RESPONSE OF DYNAMIC SYSTEMS TO A CLASS OF RENEWAL IMPULSE PROCESS EXCITATIONS: NON-DIFFUSIVE MARKOV APPROACH.

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..TO MY LITTLE LORENZO,
WHOSE BIRTH PROVES THAT
AN UNEXPECTED, UNMEASURABLE
JOY CAN STRIKE ANYBODY'S LIFE
AT AN UMPREDICTABLE,
MAGIC TIME INSTANT...

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ABSTRACT

The most suitable model that idealizes random sequences of shock and impacts on vibratory systems is that of a random train of pulses (or impulses), whose arrivals are characterized in terms of stochastic point processes. Most of the existing methods of stochastic dynamics are relevant to random impulsive excitations driven by Poisson processes and there exist some methods for Erlang renewal-driven impulse processes. Herein, two classes of random impulse processes are considered. The first one is the train of impulses whose interarrival timesare driven by an Erlang renewal process. The second class is obtained by selecting some impulses from the train driven by an Erlang renewal process. The selection is performed with the aid of the jump, zero-one, stochastic process governed by the stochastic differential equation driven by the independent Erlang renewal processes. The underlying counting process, driving the arrival times of the impulses, is fully characterized. The expressions for the probability density functions of the first and second waiting times are derived and by means of these functions it is proved that the underlying counting process is a renewal (non-Erlang) process. The probability density functions of the interarrival times are evaluated for four different cases of the driving process and the results obtained for some example sets of parameters are shown graphically.

The advantage of modeling the interarrival times using the class of non-Erlang renewal processes analyzed in the present dissertation, rather than the Poisson or Erlang distributions is that it is possible to deal with a broader class of the interarrival probability density functions. The non-Erlang renewal processes considered herein, obtained from two independent Erlang renewal processes, are characterized by four parameters that can be chosen to fit more closely the actual data on the distribution of the interarrival times.

As the renewal counting process is not the one with independent increments, the state vector of the dynamic system under a renewal impulse process excitation is not a Markov process. The non-Markov problem may be then converted into a Markov one at the expense of augmenting the state vector by auxiliary discrete stochastic variables driven by a Poisson process. Other than the existing in literature (Iwankiewicz and Nielsen), a novel technique of conversion is devised here, where the auxiliary variables are all zero-one processes. In a considered class of non-Erlang renewal impulse processes each of the driving Erlang processes is recast in terms of the Poisson process, the augmented state vector driven by two independent Poisson processes becomes a non-diffusive Markov process.

For a linear oscillator, under a considered class of non-Erlang renewal impulse process, the equations for response moments are obtained from the generalized Ito's differential rule and the mean value and variance of the response are evaluated and shown graphically for some selected sets of parameters.

For a non-linear oscillator under both Erlang renewal-driven impulses and the considered class of non-Erlang renewal impulse processes, the technique of equations for moments together with a modified closure technique is devised.

The specific physical properties of an impulsive load process allow to modify the classical cumulant-neglect closure scheme and to develop a more efficient technique for the class of excitations considered. The joint probability density of the augmented state vector is expressed as sum of contributions conditioned on the ‘on’ and ‘off’ states of the auxiliary variables. A discrete part of the joint probability density function accounts for the fact that there is a finite probability of the system being in a deterministic state (for example at rest) from the initial time to the occurrence of the first impulse. The continuous part, which is the conditional probability given that the first impulse has occurred, can be expressed in terms of functions of the displacement and velocity of the system. These functions can be viewed as unknown probability densities of a bi-variate stochastic process, each of which originates a set of ‘conditional moments’. The set of relationships between unconditional and conditional moments is derived. The ordinary cumulant neglect closure is then performed on the conditional moments pertinent to the continuous part only. The closure scheme is then formulated by expressing the ‘unconditional’ moments of order greater then the order of closure, in terms of unconditional moments of lower order.

The stochastic analysis of a Duffing oscillator under the the random train of impulses driven by an Erlang renewal processes and a non-Erlang renewal process R(t), is performed by applying the second order ordinary cumulant neglect closure and the modified second order closure approximation and the approximate analytical results are verified against direct Monte Carlo simulation. The modified closure scheme proves to give better results for highly non-Gaussian train of impulses, characterized by low mean arrival rate.

1. INTRODUCTION AND PRELIMINARY CONCEPTS

1.1 PROBABILISTIC THEORY OF VIBRATIONS

'..It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge'

Pierre-Simon Laplace, Theorie Analytique des Probabilites, 1812

The interest in the quantitative aspects of 'uncertainty' that led to quantify the idea of probability was initially confined to random events connected with games of chance. Soon, it was realized that randomness characterizes the physical features of all natural phenomena and environment, so the probability theory found application in a wide variety of physical problems and developed as a rigorous mathematical discipline with the advances in science and engineering. A considerable amount of knowledge has been hinerited from the work of physicists on the Brownian motion (*Einstein (1905)*). The most significant engineering applications of the theory of random processes have occurred in the area of communication theory and control theory since the early 1930s (*Rice (1944), Rice (1945), Middleton (1960)*). It was soon realized that the theory provides a powerful tool for a more realistic treatment of a large class of engineering problems, including analysis and design of vibratory structural/mechanical systems (*Bolotin (1969)*). The primary incentive for the adaptation of the probabilistic approach to structural dynamics was the modelling of random excitations in aerospace engineering applications (*Lin (1967), Press and Houbolt (1955), Clarkson and Mead (1973), Bendat et al. (1961)*). When an airplane flies in gusty regions, the irregularly fluctuating lifting loads due to turbulence produce high stresses in the wing structure that, for certain wing configurations, may significantly influence the structural design. This kind of excitation is irregular, lacks repeatability and cannot be treated on a conventional 'deterministic basis'. The probabilistic approach provides a rational and realistic basis for system analysis and design in a mathematical framework, through systematic treatment of uncertainty, where both the excitation and the response are modelled as "stochastic" or "random" processes, which can be viewed as an infinite "ensemble" of possible "sample functions" or "realizations".

There is a large measure of uncertainty in the analysis and design of structural and mechanical systems. The loads acting upon a ship hull in rough sea is similar to that of an airplane in a gusty atmosphere. Civil engineering structures are exposed to natural forces. An offshore structure is subjected to wind loads, ocean waves and, if in a seismic region, to earthquakes. Due to random fluctuations in wind velocity and direction and to flow separation, the loads induced by wind are random in space and time (*St Denis and Pierson (1953)*). The sea waves' height, velocity and direction are random in space and time (*Davenport and Novak (1976)*). In comparison with wind and waves, earthquakes loads are rare events produced by seismic events, random in nature (*Vanmarke (1976)*).

The most significant feature of the analytical method to the analysis and design of systems is the process of idealization. Most real structural/mechanical systems have complex geometrical and material properties and operate under complex environmental conditions. The process of idealization involves simplifying assumptions for constructing analytically tractable mathematical models for the system, its environment and the interaction between them. In order to verify the suitability of a probabilistic model it is necessary to resort to measurements and statistical inference (*Bendat (1966))* The central feature of the "probabilistic approach" is a systematic treatment of uncertainty, where both the excitation and the response are modelled as "stochastic" or "random" processes, that can be viewed as an infinite "ensemble" of possible "sample functions " or "realizations".

The complete solution of a random vibration problem implies the probabilistic characterization of the response process of the dynamical system, given the probabilistic structure of the loads.

In the case of Gaussian excitation acting on linear systems, the response process is also Gaussian and the probabilistic theory of structural dynamics enables all the statistical parameters of the response to be directly related to the corresponding parameters of the excitation. A class of important non-Gaussian excitation is that of randomly occurring short-duration loads. The random train of pulses (with arbitrary pulse shape function) is a model of loading in the cases where the system is subjected to a train of shock and impacts. Also, the problem of moving loads on a bridge is reduced to that of a random pulse train.

1.1.1 The Markov Approach equation

The probabilistic law governing the future state of a system, can be determined if its present state is known, irrespective of how the system arrived at the present state, if the corresponding random process has the so-called Markov property. The class of the Markov processes is characterized by the property that the 'future' behavior of the process is independent of the 'past 'when the 'present' is known (*Barucha-Reid (1960), Stratonovich (1963), Arnold (1974), Sobczyk (1991)*). All the relevant predictions on the future depend on the most recent known state and can neglect the past. A random process $X(t), t \in T$ is said to be Markov if the conditional probability satisfies the following equation

$$\begin{gathered} P\left[X\left(t_n\right) \le x_n \middle| X\left(t_1\right) = x_1, \ldots X\left(t_{n-1}\right) = x_{n-1} \right] = \\ P\left[X\left(t_n\right) \le x_n \middle| X\left(t_{n-1}\right) = x_{n-1} \right] \end{gathered} \tag{1.1.1}$$

for any n and t_i, with $t_1 < t_2 \ldots < t_n$. The set of all possible values x is the state space that can be discrete or continuous. The Markov process is fully described in terms of its transition probability function $P[X\left(t\right) \in E\left[X\left(s\right) = x\right]]$ which is the conditional probability that the system at the time t belongs to the set E[] given that it is in a state x at a previous time s. The Markov process has 'one-step memory'. Any process with independent increments and for which the initial value is stochastically independent of any increment, is a Markov process. For this class of processes a wide variety of analytical methods of analysis are available. The probabilistic characterization of the response to random loads can be formulated in terms of its transition probability density function, which is the solution of the Fokker-Plank-Kolmorov differential equation (*Gikhman and Skorohod,1972, Risken, (1985)*). In order to remain within the framework of the Markov approach, when the excitation process does not have independent increments, it can be regarded as the rth order differential form of an auxiliary process, which in turn is the result of filtering the generating source process with independent increments through sth order filter (s>r).The state vector of the system, augmented by the state variables of a filter, governed by a set of first order differential equations driven by the process with independent increments, is a Markov vector process.

1.1.2 Poisson driven stochastic differential equstion and Ito's rule.

Differential equations represent a basic tool in the application of mathematics to natural and engineering science. A more realistic formulation of the differential equations arising in applied science, in the attempt to investigate quantitatively the regularities of phenomena that cannot be uniquely characterized, involves stochastic differential equations. The intuitive concept of randomness is formed by the mere observation that the outcomes of experiments carried out under the same conditions do not coincide.

The theory of stochastic processes, initiated in mathematics as a method of representation of the Brownian motion in terms of Markov processes, was systematically formulated by Ito (*Ito (1951)*). The stochastic differential equations have been recently recognized as an important mathematical tool for the analysis of a great variety of engineering processes. A very extensive literature exists dealing with the mathematical formulation and applications of stochastic differential equations (*Arnold (1974), Oksendal (1995), Sobczyk (1991), Friedman (1975), Gihman and Skorohod (1972)*). The usual rules of integration and differentiation of the ordinary differential calculus fail.

For random impulse process excitations, the response vector $\mathbf{X}(t)$ of a stochastic dynamic systems is governed by a stochastic differential equation of the following type

$$d\mathbf{X}(t)=\mathbf{c}\left(\mathbf{X}(t)\right)dt+\mathbf{b}\left(P(t),\mathbf{X}(t)\right)d\mathbf{N}(t) \tag{1.1.2}$$

where $d\mathbf{N}(t)$ indicates the increment of the source vector process in the time interval [t,t+dt[, $P(t)$ denotes a random variable assigned to the event occurring in the time interval $[t,t+dt]$, while $\mathbf{b}\left(P(t),\mathbf{X}(t)\right)$ carries information of the source process up to but not including the time instant t and is stochastically independent of $d\mathbf{N}(t)$.

Consider a function $V(t,\mathbf{X})$ of the state vector of the system. A jump of unit magnitude of the α th component of the vector $d\mathbf{N}(t)$ produces a jump of magnitude $d\mathbf{X}(t)=b_\alpha\left(P(t),\mathbf{X}(t)\right)$ of the state vector $\mathbf{X}$ and a jump of magnitude (*Iwankiewicz and Nielsen (1999)*)

$$dV(t,\mathbf{X})=V\left(t,\mathbf{X}(t)+b_\alpha\left(P(t),\mathbf{X}(t)\right)\right)-V\left(t,\mathbf{X}(t)\right) \tag{1.1.3}$$

Considering the Taylor expansion of the increment $dV\left(t,\mathbf{X}\right)=V\left(t+dt,\mathbf{X}\left(t+dt\right)\right)-V\left(t,\mathbf{X}\left(t\right)\right)$ and the fact that any increment is sum of the increments due to the continuous motion and the increment due to a possible jump (1.1.3), it follows (*Iwankiewicz and Nielsen (1999))*

$$dV\left(t,\mathbf{X}\right)=\frac{\partial V\left(t,\mathbf{X}\right)}{\partial t}+ \sum_{j=1}^{N}\frac{\partial V\left(t,\mathbf{X}\right)}{\partial X_j}c_j\left(\mathbf{X},\dot{\mathbf{X}}\right)+\sum_{\alpha}\left[V\left(t,\mathbf{X}+b^{\alpha}\left(P\left(t\right),\mathbf{X}\right)\right)-V\left(t,\mathbf{X}\right)\right]dN_{\alpha} \tag{1.1.4}$$

Equation (1.1.4) is the generalized Ito differential rule for system subjected to jump processes. It is an alternative of the integro-differential equations or of the partial differential equations.

In general, the Ito's differential rule originates from the dependence of $V\left(t,\mathbf{X}\right)$ on the excitation $\mathbf{N}\left(t\right)$ that should be 'non-anticipative' (*Di Paola (1993), Sobczyk (1991), Snyder (1975), Gikhman and, Skorokhod (1972)*). $V\left(t,\mathbf{X}\right)$ can depend at most on the present and past values of $\mathbf{N}\left(t\right)$, in other words is independent of the increments $\mathbf{N}\left(t_{k+1}\right)-\mathbf{N}\left(t_k\right)$, with t_k,t_{k+1} two successive time instants in an infinitesimal interval.

1.2 RANDOMLY OCCURRING SHORT DURATION LOADS

The dynamic excitation on structural and mechanical system may consist of short duration loads occurring at random times, with random magnitudes.

Typical examples of this kind of excitations occur in structural, mechanical and industrial engineering:

Dynamic actions on vehicles due to the irregularity of the road surface (*Lingren (1981), Schielen (1985)*); Dynamic behavior of crushing machines; Loads due to the atmospheric turbulence; Random train of vehicles crossing highway bridges (*Tung (1967), Tung (1969), Gerlough (1955)*); Strong earthquake phenomena seen as impulsive change of ground motion acceleration (*Cornell (1964)*).

The most suitable model of excitations that simulates random train of shock and impacts is that of a random train of pulses (with arbitrary pulse shape function).

The random impulsive excitations are characterized in terms of stochastic point processes.

1.2.1 Stochastic Point Processes

Stochastic point processes, whose realizations consist of point events in time and space, arise in many fields of application such as statistical physics, astrophysics, astronomy, biology, communication theory, management science and mechanics. Typical problems in which the point process models are used are e. g. stochastic and non-linear response problems, theory of queues, renewal theory, reliability theory (*Srinivasan (1974), Snyder (1975), Cox (1962), Cox and Isham (1980), Gross and Harris (1985), Iwankiewicz (1995), Iwankiewicz and Nielsen (1999)*).

The central idea, in the study of this particular class of random process, is the analysis of random collections of point occurrences. Consider the points occurring along a time axis, although it is possible to consider that the points occur in some region of space. In road traffic studies, we may consider the sequence of time points at which vehicles pass a reference point. Alternatively, examining the length of road at a certain time instant, we can specify the position of a vehicle by a point, having a point process in one/dimensional space rather than in time.

A random counting process $N(t)$ is an index continuous state discrete stochastic variable specifying the number of events t_i in the interval $[0,t[$, with the assumption $\Pr\{N(0)=0\}=1$. The expected number of events in every finite time interval is finite.

The increment $dN(t)=N(t+dt)-N(t)$ of the counting process in the time interval $[t,t+dt[$ is regular if

$$\begin{aligned} &\Pr\{dN(t)=1\}=\nu(t)dt+o\left(dt^2\right) \\ &\Pr\{dN(t)>1\}=o\left(dt^2\right) \\ &\Pr\{dN(t)=0\}=1-\nu(t)dt+o\left(dt^2\right) \end{aligned} \tag{1.2.1}$$

These properties mean that the probability of occurrence of one event in the infinitesimal interval $[t,t+dt[$ is proportional to dt and the probability of occurrence of more than one event is negligibly small. It follows that for any n

$$E\left[dN(t)\right]=E\left[dN(t)^n\right]=\nu(t)dt+o\left(dt^2\right) \tag{1.2.2}$$

where $\nu(t)$ is the mean arrival rate of events.

Let us choose from the interval $(0,t[$, the disjoint infinitesimal time intervals $[t_i,t_i+dt_i[\,, i=1,2...n$. The probability of occurrence of n events in the interval $(0,t[$ can be evaluated as follows

$$\Pr\{N(t)=n\}=\frac{1}{(n)!}\int_0^t..\int_0^t \pi_n\left(t_1,t_2,...t_n\right)dt_1dt_2...dt_n \tag{1.2.3}$$

where $\pi_k\left(t_1,t_2,..t_k\right)$ is the joint density function defined as (*Srinivasan (1974), Iwankiewicz (1995), Iwankiewicz and Nielsen (1999)*)

$$\pi_k\left(t_1,t_2,..t_k\right)=\sum_{n=k}^{\infty}\frac{(-1)^{n-k}}{(n-k)!}\int_0^t..\int_0^t f_n\left(t_1,t_2,..t_k,t_{k+1}...t_n\right)dt_{k+1}..dt_n \tag{1.2.4}$$

where the n-th degree product density function $f_n\left(t_1,t_2,..t_n\right)$ represent the probability that one event occurs in each of the intervals $[t_i,t_i+dt_i[$, irrespective of other events in the interval $(0,t[$, that is

$$\begin{aligned}&f_n\left(t_1,t_2,..t_n\right)=\Pr\left\{dN\left(t_1\right)=1\wedge dN\left(t_2\right)=1\wedge...dN\left(t_n\right)=1\right\}\\&t_1\neq t_2\neq..t_n\end{aligned} \tag{1.2.5}$$

1.2.3 Poisson process

The simplest point process is one in which points occur totally randomly. The probability of finding a point in the time interval (t, t + δ] does not depend on whether there have been few or many points just before t, or whether there is a point exactly at t. This property virtually excludes the possibility of multiple simultaneous occurrences.
The Poisson process is a point process whose increments defined on disjoint intervals are independent, and is completely characterized by its first order product density function

$$f_1(t)=\nu(t) \tag{1.2.6}$$

called the intensity of the Poisson process. For a homogeneous Poisson process ($\nu(t)=\nu=\cos t$), it is

$$\begin{aligned}&f_n\left(t_1,t_2,..t_n\right)=\nu^n\\&\pi_n\left(t_1,t_2,..t_n\right)=\nu^n\exp(-\nu t)\\&\Pr\left\{N(t)=n\right\}=\frac{(\nu t)^n}{n!}\exp(-\nu t)\end{aligned} \tag{1.2.7}$$

1.2.4 Renewal processes

An important class of point processes, generalising the Poisson process, is obtained assuming that the intervals are independent but not necessarily exponential distributed.
A renewal process is a random sequence of points $t_1,t_2,...,t_n$, the intervals T_i ($t_1=T_1$ and $t_i-t_{i-1}=T_i$, with $i=2,3,...$) between the successive points, called *interarrival times* being positive, independent and identically distributed random variables.
The point process is an *ordinary renewal process* if the first waiting time T_1 has the distribution as the other intervals T_i. In this case the origin is placed in the initial event that is not counted. If the origin is placed arbitrarily, the first waiting time has another distribution than other intervals and the point process is called a *delayed renewal process.*

An *ordinary renewal density* $h_0(t)$ is defined as the probability that a random point occurs in the interval $[t, t+dt[$, given that an event occurs at the origin. A *modified renewal density* $h_m(t)$ is defined as the probability that a random point occurs in the interval $[t, t+dt[$, with arbitrarily placed origin and coincide with the first-order product density:

$$h_m(t)\,dt = \Pr\left\{dN(t)=1\right\} = f_1(t)\,dt \tag{1.2.8}$$

Due to the fact that an event occurring in the interval $[t, t+dt[$ can be either the first point or one of the subsequent, the renewal densities satisfy the renewal equations (*Cox (1962), Cox and Isham (1980), Srinivasan (1974)*).:

$$\begin{aligned} h_m(t) &= g_1(t) + \int_0^t h_m(t-u)g(u)\,du \\ h_o(t) &= g(t) + \int_0^t h_o(t-u)g(u)\,du \end{aligned} \tag{1.2..9}$$

where $g_1(t)$ denote the probability of occurrence of the first event, and $g(t)$ the probability density of the subsequent intervals T_i.

The renewal densities can be evaluated by taking the Laplace transforms of the equations (1.2..9) as follows

$$\begin{aligned} h_m(t) &= L^{-1}\left\{\frac{g_1^*(s)}{1-g_1^*(s)}\right\} \\ h_o(t) &= L^{-1}\left\{\frac{g^*(s)}{1-g^*(s)}\right\} \end{aligned} \tag{1.2.10}$$

If the probability density of the interarrival times is a gamma function, with integer parameter k,

$$g(t) = \frac{\nu^k}{(k-1)!}t^{k-1}\exp(-\nu t), \qquad t>0 \tag{1.2.11}$$

the corresponding renewal process is an Erlang renewal process. Letting $k=1$, the interarrival times are negative exponential distributed

$$g(t) = \nu\exp(-\nu t), \qquad t>0 \tag{1.2.12}$$

and the corresponding renewal process is a Poisson process.

The distribution with probability density (1.2.11) is the distribution of the sum of k independent negative-exponential distributed variables, with parameter ν. The events driven by an Erlang renewal process with parameter k, can be viewed as every k th event of the generating Poisson process with parameter ν.

2. EXTENSIVE SUMMARY OF CONTENTS AND OBJECTIVES OF RESEARCH

2.1 CONVERSION OF AN ERLANG RENEWAL PROCESS INTO A POISSON ONE

One of the main contributions of the present thesis is the derivation of a transformation rule that allows to express any Erlang renewal process in terms of the corresponding Poisson one (see *Tellier and Iwankiewicz (2006)*). This is an alternative formulation to the one given in (*Nielsen, Iwankiewicz and Skjaerbaek (1995), Iwankiewicz and Nielsen (1999), Iwankiewicz and Nielsen (2000)*).

An Erlang renewal process $R_\nu(t)$ with parameters k and ν can be exactly expressed in terms of the corresponding Poisson process $N(t)$ with parameter ν through the following transformation

$$
\begin{aligned}
& dR_\nu(t) = \rho_1^\nu(t)\, dN_\nu(t) \\
& d\rho_1^\nu(t) = \left(\rho_2^\nu(t) - \rho_1^\nu(t)\right) dN_\nu(t) \\
& d\rho_2^\nu(t) = \left(\rho_3^\nu(t) - \rho_2^\nu(t)\right) dN_\nu(t) \\
& .. \\
& d\rho_{k-2}^\nu(t) = \left(\rho_{k-1}^\nu(t) - \rho_{k-2}^\nu(t)\right) dN_\nu(t) \\
& d\rho_{k-1}^\nu(t) = \left(1 - \sum_{j=1}^{k-2} \rho_j^\nu(t) - 2\rho_{k-1}^\nu(t)\right) dN_\nu(t)
\end{aligned}
\qquad (2.1.1)
$$

In chapter III, the expectation of the renewal process $R_\nu(t)$ is found to be the solution of a linear differential equation of order $k-1$ with constant coefficients, which is equivalent to a set of first order equations, in terms of stepwise stochastic variables. Those variables are exactly recast in terms of the zero-one stochastic functions $\rho_1^\nu(t)$, $\rho_2^\nu(t)$,.., $\rho_{k-1}^\nu(t)$ appearing in eq. (2.1.1). The variable $\rho_1^\nu(t)$ equals 1 in the time interval between the (n-1)st arrival of the Poisson process $N_\nu(t)$ and the n-th arrival. A sample function of the process $R_\nu(t)$ and the correspondent zero-one variables $\rho_j^\nu(t)$ ($j = 1,..k-1$), are depicted in Fig. 2.1.1. The variable $\rho_2^\nu(t)$ equals 1 in the time interval between the (n-2)nd arrival of the Poisson process $N_\nu(t)$ and the (n-1)st arrival. The variable $\rho_{k-1}^{\nu}(t)$ equals 1 in the time interval between the 1st and the second arrivals of the Poisson process $N_\nu(t)$.

If the impulsive excitation is driven by an Erlang renewal process, it can be recast in terms of a Poisson process through the transformation (2.1.1) and the original non-Markov problem is converted into a Markov one.

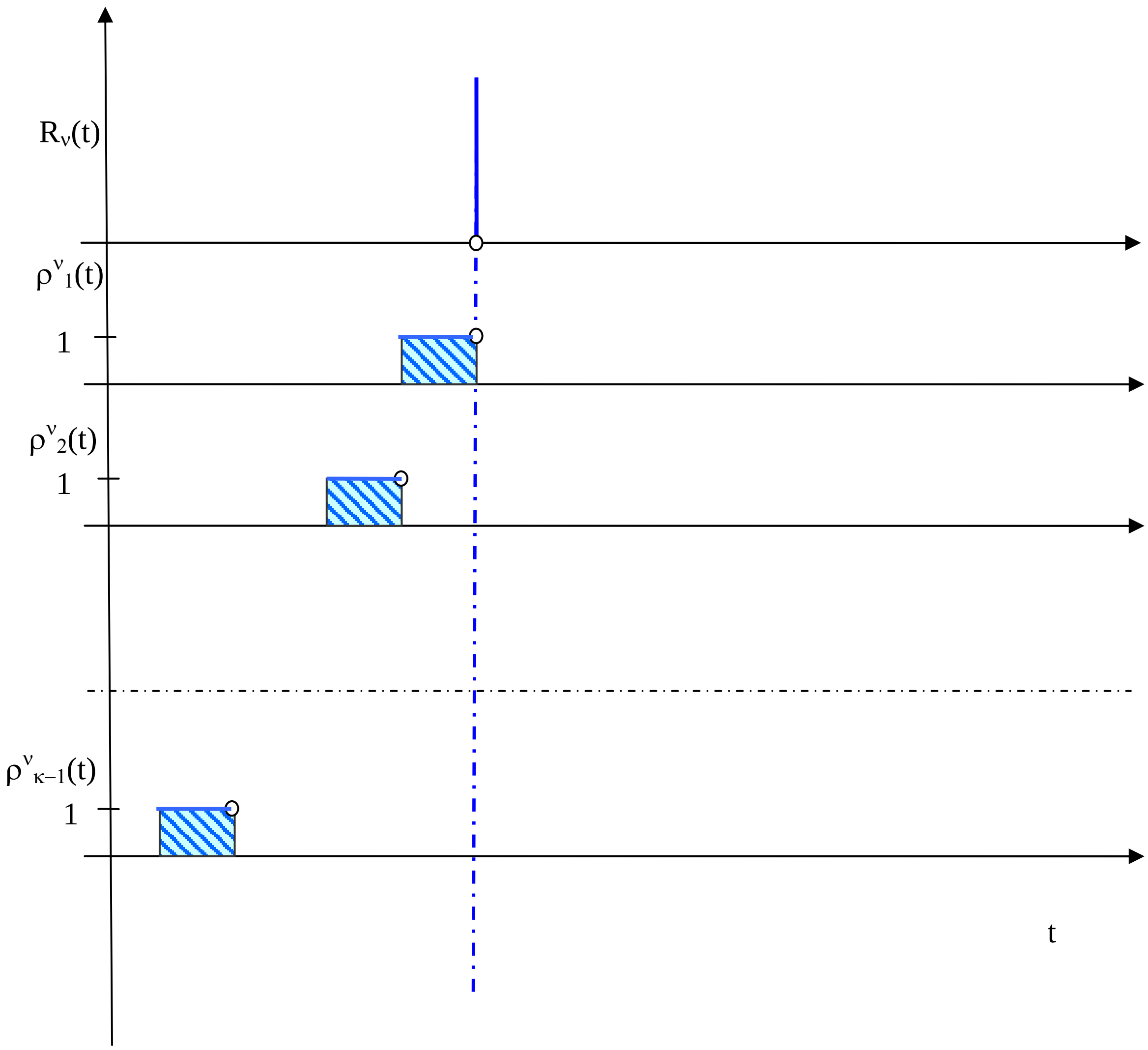

Figure 2.1.1

Sample function of an impulse process driven by an Erlang renewal process with generic parameters k and ν and auxiliary zero-one variables appearing in expressions (2.1.1).

2.2 A SPECIAL CLASS OF RENEWAL PROCESSES

Most of the existing methods of stochastic dynamics are relevant to random pulses driven by Poisson processes or Erlang renewal processes. This class of pulse problems is quite

narrow. If random occurrences of impulses are assumed to be independent, the occurrence times are described by the Poisson process. The question arises whether, and if so, to what extent the Poisson process is an adequate model of actual trains of events. In this regard, best investigated are the traffic highway phenomena.
Let us consider a road in which vehicles are driving in one direction only and all with the same constant velocity. The ‘events’ can be

- time instants when vehicles pass a certain point on the road
- considering the time scale as map of the road, the position of the vehicles at certain instant

In both cases, the form of the interarrival distribution is expected to depend on the traffic volume on the road. In a rural road the probability distribution may be taken as exponential, while on a main street in a city the vehicles tend to be equally spaced and the probability distribution should be concentrated at one point.
Gerlough paper (*Gerlough (1955)*) describes some of the applications of the Poisson distribution in highway traffic which include the analysis of arrival rates at a given point, determination of the probability of finding a vacant parking space and studies of certain accident locations.
The assumption inherent in the Poisson law, that the probability of an event remains constant, is seldom true in traffic practice. Intersection counts have shown that arrivals through the entire peak hour are not Poisson (only arrivals during the peak period within the peak hour are Poisson). The failure of the chi-square test (non acceptability of fit) indicates that the distribution of arrivals does not conform to a Poisson distribution. Gerlough concludes that the Poisson distribution cannot be assumed as an adequate description of the data.
More general is the modeling in terms of renewal processes which are defined as sequences of independent, identically distributed random variables (inter-arrival times). Different probability distributions can be assumed for the inter-arrival times, thus resorting to renewal processes allows accounting for more realistic, unimodal probability density functions of the interarrival times. One of the renewal processes widely used in traffic engineering is the Erlang process. It affords the opportunity of considering the distribution of vehicles for all the cases from independency (the special case of negative exponential distribution, the time spacing distribution between Poisson arrivals for which k=1) and complete uniformity ($k=\infty$). While the negative exponential distribution is characterized by only one parameter, the Erlang distribution has two parameters that can be estimated from the mean and the variance of the field measurements.
The advantage of modeling the interarrival times using the class of renewal processes analyzed in the present dissertation, rather than the Poisson or Erlang distributions is that it is possible to deal with a broader class of the interarrival probability density functions. The non-Erlang renewal processes considered herein, obtained from two independent Erlang renewal processes, are characterized by four parameters that can be chosen to fit more closely the actual data on the distribution of the interarrival times.

2.2.1 Characterization of the non-Erlang renewal process R(t)

The class of impulse processes here considered is obtained by selecting the events from an Erlang renewal process $R_\nu(t)$ with parameters ν and k, with the aid of the replacement $\sum_{i,R=1}^{R(t)} P_{i,R}\delta(t-t_i) = \sum_{i=1}^{R_\nu(t)} Z(t_i) P_i \delta(t-t_i)$, where the zero-one stochastic variable $Z(t)$ is governed by the following stochastic differential equation:

$$dZ(t) = (1-Z)\,dR_\mu(t) - Z\,dR_\nu(t) \qquad (2.2.1)$$

The variable $Z(t)$, a left continuous variable with right limits (see Figure 2.2.1) is zero except in the time interval between the first $R_\mu(t)$ driven event occurring after a $R_\nu(t)$ driven event and the first subsequent $\Re_\nu(t)$ driven event. In other words, $Z(t_i)$ is zero at all instants t_i driven by $R_\nu(t)$ except the first ones occurring after $R_\mu(t)$ driven events.
The increment of this class of non-Erlang processes becomes

$$dR(t) = Z(t)\,dR_\nu\,. \qquad (2.2.2)$$

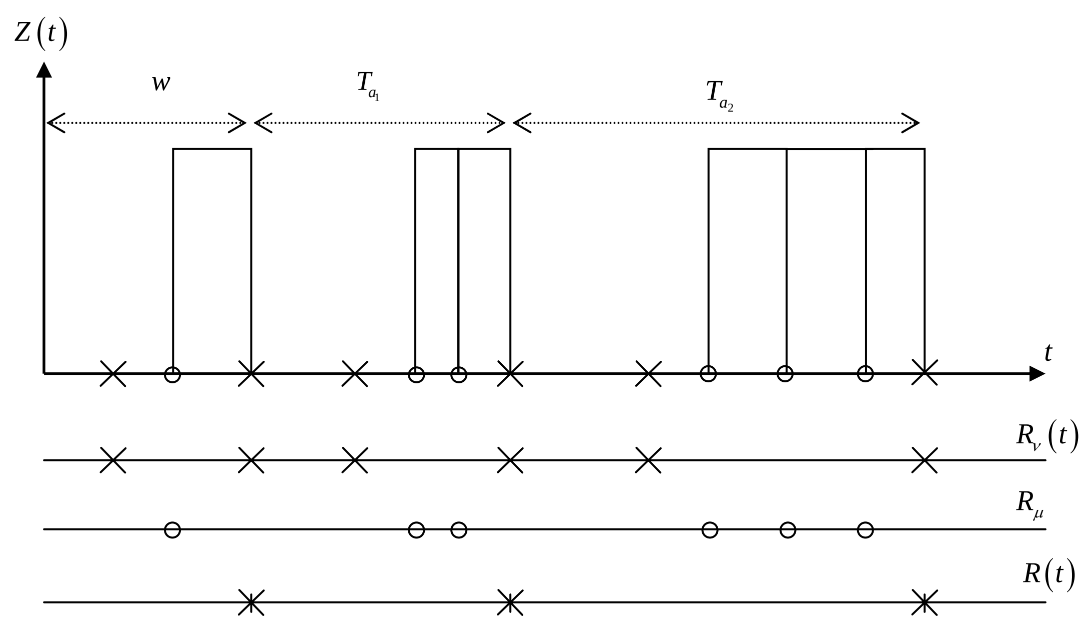

Figure 2.2.2.

Sample function of the process $Z(t)$ governed by the equation (2.2.1) and the correspondent renewal process governed by the equation (2.2.2).

2.3 ANALYSIS OF LINEAR SYSTEMS UNDER RANDOM TRAINS OF IMPULSES

For any stochastic point process, the statistical moments of the response of a linear system may be evaluated in the form of explicit integral expressions in terms of the product densities of the underlying point process (*Iwankiewicz (1995), Iwankiewicz and*

Nielsen (1999)). However, the drawback of this approach is that the evaluation of higher order response moments requires the cumbersome evaluation of multifold integrals.

Computationally more effective and applicable to non-linear systems as well, is the approach that leads to differential equations. Such an approach requires the formulation of the problem in terms of stochastic differential equations and the use of the theory of Markov processes. If the dynamic system is excited by a Poisson distributed train of impulses, the state vector of the system is a non-diffusive Markov process and the tools of the theory of Markov processes can be directly used. If however, the point process generating the impulse train is not a Poisson process the state vector of the dynamic system is not a Markov process.

One of the most effective approaches to deal with linear and non-linear systems subjected to non-Poisson trains of impulses is to convert the original problems into Markov ones. For a Poisson train of overlapping pulses, an auxiliary filter under a Poisson impulse process may be used to transform the original non-Markov problem to a Markov one (*Ricciardi (1994)*). When the excitation is a filtered Poisson process of polynomial form (*Grigoriu and Waisman (1986)*), the state vector, augmented by Poisson driven stochastic variables, is a Markov process. An exact converting technique was developed for trains of impulses driven by Erlang renewal processes (*Nielsen et al. (1995)*). The original, Erlang-driven, train of impulses is recast into a Poisson-driven one with the aid of auxiliary stochastic variables driven by a Poisson process. A random train of impulses driven by a generalised Erlang renewal process, where the interarrival times are sum of two independent, negative exponential distributed random variables, was dealt with in (*Iwankiewicz (2002)*). Recasting the original train of impulses driven by a generalised Erlang renewal process is performed with the aid of a zero-one valued auxiliary variable governed by two independent Poisson processes. Extension of this approach to more general, non-Erlang renewal processes can be done by introducing an auxiliary variable governed by a stochastic equation driven by auxiliary processes: one Poisson process and one Erlang renewal process (*Iwankiewicz*, *2003*).

In the present dissertation, the response of a linear oscillator under a random train of impulses driven by the process $R(t)$ defined by equations (2.2.1) and (2.2.2) is analyzed. The underlying process defines a class of non-Erlang renewal processes obtained by multiplying the random impulses magnitudes of an Erlang renewal process $R_\nu(t)$, by a zero-one stochastic variable $Z(t)$. This variable is driven by two independent Erlang renewal processes that are, in turn, exactly expressed, with the aid of auxiliary variables $Z_1 \quad Z_2 .. \quad Z_N$, in terms of Poisson processes. In this way, the state vector of the dynamic system augmented by auxiliary variables becomes a non-diffusive Markov process.

The stochastic equations governing the augmented state vector $\mathbf{X} = \left[X \quad \dot{X} \quad Z_1 \quad Z_2 .. \quad Z_N \right]^T$ can be written as

$$d\mathbf{X}(t) = \mathbf{c}\left(\mathbf{X}(t)\right)dt + \mathbf{b}\left(P(t), \mathbf{X}(t)\right)d\mathbf{N}(t) \tag{2.3.1}$$

One of the major contributions of the present thesis is the development of the Markov approach to non-Markov problems arising when the special class of renewal processes here considered is used in modeling impulsive loading phenomena. The use of this class of renewal counting processes also allows a better fitting to the actual data on the distribution of interarrival times than the classical models using Poisson and Erlang processes.

The equation governing the evolution of the mean values of the auxiliary variables are derived and the renewal densities of the underlying renewal process are evaluated for different excitation processes or different cases of the driving process. The general expression for the probability density of the first waiting time is derived. The probability density functions of the interarrival times are found for different cases and analyzed for various sets of parameters. Equations for the response mean values are obtained by direct averaging of the governing stochastic equations. Equations for the response variance are derived with the aid of the generalized Ito's rule. Mean value and variance of the transient response of a linear oscillator are evaluated numerically and analyzed.

2.4 NON-LINEAR SYSTEMS

2.4.1 Review of closure techniques

In engineering applications, linear models are usually accurate to reproduce the dynamic behaviour of structures under small-amplitude vibrations. However, no real system is linear and a non-linear model of analysis is certainly more adequate under large amplitude levels. The structural response to natural hazard loads may exhibit strongly non-linear characteristics.

The methods based on the Ito differential rule allow to characterize the probabilistic structure of the response of systems subjected to random loadings. When the probabilistic method is applied to non-linear systems, it is necessary to introduce some approximations in order to evaluate the statistics of the response process. Since the system is non-linear, the equations for moments involve unknown expectations of non-linear transformations of the state variables. When the non-linearity is a polynomial (of degree>1), the equations for moments form an infinite hierarchy. For non-linearities other than polynomial, the expectations appearing in the equations for moments cannot be explicitly expressed in terms of moments, but must be evaluated as integrals respect to the unknown probability density of the system.

The most natural technique consists in replacing the set of non-linear differential equations governing the non-linear problem by an equivalent set of linear equations whose parameters are obtained minimizing in a convenient way the difference between the two sets or 'error'. The statistical linearization method provides a useful analytical tool in the analysis of physical systems with weak non-linearities (*Roberts and Spanos (1990)*). Non-Gaussian closure methods have been developed in order to evaluate the stochastic response of strongly non-linear systems. The cumulant neglect closure is based on the truncation of the Taylor series expansion of the log-characteristic function (*Stratonovich,(1963)*). The quasi-moment neglect closure originates from the truncation of the A-type Gram-Charlier expansion of the probability density function of the response (*Ibrahim (1985), Wu and Lin. (1984)*). The coefficients of a given order of the two series expansions are related to the moments of the response by algebraic relationships (*Muscolino (1993), Kenney and Keeping (1951), Abramowitz and Stegun (1972)*).

2.4.2 Analysis of non-linear systems under random trains of impulses

The use of a non-linear model becomes fundamental in modeling natural impact loads such as strong ground motion acceleration due to earthquakes (*Lin (1963), Cornell (1964)*), loading caused by wind gusts associated with eddies (*Merchant (1964)*), dynamic action due to waves on offshore structures (*Madsen (1988)*), intermittent 'down-wash' exciting an airplain tail, the motion of vehicles on rough ground (*Roberts (1966)*).
The simplest model of such excitations is a Poisson-distributed train of impulses. Roberts in (*Roberts (1972)*) analysed the problem of non-linear dynamical systems under such excitation, devising a perturbation solution to the Fokker-Planc-Kolmogorov equation governing the probability density function of the response. Cai and Lin proposed an improved perturbation technique (*Cai and Lin (1982), Lin and Cai (1995)*). Tylikowski and Marowski in (*Tylikowski and Marowski (1986)*) applied the equivalent linearization to such a problem.
Another approach, to the problem of a non-linear system under a Poisson driven train of impulses based on closure approximations of the equations for moments was formulated in (*Iwankiewicz and Nielsen (1990), Iwankiewicz and Nielsen (1992), Iwankiewicz (1995)*). The same approach was then applied to the case of renewal driven impulses (*Iwankiewicz and Nielsen (1994*), *Nielsen et al.(1995), Nielsen and Iwankiewicz (1997)*).
A cell-to-cell mapping technique for Poisson impulses, renewal impulses and Poisson pulses was developed in (*Koyluoglu et al.(1994), Koyluoglu et al.(1995),* (*Iwankiewicz and Nielsen (1996), Di Paola and Falsone (1993)).* Grigoriu in (*Grigoriu (1996*)) applied the equivalent linearization technique to solve the equation governing the characteristic function of the response of a non-linear system to a Poisson impulse process.

2.4.3 Modified closure scheme

Let us assume that the excitation is a random train of impulses driven by an Erlang renewal process or driven by the non-Erlang renewal process $R(t)$ defined in equation (2.2.2). The load process can be exactly expressed, with the aid of a suitable set of auxiliary variables, in terms of Poisson processes. Thus the augmented state vector, consisting of the original state vector and of auxiliary variables, is driven by two independent Poisson processes, and becomes a Markov process. The Ito's differential rule is used to derive the differential equations governing the response statistical moments.
A novel closure scheme is here developed that takes into account the specific physical properties of impulsive load processes. The joint probability density of the augmented state vector is expressed as sum of contributions conditioned on the 'on' and 'off' states of the auxiliary variables. A discrete part accounts for the fact that there is a finite probability of the system being in a deterministic state from the initial time to the occurrence of the first impulse, the continuous part, which is the conditional probability given that the first impulse has occurred, can be expressed in terms of functions of the

displacement and velocity of the system. These functions can be viewed as unknown probability densities of a bi-variate stochastic process, each of which originates a set of ‘conditional moments’. The ordinary cumulant neglect closure is then performed on the conditional moments pertinent to the continuous part only. From the expression of the joint probability density function the relationships between unconditional and conditional moments are derived. The closure scheme is then formulated by expressing the ‘unconditional’ moments of order greater then the order of closure, in terms of unconditional moments of lower order.

The stochastic analysis of a Duffing oscillator under the the random train of impulses driven by an Erlang renewal processes or a non-Erlang renewal process R(t), is performed by applying the ordinary cumulant neglect closure and the modified closure approximation and the approximate analytical results are verified against direct Monte Carlo simulation. Departure of the excitation process from Gaussianity depends on the ratio between the mean arrival rate of the impulses and the system natural frequency. As the ratio decreases, the departure from Gaussianity increases. The modified closure scheme proves to give better results for highly non-Gaussian train of impulses, characterized by low mean arrival rate.

3. DYNAMIC RESPONSE OF LINEAR SYSTEMS

A novel transformation rule that allows recasting any Erlang renewal process in terms of the corresponding Poisson one is here devised. A more general class of renewal processes is then considered, obtained by selecting impulses from an Erlang-driven train with the aid of an auxiliary jump, zero-one, stochastic variable driven by two independent Erlang processes.
The analysis of linear systems under such excitations is then performed by using the tools of the theory of the Markov processes. Conversion of the original non-Markov problem for the original state vector driven by a renewal impulse process into a Markov problem is performed by means of augmenting the state vector by auxiliary variables which are the jump stochastic processes.

3.1 ERLANG RENEWAL IMPULSE PROCESS

3.1.1 Statement of the problem

Consider a linear oscillator governed by the equation

$$\ddot{X}(t)+2\zeta\omega\dot{X}(t)+\omega^2 X(t)=\sum_{i,R=1}^{R(t)} P_{i,R}\delta\left(t-t_{i,R}\right) \tag{3.1.1}$$

where the excitation is a random train of impulses whose interarrival times $t_{i,R}$ are driven by an Erlang renewal process $R(t)$. The impulses magnitudes $P_{i,R}$ are independent, identically distributed random variables. Each of the variables $P_{i,R}$ is assigned to a random point $t_{i,R}$. It is assumed that the counting process $R(t)$ gives the number of events in the time interval (0,t), excluding the one that possibly occurs at t. The possibility of an event at the origin is excluded, which implies $R(0)=0$ with probability 1. Hence, the sample paths of $R(t)$ are left-continuous with right limits.
If $R(t)$ is an Erlang renewal process, with parameters α and k, the original train of impulses may be replaced by a Poisson driven one with the aid of an auxiliary variable $\rho(t)$, as $\sum_{i,R=1}^{R(t)} P_{i,R}\delta\left(t-t_{i,R}\right)=\sum_{i=1}^{N(t)}\rho(t_i)P_i\delta(t-t_i)$, where $\rho(t)$ is a jump zero-one stochastic process with $\rho(0)=\rho\left(N(0)\right)=0$ with probability 1, whose sample paths are left-continuous with right limits. Therefore $\rho(t_i)=\rho\left(t_{i^-}\right)=1$ at every $n\cdot k$-th Poisson driven event ($n\geq 1$), otherwise $\rho(t_i)=0$. Then

$$dR(t)=\rho(t)\,dN(t) \tag{3.1.2}$$

where $dN(t)=N(t+dt)-N(t)$. The impulses magnitudes P_i are independent random variables, identically distributed as the variables $P_{i,R}$. Each of the variables P_i is assigned to a random point.

3.1.2 Conversion of an Erlang renewal impulse process into a Poisson one

If an Erlang renewal process $R(t)$ has parameters α and k, the time intervals between events have gamma distribution, with density function

$$g(t)=\frac{\alpha^k t^{k-1}}{(k-1)!}e^{-\alpha t}; \tag{3.1.3}$$

The renewal density of an ordinary renewal process, defined as $E\left[dR(t)\right]=h(t)dt$, can be evaluated as [Srivanisan (1974); Cox (1962); Cox and Isham (1980)]

$$h(t)=\mathcal{L}^{-1}\left\{\frac{g^*(s)}{1-g^*(s)}\right\}; \tag{3.1.4}$$

From (3.1.3) it follows that

$$g^*(s)=\frac{\alpha^k}{(\alpha+s)^k}; \tag{3.1.5}$$

then (3.1.4) becomes

$$h(t)=\mathcal{L}^{-1}\left\{\frac{\alpha^k}{(\alpha+s)^k-\alpha^k}\right\}; \tag{3.1.6}$$

hence (cf. Srivanisan (1974); Cox (1962); Cox and Isham (1980)])

$$h(t)=\frac{\alpha}{2}-\frac{\alpha}{2}e^{-2\alpha t}\ , \qquad k=2,$$

$$\begin{aligned}h(t)&=\frac{\alpha}{3}-\frac{\alpha}{2}e^{-\frac{1}{2}\left(3+i\sqrt{3}\right)\alpha t}\left(\frac{1}{3}+\frac{i}{\sqrt{3}}\right)-\frac{\alpha}{2}e^{-\frac{1}{2}\left(3-i\sqrt{3}\right)\alpha t}\left(\frac{1}{3}-\frac{i}{\sqrt{3}}\right)\\&=\frac{\alpha}{3}-\frac{\alpha}{3}e^{-\frac{3}{2}\alpha t}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\alpha t\right)+\cos\left(\frac{\sqrt{3}}{2}\alpha t\right)\right)\end{aligned}\ , \qquad k=3,$$

$$\begin{aligned}h(t)&=\frac{\alpha}{4}-\frac{\alpha}{4}e^{-2\alpha t}-i\frac{\alpha}{4}e^{-(1+i)\alpha t}+i\frac{\alpha}{4}e^{-(1-i)\alpha t}\\&=\frac{\alpha}{4}\left(1-2\sin(\alpha t)e^{-\alpha t}-e^{-2\alpha t}\right)\end{aligned}\ , \qquad k=4. \tag{3.1.7}$$

It can be shown that for an arbitrary k we obtain

$$h(t)=\frac{\alpha}{k}+\frac{\alpha}{k}\sum_{j=1}^{k-1}e^{\lambda_j t}\frac{\prod\limits_{\substack{l=1\\l\neq j}}^{k-1}\lambda_l}{\sum\limits_{\substack{l=1\\l\neq j}}^{k-1}\left(\lambda_j-\lambda_l\right)} \tag{3.1.8}$$

where λ_j is the j^{th} root of the following polynomial of order $k-1$

$$p_{k-1}(x) = (x-\lambda_1)(x-\lambda_2)...(x-\lambda_{k-1}) = \sum_{j=1}^{k-1} \binom{k}{j} \alpha^{k-j} x^{j-1} + x^{k-1} \tag{3.1.9}$$

Noting that the solution of a kth order differential equations is the sum of a constant term and of k-1 exponentials whose arguments are given by t times the corresponding root of the polynomial (3.1.9), it can be proved that the renewal density (3.1.8) is a solution of the linear differential equation of order $k-1$ with constant coefficients

$$\frac{d^{k-1}h(t)}{dt^{k-1}} + \sum_{j=1}^{k-1} \binom{k}{j} \alpha^{k-j} \frac{d^{j-1}h(t)}{dt^{j-1}} = \alpha^k, \qquad h(0) = \frac{dh(0)}{dt} = .. \frac{d^{k-2}h(0)}{dt^{k-2}} = 0 \tag{3.1.10}$$

Performing the Laplace transform of both sides of eqn. (3.1.10), and observing that

$$\begin{aligned}
&\mathcal{L}\left(\frac{dh(t)}{dt}\right) = sh^*(s) - h(0) \\
&\mathcal{L}\left(\frac{d^2h(t)}{dt^2}\right) = s^2h^*(s) - sh(0) - \frac{dh(0)}{dt} \\
&\quad .. \\
&\mathcal{L}\left(\frac{d^nh(t)}{dt^n}\right) = s^nh^*(s) - \sum_{j=1}^{n} s^{n-j} \frac{d^{j-1}h(0)}{dt^{j-1}}
\end{aligned} \tag{3.1.11}$$

we obtain

$$\left(\sum_{j=1}^{k} \binom{k}{j} \alpha^{k-j} s^{j-1}\right) h^*(s) = \frac{\alpha^k}{s} \tag{3.1.12}$$

hence

$$h^*(s) = \frac{\alpha^k}{\left(\sum_{j=1}^{k} \binom{k}{j} \alpha^{k-j} s^j\right)} = \frac{\alpha^k}{\left(\sum_{j=0}^{k} \binom{k}{j} \alpha^{k-j} s^j - \alpha^k\right)} = \frac{\alpha^k}{(\alpha+s)^k - \alpha^k} \tag{3.1.13}$$

which is indeed the Laplace transform of the renewal density $h(t)$ (cf.(3.1.6)).
The renewal density can also be obtained, from the identity (3.1.2), as

$$h(t)dt = E\left[dR(t)\right] = E\left[\rho^\alpha(t)\right]\alpha dt \tag{3.1.14}$$

hence, it follows that the differential equation governing $E\left[\rho^\alpha(t)\right]$ obtained by dividing the equation (3.1.10) through by α, is

$$
\begin{aligned}
&\frac{d^{k-1}E\left[\rho^{\alpha}(t)\right]}{dt^{k-1}}+\sum_{j=1}^{k-1}\binom{k}{j}\alpha^{k-j-1}\frac{d^{j-1}E\left[\rho^{\alpha}(t)\right]}{dt^{j-1}}=\alpha^{k-1},\\
&E\left[\rho^{\alpha}(0)\right]=\frac{dE\left[\rho^{\alpha}(0)\right]}{dt}=..\frac{d^{k-2}E\left[\rho^{\alpha}(0)\right]}{dt^{k-2}}=0
\end{aligned}
\tag{3.1.15}
$$

Equation (3.1.15) is equivalent to the set of first order equations

$$
\begin{aligned}
&\dot{x}_1=x_2,\\
&\dot{x}_2=x_3,\\
&..\\
&\dot{x}_{k-2}=x_{k-1},\\
&\dot{x}_{k-1}=\alpha^{k-1}-\sum_{j=1}^{k-1}\binom{k}{j}\alpha^{k-j}x_j.
\end{aligned}
\tag{3.1.16}
$$

where $x_1=E\left[\rho^{\alpha}(t)\right]$. After the change of variables $y_j=x_j/\alpha^{j-1}$, the differential system (3.1.16) becomes

$$
\begin{aligned}
&\dot{y}_1=\alpha y_2,\\
&\dot{y}_2=\alpha y_3,\\
&..\\
&\dot{y}_{k-2}=\alpha y_{k-1},\\
&\dot{y}_{k-1}=\alpha\left(1-\sum_{j=1}^{k-1}\binom{k}{j}y_j\right).
\end{aligned}
\tag{3.1.17}
$$

Equations (3.1.17) govern the time evolution of the expectation $E\left[\rho_{\alpha}(t)\right]$, which is also the solution to equation (3.1.15) and which equals (cf. equation (3.1.8))

$$
E\left[\rho^{\alpha}(t)\right]=\frac{1}{k}\left(1+\sum_{j=1}^{k-1}e^{\lambda_j t}\frac{\prod_{\substack{l=1\\ l\neq j}}^{k-1}\lambda_l}{\sum_{\substack{l=1\\ l\neq j}}^{k-1}\left(\lambda_j-\lambda_l\right)}\right)
\tag{3.1.18}
$$

The form of equations (3.1.17) implies that the auxiliary variable $\rho^{\alpha}(t)=\rho_1^{\alpha}(t)$ is governed by the following stochastic differential equations

$$
\begin{aligned}
&d\rho_1^{\alpha}(t)=\rho_2^{\alpha}(t)\,dN_{\alpha}(t)\\
&......\\
&d\rho_{k-2}^{\alpha}(t)=\rho_{k-1}^{\alpha}(t)\,dN_{\alpha}(t)\\
&d\rho_{k-1}^{\alpha}(t)=\left(1-\sum_{j=1}^{k-1}\binom{k}{j}\rho_j^{\alpha}(t)\right)dN_{\alpha}(t)
\end{aligned}
\tag{3.1.19}
$$

For $k=2$, equations (3.1.19) become (cf. *Nielsen, Iwankiewicz and Skjaerbaek (1995))*

$$d\rho_1^{\alpha}(t) = \left(1 - 2\rho_1^{\alpha}(t)\right) dN_{\alpha}(t)$$

This is an alternative formulation to the one given in *Iwankiewicz and Nielsen (1999), Nielsen, Iwankiewicz and Skjaerbaek (1995), Iwankiewicz and Nielsen (2000).*

For example, for $k=3$ equation (3.1.19) becomes (the superscripts α are dropped out for the sake of simplicity of notation)

$$\begin{aligned} d\rho_1(t) &= \rho_2(t)\, dN_{\alpha}(t) \\ d\rho_2(t) &= \left(1 - 3\rho_1(t) - 3\rho_2(t)\right) dN_{\alpha}(t) \end{aligned} \tag{3.1.20}$$

It can be observed from Figure 3.1.1 that

$$\begin{aligned} \rho_2(t) &= \rho_2^*(t) - \rho_1(t) \\ d\rho_2(t) &= d\rho_2^*(t) - d\rho_1(t) \end{aligned} \tag{3.1.21}$$

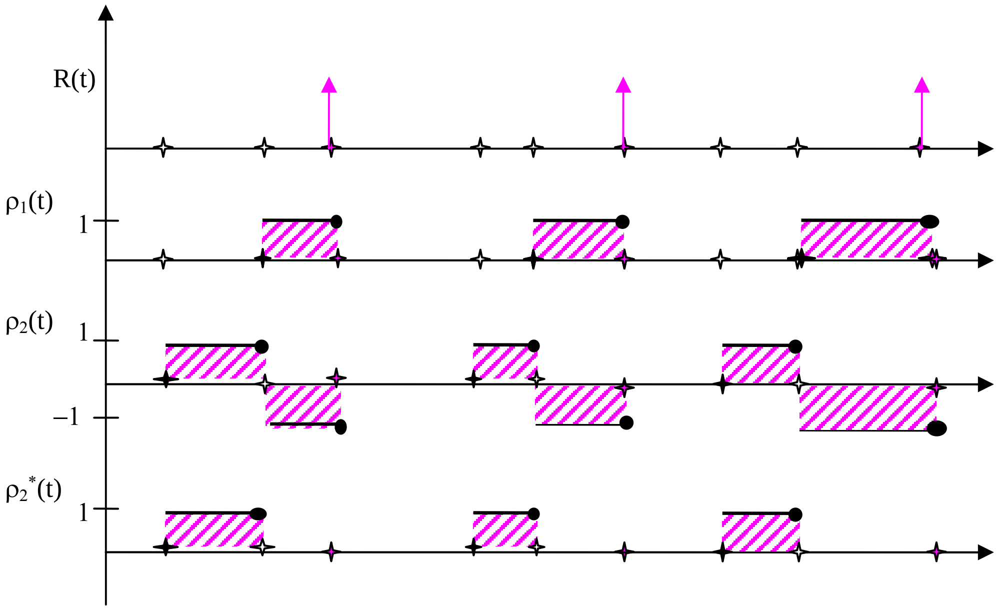

Figure 3.1.1

Sample function of the process $R(t)$ *defined in (3.1.2) for k=3 and auxiliary stochastic variables governed by (3.1.20) and (3.1.21)*

and from (3.1.20) it follows

$$\begin{aligned} d\rho_1(t) &= \left(\rho_2^*(t) - \rho_1(t)\right) dN_{\alpha} \\ d\rho_2^*(t) &= d\rho_2(t) + d\rho_1(t) = \left(1 - \rho_1(t) - 2\rho_2^*(t)\right) dN_{\alpha} \end{aligned} \tag{3.1.21}$$

Hence, equation (3.1.20) may be written as

$$
\begin{aligned}
d\rho_1(t) &= \left(\rho^*{}_2(t) - \rho_1(t)\right) dN_\alpha(t) \\
d\rho^*{}_2(t) &= \left(1 - \rho_1(t) - 2\rho^*{}_2(t)\right) dN_\alpha(t)
\end{aligned} \qquad (3.1.22)
$$

where the variables are all zero-one stochastic variables (see Figure 3.1.2).

