ALGEBRAIC CLOSURE MODELS APPLIED TO THE TWO-DIMENSIONAL TURBULENT CLASSICAL FAR WAKE

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Master of Science.

June 2020

Declaration of Authorship

I, Kendall BORN, declare that this dissertation titled, 'Algebraic closure models applied to the two-dimensional turbulent classical far wake' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this dissertation has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this dissertation is entirely my own work.
- I have acknowledged all main sources of help.
- Where the dissertation is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ____

Date: 02/06/2020

“I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic."

Horace Lamb

Abstract

The study of turbulence in fluids is of great importance because turbulence occurs in natural phenomena and has practical uses in industry. In this dissertation, a brief history of turbulence is presented. Particular attention is then paid to the two-dimensional turbulent classical wake and the various algebraic closure models used to complete the system of equations. The system of partial differential equations is reduced to a system of ordinary differential equations using the Lie point symmetry associated with the elementary conserved vector. When comparing with experimental results, the closure models considered consists of the constant eddy viscosity model, Prandtl's mixing length model, and Prandtl's improved mixing length model. The profile of the mean velocity deficit for each model is plotted on the same set of axes as the experimental profile. These profiles are compared to determine which closure model provides a better prediction of the velocity profile.

Acknowledgements

I would like to thank my supervisors Dr. A. J Hutchinson and Prof. D. P Mason.

I am deeply indebted to Dr. A. J Hutchinson. Your passion for academics continues to inspire me. Thank your for your guidance and advice, believing in me and your reassurance throughout this dissertation. I could not have picked a better mentor.

I would also like to extend my deepest gratitude to Prof. D. P Mason. Your experience and willingness to share your extensive knowledge of fluid mechanics is unparalleled. I will be forever grateful.

I very much appreciate the lecturing opportunity provided by Prof. R. J Moitsheki and the school of Computer Science and Applied Mathematics, without which I could not have funded my Masters.

My sincere thanks go to Timo Harsch for granting me permission to use his photograph in Figure 2.1c. Timo woke up one at 3am to take this photograph on the 30th of March 2014 after weeks of planning. It is a prime example of aircraft wake turbulence and it is an honour to place it in this dissertation.

This dissertation compiled smoothly in LaTeX thanks to the assistance that I received from Mr. W. A Carlson.

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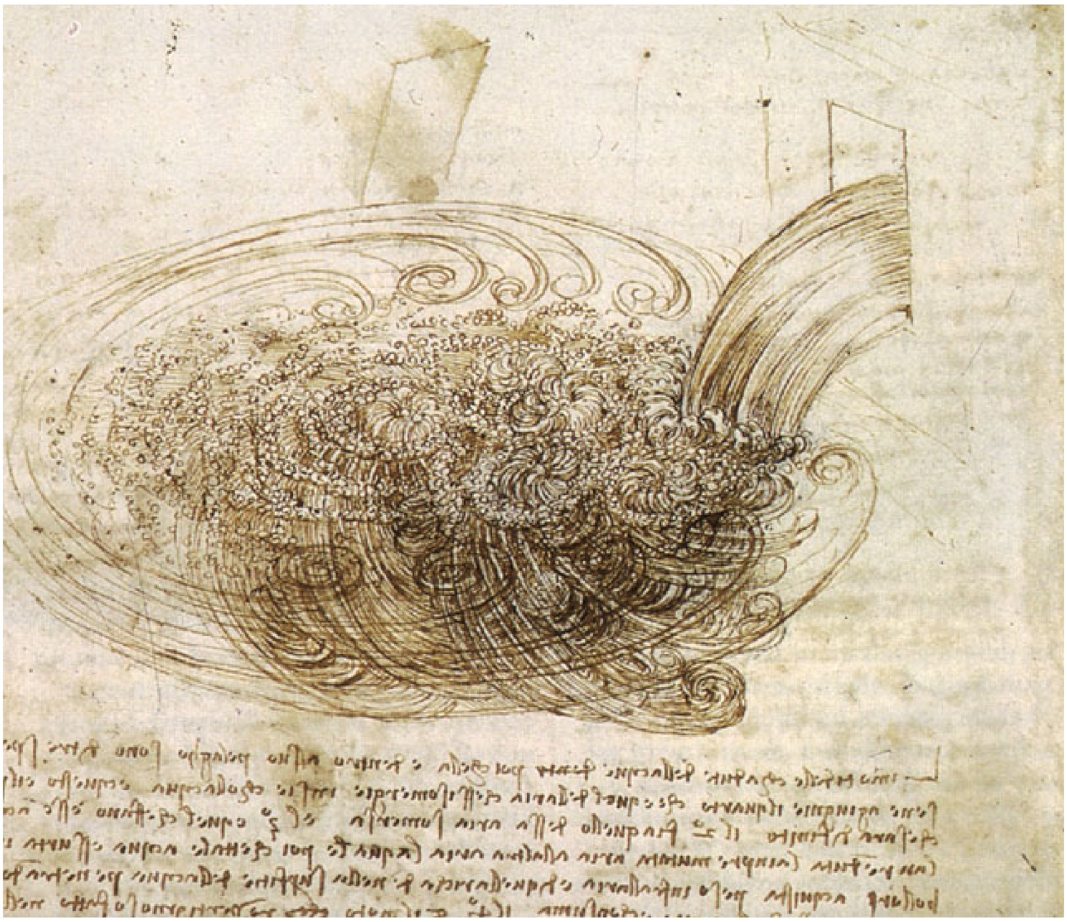


FIGURE 1: Leonardo da Vinci's sketch of turbulence and vortices generated from water falling into a pool. (circa 1508)

Chapter 1

A STUDY OF TURBULENCE

In the modern world everyone comes into contact with turbulence daily. Passengers experiencing turbulence in aeroplanes, the avid coffee drinker's turbulent vortices that mix the milk in their cup of coffee, turbulent smoke plumes rising from a smoker's cigarette, children identifying animals in clouds in the turbulent atmosphere [1] and surfers on waves in the Earth's turbulent oceans [2] are fine examples of such interactions. It comes as no surprise that the fascination to understand turbulence dates as far back as 1508 when Leonardo da Vinci studied the phenomena in his observations of vortices [3]. Darrigol [4] attributes the first usage of the word 'turbulence' as a noun back to papers in 1887 which were written by William Thomson (Lord Kelvin). In Thomson's paper 'Broad River flowing down an Inclined Plane Bed' [5], the first appearance of the word turbulence occurs as seen in Figure 1.1a. In Thomson's paper 'On the Propagation of Laminar Motion through a turbulently moving Inviscid Liquid' [6], turbulence appeared for the first time in a title as shown in Figure 1.1b. Boussinesq used the terms "eddy agitations" (agitation tourbillonnaire) [7, pp. 6] or "liquid eddy theory" (théorie des tourbillons liquides) [8]. Twenty years later Reynolds used the terms "sinuous or eddying motion" [9, pp. 124] or "relative disturbance" [9, pp. 130]. Schmitt [10] points out that the first emergence of the term into main publications was due to Horace Lamb [11] and to Frederick Lanchester [12] in 1906 and 1907 respectively. Investigating turbulence is not only important for

XXXIV. *Stability of Motion (continued from the May, June, and August Numbers).—Broad River flowing down an Inclined Plane Bed. By* Sir William Thomson, *F.R.S.**

41. CONSIDER now the second of the two cases referred to in § 27—that is to say, the case of water on an inclined plane bottom, under a fixed parallel plane cover (ice, for example), both planes infinite in all directions and gravity everywhere uniform. We shall include, as a sub-case, the icy cover moving with the water in contact with it, which is particularly interesting, because, as it annuls tangential force at the upper surface, it is, for the steady motion, the same case as that of a broad open river flowing uniformly over a perfectly smooth inclined plane bed. It is not the same, except when the motion is steadily laminar, the difference being that the surface is kept rigorously plane, but not free from tangential force, by a rigid cover, while the open surface is kept almost but not quite rigorously plane by gravity, and rigorously free from tangential force. But, provided the bottom is smooth, the smallness of the dimples and little round hollows which we see on the surface, produced by turbulence (when the motion is turbulent), seems to prove that the motion must be very nearly the same as it would be if the upper surface were kept rigorously plane, and free from tangential force.

(A) A copy of a section of the first page of 'Broad River flowing down an Inclined Plane Bed' by William Thomson [5]. The word turbulence (underlined), has been introduced for the first time into academic literature as a noun.

XLV. *On the Propagation of Laminar Motion through a turbulently moving Inviscid Liquid. By* Sir William Thomson, *LL.D., F.R.S.**

1. IN endeavouring to investigate turbulent motion of water between two fixed planes, for a promised communication to Section A of the British Association at its coming Meeting in Manchester, I have found something seemingly towards a solution (many times tried for within the last twenty years) of the problem to construct, by giving vortex motion to an incompressible inviscid fluid, a medium which shall transmit waves of laminar motion as the luminiferous æther transmits waves of light.

2. Let the fluid be unbounded on all sides, and let u, v, w be the velocity-components, and p the pressure at (x, y, z, t). We have

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \quad . \quad . \quad . \quad . \quad . \quad (1),$$

(B) A copy of a section of the first page of 'On the Propagation of Laminar Motion through a turbulently moving Inviscid Liquid' by William Thomson [6]. The word turbulently (underlined) appears for the first time in a title, and turbulent motion (underlined) is used to describe the fluid flow.

FIGURE 1.1: Two paper excerpts by Sir William Thomson

academics. Understanding turbulence has applications in industry, predicting weather patterns, and understanding ocean currents. So what is turbulence and why is turbulence such a difficult problem to model mathematically?

The behaviour of turbulence is unpredictable and is inherently nonlinear. As a consequence the flow varies considerably both spatially and temporally [13, 14]. Unfortunately for mathematicians, physicists, and engineers who study turbulence, it is an extremely difficult problem. On a large scale turbulence appears to slowly dissipate according to some type of law. However, when following individual particles and investigating the smaller scales, randomness and chaos proliferates. It is here in the smaller scales that turbulence is 'unsolved'. One of the greatest mathematicians of the 20^{th} century, Kolmogorov, has said, "I soon understood that there was little hope of developing a pure, closed theory, and because of the absence of such a theory the investigation must be based on hypotheses obtained in processing experimental data." [15].

The majority of flows are turbulent and therefore it is essential to develop accurate descriptions in order to study the behaviour of systems involving turbulence.

Reliable models are needed in order to predict complex systems such as the weather, drag in oil pipes, and atmospheric flows. Before one can construct such models, one needs to understand what type of turbulence needs to be modelled. Laminar flow transitions into turbulent flow and with increasing Reynolds number becomes increasingly turbulent. In order to define what turbulence is, fully formed turbulence at large Reynolds numbers will be considered. For fully formed turbulence, the velocity profile does not change farther downstream. An essential feature of turbulence is that without a generator (continuous supply of energy), turbulent flow will quickly dissipate back into a laminar flow.

This dissertation is outlined as follows:

- Chapter 2: In this chapter, a brief introduction to turbulent free shear flows is provided. Classical wakes, wakes generated by a self-propelled body, jets, and plumes which are fine examples of turbulent free shear flows, are discussed. Motivations for this work which focuses predominantly on the two-dimensional turbulent classical wake are given.

- Chapter 3: In this chapter, an overview of various closure models is given and the Reynolds-averaged Navier-Stokes equations, which serve as the basis for modelling turbulent flows, are derived. The physical interpretations of the algebraic closure models, which are used in this work, are provided. The models include: a constant eddy viscosity model, Prandtl's mixing length model, Prandtl's extended mixing length model, and von Kármán's mixing length model. In Prandtl's models and von Kármán's model, two cases are considered; one which includes the kinematic viscosity and one where the kinematic viscosity is neglected. Prandtl's mixing length model which is well known and holds up to experimental confirmation is derived. Von Kármán's mixing length is derived using three independent methods: von Kármán's similarity hypothesis, Betz's physical derivation, and using boundary layer theory. It is unknown to the authors if a boundary layer derivation appears in literature. Therefore, the derivation presented in this work appears to be a novel contribution.

- Chapter 4: In this chapter the Reynolds-averaged equations are used to derive the governing equations for the two-dimensional turbulent classical far wake. The boundary conditions and conserved quantity are also provided.

- Chapter 5: In this chapter, a more recent approach using Lie symmetries is used to derive the symmetry associated with the elementary conserved vector for an arbitrary effective viscosity $E = E(x, w_y, w_{yy})$. The algebraic closure models are then applied directly to the wake and the resulting ordinary differential equations governing the velocity field are found.

- Chapter 6: In this chapter, the ordinary differential equations derived in Chapter 5 are solved either analytically or numerically. The solutions are plotted against an experimental curve, that was obtained empirically, to determine the effectiveness of the closure models used. The strengths and weaknesses of each model and the wake regions in which its application is better suited is summarised.

- Chapter 7: In this chapter, concluding remarks are given.

Chapter 2

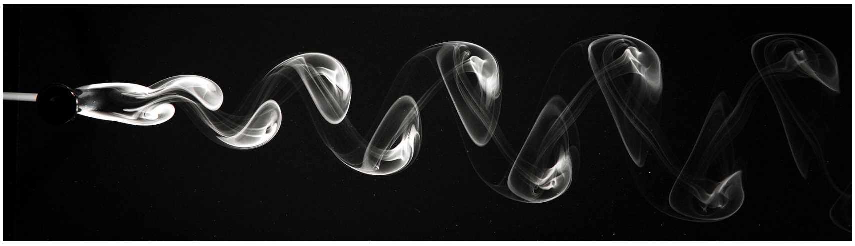
WAKES —A CLASS OF FREE SHEAR FLOWS

This chapter describes shear flows and the notable differences between shear flows and free shear flows. In order to study turbulent wakes, an in-depth understanding of shear flows is required. The two-dimensional turbulent classical wake, which is the flow considered in this dissertation, is introduced in the context of free shearing flows.

2.1 Shear flows

Consider Couette flow where fluid moves between two parallel flat plates. The upper plate moves with some velocity relative to the lower plate. This is one example of a simple shear flow which is characterised by a constant shear stress distribution. Complex shear flows also exist; both in nature and industry. Shear layers on rotating bodies such as wind turbines, and swirling flows such as tornadoes are two such examples. For laminar Couette flow the velocity profile is linear as opposed to turbulent Couette flow where the profile is S-shaped [16]. An important feature of shear flows is that the flow is induced by surface forces. In Couette flow the force inducing the flow is the drag caused by the relative motion of the two plates. This is distinguishable from Poiseuille flow which is a pressure induced flow. An important feature of a shearing flow is that adjacent layers of fluid move parallel to each other. That is, liquid particles remain on parallel surfaces which move translationally to each other. Viscous fluids will resist the shearing motion, but at high Reynolds numbers shear flows become increasingly unstable. This is due to the fluid's viscosity being too low to dampen the perturbations generated in the flow and this in turn causes turbulence. Free shear flows are a category of shear flows which have mean velocity gradients that develop in the absence of boundaries. Free shear flows that start off laminar usually always become turbulent in the absence of boundaries as the three-dimensional vorticity which is necessary for this transition from laminar flow to turbulent flow can develop significantly easier in the absence of boundaries that would usually inhibit the growth of the velocity components normal to them [17]. Some examples of free shear flows are: vortices formed behind objects as seen in Figure 2.1a which are common in the atmosphere; thermal plumes such as in Figure 2.1b which are typical of smoke from cigarettes; and wakes which are formed behind obstacles as seen in sketches by da Vinci in Figure 2.1c.

An important feature of free shear flows is entrainment. Entrainment is the process whereby fluid in the turbulent regions pulls fluid from the non-turbulent ambient fluid region and carries it along in the flow. This process converts the mean flow energy to turbulent energy and as long as there is a generator for the turbulent flow, entrainment will continuously increases the amount of fluid which is turbulent [18]. Free shear flows are useful in studying the applicability of turbulence models as, unlike in bounded flows, there are neither complicated boundary conditions nor solid surfaces to concern about. Only the fluid and its flow are studied. In this dissertation fully formed turbulent flow in the far wake region is explored. Fully formed flow implies that the velocity profile will not change as you move downstream.



(A) “Visualisation of the Von Kármán vortex sheet behind a circlular cylinder in air flow. The flow is made visible by means of the release of oil vapour near the cylinder." by Jürgen Wagner - Self-photographed is licensed under CC BY-SA 4.0.

(B) “This Schlieren photograph shows the thermal convection plume rising from an ordinary candle in still air. The plume is initially laminar, but transition to turbulence occurs in the upper 1/3 of the image. The image was made using the 1-meter-diameter schlieren mirror of Floviz Inc." by Dr. Gary Settles is licensed under CC BY-SA 3.0.

(C) Leonardo da Vinci's sketches of wakes behind obstacles. (circa 1513)

FIGURE 2.1: Three examples of different turbulent free shear flows.

2.2 Wakes

2.2.1 Classical wake

A classical wake is a disturbed flow in the region downstream of a stationary solid body. The flow has been disturbed due to the fluid flow about the body and this disturbance propagates downstream.

The original investigation on the classical near wake is credited to Goldstein. Existing work first done by Blasius [19] looked at the velocity distribution in the boundary layer from the leading edge to the trailing edge of a plate of finite length, ℓ, and the flow around a cylindrical body. Goldstein furthered this research by studying the velocity in the near wake up to half the length of the plate downstream [20]. Tollmien [21] extended Goldstein's research by investigating the region even farther downstream of the trailing edge using a first asymptotic approximation for the velocity distribution in the wake region and derived a formula for the velocity. Tollmien stated that his derived formula is valid for $X > 3\ell$, where ℓ is the length of the plate. His work is valid for any symmetrical laminar near wake. However, the following explanations were lacking: the position of X is unknown and was not defined in terms of a position measured from the origin, and no justification for his statement concerning the validity of the given formula was provided. Kaufmann [22] considered the flow with separation surfaces around an infinitely long plate, the profile resembling that of a wake. Separation surfaces appear when the boundary layer detaches from the surface of an object. Goldstein later studied the two-dimensional steady flow of an incompressible Newtonian fluid past a fixed thin symmetric body parallel to the mainstream flow [23]. He commented on the work done by Tollmien, as Tollmien was able to find the velocity in a region where Goldstein's calculations are invalid, but in particular he analysed the far wake region behind a thin plate. In order to derive analytic solutions, Goldstein assumed that the Reynolds number was large enough for Prandtl's boundary layer equations to be applied in the wake region. He further simplified the equations by supposing that the wake flow can be described as a perturbation on the mainstream flow in the far wake region. The governing equations were solved subject to appropriate boundary conditions and a conserved quantity, which in this case was the drag. Jeffreys [24] considered the flow of a far wake in a fluid past a solid object. In years to follow, the classical wake and exercises relating to wake flows became common in textbooks in applied mathematics, physics, and engineering.

