ORTHOGONAL POLYNOMIALS AND THREE–TERM RECURRENCE RELATIONS

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Declaration

I declare that this research report is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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I wish to acknowledge all the support
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CONTENTS

CHAPTER 1

CLASSICAL THEORY OF ORTHOGONAL POLYNOMIALS

1.1 Introduction.

Orthogonal polynomials have had a long history. They have featured in the work of Legendre on planetary motion, continued fractions of Stieltjes, mechanical quadrature of Gauss etc.

After the publication of 'Orthogonal Polynomials' by Gabor Szegö in 1938 relatively little was published on orthogonal polynomials. This changed in the 1970's when increased interest in approximation theory brought about by the incredible upsurge in the use of the computer in the sciences occurred.

Recently orthogonal polynomials have found uses in such diverse topics such as quantum mechanics, mathematical statistics, scattering theory, coding theory, digital signal processing and so on.

1.2 Distribution functions and 3–term recurrence relations.

Definition 1.1 Let $\alpha: R^1 \to R^1$ be non–decreasing, bounded, take on infinitely many different values on R^1 and have finite moments i.e.

$$\mu_n := \int_{-\infty}^{\infty} x^n \, d\alpha(x) < \infty, \qquad n = 0,1,2,\dots,.$$

Then $\alpha(x)$ is called a distribution function with momenta or simply an m–distribution.

Note that if $\alpha(x)$ is an m–distribution, then $\int p(x)d\alpha(x)$ exists and is finite for every polynomial $p(x)$.

If $\alpha(x)$ is absolutely continuous, then $\alpha'(x) = w(x)$ for some function $w(x)$, in which case we shall simply speak of the m–distribution $w(x)$. We call $w(x)$ a weight function.

We shall write $p \in \Pi_n$ if $p(x)$ is a polynomial of degree at most n.

In this chapter we shall assume that every m–distribution function $\alpha(x)$ is normalized i.e. $\alpha(-\infty) = 0$.

Lemma 1.2 Let $\alpha(x)$ be an m–distribution function. Let $p(x)$ be a non–identically vanishing polynomial such that $p(x) \geq 0$ for every x in the support of $d\alpha$. Then

$$\int_{-\infty}^{\infty} p(x)\, d\alpha(x) > 0.$$

Proof See Freud [5, p.13].■

Theorem 1.3 Let $\alpha(x)$ be an m–distribution function. Then there exists a unique sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ such that

(i) $p_n(x) = \gamma_n x^n + \ldots$, is a polynomial of precise degree n whose leading coefficient γ_n is positive, for $n = 0,1,2,\ldots$.

(ii) The sequence $\{p_n(x)\}_{n=0}^{\infty}$ satisfies the following orthogonality relation

$$\int_{-\infty}^{\infty} p_m(x)\, p_n(x)\, d\alpha(x) = \begin{cases} 0, & m \neq n \\ 1, & \text{otherwise} \end{cases}, \quad m,n \geq 0. \tag{1.1}$$

We say that the sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ is orthonormal with respect to the measure $d\alpha$. If we wish to emphasize this correspondence we shall write

$p_n(x) = p_n(d\alpha;x)$.

Proof We shall inductively define the sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$.

Put

$$p_0(x) := \left[\int_{-\infty}^{\infty} d\alpha(x)\right]^{-1/2}.$$

Suppose we have defined the polynomials $p_n(x) = \gamma_n x^n$ + lower degree terms, where $\gamma_n > 0$ for $n = 0,1,2,\ldots \ell-1$ such that (1.1) holds for $0 \le m,n \le \ell-1$.

Put

$$P_\ell(x) := x^\ell - \sum_{k=0}^{\ell-1} c_k\, p_k(x)\,,$$

where

$$c_k := \int_{-\infty}^{\infty} x^\ell\, p_k(x)\, d\alpha(x)\,.$$

Then for $0 \le m \le \ell-1$,

$$\int_{-\infty}^{\infty} P_\ell(x)\, p_m(x)\, d\alpha(x) = \int_{-\infty}^{\infty} x^\ell\, p_m(x)\, d\alpha(x) - \sum_{k=0}^{\ell-1} c_k \int_{-\infty}^{\infty} p_m(x)\, p_k(x)\, d\alpha(x)$$

$$= c_m - c_m = 0, \text{ by the orthogonality relation (1.1).}$$

Also, by Lemma 1.2,

$$I := \int_{-\infty}^{\infty} [P_\ell(x)]^2\, d\alpha(x) > 0.$$

Put

$$\gamma_\ell := I^{-1/2}$$

and

$$p_\ell(x) := \gamma_\ell\, P_\ell(x).$$

Then $p_\ell(x)$ is a polynomial of precise degree ℓ wi[illegible]ding coefficient γ_ℓ positive. Also, (1.1) is satisfied for $0 \le m,n \le \ell$. Continuing in this way we can construct a sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfying (i) and (ii). We now show that $\{p_n(x)\}_{n=0}^{\infty}$ is uniquely determined by the conditions (i) and (ii). We first note that if

$$p(x) = \sum_{k=0}^{\ell} c_k x^k,$$

then $p(x)$ can be uniquely expressed as

$$p(x) = \sum_{k=0}^{\ell} a_k p_k(x). \tag{1.2}$$

Indeed, $p(x) - a_\ell p_\ell(x)$ is a polynomial of degree $\le \ell-1$ iff $a_\ell = c_\ell \gamma_\ell^{-1}$. To obtain the next coefficient $a_{\ell-1}$ repeat the argument with the polynomial $(p - a_\ell p_\ell)(x)$. (Note that the orthogonality of $\{p_n(x)\}_{n=0}^{\infty}$ was not used here.)

We prove uniqueness. Let $\{P_k(x) := \Gamma_k x^k + \ldots\}_0^{\infty}$ be another sequence of polynomials satisfying (i) and (ii). If $p(x)$ is a polynomial of degree $\ell < n$ then there exists coefficient $a_0, \ldots, a_\ell$ such that

$$p(x) = \sum_{k=0}^{\ell} a_k p_k(x), \qquad \text{by (1.2)}.$$

Therefore

$$\int_{-\infty}^{\infty} p(x)\, p_n(x)\, d\alpha(x) = \sum_{k=0}^{\ell} a_k \int_{-\infty}^{\infty} p_k(x)\, p_n(x)\, d\alpha(x) = 0, \tag{1.3}$$

Let $p(x) := \Gamma_n p_n(x) - \gamma_n P_n(x)$. Then $p(x)$ is a polynomial of degree $< n$ so that

$$\begin{aligned}\int_{-\infty}^{\infty} p^2(x)\, d\alpha(x) &= \int_{-\infty}^{\infty} p(x)\{\Gamma_n p_n(x) - \gamma_n P_n(x)\}\, d\alpha(x) \\ &= 0, \qquad \text{by (1.3)}.\end{aligned}$$

Therefore by Lemma 1.2

$$(\Gamma_n p_n - \gamma_n P_n)^2(x) \equiv 0.$$

Therefore

$$P_n(x) \equiv \gamma_n^{-1} \Gamma_n p_n(x). \tag{1.4}$$

Therefore

$$1 = \int_{-\infty}^{\infty} P_n^2(x)\, d\alpha(x) = \gamma_n^{-2} \Gamma_n^2 \int_{-\infty}^{\infty} p_n^2(x)\, d\alpha(x) = \gamma_n^{-2} \Gamma_n^2 .$$

Since both Γ_n and γ_n are positive we conclude that $\Gamma_n = \gamma_n$. Therefore by (1.4),

$$P_n(x) \equiv p_n(x). \qquad \blacksquare$$

Theorem 1.4 Let $\alpha(x)$ be an m–distribution function and let $\{p_k(x) := \gamma_k x^k + \ldots\}_0^\infty$ be the corresponding system of orthonormal polynomials. Then $\{p_n(x)\}_{n=-1}^\infty$ satisfies the three term recurrence relation

$$x\, p_n(x) = a_{n+1}\, p_{n+1} + b_n\, p_n(x) + a_n\, p_{n-1}(x) \qquad n = 0,1,2,\ldots,$$

where

$$p_{-1}(x) \equiv 0, \quad a_n = \gamma_{n-1}/\gamma_n, \text{ and } \quad b_n = \int^{\infty} x\, p_n^2(x)\, d\alpha(x).$$

<u>Proof</u> Let

$$x\, p_n(x) = \sum_{k=0}^{n+1} c_k\, p_k(x), \qquad \text{by (1.2).}$$

Then

$$c_k = \int_{-\infty}^{\infty} x\, p_k(x)\, p_n(x)\, d\alpha(x), k = 0,1,2,..,n+1. \tag{1.5}$$

Therefore

$$0 = c_0 = c_1 = \ldots = c_{n-2}, \text{ by (1.3).}$$

The remaining three coefficients we compute as follows:

$$c_{n+1} = \int_{-\infty}^{\infty} x\, p_{n+1}(x)\, p_n(x)\, d\alpha(x)$$

$$= \int_{-\infty}^{\infty} [\gamma_n/\gamma_{n+1}\, p_{n+1} + p(x)]\, p_{n+1}\, d\alpha(x)\,, \text{ for some polynomial } p(x) \in \Pi_n$$

$$= \gamma_n/\gamma_{n+1} \int_{-\infty}^{\infty} p_{n+1}^2\, d\alpha(x) + \int_{-\infty}^{\infty} p_{n+1}(x)\, p(x)\, d\alpha(x)$$

$$= \gamma_n/\gamma_{n+1} = a_{n+1}.$$

Similarly,

$$c_{n-1} = \gamma_{n-1}/\gamma_n = a_n.$$

Also, by (1.5)

$$c_n = \int_{-\infty}^{\infty} x\, p_n^2\, d\alpha(x)$$

$$= b_n.$$

Therefore

$$xp_n(x) = c_{n+1}\, p_{n+1} + c_n\, p_n(x) + c_{n-1}\, p_{n-1}(x)$$

$$= a_{n+1}\, p_{n+1}(x) + b_n\, p_n(x) + a_n\, p_{n-1}(x). \qquad \blacksquare$$

1.3 Zeros of orthogonal polynomials.

Theorem 1.5 All zeros of $p_n(x)$ are real and simple. If $\mathrm{supp}(d\alpha) \subset [a,b]$ then all the zeros of $p_n(x)$ lie in (a,b).

Proof Let $\mathrm{supp}(d\alpha) \subset [a,b]$, $-\infty \leq a < b \leq \infty$. Let $\xi_1, \ldots \xi_m$ denote the zeros of $p_n(x)$ of odd multiplicity lying in (a,b). Suppose $m < n$. Then $\Pi(x - \zeta_i) \in \Pi_{n-1}$, so by orthogonality,

$$\int_a^b p_n(x) \prod_{i=1}^{m} (x - \xi_i)\, d\alpha(x)$$

$$= \int_{-\infty}^{\infty} p_n(x) \prod_{i=1}^{m} (x - \xi_i)\, d\alpha(x) = 0.$$

This contradicts Lemma 1.2, since $p_n(x) \prod_{i=1}^{m} (x - \xi_i) \geq 0$ in (a,b). Therefore $m = n$ which proves the assertion. ■

We shall denote the zeros of $p_n(x)$ by $\{x_{k,n}\}_{k=1}^{n}$,ordered such that

$$x_{1,n} > x_{2,n} > \dots > x_{n,n}.$$

Theorem 1.6

$$x_{j,n} > x_{j,n-1} > x_{j+1,n} \qquad j = 1,2,\dots,n-1,\ n = 0,1,2,\dots$$

<u>Proof</u>. See Freud,[5, p.17]■

Theorem 1.7 Let $c < d$ and $\alpha(c) = \alpha(d)$. Then $p_n(x)$ has at most one zero in the interval $[c,d]$.

<u>Proof</u>. See Freud,[5, p.18]■

We now introduce the polynomial

$$\psi_n(x,\xi) = p_{n-1}(\xi)\, p_n(x) - p_n(\xi)\, p_{n-1}(x),$$

where $\xi \in R^1$ and $x \in R^1$. $\psi_n(x,\xi)$ will play an important role in subsequent proceedings.

By Theorem 1.7, p_n and p_{n-1} cannot have a common zero. Therefore the degree of $\psi_n(\cdot,\xi)$,which we shall denote by $n^\dagger$,is given by

$$n^\dagger = \begin{cases} n, & p_{n-1}(\xi) \neq 0 \\ n-1 & \text{otherwise} \end{cases}.$$

Theorem 1.8 All zeros of the polynomial $\psi_n(\cdot,\xi)$ are real and simple. If $\mathrm{supp}(d\alpha) \subset [a,b]$ then at least $n-1$ zeros of $\psi_n(\cdot,\xi)$ lie in (a,b).

<u>Proof</u> See Freud [5, p.19] ■

We shall denote the zeros of $\psi_n(\cdot,\xi)$ by $\{\xi_i\}_{i=1}^{n^\dagger}$ ordered so that $\xi_1 > \dots > \xi_{n^\dagger}$. Note that ξ is one of the ξ_i's.

