from pre-test to post-test. The solutions in figure 4.9.1 (b) will be used to show the anomaly. Learner 34 made14 errors (13 routine errors and one blank) in the pre-test and a total of 14 routine errors in the post-test.

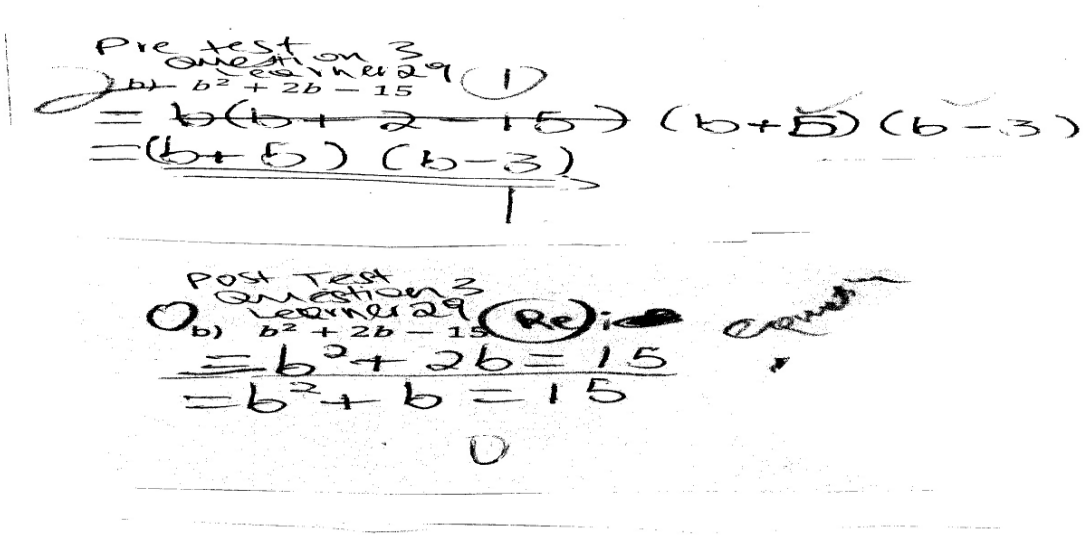
Figure 4.9.1 (b) Learner 34



There is a good mathematical communication around multiplying out two binomials and then simplifying to find an acceptable answer as endorsed by the mathematics community in the pre-test. $(m+2)(m-3)$ becomes $m^2-3m+2m-6$ and $m(m+3)$ becomes m^2+3m. Addition of the terms yield $2m-6$. However, in the post-test, after applying the distributive law correctly, incorrect addition of algebraic terms is done to introduce an error. We also have that phenomenon of a correct solution in the pre-test and partially correct solution in the post-test.

The next case to be considered will be learner 29 as shown in figure 4.9.1 (c) below. There are two errors (one routine error and one can't explain error) for learner 29 in the pre-test and 8 errors (four routine errors, two blanks as well as two can't explain errors) in the post-test

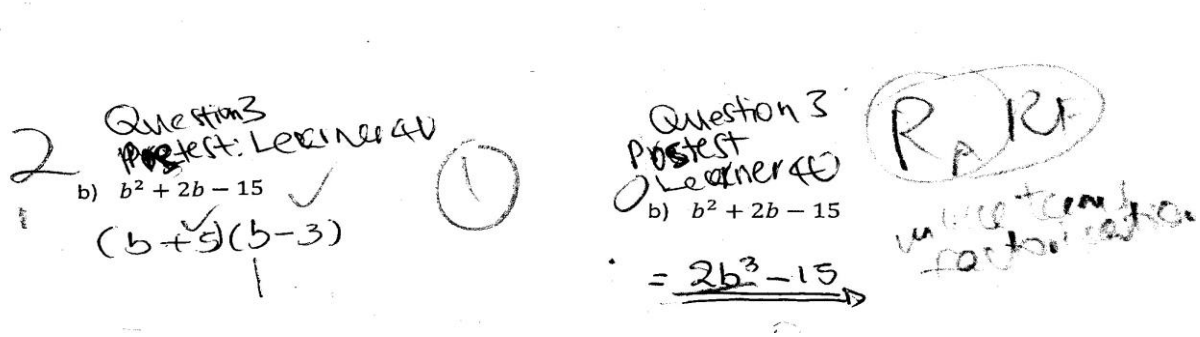
Figure 4.9.1 (c) Learner 29



Learner 29 made a routine factorization error and can't explain error in the pre-test. In the post-test four routine errors, two blanks and two can't explain errors were made. In the pre-test the learner factorized $b^2 + 2b - 15$ to yield $(b+5)(b-3)$ and that is a correct response. In the post-test, the learner factorized the same expression to get $= b^2 + 2b = 15$ and then wrote = $b^2 + b = 15$ The mere fact the learner introduced an equals sign, and in fact two when only an expression was given, suggests that the difference between an expression and an equation was not clear to this particular learner.

The meaning of an equals sign is not known to the learner hence a trend of magically introducing this sign when an expression is given is seen. Writing 2b as b could have been a slip and not knowing the difference between the two terms. Another case will be illustrated by considering learner 40's solution in the figure that follows. Learner 40 made ten pre-test errors (nine routine errors and one can't explain error) and 12 post-test errors (ten routine errors, one arithmetic error and one blanks).

Figure 4.9.1 (d) Learner 40



As learner 40 extends his/her discourse the modification is not smooth as seen in the workout above. What is not smooth is the poor recognition of the fact that b^2 and 2b represent different symbolic expressions and cannot just be added as they are unlike as done in the post-test. Here the error is adding b^2 and $2b$ to yield $2b^3$.

What is clear from Q1, Q2 and Q3 discussion is that in the post-test, after applying the distributive law correctly, incorrect addition of algebraic terms is done to introduce errors and that poor factorization as well as accompanying errors are a problem. Now all these tools are necessary in handling equations. Attention will be given to linear equations to determine the compounding effect of these tools in equations. The tables below will be used to bring matters into a better perspective:

Table 4.9 (a) Blanks and correct answers pre-test

	Question 1	Question 2	Question 3	Question 5	Total
0	13	6	5	20	44
1	36	11	15	16	78

Table 4.9 (b) Blanks and correct answers post-test

	Question 1	Question 2	Question 3	Question 5	Total
0	3	2	6	32	43
1	116	26	28	30	200

The number of 200 correct answers in the post-test compared to the number of 78 in the pre-test shows a marked improvement in getting answers right. The drop in blanks (represented by zero) from pre-test to post-test means that there is a decrease in the number of errors, although not that substantial as the difference is only one. This means that the errors do not disappear.