In order to study turbulent wakes, a common approach used is to separate the flow into a mean flow and fluctuation. This is known as the Reynolds decomposition and when applied to the Navier-Stokes equations results in the Reynolds-averaged Navier-Stokes equations. The mean velocity components then need to be solved. In the process of averaging the equations, additional Reynolds stress terms arise which results in more unknowns than equations. A model for the stress terms is therefore needed to close the system of equations. In more recent years, the turbulent classical far wake was analysed using Lie symmetry methods by Hutchinson, Mason and Mahomed [25]. The same Lie symmetry methods will be used in this dissertation. Hutchinson and Mason [26] also applied a revised version of Prandtl's mixing length model, which included the kinematic viscosity, to close the equations for the two-dimensional turbulent classical far wake. In this dissertation a two-dimensional turbulent classical far wake is used as the focus of the study.

2.2.2 Other wakes

Wakes of self-propelled bodies represent a class of wakes in which the object is no longer stationary. Instead the object propels itself through the fluid. The conserved quantity is no longer the drag but instead the second moment of the axial momentum deficit. Birkhoff and Zarantonello [27] studied the laminar wake behind a self-propelled body. Herczynski, Weidman, and Burde [28] investigated both the two-fluid wake of a self-propelled body and the two-fluid laminar classical wake. In the case of two-fluid wakes, the object is placed at the interface of two fluids which have differing densities and viscosities. Another type of common wake is the wall wake. Hunt investigated the laminar wall wake [29] where the proximity of the wake to the wall results in vortex shedding, where vortices are created and detached periodically.

The laminar cases above have been extended to include turbulence. Hutchinson and Mason [30] applied Lie symmetry methods to the turbulent wake of a self-propelled body. Hunt and Jackson furthered Hunt's original laminar wall wake investigation by considering the turbulent wall wake [31]. Of particular interest are wakes behind rotating bodies such as wind turbines. In recent years, with the rise in the necessity of renewable energy, wind generated electricity has risen almost 75 times since 1997 [32]. Betz [33] is well known for formulating the maximum efficiency of a motorised wind turbine known as the 'Betz limit' or 'Betz law' in 1920. The same results were however produced independently by two other researchers before Betz. Lanchester [34] and Joukovsky [35] both around 1915 concluded the same limit resulting in the proposed renaming of the Betz limit to the Lanchester-Betz-Joukovsky limit [36].

2.3 Other free shear flows

Jets, plumes, and free shear layers are just a few examples of different free shear flows. Jets are streams of fluid ejected from nozzles usually into quiescent fluid while plumes are columns of a fluid moving through another. Thermal free shear flows make up a subclass of flows where heating/cooling have effects on the flow. For example, heating can decrease viscosity in liquids or increase the viscosity of gases. Thermal affects can also change the buoyancy properties of a fluid. Recent studies of interest in jet flows have been undertaken by Mason [37] for the two dimensional laminar jet, Mason and Ruscic [38] for the laminar axisymmetric jet and Mason and Hill [39] for the turbulent free jet.

FIGURE 2.2: Wake turbulence behind the South African Airbus A340-642 ZS-SNG on flight SA 264 from Johannesburg landing at Munich airport. The early morning ground fog has made the wake and wingtip vortices visible. Photographed by Timo Harsch | jetpictures.de

Chapter 3

APPROACHES TO MODELLING TURBULENCE

The Reynolds-averaged Navier-Stokes equations are obtained using the Reynolds decomposition of the flow variables into the sum of a mean velocity and a fluctuation. One approach to modelling turbulence is to solve for the mean flow variables. The Reynolds-averaged equations must be solved in order to do so. However, the Reynolds-averaged equations contain unknown Reynolds stress terms. A model is needed to provide a 'closure' to express these Reynolds stresses using averaged quantities and therefore complete the system of equations. Each closure model has a certain accuracy and range of applicability associated with it. It is important to fully understand the physics behind each model, to evaluate how they work, and under which conditions they are suitable. A few alternative methods found in Figure 3.1 are briefly explored and in particular the algebraic models that will be used are derived.

3.1 Reynolds-averaged Navier-Stokes equations

The Reynolds decomposition separates the flow variables, such as the velocity, u, and pressure, p, into a mean flow component and a fluctuation with zero mean.

Let the velocity components be denoted by u_i, where $i = 1,2,3$. The Reynolds decomposition is written as

$$u_i = \overline{u}_i + u_i', \tag{3.1}$$

where $\overline{u}_i$ represents the mean flow and u_i' represents the fluctuation. The mean flow $\overline{u}_i$ is interpreted as a time average, defined by [40, pp. 558]

$$\overline{u}_i(x,y,z) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} u_i(x,y,z,t)\,dt, \tag{3.2}$$

where t_0 is some reference time. In order for the time average in Equation (3.2) to be independent of time, it is required that the time interval T be sufficiently large. Flows in which the mean flow is independent of time are called steady turbulent flows [41]. The time averages of the mean flow components are equal to the mean flow. The time average of a fluctuation is zero. This means that $\overline{u_i'} = 0$ which is easily shown from the result $u_i' = u_i - \overline{u}_i$ and

$$\overline{u_i'} = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} (u_i - \overline{u}_i)\,dt = \overline{u}_i - \overline{u}_i \equiv 0. \tag{3.3}$$

The time average of a product of fluctuations is non-zero. It will be shown that additional terms arise which are the averaged products of fluctuations. These are referred to as the Reynolds shear stresses.

The fluid is incompressible and therefore Equation (3.1) must satisfy the continuity equation. This results in

$$\nabla\cdot u_i = \nabla\cdot\overline{u_i} + \nabla\cdot {u_i}' = 0. \tag{3.4}$$

Taking the mean of Equation (3.4) leaves

$$\nabla\cdot\overline{u_i} = 0, \tag{3.5}$$

and therefore, by substituting Equation (3.5) into Equation (3.4) the following is obtained

$$\nabla\cdot {u_i}' = 0. \tag{3.6}$$

Equations (3.5) and (3.6) show respectively that the mean flow and the turbulent velocity fluctuations are incompressible. The body force due to gravity is neglected. The i-th component of the Navier-Stokes equation with zero body force is given by

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i, \tag{3.7}$$

where ρ is the density, μ is the shear viscosity, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}, \tag{3.8}$$

is the substantial derivative. Inserting Equation (3.1) into Equation (3.7) and taking the mean gives the Reynolds-averaged equations [13, pp. 85]

$$\rho \frac{\overline{D}\overline{u}_i}{\overline{D}t} = -\frac{\partial \overline{p}}{\partial x_i} + \mu \nabla^2 \overline{u}_i - \rho \frac{\partial \overline{u'_j u'_i}}{\partial x_j}, \tag{3.9}$$

where

$$\frac{\overline{D}}{\overline{D}t} \equiv \frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla,$$

is the mean substantial derivative. It is important to note that the mean of the substantial derivative is not equal to the mean substantial derivative of the mean. That is,

$$\overline{\frac{Du_i}{Dt}} \neq \frac{\overline{D}\overline{u}_i}{\overline{D}t}.$$

These Reynolds-averaged equations are similar to the Navier-Stokes equations with the exception of the presence of the term $\frac{\partial \overline{u'_j u'_i}}{\partial x_j}$. Rewriting Equation (3.9) in the general form of a momentum conservation equation yields

$$\rho \frac{\overline{D}}{\overline{D}t} \overline{u}_i = \frac{\partial}{\partial x_j} \left[-\overline{p}\delta_{ji} + \mu \left(\frac{\partial \overline{u}_j}{\partial x_i} + \frac{\partial \overline{u}_i}{\partial x_j} \right) - \rho \overline{u'_j u'_i} \right]. \tag{3.10}$$

The term in square brackets represents the sum of three stresses: the isotropic stress $-\overline{p}\delta_{ji}$, the viscous stress $\mu \left(\frac{\partial \overline{u}_j}{\partial x_i} + \frac{\partial \overline{u}_i}{\partial x_j} \right)$, and the stress from the fluctuating velocity field $-\rho \overline{u'_j u'_i}$. The terms $\overline{u'_j u'_i}$ are simply referred to as Reynolds stresses.

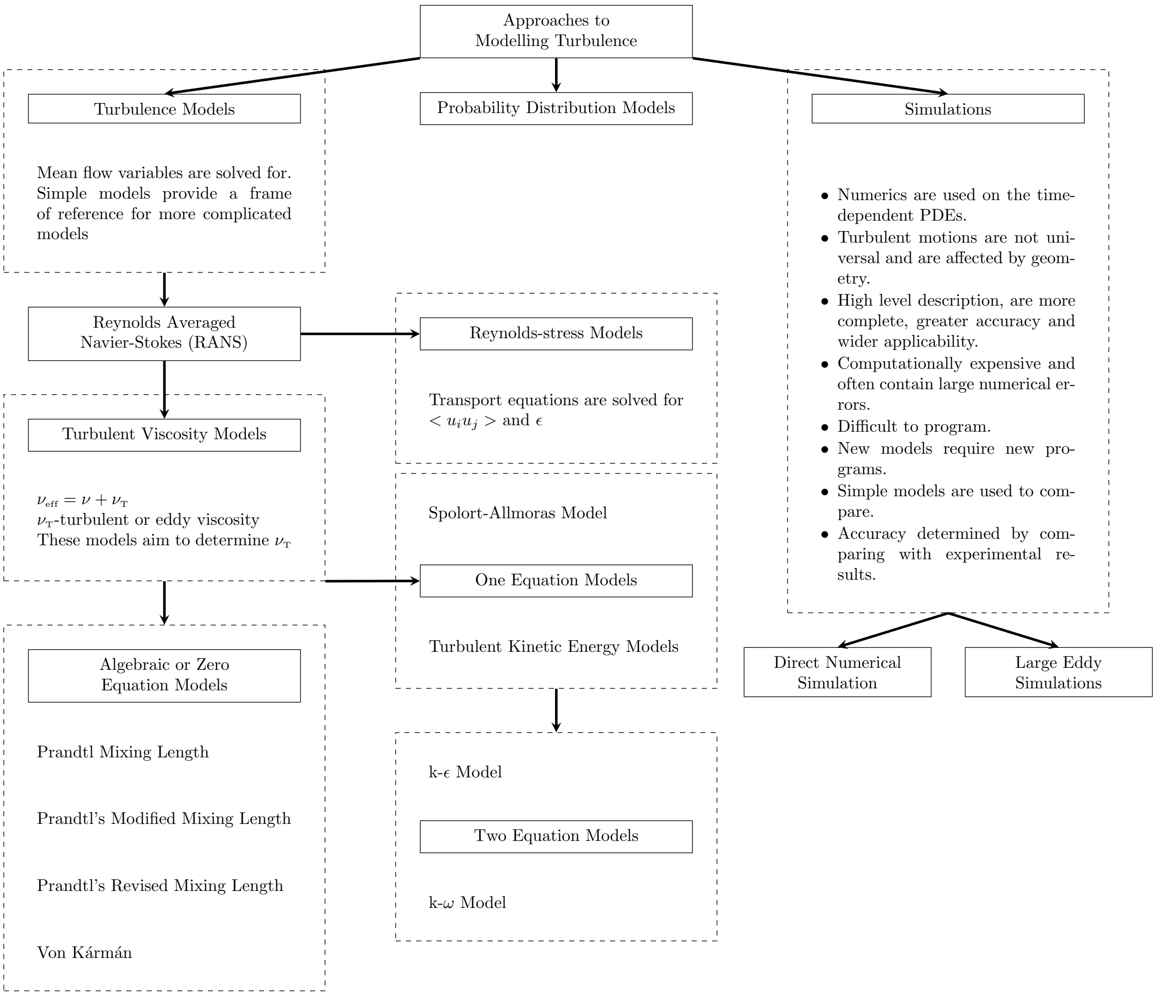


FIGURE 3.1: Flow chart summarising some of the methods used to model turbulence.

3.2 Turbulence modelling techniques

There are three main methods of approaching turbulent flows: simulations, probability distribution models, and turbulence models. The two main simulation methods are Direct Numerical Simulation (DNS) and Large Eddy Simulation (LES). DNS solves the Navier-Stokes equations, subject to initial and boundary conditions using numerical techniques. DNS is computationally expensive as the amount of operations needed is proportional to Re^3 [13, pp. 349] and as such is only computationally viable for lower Reynolds numbers. LES seeks to improve on DNS by reducing the computational cost. This is achieved by removing small length scales from the computations which consume most of the computational time.

Probability density functions (PDFs) serve as the foundation for probability distribution models used in modelling turbulent flows. Turbulent flows are statistical and stochastic in nature. Kolmogorov's [42] first hypothesis for similarity stated that for locally isotropic turbulence the PDF is uniquely determined by the kinematic viscosity and the average dispersion of energy.

Turbulence models are vast and vary from simple algebraic models to complex models involving one or more transport equations. In turbulent kinetic energy models, Kolmogorov and Prandtl [13, pp. 369] suggested that the velocity scale should be based on the turbulent kinetic energy, k. A model transport equation therefore needs to be solved for k. This is known as a one-equation model. Two-equation models such as the $k-\varepsilon$ and $k-\omega$ models which include the dissipative rate of turbulent kinetic energy and turbulence frequency respectively require solving two transport equations. In this dissertation turbulent viscosity models that aim to determine the kinematic eddy viscosity, ν_T, are explored.

3.3 Zero equation closure models

Zero equation closure models are otherwise known as algebraic closure models. The naming convention 'zero' implies that there is no partial differential equation (PDE) to describe the transport of the stresses but rather a simple algebraic relation is used. It is often mistakenly reported that Boussinesq was one of the first to address the Reynolds stresses where his 1877 work titled 'Essai sur la théorie des eaux courantes' is cited [7]. However, it was not until 1895 that Reynolds averaged the Navier-Stokes equations [9]. Schmitt [43] found it essential to address the apparent inconsistency by considering Boussinesq's original publication. Schmitt explains that Boussinesq not only averaged the Navier-Stokes equations but also performed a closure, using an analogy to the kinetic theory of gases.

Boussinesq [7, pp. 23-46] proposed the eddy viscosity approach, which is used to relate the turbulent shear stresses to the mean rates of deformation by

$$-\rho\overline{u_i'u_j'} = -\frac{2}{3}\rho k\delta_{ij} + 2\mu_T\overline{S}_{ij} = -\frac{2}{3}\rho k\delta_{ij} + \mu_T\left[\frac{\partial\overline{u}_i}{\partial x_j} + \frac{\partial\overline{u}_j}{\partial x_i}\right], \tag{3.11}$$

where ρ is the fluid density, μ_T is the eddy viscosity, and $k = \frac{1}{2}\overline{u_i'u_i'}$ is the average turbulent kinetic energy. It should be noted that the eddy viscosity is a property of the flow and not of the fluid. In subsections to follow the kinematic eddy viscosity ν_T, which is also a property of the flow, is specified where

$$\nu_T = \frac{\mu_T}{\rho}.$$

3.3.1 Constant eddy viscosity and uniform turbulent viscosity

The simplest type of eddy viscosity closure model is when the eddy viscosity is constant. The effective viscosity E has the form $E = \nu + \nu_T =$ constant. It is already known that a constant eddy viscosity is not a realistic closure model for turbulent flows, as eddies vary in size across the wake [44]. The goal is to compare various models and a constant eddy is used as a baseline for comparisons.

In order to improve on the constant eddy viscosity model, the uniform turbulent viscosity model was proposed. For the majority of flows, the accuracy of the uniform turbulent viscosity closure model is very poor. In this model, applied to a planar two-dimensional free shear flow, the turbulent kinematic viscosity is given by

$$\nu_T(x) = \frac{U_0(x)\delta(x)}{R_T},$$

where U_0 and δ are the characteristic velocity scale and length scale of the mean flow respectively, and R_T is a flow-dependent constant, similar to the Reynolds number. Pope [13, pp 365-366] summarised the model as being 'incomplete with a very limited range of applicability'. A case where the model could be applied is that of self-similar free shear flows where the basic descriptions of the mean velocity profiles are being investigated. The uniform turbulent viscosity model will not be considered further.

3.3.2 Prandtl's mixing length

In this section the class of models based on Prandtl's work is discussed. Prandtl's mixing length model and two versions of Prandtl's extended mixing length model are derived.

3.3.2.1 Prantl's mixing length model

In order to improve on the accuracy of a constant or uniform eddy viscosity model, Prandtl proposed a mixing length concept. When introducing Prandtl's mixing length theory, 'Über die ausgebildete Turbulenz', Prandtl's lecture that was held in Zurich in 1926 is often cited [45]. However he first published this work, along with his extended model (see Section 3.3.2.2), in 1925 [46]. Before introducing and deriving the model a few deficiencies should be noted. The proposed model in this subsection implies that $\nu_T = 0$ on the wake axis. Prandtl addressed this issue in his original publication. The model also implies that there is just one length scale. It has already been established that the large scale eddies are responsible for mixing, which corresponds to the lateral spreading of the wake. The smaller scale eddies transfer momentum and are responsible for the dissipation of the momentum. The Reynolds number or viscosity also do not feature in the formulation of the mixing length and therefore the mixing length is usually found experimentally.