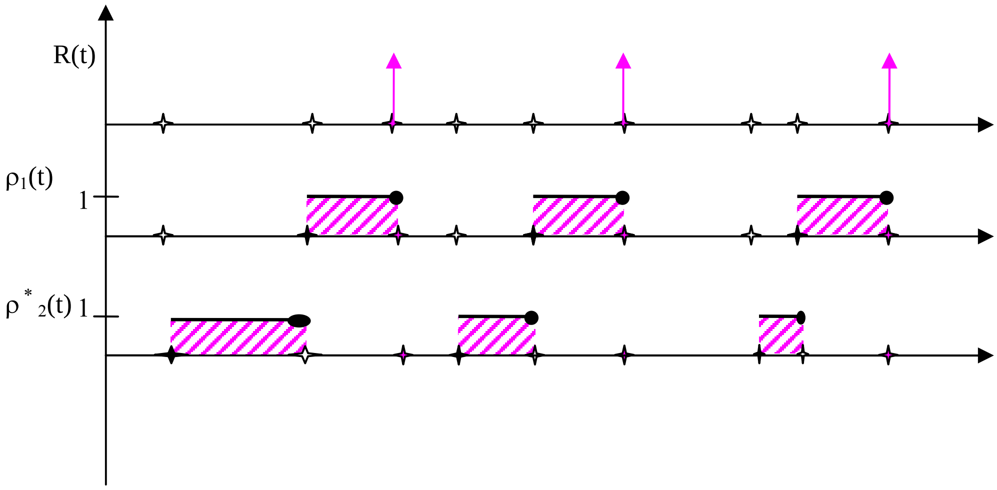

Figure 3.1.2

Sample function of the process $R(t)$ *defined in (3.1.2) for k=3 and auxiliary stochastic variables governed by (3.1.22)*

For $k = 4$, the renewal process can be expressed as follows

$$
\begin{aligned}
d\rho_1(t) &= \rho_2(t) dN_\alpha(t) \\
d\rho_2(t) &= \rho_3(t) dN_\alpha(t) \\
d\rho_3(t) &= \left(1 - 4\rho_1(t) - 6\rho_2(t) - 4\rho_3(t)\right) dN_\alpha(t)
\end{aligned} \qquad (3.1.23)
$$

where the auxiliary variables $\rho_1(t), \rho_2(t)$ and $\rho_3(t)$ are depicted in Figure 3.1.3.
It can be observed that

$$
\begin{aligned}
\rho_2(t) &= \rho_2^*(t) - \rho_1(t) \\
\rho_3(t) &= \rho_3^*(t) - 2\rho_2^*(t) + \rho_1(t) \\
d\rho_2(t) &= d\rho_2^*(t) - d\rho_1(t) \\
d\rho_3(t) &= d\rho_3^*(t) - 2d\rho_2^*(t) + d\rho_1(t)
\end{aligned} \qquad (3.1.24)
$$

and from (3.1.23) and (3.1.24) it follows

$$
\begin{aligned}
d\rho_1(t) &= \left(\rho_2^*(t) - \rho_1(t)\right) dN_\alpha \\
d\rho_2^*(t) &= d\rho_2(t) + d\rho_1(t) = \left(\rho_3^*(t) - \rho_2^*(t)\right) dN_\alpha \\
d\rho_3^*(t) &= d\rho_3(t) + 2d\rho_2^*(t) - d\rho_1(t) = \left(1 - \rho_1(t) - \rho_2^*(t) - 2\rho_3^*(t)\right) dN_\alpha
\end{aligned} \qquad (3.1.25)
$$

where $\rho_1(t)$, $\rho^*{}_2(t)$, $\rho^*{}_3(t)$, as depicted in figure 3.1.4, are zero-one stochastic variables

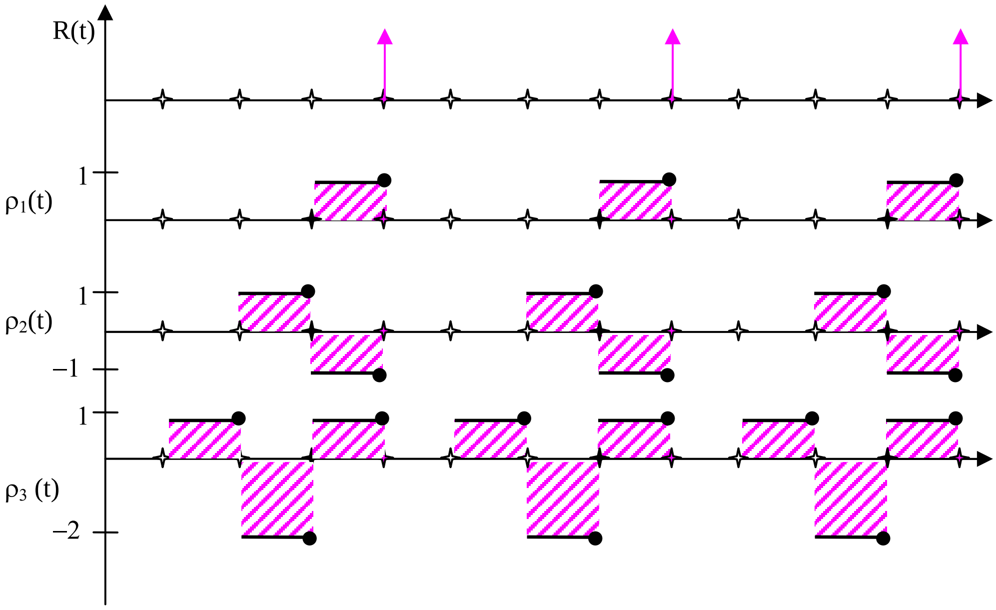

Figure 3.1.3

Sample function of the process $R(t)$ *defined in (3.1.2) for k=4 and auxiliary stochastic variables governed by (3.1.23) and (3.1.24)*

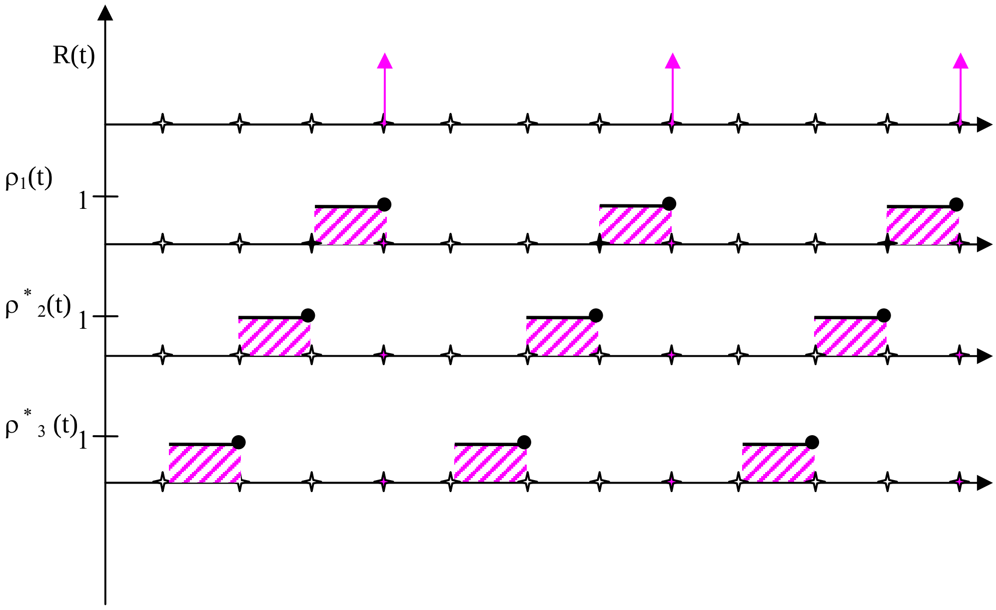

Figure 3.1.4

Sample function of the process $R(t)$ *defined in (3.1.2) for k=4 and auxiliary stochastic variables governed by (3.1.25)*

In general, an Erlang renewal process characterized by an arbitrary couple of parameters α and k can be expressed in terms of the correspondent Poisson process, through the following transformation:

$$
\begin{aligned}
&d\rho_1^\alpha(t) = \left(\rho_2^\alpha(t) - \rho_1^\alpha(t)\right) dN_\alpha(t) \\
&d\rho_2^\alpha(t) = \left(\rho_3^\alpha(t) - \rho_2^\alpha(t)\right) dN_\alpha(t) \\
&.. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.1.26) \\
&d\rho_{k-2}^\alpha(t) = \left(\rho_{k-1}^\alpha(t) - \rho_{k-2}^\alpha(t)\right) dN_\alpha(t) \\
&d\rho_{k-1}^\alpha(t) = \left(1 - \sum_{j=1}^{k-2} \rho_j^\alpha(t) - 2\rho_{k-1}^\alpha(t)\right) dN_\alpha(t)
\end{aligned}
$$

where the stars have been dropped for the sake of simplicity.
The variables $\rho_1^\alpha(t), \rho_2^\alpha(t), \ldots.., \rho_{k-1}^\alpha(t)$ only take values 0 or 1. The variable $\rho_1(t)$ equals 1 in the time interval between the (n-1)st arrival of the Poisson process $N_\alpha(t)$ and the n-th arrival. The variable $\rho_2^\alpha(t)$ equals 1 in the time interval between the (n-2)nd arrival of the Poisson process $N_\alpha(t)$ and the (n-1)st arrival. …..
The variable $\rho_{k-1}^\alpha(t)$ equals 1 in the time interval between the 1st and the second arrivals of the Poisson process $N_\alpha(t)$.

3.2 A CLASS OF NON-ERLANG RENEWAL IMPULSE PROCESSES

3.2.1 Statement of the problem

The class of impulse processes here considered is obtained by selecting the events from an Erlang renewal process $R_\nu(t)$ with parameters ν and k, with the aid of the replacement $\sum_{i,R=1}^{R(t)} P_{i,R}\delta(t-t_i) = \sum_{i=1}^{R_\nu(t)} Z(t_i) P_i \delta(t-t_i)$, where the zero-one stochastic variable $Z(t)$ is governed by the following stochastic differential equation:

$$dZ(t) = (1-Z)\, dR_\mu(t) - Z dR_\nu(t) \tag{3.2.1}$$

The variable $Z(t)$, a left continuous variable with right limits (see Figure 3.2.1) is zero except in the time interval between the first $R_\mu(t)$ driven event occurring after a $R_\nu(t)$ driven event and the first subsequent $\Re_\nu(t)$ driven event. In other words, $Z(t_i)$ is zero at all instants t_i driven by $R_\nu(t)$ except the first ones occurring after $R_\mu(t)$ driven events.
The increment of this class of non-Erlang processes becomes

$$dR(t) = Z(t)\, dR_\nu\,. \tag{3.2.2}$$

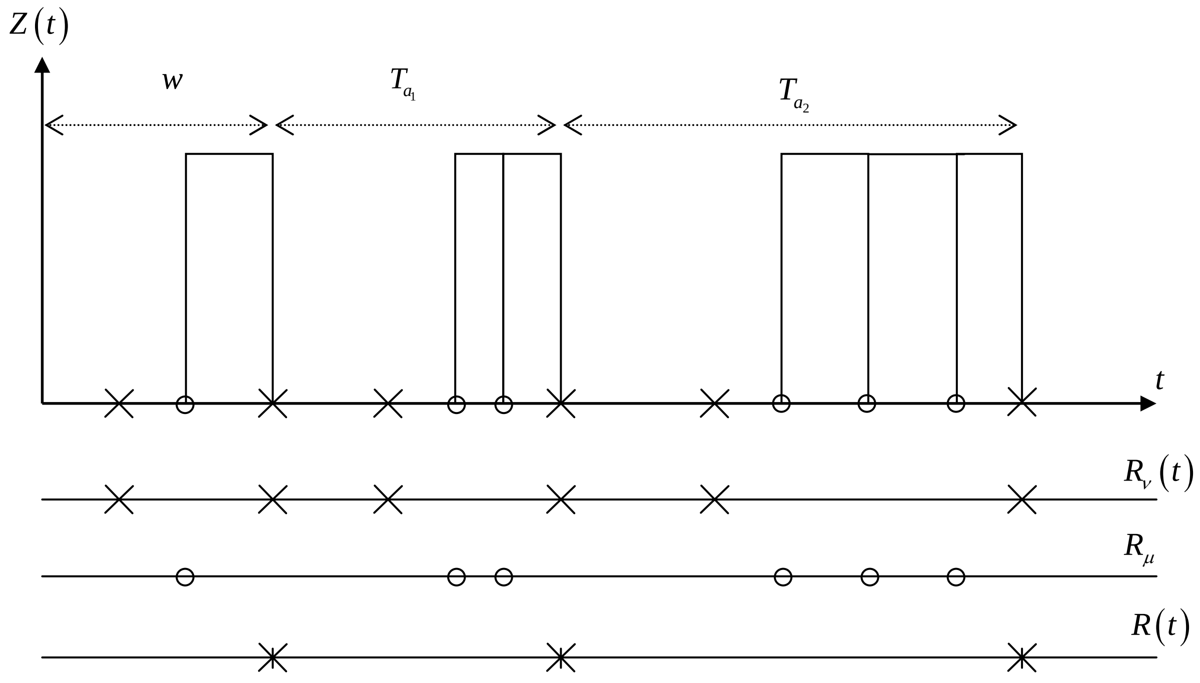

Figure 3.2.1.

Sample function of the process $Z(t)$ governed by the equation (3.2.1) and the correspondent renewal process governed by the equation (3.2.2).

If the interarrival times between events are sum of two independent, exponential distributed variates with parameters μ, ν the counting process $R(t)$ can be exactly obtained by selecting the events from the Poisson process $N_\nu(t)$, with the aid of a zero-one stochastic variable $Z(t)$ governed by the following stochastic differential equation (Iwankiewicz (2002), Iwankiewicz (2003)):

$$dZ(t) = (1 - Z)\,dN_\mu(t) - Z\,dN_\nu(t) \tag{3.2.3}$$

where $N_\mu(t)$ and $N_\nu(t)$ are two homogeneous Poisson processes with parameters μ and ν respectively. The increment of the renewal process becomes

$$dR(t) = Z(t)\,dN_\nu \tag{3.2.4}$$

3.2.2 Probability density function of the first and second waiting time

First waiting time

Let us consider the impulse process generated with the aid of the stochastic equation (3.2.1) driven by Erlang processes $R_\mu(t)$, with parameters μ, l and $R_\nu(t)$, with parameters ν, k and let T_μ and T_ν be the corresponding interarrival times. The probability density of the first waiting time w_1, that is the time elapsed from the origin to the first impulse driven by $R(t)$, is expressed as (see Figure 3.2.2).

$$
\begin{aligned}
& f_{w_1}(t)dt = \Pr\left\{w_1 \in (t, t+dt)\right\} = \sum_{u=0}^{t} \Pr\left\{T_\mu \in (u, u+du) \wedge T_\nu \in (t, t+dt)\right\} + \\
& \sum_{u=0}^{t} \Pr\left\{T_\mu \in (u, u+du) \wedge \sum_{\xi=0}^{u} dR_\nu \in (\xi, \xi+d\xi) \wedge T_\nu \in (t-\xi, t-\xi+dt)\right\} = \\
& \int_0^t g_{T_\mu}(u)du \; g_{T_\nu}(t)dt + \int_0^t g_{T_\mu}(u)du \int_0^u h_\nu(\xi)d\xi \; g_{T_\nu}(t-\xi)dt
\end{aligned}
\tag{3.2.5}
$$

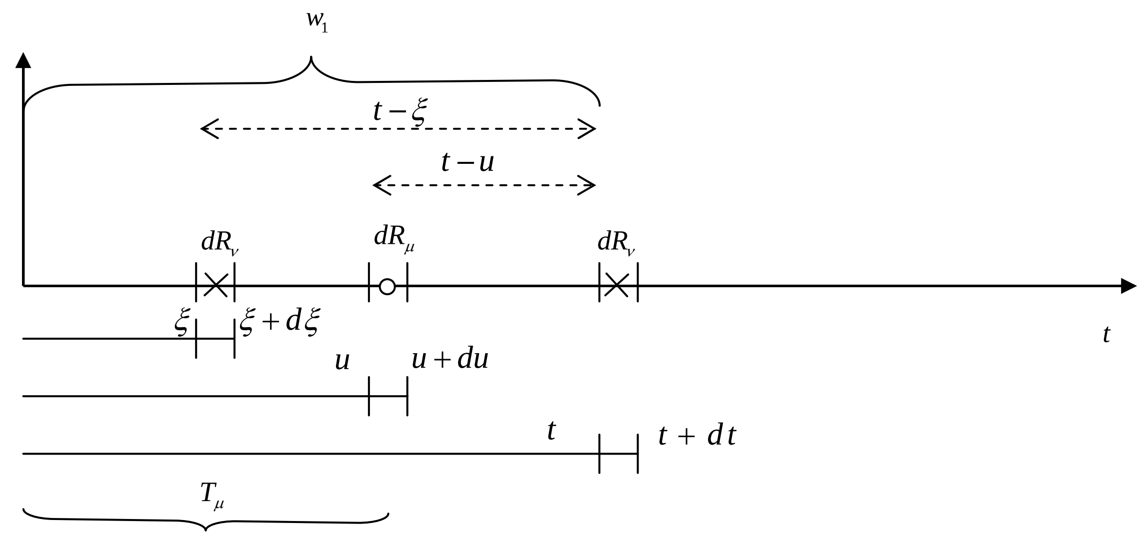

Figure 3.2.2.

Definition of the variables appearing in the expression of the probability function of the waiting time (equation (3.2.5))

The contributions at the right hand side of equation (3.2.5) account for the probabilities that the first $R_\nu(t)$ driven event occurring after the first $R_\mu(t)$ driven event has occurred, may be either the first $R_\nu(t)$ driven event at all, or the subsequent event.

Then the probability density $f_{w_1}(t)$ of the waiting time w_1 is given by the following expression:

$$
f_{w_1}(t)dt = \int_0^t g_{T_\mu}(u)\left(g_{T_\nu}(t) + \int_0^u h_\nu(\xi)\, g_{T_\nu}(t-\xi)\, d\xi\right)dudt
\tag{3.2.6}
$$

where $g_{T_\alpha}(t)$ is the probability density function of the interarrival times T_α and $h_\nu(t)$ is the renewal density function of the process $R_\nu(t)$.

Second waiting time

Let us consider the impulse process generated with the aid of the stochastic equation (3.2.1) driven by Erlang processes $R_\mu(t)$, with parameters μ, l and $R_\nu(t)$, with parameters ν, k and let T_μ and T_ν be the corresponding interarrival times. The probability density of the second arrival, that is the time elapsed from the origin to the second event, is expressed as (see Figure 3.2.3).

$$
\begin{aligned}
& f_{w_2}(t_1)\,dt = \Pr\left\{ w_2 \in (t_1, t_1 + dt_1) \right\} = \\
& \sum_{\tau=0}^{t_1}\sum_{t=0}^{\tau}\sum_{u=0}^{t} \Pr\left\{ T_\mu \in (u, u+du) \wedge T_\mu \in (\tau - u, \tau - u + d\tau) \wedge T_\nu \in (t, t+dt) \wedge T_\nu \in (t_1 - t, t_1 - t + dt_1) \right\} + \\
& \sum_{\tau=0}^{t_1}\sum_{t=0}^{\tau}\sum_{u=0}^{t}\sum_{\xi=0}^{u} \Pr\left\{ \begin{array}{l} T_\mu \in (u, u+du) \wedge T_\mu \in (\tau - u, \tau - u + d\tau) \wedge dR_\nu \in (\xi, \xi + d\xi) \wedge T_\nu \in (t - \xi, t - \xi + dt) \\ \wedge T_\nu \in (t_1 - t, t_1 - t + dt_1) \end{array} \right\} + \\
& \sum_{\tau=0}^{t_1}\sum_{t=0}^{\tau}\sum_{u=0}^{t}\sum_{u_1=u}^{t} \Pr\left\{ \begin{array}{l} T_\mu \in (u, u+du) \wedge T_\nu \in (t, t+dt) \wedge dR_\mu \in (u_1, u_1 + du_1) \wedge T_\mu \in (\tau - u, \tau - u + d\tau) \\ \wedge T_\nu \in (t_1 - t, t_1 - t + dt_1) \end{array} \right\} + \\
& \sum_{\tau=0}^{t_1}\sum_{t=0}^{\tau}\sum_{u=0}^{t}\sum_{u_1=u}^{t}\sum_{\xi=0}^{u} \Pr\left\{ \begin{array}{l} T_\mu \in (u, u+du) \wedge dR_\nu \in (\xi, \xi + d\xi) \wedge T_\nu \in (t - \xi, t - \xi + dt) \wedge dR_\mu \in (u_1, u_1 + du_1) \\ \wedge T_\mu \in (\tau - u, \tau - u + d\tau) \wedge T_\nu \in (t_1 - t, t_1 - t + dt_1) \end{array} \right\} + \\
& \sum_{\tau=0}^{t_1}\sum_{t=0}^{\tau}\sum_{u=0}^{t}\sum_{\xi_1=t}^{\tau} \Pr\left\{ \begin{array}{l} T_\mu \in (u, u+du) \wedge T_\mu \in (\tau - u, \tau - u + d\tau) \wedge T_\nu \in (t, t+dt) \wedge dR_\nu \in (\xi_1, \xi_1 + d\xi_1) \\ \wedge T_\nu \in (t_1 - \xi_1, t_1 - \xi_1 + dt_1) \end{array} \right\} + \\
& \sum_{\tau=0}^{t_1}\sum_{t=0}^{\tau}\sum_{u=0}^{t}\sum_{\xi=0}^{u}\sum_{\xi_1=t}^{\tau} \Pr\left\{ \begin{array}{l} T_\mu \in (u, u+du) \wedge T_\mu \in (\tau - u, \tau - u + d\tau) \wedge dR_\nu \in (\xi, \xi + d\xi) \wedge T_\nu \in (t - \xi, t - \xi + dt) \\ \wedge dR_\nu \in (\xi_1, \xi_1 + d\xi_1) \wedge T_\nu \in (t_1 - \xi_1, t_1 - \xi_1 + dt_1) \end{array} \right\} + \\
& \sum_{\tau=0}^{t_1}\sum_{t=0}^{\tau}\sum_{u=0}^{t}\sum_{u_1=u}^{t}\sum_{\xi_1=t}^{\tau} \Pr\left\{ \begin{array}{l} T_\mu \in (u, u+du) \wedge T_\nu \in (t, t+dt) \wedge dR_\mu \in (u_1, u_1 + du_1) \\ \wedge dR_\nu \in (\xi_1, \xi_1 + d\xi_1) \wedge T_\nu \in (t_1 - \xi_1, t_1 - \xi_1 + dt_1) \end{array} \right\} \wedge T_\mu \in (\tau - u_1, \tau - u_1 + d\tau) + \\
& \sum_{\tau=0}^{t_1}\sum_{t=0}^{\tau}\sum_{u=0}^{t}\sum_{u_1=u}^{t}\sum_{\xi=0}^{u}\sum_{\xi_1=t}^{\tau} \Pr\left\{ \begin{array}{l} T_\mu \in (u, u+du) \wedge dR_\nu \in (\xi, \xi + d\xi) \wedge T_\nu \in (t - \xi, t - \xi + dt) \wedge dR_\mu \in (u_1, u_1 + du_1) \\ \wedge T_\mu \in (\tau - u, \tau - u + d\tau) \wedge dR_\nu \in (\xi_1, \xi_1 + d\xi_1) \wedge T_\nu \in (t_1 - \xi_1, t_1 - \xi_1 + dt_1) \end{array} \right\} +
\end{aligned}
\tag{3.2.7}
$$

The contributions at the right hand side of equation (3.2.7) account for the probabilities that the second impulse, a $R_\nu(t)$ driven event, may be either the second $R_\nu(t)$ driven event occurring after the first impulse, or a subsequent event.

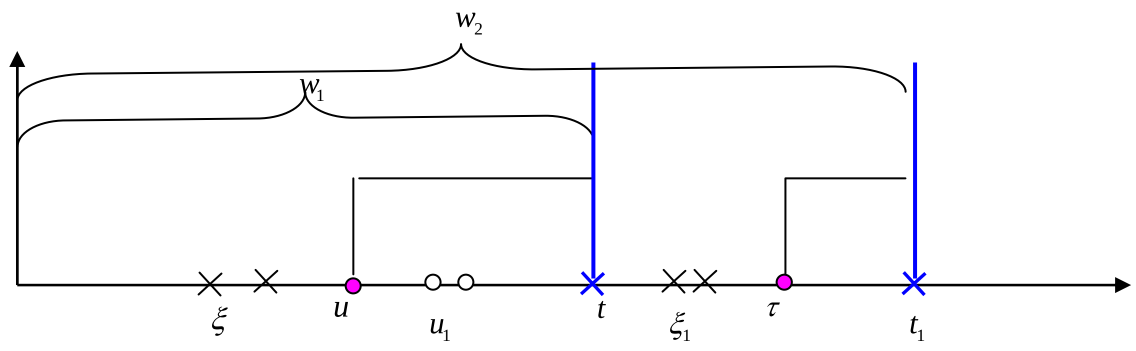

Figure 3.2.3

Definition of the variables appearing in the expression of the probability function of the time elapsed between the origin and the second impulse (equation (3.2.7))

Then the probability density $f_{w_2}(t_1)$ of the time elapsed between the origin and the second event is given by the following expression:

$$f_{w_2}(t_1)=\int_{\tau=0}^{t_1}\int_{t=0}^{\tau}\int_{u=0}^{t} g_{T_\mu}(u)\,g_{T_\mu}(\tau-u)\,g_{T_\nu}(t)\,g_{T_\nu}(t_1-t)\,dudtd\tau+$$

$$\int_{\tau=0}^{t_1}\int_{t=0}^{\tau}\int_{u=0}^{t}\int_{\xi=0}^{u} g_{T_\mu}(u)\,g_{T_\mu}(\tau-u)\,h_\nu(\xi)\,g_{T_\nu}(t-\xi)\,g_{T_\nu}(t_1-t)\,dudtd\tau d\xi+$$

$$\int_{\tau=0}^{t_1}\int_{t=0}^{\tau}\int_{u=0}^{t}\int_{u_1=u}^{t} g_{T_\mu}(u)\,g_{T_\nu}(t)\,h_\mu(u_1-u)\,g_{T_\mu}(\tau-u_1)\,g_{T_\nu}(t_1-t)\,dudtdu_1d\tau+$$

$$\int_{\tau=0}^{t_1}\int_{t=0}^{\tau}\int_{u=0}^{t}\int_{u_1=u}^{t}\int_{\xi=0}^{u} g_{T_\mu}(u)\,h_\mu(u_1-u)\,g_{T_\mu}(\tau-u_1)\,h_\nu(\xi)\,g_{T_\nu}(t-\xi)\,g_{T_\nu}(t_1-t)\,dudtd\tau du_1d\xi+$$

$$\int_{\tau=0}^{t_1}\int_{t=0}^{\tau}\int_{u=0}^{t}\int_{\xi_1=t}^{\tau} g_{T_\mu}(u)\,g_{T_\mu}(\tau-u)\,g_{T_\nu}(t)\,h_\nu(\xi_1-t)\,g_{T_\nu}(t_1-\xi)\,dudtd\tau d\xi_1+$$

$$\int_{\tau=0}^{t_1}\int_{t=0}^{\tau}\int_{u=0}^{t}\int_{\xi=0}^{u}\int_{\xi_1=t}^{\tau} g_{T_\mu}(u)\,g_{T_\mu}(\tau-u)\,h_\nu(\xi)\,g_{T_\nu}(t-\xi_1)\,h_\nu(\xi_1-t)\,g_{T_\nu}(t_1-\xi_1)\,dudtd\tau d\xi_1 d\xi+$$

$$\int_{\tau=0}^{t_1}\int_{t=0}^{\tau}\int_{u=0}^{t}\int_{u_1=u}^{t}\int_{\xi_1=t}^{\tau} g_{T_\mu}(u)\,h_\mu(u_1-u)\,g_{T_\nu}(t)\,h_\nu(\xi_1-t)\,g_{T_\mu}(\tau-u_1)\,g_{T_\nu}(t_1-\xi_1)\,dudtd\tau d\xi_1 d\xi+$$

$$\int_{\tau=0}^{t_1}\int_{t=0}^{\tau}\int_{u=0}^{t}\int_{u_1=u}^{t}\int_{\xi=0}^{u}\int_{\xi_1=t}^{\tau} g_{T_\mu}(u)\,h_\mu(u_1-u)\,h_\nu(\xi)\,g_{T_\nu}(t-\xi)\,h_\nu(\xi_1-t)\,g_{T_\mu}(\tau-u_1)\,g_{T_\nu}(t_1-\xi_1)\,dudtd\tau du_1d\xi_1d\xi. \tag{3.2.8}$$

where $g_{T_\alpha}(t)$ is the probability density function of the interarrival times T_α and $h_\alpha(t)$ is the renewal density function of the process $R_\alpha(t)$.

Characterization of the excitation process

In the derivations that follow the analytical expressions of the probability density of the first and second arrival for the process defined by (3.2.2) will be found.
Then, four different processes obtained from (3.2.2) by substituting in (3.2.1) Poisson or Erlang processes, will be analysed

Process I - $dZ(t)=(1-Z)\,dN_\mu(t)-ZdN_\nu(t)$,
Process II - $dZ(t)=(1-Z)\,dR_\mu(t)-ZdN_\nu(t)$
Process III - $dZ(t)=(1-Z)\,dN_\mu(t)-ZdR_\nu(t)$,
Process IV- $dZ(t)=(1-Z)\,dR_\mu(t)-ZdR_\nu(t)$

It will be assessed weather or not their interarrival times are identically distributed and weather or not the underlying processes $R(t)$ are renewal.

3.2.3 Characterization of the Process I : $R_{\mu}(t)=P(\mu)$ = and $R_{\nu}(t)=P(\nu)$.

Let us consider the process obtained from equation (3.2.1), when $R_{\mu}(t)=P(\mu)$ and $R_{\nu}(t)=P(\nu)$ (two Poisson processes with parameters μ and ν respectively). In Iwankiewicz (2002) it was proved that the generated renewal process is an ordinary renewal process with interarrival times which are the sum of two independent negative exponential distributed variables with parameters μ and ν The equation for the mean value of the stochastic variable $Z(t)$ is

$$\frac{d}{dt}E[Z(t)]=-(\mu+\nu)E[Z(t)]+\mu; \tag{3.2.9}$$

The renewal density is given by

$$h(t)dt=E\left[dR(t)\right]=E\left[Z(t)\right]\nu dt=\frac{\nu\mu}{\nu+\mu}\left(1-\exp\left(-(\nu+\mu)t\right)\right)dt \tag{3.2.10}$$

This is exactly the renewal density of the process above mentioned. Let us perform the demonstration in an alternative way from (3.3.8). The probability density of the first waiting time is obtained as:

$$f_w(t)=\frac{\nu\mu}{\mu-\nu}\left(e^{-t\nu}-e^{-t\mu}\right); \tag{3.2.11}$$

and it is the probability density function of the sum of two independent negative-exponential distributed variables T_{μ} and T_{ν} with parameters μ and ν. Let us assume that the underlying process is renewal and let us use of the relationship

$$h^*(s)=\frac{f_w^*(s)}{1-g_{T_a}^*(s)} \tag{3.2.12}$$

inserting

$$h^*(s)=\mathcal{L}\left\{h(t)\right\}=\frac{\mu\nu}{(\mu+\nu)}\frac{1}{s+s^2} \tag{3.2.13}$$

and

$$f_w^{\ *}(s)=\mathcal{L}\left\{f_w(t)\right\}=\frac{\mu\nu}{(\mu+s)(\nu+s)} \tag{3.2.14}$$

and solving for $g_{T_a}^*(s)$ we find

$$g_{T_a}^*(s)=\frac{\mu\nu}{(\mu+s)(\nu+s)} \tag{3.2.15}$$

and

$$g_{T_a}(t)=\frac{\nu\mu}{\mu-\nu}\left(e^{-t\nu}-e^{-t\mu}\right) \tag{3.2.16}$$

which means that the interarrival times have the same distribution as the first waiting time.
The probability density of the time elapsed from the origin to the second event is obtained from eq (3.2.8) as:

$$f_{w_2}(t)=\frac{e^{-t(\mu+\nu)}\nu^2\mu^2\left(e^{t\mu}\left(-2+t(\mu-\nu)\right)+e^{t\nu}\left(2+t(\mu-\nu)\right)\right)}{(\mu-\nu)^3}; \tag{3.2.17}$$

If the underlying counting process is a renewal process, with a probability density of the interarrival time as given by equation (3.2.16), then the probability density $f_{w_2}(t)$ can be found as

$$f_{w_2}(t)=\mathcal{L}^{-1}\left\{f_{w_1}{}^*(s)\,g_{T_a}{}^*(s)\right\}=\frac{e^{-t(\mu+\nu)}\nu^2\mu^2\left(e^{t\mu}\left(-2+t(\mu-\nu)\right)+e^{t\nu}\left(2+t(\mu-\nu)\right)\right)}{(\mu-\nu)^3} \tag{3.2.18}$$

Since the expressions of the probability density $f_{w_2}(t)$, independently obtained, coincide, as it was stated, the underlying counting process is indeed an ordinary renewal process. From (3.2.16) it follows that the stationary mean value and variance of the interarrival times are given, respectively, by

$$E[T_a]=\frac{\nu+\mu}{\nu\mu}; \qquad \sigma_{T_a}^2=\frac{\nu^2+\mu^2}{\nu^2\mu^2}; \tag{3.2.19}$$

3.2.3 Characterization of the Process II : $R_\mu(t)=E(\mu,l)$ and $R_\nu(t)=P(\nu)$.

If $R_\mu(t)=E(\mu,l)$ and $R_\nu(t)=P(\nu)$, where $E(\mu,l)$ is an Erlang process with parameters μ and l, while $P(\nu)$ is a Poisson process with mean arrival rate ν, the increment of the process $dR(t)$ is

$$dR(t)=Z(t)\,dN_\nu(t) \tag{3.2.20}$$

where the variable $Z(t)$ is governed by the equation:

$$dZ(t)=(1-Z)\,dR_\mu(t)-Z\,dN_\nu(t) \tag{3.2.21}$$

The Erlang renewal process $R_\mu(t)$ can be expressed in terms of the Poisson process $N_\mu(t)$ as follows

$$dR_\mu(t)=\rho_\mu(t)\,dN_\mu(t) \tag{3.2.22}$$

If $k=2$, the equation governing the auxiliary variable $\rho_\mu(t)$ becomes

$$d\rho_\mu(t) = \left(1-2\rho_\mu(t)\right)dN_\mu(t) \tag{3.2.23}$$

Introducing new stochastic variables: $Z_1(t)=Z$ and $Z_2(t)=Z\rho_\mu$, the equations for the mean values become:

$$\begin{aligned}
&\frac{d}{dt}E[Z_1(t)] = -E[Z_1(t)]\nu - \mu E[Z_2(t)] + E[\rho_\mu]\mu;\\
&\frac{d}{dt}E[Z_2(t)] = E[Z_1(t)]\mu - E[Z_2(t)](\nu+2\mu);
\end{aligned} \tag{3.2.24}$$

Let us assume that the process $R(t)$ is a renewal process. The renewal density is obtained as:

$$h(t)dt = E[dR(t)] = E[Z_1(t)]\nu dt \tag{3.2.25}$$

hence

$$\begin{aligned}
&h(t) = E[Z_1(t)]\nu = \\
&\left(\frac{(2\mu+\nu)\mu}{2(\nu+\mu)^2} - \mathrm{e}^{-2t\mu}\frac{\nu\mu}{2(\mu-\nu)^2} + \mathrm{e}^{-t(\mu+\nu)}\frac{\mu^2}{(\mu^2-\nu^2)}\left(1+t\mu-\frac{2\nu\mu}{(\mu^2-\nu^2)}\right)\right)\nu;
\end{aligned} \tag{3.2.26}$$

From equation (3.2.6) the probability density of the first waiting time w_1, which is the sum of an Erlang variable with parameter k=2 and a negative exponential distributed variable, is obtained as

$$f_w(t) = \frac{\nu\mu^2}{(\mu-\nu)^2}e^{-t(\mu+\nu)}\left(e^{t\mu}-e^{t\nu}\left(1+t(\mu-\nu)\right)\right); \tag{3.2.27}$$

Under the assumption that the underlying counting process is renewal, the probability density of the interarrival times is obtained as:

$$\begin{aligned}
&g_{T_a}(t) = \mathcal{L}^{-1}\left\{1-\frac{f_w^*(s)}{h(s)}\right\} = \\
&\frac{\mu\nu(2\mu-\nu)}{2(\mu-\nu)^2}\mathrm{e}^{-t\nu} + \frac{\mu\nu(2\mu+\nu)}{2(\nu+\mu)^2}\mathrm{e}^{-t(2\mu+\nu)} + \frac{\mu^2\nu\left(t\nu^3-\mu^2(2+t\nu)\right)}{(\mu-\nu)^2(\nu+\mu)^2}\mathrm{e}^{-t\mu}
\end{aligned} \tag{3.2.28}$$

The probability density of the time elapsed from the origin to the second event is obtained from eq (3.2.8) as:

$f_{w2}(t) =$

$$\frac{1}{12}\mu^4\nu^2\left(-\frac{3e^{-t(2\mu+\nu)}(2\mu+\nu)}{\mu^2(\mu+\nu)^4}+\frac{3e^{-t\nu}(4t\mu^3-\nu^2+\mu\nu(7+2t\nu)-2\mu^2(7+3t\nu))}{\mu^2(\mu-\nu)^5}+\frac{1}{(\mu-\nu)^5(\mu+\nu)^4}(2e^{-t\mu}(3\mu\nu^3(-2+t\nu)^2+3t\mu^5(4+t\nu)+t^2\mu^6(6+t\nu)+\mu^3(60\nu-6t^2\nu^3)+3\mu^2\nu^2(36-16t\nu+t^3\nu^3)-\nu^4(12-3t^2\nu^2+t^3\nu^3)-3\mu^4(-8-16t\nu+3t^2\nu^2+t^3\nu^3)))\right); \tag{3.2.29}$$

If the underlying counting process is renewal, with probability density of the interarrival time $g_{T_a}(t)$ given by (3.2.28), the probability density of the second waiting time $f_{w_2}(t)$ can be obtained through the inverse Laplace transform $\mathcal{L}^{-1}\left\{f_{w_1}{}^*(s)\,g_{T_a}{}^*(s)\right\}$ as

$$f_{w_2}(t)=\mathcal{L}^{-1}\left\{f_{w_1}{}^*(s)\,g_{T_a}{}^*(s)\right\}=$$

$$\mathcal{L}^{-1}\left\{\frac{\mu^2\nu}{(s+\mu)^2(s+\nu)}\,\frac{\mu^2\nu(2s+2\mu+\nu)}{(s+\mu)^2(s+\nu)(s+2\mu+\nu)}\right\}=\frac{1}{12}\mu^4\nu^2\left(-\frac{3e^{-t(2\mu+\nu)}(2\mu+\nu)}{\mu^2(\mu+\nu)^4}+\frac{3e^{-t\nu}(4t\mu^3-\nu^2+\mu\nu(7+2t\nu)-2\mu^2(7+3t\nu))}{\mu^2(\mu-\nu)^5}+\frac{1}{(\mu-\nu)^5(\mu+\nu)^4}\left(2e^{-t\mu}\left(3\mu\nu^3(-2+t\nu)^2+3t\mu^5(4+t\nu)+t^2\mu^6(6+t\nu)+\mu^3\left(60\nu-6t^2\nu^3\right)+3\mu^2\nu^2\left(36-16t\nu+t^3\nu^3\right)-\nu^4\left(12-3t^2\nu^2+t^3\nu^3\right)-3\mu^4\left(-8-16t\nu+3t^2\nu^2+t^3\nu^3\right)\right)\right)\right) \tag{3.2.30}$$

This is the same probability density function as in (3.2.29), hence the underlying counting process $R(t)$ is renewal. Moreover, since $g_{T_a}(t)$ is different from $f_w(t)$, $R(t)$ is a delayed renewal process.

From (3.2.28) it follows that the expression for the stationary mean value and variance of the interarrival times take on, respectively, he forms:

$$E[T_a]=\frac{2(\nu+\mu)^2}{(\mu+2\nu)\mu\nu};\qquad \sigma_{T_a}^2=\frac{2\left(\mu^4+4\mu^3\nu+3\mu^2\nu^2+2\mu\nu^3+2\nu^4\right)}{(\mu+2\nu)^2\,\mu^2\nu^2}; \tag{3.2.31}$$

3.2.3 Characterization of the Process III : $R_\mu(t)=P(\mu)$ *and* $R_\nu(t)=E(\nu,k)$.

If in equations (3.2.1) and (3.2.2) $R_\mu(t)=P(\mu)$ and $R_\nu(t)=E(\nu,k)$, the increment of the renewal process $dR(t)$ becomes

$$dR(t)=Z(t)\,dN_\nu(t) \tag{3.2.32}$$

where the variable $Z(t)$ is governed by the equation:

$$dZ(t)=(1-Z)\,dN_\mu(t)-Z\,dR_\nu(t) \tag{3.2.33}$$

If $R_\nu(t)$ is an Erlang renewal process with parameter $k=2$, it can be expressed in terms of the Poisson process $N_\nu(t)$ as follows

$$dR_\nu(t)=\rho_\nu(t)\,dN_\nu(t) \tag{3.2.34}$$

with

$$d\rho_\nu(t)=\left(1-2\rho_\nu(t)\right)dN_\nu(t) \tag{3.2.35}$$

The equations for the mean values of the stochastic variables $Z_1(t)=Z$ and $Z_2(t)=Z\rho_\nu$ are

$$\begin{aligned}\frac{d}{dt}E[Z_1(t)]&=-E[Z_1(t)]\mu-\nu E[Z_2(t)]+\mu;\\ \frac{d}{dt}E[Z_2(t)]&=E[Z_1(t)]\nu-E[Z_2(t)](2\nu+\mu)+\mu E[\rho_\nu(t)];\end{aligned} \tag{3.2.36}$$

Let us assume that the underlying process is renewal. The renewal density is obtained as:

$$h(t)dt=E[dR(t)]=E[Z_2(t)]\nu dt \tag{3.2.37}$$

hence

$$h(t)=E[Z_2(t)]\nu=\left(\frac{(\mu+2\nu)\mu}{2(\nu+\mu)^2}-\mathrm{e}^{-2t\nu}\frac{(\mu-2\nu)\mu}{2(\mu-\nu)^2}-\mathrm{e}^{-t(\mu+\nu)}\frac{\mu}{(\mu^2-\nu^2)}\left(t\mu+\frac{2\nu^3}{(\mu^2-\nu^2)}\right)\right)\nu; \tag{3.2.38}$$

From equation (3.2.6) the probability density of the waiting time is obtained as

$$f_{w_1}(t)=\frac{\mu\nu(2\nu-\mu)}{2(\mu-\nu)^2}\mathrm{e}^{-t\mu}+\frac{\mu\nu(\mu+2\nu)}{2(\nu+\mu)^2}\mathrm{e}^{-t(2\nu+\mu)}+\frac{\mu\nu^2\left(t\mu^3-\nu^2(2+t\mu)\right)}{(\mu-\nu)^2(\nu+\mu)^2}\mathrm{e}^{-t\nu}; \tag{3.2.39}$$

Under the assumption that the underlying counting process is renewal, the probability density of the interarrival times is given by:

$$g_{T_a}(t) = \mathcal{L}^{-1}\left\{1 - \frac{f_w^*(s)}{h(s)}\right\} =$$

$$\frac{\mu\nu(2\nu-\mu)}{2(\mu-\nu)^2}e^{-t\mu} + \frac{\mu\nu(\mu+2\nu)}{2(\nu+\mu)^2}e^{-t(2\nu+\mu)} + \frac{\mu\nu^2\left(t\mu^3 - \nu^2(2+t\mu)\right)}{(\mu-\nu)^2(\nu+\mu)^2}e^{-t\nu} = f_{w_1}(t) \tag{3.2.40}$$

The probability density of the time elapsed from the origin to the second event is obtained from eq (3.2.8) as:

$$f_{w2}(t) = \frac{1}{12}\mu^2\nu^4\left(\frac{3e^{-t\mu}(\mu-2\nu)(2\nu^2(-3+t\nu)+\mu^2(-1+t\nu)-3\mu\nu(-1+t\nu))}{(\mu-\nu)^5\nu^3} + \frac{3e^{-t(\mu+2\nu)}(\mu+2\nu)(\mu^2(1+t\nu)+3\mu\nu(1+t\nu)+2\nu^2(3+t\nu))}{\nu^3(\mu+\nu)^5} + \left(2e^{-t\nu}\left(-t^3\mu^8+24t\nu^6-24\mu^3\nu^2(-3+t^2\nu^2)+12\mu^5(-2+t^2\nu^2)+3t\mu^6(4+t^2\nu^2)+12\mu\nu^4(12+t^2\nu^2)-3t\mu^4\nu^2(24+t^2\nu^2)+t\mu^2\nu^4(36+t^2\nu^2)\right)\right)\Big/\left((-\mu+\nu)^5(\mu+\nu)^5\right)\right) \tag{3.2.41}$$

If the underlying counting process is renewal, with probability density of the interarrival time $g_{T_a}(t)$ given by (3.2.40), the probability density of the second waiting time $f_{w_2}(t)$ can be obtained through the inverse Laplace transform $\mathcal{L}^{-1}\left\{f_{w_1}{}^*(s)\,g_{T_a}{}^*(s)\right\}$ as

$$f_{w_2}(t) = \mathcal{L}^{-1}\left\{f_{w_1}{}^*(s)\,g_{T_a}{}^*(s)\right\} =$$

$$\frac{1}{12}\mu^2\nu^4\left(\frac{3e^{-t\mu}(\mu-2\nu)(2\nu^2(-3+t\nu)+\mu^2(-1+t\nu)-3\mu\nu(-1+t\nu))}{(\mu-\nu)^5\nu^3} + \frac{3e^{-t(\mu+2\nu)}(\mu+2\nu)(\mu^2(1+t\nu)+3\mu\nu(1+t\nu)+2\nu^2(3+t\nu))}{\nu^3(\mu+\nu)^5} + \left(2e^{-t\nu}\left(-t^3\mu^8+24t\nu^6-24\mu^3\nu^2(-3+t^2\nu^2)+12\mu^5(-2+t^2\nu^2)+3t\mu^6(4+t^2\nu^2)+12\mu\nu^4(12+t^2\nu^2)-3t\mu^4\nu^2(24+t^2\nu^2)+t\mu^2\nu^4(36+t^2\nu^2)\right)\right)\Big/\left((-\mu+\nu)^5(\mu+\nu)^5\right)\right) \tag{3.2.42}$$

This probability density function is exactly the same as (3.2.41) hence the underlying counting process $R(t)$ is renewal. As $g_{T_a}(t) = f_{w_1}(t)$ this renewal process is an ordinary one.
The stationary values of mean and variance of the interarrival times take on the form:

$$E[T_a] = \frac{2(\nu+\mu)^2}{\nu\mu(\nu+2\mu)}; \qquad \sigma_{T_a}^2 = \frac{2\left(2\mu^4+2\mu^3\nu+3\mu^2\nu^2+4\mu\nu^3+\nu^4\right)^2}{\nu^2\mu^2(\nu+2\mu)^2}; \tag{3.2.43}$$

3.2.3 Characterization of the Process IV : $R_\mu(t)=E(\mu,l)$ and $R_\nu(t)=E(\nu,k)$.

If in equations (3.2.1) and (3.2.2) $R_\mu(t)=E(\mu,l)$ and $R_\nu(t)=E(\nu,k)$, the increment of the non-Erlang renewal process is expressed as

$$dR(t)=Z(t)\,dR_\nu(t) \tag{3.2.44}$$

and the variable $Z(t)$ is governed by the equation:

$$dZ(t)=(1-Z)\,dR_\mu(t)-Z\,dR_\nu(t), \tag{3.2.45}$$

if $R_\mu(t)$ and $R_\nu(t)$ are Erlang renewal processes with parameter $k=2$, they can be expressed in terms of the Poisson processes $N_\nu(t)$ and $N_\mu(t)$ as follows

$$dR_\nu(t)=\rho_\nu(t)\,dN_\nu(t),\, dR_\mu(t)=\rho_\mu(t)\,dN_\mu(t) \tag{3.2.46}$$

with

$$d\rho_\nu(t)=\left(1-2\rho_\nu(t)\right)dN_\nu(t),\, d\rho_\mu(t)=\left(1-2\rho_\mu(t)\right)dN_\mu(t) \tag{3.2.47}$$

Introducing the stochastic variables $Z_1(t)=Z$, $Z_2(t)=Z\rho_\nu$, $Z_3(t)=Z\rho_\mu$ and $Z_4(t)=Z\rho_\nu\rho_\mu$ the corresponding equations for the mean values are

$$\begin{aligned}
&\frac{d}{dt}E[Z_1(t)]=-E[Z_3(t)]\mu-\nu E[Z_2(t)]+E[\rho_\mu]\mu;\\
&\frac{d}{dt}E[Z_2(t)]=E[Z_1(t)]\nu-2E[Z_2(t)]\nu-E[Z_4(t)]\mu+E[\rho_\mu]E[\rho_\nu]\mu;\\
&\frac{d}{dt}E[Z_3(t)]=E[Z_1(t)]\mu-2E[Z_3(t)]\mu-E[Z_4(t)]\nu;\\
&\frac{d}{dt}E[Z_4(t)]=E[Z_2(t)]\mu+E[Z_3(t)]\nu-2E[Z_4(t)](\nu+\mu);
\end{aligned} \tag{3.2.48}$$

Let us assume that the underlying process is a renewal one. The renewal density is obtained as:

$$h(t)\,dt=E[dR(t)]=E[Z_2(t)]\nu\,dt \tag{3.2.49}$$

hence

$$\begin{aligned}
h(t)=E[Z_2(t)]\nu=&\frac{\left(\mu^2+3\mu\nu+\nu^2\right)\mu}{2(\nu+\mu)^3}-\mathrm{e}^{-2(\nu+\mu)t}\frac{\mu\nu^2}{2(\mu+\nu)^3}+\\
&+\mathrm{e}^{-2t\mu}\frac{\mu\nu^2}{2(\mu-\nu)^3}-\mathrm{e}^{-2t\nu}\frac{\left(\mu^2-3\mu\nu+\nu^2\right)\mu}{2(\mu-\nu)^3}-\mathrm{e}^{-(\nu+\mu)t}\frac{\mu^2\nu}{\left(\mu^2-\nu^2\right)}\left(t(1+t\mu)+\frac{8\mu\nu^2}{\left(\mu^2-\nu^2\right)^2}\right);
\end{aligned} \tag{3.2.50}$$

From equation (3.2.6) the probability density of the waiting time is obtained as

$$f_{w_1}(t) = \frac{e^{-t(2\nu+\mu)}\,\mu\nu\left(2\left(1+e^{2\nu t}-2e^{t(\nu+\mu)}\right)\nu^3-\mu\nu^2\left(3-3e^{2\nu t}+2e^{t(\nu+\mu)}\,t\nu\right)+\mu^3\left(1-e^{2\nu t}+2e^{t(\nu+\mu)}\,t\nu\right)\right)}{2(\mu-\nu)^2(\nu+\mu)^2} \qquad (3.2.51)$$

Under the assumption that the underlying process is a renewal one, the probability density of the interarrival times can be obtained through the following relationship:

$$g_{T_a}(t)=\mathcal{L}^{-1}\left\{1-\frac{f_w^*(s)}{h(s)}\right\}=$$

$$\mathcal{L}^{-1}\left\{1-\frac{s(s+2\mu)(s+\mu+\nu)^3(s+2\nu)(s+2(\mu+\nu))(3s^2+(\mu+2\nu)^2+s(4\mu+6\nu))}{(s+\mu)^2(s+\nu)^2(s+\mu+2\nu)^2((3s+\mu)(s+2\mu)^2+(s+2\mu)(9s+8\mu)\nu+2(5s+8\mu)\nu^2+4\nu^3)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{B_1}{(s+\mu)}+\frac{B_2}{(s+\mu)^2}+\frac{B_3}{(s+\nu)}+\frac{B_4}{(s+\nu)^2}+\frac{B_5}{(s+\mu+2\nu)}+\frac{B_6}{(s+\mu+2\nu)^2}+\frac{B_7}{(s-b_1)}+\frac{B_8 s+B_9}{(s^2+b_2 s+b_3)}\right\}$$

$$= e^{-t(\mu+2\nu)}\left(e^{2t\nu}B_1+e^{2t\nu}tB_2+e^{t(\mu+\nu)}B_3+e^{t(\mu+\nu)}tB_4+B_5+tB_6+e^{t(\mu+2\nu+b_1)}B_7\right)+$$
$$\frac{1}{2}e^{-\frac{1}{2}t\left(b_2+\sqrt{b_2^2-4b_3}\right)}\left(1+e^{t\sqrt{b_2^2-4b_3}}\right)B_8-\frac{e^{-\frac{1}{2}t\left(b_2+\sqrt{b_2^2-4b_3}\right)}\left(-1+e^{t\sqrt{b_2^2-4b_3}}\right)(b_2B_8-2B_9)}{2\sqrt{b_2^2-4b_3}} \qquad (3.2.52)$$

where the coefficients B_i ($i=1,2,..9$) and b_k ($k=1,2,3$) are functions of the parameters μ and ν and their expressions are reported in Appendix.
The probability density of the time elapsed from the origin to the second event can be obtained from eq (3.2.8)

$$\begin{aligned}
f_{w2}(t) = &\frac{1}{12(\mu-\nu)^7(\mu+\nu)^7}\\
&\Big(e^{-2t(\mu+\nu)}\mu^2\nu\big(3e^{t\nu}\nu^3(\mu^2-\nu^2)^2(12\mu^5+4t\mu^6+44\mu^3\nu^2-t\mu^4\nu^2-8\mu\nu^4-4t\mu^2\nu^4+t\nu^6)+\\
&\quad e^{t\mu}\mu^2(\mu-\nu)^2(3t\mu^9(1+t\nu)+\mu^8(3+9t\nu+6t^2\nu^2)+\nu^8(60-9t^2\nu^2-2t^3\nu^3)+3\mu^4\nu^4(-21+109t\nu+17t^2\nu^2-2t^3\nu^3)+\\
&\qquad \mu\nu^7(120+60t\nu+6t^2\nu^2-t^3\nu^3)+\mu^7\nu(6-3t\nu-21t^2\nu^2+t^3\nu^3)-3\mu^5\nu^3(36-15t\nu-13t^2\nu^2+t^3\nu^3)+\\
&\qquad 3\mu^3\nu^5(58-35t\nu-9t^2\nu^2+t^3\nu^3)+\mu^6\nu^2(-3-63t\nu-39t^2\nu^2+2t^3\nu^3)+3\mu^2\nu^6(-255-91t\nu-3t^2\nu^2+2t^3\nu^3))+\\
&\quad e^{t(\mu+2\nu)}\mu^2(\mu+\nu)^2(3t\mu^9(-1+t\nu)+\mu^8(-3+9t\nu-6t^2\nu^2)+3\mu^4\nu^4(21+109t\nu-17t^2\nu^2-2t^3\nu^3)+\nu^8(-60+9t^2\nu^2-2t^3\nu^3)+\\
&\qquad \mu^7\nu(6+3t\nu-21t^2\nu^2-t^3\nu^3)+\mu\nu^7(120-60t\nu+6t^2\nu^2+t^3\nu^3)-3\mu^3\nu^5(-58-35t\nu+9t^2\nu^2+t^3\nu^3)+\\
&\qquad 3\mu^5\nu^3(-36-15t\nu+13t^2\nu^2+t^3\nu^3)+3\mu^2\nu^6(255-91t\nu+3t^2\nu^2+2t^3\nu^3)+\mu^6\nu^2(3-63t\nu+39t^2\nu^2+2t^3\nu^3))+\\
&\quad e^{t(2\mu+\nu)}\nu^3(2t^3\mu^{12}+483t\mu^8\nu^2+90t\mu^2\nu^8-3t\nu^{10}-78\mu^9(-2+t^2\nu^2)+6\mu\nu^8(4+t^2\nu^2)-36\mu^3\nu^6(5+t^2\nu^2)-4t\mu^{10}(15+t^2\nu^2)-\\
&\qquad 2t\mu^4\nu^6(384+t^2\nu^2)-12\mu^5\nu^4(283+2t^2\nu^2)+12\mu^7\nu^2(-37+11t^2\nu^2)+\mu^6(258t\nu^4+4t^3\nu^6)))\Big)
\end{aligned} \qquad (3.2.53)$$

Under the assumption that the underlying process is renewal, the probability density of the second arrival can be obtained through the inverse Laplace transform

$$g_{w_2}(t)=\mathcal{L}^{-1}\left\{ f_{w_1}^{*}(s)\, g_{T_a}^{*}(s)\right\} =$$

$$\mathcal{L}^{-1}\{(\mu^6 v^6 (3s^2+(\mu+2v)^2+s(4\mu+6v))^2 (17s^5+s^4(75\mu+85v)+4(\mu+v)(\mu+2v)^2(\mu^2+3\mu v+v^2)+2s(\mu+v)(\mu+2v)(16\mu^2+45\mu v+22v^2)+ s^3(123\mu^2+300\mu v+172v^2)+s^2(93\mu^3+369\mu^2 v+456\mu v^2+176v^3)))/ ((s+\mu)^4(s+v)^4(s+\mu+2v)^4((3s+\mu)(s+2\mu)^2+(s+2\mu)(9s+8\mu)v+2(5s+8\mu)v^2+4v^3))\} =$$

$$\mathcal{L}^{-1}\left\{\frac{C_1}{(s+\mu)}+\frac{C_2}{(s+\mu)^2}+\frac{C_3}{(s+\mu)^3}+\frac{C_4}{(s+\mu)^4}+\frac{C_5}{(s+v)}+\frac{C_6}{(s+v)^2}+\frac{C_7}{(s+v)^3}+\frac{C_8}{(s+v)^4}+\frac{C_9}{(s+\mu+2v)}+\frac{C_{10}}{(s+\mu+2v)^2}+\frac{C_{11}}{(s+\mu+2v)^3}+\frac{C_{12}}{(s+\mu+2v)^4}+\frac{C_{13}}{(s-b1)}+\frac{C_{14}s+C_{15}}{(s^2+b_2 s+b_3)}\right\}=$$

$$\frac{1}{6}\Bigg(6e^{-t\mu}C_1+6e^{-t\mu}tC_2+3e^{-t\mu}t^2C_3+e^{-t\mu}t^3C_4+6e^{-tv}C_5+6e^{-tv}tC_6+3e^{-tv}t^2C_7+ e^{-tv}t^3C_8+6e^{-t(\mu+2v)}C_9+6e^{-t(\mu+2v)}tC_{10}+3e^{-t(\mu+2v)}t^2C_{11}+e^{-t(\mu+2v)}t^3C_{12}+6e^{tb1}C_{13}+ \frac{3e^{-\frac{1}{2}t\left(b_2+\sqrt{b_2^2-4b_3}\right)}\left(-\left(-1+e^{t\sqrt{b_2^2-4b_3}}\right)b_2C_{14}+\left(1+e^{t\sqrt{b_2^2-4b_3}}\right)\sqrt{b_2^2-4b_3}\,C_{14}+2\left(-1+e^{t\sqrt{b_2^2-4b_3}}\right)C_{15}\right)}{\sqrt{b_2^2-4b_3}}\Bigg)$$

(3.2.54)

The coefficients C_i ($i=1,2,..15$) are functions of the parameters μ and v. Expressions (3.2.53) and (3.2.54) are identical (the symbolic difference between the two functions has been evaluated as the null function with the aid of the program Mathematica) and can be proved by comparing them for one set of parameters v and μ, e.g. $v=10$ $\mu=1$:

$$f_{w2}(t)=g_{w2}(t)= -0.1829e^{-2.6905t}+5.301116664131303`e^{-2.2233t}-4.9862e^{-2t}-0.9999e^{-t}+6.62015e^{-0.9194t} -5.7520e^{-0.5t}+1.1934e^{-2t}t+2.3386e^{-0.5t}t-0.0493e^{-2.t}t^2-0.4411e^{-0.5t}t^2+0.0406e^{-0.5t}t^3; \quad (3.2.55)$$

It follows that the underlying counting process is a renewal process. Since $f_{w_1}(t)\neq g_{T_a}(t)$ (here again the difference between the two functions has been symbolically evaluated with the aid of Mathematica and it results different from the null function)

$$f_{w_1}(t)-g_{T_a}(t)=$$

$$
\begin{aligned}
&\frac{1}{12(\mu-\nu)^7(\mu+\nu)^7}\Big(\mu^2\nu\Big(-6e^{-t(\mu+2\nu)}(\mu-\nu)^4(\mu+\nu)^4\big(\big(-1+e^{2t\nu}\big)t\mu^5-\\
&\quad 4\big(-1+e^{2t\nu}\big)t\mu^3\nu^2+\mu^4\big(-1+e^{2t\nu}-2e^{t(\mu+\nu)}t\nu\big)-\\
&\quad 2\mu^2\nu^2\big(-3+t\nu+e^{2t\nu}(3+t\nu)\big)+\nu^4\big(3+2t\nu+2e^{t(\mu+\nu)}t\nu+e^{2t\nu}(-3+2t\nu)\big)+\\
&\quad \mu\nu^3\big(-8+16e^{t(\mu+\nu)}-3t\nu+e^{2t\nu}(-8+3t\nu)\big)\big)-\\
&e^{-2t(\mu+\nu)}\Big(3e^{t\nu}\nu^3\big(\mu^2-\nu^2\big)^2\big(12\mu^5+4t\mu^6+44\mu^3\nu^2-t\mu^4\nu^2-8\mu\nu^4-4t\mu^2\nu^4+t\nu^6\big)+\\
&\quad e^{t\mu}\mu^2(\mu-\nu)^2\big(3t\mu^9(1+t\nu)+\mu^8\big(3+9t\nu+6t^2\nu^2\big)+\nu^8\big(60-9t^2\nu^2-2t^3\nu^3\big)+\\
&\qquad 3\mu^4\nu^4\big(-21+109t\nu+17t^2\nu^2-2t^3\nu^3\big)+\mu\nu^7\big(120+60t\nu+6t^2\nu^2-t^3\nu^3\big)+\\
&\qquad \mu^7\nu\big(6-3t\nu-21t^2\nu^2+t^3\nu^3\big)-3\mu^5\nu^3\big(36-15t\nu-13t^2\nu^2+t^3\nu^3\big)+\\
&\qquad 3\mu^3\nu^5\big(58-35t\nu-9t^2\nu^2+t^3\nu^3\big)+\mu^6\nu^2\big(-3-63t\nu-39t^2\nu^2+2t^3\nu^3\big)+\\
&\qquad 3\mu^2\nu^6\big(-255-91t\nu-3t^2\nu^2+2t^3\nu^3\big)\big)+\\
&\quad e^{t(\mu+2\nu)}\mu^2(\mu+\nu)^2\big(3t\mu^9(-1+t\nu)+\mu^8\big(-3+9t\nu-6t^2\nu^2\big)+3\mu^4\nu^4\big(21+109t\nu-17t^2\nu^2-2t^3\nu^3\big)+\\
&\qquad \nu^8\big(-60+9t^2\nu^2-2t^3\nu^3\big)+\mu^7\nu\big(6+3t\nu-21t^2\nu^2-t^3\nu^3\big)+\mu\nu^7\big(120-60t\nu+6t^2\nu^2+t^3\nu^3\big)-\\
&\qquad 3\mu^3\nu^5\big(-58-35t\nu+9t^2\nu^2+t^3\nu^3\big)+3\mu^5\nu^3\big(-36-15t\nu+13t^2\nu^2+t^3\nu^3\big)+\\
&\qquad 3\mu^2\nu^6\big(255-91t\nu+3t^2\nu^2+2t^3\nu^3\big)+\mu^6\nu^2\big(3-63t\nu+39t^2\nu^2+2t^3\nu^3\big)\big)+\\
&\quad e^{t(2\mu+\nu)}\nu^3\big(2t^3\mu^{12}+483t\mu^8\nu^2+90t\mu^2\nu^8-3t\nu^{10}-78\mu^9\big(-2+t^2\nu^2\big)+6\mu\nu^8\big(4+t^2\nu^2\big)-\\
&\qquad 36\mu^3\nu^6\big(5+t^2\nu^2\big)-4t\mu^{10}\big(15+t^2\nu^2\big)-2t\mu^4\nu^6\big(384+t^2\nu^2\big)-12\mu^5\nu^4\big(283+2t^2\nu^2\big)+\\
&\qquad 12\mu^7\nu^2\big(-37+11t^2\nu^2\big)+\mu^6\big(258t\nu^4+4t^3\nu^6\big)\big)\Big)\Big)\Big)\neq 0
\end{aligned}
\tag{3.2.56}
$$

it follows that the renewal process is a delayed one.
The stationary values of mean and variance of the interarrival times take on the form:

$$
\begin{aligned}
E[T_a] &= \frac{2(\nu+\mu)^3}{\mu\nu\left(\mu^2+3\mu\nu+\nu^2\right)};\\
\sigma_{T_a}^2 &= \frac{2(\nu+\mu)^2\left(\mu^6+8\mu^5\nu+20\mu^4\nu^2+21\mu^3\nu^3+17\mu^2\nu^4+20\mu\nu^5+4\nu^6\right)}{\mu^2\nu^2(2\nu+\mu)^2\left(\mu^2+3\mu\nu+\nu^2\right)^2};
\end{aligned}
\tag{3.2.57}
$$

3.3 LINEAR OSCILLATOR UNDER A RANDOM TRAIN OF IMPULSES DRIVEN BY A SPECIAL CLASS OF NON-ERLANG RENEWAL PROCESSES

3.3.1 Equations for moments

Consider a linear oscillator governed by the equation

$$
\ddot{X}(t)+2\zeta\omega\dot{X}(t)+\omega^2X(t)=\sum_{i=1}^{R_\nu(t)}Z(t_i)P_i\delta(t-t_i) \tag{3.3.1}
$$

where the arrival times t_i are driven by an Erlang renewal process $R_\nu(t)$ with parameters ν and k and $Z(t_i)$ is a value at t_{i-} of an intermittent, zero-one stochastic variable $Z(t)$ governed by the stochastic equation

$$
dZ(t)=(1-Z)dR_\mu(t)-ZdR_\nu(t) \tag{3.3.2}
$$

The stochastic equations governing the system state vector $\mathbf{X}$ can be written as

$$d\mathbf{X}(t)=\mathbf{c}\left(\mathbf{X}(t)\right)dt+\mathbf{b}dR(t) \qquad (3.3.3)$$

where $\mathbf{X}=\left[X,\dot{X}\right]^T$, $\mathbf{c}\left(X(t)\right)=\left(\dot{X},-2\zeta\omega\dot{X}(t)-\omega^2X(t)\right)$, $\mathbf{b}=\left(0,P(t)\right)$.