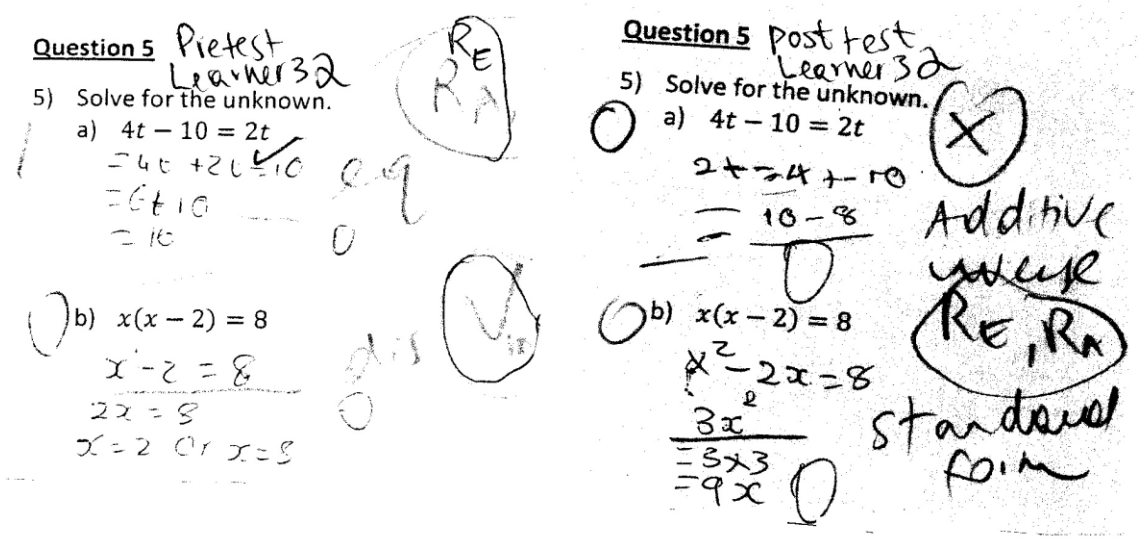
So going from post-test to pre-test we still have a substantial number of errors and question 5 contributes substantially to these errors. The reason Question 5 contributes substantially is that it has most errors in each case as compared to other questions. It seems the compounding effect of errors made in other questions relate to the many errors made in equations. So question 5 which is mainly algebraic and deals with linear equations will then be given more attention to determine the compounding effect.

What is emphasized here is that as learners modify and extend their discourses from working with arithmetic and basic algebra to handle linear equations, errors do not disappear in the learners' discourse. This is shown by the substantial number of errors made when equations are tackled. It is hoped that more focus on this question which is algebraic will give clues about the shift from arithmetic to algebra. The question will be called linear equations as even the one quadratic equation given has two linear equations after factorizing and employing a multiplication $ab = 0$ implies $a = 0$ or $b = 0$. Now in dealing with $a = 0$ and $b = 0$ linear equations are dealt with in this particular case.

4.10 Linear equations

Looking at how learner 32 performed in the pre-test and in the post-test we see interesting findings. Learner 32 made 11 routine errors, one can't explain error and one visual mediator error in the pre-test and made four routine errors and two blanks in the post-test. Here clearly the number of errors decreases from pre-test to post-test. Figure 4.10 (a) below illustrates errors made by learner 32:

Figure 4.10 (a) **Learner 32**

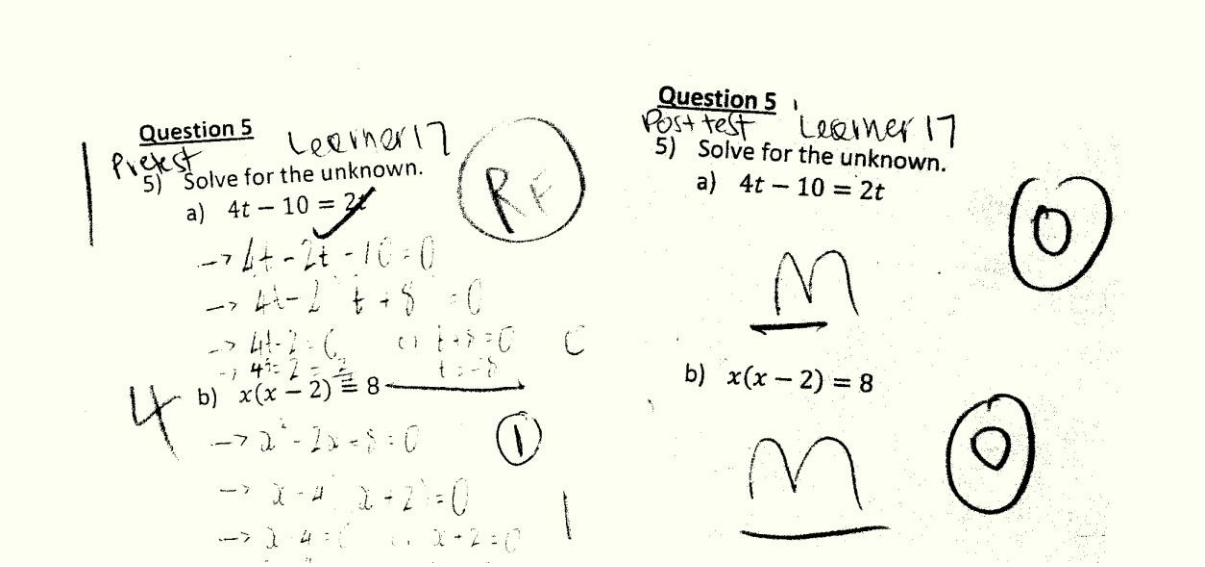


In question 5(a) we see picking the term 2t from the right hand side, and moving the terms to the other side of an equation introducing errors. Equivalence is not employed when in the third step the equation becomes an expression with the given equals sign disappearing. The concepts of equivalence and inverse is not known. The post-test performance shows no improvement. So the errors from pre-test to post-test as learners try to modify and extend their discourses do not disappear. The mark from pre-test to post-test drops as this question was answered well in the pre-test but is not well answered in the post-test.

In question 2(b) we see the only visual mediator in the scripts. Here $a \times b = 0$ implies $a = 0$ or $b = 0$ is employed wrongly as the equation does not have zero on the right hand side. This helps to justify the existence of visual mediators. In the post-test the learner shows an improvement in the ability to multiply using the distributive law but does not understand the importance of writing a quadratic equation in standard form and also does not see the importance of the equals sign. The purpose of discussing this question is to show the presence of a visual mediator.

Our next discussion is about learner 17. What is interesting about this learner is that in two questions a pre-test answer is given some marks but in the post-test there is no mark given. That is arrived at by looking at question 5(a) in which a mark is given for pre-test and for post-test there is no mark given. An error of one blank is caused by the fact that the learner does not write anything and so we cannot have two or three errors. Another example is 5(b). In the case of 5(b) the learner demonstrates ability to use the additive inverse in the pre-test and yet the same skill seems to disappear in the post-test.