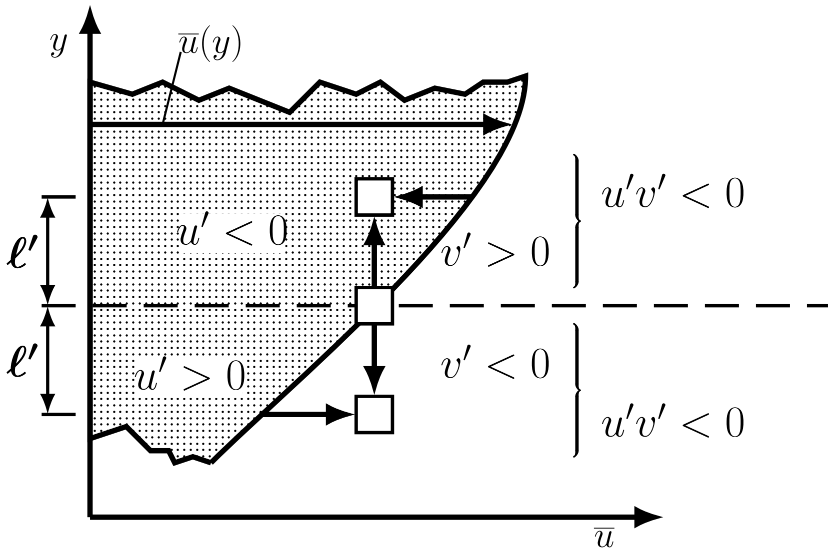


FIGURE 3.2: Explanation of the mixing length concept. Diagram reproduced from Schlichting [40, pp.560].

A derivation of Prandtl's mixing length model is adapted from Schlichting [40]. Consider a fluid element at position y. The fluid element moves with the mainstream flow with velocity $(\overline{u}(y), 0)$.

The fluid element is displaced by the fluctuations to position $y + \ell'$. During the jump, momentum is conserved, and hence the velocity of the fluid element remains as $(\overline{u}(y), 0)$. The velocity of the fluid surrounding the fluid element at its new position of $y + \ell'$ is $(\overline{u}(y + \ell'), 0)$.

The size of ℓ' gives an estimate as to the strength of the fluctuations that resulted in the fluid element moving from y to $y + \ell'$. In other words, stronger fluctuations will cause a larger displacement, and hence a larger ℓ'. In order to estimate the strength of these fluctuations let

$$\triangle u = \overline{u}(y + \ell') - \overline{u}(y), \tag{3.12}$$

and assume that the fluctuations are proportional to this difference. Consider a mean flow where $\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}y} > 0$ as seen in Figure 3.2, then $\triangle u > 0$. Expanding $\overline{u}(y + \ell')$

using a Taylor series gives

$$\begin{aligned}\triangle u &= \overline{u}(y+\ell') - \overline{u}(y) \\ &= \overline{u}(y) + \ell' \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}(y) + \mathcal{O}\left(\ell'^2\right) - \overline{u}(y) \\ &\approx \ell' \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}(y), \end{aligned} \tag{3.13}$$

where terms $\mathcal{O}(\ell'^2)$ and higher are neglected. In the y-direction, assume the fluctuations have the same order of magnitude. Thus

$$\triangle v = \ell' \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}(y). \tag{3.14}$$

Now, considering an upward movement from y to $y+\ell'$, then $v' > 0$. From conservation of mass,

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ \Longrightarrow\ &\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0, \\ \Longrightarrow\ &\frac{\partial u'}{\partial x} = -\frac{\partial v'}{\partial y} \sim -\frac{v'}{\ell'} < 0, \end{aligned} \tag{3.15}$$

then $u' < 0$. Since $\triangle u$ and $\triangle v$ are measures of u' and v' respectively, let

$$u' = -K_1 \ell' \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} = -K_1 \triangle u, \quad K_1 > 0, \tag{3.16}$$

$$v' = +K_2 \ell' \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} = +K_2 \triangle v, \quad K_2 > 0. \tag{3.17}$$

Then

$$\begin{aligned}\tau_{xy} &= -\rho \overline{u'v'}, \\ &= \rho K_1 K_2 \overline{\ell'\ell'} \left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right)^2, \\ &= \rho \ell^2 \left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right)^2, \end{aligned} \tag{3.18}$$

where $\ell^2 = K_1 K_2 \overline{\ell' \ell'}$. Finally,

$$\tau_{xy} = \rho \ell^2 \left| \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} \right| \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}, \tag{3.19}$$

where the absolute value has been taken to ensure that negative $\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}$ also implies negative τ.

3.3.2.2 Prandtl's extended model

As aforementioned, Prandtl foresaw the shortcomings of his original model and provided an extended version. Consider the Reynolds decomposition of the velocity into a mean and fluctuation component,

$$u = \overline{u} + u' = \overline{u} + \ell' \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}, \tag{3.20}$$

where the fluctuation is defined by Equation (3.16). Differentiating Equation (3.20) with respect to y gives

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - K_1 \ell' \frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2}. \tag{3.21}$$

The following properties will be used:

$$\left| \frac{\mathrm{d}u}{\mathrm{d}y} \right| = \sqrt{\left(\frac{\mathrm{d}u}{\mathrm{d}y} \right)^2}, \tag{3.22}$$

$$\overline{\frac{\mathrm{d}u}{\mathrm{d}y}} = \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}, \tag{3.23}$$

$$\overline{\sqrt{\frac{\mathrm{d}u}{\mathrm{d}y}}} = \sqrt{\overline{\frac{\mathrm{d}u}{\mathrm{d}y}}}. \tag{3.24}$$

Substituting Equation (3.21) into $\left|\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right|$ yields

$$\begin{aligned}
\left|\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right| &= \sqrt{\overline{\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2}} = \sqrt{\overline{\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - K_1\ell'\frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2}\right)^2}} \\
&= \sqrt{\overline{\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right)^2 - 2K_1\ell'\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2} + K_1^2\ell'^2\left(\frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2}\right)^2}} \\
&= \sqrt{\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right)^2 + K_1^2\overline{\ell'^2}\left(\frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2}\right)^2} \quad \text{as} \quad \overline{2\ell'\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2}} = 0, \\
&= \sqrt{\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right)^2 + \frac{K_1}{K_2}\ell^2\left(\frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2}\right)^2}.
\end{aligned} \tag{3.25}$$

Thus

$$\nu_T = \ell^2\sqrt{\left(\frac{\partial\overline{u}}{\partial y}\right)^2 + \frac{K_1}{K_2}\ell^2\left(\frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2}\right)^2}. \tag{3.26}$$

Since $\frac{K_1}{K_2} = \frac{-u'}{v'} > 0$ and v' and u' are assumed to have the same order of magnitude, it follows that $\frac{K_1}{K_2}$ is order of magnitude unity. Prandtl's two mixing lengths are therefore of the same order of magnitude.

The derivation of Equation (3.26) is unsatisfactory and a better method arriving at a much more intuitive answer is provided. Schlichting [40, pp 581.] defines the mean of the absolute fluctuation as

$$\overline{|u'|} = \ell\left|\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right)\right|. \tag{3.27}$$

The same argument made in Equation (3.13) is made however the terms $\mathcal{O}(\ell'^2)$ are no longer neglected resulting in

$$
\begin{aligned}
u'(y) &= \overline{u}(y+\ell') - \overline{u}(y) \\
&= \overline{u}(y) + \ell' \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}(y) + \frac{1}{2}\ell'^2 \frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2}(y) + \mathscr{O}\left(\ell'^3\right) - \overline{u}(y) \\
&= \ell' \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}(y) + \frac{1}{2}\ell'^2 \frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2}(y),
\end{aligned}
\tag{3.28}
$$

where terms $\mathscr{O}(\ell'^3)$ are neglected. Substituting the version of u' given by Equation (3.28) into Equation (3.27) gives

$$
\overline{|u'|} = \left| \ell \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} + \frac{1}{2}\ell^2 \frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2} \right|. \tag{3.29}
$$

Keeping v' as $\ell \dfrac{\partial \overline{u}}{\partial y}$ and using Equation (3.29) gives the result

$$
\overline{u'v'} = \ell \left| \ell \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} + \frac{1}{2}\ell^2 \frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2} \right| \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}. \tag{3.30}
$$

The form of Equation (3.30) appears in literature when modelling turbulence in circular tubes and the results of the model were compared to empirical values [47].

3.4 Von Kármán's mixing length

3.4.1 Von Kármán's similarity hypothesis

An expression for Prandtl's mixing length can be derived using von Kármán's similarity hypothesis [48]. Define the stream function Ψ for the velocity components by

$$
u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. \tag{3.31}
$$

Von Kármán originally derived the expression using the vortex transport equation for the stream function Ψ

$$\frac{\partial \Psi}{\partial y}\frac{\partial \Delta\Psi}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \Delta\Psi}{\partial y} = \nu\Delta\Delta\Psi, \tag{3.32}$$

where

$$\Delta\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}$$

is the vortex intensity and ν is the kinematic viscosity. Von Kármán assumed that there was similitude of oscillatory motion. That is, he assumed the mean flow to be parallel and in the x-direction. He also assumed that the fluctuation flow consists of disturbances which have a relatively small extent in the y-direction and -at least for a certain period of time- are taken along as a steady stream of the main stream flow.

Von Kármán wished to investigate under what conditions the perturbations may be similar. In this way, the states of flow close to two points corresponding to the different values of y will differ only by a multiplicative factor of the fluctuation velocity and in the length scale of the flow field. He therefore assumed similarity of the state of fluctuation, regardless of the position of the place near which he examined the state of fluctuation.

A coordinate system is chosen so that the point under investigation comes to lie on the axis $y = 0$, and will develop next to the mean velocity of y.

$$U = U_0'y + U_0''\frac{y^2}{2} + \ldots.$$

where $U(0) = U_0$. The stream function is examined in the vicinity of the chosen point according to

$$\Psi(x,y) = U_0'\frac{y^2}{2} + U_0''\frac{y^3}{6} + \ldots + \psi(x,y), \tag{3.33}$$

where $\psi(x,y)$ is the stream function of the fluctuations defined by

$$u' = \frac{\partial\psi}{\partial y}, \quad v' = -\frac{\partial\psi}{\partial x}. \tag{3.34}$$

Set

$$x = \ell\xi, \quad y = \ell\eta, \quad \psi = Af(\xi,\eta), \tag{3.35}$$

where ℓ is a characteristic length and a convenient choice for the parameter A, having the dimensions of kinematic viscosity, is to be determined.

The viscosity terms in Equation (3.32) are neglected. In agreement with the assumption that the fluctuation field near the axis $y = 0$ is to be investigated, only the first terms in the terms that derive from the main motion in Equation (3.33) are retained. Von Kármán acknowledged the limitation in restricting the analysis to only the first terms.

Applying Equation (3.33) to Equation (3.32) yields

$$\left(U_0'y + \frac{\partial\psi}{\partial y}\right)\frac{\partial\Delta\psi}{\partial x} - \left(U_0'' + \frac{\partial\Delta\psi}{\partial y}\right)\frac{\partial\psi}{\partial x} = 0. \tag{3.36}$$

Substituting Equation (3.35) into Equation (3.36) gives

$$\frac{U_0'\ell^2}{A}\eta\frac{\partial}{\partial\xi}(\Delta f) - \frac{U_0''\ell^3}{A}\frac{\partial f}{\partial\xi} + \frac{\partial f}{\partial\eta}\frac{\partial\Delta f}{\partial\xi} - \frac{\partial f}{\partial\xi}\frac{\partial\Delta f}{\partial\eta} = 0, \tag{3.37}$$

where Δ is now defined by

$$\Delta = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}.$$

It is required that only ℓ and A are dependent on U_0', U_0'', ..., in the region around the point whereas the function $f(\xi,\eta)$ remains independent. Equation (3.37) is independent of A, ℓ and U_0', U_0'' given the following relationships:

$$U_0' \cong \frac{A}{\ell^2}, \tag{3.38}$$

$$U_0'' \cong \frac{A}{\ell^3}, \tag{3.39}$$

where $\cong$ is used to indicate a level of similarity or proportionality. Dividing corresponding terms of Equation (3.38) by those of Equation (3.39) results in

$$\ell \cong \frac{U_0'}{U_0''}. \tag{3.40}$$

The relationship for A is also realised by

$$A \cong \frac{{U_0'}^3}{{U_0''}^2} \cong \ell^2 U_0' \tag{3.41}$$

In order to show the importance of the above relationship for A, Equation (3.35) and the stream function for the fluctuations are substituted into the stress tensor τ to yield

$$\tau = -\rho\overline{u'v'} = -\rho\overline{\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}} = -\rho\frac{A^2}{\ell^2}\overline{\frac{\partial f}{\partial\xi}\frac{\partial f}{\partial\eta}}. \tag{3.42}$$

Using Equation (3.41) it is found that

$$\tau \cong \rho\ell^2{U_0'}^2. \tag{3.43}$$

Von Kármán summarised his findings for the conditions of similitude as follows.

(1) The fluctuation field has a characteristic length of the length scale of the disturbances which is determined by

$$\ell \cong \frac{U_0'}{U_0''} = k\frac{\frac{dU}{dy}}{\frac{d^2U}{dy^2}}. \tag{3.44}$$

(2) The shear stress is proportional to the density, the square of the characteristic length ℓ and to the square of the velocity gradient $\frac{dU}{dy}$.

In conclusion, von Kármán introduced the first nondimensional constant in turbulence, k, in the relationship

$$\ell = k\frac{U'}{U''}, \tag{3.45}$$

in which k is dependent only on the kind of fluctuation mechanism and the solution for $f(\xi,\eta)$. There has been some debate about whether or not von Kármán's constant, k, is a universal constant. It is generally agreed k takes on some value in a region around 0.4, which is supported by results obtained by Bailey, et al [49].

3.4.2 Physical interpretation and derivation

Betz provided a physical interpretation of von Kármán's mixing length concept [50]. Betz was able to expand the concept graphically and deduce the same result. By the date of the submission of this dissertation, it is unknown to the authors if a translation of Betz's paper exists. Below a translation of the main argument of Betz's paper has been attempted with commentary found in the footnotes.

A flow with mean velocity U is considered. The concept of a mixing length is described as follows; fluid components/parts are displaced by the turbulent

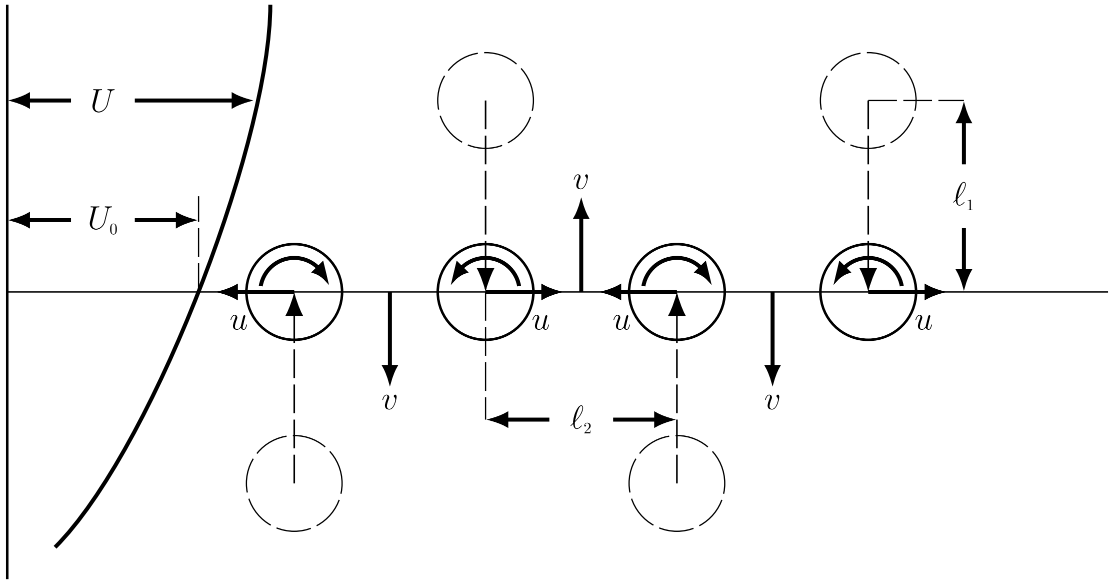


FIGURE 3.3: Displacement of liquid parts and the resulting disturbance velocities. Diagram reproduced from "Die v. Kármánsche Ähnlichkeitsüberlegung für turbulente Vorgänge in physikalischer Auffassung." [50]

transverse movements. Consider a layer of fluid which has its own mean velocity given by U_0. In Figure 3.3, the fluid parts inside the considered layer are represented by the solid circles, while fluid parts in the adjacent layers are represented by dashed circles. In this case, the layer becomes slower on one side and faster on the other side. That is to say, that these fluid parts have different velocities removed from the mean velocity U_0 by some disturbance velocity u. Consider areas outside of the chosen layer at a distance of ℓ_1. Parts from these areas have velocity

$$U = U_0 \pm \ell_1 U_0',$$

and a disturbance velocity given by

$$u = \pm \ell_1 U_0'. \tag{3.46}$$

Extended further, these parts bring with them not only their velocity, but also their mean rotation

$$\zeta = \mathrm{rot}U^{\S} = U'. \tag{3.47}$$

If the average rate of rotation in the considered layer is given by ζ_0, then the mean rate of rotation as in Equation (3.47) is at a distance ℓ_1 from the layer and is given by

$$\zeta = \zeta_0 \pm \ell_1 U_0''.^{\dagger\ddagger}$$

Therefore, these parts have a rate of rotation that divides from the mean rate of rotation in the relevant layer

$$\zeta - \zeta_0 = \pm\ell_1 U_0''.$$

These parts represent right and left rotating vortices which are embedded in the undisturbed flow. The turbulent disturbance velocities can now be interpreted as a field of these eddies. Define the distance between the centres of the right and left rotating parts to be ℓ_2, then the diameter of these parts is $\sim \ell_2$. The transverse speed v between opposing rotating parts is now defined as

$$v \sim (\zeta - \zeta_0)\,\ell_2{}^{\P} = \pm\ell_1\ell_2 U_0''. \tag{3.48}$$

If we assume von Kármán's hypothesis, that there is mechanical similarity of the turbulent processes in different areas of the flow, then it must be that

[§] Here rot is simply the curl.