As explained in section 3.1, an Erlang renewal process $R_\alpha(t)$ $(\alpha=\mu,\nu)$ may be expressed in terms of the Poisson process $N_\alpha(t)$ $(\alpha=\mu,\nu)$ at the expense of introducing auxiliary variables, for any integer parameter k or l. For any α the following replacement is valid (cf. 3.1.2)

$$dR_\alpha(t)=\rho^\alpha(t)dN_\alpha(t) \qquad (3.3.4)$$

where the ρ^α is a variable which only takes values 0 or 1 and is governed by

$$\begin{aligned}
d\rho_1^\alpha(t)&=\left(\rho_2^\alpha(t)-\rho_1^\alpha(t)\right)dN_\alpha(t)\\
d\rho_2^\alpha(t)&=\left(\rho_3^\alpha(t)-\rho_2^\alpha(t)\right)dN_\alpha(t)\\
&..\\
d\rho_{k-2}^\alpha(t)&=\left(\rho_{k-1}^\alpha(t)-\rho_{k-2}^\alpha(t)\right)dN_\alpha(t)\\
d\rho_{k-1}^\alpha(t)&=\left(1-\sum_{j=1}^{k-2}\rho_j^\alpha(t)-2\rho_{k-1}^\alpha(t)\right)dN_\alpha(t)
\end{aligned} \qquad (3.3.5)$$

where $\rho_1^\alpha(t)=\rho^\alpha(t)$.

The variables $\rho_1^\alpha(t),\rho_2^\alpha(t)..\rho_{k-1}^\alpha(t)$ only take values 0 or 1 (see chapter ..).

It is convenient to augment the state vector by new combined variables

$$\begin{aligned}
&Z_1=Z, Z_2=\rho_1^\nu,..Z_k=\rho_{k-1}^\nu, Z_{k+1}=\rho_1^\mu,..Z_{k+l-1}=\rho_{l-1}^\mu,\\
&Z_{k+l}=\rho_1^\nu\rho_1^\mu,..Z_{kl}=\rho_{k-1}^\nu\rho_{l-1}^\mu, Z_{kl+1}=Z\rho_1^\nu,..Z_{kl+k-1}=Z\rho_{k-1}^\nu,\\
&Z_{kl+k}=Z\rho_1^\mu,..Z_{kl+k+l-2}=Z\rho_{l-1}^\mu, Z_{kl+k+l-1}=Z\rho_1^\nu\rho_1^\mu,..Z_{2kl-1}=Z\rho_{k-1}^\nu\rho_{l-1}^\mu.
\end{aligned}$$

The stochastic equations governing the augmented state vector $\mathbf{X}=\begin{bmatrix}X & \dot{X} & Z_1 & Z_2 .. & Z_{2kl-1}\end{bmatrix}^T$ can be written as

$$d\mathbf{X}(t)=\mathbf{c}\left(\mathbf{X}(t)\right)dt+\mathbf{b}\left(P(t),\mathbf{X}(t)\right)d\mathbf{N}(t) \qquad (3.3.6)$$

where

$$\begin{aligned}
&\mathbf{X}(t)=\begin{bmatrix}X & \dot{X} & Z_1 & Z_2 & .. & Z_{2kl-1}\end{bmatrix}^T;\\
&\mathbf{c}\left(\mathbf{X}(t),t\right)=\begin{bmatrix}\dot{X} & -2\zeta\omega\dot{X}(t)-\omega^2X(t) & 0 & 0 & .. & 0\end{bmatrix}^T;\\
&b^\nu=\begin{bmatrix}b_1^\nu & b_2^\nu & .. & b_{2kl+1}^\nu\end{bmatrix}^T;\\
&b^\mu=\begin{bmatrix}b_1^\mu & b_2^\mu & .. & b_{2kl+1}^\mu\end{bmatrix}^T;\\
&d\mathbf{N}(t)=\begin{bmatrix}dN_\nu & dN_\mu\end{bmatrix}
\end{aligned} \qquad (3.3.7)$$

with

$$b_1^\nu = 0, b_2^\nu = PZ\rho_1^\nu, b_3^\nu = -Z\rho_1^\nu, b_4^\nu = \rho_2^\nu - \rho_1^\nu, ..b_{k+2}^\nu = 1 - \sum_{j=1}^{k-2} \rho_j^\nu - 2\rho_{k-1}^\nu,$$

$$b_{k+3}^\nu = \rho_2^\mu - \rho_1^\mu, ..b_{k+l+1}^\nu = 1 - \sum_{j=1}^{l-2} \rho_j^\mu - 2\rho_{l-1}^\mu, b_{k+l+2}^\nu = \rho_2^\nu \rho_1^\mu - \rho_1^\nu \rho_1^\mu,$$

$$b_{kl+3}^\nu = \rho_{l-1}^\mu - \sum_{j=1}^{k-2} \rho_j^\nu \rho_{l-1}^\mu - 2\rho_{k-1}^\nu \rho_{l-1}^\mu, b_{kl+4}^\nu = Z\rho_2^\nu - Z\rho_1^\nu, ..$$

$$b_{kl+k+3}^\nu = Z - Z\rho_1^\nu - 2Z\rho_{k-1}^\nu, b_{kl+k+4}^\nu = -Z\rho_1^\nu \rho_1^\mu, ..b_{kl+k+l+3}^\nu = -Z\rho_1^\nu \rho_{l-1}^\mu,$$

$$b_{kl+k+l+4}^\nu = Z\rho_2^\nu \rho_1^\mu - Z\rho_1^\nu \rho_1^\mu, ..b_{2kl+5}^\nu = Z\rho_{l-1}^\mu - Z\rho_1^\nu \rho_{l-1}^\mu - \sum_{j=2}^{k-2} Z\rho_j^\nu \rho_{l-1}^\mu - 2Z\rho_{k-1}^\nu \rho_{l-1}^\mu.$$

$$b_1^\mu = 0, b_2^\mu = 0, b_3^\mu = \rho_1^\mu - Z\rho_1^\mu, b_4^\mu = 0, ..b_{k+2}^\mu = 0, b_{k+3}^\mu = 0, ..b_{k+l+2}^\mu = 0, b_{k+l+3}^\mu = 0,$$

$$b_{kl+4}^\mu = \rho_{k-1}^\nu - \sum_{j=1}^{l-2} \rho_{k-1}^\nu \rho_j^\mu - 2\rho_{k-1}^\nu \rho_{l-1}^\mu, b_{kl+5}^\mu = \rho_1^\nu \rho_1^\mu - Z\rho_1^\nu \rho_1^\mu,$$

$$..b_{kl+k+4}^\mu = \rho_{k-1}^\nu \rho_1^\mu - Z\rho_{k-1}^\nu \rho_1^\mu, b_{kl+k+5}^\mu = Z\rho_2^\mu - Z\rho_1^\mu, ..$$

$$b_{kl+k+l+4}^\mu = Z + \rho_1^\mu - 2Z\rho_1^\mu - \sum_{j=2}^{l-2} Z\rho_j^\mu - 2Z\rho_{l-1}^\mu,$$

$$b_{kl+k+l+5}^\mu = Z\rho_1^\nu \rho_2^\mu - Z\rho_1^\nu \rho_1^\mu, ..b_{2kl+6}^\mu = Z\rho_{k-1}^\nu - \rho_{k-1}^\nu \rho_1^\mu - 2Z\rho_{k-1}^\nu \rho_1^\mu - \sum_{j=2}^{l-2} Z\rho_{k-1}^\nu \rho_j^\mu - 2Z\rho_{k-1}^\nu \rho_{l-1}^\mu.$$

The number of auxiliary zero-one stochastic variables is $2kl-1$.
While the original state vector consisting of X and $\dot{X}$ is not a Markov process, the augmented state vector $\mathbf{X}$, governed by eq. (3.3.7) driven by two independent Poisson processes N_ν and N_μ , is a non-diffusive Markov process.
By means of the generalized Ito's differential rule (see section 1.1.2) the equations for the mean values and for 2nd, 3rd and 4th order moments can be written as

$$\dot{m}_i(t) = E\left[c_i\right] + \sum_{\alpha=\nu,\mu} \alpha E\left[b_i^\alpha\right],$$

$$\dot{\mu}_{ij}(t) = 2\left\{E\left[X_i\left(c_j + \sum_{\alpha=\nu,\mu} \alpha E\left[b_j^\alpha\right]\right)\right]\right\}_s + \sum_{\alpha=\nu,\mu} \alpha E\left[b_i^\alpha b_j^\alpha\right],$$

$$\dot{\mu}_{ijk}(t) = 3\left\{E\left[X_i X_j\left(c_k + \sum_{\alpha=\nu,\mu} \alpha E\left[b_k^\alpha\right]\right)\right]\right\}_s + 3\sum_{\alpha=\nu,\mu} \alpha\left\{E\left[X_i b_j^\alpha b_k^\alpha\right]\right\}_s +$$
$$\sum_{\alpha=\nu,\mu} \alpha E\left[b_i^\alpha b_j^\alpha b_k^\alpha\right],$$

$$\dot{\mu}_{ijkl}(t) = 4\left\{E\left[X_i X_j X_k\left(c_l + 4\sum_{\alpha=\nu,\mu} \alpha E\left[b_l^\alpha\right]\right)\right]\right\}_s + 6\sum_{\alpha=\nu,\mu} \alpha\left\{E\left[X_i X_j b_k^\alpha b_l^\alpha\right]\right\}_s +$$
$$4\sum_{\alpha=\nu,\mu} \alpha\left\{E\left[X_i b_j^\alpha b_k^\alpha b_l^\alpha\right]\right\}_s + \sum_{\alpha=\nu,\mu} \alpha E\left[b_i^\alpha b_j^\alpha b_k^\alpha b_l^\alpha\right], \tag{3.3.8}$$

where $m_i(t) = E[X_i]$ and $\mu_{ij}(t) = E\left[X_i X_j\right]$.and where $\{..\}_s$ denotes the Stratonovich symmetrizing operator (e.g. $\left\{a_i a_j a_k\right\}_s = \frac{1}{3}\left(a_i a_j a_k + a_j a_k a_i + a_k a_i a_j\right)$.)

3.3.2 Process I (Generalised Erlang renewal process): $R_\mu(t) = P(\mu)$ and $R_\nu(t) = P(\nu)$

Consider the system excited by the renewal train of impulses (3.1.1), where the renewal counting process is given by (3.2.1) and (3.2.2), with $R_\mu(t) = P(\mu)$ and $R_\nu(t) = P(\nu)$, $P(\mu)$ and $P(\nu)$ being Poisson processes with mean arrival rates μ and ν respectively. The stochastic equation of motion (3.3.7) becomes

$$\mathbf{X}(t) = \begin{bmatrix} X(t) \\ \dot{X}(t) \\ Z(t) \end{bmatrix}; \qquad \mathbf{c}(\mathbf{X}(t)) = \begin{bmatrix} \dot{X}(t) \\ -\omega^2 X(t) - 2\zeta\omega \dot{X}(t) \\ 0 \end{bmatrix};$$

$$\mathbf{b}(P(t), \mathbf{X}(t)) = \begin{bmatrix} b^\nu & b^\mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ P(t)Z(t) & 0 \\ -Z(t) & 1 - Z(t) \end{bmatrix}; \tag{3.3.9}$$

$$d\mathbf{N}(t)^T = \begin{bmatrix} dN_\nu & dN_\mu \end{bmatrix};$$

The equations for the mean values $m_1 = E\left[X(t)\right], m_2 = E\left[\dot{X}(t)\right], m_3 = E\left[Z(t)\right]$ are (cf. *Iwankiewicz (2002)*)

$$\begin{aligned} \dot{m}_1(t) &= m_2(t); \\ \dot{m}_2(t) &= -\omega^2 m_1(t) - 2\zeta\omega m_2(t) + E[P(t)] m_3(t)\nu; \\ \dot{m}_3(t) &= -(\mu + \nu) m_3(t) + \mu; \end{aligned} \tag{3.3.10}$$

The stochastic equations for centralised variables $\mathbf{Y}(t) = \mathbf{X}(t) - E\left[\mathbf{X}(t)\right]$ are

$$d\mathbf{Y}(t) = c^0(\mathbf{Y}(t))dt + b(P(t), \mathbf{Y}(t))d\mathbf{N}(t) \tag{3.3.11}$$

where

$$\mathbf{c}^0(\mathbf{X}(t)) = \begin{bmatrix} Y_2(t) \\ -\omega^2 Y_1(t) - 2\zeta\omega Y_2(t) - \nu E\left[P(t)\right] E\left[Z(t)\right] \\ \nu E\left[Z(t)\right] - \mu + \mu E\left[Z(t)\right] \end{bmatrix}; \tag{3.3.12}$$

The equations for the second-order central moments of the response, are derived from Ito's rule as (cf. *Iwankiewicz (2002)*)

$$
\begin{aligned}
&\dot{\mu}_{11}(t) = 2\mu_{12}(t)\\
&\dot{\mu}_{12}(t) = \mu_{22}(t) - \omega^2\mu_{11}(t) - 2\zeta\omega\mu_{12}(t) + \nu E\left[P(t)\right]\mu_{13}(t);\\
&\dot{\mu}_{22}(t) = -2\omega^2\mu_{12}(t) - 4\zeta\omega\mu_{22}(t) + \nu E\left[P^2(t)\right]m_3(t) + 2\nu E\left[P(t)\right]\mu_{23}(t);\\
&\dot{\mu}_{13}(t) = \mu_{23}(t) - \mu_{13}(t)(\nu+\mu);\\
&\dot{\mu}_{23}(t) = -\omega^2\mu_{13}(t) - \mu_{23}(t)(2\zeta\omega+\nu+\mu) - \nu E\left[P(t)\right](m_3(t))^2
\end{aligned}
\tag{3.3.13}
$$

The stationary solutions for $m_1 = E\left[X\right]$ and $\mu_{11} = E\left[Y_1^2\right]$ give the stationary mean value and the steady-state variance of the response, respectively: (cf. *Iwankiewicz (2002)*)

$$
m_1 = \frac{\nu\mu}{\nu+\mu}\frac{E[P]}{\omega^2};\ \mu_{11} = \frac{\nu\mu}{(\nu+\mu)}\frac{E[P^2]}{4\zeta\omega^3} - \frac{\nu^2\mu^2}{(\nu+\mu)^2}\frac{(E[P])^2(2\zeta\omega+\nu+\mu)}{2\zeta\omega^3\left[\omega^2+(\nu+\mu)(2\zeta\omega+\nu+\mu)\right]};
\tag{3.3.14}
$$

3.3.3 Process II : $R_\mu(t) = E(\mu, l)$ and $R_\nu(t) = P(\nu)$.

Consider the system excited by the renewal train of impulses (3.1.1), where the renewal counting process is given by (3.2.1) and (3.2.2), with $R_\mu(t) = E(\mu, l)$ and $R_\nu(t) = P(\nu)$, where $E(\mu, l)$ denotes an Erlang process with parameters μ and l. The stochastic equation of motion (3.3.7) is specified by (cf. *Tellier and Iwankiewicz (2005)*)

$$
\mathbf{X}(t) = \begin{bmatrix} X(t)\\ \dot{X}(t)\\ Z_1(t)\\ Z_2(t)\end{bmatrix};\mathbf{c}(\mathbf{X}(t)) = \begin{bmatrix} \dot{X}(t)\\ -\omega^2X(t) - 2\zeta\omega\dot{X}(t)\\ 0\\ 0\end{bmatrix};
$$

$$
\mathbf{b}(P(t),\mathbf{X}(t)) = \begin{bmatrix} b^\nu & b^\mu\end{bmatrix} = \begin{bmatrix} 0 & 0\\ P(t)Z(t) & 0\\ -Z_1(t) & \rho_\mu - Z_2(t)\\ -Z_2(t) & Z_1(t) - 2Z_2(t)\end{bmatrix};
\tag{3.3.15}
$$

where $Z_1(t) = Z$ and $Z_2(t) = Z\rho_\mu$.

The equations for the mean values ($m_1 = E\left[X(t)\right], m_2 = E\left[\dot{X}(t)\right], m_3 = E\left[Z_1(t)\right], m_4 = E\left[Z_2(t)\right]$) are

$$
\begin{aligned}
&\dot{m}_1(t) = m_2(t);\\
&\dot{m}_2(t) = -\omega^2 m_1(t) - 2\zeta\omega m_2(t) + E[P(t)]m_3(t)\nu;\\
&\dot{m}_3(t) = -m_3(t)\nu - \mu m_4(t) + E[\rho_\mu]\mu;\\
&\dot{m}_4(t) = m_3(t)\mu - m_4(t)(\nu+2\mu);
\end{aligned}
\tag{3.3.16}
$$

In the stochastic equations for centralised variables the vector $\mathbf{c}^0(\mathbf{X}(t))$ is given by

$$\mathbf{c}^0\left(\mathbf{X}(t)\right)=\begin{bmatrix} Y_2(t) \\ -\omega^2 Y_1(t)-2\zeta\omega Y_2(t)-\nu E\left[P(t)\right]E\left[Z_1(t)\right] \\ \nu E\left[Z_1(t)\right]-\mu E\left[\rho_\mu\right]+\mu E\left[Z_2(t)\right] \\ \nu E\left[Z_2(t)\right]-\mu E\left[Z_1(t)\right]+2\mu E\left[Z_2(t)\right] \end{bmatrix}; \tag{3.3.17}$$

The equations for the second-order central moments of the response are

$$\begin{aligned}
&\dot{\mu}_{11}(t)=2\mu_{11}(t) \\
&\dot{\mu}_{12}(t)=\mu_{22}(t)-\omega^2\mu_{11}(t)-2\zeta\omega\mu_{12}(t)+\nu E\left[P(t)\right]\mu_{13}(t); \\
&\dot{\mu}_{13}(t)=\mu_{23}(t)-\mu_{13}(t)\nu-\mu_{14}(t)\mu; \\
&\dot{\mu}_{14}(t)=\mu_{24}(t)-\mu_{14}(t)\nu+\mu_{13}(t)\mu-2\mu_{14}(t)\mu; \\
&\dot{\mu}_{22}(t)=-2\omega^2\mu_{12}(t)-4\zeta\omega\mu_{22}(t)+\nu E\left[P^2(t)\right]m_3(t)+2\nu E\left[P(t)\right]\mu_{23}(t); \\
&\dot{\mu}_{23}(t)=-\omega^2\mu_{13}(t)-\mu_{23}(t)\left(2\zeta\omega+\nu\right)-\mu_{24}(t)\mu-\nu E\left[P(t)\right]\left(m_3(t)\right)^2; \\
&\dot{\mu}_{24}(t)=-\omega^2\mu_{14}(t)-\mu_{24}(t)\left(2\zeta\omega+\nu+2\mu\right)+\mu_{23}(t)\mu-\nu E\left[P(t)\right]m_3(t)m_4(t);
\end{aligned} \tag{3.3.18}$$

The stationary solution for $m_1=E\left[X\right]$ and $\mu_{11}=E\left[Y_1^2\right]$ give the stationary mean value and the steady-state variance of the response, respectively

$$m_1=\frac{\nu\mu\left(2\nu+\mu\right)}{2\left(\nu+\mu\right)^2}\frac{E[P]}{\omega^2};$$

$$\mu_{11}=\frac{\nu\mu\left(2\nu+\mu\right)}{\left(\nu+\mu\right)^2}\frac{E[P^2]}{8\zeta\omega^3}+$$

$$\frac{\nu^3\mu\left(2\nu+\mu\right)\left(E[P]\right)^2\left(-\mu\left(\nu+\mu\right)^3-4\zeta\mu\omega\left(\nu+\mu\right)^2-\left(\left(-3+4\zeta^2\right)\mu-4\nu\right)\left(\nu+\mu\right)\omega^2+2\zeta\omega^3+\nu\left(2\nu+\mu\right)\right)}{8\zeta\omega^3\left[\omega^2+\left(\nu+\mu\right)\left(2\zeta\omega+\nu+\mu\right)\right]^2\left(\nu+\mu\right)^4} \tag{3.3.19}$$

3.3.4 Process III: $R_\mu(t)=P(\mu)$ and $R_\nu(t)=E(\nu,k)$.

Consider the system excited by the renewal train of impulses (3.1.1), where the renewal counting process is given by (3.2.1) and (3.2.2), with $R_\mu(t)=P(\mu)$ and $R_\nu(t)=E(\nu,k)$. The stochastic equation of motion (3.3.7) is specified by (cf. *Tellier and Iwankiewicz (2005), Iwankiewicz (2003*)

$$\mathbf{X}(t)=\begin{bmatrix} X(t) \\ \dot{X}(t) \\ Z_1(t) \\ Z_2(t) \end{bmatrix};\mathbf{c}\left(\mathbf{X}(t)\right)=\begin{bmatrix} \dot{X}(t) \\ -\omega^2 X(t)-2\zeta\omega\dot{X}(t) \\ 0 \\ 0 \end{bmatrix};$$

$$\mathbf{b}\left(P(t),\mathbf{X}(t)\right)=\left[b^{\nu} \quad b^{\mu}\right]=\begin{bmatrix} 0 & 0 \\ P(t)Z_2(t) & 0 \\ -Z_2(t) & 1-Z_1(t) \\ Z_1(t)-2Z_2(t) & \rho_\nu - Z_2(t) \end{bmatrix}; \tag{3.3.20}$$

where $Z_1(t)=Z, Z_2(t)=Z\rho_\nu$.

The equations for the mean values ($m_1=E\left[X(t)\right], m_2=E\left[\dot{X}(t)\right], m_3=E\left[Z_1(t)\right], m_4=E\left[Z_2(t)\right]$) are

$$\begin{aligned}
&\dot{m}_1(t)=m_2(t); \\
&\dot{m}_2(t)=-\omega^2 m_1(t)-2\zeta\omega m_2(t)+E[P(t)]m_4(t)\nu; \\
&\dot{m}_3(t)=-m_3(t)\mu-\nu m_4(t)+\mu; \\
&\dot{m}_4(t)=m_3(t)\nu-m_4(t)(2\nu+\mu)+E[\rho_\nu]\mu;
\end{aligned} \tag{3.3.21}$$

In the stochastic equations for centralised variables the vector $\mathbf{c}^0\left(\mathbf{X}(t)\right)$ is given by

$$\mathbf{c}^0\left(\mathbf{X}(t)\right)=\begin{bmatrix} Y_2(t) \\ -\omega^2 Y_1(t)-2\zeta\omega Y_2(t)-\nu E\left[P(t)\right]E\left[Z_2(t)\right] \\ \nu E\left[Z_2(t)\right]+\mu-\mu E\left[Z_1(t)\right] \\ -\nu E\left[Z_1(t)\right]-\mu E\left[\rho_\nu\right]+(2\nu+\mu)E\left[Z_2(t)\right] \end{bmatrix} \tag{3.3.22}$$

The equations for the second-order central moments of the response are

$$\begin{aligned}
&\dot{\mu}_{11}(t)=2\mu_{12}(t) \\
&\dot{\mu}_{12}(t)=\mu_{22}(t)-\omega^2\mu_{11}(t)-2\zeta\omega\mu_{12}(t)+\nu E\left[P(t)\right]\mu_{14}(t); \\
&\dot{\mu}_{13}(t)=\mu_{23}(t)-\mu_{14}(t)\nu-\mu_{13}(t)\mu; \\
&\dot{\mu}_{14}(t)=\mu_{24}(t)+\mu_{13}(t)\nu-\mu_{14}(t)(2\nu+\mu); \\
&\dot{\mu}_{22}(t)=-2\omega^2\mu_{12}(t)-4\zeta\omega\mu_{22}(t)+\nu E\left[P^2(t)\right]m_4(t)+2\nu E\left[P(t)\right]\mu_{24}(t); \\
&\dot{\mu}_{23}(t)=-\omega^2\mu_{13}(t)-(2\zeta\omega+\mu)\mu_{23}(t)-\mu_{24}(t)\nu-E\left[P(t)\right]m_3(t)m_4(t); \\
&\dot{\mu}_{24}(t)=-\omega^2\mu_{14}(t)-\mu_{24}(t)(2\zeta\omega+2\nu+\mu)+\mu_{23}(t)\nu-\nu E\left[P(t)\right]\left(m_4(t)\right)^2;
\end{aligned} \tag{3.3.23}$$

The stationary solution for $m_1=E\left[X\right]$ and $\mu_{11}=E\left[Y_1^2\right]$ give the stationary mean value and the steady-state variance of the response, respectively

$$m_1=\frac{\nu\mu(\nu+2\mu)}{2(\nu+\mu)^2}\frac{E[P]}{\omega^2}; \tag{3.3.24}$$

$$\mu_{11}=\frac{\nu\mu(2\mu+\nu)}{(\nu+\mu)^2}\frac{E[P^2]}{8\zeta\omega^3}+$$

$$\frac{\mu^3\nu(2\mu+\nu)\left(E[P]\right)^2\left(-\nu(\nu+\mu)^3-4\zeta\nu\omega(\nu+\mu)^2-\left(\left(-3+4\zeta^2\right)\nu-4\mu\right)(\nu+\mu)\omega^2+2\zeta\omega^3+\mu(\nu+2\mu)\right)}{8\zeta\omega^3\left[\omega^2+(\nu+\mu)(2\zeta\omega+\nu+\mu)\right]^2(\nu+\mu)^4}$$

3.3.5 Process IV: $R_\mu(t)=E(\mu,l)$ and $R_\nu(t)=E(\nu,k)$.

Consider the system excited by the train of impulses (3.1.1), where the driving counting process is given by (3.3.1) and (3.3.2), with $R_\mu(t)=E(\mu,l)$ and $R_\mu(t)=E(\mu,l)$.The stochastic equation of motion (3.3.7) is specified by

$$\mathbf{X}(t)=\begin{bmatrix} X(t)\\ \dot{X}(t)\\ Z_1(t)\\ Z_2(t)\\ Z_3(t)\\ Z_4(t)\end{bmatrix};\mathbf{c}(\mathbf{X}(t))=\begin{bmatrix} \dot{X}(t)\\ -\omega^2X(t)-2\zeta\omega\dot{X}(t)\\ 0\\ 0\\ 0\\ 0\end{bmatrix};$$

$$\mathbf{b}(P(t),\mathbf{X}(t))=\begin{bmatrix} b^\nu & b^\mu\end{bmatrix}=\begin{bmatrix} 0 & 0\\ P(t)Z_2(t) & 0\\ -Z_2(t) & \rho_\mu-Z_3(t)\\ Z_1(t)-2Z_2(t) & \rho_\nu\rho_\mu-Z_4(t)\\ -Z_4(t) & Z_1(t)-2Z_3(t)\\ Z_3(t)-2Z_4(t) & Z_2(t)-2Z_4(t)\end{bmatrix}; \tag{3.3.25}$$

where $Z_1(t)=Z, Z_2(t)=Z\rho_\nu, Z_3(t)=Z\rho_\mu, Z_4(t)=Z\rho_\nu\rho_\mu$.
The equations for the mean values
($m_1=E\left[X(t)\right], m_2=E\left[\dot{X}(t)\right], m_3=E\left[Z_1(t)\right]$, $m_4=E\left[Z_2(t)\right]$, $m_5=E\left[Z_3(t)\right]$ $m_6=E\left[Z_4(t)\right]$) are

$$\begin{aligned}
&\dot{m}_1(t)=m_2(t);\\
&\dot{m}_2(t)=-\omega^2m_1(t)-2\zeta\omega m_2(t)+E[P(t)]m_4(t)\nu;\\
&\dot{m}_3(t)=-m_5(t)\mu-\nu m_4(t)+E[\rho_\mu]\mu;\\
&\dot{m}_4(t)=m_3(t)\nu-2m_4(t)\nu-m_6(t)\mu+E[\rho_\mu]E[\rho_\nu]\mu;\\
&\dot{m}_5(t)=m_3(t)\mu-2m_5(t)\mu-m_6(t)\nu;\\
&\dot{m}_6(t)=m_4(t)\mu+m_5(t)\nu-2m_6(t)(\nu+\mu);
\end{aligned} \tag{3.3.26}$$

In the stochastic equations for centralised variables the vector $\mathbf{c}^0(\mathbf{X}(t))$ is given by

$$\mathbf{c}^0(\mathbf{X}(t))^T=\begin{bmatrix} Y_2(t)\\ -\omega^2Y_1(t)-2\zeta\omega Y_2(t)-\nu E\left[P(t)\right]E\left[Z_2(t)\right]\\ \nu E\left[Z_2(t)\right]+\mu-\mu E\left[Z_1(t)\right]\\ -\nu E\left[Z_1(t)\right]-\mu E\left[\rho_\nu\right]+(2\nu+\mu)E\left[Z_2(t)\right]\\ \nu E\left[Z_4(t)\right]-\mu E\left[Z_1(t)\right]+2\mu E\left[Z_3(t)\right]\\ -\nu E\left[Z_3(t)\right]-2(\nu-\mu)E\left[Z_4(t)\right]-\mu E\left[Z_2(t)\right]\end{bmatrix} \tag{3.3.27}$$

The equations for the second-order central moments of the response are

$$
\begin{aligned}
&\dot{\mu}_{11}(t) = 2\mu_{12}(t) \\
&\dot{\mu}_{12}(t) = \mu_{22}(t) - \omega^2\mu_{11}(t) - 2\zeta\omega\mu_{12}(t) + \nu E\left[P(t)\right]\mu_{14}(t); \\
&\dot{\mu}_{13}(t) = \mu_{23}(t) - \mu_{14}(t)\nu - \mu_{13}(t)\mu; \\
&\dot{\mu}_{14}(t) = \mu_{24}(t) - \mu_{13}(t)\nu - \mu_{14}(t)(2\nu + \mu); \\
&\dot{\mu}_{15}(t) = \mu_{25}(t) - \mu_{16}(t)\nu - \mu_{14}(t)\mu - 2\mu_{16}(t)\mu; \\
&\dot{\mu}_{16}(t) = \mu_{26}(t) + \mu_{15}(t)\nu + \mu_{14}(t)\mu - 2\mu_{16}(t)(\nu + \mu); \\
&\dot{\mu}_{22}(t) = -2\omega^2\mu_{12}(t) - 4\zeta\omega\mu_{22}(t) + \nu E\left[P^2(t)\right]m_4(t) + 2\nu E\left[P(t)\right]\mu_{24}(t); \\
&\dot{\mu}_{23}(t) = -\omega^2\mu_{13}(t) - 2\zeta\omega\mu_{23}(t) - \mu_{24}(t)\nu + E\left[P(t)\right]\mu_{34}(t)\mu - \nu E\left[P(t)\right]m_4(t); \\
&\dot{\mu}_{24}(t) = -\omega^2\mu_{14}(t) - \mu_{24}(t)(2\zeta\omega + 2\nu + \mu) + \mu_{23}(t)\nu + \nu E\left[P(t)\right]m_4(t)\left(m_3(t) - m_4(t) - 1\right); \\
&\dot{\mu}_{25}(t) = -\omega^2\mu_{15}(t) - \mu_{25}(t)(2\zeta\omega) - \mu_{26}(t)(\nu + 2\mu) + \nu\mu_{45}(t)E\left[P(t)\right] \\
&-\nu\mu_{46}(t)E\left[P(t)\right] + \mu_{24}(t)\mu - \nu E\left[P(t)\right]m_4(t)m_6(t); \\
&\dot{\mu}_{26}(t) = -\omega^2\mu_{16}(t) - \mu_{26}(t)(2\zeta\omega + 2\mu + 2\nu) - \mu_{25}(t)(\nu) + \nu\left(\mu_{45}(t) - \mu_{46}(t)\right)E\left[P(t)\right] \\
&+\mu_{24}(t)\mu - \nu E\left[P(t)\right]m_4(t)\left(m_5(t) + 2m_6(t)\right);
\end{aligned}
\tag{3.3.28}
$$

The stationary solution for $m_1 = E\left[X\right]$ and $\mu_{11} = E\left[Y_1^2\right]$ give the stationary mean value and the steady-state variance of the response, respectively

$$
\begin{aligned}
&m_1 = \frac{\left(\mu^2 + 3\mu\nu + \nu^2\right)\mu\nu E[P]}{2\omega^2(\nu + \mu)^3}; \\
&\mu_{11} = \frac{\left(\mu^2 + 3\mu\nu + \nu^2\right)\mu\nu}{(\nu + \mu)^3}\frac{E[P^2]}{8\zeta\omega^3} + \\
&\frac{\left(E[P]\right)^2\nu^2\mu^2\left(\mu^2 + 3\mu\nu + \nu^2\right)}{16\zeta\omega^3\left[\omega^2 + (\nu + \mu)(2\zeta\omega + \nu + \mu)\right]^3(\nu + \mu)^6} * P(\mu, \nu);
\end{aligned}
\tag{3.3.29}
$$

where

$$
\begin{aligned}
&P(\mu, \nu) = \\
&-3(\nu + \mu)^5\left(\mu^2 + 4\mu\nu + \nu^2\right) - 18\zeta\omega(\nu + \mu)^4\left(\mu^2 + 4\mu\nu + \nu^2\right) - \\
&2\omega^2(\nu + \mu)^3\left(2\left(1 + 9\zeta^2\right)\mu^2 + 2\left(1 + 9\zeta^2\right)\nu^2 + 3\mu\nu\left(1 + 24\zeta^2\right)\right) - \\
&6\zeta\omega^3(\nu + \mu)^2\left(\left(3 + 4\zeta^2\right)\mu^2 + \left(3 + 4\zeta^2\right)\nu^2 + 8\mu\nu\left(1 + 2\zeta^2\right)\right) - \\
&(\nu + \mu)\omega^4\left(\left(1 + 20\zeta^2\right)\mu^2 + \left(1 + 20\zeta^2\right)\nu^2 + 2\mu\nu\left(1 + 32\zeta^2\right)\right) - \\
&4\zeta\left(\mu^2 + 3\mu\nu + \nu^2\right)\omega^5
\end{aligned}
\tag{3.3.30}
$$

3.3.6 Numerical analysis

In order to illustrate the stochastic behaviour of a linear system under a non-Erlang renewal train of impulses, using the method developed, consider a linear oscillator governed by the differential equation (3.1.1).
In order for the four different cases of renewal processes examined to be comparable, the couples of parameters μ_l and ν_l with $l = I,..IV$ are chosen in such a way as to maintain the same stationary mean value $E[T]$ of the interarrival times and hence the same stationary mean value of the response. In fact, from equations (3.2.19), (3.2.31), (3.2.43), (3.2.57), (3.3.14), (3.3.19), (3.3.24) and (3.3.29), it appears that the stationary mean response of a linear oscillator under the three types of renewal driven impulsive excitations considered in this paper is

$$E[X] = \frac{E[P]}{E[T_l]\omega^2} \qquad l = I,..,IV \tag{3.3.31}$$

where $E[T]$ is the stationary mean value of the interarrival times.
Assuming the first process as reference, for different choices of parameters ($\mu_I = 1, \nu_I = 0.1$; $\mu_I = 1, \nu_I = 1$; $\mu_I = 1, \nu_I = 10$), the corresponding values of the parameter ν_l with $l = II,..IV$, arbitrarily fixing μ_l ($l = II,..IV$), are derived from the condition (see tab 1):

$$E[T_l] = E[T_I] \qquad l = II,..IV \tag{3.3.32}$$

The data assumed for the oscillator is: $\omega = 1s^{-1}, \zeta = 0.05$. The impulses magnitudes are assumed to be Gaussian distributed random variables with $E[P] = 1\frac{m}{s^{-1}}$ and $E[P^2] = 2\frac{m}{s^{-2}}$.
In fig. 3.3.1 are depicted the probability density functions of the interarrival times for the different types of impulse processes considered. It should be noted that, although the mean values are the same, due to the different shapes of the probability density functions of the processes II,.. IV, the corresponding peaks are shifted to the right with respect to that of the first one, and the tails go rapidly below that of the first process.
The stochastic response of a linear oscillator under a train of impulses driven by the renewal processes I,..IV is analysed.
The results for the mean values and the variances of the response are shown in figures 3.3.2 and 3.3.3.
The mean responses of the linear oscillator for the different processes show slight differences in the transient phase, except when the ratio $r_I = \nu_I / \mu_I$ of the reference process parameters is small.
From equations equations (3.2.19), (3.2.31), (3.2.43), (3.2.57), (3.3.14), (3.3.19), (3.3.24) and (3.3.29), it appears that the stationary variance of the response is composed of two contributions, respectively proportional to the second order moment $E[P^2]$ and the square of the mean value $(E[P])^2$ of the impulses amplitudes. The first contribution, for all the processes considered, is $E[P^2]/(4\zeta\omega^3 E[T_l])$, considering that the parameters μ_l and ν_l with $l = I,..IV$ are chosen in such a way as to maintain the same stationary mean value $E[T]$ of the interarrival times (cf. equation (3.3.31)), the differences in the behaviour of the variances of the responses are due to the second contribution.

It may be noted that in the transient phase, for each of the three examples considered the curve corresponding to the reference process is always above all the others. However the stationary behaviour, parameters μ_k being equal, seems to be influenced by the parameters ν_k, with $k = II, III, IV$, in such a way that $\sigma^2_{X_k} > \sigma^2_{X_j}$ if $\nu_k > \nu_j$, with $k = II, III, IV$ and $j = II, III, IV$, except when the ratio $r_I = \nu_I / \mu_I$ of the reference process parameters is small.

μ1	ν1	μ2	ν2	μ3	ν3	μ4	ν4
1	1	1.2	1.73939	1.2	1.41343	1.2	2.61617
1	10	5	1.26783	5	1.9769	5	2.14774
1	0.1	0.2	0.463325	0.2	0.272379	0.2	0.683394

Tab.1

Parameters μ_i and ν_i ($i = II, III, IV$) derived choosing the process I as reference

($\mu_I = 1, \nu_I = 1$; $\mu_I = 1, \nu_I = 10$; $\mu_I = 1\ \nu_I = 0.1$), arbitrarily fixing μ_i and using the condition

$E[T_i] = E[T_I]$

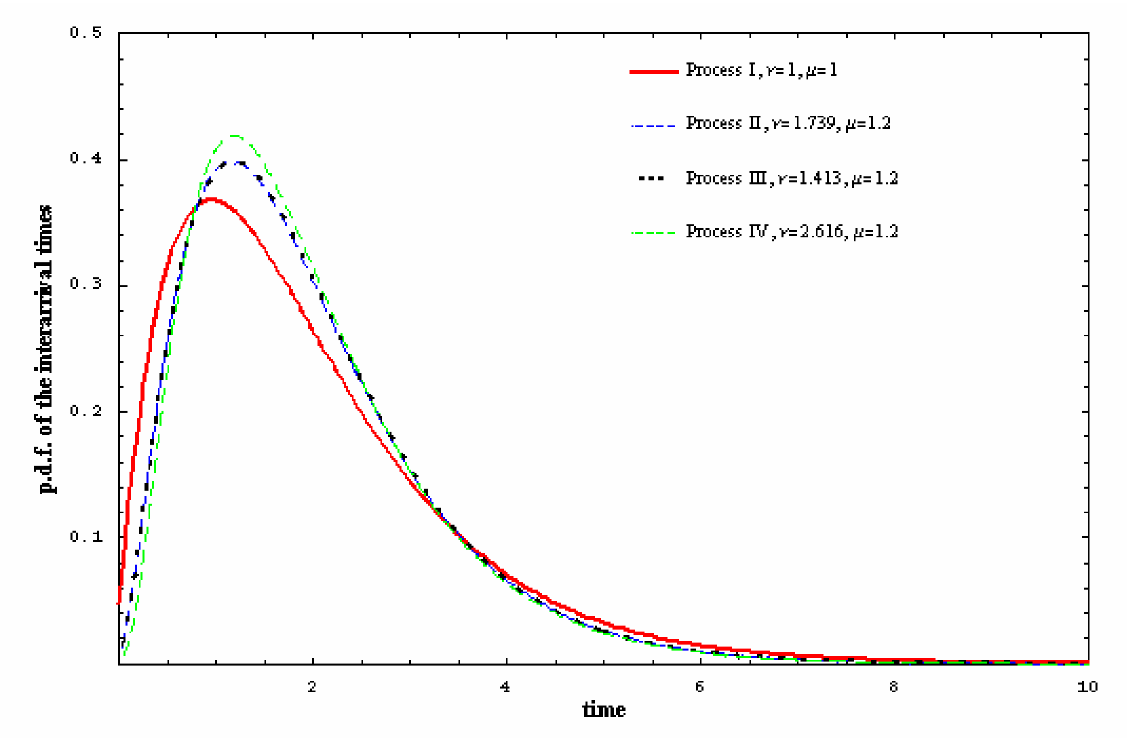

Fig.3.3.1-(a)

Probability density of the interarrival times- non-Erlang renewal processes with $E[T] = 2$

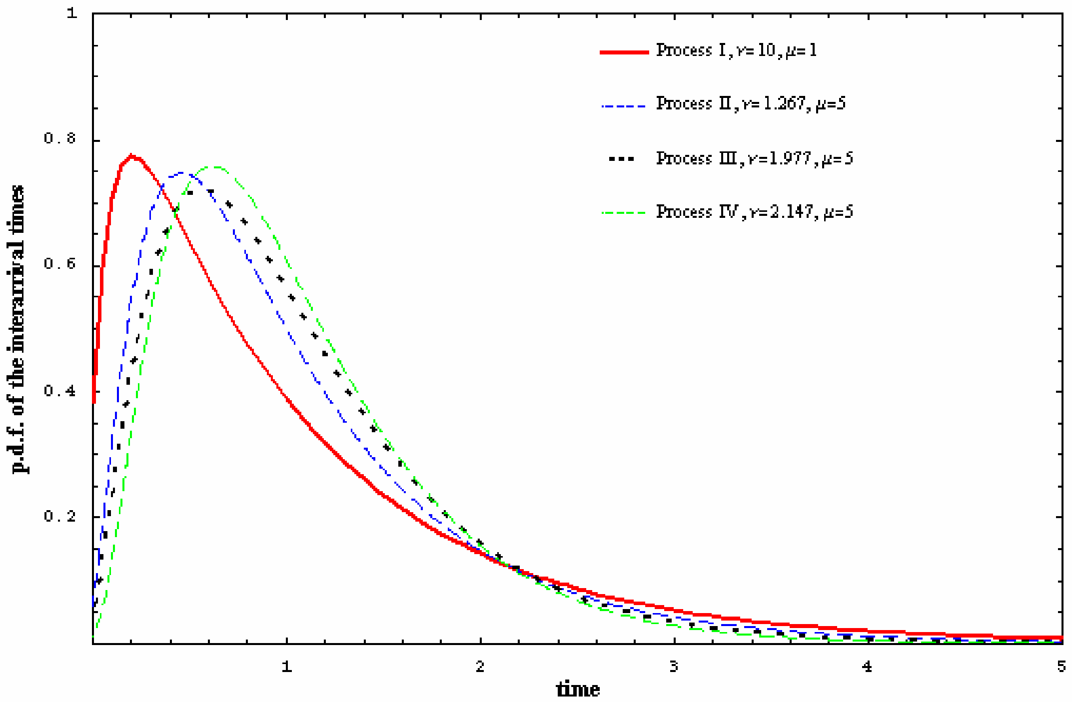

Fig.3.3.1-(b)

Probability density of the interarrival times- non-Erlang renewal processes with $E[T]=1.1$

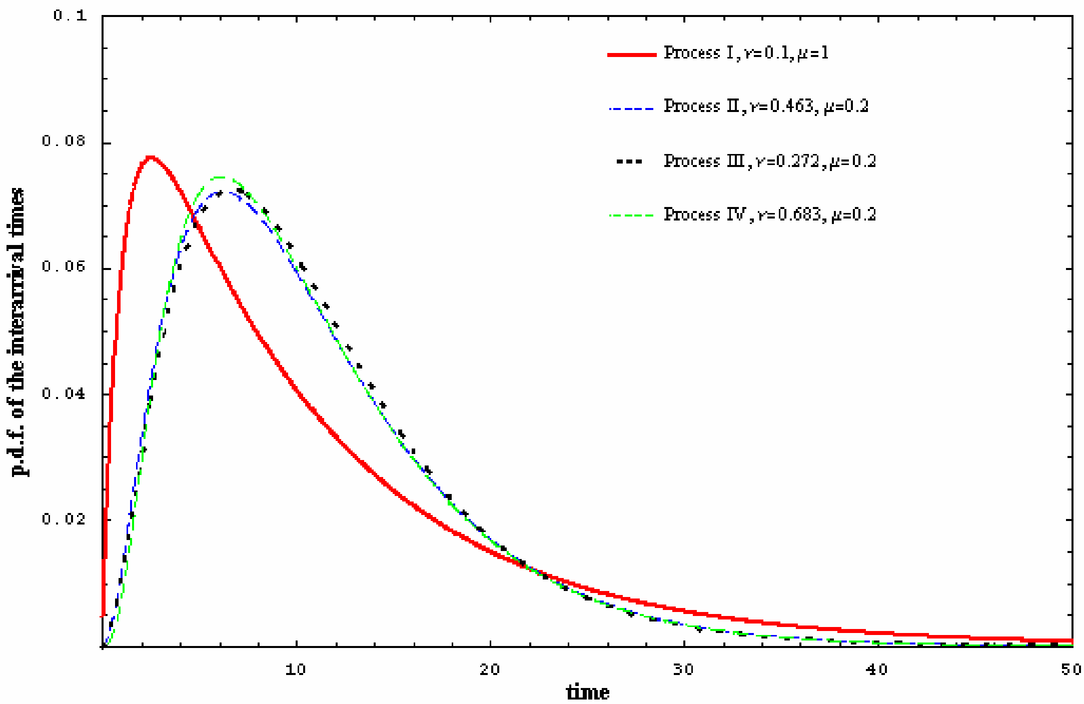

Fig.3.3.1-(c)

Probability density of the interarrival times- non-Erlang renewal processes with $E[T]=11$

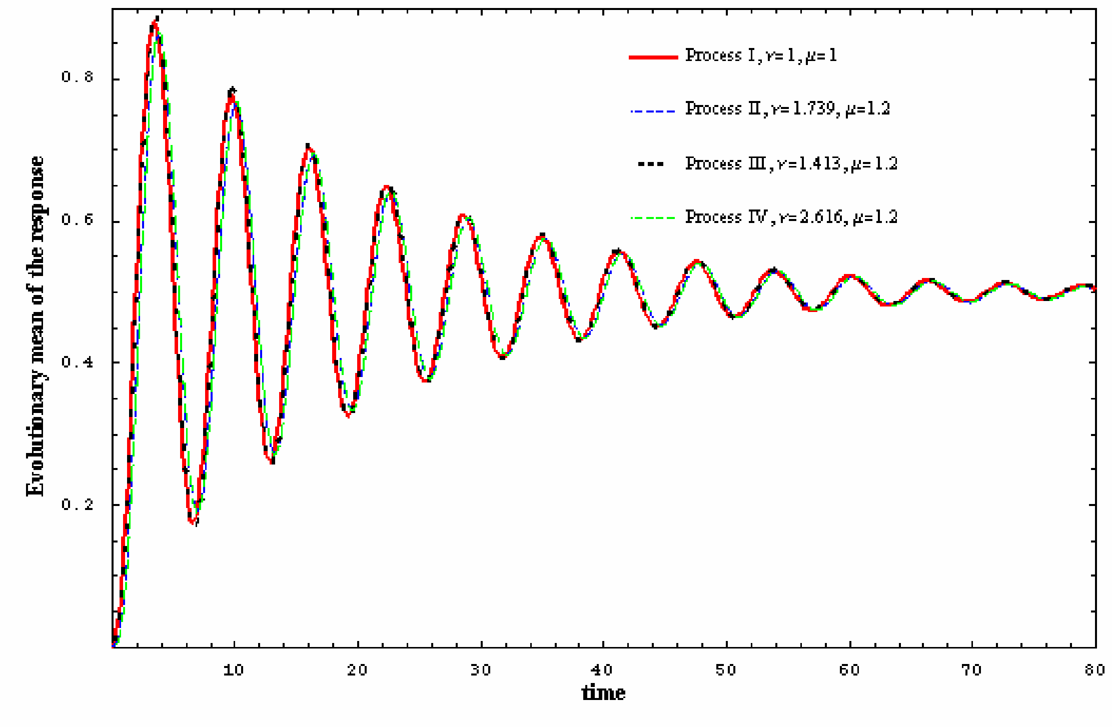

Fig. 3.3.2-(a)

Evolutionary mean of the response of a linear oscillator($E[T]=2$)

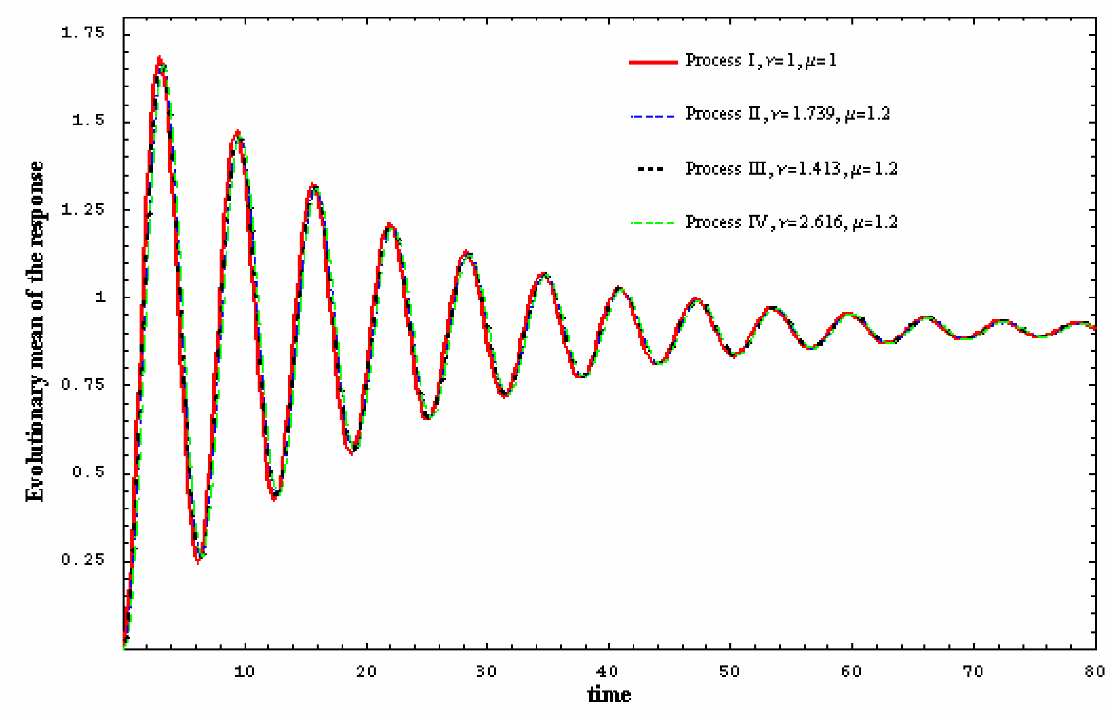

Fig. .3.3.2-(b)

Evolutionary mean of the response of a linear oscillator ($E[T]=1.1$)

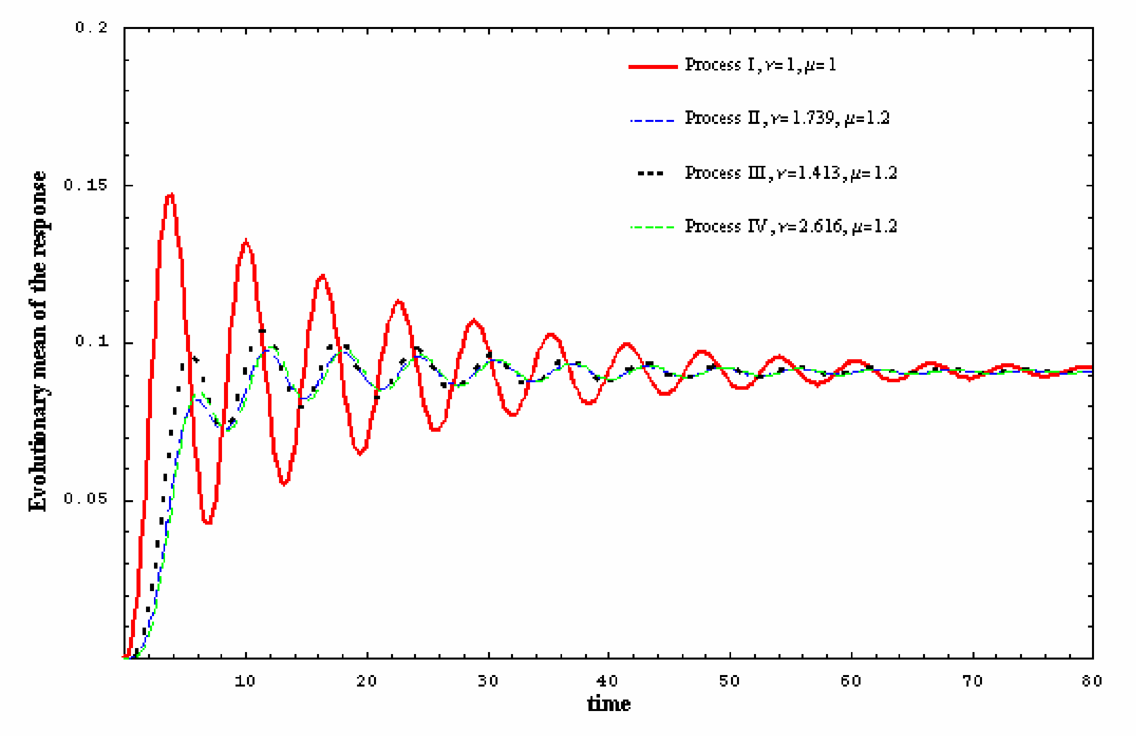

Fig. .3.3.2-(c)

Evolutionary mean of the response of a linear oscillator ($E[T]=11$)

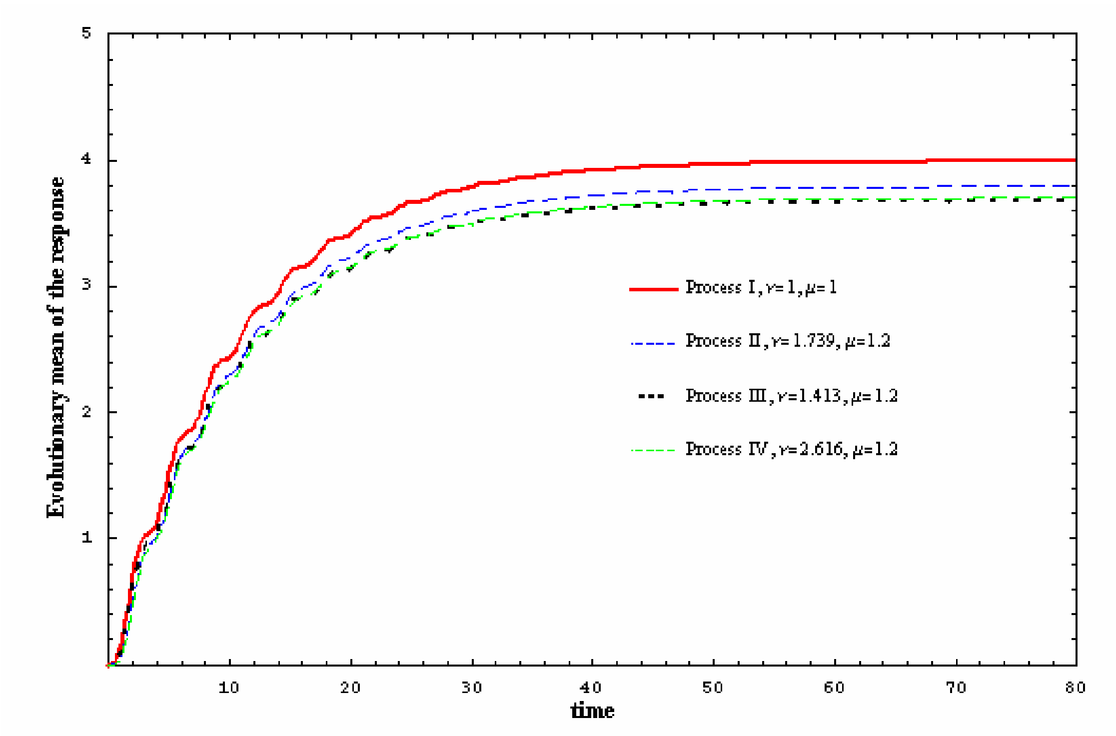

Fig. .3.3.3-(a)

Evolutionary variance of the response of a linear oscillator ($E[T]=2$)

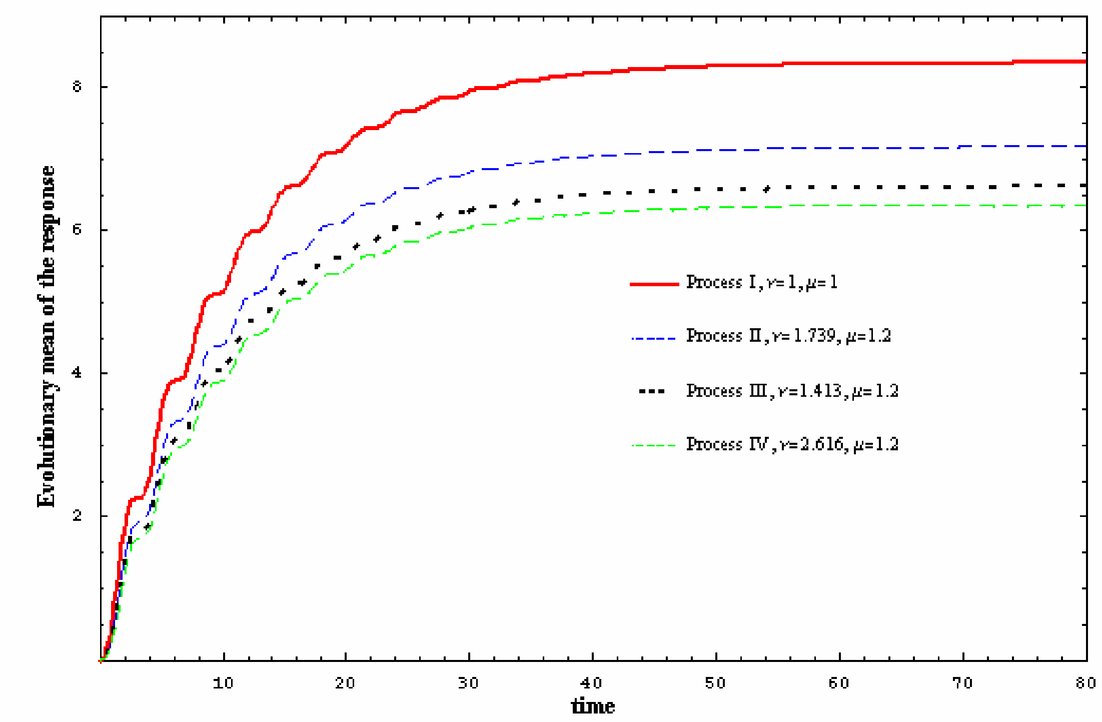

Fig. .3.3.3-(b)

Evolutionary variance of the response of a linear oscillator ($E[T]=1.1$)

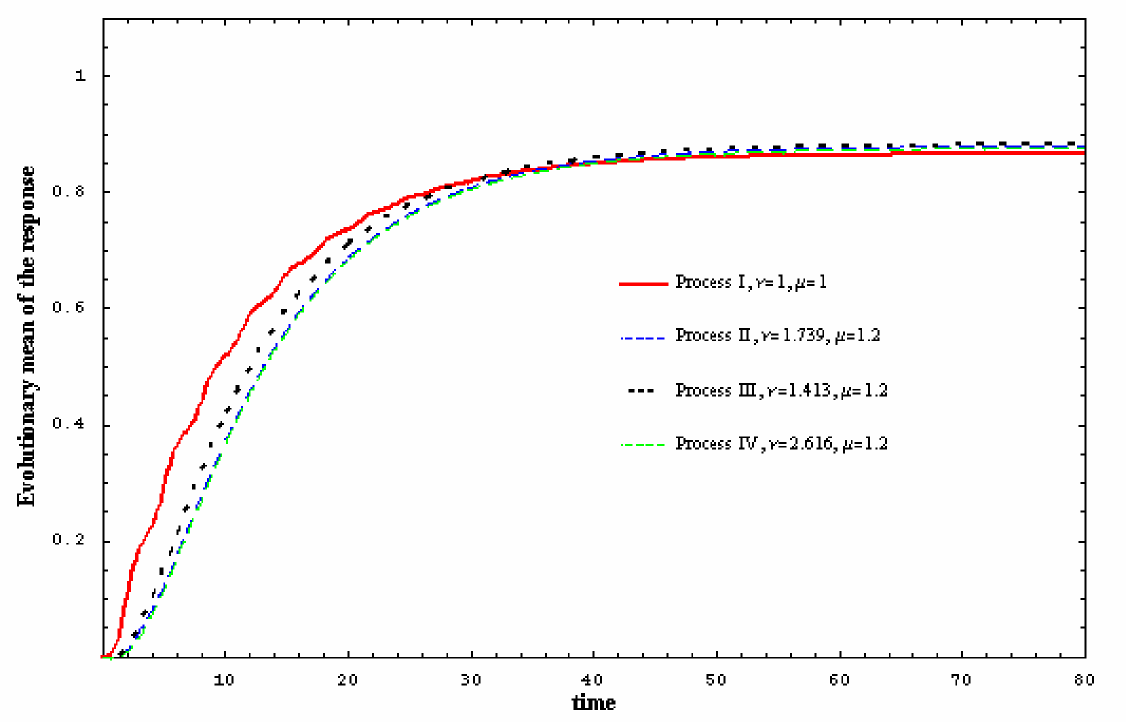

Fig. .3.5.3-(c)

Evolutionary variance of the response of a linear oscillator ($E[T]=11$)

4. DYNAMIC RESPONSE OF NON-LINEAR SYSTEMS

4.1 NON-LINEAR OSCILLATOR UNDER AN ERLANG RENEWAL DRIVEN TRAIN OF IMPULSES

4.1.1 Statement of the problem

Consider a non-linear, non-hysteretic oscillator under purely external excitation, governed by the equation

$$\ddot{X}(t) + f\left(X(t), \dot{X}(t)\right) = \sum_{i,R=1}^{R_\nu(t)} P_{i,R}\delta\left(t - t_{i,R}\right) \tag{4.1.1}$$

where $f\left(X(t), \dot{X}(t)\right)$ is the function of instantaneous values of $X(t)$ and $\dot{X}(t)$ and the stochastic excitation is the random train of impulses whose arrival times $t_{i,R}$ are driven by an Erlang renewal process $R_\nu(t)$ with parameters k and ν.