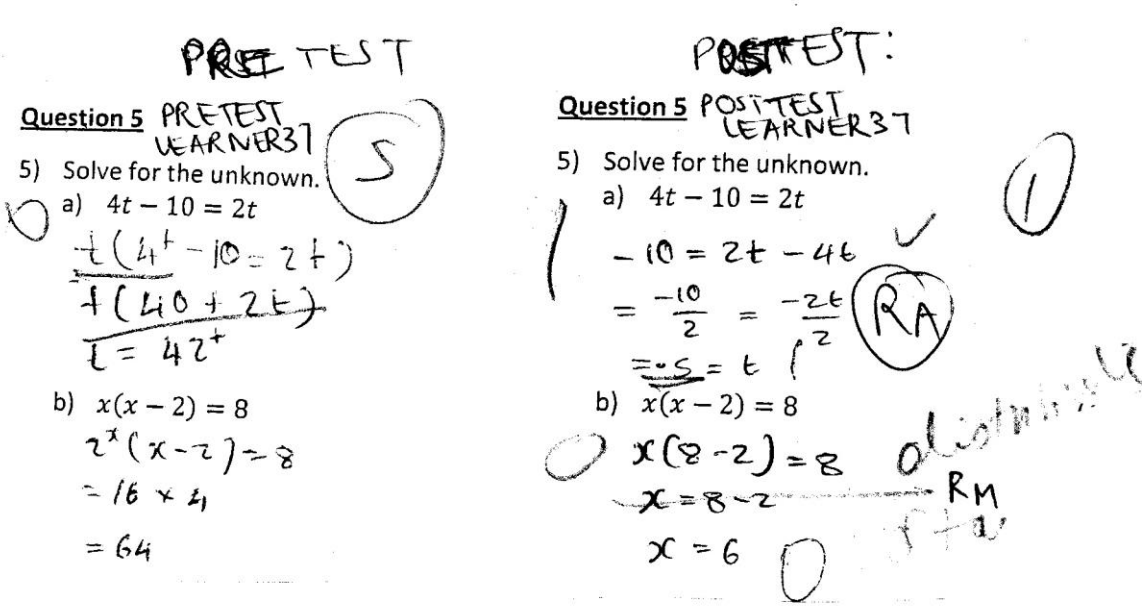
Figure 4.10 (b) **Learner 17**



In the pre-test the learner solved for the unknown in $x(x\text{-}2) = 8$ as follows, $x^2\text{-}2x\text{-}8 = 0$, $(x\text{-}4)\ (x\text{+}2) = 0$ and got an answer of $x = 4$ or $x = 2$. So the pre-test solution is correct. However in the post-test the same learner left a blank for the same question. It will be interesting to find out why this phenomenon occurred. What is clear though is that the learner no longer knows how to solve the problem. The modification of existing tools could only lead to blanks for the two scripts of learner 17. As blanks are errors then it means that errors from pre-test to post-test do not just disappear.

Learner 37 made a considerable number of routine errors and some signifier errors- the highest number of signifier errors made by a single learner. But those are to be found in the pre-test only where 2c becomes 2^c, 3b becomes 3^b and 4t becomes 4^t.

Figure 4.10 (c) Learner 37

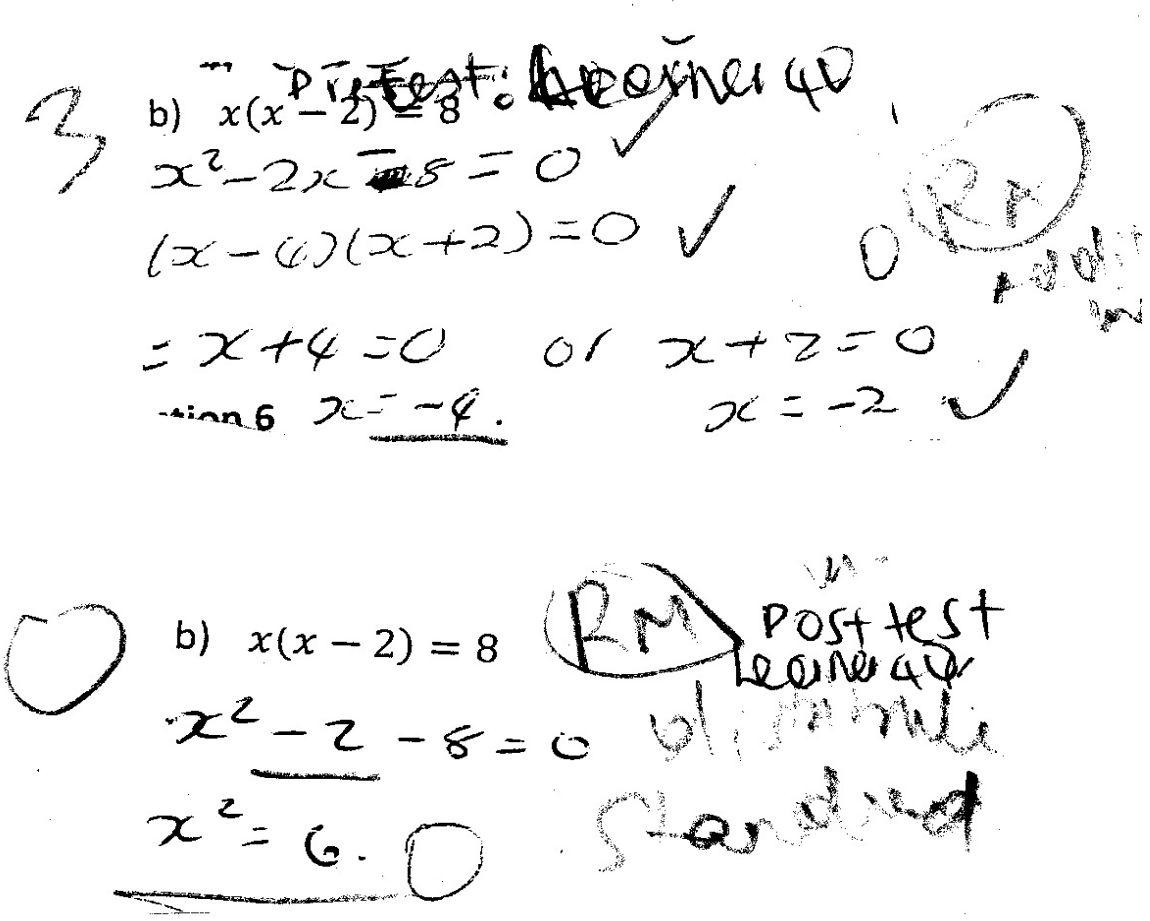


The learner here is struggling to move from a unique discourse of exponents to a discourse of dealing with algebraic terms. The discourse of exponents is so tenacious that it simply does not disappear as a shift to working with other algebraic discourses happen. The script has been specifically selected to show the presence of signifiers.

Another interesting case involves learner 40. There were zero correct answers in the post-test compared to two in the pre-test for learner 40. What is interesting about the two correct pre-test answers is that the learner get these answers wrong in the post-test.

The working out of the learner is shown below.