[†] $\zeta = \zeta\left(y_0 + \left(y - y_0\right)\right) = \zeta\left(y_0 + \ell_1\right) = \zeta_0 + \ell_1 \frac{\partial \zeta_0}{\partial y} = \zeta_0 + \ell_1 \frac{\partial U_0'}{\partial y} = \zeta_0 + \ell_1 U_0''$.

[‡] The $\pm$ is used to differentiate between a layer above, $+\ell_1$, from a layer below, $-\ell_1$.

[¶] $v = \omega r$ where $\omega = \zeta - \zeta_0$ and $r = \ell_2$.

$$\ell_2 \sim \ell_1 \qquad \text{and} \qquad v \sim u.$$

Define a common proportional length for ℓ_1 and ℓ_2, say ℓ and substituting into Equations (3.46) and (3.48) yields

$$\ell^2 U'' \sim \ell U' \qquad \text{or} \qquad \ell \sim \frac{U'}{U''}, \tag{3.49}$$

which is in agreement with von Kármán's result.

3.4.3 Von Kármán's similarity hypothesis adapted to boundary layer flow

Von Kármán's mixing length is applicable in pipe flows. A motivation is therefore required in order to apply von Kármán's mixing length to free shear flows such as the wake. It will be demonstrated that the mixing length can be derived from the boundary layer equation. It is currently unknown to the authors if such a derivation exists in literature.

Consider the two dimensional-boundary layer equations given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial y}\right) + U\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{3.50}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.51}$$

The viscosity term was neglected in Equation (3.32) by von Kármán. Therefore setting $\nu = 0$ and when $U(x) = U$ is a constant mainstream velocity, Equation (3.50) reduces to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0. \tag{3.52}$$

The velocity components are separated into a mean velocity and a fluctuation given by

$$u(x, y, t) = \overline{u}(x, y) + u'(x, y, t), \tag{3.53}$$

$$v(x, y, t) = \overline{v}(x, y) + v'(x, y, t). \tag{3.54}$$

Substituting the components defined in Equations (3.53) and (3.54) into Equations (3.51) and (3.52) yields

$$\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \overline{u}\frac{\partial \overline{u}}{\partial x} + u'\frac{\partial \overline{u}}{\partial x} + u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + v'\frac{\partial \overline{u}}{\partial y} + \overline{v}\frac{\partial u'}{\partial y} + \overline{v}\frac{\partial \overline{u}}{\partial y} = 0. \tag{3.55}$$

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial v'}{\partial y} = 0. \tag{3.56}$$

Now, the fluctuations have the characteristic length scale ℓ and characteristic velocity scale B. The scales for length in the x- and y- directions are given respectively by L and H, where for a boundary layer

$$\frac{H}{L} \ll 1. \tag{3.57}$$

Similarly, the scales for the mean velocity in the x- and y- directions are given respectively by U and V. Equation (3.56) must satisfy both

$$\frac{U}{L}\frac{\partial \overline{u}}{\partial x} + \frac{V}{H}\frac{\partial \overline{v}}{\partial y} = 0, \tag{3.58}$$

and

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0. \tag{3.59}$$

The magnitude of the mean velocity in the y-direction follows directly from the continuity equation for the mean flow Equation (3.58), thus

$$V = \frac{UH}{L}. \tag{3.60}$$

The characteristic time for the mean flow is $\frac{L}{U}$ and for the fluctuation it is $\frac{\ell}{B}$. Since the characteristic fluctuation time is much less than the characteristic time for the mean flow,

$$\frac{\ell}{B} \ll \frac{L}{U} \quad \text{or} \quad \frac{\ell}{L}\frac{U}{B} \ll 1. \tag{3.61}$$

Assume that B and U are of the same order of magnitude and that ℓ and H are also the same order of magnitude. Thus

$$\frac{U}{B} \sim 1, \quad \frac{\ell}{H} \sim 1, \tag{3.62}$$

and from Equation (3.57)

$$\frac{\ell}{L} \ll 1. \tag{3.63}$$

$$\frac{\ell}{L} \ll 1 \quad \text{and} \quad \frac{H}{L} \ll 1.$$

In Table 3.1 the magnitude of the terms in Equation (3.55) are evaluated using the boundary layer approximation. In this approach, it can be determined which terms should be neglected. The magnitude of the terms in Equation (3.55) are compared with the magnitude of $\frac{\partial u'}{\partial t}$. Equation (3.55) is therefore divided by $\frac{B^2}{\ell}$.

Equation (3.55) is reduced to

TABLE 3.1: Table containing the magnitude of the terms in Equation (3.55).

Term	**Magnitude**	**Multiplied by** $\frac{\ell}{B^2}$	**Outcome**
$\frac{\partial u'}{\partial t}$	$\frac{B^2}{\ell}$	1	Keep
$\overline{u}\frac{\partial \overline{u}}{\partial x}$	$\frac{U^2}{\ell}$	$\frac{\ell}{L}\left(\frac{U}{B}\right)^2$	Neglect
$\overline{u}\frac{\partial u'}{\partial x}$	$\frac{UB}{\ell}$	$\frac{U}{B}$	Keep
$u'\frac{\partial \overline{u}}{\partial x}$	$\frac{UB}{L}$	$\frac{\ell}{L}\frac{U}{B}$	Neglect
$u'\frac{\partial u'}{\partial x}$	$\frac{B^2}{\ell}$	1	Keep
$\overline{v}\frac{\partial \overline{u}}{\partial y}$	$\frac{U^2}{L}$	$\frac{\ell}{L}\left(\frac{U}{B}\right)^2$	Neglect
$\overline{v}\frac{\partial u'}{\partial y}$	$\frac{HUB}{L\ell}$	$\frac{HU}{LB}$	Neglect
$v'\frac{\partial \overline{u}}{\partial y}$	$\frac{UB}{H}$	$\frac{\ell}{H}\frac{U}{B}$	Keep
$v'\frac{\partial u'}{\partial y}$	$\frac{B^2}{\ell}$	1	Keep

$$\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x} + v'\frac{\partial \overline{u}}{\partial y} + v'\frac{\partial u'}{\partial y} = 0. \tag{3.64}$$

Since the fluctuations satisfy the continuity equation Equation (3.59), the stream function for the fluctuations can be introduced and is defined by

$$u' = \frac{\partial \psi'}{\partial y} \quad \text{and} \quad v' = -\frac{\partial \psi'}{\partial x}. \tag{3.65}$$

Equation (3.55) in terms of the stream function is now

$$\frac{\partial^2 \psi}{\partial t \partial y} + \overline{u}\frac{\partial^2 \psi'}{\partial x \partial y} + \frac{\partial \psi'}{\partial y}\frac{\partial^2 \psi'}{\partial x \partial y} - \frac{\partial \psi'}{\partial x}\frac{\partial \overline{u}}{\partial y} - \frac{\partial^2 \psi'}{\partial y^2}\frac{\partial \psi'}{\partial x} = 0. \tag{3.66}$$

Consider an expansion of $\overline{u}(x, y)$ around the point (x_0, y_0) up to second order given by

$$\begin{aligned}\overline{u}(x,y) = {}& \overline{u}(x_0,y_0) + (y-y_0)\frac{\partial \overline{u}}{\partial y}(x_0,y_0) + (x-x_0)\frac{\partial \overline{u}}{\partial x}(x_0,y_0) \\ & + \frac{1}{2}(y-y_0)^2\frac{\partial^2 \overline{u}}{\partial y^2}(x_0,y_0) + \frac{1}{2}(x-x_0)^2\frac{\partial^2 \overline{u}}{\partial x^2}(x_0,y_0) \\ & + (x-x_0)(y-y_0)\frac{\partial^2 \overline{u}}{\partial x \partial y}(x_0,y_0) + \mathcal{O}((x-x_0)^3 + (y-y_0)^3).\end{aligned}$$

Equation (3.66) now becomes

$$\begin{aligned}&\frac{\partial^2 \psi'}{\partial t \partial y} + \frac{\partial^2 \psi'}{\partial x \partial y}\left[\overline{u}(x_0,y_0) + (x-x_0)\frac{\partial \overline{u}}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial \overline{u}}{\partial y}(x_0,y_0) + \frac{\partial \psi'}{\partial y}\right] \\ &- \frac{\partial \psi'}{\partial x}\left[\frac{\partial \overline{u}}{\partial y}(x_0,y_0) + (y-y_0)\frac{\partial^2 \overline{u}}{\partial y^2}(x_0,y_0) + (x-x_0)\frac{\partial^2 \overline{u}}{\partial x \partial y}(x_0,y_0) + \frac{\partial^2 \psi'}{\partial y^2}\right] = 0. \end{aligned}\tag{3.67}$$

Define a new coordinate system

$$\begin{aligned} x^* &= x - \overline{u}(x_0,y_0)\,t, \\ y^* &= y - \overline{v}(x_0,y_0)\,t \end{aligned}$$

and assume the fluctuations are steady with respect to this coordinate system. That is,

$$u'(x,y,t) = u'(x^*,y^*) \tag{3.68}$$

$$v'(x,y,t) = v'(x^*,y^*) \tag{3.69}$$

$$\psi'(x,y,t) = \psi^{*\prime}(x^*,y^*). \tag{3.70}$$

Equation (3.70) necessitates that the fluctuations in the old coordinate system are equal to the fluctuations in the new coordinate system which is independent of time and are thus steady. Equation (3.67) in the new coordinate system is now

$$(x^* - {x_0}^*)\frac{\partial^2 \psi^{*\prime}}{\partial x^* \partial y^*}\frac{\partial \overline{u}}{\partial x}(x_0,y_0) + (y^* - {y_0}^*)\frac{\partial^2 \psi^{*\prime}}{\partial x^* \partial y^*}\frac{\partial \overline{u}}{\partial y}(x_0,y_0) + \frac{\partial \psi^{*\prime}}{\partial y^*}\frac{\partial^2 \psi^{*\prime}}{\partial x^* \partial y^*}$$
$$-\overline{v}(x_0,y_0)\frac{\partial^2 \psi^{*\prime}}{\partial {y^*}^2} - \frac{\partial \psi^{*\prime}}{\partial x^*}\frac{\partial \overline{u}}{\partial y}(x_0,y_0) - (x^* - {x_0}^*)\frac{\partial \psi^{*\prime}}{\partial x^*}\frac{\partial^2 \overline{u}}{\partial x \partial y}(x_0,y_0)$$
$$-(y^* - {y_0}^*)\frac{\partial \psi^{*\prime}}{\partial x^*}\frac{\partial^2 \overline{u}}{\partial y^2}(x_0,y_0) - \frac{\partial \psi^{*\prime}}{\partial x^*}\frac{\partial^2 \psi^{*\prime}}{\partial {y^*}^2} = 0. \tag{3.71}$$

In Table 3.2 the terms in Equation (3.71) are again evaluated using the boundary layer approximation. The characteristic stream function for the fluctuations is $B\ell$.

Equation (3.71) reduces to

$$(y^* - {y_0}^*)\frac{\partial^2 \psi^{*\prime}}{\partial x^* \partial y^*}\frac{\partial \overline{u}}{\partial y}(x_0,y_0) + \frac{\partial \psi^{*\prime}}{\partial y^*}\frac{\partial^2 \psi^{*\prime}}{\partial x^* \partial y^*} - \frac{\partial \psi^{*\prime}}{\partial x^*}\frac{\partial \overline{u}}{\partial y}(x_0,y_0)$$
$$-(y^* - {y_0}^*)\frac{\partial \psi^{*\prime}}{\partial x^*}\frac{\partial^2 \overline{u}}{\partial y^2}(x_0,y_0) - \frac{\partial \psi^{*\prime}}{\partial x^*}\frac{\partial^2 \psi^{*\prime}}{\partial {y^*}^2} = 0. \tag{3.72}$$

By von Kármán's similarity hypothesis the stream function of the fluctuation, $\psi^{*\prime}$, must depend only on the length scale ℓ and the velocity scale B at the point (x_0, y_0). Therefore the variables ξ, η, f' are introduced for the fluctuation defined by

$$\xi = \frac{x^* - x_0^*}{\ell}, \tag{3.73}$$

$$\eta = \frac{y^* - y_0^*}{\ell}, \tag{3.74}$$

$$f'(\xi,\eta) = \frac{\psi^{*\prime}(x^*, y^*)}{\ell B}. \tag{3.75}$$

Using Equations (3.73) to (3.75), Equation (3.72) becomes

$$\frac{\ell}{B}\eta\frac{\partial\overline{u}}{\partial y}(x_0,y_0)\frac{\partial^2 f'}{\partial\xi\partial\eta} + \frac{\partial f'}{\partial\eta}\frac{\partial^2 f'}{\partial\xi\partial\eta} - \frac{\ell}{B}\frac{\partial\overline{u}}{\partial y}(x_0,y_0)\frac{\partial f'}{\partial\xi} - \frac{\ell^2}{B}\eta\frac{\partial^2\overline{u}}{\partial y^2}(x_0,y_0)\frac{\partial f'}{\partial\xi} - \frac{\partial f'}{\partial\xi}\frac{\partial^2 f'}{\partial\eta^2} = 0. \tag{3.76}$$

By von Kármán's similarity hypothesis, Equation (3.76) must be independent of $\ell, B, \dfrac{\partial\overline{u}}{\partial y}(x_0,y_0)$ and $\dfrac{\partial^2\overline{u}}{\partial y^2}(x_0,y_0)$. Thus it is necessary that

$$\frac{\ell}{B}\frac{\partial\overline{u}}{\partial y}(x_0,y_0) = c_1, \tag{3.77}$$

where c_1 is a constant and that

$$\frac{\ell^2}{B}\frac{\partial^2\overline{u}}{\partial y^2}(x_0,y_0) = c_2, \tag{3.78}$$

where c_2 is a constant. Dividing corresponding terms of Equation (3.77) by those of Equation (3.78) yields von Kármán's mixing length

$$\frac{\dfrac{\partial\overline{u}}{\partial y}}{\dfrac{\partial^2\overline{u}}{\partial y^2}} = \frac{c_1}{c_2}\ell. \tag{3.79}$$

It is important to note that Equation (3.79) was derived from the boundary layer equation, thus motivating its application to the classical wake.

TABLE 3.2: Table containing the magnitude of the terms in Equation (3.71).

Term	**Magnitude**	**Multiplied by** $\frac{\ell}{B^2}$	**Outcome**
$(x^* - x_0{}^*)\frac{\partial^2 \psi^{*\prime}}{\partial x^* \partial y^*}\frac{\partial \overline{u}}{\partial x}(x_0, y_0)$	$\frac{BU}{L}$	$\frac{\ell}{L}\frac{U}{B}$	Neglect
$(y^* - y_0{}^*)\frac{\partial^2 \psi^{*\prime}}{\partial x^* \partial y^*}\frac{\partial \overline{u}}{\partial y}(x_0, y_0)$	$\frac{BU}{H}$	$\frac{\ell}{H}\frac{U}{B}$	Keep
$\frac{\partial \psi^{*\prime}}{\partial y^*}\frac{\partial^2 \psi^{*\prime}}{\partial x^* \partial y^*}$	$\frac{B^2}{\ell}$	1	Keep
$\overline{v}(x_0, y_0)\frac{\partial^2 \psi^{*\prime}}{\partial y^{*2}}$	$\frac{BUH}{L\ell}$	$\frac{H}{L}\frac{U}{B}$	Neglect
$\frac{\partial \psi^{*\prime}}{\partial x^*}\frac{\partial \overline{u}(x_0, y_0)}{\partial y}$	$\frac{UB}{H}$	$\frac{\ell}{H}\frac{U}{B}$	Keep
$(x^* - x_0{}^*)\frac{\partial \psi^{*\prime}}{\partial x^*}\frac{\partial^2 \overline{u}}{\partial x \partial y}(x_0, y_0)$	$\frac{\ell BU}{LH}$	$\frac{\ell}{L}\frac{\ell}{H}\frac{U}{B}$	Neglect
$(y^* - y_0{}^*)\frac{\partial \psi^{*\prime}}{\partial x^*}\frac{\partial^2 \overline{u}}{\partial y^2}(x_0, y_0)$	$\frac{\ell BU}{H^2}$	$\left(\frac{\ell}{H}\right)^2\frac{U}{B}$	Keep
$\frac{\partial \psi^{*\prime}}{\partial x^*}\frac{\partial^2 \psi^{*\prime}}{\partial y^{*2}}$	$\frac{B^2}{\ell}$	1	Keep

Chapter 4

DERIVATION OF THE GOVERNING EQUATIONS FOR THE TURBULENT CLASSICAL FAR WAKE

Using the Reynolds-averaged equations, the governing equations for the two-dimensional turbulent classical far wake are derived. The governing equations are solved subject to boundary conditions and a conserved quantity, which for the classical wake is the drag.

The classical wake refers to the flow past a slender symmetric body aligned with a uniform mainstream flow. The obstructing body is stationary. The origin of the coordinate system is defined to be at the trailing edge of the body. The constant speed of the mainstream flow is denoted by U and consists only of a component in the x-direction. Wakes with an infinite boundary and finite boundaries will be considered. The type of wake boundary obtained is a result of the closure model used to complete the system of equations. For high Reynolds number flows, the obstructing object causes instabilities resulting in the generation of turbulence in the wake downstream of the body.