1.4 The Quadrature formula and the Christoffel–Darboux summation formula.

Theorem 1.9 <u>The Quadrature Formula</u>

Let $p(x)$ be a polynomial of degree at most n' where $n' = n^\dagger + n - 2$. Then

$$\int_{-\infty}^{\infty} p(x)\, d\alpha(x) = \sum_{k=1}^{n^\dagger} p(\xi_k)\, \lambda_n(\xi_k),$$

where

$$\lambda_n(\xi_k) = \int_{-\infty}^{\infty} l_n^2(x,\xi_k)\, d\alpha(x), \qquad k = 1,2,\ldots n^\dagger,$$

and

$$l_n(x,\xi_k) = \frac{\psi_n(x,\xi)}{\psi_n'(\xi_k,\xi)(x - \xi_k)}, \qquad k = 1,2,\ldots n^\dagger,$$

are the fundamental polynomials of Lagrange interpolation of degree $\le n-1$, i.e.

$$l_n(\xi_i,\xi_k) = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \ne k \end{cases}.$$

We call $\lambda_n(\xi_k)$, $k = 1,2,\ldots,n^\dagger$, the nth Christoffel numbers.

<u>Proof</u> We shall first show that if $p(x)$ and $q(x)$ are polynomials of degree $\le n'$ and

$p(\xi_i) = q(\xi_i)$ for $i = 1,2,\ldots n^\dagger$,

then

$$\int_{-\infty}^{\infty} p(x)\, d\alpha(x) = \int_{-\infty}^{\infty} q(x)\, d\alpha(x). \tag{1.6}$$

Indeed, every zero of $\psi_n(\cdot,\xi)$ is a zero of $(p-q)(x)$ and so $\deg((p-q)/\psi_n) \le n' - n^\dagger = n-2$.

Therefore there exists a polynomial $r(x) \in \Pi_{n-2}$ s.t. $(p-q)(x) = r(x)\, \psi_n(x,\xi)$.

By orthogonality,

$$\begin{aligned}\int_{-\infty}^{\infty} (p-q)(x)\, d\alpha(x) &= \int_{-\infty}^{\infty} r(x)\, \psi_n(x,\xi)\, d\alpha(x) \\ &= p_{n-1}(\xi) \int_{-\infty}^{\infty} r(x)\, p_n(x)\, d\alpha(x) + p_n(\xi) \int_{-\infty}^{\infty} r(x)\, p_{n-1}(x)\, d\alpha(x) \\ &= 0.\end{aligned}$$

This proves (1.6). Next let $p(x) \in \Pi_{n'}$. Then $\sum_{k=1}^{n\dagger} p(\xi_k)\, l_{n\dagger}(x,\xi_k)$ is a polynomial of degree $\le$ $n\dagger - 1 \le n'$ which agrees with $p(x)$ at the points $\xi_1,\dots,\xi_{n\dagger}$. Therefore

$$\int_{-\infty}^{\infty} p(x)\, d\alpha(x) = \int_{-\infty}^{\infty} \left\{ \sum_{k=1}^{n\dagger} p(\xi_k)\, l_{n\dagger}(x,\xi_k) \right\} d\alpha(x), \quad \text{by (1.6)}$$

$$= \sum_{k=1}^{n\dagger} \left\{ p(\xi_k) \int_{-\infty}^{\infty} l_{n\dagger}(x,\xi_k)\, d\alpha(x) \right\}$$

$$= \sum_{k=1}^{n\dagger} \left\{ p(\xi_k) \int_{-\infty}^{\infty} l^2_{n\dagger}(x,\xi_k)\, d\alpha(x) \right\}$$

(since $\deg(l^2_{n\dagger}(\cdot,\xi_k)) \le 2(n\dagger - 1) \le n'$ and $l^2_{n\dagger}(x,\xi_k) = l_{n\dagger}(x,\xi_k)$ for $1 \le i,k \le n\dagger$).

$$= \sum_{k=1}^{n\dagger} p(\xi_k)\, \lambda_n(\xi_k). \qquad \blacksquare$$

Theorem 1.10 Let

$$K_n(x,\xi) = \sum_{k=0}^{n-1} p_k(\xi)\, p_k(x).$$

Then

$$K_n(x,\xi) = \frac{\gamma_{n-1}}{\gamma_n} \frac{\psi_n(x,\xi)}{x - \xi}. \tag{1.7}$$

Equation (1.7) is called the Christoffel – Darboux summation formula.

Proof We may write $l_{n\dagger}(\cdot,\xi)$ – a fundamental polynomial of Lagrange interpolation at the points $\{\xi_i\}_1^{n\dagger}$ as

$$l_{n\dagger}(x,\xi) = \sum_{k=0}^{n\dagger - 1} a_k\, p_k(x), \qquad \text{by (1.2)},$$

where

$$a_k = \int_{-\infty}^{\infty} p_k(x)\, l_{n\dagger}(x,\xi)\, d\alpha(x), \;\; k = 0,1,\ldots,n\dagger-1.$$

For $k \le n\dagger - 1$ we have by the quadrature formula

$$a_k = \int_{-\infty}^{\infty} p_k(x)\, l_{n\dagger}(x,\xi)\, d\alpha(x) = \sum_{i=1}^{n\dagger} p_k(\xi_i) l_{n\dagger}(\xi_i,\xi)\, \lambda_n(\xi_i)$$

$$= p_k(\xi)\, \lambda_n(\xi).$$

Therefore

$$l_{n\dagger}(x,\xi) = \sum_{k=0}^{n\dagger-1} p_k(\xi)\, \lambda_n(\xi)\, p_k(x)$$

$$= \lambda_n(\xi)\, K_n(x,\xi). \tag{1.8}$$

Recalling that

$$l_{n\dagger}(x,\xi) = \frac{\psi_n(x,\xi)}{\psi_n'(\xi,\xi)(x-\xi)}$$

we obtain

$$K_n(x,\xi) = \frac{\psi_n(x,\xi)}{\psi_n'(\xi,\xi)(x-\xi)} \frac{1}{\lambda_n(\xi)}$$
$$= \frac{p_{n-1}(\xi)\;\; p_n(x) - p_n(\xi)\;\; p_{n-1}(x)}{\psi_n'(\xi,\xi)\;\; \lambda_n(\xi)\;\; (x-\xi)}.$$

By comparing the coefficient of x^{n-1} on both sides of the above equation we get

$$\gamma_{n-1}\, p_{n-1}(\xi) = \{\lambda_n(\xi)\, \psi_n'(\xi,\xi)\}^{-1}\, \gamma_n\, p_{n-1}(\xi).$$

Therefore

$$K_n(x,\xi) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_{n-1}(\xi)\;\; p_n(x) - p_n(\xi)\;\; p_{n-1}(x)}{x-\xi} \qquad \blacksquare$$

We now derive an important relation linking $\lambda_n(\cdot)$ and the sums of squares of orthogonal polynomials.

Since $K_n(\cdot,\xi) \in \Pi_{n-1}$,

$$K_n(x,\xi) = \sum_{k=1}^{n\dagger} K_n(\xi_k,\xi)\, l_{n\dagger}(x,\xi_k)$$

$$= \sum_{k=1}^{n\dagger} K_n(\xi,\xi_k)\, l_{n\dagger}(x,\xi_k)$$

$$= \sum_{k=1}^{n\dagger} \frac{l_{n\dagger}(x,\xi_k)\, l_{n\dagger}(\xi,\xi_k)}{\lambda_n(\xi_k)}, \quad \text{by (1.8)}.$$

Putting $x = \xi$ in the above equation and using (1.8) yields

$$\lambda_n^{-1}(\xi) = \lambda_n^{-1}(\xi)\, l_n(\xi,\xi) = K_n(\xi,\xi) = \sum_{k=0}^{n-1} p_k^2(\xi) = \sum_{k=0}^{n-1} \frac{l_n^2(\xi,\xi k)}{\lambda_n(\xi_k)}. \blacksquare$$

Theorem 1.11

$$\lambda_n(\xi) = \min \int_{-\infty}^{\infty} p^2(x)\, d\alpha(x),$$

where the infimum is taken over all polynomials $p(x)$ of degree $\leq n-1$ s.t. $p(\xi) = 1$.

<u>Proof Case 1</u>. Assume $p_{n-1}(\xi) \neq 0$. Then $n\dagger = n$, and $n' = n+n-2 = 2(n-1)$.
By the quadrature formula , for any polynomial $p(x)$ of degree $\leq$ -1 satisfying $p(\xi) = 1$ we have

$$\int_{-\infty}^{\infty} p^2(x)\, d\alpha(x) = \sum_{k=1}^{n} p^2(\xi_k)\, \lambda_n(\xi_k)$$

$$\geq p^2(\xi)\, \lambda_n(\xi)$$

$$= \lambda_n(\xi).$$

Also, $l_n(\cdot,\xi)$ is a polynomial of degree $\leq n-1$ and $l_n(\xi,\xi) = 1$. Therefore

$$\lambda_n(\xi) = \int_{-\infty}^{\infty} l_n^2(x,\xi)\, d\alpha(x) = \min \int_{-\infty}^{\infty} p^2(x)\, d\alpha(x)$$

where the infimum is taken over all polynomials $p(x)$ of degree $\leq n-1$ s.t. $p(\xi) = 1$.

<u>Case 2</u> Assume $p_{n-1}(\xi) = 0$. Then

$$\lambda_n(\xi) = \left[\sum_{k=0}^{n-1} p_k^2(\xi)\right]^{-1} = \left[\sum_{k=0}^{n-2} p_k^2(\xi)\right]^{-1} = \lambda_{n-1}(\xi).$$

Now let $p(x)$ be a polynomial of degree $\le n-1$ such that $p(\xi) = 1$. Then there exists a constant A and a polynomial $q(x)$ of degree at most $n-2$ s.t.

$$p(x) = A\, p_{n-1}(x) + q(x).$$

Then $q(\xi) = 1$. Since $p_{n-2}(\xi) \neq 0$, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} p^2(x)\, d\alpha(x) &= \int_{-\infty}^{\infty} (A\, p_{n-1} + q)^2(x)\, d\alpha(x) \\
&= A^2 + \int q^2(x)\, d\alpha(x) \\
&\ge A^2 + \lambda_{n-1}(\xi), \text{ (by Case 1)} \\
&\ge \lambda_{n-1}(\xi) \\
&= \lambda_n(\xi),
\end{aligned}$$

and again we obtain

$$\lambda_n(\xi) = \int_{-\infty}^{\infty} l_n^2(x,\xi)\, d\alpha(x) = \min \int_{-\infty}^{\infty} p^2(x)\, d\alpha(x)$$

where the infimum is taken over all polynomials $p(x)$ of degree $\le n-1$ s.t. $p(\xi) = 1$ ■

Theorem 1.12. Let $\alpha_1(x)$ and $\alpha_2(x)$ be two m–distribution functions s.t.

$$x < y \implies \alpha_1(y) - \alpha_1(x) \le \alpha_2(y) - \alpha_2(x). \tag{1.9}$$

Then for every $\xi \in R^1$,

$$\lambda_n(d\alpha_1,\xi) \le \lambda_n(d\alpha_2,\xi).$$

<u>Proof</u>
$$\begin{aligned}
\lambda_n(d\alpha_2,\xi) &= \int_{-\infty}^{\infty} l_n^2(d\alpha_2,x,\xi)\, d\alpha_2(x) \\
&\ge \int_{-\infty}^{\infty} l_n^2(d\alpha_2,x,\xi)\, d\alpha_1(x), \quad \text{by (1.9)} \\
&\ge \lambda_n(d\alpha_1,\xi), \quad \text{by Theorem 1.11.}
\end{aligned}$$
■

1.5 The Markov–Stieltjes inequality.

We shall denote the zeros of $\psi_n(\cdot,\xi)$ by $\{\xi_i\}_{i=1}^{n^\dagger}$ s.t. $\xi_1 > \xi_2 ... > \xi_{n^\dagger}$ and let $\xi = \xi_j$. Let $\phi_n(\cdot,\xi)$ and $\Phi_n(\cdot,\xi)$ denote the polynomials of degree $\le 2n^\dagger - 2$ satisfying the $n^\dagger + (n^\dagger - 1) = 2n^\dagger - 1$ conditions

$$\phi_n(\xi_i,\xi) = \begin{cases} 1, & i > j \\ 0, & i \le j \end{cases} \text{ and } \phi_n'(\xi_j,\xi) = 0 \text{ if } i \ne j,$$

$$\Phi_n(\xi_i,\xi) = \begin{cases} 1, & i \ge j \\ 0, & i < j \end{cases} \text{ and } \Phi_n'(\xi_j,\xi) = 0 \text{ if } i \ne j.$$

We shall also need

$$\Gamma(x,\xi) = \begin{cases} 1, & x \le \xi \\ 0, & x > \xi \end{cases}.$$

Then the following inequalities hold:

Theorem 1.13

$$\phi_n(x,\xi) \le \Gamma(x,\xi) \le \Phi_n(x,\xi) \quad \text{for every } x \in R^1.$$

Proof See Freud [5, p.27] ■

Define

$$\begin{aligned} \underline{\alpha}_n(\xi) &:= \sum_{\xi_i < \xi} \lambda_n(\xi) = \sum_{i=1}^{n^\dagger} \phi_n(\xi_i,\xi)\, \lambda_n(\xi) \\ &= \int_{-\infty}^{\infty} \phi_n(x,\xi)\, d\alpha(x), \text{ by the quadrature formula} \\ &\le \int_{-\infty}^{\infty} \Gamma(x,\xi)\, d\alpha(x) = \alpha(\xi - 0), \end{aligned}$$

and define similarly

$$\overline{\alpha}_n(\xi) := \sum_{\xi_i \le \xi} \lambda_n(\xi) \ge \alpha(\xi + 0).$$

Then

$$\underline{\alpha}_n(\xi) \le \alpha(\xi - 0) \le \alpha(\xi + 0) \le \overline{\alpha}_n(\xi), \tag{1.10}$$

since $\alpha(x)$ is non–decreasing.