The stochastic equations governing the system state vector $\mathbf{X} = \left[X, \dot{X}\right]^T$ can be written as

$$d\mathbf{X}(t) = \mathbf{c}\left(X(t)\right)dt + \mathbf{b}dR_\nu(t) \tag{4.1.2}$$

where $\mathbf{c}\left(X(t)\right) = \left[\dot{X}, -f\left(X, \dot{X}\right)\right]^T$, $\mathbf{b} = \left[0, P(t)\right]^T$.

As before, the process $R_\nu(t)$ can be expressed in terms of the Poisson process as

$$dR_\nu = \rho_1^\nu(t)\,dN_\nu(t) \tag{4.1.3}$$

where the ρ_1^ν is a variable which only takes values 0 or 1 and is governed by (cf. equation (3.1.26)

$$\begin{aligned}
d\rho_1^\nu(t) &= \left(\rho_2^\nu(t) - \rho_1^\nu(t)\right)dN_\nu(t) \\
d\rho_2^\nu(t) &= \left(\rho_3^\nu(t) - \rho_2^\nu(t)\right)dN_\nu(t) \\
&\ldots \\
d\rho_{k-2}^\nu(t) &= \left(\rho_{k-1}^\nu(t) - \rho_{k-2}^\nu(t)\right)dN_\nu(t) \\
d\rho_{k-1}^\nu(t) &= \left(1 - \sum_{j=1}^{k-2}\rho_j^\nu(t) - 2\rho_{k-1}^\nu(t)\right)dN_\nu(t)
\end{aligned} \tag{4.1.4}$$

The auxiliary variables $\rho_1^\nu(t), \rho_2^\nu(t) .. \rho_{k-1}^\nu(t)$ only take values 0 or 1. The variable ρ_1^ν is equal to 1 in the time interval between the (n-1)st arrival of the Poisson process $N_\nu(t)$ and the n-th arrival. The variable $\rho_2^\nu(t)$ is equal to 1 in the time interval between the (k-2)nd arrival of the Poisson process $N_\nu(t)$ and the (k-1)st arrival.

The variable $\rho_{k-1}^\nu(t)$ is different from zero in the time interval between the 1st and the second arrivals of the Poisson process $N_\nu(t)$ (see Figure 4.1.1).

The time evolution of the augmented state vector $\mathbf{X} = \left[X \quad \dot{X} \quad \rho_1^\nu \quad \rho_2^\nu .. \rho_{k-1}^\nu \right]^T$ is governed by the stochastic equation

$$d\mathbf{X}(t) = \mathbf{c}(\mathbf{X}(t))dt + \mathbf{b}(P(t), \mathbf{X}(t))dN_\nu(t) \tag{4.1.5}$$

with

$$\mathbf{X}(t) = \begin{bmatrix} X(t) \\ \dot{X}(t) \\ \rho_1^\nu(t) \\ \rho_2^\nu(t) \\ .. \\ \rho_{k-1}^\nu(t) \end{bmatrix} = \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ X_4(t) \\ .. \\ X_{k+1}(t) \end{bmatrix}; \mathbf{c}(\mathbf{X}(t)) = \begin{bmatrix} X_2(t) \\ -\omega^2 X_1(t) - 2\zeta\omega X_2(t) - \varepsilon\omega^2 X_1^3(t) \\ 0 \\ 0 \\ ... \\ 0 \end{bmatrix};$$

and
$$\mathbf{b}(P(t), \mathbf{X}(t)) = \begin{bmatrix} 0 \\ P(t)\rho_1^\nu \\ \rho_2^\nu \\ ... \\ ... \\ 1 - \sum_{j=1}^{k-2} \rho_j^\nu - 2\rho_{k-1}^\nu \end{bmatrix}; \tag{4.1.6}$$

While the original state vector consisting of X and $\dot{X}$ is not a Markov process, the augmented state vector $\mathbf{X}$ is a non-diffusive Markov process.

With the aid of the generalized Ito's differential rule (cf. section 2), that becomes

$$dV(t, \mathbf{X}) = \frac{\partial V(t, \mathbf{X})}{\partial t} + \sum_{j=1}^{k+1} \frac{\partial V(t, \mathbf{X})}{\partial X_j} c_j(\mathbf{X}, \dot{\mathbf{X}}) + \left[V(t, \mathbf{X} + b^\nu(t, \mathbf{X})) - V(t, \mathbf{X}) \right] dN_\nu \tag{4.1.7}$$

it is possible to derive the equations for the mean values $m_i(t) = E[X_i]$ and for 2nd, 3rd and 4th order moments as

($\mu_{ij}(t) = E\left[X_i X_j \right], \mu_{ijk}(t) = E\left[X_i X_j X_k \right], \mu_{ijkl}(t) = E\left[X_i X_j X_k X_l \right]$)

$$
\begin{aligned}
&\dot{m}_i(t) = E\left[c_i\right] + \nu E\left[b_i^\nu\right],\\
&\dot{\mu}_{ij}(t) = 2\left\{E\left[X_i\left(c_j + \nu E\left[b_j^\nu\right]\right)\right]\right\}_s + \nu E\left[b_i^\nu b_j^\nu\right],\\
&\dot{\mu}_{ijk}(t) = 3\left\{E\left[X_i X_j\left(c_k + \nu E\left[b_k^\nu\right]\right)\right]\right\}_s + 3\nu\left\{E\left[X_i b_j^\nu b_k^\nu\right]\right\}_s +\\
&\nu E\left[b_i^\nu b_j^\nu b_k^\nu\right],\\
&\dot{\mu}_{ijkl}(t) = 4\left\{E\left[X_i X_j X_k\left(c_l + 4\nu E\left[b_l^\nu\right]\right)\right]\right\}_s + 6\nu\left\{E\left[X_i X_j b_k^\nu b_l^\nu\right]\right\}_s +\\
&4\nu\left\{E\left[X_i b_j^\nu b_k^\nu b_l^\nu\right]\right\}_s + \nu E\left[b_i^\nu b_j^\nu b_k^\nu b_l^\nu\right],
\end{aligned}
\tag{4.1.8}
$$

where $\{..\}_s$ denotes the Stratonovich symmetrizing operator, e.g.

$$\left\{a_i a_j a_k\right\}_s = \frac{1}{3}\left(a_i a_j a_k + a_j a_k a_i + a_k a_i a_j\right). \tag{4.1.9}$$

For the non-linear oscillators with terms $f\left(X, \dot{X}\right)$ which are polynomials in X and $\dot{X}$, the right-hand sides of the equations for moments (4.1.8) involve the unknown expectations of the non-linear transformations of state variables. If the non-linearities are polynomial the equations for moments form an infinite hierarchy and cannot be directly solved. The unknown moments can only be evaluated approximately, using suitable closure approximations.

For example, for the 3rd order polynomial non linearity (Duffing oscillator), if the set of moments equations is truncated at the 4th order moments level, the redundant moments are of the 5th and 6th order.

If the non-linearities are other than polynomial, the expectation of the non-linear transformations of the state variables cannot be expressed directly in terms of moments.

4.1.2 Modified closure scheme

A novel closure scheme is here developed that takes into account the specific physical properties of impulsive load processes (*Iwankiewicz, Nielsen and Christensen (1990), Iwankiewicz and Nielsen, (1999)*). Assume that the system, subjected to a random train of impulses and to zero initial conditions, has been at rest in the time interval [0,t[where no impulse has yet occurred. The joint probability density of the augmented state vector is expressed as sum of contributions conditioned on the ‘on’ and ‘off’ states of the auxiliary variables. A discrete part account for the fact that there is a finite probability of the system being at rest from the initial time to the occurrence of the first impulse with Dirac delta spike. The continuous part which is the conditional probability given that the first impulse has occurred, can be expressed in terms of functions depending only on displacement and velocity .

Before the first impulse occurrence, the variables $\rho_1^\nu(t), \rho_2^\nu(t) .. \rho_{k-1}^\nu(t)$ can be in their first ‘off’ state with probability $P^1_{\rho_1^\nu off, \rho_2^\nu off .. \rho_{k-1}^\nu off} = \Pr\left\{\left(N_\nu(t) = 0\right)\right\} = e^{-\nu t}$.

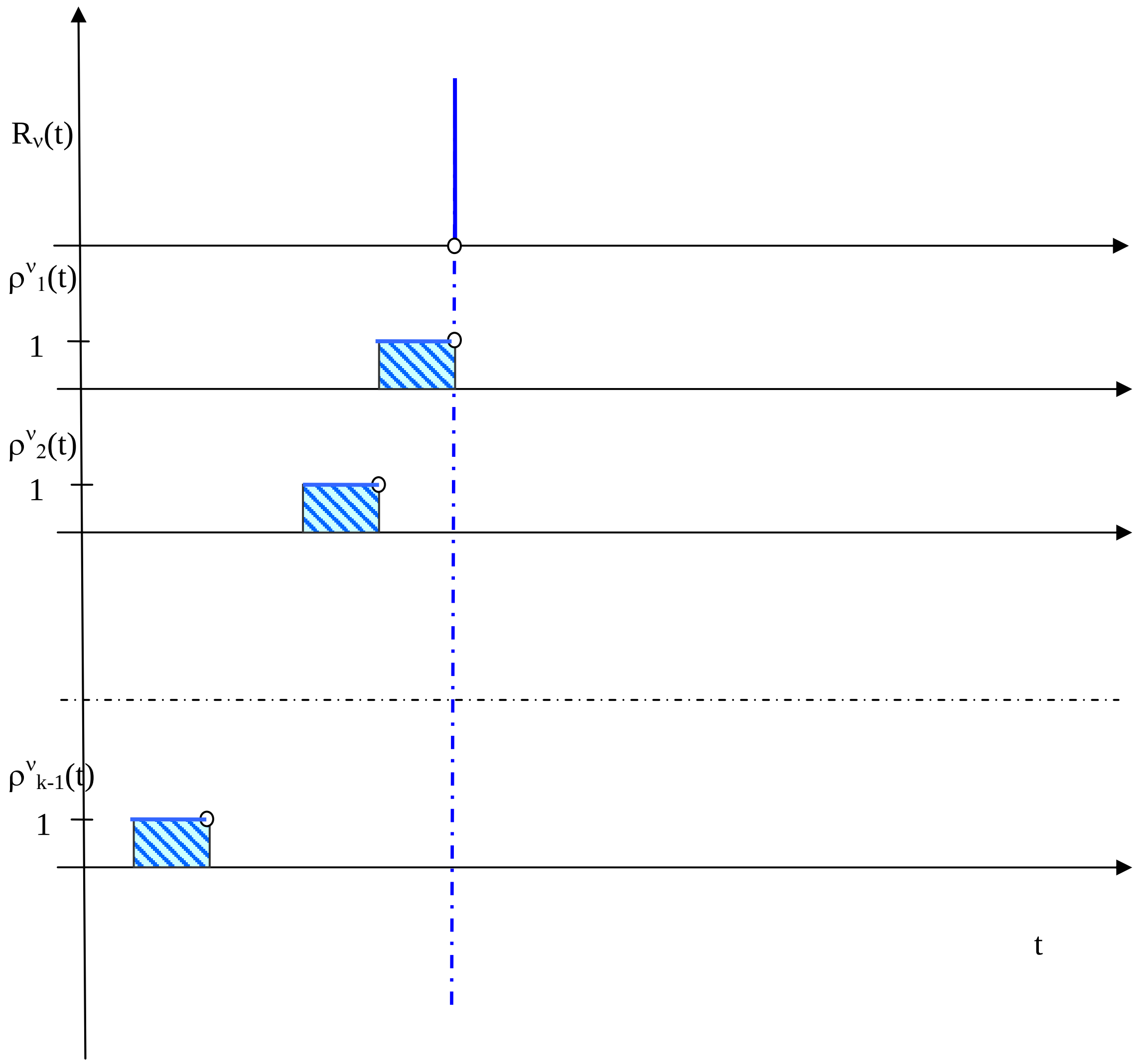

Figure 4.1.1

Sample function of an impulse process driven by an Erlang renewal process with generic parameters k and ν and auxiliary zero-one variables appearing in expressions (4.1.4).

Let us consider the following events.

The variable $\rho_{k-1}^{\nu}(t)$ is in its first 'on' state while the remaining variables are 'off', with probability $P^{1}_{\rho_1^{\nu}off,\rho_2^{\nu}off..\rho_{k-2}^{\nu}off,\rho_{k-1}^{\nu}on} = \Pr\{(N_{\nu}(t)=1)\} = \nu t e^{-\nu t}$.

The variable $\rho_{k-2}^{\nu}(t)$ can be in its first 'on' state while the remaining variables are 'off' with probability $P^{1}_{\rho_1^{\nu}off,\rho_2^{\nu}off..\rho_{k-2}^{\nu}on,\rho_{k-1}^{\nu}off} = \Pr\{(N_{\nu}(t)=2)\} = \frac{(\nu t)^2}{2}e^{-\nu t}$.

.

.

etc.

The variable $\rho_1^\nu\left(t\right)$ can be in its first 'on' state while the remaining variables are 'off' with probability $P^1_{\rho_1^\nu on,\rho_2^\nu off..\rho_{k-1}^\nu off} = \Pr\left\{\left(N_\nu\left(t\right)=k-1\right)\right\} = \frac{\left(\nu t\right)^{k-1}}{\left(k-1\right)!}e^{-\nu t}$.

Consequently, the joint probability density function of the state variables can be expressed as

$$p\left(x,\dot{x},\rho_1^\nu,..,\rho_{k-1}^\nu\right) = p^{(1)}\left(x,\dot{x},\rho_1^\nu,..,\rho_{k-1}^\nu\right) + p^{(*)}\left(x,\dot{x},\rho_1^\nu,..,\rho_{k-1}^\nu\right) \tag{4.1.10}$$

The function $p^{(1)}\left(x,\dot{x},\rho_1^\nu,..,\rho_{k-1}^\nu\right)$ is expressed in terms of the state distribution probabilities P_j^1 of the auxiliary variables given that the first Erlang driven impulse has not occurred, multiplied by the pertinent conditional probability densities

$$\begin{aligned}&p^{(1)}\left(x,\dot{x},\rho_1^\nu,..,\rho_{k-1}^\nu\right) = \\ &P_1^1\delta\left(x\right)\delta\left(\dot{x}\right)\delta\left(\rho_1^\nu\right)..\delta\left(\rho_{k-1}^\nu\right) + P_2^1\delta\left(x\right)\delta\left(\dot{x}\right)\delta\left(\rho_1^\nu\right)..\delta\left(\rho_{k-1}^\nu-1\right) + \\ &..+P_k^1\delta\left(x\right)\delta\left(\dot{x}\right)\delta\left(\rho_1^\nu-1\right)..\delta\left(\rho_{k-1}^\nu\right)\end{aligned} \tag{4.1.11}$$

with

$$\begin{aligned}&P_1^1 = P^1_{\rho_1^\nu off,\rho_2^\nu off..\rho_{k-1}^\nu off} = e^{-\nu t};\\ &P_2^1 = P^1_{\rho_1^\nu off,\rho_2^\nu off..\rho_{k-1}^\nu on} = \nu t e^{-\nu t};\\ &...\\ &P_k^1 = P^1_{\rho_1^\nu on,\rho_2^\nu off..\rho_{k-1}^\nu off} = \frac{\left(\nu t\right)^{k-1}}{\left(k-1\right)!}e^{-\nu t};\end{aligned} \tag{4.1.12}$$

The function $p^{(*)}\left(x,\dot{x},\rho_1^\nu,..,\rho_{k-1}^\nu\right)$ is expressed in terms of the conditional state probabilities P_j^* of the auxiliary variables given the first Erlang driven impulse has occurred, multiplied by the pertinent conditional probability densities

$$\begin{aligned}&p^{(*)}\left(x,\dot{x},\rho_1^\nu,..,\rho_{k-1}^\nu\right) = \\ &P_1^* f^1\left(x,\dot{x}\right)\delta\left(\rho_1^\nu\right)..\delta\left(\rho_{k-1}^\nu\right) + P_2^* f^2\left(x,\dot{x}\right)\delta\left(\rho_1^\nu\right)..\delta\left(\rho_{k-1}^\nu-1\right) + \\ &..+P_k^* f^k\left(x,\dot{x}\right)\delta\left(\rho_1^\nu-1\right)..\delta\left(\rho_{k-1}^\nu\right).\end{aligned} \tag{4.1.13}$$

with

$$P_1^* = \sum_{j>1} P_1^j = E\left[\rho_{k-1}^{\nu}\right] - P_1^1;$$
$$P_2^* = \sum_{j>1} P_2^j = E\left[\rho_{k-2}^{\nu}\right] - P_2^1; \qquad (4.1.14)$$
..
$$P_k^* = \sum_{j>1} P_k^j = E\left[\rho_1^{\nu}\right] - P_k^1;$$

The joint probability density function has to satisfy the equation

$$\int_{-\infty}^{\infty} p\left(x, \dot{x}, \rho_1^{\nu}, ..\rho_{k-1}^{\nu}\right) dx d\dot{x} d\rho_1^{\nu} .. d\rho_{k-1}^{\nu} =$$
$$\sum_{j=1}^{k} P_j^1 + \sum_{j=1}^{k} P_j^* \int_{-\infty}^{\infty} f^{(j)}\left(x, \dot{x}\right) dx d\dot{x} = 1 \qquad (4.1.15)$$

considering that

$$\sum_{j=1}^{k} P_j^1 = P_R; \qquad \sum_{j=1}^{k} P_j^* = 1 - P_R. \qquad (4.1.16)$$

the following relationships hold

$$\int_{-\infty}^{\infty} f^{(1)}\left(x, \dot{x}\right) dx d\dot{x} = 1, ... \int_{-\infty}^{\infty} f^{(k)}\left(x, \dot{x}\right) dx d\dot{x} = 1 \qquad (4.1.17)$$

It can also be proved that the following identities hold

$$p\left(\rho_1^{\nu}\right) = \int_{-\infty}^{\infty} p\left(x, \dot{x}, \rho_1^{\nu}, ..\rho_{k-1}^{\nu}\right)\left(dx\right)\left(d\dot{x}\right)\left(\rho_2^{\nu}\right)..\left(\rho_{k-1}^{\nu}\right) =$$
$$\delta\left(\rho_1^{\nu} - 1\right)\left(E\left[\rho_1^{\nu}\right]\right) + \delta\left(\rho_1^{\nu}\right)\left(1 - E\left[\rho_1^{\nu}\right]\right),$$
... (4.1.18)
$$p\left(\rho_{k-1}^{\nu}\right) = \int_{-\infty}^{\infty} p\left(x, \dot{x}, \rho_1^{\nu}, ..\rho_{k-1}^{\nu}\right)\left(dx\right)\left(d\dot{x}\right)\left(\rho_1^{\nu}\right)..\left(\rho_{k-2}^{\nu}\right) =$$
$$\delta\left(\rho_{k-1}^{\nu} - 1\right)\left(E\left[\rho_{k-1}^{\nu}\right]\right) + \delta\left(\rho_{k-1}^{\nu}\right)\left(1 - E\left[\rho_{k-1}^{\nu}\right]\right).$$

Taking for instance the marginal probability density $p\left(\rho_1^{\nu}\right)$, it can be derived as

$$p\left(\rho_1^{\nu}\right) = \int_{-\infty}^{\infty} p\left(x, \dot{x}, \rho_1^{\nu}, ..\rho_{k-1}^{\nu}\right) dx d\dot{x} d\rho_2^{\nu} .. d\rho_{k-1}^{\nu} =$$
$$\sum_{j=1}^{k-1} P_j^1 \delta\left(\rho_1^{\nu}\right) + P_k^1 \delta\left(\rho_1^{\nu} - 1\right) + \sum_{j=1}^{k-1} P_j^* \delta\left(\rho_1^{\nu}\right) + P_k^* \delta\left(\rho_1^{\nu} - 1\right) \qquad (4.1.19)$$

observing that

$$P_k^1 + P_k^* = E\left[\rho_1^\nu\right];$$
$$\sum_{j=1}^{k-1} P_j^1 + \sum_{j=1}^{k-1} P_j^* = 1 - E\left[\rho_1^\nu\right] \tag{4.1.20}$$

equation (4.1.19) becomes

$$p\left(\rho_1^\nu\right) = \delta\left(\rho_1^\nu - 1\right)\left(E\left[\rho_1^\nu\right]\right) + \delta\left(\rho_1^\nu\right)\left(1 - E\left[\rho_1^\nu\right]\right) \tag{4.1.21}$$

Since the auxiliary variables are zero-one processes, the following relationships hold

$$E\left[X^m \dot{X}^n \rho_1^{\nu o}\right] = E\left[X^m \dot{X}^n \rho_1^\nu\right]$$
$$\ldots$$
$$E\left[X^m \dot{X}^n \rho_{k-1}^{\nu\ o}\right] = E\left[X^m \dot{X}^n \rho_{k-1}^\nu\right] \tag{4.1.22}$$
$$E\left[X^m \dot{X}^n \rho_i^{\nu o} \rho_j^{\nu p}\right] = 0$$

The unconditional moment of order $p = m + n$ involving displacements and velocity can be expressed in terms of the conditional moments of the same order as follows

$$E[X^m \dot{X}^n] = P_1^* E^{(1)}[X^m \dot{X}^n] + P_2^* E^{(2)}[X^m \dot{X}^n] + \ldots P_k^* E^{(k)}[X^m \dot{X}^n] \tag{4.1.23}$$

where the conditional moments are evaluated respect to the conditional p.d.f. (4.1.13).
The unconditional moment of order $p + 1 = m + n + 1$ involving also the auxiliary variables can be expressed in terms of the conditional moments of order p as follows

$$E[X^m \dot{X}^n \rho_1^\nu] = P_k^* E^{(k)}[X^m \dot{X}^n];$$
$$E[X^m \dot{X}^n \rho_2^\nu] = P_{k-1}^* E^{(k-1)}[X^m \dot{X}^n]; \tag{4.1.24}$$
$$..$$
$$E[X^m \dot{X}^n \rho_{k-1}^\nu] = P_2^* E^{(2)}[X^m \dot{X}^n].$$

The following relationships relating the conditional moments of order $p = m + n$ to the unconditional ones can also be derived:

$$E^{(1)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^n\right]-E\left[X^m\dot{X}^n\rho_1^\nu\right]-E\left[X^m\dot{X}^n\rho_2^\nu\right]-...E\left[X^m\dot{X}^n\rho_{k-1}^\nu\right]}{P_1^*};$$

$$E^{(2)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^n\rho_{k-1}^\nu\right]}{P_2^*};$$

..

$$E^{(k)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^n\rho_1^\nu\right]}{P_k^*}.$$

(4.1.25)

The Gram-Charlier expansion can be applied to the conditional density functions $f^{(1)}(x,\dot{x})$,.. $f^{(k)}(x,\dot{x})$ (cf. (4.1.13)) that can be viewed as probability densities of a bi-dimensional stochastic variable.

The conditional cumulants are related to the conditional moments by the following formula (Abramowitz and Stegun (1972), Kenney and Keeping (1951)):

$$\kappa^{(k)}\left[X_1...X_s\right]=\sum_\pi\prod_{B\in\pi}\left(|B|-1\right)!(-1)^{|B|-1}E^{(k)}\left[\prod_{i\in B}X_i\right] \tag{4.1.26}$$

Where π runs through the list of all partitions of $\{1,2,...s\}$, B runs through the list of blocks of the partition π and $|B|$ is the size of the set B.

The conditional moments are related to the conditional cumulants through the following formula

$$E^{(k)}\left[X_1..X_s\right]=\sum_\pi\prod_{B\in\pi}\kappa^{(k)}\left[X_i:i\in B\right] \tag{4.1.27}$$

Applying a cumulant neglect closure of order r, the conditional moments of order s>r can be expressed in terms of the lower order conditional moments through the following equation

$$\begin{aligned}E^{(k)}\left[X_1..X_s\right]&=\sum_{\pi^r}\prod_{B^r\in\pi^r}\kappa^{(k)}\left[X_i:i\in B^N\right]\\&=\sum_{\pi^r}\prod_{B^r\in\pi^r}\left(\sum_{\substack{\pi^i\\ i\in B^{rr}}}\prod_{B^i\in\pi^i}\left(|B^i|-1\right)!(-1)^{|B^i|-1}E^{(k)}\left[\prod_{j\in B^i}X_j\right]\right)\end{aligned} \tag{4.1.28}$$

Where π^r runs through the list of the partitions of $\{1,2,...s\}$ in blocks of maximum dimension r and B^r runs through the list of blocks of the partition π^r .

Inserting equations (4.1.28) in (4.1.25), the conditional moments of order s>r are expressed in terms of the unconditional moments of lower order through the following relations

$$E^{(1)}\left[X_1..X_s\right]=$$

$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_j\rho_1^\nu\right]-...E\left[\prod_{j\in B^i}X_j\rho_{k-1}^\nu\right]}{P_1^*}\right)\right]\right\};$$

...

$$E^{(k)}\left[X_1..X_s\right]=$$

$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_1^\nu\right]}{P_k^*}\right)\right]\right\};$$

(4.1.29)

The unconditional moments of order s involving displacement and velocity only can be expressed in terms of unconditional moments of order up to r, by inserting equations (4.1.29) in (4.1.23), as follows

$$E\left[X_1..X_s\right]=$$

$$P_1^*\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_j\rho_1^\nu\right]-...E\left[\prod_{j\in B^i}X_j\rho_{k-1}^\nu\right]}{P_1^*}\right)\right]\right\}+$$

$$..P_k^*\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_1^\nu\right]}{P_k^*}\right)\right]\right\};$$

(4.1.30)

The unconditional moments of order s+1 involving also the auxiliary variables can be expressed in terms of unconditional moments of order up to r, by inserting equations (4.1.29) in (4.1.24), as follows

$$E\left[X_1..X_s\rho_1^{\nu}\right]=P_k^*\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_1^{\nu}\right]}{P_k^*}\right)\right]\right\};$$

...

$$E[X_1..X_s\rho_{k-1}^{\nu}]=P_2^*\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_{k-1}^{\nu}\right]}{P_2^*}\right)\right]\right\}.$$

(4.1.31)

4.1.3 Erlang driven train of impulses (k=2, ν)

Consider the response of a Duffing oscillator to the random train of impulses driven by an Erlang renewal process with k=2, ν $dR_\nu(t)=\rho_1^{\nu}(t)dN_\nu(t)$, with $d\rho_1^{\nu}(t)=\left(1-2\rho_1^{\nu}(t)\right)dN_\nu(t)$, (see Figure 4.1.2).
The stochastic equation of motion is specified by

$$d\mathbf{X}(t)=\mathbf{c}\left(\mathbf{X}(t)\right)dt+\mathbf{b}\left(P(t),\mathbf{X}(t)\right)dN_\nu(t) \qquad (4.1.32)$$

with

$$\mathbf{X}(t)=\begin{bmatrix}X\\ \dot{X}\\ \rho_1^{\nu}\end{bmatrix}=\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix};\mathbf{c}\left(\mathbf{X}(t)\right)=\begin{bmatrix}X_2\\ -\omega^2X_1-2\zeta\omega X_2-\varepsilon\omega^2X_1^3\\ 0\end{bmatrix};$$
$$\mathbf{b}\left(P(t),\mathbf{X}(t)\right)=\begin{bmatrix}0\\ PX_3\\ 1-2X_3\end{bmatrix} \qquad (4.1.33)$$

Before the first impulse occurrence, the variable ρ_1^{ν} can be in its first 'off' state with probability $P_{\rho 1off}^{(1)}$ or in its first 'on' state with probability $P_{\rho 1on}^{(1)}$. After the first impulse occurrence the variable ρ_1^{ν} can be 'off' with probability $P_{\rho off}^{(*)}=\sum_{j\geq 2}P_{\rho off}^{(j)}$ or 'with probability $P_{\rho on}^{(*)}=\sum_{j\geq 2}P_{\rho on}^{(j)}$.

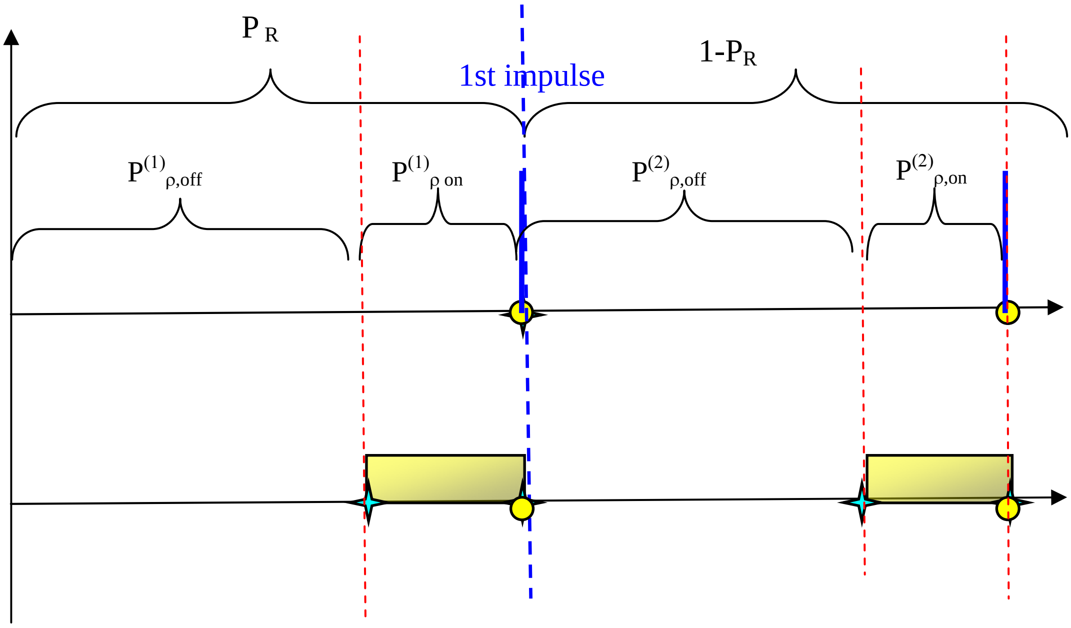

Figure 4.1.2

Sample function of an impulse process driven by an Erlang renewal process with parameters $k=2$ and ν and the auxiliary zero-one variable appearing in expressions (4.1.33).

The equations governing the state probabilities of the auxiliary variable are

$$
\begin{aligned}
P^{(1)}_{\rho off} + P^{(*)}_{\rho off} &= 1 - E\left[\rho_1^{\nu}\right] \\
P^{(1)}_{\rho on} + P^{(*)}_{\rho on} &= E\left[\rho_1^{\nu}\right]
\end{aligned}
\tag{4.1.34}
$$

Therefore

$$
P^{(*)}_{\rho on} = E\left[\rho_1^{\nu}\right] - P^{(1)}_{\rho on};\, P^{(*)}_{\rho off} = 1 - E\left[\rho_1^{\nu}\right] - P^{(1)}_{\rho off} \tag{4.1.35}
$$

Considering that the probability that the number of Poisson events in the time interval (0,t) is n is $\Pr\left\{dN_{\nu}\left(t\right)=n\right\} = \dfrac{\left(\nu t\right)^{n}}{n!}e^{-\nu t}$, it follows that

$$
P^{(1)}_{\rho off} = e^{-\nu t};\, P^{(1)}_{\rho on} = \nu t e^{-\nu t}. \tag{4.1.36}
$$

Let us express the unknown joint probability density of the state vector **X** in terms of the conditional probability functions:

- $p^{1}\left(x,\dot{x},\rho_1^{\nu}\right)dxd\dot{x} =$
  $\Pr\left\{X \in \left(x, x+dx\right) \wedge \dot{X} \in \left(\dot{x}, \dot{x}+d\dot{x}\right)\right\} \wedge \rho_1^{\nu}\ is\, in\, its\ first\ 'off'\ phase$
  given that no impulses have occurred and the variable ρ_1^{ν} is in its first 'off' phase.

- $p^2\left(x,\dot{x},\rho_1^\nu\right)dxd\dot{x} =$
  $\Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\} \wedge \rho_1^\nu \; is\, in\, its\;\; first\, 'on'\; phase$
  given that no impulses have occurred and the variable ρ_1^ν is in its first 'on' phase.
- $p^3\left(x,\dot{x},\rho_1^\nu\right)dxd\dot{x} =$
  $\Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\} \wedge \rho_1^\nu \; is\, in\, any\, 'off\,'\; phase\;\; following\, the\;\; first$
  given that after the first impulse occurrence the variable is'off'.
- $p^4\left(x,\dot{x},\rho_1^\nu\right)dxd\dot{x} =$
  $\Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\} \wedge \rho_1^\nu \; is\, in\, any\, 'on'\; phase\;\; following\, the\;\; first$
  given that after the first impulse occurrence the variable ρ_1^ν is 'on'.

Hence, the expression for the probability density function becomes

$$p\left(x,\dot{x},\rho_1^\nu\right) = \sum_{j=1}^{4} p^j\left(x,\dot{x},\rho_1^\nu\right) \tag{4.1.37}$$

Due to the fact that there is a finite probability of the system being in a deterministic state from the initial time to the first impulse, the probability densities given that no impulse have occurred ($p^1\left(x,\dot{x},\rho_1^\nu\right)$ and $p^2\left(x,\dot{x},\rho_1^\nu\right)$) can be expressed as

$$\begin{aligned} p^1\left(x,\dot{x},\rho_1^\nu\right) &= P_{\rho off}^{(1)}\delta\left(x\right)\delta\left(\dot{x}\right)\delta\left(\rho_1^\nu\right) \\ p^2\left(x,\dot{x},\rho_1^\nu\right) &= P_{\rho on}^{(1)}\delta\left(x\right)\delta\left(\dot{x}\right)\delta\left(\rho_1^\nu - 1\right) \end{aligned} \tag{4.1.38}$$

Let us express the contribution taking into account that the first Erlang impulse has occurred as

$$\begin{aligned} p^3\left(x,\dot{x},\rho_1^\nu\right) &= P_{\rho off}^{(*)} f^{(1)}\left(x,\dot{x}\right)\delta\left(\rho_1^\nu\right); \\ p^4\left(x,\dot{x},\rho_1^\nu\right) &= P_{\rho on}^{(*)} f^{(2)}\left(x,\dot{x}\right)\delta\left(\rho_1^\nu - 1\right) \end{aligned} \tag{4.1.39}$$

Considering that the joint probability density function has to satisfy the equation

$$\begin{aligned} &\int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu\right) dxd\dot{x}d\rho_1^\nu = \\ &P_{\rho off}^{(1)} + P_{\rho on}^{(1)} + P_{\rho off}^{(*)}\int_{-\infty}^{+\infty} f^{(1)}\left(x,\dot{x}\right)dxd\dot{x} + P_{\rho on}^{(*)}\int_{-\infty}^{+\infty} f^{(2)}\left(x,\dot{x}\right)dxd\dot{x} = 1 \end{aligned} \tag{4.1.40}$$

and that it is possible to write the following relationships between the state probabilities of the auxiliary variable

$$P_{\rho off}^{(1)} + P_{\rho on}^{(1)} = P_R ; P_{\rho off}^{(*)} + P_{\rho on}^{(*)} = 1 - P_R \tag{4.1.41}$$

where P_R is the probability that no Erlang driven impulse has occurred. From (4.1.40) and (4.1.41) it follows that

$$\int_{-\infty}^{+\infty} f^{(1)}\left(x,\dot{x}\right)dxd\dot{x}=1; \int_{-\infty}^{+\infty} f^{(2)}\left(x,\dot{x}\right)dxd\dot{x}=1. \tag{4.1.42}$$

The marginal probability density of the variable ρ_1^ν can be derived as follows

$$\begin{aligned} p\left(\rho_1^\nu\right) &= \int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu\right)dxd\dot{x} = \\ &P_{\rho off}^{(1)}\delta\left(\rho_1^\nu\right)+P_{\rho on}^{(1)}\delta\left(\rho_1^\nu-1\right)+P_{\rho off}^{(*)}\delta\left(\rho_1^\nu\right)+P_{\rho on}^{(*)}\delta\left(\rho_1^\nu-1\right). \end{aligned} \tag{4.1.43}$$

Taking into account equation (4.1.34) it follows

$$p\left(\rho_1^\nu\right) = \int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu\right)dxd\dot{x} = \delta\left(\rho_1^\nu\right)\left(1-E\left[\rho_1^\nu\right]\right)+\delta\left(\rho_1^\nu-1\right)E\left[\rho_1^\nu\right] \tag{4.1.44}$$

Since the auxiliary variable is a zero-one process, the following relationship holds

$$E\left[X^m\dot{X}^n\rho_1^{\nu o}\right]=E\left[X^m\dot{X}^n\rho_1^\nu\right] \tag{4.1.45}$$

The unconditional moments of order $p=m+n$ involving displacement and velocity and the moments of order $p+1$ involving the auxiliary variable, can be expressed in terms of the conditional moments of the same order, as follows

$$\begin{aligned} &E\left[X^m\dot{X}^n\right]=P_{\rho off}^{(*)}E^{(1)}\left[X^m\dot{X}^n\right]+P_{\rho on}^{(*)}E^{(2)}\left[X^m\dot{X}^n\right]; \\ &E\left[X^m\dot{X}^n\rho_1^\nu\right]=P_{\rho on}^{(*)}E^{(2)}\left[X^m\dot{X}^n\right]. \end{aligned} \tag{4.1.46}$$

The conditional moments can be expressed in terms of unconditional ones as follows

$$\begin{aligned} &E^{(1)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^n\right]-E\left[X^m\dot{X}^n\rho_1^\nu\right]}{P_{\rho off}^{(*)}}; \\ &E^{(2)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^n\rho_1^\nu\right]}{P_{\rho on}^{(*)}} \end{aligned} \tag{4.1.47}$$

Applying a cumulant neglect closure of order r, the conditional moments of order s>r can be expressed in terms of the lower order conditional moments. The conditional moments of order s>r are then expressed in terms of the unconditional moments of lower order through the following relations

$$E^{(1)}\left[X_1..X_s\right]=$$
$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_j\rho_1^\nu\right]}{P_{\rho off}^{(*)}}\right)\right]\right\};$$
$$E^{(2)}\left[X_1..X_s\right]=$$
$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_1^\nu\right]}{P_{\rho on}^{(*)}}\right)\right]\right\};\tag{4.1.48}$$

The unconditional moments of order s involving displacement and velocity only and the moments of order s+1 involving also the auxiliary variables can be expressed in terms of unconditional moments of order up to r, by inserting equations (4.1.48) in (4.1.46), as follows

$$E\left[X_1..X_s\right]=$$
$$P_{\rho off}^{(*)}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_j\rho_1^\nu\right]}{P_{\rho off}^{(*)}}\right)\right]\right\}+$$
$$P_{\rho on}^{(*)}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_1^\nu\right]}{P_{\rho on}^{(*)}}\right)\right]\right\};$$
$$E\left[X_1..X_s\rho_1^\nu\right]=P_{\rho on}^{(*)}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_1^\nu\right]}{P_{\rho on}^{(*)}}\right)\right]\right\};\tag{4.1.49}$$

4.1.4 Erlang driven train of impulses (k=3, ν)

Consider the response of a Duffing oscillator to the random train of impulses driven by an Erlang renewal process with k=3, ν $dR_\nu(t)=\rho_1^\nu(t)\,dN_\nu(t)$, with $d\rho_1^\nu(t)=\left(\rho_2^\nu(t)-\rho_1^\nu(t)\right)dN_\nu(t)$, $d\rho_2^\nu(t)=\left(1-\rho_1^\nu(t)-2\rho_2^\nu(t)\right)dN_\nu(t)$ (see figure 4.1.3). The stochastic equation of motion is given as

$$d\mathbf{X}(t)=\mathbf{c}\left(\mathbf{X}(t)\right)dt+\mathbf{b}\left(P(t),\mathbf{X}(t)\right)dN_\nu(t)\tag{4.1.50}$$

with

$$\mathbf{X}(t)=\begin{bmatrix} X \\ \dot{X} \\ \rho_1^{\nu} \\ \rho_2^{\nu} \end{bmatrix}=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}; \qquad \mathbf{c}\left(\mathbf{X}(t)\right)=\begin{bmatrix} X_2 \\ -\omega^2 X_1-2\zeta\omega X_2-\varepsilon\omega^2 X_1^3 \\ 0 \\ 0 \end{bmatrix};$$

$$\mathbf{b}\left(P(t),\mathbf{X}(t)\right)=\begin{bmatrix} 0 \\ PX_3 \\ X_4 \\ 1-3X_3-3X_4 \end{bmatrix} \tag{4.1.51}$$

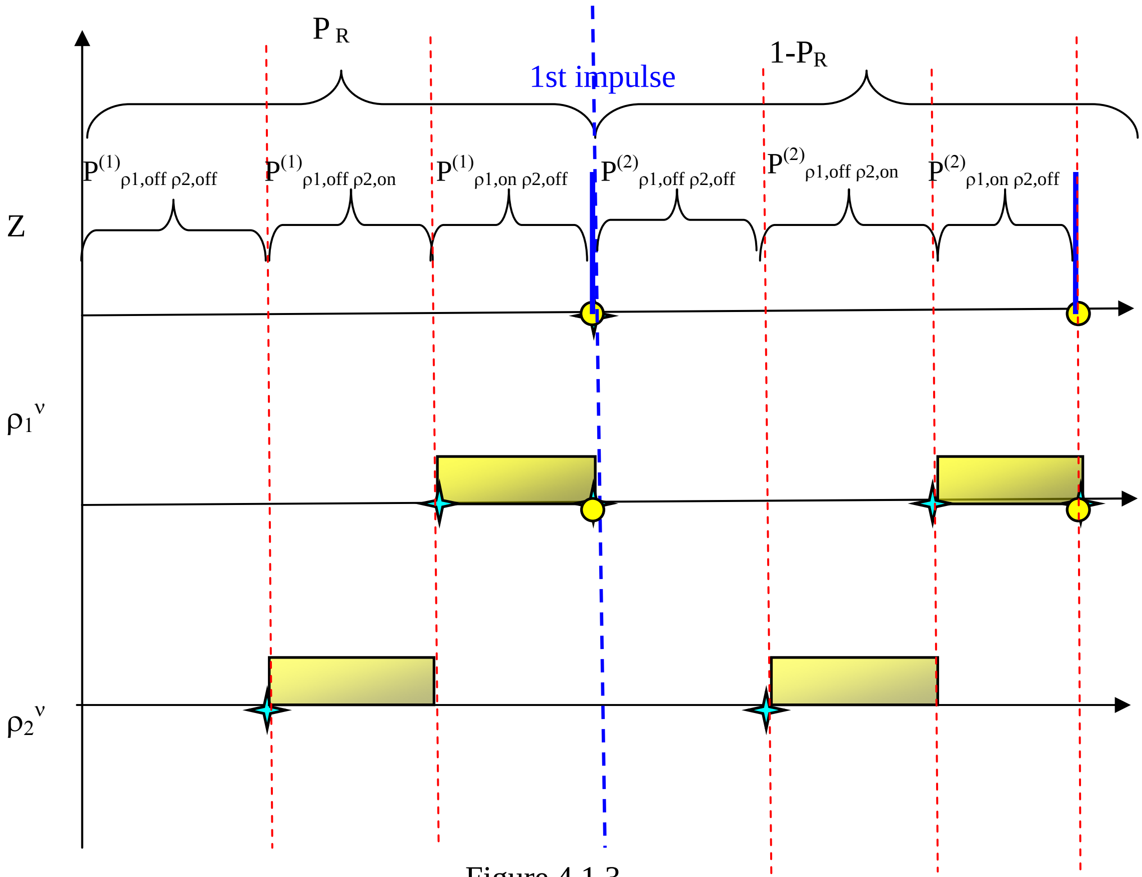

Figure 4.1.3

Sample function of an impulse process driven by an Erlang renewal process with parameters $k=3$ and ν and the auxiliary zero-one variables appearing in expressions (4.1.51).

Before the first impulse occurrence, the variables ρ_1^{ν} and ρ_2^{ν} can be in their first 'off' state with probability $P^{(1)}_{\rho 1off,\rho 2off}$. The variable ρ_2^{ν} can be in its first 'on' state, while ρ_1^{ν} is still 'off' with probability $P^{(1)}_{\rho 1off,\rho 2on}$. The variable ρ_1^{ν} can be in its first 'on' state, while ρ_2^{ν} is 'off' with probability $P^{(1)}_{\rho 1off,\rho 2on}$. After the first impulse occurrence the variables ρ_1^{ν} and ρ_2^{ν} can be simultaneously 'off' with probability $P^{(*)}_{\rho 1off,\rho 2off} = \sum_{j\geq 2} P^{(j)}_{\rho 1off,\rho 2off}$, ρ_2^{ν} can be 'on' while ρ_1^{ν} is 'off' with probability $P^{(*)}_{\rho 1off,\rho 2on} = \sum_{j\geq 2} P^{(j)}_{\rho 1off,\rho 2on}$, ρ_1^{ν} can be 'on' while ρ_2^{ν} is 'off' with probability $P^{(*)}_{\rho 1on,\rho 2off} = \sum_{j\geq 2} P^{(j)}_{\rho 1on,\rho 2off}$.

The equations governing the state probabilities of the auxiliary variables are

$$\begin{aligned}
&P^{(1)}_{\rho 1on,\rho 2off} + P^{(1)}_{\rho 1off,\rho 2on} + P^{(*)}_{\rho 1off,\rho 2off} + P^{(*)}_{\rho 1off,\rho 2on} = 1 - E\left[\rho_1^{\nu}\right] \\
&P^{(1)}_{\rho 1off,\rho 2off} + P^{(1)}_{\rho 1on,\rho 2off} + P^{(*)}_{\rho 1off,\rho 2off} + P^{(*)}_{\rho 1on,\rho 2off} = 1 - E\left[\rho_2^{\nu}\right] \\
&P^{(1)}_{\rho 1on,\rho 2off} + P^{(*)}_{\rho 1on,\rho 2off} = E\left[\rho_1^{\nu}\right] \\
&P^{(1)}_{\rho 1off,\rho 2on} + P^{(*)}_{\rho 1off,\rho 2on} = E\left[\rho_2^{\nu}\right]
\end{aligned} \tag{4.1.52}$$

From the Poisson law $\Pr\left\{dN_{\nu}\left(t\right) = n\right\} = \frac{\left(\nu t\right)^n}{n!} e^{-\nu t}$, it follows that

$$P^{(1)}_{\rho 1off,\rho 2off} = e^{-\nu t}; P^{(1)}_{\rho 1off,\rho 2on} = \nu t e^{-\nu t}; P^{(1)}_{\rho 1on,\rho 2off} = \frac{\left(\nu t\right)^2}{2} e^{-\nu t}. \tag{4.1.53}$$

Therefore

$$\begin{aligned}
&P^{(*)}_{\rho 1on,\rho 2off} = E\left[\rho_1^{\nu}\right] - P^{(1)}_{\rho 1on,\rho 2off} \\
&P^{(*)}_{\rho 1off,\rho 2on} = E\left[\rho_2^{\nu}\right] - P^{(1)}_{\rho 1off,\rho 2on} \\
&P^{(*)}_{\rho 1off,\rho 2off} = 1 - E\left[\rho_2^{\nu}\right] - E\left[\rho_1^{\nu}\right] - P^{(1)}_{\rho 1off,\rho 2off}
\end{aligned} \tag{4.1.54}$$

Let us express the unknown joint probability density of the state vector $\mathbf{X}$ in terms of the conditional probability functions

$p^1\left(x, \dot{x}, \rho_1^{\nu}, \rho_2^{\nu}\right) dx d\dot{x} =$

- $\Pr\left\{X \in \left(x, x+dx\right) \wedge \dot{X} \in \left(\dot{x}, \dot{x}+d\dot{x}\right)\right\} \wedge \rho_1^{\nu}\ is\,in\,its\ first\,'off\,'\ phase$
  $\wedge \rho_2^{\nu}\ is\,in\,its\ first\,'off\,'\ phase$
  given that no impulses have occurred and the variables ρ_1^{ν} and ρ_2^{ν} are in their first 'off' phase.

$p^2\left(x, \dot{x}, \rho_1^{\nu}, \rho_2^{\nu}\right) dx d\dot{x} =$

- $\Pr\left\{X \in \left(x, x+dx\right) \wedge \dot{X} \in \left(\dot{x}, \dot{x}+d\dot{x}\right)\right\} \wedge \rho_1^{\nu}\ is\,in\,its\ first\,'off\,'\ phase$
  $\wedge \rho_2^{\nu}\ is\,in\,its\ first\,'on'\ phase$
  given that no impulses have occurred and the variable ρ_1^{ν} is still 'off' while ρ_2^{ν} is in its first 'on' phase.

$p^3\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)dxd\dot{x}=$

- $\Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}\wedge\rho_1^\nu\ is\, in\, its\ first\, 'on'\, phase$

$\wedge\rho_2^\nu\ is\, 'off'$

given that no impulses have occurred and the variable ρ_1^ν is in its first 'on' phase while ρ_2^ν is 'off'.

$p^4\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)dxd\dot{x}=$

- $\Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$

$\wedge\rho_1^\nu\ is\, in\, any\, 'off'\ phase\ following\, the\ first$

$\wedge\rho_2^\nu\ is\, in\, any\, 'off'\ phase\ following\, the\ first$

given that after the first impulse occurrence the variables ρ_1^ν and ρ_2^ν are simultaneously 'off'.

$p^5\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)dxd\dot{x}=$

- $\Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$

$\wedge\rho_1^\nu\ is\, in\, any\, 'off'\ phase\ following\, the\ first\wedge\rho_2^\nu\ is\, 'on'$

given that after the first impulse occurrence the variable ρ_1^ν is 'off' and ρ_2^ν is 'on'.

$p^5\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)dxd\dot{x}=$

- $\Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}\wedge\rho_1^\nu\ is\, 'on'$

$\wedge\rho_2^\nu\ is\, in\, any\, 'off'\ phase\ following\, the\ first$

given that after the first impulse occurrence the variable ρ_1^ν is 'on' and ρ_2^ν is 'off'.