Figure 4.10 (d) Learner 40

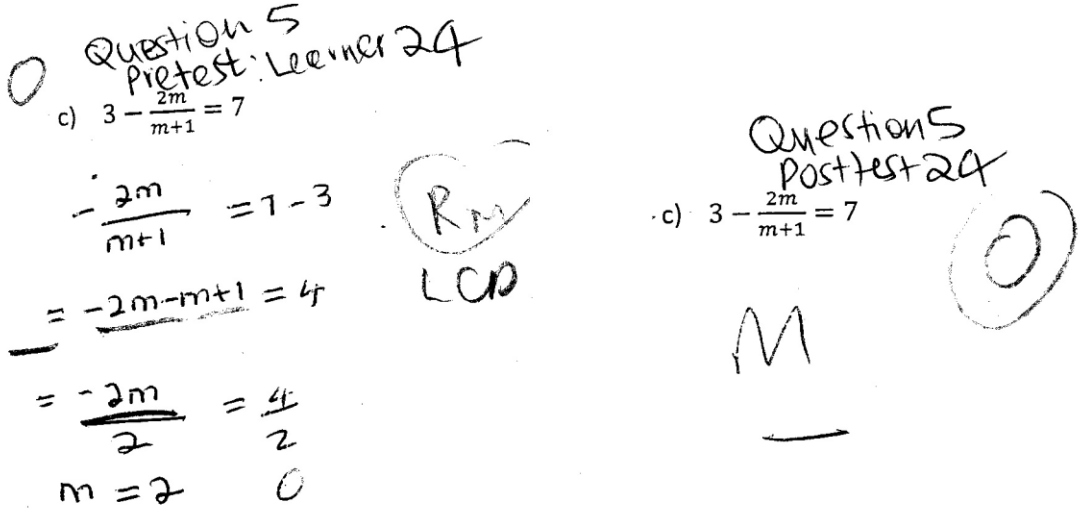


In the pre-test the learner solution is correct up to a point where the answer must be obtained by merely using the additive inverse. Employing the distributive law and writing the equation in standard form, factorizing the trinomial obtained and employing $a \times b = 0$ implies $a = 0$ or $b = 0$ are all done well in the pre-test. In the pre-test what comes out clearly is that as the learner modifies his/her own discourses the road is not smooth. Errors are made when using the distributive law resulting in an equation that involves a square root and is more complex to the learner as it is not like just finding factors by inspection. My classroom experience has taught me that few learners would be able to see that $x^2 - 6 = (x+\sqrt{6})(x-\sqrt{6})$ so old habits of thinking of 6 in terms of a perfect square would just linger on, in trying to extend their discourse to numbers that are not perfect squares. A question that could be solved by a learner in the pre-test becomes difficult to solve in the post-test.

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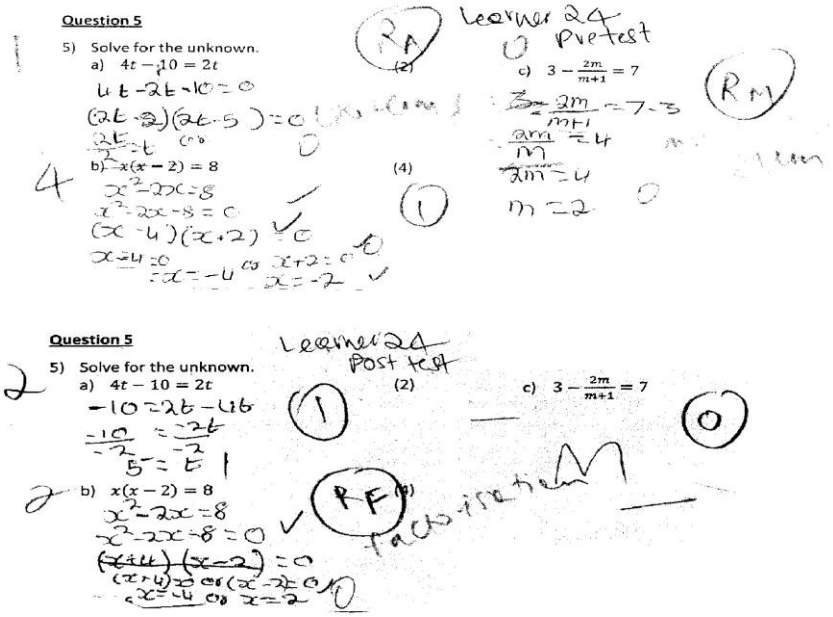
Learner 24 also helps us to generalise about what happens as learners try to modify and extend their own discourse from just working with numbers to working with algebraic terms and generalisations.

Figure 4.10 (e) Learner 24



In the pre-test the inability to deal with a fraction by using the distributive law is observed. The fact that a fraction has a numerator and denominator is clearly not dealt with the correct way by the learner. This is evidenced by combining the numerator of -2m and denominator of m+1 to yield -2m-m+1. The symbols that are part of the fractions seem to be an impediment. The post-test solution is just blank indicating that errors do not just disappear as a blank is an error here. The reason the question was chosen is to show that there are instances whereby an attempt is made in the pre-test but in the post-test there is no attempt done. The learner could at least add an additive inverse to both sides of an equation in the pre-test but can't do that in the post-test.

Figure 4.10 (f) Learner 24



What is interesting about learner 34 is that in the pre-test question 5(b) correct factorization is employed in solving the equation $x(x\text{-}2) = 8$ to yield $(x\text{-}4)\ (x\text{+}2) = 0$ and in the post-test an error in factorization yields $(x\text{+}4)\ (x\text{-}2) = 0$. That is indicative of a shift in dialogue not yielding desired results. A mark for factorization is obtained in the pre-test but not in the post-test.

The general finding here is that the shift from arithmetic to algebra is not smooth. The mathematical concepts that are being talked about like errors discussed above arise as by-products of the ongoing mathematical discourse. Extending the existing assortment of routines and endorsed narratives that currently exist in the learners mathematical practices has its challenges. The lingering old habits do not just become automatically adapted. Routine errors, Visual mediator errors and signifiers do not just disappear. Their numbers are reduced but they definitely do not disappear.

So errors made by grade 10 learners when dealing with equations stem from errors in arithmetic is quite substantial due to these lingering old habits of errors discussed here not disappearing.

The extent to which errors made by grade 10 learners stem from errors in basic algebra is also quite significant as adaptation of preexisting assortment of mathematical discourse to a new assortment of discourses does not occur without its own impediments.

4.11 Conclusion

In this chapter it was shown how some few innovations are added to the study done by Brodie and Berger (2010) to analyse errors. The innovations mainly included marking the test and dividing routine errors into subgroups to have further insights about errors made by learners so that working with them can be enhanced. Getting a teacher to mark and a teacher to code were innovations that were meant to strengthen reliability of the study.

Marking was done so as to aid the discursive framework in terms of informing the codes. This was enhanced through looking at performance of the learners in the pre-test and in the post-test.

The challenges encountered by learners of making errors when dealing with arithmetic and basic algebra were found to some extent to manifest when the marks were analysed. This informed the subcategories that were used in the discursive approach. Analysis of the codes confirmed the presence of the three different types of errors, with routine errors being prevalent. So the difference was the degree of prevalence. This is, however, expected as the weighting of the questions were different and the nature of the questions differed.