4.1 Two-dimensional turbulent classical far wake

Consider now a two-dimensional turbulent classical far wake. An incompressible Newtonian fluid flows past a thin symmetric body aligned with the mainstream flow. The Cartesian coordinates (x, y) are defined with the trailing edge of the body chosen as the origin as illustrated in Figure 4.1. The x-axis runs through the axis of symmetry of the body. Let $u = \overline{u} + u'$ and $v = \overline{v} + v'$ denote the mean

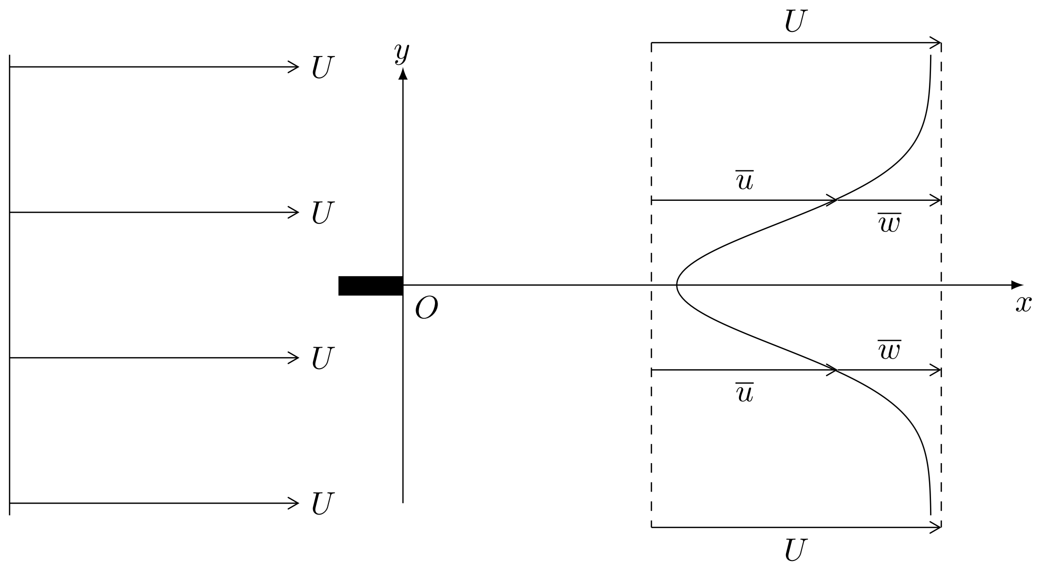


FIGURE 4.1: Two-dimensional turbulent classical wake behind a thin symmetric body.

velocities in the x- and y- directions respectively. Then using Equation (3.10) the x- and y- components may be expressed respectively as

$$\rho\left[\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}\right]=\frac{\partial}{\partial x}\left[-\overline{p}-\rho\overline{u'u'}+2\mu\frac{\partial\overline{u}}{\partial x}\right]+\frac{\partial}{\partial y}\left[-\rho\overline{u'v'}+\mu\left(\frac{\partial\overline{u}}{\partial y}+\frac{\partial\overline{v}}{\partial x}\right)\right], \quad (4.1)$$

$$\rho\left[\overline{u}\frac{\partial\overline{v}}{\partial x}+\overline{v}\frac{\partial\overline{v}}{\partial y}\right]=\frac{\partial}{\partial x}\left[-\rho\overline{u'v'}+\mu\left(\frac{\partial\overline{v}}{\partial x}+\frac{\partial\overline{u}}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[-\overline{p}-\rho\overline{v'v'}+2\mu\frac{\partial\overline{v}}{\partial y}\right]. \quad (4.2)$$

The viscosity μ and the density ρ of the fluid are considered to be constant. Define the stress tensor for the mean flow by

$$\overline{\tau}_{ij} = -\overline{p}\delta_{ij} + 2\mu\overline{S}_{ij} - \rho\overline{u'_i u'_j}, \tag{4.3}$$

where $\overline{S}_{ij}$ is the mean rate-of-strain tensor defined by

$$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right). \tag{4.4}$$

Using Equation (4.3), Equations (4.1) and (4.2) can be written as

$$\rho\left[\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y}\right] = \frac{\partial}{\partial x}\overline{\tau}_{xx} + \frac{\partial}{\partial y}\overline{\tau}_{yx}, \tag{4.5}$$

$$\rho\left[\overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y}\right] = \frac{\partial}{\partial x}\overline{\tau}_{xy} + \frac{\partial}{\partial y}\overline{\tau}_{yy}. \tag{4.6}$$

Using Equation (3.11), Equation (4.3) becomes

$$\overline{\tau}_{ij} = -\overline{p}\delta_{ij} + 2(\mu + \mu_T)\overline{S}_{ij}. \tag{4.7}$$

The viscosity μ_T is a property of the flow. Therefore the eddy viscosity can depend on the spacial coordinates, x and y, as well as on the mean velocity components, $\bar{u}$ and $\bar{v}$, and their spacial derivatives. It will be assumed that

$$\mu_T = \mu_T\left(x, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}\right).$$

The effective kinematic viscosity is then defined by [13, pp. 93]

$$E = \nu + \nu_T. \tag{4.8}$$

Using Equation (4.8), Equations (4.5) and (4.6) become

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \frac{\partial}{\partial x}\left(2E\left(x, \frac{\partial \overline{u}}{\partial y}, \frac{\partial^2 \overline{u}}{\partial y^2}\right)\frac{\partial \overline{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(E\left(x, \frac{\partial \overline{u}}{\partial y}, \frac{\partial^2 \overline{u}}{\partial y^2}\right)\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x}\right)\right), \tag{4.9}$$

$$\overline{u}\frac{\partial \overline{v}}{\partial x}+\overline{v}\frac{\partial \overline{v}}{\partial y}=-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y}+\frac{\partial}{\partial x}\left(E\left(x,\frac{\partial \overline{u}}{\partial y},\frac{\partial^2 \overline{u}}{\partial y^2}\right)\left(\frac{\partial \overline{u}}{\partial y}+\frac{\partial \overline{v}}{\partial x}\right)\right)+\frac{\partial}{\partial y}\left(2E\left(x,\frac{\partial \overline{u}}{\partial y},\frac{\partial^2 \overline{u}}{\partial y^2}\right)\frac{\partial \overline{v}}{\partial y}\right). \tag{4.10}$$

Although there is no solid boundary in the far wake, there is a region of sharp change perpendicular to the axis of the wake. In this region the boundary layer approximation can be applied. Equations (4.9) and (4.10) need to be non-dimensionalised before applying the boundary layer approximation. The characteristic variables in the x-direction are defined as follows:

- For the classical wake, the uniform mainstream flow upstream of the body is given by U. The characteristic speed in the x-direction is therefore chosen to be U.
- At some distance downstream of the body, which is estimated to have length L, the reduction of velocity in this wake region is not negligible. The characteristic length in the x-direction is therefore chosen to be L.

The characteristic variables in the y-direction are defined as follows:

- The characteristic length is δ, which is obtained during the analysis.
- The characteristic speed, which is derived from the conservation of mass equation, is given by
$$V=\frac{\delta U}{L}$$

Lastly, in order to balance the inertia term with the pressure gradient term, the characteristic pressure is chosen to be ρU^2. The dimensionless variables are defined by:

$$x^*=\frac{x}{L} \qquad y^*=\frac{y}{\delta} \qquad \overline{u}^*=\frac{\overline{u}}{U}$$
$$\overline{p}^*=\frac{\overline{p}}{\rho U^2} \qquad E^*=\frac{E}{E_C} \qquad \overline{v}^*=\frac{\overline{v}}{V}=\frac{L}{\delta}\frac{\overline{v}}{U}, \tag{4.11}$$

where the characteristic effective kinematic viscosity E_C is defined in terms of the characteristic kinematic eddy viscosity ν_{TC} by [26]

$$E_C=\nu+\nu_{TC}. \tag{4.12}$$

Expressed in terms of the dimensionless variables, Equations (4.9) and (4.10) become

$$\overline{u}^*\frac{\partial\overline{u}^*}{\partial x^*}+\overline{v}^*\frac{\partial\overline{u}^*}{\partial y^*}=-\frac{\partial\overline{p}^*}{\partial x^*}+\frac{2}{Re_T}\frac{\partial}{\partial x^*}\left[E^*\frac{\partial\overline{u}^*}{\partial x^*}\right]+\frac{1}{Re_T}\left(\frac{L}{\delta}\right)^2\frac{\partial}{\partial y^*}\left[E^*\frac{\partial\overline{u}^*}{\partial y^*}\right]$$
$$+\frac{1}{Re_T}\frac{\partial}{\partial y^*}\left[E^*\frac{\partial\overline{v}^*}{\partial x^*}\right], \tag{4.13}$$

$$\left(\frac{\delta}{L}\right)^2\left[\overline{u}^*\frac{\partial\overline{v}^*}{\partial x^*}+\overline{v}^*\frac{\partial\overline{v}^*}{\partial y^*}\right]=-\frac{\partial\overline{p}^*}{\partial y^*}+\frac{1}{Re_T}\frac{\partial}{\partial x^*}\left[E^*\frac{\partial\overline{u}^*}{\partial y^*}\right]+\left(\frac{\delta}{L}\right)^2\frac{1}{Re_T}\frac{\partial}{\partial x^*}\left[E^*\frac{\partial\overline{v}^*}{\partial x^*}\right]$$
$$+\frac{2}{Re_T}\frac{\partial}{\partial y^*}\left[E^*\frac{\partial\overline{v}^*}{\partial y^*}\right], \tag{4.14}$$

where Re_T is the turbulent Reynolds number defined by

$$Re_T=\frac{UL}{E_C}. \tag{4.15}$$

The continuity equation for the mean flow, Equation (3.4), becomes

$$\frac{\partial\overline{u}^*}{\partial x^*}+\frac{\partial\overline{v}^*}{\partial y^*}=0. \tag{4.16}$$

The rapid change in $\overline{u}^*$ with respect to y^* should be just large enough to balance the inertia term despite Re_T being large. The viscous term in Equation (4.13) will be the same order of magnitude as the inertia term and pressure gradient term provided

$$\frac{1}{Re_T}\left(\frac{L}{\delta}\right)^2\sim 1, \tag{4.17}$$

that is, provided

$$\frac{\delta}{L}\sim\frac{1}{\sqrt{Re_T}}. \tag{4.18}$$

The characteristic length in the y-direction is chosen as

$$\frac{\delta}{L}=\frac{1}{\sqrt{Re_T}}. \tag{4.19}$$

Equations (4.13) and (4.14) become

$$\overline{u}^{*}\frac{\partial\overline{u}^{*}}{\partial x^{*}}+\overline{v}^{*}\frac{\partial\overline{u}^{*}}{\partial y^{*}}=-\frac{\partial\overline{p}^{*}}{\partial x^{*}}+\frac{2}{Re_{T}}\frac{\partial}{\partial x^{*}}\left[E^{*}\frac{\partial\overline{u}^{*}}{\partial x^{*}}\right]+\frac{\partial}{\partial y^{*}}\left[E^{*}\frac{\partial\overline{u}^{*}}{\partial y^{*}}\right]$$
$$+\frac{1}{Re_{T}}\frac{\partial}{\partial y^{*}}\left[E^{*}\frac{\partial\overline{v}^{*}}{\partial x^{*}}\right], \tag{4.20}$$

$$\frac{1}{Re_{T}}\left[\overline{u}^{*}\frac{\partial\overline{v}^{*}}{\partial x^{*}}+\overline{v}^{*}\frac{\partial\overline{v}^{*}}{\partial y^{*}}\right]=-\frac{\partial\overline{p}^{*}}{\partial y^{*}}+\frac{1}{Re_{T}}\frac{\partial}{\partial x^{*}}\left[E^{*}\frac{\partial\overline{u}^{*}}{\partial y^{*}}\right]+\frac{1}{(Re_{T})^{2}}\frac{\partial}{\partial x^{*}}\left[E^{*}\frac{\partial\overline{v}^{*}}{\partial x^{*}}\right]$$
$$+\frac{2}{Re_{T}}\frac{\partial}{\partial y^{*}}\left[E^{*}\frac{\partial\overline{v}^{*}}{\partial y^{*}}\right]. \tag{4.21}$$

Terms of order $\dfrac{1}{\sqrt{Re_{T}}}$ and smaller will be neglected, and the stars($*$) are dropped for convenience. Equations (4.20) and (4.21) reduce to

$$\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}=-\frac{\partial\overline{p}}{\partial x}+\frac{\partial}{\partial y}\left(E\frac{\partial\overline{u}}{\partial y}\right), \tag{4.22}$$

$$\frac{\partial\overline{p}}{\partial y}=0. \tag{4.23}$$

It is immediately seen by Equation (4.23) that the mean pressure, $\overline{p}$, is independent of y. Its value at any position x is determined by the corresponding mainstream conditions of the flow which are satisfied at $y=\pm y_{b}(x)$. If the wake extends to infinity in the x- and y- directions then $y_{b}(x)=\infty$. The uniform flow in the mainstream is in the x-direction with constant speed U. It is inviscid and satisfies Euler's equation, the x-component of which gives

$$0=-\frac{d\overline{p}}{dx}. \tag{4.24}$$

As a result Equation (4.22) reduces to

$$\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}=\frac{\partial}{\partial y}\left(E\left(x,\frac{\partial\overline{u}}{\partial y},\frac{\partial^{2}\overline{u}}{\partial y^{2}}\right)\frac{\partial\overline{u}}{\partial y}\right). \tag{4.25}$$

A two-dimensional turbulent wake with mean components $\overline{u}$ and $\overline{v}$ is completely described by Equations (4.16) and (4.25) in dimensionless form.

As a consequence of choosing U to non-dimensionalise $\overline{u}$, the mainstream velocity is now unity in the x-direction and it is zero in the y-direction. That is, far downstream from the body, let

$$\overline{u}(x,y) = 1 - \overline{w}(x,y) \qquad \text{and} \qquad \overline{v}(x,y) = 0 + \overline{v}(x,y), \tag{4.26}$$

where $\overline{w}(x,y)$ is the mean velocity deficit in the wake aligned with the mainstream flow. Substituting the new variables given by Equation (4.26) into Equations (4.16) and (4.25) gives

$$-\frac{\partial \overline{w}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0, \tag{4.27}$$

$$\frac{\partial \overline{w}}{\partial x} - \overline{w}\frac{\partial \overline{w}}{\partial x} + \overline{v}\frac{\partial \overline{w}}{\partial y} = \frac{\partial}{\partial y}\left(E\left(x, \frac{\partial \overline{w}}{\partial y}, \frac{\partial^2 \overline{w}}{\partial y^2}\right)\frac{\partial \overline{w}}{\partial y}\right). \tag{4.28}$$

Select a region that is sufficiently far downstream such that $\overline{w}$ and $\overline{v}$ are small and that any products or powers thereof can be neglected. Equation (4.28) now becomes

$$\frac{\partial \overline{w}}{\partial x} = \frac{\partial}{\partial y}\left(E\left(x, \frac{\partial \overline{w}}{\partial y}, \frac{\partial^2 \overline{w}}{\partial y^2}\right)\frac{\partial \overline{w}}{\partial y}\right). \tag{4.29}$$

4.1.1 Boundary conditions and the conserved quantity

The boundary conditions for $\overline{w}$ and $\overline{v}$ are required. Consider first the mean velocity deficit $\overline{w}$ in the x-direction. While y tends to $\pm y_b(x)$ where $y_b = \infty$ is possible, the mean velocity $\overline{u}$ tends to unity. Therefore the deficit $\overline{w}$ will tend to zero. The second boundary condition is given by Hutchinson [51]. At the boundary of the wake, $y = \pm y_b(x)$, the conditions must match that of the uniform mainstream flow. The vorticity in the wake is given by the curl

$$\omega = \frac{\partial \overline{u}}{\partial y} = -\frac{\partial \overline{w}}{\partial y}. \tag{4.30}$$

As the uniform mainstream flow has no vorticity, the vorticity at the boundary of the wake must vanish. The two boundary conditions for $\overline{w}$ and it's derivative with respect to y are given by

$$\overline{w}(x, \pm y_b) = 0 \qquad \text{and} \qquad \frac{\partial \overline{w}}{\partial y}(x, \pm y_b) = 0. \tag{4.31}$$

The velocity deficit $\overline{w}(x, y)$ is a maximum with respect to y at each point of the positive x-axis and therefore

$$\frac{\partial \overline{w}}{\partial y}(x, 0) = 0, \quad x > 0. \tag{4.32}$$

Consider now $\overline{v}$, which is zero along the positive x-axis:

$$\overline{v}(x, 0) = 0, \quad x > 0. \tag{4.33}$$

Lastly a conserved quantity exists. The conserved quantity for the classical wake is the drag force [23]. The conserved quantity is calculated by integrating Equation (4.29) across the wake at some fixed point in x. This gives

$$\int_{-y_b(x)}^{+y_b(x)} \frac{\partial \overline{w}}{\partial x}(x, y)\,dy = E\frac{\partial \overline{w}}{\partial y}\Big|_{-y_b(x)}^{y_b(x)} = 0, \tag{4.34}$$

which from the second boundary condition in Equation (4.31) equates to zero. Also using the formula for differentiation under the integral sign [52]

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-y_b(x)}^{y_b(x)} \overline{w}(x,y)\,dy = \int_{-y_b(x)}^{y_b(x)} \frac{\partial \overline{w}}{\partial x}(x,y)\,dy + \overline{w}(x,y_b(x))\frac{\mathrm{d}y_b(x)}{\mathrm{d}x}$$
$$- \overline{w}(x,-y_b(x))\frac{\mathrm{d}(-y_b(x))}{\mathrm{d}x}, \tag{4.35}$$

and hence using the first boundary condition in Equation (4.31), Equation (4.34) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-y_b(x)}^{y_b(x)} \overline{w}(x,y)\,dy = 0. \tag{4.36}$$

Thus

$$\int_{-y_b(x)}^{y_b(x)} \overline{w}(x,y)\,dy = D, \tag{4.37}$$

where D is a constant independent of x and is the dimensionless surface drag on the fixed body.