Inequality (1.10) is known as the Markov–Stieltjes inequality. It plays a fundamental role in moment problems. $\underline{\alpha}_n(\xi)$ and $\overline{\alpha}_n(\xi)$ have the following extremal properties:

(i) $\underline{\alpha}_n(\xi) = \max \int_{-\infty}^{\infty} p(x)\, d\alpha(x)$

where the supremum is taken over all polynomials p(x) of degree at most $2n^\dagger - 2$ satisfying the inequality

$$p(x) \leq \begin{cases} 1, & x < \xi \\ 0, & x \geq \xi \end{cases}.$$

(ii) $\overline{\alpha}_n(\xi) = \min \int_{-\infty}^{\infty} p(x)\, d\alpha(x)$

where the infimum is taken over all polynomials p(x) of degree at most $2n^\dagger - 2$ satisfying the inequality

$$p(x) \geq \begin{cases} 1, & x \leq \xi \\ 0, & x > \xi \end{cases}$$

Theorem 1.14 The Posse–Markov–Stieltjes Inequality

Let $\operatorname{supp}(d\alpha) \subset (a,b)$, $-\infty \leq a < b \leq \infty$. Let $f:(a,b) \longrightarrow R^1$ be absolutely monotone in (a,b), i.e.

$$f^{(j)}(x) \geq 0\ ,\ x \in (a,b),\ j = 0,1,2,\ldots$$

Then for $1 \leq k \leq n^\dagger$

$$\sum_{j=k+1}^{n^\dagger} \lambda_n(\xi_j)\, f(\xi_j) \leq \int_a^{\xi_k} f(t)\,d\alpha(t)$$

and

$$\sum_{j=k}^{n^\dagger} \lambda_n(\xi_j)\, f(\xi_j) \geq \int_a^{\xi_k} f(t)\,d\alpha(t).$$

Proof See Freud [5, p.32–33].■

1.6 The Hamburger–Stieltjes moment problem.

We now discuss the Hamburger–Stieltjes moment problem. Namely, given $\mu_0, \mu_1, \mu_2, \ldots \in \mathbb{R}$,to find an m–distribution function $\alpha(x)$ such that

$$\mu_n = \int_{-\infty}^{\infty} x^n \, d\alpha(x), \quad n = 0,1,2,\ldots, . \tag{1.11}$$

Given a sequence of real numbers $\{\mu_n\}$ we define a function μ as follows:

$\mu(\Sigma\, a_k x^k) = \Sigma\, a_k \mu_k$ for any polynomial $p(x) = \Sigma\, a_k x^k$. Notice that μ is linear i.e. $\mu(\alpha p + \beta q) = \alpha\mu(p) + \beta\mu(q)$.

We shall call a polynomial $p(x)$ positive if $p \neq 0$ and $p(x) \geq 0$ for all $x \in \mathbb{R}$.

If $\mu(p) > 0$ for every positive polynomial,then the corresponding sequence of real numbers $\{\mu_n\}$ is said to be positive definite.

With the following theorem the moment problem is solved.

Theorem 1.15 Let $\{\mu_n\}$ be a sequence of real numbers. Then the m–distribution $d\alpha$ is a solution to the moment problem (1.11) if and only if all the following determinants

$$\mu_0, \quad \begin{vmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{vmatrix}, \ldots\ldots, \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}, \ldots\ldots,$$

are positive. This condition is equivalent to

$$\mu(p) = \sum_{k=0}^{\ell} a_k \mu_k > 0$$

for every positive polynomial $p(x) = \sum_{k=0}^{\ell} a_k x^k$. i.e. $\{\mu_n\}$ is positive definite.

<u>Proof.</u> See Freud [5, p 60] ■

Theorem 1.16 Let $\{\tilde{p}_n(x) = x^n + \ldots\}_0^{\infty}$ be a sequence of polynomials with real

coefficients,satisfying the recursion formula

$$\tilde{p}_{n+1}(x) = (x - \beta_{n+1})\, \tilde{p}_n(x) - \Lambda_n\, \tilde{p}_{n-1}(x),\ n = 2,3,..., \tag{1.12}$$

with positive Λ_n and real β_n.Then there exists an m–distribution $d\alpha$ such that

$$\int_{-\infty}^{\infty} \tilde{p}_n(x)\, \tilde{p}_m(x)\, d\alpha(x) = 0\ , \qquad m \neq n.$$

Proof Let $\mu_0 = 1$ and $\mu\,(c) = c$ if c is a constant polynomial.Suppose $\mu_0,\mu_1,...,\mu_n$ have been defined.If $p(x) = \Sigma\, c_k x^k$ is an arbitrary polynomial of degree at most n,we set

$$\mu(p) = \sum_{k=0}^{n} c_k \mu_k .$$

Define $\mu_{n+1} = \mu(x^{n+1} - \tilde{p}_{n+1}(x))$.

We prove by induction on m that

$$\mu(x^m\, \tilde{p}_n(x)) = 0 \text{ if } m < n. \tag{1.13}$$

For m = 0 we have

$$\mu(x^0\, \tilde{p}_n(x)) = \mu(\tilde{p}_n(x)) = \mu(x^n) - \mu(x^n - \tilde{p}_n(x)) = \mu_n - \mu_n = 0.$$

Suppose that the assertion is true for $m < n$.If $m+1 < n$,then by the recurrence relation (1.12),

$$\begin{aligned}
&\mu(x^{m+1}\, \tilde{p}_n(x)) \\
&\quad = \mu(x^m\, x\, \tilde{p}_n(x)) \\
&\quad = \mu(x^m\, \tilde{p}_{n+1}(x)) + \beta_{n+1}\mu(x^m\, \tilde{p}_n(x)) + \Lambda_n \mu(x^m\, \tilde{p}_{n-1}(x)) \\
&\quad = 0, \text{ by the induction hypothesis.}
\end{aligned} \tag{1.14}$$

Therefore $\mu(x^m\, \tilde{p}_n) = 0$ if $m < n$. Also from (1.14) we see that $\mu(x^m\, \tilde{p}_m) = \Lambda_m \mu(x^{m-1}\, \tilde{p}_{m-1})$.

Since $\tilde{p}_0 = x^0 = 1$, $\mu(\tilde{p}_0) = \mu_0 = 1$ and since $\Lambda_n > 0$ for all n,we see that

$$\mu(x^n\, \tilde{p}_n(x)) > 0 \ \text{ for all } n \geq 0. \tag{1.15}$$

Now $x^n - \tilde{p}_n(x) = p(x)$ for some $p \in \Pi_{n-1}$.Therefore $\mu((x^n - \tilde{p}_n)\tilde{p}_n) = \mu(p\, \tilde{p}_n) = 0$ by(1.14).

Therefore by (1.15)

$$\mu(\tilde{p}_n^2) = \mu(x^n\, \tilde{p}_n) > 0. \tag{1.16}$$

We show next that the sequence $\{\mu_n\}_0^\infty$ is positive definite.

Accordingly, let $P(x) = \sum_{k=0}^{n} c_k \tilde{p}_k(x)$ be an arbitrary polynomial. Then

$$P^2(x) = \sum_{k=0}^{n} \sum_{\ell=0}^{n} c_k c_l \tilde{p}_k \tilde{p}_l,$$

so that $\mu(P^2) = \sum_{k=0}^{n} \sum_{\ell=0}^{n} c_k c_l \mu(\tilde{p}_k \tilde{p}_l) = \sum_{k=0}^{n} c_k^2 \mu(\tilde{p}_k^2) > 0$ by (1.16), if P is not identically zero.

Hence the sequence $\{\mu_n\}$ is positive definite and by Theorem 1.15 there exists an m–distribution $d\alpha$ such that $\mu_n = \int_{-\infty}^{\infty} x^n \, d\alpha(x), \quad n = 0,1,2,\ldots,$ .Also,

$$\int_{-\infty}^{\infty} p(x)\tilde{p}_n(x)\, d\alpha(x) = 0 \text{ for every } p \in \Pi_{n-1}$$

and

$$\int_{-\infty}^{\infty} \tilde{p}_n^2(x)\, d\alpha(x) > 0 \quad \text{for } n = 0,1,2,\ldots, . \qquad \blacksquare$$

1.7 Uniqueness of solutions to the Hamburger moment problem.

Definition 1.17 Let $\mathcal{E}$ denote the set of all m–distributions which are uniquely determined by their moments i.e. if $d\alpha \in \mathcal{E}$, then whenever $d\beta$ is another m–distribution such that

$$\int_{-\infty}^{\infty} x^n \, d\alpha(x) = \int_{-\infty}^{\infty} x^n \, d\beta(x) \quad \text{for all } n = 0,1,2,\ldots, .$$

Then $\alpha(x) - \alpha(-\infty) = \beta(x) - \beta(-\infty)$, except perhaps on at most countably many points of discontinuity.

If $\alpha(x)$ is absolutely continuous, $\alpha'(x) = w(x)$, then we shall write $w \in \mathcal{E}$ instead of $d\alpha \in \mathcal{E}$.

As done in the earlier sections we fix a real number ξ, denote by n' the degree of $\psi_n(\cdot,\xi)$ and denote the zeros of the function $\psi_n(x,\xi)$ by $\{\xi_{in}\}_1^{n^\dagger}$ where

$\xi_{1n} > \xi_{2n} > \xi_{3n} > \dots > \xi_{n^\dagger,n}$ and $\lambda_n(\xi_{in})$ denotes the corresponding Christoffel numbers,ie

$$\lambda_n(\xi_{in}) = \int_{-\infty}^{\infty} l_n^2(x,\xi_{in})\, d\alpha(x), \quad i = 1,2,\dots,n^\dagger.$$

Define

$$\alpha_n(x,\xi) := \sum_{\xi_{in}<x} \lambda_n(\xi_{in}).$$

By the quadrature formula,for $m \leq n' \leq n + (n-1)-2 = 2n-3$, we obtain

$$\mu_m = \int_{-\infty}^{\infty} x^m\, d\alpha(x) = \sum_{i=1}^{n^\dagger} (\xi_{in})^m\, \lambda_n(\xi_{in}) = \int_{-\infty}^{\infty} x^m\, d\alpha_n(x).$$

Since each $\alpha_n(x,\xi)$ is non–decreasing, there exists a subsequence $\{\alpha_{n_j}(x,\xi)\}_{j=1}^{\infty}$ and a function $\alpha(x,\xi)$ such that

$\alpha_{n_j}(x,\xi) \to \alpha(x,\xi)$ as $j \to \infty$, and it is not difficult to show that

$$\int_{-\infty}^{\infty} x^m\, d\alpha(x,\xi) = \lim_{j\to\infty} \int_{-\infty}^{\infty} x^m\, d\alpha_{n_j}(x,\xi) = \mu_m. \tag{1.17}$$

Since

$$\lambda_n(\xi) = \min\left\{\int_{-\infty}^{\infty} p^2(x)\, d\alpha(x) : p \in \Pi_{n-1},\ p(\xi)=1\right\},$$

the sequence $\{\lambda_n(\xi)\}$ is non–increasing and therefore has a limit $\lambda(\xi)$.

Therefore

$$\alpha(\xi+0,\xi) - \alpha(\xi-0,\xi) \geq \lim_{n\to\infty} \{\alpha_n(\xi+0,\xi) - \alpha_n(\xi-0,\xi)\}$$
$$= \lim_{n\to\infty} \{\lambda_n(\xi)\} = \lambda(\xi). \tag{1.18}$$

Theorem 1.18 If $d\alpha \in \mathscr{E}$ then apart from points of discontinuity of $\alpha(x)$,

$$\lambda(\xi) = 0, \tag{1.19}$$

for $\xi \in R^1$. Conversely, if (1.19) is satisfied for $\xi \in R^1$, then $\alpha(\xi)$ is uniquely determined by the momentum sequence $\{\mu_n\}$ and $\alpha(x)$ is continuous at the point $x=\xi$.

Proof. (Necessity) Suppose that $d\alpha \in \mathscr{E}$ and let ξ be a point of continuity of $\alpha(x)$. By (1.17) $\alpha(x,\xi)$ generates the same momentum sequence as $\alpha(x)$.

Therefore

$$\alpha(\xi+0) = \alpha(\xi+0,\xi) \text{ and } \alpha(\xi-0) = \alpha(\xi-0,\xi).$$

Therefore

$$\lambda(\xi) \le \alpha(\xi+0,\xi) - \alpha(\xi-0,\xi) = \alpha(\xi+0) - \alpha(\xi-0) = 0\ ,$$ since ξ is a point of continuity of $\alpha(x)$. This proves necessity. Conversely, let $\xi \in R^1$ s.t. $\lambda(\xi) = 0$. Suppose that $\alpha^*(x)$ is an arbitrary (normed) m–distribution generating the same moment sequence as $\alpha(x)$. We wish to show that $\alpha^*(x) = \alpha(x)$. Let ξ be a point of continuity of $\alpha(x)$.