The question of making progress (as evidenced by the correct response) or lack thereof in the shift (as evidenced by a blank) from pre-test to post-test and hopefully also from arithmetic to basic algebra also informed the choice of categories. The two situations respectively led to the codes of 1 and 0.

The analysis started with investigating concepts like substitution, basic operations, order of operations and factorization. The relationship of the concepts mentioned with basic algebra led to a more focused attention on linear equations. The general finding here is that the shift from arithmetic to algebra has its own impediments.

The mathematical errors arise as by-products of the ongoing mathematical discourse. Extending the existing assortment of routines and its challenges. The lingering old habits do not just become automatically adapted. Routine errors, visual mediator errors and errors do not just disappear. Their numbers are reduced but they definitely do not disappear. So the extent to which errors made by grade 10 learners when dealing with equations stem from errors in arithmetic is quite substantial due to these lingering old habits of errors discussed here not disappearing.

The extent to which errors made by grade 10 learners stem from errors in basic algebra is also quite substantial as adaptation of pre-existing assortment of mathematical discourse to a new assortment of discourses does not does not occur without impediments.

The implications of the findings mentioned will be discussed in the next chapter. Limitations of the study will also be discussed.

Chapter 5 Conclusion

5.1 Introduction

This study has explored errors in basic algebra, arithmetic so as to inform linear equations errors using a discursive approach. The literature review was initially to aid the exploration of determining whether the errors were related to equations. The commognitive and discursive frameworks were selected as a preferred framework to aid the analysis.

The literature review pointed to issues in basic operations, substitution, multiplication and factorization being important in analyzing the relationship between arithmetic, basic algebra and linear equations. The discursive approach informed the categories used to carry out the study.

Marking of the analyzed scripts was used as an innovation that could, together with literature review, strengthen the discursive categories of Berger and Brodie (2010): routine, visual mediators and signifiers. It turned out that dividing routines into subcategories was helpful. The research problem was that that learners perform poorly in linear equations in grade 10, and the research questions were:

- To what extent do errors made by grade 10 learners when dealing with equations stem from errors in arithmetic?
- To what extent do errors made by grade 10 learners stem from errors in basic algebra?

The data for the study was made out of Grade 10 learners' responses to a pre-test and post-test. It is part of a much larger dataset collected by the Wits Maths Connect Secondary Project (WMCS) in 2013. Although the learner responses had already been analyzed by the WMCS at the level of correct/incorrect, this analysis focused on the errors themselves. In developing marking memo the allocation of marks in the test was adapted to give attention to issues raised in the literature review. Those issues include errors made by learners when working with equivalence, basic operations and factorization. The study analyzed in more detail the errors of one class (N=45) selected from the larger data set. The class was selected as a typical case.

Codes were developed and used to code the scripts according to the three categories of routine, visual mediators and signifiers, and the notion of dividing routine errors into subcategories. Initial coding of 10 scripts resulted in a necessity to divide the category of routines into sub categories, as that category had the most errors. All 45 scripts, including those 10, were coded. Multiple coding of at most three errors per sub question was used to accommodate learners making multiple errors in a single sub-question. Since it was anticipated that they would not get a question correct after three errors, multiple coding was limited to at most three errors.

The patterns in errors made in the pre-test which was written in February were compared to the patterns of the post-test which was written in October. Further algebra would have been done by the learners in that time, so to some extent clues about the shift from arithmetic to basic algebra could have been obtained. It has to be emphasized here that the shift starts as early as when learners are in grade six.

5.2 Summary of main findings

5.2 1 The shift from arithmetic to algebra

The general findings were that the shift from arithmetic to algebra is not smooth so the extent to which errors made by grade 10 learners when dealing with equations stem from errors in arithmetic is substantial, due to these lingering habits not disappearing.

The errors that related to equations arose as by-products of the ongoing mathematical discourse. Extending the existing assortment of routines and endorsed narratives in the learners' mathematical practice has challenges that involve errors.

The extent to which errors made by grade 10 learners stem from errors in basic algebra is also substantial as adaptation of the existing assortment of mathematical discourse to a new assortment of discourses does not does not occur without impediments.

5.2.2 Learner performance

The study has shown that only one learner in the pre-test did not answer all the questions, and this means that all the learners attempted at least one question in the pre-test. One learner 2% of the 45) got every answer correct in the post-test; all of the others made at least one error. Thus 100% of the learners in the pre-test, and 98% in the post-test made errors.

Question-by-question analysis was done for further understanding of the errors. The approach was to determine whether the type of problem and performance in the particular question tells one about what could be anticipated to inform classroom practice.Question 5 was discussed in detail, as it is at the core of this study. 5 (c) was least answered in the pre-test, and all the post-test question 5 sub questions had most zeros as compared to other questions.

The low pass rate in 5 (c) in both pre-test and post-test, as well as an insignificant improvement in performance, means that this question requires serious attention

The three representational expressions of question 1 showed that translating symbols to numbers and understanding the order of operations was a challenge, as evidenced by the errors. The importance of the question is its helpfulness in determining the transition from just working with numbers to seeing that symbols are used to represent numbers. This skill is useful in checking whether the left hand side of an equation is equal to the right hand side by putting numbers in the place of symbols. Errors made here are indicative of progress (or lack of progress) made in the transition from numbers to abstraction that involves symbols.

5.2.3 Learner errors

100% of the learners in the pre-test, and 98% in the post-test made errors, that point was argued in the previous section. So the errors are significant. This tendency to make errors, and deviating from practices that are endorsed by the community of mathematics, cannot therefore be ignored. It could provide clues about why there are problems in the learning and teaching of mathematics, and about how to work with those errors.

The most common errors, namely, routine multiplication errors that have to do with multiplication, division and multiplicative inverses and the distributive law and routine addition errors that have to do with addition subtraction and additive inverses, suggest that there are serious problems involving multiplication, division, the distributive law, addition and subtraction of algebraic expressions. It is clear that the repetitive patterns used by each learner in their discourse as they engaged with the questions were mainly influenced by basic operations in algebra and arithmetic.

5 (c) was least answered in the pre-test, and all the post-test question 5 sub questions had most zeros as compared to other questions. This meant that the errors were not as high as had been anticipated, as the learners were not writing - although the number of errors made is sufficient. The number of errors means that learners are having problem with this question. So these errors are necessary in informing interventions.

From the tables below it can be seen that equation stem mainly from routine errors.

Table 5.2.3 (a) Pre-test errors

N	S	V_M	X	O	R_M	R_A
31	3	22	49	44	382	98

Table 5.2.3 (b) Post-test errors

N	S	V_M	X	0	R_M	R_A
36	2	1	27	43	282	65

N = Arithmetic errors, S = Signifier errors, V_M = Visual mediator errors,

0 = Blanks, R_M = Routine multiplication errors.