Chapter 5

SOLUTIONS FOR THE TURBULENT CLASSICAL WAKE USING LIE SYMMETRY METHODS

In this dissertation various effective kinematic viscosities, E, depending on x, $\overline{w}_y$, and $\overline{w}_{yy}$ or any combination thereof are used. It is therefore beneficial to derive a Lie point symmetry associated with the conserved vector as far as possible for arbitrary E. The notation as given in [25] is used. That is, when x, y, $\overline{w}$, and the partial derivatives of $\overline{w}$ are regarded as independent variables, the suffix notation $\overline{w}_x$, $\overline{w}_y$, $\overline{w}_{xy}$, ... is used to denote partial derivatives. The notation $\dfrac{\partial \overline{w}}{\partial x}$, $\dfrac{\partial \overline{w}}{\partial y}$, $\dfrac{\partial^2 \overline{w}}{\partial x \partial y}$, ... is used when the partial derivatives of $\overline{w}$ are regarded as dependent variables which are functions of the independent variables x and y.

5.1 Derivation of a general Lie point symmetry for $E = E(x, w_y, w_{yy})$

In this section the bars are neglected to keep the notation simple.

The partial differential equation for the turbulent classical wake in terms of the velocity deficit w is from Equation (4.29)

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial y}\left(E(x, w_y, w_{yy})\, w_y\right). \tag{5.1}$$

A conservation law for the partial differential equation (5.1) satisfies

$$D_1 T^1 + D_2 T^2|_{(5.1)} = 0, \tag{5.2}$$

where the total derivative operators, D_1 and D_2, are given by

$$D_1 = D_x = \frac{\partial}{\partial x} + w_x \frac{\partial}{\partial w} + w_{xx}\frac{\partial}{\partial w_x} + w_{xy}\frac{\partial}{\partial w_y} + \ldots, \tag{5.3}$$

$$D_2 = D_y = \frac{\partial}{\partial y} + w_y \frac{\partial}{\partial w} + w_{yy}\frac{\partial}{\partial w_y} + w_{yx}\frac{\partial}{\partial w_x} + \ldots. \tag{5.4}$$

The conserved vector $T = (T^1, T^2)$ is invariant under the action of the Lie point symmetry

$$X = \xi^1(x, y, w)\frac{\partial}{\partial x} + \xi^2(x, y, w)\frac{\partial}{\partial y} + \eta(x, y, w)\frac{\partial}{\partial w} \tag{5.5}$$

provided [53, 54]

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0. \tag{5.6}$$

Equation (5.6) is broken into its two components:

$$X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0, \tag{5.7}$$

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0. \tag{5.8}$$

The components of the corresponding conserved vector $T = (T^1, T^2)$ are

$$T^1 = w \qquad \text{and} \qquad T^2 = -E(x, w_y, w_{yy})\, w_y. \tag{5.9}$$

It is readily verified that Equation (5.9) satisfies Equation (5.2).

5.1.1 Case 1: $\dfrac{\partial E}{\partial w_{yy}} \neq 0$

5.1.1.1 The first invariance condition

Consider the first invariance condition given by Equation (5.7):

$$\eta + w\left[\frac{\partial \xi^2}{\partial y} + w_y \frac{\partial \xi^2}{\partial w}\right] + E w_y \left[\frac{\partial \xi^1}{\partial y} + w_y \frac{\partial \xi^1}{\partial w}\right] = 0. \tag{5.10}$$

Differentiating Equation (5.10) with respect to w_{yy} leaves

$$\frac{\partial E}{\partial w_{yy}}\left[\frac{\partial \xi^1}{\partial y} + w_y \frac{\partial \xi^1}{\partial w}\right] = 0. \tag{5.11}$$

Thus, either

$$\frac{\partial E}{\partial w_{yy}} \neq 0 \quad \text{and} \quad \frac{\partial \xi^1}{\partial y} + w_y \frac{\partial \xi^1}{\partial w} = 0, \tag{5.12}$$

or

$$\frac{\partial E}{\partial w_{yy}} = 0, \tag{5.13}$$

must hold. The latter is to be considered in Case 2.

Consider Case 1 where

$$\frac{\partial E}{\partial w_{yy}} \neq 0.$$

Then Equation (5.12) implies that

$$\frac{\partial \xi^1}{\partial y} + w_y \frac{\partial \xi^1}{\partial w} = 0. \tag{5.14}$$

Separating Equation (5.14) by w_y results in

$$w_y^0: \qquad \frac{\partial \xi^1}{\partial y} = 0, \tag{5.15}$$

$$w_y: \qquad \frac{\partial \xi^1}{\partial w} = 0, \tag{5.16}$$

and therefore

$$\xi^1(x, y, w) = \xi^1(x). \tag{5.17}$$

Using $\xi^1 = \xi^1(x)$, Equation (5.10) reduces to

$$\eta + w\left[\frac{\partial \xi^2}{\partial y} + w_y \frac{\partial \xi^2}{\partial w}\right] = 0, \tag{5.18}$$

which is independent of E.

Separate Equation (5.18) by w_y

$$w_y^0: \qquad \eta + w\frac{\partial \xi^2}{\partial y} = 0, \tag{5.19}$$

$$w_y: \qquad \frac{\partial \xi^2}{\partial w} = 0. \tag{5.20}$$

Thus

$$\xi^2 = \xi^2(x, y) \tag{5.21}$$

and

$$\eta(x, y, w) = -w\frac{\partial \xi^2}{\partial y}(x, y). \tag{5.22}$$

SUMMARY FOR THE FIRST INVARIANCE CONDITION (S)

- $\xi^1 = \xi^1(x)$,
- $\xi^2 = \xi^2(x, y)$,
- $\eta(x, y, w) = -w\dfrac{\partial \xi^2}{\partial y}(x, y)$.

5.1.1.2 The second invariance condition

Consider now the second invariance condition given by Equation (5.8). The second prolongation of X is denoted by $X^{[2]}$. Since $T^2 = T^2\left(x, w_y, w_{yy}\right)$ the only terms in $X^{[2]}$ which are required are

$$X^{[2]} = \xi^1(x)\frac{\partial}{\partial x} + \xi^2(x,y)\frac{\partial}{\partial y} - w\frac{\partial \xi^2}{\partial y}(x,y)\frac{\partial}{\partial w} + \zeta_2\frac{\partial}{\partial w_y} + \zeta_{22}\frac{\partial}{\partial w_{yy}} \tag{5.23}$$

where

$$\zeta_2 = D_2(\eta) - w_k D_2(\xi^k), \tag{5.24}$$

$$\zeta_{22} = D_2(\zeta_2) - w_{2k} D_2(\xi^k). \tag{5.25}$$

It can be verified that

$$\zeta_2 = -w\frac{\partial^2 \xi^2}{\partial y^2} - 2w_y\frac{\partial \xi^2}{\partial y}, \tag{5.26}$$

$$\zeta_{22} = -w\frac{\partial^3 \xi^2}{\partial y^3} - 3w_y\frac{\partial^2 \xi^2}{\partial y^2} - 3w_{yy}\frac{\partial \xi^2}{\partial y}. \tag{5.27}$$

Using Equations (5.23), (5.26) and (5.27) on Equation (5.8) gives

$$\begin{aligned} 0 = &\left[w\frac{\partial^3 \xi^2}{\partial y^3} + 3w_y\frac{\partial^2 \xi^2}{\partial y^2} + 3w_{yy}\frac{\partial \xi^2}{\partial y}\right] w_y\frac{\partial E}{\partial w_{yy}} + E\left[w\frac{\partial^2 \xi^2}{\partial y^2} + 2w_y\frac{\partial \xi^2}{\partial y}\right] \\ &- Ew_y\frac{\mathrm{d}\xi^1}{\mathrm{d}x} - w\frac{\partial \xi^2}{\partial x} - \xi^1\frac{\partial E}{\partial x}w_y + \left[w\frac{\partial^2 \xi^2}{\partial y^2} + 2w_y\frac{\partial \xi^2}{\partial y}\right] w_y\frac{\partial E}{\partial w_y}. \end{aligned} \tag{5.28}$$

Separate Equation (5.28) by powers of w

$$\begin{aligned} w^0: \quad & 2\left[E + w_y\frac{\partial E}{\partial w_y}\right]\frac{\partial \xi^2}{\partial y} + 3\frac{\partial E}{\partial w_{yy}}\left[w_y\frac{\partial^2 \xi^2}{\partial y^2} + w_{yy}\frac{\partial \xi^2}{\partial y}\right] \\ & - E\frac{\mathrm{d}\xi^1}{\mathrm{d}x} - \xi^1\frac{\partial E}{\partial x} = 0, \end{aligned} \tag{5.29}$$

$$w^1: \qquad \frac{\partial^2 \xi^2}{\partial y^2}\left[E + w_y \frac{\partial E}{\partial w_y}\right] + w_y \frac{\partial E}{\partial w_{yy}} \frac{\partial^3 \xi^2}{\partial y^3} - \frac{\partial \xi^2}{\partial x} = 0. \tag{5.30}$$

SUMMARY FOR $E = E(x, w_y, w_{yy})$

- $\xi^1 = \xi^1(x),$ (5.31)

- $\xi^2 = \xi^2(x, y),$ (5.32)

- $\eta(x, y, w) = -w \dfrac{\partial \xi^2}{\partial y}(x, y),$ (5.33)

- $\dfrac{\partial^2 \xi^2}{\partial y^2}\left[E + w_y \dfrac{\partial E}{\partial w_y}\right] + w_y \dfrac{\partial E}{\partial w_{yy}} \dfrac{\partial^3 \xi^2}{\partial y^3} - \dfrac{\partial \xi^2}{\partial x} = 0,$ (5.34)

- $\left[2E + 2w_y \dfrac{\partial E}{\partial w_y} + 3w_{yy} \dfrac{\partial E}{\partial w_{yy}}\right] \dfrac{\partial \xi^2}{\partial y} + 3w_y \dfrac{\partial E}{\partial w_{yy}} \dfrac{\partial^2 \xi^2}{\partial y^2} - E \dfrac{\mathrm{d}\xi^1}{\mathrm{d}x} - \xi^1 \dfrac{\partial E}{\partial x} = 0.$ (5.35)

5.1.2 Case 2: $\dfrac{\partial E}{\partial w_{yy}} = 0$

In this section the second case is considered. This is equivalent to considering an effective viscosity of the form $E = E(x, w_y)$.

5.1.2.1 The first invariance condition

When $\dfrac{\partial E}{\partial w_{yy}} = 0$ the conditions given by Equation (5.14) are lost and the effective viscosity is a function of x and w_y only. Return to Equation (5.10) where

$$\xi^1 = \xi^1(x, y, w), \quad \xi^2 = \xi^2(x, y, w), \quad \eta = \eta(x, y, w) \quad \text{and} \quad E = E(x, w_y). \tag{5.36}$$

The models considered in Chapter 4 have the form

$$E = \nu + \nu_T,$$

where

$$\nu_T = \nu_T(x, w_y, w_{yy}) \quad \text{and} \quad \nu = \text{constant}. \tag{5.37}$$

In order to cover the cases where ν_T is independent of w_{yy}, consider an effective viscosity given by a polynomial of the form

$$E(x, w_y) = E_0(x) + E_1(x)\, w_y + E_2(x)\, w_y^2 + \ldots + E_n(x)\, w_y^n. \tag{5.38}$$

The effective viscosity is taken only up to the second term as given by

$$E(x, w_y) = E_0(x) + E_1(x)\, w_y, \tag{5.39}$$

which corresponds to the highest power of w_y appearing in ν_T for Prandtl's mixing length model.

Substituting Equation (5.39) into Equation (5.10) yields

$$\eta + w\frac{\partial \xi^2}{\partial y} + \left[E_0(x)\, w_y + E_1(x)\, w_y^2\right]\frac{\partial \xi^1}{\partial y} + \left[E_0(x)\, w_y^2 + E_1(x)\, w_y^3\right]\frac{\partial \xi^1}{\partial w} + w\, w_y\frac{\partial \xi^2}{\partial w} = 0. \tag{5.40}$$

Equation (5.40) can be separated by powers of w_y

$$w_y^0: \qquad \eta + w\frac{\partial \xi^2}{\partial y} = 0, \tag{5.41}$$

$$w_y^1: \qquad w\frac{\partial \xi^2}{\partial w} + E_0(x)\frac{\partial \xi^1}{\partial y} = 0, \tag{5.42}$$

$$w_y^2: \qquad E_1(x)\frac{\partial \xi^1}{\partial y} + E_0(x)\frac{\partial \xi^1}{\partial w} = 0, \tag{5.43}$$

$$w_y^3: \qquad E_1(x)\frac{\partial \xi^1}{\partial w} = 0. \tag{5.44}$$

There are three cases to consider.

Case (i): $\quad E_0(x) \neq 0 \quad \text{and} \quad E_1(x) \neq 0$

Equations (5.43) and (5.44) give that $\xi^1 = \xi^1(x)$. Equation (5.42) requires that $\xi^2 = \xi^2(x,y)$. From Equation (5.41),

$$\eta(x,y,w) = -w\frac{\partial \xi^2}{\partial y}(x,y). \tag{5.45}$$

Thus

$$\xi^1 = \xi^1(x), \quad \xi^2 = \xi^2(x,y) \quad \text{and} \quad \eta(x,y,w) = -w\frac{\partial \xi^2}{\partial y}(x,y). \tag{5.46}$$

The equations in Equation (5.46) have the same form as the summary S when E depends also on w_{yy}. Thus Equations (5.31) to (5.35) can be used.

Case (ii): $E_0(x) = 0$ and $E_1(x) \neq 0$

Equations (5.41) to (5.44) reduce to

$$w_y^0: \qquad \eta + w\frac{\partial \xi^2}{\partial y} = 0, \tag{5.47}$$

$$w_y^1: \qquad w\frac{\partial \xi^2}{\partial w} = 0, \tag{5.48}$$

$$w_y^2: \qquad E_1(x)\frac{\partial \xi^1}{\partial y} = 0, \tag{5.49}$$

$$w_y^3: \qquad E_1(x)\frac{\partial \xi^1}{\partial w} = 0. \tag{5.50}$$

The same result as shown in Equation (5.46) is obtained and so Equations (5.31) to (5.35) are again used.

Case (iii): $E_0(x) \neq 0$ and $E_1(x) = 0$

Equations (5.41) to (5.44) reduce to

$$w_y^0: \qquad \eta + w\frac{\partial \xi^2}{\partial y} = 0, \tag{5.51}$$

$$w_y^1: \qquad w\frac{\partial \xi^2}{\partial w} + E_0(x)\frac{\partial \xi^1}{\partial y} = 0, \tag{5.52}$$

$$w_y^2: \qquad E_0(x)\frac{\partial \xi^1}{\partial w} = 0. \tag{5.53}$$

From Equation (5.53), $\xi^1 = \xi^1(x, y)$ and from Equation (5.51)

$$\eta(x, y, w) = -w\frac{\partial \xi^2}{\partial y}(x, y, w). \tag{5.54}$$

Hence

$$\xi^1 = \xi^1(x, y), \quad \xi^2 = \xi^2(x, y, w), \quad \eta(x, y, w) = -w\frac{\partial \xi^2}{\partial y}(x, y, w), \tag{5.55}$$

and

$$w\frac{\partial \xi^2}{\partial w} + E_0(x)\frac{\partial \xi^1}{\partial y} = 0 \tag{5.56}$$

is an additional condition.

INVARIANCE CONDITION ONE				
$E = E_0(x) + E_1(x)w_y$	ξ^1	ξ^2	η	REMAINING CONDITIONS
CASE (i) $E_0(x) \neq 0$ $E_1(x) \neq 0$	$\xi^1(x)$	$\xi^2(x,y)$	$-w\dfrac{\partial \xi^2}{\partial y}(x,y)$	NONE
CASE (ii) $E_0(x) = 0$ $E_1(x) \neq 0$	$\xi^1(x)$	$\xi^2(x,y)$	$-w\dfrac{\partial \xi^2}{\partial y}(x,y)$	NONE
CASE (iii) $E_0(x) \neq 0$ $E_1(x) = 0$	$\xi^1(x,y)$	$\xi^2(x,y,w)$	$-w\dfrac{\partial \xi^2}{\partial y}(x,y,w)$	$w\dfrac{\partial \xi^2}{\partial w} + E_0(x)\dfrac{\partial \xi^1}{\partial y} = 0$

TABLE 5.1: Summary of the first invariance condition using $E = E_0(x) + E_1(x)w_y$

5.1.2.2 The second invariance condition

Case (i):

Case (i A): $E_0(x) \neq 0$ and $E_1(x) \neq 0$

Since the Lie point symmetry in this case is the same as when $\dfrac{\partial E}{\partial w_{yy}} \neq 0$, $E = E_0(x) + E_1(x) w_y$ is substituted into Equations (5.31) to (5.35). Equations (5.34) and (5.35) are now given by

$$\frac{\partial^2 \xi^2}{\partial y^2}\left[E_0(x) + 2w_y E_1(x)\right] - \frac{\partial \xi^2}{\partial x} = 0, \tag{5.57}$$

$$\begin{aligned}\left[2E_0(x) + 4w_y E_1(x)\right]\frac{\partial \xi^2}{\partial y} - \left[E_0(x) + w_y E_1(x)\right]\frac{\mathrm{d}\xi^1}{\mathrm{d}x}&\\ - \xi^1\left[\frac{\mathrm{d}E_0(x)}{\mathrm{d}x} + w_y\frac{\mathrm{d}E_1(x)}{\mathrm{d}x}\right] &= 0.\end{aligned} \tag{5.58}$$

Separating Equation (5.57) by w_y gives

$$w_y^0: \qquad E_0(x)\frac{\partial^2 \xi^2}{\partial y^2} - \frac{\partial \xi^2}{\partial x} = 0, \tag{5.59}$$

$$w_y^1: \qquad 2E_1(x)\frac{\partial^2 \xi^2}{\partial y^2} = 0. \tag{5.60}$$

Using Equation (5.60) in Equation (5.59) results in $\xi^2 = \xi^2(y)$. Thus Equation (5.60) can be integrated twice resulting in

$$\xi^2(y) = Ay + B, \tag{5.61}$$

where A and B are constants.