Hence, the joint probability density function can be expressed as

$$p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)=\sum_{j=1}^{6}p^j\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right) \qquad (4.1.55)$$

Due to the fact that there is a finite probability of the system being in a deterministic state from the initial time to the first impulse, the contributions accounting for the fact that no impulse have occurred ($p^1\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)$ to $p^3\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)$) can be expressed as

$$\begin{aligned}
p^1\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)&=P^{(1)}_{\rho 1off,\rho 2off}\delta\left(x\right)\delta\left(\dot{x}\right)\delta\left(\rho_1^\nu\right)\delta\left(\rho_2^\nu\right)\\
p^2\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)&=P^{(1)}_{\rho 1off,\rho 2on}\delta\left(x\right)\delta\left(\dot{x}\right)\delta\left(\rho_1^\nu\right)\delta\left(\rho_2^\nu-1\right)\\
p^3\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)&=P^{(1)}_{\rho 1on,\rho 2off}\delta\left(x\right)\delta\left(\dot{x}\right)\delta\left(\rho_1^\nu-1\right)\delta\left(\rho_2^\nu\right)
\end{aligned} \qquad (4.1.56)$$

Let us express the conditional probabilities given that the first Erlang impulse has occurred as

$$
\begin{aligned}
&p^4\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)=P^{(*)}_{\rho 1off,\rho 2off}\, f^{(1)}\left(x,\dot{x}\right)\delta\left(\rho_1^\nu\right)\delta\left(\rho_2^\nu\right)\\
&p^5\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)=P^{(*)}_{\rho 1off,\rho 2on}\, f^{(2)}\left(x,\dot{x}\right)\delta\left(\rho_1^\nu\right)\delta\left(\rho_2^\nu-1\right)\\
&p^6\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)=P^{(*)}_{\rho 1on,\rho 2off}\, f^{(3)}\left(x,\dot{x}\right)\delta\left(\rho_1^\nu-1\right)\delta\left(\rho_2^\nu\right)
\end{aligned}
\qquad (4.1.57)
$$

Considering that the joint probability density function has to satisfy the equation

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)dxd\dot{x}d\rho_1^\nu d\rho_2^\nu = P^{(1)}_{\rho 1off,\rho 2off}+P^{(1)}_{\rho 1off,\rho 2on}+P^{(1)}_{\rho 1on,\rho 2off}+\\
&P^{(*)}_{\rho 1off,\rho 2off}\int_{-\infty}^{+\infty} f^{(1)}\left(x,\dot{x}\right)dxd\dot{x}+P^{(*)}_{\rho 1off,\rho 2on}\int_{-\infty}^{+\infty} f^{(2)}\left(x,\dot{x}\right)dxd\dot{x}+\\
&P^{(*)}_{\rho 1on,\rho 2off}\int_{-\infty}^{+\infty} f^{(3)}\left(x,\dot{x}\right)dxd\dot{x}=1
\end{aligned}
\qquad (4.1.58)
$$

and the following equations relating the state probabilities of the auxiliary variables

$$
\begin{aligned}
&P^{(1)}_{\rho 1off,\rho 2off}+P^{(1)}_{\rho 1off,\rho 2on}+P^{(1)}_{\rho 1on,\rho 2off}=P_R;\\
&P^{(*)}_{\rho 1off,\rho 2off}+P^{(*)}_{\rho 1off,\rho 2on}+P^{(*)}_{\rho 1on,\rho 2off}=1-P_R
\end{aligned}
$$

it follows that

$$
\int_{-\infty}^{+\infty} f^{(1)}\left(x,\dot{x}\right)dxd\dot{x}=1;\int_{-\infty}^{+\infty} f^{(2)}\left(x,\dot{x}\right)dxd\dot{x}=1;\int_{-\infty}^{+\infty} f^{(3)}\left(x,\dot{x}\right)dxd\dot{x}=1 \qquad (4.1.59)
$$

It can also be proved that the following identities hold

$$
\begin{aligned}
&p\left(\rho_1^\nu\right)=\int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)dxd\dot{x}d\rho_2^\nu=\\
&\delta\left(\rho_1^\nu\right)\left(1-E\left[\rho_1^\nu\right]\right)+\delta\left(\rho_1^\nu-1\right)E\left[\rho_1^\nu\right];\\
&p\left(\rho_2^\nu\right)=\int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)dxd\dot{x}d\rho_1^\nu=\\
&\delta\left(\rho_2^\nu\right)\left(1-E\left[\rho_2^\nu\right]\right)+\delta\left(\rho_2^\nu-1\right)E\left[\rho_2^\nu\right];
\end{aligned}
\qquad (4.1.60)
$$

Let us consider for instance the marginal probability density of the variable ρ_1^ν

$$
\begin{aligned}
&p\left(\rho_1^\nu\right)=\int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)dxd\dot{x}d\rho_2^\nu=\\
&P^{(1)}_{\rho 1off,\rho 2off}\delta\left(\rho_1^\nu\right)+P^{(1)}_{\rho 1off,\rho 2on}\delta\left(\rho_1^\nu\right)+P^{(1)}_{\rho 1on,\rho 2off}\delta\left(\rho_1^\nu-1\right)+\\
&P^{(*)}_{\rho 1off,\rho 2off}\delta\left(\rho_1^\nu\right)+P^{(*)}_{\rho 1off,\rho 2on}\delta\left(\rho_1^\nu\right)+P^{(*)}_{\rho 1on,\rho 2off}\delta\left(\rho_1^\nu-1\right).
\end{aligned}
\qquad (4.1.61)
$$

Taking into account the following relationships between the state probabilities of the variable ρ_1^ν

$$P^{(1)}_{\rho 1off,\rho 2off}+P^{(1)}_{\rho 1off,\rho 2on}+P^{(*)}_{\rho 1off,\rho 2off}+P^{(*)}_{\rho 1off,\rho 2on}= \\ P^{(1)}_{\rho 1on,\rho 2off}+P^{(*)}_{\rho 1on,\rho 2off}= \tag{4.1.62}$$

it follows

$$p\left(\rho_1^\nu\right)=\int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu\right)dxd\dot{x}d\rho_2^\nu= \\ \delta\left(\rho_1^\nu\right)\left(1-E\left[\rho_1^\nu\right]\right)+\delta\left(\rho_1^\nu-1\right)E\left[\rho_1^\nu\right] \tag{4.1.63}$$

Since the auxiliary variables are zero-one processes, the following relationships hold

$$E\left[X^m\dot{X}^n\rho_1^{\nu o}\right]=E\left[X^m\dot{X}^n\rho_1^{\nu}\right] \\ E\left[X^m\dot{X}^n\rho_2^{\nu o}\right]=E\left[X^m\dot{X}^n\rho_2^{\nu}\right] \\ E\left[X^m\dot{X}^n\rho_1^{\nu o}\rho_2^{\nu p}\right]=0 \tag{4.1.64}$$

The unconditional moments of order $p=m+n$ involving displacement and velocity and the moments of order $p+1$ involving the auxiliary variables, can be expressed in terms of the conditional moments of the same order, as follows

$$E\left[X^m\dot{X}^n\right]= \\ P^{(*)}_{\rho 1off,\rho 2off}E^{(1)}\left[X^m\dot{X}^n\right]+P^{(*)}_{\rho 1off,\rho 2on}E^{(2)}\left[X^m\dot{X}^n\right]+P^{(*)}_{\rho 1on,\rho 2off}E^{(3)}\left[X^m\dot{X}^n\right]; \\ E\left[X^m\dot{X}^n\rho_1^\nu\right]=P^{(*)}_{\rho 1on,\rho 2off}E^{(3)}\left[X^m\dot{X}^n\right]; \\ E\left[X^m\dot{X}^n\rho_2^\nu\right]=P^{(*)}_{\rho 1off,\rho 2on}E^{(2)}\left[X^m\dot{X}^n\right]. \tag{4.1.65}$$

The conditional moments can be expressed in terms of unconditional ones as follows

$$E^{(1)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^n\right]-E\left[X^m\dot{X}^n\rho_1^\nu\right]-E\left[X^m\dot{X}^n\rho_2^\nu\right]}{P^{(*)}_{\rho 1off,\rho 2off}} \\ E^{(2)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^n\rho_1^\nu\right]}{P^{(*)}_{\rho 1on,\rho 2off}} \\ E^{(3)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^n\rho_2^\nu\right]}{P^{(*)}_{\rho 1off,\rho 2on}} \tag{4.1.66}$$

Applying a cumulant neglect closure of order r, the conditional moments of order s>r can be expressed in terms of the lower order conditional moments. The conditional moments of order s>r are then expressed in terms of the unconditional moments of lower order through the following relations

$$E^{(1)}\left[X_1..X_s\right]=$$
$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_j\rho_1^v\right]-E\left[\prod_{j\in B^i}X_j\rho_2^v\right]}{P^{(*)}_{\rho 1off,\rho 2off}}\right)\right]\right\};$$
$$E^{(2)}\left[X_1..X_s\right]=$$
$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_2^v\right]}{P^{(*)}_{\rho 1off,\rho 2on}}\right)\right]\right\};$$
$$E^{(3)}\left[X_1..X_s\right]=$$
$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_1^v\right]}{P^{(*)}_{\rho 1on,\rho 2off}}\right)\right]\right\};$$

(4.1.67)

The unconditional moments of order s involving displacement and velocity only and the moments of order s+1 involving also the auxiliary variables can be expressed in terms of unconditional moments of order up to r, by inserting equations (4.1.67) in (4.1.65), as follows

$$E\left[X_1..X_s\right]=$$

$$P^{(*)}_{\rho 1off,\rho 2off}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_j\rho_1^{\nu}\right]-E\left[\prod_{j\in B^i}X_j\rho_2^{\nu}\right]}{P^{(*)}_{\rho 1off,\rho 2off}}\right)\right]\right\}+$$

$$P^{(*)}_{\rho 1off,\rho 2on}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_2^{\nu}\right]}{P^{(*)}_{\rho 1off,\rho 2on}}\right)\right]\right\}+$$

$$P^{(*)}_{\rho 1on,\rho 2off}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_1^{\nu}\right]}{P^{(*)}_{\rho 1on,\rho 2off}}\right)\right]\right\};$$

$$E\left[X_1..X_s\rho_1^{\nu}\right]=P^{(*)}_{\rho 1on,\rho 2off}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_1^{\nu}\right]}{P^{(*)}_{\rho 1on,\rho 2off}}\right)\right]\right\};$$

$$E\left[X_1..X_s\rho_2^{\nu}\right]=P^{(*)}_{\rho 1off,\rho 2on}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_2^{\nu}\right]}{P^{(*)}_{\rho 1off,\rho 2on}}\right)\right]\right\};$$

(4.1.68)

4.1.5 Erlang driven train of impulses (k=4,ν)

Consider the response of a Duffing oscillator to the random train of impulses driven by an Erlang renewal process with k=3,ν $dR_\nu(t)=\rho_1^\nu(t)\,dN_\nu(t)$, with $d\rho_1^\nu(t)=\left(\rho_2^\nu(t)-\rho_1^\nu(t)\right)dN_\nu(t), d\rho_2^\nu(t)=\left(\rho_3^\nu(t)-\rho_2^\nu(t)\right)dN_\nu(t)$, $d\rho_3^\nu(t)=\left(1-\rho_1^\nu(t)-\rho_2^\nu(t)-2\rho_3^\nu(t)\right)dN_\nu(t)$ (see figure 4.1.4). The stochastic equation of motion is given as

$$d\mathbf{X}(t)=\mathbf{c}\left(\mathbf{X}(t)\right)dt+\mathbf{b}\left(P(t),\mathbf{X}(t)\right)dN_\nu(t) \tag{4.1.69}$$

with

$$\mathbf{X}(t)=\begin{bmatrix} X \\ \dot{X} \\ \rho_1^\nu \\ \rho_2^\nu \\ \rho_3^\nu \end{bmatrix}=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix};\ \mathbf{c}\left(\mathbf{X}(t)\right)=\begin{bmatrix} X_2 \\ -\omega^2X_1-2\zeta\omega X_2-\varepsilon\omega^2X_1^3 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$
$$\mathbf{b}\left(P(t),\mathbf{X}(t)\right)=\begin{bmatrix} 0 \\ PX_3 \\ X_4-X_3 \\ X_5-X_4 \\ 1-X_3-X_4-2X_5 \end{bmatrix} \tag{4.1.70}$$

Before the first impulse occurrence, the auxiliary variables can be in their first 'off' state with probability $P^{(1)}_{\rho1off,\rho2off,\rho3off}$. The variable ρ_3^ν can be in its first 'on' state, while ρ_1^ν and ρ_2^ν are still 'off' with probability $P^{(1)}_{\rho1off,\rho2off,\rho3on}$. The variable ρ_2^ν can be in its first 'on' state, while ρ_2^ν and ρ_3^ν are 'off' with probability $P^{(1)}_{\rho1off,\rho2on,\rho3off}$. After the first impulse occurrence all the variables can be simultaneously 'off' with probability $P^{(*)}_{\rho1off,\rho2off,\rho3off}=\sum_{j\ge2}P^{(j)}_{\rho1off,\rho2off,\rho3off}$, ρ_3^ν can be 'on' while ρ_1^ν and ρ_2^ν are 'off' with probability $P^{(*)}_{\rho1off,\rho2off,\rho3on}=\sum_{j\ge2}P^{(j)}_{\rho1off,\rho2off,\rho3on}$; ρ_2^ν can be 'on' while ρ_1^ν and ρ_3^ν are 'off' with probability $P^{(*)}_{\rho1off,\rho2on,\rho3off}=\sum_{j\ge2}P^{(j)}_{\rho1off,\rho2on,\rho3off}$; ρ_1^ν can be 'on' while ρ_2^ν and ρ_3^ν are 'off' with probability $P^{(*)}_{\rho1on,\rho2off,\rho3off}=\sum_{j\ge2}P^{(j)}_{\rho1on,\rho2off,\rho3off}$. The equations governing the state probabilities of the auxiliary variables are

$$
\begin{aligned}
&P^{(1)}_{\rho 1off,\rho 2off,\rho 3off}+P^{(1)}_{\rho 1off,\rho 2off,\rho 3on}+P^{(1)}_{\rho 1off,\rho 2on,\rho 3off}+\\
&P^{(*)}_{\rho 1off,\rho 2off,\rho 3off}+P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}+P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}=1-E\left[\rho_1^{\nu}\right],\\
&P^{(1)}_{\rho 1off,\rho 2off,\rho 3off}+P^{(1)}_{\rho 1off,\rho 2off,\rho 3on}+P^{(1)}_{\rho 1on,\rho 2off,\rho 3off}+\\
&P^{(*)}_{\rho 1off,\rho 2off,\rho 3off}+P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}+P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}=1-E\left[\rho_2^{\nu}\right],\\
&P^{(1)}_{\rho 1off,\rho 2off,\rho 3off}+P^{(1)}_{\rho 1off,\rho 2on,\rho 3off}+P^{(1)}_{\rho 1on,\rho 2off,\rho 3off}+\\
&P^{(*)}_{\rho 1off,\rho 2off,\rho 3off}+P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}+P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}=1-E\left[\rho_3^{\nu}\right],\\
&P^{(1)}_{\rho 1on,\rho 2off,\rho 3off}+P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}=E\left[\rho_1^{\nu}\right],\\
&P^{(1)}_{\rho 1off,\rho 2on,\rho 3off}+P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}=E\left[\rho_2^{\nu}\right],\\
&P^{(1)}_{\rho 1off,\rho 2off,\rho 3on}+P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}=E\left[\rho_2^{\nu}\right].
\end{aligned}
\tag{4.1.71}
$$

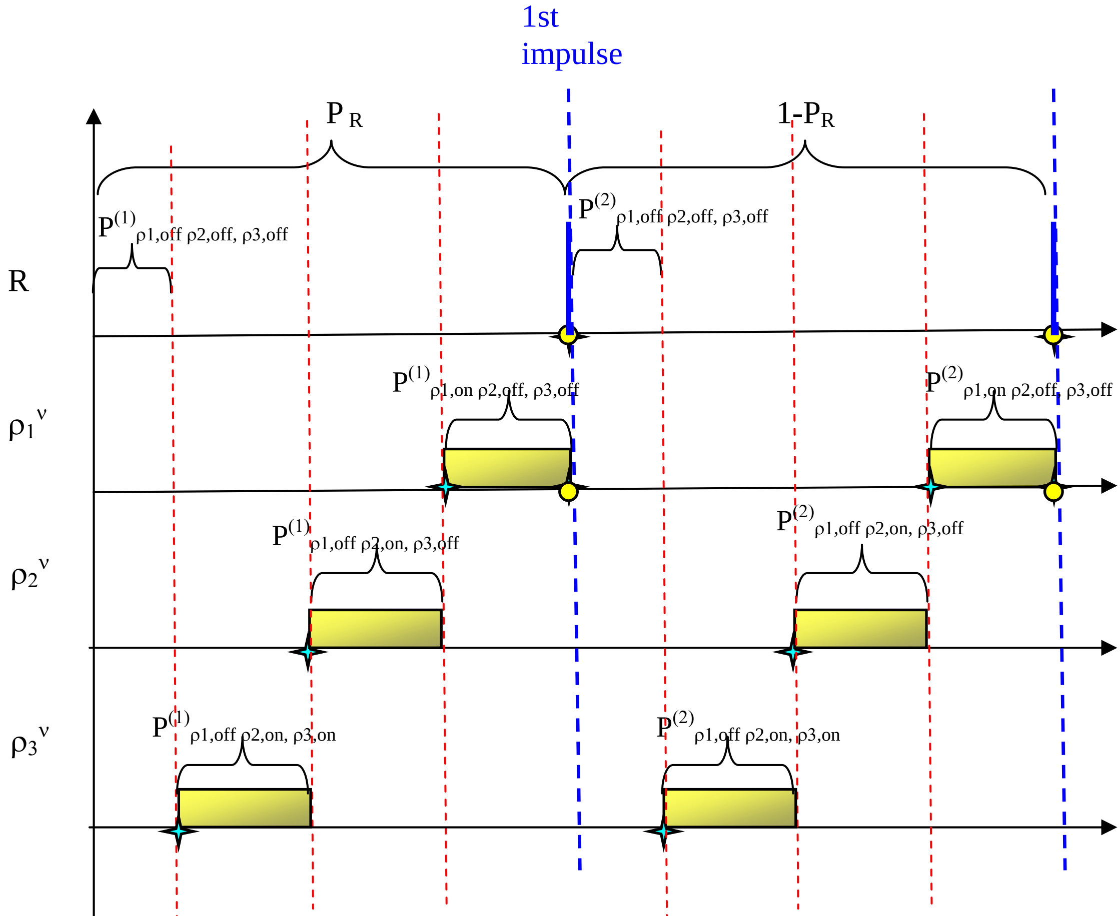

Figure 4.1.4

Sample function of an impulse process driven by an Erlang renewal process with parameters $k=4$ and ν and the auxiliary zero-one variables appearing in expressions (4.1.70).

From the Poisson law $\Pr\left\{dN_{\nu}(t)=n\right\}=\dfrac{(\nu t)^n}{n!}e^{-\nu t}$, it follows that

$$
\begin{aligned}
&P^{(1)}_{\rho 1off,\rho 2off,\rho 3off} = e^{-\nu t},\\
&P^{(1)}_{\rho 1off,\rho 2off,\rho 3on} = \nu t e^{-\nu t},\\
&P^{(1)}_{\rho 1off,\rho 2on,\rho 3off} = \frac{(\nu t)^2}{2} e^{-\nu t},\\
&P^{(1)}_{\rho 1on,\rho 2off,\rho 3off} = \frac{(\nu t)^3}{6} e^{-\nu t}
\end{aligned}
\tag{4.1.72}
$$

Therefore

$$
\begin{aligned}
&P^{(*)}_{\rho 1on,\rho 2off,\rho 3off} = E\left[\rho_1^\nu\right] - P^{(1)}_{\rho 1on,\rho 2off,\rho 3off},\\
&P^{(*)}_{\rho 1off,\rho 2on,\rho 3off} = E\left[\rho_2^\nu\right] - P^{(1)}_{\rho 1off,\rho 2on,\rho 3off},\\
&P^{(*)}_{\rho 1off,\rho 2off,\rho 3on} = E\left[\rho_3^\nu\right] - P^{(1)}_{\rho 1off,\rho 2off,\rho 3on},\\
&P^{(*)}_{\rho 1off,\rho 2off,\rho 3off} = 1 - E\left[\rho_1^\nu\right] - E\left[\rho_2^\nu\right] - E\left[\rho_3^\nu\right] - P^{(1)}_{\rho 1off,\rho 2off,\rho 3off}.
\end{aligned}
\tag{4.1.73}
$$

Let us express the unknown joint probability density of the state vector **X** in terms of the conditional probability functions

$p^1\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x} =$

- $\Pr\left\{X \in (x,x+dx) \wedge \dot{X} \in (\dot{x},\dot{x}+d\dot{x})\right\} \wedge \rho_1^\nu \; is\, in\, its \; first\,'off\,' \; phase$

  $\wedge \rho_2^\nu \; is\, in\, its \; first\,'off\,' \; phase \wedge \rho_3^\nu \; is\, in\, its \; first\,'off\,' \; phase$

  given that no impulses have occurred and all the auxiliary are in their first 'off' phase.

  $p^2\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x} =$

- $\Pr\left\{X \in (x,x+dx) \wedge \dot{X} \in (\dot{x},\dot{x}+d\dot{x})\right\} \wedge \rho_1^\nu \; is\, in\, its \; first\,'off\,' \; phase$

  $\wedge \rho_2^\nu \; is\, in\, its \; first\,'off\,' \; phase \wedge \rho_3^\nu \; is\, in\, its \; first\,'on'\, phase$

  given that no impulses have occurred and the variable ρ_3^ν is in its first 'on' phase.

  $p^3\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x} =$

- $\Pr\left\{X \in (x,x+dx) \wedge \dot{X} \in (\dot{x},\dot{x}+d\dot{x})\right\} \wedge \rho_1^\nu \; is\, in\, its \; first\,'off\,' \; phase$

  $\wedge \rho_2^\nu \; is\, in\, its \; first\,'on'\, phase \wedge \rho_3^\nu \; is\,'off\,'$

  given that no impulses have occurred and the variable ρ_2^ν is in its first 'on' phase.

  $p^4\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x} =$

- $\Pr\left\{X \in (x,x+dx) \wedge \dot{X} \in (\dot{x},\dot{x}+d\dot{x})\right\} \wedge \rho_1^\nu \; is\, in\, its \; first\,'on'\, phase$

  $\wedge \rho_2^\nu \; is\,'off\,' \wedge \rho_3^\nu \; is\,'off\,'$

  given that no impulses have occurred and the variable ρ_1^ν is in its first 'on' phase.

$$p^5\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x}=$$

$$\Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$

$$\wedge\rho_1^\nu\ is\,in\,any\,'off\,'\ phase\ following\,the\ first\,impulse$$

$$\wedge\rho_2^\nu\ is\,in\,any\,'off\,'\ phase\ following\,the\ first\,impulse$$

$$\wedge\rho_3^\nu\ is\,in\,any\,'off\,'\ phase\ following\,the\ first\,impulse$$

given that after the first impulse occurrence the auxiliary variables are simultaneously ‘off’.

$$p^6\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x}=$$

$$\Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$

- $\wedge\rho_1^\nu\ is\,in\,any\,'off\,'\ phase\ following\,the\ first\,impulse$

$$\wedge\rho_2^\nu\ is\,in\,any\,'off\,'\ phase\ following\,the\ first\,impulse$$

$$\wedge\rho_3^\nu\ is\,in\,any\,'on'\ phase\ following\,the\ first\,impulse$$

given that after the first impulse occurrence the variable ρ_3^ν is ‘on’.

$$p^7\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x}=$$

$$\Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$

- $\wedge\rho_1^\nu\ is\,in\,any\,'off\,'\ phase\ following\,the\ first\,impulse$

$$\wedge\rho_2^\nu\ is\,in\,any\,'on'\ phase\ following\,the\ first\,impulse$$

$$\wedge\rho_3^\nu\ is\,in\,any\,'off\,'\ phase\ following\,the\ first\,impulse$$

given that after the first impulse occurrence the variable ρ_2^ν is ‘on’.

$$p^8\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x}=$$

$$\Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$

- $\wedge\rho_1^\nu\ is\,in\,any\,'on'\ phase\ following\,the\ first\,impulse$

$$\wedge\rho_2^\nu\ is\,in\,any\,'off\,'\ phase\ following\,the\ first\,impulse$$

$$\wedge\rho_3^\nu\ is\,in\,any\,'off\,'\ phase\ following\,the\ first\,impulse$$

given that after the first impulse occurrence the variable ρ_1^ν is ‘on’.

Hence, the joint probability density function can be expressed as

$$p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)=\sum_{j=1}^{8}p^j\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right) \qquad (4.1.74)$$

Due to the fact that there is a finite probability of the system being in a deterministic state from the initial time to the first impulse, the contributions accounting for the fact that no impulse have occurred ($p^1\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)$ to $p^4\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)$) can be expressed as

$$
\begin{aligned}
&p^1\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)=P^{(1)}_{\rho 1off,\rho 2off,\rho 3off}\delta(x)\delta(\dot{x})\delta\left(\rho_1^\nu\right)\delta\left(\rho_2^\nu\right)\delta\left(\rho_3^\nu\right),\\
&p^2\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)=P^{(1)}_{\rho 1off,\rho 2off,\rho 3on}\delta(x)\delta(\dot{x})\delta\left(\rho_1^\nu\right)\delta\left(\rho_2^\nu\right)\delta\left(\rho_3^\nu-1\right),\\
&p^3\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)=P^{(1)}_{\rho 1off,\rho 2on,\rho 3off}\delta(x)\delta(\dot{x})\delta\left(\rho_1^\nu\right)\delta\left(\rho_2^\nu-1\right)\delta\left(\rho_3^\nu\right),\\
&p^4\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)=P^{(1)}_{\rho 1on,\rho 2off,\rho 3off}\delta(x)\delta(\dot{x})\delta\left(\rho_1^\nu-1\right)\delta\left(\rho_2^\nu\right)\delta\left(\rho_3^\nu\right)
\end{aligned}
\tag{4.1.75}
$$

Let us express the conditional probabilities given that the first Erlang impulse has occurred as

$$
\begin{aligned}
&p^5\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)=P^{(*)}_{\rho 1off,\rho 2off,\rho 3off}f^{(1)}(x,\dot{x})\delta\left(\rho_1^\nu\right)\delta\left(\rho_2^\nu\right)\delta\left(\rho_3^\nu\right),\\
&p^6\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)=P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}f^{(2)}(x,\dot{x})\delta\left(\rho_1^\nu\right)\delta\left(\rho_2^\nu\right)\delta\left(\rho_3^\nu-1\right),\\
&p^7\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)=P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}f^{(3)}(x,\dot{x})\delta\left(\rho_1^\nu\right)\delta\left(\rho_2^\nu-1\right)\delta\left(\rho_3^\nu\right),\\
&p^8\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)=P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}f^{(4)}(x,\dot{x})\delta\left(\rho_1^\nu-1\right)\delta\left(\rho_2^\nu\right)\delta\left(\rho_3^\nu\right)
\end{aligned}
\tag{4.1.76}
$$

Considering that the joint probability density function has to satisfy the equation

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x}d\rho_1^\nu d\rho_2^\nu d\rho_3^\nu=\\
&P^{(1)}_{\rho 1off,\rho 2off,\rho 3off}+P^{(1)}_{\rho 1off,\rho 2off,\rho 3on}+P^{(1)}_{\rho 1off,\rho 2on,\rho 3off}+P^{(1)}_{\rho 1on,\rho 2off,\rho 3off}+\\
&P^{(*)}_{\rho 1off,\rho 2off,\rho 3off}\int_{-\infty}^{+\infty} f^{(1)}(x,\dot{x})dxd\dot{x}+P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}\int_{-\infty}^{+\infty} f^{(2)}(x,\dot{x})dxd\dot{x}+\\
&P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}\int_{-\infty}^{+\infty} f^{(3)}(x,\dot{x})dxd\dot{x}+P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}\int_{-\infty}^{+\infty} f^{(4)}(x,\dot{x})dxd\dot{x}=1
\end{aligned}
\tag{4.1.77}
$$

and the following equations relating the state probabilities of the auxiliary variables

$$
\begin{aligned}
&P^{(1)}_{\rho 1off,\rho 2off,\rho 3off}+P^{(1)}_{\rho 1off,\rho 2off,\rho 3on}+P^{(1)}_{\rho 1off,\rho 2on,\rho 3off}+P^{(1)}_{\rho 1on,\rho 2off,\rho 3off}=P_R;\\
&P^{(*)}_{\rho 1off,\rho 2off,\rho 3off}+P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}+P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}+P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}=1-P_R
\end{aligned}
$$

it follows that

$$
\int_{-\infty}^{+\infty} f^{(1)}(x,\dot{x})dxd\dot{x}=1;\ldots;\int_{-\infty}^{+\infty} f^{(4)}(x,\dot{x})dxd\dot{x}=1 \tag{4.1.78}
$$

It can also be proved that the following identities hold

$$
\begin{aligned}
&p\left(\rho_1^\nu\right)=\int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x}d\rho_2^\nu d\rho_3^\nu = \\
&\delta\left(\rho_1^\nu\right)\left(1-E\left[\rho_1^\nu\right]\right)+\delta\left(\rho_1^\nu-1\right)E\left[\rho_1^\nu\right]; \\
&p\left(\rho_2^\nu\right)=\int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x}d\rho_1^\nu d\rho_3^\nu = \\
&\delta\left(\rho_2^\nu\right)\left(1-E\left[\rho_2^\nu\right]\right)+\delta\left(\rho_2^\nu-1\right)E\left[\rho_2^\nu\right]; \\
&p\left(\rho_3^\nu\right)=\int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x}d\rho_1^\nu d\rho_2^\nu = \\
&\delta\left(\rho_3^\nu\right)\left(1-E\left[\rho_3^\nu\right]\right)+\delta\left(\rho_3^\nu-1\right)E\left[\rho_3^\nu\right];
\end{aligned}
\tag{4.1.79}
$$

Let us consider for instance the marginal probability density of the variable ρ_1^ν

$$
\begin{aligned}
&p\left(\rho_1^\nu\right)=\int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x}d\rho_2^\nu d\rho_3^\nu = \\
&P^{(1)}_{\rho 1off,\rho 2off,\rho 3off}\delta\left(\rho_1^\nu\right)+P^{(1)}_{\rho 1off,\rho 2off,\rho 3on}\delta\left(\rho_1^\nu\right)+ \\
&P^{(1)}_{\rho 1off,\rho 2on,\rho 3off}\delta\left(\rho_1^\nu\right)+P^{(1)}_{\rho 1on,\rho 2off,\rho 3off}\delta\left(\rho_1^\nu-1\right)+ \\
&P^{(*)}_{\rho 1off,\rho 2off,\rho 3off}\delta\left(\rho_1^\nu\right)+P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}\delta\left(\rho_1^\nu\right)+ \\
&P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}\delta\left(\rho_1^\nu\right)+P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}\delta\left(\rho_1^\nu-1\right).
\end{aligned}
\tag{4.1.80}
$$

Taking into account the relationships between the state probabilities of the variable ρ_1^ν in (4.1.71), it follows

$$
\begin{aligned}
&p\left(\rho_1^\nu\right)=\int_{-\infty}^{+\infty} p\left(x,\dot{x},\rho_1^\nu,\rho_2^\nu,\rho_3^\nu\right)dxd\dot{x}d\rho_2^\nu d\rho_3^\nu = \\
&\delta\left(\rho_1^\nu\right)\left(1-E\left[\rho_1^\nu\right]\right)+\delta\left(\rho_1^\nu-1\right)E\left[\rho_1^\nu\right]
\end{aligned}
\tag{4.1.81}
$$

Since the auxiliary variables are zero-one processes, the following relationships hold

$$
\begin{aligned}
&E\left[X^m\dot{X}^n\rho_1^{\nu o}\right]=E\left[X^m\dot{X}^n\rho_1^{\nu}\right], \\
&E\left[X^m\dot{X}^n\rho_2^{\nu o}\right]=E\left[X^m\dot{X}^n\rho_2^{\nu}\right], \\
&E\left[X^m\dot{X}^n\rho_3^{\nu o}\right]=E\left[X^m\dot{X}^n\rho_3^{\nu}\right], \\
&E\left[X^m\dot{X}^n\rho_1^{\nu o}\rho_2^{\nu p}\rho_3^{\nu q}\right]=0
\end{aligned}
\tag{4.1.82}
$$

The unconditional moments of order $p=m+n$ involving displacement and velocity and the moments of order $p+1$ involving the auxiliary variables, can be expressed in terms of the conditional moments of the same order, as follows

$$
\begin{aligned}
&E\left[X^m\dot{X}^n\right]=\\
&P^{(*)}_{\rho 1off,\rho 2off,\rho 3off}E^{(1)}\left[X^m\dot{X}^n\right]+P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}E^{(2)}\left[X^m\dot{X}^n\right]+\\
&P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}E^{(3)}\left[X^m\dot{X}^n\right]+P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}E^{(4)}\left[X^m\dot{X}^n\right];\\
&E\left[X^m\dot{X}^n\rho_1^\nu\right]=P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}E^{(4)}\left[X^m\dot{X}^n\right];\\
&E\left[X^m\dot{X}^n\rho_2^\nu\right]=P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}E^{(3)}\left[X^m\dot{X}^n\right];\\
&E\left[X^m\dot{X}^n\rho_3^\nu\right]=P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}E^{(2)}\left[X^m\dot{X}^n\right];
\end{aligned}
\tag{4.1.83}
$$

The conditional moments can be expressed in terms of unconditional ones as follows

$$
\begin{aligned}
&E^{(1)}\left[X^m\dot{X}^n\right]=\\
&\frac{E\left[X^m\dot{X}^n\right]-E\left[X^m\dot{X}^n\rho_1^\nu\right]-E\left[X^m\dot{X}^n\rho_2^\nu\right]-E\left[X^m\dot{X}^n\rho_3^\nu\right]}{P^{(*)}_{\rho 1off,\rho 2off,\rho 3off}};\\
&E^{(2)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^n\rho_3^\nu\right]}{P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}};\\
&E^{(3)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^n\rho_2^\nu\right]}{P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}};\\
&E^{(4)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^n\rho_1^\nu\right]}{P^{(*)}_{\rho 1on,\rho 2on,\rho 3off}}.
\end{aligned}
\tag{4.1.84}
$$

Applying a cumulant neglect closure of order r, the conditional moments of order s>r can be expressed in terms of the lower order conditional moments. The conditional moments of order s>r are then expressed in terms of the unconditional moments of lower order through the following relationships

$$
\begin{aligned}
&E^{(1)}\left[X_1..X_s\right]=\\
&\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_j\rho_1^\nu\right]-E\left[\prod_{j\in B^i}X_j\rho_2^\nu\right]-E\left[\prod_{j\in B^i}X_j\rho_3^\nu\right]}{P^{(*)}_{\rho 1off,\rho 2off,\rho 3off,}}\right)\right]\right\};\\
&E^{(2)}\left[X_1..X_s\right]=\\
&\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_3^\nu\right]}{P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}}\right)\right]\right\};
\end{aligned}
$$

$$E^{(3)}\left[X_1..X_s\right]=$$
$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_2^{\nu}\right]}{P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}}\right)\right]\right\};$$
.
$$E^{(4)}\left[X_1..X_s\right]=\tag{4.1.85}$$
$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_1^{\nu}\right]}{P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}}\right)\right]\right\};$$

The unconditional moments of order s involving displacement and velocity only and the moments of order s+1 involving also the auxiliary variables can be expressed in terms of unconditional moments of order up to r, by inserting equations (4.1.67) in (4.1.65), as follows

$$E\left[X_1..X_s\right]=P^{(*)}_{\rho 1off,\rho 2off,\rho 3off}$$
$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_j\rho_1^{\nu}\right]-E\left[\prod_{j\in B^i}X_j\rho_2^{\nu}\right]-E\left[\prod_{j\in B^i}X_j\rho_3^{\nu}\right]}{P^{(*)}_{\rho 1off,\rho 2off,\rho 3off}}\right)\right]\right\}+$$
$$P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_3^{\nu}\right]}{P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}}\right)\right]\right\}+$$
$$P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_2^{\nu}\right]}{P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}}\right)\right]\right\}+$$
$$P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_1^{\nu}\right]}{P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}}\right)\right]\right\}.$$

$$E\left[X_1..X_s\rho_1^{\nu}\right]=$$

$$P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_1^{\nu}\right]}{P^{(*)}_{\rho 1on,\rho 2off,\rho 3off}}\right)\right]\right\};$$

$$E\left[X_1..X_s\rho_2^{\nu}\right]=$$

$$P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_2^{\nu}\right]}{P^{(*)}_{\rho 1off,\rho 2on,\rho 3off}}\right)\right]\right\};$$

$$E\left[X_1..X_s\rho_3^{\nu}\right]=$$

$$P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_3^{\nu}\right]}{P^{(*)}_{\rho 1off,\rho 2off,\rho 3on}}\right)\right]\right\}. \qquad (4.1.86)$$

4.1.6 Numerical analysis

In order to illustrate the modified moment closure scheme devised, consider a Duffing oscillator under a train of impulses driven by an Erlang renewal process $E(\nu,k)$. The response is governed by the stochastic differential equation (4.1.5). The data assumed for the Duffing oscillator is: $\omega_0 = 1s^{-1}, \zeta = 0.05$. A substantial non linear effect has been taken into account by assuming the non-linearity coefficient to be $\varepsilon = 0.5$ (cf. *Wen (1975)).*

Departure of the excitation process from Gaussianity depends on the value of the mean arrival rate of the impulses, compared to the system natural frequency ω_o. When the mean arrival rate is $0.1\omega_o$ or lower, the departure from Gaussianity is expected to be substantial (cf. *Iwankiewicz and Nielsen (1989), Janssen and Lambert (1967)*).

Computations have been performed for the cases $k=2, k=3$ and $k=4$. The values of the parameter ν have been assumed in such a way that the mean arrival rate of the correspondent Poisson process is $\frac{\nu}{k} = 0.1\omega_o$ ($\nu = 0.2\omega_o$, $\nu = 0.3\omega_o$ and $\nu = 0.4\omega_o$, respectively).

The random magnitudes of impulses have been assumed as centralized, Rayleigh distributed random variables. The values of the parameter $\sigma_r = E\left[P^2\right]/\sqrt{2}$ for each case have been chosen so that the stationary value of the variance of the response of the corresponding linear oscillator $\sigma_X^2 = \frac{\nu}{k}\frac{E\left[P^2\right]}{4\zeta\omega_o^3}$ has a unit value. Therefore it results $\sigma_r = \sqrt{\frac{4k\zeta\omega_o^3}{2\nu}}$.

To verify the approximate analytical results, the response moments have been obtained from Monte Carlo simulations based on the ensemble of 30000 of the response sample functions, obtained by numerical integration of the equation of motion (4.1.1) with the aid of the computer program Mathematica.

Transient response statistics of the non-linear oscillator are shown in Figures 4.1.5.to 4.1.7. The analytical results are obtained by applying the ordinary and the modified cumulant-neglect closure techniques, neglecting in both schemes the cumulants above the second order.

It is seen that the agreement between the analytical and simulation results is very good in the first part of the transient mean value and variance of the response.

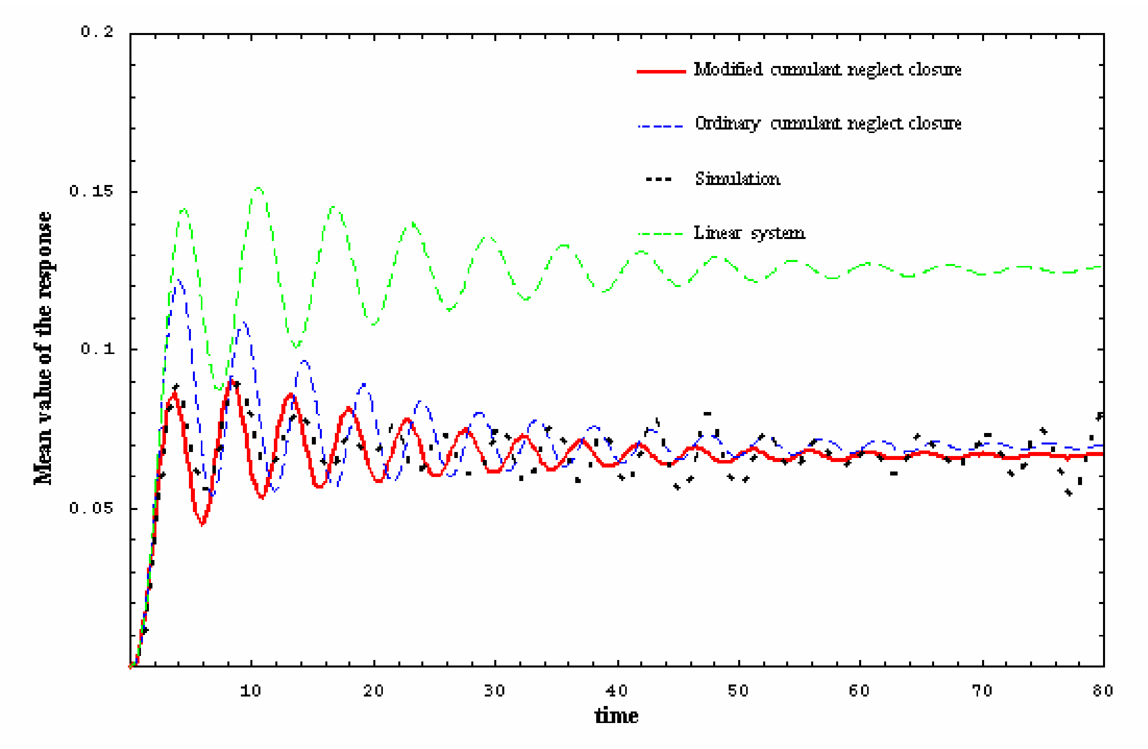

Figure 4.1.5 (a)
Mean value of the response of a Duffing oscillator to a random train of impulses driven by an Erlang process with parameters k=2 and ν=0.2

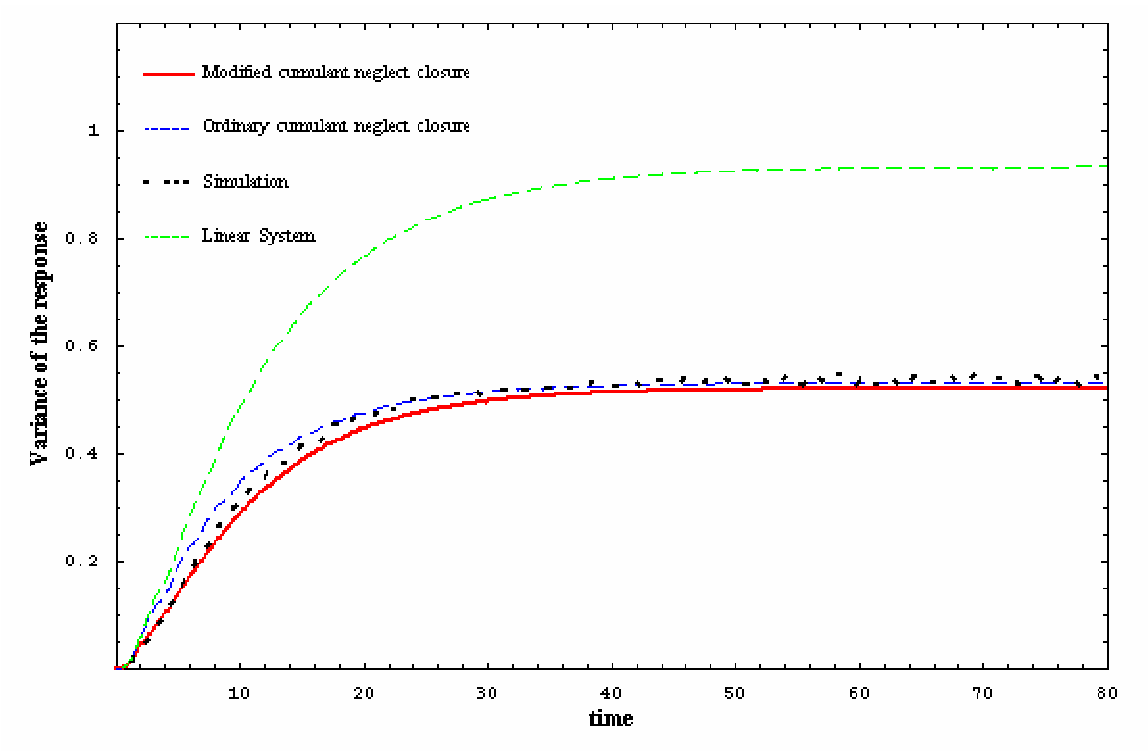

Figure 4.1.5 (b)
Variance of the response of a Duffing oscillator to a random train of impulses driven by an Erlang process with parameters k=2 and ν=0.2

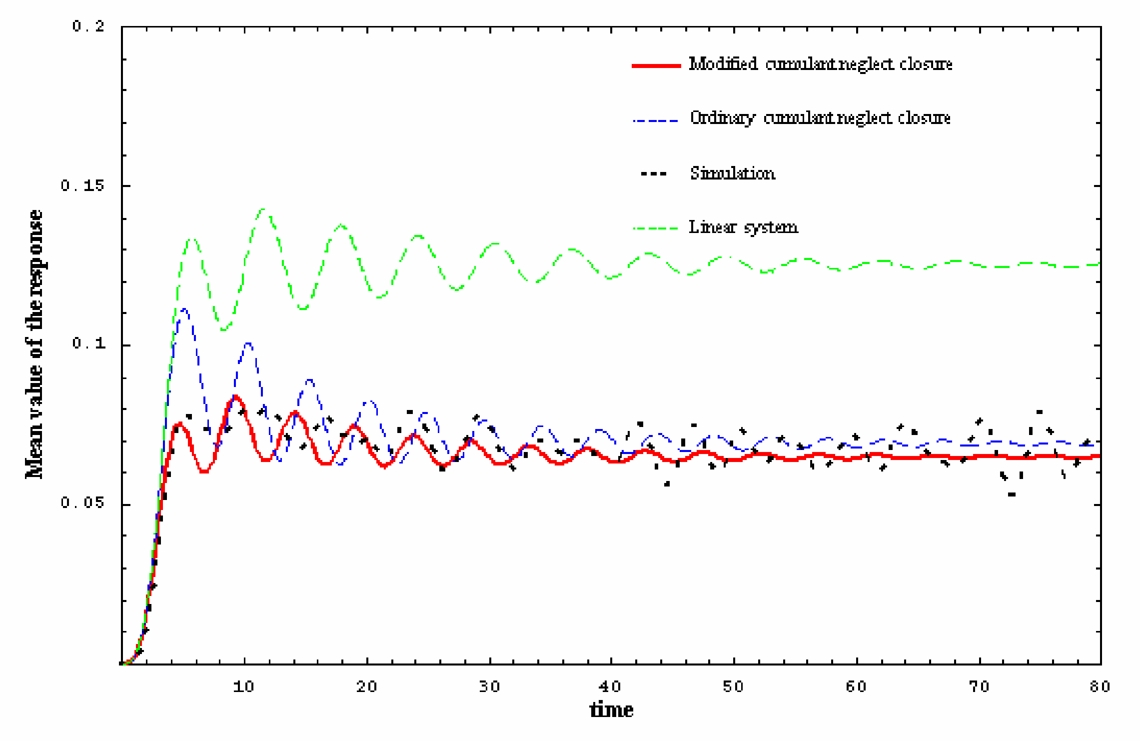

Figure 4.1.6 (a)
Mean value of the response of a Duffing oscillator to a random train of impulses driven by an Erlang process with parameters k=3 and ν=0.3

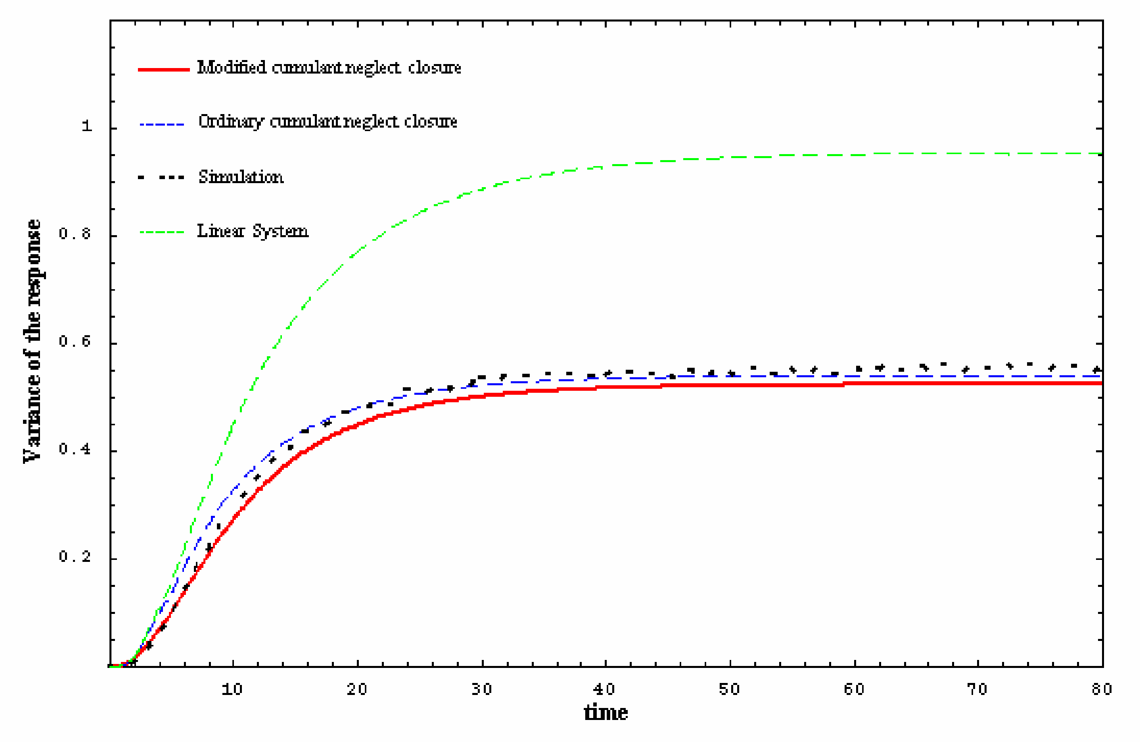

Figure 4.1.6 (b)
Variance of the response of a Duffing oscillator to a random train of impulses driven by an Erlang process with parameters k=3 and ν=0.3

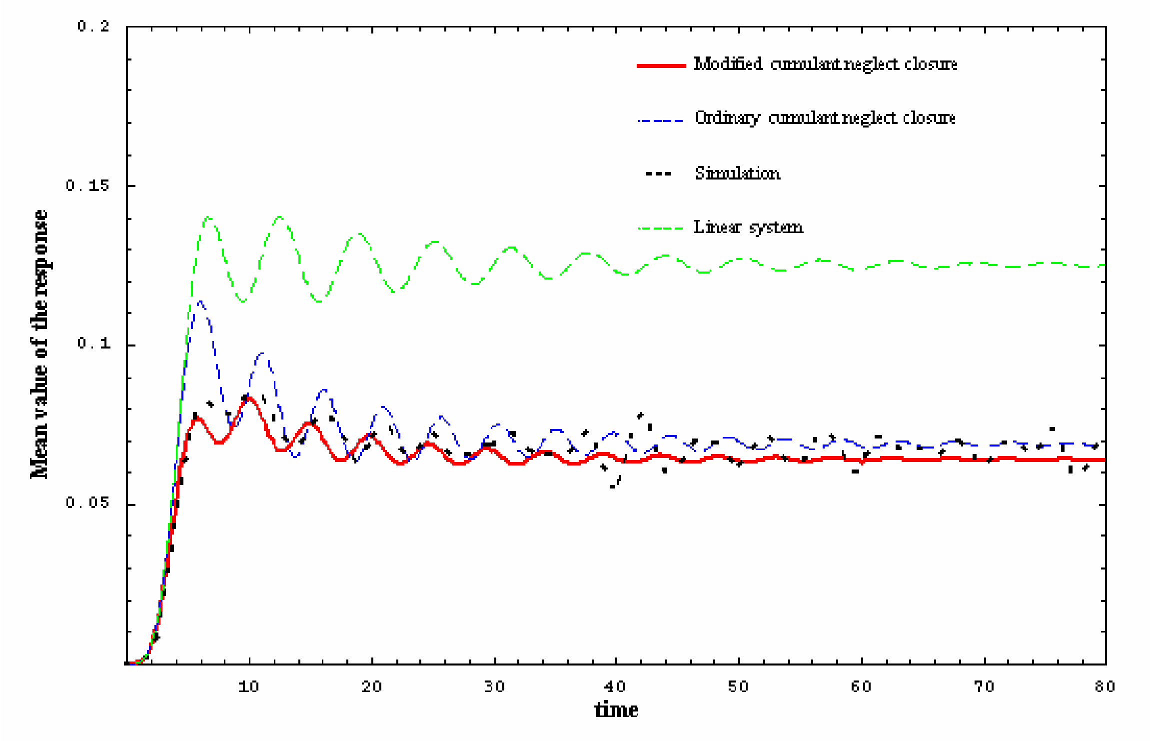

Figure 4.1.7 (a)
Mean value of the response of a Duffing oscillator to a random train of impulses driven by an Erlang process with parameters k=4 and ν=0.4

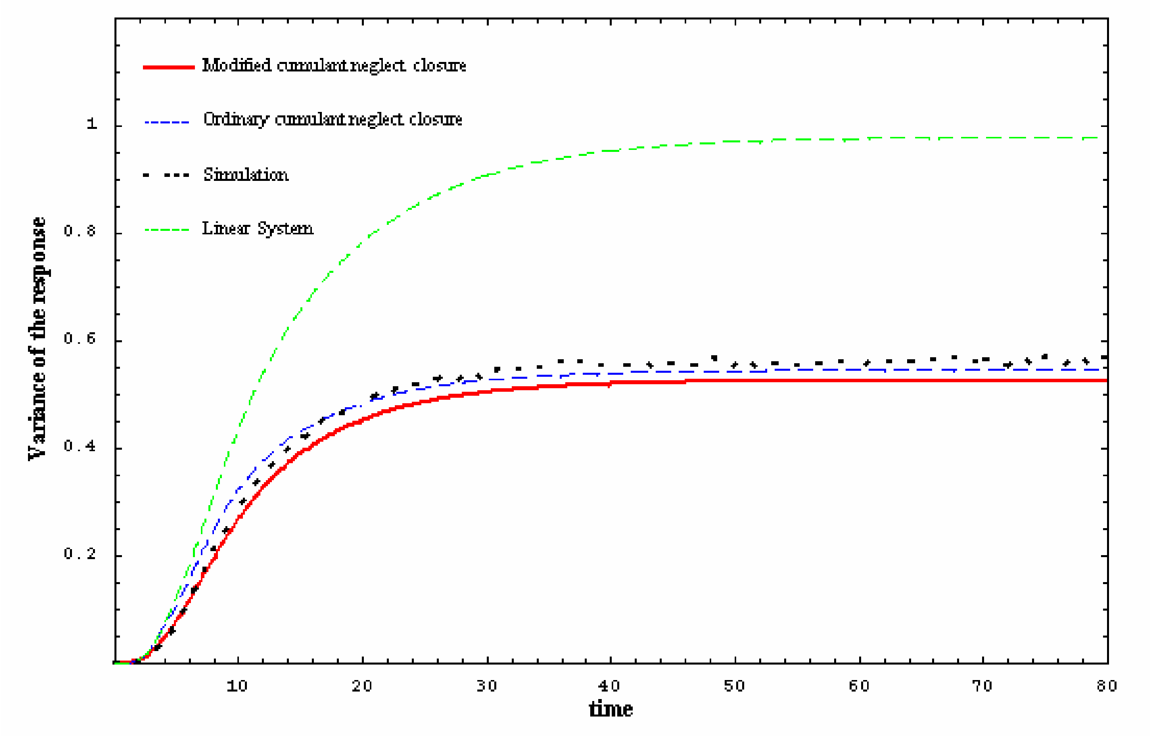

Figure 4.1.7 (b)
Variance of the response of a Duffing oscillator to a random train of impulses driven by an Erlang process with parameters k=4 and ν=0.4

4.2. NON-LINEAR OSCILLATOR UNDER TRAINS OF IMPULSES DRIVEN BY A SPECIAL CLASS OF NON-ERLANG RENEWAL PROCESSES

4.2.1 Statement of the problem

Consider a non-linear, non-hysteretic oscillator governed by the equation

$$\ddot{X}(t) + f\left(X(t), \dot{X}(t)\right) = \sum_{i,R=1}^{R(t)} P_{i,R}\delta\left(t - t_{i,R}\right), \tag{4.2.1}$$

where $f\left(X(t), \dot{X}(t)\right)$ is the function of instantaneous values of $X(t)$ and $\dot{X}(t)$ and the stochastic excitation is the random train of impulses whose arrival times $t_{i,R}$ are driven by the renewal process $R(t)$. The stochastic equations governing the system state vector $\mathbf{X}$ can be written as

$$d\mathbf{X}(t) = \mathbf{c}\left(X(t)\right)dt + \mathbf{b}dR(t) \tag{4.2.2}$$

where $\mathbf{X} = \left[X, \dot{X}\right]^T$, $\mathbf{c}\left(X(t)\right) = \left[\dot{X}, -f\left(X, \dot{X}\right)\right]^T$, $\mathbf{b} = \left[0, P(t)\right]^T$,

Let us consider a class of impulse processes which may be represented as follows

$$\sum_{i,R=1}^{R(t)} P_{i,R}\delta\left(t - t_{i,R}\right) = \sum_{i=1}^{R_\nu(t)} Z(t_i)P_i\delta\left(t - t_i\right) \tag{4.2.3}$$

where the arrival times t_i are driven by an Erlang renewal process $R_\nu(t)$ with parameters ν and k and $Z(t_i)$ is a value at t_{i-} of an intermittent, zero-one stochastic variable $Z(t)$ governed by the stochastic equation

$$dZ(t) = (1 - Z)dR_\mu(t) - ZdR_\nu(t) \tag{4.2.4}$$

sample functions of the counting process $Z(t)$ are assumed to be left continuous with right limits. $R_\mu(t)$ is an Erlang renewal process with parameters μ and l. The processes $R_\mu(t)$ and $R_\nu(t)$ are assumed to be independent.
The following replacement is used (cf. section 3.1.1)

$$dR_\alpha(t) = \rho^\alpha(t)dN_\alpha(t) \tag{4.2.5}$$

where the $\rho^\alpha(t)$ is a variable which only takes values 0 or 1 and is governed by

$$
\begin{aligned}
& d\rho^{\alpha}(t) = \left(\rho_2^{\alpha}(t) - \rho^{\alpha}(t)\right) dN_{\alpha}(t) \\
& d\rho_2^{\alpha}(t) = \left(\rho_3^{\alpha}(t) - \rho_2^{\alpha}(t)\right) dN_{\alpha}(t) \\
& .. \\
& d\rho_{k-2}^{\alpha}(t) = \left(\rho_{k-1}^{\alpha}(t) - \rho_{k-2}^{\alpha}(t)\right) dN_{\alpha}(t) \\
& d\rho_{k-1}^{\alpha}(t) = \left(1 - \sum_{j=1}^{k-2} \rho_j^{\alpha}(t) - 2\rho_{k-1}^{\alpha}(t)\right) dN_{\alpha}(t)
\end{aligned}
\tag{4.2.6}
$$

where $\rho_1^{\alpha}(t) = \rho^{\alpha}(t)$.

The variables $\rho^{\alpha}(t), \rho_2^{\alpha}(t) .. \rho_{k-1}^{\alpha}(t)$ only take values 0 or 1 (see chapter ..).
It is convenient to augment the state vector by new combined variables

$$
\begin{aligned}
& Z_1 = Z, Z_2 = \rho_1^{\nu}, ..Z_k = \rho_{k-1}^{\nu}, Z_{k+1} = \rho_1^{\mu}, ..Z_{k+l-1} = \rho_{l-1}^{\mu}, \\
& Z_{k+l} = \rho_1^{\nu}\rho_1^{\mu}, ..Z_{kl} = \rho_{k-1}^{\nu}\rho_{l-1}^{\mu}, Z_{kl+1} = Z\rho_1^{\nu}, ..Z_{kl+k-1} = Z\rho_{k-1}^{\nu}, \\
& Z_{kl+k} = Z\rho_1^{\mu}, ..Z_{kl+k+l-2} = Z\rho_{l-1}^{\mu}, Z_{kl+k+l-1} = Z\rho_1^{\nu}\rho_1^{\mu}, ..Z_{2kl-1} = Z\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}.
\end{aligned}
$$

The stochastic equations governing the augmented state vector $\mathbf{X} = \begin{bmatrix} X & \dot{X} & Z_1 & Z_2 .. & Z_{2kl-1} \end{bmatrix}^T$ can be written as

$$
d\mathbf{X}(t) = \mathbf{c}\left(\mathbf{X}(t)\right) dt + \mathbf{b}\left(P(t), \mathbf{X}(t)\right) d\mathbf{N}(t) \tag{4.2.7}
$$

where

$$
\begin{aligned}
& \mathbf{X}(t) = \begin{bmatrix} X & \dot{X} & Z_1 & Z_2 & .. & Z_{2kl-1} \end{bmatrix}^T; \\
& \mathbf{c}\left(\mathbf{X}(t), t\right) = \begin{bmatrix} \dot{X} & -f\left(X, \dot{X}\right) & 0 & 0 & .. & 0 \end{bmatrix}^T; \\
& b^{\nu} = \begin{bmatrix} b_1^{\nu} & b_2^{\nu} & .. & b_{2kl+1}^{\nu} \end{bmatrix}^T; \\
& b^{\mu} = \begin{bmatrix} b_1^{\mu} & b_2^{\mu} & .. & b_{2kl+1}^{\mu} \end{bmatrix}^T; \\
& d\mathbf{N}(t) = \begin{bmatrix} dN_{\nu} & dN_{\mu} \end{bmatrix}
\end{aligned}
\tag{4.2.8}
$$

with

$$
\begin{aligned}
& b_1^{\nu} = 0, b_2^{\nu} = PZ\rho_1^{\nu}, b_3^{\nu} = -Z\rho_1^{\nu}, b_4^{\nu} = \rho_2^{\nu} - \rho_1^{\nu}, ..b_{k+2}^{\nu} = 1 - \sum_{j=1}^{k-2} \rho_j^{\nu} - 2\rho_{k-1}^{\nu}, \\
& b_{k+3}^{\nu} = \rho_2^{\mu} - \rho_1^{\mu}, ..b_{k+l+1}^{\nu} = 1 - \sum_{j=1}^{l-2} \rho_j^{\mu} - 2\rho_{l-1}^{\mu}, b_{k+l+2}^{\nu} = \rho_2^{\nu}\rho_1^{\mu} - \rho_1^{\nu}\rho_1^{\mu}, \\
& b_{kl+3}^{\nu} = \rho_{l-1}^{\mu} - \sum_{j=1}^{k-2} \rho_j^{\nu}\rho_{l-1}^{\mu} - 2\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}, b_{kl+4}^{\nu} = Z\rho_2^{\nu} - Z\rho_1^{\nu}, .. \\
& b_{kl+k+3}^{\nu} = Z - Z\rho_1^{\nu} - 2Z\rho_{k-1}^{\nu}, b_{kl+k+4}^{\nu} = -Z\rho_1^{\nu}\rho_1^{\mu}, ..b_{kl+k+l+3}^{\nu} = -Z\rho_1^{\nu}\rho_{l-1}^{\mu}, \\
& b_{kl+k+l+4}^{\nu} = Z\rho_2^{\nu}\rho_1^{\mu} - Z\rho_1^{\nu}\rho_1^{\mu}, ..b_{2kl+5}^{\nu} = Z\rho_{l-1}^{\mu} - Z\rho_1^{\nu}\rho_{l-1}^{\mu} - \sum_{j=2}^{k-2} Z\rho_j^{\nu}\rho_{l-1}^{\mu} - 2Z\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}.
\end{aligned}
$$

$$
\begin{aligned}
& b_1^\mu = 0, b_2^\mu = 0, b_3^\mu = \rho_1^\mu - Z\rho_1^\mu, b_4^\mu = 0, ..b_{k+2}^\mu = 0, b_{k+3}^\mu = 0, ..b_{k+l+2}^\mu = 0, b_{k+l+3}^\mu = 0, \\
& b_{kl+4}^\mu = \rho_{k-1}^\nu - \sum_{j=1}^{l-2} \rho_{k-1}^\nu \rho_j^\mu - 2\rho_{k-1}^\nu \rho_{l-1}^\mu, b_{kl+5}^\mu = \rho_1^\nu \rho_1^\mu - Z\rho_1^\nu \rho_1^\mu, \\
& ..b_{kl+k+4}^\mu = \rho_{k-1}^\nu \rho_1^\mu - Z\rho_{k-1}^\nu \rho_1^\mu, b_{kl+k+5}^\mu = Z\rho_2^\mu - Z\rho_1^\mu, .. \\
& b_{kl+k+l+4}^\mu = Z + \rho_1^\mu - 2Z\rho_1^\mu - \sum_{j=2}^{l-2} Z\rho_j^\mu - 2Z\rho_{l-1}^\mu, \\
& b_{kl+k+l+5}^\mu = Z\rho_1^\nu \rho_2^\mu - Z\rho_1^\nu \rho_1^\mu, ..b_{2kl+6}^\mu = Z\rho_{k-1}^\nu - \rho_{k-1}^\nu \rho_1^\mu - 2Z\rho_{k-1}^\nu \rho_1^\mu - \sum_{j=2}^{l-2} Z\rho_{k-1}^\nu \rho_j^\mu - 2Z\rho_{k-1}^\nu \rho_{l-1}^\mu.
\end{aligned}
$$

The number of auxiliary zero-one stochastic variables is $2kl-1$.
While the original state vector consisting of X and $\dot{X}$ and governed by (4.2.2) is not a Markov process, the augmented state vector $\mathbf{X}$, governed by equation (4.2.7) driven by two independent Poisson processes N_ν and N_μ , is a non-diffusive Markov process.
By applying the Ito differential rule, the equations for the mean values and for 2nd, 3rd and 4th order moments are obtained as (cf. *Iwankiewicz and Nielsen (1999)*)

$$
\begin{aligned}
& \dot{m}_i(t) = E\left[c_i\right] + \sum_{\alpha=\nu,\mu} \alpha E\left[b_i^\alpha\right], \\
& \dot{\mu}_{ij}(t) = 2\left\{E\left[X_i\left(c_j + \sum_{\alpha=\nu,\mu} \alpha E\left[b_j^\alpha\right]\right)\right]\right\}_s + \sum_{\alpha=\nu,\mu} \alpha E\left[b_i^\alpha b_j^\alpha\right], \\
& \dot{\mu}_{ijk}(t) = 3\left\{E\left[X_i X_j\left(c_k + \sum_{\alpha=\nu,\mu} \alpha E\left[b_k^\alpha\right]\right)\right]\right\}_s + 3\sum_{\alpha=\nu,\mu} \alpha\left\{E\left[X_i b_j^\alpha b_k^\alpha\right]\right\}_s + \\
& \sum_{\alpha=\nu,\mu} \alpha E\left[b_i^\alpha b_j^\alpha b_k^\alpha\right], \\
& \dot{\mu}_{ijkl}(t) = 4\left\{E\left[X_i X_j X_k\left(c_l + 4\sum_{\alpha=\nu,\mu} \alpha E\left[b_l^\alpha\right]\right)\right]\right\}_s + 6\sum_{\alpha=\nu,\mu} \alpha\left\{E\left[X_i X_j b_k^\alpha b_l^\alpha\right]\right\}_s + \\
& 4\sum_{\alpha=\nu,\mu} \alpha\left\{E\left[X_i b_j^\alpha b_k^\alpha b_l^\alpha\right]\right\}_s + \sum_{\alpha=\nu,\mu} \alpha E\left[b_i^\alpha b_j^\alpha b_k^\alpha b_l^\alpha\right],
\end{aligned}
\tag{4.2.9}
$$

where $m_i(t) = E\left[X_i\right]$ and $\mu_{ij}(t) = E\left[X_i X_j\right]$.
The right-hand sides of the equations for moments (4.2.9) involve expectations of non-linear transformations of state variables respect to the unknown joint probability density function. For polynomial non-linearities, the equations for moments form an infinite hierarchy and cannot be directly solved. The unknown moments can only be evaluated approximately, using suitable closure approximations.
If the non-linearities are other than polynomial, the expectation of the non-linear transformations of the state variables cannot be expressed directly in terms of moments.