If $p(x) = \Sigma c_k x^k$ is an arbitrary polynomial, then

$$\int_{-\infty}^{\infty} p(x)\, d\alpha(x) = \Sigma c_k \mu_k = \int_{-\infty}^{\infty} p(x)\, d\alpha^*(x).$$

Therefore for the polynomials $\phi_n(x,\xi)$ and $\Phi_n(x,\xi)$ defined earlier we see that

$$\begin{aligned}
\underline{\alpha}_n(\xi) &= \int_{-\infty}^{\infty} \phi_n(x,\xi)\, d\alpha(x) \\
&= \int_{-\infty}^{\infty} \phi_n(x,\xi)\, d\alpha^*(x) \\
&\le \alpha^*(\xi-0) \\
&\le \alpha^*(\xi+0) \\
&\le \int_{-\infty}^{-\infty} \Phi_n(x,\xi)\, d\alpha^*(x) \\
&\le \int_{-\infty}^{-\infty} \Phi_n(x,\xi)\, d\alpha(x) \\
&\le \overline{\alpha}_n(\xi).
\end{aligned}$$

Similarly we can prove the inequalities

$$\underline{\alpha}_n(\xi) \le \alpha(\xi-0) \le \alpha(\xi+0) \le \overline{\alpha}_n(\xi).$$

Combining these inequalities and using the fact that

$$0 = \lambda_n(\xi) = \overline{\alpha}_n(\xi) - \underline{\alpha}_n(\xi),$$

we obtain

$$\alpha(\xi-0) = \alpha^*(\xi-0) = \alpha(\xi+0) = \alpha^*(\xi+0).$$

Thus we have shown that, apart from points of discontinuity of $\alpha(x)$, $\alpha(x) = \alpha^*(x)$, which proves that $\alpha(x)$ is uniquely determined by its moment sequence. ∎

We list some more results concerning the uniqueness of solutions to the Hamburger moment problem.

Theorem 1.19 Let $d\alpha$ be an m–distribution with finitely or infinitely many points of increase s.t. $\mathrm{supp}(d\alpha) \subset [a,b]$, $-\infty < a < b < \infty$, then $d\alpha \in \mathcal{E}$.

<u>Proof.</u> See Freud [5, p.64] ∎

Theorem 1.20 Let $\alpha(x)$ be an m–distribution function generated by the moments $\{\mu_n\}$. If there exists $\xi \in \mathbb{R}$ s.t. $\lambda(\zeta) = 0$, then $d\alpha \in \mathcal{E}$.

<u>Proof.</u> See Freud [5, p.66] ∎

1.8 The connection between the uniqueness of the solution of the Hamburger moment problem and approximation by polynomials.

Definition 1.21 Let $\{f_i\}$ be an orthogonal system in $L^2(d\alpha)$ i.e.

$$\int_{-\infty}^{\infty} f_i f_j \, d\alpha = 0 \quad \text{whenever } i \neq j.$$

Then $\{f_i\}$ is said to be complete if there does not exist $g \in L^2(d\alpha)\backslash\{f_i\}$ s.t. $g(x)$ is not vanishing almost everywhere and $\{f_i\} \cup \{g\}$ is orthogonal.

It is well known that the space spanned by a complete orthogonal system is dense in $L^2(d\alpha)$ with respect to the L^2–norm. i.e. given $g \in L^2(d\alpha)$ and $\epsilon > 0 \; \exists \; h = \Sigma\alpha_i f_i$ such that

$$\int_{-\infty}^{\infty} |g - h|^2 \, d\alpha < \epsilon.$$

Definition 1.22 Let $d\alpha$ be an m – distribution function and $\{p_n(d\alpha)\}$ the corresponding set of orthogonal polynomials. Then $d\alpha$ is called a Riesz m–distribution if $\{p_n(d\alpha)\}$ is complete.

Lemma 1.23 Suppose $\dfrac{d\alpha(x)}{1+x^2} \in \mathfrak{F}$ and suppose that a is a point of continuity of $\alpha(x)$. Then

$$p(x) := \frac{1}{x - a - ib}, \qquad b \neq 0,$$

can be approximated by polynomials in the $L^2(d\alpha)$ – norm. i.e. given $\epsilon > 0$ there exists a polynomial $q(x)$ such that

$$\int_{-\infty}^{\infty} |p - q|^2(x) \, d\alpha(x) < \epsilon.$$

Proof. See Freud [5, p.74] ■

Theorem 1.24 Let $f \in L^2(d\alpha)$ and suppose that $\dfrac{d\alpha(x)}{1+x^2} \in \mathfrak{F}$. Let $\epsilon > 0$. Then there exists a polynomial q(x) such that

$$\int_{-\infty}^{\infty} |f-q|^2(x)\, d\alpha(x) < \infty.$$

Proof. We shall construct a function

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{d_k}{x-a-ib_k} + A + Bx\ ,b_k \neq 0, \tag{1.20}$$

such that

$$\int_{-\infty}^{\infty} |f(x)-\varphi(x)|^2\, d\alpha(x) < \epsilon.$$

The assertion will then follow by Lemma 1.23.

Let $\epsilon_1 > 0$.There exists a function $f^*(x)$ continuous on R^1 and differentiable at a such that

$$\int_{-\infty}^{\infty} |f(x)-f^*(x)|^2\, d\alpha(x) < \epsilon_1$$

and

$$\lim_{x\to\infty} f^*(x) = \lim_{x\to-\infty} f^*(x) = 0.$$

Put $f_1(a+x) = \frac{1}{2}[f^*(a+x) + f^*(a-x)]$;

$f_2(a+x) = \frac{1}{2x}[f^*(a+x) - f^*(a-x)]$.

Then

$$f^*(x) = f_1(x) + (x-a)f_2(x).$$

Let $b > 0$ and consider the change of variable $x: \to x' = \dfrac{1}{(x-a)^2+b^2}$ which maps R^1 into the interval $[0,b^{-2}]$.

Let $f_1(x) = g_1(x')$ and $f_2(x) = g_2(x')$.

Define $g_1(0) = g_2(0) = 0$,then g_1 and g_2 are continuous on $[0,b^{-2}]$.Since $[0,b^{-2}]$is compact,Weierstrass' theorem guarantees the existence of polynomials P_1 and P_2 such that

$$|g_1(x')-P_1(x')| < \epsilon_1 \text{ and } |g_2(x')-P_2(x')| < \epsilon_1, \text{ for } x' \in [0,b^{-2}].$$

Therefore,

$$|f^*(x)-P_1(x')-(x-a)P_2(x')|$$
$$= |f_1(x)+(x-a)f_2(x)-P_1(x')-(x-a)P_2(x')|$$

$$\begin{aligned}
&\le |f_1(x)-P_1(x')|+|x-a|\,|f_2(x)-P_2(x')| \\
&= |g_1(x')-P_1(x')|+|x-a|\,|g_2(x')-P_2(x')| \qquad (1.21)\\
&\le \epsilon'+|x-a|\,\epsilon' = \epsilon'(1+|x-a|) \qquad \text{for } x \in (-\infty,\infty).
\end{aligned}$$

Also, $P_1(x')+(x-a)P_2(x')$

$$= P_1\left[\frac{1}{(x-a)^2+b^2}\right] + (x-a)\, P_2\left[\frac{1}{(x-a)^2+b^2}\right]$$

$$= \frac{p^*(x)}{[(x-a)^2+b^2]^M} \text{ ,for some polynomial } p^*(x).$$

Since $\lim_{x\to\infty} f^*(x) = 0$ the degree of $p^*(x)$ cannot exceed 2M+1, there exist distinct positive numbers b_k such that

$$\left| \frac{p^*(x)}{[(x-a)^2+b^2]^M} - \frac{p^*(x)}{\prod_{k=1}^{M} [(x-a)^2+b_k^2]} \right| \le \epsilon_1(1+|x-a|).$$

By expanding $\varphi(x) := \dfrac{p^*(x)}{\prod_{k=1}^{M} [(x-a)^2+b_k^2]}$ into partial fractions,it can be seen that $\varphi(x)$ has the form (1.20).

Therefore, $\int_{-\infty}^{\infty} |f(x)-\varphi(x)|^2\, d\alpha(x) \le \epsilon_1\{1+2\int_{-\infty}^{\infty} (1+|x-a|)\, d\alpha(x)\}$.

Since $\epsilon_1 > 0$ is arbitrary, the assertion follows. ∎

We now prove the main result of this section:

Theorem 1.25 If the m–distribution function $\alpha(x)$ is continuous on R^1, then $d\alpha$ is a Riesz m–distribution if and only if it is an $\mathcal{E}$–m–distribution.

<u>Proof.</u> [⇐] $d\alpha \in \mathcal{E} \Rightarrow \dfrac{d\alpha}{1+x^2} \in \mathcal{E} \Rightarrow d\alpha$ is a Riesz m–distribution,by Theorem 1.24.

[⇒] We argue by contradiction. We shall assume that $d\alpha$ is a Riesz m–distribution but suppose that $d\alpha$ is not an $\mathcal{E}$–m–distribution. Then

$$\lambda^{-1}(x) = \sum_{k=0}^{\infty} p_k^2(x) < \infty, \quad \text{for every } x \in R^1, \tag{1.22}$$

by Theorem 1.18.

For $\rho > 0$ define the measurable sets

$$M_\rho := \{x \in R^1 : \lambda^{-1}(x) < \rho\}.$$

Then $\rho_1 < \rho_2 \Rightarrow M_{\rho_1} \subset M_{\rho_2}$. Therefore, by (1.22),

$$\bigcup_{\rho \to \infty} M_\rho = R^1.$$

Since $d\alpha$ is not identically zero, there exists ρ such that $d\alpha \neq 0$ on M_ρ. Choose a finite subset $M'_\rho \subset M_\rho$ such that $0 < \int_{M'_\rho} d\alpha < \frac{1}{4\rho}$.

Let $\varphi(x) := \chi_{M'_\rho}(x) := \begin{cases} 0, & x \notin M'_\rho \\ 1, & x \in M'_\rho \end{cases}$.

$$\text{Let } \varphi(x) = \sum_{k=0}^{\infty} a_k(d\alpha,\varphi)\, p_k(d\alpha;x), \tag{1.23}$$

be the orthogonal expansion of $\varphi(x)$, where,

$$a_k = \int_{-\infty}^{\infty} \varphi(x)\, p_k(x)\, d\alpha(x).$$

We have to show that the infinite sum (1.23) exists for every x. Firstly,

$$\varphi(x)^2 = \left[\sum_{k=0}^{\infty} a_k(d\alpha;\varphi)\, p_k(d\alpha;x)\right]^2$$

$$\leq \sum_{k=0}^{\infty} a_k^2(d\alpha,\varphi) \sum_{k=0}^{\infty} p_k^2(d\alpha;x).$$

Also

$$\sum_{k=0}^{\infty} a_k^2(d\alpha,\varphi) \leq \int_{-\infty}^{\infty} \varphi^2(x)\, d\alpha(x) \quad \text{(by the Bessel inequality)}$$

$$= \int_{M'_\rho} d\alpha(x) \leq \frac{1}{4\rho} < \infty. \tag{1.24}$$

Also, $\sum_{k=0}^{\infty} p_k^2(d\alpha;x) < \infty$, by (1.22).

Hence the infinise sum (1.23) exists.Put

$$\varphi^*(x) = \sum_{k=0}^{\infty} a_k(d\alpha,\varphi)\, p_k(d\alpha;x) \quad \text{for } x \in R^1.$$

Since partial sums of this series tend to $\varphi^*(x)$ in $L^2(d\alpha)$ norm as well, we have

$$a_k(d\alpha,\varphi^*) = a_k(d\alpha,\varphi) \ \ k = 0,1,2,\ldots, . \tag{1.25}$$

Also, since $d\alpha(x)$ is a Riesz distribution, we have by (1.25) that $\varphi^*(x) = \varphi(x)$ a.e.$[d\alpha]$. Choose $x_0 \in M_\rho^1$ such that $\varphi^*(x_0) = \varphi(x_0) = 1$.Then $x_0 \in M_\rho^1 \subset M_\rho$ so that $\lambda^{-1}(x_0) < \rho$. Therefore

$$1 = (\varphi^*(x_0))^2 = \Big[\sum_{k=0}^{\infty} a_k(d\alpha,\varphi)\, p_k(d\alpha;x_0)\Big]^2$$

$$\leq \sum_{k=0}^{\infty} a_k^2(d\alpha,\varphi) \sum_{k=0}^{\infty} p_k^2(d\alpha;x_0)$$

$$\leq \frac{1}{4\rho}\lambda^{-1}(x_0) < 1/4. \text{ Contradiction.} \qquad \blacksquare$$

CHAPTER 2

A SURVEY OF SOME RESULTS CONCERNING PROPERTIES OF RECURRENCE COEFFICIENTS AND THE CORRESPONDING ORTHOGONAL MEASURE

2.1 Introduction.

In this chapter we survey some results pertaining to the following problem: Given recurrence coefficients $\{a_n > 0, b_n \epsilon R^1\}$ and polynomials $\{p_n(x)\}$ satisfying the 3–term recurrence relation

$$x\, p_n(x) = a_{n+1}\, p_{n+1}(x) + b_n\, p_n(x) + a_n\, p_{n-1}(x), \tag{2.1}$$

where $p_{-1}(x) = 0$, $p_0(x) = \text{const} > 0$ and $\alpha(x)$ an m–distribution function such that

$$\int_{-\infty}^{\infty} p_m(x)\, p_n(x)\, d\alpha(x) = 0 \text{ if } m \neq n,$$

we ask what properties $\alpha(x)$ possesses. For instance, in 1898 Blumenthal proved that if $a_n \to 1/2$, $b_n \to 0$ as $n \to \infty$, then $\text{supp}(d\alpha) = [-1,1] \cup S$ where S is at most countable with only possible limit points at ± 1.