These tables show that routine multiplication errors and routine addition errors are the most prevalent in both pre-test and post-test. Factorisation should also be given sufficient attention, as it is the third-most prevalent category in both pre-test and

post-test, as shown in the tables above. Arithmetic and substitution errors should be next.

These arithmetic errors require attention, as they would impact negatively in dealing with algebraic equations, as algebraic equations also involve numbers

5.2.4 Learner performance and errors

The initial analysis shows that students' prior knowledge of arithmetic and basic algebra seem to have impacted negatively on how they solved equations and simplified expressions. The high number of errors vindicates this.

5.2.5 Findings that relate to linear equations

The global equation $4t - 10 = 2t$ seemed challenging, as the equation involved more than additive inverse. But even working with additive inverse on both sides was not easy for some learners, as one of their errors was to add an additive inverse on only one side of the equation; that is, they did not do the same thing on both sides of the equation. Others just performed operations on the symbols that enabled them to mentally pick up symbols from one side of an equation to the other side of the equation, and did not note that on the other side its sign would be negative.

The transformational equation $x(x-2) = 8$ was included as it involves dealing with solving linear equations, and links up with well with the visual mediator example used by Brodie and Berger (2010). The fact that only one learner made the visual mediator of putting $x = 8$ or $x-2 = 8$ indicates that the problem is not rife at this level. Yet, although visual mediator errors were not made, other types of errors were made. Changing the form of an expression to an equivalent one introduced errors of multiplication, factorization and addition.

The transformational equation $3 - \frac{2m}{2m+1} = 7$ provided evidence that the ability to deal with fractions in equations was a challenge. Errors included learners ignoring the $m+1$ in the fraction $\frac{2m}{m+1}$ and treating the fraction as $2m$. Other errors include inability to find the correct lowest common denominator, ignoring the minus sign and not following the order of operations when simplifying. So equation errors in this case stem mainly from routine errors.

There was a tendency for the learners to construe an equation as a command to execute what is on a certain side of that equation, and not to mean that the left hand side of that particular equation is equal to the right hand side. This means that the role played by an equal sign, as endorsed by the mathematics community, was not well understood by a considerable number of learners.

Another explanation could have been an inability to accept that expressions are complete algebraic objects and do not require one to add a 'magical' equal sign (and sometimes even a zero) when that is not given. The fact that there is not a single learner who checked the answer(s) by substituting the answer(s) in the given equation could have led to some learners not realizing that they are dealing with two items that are connected by being equal. They would also not have seen that the equals sign is meant to connect the items because they are equal. They would have struggled to modify and extend their mathematical discourse. An important fact emphasized here is that equations form an important part of algebra, and are not only about symbols. Although symbolization is important in the development of algebra, it is only a part of algebraic processes.

The fact that arithmetic errors tended to be more pronounced in the substitution question suggests that arithmetic error could creep in when checking the answers after solving linear equations, as that involves substitution. Multiplication and addition errors tended to be more pronounced in the multiply and simplify question. That should be anticipated and planned for when linear equations are tackled. Factorization errors were more pronounced in the factorisation question. These errors should be anticipated and planned for when dealing with the part of linear equations that require factorisation.

The fact that R_E constituted 10% of the routine errors in the pre-test and 6% in the post-test means that this error also impacted negatively in a smooth transition from numbers to solving an algebraic equation. This cannot be ignored as, in some instances, some learners changed equations to expressions by dropping the equal sign. In some cases, some learners introduced an equal sign when dealing with an expression.

Identifying the errors so as to inform practice is necessary, as in the transition from arithmetic to algebra learners make adjustments which could be related to these errors. The questions are important in understanding the adjustment from just working with calculations to giving attention to their representations as mathematical objects.

The importance of the representation is that it has equivalent forms, and those are important in dealing with equations. Errors related to, for instance, wrong order of operations, such as adding a and 4 before multiplying 4 by c- b indicate whether there is a smooth transition from just calculating to transforming to equivalent forms.

The rule-based Question 2 is mainly concerned with changing the form of an expression to obtain an equivalent one. These transformational expressions were included as they involve changing the form of an expression to an equivalent one using rules for manipulating algebraic symbols. However, the actual shift from

process to object seems difficult for many students, as many errors occurred in finding an equivalent expression.

The ability to use the distributive law was compromised by mainly routine errors when simplifying expressions into equivalent ones. These questions are more cognitively demanding as they do not have numeric answers, and are meant to test whether learners are making a transition towards working with abstraction in terms of symbols. Possible errors, which include the addition of unlike terms after application of the distributive law and applying the distributive law incorrectly, provided evidence that prioritizing these errors in an intervention agenda is necessary in "modifying and extending" discourses.

The rule-based Question 3 provided evidence that errors are made in communication of transforming expressions into equivalent ones using the rules of factorisation. The transformational expressions were included as solving equations often involves transforming an equation written in standard form into a product, using factorisation. In applying rules of factorisation to write this transformational equation into an equivalent one the errors made provide evidence that manipulating algebraic symbols is not a smooth procedure.

The minus sign in a) made the problem slightly more difficult, as learners encountered difficulties in noting that +15 could be written as -(-15). So the process of factoring out the minus sign was not effective. In Question 3 (c) ignorance of finding factors led to not allocating the minus sign to 3. The minus sign in 15 makes the question slightly interesting because of the allocation of the minus sign to factors, but the question mainly tested procedure. So inability to deal with factors which could be seen to be stemming from errors in basic algebra has a negative impact on dealing successfully with equations. The role played by factors in equations is what introduces errors if there is no mastery of the discourse of finding right factors.

The fact that the research had been done some years previously made it difficult to get further information from the school as learners and also some teachers had left the school. So I chose to focus on the scripts, as that was the only way explore performance of that cohort of learners.

5.3 Limitations

The current study has a limitation that it is concerned with the existence of errors, and does not investigate what could have led to the reasons. An analysis of classroom practice that involves teacher and learner could be helpful in providing further reasons. This analysis could provide further reasons responsible for these errors, and enhance further classroom planning, strategizing and delivery of mathematics teaching.

An introduction of special symbols, word problems and a consideration of graphical equations and equation problems linked to diagrams and a table of values could assist in providing further information about visual mediator errors.

5.4 Recommendations for teaching, planning and further research

5.4.1 Accommodating learners' different backgrounds

For a smoother transition learners need to focus on the processes that lead to a smooth transition from the level that involves numbers to a higher level that involves algebra and its abstraction, without being distracted by the demands of number operations.

Based on the fact that learners from different backgrounds make different kind of errors it is necessary to accommodate them. The errors should be accommodated so that learners can be given a chance to reflect on their experiences and build on them.

Attention should not only be given to processes that have been shown to be part of arithmetic processes, but also to algebraic structures.

Clearly, an adjustment that involves a discourse that is not limited to just shifting symbols from one side of the equation to the other is required. The process of doing the same thing on both sides of the equation so that equivalence is maintained needs to also be part of the discourse.