4.2.2 Modified closure scheme

As before (see section 4.1.2) the modified cumulant-neglect closure technique can be devised by expressing the joint probability density function of the state vector as the sum of discrete and continuous parts. Let us assume that the system is at rest at t=0.
In the discrete parts, Dirac delta functions represent the finite probability that the system is at rest before the occurrence of the first impulse, while the auxiliary

variables can take the values zero or one.
The continuous parts are expressed in terms of the conditional probabilities of the system vector, given that the first impulse has occurred.
Before the occurrence of the first impulse, the variable Z can be in its first 'off' state with probability $P^{(1)}_{Z,off}$ or in its first 'on' state with probability $P^{(1)}_{Z,on}$. Meanwhile the variables $Z_2 .. Z_N$ can be 'off' or 'on' with probabilities

$$\begin{aligned} P^{(1)}_{Z_i,off} &= P^{(1)}_{Z_i,off/Z,off} + P^{(1)}_{Z_i,off/Z,on}; \\ P^{(1)}_{Z_i,on} &= P^{(1)}_{Z_i,on/Z,off} + P^{(1)}_{Z_i,on/Z,on}. \end{aligned} \tag{4.2.10}$$

with $i = 2,...,N$.
After the occurrence of the first impulse the variable Z can be 'off' with probability $P^{\#}_{Z,off} = \sum_{j\geq 2} P^{(j)}_{Z,off}$ or 'on' with probability $P^{\#}_{Z,on} = \sum_{j\geq 2} P^{(j)}_{Z,on}$ .The variables $Z_2 .. Z_N$ can be 'off' or 'on' with probabilities

$$\begin{aligned} P^{\#}_{Z_i,off} &= \sum_{j\geq 2} P^{(j)}_{Z_i,off/Z,off} + \sum_{j\geq 2} P^{(j)}_{Z_i,off/Z,on}; \\ P^{\#}_{Z_i,on} &= \sum_{j\geq 2} P^{(j)}_{Z_i,on/Z,off} + \sum_{j\geq 2} P^{(j)}_{Z_i,on/Z,on}. \end{aligned} \tag{4.2.11}$$

The following equations governing the state probabilities of the auxiliary variables can be written

$$P^{(1)}_{Z_i,on} + P^{(1)}_{Z_i,off} = P_R \tag{4.2.12}$$

where P_R is the probability that no impulse has occurred, and

$$P^{(1)}_{Z_i,on} + P^{\#}_{Z_i,on} = E\left[Z_i\right] \tag{4.2.13}$$

with $i = 1,...,N$. It is also possible mutually relate the conditional probabilities of different auxiliary variables.
The conditional probabilities governing the 'on' and 'off' states of the auxiliary variables, before and after the occurrence of the first impulse, can always be found in closed form.
Let us express the unknown probability density of the augmented state vector $X = \begin{bmatrix} X & \dot{X} & Z & .. & Z_N \end{bmatrix}^T$ in terms of joint probabilities, as follows

$$p\left(x, \dot{x}, z, ..z_N\right) = \sum_j p^{(j)}\left(x, \dot{x}, z, ..z_N\right) \tag{4.2.14}$$

Due to the fact that there is a finite probability of the system being in a deterministic state from the initial time to the occurrence of the first impulse, the joint densities given that no impulse has occurred can be expressed as

$$
\begin{aligned}
&p^{(1)}\left(x,\dot{x},z,..z_N\right)=\bar{P}_1\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta\left(z\right)..\delta\left(z\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}\right),\\
&p^{(2)}\left(x,\dot{x},z,..z_N\right)=\bar{P}_2\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta\left(z\right)\delta\left(\rho_1^{\nu}-1\right)..\delta\left(z\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}\right),\\
&..\\
&p^{(kl)}\left(x,\dot{x},z,..z_N\right)=\bar{P}_{kl}\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta\left(z\right)..\delta\left(\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}-1\right)..\delta\left(z\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}\right),\\
&p^{(kl+1)}\left(x,\dot{x},z,..z_N\right)=\bar{P}_{kl+1}\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta\left(z-1\right)..\delta\left(z\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}\right),\\
&...\\
&p^{(2kl)}\left(x,\dot{x},z,..z_N\right)=\\
&\bar{P}_{2kl}\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta\left(z-1\right)..\delta\left(\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}-1\right)..\delta\left(z\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}-1\right).
\end{aligned}
$$

(4.2.15)

If the joint probabilities given that the first impulse has occurred are expressed as

$$
\begin{aligned}
&p^{(2kl+1)}\left(x,\dot{x},z,..z_N\right)=\bar{P}_{2kl+1}f^{(1)}\left(x,\dot{x}\right)\delta\left(z\right)..\delta\left(z\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}\right),\\
&p^{(2kl+2)}\left(x,\dot{x},z,..z_N\right)=\bar{P}_{2kl+2}f^{(2)}\left(x,\dot{x}\right)\delta\left(z\right)\delta\left(\rho_1^{\nu}-1\right)..\delta\left(z\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}\right),\\
&..\\
&p^{(3kl)}\left(x,\dot{x},z,..z_N\right)=\bar{P}_{3kl}f^{(kl)}\left(x,\dot{x}\right)\delta\left(z\right)..\delta\left(\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}-1\right)..\delta\left(z\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}\right),\\
&p^{(3kl+1)}\left(x,\dot{x},z,..z_N\right)=\bar{P}_{3kl+1}f^{(kl+1)}\left(x,\dot{x}\right)\delta\left(z-1\right)..\delta\left(z\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}\right),\\
&...\\
&p^{(4kl)}\left(x,\dot{x},z,..z_N\right)=\\
&\bar{P}_{4kl}f^{(2kl)}\left(x,\dot{x}\right)\delta\left(z-1\right)..\delta\left(\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}-1\right)..\delta\left(z\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}-1\right).
\end{aligned}
\tag{4.2.16}
$$

The joint probability density function has to satisfy the equation

$$
\begin{aligned}
&\int_{-\infty}^{\infty}p\left(x,\dot{x},z,..z_N\right)\left(dx\right)\left(d\dot{x}\right)\left(dz\right)=\\
&\sum_{j=1}^{2kl}\bar{P}_j+\sum_{j=1}^{2kl}\bar{P}_{2kl+j}\int_{-\infty}^{\infty}f^{(j)}\left(x,\dot{x}\right)\left(dx\right)\left(d\dot{x}\right)=1
\end{aligned}
\tag{4.2.17}
$$

By observing that

$$\sum_{j=1}^{4kl}\bar{P}_j=1 \tag{4.2.18}$$

It follows that

$$\int_{-\infty}^{\infty}f^{(j)}\left(x,\dot{x}\right)dxd\dot{x}=1 \qquad j=1,...2kl \tag{4.2.19}$$

It is possible to prove that the following identity holds

$$p\left(z_j\right)=\int_{-\infty}^{\infty} p\left(x,\dot{x},..z_N\right)dxd\dot{x}dz_1..dz_{j-1}dz_{j+1}..dz_N =$$
$$\left(\delta\left(z_j\right)\left(1-E\left[z_j\right]\right)+\delta\left(z_j-1\right)\left(E\left[z_j\right]\right)\right) \tag{4.2.20}$$
$$j=1,..,N$$

For j=1, for instance, equation (4.2.20) becomes

$$p\left(z\right)=\int_{-\infty}^{\infty} p\left(x,\dot{x},..z_N\right)dxd\dot{x}dz_2...dz_N =$$
$$\sum_{j=1}^{kl}\overline{P}_j\delta\left(z\right)+\sum_{j=kl+1}^{2kl}\overline{P}_j\delta\left(z-1\right)+\sum_{j=2kl}^{3kl}\overline{P}_j\delta\left(z\right)+\sum_{j=3kl+1}^{4kl}\overline{P}_j\delta\left(z-1\right) \tag{4.2.21}$$

Observing that

$$\sum_{j=1}^{kl}\overline{P}_j=P_{Z,off}^{(1)},\ \sum_{j=kl+1}^{2kl}\overline{P}_j=P_{Z,on}^{(1)},\ \sum_{j=2kl}^{3kl}\overline{P}_j=P_{Z,off}^{\#},\ \sum_{j=3kl+1}^{4kl}\overline{P}_j=P_{Z,on}^{\#}. \tag{4.2.22}$$

and from equations (4.2.12) and (4.2.13),it follows that

$$\sum_{j=1}^{kl}\overline{P}_j+\sum_{j=2kl}^{3kl}\overline{P}_j=1-E\left[Z\right];$$
$$\sum_{j=kl+1}^{2kl}\overline{P}_j+\sum_{j=3kl+1}^{4kl}\overline{P}_j=E\left[Z\right]. \tag{4.2.23}$$

Therefore the equation (4.2.21) becomes

$$p\left(z\right)=\left(1-E\left[z\right]\right)\delta\left(z\right)+\left(E\left[z\right]\right)\delta\left(z-1\right) \tag{4.2.24}$$

Let us consider the case of zero initial conditions. Since the auxiliary variables are zero-one processes, the following relationships hold

$$E\left[X^k\dot{X}^lZ_j^{\ m}\right]=E\left[X^k\dot{X}^lZ_j\right],\ j=1,..,N;$$
$$E\left[X^k\dot{X}^l\left(\rho_i^{\nu}\right)^m\left(\rho_j^{\nu}\right)^n\right]=\begin{matrix}E\left[X^k\dot{X}^l\rho_i^{\nu}\right],if\quad i=j;\\ 0,if\quad i\neq j;\end{matrix}$$
$$E\left[X^k\dot{X}^l\left(\rho_i^{\mu}\right)^m\left(\rho_j^{\mu}\right)^n\right]=\begin{matrix}E\left[X^k\dot{X}^l\rho_i^{\mu}\right],if\quad i=j;\\ 0,if\quad i\neq j;\end{matrix}$$

$$E\left[X^k \dot{X}^l Z^m \left(\rho_j^\nu\right)^n\right] = E\left[X^k \dot{X}^l Z \rho_j^\nu\right],\ j = 1,..,k-1;$$
$$E\left[X^k \dot{X}^l Z^m \left(\rho_j^\mu\right)^n\right] = E\left[X^k \dot{X}^l Z \rho_j^\mu\right],\ j = 1,..,l-1; \tag{4.2.25}$$
$$E\left[X^k \dot{X}^l Z^m \left(\rho_i^\nu\right)^n \left(\rho_j^\mu\right)^o\right] = E\left[X^k \dot{X}^l Z \rho_i^\nu \rho_j^\mu\right], i = 1,..,k-1,\ j = 1,..,l-1;$$

The unconditional moment of order $p = m + n$ involving displacements and velocity can be expressed in terms of the conditional moments of the same order as follows

$$E[X^m \dot{X}^n] = \sum_{j=1}^{2kl} \bar{P}_{j+2kl} E^{(j)}[X^m \dot{X}^n] \tag{4.2.26}$$

The unconditional moment of order $p+1 = m+n+1$ involving also the auxiliary variables can be expressed in terms of the conditional moments of order p as follows

$$E[X^m \dot{X}^n Z] = \sum_{j=kl+1}^{2kl} \bar{P}_{j+2kl} E^{(j)}[X^m \dot{X}^n],$$
$$E[X^m \dot{X}^n \rho_1^\nu] = \bar{P}_{2kl+2} E^{(2)}[X^m \dot{X}^n] + \bar{P}_{3kl+2} E^{(kl+1)}[X^m \dot{X}^n], \tag{4.2.27}$$
$$..$$
$$E[X^m \dot{X}^n Z \rho_{k-1}^\nu \rho_{l-1}^\mu] = \bar{P}_{4kl} E^{(2kl)}[X^m \dot{X}^n].$$

The following relationships between the conditional moments of order p=m+n and the unconditional ones can be derived:

$$E^{(1)}\left[X^m \dot{X}^n\right] = \frac{E\left[X^m \dot{X}^n\right] - \sum_{j=1}^{N} E\left[X^m \dot{X}^n Z_j\right]}{\bar{P}_{2kl}};$$
$$E^{(2)}\left[X^m \dot{X}^n\right] = \frac{E\left[X^m \dot{X}^n \rho_1^\nu\right] - E\left[X^m \dot{X}^n Z \rho_1^\nu\right]}{\bar{P}_{2kl+2}};$$
$$..$$
$$E^{(kl+1)}\left[X^m \dot{X}^n\right] = \frac{E\left[X^m \dot{X}^n \rho_{k-1}^\nu \rho_{l-1}^\mu\right] - E\left[X^m \dot{X}^n Z \rho_{k-1}^\nu \rho_{l-1}^\mu\right]}{\bar{P}_{3kl+1}}; \tag{4.2.28}$$
$$E^{(kl+2)}\left[X^m \dot{X}^n\right] = \frac{E\left[X^m \dot{X}^n Z \rho_1^\nu\right]}{\bar{P}_{3kl+2}};$$
$$..$$
$$E^{(2kl)}\left[X^m \dot{X}^n\right] = \frac{E\left[X^m \dot{X}^n Z \rho_{k-1}^\nu \rho_{l-1}^\mu\right]}{\bar{P}_{4kl}};$$

The Gram-Charlier expansion can be applied to the conditional density functions $f^{(1)}(x,\dot{x})$ .. $f^{(2kl)}(x,\dot{x})$ that can be viewed as probability densities of a bi-dimensional stochastic variable.

A modified closure scheme can be constructed by performing the ordinary cumulant neglect closure on each set of conditional moments. The conditional moments of order s higher than the closure order r are expressed in terms of conditional moments of order lower than r through the following relationships

$$E^{(k)}[X_1....X_s]=\sum_{\pi^r}\prod_{B^r\in\pi^r}\kappa\left(\underset{i\in B^r}{X_i}\right)=$$

$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(E^{(k)}\left[\prod_{j\in B^i}X_j\right]\right)\right]\right\}, \tag{4.2.29}$$

$k=1,...2kl$

The conditional moments of order up to r appearing at the right hand side of equation (4.2.29) can be expressed in terms of unconditional ones through equation (4.2.28) as follows

$$E^{(1)}[X_1....X_s]=$$

$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-\sum_{q=1}^{N}E\left[\prod_{j\in B^i}X_jZ_q\right]}{\bar{P}_{2kl}}\right)\right]\right\},$$

..

$$E^{(2kl)}[X_1....X_s]=$$

$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_jZ\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}\right]}{\bar{P}_{4kl}}\right)\right]\right\}.$$

(4.2.30)

From equations (4.2.26) and (4.2.27) the unconditional moments of order s>r containing displacement and velocity only and the unconditional moments of order s+1 containing each of the auxiliary variables, are expressed in terms of the unconditional moments of order up to the rth through the following relationships

$$E\left[X_1....X_s\right]=$$

$$\bar{P}_{2kl}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-\sum_{jj=1}^{N}E\left[\prod_{j\in B^i}X_jZ_{jj}\right]}{\bar{P}_{2kl}}\right)\right]\right\}+$$

..

$$\bar{P}_{4kl}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_jZ\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}\right]}{\bar{P}_{4kl}}\right)\right]\right\},$$

..

$$E\left[X_1....X_nZ\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}\right]=$$

$$\bar{P}_{4kl}\sum_{\pi^N}\prod_{B^N\in\pi^{*N}}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^N}}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_jZ\rho_{k-1}^{\nu}\rho_{l-1}^{\mu}\right]}{\bar{P}_{4kl}}\right)\right]\right\}. \qquad (4.2.31)$$

Where π^r runs through the list of the partitions of $\{1,2,...s\}$ in blocks of maximum dimension r , B^r runs through the list of blocks of the partition π^r .

For cubic non-linearity, performing a cumulant neglect closure of order r on the 2kl sets of conditional moments, the conditional moments of the order r+1 and r+2 involving displacement and velocity only and the moments of order r+2 and r+3 involving also each of the auxiliary variables are expressed in terms of moments up to the order r.

4.2.3 Process I: (Generalised Erlang renewal process) $R_\mu(t)=P(\mu)$ and $R_\nu(t)=P(\nu)$

Consider the response of a Duffing oscillator $f\left(X,\dot{X}\right)=-2\zeta\omega\dot{X}-\omega^2X-\varepsilon X^3$ to the random train of impulse $R(t)$, defined by equation (4.2.5) where the stochastic process $Z(t)$ in (4.2.4) is obtained by choosing $R_\mu(t)=P(\mu)$ (a Poisson process with parameter μ) and $R_\nu(t)=P(\nu)$ (a Poisson process with parameter ν).

The stochastic equation of motion (4.2.7) is specified by

$$\mathbf{X}(t)=\begin{bmatrix}X(t)\\ \dot{X}(t)\\ Z(t)\end{bmatrix}=\begin{bmatrix}X_1(t)\\ X_2(t)\\ X_3(t)\end{bmatrix};$$

$$
\mathbf{c}(\mathbf{X}(t)) = \begin{bmatrix} X_2(t) \\ -\omega^2 X_1(t) - 2\zeta\omega X_2(t) - \varepsilon\omega^2 X_1^{\,3}(t) \\ 0 \end{bmatrix};
$$
$$
\mathbf{b}(P(t), \mathbf{X}(t)) = \begin{bmatrix} b^\mu & b^\nu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & P(t) X_3(t) \\ 1 - X_3(t) & -X_3(t) \end{bmatrix}; \tag{4.2.32}
$$

Before the occurrence of the first impulse, the variable Z can be in its first 'off' state (with probability $P^{(1)}_{Z,off}$) or in its first 'on' state (with probability $P^{(1)}_{Z,on}$). After the first impulse occurrence, the auxiliary variable can be 'on' or 'off' with probabilities $P^{\#}_{Z,on} = \sum_{j\geq 2} P^{(j)}_{Z,on}$, $P^{\#}_{Z,off} = \sum_{j\geq 2} P^{(j)}_{Z,off}$, respectively (see Fig.4.2.1).
For the variable Z the following equations governing the state probabilities can be written:

$$
\begin{aligned}
&P^{(1)}_{z,off} + P^{(1)}_{z,on} = P_R ; P^{\#}_{z,off} + P^{\#}_{z,on} = 1 - P_R \\
&P^{(1)}_{z,off} + P^{\#}_{z,off} = 1 - E[Z]; P^{(1)}_{z,on} + P^{\#}_{z,on} = E[Z]
\end{aligned} \tag{4.2.33}
$$

with

$$
\begin{aligned}
&P^{(1)}_{z,off} = P_{N_\mu} ; P^{(1)}_{z,on} = P_R - P_{N_\mu} ; \\
&P^{\#}_{z,on} = E[Z] - P^{(1)}_{z,on} ; P^{\#}_{z,off} = 1 - E[Z] - P^{(1)}_{z,off} .
\end{aligned} \tag{4.2.34}
$$

where $P_{N_\mu} = e^{-\mu t}$ is the probability that the first N_μ driven event has not occurred, $P_R = (1 - F_{w_1})$, with $F_{w_1} = \int_0^t f_{w_1}(x)\,dx$, is the probability that the first event driven by the process $R(t)$ has not occurred.

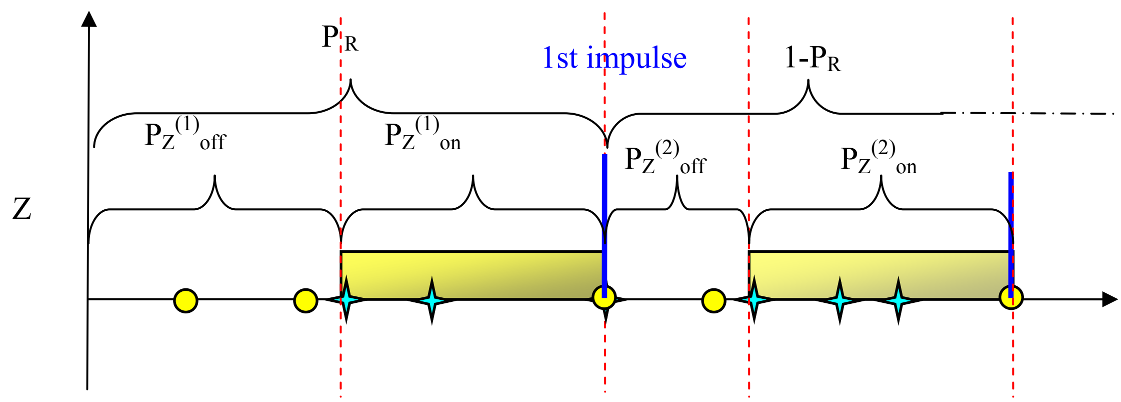

Figure 4.2.1
Sample function of the train of impulses driven by the non-Erlang renewal process $R(t)$

Let us express the unknown probability density of the state vector $\mathbf{X}$ in terms of the conditional probabilities

- $$p^{(1)}(x,\dot{x},z)\,dxd\dot{x} = \Pr\left\{X \in (x, x+dx) \wedge \dot{X} \in (\dot{x}, \dot{x}+d\dot{x})\right\} \wedge Z\ is\ in\ its\ first\ 'off'\ phase,$$
  given that no impulses have occurred and the variable Z is in its first "off" state.

- $$p^{(2)}(x,\dot{x},z)\,dxd\dot{x} = \Pr\left\{X \in (x, x+dx) \wedge \dot{X} \in (\dot{x}, \dot{x}+d\dot{x})\right\} \wedge Z\ is\ in\ its\ first\ 'on'\ phase$$
  given that no impulses have occurred and the variable Z is in its first "on" state.

- $$p^{(3)}(x,\dot{x},z)\,dxd\dot{x} = \Pr\left\{X \in (x, x+dx) \wedge \dot{X} \in (\dot{x}, \dot{x}+d\dot{x})\right\} \wedge Z\ is\ in\ any\ 'off'\ phase\ following\ the\ first$$
  given that no impulses have occurred and the variable Z is 'off'" .

- $$p^{(4)}(x,\dot{x},z)\,dxd\dot{x} = \Pr\left\{X \in (x, x+dx) \wedge \dot{X} \in (\dot{x}, \dot{x}+d\dot{x})\right\} \wedge Z\ is\ in\ any\ 'on'\ phase\ following\ the\ first$$
  given that no impulses have occurred and the variable Z is "on".

The joint probability density function can be expressed as

$$p(x,\dot{x},z) = \sum_{j=1}^{4} p^{(j)}(x,\dot{x},z) \tag{4.2.35}$$

Due to the fact that there is a finite probability of the system being in a deterministic state from the initial time to the first impulse, the terms $p^{(1)}(x,\dot{x},z)$ and $p^{(2)}(x,\dot{x},z)$ that contain the conditional probability of the state vector given that no impulses have occurred, can be respectively expressed as

$$\begin{aligned} p^{(1)}(x,\dot{x},z) &= P^{(1)}_{z,off}\,\delta(x-x_0)\,\delta(\dot{x}-\dot{x}_0)\,\delta(z), \\ p^{(2)}(x,\dot{x},z) &= P^{(1)}_{z,on}\,\delta(x-x_0)\,\delta(\dot{x}-\dot{x}_0)\,\delta(z-1). \end{aligned} \tag{4.2.36}$$

Let us express the terms $p^{(3)}(x,\dot{x},z)$ and $p^{(4)}(x,\dot{x},z)$ containing the conditional probabilities of the state vector given that the first impulse has occurred as follows

$$\begin{aligned} p^{(3)}(x,\dot{x},z) &= P^{\#}_{z,off}\, f^{(1)}(x,\dot{x})\,\delta(z) \\ p^{(4)}(x,\dot{x},z) &= P^{\#}_{z,on}\, f^{(2)}(x,\dot{x})\,\delta(z-1) \end{aligned} \tag{4.2.37}$$

Considering that the joint probability density function has to satisfy the equation

$$\int_{-\infty}^{\infty} p\left(x,\dot{x},z\right)dxd\dot{x}dz = P_{Z,off}^{(1)} + P_{Z,on}^{(1)} + P_{Z,off}^{(\#)} \int_{-\infty}^{\infty} f^{(1)}\left(x,\dot{x}\right)dxd\dot{x} + P_{Z,on}^{(\#)} \int_{-\infty}^{\infty} f^{(2)}\left(x,\dot{x}\right)dxd\dot{x} = 1 \tag{4.2.38}$$

from equations (4.2.34) it follows that the functions $f^{(1)}\left(x,\dot{x}\right)$ and $f^{(2)}\left(x,\dot{x}\right)$ have to satisfy the following relationships

$$\int_{-\infty}^{\infty} f^{(1)}\left(x,\dot{x}\right)dxd\dot{x} = 1, \int_{-\infty}^{\infty} f^{(2)}\left(x,\dot{x}\right)dxd\dot{x} = 1 \tag{4.2.39}$$

By integrating the joint probability density function (4.2.35), together with (4.2.36) and (4.2.37), with respect to the variables displacement and velocity, the marginal probability density of the auxiliary variable Z is derived as follows

$$\begin{aligned} p\left(Z\right) &= \int_{-\infty}^{\infty} p\left(x,\dot{x},z\right)\left(dx\right)\left(d\dot{x}\right) = \\ & P_{Z,off}^{(1)}\delta\left(z\right) + P_{Z,on}^{(1)}\delta\left(z-1\right) + P_{Z,off}^{(\#)}\delta\left(z\right) + P_{Z,on}^{(\#)}\delta\left(z-1\right) = \\ & \delta\left(z\right)\left(1-E\left[z\right]\right) + \delta\left(z-1\right)\left(E\left[z\right]\right) \end{aligned} \tag{4.2.40}$$

which is the known expression of the probability density of a zero-one stochastic variable.
Let us consider the case of zero initial conditions. Since the variable Z is a zero-one process, the following relationships hold

$$E\left[X^k \dot{X}^l Z^m\right] = E\left[X^k \dot{X}^l Z\right] \tag{4.2.41}$$

The unconditional moment of order $p = m + n$ involving displacements and velocity only can be expressed in terms of the conditional moments of the same order as follows

$$E[X^m \dot{X}^n] = P_{z,off}^{\#} E^{(1)}[X^m \dot{X}^n] + P_{z,on}^{\#} E^{(2)}[X^m \dot{X}^n]. \tag{4.2.42}$$

The unconditional moment of order $p + 1 = m + n + 1$ involving also the auxiliary variable can be expressed in terms of the conditional moments of order p as follows

$$E[X^m \dot{X}^n Z] = P_{z,on}^{\#} E^{(2)}[X^m \dot{X}^n]. \tag{4.2.43}$$

The following relationships between the conditional moments of order p=m+n to the unconditional ones can be derived:

$$E^{(1)}\left[X^m \dot{X}^n\right] = \frac{E\left[X^m \dot{X}^n\right] - E\left[X^m \dot{X}^n Z\right]}{P^{\#}_{z,off}};$$
$$E^{(2)}\left[X^m \dot{X}^n\right] = \frac{E\left[X^m \dot{X}^n Z\right]}{P^{\#}_{z,on}}; \tag{4.2.44}$$

Let us perform the ordinary cumulant neglect closure on the two sets of conditional moments. The conditional moments of order s higher than the closure order r are expressed in terms of conditional moments of order lower than r through the following relationships

$$E^{(k)}\left[X_1....X_s\right] = \sum_{\pi^r} \prod_{B^r \in \pi^r} \kappa\left(\underset{i\in B^r}{X_i}\right) =$$
$$\sum_{\pi^r} \prod_{B^r \in \pi^r} \left\{ \sum_{\pi^i} \prod_{\substack{B^i \in \pi^i \\ i\in B^r}} \left[\left(\left|B^i\right| - 1\right)! (-1)^{\left|B^i\right|-1} \left(E^{(k)}\left[\prod_{j\in B^i} X_j \right] \right) \right] \right\}, \tag{4.2.45}$$
$$k = 1, 2$$

The conditional moments of order up to r appearing at the right hand side of equation (4.2.45) can be expressed in terms of unconditional ones through equation (4.2.44) as follows

$$E^{(1)}\left[X_1....X_s\right] =$$
$$\sum_{\pi^r} \prod_{B^r \in \pi^r} \left\{ \sum_{\pi^i} \prod_{\substack{B^i \in \pi^i \\ i\in B^r}} \left[\left(\left|B^i\right| - 1\right)! (-1)^{\left|B^i\right|-1} \left(\frac{E\left[\prod_{j\in B^i} X_j \right] - E\left[\prod_{j\in B^i} X_j Z \right]}{P^{\#}_{z,off}} \right) \right] \right\},$$
$$E^{(2)}\left[X_1....X_s\right] = \tag{4.2.46}$$
$$\sum_{\pi^r} \prod_{B^r \in \pi^r} \left\{ \sum_{\pi^i} \prod_{\substack{B^i \in \pi^i \\ i\in B^r}} \left[\left(\left|B^i\right| - 1\right)! (-1)^{\left|B^i\right|-1} \left(\frac{E\left[\prod_{j\in B^i} X_j Z \right]}{P^{\#}_{z,on}} \right) \right] \right\}.$$

From equations (4.2.42) and (4.2.43) the unconditional moments of order s>r containing displacement and velocity only and the unconditional moments of order s+1 containing each of the auxiliary variables, are expressed in terms of the unconditional moments of order up to the rth through the following relationships

$$E\left[X_1....X_s\right]=$$

$$P^{\#}_{z,off}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_jZ\right]}{P^{\#}_{z,off}}\right)\right]\right\}+$$

$$P^{\#}_{z,on}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_jZ\right]}{P^{\#}_{z,on}}\right)\right]\right\},$$

$$E\left[X_1....X_sZ\right]=P^{\#}_{z,on}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_jZ\right]}{P^{\#}_{z,on}}\right)\right]\right\}. \qquad (4.2.47)$$

Where π^r runs through the list of the partitions of $\{1,2,...s\}$ in blocks of maximum dimension r, B^r runs through the list of blocks of the partition π^r.

4.2.4 Process II: $R_\mu(t)=E(\mu,l)$ and $R_\nu(t)=P(\nu)$

Consider the response of a Duffing oscillator $f\left(X,\dot{X}\right)=-2\zeta\omega\dot{X}-\omega^2X-\varepsilon X^3$ to the random train of impulses $R(t)$, derived from equations (4.2.4) and (4.2.5) with $R_\mu(t)=E(\mu,l)$ (an Erlang process with parameters μ and l) and $R_\nu(t)=P(\nu)$ (a Poisson process with parameter ν).
Let us assume that the Erlang renewal process $R_\mu(t)$ is defined by the parameters μ and $l=2$. Equations (4.2.5) and (4.2.6) become

$$\begin{aligned} dR_\mu(t)&=\rho_\mu(t)dN_\mu(t)\\ d\rho_\mu(t)&=\left(1-2\rho_\mu(t)\right)dN_\mu(t)\end{aligned} \qquad (4.2.48)$$

The stochastic equation of motion (4.2.7) is specified by

$$\mathbf{X}(t)=\begin{bmatrix} X(t)\\ \dot{X}(t)\\ Z(t)\\ \rho_\mu(t)\\ Z(t)\rho_\mu(t)\end{bmatrix}=\begin{bmatrix} X_1(t)\\ X_2(t)\\ X_3(t)\\ X_4(t)\\ X_5(t)\end{bmatrix};$$

$$\mathbf{c}(\mathbf{X}(t)) = \begin{bmatrix} X_2(t) \\ -\omega^2 X_1(t) - 2\zeta\omega X_2(t) - \varepsilon\omega^2 X_1^3(t) \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$\mathbf{b}(P(t),\mathbf{X}(t)) = \begin{bmatrix} b^{\mu} & b^{\nu} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & P(t)X_3(t) \\ Z_4(t) - X_5(t) & -X_3(t) \\ 1-2X_4(t) & 0 \\ X_3(t) - 2X_5(t) & -X_5(t) \end{bmatrix}; \tag{4.2.49}$$

Before the occurrence of the first impulse, the variable Z can be

- in its first 'off' state with probability $P^{(1)}_{Z,off}$ (while the variable ρ_μ can be in its first 'off' phase or in its first 'on' phase with probabilities, respectively $P^{(1)}_{\rho_\mu,off/Z,off}$ and $P^{(1)}_{\rho_\mu,on/Z,off}$ and the variable $Z\rho_\mu$ is in its first 'off' phase with probability $P^{(1)}_{Z\rho_\mu,off/Z,off}$)

or

- or in its first 'on' state with probability $P^{(1)}_{Z,on}$ (while the variable ρ_μ can be 'off' or 'on' with probabilities, respectively $P^{(1)}_{\rho_\mu,off/Z,on}$ and $P^{(1)}_{\rho_\mu,on/Z,on}$ and the variable $Z\rho_\mu$ can be 'off' or 'on' with probabilities $P^{(1)}_{Z\rho_\mu,off/Z,on}$ and $P^{(1)}_{Z\rho_\mu,on/Z,on}$).

After the first impulse occurrence, the auxiliary variables can be 'on' or 'off' with probabilities (see Figure 4.2.2)

$$\begin{aligned} &P^{\#}_{Z,on} = \sum_{j\geq 2} P^{(j)}_{Z,on}; P^{\#}_{\rho_\mu,on} = \sum_{j\geq 2} P^{(j)}_{\rho_\mu,on} = \sum_{j\geq 2}\left(P^{(j)}_{\rho_\mu,on/Z,off} + P^{(j)}_{\rho_\mu,on/Z,on}\right); \\ &P^{\#}_{Z\rho_\mu,on} = \sum_{j\geq 2} P^{(j)}_{Z\rho_\mu,on} = \sum_{j\geq 2}\left(P^{(j)}_{Z\rho_\mu,on/Z,off} + P^{(j)}_{Z\rho_\mu,on/Z,on}\right) \end{aligned} \tag{4.2.50}$$

The state probabilities for the variable Z satisfy the following equations:

$$\begin{aligned} &P^1_{z,off} + P^1_{z,on} = P_R \\ &P^{\#}_{z,off} + P^{\#}_{z,on} = 1 - P_R \\ &P^1_{z,off} + P^{\#}_{z,off} = 1 - E[Z] \\ &P^1_{z,on} + P^{\#}_{z,on} = E[Z] \end{aligned} \tag{4.2.51}$$

with

$$
\begin{aligned}
&P^{1}_{z,off} = P_{R_\mu} \\
&P^{1}_{z,on} = P_R - P_{R_\mu}, \\
&P^{\#}_{z,on} = E[Z] - P_R + P_{R_\mu}, \\
&P^{\#}_{z,off} = 1 - E[Z] - P_{R_\mu}.
\end{aligned}
\tag{4.2.52}
$$

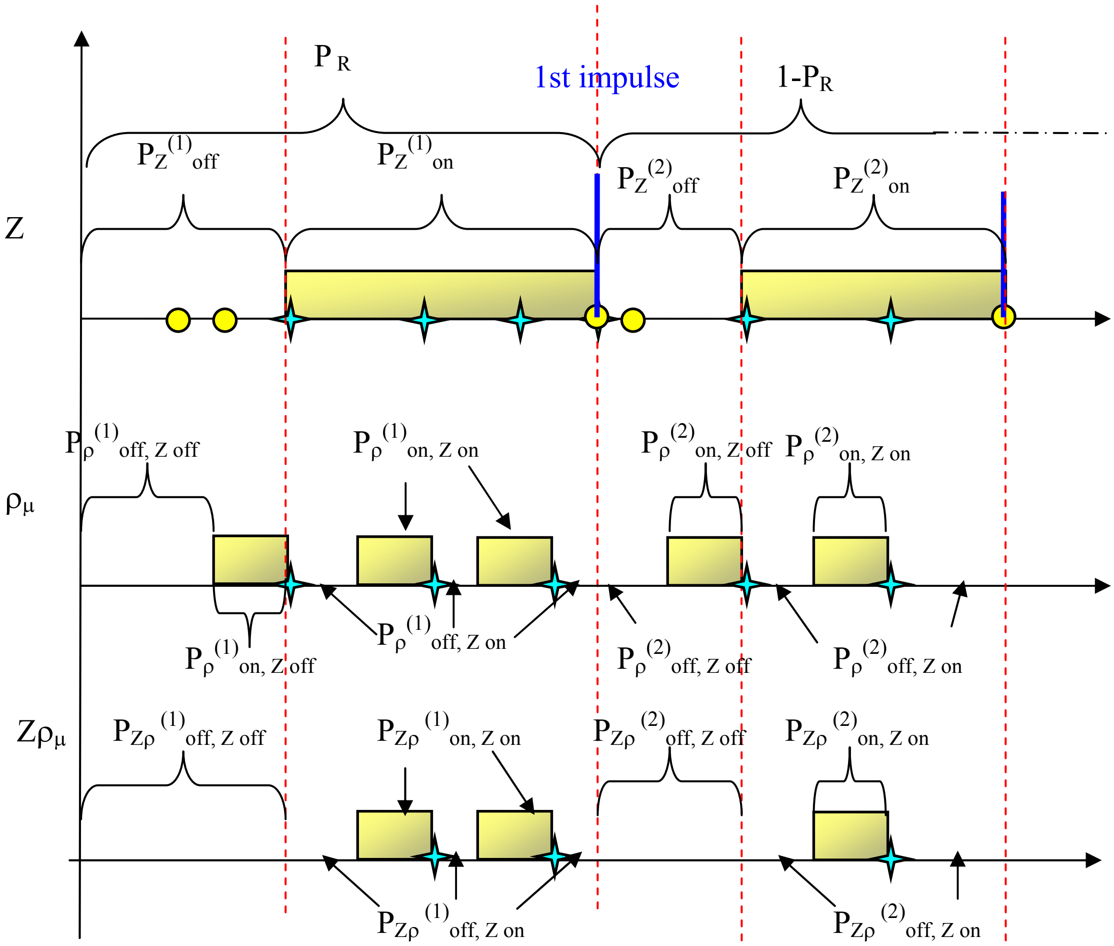

Figure 4.2.2
Sample functions of the train of impulses driven by the non-Erlang renewal process R, and the zero-one processes Z, ρ_μ and $Z\rho_\mu$ appearing in the stochastic equation (4.2.49).

where $P_{N_\mu} = e^{-\mu t}$ is the probability that the first N_μ driven event has not occurred, $P_{R_\mu} = (1+\mu t)te^{-\mu t}$ is the probability that the first R_μ driven event has not occurred and $P_R = 1 - F_{w_1}$, with $F_{w_1} = \int_0^t f_{w_1}(x)\,dx$, is the probability that no impulse driven by $R(t)$ has occurred.

The equations governing the state probabilities of the variable ρ_μ are:

$$P^{(1)}_{\rho_\mu,off/Z,off} + P^{(1)}_{\rho_\mu,off/Z,on} + P^{\#}_{\rho_\mu,off/Z,off} + P^{\#}_{\rho_\mu,off/Z,on} = 1 - E[\rho_\mu]$$
$$P^{(1)}_{\rho_\mu,on/Z,off} + P^{(1)}_{\rho_\mu,on/Z,on} + P^{\#}_{\rho_\mu,on/Z,off} + P^{\#}_{\rho_\mu,on/Z,on} = E[\rho_\mu] \qquad (4.2.53)$$
$$P^{(1)}_{\rho_\mu,off/Z,off} + P^{(1)}_{\rho_\mu,off/Z,on} + P^{(1)}_{\rho_\mu,on/Z,off} + P^{(1)}_{\rho_\mu,on/Z,on} = P_R$$

where

$$P^{(1)}_{\rho_\mu,off/Z,off} = P_{N_\mu}; P^{(1)}_{\rho_\mu,on/Z,off} = P_{R_\mu} - P_{N_\mu};$$
$$P^{(1)}_{\rho_\mu,off/Z,on} = P_1; P^{(1)}_{\rho_\mu,on/Z,on} = P_2;$$
$$P^{\#}_{\rho_\mu,off/Z,off} = P_3; P^{\#}_{\rho_\mu,on/Z,off} = P_4; \qquad (4.2.54)$$
$$P^{\#}_{\rho_\mu,off/Z,on} = P_5; P^{\#}_{\rho_\mu,on/Z,on} = P_6.$$

The equations governing the state probabilities of the variable $Z\rho_\mu$ can be written as

$$P^{(1)}_{Z\rho_\mu,off/Z,off} + P^{(1)}_{Z\rho_\mu,off/Z,on} + P^{\#}_{Z\rho_\mu,off/Z,off} + P^{\#}_{Z\rho_\mu,off/Z,on} = 1 - E[Z\rho]$$
$$P^{(1)}_{Z\rho_\mu,on/Z,on} + P^{\#}_{Z\rho_\mu,on/Z,on} = E[Z\rho] \qquad (4.2.55)$$
$$P^{(1)}_{Z\rho_\mu,off/Z,off} + P^{(1)}_{Z\rho_\mu,on/Z,on} + P^{(1)}_{Z\rho_\mu,off/Z,on} = P_R$$

where

$$P^{(1)}_{Z\rho_\mu,off/Z,off} = P_{R_\mu}, P^{(1)}_{Z\rho_\mu,off/Z,on} = P^{(1)}_{\rho_\mu,off/Z,on} = P_1, P^{(1)}_{Z\rho_\mu,on/Z,on} = P^{(1)}_{\rho_\mu,on/Z,on} = P_2,$$
$$P^{\#}_{Z\rho_\mu,off/Z,off} = P^{\#}_{Z,off} = 1 - E\left[Z\right] - P_{R_\mu}, P^{\#}_{Z\rho_\mu,off/Z,on} = P^{\#}_{\rho_\mu,off/Z,on} = P_5,$$
$$P^{\#}_{Z\rho_\mu,on/Z,on} = P^{\#}_{\rho_\mu,on/Z,on} = P_6.$$
(4.2.56)

The following equations show the relationships between the state probabilities of the variable Z and the state probabilities of the variable ρ_μ

$$P^{(1)}_{\rho_\mu,off/Z,off} + P^{(1)}_{\rho_\mu,on/Z,off} + P^{\#}_{\rho_\mu,off/Z,off} + P^{\#}_{\rho_\mu,on/Z,off} = 1 - E[Z]$$
$$P^{(1)}_{\rho_\mu,on/Z,on} + P^{(1)}_{\rho_\mu,off/Z,on} + P^{\#}_{\rho_\mu,on/Z,on} + P^{\#}_{\rho_\mu,off/Z,on} = E[Z] \qquad (4.2.57)$$

It can be proved that equations (4.2.53), (4.2.55) and (4.2.57) lead to the linear system of equations in the unknowns P_1,..P_6. The state probabilities P_3 and P_4 can be resolved as

$$P_3 = 1 - E[Z] + E[Z\rho] - E[\rho] - P_{N_\mu},$$
$$P_4 = -E[Z\rho] + E[\rho] - P_{R_\mu} + P_{N_\mu}. \qquad (4.2.58)$$

the remaining unknowns can be expressed in terms of P_1 as follows

$$P_2 = P_R - P_{R_\mu} - P_1,$$
$$P_5 = E[Z] - E[Z\rho] - P_1, \qquad (4.2.59)$$
$$P_6 = E[Z\rho] + P_{R_\mu} - P_R + P_1.$$

State probability $P_1(t)$

The probability density P_1 that the variable ρ is 'off' during Z first 'on' phase can be expressed as

$$P_1(t)dt =$$
$$\sum_{u=0}^{t} \Pr\{T_\mu \in (u, u+du) \wedge N_\mu(t-u) = 0,2,4,..\} \wedge T_\nu > t +$$
$$\sum_{u=0}^{t} \Pr\{T_\mu \in (u, u+du) \wedge N_\mu(t-u) = 0,2,4,..\} \wedge \sum_{\xi=0}^{\mu} dR_\nu \in (\xi, \xi + d\xi) \wedge T_\nu > t - \xi$$
$$(4.2.60)$$

where the terms at the right hand side account for the probability that the variable Z is 'on', that is the time variable t is between the first R_μ event and the subsequent N_ν event and simultaneously between an even-number and the subsequent odd-number Poisson-driven event with parameter μ.
The probability density P_1 can be expressed as

$$P_1(t) = \int_0^t g_{T_\mu}(u)\frac{1}{2}\left(1+e^{-2\mu(t-u)}\right)du\left(1-F_{T_\nu}(t)\right)du +$$
$$\int_0^t g_{T_\mu}(u)\frac{1}{2}\left(1+e^{-2\mu(t-u)}\right)\int_0^u h_\nu(\xi)\left(1-F_{T_\nu}(t-\xi)\right)d\xi du; \qquad (4.2.61)$$

The variables appearing at the right hand side of the equation above are defined in Figure 4.2.3.

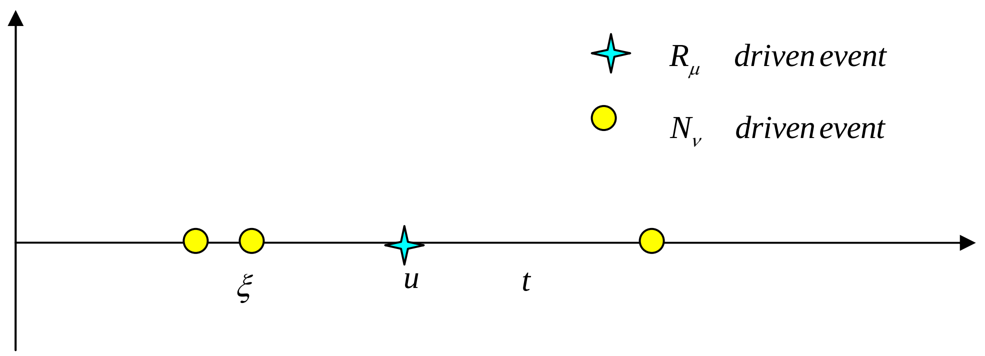

Figure 4.2.3
Definition of the variables appearing in the expression of the probability function $P_1(t)$

Let us express the unknown probability density of the state vector $\mathbf{X}$ in terms of the conditional probabilities

- $$p^{(1)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)dxd\dot{x} = \Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$
  $$\wedge Z\ is\ in\ its\ first\ 'off'\ phase \wedge \rho_\mu\ is\ 'off' \wedge Z\rho_\mu\ is\ 'off'$$
  given that no impulses have occurred and the variables ρ_μ is 'off' while Z and $Z\rho_\mu$ are in their first "off" state.

- $$p^{(2)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)dxd\dot{x} = \Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$
  $$\wedge Z\ is\ in\ its\ first\ 'off'\ phase \wedge \rho_\mu\ is\ in\ its\ first\ 'on'\ phase \wedge Z\rho_\mu\ is\ 'off'$$
  given that no impulses have occurred and the variable ρ_μ is in its first 'on' phase while Z and $Z\rho_\mu$ are still in their first "off" state.

- $$p^{(3)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)dxd\dot{x} = \Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$
  $$\wedge Z\ is\ in\ its\ first\ 'on'\ phase \wedge \rho_\mu\ is\ 'off' \wedge Z\rho_\mu\ is\ 'off'$$
  given that no impulses have occurred and the variable Z is in its first on state while ρ_μ and $Z\rho_\mu$ are "off".

- $$p^{(4)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)dxd\dot{x} = \Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$
  $$\wedge Z\ is\ in\ its\ first\ 'on'\ phase \wedge \rho_\mu\ is\ 'on' \wedge Z\rho_\mu\ is\ 'on'$$
  given that no impulses have occurred and the variables Z, ρ_μ and $Z\rho_\mu$ are simultaneously "on".

- $$p^{(5)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)dxd\dot{x} = \Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$
  $$\wedge Z\ is\ in\ any\ 'off'\ phase\ following\ the\ first\ impulse$$
  $$\wedge \rho_\mu\ is\ in\ any\ 'off'\ phase\ following\ the\ first\ impulse$$
  $$\wedge Z\rho_\mu\ is\ in\ any\ 'off'\ phase\ following\ the\ first\ impulse$$
  given that the first impulse has occurred and the three variables are simultaneously 'off'.

$$p^{(6)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)dxd\dot{x} =$$
$$\Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$

- $\wedge Z\ is\ in\ any\ 'off'\ phase\ following\ the\ first\ impulse$
  $\wedge \rho_\mu\ is\ in\ any\ 'on'\ phase\ following\ the\ first\ impulse$
  $\wedge Z\rho_\mu\ is\ in\ any\ 'off'\ phase\ following\ the\ first\ impulse$
  given that the first impulse has occurred and ρ_μ is 'on' while Z and $Z\rho_\mu$ are "off".

$$p^{(7)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)dxd\dot{x} =$$
$$\Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$

- $\wedge Z\ is\ in\ any\ 'on'\ phase\ following\ the\ first\ impulse$
  $\wedge \rho_\mu\ is\ in\ any\ 'off'\ phase\ following\ the\ first\ impulse$
  $\wedge Z\rho_\mu\ is\ in\ any\ 'off'\ phase\ following\ the\ first\ impulse$
  given that the first impulse has occurred and Z is in an 'on' state while ρ_μ and $Z\rho_\mu$ are "off".

$$p^{(8)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)dxd\dot{x} =$$
$$\Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$

- $\wedge Z\ is\ in\ any\ 'on'\ phase\ following\ the\ first\ impulse$
  $\wedge \rho_\mu\ is\ in\ any\ 'on'\ phase\ following\ the\ first\ impulse$
  $\wedge Z\rho_\mu\ is\ in\ any\ 'on'\ phase\ following\ the\ first\ impulse$
  given that the first impulse has occurred and the variables Z, ρ_μ and $Z\rho_\mu$ are simultaneously "on".

Hence, the joint probability density function can be expressed as

$$p\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right) = \sum_{j=1}^{8} p^{(j)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right) \tag{3.2.62}$$

Due to the fact that there is a finite probability of the system being in a deterministic state from the initial time to the first impulse, the terms $p^{(1)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)$ to $p^{(4)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)$ containing the conditional probability densities given that no impulses have occurred, can be respectively expressed as

$$
\begin{aligned}
&p^{(1)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)= \\
&P^{(1)}_{\rho_\mu,off/Z,off}\left(t\right)\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta\left(z\right)\delta\left(z\rho_\mu\right)\delta\left(\rho_\mu\right), \\
&p^{(2)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)= \\
&P^{(1)}_{\rho_\mu,on/Z,off}\left(t\right)\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta\left(z\right)\delta\left(z\rho_\mu\right)\delta\left(\rho_\mu-1\right), \\
&p^{(3)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)= \\
&P^{(1)}_{\rho_\mu,off/Z,on}\left(t\right)\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta\left(z-1\right)\delta\left(z\rho_\mu-1\right)\delta\left(\rho_\mu\right), \\
&p^{(4)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)= \\
&P^{(1)}_{\rho_\mu,on/Z,on}\left(t\right)\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta\left(z-1\right)\delta\left(z\rho_\mu-1\right)\delta\left(\rho_\mu-1\right).
\end{aligned}
\tag{4.2.63}
$$

Let us express the terms $p^{(5)}\left(x,\dot{x},z,\rho,z\rho\right)$ to $p^{(8)}\left(x,\dot{x},z,\rho,z\rho\right)$ corresponding to the conditional probabilities given that the first impulse has occurred as follows

$$
\begin{aligned}
&p^{(5)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)=P_3 f^{(1)}\left(x,\dot{x}\right)\delta\left(z\right)\delta\left(z\rho_\mu\right)\delta\left(\rho_\mu\right) \\
&p^{(2)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)=P_4 f^{(2)}\left(x,\dot{x}\right)\delta\left(z\right)\delta\left(z\rho_\mu\right)\delta\left(\rho_\mu-1\right) \\
&p^{(3)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)=P_5 f^{(3)}\left(x,\dot{x}\right)\delta\left(z-1\right)\delta\left(z\rho_\mu\right)\delta\left(\rho_\mu\right) \\
&p^{(3)}\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)=P_6 f^{(4)}\left(x,\dot{x}\right)\delta\left(z-1\right)\delta\left(z\rho_\mu-1\right)\delta\left(\rho_\mu-1\right)
\end{aligned}
\tag{4.2.64}
$$

The joint probability density function has to satisfy the equation

$$
\begin{aligned}
&\int_{-\infty}^{\infty} p\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)dxd\dot{x}dzd\rho_\mu dz\rho_\mu = \\
&P^{(1)}_{\rho_\mu,off/Z,off}+P^{(1)}_{\rho_\mu,off/Z,off}+P^{(1)}_{\rho_\mu,off/Z,off}+P^{(1)}_{\rho_\mu,off/Z,off}+ \\
&\sum_{j=1}^{4} P_{j+2}\int_{-\infty}^{\infty} f^{(j)}\left(x,\dot{x}\right)dxd\dot{x}=1
\end{aligned}
\tag{4.2.65}
$$

observing that

$$
\begin{aligned}
&P^{(1)}_{\rho_\mu,off/Z,off}+P^{(1)}_{\rho_\mu,off/Z,off}+P^{(1)}_{\rho_\mu,off/Z,off}+P^{(1)}_{\rho_\mu,off/Z,off}=P_R, \\
&\sum_{j=1}^{4} P_{j+2}=1-P_R.
\end{aligned}
\tag{4.2.66}
$$

the functions $f^{(1)}\left(x,\dot{x}\right)$ to $f^{(4)}\left(x,\dot{x}\right)$ satisfy the following relationships

$$
\int_{-\infty}^{\infty} f^{(1)}\left(x,\dot{x}\right)dxd\dot{x}=1,\ldots\int_{-\infty}^{\infty} f^{(4)}\left(x,\dot{x}\right)dxd\dot{x}=1
\tag{4.2.67}
$$

It can also be proved that the following identities hold

$$p(Z)=\int_{-\infty}^{\infty} p\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)(dx)(d\dot{x})\left(dz\rho_\mu\right)\left(d\rho_\mu\right)=$$
$$\left(\delta(z)\left(1-E[Z]\right)+\delta(z-1)\left(E[Z]\right)\right)$$
$$p\left(\rho_\mu\right)=\int_{-\infty}^{\infty} p\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)(dx)(d\dot{x})(dz)\left(dz\rho_\mu\right)=$$
$$\left(\delta\left(\rho_\mu\right)\left(1-E\left[\rho_\mu\right]\right)+\delta\left(\rho_\mu-1\right)\left(E\left[\rho_\mu\right]\right)\right) \tag{4.2.68}$$
$$p\left(Z\rho_\mu\right)=\int_{-\infty}^{\infty} p\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)(dx)(d\dot{x})(dz)\left(d\rho_\mu\right)=$$
$$\left(\delta\left(z\rho_\mu\right)\left(1-E\left[Z\rho_\mu\right]\right)+\delta\left(z\rho_\mu-1\right)\left(E\left[Z\rho_\mu\right]\right)\right)$$

Considering as an example the marginal density of the variable ρ_μ

$$\begin{aligned}
p\left(\rho_\mu\right)&=\int_{-\infty}^{\infty} p\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)dxd\dot{x}dzdz\rho_\mu=\\
&P^{(1)}_{\rho_\mu,off/Z,off}\delta\left(\rho_\mu\right)+P^{(1)}_{\rho_\mu,off/Z,off}\delta\left(\rho_\mu-1\right)+\\
&P^{(1)}_{\rho_\mu,off/Z,off}\delta\left(\rho_\mu\right)+P^{(1)}_{\rho_\mu,off/Z,off}\delta\left(\rho_\mu-1\right)+\\
&P_3\delta\left(\rho_\mu\right)+P_4\delta\left(\rho_\mu-1\right)+P_5\delta\left(\rho_\mu\right)+P_6\delta\left(\rho_\mu-1\right).
\end{aligned} \tag{4.2.69}$$

and observing that

$$\begin{aligned}
&P^{(1)}_{\rho_\mu,off/Z,off}+P^{(1)}_{\rho_\mu,off/Z,off}+P_3+P_5=1-E\left[\rho_\mu\right];\\
&P^{(1)}_{\rho_\mu,off/Z,off}+P^{(1)}_{\rho_\mu,off/Z,off}+P_4+P_6=E\left[\rho_\mu\right].
\end{aligned} \tag{4.2.70}$$

it follows

$$\begin{aligned}
p\left(\rho_\mu\right)&=\int_{-\infty}^{\infty} p\left(x,\dot{x},z,\rho_\mu,z\rho_\mu\right)dxd\dot{x}dzdz\rho_\mu=\\
&\left(1-E\left[\rho_\mu\right]\right)\delta\left(\rho_\mu\right)+E\left[\rho_\mu\right]\delta\left(\rho_\mu-1\right)
\end{aligned} \tag{4.2.71}$$

Let us consider the case of zero initial conditions. Since the variables Z, ρ_μ and $Z\rho_\mu$ are zero-one processes, the following relationships hold

$$\begin{aligned}
&E\left[X^k\dot{X}^lZ^m\right]=E\left[X^k\dot{X}^lZ\right];E\left[X^k\dot{X}^l\rho_\mu{}^m\right]=E\left[X^k\dot{X}^l\rho_\mu\right];\\
&E\left[X^k\dot{X}^l\left(Z\rho_\mu\right)^m\right]=E\left[X^k\dot{X}^lZ\rho_\mu\right];
\end{aligned} \tag{4.2.72}$$

The unconditional moment of order $p = m + n$ involving displacements and velocity can be expressed in terms of the conditional moments of the same order as follows

$$E[X^m \dot{X}^n] = P_3 E^{(1)}[X^m \dot{X}^n] + P_4 E^{(2)}[X^m \dot{X}^n] + P_5 E^{(3)}[X^m \dot{X}^n] + P_6 E^{(4)}[X^m \dot{X}^n] \tag{4.2.73}$$

The unconditional moment of order $p+1 = m+n+1$ involving also the auxiliary variables can be expressed in terms of the conditional moments of order p as follows

$$\begin{aligned} E[X^m \dot{X}^n Z] &= P_5 E^{(3)}[X^m \dot{X}^n] + P_6 E^{(4)}[X^m \dot{X}^n]; \\ E[X^m \dot{X}^n \rho_\mu] &= P_4 E^{(2)}[X^m \dot{X}^n] + P_6 E^{(4)}[X^m \dot{X}^n]; \\ E[X^m \dot{X}^n Z \rho_\mu] &= P_6 E^{(4)}[X^m \dot{X}^n]; \end{aligned} \tag{4.2.74}$$

The following relationships between the conditional moments of order p=m+n and the unconditional ones can be derived:

$$\begin{aligned} E^{(1)}\left[X^m \dot{X}^n\right] &= \frac{E\left[X^m \dot{X}^n\right] - E\left[X^m \dot{X}^n \rho_\mu\right] + E\left[X^m \dot{X}^n Z \rho_\mu\right] - E\left[X^m \dot{X}^n Z\right]}{P_3}; \\ E^{(2)}\left[X^m \dot{X}^n\right] &= \frac{E\left[X^m \dot{X}^n \rho_\mu\right] - E\left[X^m \dot{X}^n Z \rho_\mu\right]}{P_4}; \\ E^{(3)}\left[X^m \dot{X}^n\right] &= \frac{E\left[X^m \dot{X}^n Z\right] - E\left[X^m \dot{X}^n Z \rho_\mu\right]}{P_5}; \\ E^{(4)}\left[X^m \dot{X}^n\right] &= \frac{E\left[X^m \dot{X}^n Z \rho_\mu\right]}{P_6}. \end{aligned} \tag{4.2.75}$$

Let us perform the ordinary cumulant neglect closure on the two sets of conditional moments. The conditional moments of order s higher than the closure order r are expressed in terms of conditional moments of order lower than r through the following relationships

$$\begin{aligned} & E^{(k)}\left[X_1 .... X_s\right] = \sum_{\pi^r} \prod_{B^r \in \pi^r} \kappa\left(\underset{i \in B^r}{X_i}\right) = \\ & \sum_{\pi^r} \prod_{B^r \in \pi^r} \left\{ \sum_{\pi^i} \prod_{\substack{B^i \in \pi^i \\ i \in B^r}} \left[\left(\left|B^i\right| - 1\right)! (-1)^{\left|B^i\right| - 1} \left(E^{(k)} \left[\prod_{j \in B^i} X_j \right] \right) \right] \right\}, \\ & k = 1, ..4 \end{aligned} \tag{4.2.76}$$

The conditional moments of order up to r appearing at the right hand side of equation (4.2.76) can be expressed in terms of unconditional ones through equation (4.2.75) as follows

$$E^{(1)}\left[X_1....X_s\right]=$$
$$\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_j\rho_\mu\right]+E\left[\prod_{j\in B^i}X_jZ\rho_\mu\right]-E\left[\prod_{j\in B^i}X_jZ\right]}{P_3}\right)\right]\right\};$$
$$E^{(2)}\left[X_1....X_s\right]=\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_\mu\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\mu\right]}{P_4}\right)\right]\right\};$$
$$E^{(3)}\left[X_1....X_s\right]=\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_jZ\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\mu\right]}{P_5}\right)\right]\right\};$$
$$E^{(4)}\left[X_1....X_s\right]=\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_jZ\rho_\mu\right]}{P_6}\right)\right]\right\}.$$
(4.2.77)

The closure scheme for the unconditional moments becomes

$$E\left[X_1....X_s\right]=$$
$$P_3\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_j\rho_\mu\right]+E\left[\prod_{j\in B^i}X_jZ\rho_\mu\right]-E\left[\prod_{j\in B^i}X_jZ\right]}{P_3}\right)\right]\right\}+$$
$$P_4\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_\mu\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\mu\right]}{P_4}\right)\right]\right\}+$$
$$P_5\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_jZ\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\mu\right]}{P_5}\right)\right]\right\}+$$
$$P_6\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_jZ\rho_\mu\right]}{P_6}\right)\right]\right\};$$

$$E\left[X_1....X_s Z\right]=P_5\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j Z\right]-E\left[\prod_{j\in B^i}X_j Z\rho_\mu\right]}{P_5}\right)\right]\right\}+$$

$$P_6\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j Z\rho_\mu\right]}{P_6}\right)\right]\right\};$$

$$E\left[X_1....X_s\rho_\mu\right]=P_4\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_\mu\right]-E\left[\prod_{j\in B^i}X_j Z\rho_\mu\right]}{P_4}\right)\right]\right\}+$$

$$P_6\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j Z\rho_\mu\right]}{P_6}\right)\right]\right\};$$

$$E\left[X_1....X_n Z\rho_\mu\right]=P_6\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!\left(-1\right)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j Z\rho_\mu\right]}{P_6}\right)\right]\right\}.$$

(4.2.78)

Where π^r runs through the list of the partitions of $\{1,2,...s\}$ in blocks of maximum dimension r, B^r runs through the list of blocks of the partition π^r.