2.2 Szegő's class.

Definition 2.1 An m–distribution $d\alpha(x)$ is said to belong to Szegő's class if $\log \alpha'(\cos x) \epsilon L^1$, i.e. if

$$\int_0^{\Pi} |\log \alpha'(\cos t)|\, dt < \infty .$$

One of the earliest results in this connection is

Theorem 2.2 (Shohat [15])

Assume $\text{supp}(d\alpha) \subset [-1,1]$. Then $\log \alpha'(\cos x) \epsilon L^1$ if and only if

$$\sum_{n=1}^{\infty}[2a_n - 1] < \infty, \quad \sum_{n=1}^{\infty}[2a_n - 1]^2 < \infty, \quad \sum_{n=1}^{\infty} b_n < \infty, \text{ and } \sum_{n=1}^{\infty} b_n^2 < \infty$$

It has been conjectured that

$$\sum_{n=0}^{\infty}\{|2a_n - 1| + |b_n|\} < \infty \Rightarrow \log \alpha'(\cos(\cdot)) \in L^1.$$

So far the following has been proven.

Theorem 2.3 (Nevai [14], Geronimo and Van Assche [6])

If $\{a_n > 0, b_n \in R^1\}$ satisfy

$$\sum_{n=0}^{\infty} \log n \, \{|2a_n - 1| + |b_n|\} < \infty,$$

then

$\alpha(x)$ belongs to Szegö's class.

We define the partial sums

$$S_n(x) := \sum_{k=0}^{n}\{[a_{k+1}^2 - a_k^2]\, p_k^2(x) + a_k\, [b_k - b_{k-1}]\, p_{k-1}(x)\, p_k(x)\}.$$

These partial sums were introduced by Dombrowski and are useful because they link the recurrence coefficients and orthogonal polynomials to the corresponding measure.

Theorem 2.4 (Dombrowski and Fricke [2]).

$$S_n(x) = a_{n+1}^2 \left\{p_n^2(x) - \frac{x - b_n}{a_{n+1}}\, p_{n+1}(x)\, p_n(x) + p_{n+1}^2(x)\right\}.$$

Proof. We shall use induction. We may assume, without loss of generality that $a_0 = 0$, as can be readily seen from the recurrence relation (2.1).

$$S_0(x) = (a_1^2 - a_0^2)\, p_0^2(x) + a_0(b_0 - b_{-1})\, p_{-1}\, p_0$$
$$= a_1^2\, p_0^2(x),$$
$$= a_1^2 \left\{p_0^2(x) - \frac{x - b_0}{a_1}\, p_0(x)\, p_1(x) + p_1^2(x)\right\},$$ by the three–term recurrence

(2.1). Thus the assertion is true for $n = 0$. Assume that the assertion is true for some

$n \geq 1$. Then

$$S_{n+1}(x) = S_n(x) + \left\{[a_{n+2}^2 - a_{n+1}^2]\, p_{n+1}^2(x) + a_{n+1}[b_{n+1} - b_n]\, p_n(x)\, p_{n+1}(x)\right\}$$
$$= a_{n+1}^2 \left\{p_n^2(x) - \frac{x - b_n}{a_{n+1}}\, p_{n+1}(x)\, p_n(x) + p_{n+1}^2(x)\right\} +$$
$$+ \left\{[a_{n+2}^2 - a_{n+1}^2]\, p_{n+1}^2(x) + a_{n+1}[b_{n+1} - b_n]\, p_n(x)\, p_{n+1}(x)\right\}$$
$$= a_{n+2}^2\, p_{n+1}^2(x) + \{a_{n+1}^2\, p_n^2(x) + a_{n+1}(b_{n+1} - x)p_n(x)\, p_{n+1}(x)\}$$
$$= a_{n+2}^2\, p_{n+1}^2(x) +$$
$$+ a_{n+2}^2 \left[\frac{a_{n+1}^2}{a_{n+2}^2}\, p_n^2(x) + a_{n+1}\left(\frac{b_{n+1} - x}{a_{n+2}^2}\right) p_n(x)\, p_{n+1}(x)\right]. \tag{2.2}$$

Now

$$\frac{b_{n+1} - x}{a_{n+2}}\, p_{n+1}(x)\, p_{n+2}(x) + p_{n+2}^2(x)$$
$$= \left\{- p_{n+2} - \frac{a_{n+1}}{a_{n+2}}\, p_n(x)\right\} p_{n+2}(x) + p_{n+2}^2$$
$$= - \frac{a_{n+1}}{a_{n+2}}\, p_n(x)\, p_{n+2}(x)$$
$$= - \frac{a_{n+1}}{a_{n+2}}\, p_n(x)\left[\left(\frac{x - b_{n+1}}{a_{n+2}}\right) p_{n+1}(x) - \frac{a_{n+1}}{a_{n+2}}\, p_n(x)\right]$$
$$= \frac{a_{n+1}^2}{a_{n+2}^2}\, p_n^2(x) + a_{n+1}\left(\frac{b_{n+1} - x}{a_{n+2}^2}\right) p_n(x)\, p_{n+1}(x).$$

Substituting this into (2.2) yields

$$S_{n+1}(x) = a_{n+2}^2 \left\{p_{n+1}^2(x) + \frac{b_{n+1} - x}{a_{n+2}}\, p_{n+1}(x)\, p_{n+2}(x) + p_{n+2}^2(x)\right\}.$$

Which completes the induction step. ∎

Theorem 2.5 (Nevai and Maté [11])

If

$$\sum_{n=1}^{\infty}\{|a_{n+1}-a_n|+|b_{n+1}-b_n|\}<\infty,$$

then

$$\lim_{n\to\infty}\left\{p^2(x)-\frac{x-b_n}{a_{n+1}}p_n(x)\,p_{n+1}(x)+p_{n+1}^2(x)\right\}=\frac{2}{\pi}\frac{\sqrt{1-x^2}}{\alpha'(x)},\quad -1<x<1,$$

which holds uniformly on ever, closed subinterval of (1,1).

Lemma 2.6 (Nevai–Dombrowski [3])

Let $\{a_k, b_k\}$ satisy $a_{n+1}\geq\frac{1}{2}(1+|b_n|)$ for $n>N$. Then

$$(1-x^2)\,p_{n+1}^2(x)\leq 4\,S_n(x),\quad -1\leq x\leq 1. \tag{2.3}$$

$$(1-x^2)\,p_n^2(x)\leq 4\,S_n(x),\quad -1\leq x\leq 1. \tag{2.4}$$

$$\max_{|x|\leq 1} p_{n+1}^2(x)\leq 4\,(n+2)\max_{|x|\leq 1}|S_n(x)|,\quad -1\leq x\leq 1. \tag{2.5}$$

$$\max_{|x|\leq 1} p_n^2(x)\leq 4\,(n+1)\max_{|x|\leq 1}|S_n(x)|,\quad -1\leq x\leq 1. \tag{2.6}$$

$$0\leq S_{n+1}(x)\leq$$

$$\leq S_n(x)\exp\left\{4\,\frac{(a_{n+2}-a_{n+1})+a_{n+1}\,|b_{n+1}-b_n|}{1-x^2}\right\},-1\leq x\leq 1. \tag{2.7}$$

$$0\leq\max_{|x|\leq 1}S_{n+1}(x)\leq$$

$$\leq\max_{x|\leq 1}S_n(x)\exp\left\{4(n+2)^2\,\frac{(a_{n+2}-a_{n+1})+a_{n+1}\,|b_{n+1}-b_n|}{1-x^2}\right\}, \tag{2.8}$$

$-1\leq x\leq 1.$

Theorem 2.7 (Nevai–Dombrowski [3])

Let $a_n\to 1/2$, $b_n\to 0$, as $n\to\infty$ and,

$$\sum_{n=1}^{\infty}\{|a_{n+1}-a_n| + |b_{n+1}-b_n|\} < \infty. \tag{2.9}$$

Then

$$\lim_{n\to\infty} S_n(x) = \sum_{k=0}^{\infty}\{[a_{k+1}^2 - a_k^2]\, p_k^2(x) + a_k\,[b_k - b_{k-1}]\, p_{k-1}(x)\, p_k(x)\}$$

$$= \frac{\sqrt{1-x^2}}{2\pi\ \alpha'(x)}, \quad -1 \le x \le 1,$$

and convergence is uniform on every closed subinterval of $(-1,1)$.

Theorem 2.8 (Nevai–Dombrowski [3])

Let $\{a_n, b_n\}$ satisfy $a_{n+1} \ge \frac{1}{2}(1 + |b_n|)$ for $n \ge N$, let $a_n \to 1/2$, $b_n \to 0$, as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n^2 \{|a_{n+1}-a_n| + |b_{n+1}-b_n|\} < \infty. \tag{2.10}$$

Then there exists a constant $K > 0$ such that

$$\sqrt{1-x^2}\,|p_n(x)| \le K, \ -1 \le x \le 1, \ n = 1,2,\ldots, \tag{2.11}$$

and

$$\alpha'(x) \ge K^{-1}\sqrt{1-x^2}, \ -1 \le x \le 1. \tag{2.12}$$

<u>Proof.</u> Repeated application of (2.8) shows that the sequence $\{S_n(x)\}$ is uniformly bounded in $[-1,1]$ and hence (2.11) follows by (2.4). Also, (2.12) follows from Theorem 2.7. ∎

Theorem 2.9 (Chihara and Nevai [1])

If $\sum_{n=1}^{\infty} n\{|a_n - 1/2| + |b_n|\} < \infty$, then the number of mass points is finite and are all located outside $[-1,1]$.

Askey and Ismail have posed the following question: Does

$$a_n = \frac{1}{2} + \frac{c}{n} + O(n^{-2}), \; b_n = 0, \; n = 1,2,\ldots,$$

imply that the absolutely continuous portion of $d\alpha$ belongs to Szego's class? By Theorem 2.8, it follows that the question is partially answered in the affirmative if the coefficients $\{a_n\}$ satisy the slightly stronger condition

$$a_n = \frac{1}{2} + \frac{c}{n} + \frac{d}{n^2} + o(n^{-2}), \; n = 1,2,\ldots,$$

2.3 Regularity.

Definition 2.10 The measure $d\alpha$ on $[-1,1]$ is said to be regular if

$$\lim_{n\to\infty} \sup \gamma_n(\alpha)^{1/n} = 2.$$

We have the following characterization theorem for nth root asymptotics:

Theorem 2.11 (Erdős–Turán [4])

$d\alpha(x)$ is regular iff $\lim\limits_{n\to\infty} |\, p_n(d\alpha,z)|^{1/n} = \dfrac{|\, z + \sqrt{z^2 - 1}\, |}{2}$,

locally uniformly on $\mathbb{C}\backslash[-1,1]$.

A result in similar vein for ratio asymptotics is:

Theorem 2.12 (Nevai [12], Máté–Nevai–Totik [9])

$$a_n \longrightarrow 1/2, \; b_n \longrightarrow 0, \text{ as } n \longrightarrow \infty \text{ iff}$$

$$\lim_{n\to\infty} \frac{p_{n+1}(\alpha,z)}{p_n(\alpha,z)} = z + \sqrt{z^2 - 1},$$

locally uniformly for $z \in \mathbb{C}\backslash\mathrm{supp}(d\alpha)$.

2.4 Asymptotic N–periodicity.

Definition 2.13 Let $\{a_n^0\}$, $\{b_n^0\}$ be two sequences of period $N \geq 1$. Then we say that the sequences $\{a_n, b_n\}$ are asymptotically N–periodic if

$$\lim_{n\to\infty}\{|a_n - a_n^0| + |b_n - b_n^0|\} = 0.$$

Researchers such as Aptekarev, Geronimus, Nevai, Van Assche have deduced properties which the corresponding measure $d\alpha$ of the asymptotically N–periodic sequence $\{a_n, b_n\}$ must satisfy. For example:

Theorem 2.14 (Geronimus–Van Assche [6,7])
Let $\{a_n > 0, b_n \in R^1\}$ satisfy

$$\sum_{n=1}^{\infty}\{|a_n - a_n^0| + |b_n - b_n^0|\} < \infty,$$

then the spectral measure $d\alpha$ is absolutely continuous on the essential spectrum of the Jacobi matrix J. In addition α' is also continuous and positive inside the essential spectrum of J.

In chapter 3 we shall survey two papers which shall have more on asymptotic N–periodicity.

2.5 Turán Determinants.

We saw in section 2.2 that under certain conditions

$$\sum_{k=0}^{\infty}\{[a_{k+1}^2 - a_k^2]\, p_k^2(x) + a_k\, [b_k - b_{k-1}]\, p_{k-1}(x)\, p_k(x)\} = \frac{2}{\pi}\frac{\sqrt{1 - x^2}}{\alpha'(x)}.$$

This allows us to obtain the orthogonality measure $d\alpha$ with relative ease. Similar results are possible with Turan determinants.

Definition 2.15 The nth Turán determinant is defined to be

$$D_n = p_n^2 - p_{n-1}\, p_{n+1}.$$

Theorem 2.16 (Mate–Nevai–Totik [9])

If $\operatorname{supp}(d\alpha) = [-1,1]$, and $\alpha' > 0$ a.e. in $(-1,1)$, then

$$\lim_{n\to\infty} \int_{-1}^{1} \left| D_n(\alpha,x)\alpha'(x) - \tfrac{2}{\pi}\sqrt{1 - x^2} \right| dx = 0.$$

Theorem 2.17 (Mate–Nevai–Totik [10])

If $a_n \longrightarrow 1/2$, $b_n \longrightarrow 0$, as $n \longrightarrow \infty$, and

$$\sum \{|a_{n+1} - a_n| + |b_{n+1} - b_n|\} < \infty,$$

then

$$\lim_{n\to\infty} D_n(x) = \frac{2}{\pi}\frac{\sqrt{1 - x^2}}{\alpha'(x)},$$

uniformly on all compact subsets of $(-1,1)$.