Separating Equation (5.58) by w_y gives

$$w_y^0: \qquad 2E_0(x)\frac{\partial \xi^2}{\partial y} - E_0(x)\frac{\mathrm{d}\xi^1}{\mathrm{d}x} - \xi^1\frac{\mathrm{d}E_0(x)}{\mathrm{d}x} = 0, \tag{5.62}$$

$$w_y^1: \qquad 4E_1(x)\frac{\partial \xi^2}{\partial y} - E_1(x)\frac{\mathrm{d}\xi^1}{\mathrm{d}x} - \xi^1\frac{\mathrm{d}E_1(x)}{\mathrm{d}x} = 0. \tag{5.63}$$

Equation (5.62) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\xi^1 E_0(x)\right) = 2AE_0(x), \tag{5.64}$$

which, after integrating, yields

$$\xi^1(x) = \frac{1}{E_0(x)}\left[2A\int^x E_0(x)dx + F_0\right], \tag{5.65}$$

where F_0 is a constant. Similarly, Equation (5.63) gives

$$\xi^1(x) = \frac{1}{E_1(x)}\left[4A\int^x E_1(x)dx + F_1\right], \tag{5.66}$$

where F_1 is a constant.

For the Lie point symmetry to exist Equations (5.65) and (5.66) yield a consistency condition given by

$$\frac{1}{E_0(x)}\left[2A\int E_0(x)dx + F_0\right] = \frac{1}{E_1(x)}\left[4A\int E_1(x)dx + F_1\right], \tag{5.67}$$

where F_0 and F_1 are constants. From Equation (5.33),

$$\eta(w) = -Aw. \tag{5.68}$$

Thus when

$$E(x, w_y) = E_0(x) + E_1(x)w_y, \tag{5.69}$$

where $E_0(x) \neq 0$ and $E_1(x) \neq 0$, the Lie point symmetry is given by

$$X = \xi^1(x)\frac{\partial}{\partial x} + (Ay + B)\frac{\partial}{\partial y} - Aw\frac{\partial}{\partial w}, \tag{5.70}$$

where A and B are constants and where $\xi^1(x)$ is given by Equations (5.65) and (5.66), provided the consistency condition in Equation (5.67) is satisfied.

Case (i B): $E_0(x) = E_0 \neq 0$ and $E_1(x) \neq 0$

Here a special case of Case (i A) is considered, where $E_0(x) = E_0$ = constant is a non-zero constant. Substituting E_0 into Equation (5.62) and using $\xi^2(y) = Ay + B$ yields

$$2A - \frac{d\xi^1}{dx} = 0, \tag{5.71}$$

which can be integrated to give

$$\xi^1(x) = 2Ax + C, \tag{5.72}$$

where C is a constant. Equation (5.63) is now given by

$$2AE_1(x) - (2Ax + C)\frac{dE_1(x)}{dx} = 0. \tag{5.73}$$

Equation (5.73) can be solved using separation of variables to yield

$$E_1(x) = (2Ax + C)\,k, \tag{5.74}$$

where k is a constant. Thus when

$$E(x, w_y) = E_0 + E_1(x)w_y, \tag{5.75}$$

where E_0 is a non-zero constant a Lie point symmetry exists of the form

$$X = (2Ax + C)\frac{\partial}{\partial x} + (Ay + B)\frac{\partial}{\partial y} - Aw\frac{\partial}{\partial w}, \tag{5.76}$$

where A,B and C are constants.

Case (ii): $E_0(x) = 0$ and $E_1(x) \neq 0$

Since the Lie point symmetry in this case is the same as when $\dfrac{\partial E}{\partial w_{yy}} \neq 0$, $E = E_1(x)\, w_y$ is substituted into Equations (5.31) to (5.35). Equations (5.34) and (5.35) are now given by

$$2w_y E_1(x)\frac{\partial^2 \xi^2}{\partial y^2} - \frac{\partial \xi^2}{\partial x} = 0, \tag{5.77}$$

$$4E_1(x)\frac{\partial \xi^2}{\partial y} - E_1(x)\frac{\mathrm{d}\xi^1}{\mathrm{d}x} - \xi^1\frac{\mathrm{d}E_1(x)}{\mathrm{d}x} = 0. \tag{5.78}$$

Separating Equation (5.77) by w_y gives

$$w_y^0: \qquad \frac{\partial \xi^2}{\partial x} = 0, \tag{5.79}$$

$$w_y^1: \qquad 2E_1(x)\frac{\partial^2 \xi^2}{\partial y^2} = 0. \tag{5.80}$$

Equation (5.79) necessitates that $\xi^2 = \xi^2(y)$ and thus integrating Equation (5.80) twice gives

$$\xi^2 = Ay + B, \tag{5.81}$$

where A and B are constants.

Equation (5.78) now becomes

$$4AE_1(x) - E_1(x)\frac{\mathrm{d}\xi^1}{\mathrm{d}x} - \xi^1\frac{\mathrm{d}E_1(x)}{\mathrm{d}x} = 0. \tag{5.82}$$

If $E_1(x)$ is prescribed then writing Equation (5.82) as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(E_1(x)\xi^1(x)\right) = 4AE_1(x) \tag{5.83}$$

gives

$$\xi^1(x) = \frac{1}{E_1(x)}\left[4A\int^x E_1(x)dx + F_1\right] \tag{5.84}$$

where F_1 is a constant. If $\xi^1(x)$ and $E_1(x)$ are both unknown then a second condition in addition to Equation (5.82) is required to determine $\xi^1(x)$ and $E_1(x)$ in an invariant solution. From Equations (5.33) and (5.81) it follows again that

$$\eta(w) = -Aw. \tag{5.85}$$

Hence when $E(x, w_y) = E_1(x)w_y$ the Lie point symmetry is

$$X = \xi^1(x)\frac{\partial}{\partial x} + (Ay+B)\frac{\partial}{\partial y} - Aw\frac{\partial}{\partial w}, \tag{5.86}$$

where $\xi^1(x)$ and $E_1(x)$ satisfy Equation (5.82). If $E_1(x)$ is prescribed then $\xi^1(x)$ is given by Equation (5.84)

Case (iii):

Case (iii A): $E_0(x) \neq 0$ and $E_1(x) = 0$

The Lie point symmetry is given by Equation (5.55) provided the condition in Equation (5.56) is satisfied. The conserved vector is of the form Equation (5.9). The first prolongation is

$$X^{[1]} = \xi^1(x,y)\frac{\partial}{\partial x} + \xi^2(x,y,w)\frac{\partial}{\partial y} - w\frac{\partial \xi^2}{\partial y}(x,y,w)\frac{\partial}{\partial w} + \zeta_2\frac{\partial}{\partial w_y},$$

where from Equation (5.26)

$$\zeta_2 = -w\frac{\partial^2\xi^2}{\partial y^2} - ww_y\frac{\partial^2\xi^2}{\partial y\partial w} - 2w_y\frac{\partial \xi^2}{\partial y} - w_y^2\frac{\partial \xi^2}{\partial w} - w_x\frac{\partial \xi^1}{\partial y}.$$

The second invariance condition becomes

$$\begin{aligned} 0 = E_0(x)&\left[w\frac{\partial^2\xi^2}{\partial y^2} + \left(2\frac{\partial \xi^2}{\partial y} + w\frac{\partial^2\xi^2}{\partial y\partial w}\right)w_y + w_x\frac{\partial \xi^1}{\partial y} + w_y^2\frac{\partial \xi^2}{\partial w}\right] \\ &- w_y\xi^1\frac{\mathrm{d}E_0(x)}{\mathrm{d}x} - w_yE_0(x)\frac{\partial \xi^1}{\partial x} - w\left[\frac{\partial \xi^2}{\partial x} + w_x\frac{\partial \xi^2}{\partial w}\right]. \end{aligned} \tag{5.87}$$

Equation (5.87) is separated in powers of the independent partial derivatives of w

$$w_y^0: \qquad E_0(x)\frac{\partial^2\xi^2}{\partial y^2} - \frac{\partial \xi^2}{\partial x} = 0, \tag{5.88}$$

$$w_y: \qquad E_0(x)\left[w\frac{\partial^2\xi^2}{\partial y\partial w} + 2\frac{\partial \xi^2}{\partial y} - \frac{\partial \xi^1}{\partial x}\right] - \xi^1\frac{\mathrm{d}E_0}{\mathrm{d}x} = 0, \tag{5.89}$$

$$w_y^2: \qquad E_0(x)\frac{\partial \xi^2}{\partial w} = 0, \tag{5.90}$$

$$w_x: \qquad E_0(x)\frac{\partial \xi^1}{\partial y} - w\frac{\partial \xi^2}{\partial w} = 0. \tag{5.91}$$

There is also the remaining condition Equation (5.56)

$$w\frac{\partial \xi^2}{\partial w} + E_0(x)\frac{\partial \xi^1}{\partial y} = 0. \tag{5.92}$$

From Equation (5.90),

$$\xi^2 = \xi^2(x, y), \tag{5.93}$$

and from Equation (5.91)

$$\xi^1 = \xi^1(x). \tag{5.94}$$

The condition given by Equation (5.92) is identically satisfied and Equation (5.89) reduces to

$$2E_0(x)\frac{\partial \xi^2}{\partial y}(x, y) - E_0(x)\frac{\mathrm{d}\xi^1}{\mathrm{d}x} - \xi^1(x)\frac{\mathrm{d}E_0(x)}{\mathrm{d}x} = 0. \tag{5.95}$$

Differentiating Equation (5.95) with respect to y gives

$$\frac{\partial^2 \xi^2}{\partial y^2} = 0, \tag{5.96}$$

and Equation (5.88) reduces to

$$\frac{\partial \xi^2}{\partial x} = 0. \tag{5.97}$$

From Equations (5.96) and (5.97),

$$\xi^2 = \xi^2(y) = Ay + B, \tag{5.98}$$

where A and B are constants. Thus Equation (5.95) becomes

$$E_0(x)\frac{\mathrm{d}\xi^1}{\mathrm{d}x} + \xi^1(x)\frac{\mathrm{d}E_0(x)}{\mathrm{d}x} = 2E_0(x)A \tag{5.99}$$

and if $E_0(x)$ is prescribed then

$$\xi^1(x) = \frac{1}{E_1(x)}\left[2A\int^x E_1(x)dx + F_0\right]. \tag{5.100}$$

From Equations (5.33) and (5.98)

$$\eta(w) = -Aw. \tag{5.101}$$

Hence if $E = E_0(x)$ then the Lie point symmetry is

$$X = \xi^1(x)\frac{\partial}{\partial x} + (Ay+B)\frac{\partial}{\partial y} - Aw\frac{\partial}{\partial w}, \tag{5.102}$$

provided $\xi^1(x)$ and $E_0(x)$ satisfy the condition given by Equation (5.99). If $E_0(x)$ is prescribed then $\xi^1(x)$ is given by Equation (5.100).

Case (iii B): $E_0(x) = E_0$ and $E_1(x) = 0$

A special case of Case (iii A) is considered where $E = E_0$ = a non-zero constant. Substituting this constant value into Equation (5.99) gives

$$\frac{\mathrm{d}\xi^1}{\mathrm{d}x} = 2A \tag{5.103}$$

and therefore

$$\xi^1 = 2Ax + C, \tag{5.104}$$

where C is a constant. Thus if $E = E_0$ where E_0 is a non-zero constant then the Lie point symmetry is

$$X = (2Ax + C)\frac{\partial}{\partial x} + (Ay + B)\frac{\partial}{\partial y} - Aw\frac{\partial}{\partial w}, \tag{5.105}$$

where A, B and C are constants.

The results for the Lie point symmetry when $E = E_0(x) + E_1(x)\,w_y$ obtained from the first and second invariance conditions are summarised in Table 5.2.

INVARIANCE CONDITIONS ONE AND TWO				
$E = E_0(x) + E_1(x)\,w_y$	ξ^1	ξ^2	η	REMAINING CONDITIONS
CASE (iA) $E_0(x) \neq 0$ $E_1(x) \neq 0$	$\xi^1(x)$	$Ay + B$	$-Aw$	$\xi^1 \dfrac{\mathrm{d}E_0(x)}{\mathrm{d}x} + E_0(x)\dfrac{\mathrm{d}\xi^1}{\mathrm{d}x} = 2AE_0(x)$ $\xi^1 \dfrac{\mathrm{d}E_1(x)}{\mathrm{d}x} + E_1(x)\dfrac{\mathrm{d}\xi^1}{\mathrm{d}x} = 4AE_1(x)$ $\dfrac{1}{E_0(x)}\left[2A\int E_0(x)dx + F_0\right] = \dfrac{1}{E_1(x)}\left[4A\int E_1(x)dx + F_1\right]$
CASE (iB) $E_0(x) = E_0 \neq 0$ $E_1(x) \neq 0$	$2Ax + C$	$Ay + B$	$-Aw$	$E_1(x) = (2Ax + C)\,k$
CASE (ii) $E_0(x) = 0$ $E_1(x) \neq 0$	$\xi^1(x)$	$Ay + B$	$-Aw$	$\xi^1 \dfrac{\mathrm{d}E_1(x)}{\mathrm{d}x} + E_1(x)\dfrac{\mathrm{d}\xi^1}{\mathrm{d}x} = 4AE_1(x)$
CASE (iiiA) $E_0(x) \neq 0$ $E_1(x) = 0$	$\xi^1(x)$	$Ay + B$	$-Aw$	$\xi^1 \dfrac{\mathrm{d}E_0(x)}{\mathrm{d}x} + E_0(x)\dfrac{\mathrm{d}\xi^1}{\mathrm{d}x} = 2AE_0(x)$
CASE (iiiB) $E_0(x) = E_0 \neq 0$ $E_1(x) = 0$	$2Ax + C$	$Ay + B$	$-Aw$	NONE

TABLE 5.2: Summary of the invariance conditions using $E = E_0(x) + E_1(x)\,w_y$

5.2 Constant eddy viscosity

In this section a constant eddy viscosity is applied to the wake. The effective viscosity is denoted by $E = \nu + \nu_T$ = a non-zero constant. For simplicity, the constant is chosen to be $\mathscr{E}$. This corresponds to Case (iii A) when $E_0(x) = E_0$ and $E_1(x) = 0$.

Thus

$$\xi^1 = 2Ax + C, \quad \xi^2 = Ay + B \quad \text{and} \quad \eta = -Aw. \tag{5.106}$$

The Lie point symmetry associated with the conserved vector with a constant eddy viscosity is given by

$$X = (2Ax + C)\frac{\partial}{\partial x} + (Ay + B)\frac{\partial}{\partial y} - Aw\frac{\partial}{\partial w}. \tag{5.107}$$

Consider the general case for which $A \neq 0$, Equation (5.107) is divided through by A and defining $\alpha = \dfrac{C}{A}$ and $\beta = \dfrac{B}{A}$ results in

$$X = (2x + \alpha)\frac{\partial}{\partial x} + (y + \beta)\frac{\partial}{\partial y} - w\frac{\partial}{\partial w}. \tag{5.108}$$

Now, $w = W(x, y)$ is an invariant solution of the partial differential equation given by Equation (5.1) with $E = \mathscr{E}$ a constant, generated by the Lie point symmetry associated with the elementary conserved vector provided that

$$X(w - W(x, y))|_{w=W} = 0, \tag{5.109}$$

that is, provided

$$(2x + \alpha)\frac{\partial W}{\partial x} + (y + \beta)\frac{\partial W}{\partial y} = -W. \tag{5.110}$$

The differential equations of the characteristic curves of Equation (5.110) are

$$\frac{\mathrm{d}x}{2x+\alpha} = \frac{\mathrm{d}y}{y+\beta} = \frac{\mathrm{d}W}{-W}. \tag{5.111}$$

The first pair of terms gives

$$\frac{y+\beta}{\left(x+\dfrac{\alpha}{2}\right)^{1/2}} = C_1, \tag{5.112}$$

where C_1 is a constant. The first and last terms yield

$$W = \frac{C_2}{\left(x+\dfrac{\alpha}{2}\right)^{1/2}}, \tag{5.113}$$

where C_2 is a constant. A general solution of the partial differential equation given by Equation (5.110) is

$$C_2 = F(C_1), \tag{5.114}$$

where F is an arbitrary function. Since $w = W(x,y)$ this results in

$$w = \frac{F(\xi)}{\left(x+\dfrac{\alpha}{2}\right)^{1/2}}, \tag{5.115}$$

where

$$\xi = \frac{y+\beta}{\left(x+\dfrac{\alpha}{2}\right)^{1/2}}. \tag{5.116}$$

The boundary conditions given in Equation (4.31) transform to

$$F(\pm\xi_b) = 0 \qquad \text{and} \qquad \frac{\mathrm{d}F}{\mathrm{d}\xi}(\pm\xi_b) = 0, \tag{5.117}$$

where

$$\xi_b(x) = \frac{y_b(x) + \beta}{\left(x + \dfrac{\alpha}{2}\right)^{1/2}}. \tag{5.118}$$

The boundary condition given in Equation (4.32) transforms to

$$\frac{\mathrm{d}F}{\mathrm{d}\xi}(\xi|_{y=0}) = 0. \tag{5.119}$$

It can be demonstrated that the constants α and β can be determined from physical properties and are not arbitrary as shown by Hutchinson et al [25].