The fact that the process could be applied several times also needs to be emphasized in interventions. It is suggested that such a mathematically endorsed discourse could contribute towards a smoother transition to modifying and extending discourses of learners from doing arithmetic to operating in algebra.

Manipulatives could be helpful in modelling algebraic structures for learners who prefer hands on learning. Functional machines could also be helpful. Some computer spreadsheets could help learners in setting up equations.

5.4.2 Giving students an opportunity to work with functions and tables

Giving learners an opportunity to work with functions and tables when doing equations could be helpful, as they have been seen to be not using visual mediators when tackling equations. Evidence was provided by the lack of tables and graphs in responding to questions on equations.

The notion of functions could also be used to introduce equations. This idea ties in with gradually increasing the learners' participation, so that they are not left behind as mathematics will then be phased in gradually.

5.4.3 Planning

The fact that R_M (multiplication and division errors), R_A (addition and subtraction errors) and R_F (factorisation errors) were most prevalent amongst routine errors means that that these require prioritisation in interventions and planning. (That does not mean that the other errors should not be incorporated in planning and intervention. The point made is about prioritisation).

Prioritising is enormously important in planning for school interventions, as there is often a competition of time allocation due to time constraints.

Interventions for addressing highlighted errors is important so as to have sufficient dialogue about them. Practices such as lesson plans and preparation could then be structured such that prioritisation is given attention. Confronting errors picked up in material development and assessment could also be helpful.

However, school teaching should not be the only intervention. Learners could have extra lessons, or their parents could play a greater role by ensuring that homework is done by their kids. All these should be informed by the errors made by learners.

The fact that learners struggled with question 1 (substitution) and question 2 (simplify and multiply), as well as question 3 (factorization) means that the skills tested in these questions could not be used in the question on equations, which require mastery of these skills. Teachers must work towards ensuring that the knowledge gained in engaging substitution, multiplying and factorization is transferred to working with equations.

Mathematical discourse needs to be enhanced though gradual introduction of concepts so that learners are provided with opportunities that enhance their ability. In establishing the equivalence of expressions, and using a function as an object to use visual mediators like graphs and tables in dealing with equations, learners could improve their understanding.

Too much emphasis on symbols without understanding led to many of the errors seen. Emphasis should be given by teachers to reading numerical and algebraic operations. They need to resist emphasizing symbols without giving attention to understanding the relations it represents. ICT needs to be utilized in providing a new window for understanding differences in symbols.

5.4.4 An opportunity for further study

There were errors that could not be explained using my codes, and that is an opportunity for another study. Other categories in the discursive framework, namely visual mediators could be broken up into subcategories to get deeper into errors related to basic algebra and equations. The codes from pre-test to post-test could be compared to investigate shifts in more detail.

5.5 Concluding remarks

This study has provided suggestions around how the discursive framework can be strengthened to analyse scripts of learners by relying on scripts only. Such a skill is necessary in analysing grade 12 scripts who after writing their exams exit the system. It is envisaged that even when dealing with higher grades and with more grades the categories of visual mediators and signifiers may have to be split up into subgroup for a better quality of data. Collection of secondary data is often helpful, but where that is not possible the researcher has to make do with what is available.

It is hoped that all the schools including the one whose scripts are part of the project will benefit from reading results although the sample was not too big enough for generalisation. The finding is that errors in linear equations stem from errors related to arithmetic and basic algebra.

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APPENDIX 1 SAMPLE PRE-TEST CODED SHEET

			QUESTION 1						
		A		B			C		
	1A1	1A2	1A3	1B1	1B2	1B3	1C1	1C2	
1	1			1			1		
2	X			R_S			R_M		
3	N			1			N		
4	R_M			R_M			R_M		
5	N			1			R_S	N	
6	V_M			M_V	N		R_M		
7	R_A			R_S			R_A		
8	1			1			N		
9	R_M			R_M			R_S	N	
10	V_M			V_M			V_M		
11	R_F			R_F			R_A	R_M	
12	X			X			R_M	V_M	
13	R_M			1			N		
14	V_M			V_M			R_M	V_M	
15	X			1			R_S	N	
16	X			X			R_M		
17	0			0			0		
18	N			N			R_S	N	
19	R_E	S		0			0		
20	N			1			1		
21	0			0			0		
22	R_S	R_A		R_S	R_A		R_A		
23	X			X			X		
24	1			1			1		
25	1			1			1		
26	R_M	V_M		R_M	V_M		R_M	V_M	
27	1			1			R_S	N	
28	0			R_M			0		
29	R_A			R_A			V_M		
30	R_E			R_E			R_E	R_A	
31	R_A			R_F			N		

32	R_A	R_M		R_A			R_M		
33	0			0			0		
34	R_A			R_A			N		
35	R_E	R_A		R_E	R_A		R_M		
36	V_M			R_F			V_M		
37	R_E	V_M		R_E			R_M		
38	R_E	V_M		R_E			R_M		
39	R_M			R_S	N		R_S	N	
40	R_A			R_A			R_M		
41	1			R_M			R_S	N	
42	R_S			X			X		
43	R_A			R_A			N		
44	R_M			R_S	N		R_S	R_A	
45	X			V_M			V_M		
				QUESTION 1					
R_A	7	2	0	5	2	0	3	2	
R_E	5	0	0	4	0	0	1	0	
R_F	1	0	0	3	0	0	0	0	
R_M	6	1	0	5	0	0	12	1	
R_S	2	0	0	5	0	0	8	0	
R_V	0	0	0	0	0	0	0	0	
TOTAL	**21**	**3**	**0**	**22**	**2**	**0**	**24**	**3**	
N	4	0	0	1	3	0	6	7	
S	0	1	1	0	0	0	0	0	
V_M	4	3	0	3	1	0	4	3	
TOTAL	**8**	**4**	**1**	**4**	**4**	**0**	**10**	**10**	
X	6	0	0	4	0	0	2	0	
TOTAL	**6**	**0**	**0**	**4**	**0**	**0**	**2**	**0**	
1	6	0	0	10	16	0	4	0	
0	4	0	0	4	0	0	5	0	
TOTAL	**10**	**0**	**0**	**14**	**16**	**0**	**9**	**0**	

APPENDIX 2 SAMPLE POST-TEST CODE SHEET

	QUESTION 1										
	A			B			C				
	1A1	1A2	1A3	1B1	1B2	1B3	1C1	1C2	1C3	2A1	
1	1			1			1			1	
2	V_M			R_S	R_A		R_S			N	
3	1			1			1			R_A	
4	1			1			N			1	
5	1			1			N	R_S		1	
6	R_S			N			N			R_M	
7	1			R_A			1			R_A	
8	1			1			N			1	
9	1			N			1			1	
10	N			1			N			R_M	
11	X			R_S	N		R_S			1	
12	N			R_A			R_S			1	
13	1			1			1			1	
14	1			N			N			R_M	
15	N			1			N			1	
16	1			1			R_S			R_A	
17	1			1			N			R_M	
18	1			1			R_S			R_A	
19	R_S			R_E	S_R		R_E	N		R_M	
20	1			1			N			1	
21	1			1			R_S			R_M	
22	1			1			1			1	
23	1			1			1			R_M	
24	1			1			N			1	
25	1			1			1			1	
26	1			1			N			1	
27	1			1			1			N	
28	1			1			1			0	
29	X			X			0			R_A	
30	N			R_S	R_V	N	R_S	N		1	

