WISHART LAWS ON CONVEX CONES

A PhD thesis

by

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Summary

In high dimensional settings, Wishart distributions defined within the framework of graphical models are of particular importance.

A graphical model for a random vector $(X_1, \ldots, X_n)$ is a family $\{f_\theta : \theta \in \Theta\}$ of probability distributions such that each f_θ satisfies a set of conditional independence relations encoded in graph G: each random variable is represented by a node and the absence of an edge between two nodes symbolizes conditional independence of the corresponding random variables given the remaining random variables. For a Gaussian graphical model, with covariance matrix Σ and concentration matrix $K = \Sigma^{-1}$, the conditional independence constraints are equivalent to $K_{ij} = 0$ for all non-adjacent vertices i and j. This implies that the entries of Σ corresponding to non-adjacent vertices are not free parameters and the model can be alternatively parametrized by an incomplete matrix with entries corresponding to non-adjacent vertices left out. In the framework of Gaussian graphical models, Wishart distributions are therefore defined on two alternative restrictions of the cone of symmetric positive definite matrices: the cone P_G of symmetric positive definite matrices x satisfying $x_{ij} = 0$ for all non-adjacent vertices i and j and its dual cone Q_G. The existing construction of Wishart exponential families on Q_G and P_G used two different approaches for homogeneous and non-homogeneous cones and as a result, provides two different presentations of the distributions on the two classes of cones. Also, it does not specify the set of parameters of these families for non-homogeneous graphs. Only a conjecture was made about this parameter set.

In this thesis, we propose some parameter parsimonious models which are of great importance in high dimensional data analysis. We first present a background of classical Wishart distributions and multiparameter Wishart distributions in terms of the canonical measures of cones. Then, we provide a harmonious construction of Wishart exponential families in nearest neighbour interaction graphical models, in other terms governed by the graph A_n : $\overset{1}{\bullet} - \overset{2}{\bullet} - \cdots - \overset{n}{\bullet}$. Our simple method is based on analysis on convex cones compared to existent work which relies more on graph theory. The focus is on nearest neighbours interactions graphical models which have the advantage of being relatively simple while including all particular cases of interest such as the univariate case, a symmetric cone case, a non-symmetric homogeneous cone case and an infinite number of non-homogeneous cone cases. We derive the Laplace transforms of the Riesz generating measures. Next, we define the Wishart distributions and explicitly determine their classical objects such as the Wishart densities, the Laplace transforms and the mean functions. The Wishart distributions on Q_{A_n} are constructed as the exponential family generated from the gamma function on Q_{A_n}. The Wishart distributions on P_{A_n} are then constructed as the Diaconis-Ylvisaker conjugate family for the exponential family of Wishart distributions on Q_{A_n}. For Wishart distributions on Q_{A_n}, explicit formulas for the inverse mean map and the variance function are derived. Later, the methods of construction of Wishart laws introduced in this thesis are used to solve the Letac-Massam Conjecture on the set of parameters of type I Wishart distributions on Q_{A_n}. Finally, we introduce and study exponential families of distributions parametrized by a segment of means with an emphasis on their Fisher information. This class of models will be useful in high-dimensional data analysis, particularly when one is hesitating between two parameter values. We derive the mean function, the variance function and the Fisher information of the model. We also propose some estimators and explore their properties. The particular cases of Gaussian and Wishart exponential families parametrized by a segment of means are examined.