In the case of multiple intervals we need a generalization of Turan determinants, namely:

Definition 2.18 Geronimo and Van Assche determinants [6].

$$D_n(N) = p_n\, p_{n-N+1} - \frac{a_{n+1}}{a_{n-N+1}}\, p_{n+1}\, p_{n-N}.$$

Theorem 2.19 (Geronomo–Van Assche [6])

If J is asymptotic N–periodic and of bounded N–variation, i.e.

$$\sum_{n=1}^{\infty} \{|a_{n+N} - a_n| + |b_{n+N} - b_n|\} < \infty,$$

then for every $j = 0,1,2,\ldots,N-1$,

$$\lim_{n\to\infty} D_{nN+j}(x) = C_j \frac{V(x)}{\alpha'(x)},$$

uniformly on all compact sets inside the essential spectrum of $J \setminus \{\text{exceptional points}\}$. Here $C_j = 1/a^o_{j+1}$, $j = 0,1,2,\ldots,N-1$, and $V(x)$ is a function which depends on the spectral measure of the periodic Jacobi matrix $J_0 = J(\{a^o_n, b^o_n\})$ and can be computed explicitly.

Theorem 2.20 (Mate–Nevai–Totik [10])

Let $d\alpha$ be an m–distribution such that $a_n \to 0$, $b_n \to 0$ as $n \to \infty$ and such that

$$\sum_{n=1}^{\infty}\{|a_{n+1}-a_n| + |b_{n+1}-b_n|\} < \infty,$$

then

$$\alpha'(x)^{1/2}(1-x^2)^{1/4}\, p_n(x) =$$

$$= \sqrt{2/\pi}\,\sin\left[\sum_{k=1}^{n} \arg t_{1k}(x) + \arg g(x)\right] + 0\left[\sum_{k=n+1}^{\infty}\{|a_{n+1}-a_n| + |b_{n+1}-b_n|\}\right],$$

uniformly on closed subintervals of $(-1,1)$, where [illegible](x) is continuous and non–vanishing and

$t_{1k}(x) = (a_k/a_{k+1})\ ((x-b_n)(a_k\, a_{k+1})$ [illegible] $/2)$ is a root of the characteristic equation

$$a_{k+1}\, t^2 + (b_k - x)\, t + a_k = 0.$$

CHAPTER 3

A SURVEY OF THE PAPER "SUB–EXPONENTIAL GROWTH OF SOLUTIONS OF DIFFERENCE EQUATIONS BY P.NEVAI AND D.LUBINSKY

3.1 Fundamental Results for 3–term difference equations.

Definition 3.1 Let $\{Y_n\}$ be a solution to the difference equation

$$Y_n = (A_n + B_n)\, Y_{n-1} + Z_n, \quad n = 1,2,3,..., \tag{3.1}$$

where $\{Y_n\}$, $\{Z_n\}$, are sequences in a normed linear space X and $\{A_n\}$, $\{B_n\}$ are sequences of linear operators on X. We say that the sequence $\{Y_n\}$ has sub–exponential growth if for $0 < p < \infty$,

$$\lim_{n\to\infty} \frac{\|Y_k\|^p}{\sum_{k=0}^{n-1} \| Y_k\|^p} = 0.$$

Lemma 3.2 Let $\{Y_n\}_{n=\ell-1}^{m}$ satisfy (3.1) for $n = \ell,\ell+1,...m$. Then

$$Y_m = A_m A_{m-1} \cdots A_\ell Y_{\ell-1} + \sum_{k=\ell}^{m} A_m A_{m-1} \cdots A_{k+1} B_k Y_{k-1} + \\ + \sum_{k=\ell}^{m} A_m A_{m-1} \cdots A_{k+1} Z_k. \tag{3.2}$$

<u>Proof.</u> $Y_m = (A_m + B_m)\, Y_{m-1} + Z_m$, by (3.1)

$$= A_m\{(A_{m-1} + B_{m-1})\, Y_{m-2} + Z_{m-1}\} + B_m Y_{m-1} + Z_m$$

$$= A_m A_{m-1} Y_{m-2} + A_m B_{m-1} Y_{m-2} + A_m Z_{m-1} + B_m Y_{m-1} + Z_m,$$

which has the correct form (3.2) for $\ell = m - 1$. Continued substitution for Y_{m-2}, $Y_{m-3}, \cdots, Y_\ell$ using (3.1) gives the general form of (3.2). ■

Lemma 3.3 Let $0 < p < \infty$, $2 \leq N < m$ and assume that the difference equation (3.1) holds for $m - N + 1 \leq n \leq m$.

Let

$$\beta_{m,N} = \sup_{m-N < k \leq m} \| B_k \| , \tag{3.3}$$

$$\zeta_{m,N} = \sup_{m-N < k \leq m} \| Z_k \| , \tag{3.4}$$

and

$$\alpha_{m,N} = \sup_{m-N < k \leq m} \| A_m A_{m-1} \cdots A_{k+1} \| . \tag{3.5}$$

Then

$$\| Y_m \|^p = \leq 2\, 3^p \frac{\alpha^p_{m,N}}{N} \Big[(1 + N^{p+1} \beta^p_{m,N}) \sum_{k=m-N}^{m-1} \| Y_k \|^p + N^{p+1} \zeta^p_{m,N} \Big] .$$

<u>Proof</u> Let $m - N + 1 \leq \ell \leq m - 1$. By Lemma 3.2,

$$Y_m = A_m A_{m-1} \cdots A_\ell Y_{\ell-1} + \sum_{k=\ell}^{m} A_m A_{m-1} \cdots A_{k+1} B_k Y_{k-1} + $$

$$+ \sum_{k=\ell}^{m} A_m A_{m-1} \cdots A_{k+1} Z_k .$$

Therefore

$$\| Y_m \| \leq \| A_m A_{m-1} \cdots A_\ell \| \, \| Y_{\ell-1} \| + \sum_{k=\ell}^{m} \| A_m A_{m-1} \cdots A_{k+1} \| \, \| B_k \| \, \| Y_{k-1} \| + $$

$$+ \sum_{k=\ell}^{m} \| A_m A_{m-1} \cdots A_{k+1} \| \, \| Z_k \| .$$

Therefore by (3.3), (3.4) and (3.5),

$$\|Y_m\| \le \alpha_{m,N}\Big\{\|Y_{\ell-1}\| + \beta_{m,N}\sum_{k=\ell}^{m}\|Y_{k-1}\| + (m-\ell+1)\,\zeta_{m,N}\Big\}$$

$$\le \alpha_{m,N}\Big\{\|Y_{\ell-1}\| + \beta_{m,N}\sum_{k=\ell}^{m}\|Y_{k-1}\| + N\,\zeta_{m,N}\Big\}. \tag{3.6}$$

Using the elementary inequality (s=3)

$$\Big[\sum_{k=1}^{s} x_k\Big]^p \le s^p \sum_{k=1}^{s} x_k^p, \quad x_j \ge 0$$

in the inequality (3.6) yields

$$\|Y_m\|^p \le (3\,\alpha_{m,N})^p\Big\{\|Y_{\ell-1}\|^p + \beta_{m,N}^p\Big[\sum_{k=\ell}^{m}\|Y_{k-1}\|\Big]^p + (N\,\zeta_{m,N})^p\Big\}$$

$$\le (3\,\alpha_{m,N})^p\Big\{\|Y_{\ell-1}\|^p + \beta_{m,N}^p\,(m-\ell+1)^p\sum_{k=\ell}^{m}\|Y_{k-1}\|^p + (N\,\zeta_{m,N})^p\Big\}$$

$$\le (3\,\alpha_{m,N})^p\Big\{\|Y_{\ell-1}\|^p + \beta_{m,N}^p\,N^p\sum_{k=\ell}^{m}\|Y_{k-1}\|^p + (N\,\zeta_{m,N})^p\Big\}.$$

Adding these inequalities for $m - N + 1 \le \ell \le m - 1$ yields

$$(N-1)\,\|Y_m\|^p \le (3\,\alpha_{m,N})^p\Big\{\sum_{\ell=m-N}^{m-2}\|Y_\ell\|^p + \beta_{m,N}^p\,N^{p+1}\sum_{k=m-N}^{m-1}\|Y_k\|^p + N^{p+1}\,\zeta_{m,N}^p\Big\}.$$

Since $N-1 \ge N/2$, if $N > 1$, dividing the left hand side by $N-1$ and the right hand side by $N/2$ gives

$$\|Y_m\|^p \le \tfrac{2}{N}\,(3\,\alpha_{m,N})^p\Big\{(1 + \beta_{m,N}^p\,N^{p+1})\sum_{k=m-N}^{m-1}\|Y_k\|^p + N^{p+1}\,\zeta_{m,N}^p\Big\}. \qquad \blacksquare$$

Theorem 3.4 Let $\{Y_n\}$, $\{Z_n\}$ be sequences of a normed space X and let $\{A_n\}$, $\{B_n\}$ be linear operators acting on X such that

$$\lim_{n\to\infty}\|B_n\| = 0, \text{ and } \lim_{n\to\infty}\|Z_n\| = 0. \tag{3.7}$$

Let $0 < p < \infty$ be fixed. Assume that for every $\epsilon > 0$ there exists positive integers N and n_0

such that

$$\max_{1\le k\le N} \| A_n A_{n-1} \cdots A_{n-k+1} \|^p \le \epsilon N, \tag{3.8}$$

for $n \ge N_0$. Then every non–identically vanishing solution $\{Y_n\}$ of the difference equation

$$Y_n = (A_n + B_n) Y_{n-1} + Z_n, \quad n=1,2,..., \tag{3.9}$$

satisfies

$$\lim_{n\to\infty} \frac{\| Y_n\|^p}{\sum_{k=0}^{n-1} \| Y_k\|^p} = 0. \tag{3.10}$$

Also, if $\{A_n\}$, $\{B_n\}$ and $\{Z_n\}$ depend on a family of parameters and (3.7) and (3.8) are valid uniformly for these parameters, then the conclusion holds uniformly as well.

<u>Proof.</u> Let $\epsilon > 0$. By (3.8) there exists positive integers n_0,N such that for $m \ge n_0$ we have $\alpha^p_{m,N} \le \epsilon N$. Then by Lemma 3.3

$$\| Y_n\|^p \le 2\,(3\,\alpha_{n,N})^p / N \left\{(1 + \beta^p_{n,N} N^{p+1}) \sum_{k=n-N}^{n-1} \| Y_k\|^p + N^{p+1} \zeta^p_{n,N} \right\}.$$

Since $\{Y_n\}$ is non–identically vanishing there exists $j \ge 1$ such that $\| Y_j\| > 0$. Therefore

$\sum_{k=1}^{n-1} \| Y_k\|^p \ge C$ for all n sufficiently large; say $n \ge n_1 \ge 1$.

Therefore

$$\frac{\| Y_k\|^p}{\sum_{k=0}^{n-1} \| Y_k\|^p} \le 2\,3^p\,\epsilon \left\{1 + \beta^p_{n,N} N^{p+1} + C^{-1} N^{p+1} \zeta^p_{n,N}\right\},$$

for $n \ge \max\{n_0, n_1\}$. By (3.7), keeping N fixed,

$$\limsup_{n\to\infty} \frac{\| Y_k\|^p}{\sum_{k=0}^{n-1} \| Y_k\|^p} \le 2\,3^p\,\epsilon.$$

Since $\epsilon > 0$ is arbitary we conclude that

$$\lim_{n\to\infty} \frac{\|Y_n\|^p}{\sum_{k=0}^{n-1} \|Y_k\|^p} = 0. \qquad \blacksquare$$

3.2 Higher order scalar difference equations

Definition 3.5 A norm $\|\cdot\|$ on $\mathbb{C}^{k\times k}$ is said to be consistent if $\|DX\| \le \|D\|\,\|X\|$ for every $D \in \mathbb{C}^{k\times k}$ and $X \in \mathbb{C}^k$.

Theorem 3.6 For the kth order difference equation

$$x_n + \sum_{j=1}^{k} \left[a_n^{(j)} + b_n^{(j)}\right] x_{n-j} = z_n, \quad n = 1,2,..., \tag{3.11}$$

assume that

$$\rho := \inf_{n\ge 1} \left\{\prod_{\ell=1}^{n} |a_\ell^{(k)}|\right\} > 0, \tag{3.12}$$

and

$$\lim_{n\to\infty} \max_{1\le j\le k} |b_n^{(j)}| = 0 \text{ and } \lim_{n\to\infty} |z_n| = 0, \tag{3.13}$$

and that all solutions of the unperturbed homogeneous equation

$$x_n + \sum_{j=1}^{k} a_n^{(j)} x_{n-j} = 0\,, \quad n = 1,2,..., \tag{3.14}$$

are bounded. Then every non–identically vanishing solution of (3.12) satisfies

$$\lim_{n\to\infty} \frac{|x_n|^p}{\sum_{k=0}^{n-1} |x_k|^p} = 0,$$

for every $0 < p < \infty$.