31	R_S			N			X			X
32	X			X			R_A			N
33	1			1			N			R_M
34	N			N			N			1
35	X			R_A			R_M			R_M
36	1			1			N			1
37	1			1			R_M	N		R_A
38	R_M	R_E		X			R_M			R_A
39	1			1			R_S			R_M
40	R_A			R_A			N			R_M
41	1			1			1			1
42	0			R_M	R_S		0			1
43	R_A			R_A			R_M			R_A
44	1			N			N			1
45	X			R_S			1			R_M
	QUESTION 1									
R_A	2	0	0	5	1	0	1	0	0	8
R_E	0	1	0	1	0	0	1	0	0	0
R_F	0	0	0	0	0	0	0	0	0	0
R_M	1	0	0	1	0	0	4	0	0	12
R_S	3	0	0	4	1	0	8	1	0	0
R_V	0	0	0	0	1	0	0	0	0	0
TOTAL	**6**	**1**	**0**	**11**	**3**	**0**	**14**	**1**	**0**	**20**
N	5	0	0	6	1	1	16	3	0	3
S	0	0	1	0	0	0	0	0	0	0
V_M	1	0	0	0	0	0	0	0	0	0
TOTAL	**6**	**0**	**1**	**6**	**1**	**1**	**16**	**3**	**0**	**3**
X	5	0	0	h	0	0	1	0	0	1
TOTAL	**5**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**1**
1	27	0	0	25	52	0	12	0	0	20
0	1	0	0	0	0	0	2	0	0	1
TOTAL	**28**	**0**	**0**	**25**	**52**	**0**	**14**	**0**	**0**	**21**

APPENDIX 3: PRE-TEST AND POST-TEST SIMILARITIES AND DIFFERENCES.

Note: The numbers in the first column represent individual learners. The comments about errors refer only to four questions selected for analysis. In the table $\mathbf{R_E}$ = Errors related to wrong usage of equal sign, $\mathbf{R_A}$ = Errors related to addition and subtraction, $\mathbf{R_F}$ = Errors related to factorization, $\mathbf{R_M}$ = Errors related to multiplication and division, $\mathbf{R_S}$ = Errors related to substitution, $\mathbf{R_V}$ = Errors related inability to handle algebraic terms, **X** = Errors I could not explain using my codes, **1** = Correct answers, **0** = Blank and repeating questions, $\mathbf{V_M}$ = Visual mediators, **S** = Signifiers.

	PRESET	**POST-TEST AND COMPARISON** **WITH PRE-TEST**
1	R_E = one error, R_A = one error and R_V = one error. A total of ten errors were made. 1= eight correct answers.	There were no errors made in the post-test questions chosen for analysis and this means that there was a decrease in the number of errors. Student 1 chose to leave question 5 (c) blank. Nine correct answers and one incorrect answer were obtained.
2	R_A= four errors, R_E =two errors, R_F= one error, R_M= one error, R_S= one error, V_M= one error, X= one error 0= one answer Total=12 errors	R_A= three errors, R_F= two errors, R_S= two errors, V_M= one error, X= one error 0 = one answers. Total = ten answers There were no correct answers in the learners work in both cases and a lot of errors are made. The errors were mainly routine errors
3	R_A = two errors, R_E = two errors, R_F = two errors, R_M = two errors, N = two errors. Total errors = 10 errors 1= one answer	R_A = two errors, R_F = two errors, R_M = one error, R_S = one error, X = one error 0= two answers. 1= three answers. There were three 1s (correct answers) in the post-test compared to in the pre-test which has only one correct answer.

		There were more errors in the pre-test as compared to the post-test
4	R_A = three errors, R_E = three errors, R_F = two errors, R_M = five errors, N= one error. Total errors = 12 errors.	R_A = three errors, R_E = one error, R_F= one error, R_M = one error. N = one error. Total=7 errors. There were four 1s (correct answers) in the post-test compared to in the pre-test which has no correct answer. There were more errors in the pre-test as compared to the post-test. The errors were mainly routine errors
5	R_A = two errors, R_E = three errors, R_F = two errors, R_M = one error, R_S = one error, N = two errors. 1 = four answers Total=11 errors.	R_E = one error, R_F = two errors, RS= one error, N = one error. 1 = 6 answers. Total= 5 errors. There were six 1s (correct answers) in the post-test compared to in the pre-test which has four correct answers. There were more errors in the pre-test as compared to the post-test. Most of the errors are routine errors Most of the errors are routine errors.
6	R_A = one error, R_E = three errors, R_M = three errors, R_S = one error, N = one error. Total=9 errors.	R_A = one error, R_E = one error, R_M = one error, R_S = two errors, N = two errors. 0 = one error. Total= seven errors. There were more errors in the pre-test as compared to the post-test There were no correct answers in the learners work and a lot of routine errors are made
7	R_A = three errors, R_F = two errors, R_M = two errors, RS = one error, X = one error, 0 = one error. Total=10 errors.	R_A = four errors, R_E = three errors, R_F = three errors, R_M = two error, R_S = one error, S = one error.1 = three answers. Total=14 errors. There were three 1s (correct answers) in the post-test compared to in the pre-test

		which had no correct answer. There more errors in the post-test as compared to in the pre-test. Most of the errors are routine errors. There is one signifier error in the learners work
8	R_A = one error, R_M = one error, R_S = one error, N = one error Total = four errors. 1 = seven answers	R_A = one error, R_E = one error, R_F = one error, N = one error.1 = seven answers Total = four errors. The number of errors and correct answers in the pre-test is the same as the number of errors and correct answers in the post-test. There were seven 1s (correct answers) in the post-test compared to in the pre-test which has seven correct answer.
9	R_A = one error, R_E = one error, R_F = one error, R_M = three errors, R_S = one error, N = one error. X = one error. Total = 9 errors. 1 = three answers. There were few mainly routine errors, and there was one arithmetic error made.	N = one error. Total = one error 1 = nine answers. There are nine 1s (correct answers) in the post-test compared to in the pre-test which has three correct answers. The number of errors from pre-test to post-test decreases, and the number of correct answers from pre-test to post-test increases.
10	R_A = four errors error, R_E = one error, R_F = one error, R_M = three errors, R_V = one error, V_M = three errors. N = one error. X = one error. Total = 15 errors.	R_A = three errors, R_E = three errors, R_M = four errors, N = two errors. X = one error. Total = 13 errors. 1 = one answers. There were few mainly routine errors, There was one correct answers in the post-test compared to in the pre-test which has no correct answer. The number of errors from pre-test to post-test decreases. The errors are mainly routine errors. .