4.2.5 Process III: $R_\mu(t)=P(\mu)$ and $R_\nu(t)=E(\nu,k)$

Consider the response of a Duffing oscillator $f(X,\dot{X})=-2\zeta\omega\dot{X}-\omega^2X-\varepsilon X^3$ to the random train of impulses $R(t)$, obtained from equations (4.2.4) and (4.2.5) with $R_\mu(t)=P(\mu)$ (a Poisson process with parameter μ), $R_\nu(t)=E(\nu,k)$ (an Erlang process with parameters ν and $k=2$), with

$dR_\nu(t)=\rho_\nu dN_\nu(t)$,

$d\rho_\nu(t)=(1-2\rho_\nu)dN_\nu(t)$.

The stochastic equation of motion (4.2.7) is specified by

$$
\mathbf{X}(t)=\begin{bmatrix} X(t) \\ \dot{X}(t) \\ Z(t) \\ \rho_\nu(t) \\ Z(t)\rho_\nu(t) \end{bmatrix}=\begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ X_4(t) \\ X_5(t) \end{bmatrix};
$$

$$
\mathbf{c}(\mathbf{X}(t))=\begin{bmatrix} X_2(t) \\ -\omega^2 X_1(t)-2\zeta\omega X_2(t)-\varepsilon\omega^2 X_1^{\,3}(t) \\ 0 \\ 0 \\ 0 \end{bmatrix};
$$

$$
\mathbf{b}(P(t),\mathbf{X}(t))=\begin{bmatrix} b^\mu & b^\nu \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & P(t)X_5(t) \\ 1-X_3(t) & -X_5(t) \\ 0 & 1-2X_4(t) \\ X_4(t)-X_5(t) & X_3(t)-2X_5(t) \end{bmatrix}; \tag{4.2.79}
$$

Before the occurrence of the first impulse, the variable Z can be in its first 'off' state (with probability $P^{(1)}_{Z,off}$) or in its first 'on' state (with probability $P^{(1)}_{Z,on}$). Meanwhile the variable ρ_ν. can be in any 'off' state or in any 'on' state before the occurrence of the first impulse (with probabilities $P^{(1)}_{\rho_\nu,on}=P^{(1)}_{\rho_\nu,on;Z,off}+P^{(1)}_{\rho_\nu,on;Z,on}$ and $P^{(1)}_{\rho_\nu,off}=P^{(1)}_{\rho_\nu,off;Z,off}+P^{(1)}_{\rho_\nu,off;Z,on}$ respectively). In a similar way, the variable $Z\rho_\nu$. can be found in its first 'off' state or in its first 'on' state before the occurrence of the first impulse with probabilities $P^{(1)}_{Z\rho_\nu,off}$ and $P^{(1)}_{Z\rho_\nu,on}$ respectively.

After the first impulse occurrence, the auxiliary variables can be 'on' or 'off' with probabilities $P^{\#}_{Z,on}=\sum_{j\geq 2}P^{(j)}_{Z,on}$, $P^{\#}_{Z,off}=\sum_{j\geq 2}P^{(j)}_{Z,off}$, $P^{\#}_{\rho_\nu,on}=\sum_{j\geq 2}P^{(j)}_{\rho_\nu,on}$, $P^{\#}_{\rho_\nu,off}=\sum_{j\geq 2}P^{(j)}_{\rho_\nu,off}$ or $P^{\#}_{Z\rho_\nu,on}=\sum_{j\geq 2}P^{(j)}_{Z\rho_\nu,on}$, $P^{\#}_{Z\rho_\nu,off}=\sum_{j\geq 2}P^{(j)}_{Z\rho_\nu,off}$ respectively (see Fig.4.2.4).

For the variable Z the following equations governing the state probabilities can be written:

$$
\begin{aligned}
&P^{(1)}_{z,off}+P^{(1)}_{z,on}=P_R;P^{\#}_{z,off}+P^{\#}_{z,on}=1-P_R \\
&P^{(1)}_{z,off}+P^{\#}_{z,off}=1-E[Z];P^{(1)}_{z,on}+P^{\#}_{z,on}=E[Z]
\end{aligned} \tag{4.2.80}
$$

with

$$
\begin{aligned}
&P^{(1)}_{z,off}=P_{N_\mu};P^{(1)}_{z,on}=P_R-P_{N_\mu}; \\
&P^{\#}_{z,on}=E[Z]-P^{(1)}_{z,on};P^{\#}_{z,off}=1-E[Z]-P^{(1)}_{z,off}.
\end{aligned} \tag{4.2.81}
$$

where $P_{N_\mu} = e^{-\mu t}$ is the probability that the first N_μ driven event has not occurred, $P_R = (1 - F_{w_1})$ (with $F_{w_1} = \int_0^t f_{w_1}(x)\,dx$) is the probability that the first event driven by the process $R(t)$ has not occurred.

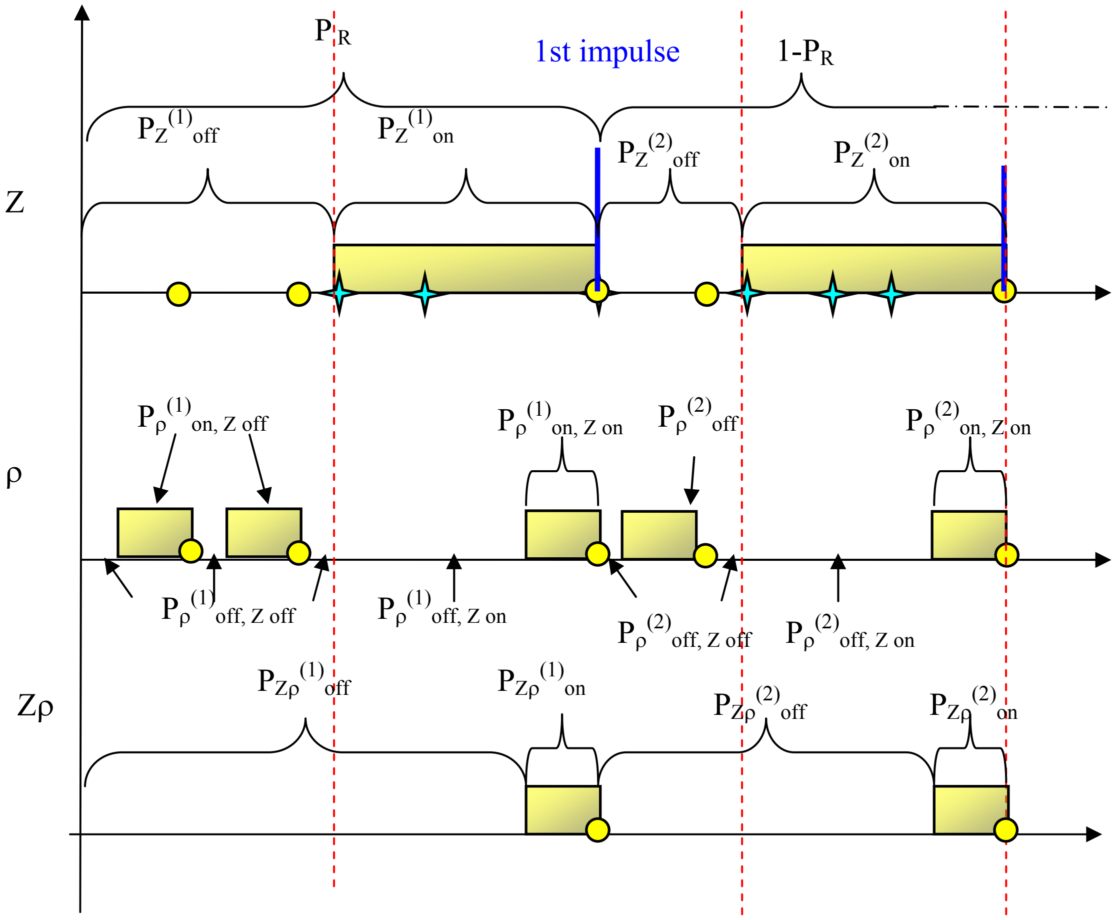

Figure 4.2.4

Sample functions of the train of impulses driven by the non-Erlang renewal process R, and the zero-one processes Z, ρ_ν and $Z\rho_\nu$ appearing in the stochastic equation (4.2.79).

The equations governing the state probabilities of the variable ρ_ν. are:

$$
\begin{aligned}
&P^{(1)}_{\rho_\nu,off/Z,off} + P^{(1)}_{\rho_\nu,off/Z,on} + P^{\#}_{\rho_\nu,off/Z,off} + P^{\#}_{\rho_\nu,off/Z,on} = 1 - E[\rho_\nu] \\
&P^{(1)}_{\rho_\nu,on/Z,off} + P^{(1)}_{\rho_\nu,on/Z,on} + P^{\#}_{\rho_\nu,on/Z,off} + P^{\#}_{\rho_\nu,on/Z,on} = E[\rho_\nu] \\
&P^{(1)}_{\rho_\nu,off/Z,off} + P^{(1)}_{\rho_\nu,off/Z,on} + P^{(1)}_{\rho_\nu,on/Z,off} + P^{(1)}_{\rho_\nu,on/Z,on} = P_R
\end{aligned}
\tag{4.2.82}
$$

where

$$
\begin{aligned}
&P^{(1)}_{\rho_\nu,off/Z,off} = P_1; P^{(1)}_{\rho_\nu,off/Z,on} = P_2;\\
&P^{\#}_{\rho_\nu,off/Z,off} = P_3; P^{\#}_{\rho_\nu,off/Z,on} = P_4;\\
&P^{(1)}_{\rho_\nu,on/Z,off} = -P_1 + P^{(1)}_{Z,off}; P^{(1)}_{\rho_\nu on/Z,on} = -P_2 + P^{(1)}_{Z,on};\\
&P^{\#}_{\rho_\nu,on/Z,off} = P^{\#}_{Z,off} - P_3; P^{\#}_{\rho_\nu,on/Z,on} = P^{\#}_{Z,on} - P_4;
\end{aligned}
\tag{4.2.83}
$$

the equations governing the state probabilities of the variable $Z\rho_\nu$ can be written as

$$
\begin{aligned}
&P^{(1)}_{Z,off} + P^{\#}_{Z,off} + P^{(1)}_{Z\rho,off/Z,on} + P^{\#}_{Z\rho,off/Z,on} = 1 - E[Z\rho_\nu]\\
&P^{(1)}_{Z\rho,on/Z,on} + P^{\#}_{Z\rho,on/Z,on} = E[Z\rho_\nu]\\
&P^{(1)}_{Z\rho,on} + P^{(1)}_{Z\rho,off} = P_R
\end{aligned}
\tag{4.2.84}
$$

where

$$
\begin{aligned}
&P^{(1)}_{Z\rho,off} = P^{(1)}_{Z,off} + P^{(1)}_{\rho_\nu,on/Z,on} = P^{(1)}_{Z,off} + P^{\#}_{Z,on} - P_2,\\
&P^{(1)}_{Z\rho,on} = P^{(1)}_{\rho_\nu,on/Z,on} = -P_2 + P^{(1)}_{Z,on},\\
&P^{\#}_{Z\rho,on} = P^{\#}_{\rho_\nu,on/Z,on} = P^{\#}_{Z,on} - P_4,\\
&P^{\#}_{Z\rho,off} = P^{\#}_{Z,off} - P^{\#}_{\rho_\nu,on/Z,on} = P^{\#}_{Z,off} - P^{\#}_{Z,on} + P_4,
\end{aligned}
\tag{4.2.85}
$$

It can be proved that the equations above lead to the linear system of equations in the unknown P_1,..P_4

$$
\begin{aligned}
&P_1 + P_2 + P_3 + P_4 = 1 - E[\rho_\nu]\\
&P_4 + P_2 = E[Z] - E[Z\rho_\nu]
\end{aligned}
\tag{4.2.86}
$$

The unknowns P_3, P_4 can be expressed in terms of P_1 and P_2 as follows

$$
\begin{aligned}
&P_3 = 1 - E[Z] - E[\rho_\nu] + E[Z\rho_\nu] - P_1;\\
&P_4 = E[Z] - E[Z\rho_\nu] - P_2;
\end{aligned}
\tag{4.2.87}
$$

State probability $P_1(t)$

The probability P_1 that the variable ρ_ν is ‘off’ during Z first ‘off’ state can be expressed as

$$
P_1(t) = \Pr\{T_\mu > t \wedge N_\nu(t) = 0, 2, ..\}
\tag{4.2.88}
$$

where the terms at the right hand side account for the probability that the variable Z is ‘off’, that is the time variable t is before the first R_μ event event and simultaneously between an even-number and the subsequent odd-number Poisson-driven event with parameter ν. The probability P_1 can be expressed as

$$P_1(t)=\left(1-F_{T_\mu}(t)\right)\frac{1}{2}\left(1+e^{-2\nu t}\right); \tag{4.2.89}$$

State probability $P_2(t)$

The probability P_2 that the variable ρ_ν is 'off' during Z first 'on' state can be expressed as

$$P_2(t)=$$
$$\sum_{u=0}^{t}\Pr\left\{T_\mu\in(u,u+du)\wedge N_\nu(t)=0\wedge T_\nu>t\right\}+$$
$$\sum_{u=0}^{t}\Pr\left\{T_\mu\in(u,u+du)\wedge\sum_{\xi=0}^{u}dR_\nu\in(\xi,\xi+d\xi)\wedge N_\nu(t-\xi)=0\wedge T_\nu>t-\xi\right\} \tag{4.2.100}$$

where the terms at the right hand side account for the probability that the variable Z is 'on', that is the time variable t is between the first R_μ event and the subsequent R_ν event and simultaneously between an even-number and the subsequent odd-number Poisson-driven event with parameter ν. The probability P_2 can be expressed as

$$P_2(t)=$$
$$\int_0^t g_{T_\mu}(u)e^{-\nu t}du\left(1-F_{T_\nu}(t)\right)du+ \tag{4.2.101}$$
$$\int_0^t g_{T_\mu}(u)\int_0^u h_\nu(\xi)\left(1-F_{T_\nu}(t-\xi)\right)e^{-\nu(t-\xi)}d\xi du;$$

The variables appearing at the right hand side of the expression above are defined in Figure 4.2.5

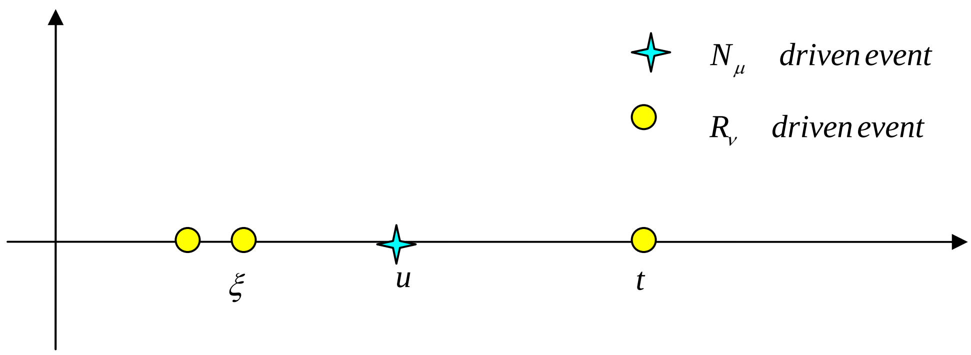

Figure 4.2.5
Definition of the variables appearing in the expression of the state probability $P_2(t)$

Let us express the unknown probability density of the state vector $\mathbf{X}$ in terms of the conditional probabilities

- $$p^{(1)}\left(x,\dot{x},z,\rho_v,z\rho_v\right)dxd\dot{x} = \Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$
  $$\wedge Z\, is\, in\, its\;\, first\,'off\,'\, phase\wedge\rho_v\, is\,'off\,'\wedge Z\rho_v\, is\,'off\,'$$
  given that no impulses have occurred and the variables ρ_v is 'off' while Z and $Z\rho_v$ are in their first "off" state.

- $$p^{(2)}\left(x,\dot{x},z,\rho_v,z\rho_v\right)dxd\dot{x} = \Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$
  $$\wedge Z\, is\, in\, its\;\, first\,'off\,'\, phase\wedge\rho_v\, is\, in\, its\;\, first\,'on'\, phase\wedge Z\rho_v\, is\,'off\,'$$
  given that no impulses have occurred and the variable ρ_v is in its first 'on' phase while Z and $Z\rho_v$ are still in their first "off" state.

- $$p^{(3)}\left(x,\dot{x},z,\rho_v,z\rho_v\right)dxd\dot{x} = \Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$
  $$\wedge Z\, is\, in\, its\;\, first\,'on'\, phase\wedge\rho_v\, is\,'off\,'\wedge Z\rho_v\, is\,'off\,'$$
  given that no impulses have occurred and the variable Z is in its first on state while ρ_v and $Z\rho_v$ are "off".

- $$p^{(4)}\left(x,\dot{x},z,\rho_v,z\rho_v\right)dxd\dot{x} = \Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$
  $$\wedge Z\, is\, in\, its\;\, first\,'on'\, phase\wedge\rho_v\, is\,'on'\wedge Z\rho_v\, is\,'on'$$
  given that no impulses have occurred and the variables Z, ρ_v and $Z\rho_v$ are simultaneously "on".

- $$p^{(5)}\left(x,\dot{x},z,\rho_v,z\rho_v\right)dxd\dot{x} = \Pr\left\{X\in\left(x,x+dx\right)\wedge\dot{X}\in\left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$
  $$\wedge Z\, is\, in\, any\,'off\,'\, phase\;\, following\, the\;\, first\, impulse$$
  $$\wedge\rho_v\, is\, in\, any\,'off\,'\, phase\;\, following\, the\;\, first\, impulse$$
  $$\wedge Z\rho_v\, is\, in\, any\,'off\,'\, phase\;\, following\, the\;\, first\, impulse$$
  given that the first impulse has occurred and the three variables are simultaneously 'off'.

$$p^{(6)}\left(x,\dot{x},z,\rho_v,z\rho_v\right)dxd\dot{x} =$$
$$\Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$

- $\wedge Z$ *is in any 'off' phase following the first impulse*
  $\wedge \rho_v$ *is in any 'on' phase following the first impulse*
  $\wedge Z\rho_v$ *is in any 'off' phase following the first impulse*
  given that the first impulse has occurred and ρ_v is 'on' while Z and $Z\rho_v$ are "off".

$$p^{(7)}\left(x,\dot{x},z,\rho_v,z\rho_v\right)dxd\dot{x} =$$
$$\Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$

- $\wedge Z$ *is in any 'on' phase following the first impulse*
  $\wedge \rho_v$ *is in any 'off' phase following the first impulse*
  $\wedge Z\rho_v$ *is in any 'off' phase following the first impulse*
  given that the first impulse has occurred and Z is in an 'on' state while ρ_v and $Z\rho_v$ are "off".

$$p^{(8)}\left(x,\dot{x},z,\rho_v,z\rho_v\right)dxd\dot{x} =$$
$$\Pr\left\{X \in \left(x,x+dx\right) \wedge \dot{X} \in \left(\dot{x},\dot{x}+d\dot{x}\right)\right\}$$

- $\wedge Z$ *is in any 'on' phase following the first impulse*
  $\wedge \rho_v$ *is in any 'on' phase following the first impulse*
  $\wedge Z\rho_v$ *is in any 'on' phase following the first impulse*
  given that the first impulse has occurred and the variables Z, ρ_v and $Z\rho_v$ are simultaneously "on".

Hence, the joint probability density function can be expressed as

$$p\left(x,\dot{x},z,\rho_v,z\rho_v\right) = \sum_{j=1}^{8} p^{(j)}\left(x,\dot{x},z,\rho_v,z\rho_v\right) \tag{4.2.102}$$

Due to the fact that there is a finite probability of the system being in a deterministic state from the initial time to the first impulse, the terms $p^{(1)}\left(x,\dot{x},z,\rho_v,z\rho_v\right)$ to $p^{(4)}\left(x,\dot{x},z,\rho_v,z\rho_v\right)$ corresponding to the conditional probability densities given that no impulses have occurred, can be respectively expressed as

$$\begin{aligned}
p^{(1)}\left(x,\dot{x},z,\rho_v,z\rho_v\right) &= P_1(t)\,\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta(z)\,\delta\left(z\rho_v\right)\delta\left(\rho_v\right),\\
p^{(2)}\left(x,\dot{x},z,\rho_v,z\rho_v\right) &= P_2(t)\,\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta(z)\,\delta\left(z\rho_v\right)\delta\left(\rho_v-1\right),\\
p^{(3)}\left(x,\dot{x},z,\rho_v,z\rho_v\right) &= P_3(t)\,\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta(z-1)\,\delta\left(z\rho_v-1\right)\delta\left(\rho_v-1\right),\\
p^{(4)}\left(x,\dot{x},z,\rho_v,z\rho_v\right) &= P_4(t)\,\delta\left(x-x_0\right)\delta\left(\dot{x}-\dot{x}_0\right)\delta(z-1)\,\delta\left(z\rho_v-1\right)\delta\left(\rho_v-1\right),
\end{aligned} \tag{4.2.103}$$

The terms $p^{(5)}(x,\dot{x},z,\rho,z\rho)$ to $p^{(8)}(x,\dot{x},z,\rho,z\rho)$ corresponding to the conditional probabilities given that the first impulse has occurred are expressed as

$$\begin{aligned}
p^{(5)}(x,\dot{x},z,\rho_v,z\rho_v) &= P_5 f^{(1)}(x,\dot{x})\delta(z)\delta(z\rho_v)\delta(\rho_v)\\
p^{(2)}(x,\dot{x},z,\rho_v,z\rho_v) &= P_6 f^{(2)}(x,\dot{x})\delta(z)\delta(z\rho_v)\delta(\rho_v-1)\\
p^{(3)}(x,\dot{x},z,\rho_v,z\rho_v) &= P_7 f^{(3)}(x,\dot{x})\delta(z-1)\delta(z\rho_v)\delta(\rho_v)\\
p^{(4)}(x,\dot{x},z,\rho_v,z\rho_v) &= P_8 f^{(4)}(x,\dot{x})\delta(z-1)\delta(z\rho_v-1)\delta(\rho_v-1)
\end{aligned} \tag{4.2.104}$$

The joint probability density function has to satisfy the equation

$$\begin{aligned}
&\int_{-\infty}^{\infty} p(x,\dot{x},z,\rho_v,z\rho_v)dxd\dot{x}dzd\rho_v dz\rho_v = \\
&P_1(t)+P_2(t)+P_3(t)+P_4(t)+P_5\int_{-\infty}^{\infty} f^{(1)}(x,\dot{x})dxd\dot{x}+ \\
&P_6\int_{-\infty}^{\infty} f^{(2)}(x,\dot{x})dxd\dot{x}+P_7\int_{-\infty}^{\infty} f^{(3)}(x,\dot{x})dxd\dot{x}+P_8\int_{-\infty}^{\infty} f^{(4)}(x,\dot{x})dxd\dot{x}=1
\end{aligned} \tag{4.2.105}$$

Considering that

$$\sum_{j=1}^{4} P_j(t) = P_R(t), \sum_{j=4}^{8} P_j(t) = 1-P_R(t) \tag{4.2.106}$$

the following relationships hold

$$\begin{aligned}
&\int_{-\infty}^{\infty} f^{(1)}(x,\dot{x})dxd\dot{x}=1, \int_{-\infty}^{\infty} f^{(2)}(x,\dot{x})dxd\dot{x}=1, \\
&\int_{-\infty}^{\infty} f^{(3)}(x,\dot{x})dxd\dot{x}=1, \int_{-\infty}^{\infty} f^{(4)}(x,\dot{x})dxd\dot{x}=1
\end{aligned} \tag{4.2.107}$$

It can also be proved that the following identities hold

$$p(Z)=\int_{-\infty}^{\infty} p(x,\dot{x},z,\rho_v,z\rho_v)(dx)(d\dot{x})(dz\rho_v)(d\rho_v)=$$
$$(\delta(Z)(1-E[Z])+\delta(Z-1)(E[Z]))$$
$$p(\rho_v)=\int_{-\infty}^{\infty} p(x,\dot{x},z,\rho_v,z\rho_v)(dx)(d\dot{x})(dz)(dz\rho_v)= \tag{4.2.108}$$
$$(\delta(\rho_v)(1-E[\rho_v])+\delta(\rho_v-1)(E[\rho_v]))$$
$$p(Z\rho_v)=\int_{-\infty}^{\infty} p(x,\dot{x},z,\rho_v,z\rho_v)(dx)(d\dot{x})(dz)(d\rho_v)=$$
$$(\delta(Z\rho_v)(1-E[Z\rho_v])+\delta(Z\rho_v-1)(E[Z\rho_v]))$$

Let us consider the case of zero initial conditions. Since the variables Z , ρ_v and $Z\rho_v$ are zero-one processes, the following relationships hold

$$\begin{aligned}
&E\left[X^k\dot{X}^lZ^m\right]=E\left[X^k\dot{X}^lZ\right];\\
&E\left[X^k\dot{X}^l\rho_v{}^m\right]=E\left[X^k\dot{X}^l\rho_v\right];\\
&E\left[X^k\dot{X}^l(Z\rho_v)^m\right]=E\left[X^k\dot{X}^lZ\rho_v\right];
\end{aligned} \tag{4.2.109}$$

The unconditional moment of order $p=m+n$ involving displacements and velocity can be expressed in terms of the conditional moments of the same order as follows

$$\begin{aligned}
&E[X^m\dot{X}^n]=\\
&P_5E^{(1)}[X^m\dot{X}^n]+P_6E^{(2)}[X^m\dot{X}^n]+P_7E^{(3)}[X^m\dot{X}^n]+P_8E^{(4)}[X^m\dot{X}^n]
\end{aligned} \tag{4.2.110}$$

The unconditional moment of order $p+1=m+n+1$ involving also the auxiliary variables can be expressed in terms of the conditional moments of order p as follows

$$\begin{aligned}
&E[X^m\dot{X}^nZ]=P_7E^{(3)}[X^m\dot{X}^n]+P_8E^{(4)}[X^m\dot{X}^n];\\
&E[X^m\dot{X}^n\rho_v]=P_6E^{(2)}[X^m\dot{X}^n]+P_8E^{(4)}[X^m\dot{X}^n];\\
&E[X^m\dot{X}^nZ\rho_v]=P_8E^{(4)}[X^m\dot{X}^n];
\end{aligned} \tag{4.2.111}$$

The following relationships relating the conditional moments of order p=m+n to the unconditional ones can be derived:

$$
\begin{aligned}
E^{(1)}\left[X^m\dot{X}^n\right] &= \frac{E\left[X^m\dot{X}^n\right]-E\left[X^m\dot{X}^n\rho_\nu\right]+E\left[X^m\dot{X}^n Z\rho_\nu\right]-E\left[X^m\dot{X}^n Z\right]}{P_5};\\
E^{(2)}\left[X^m\dot{X}^n\right] &= \frac{E\left[X^m\dot{X}^n\rho_\nu\right]-E\left[X^m\dot{X}^n Z\rho_\nu\right]}{P_6};\\
E^{(3)}\left[X^m\dot{X}^n\right] &= \frac{E\left[X^m\dot{X}^n Z\right]-E\left[X^m\dot{X}^n Z\rho_\nu\right]}{P_7};\\
E^{(4)}\left[X^m\dot{X}^n\right] &= \frac{E\left[X^m\dot{X}^n Z\rho_\nu\right]}{P_8}.
\end{aligned}
\tag{4.2.112}
$$

Let us perform the ordinary cumulant neglect closure on the four sets of conditional moments. The conditional moments of order s higher than the closure order r are expressed in terms of conditional moments of order lower than r, which in turn can be derived as functions of the unconditional moments of order lower than r through equation (4.2.112). From equations (4.2.110) and (4.2.111), the modified closure scheme for the unconditional moments becomes

$$
\begin{aligned}
&E\left[X_1....X_s\right]=\\
&P_5\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_j\rho_\nu\right]+E\left[\prod_{j\in B^i}X_jZ\rho_\nu\right]-E\left[\prod_{j\in B^i}X_jZ\right]}{P_5}\right)\right]\right\}+\\
&P_6\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_j\rho_\nu\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\nu\right]}{P_6}\right)\right]\right\}+\\
&P_7\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_jZ\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\nu\right]}{P_7}\right)\right]\right\}+\\
&P_8\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\pi^i}\prod_{\substack{B^i\in\pi^i\\ i\in B^r}}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\left(\frac{E\left[\prod_{j\in B^i}X_jZ\rho_\nu\right]}{P_8}\right)\right]\right\};
\end{aligned}
$$

$$E\left[X_1....X_s Z\right]=$$

$$P_7 \sum_{\pi^r} \prod_{B^r \in \pi^r} \left\{ \sum_{\pi^i} \prod_{\substack{B^i \in \pi^i \\ i \in B^r}} \left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1} \left(\frac{E\left[\prod_{j\in B^i} X_j Z\right] - E\left[\prod_{j\in B^i} X_j Z \rho_\nu\right]}{P_7} \right) \right] \right\} +$$

$$P_8 \sum_{\pi^r} \prod_{B^r \in \pi^r} \left\{ \sum_{\pi^i} \prod_{\substack{B^i \in \pi^i \\ i \in B^r}} \left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1} \left(\frac{E\left[\prod_{j\in B^i} X_j Z \rho_\nu\right]}{P_8} \right) \right] \right\};$$

$$E\left[X_1....X_s \rho_\nu\right]=$$

$$P_6 \sum_{\pi^r} \prod_{B^r \in \pi^r} \left\{ \sum_{\pi^i} \prod_{\substack{B^i \in \pi^i \\ i \in B^r}} \left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1} \left(\frac{E\left[\prod_{j\in B^i} X_j \rho_\nu\right] - E\left[\prod_{j\in B^i} X_j Z \rho_\nu\right]}{P_6} \right) \right] \right\} +$$

$$P_8 \sum_{\pi^r} \prod_{B^r \in \pi^r} \left\{ \sum_{\pi^i} \prod_{\substack{B^i \in \pi^i \\ i \in B^r}} \left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1} \left(\frac{E\left[\prod_{j\in B^i} X_j Z \rho_\nu\right]}{P_8} \right) \right] \right\};$$

$$E\left[X_1....X_s Z \rho_\nu\right]=$$

$$P_8 \sum_{\pi^r} \prod_{B^r \in \pi^r} \left\{ \sum_{\pi^i} \prod_{\substack{B^i \in \pi^i \\ i \in B^r}} \left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1} \left(\frac{E\left[\prod_{j\in B^i} X_j Z \rho_\nu\right]}{P_8} \right) \right] \right\}. \tag{4.2.113}$$

Where π^r runs through the list of the partitions of $\{1,2,...s\}$ in blocks of maximum dimension r, B^r runs through the list of blocks of the partition π^r.

4.2.6 Process IV: $R_\mu(t)=E(\mu,l)$ and $R_\nu(t)=E(\nu,k)$

Consider the response of a Duffing oscillator $f\left(X,\dot{X}\right)=-2\zeta\omega\dot{X}-\omega^2 X-\varepsilon X^3$ to the random train of impulses $R(t)$, derived from equations (3.2.4) and (3.2.5) with $R_\mu(t)=E(\mu,l)$ and $R_\nu(t)=E(\nu,k)$ (Erlang processes with parameters $\mu,l=2$ and $\nu,k=2$, respectively) and with

$$dR_\mu(t)=\rho_\mu dN_\mu(t),$$
$$d\rho_\mu(t)=\left(1-2\rho_\mu\right)dN_\mu(t)$$

and

$$dR_v(t) = \rho_v dN_v(t),$$
$$d\rho_v(t) = (1-2\rho_v)dN_v(t).$$

The stochastic equation of motion (4.2.7) is specified by

$$\mathbf{X}(t) = \begin{bmatrix} X(t) \\ \dot{X}(t) \\ Z(t) \\ \rho_\mu(t) \\ \rho_v(t) \\ \rho_\mu(t)\rho_v(t) \\ Z(t)\rho_\mu(t) \\ Z(t)\rho_v(t) \\ Z(t)\rho_\mu(t)\rho_v(t) \end{bmatrix} = \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ X_4(t) \\ X_5(t) \\ X_6(t) \\ X_7(t) \\ X_8(t) \\ X_9(t) \end{bmatrix};$$

$$\mathbf{c}(\mathbf{X}(t)) = \begin{bmatrix} X_2(t) \\ -\omega^2 X_1(t) - 2\zeta\omega X_2(t) - \varepsilon\omega^2 X_1^{\,3}(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$\mathbf{b}(P(t),\mathbf{X}(t)) = \begin{bmatrix} b^\mu & b^v \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & P(t)X_8(t) \\ X_4(t) - X_7(t) & -X_8(t) \\ 1-2X_4(t) & 0 \\ 0 & 1-2X_5(t) \\ X_5(t)-2X_6(t) & X_4(t)-2X_6(t) \\ X_3(t)-2X_7(t) & -X_9(t) \\ X_6(t)-X_9(t) & X_3(t)-2X_8(t) \\ X_8(t)-2X_9(t) & X_7(t)-2X_9(t) \end{bmatrix}; \qquad (4.2.114)$$

Before the occurrence of the first impulse, the variable Z can be in its first 'off' state (with probability $P^{(1)}_{Z,off}$) or in its first 'on' state (with probability $P^{(1)}_{Z,on}$). Meanwhile the variable ρ_v. can be in any 'off' state or in any 'on' state before the occurrence of the first impulse (with probabilities $P^{(1)}_{\rho_\nu,on} = P^{(1)}_{\rho_\nu,on;Z,off} + P^{(1)}_{\rho_\nu,on;Z,on}$ and $P^{(1)}_{\rho_\nu,off} = P^{(1)}_{\rho_\nu,off;Z,off} + P^{(1)}_{\rho_\nu,off;Z,on}$ respectively). The variable ρ_μ. can be in its first 'off' state or in its first 'on' state while Z is in its first 'off' state before the occurrence of the first impulse (with probabilities $P^{(1)}_{\rho_\mu,off;Z,off}$ and $P^{(1)}_{\rho_\mu,on;Z,off}$ respectively) or can be in any subsequent 'off' state or 'on' state during Z first 'on' state (with probabilities $P^{(1)}_{\rho_\mu,off;Z,on}$ and $P^{(1)}_{\rho_\mu,on;Z,off}$ respectively). In a similar way, the variables $\rho_\mu\rho_\nu$, $Z\rho_\mu$. $Z\rho_\nu$ and $Z\rho_\mu\rho_\nu$ can be found in the 'off' or 'on' states shown in Figure 4.2.6 with the correspondent probabilities in Table 4.1.

After the first impulse occurrence, the auxiliary variables can be 'on' or 'off' with probabilities $P^{\#}_{Z,on} = \sum_{j\geq 2} P^{(j)}_{Z,on}$, $P^{\#}_{Z,off} = \sum_{j\geq 2} P^{(j)}_{Z,off}$, $P^{\#}_{\rho_\nu,on} = \sum_{j\geq 2} P^{(j)}_{\rho_\nu,on}$, $P^{\#}_{\rho_\nu,off} = \sum_{j\geq 2} P^{(j)}_{\rho_\nu,off}$ or $P^{\#}_{Z\rho_\nu,on} = \sum_{j\geq 2} P^{(j)}_{Z\rho_\nu,on}$, $P^{\#}_{Z\rho_\nu,off} = \sum_{j\geq 2} P^{(j)}_{Z\rho_\nu,off}$ respectively (see Fig.4.2.6 and Table 4.1)

The following equations governing the state probabilities for the variable Z can be written:

$$
\begin{aligned}
&P^{(1)}_{z,off} + P^{(1)}_{z,on} = P_R ; P^{\#}_{z,off} + P^{\#}_{z,on} = 1 - P_R \\
&P^{(1)}_{z,off} + P^{\#}_{z,off} = 1 - E[Z] ; P^{(1)}_{z,on} + P^{\#}_{z,on} = E[Z]
\end{aligned}
\tag{4.2.115}
$$

with

$$
\begin{aligned}
&P^{(1)}_{z,off} = P_{R_\mu} ; P^{(1)}_{z,on} = P_R - P_{R_\mu} ; \\
&P^{\#}_{z,on} = E[Z] - P^{(1)}_{z,on} ; P^{\#}_{z,off} = 1 - E[Z] - P^{(1)}_{z,off} .
\end{aligned}
\tag{4.2.116}
$$

where $P_{R_\mu} = \mu t e^{-\mu t}$ is the probability that the first R_μ driven event has not occurred, $P_{N_\mu} = e^{-\mu t}$ is the probability that the first N_μ driven event has not occurred, $P_R = (1 - F_{w_1})$ (with $F_{w_1} = \int_0^t f_{w_1}(x)\,dx$) is the probability that the first event driven by the process $R(t)$ has not occurred.

The equations governing the state probabilities of the variable ρ_ν. are:

$$
\begin{aligned}
&P^{(1)}_{\rho_\nu,off/Z,off} + P^{(1)}_{\rho_\nu,off/Z,on} + P^{\#}_{\rho_\nu,off/Z,off} + P^{\#}_{\rho_\nu,off/Z,on} = 1 - E[\rho_\nu] \\
&P^{(1)}_{\rho_\nu,on/Z,off} + P^{(1)}_{\rho_\nu,on/Z,on} + P^{\#}_{\rho_\nu,on/Z,off} + P^{\#}_{\rho_\nu,on/Z,on} = E[\rho_\nu] \\
&P^{(1)}_{\rho_\nu,off/Z,off} + P^{(1)}_{\rho_\nu,off/Z,on} + P^{(1)}_{\rho_\nu,on/Z,off} + P^{(1)}_{\rho_\nu,on/Z,on} = P_R
\end{aligned}
\tag{4.2.117}
$$

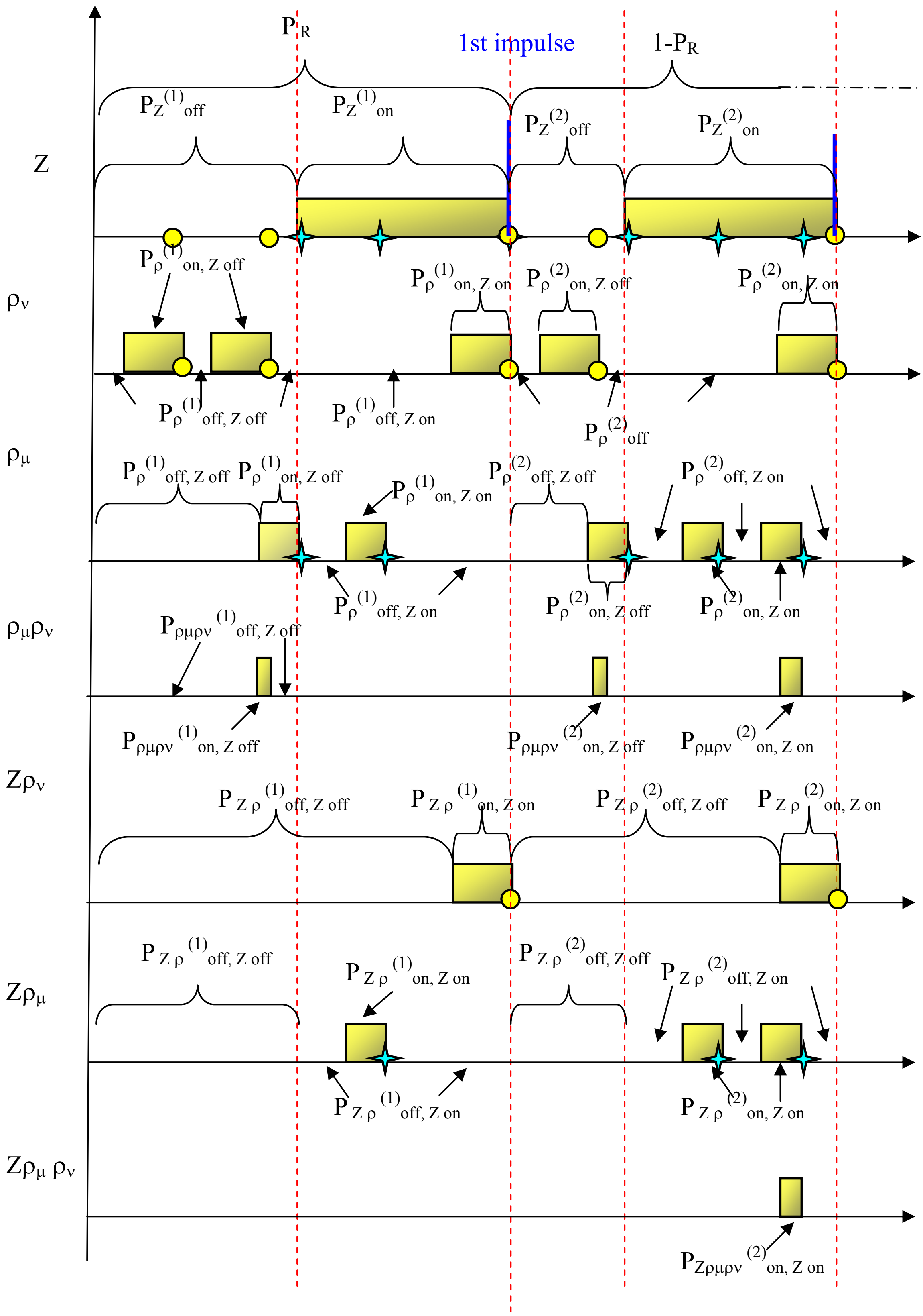

Figure 4.2.6

Sample functions of the train of impulses driven by the non-Erlang renewal process R, and the zero-one processes Z, ρ_ν, ρ_μ, $\rho_\mu\rho_\nu$, $Z\rho_\mu$, $Z\rho_\nu$ and $Z\rho_\mu\rho_\nu$ appearing in the stochastic equation (4.2.102).

where

$$
\begin{aligned}
&P^{(1)}_{\rho_\nu,off/Z,off} = P_1\\
&P^{(1)}_{\rho_\nu,off/Z,on} = P_2\\
&P^{\#}_{\rho_\nu,off/Z,off} = P_3\\
&P^{\#}_{\rho_\nu,off/Z,on} = P_4\\
&P^{(1)}_{\rho_\nu,on/Z,off} = P^{(1)}_{Z,off} - P_1\\
&P^{(1)}_{\rho_\nu,on/Z,on} = P^{(1)}_{Z,on} - P_2\\
&P^{\#}_{\rho_\nu,on/Z,off} = P^{\#}_{Z,off} - P_3\\
&P^{\#}_{\rho_\nu,on/Z,on} = P^{\#}_{Z,on} - P_4
\end{aligned}
\tag{4.2.118}
$$

The equations governing the state probabilities of the variable ρ_μ. are:

$$
\begin{aligned}
&P^{(1)}_{\rho_\mu,off/Z,off} + P^{(1)}_{\rho_\mu,off/Z,on} + P^{\#}_{\rho_\mu,off/Z,off} + P^{\#}_{\rho_\mu,off/Z,on} = 1 - E[\rho_\mu]\\
&P^{(1)}_{\rho_\mu,on/Z,off} + P^{(1)}_{\rho_\mu,on/Z,on} + P^{\#}_{\rho_\mu,on/Z,off} + P^{\#}_{\rho_\mu,on/Z,on} = E[\rho_\mu]\\
&P^{(1)}_{\rho_\mu,off/Z,off} + P^{(1)}_{\rho_\mu,off/Z,on} + P^{(1)}_{\rho_\mu,on/Z,off} + P^{(1)}_{\rho_\mu,on/Z,on} = P_R
\end{aligned}
\tag{4.2.119}
$$

where

$$
\begin{aligned}
&P^{(1)}_{\rho_\mu,off/Z,off} = P_{N_\mu}\\
&P^{(1)}_{\rho_\mu,off/Z,on} = P_5\\
&P^{\#}_{\rho_\mu,off/Z,off} = P_6\\
&P^{\#}_{\rho_\mu,off/Z,on} = P_7\\
&P^{(1)}_{\rho_\mu,on/Z,off} = P^{(1)}_{Z,off} - P_{N_\mu}\\
&P^{(1)}_{\rho_\mu,on/Z,on} = P^{(1)}_{Z,on} - P_5\\
&P^{\#}_{\rho_\mu,on/Z,off} = P^{\#}_{Z,off} - P_6\\
&P^{\#}_{\rho_\mu,on/Z,on} = P^{\#}_{Z,on} - P_7
\end{aligned}
\tag{4.2.120}
$$

The equations governing the state probabilities of the variable $\rho_\nu\rho_\mu$. are:

$$
\begin{aligned}
&P^{(1)}_{\rho_\mu\rho_\nu,off/Z,off} + P^{(1)}_{\rho_\mu\rho_\nu,off/Z,on} + P^{\#}_{\rho_\mu\rho_\nu,off/Z,off} + P^{\#}_{\rho_\mu\rho_\nu,off/Z,on} = 1 - E[\rho_\mu\rho_\nu]\\
&P^{(1)}_{\rho_\mu\rho_\nu,on/Z,off} + P^{(1)}_{\rho_\mu\rho_\nu,on/Z,on} + P^{\#}_{\rho_\mu\rho_\nu,on/Z,off} + P^{\#}_{\rho_\mu\rho_\nu,on/Z,on} = E[\rho_\mu\rho_\nu]\\
&P^{(1)}_{\rho_\mu\rho_\nu,off/Z,off} + P^{(1)}_{\rho_\mu\rho_\nu,off/Z,on} + P^{(1)}_{\rho_\mu\rho_\nu,on/Z,off} + P^{(1)}_{\rho_\mu\rho_\nu,on/Z,on} = P_R
\end{aligned}
\tag{4.2.121}
$$

where

$$
\begin{aligned}
&P^{(1)}_{\rho_\mu\rho_\nu,off/Z,off} = P_8 \\
&P^{(1)}_{\rho_\mu\rho_\nu,off/Z,on} = P_9 \\
&P^{\#}_{\rho_\mu\rho_\nu,off/Z,off} = P_{10} \\
&P^{\#}_{\rho_\mu\rho_\nu,off/Z,on} = P_{11} \\
&P^{(1)}_{\rho_\mu\rho_\nu,on/Z,off} = P^{(1)}_{Z,off} - P_8 \\
&P^{(1)}_{\rho_\mu\rho_\nu,on/Z,on} = P^{(1)}_{Z,on} - P_9 \\
&P^{\#}_{\rho_\mu\rho_\nu,on/Z,off} = P^{\#}_{Z,off} - P_{10} \\
&P^{\#}_{\rho_\mu\rho_\nu,on/Z,on} = P^{\#}_{Z,on} - P_{11}
\end{aligned}
\tag{4.2.122}
$$

The equations governing the state probabilities of the variable $Z\rho_\nu$. are:

$$
\begin{aligned}
&P^{(1)}_{Z\rho_\nu,off/Z,off} + P^{(1)}_{Z\rho_\nu,off/Z,on} + P^{\#}_{Z\rho_\nu,off/Z,off} + P^{\#}_{Z\rho_\nu,off/Z,on} = 1 - E[Z\rho_\nu] \\
&P^{(1)}_{Z\rho_\nu,on/Z,off} + P^{(1)}_{Z\rho_\nu,on/Z,on} + P^{\#}_{Z\rho_\nu,on/Z,off} + P^{\#}_{Z\rho_\nu,on/Z,on} = E[Z\rho_\nu] \\
&P^{(1)}_{Z\rho_\nu,off/Z,off} + P^{(1)}_{Z\rho_\nu,off/Z,on} + P^{(1)}_{Z\rho_\nu,on/Z,off} + P^{(1)}_{Z\rho_\nu,on/Z,on} = P_R
\end{aligned}
\tag{4.2.123}
$$

where

$$
\begin{aligned}
&P^{(1)}_{Z\rho_\nu,off/Z,off} = P^{(1)}_{Z,off} \\
&P^{(1)}_{Z\rho_\nu,off/Z,on} = P^{(1)}_{\rho_\nu,off/Z,on} = P_2 \\
&P^{\#}_{Z\rho_\nu,off/Z,off} = P^{\#}_{Z,off} \\
&P^{\#}_{Z\rho_\nu,off/Z,on} = P^{\#}_{\rho_\nu,off/Z,on} = P_4 \\
&P^{(1)}_{Z\rho_\nu,on/Z,off} = 0 \\
&P^{(1)}_{Z\rho_\nu,on/Z,on} = P^{(1)}_{\rho_\nu,on/Z,on} = P^{(1)}_{Z,on} - P_2 \\
&P^{\#}_{Z\rho_\nu,on/Z,off} = 0 \\
&P^{\#}_{Z\rho_\nu,on/Z,on} = P^{\#}_{\rho_\nu,on/Z,on} = P^{\#}_{Z,on} - P_4
\end{aligned}
\tag{4.2.124}
$$

The equations governing the state probabilities of the variable $Z\rho_\mu$. are:

$$
\begin{aligned}
&P^{(1)}_{Z\rho_\mu,off/Z,off} + P^{(1)}_{Z\rho_\mu,off/Z,on} + P^{\#}_{Z\rho_\mu,off/Z,off} + P^{\#}_{Z\rho_\mu,off/Z,on} = 1 - E[Z\rho_\mu] \\
&P^{(1)}_{Z\rho_\mu,on/Z,off} + P^{(1)}_{Z\rho_\mu,on/Z,on} + P^{\#}_{Z\rho_\mu,on/Z,off} + P^{\#}_{Z\rho_\mu,on/Z,on} = E[Z\rho_\mu] \\
&P^{(1)}_{Z\rho_\mu,off/Z,off} + P^{(1)}_{Z\rho_\mu,off/Z,on} + P^{(1)}_{Z\rho_\mu,on/Z,off} + P^{(1)}_{Z\rho_\mu,on/Z,on} = P_R
\end{aligned}
\tag{4.2.125}
$$

where

$$P^{(1)}_{Z\rho_\mu,off/Z,off} = P^{(1)}_{Z,off}$$
$$P^{(1)}_{Z\rho_\mu,off/Z,on} = P^{(1)}_{\rho_\mu,off/Z,on} = P_5$$
$$P^{\#}_{Z\rho_\mu,off/Z,off} = P^{\#}_{Z,off}$$
$$P^{\#}_{Z\rho_\mu,off/Z,on} = P^{\#}_{\rho_\mu,off/Z,on} = P_7$$
$$P^{(1)}_{Z\rho_\mu,on/Z,off} = 0 \qquad (4.2.126)$$
$$P^{(1)}_{Z\rho_\mu,on/Z,on} = P^{(1)}_{\rho_\mu,on/Z,on} = P^{(1)}_{Z,on} - P_5$$
$$P^{\#}_{Z\rho_\mu,on/Z,off} = 0$$
$$P^{\#}_{Z\rho_\mu,on/Z,on} = P^{\#}_{\rho_\mu,on/Z,on} = P^{\#}_{Z,on} - P_7$$

The equations governing the state probabilities of the variable $Z\rho_\mu\rho_\nu$. are:

$$P^{(1)}_{Z\rho_\mu\rho_\nu,off/Z,off} + P^{(1)}_{Z\rho_\mu\rho_\nu,off/Z,on} + P^{\#}_{Z\rho_\mu\rho_\nu,off/Z,off} + P^{\#}_{Z\rho_\mu\rho_\nu,off/Z,on} = 1 - E[Z\rho_\mu\rho_\nu]$$
$$P^{(1)}_{Z\rho_\mu\rho_\nu,on/Z,off} + P^{(1)}_{Z\rho_\mu\rho_\nu,on/Z,on} + P^{\#}_{Z\rho_\mu\rho_\nu,on/Z,off} + P^{\#}_{Z\rho_\mu\rho_\nu,on/Z,on} = E[Z\rho_\mu\rho_\nu] \qquad (4.2.127)$$
$$P^{(1)}_{Z\rho_\mu\rho_\nu,off/Z,off} + P^{(1)}_{Z\rho_\mu\rho_\nu,off/Z,on} + P^{(1)}_{Z\rho_\mu\rho_\nu,on/Z,off} + P^{(1)}_{Z\rho_\mu\rho_\nu,on/Z,on} = P_R$$

where

$$P^{(1)}_{Z\rho_\mu\rho_\nu,off/Z,off} = P^{(1)}_{Z,off}$$
$$P^{(1)}_{Z\rho_\mu\rho_\nu,off/Z,on} = P^{(1)}_{\rho_\mu\rho_\nu,off/Z,on} = P_9$$
$$P^{\#}_{Z\rho_\mu\rho_\nu,off/Z,off} = P^{\#}_{Z,off}$$
$$P^{\#}_{Z\rho_\mu\rho_\nu,off/Z,on} = P^{\#}_{\rho_\mu\rho_\nu,off/Z,on} = P_{11}$$
$$P^{(1)}_{Z\rho_\mu\rho_\nu,on/Z,off} = 0 \qquad (4.2.128)$$
$$P^{(1)}_{Z\rho_\mu\rho_\nu,on/Z,on} = P^{(1)}_{\rho_\mu\rho_\nu,on/Z,on} = P^{(1)}_{Z,on} - P_9$$
$$P^{\#}_{Z\rho_\mu\rho_\nu,on/Z,off} = 0$$
$$P^{\#}_{Z\rho_\mu\rho_\nu,on/Z,on} = P^{\#}_{\rho_\mu\rho_\nu,on/Z,on} = P^{\#}_{Z,on} - P_{11}$$

The conditional probability P_6 can be directly determined from the equations above, the others can be expressed in terms of the probabilities P_1, P_2, P_5, P_8, P_9 as follows

$$P_3 = 1 - E[Z] - E[\rho_\nu] + E[Z\rho_\nu] - P_1;$$
$$P_4 = -E[Z\rho_\nu] + E[Z] - P_2;$$
$$P_6 = 1 - E[\rho_\mu] - E[Z] + E[Z\rho_\mu] - P_{N_\mu};$$
$$P_7 = E[Z\rho_\mu] - E[Z] + P_5; \qquad (4.2.129)$$
$$P_{10} = 1 + E[Z\rho_\mu\rho_\nu] - E[\rho_\mu\rho_\nu] - E[Z] - P_8;$$
$$P_{11} = E[Z] - E[Z\rho_\mu\rho_\nu] - P_9;$$

Z	ρ_ν	ρ_μ	$\rho_\nu\ \rho_\mu$	$Z\rho_\nu$	$Z\rho_\mu$	$Z\rho_\nu\ \rho_\mu$
$P^{(1)}_{Z\ off}$	$P^{(1)}_{\rho\nu\ off,\ Z\ off}$	$P^{(1)}_{\rho\mu\ off,\ Z\ off}$	$P^{(1)}_{\rho\nu\,\rho\mu\ off,\ Z\ off}$	$P^{(1)}_{Z\rho\nu\ off,\ Z\ off}$	$P^{(1)}_{Z\rho\mu\ off,\ Z\ off}$	$P^{(1)}_{Z\rho\nu\,\rho\mu\ off,\ Z\ off}$
	$P^{(1)}_{\rho\nu\ on,\ Z\ off}$	$P^{(1)}_{\rho\mu\ on,\ Z\ off}$	$P^{(1)}_{\rho\nu\,\rho\mu\ on,\ Z\ off}$	$P^{(1)}_{Z\rho\nu\ on,\ Z\ off}$	$P^{(1)}_{Z\rho\mu\ on,\ Z\ off}$	$P^{(1)}_{Z\rho\nu\,\rho\mu\ on,\ Z\ off}$
$P^{(1)}_{Z\ on}$	$P^{(1)}_{\rho\nu\ off,\ Z\ on}$	$P^{(1)}_{\rho\mu\ off,\ Z\ on}$	$P^{(1)}_{\rho\nu\,\rho\mu\ off,\ Z\ on}$	$P^{(1)}_{Z\rho\nu\ off,\ Z\ on}$	$P^{(1)}_{Z\rho\mu\ off,\ Z\ on}$	$P^{(1)}_{Z\rho\nu\,\rho\mu\ off,\ Z\ on}$
	$P^{(1)}_{\rho\nu\ on,\ Z\ on}$	$P^{(1)}_{\rho\mu\ on,\ Z\ on}$	$P^{(1)}_{\rho\nu\,\rho\mu\ on,\ Z\ on}$	$P^{(1)}_{Z\rho\nu\ on,\ Z\ on}$	$P^{(1)}_{Z\rho\mu\ on,\ Z\ on}$	$P^{(1)}_{Z\rho\nu\,\rho\mu\ on,\ Z\ on}$
$P^{\#}_{Z\ off}$	$P^{\#}_{\rho\nu\ off,\ Z\ off}$	$P^{\#}_{\rho\mu\ off,\ Z\ off}$	$P^{\#}_{\rho\nu\,\rho\mu\ off,\ Z\ off}$	$P^{\#}_{Z\rho\nu\ off,\ Z\ off}$	$P^{\#}_{Z\rho\mu\ off,\ Z\ off}$	$P^{\#}_{Z\rho\nu\rho\mu\ off,\ Z\ off}$
	$P^{\#}_{\rho\nu\ on,\ Z\ off}$	$P^{\#}_{\rho\mu\ on,\ Z\ off}$	$P^{\#}_{\rho\nu\,\rho\mu\ on,\ Z\ off}$	$P^{\#}_{Z\rho\nu\ on,\ Z\ off}$	$P^{\#}_{Z\rho\mu\ on,\ Z\ off}$	$P^{\#}_{Z\rho\nu\rho\mu\ on,\ Z\ off}$
$P^{\#}_{Z\ on}$	$P^{\#}_{\rho\nu\ off,\ Z\ on}$	$P^{\#}_{\rho\mu\ off,\ Z\ on}$	$P^{\#}_{\rho\nu\,\rho\mu\ off,\ Z\ on}$	$P^{\#}_{Z\rho\nu\ off,\ Z\ on}$	$P^{\#}_{Z\rho\mu\ off,\ Z\ on}$	$P^{\#}_{Z\rho\nu\,\rho\mu\ off,\ Z\ on}$
	$P^{\#}_{\rho\nu\ on,\ Z\ on}$	$P^{\#}_{\rho\mu\ on,\ Z\ on}$	$P^{\#}_{\rho\nu\,\rho\mu\ on,\ Z\ on}$	$P^{\#}_{Z\rho\nu\ on,\ Z\ on}$	$P^{\#}_{Z\ \mu\ on,\ Z\ on}$	$P^{\#}_{Z\rho\nu\,\rho\mu\ on,\ Z\ on}$

Table 4.1
State probabilities of the variables Z, ρ_ν , ρ_μ , $\rho_\mu\rho_\nu$, $Z\rho_\mu.$, $Z\rho_\nu$ and $Z\rho_\mu\rho_\nu$.