<u>Remarks.</u> Let

$$A_n := \begin{bmatrix} -a_n^{(1)} & -a_n^{(2)} & \cdots\cdots & -a_n^{(k-2)} & -a_n^{(k-1)} & -a_n^{(k)} \\ 1 & 0 & \cdots\cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots\cdots & 0 & 1 & 0 \end{bmatrix};$$

$$Y_n := (x_n, x_{n-1}, \cdots\cdots, x_{n-k+1})^T.$$

Then (3.14) can be written in matrix form as

$$Y_n = A_n\, Y_{n-1}.$$

With this notation it is well known that all solutions of (3.14) are bounded iff $\{A_n\}$ satisfies

$$\tau := \sup_{n\geq 1} \| A_n\, A_{n-1} \cdots A_1 \| < \infty,$$

for some consictent matrix norm $\|\cdot\|$. We shall also need the following properties of consistent matrix norms: Firstly, since in finite dimensional spaces all norms are equivalent, if $D = (d_{ij}) \in \mathbb{C}^{k\times k}$, then there exists two positive numbers α and β depending on k only such that

$$\alpha \max_{1\leq i,j\leq k} |d_{ij}| \leq \| D \| \leq \beta \max_{1\leq i,j\leq k} |d_{ij}|.$$

Also, if D is invertible, then it follows from the cofactor expansion of D^{-1} that

$$\| D^{-1} \| \leq \gamma \frac{\left[\max_{1\leq i,j\leq k} |d_{ij}|\right]^{k-1}}{|\det D|},$$

where γ depends only on k.

<u>Proof of Theorem 3.6</u> Define the following matrices:

$$B_n = \begin{bmatrix} -b_n^{(1)} & -b_n^{(2)} & \cdots\cdots & -b_n^{(k-2)} & -b_n^{(k-1)} & -b_n^{(k)} \\ 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots\cdots & 0 & 0 & 0 \end{bmatrix}.$$

Then (3.11) can be written in matrix form as

$$Y_n = (A_n + B_n)\, Y_{n-1} + Z_n,\ \ n = 1,2,3,\ldots, \tag{3.15}$$

By (3.13) and remarks,

$$\|B_n\| \le \alpha \max_{1\le i\le k} |b_n^{(j)}| \longrightarrow 0, \text{ as } n \longrightarrow \infty, \text{ and}$$

$$\|Z_n\| \le \alpha \max_{1\le i\le k} |z_{n-i}| \longrightarrow 0, \text{ as } n \longrightarrow \infty\,.$$

Also, expanding $\det(A_n)$ in its last column, we get

$$|\det(A_n)| = |a_n^{(k)}| \ne 0.$$

Therefore

$$|\det(A_m A_{m-1} \cdots \cdots A_1)| = \prod_{\ell=1}^{m} |a_\ell^{(k)}| \ge \rho > 0.$$

Therefore

$$\|(A_m\, A_{m-1} \cdots\cdots A_1)^{-1}\| \le \gamma \frac{\left[\max\limits_{1\le i,j\le k} |d_{ij}|\right]^{k-1}}{\rho}$$

$$\le \frac{\gamma}{\rho}\left[\frac{\|A_{m-1}\ A_{m-2}\ \cdots\ A_1\|}{\alpha}\right]^{k-1}$$

$$\le \frac{\gamma}{\rho}\left[\frac{r}{\alpha}\right]^{k-1}.$$

Hence

$$\|A_n\, A_{n-1} \ldots A_m\| \le \|(A_n\, A_{n-1} \ldots A_1)(A_{m-1}\, A_{m-2} \ldots A_1)^{-1}\| \le r\frac{\gamma}{\rho}\left[\frac{r}{\alpha}\right]^{k-1}.$$

Therefore

$$\sup_{1\le m\le n} \|A_n\, A_{n-1} \cdots\cdots A_m\| < \infty\,,$$

which is stronger than hypothesis (3.8) of Theorem 3.4. Hence we may apply Theorem 3.4 to the difference equation (3.15) to obtain

$$\lim_{n\to\infty} \frac{\|Y_n\|^p}{\sum_{k=0}^{n-1} \|Y_k\|^p} = 0.$$

By the equivalence of norms on $\mathbb{C}^{k\times k}$, we immediately get

$$\lim_{n\to\infty} \frac{|x_n|^p}{\sum_{k=0}^{n-1} |x_k|^p} = 0.$$ ■

3.3 Orthogonal polynomials and three–term recurrences.

Theorem 3.7 Let $\{a_n^*\}$ be a non–vanishing sequence of complex numbers satisfying $\sup\{|a_n^*|^{\pm 1}: n=0,1,2,\cdots,\} < \infty$ and let $\{b_n^*\}$ be an arbitrary sequence of complex numbers such that every solution of

$$x\, p_n(x) = a_{n+1}^*\, p_{n+1} + b_n^*\, p_n(x) + a_n^*\, p_{n-1}(x), \quad n = 0,1,2,\ldots,$$

where $p_{-1}(x) = C_1$, $p_0 = C_2$, is uniformly bounded on a set $K \subset \mathbb{C}$. Let $\{a_n \neq 0\}$ and $\{b_n\}$ satisfy

$$\lim_{n\to\infty} \{|a_n - a_n^*| + |b_n - b_n^*|\} = 0.$$

Let $\{p_n(x)\}$ be a sequence of polynomials satisfying

$$x\, p_n(x) = a_{n+1}\, p_{n+1} + b_n\, p_n(x) + a_n\, p_{n-1}(x),\ n = 0,1,2,.., \tag{3.16}$$

where $p_{-1}(x) = 0$, $p_0 = C \neq 0$. Then for every $0 < p < \infty$

$$\lim_{n\to\infty} \max_{x\in K} \frac{|p_n(x)|^p}{\sum_{k=0}^{n-1} |p_k(x)|^p} = 0. \tag{3.17}$$

<u>Proof.</u> Rewrite the recurrence relation (3.16) in the form

$$0 = p_n(x) + \left[\frac{b_{n-1} - x}{a_n}\right] p_{n-1}(x) + \frac{a_{n-1}}{a_n} p_{n-2}(x)$$

$$= p_n(x) + \left\{\frac{b^*_{n-1} - x}{a^*_n} + \left[\frac{b_{n-1} - x}{a_n} - \frac{b^*_{n-1} - x}{a^*_n}\right]\right\} p_{n-1}(x) +$$

$$+ \left\{\frac{a^*_{n-1}}{a^*_n} + \left[\frac{b_{n-1}}{a_n} - \frac{a^*_{n-1}}{a^*_n}\right]\right\} p_{n-2}(x), \quad n = 1,2,\ldots$$

We show that the above difference relation satisfies the conditions of Theorem 3.6

$$\rho := \inf_{n\geq 1} \prod_{\ell=1}^{n} |a_\ell^{(2)}| = \inf_{n\geq 1} \prod_{\ell=1}^{n} \frac{a^*_{\ell-1}}{a^*_\ell} \inf_{n\geq 1} \frac{a^*_0}{a^*_n} > 0,$$

since $\sup\{|a^*_n|^{\pm}: n=0,1,2,\cdots,\} < \infty$. Also,it is easy to show that

$b_n^{(1)}$ and $b_n^{(2)} \longrightarrow 0$, as $n \longrightarrow \infty$. The assertion now follows by Theorem 3.6.∎

In [13] Nevai, Totik and Zhang prove the following theorem:

Theorem 3.8 (Nevai–Totik–Zhang [13])

Let $\{a_n \neq 0, b_n\} \subset \mathbb{C}$ be sequences such that

$$\lim_{n\to\infty} 2\, a_n = a \text{ and } \lim_{n\to\infty} b_n = b. \tag{3.18}$$

Then for every $0 < p < \infty$

$$\lim_{n\to\infty} \max_{x\in[b-a,b+a]} \frac{|p_n(x)|^p}{\sum_{k=0}^{n-1} |p_k(x)|^p} = 0. \tag{3.19}$$

The proof of this theorem relies on a difficult lemma. The following theorem, however, has a much simpler proof and although it does not yield the full strength of (3.19) of Theorem 3.8, it allows for unbounded recurrence coefficients and thus is applicable to a much larger class of orthogonal polynomials than Theorem 3.8.

Theorem 3.9 Let $\{a_n \neq 0, b_n\} \subset \mathbb{C}$ be sequences such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \text{ and } \lim_{n\to\infty} \frac{b_n}{a_n} = \beta \in R^1. \tag{3.20}$$

Let $\{p_n(x)\}$ be a sequence of polynomials satisfying the usual 3–term recurrence relation

$$x\, p_n(x) = a_{n+1}\, p_{n+1} + b_n\, p_n(x) + a_n\, p_{n-1}(x), \quad n \geq 0, \tag{3.21}$$

with $p_{-1}(x) = 0$ and $p_0 = c \neq 0$. Then given $s \in (0,1)$ and $0 < p < \infty$,

$$\lim_{n\to\infty} \max_{x \in [b_n - 2sa_n, b_n + 2sa_n]} \frac{|p_n(x)|^p}{\sum_{k=0}^{n-1} |p_k(x)|^p} = 0, \tag{3.22}$$

where $[b_n - 2\,s\,a_n, b_n + 2\,s\,a_n]$ denotes the straight line segment connecting the complex numbers $b_n - 2\,s\,a_n$ and $b_n + 2\,s\,a_n$.

<u>Proof.</u> Define

$$Y_n(x) = [p_n(x), p_{n-1}(x)]^T, \quad n = 1,2,\ldots,$$

with $p_{-1}(x) = 0$ and $p_0(x) = \text{const} \neq 0$.

Then we can rewrite the three term recurrence (3.21) as

$$Y_{n+1}(x) = \begin{bmatrix} \dfrac{x - b_n}{a_{n+1}} & -\dfrac{a_n}{a_{n+1}} \\ 1 & 0 \end{bmatrix} Y_n(x), \quad n = 1,2,\ldots,$$

Let $\|\cdot\|$ denote the max norm on $\mathbb{C}^2$ so that $\|(z,w)^T\| = \max\{|z|,|w|\}$. This induces a norm on the set of 2×2 matrices as follows: If $D = (d_{ij})_{i,j=1}^2$,

then $\|D\| = \max_{i=1,2} \{|d_{i1}| + |d_{i2}|\}$. It follows that

$$\max_{1\leq i,j\leq 2} \{|d_{ij}|\} \leq \|D\| \leq 2 \max_{1\leq i,j\leq 2} \{|d_{ij}|\}.$$

Let

$$A(x) = \begin{bmatrix} 2x & -1 \\ 1 & 0 \end{bmatrix},$$

then

$$A^k(x) = \begin{bmatrix} U_k(x) & -U_{k-1}(x) \\ U_{k-1}(x) & -U_{k-2}(x) \end{bmatrix},$$

where as usual $U_k(x)$ denotes the Chebyshev polynomial of the second kind of degree k.

Also,

$$\| A^k(x) \| \le \frac{2}{\sqrt{1-x^2}}, \quad -1 < x < 1, \ k = 1,2,\dots, .$$

We shall use Lemma 3.3. Fix $m \ge N > 2$. Then

$$Y_n(x) = \begin{bmatrix} \dfrac{x - b_{n-1}}{a_n} & -\dfrac{a_{n-1}}{a_n} \\ 1 & 0 \end{bmatrix} Y_{n-1}(x)$$

$$= \left[A\left(\left[\frac{x - b_m}{2\ a_m}\right]\right) + B_n(x) \right] Y_{n-1}(x),$$

where $B_n = B_{n,m}$ is given by

$$B_n(x) = \begin{bmatrix} \dfrac{x - b_{n-1}}{a_n} & -\dfrac{a_{n-1}}{a_n} \\ 1 & 0 \end{bmatrix} - A\left(\left[\frac{x - b_m}{2\ a_m}\right]\right)$$

$$= \begin{bmatrix} \dfrac{x}{a_n}\left[1 - \dfrac{a_n}{a_m}\right] + \left[\dfrac{b_m}{a_m} - \dfrac{b_{n-1}}{a_n}\right] & 1 - \dfrac{a_{n-1}}{a_n} \\ 0 & 0 \end{bmatrix}.$$

As before we define

$$\alpha_{m,N}(x) = \sup_{0 \le k \le N} \left\| A^k\left[\frac{x - b_m}{2\ a_m}\right] \right\| \text{ and } \beta_{m,N}(x) = \sup_{m-N \le k \le m} \| B_k(x) \| .$$

Suppose that $s \in (0,1)$ and $x \in [b_m - 2\,s\,a_m,\ b_m + 2\,s\,a_m]$. Then $|x - b_m| \le 2\,s\,a_m$ or

$|x - b_m| \ / \ 2\,a_m \le s$. By (3.20),

$$\left|\frac{x}{a_n}\right| \le \left|\frac{x - b_m}{a_n}\right| + \left|\frac{b_m}{a_n}\right| = \frac{a_m}{a_n}\left\{\left|\frac{x - b_m}{a_m}\right| + \left|\frac{b_m}{a_m}\right|\right\}$$

$$\leq \frac{a_m}{a_n}\left[2\,s + |\beta| + o(1)\right], \text{ as } m \to \infty.$$

Therefore

$$\alpha_{m,N}(x) \leq \frac{2}{\sqrt{1 - s^2}}.$$

Also by assumptions in (3.20),

$$\lim_{m\to\infty} \max_{x\epsilon[b_m-2sa_m,b_m+2sa_m]} \beta_{m,N}(x) = 0.$$

Hence by Lemma 3.3,

$$\|Y_m(x)\|^p \leq 2.3^p \left[\frac{2^p}{N}(1-s^2)^{-p/2}\right]\left[1 + N^{p+1} \max \beta^p_{m,N}(x)\right] \sum_{k=0}^{m-1} |p_k(x)|^p,$$

for every $x \epsilon [b_m-2\,s\,a_m\,,\,b_m+2\,s\,a_m]$. Dividing both sides by $\sum_{k=0}^{m-1} |p_k(x)|^p$, and letting first m and then N tend to ∞, yields formula (3.22). ■

CHAPTER 4

A SURVEY OF "ORTHOGONAL POLYNOMIALS: THEIR GROWTH RELATIVE TO THEIR SUMS"

BY NEVAI,TOTIK & ZHANG

4.1 Introduction.

Let $\{a_n,b_n\}$ be sequences of complex numbers, $a_0 \neq 0$. Define a sequence of functions $\{p_n(x)\}$ by the three term recurrence relation

$$x\, p_n(x) = a_{n+1}\, p_{n+1}(x) + b_n\, p_n(x) + a_n\, p_{n-1}(x),\ n \geq 0, \tag{4.1}$$

where $p_0(x)$ and $p_{-1}(x)$ are given initial functions.

If

$$p_0(x) = \text{const} > 0 \text{ and } p_{-1}(x) = 0, \tag{4.2}$$

then $p_n(x)$ is a polynomial of precise degree n. In general,however,$\{p_n(x)\}$ will not be a sequence of polynomials.

Definition 4.1 For $a \geq 0$ and $b \in R^1$,define

$M(b,a) =$

$$\{\{a_n,b_n\}_{n=0}^{\infty} : a_n > 0\ ,\ b_n \in R^1,\ a_n \to a/2,\ b_n \to b, \text{ as } n \to \infty\}. \tag{4.3}$$

We shall say that the m–distribution $d\alpha(x)$ is in $M(b,a)$ if the corresponding recurrence coefficients are in $M(b,a)$.

By Blumenthal's Theorem, if $d\alpha(x) \in M(b,a)$, and $a > 0$, then

$$\text{supp}(d\alpha) = [b-a,b+a] \cup S,$$

where S is at most countable with only possible limit points $b \pm a$. Also it is well known that

$$\sum_{k=0}^{\infty} p_k^2(d\alpha,x) \le \frac{1}{\alpha(\{x\})}, \text{ for } x \in R^1.$$

Therefore for $x \in S\backslash\{b\pm a\}$, $\alpha(\{x\}) > 0$ and hence

$$\sum_{k=0}^{\infty} p_k^2(d\alpha,x) < \infty, \text{ and so } p_k(d\alpha,x) \to 0, \text{ as } n \to \infty.$$

Therefore for every $0 < p < \infty$ and $x \in \mathrm{supp}(d\alpha)\backslash[b-a,b+a]$, we have

$$\lim_{n\to\infty} \frac{|p_n(d\alpha,x)|^p}{\sum_{k=0}^{n-1} |p_k(d\alpha,x)|^p} = 0.$$

It is natural to ask what can be said about uniform convergence. Nevai had already proved as early as 1979 [12, p.11] that

$$\lim_{n\to\infty} \max_{x\in[b-a,b+a]} [a^2 - (x-b)^2] \frac{p_n^2(d\alpha,x)}{\sum_{k=0}^{n-1} p_k^2(d\alpha,x)} = 0,$$

where $d\alpha(x) \in M(b,a)$, $a > 0$.

The paper to be surveyed deals with generalizing the above results, allowing complex coefficients, arbitrary $p \in (0,\infty)$ and allowing arbitrary initial functions $p_0(x)$ and $p_{-1}(x)$.

4.2 Fundamental results.

Definition 4.2 For $a, b \in \mathbb{C}$, define

$$CM(b,a) = \{\{a_n,b_n\}_{n=0}^{\infty} : a_n \in \mathbb{C}\backslash\{0\}, b_n \in \mathbb{C}, a_n \to a/2, b_n \to b, \text{ as } n \to \infty\}.$$

They prove:

Theorem 4.3 Let $\{a_n,b_n\} \in CM(b,a)$, $a \neq 0$. Let the sequence of functions $\{p_n(x)\}$ be generated by

$$x\, p_n(x) = a_{n+1}\, p_{n+1}(x) + b_n\, p_n(x) + a_n\, p_{n-1}(x), \tag{4.4}$$

where $p_0(x)$ and $p_{-1}(x)$ are given initial functions. Then for every $p \in (0,\infty)$,

(i) if the initial functions $p_0(x)$ and $p_{-1}(x)$ are finite in $[b-a,b+a]$ and

$$Z := \{x \in R^1 : |p_0(x)| + |p_{-1}(x)| = 0\},$$

then

$$\lim_{n\to\infty} \sup_{x\in[b-a,b+a]\setminus Z} \frac{|p_n(d\alpha,x)|^p}{\Sigma_{k=0}^{n-1} |p_k(d\alpha,x)|^p} = 0, \tag{4.5}$$

(if the initial functions $p_0(x)$ and $p_1(x)$ are polynomials, then we can take $Z = \phi$ in (4.5)).

(ii) if $\{p_n(x)\}$ is an orthogonal polynomial system associated with a positive measure $d\alpha \in M(b,a)$, where $a > 0$ and $b \in R^1$, then

$$\lim_{n\to\infty} \max_{x \in \operatorname{supp}(d\alpha)} \frac{|p_n(d\alpha,x)|^p}{\Sigma_{k=0}^{n-1} |p_k(d\alpha,x)|^p} = 0. \tag{4.6}$$

Theorem 4.4 Let the positive measure $d\alpha(x)$ have compact support and let $\{p_n(x)\}$ be a corresponding system of orthogonal polynomials. Let $0 < p < \infty$, $a > 0$, $b \in R^1$. Then $d\alpha \in M(b,a)$ if and only if

$$\lim_{n\to\infty} \frac{|p_n(d\alpha,x)|^p}{\Sigma_{k=0}^{n-1} |p_k(d\alpha,x)|^p} = \left|\frac{x-b}{a} + \sqrt{\left[\frac{x-b}{a}\right]^2 - 1}\right|^p - 1, \tag{4.7}$$

holds uniformly for every compact set $K \subset \mathbb{C}\setminus\operatorname{supp}(d\alpha)$. When p is a positive integer (4.7) remains true with the absolute value signs dropped on both sides.

4.3 Applications.

Recall that the nth Christoffel function $\lambda_n(d\alpha,x)$ is defined by

$$\lambda_n(d\alpha,x) = \left[\Sigma_{k=0}^{n-1} p_k^2(d\alpha,x)\right]^{-1}. \tag{4.8}$$

The limit relation

$$\lim_{n\to\infty} \lambda_n(d\alpha,x)\, p_n^2(d\alpha,x) = 0,$$

has important applications in the theory of orthogonal polynomials. In this section we will discuss some sufficient conditions for the above limit to hold uniformly over an interval.

Definition 4.5 A measure $d\alpha(x)$ on the closed interval I is called a Erdös measure on I if $\mathrm{supp}(d\alpha) = I$ and $\alpha'(x) > 0$ a.e. on I.

Corollary 4.6 Let $I \subset R^1$ be a closed interval. Let $d\alpha(x)$ be an Erdös measure on I. Then the corresponding system of orthogonal polynomials $\{p_n(d\alpha)\}$ satisfy

$$\lim_{n\to\infty} \max_{x \in I} \lambda_n(d\alpha,x)\, p_n(d\alpha,x) = 0. \tag{4.9}$$

<u>Proof.</u> Let $I = [b-a,b+a]$. By a theorem of Rahmanov, $a_n \to 1/2$, $b_n \to b$, as $n \to \infty$. i.e. $d\alpha \in M(b,a)$. Let $p = 2$. Then by Theorem 4.3 with $Z = \phi$, the assertion follows immediately if we recall the definition of the nth Christoffel function (4.8).

The following results concern estimates of orthogonal polynomials and Lebesgue functions.

Lemma 4.7 Let $a > 0$ and $b \in R^1$. If $d\alpha(x)$ satisfies

$$\alpha'(x) \geq C[a^2 - (x-b)^2]^{-1/2}, \text{ for } x \in [a-b,a+b], \tag{4.10}$$

then

$$\lambda_n^{-1}(d\alpha,x) \leq \frac{2n}{\pi C}, \text{ for } x \in [a-b,a+b].$$

Proof See [5, Theorem 1.4.1,p.25, Theorem3.3.4, p.105]■

Corollary 4.8 Let $a > 0$ and let $b \in R^1$. Let either $\mathrm{supp}(d\alpha) = [b-a,b+a]$ or $d\alpha \in M(b,a)$ and assume $d\alpha$ satisfies (4.10). Then

$$p_n(d\alpha,x) = o(\sqrt{n}), \quad n = 1,2,\ldots,$$

uniformly for $x \in [b-a,b+a]$.

The above Corollary gives a nice bound for orthogonal polynomials. In 1921 Steklov made the following famous conjecture:

Steklov's Conjecture. Let $w(x)$ be a weight function on $[-1,1]$ such that

$$w(x) \geq \frac{C}{\sqrt{1-x^2}}, \text{ for every } x \in [-1,1].$$

Then for every closed subinterval $[a,b]$ of $(-1,1)$ there exists a constant $C > 0$ such that

$$|p_n(w,x)| \leq C, \text{ for } x \in [a,b],\ n = 1,2,\ldots,.$$

It withstood the efforts of many researchers and was only resolved by E.A. Rahmanov in 1979, who furnished a counter example.

Define the Lebesgue function $\Omega_n(d\alpha,x)$ by

$$\Omega_n(d\alpha,x) = \sup_{\|f\|_\infty \leq 1} |S_n(d\alpha,f,x)|,$$

for partial sums

$$S_n(d\alpha,f) = \Sigma_{k=0}^{n-1} c_k(d\alpha,f)\, p_k(d\alpha),$$

where the coefficients $c_k(d\alpha,x)$ are given by

$$c_k(d\alpha,f) = \int_{-\infty}^{\infty} p_k(d\alpha,x)\, d\alpha(x).$$

Define

$$\hat{\alpha}(t) = \int_{-\infty}^{t} d\alpha(x).$$

Corollary 4.9 Let $a > 0$ and $b \in R^1$. Let $d\alpha \in M(b,a)$. If $\hat{\alpha}(x)$ is uniformly continuous on a set $\Delta \subset [b-a,b+a]$, then

$$\lim_{n\to\infty} \sup_{x \in \Delta} \lambda_n(d\alpha,x)\, \Omega_n^2(d\alpha,x) = 0.$$

If, in addition, $d\alpha(x)$ satisfies (4.10), then

$$\Omega_n(d\alpha,x) = o(\sqrt{n}), \quad n = 1,2,\ldots,$$

uniformly for $x \in \Delta$.

Definition 4.10 The G–operator is defined by

$$G_n(d\alpha,f,x) = \lambda_n(d\alpha,x) \int_{-\infty}^{\infty} f(t)\, K_n^2(d\alpha,x,t)\, d\alpha(t),$$

where $\lambda_n(d\alpha)$ is the nth Christoffel function and $K_n(d\alpha)$ is the nth reproducing kernel, i.e.

$$K_n(d\alpha,x,t) = \Sigma_{k=0}^{n-1}\, p_k(d\alpha,x)\, p_k(d\alpha,t).$$

We have seen in Chapter 1 that

$$K_n(d\alpha,x,t) = a_n(d\alpha)\, \frac{p_n(d\alpha,x)\ p_{n-1}(d\alpha,t) - p_n(d\alpha,t)\ p_{n-1}(d\alpha,x)}{x - t}.$$

Corollary 4.11 Let $a > 0$, $b \in R^1$. If $d\alpha \in M(b,a)$ and $f \in L^\infty(R^1)$ is uniformly continuous on a set $\Delta \subset [b-a,b+a]$, then

$$\lim_{n\to\infty} \sup_{x\in\Delta} | G_n(d\alpha,f,x) - f(x) | = 0.$$

The following results concern Christoffel functions and Cotes numbers.

Corollary 4.12 If $d\alpha \in M(b,a)$, $a > 0$, $b \in R^1$, then for every fixed integer m,

$$\lim_{n\to\infty} \max_{x\in[b-a,\,b+a]} \left| \frac{\lambda_{n+m}(d\alpha,x)}{\lambda_n(d\alpha,x)} - 1 \right| = 0.$$

For a function $g(x)$ and measure $d\alpha(x)$ define

$$d\alpha_g := g\, d\alpha.$$

The following Corollary concerns ratio asymptotics of corresponding Christoffel functions $\lambda_n(d\alpha)$ and $\lambda_n(d\alpha_g)$.

Corollary 4.13 Let $d\alpha \in M(b,a)$, $a > 0$ and $b \in R^1$ and let $g \geq 0$ on R^1. Assume that there is a polynomial $p(x)$ such that $pg(x)$ and $pg^{-1}(x)$ are both uniformly continuous in a set $\Delta \subset [b-a,b+a]$, and that they are both uniformly bounded in R^1. Then

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)} = g(x),$$

uniformly in every compact subset of Δ void of zeros of $p(x)$. In particular, if $g(x)$ is bounded and strictly positive in R^1 and is continuous in $[b-a, b+a]$, then the convergence is uniform in the entire interval $[b-a, b+a]$.

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