State Probability $P_1(t)$

The probability $P_1 = P^{(1)}_{\rho_\nu,off/Z,off}$ that the variable ρ_ν is ‘off’ during Z first ‘off’ state can be expressed as

$$P_1(t) = \Pr\left\{T_\mu > t \wedge N_\nu(t) = 0, 2, 4, ...\right\} \tag{4.2.130}$$

where the term at the right hand side accounts for the probability that the variable Z is ‘off’, that is the time variable t is before the first R_μ driven event and simultaneously between an even-number and the subsequent odd-number Poisson-driven event with parameter ν .
The probability density P_1 can be expressed as

$$P_1(t) = \left(1 - F_{T_\mu}(t)\right)\frac{1}{2}\left(1 + e^{-2\nu t}\right) \tag{4.2.131}$$

The variables appearing at the right hand side of the above expression are defined in Figure 4.2.7

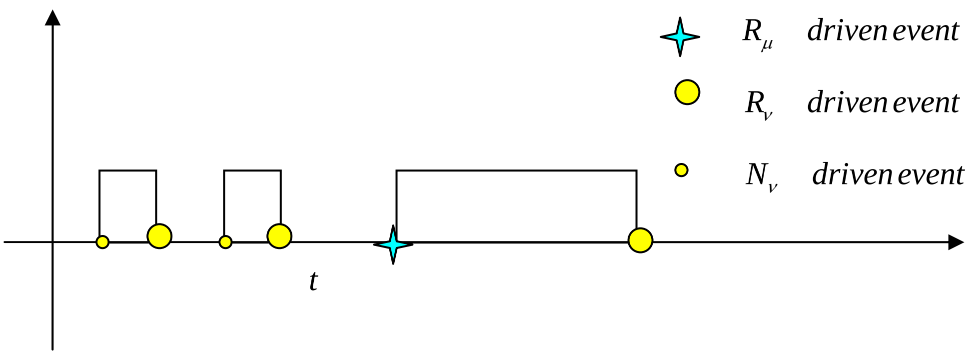

Figure 4.2.7
Definition of the variable t appearing in equation (4.2.131).

State Probability $P_2(t)$

The probability $P_2 = P^{(1)}_{\rho_\nu,off/Z,on}$ that the variable ρ_ν is 'off' during Z first 'on' state can be expressed as

$$P_2(t) =$$
$$\sum_{u=0}^{t} \Pr\{T_\mu \in (u,u+du) \wedge N_\nu(t) = 0 \wedge T_\nu > t\} +$$
$$\sum_{u=0}^{t} \Pr\left\{T_\mu \in (u,u+du) \wedge \sum_{\xi=0}^{u} dR_\nu \in (\xi,\xi+d\xi) \wedge N_\nu(t-\xi) = 0 \wedge T_\nu > t-\xi\right\} \tag{4.2.132}$$

where the terms at the right hand side account for the probability that the variable Z is in its first 'on' state, that is the time variable t is between the first R_μ event and the subsequent R_ν event and simultaneously no events driven by the Poisson process with parameter ν has occurred.
The probability density P_2 can be expressed as

$$P_2(t) =$$
$$\int_0^t g_{T_\mu}(u) e^{-\nu t} du \left(1 - F_{T_\nu}(t)\right) du + \tag{4.2.133}$$
$$\int_0^t g_{T_\mu}(u) \int_0^u h_\nu(\xi) e^{-\nu(t-\xi)} \left(1 - F_{T_\nu}(t-\xi)\right) d\xi du;$$

The variables appearing at the right hand side of the above expression are defined in Figure 4.2.8

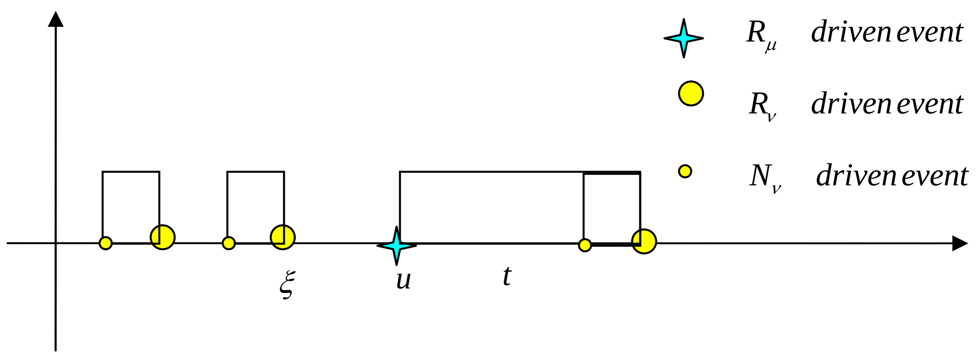

Figure 4.2.8
Definition of the variables appearing in equation (4.2.133).

State Probability $P_5(t)$

The probability $P_5 = P^{(1)}_{\rho_\mu,off/Z,on}$ that the variable ρ_μ is 'off' during Z first 'on' state can be derived as

$$P_5(t) =$$
$$\sum_{u=0}^{t} \Pr\left\{T_\mu \in (u,u+du) \wedge T_\nu > t \wedge N_\mu(t-u) = 0,2,4..\right\} +$$
$$\sum_{u=0}^{t} \Pr\left\{T_\mu \in (u,u+du) \wedge \sum_{\xi=0}^{u} dR_\nu \in (\xi,\xi+d\xi) \wedge T_\nu > t-\xi \wedge N_\mu(t-u) = 0,2,4..\right\}$$
(4.2.134)

where the terms at the right hand side account for the probability that the variable Z is in its first 'on' state, that is the time variable t is between the first R_μ driven event and the subsequent R_ν event and simultaneously between an even–number and the subsequent odd-number Poisson-driven event with parameter μ.
The state probability P_5 can be expressed as

$$P_5(t) =$$
$$\int_0^t g_{T_{N_\nu}}(u) \frac{1}{2}\left(1+e^{-2\mu(t-u)}\right)\left(1-F_{T_\nu}(t)\right) du + \qquad (4.2.135)$$
$$\int_0^t g_{T_{N_\nu}}(u) \int_0^u h_\nu(\xi)\left(1-F_{T_\nu}(t-\xi)\right)\frac{1}{2}\left(1+e^{-2\mu(t-u)}\right) d\xi du.$$

The variables appearing at the right hand side of the expression above are defined in Figure 4.2.9

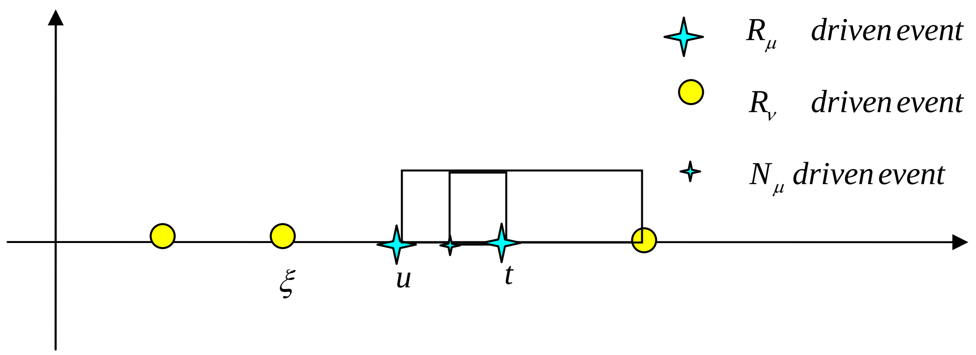

Figure 4.2.9

Definition of the variables appearing in equation (4.2.135).

State Probability $P_8(t)$

The probability $P_8 = P^{(1)}_{\rho_\mu\rho_\nu,off/Z,off}$ that the variable $\rho_\mu\rho_\nu$ is 'off' during Z first 'off' state can be expressed as

$$P_7 = P^{(1)}_{\rho_\mu\rho_\nu,off/Z,off} = P^{(1)}_{Z,off} - P^{(1)}_{\rho_\mu\rho_\nu,on/Z,off} \tag{4.2.136}$$

The conditional probability $P^{(1)}_{\rho_\mu\rho_\nu,on/Z,off}$ can be found as

$$\begin{aligned} &P^{(1)}_{\rho_\mu\rho_\nu,on/Z,off}(t)\,dt = \\ &\sum_{u=0}^{t} \Pr\left\{N_\mu \in (u,u+du) \wedge T_\mu > t \wedge N_\nu(t) = 1\right\} + \\ &\sum_{u=0}^{t} \Pr\left\{N_\mu \in (u,u+du) \wedge T_\mu > t \wedge \sum_{\xi=0}^{u} dR_\nu \in (\xi,\xi+d\xi) \wedge N_\nu(t-\xi) = 1,3,..\right\} \end{aligned} \tag{4.2.137}$$

where the terms at the right hand side account for the probability that the variable Z is in its first 'off' state and simultaneously $\rho_\mu\rho_\nu$ is 'on', that is the time variable t is between the first N_μ driven event and the subsequent R_μ event and simultaneously between an odd-number and the subsequent even-number Poisson-driven event with parameter ν.

The probability density $P^{(1)}_{\rho_\mu\rho_\nu,on/Z,off}$ can be expressed as

$$\begin{aligned} &P^{(1)}_{\rho_\mu\rho_\nu,on/Z,off}(t) = \\ &\int_0^t g_{T_{N_\nu}}(u)\,du\,\nu e^{-\nu t}\left(1 - F_{T_\mu}(t)\right) + \\ &\int_0^t g_{T_{N_\nu}}(u) \int_0^u h_\nu(\xi) \frac{1}{2}\left(1 - e^{-2\nu(t-\xi)}\right) d\xi\,du \left(1 - F_{T_\mu}(t)\right) \end{aligned} \tag{4.2.138}$$

The variables appearing at the right hand side of the expression above are defined in Figure 4.2.10

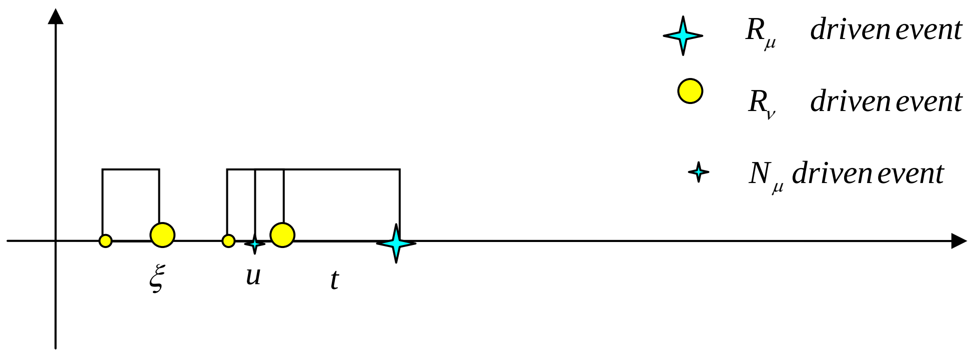

Figure 4.2.10

Definition of the variables appearing in equation (4.2.138).

State Probability $P_9(t)$

The state probability $P_9 = P^{(1)}_{\rho_\mu\rho_\nu,off/Z,on}$ that the variable $\rho_\mu\rho_\nu$ is 'off' during Z first 'on' state can be expressed as

$$P_8 = P^{(1)}_{\rho_\mu\rho_\nu,off/Z,on} = P^{(1)}_{Z,on} - P^{(1)}_{\rho_\mu\rho_\nu,on/Z,on} \tag{4.2.139}$$

the state probability $P^{(1)}_{\rho_\mu\rho_\nu,on/Z,on}$ can be found as

$$P^{(1)}_{\rho_\mu\rho_\nu,on/Z,on}(t) =$$
$$\sum_{u=0}^{t} \Pr\left\{ \begin{array}{l} T_\mu \in (u,u+du) \wedge \\ \sum_{u_1=u}^{\xi_1} dR_\mu \in (u_1,u_1+du_1) \wedge T_{N_\nu}(\xi_1,\xi_1+d\xi_1) \wedge T_\nu > t \wedge N_\mu(t-u_1) = 1,3,.. \end{array} \right\} +$$
$$\sum_{u=0}^{t} \Pr\left\{ \begin{array}{l} T_\mu \in (u,u+du) \wedge \sum_{\xi=0}^{u} dR_\nu \in (\xi,\xi+d\xi) \wedge T_{N_\nu}(\xi_1-\xi,\xi_1-\xi+d\xi_1) \\ \sum_{u_1=u}^{\xi_1} dR_\mu \in (u_1,u_1+du_1) \wedge T_\nu > t-\xi \wedge N_\mu(t-u_1) = 1,3,.. \end{array} \right\}$$
$$\tag{4.2.140}$$

where the terms at the right hand side account for the probability that the time variable t is between the first N_ν driven event following the first R_μ driven event and the subsequent N_ν event and simultaneously between an even-number and the subsequent odd-number Poisson-driven event with parameter μ.

The probability density $P^{(1)}_{\rho_\mu\rho_\nu,on/Z,on}$ can be expressed as

$$P^{(1)}_{\rho_\mu \rho_\nu, on/Z,on}(t) =$$

$$\left(1-F_{T_\nu}(t)\right)\int_0^t g_{T_\mu}(u)\left(\int_u^t g_{T_{N\nu}}(\xi_1)\left(\int_u^{\xi_1} h_\mu(u_1)\frac{1}{2}\left(1-e^{-2\mu(t-u_1)}\right)du_1\right)d\xi_1\right)du$$

$$\int_0^t g_{T_\mu}(u)\left(\int_0^u h_\nu(\xi)\, g_{T_{N\nu}}(\xi_1-\xi)\left(1-F_{T_\nu}(t-\xi)\right)d\xi\left(\int_u^{\xi_1} h_\mu(u_1)\frac{1}{2}\left(1-e^{-2\mu(t-u_1)}\right)d\xi_1\right)\right)du$$

(4.2.141)

The variables appearing at the right hand side of the expression above are defined in Figure 4.2.11

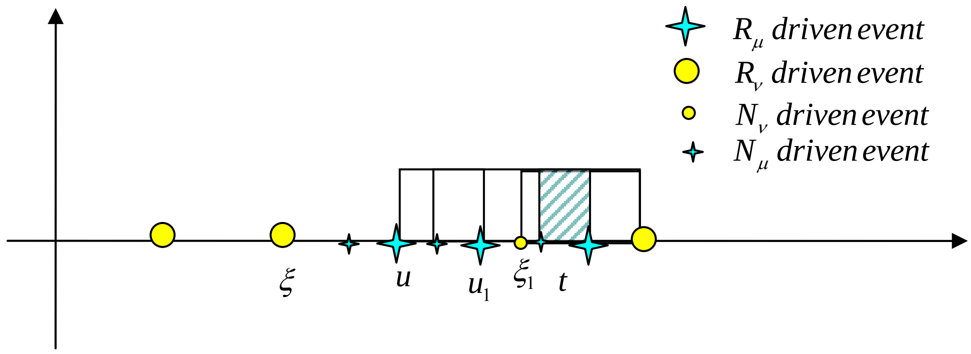

Figure 4.2.11

Definition of the variables appearing in equation (4.2.129).

Let us express the joint probability density of the state vector $\mathbf{X}$ in terms of the joint probabilities

$$p(x,\dot{x},\mathbf{z}) = \sum_{j=1}^{16} p^{(j)}(x,\dot{x},\mathbf{z}) \qquad (4.2.142)$$

where $\mathbf{z} = \left(z, \rho_\nu, \rho_\mu, \rho_\nu\rho_\mu, z\rho_\nu, z\rho_\mu, z\rho_\nu\rho_\mu\right)$.

Due to the fact that there is a finite probability of the system being in a deterministic state from the initial time to the first impulse, the joint probabilities $p^{(1)}(x,\dot{x},\mathbf{z})$ to $p^{(8)}(x,\dot{x},\mathbf{z})$ given that no impulses have occurred, can be expressed as

$$p^{(1)}(x,\dot{x},\mathbf{z}) = \bar{P}_1(t)$$
$$\delta(x-x_0)\delta(\dot{x}-\dot{x}_0)\delta(z)\delta(\rho_\nu)\delta(\rho_\mu)\delta(\rho_\mu\rho_\nu)\delta(z\rho_\nu)\delta(z\rho_\mu)\delta(z\rho_\mu\rho_\nu),$$
$$p^{(2)}(x,\dot{x},\mathbf{z}) = \bar{P}_2(t)$$
$$\delta(x-x_0)\delta(\dot{x}-\dot{x}_0)\delta(z)\delta(\rho_\nu-1)\delta(\rho_\mu)\delta(\rho_\mu\rho_\nu)\delta(z\rho_\nu)\delta(z\rho_\mu)\delta(z\rho_\mu\rho_\nu),$$
$$p^{(3)}(x,\dot{x},\mathbf{z}) = \bar{P}_3(t)$$
$$\delta(x-x_0)\delta(\dot{x}-\dot{x}_0)\delta(z)\delta(\rho_\nu)\delta(\rho_\mu-1)\delta(\rho_\mu\rho_\nu)\delta(z\rho_\nu)\delta(z\rho_\mu)\delta(z\rho_\mu\rho_\nu),$$
$$p^{(4)}(x,\dot{x},\mathbf{z}) = \bar{P}_4(t)$$
$$\delta(x-x_0)\delta(\dot{x}-\dot{x}_0)\delta(z)\delta(\rho_\nu-1)\delta(\rho_\mu-1)\delta(\rho_\mu\rho_\nu-1)\delta(z\rho_\nu)\delta(z\rho_\mu)\delta(z\rho_\mu\rho_\nu),$$
$$p^{(5)}(x,\dot{x},\mathbf{z}) = \bar{P}_5(t)$$
$$\delta(x-x_0)\delta(\dot{x}-\dot{x}_0)\delta(z-1)\delta(\rho_\nu)\delta(\rho_\mu)\delta(\rho_\mu\rho_\nu)\delta(z\rho_\nu)\delta(z\rho_\mu)\delta(z\rho_\mu\rho_\nu),$$
$$p^{(6)}(x,\dot{x},\mathbf{z}) = \bar{P}_6(t)$$
$$\delta(x-x_0)\delta(\dot{x}-\dot{x}_0)\delta(z-1)\delta(\rho_\nu-1)\delta(\rho_\mu)\delta(\rho_\mu\rho_\nu)\delta(z\rho_\nu-1)\delta(z\rho_\mu)\delta(z\rho_\mu\rho_\nu),$$
$$p^{(7)}(x,\dot{x},\mathbf{z}) = \bar{P}_7(t)$$
$$\delta(x-x_0)\delta(\dot{x}-\dot{x}_0)\delta(z-1)\delta(\rho_\nu)\delta(\rho_\mu-1)\delta(\rho_\mu\rho_\nu)\delta(z\rho_\nu)\delta(z\rho_\mu-1)\delta(z\rho_\mu\rho_\nu),$$
$$p^{(8)}(x,\dot{x},\mathbf{z}) = \bar{P}_8(t)$$
$$\delta(x-x_0)\delta(\dot{x}-\dot{x}_0)\delta(z-1)\delta(\rho_\nu-1)\delta(\rho_\mu-1)\delta(\rho_\mu\rho_\nu-1)\delta(z\rho_\nu-1)\delta(z\rho_\mu-1)\delta(z\rho_\mu\rho_\nu-1),$$

(4.2.143)

where

$$
\begin{aligned}
\bar{P}_1(t) &= P^{(1)}_{Z,off} - P^{(1)}_{\rho_\nu,on/Z,off} - P^{(1)}_{\rho_\mu,on/Z,off} + P^{(1)}_{\rho_\mu\rho_\nu,on/Z,off} \\
\bar{P}_2(t) &= P^{(1)}_{\rho_\nu,on/Z,off} - P^{(1)}_{\rho_\mu\rho_\nu,on/Z,off} \\
\bar{P}_3(t) &= P^{(1)}_{\rho_\mu,on/Z,off} - P^{(1)}_{\rho_\mu\rho_\nu,on/Z,off} \\
\bar{P}_4(t) &= P^{(1)}_{\rho_\mu\rho_\nu,on/Z,off} \\
\bar{P}_5(t) &= P^{(1)}_{Z,on} - P^{(1)}_{\rho_\nu,on/Z,on} - P^{(1)}_{\rho_\mu,on/Z,on} + P^{(1)}_{\rho_\mu\rho_\nu,on/Z,on} \\
\bar{P}_6(t) &= P^{(1)}_{\rho_\nu,on/Z,on} - P^{(1)}_{\rho_\mu\rho_\nu,on/Z,on} \\
\bar{P}_7(t) &= P^{(1)}_{\rho_\mu,on/Z,on} - P^{(1)}_{\rho_\mu\rho_\nu,on/Z,on} \\
\bar{P}_8(t) &= P^{(1)}_{\rho_\mu\rho_\nu,on/Z,on}
\end{aligned}
\qquad (4.2.144)
$$

If the joint probabilities $p^{(9)}(x,\dot{x},\mathbf{z})$ to $p^{(16)}(x,\dot{x},\mathbf{z})$ given that the first impulse has occurred are expressed as

$$p^{(9)}\left(x,\dot{x},\mathbf{z}\right)=\overline{P}_9\left(t\right)f^{(1)}\left(x,\dot{x}\right)$$
$$\delta\left(z\right)\delta\left(\rho_\nu\right)\delta\left(\rho_\mu\right)\delta\left(\rho_\mu\rho_\nu\right)\delta\left(z\rho_\nu\right)\delta\left(z\rho_\mu\right)\delta\left(z\rho_\mu\rho_\nu\right),$$
$$p^{(10)}\left(x,\dot{x},\mathbf{z}\right)=\overline{P}_{10}\left(t\right)f^{(2)}\left(x,\dot{x}\right)$$
$$\delta\left(z\right)\delta\left(\rho_\nu-1\right)\delta\left(\rho_\mu\right)\delta\left(\rho_\mu\rho_\nu\right)\delta\left(z\rho_\nu\right)\delta\left(z\rho_\mu\right)\delta\left(z\rho_\mu\rho_\nu\right),$$
$$p^{(11)}\left(x,\dot{x},\mathbf{z}\right)=\overline{P}_{11}\left(t\right)f^{(3)}\left(x,\dot{x}\right)$$
$$\delta\left(z\right)\delta\left(\rho_\nu\right)\delta\left(\rho_\mu-1\right)\delta\left(\rho_\mu\rho_\nu\right)\delta\left(z\rho_\nu\right)\delta\left(z\rho_\mu\right)\delta\left(z\rho_\mu\rho_\nu\right),$$
$$p^{(12)}\left(x,\dot{x},\mathbf{z}\right)=\overline{P}_{12}\left(t\right)f^{(4)}\left(x,\dot{x}\right)$$
$$\delta\left(z\right)\delta\left(\rho_\nu-1\right)\delta\left(\rho_\mu-1\right)\delta\left(\rho_\mu\rho_\nu-1\right)\delta\left(z\rho_\nu\right)\delta\left(z\rho_\mu\right)\delta\left(z\rho_\mu\rho_\nu\right),$$
$$p^{(13)}\left(x,\dot{x},\mathbf{z}\right)=\overline{P}_{13}\left(t\right)f^{(5)}\left(x,\dot{x}\right)$$
$$\delta\left(z-1\right)\delta\left(\rho_\nu\right)\delta\left(\rho_\mu\right)\delta\left(\rho_\mu\rho_\nu\right)\delta\left(z\rho_\nu\right)\delta\left(z\rho_\mu\right)\delta\left(z\rho_\mu\rho_\nu\right),$$
$$p^{(14)}\left(x,\dot{x},\mathbf{z}\right)=\overline{P}_{14}\left(t\right)f^{(6)}\left(x,\dot{x}\right)$$
$$\delta\left(z-1\right)\delta\left(\rho_\nu-1\right)\delta\left(\rho_\mu\right)\delta\left(\rho_\mu\rho_\nu\right)\delta\left(z\rho_\nu-1\right)\delta\left(z\rho_\mu\right)\delta\left(z\rho_\mu\rho_\nu\right),$$
$$p^{(15)}\left(x,\dot{x},\mathbf{z}\right)=\overline{P}_{15}\left(t\right)f^{(7)}\left(x,\dot{x}\right)$$
$$\delta\left(z-1\right)\delta\left(\rho_\nu\right)\delta\left(\rho_\mu-1\right)\delta\left(\rho_\mu\rho_\nu\right)\delta\left(z\rho_\nu\right)\delta\left(z\rho_\mu-1\right)\delta\left(z\rho_\mu\rho_\nu\right), \tag{4.2.145}$$
$$p^{(16)}\left(x,\dot{x},\mathbf{z}\right)=\overline{P}_{16}\left(t\right)f^{(8)}\left(x,\dot{x}\right)$$
$$\delta\left(z-1\right)\delta\left(\rho_\nu-1\right)\delta\left(\rho_\mu-1\right)\delta\left(\rho_\mu\rho_\nu-1\right)\delta\left(z\rho_\nu-1\right)\delta\left(z\rho_\mu-1\right)\delta\left(z\rho_\mu\rho_\nu-1\right),$$

where

$$\begin{aligned}
&\overline{P}_9\left(t\right)=P^{\#}_{Z,off}-P^{\#}_{\rho_\nu,on/Z,off}-P^{\#}_{\rho_\mu,on/Z,off}+P^{\#}_{\rho_\mu\rho_\nu,on/Z,off}=P_3\left(t\right)+P_6\left(t\right)-P_{10}\left(t\right)\\
&\overline{P}_{10}\left(t\right)=P^{\#}_{\rho_\nu,on/Z,off}-P^{\#}_{\rho_\mu\rho_\nu,on/Z,off}=-P_3\left(t\right)+P_{10}\left(t\right)\\
&\overline{P}_{11}\left(t\right)=P^{\#}_{\rho_\mu,on/Z,off}-P^{\#}_{\rho_\mu\rho_\nu,on/Z,off}=-P_6\left(t\right)+P_{10}\left(t\right)\\
&\overline{P}_{12}\left(t\right)=P^{\#}_{\rho_\mu\rho_\nu,on/Z,off}=-P_{10}\left(t\right)+P^{\#}_{Z,off}\left(t\right)\\
&\overline{P}_{13}\left(t\right)=P^{\#}_{Z,on}-P^{\#}_{\rho_\nu,on/Z,on}-P^{\#}_{\rho_\mu,on/Z,on}+P^{\#}_{\rho_\mu\rho_\nu,on/Z,on}=P_4\left(t\right)+P_7\left(t\right)-P_{11}\left(t\right)\\
&\overline{P}_{14}\left(t\right)=P^{\#}_{\rho_\nu,on/Z,on}-P^{\#}_{\rho_\mu\rho_\nu,on/Z,on}=P_{11}\left(t\right)-P_4\left(t\right)\\
&\overline{P}_{15}\left(t\right)=P^{\#}_{\rho_\mu,on/Z,on}-P^{\#}_{\rho_\mu\rho_\nu,on/Z,on}=P_{11}\left(t\right)-P_7\left(t\right)\\
&\overline{P}_{16}\left(t\right)=P^{\#}_{\rho_\mu\rho_\nu,on/Z,on}=-P_{11}\left(t\right)+P^{\#}_{Z,on}\left(t\right)
\end{aligned} \tag{4.2.146}$$

Let us consider the case of zero initial conditions. Since the auxiliary variables are zero-one processes, the following relationships hold

$$E\left[X^k\dot{X}^l Z^m\right]=E\left[X^k\dot{X}^l Z\right];E\left[X^k\dot{X}^l \rho_\nu{}^m\right]=E\left[X^k\dot{X}^l \rho_\nu\right];$$
$$E\left[X^k\dot{X}^l \rho_\mu{}^m\right]=E\left[X^k\dot{X}^l \rho_\mu\right];E\left[X^k\dot{X}^l\left(Z\rho_\nu\right)^m\right]=E\left[X^k\dot{X}^l Z\rho_\nu\right];$$
$$E\left[X^k\dot{X}^l\left(Z\rho_\mu\right)^m\right]=E\left[X^k\dot{X}^l Z\rho_\mu\right];E\left[X^k\dot{X}^l\left(Z\rho_\nu\rho_\mu\right)^m\right]=E\left[X^k\dot{X}^l Z\rho_\nu\rho_\mu\right];$$

(4.2.147)

The unconditional moment of order $p=m+n$ involving displacements and velocity can be expressed in terms of the conditional moments of the same order as follows

$$E[X^m\dot{X}^n]=\sum_{j=1}^{8}P_{j+8}E^{(j)}[X^m\dot{X}^n] \tag{4.2.148}$$

The unconditional moment of order $p+1=m+n+1$ involving also the auxiliary variables can be expressed in terms of the conditional moments of order p as follows

$$\begin{aligned}
&E[X^m\dot{X}^n Z]=\sum_{j=1}^{4}P_{12+j}E^{(j+4)}[X^m\dot{X}^n];\\
&E[X^m\dot{X}^n\rho_\nu]=P_{10}E^{(2)}[X^m\dot{X}^n]+P_{12}E^{(4)}[X^m\dot{X}^n]+\\
&P_{14}E^{(6)}[X^m\dot{X}^n]+P_{16}E^{(8)}[X^m\dot{X}^n];\\
&E[X^m\dot{X}^n\rho_\mu]=P_{11}E^{(3)}[X^m\dot{X}^n]+P_{13}E^{(5)}[X^m\dot{X}^n]+\\
&P_{15}E^{(7)}[X^m\dot{X}^n]+P_{16}E^{(8)}[X^m\dot{X}^n];\\
&E[X^m\dot{X}^n\rho_\nu\rho_\mu]=P_{12}E^{(4)}[X^m\dot{X}^n]+P_{16}E^{(8)}[X^m\dot{X}^n];\\
&E[X^m\dot{X}^n Z\rho_\nu]=P_{14}E^{(6)}[X^m\dot{X}^n]+P_{16}E^{(8)}[X^m\dot{X}^n];\\
&E[X^m\dot{X}^n Z\rho_\mu]=P_{15}E^{(7)}[X^m\dot{X}^n]+P_{16}E^{(8)}[X^m\dot{X}^n];\\
&E[X^m\dot{X}^n Z\rho_\nu\rho_\mu]=P_{16}E^{(8)}[X^m\dot{X}^n].
\end{aligned} \tag{4.2.149}$$

The following relationships between the conditional moments of order p=m+n and the unconditional ones can be derived:

$$E^{(1)}\left[X^m\dot{X}^n\right]=$$
$$\frac{E\left[X^m\dot{X}^n\right]-E\left[X^m\dot{X}^n\rho_\nu\right]-E\left[X^m\dot{X}^n\rho_\mu\right]+E\left[X^m\dot{X}^nZ\right]-E\left[X^m\dot{X}^nZ\rho_\mu\right]+E\left[X^m\dot{X}^nZ\rho_\nu\rho_\mu\right]}{P_9};$$
$$E^{(2)}\left[X^m\dot{X}^n\right]=$$
$$\frac{E\left[X^m\dot{X}^n\rho_\nu\right]-E\left[X^m\dot{X}^nZ\rho_\nu\right]-E\left[X^m\dot{X}^n\rho_\nu\rho_\mu\right]+E\left[X^m\dot{X}^nZ\rho_\nu\rho_\mu\right]}{P_{10}};$$
$$E^{(3)}\left[X^m\dot{X}^n\right]=$$
$$\frac{E\left[X^m\dot{X}^n\rho_\mu\right]-E\left[X^m\dot{X}^nZ\right]+E\left[X^m\dot{X}^nZ\rho_\nu\right]+E\left[X^m\dot{X}^nZ\rho_\mu\right]-E\left[X^m\dot{X}^nZ\rho_\nu\rho_\mu\right]}{P_{11}};$$
$$E^{(4)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^n\rho_\nu\rho_\mu\right]-E\left[X^m\dot{X}^nZ\rho_\nu\rho_\mu\right]}{P_{12}};$$
$$E^{(5)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^nZ\right]-E\left[X^m\dot{X}^nZ\rho_\nu\right]-E\left[X^m\dot{X}^nZ\rho_\mu\right]+E\left[X^m\dot{X}^nZ\rho_\nu\rho_\mu\right]}{P_{13}};$$
$$E^{(6)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^nZ\rho_\nu\right]-E\left[X^m\dot{X}^nZ\rho_\nu\rho_\mu\right]}{P_{14}};$$
$$E^{(7)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^nZ\rho_\mu\right]-E\left[X^m\dot{X}^nZ\rho_\nu\rho_\mu\right]}{P_{15}};$$
$$E^{(8)}\left[X^m\dot{X}^n\right]=\frac{E\left[X^m\dot{X}^nZ\rho_\nu\rho_\mu\right]}{P_{16}}.$$
(4.2.150)

Let us perform the ordinary cumulant neglect closure on the eight sets of conditional moments. The conditional moments of order s higher than the closure order r are expressed in terms of conditional moments of order lower than r, which in turn can be derived as functions of the unconditional moments of order lower than r through equation (4.2.150). From equations (4.2.148) and (4.2.149), the modified closure scheme for the unconditional moments becomes

$$E\left[X_1...X_s\right]=\sum_{q=1}^{8}P_{q+8}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\frac{1}{P_{q+8}}\left(\bar{E}_q\right)\right]\right\};$$
$$E\left[X_1...X_sZ\right]=\sum_{q=1}^{4}P_{q+12}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\frac{1}{P_{q+12}}\left(\bar{E}_{q+4}\right)\right]\right\};$$
$$E\left[X_1...X_s\rho_\nu\right]=\sum_{q=1}^{4}P_{2q+8}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\frac{1}{P_{2q+8}}\left(\bar{E}_{2q}\right)\right]\right\};$$

$$E[X_1...X_s\rho_\mu]=\sum_{q=1}^{3}P_{2q+9}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\frac{1}{P_{2q+9}}\left(\bar{E}_{2q+1}\right)\right]\right\}+$$

$$P_{16}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\frac{1}{P_{16}}\left(\bar{E}_8\right)\right]\right\};$$

$$E\left[X_1...X_s\rho_\nu\rho_\mu\right]=\sum_{q=1}^{2}P_{4q+8}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\frac{1}{P_{4q+8}}\left(\bar{E}_{4q}\right)\right]\right\};$$

$$E\left[X_1...X_sZ\rho_\nu\right]=P_{14}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\frac{1}{P_{14}}\left(\bar{E}_6\right)\right]\right\}+$$

$$P_{16}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\frac{1}{P_{16}}\left(\bar{E}_8\right)\right]\right\};$$

$$E\left[X_1...X_sZ\rho_\mu\right]=P_{15}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\frac{1}{P_{15}}\left(\bar{E}_7\right)\right]\right\}+$$

$$P_{16}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\frac{1}{P_{16}}\left(\bar{E}_8\right)\right]\right\}; \qquad (4.2.151)$$

$$E\left[X_1...X_sZ\rho_\nu\rho_\mu\right]=P_{16}\sum_{\pi^r}\prod_{B^r\in\pi^r}\left\{\sum_{\substack{\pi^i\\ i\in B^r}}\prod_{B^i\in\pi^i}\left[\left(\left|B^i\right|-1\right)!(-1)^{\left|B^i\right|-1}\frac{1}{P_{16}}\left(\bar{E}_8\right)\right]\right\};$$

where

$$\bar{E}_1=E\left[\prod_{j\in B^i}X_j\right]-E\left[\prod_{j\in B^i}X_j\rho_\nu\right]-E\left[\prod_{j\in B^i}X_j\rho_\mu\right]+E\left[\prod_{j\in B^i}X_jZ\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\mu\right]+E\left[\prod_{j\in B^i}X_jZ\rho_\nu\rho_\mu\right];$$

$$\bar{E}_2=E\left[\prod_{j\in B^i}X_j\rho_\nu\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\nu\right]-E\left[\prod_{j\in B^i}X_j\rho_\nu\rho_\mu\right]+E\left[\prod_{j\in B^i}X_jZ\rho_\nu\rho_\mu\right];$$

$$\bar{E}_3=E\left[\prod_{j\in B^i}X_j\rho_\mu\right]-E\left[\prod_{j\in B^i}X_jZ\right]+E\left[\prod_{j\in B^i}X_jZ\rho_\nu\right]+E\left[\prod_{j\in B^i}X_jZ\rho_\mu\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\nu\rho_\mu\right];$$

$$\bar{E}_4=E\left[\prod_{j\in B^i}X_j\rho_\nu\rho_\mu\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\nu\rho_\mu\right];$$

$$\bar{E}_5=E\left[\prod_{j\in B^i}X_jZ\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\nu\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\mu\right]+E\left[\prod_{j\in B^i}X_jZ\rho_\nu\rho_\mu\right];$$

$$\bar{E}_6=E\left[\prod_{j\in B^i}X_jZ\rho_\nu\right]-E\left[\prod_{j\in B^i}X_jZ\rho_\nu\rho_\mu\right];$$

$$\bar{E}_7 = E\left[\prod_{j\in B^i} X_j Z \rho_\mu\right] - E\left[\prod_{j\in B^i} X_j Z \rho_\nu \rho_\mu\right];$$

$$\bar{E}_8 = E\left[\prod_{j\in B^i} X_j Z \rho_\nu \rho_\mu\right];$$

4.2.7 Numerical analysis

Consider a Duffing oscillator governed by the stochastic differential equation (4.2.1). The data assumed for the Duffing oscillator is: $\omega_0 = 1s^{-1}$, $\zeta = 0.05$ and $\varepsilon = 0.5$. Computations have been performed for the load processes I and II characterized in sections (4.2.3) and (4.2.4). The values of the parameters of the driving Erlang and Poisson processes have been assumed in such a way that the mean arrival rate of the impulses $\frac{1}{E[T]} = 0.008\frac{2\pi}{\omega_o}$.

The random magnitudes of impulses have been assumed as centralized, Rayleigh distributed random variables. The values of the parameter $\sigma_r = E\left[P^2\right]/\sqrt{2}$ for each case have been chosen so that the stationary value of the variance of the response of the corresponding linear oscillator has a unit value.

To verify the approximate analytical results, the response moments have been obtained from Monte Carlo simulations based on averaging 30000 of the response sample functions, obtained by numerical integration of the equation of motion (4.2.1) with the aid of the computer program Mathematica.

The analytical results are obtained by applying the ordinary and the modified cumulant-neglect closure techniques, neglecting in both schemes the cumulants above the second order. Transient response statistics of the non-linear oscillator are shown in Figures 4.2.12.and 4.2.13.

In the case of low mean arrival rate of impulses, the application of higher order ordinary cumulant neglect closure does not lead to improved results since it becomes numerically unstable. The modified scheme, on the other hand, provides very good predictions of the transient mean value and variance with a second order closure.

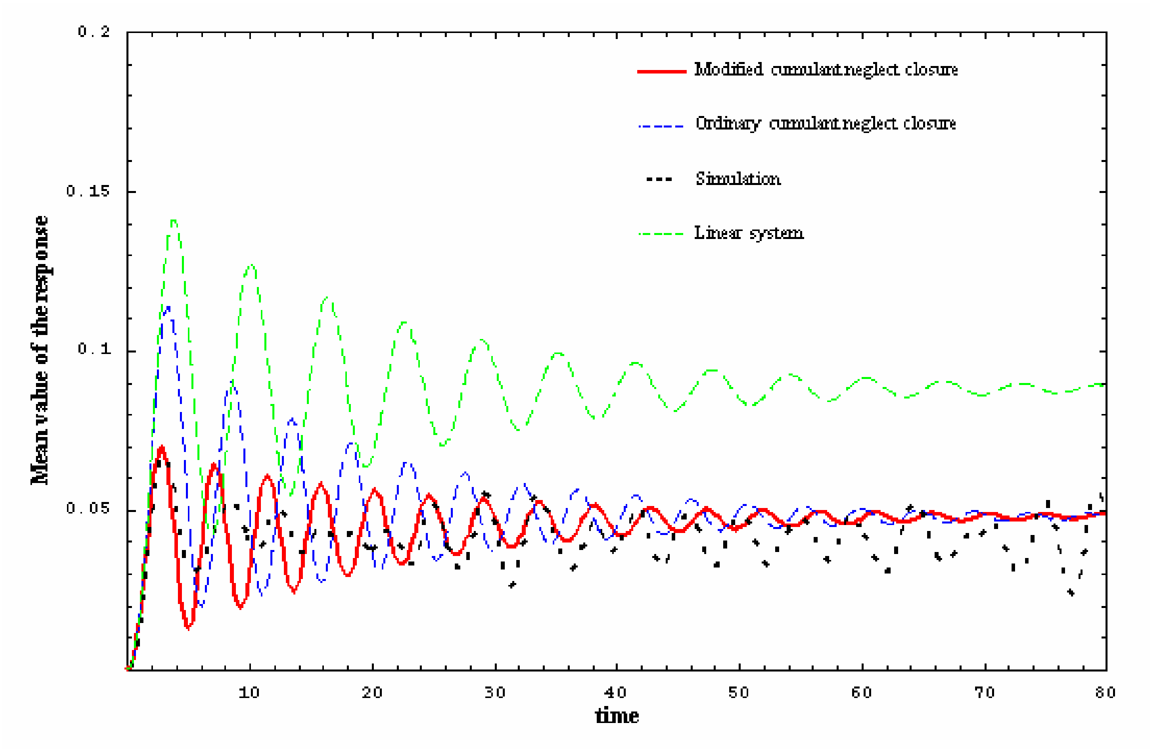

Figure 4.2.12 (a)
Mean value of the response of a Duffing oscillator to a random train of impulses driven by the non-Erlang process I (section 4.2.3) driven by two Poisson processes with parameters μ=0.05 and ν=1

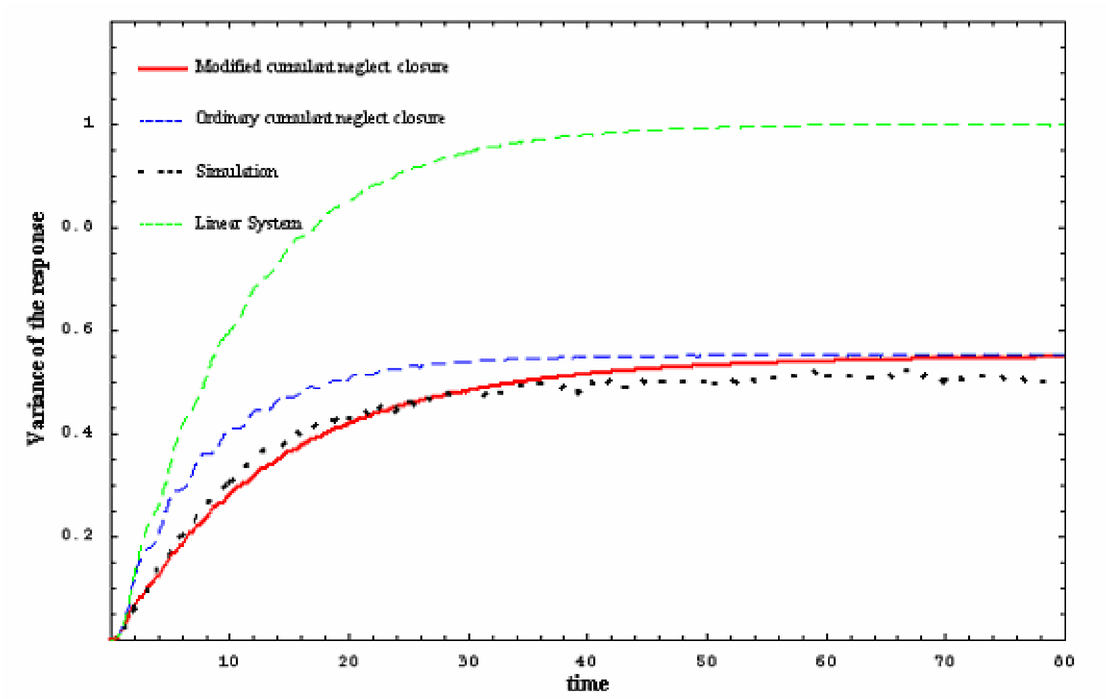

Figure 4.2.12 (b)
Variance of the response of a Duffing oscillator to a random train of impulses driven by the non-Erlang process I (section 4.2.3) driven by two Poisson processes with parameters μ=0.05 and ν=1

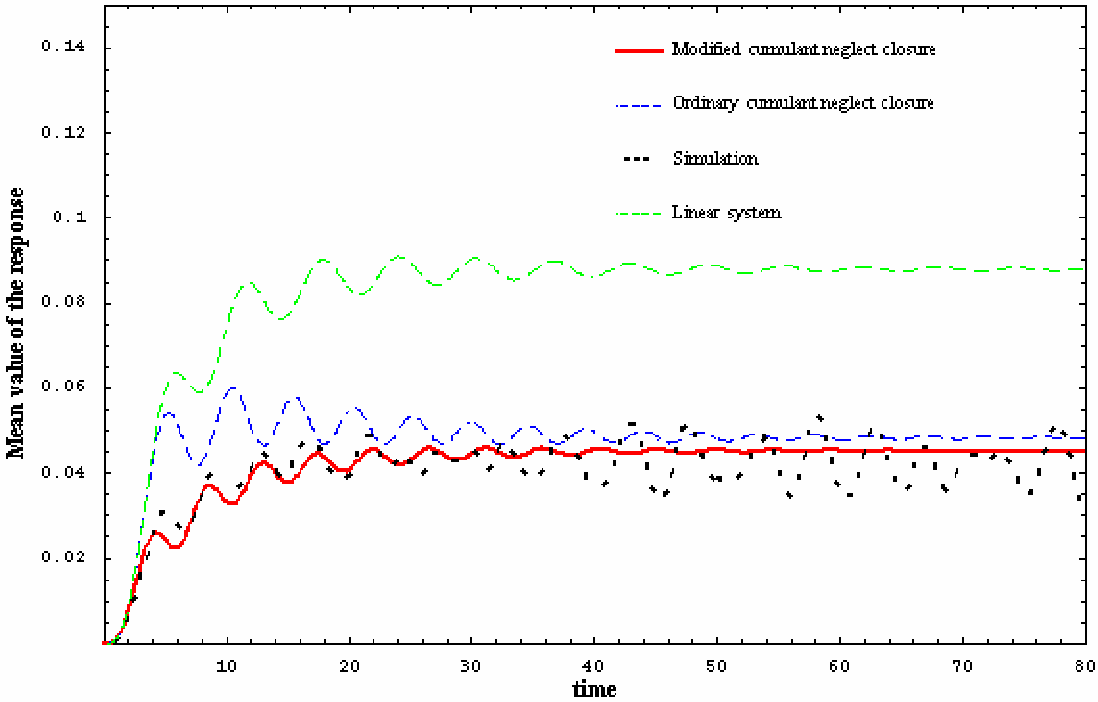

Figure 4.2.13 (a)
Mean value of the response of a Duffing oscillator to a random train of impulses driven by the non-Erlang process II (section 4.2.4) driven by an Erlang process with parameters μ=0.1 and l=2 and a Poisson process with parameter ν=1

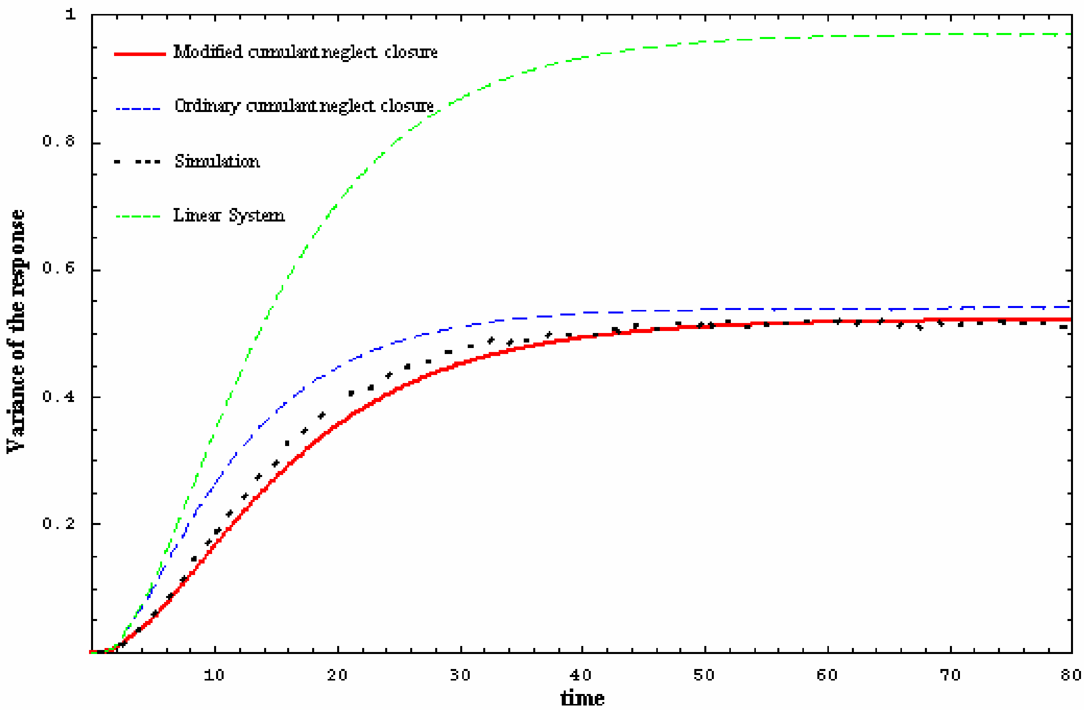

Figure 4.2.13 (b)
Variance of the response of a Duffing oscillator to a random train of impulses driven by the non-Erlang process II (section 4.2.4) driven by an Erlang process with parameters μ=0.1 and l=2 and a Poisson process with parameter ν=1

CONCLUSIONS

A non-diffusive Markov processes approach has been developed for dynamic systems under a class of renewal-driven trains of impulses. The considered class embraces Erlang-driven impulse processes as well as the impulse processes obtained by selecting impulses from an Erlang-driven train. In the latter model the impulses are selected with the aid of an auxiliary jump, zero-one, stochastic variable governed by a stochastic differential equation driven by two independent Erlang processes. The underlying counting process has been proved to be a renewal (non-Erlang) process. The proof hinges on the evaluated probability density functions of the first and second waiting times.

Conversion of the non-Markov problem for the original state vector driven by a renewal impulse process into a Markov problem is performed by means of augmenting the state vector by auxiliary variables which are the jump stochastic processes. A novel technique of recasting an Erlang renewal process in terms of the Poisson process has been developed, where the jump processes are zero-one. Thus for the Erlang renewal impulse processes the augmented state vector is driven by single Poisson process and for the non-Erlang impulse process it is driven by the independent Poisson processes. Consequently the augmented state vectors are non-diffusive Markov processes.

For non-linear dynamic systems with polynomial non-linearities under both Erlang renewal impulses and the considered class of non-Erlang impulses the technique of equations for moments combined with a novel modified cumulant neglect closure technique has been devised. This technique is based on conditioning the joint probability density function of the augmented state vector on the ‘on’ and ‘off’ states of the auxiliary zero-one variables. The form of the joint probability density function allows to derive the relationships between the unconditional and conditional moments. Application of the ordinary cumulant neglect closure scheme to the conditional moments leads to the modified cumulant neglect closure technique. The validity and accuracy of the developed technique has been examined at the example of the Duffing oscillator. The equations for moments have been closed at the second-order moments level, with the aid of the ordinary and modified cumulant neglect closure techniques and the results have been verified against Monte Carlo simulations. The results have shown that for highly non-Gaussian case of sparse trains of impulses (low mean arrival rate) the modified closure scheme provides more accurate results than the ordinary cumulant neglect closure.

The coefficients appearing in the equation (3.2.52), giving the probability density of the interarrival times for the non-Erlang renewal process IV, take on the following form

$$B_1 = -\frac{\mu^4 (\mu - 2\nu)\nu \left(6\mu^4 - 2\mu^3\nu - 45\mu^2\nu^2 + 9\mu\nu^3 + 74\nu^4\right)}{2(\mu-\nu)^3 \left(2\mu^3 + \mu^2\nu - 6\mu\nu^2 - 4\nu^3\right)^2};\ B_2 = -\frac{\mu^2(\mu - 2\nu)^2\nu^2(\mu + 2\nu)}{2(\mu-\nu)^2\left(2\mu^3 + \mu^2\nu - 6\mu\nu^2 - 4\nu^3\right)};$$

$$B_3 = -\frac{8\mu\nu^4\left(10\mu^6 + 9\mu^4\nu^2 + 3\mu^2\nu^4 - \nu^6\right)}{(\mu-\nu)^3(\mu+\nu)^3\left(4\mu^2 + 3\nu^2\right)^2};\ B_4 = \frac{4\mu^6\nu^2 + 3\mu^4\nu^4 - \mu^2\nu^6}{(\mu-\nu)^2(\mu+\nu)^2\left(4\mu^2 + 3\nu^2\right)};$$

$$B_5 = \frac{\mu^4\nu(\mu+2\nu)\left(6\mu^4 + 2\mu^3\nu - 45\mu^2\nu^2 - 9\mu\nu^3 + 74\nu^4\right)}{2(\mu+\nu)^3\left(-2\mu^3 + \mu^2\nu + 6\mu\nu^2 - 4\nu^3\right)^2};\ B_6 = -\frac{\mu^2(\mu - 2\nu)\nu^2(\mu+2\nu)^2}{2(\mu+\nu)^2\left(2\mu^3 - \mu^2\nu - 6\mu\nu^2 + 4\nu^3\right)};$$

$$\begin{aligned} B_8 = {} & \\ & \frac{1}{b_1(b_1+b_2)+b_3}\Big(b_2\left(\mu^2 B_1 - 2\mu B_2 + \nu(\nu B_3 - 2B_4) + (\mu+2\nu)((\mu+2\nu)B_5 - 2B_6)\right) + b_1\left(\left(\mu^2 - b_2^2 + b_3\right)B_1 - 2\mu B_2 + \nu(\nu B_3 - 2B_4) - \left(b_2^2 - b_3\right)(B_3+B_5) + (\mu+2\nu)((\mu+2\nu)B_5 - 2B_6)\right) + \\ & \quad b_3(\mu B_1 - B_2 + \nu B_3 - B_4 + (\mu+2\nu)B_5 - B_6) + b_1^2((\mu - b_2)B_1 - B_2 + \nu B_3 - B_4 + (\mu+2\nu)B_5 - b_2(B_3+B_5) - B_6) + b_2^2(-\mu B_1 + B_2 - \nu B_3 + B_4 - (\mu+2\nu)B_5 + B_6)\Big); \end{aligned}$$

$$\begin{aligned} B_7 = {} & \\ & -\frac{\left(\mu^2 - \mu b_2 + b_3\right)B_1 + (-2\mu + b_2)B_2 + \nu^2 B_3 - \nu b_2 B_3 + b_3 B_3 - 2\nu B_4 + b_2 B_4 + \mu^2 B_5 + 4\mu\nu B_5 + 4\nu^2 B_5 - \mu b_2 B_5 - 2\nu b_2 B_5 + b_3 B_5 - (2\mu + 4\nu - b_2)B_6}{b_1(b_1+b_2)+b_3}; \end{aligned}$$

$$B_9 = \frac{1}{b_1(b_1+b_2)+b_3}\left((\mu+b_1)(\mu - b_1 - b_2)B_1 + (-2\mu + b_2)B_2 + \left(\nu^2 - b_1^2\right)B_3 - \nu b_2 B_3 - b_1 b_2 B_3 - 2\nu B_4 + b_2 B_4 + \mu^2 B_5 + 4\mu\nu B_5 + 4\nu^2 B_5 - b_1^2 B_5 - \mu b_2 B_5 - 2\nu b_2 B_5 - b_1 b_2 B_5 - (2\mu + 4\nu - b_2)B_6\right);$$

$$b_1 = \frac{1}{9}\left(-13\mu - 9\nu + \frac{25\mu^2 - 9\nu^2}{\left(125\mu^3 - 189\mu\nu^2 + 9\sqrt{-375\mu^4\nu^2 + 366\mu^2\nu^4 + 9\nu^6}\right)^{1/3}} + \left(125\mu^3 - 189\mu\nu^2 + 9\sqrt{-375\mu^4\nu^2 + 366\mu^2\nu^4 + 9\nu^6}\right)^{1/3}\right);$$

$$b_2 = \frac{1}{9}\left(18\nu + \frac{-9\nu^2 + \left(\mu + \left(125\mu^3 - 189\mu\nu^2 + 9\sqrt{-375\mu^4\nu^2 + 366\mu^2\nu^4 + 9\nu^6}\right)^{1/3}\right)\left(25\mu + \left(125\mu^3 - 189\mu\nu^2 + 9\sqrt{-375\mu^4\nu^2 + 366\mu^2\nu^4 + 9\nu^6}\right)^{1/3}\right)}{\left(125\mu^3 - 189\mu\nu^2 + 9\sqrt{-375\mu^4\nu^2 + 366\mu^2\nu^4 + 9\nu^6}\right)^{1/3}}\right);$$

$$\begin{aligned} b_3 = {} & \left(12(\mu+\nu)\left(25\mu^2 - 9\nu^2\right)^2\left(\mu^2 + 3\mu\nu + \nu^2\right)\right) \Big/ \\ & \Big((13\mu + 9\nu)\left(25\mu^2 - 9\nu^2\right)^2 - \left(25\mu^2 - 9\nu^2\right)^2\left(125\mu^3 - 189\mu\nu^2 + 9\sqrt{-375\mu^4\nu^2 + 366\mu^2\nu^4 + 9\nu^6}\right)^{1/3} - \mu\left(125\mu^2 - 189\nu^2\right)\left(125\mu^3 - 189\mu\nu^2 + 9\sqrt{-375\mu^4\nu^2 + 366\mu^2\nu^4 + 9\nu^6}\right)^{2/3} + \\ & \quad 9\sqrt{-375\mu^4\nu^2 + 366\mu^2\nu^4 + 9\nu^6}\left(125\mu^3 - 189\mu\nu^2 + 9\sqrt{-375\mu^4\nu^2 + 366\mu^2\nu^4 + 9\nu^6}\right)^{2/3}\Big); \end{aligned}$$

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