Consider β first and differentiating Equation (5.119) with respect to x yields

$$\frac{\mathrm{d}^2F}{\mathrm{d}\xi^2}(\xi|_{y=0})\frac{\mathrm{d}\xi}{\mathrm{d}x} = -\left(\frac{\beta}{\left(x + \dfrac{\alpha}{2}\right)^{3/2}}\right)\frac{\mathrm{d}^2F}{\mathrm{d}\xi^2}(\xi|_{y=0}) = 0. \tag{5.120}$$

Recall, however, from Equation (5.115) that

$$\frac{\partial^2 w}{\partial y^2} = \frac{\mathrm{d}^2F}{\mathrm{d}\xi^2}\frac{1}{\left(x + \dfrac{\alpha}{2}\right)^{3/2}}.$$

Now at $y = 0$, a local maximum exists for w and thus

$$\frac{\partial^2 w}{\partial y^2}(x,0) < 0 \quad \text{and therefore} \quad \frac{\mathrm{d}^2F}{\mathrm{d}\xi^2}(\xi|_{y=0}) \neq 0.$$

It is therefore necessary that $\beta = 0$ to satisfy Equation (5.120) and as a result $\xi = 0$ when $y = 0$.

Let $\mathscr{W}(x)$ denote the effective width of the wake. The effective width is defined to be the range in which changes of the mean velocity are significant and where outside of this range the fluctuations are small enough to be neglected. If the wake is finite, then the effective width is up to the boundary. To find α, consider the effective half-width of the wake. If it is assumed that the effective width of the wake is valid directly behind the obstacle, that is, as $x \to 0$, then it is required that the effective width tends to zero as $x \to 0$. Now, a good estimate of the effective half-width of the wake is given in order of magnitude by $\xi = \mathscr{O}(1)$ and therefore

$$\mathscr{W}(x) = \mathscr{O}\left(\sqrt{x + \frac{\alpha}{2}}\right). \tag{5.121}$$

Then as $x \to 0$, it is expected that $\mathscr{W} \to 0$ and therefore $\alpha = 0$.

The conserved quantity given by Equation (4.34) is now expressed in terms of the similarity variables:

$$\int_{-\xi_b}^{\xi_b} F(\xi)\, \mathrm{d}\xi = D, \tag{5.122}$$

where

$$F(\xi) = w\sqrt{x}, \tag{5.123}$$

and

$$\xi = \frac{y}{\sqrt{x}}. \tag{5.124}$$

Equation (5.122) for D does not depend on x or y separately because the transformation was generated by a Lie point symmetry associated with the elementary conserved vector from which the conserved quantity D was derived. As D is a constant, the limits in the integral must also be constant. That is, ξ_b is a constant. Equation (5.108), the Lie Symmetry associated with the conserved vector reduces to

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - w\frac{\partial}{\partial w}. \tag{5.125}$$

Finally, the boundary conditions in Equations (5.117) and (5.118) become

$$F(\pm\xi_b) = 0, \qquad \frac{\mathrm{d}F}{\mathrm{d}\xi}(\pm\xi_b) = 0, \tag{5.126}$$

$$\xi_b(x) = \frac{y_b}{\sqrt{x}}$$

and

$$\frac{\mathrm{d}F}{\mathrm{d}\xi}(0) = 0, \tag{5.127}$$

respectively.

Substituting Equations (5.123) and (5.124) into Equation (4.29) yields

$$\mathscr{E}\frac{\mathrm{d}^2F}{\mathrm{d}\xi^2} + \frac{\mathrm{d}}{\mathrm{d}\xi}(\xi F) = 0. \tag{5.128}$$

Equation (5.128) is an example of the double reduction theorem of Sjoberg [55] that if a PDE is reduced to an ODE by a Lie point symmetry associated with a conserved vector of the PDE, then the ODE can be integrated at least once. Equation (5.128) can be integrated once to yield

$$\mathscr{E}\frac{\mathrm{d}F}{\mathrm{d}\xi} + \xi F(\xi) = K_1, \tag{5.129}$$

where K_1 is a constant of integration. From the boundary condition in Equation (5.127), at $\xi = 0$, K_1 is found to be zero.

Equation (5.129) can be solved by separation of variables resulting in the following solution for F:

$$F(\xi) = K_2 \exp\left[-\frac{1}{2\mathscr{E}}\xi^2\right], \tag{5.130}$$

where K_2 is a constant.

For the constant eddy viscosity, only the value of $\mathscr{E}$ changes. That is, $E = \nu + \nu_T$ when the kinematic viscosity is not neglected and $E = \nu_T$ when $\nu = 0$. The solution is of the same form and differs only by the value of the constant $\mathscr{E}$. To solve for K_2, the conserved quantity given by Equation (5.122) is used. Substituting Equation (5.130) into Equation (5.122) requires that

$$D = 2K_2 \int_0^\infty \exp\left[-\frac{1}{2\mathscr{E}}\xi^2\right] d\xi, \tag{5.131}$$

which results in

$$D = K_2 \left. \frac{\sqrt{\pi}\,\mathrm{erf}\left[\sqrt{\frac{1}{2\mathscr{E}}}\,\xi\right]}{\sqrt{\frac{1}{2\mathscr{E}}}} \right|_0^\infty . \tag{5.132}$$

But

$$\mathrm{erf}(\infty) = 1 \quad \text{and} \quad \mathrm{erf}(0) = 0,$$

and therefore

$$K_2 = \frac{D}{\sqrt{2\pi\mathscr{E}}}.$$

Finally,

$$F(\xi) = \frac{D}{\sqrt{2\pi\mathscr{E}}} \exp\left[-\frac{1}{2\mathscr{E}}\xi^2\right]. \tag{5.133}$$

5.3 Prandtl's mixing length model

In this section Prandtl's mixing length model is applied to the wake. The effective viscosity E is given by $E = \nu + \nu_T$ where $\nu_T = \ell^2(x)\left|w_y\right|$. Comparing this with Equation (5.39) it is easily seen that $E_0(x) = \nu$ and $E_1(x)$ still needs to be determined.

Hutchinson and Mason [26] gave the following argument to determine which sign of $\left|w_y\right|$ should be used in the upper half of the wake. The first boundary condition in Equation (4.31) gives that the velocity deficit is zero when $y = y_b(x)$. The velocity deficit is a positive maximum along the $x-$axis as given by Equation (4.32). Therefore the velocity deficit is a decreasing function of y. For the upper half of the wake

$$\frac{\partial w}{\partial y} \le 0, \quad 0 \le y \le y_b(x). \tag{5.134}$$

Thus,

$$\left|w_y\right| = -w_y,$$

giving $E_1(x) = -\ell^2(x)$. The turbulent viscosity, in terms of the velocity deficit and dimensionless variables

$$w^* = \frac{w}{U}, \quad y^* = \frac{y}{\delta}, \quad \ell^* = \frac{\ell}{\ell_{10}} \tag{5.135}$$

is given by

$$\nu_T = -\frac{\nu_{T0}}{\nu_C + \nu_{TC}} \ell^{*2}(x^*) \frac{\partial w^*}{\partial y^*}, \tag{5.136}$$

where ν_{T0} is defined as

$$\nu_{T0} = \ell_{10}^2 \frac{U}{\delta}. \tag{5.137}$$

5.3.1 The case $\nu \neq 0$

When $\nu \neq 0$ is a constant, Case (i B) is applied. Substituting $E_1(x) = -\ell^2(x)$ into the remaining condition given by Equation (5.74) gives

$$\ell(x) = \ell_{10}\sqrt{(2Ax + C)}. \tag{5.138}$$

where $\ell_{10} = \sqrt{-k}$.

The Lie point symmetry associated with the conserved vector using Prandtl's mixing length hypothesis is given by

$$X = (2Ax + C)\frac{\partial}{\partial x} + (Ay + B)\frac{\partial}{\partial y} - Aw\frac{\partial}{\partial w}. \tag{5.139}$$

Consider the general case when $A \neq 0$, Equation (5.139) is divided through by A and define $\alpha = \dfrac{C}{A}$ and $\beta = \dfrac{B}{A}$. In a similar matter to the case of a constant eddy viscosity, it can be shown that $\alpha = 0 = \beta$ and therefore Equation (5.139) reduces to

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - w\frac{\partial}{\partial w}. \tag{5.140}$$

It is useful to note that any Lie point symmetry of the form given by Equation (5.140) has the same similarity variables, Equations (5.123) and (5.124), and boundary conditions, Equations (5.126) and (5.127), as calculated in the case of constant eddy viscosity. For Prandtl's mixing length model with non-zero kinematic viscosity, Equation (4.29) is

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial y}\left[\left(\frac{\nu}{\nu_C + \nu_{TC}} - 2\frac{\nu_{T0}}{\nu_C + \nu_{TC}}\ell_{10}^2 x\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial y}\right]. \tag{5.141}$$

Equation (5.141) is reduced to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(\frac{\nu}{\nu_C+\nu_{TC}}-\frac{\nu_{T0}}{\nu_C+\nu_{TC}}\ell_{10}^2\frac{\mathrm{d}F}{\mathrm{d}\xi}\right)\frac{\mathrm{d}F}{\mathrm{d}\xi}\right]+\frac{\mathrm{d}}{\mathrm{d}\xi}[\xi F]=0. \tag{5.142}$$

Equation (5.142) is another example of the double reduction theorem of Sjoberg [55]. Integrating Equation (5.142) once and implementing the boundary condition given by Equation (5.127), at $\xi=0$ gives

$$\left(\frac{\nu}{\nu_C+\nu_{TC}}-\frac{\nu_{T0}}{\nu_C+\nu_{TC}}\ell_{10}^2\frac{\mathrm{d}F}{\mathrm{d}\xi}\right)\frac{\mathrm{d}F}{\mathrm{d}\xi}+\xi F=0. \tag{5.143}$$

Making the transformation

$$F=F^*A \quad \text{and} \quad \xi=\xi^*B$$

where A and B are given by

$$A=\frac{\nu}{\nu_{T0}{\ell_{10}}^2}\left[\frac{\nu}{\nu_C+\nu_{TC}}\right]^{1/2} \quad \text{and} \quad B=\left[\frac{\nu}{\nu_C+\nu_{TC}}\right]^{1/2}, \tag{5.144}$$

Equation (5.143) reduces to

$$\left(1-\frac{\mathrm{d}F^*}{\mathrm{d}\xi^*}\right)\frac{\mathrm{d}F^*}{\mathrm{d}\xi^*}+\xi^*F^*=0. \tag{5.145}$$

Equation (5.145) is a quadratic equation in $\dfrac{\mathrm{d}F^*}{\mathrm{d}\xi^*}$ with solution

$$\frac{\mathrm{d}F^*}{\mathrm{d}\xi^*}=\frac{1\pm\sqrt{1+4\xi^*F^*}}{2}.$$

Since the upper half of the wake is being considered Equation (5.134) requires that

$$\frac{\partial w}{\partial y}=\frac{1}{2x}\frac{\mathrm{d}F}{\mathrm{d}\xi}<0 \tag{5.146}$$

and hence the negative root is taken leaving

$$\frac{\mathrm{d}F}{\mathrm{d}\xi} = \frac{1-\sqrt{1+4\xi F}}{2}, \tag{5.147}$$

where the stars $*$ have been dropped for convenience.

A numerical solution of the first order differential equation given by Equation (5.147) is derived later.

5.3.2 The case $\nu = 0$

When the kinematic viscosity is neglected Case (ii) is applied. As $E_1(x)$ is prescribed, $\xi^1(x)$ is given by Equation (5.84). Substituting $E_1(x) = -\ell^2(x)$ into Equation (5.84) results in

$$\xi^1 = \frac{1}{\ell^2(x)}\left[4A\int^x \ell^2(\alpha)\,\mathrm{d}\alpha + F_1\right]. \tag{5.148}$$

The Lie point symmetry associated with the fundamental conserved vector using Prandtl's mixing length model with no kinematic viscosity is given by

$$X = \left(\frac{1}{\ell^2(x)}\left[4A\int^x \ell^2(\alpha)\,\mathrm{d}\alpha + F_1\right]\right)\frac{\partial}{\partial x} + (Ay+B)\frac{\partial}{\partial y} - Aw\frac{\partial}{\partial w}. \tag{5.149}$$

In the same manner as in Section 5.2, the constants B and F_1 can be shown to be zero and the Lie point symmetry reduces to

$$X = \frac{4}{\ell^2(x)}\int^x \ell^2(\alpha)\,\mathrm{d}\alpha\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - w\frac{\partial}{\partial w}. \tag{5.150}$$

Substituting $E_1(x) = -\ell^2(x)$ into Equation (5.82) gives

$$\frac{\mathrm{d}\xi^1}{\mathrm{d}x} + \frac{2}{\ell(x)}\frac{\mathrm{d}\ell}{\mathrm{d}x}\xi^1(x) = 4. \tag{5.151}$$

Equation (5.151) is one equation for the two unknowns $\xi^1(x)$ and $\ell(x)$. A second equation is required.

Now, $w = W(x, y)$ is an invariant solution of the PDE given by Equation (5.1) with $\nu = 0$ generated by the Lie point symmetry in Equation (5.150) provided Equation (5.109) is satisfied, that is, provided

$$\xi^1(x)\frac{\partial W}{\partial x} + y\frac{\partial W}{\partial y} = -W. \tag{5.152}$$

The differential equations of the characteristic curves of Equation (5.152) are

$$\frac{\mathrm{d}x}{\xi^1(x)} = \frac{\mathrm{d}y}{y} = -\frac{\mathrm{d}W}{W}. \tag{5.153}$$

The first pair in Equation (5.153) gives

$$\frac{y}{H(x)} = C_1, \tag{5.154}$$

where C_1 is a constant and

$$H(x) = \exp\left[\int \frac{\mathrm{d}x}{\xi^1(x)}\right]. \tag{5.155}$$

The first and last terms in Equation (5.153) give

$$WH(x) = C_2, \tag{5.156}$$

where C_2 is a constant. The general solution of the PDE in Equation (5.152) is

$$C_2 = F(C_1),$$

where F is an arbitrary function. Since $w = W(x, y)$

$$w = \frac{F(\xi)}{H(x)} \quad \text{where} \quad \xi = \frac{y}{H(x)}, \tag{5.157}$$

is obtained.

Consider now the conserved quantity for a classical wake

$$D = 2\int_0^{y_b(x)} w(x, y)\mathrm{d}y, \tag{5.158}$$

where D is a constant independent of x. Express the conserved quantity in terms of the variables ξ and $F(\xi)$ of the invariant solution. Equation (5.158) becomes

$$D = 2\int_0^{\frac{y_b(x)}{H(x)}} F(\xi)d\xi. \tag{5.159}$$

Since D is independent of x,

$$\frac{y_b}{H(x)} = \text{constant} = \xi_b, \tag{5.160}$$

where ξ_b has still to be determined. The conserved quantity in Equation (5.159) becomes

$$D = 2\int_0^{\xi_b} F(\xi)d\xi. \tag{5.161}$$

The boundary of the wake in the transverse y-direction is

$$y = y_b(x), \tag{5.162}$$

where

$$y_b(x) = \xi_b H(x). \tag{5.163}$$

The second equation relating $\xi^1(x)$ and $\ell(x)$ is obtained from Prandtl's hypothesis [45]. Prandtl made the following statement: "In these cases, of 'free turbulence' one can accept, at least for high enough Reynolds numbers, that in comparable cases the processes always run geometrically and mechanically similar in a cross section drawn transversely to the longitudinal extension. This means that the mixing lengths always remain proportional to the width of the stream as the width of the jet or wake flow increases..."

Prandtl's hypothesis that the mixing length and the breadth are proportional is used:

$$\ell = K y_b, \tag{5.164}$$

where K is a constant. From Equations (5.155), (5.163) and (5.164),

$$\ell(x) = K\xi_b \exp\left[\int^x \frac{dx}{\xi^1(x)}\right]. \tag{5.165}$$

Differentiate Equation (5.165) with respect to x. This gives

$$\frac{\mathrm{d}\ell}{\mathrm{d}x} = \frac{\ell(x)}{\xi^1(x)}. \tag{5.166}$$

Substituting Equation (5.166) for $\dfrac{\mathrm{d}\ell(x)}{\mathrm{d}x}$ into Equation (5.151) gives

$$\frac{\mathrm{d}\xi^1}{\mathrm{d}x} = 2, \tag{5.167}$$

and therefore

$$\xi^1(x) = 2x + C, \tag{5.168}$$

where C is a constant. The constant C can be shown to be zero as in Section 5.2. Thus

$$\xi^1(x) = 2x, \tag{5.169}$$

and from Equation (5.165)

$$\ell(x) = \ell_{10}\sqrt{x}, \tag{5.170}$$

where $\ell_{10} = K\xi_b$.

Also, from Equation (5.155) and Equation (5.169)

$$H(x) = \exp\left[\int^x \frac{dx}{\xi^1(x)}\right] = \sqrt{x}, \tag{5.171}$$

and therefore from Equation (5.157)

$$w = \frac{F(\xi)}{\sqrt{x}} \quad \text{where} \quad \xi = \frac{y}{\sqrt{x}}, \tag{5.172}$$

which is the same as the case for $\nu \neq 0$. When $\nu = 0$ the dimensionless effective viscosity is

$$E(x, w_y) = -\frac{\nu_{T0}}{\nu_C + \nu_{TC}}\ell^2(x)\frac{\partial w}{\partial y}. \tag{5.173}$$

Although $\nu = 0$, the characteristic eddy viscosity is still defined as $\dfrac{\nu_{T0}}{\nu_C + \nu_{TC}}$. Using Equation (5.170) for $\ell(x)$ and Equation (5.173) the PDE Equation (4.29) for w becomes

$$\frac{\partial w}{\partial x} = -\frac{\nu_{T0}}{\nu_C + \nu_{TC}}\ell_{10}^2 x\frac{\partial}{\partial y}\left[\left(\frac{\partial w}{\partial y}\right)^2\right]. \tag{5.174}$$

Expressed in terms of the invariant solution in Equation (5.172) the PDE given by Equation (5.174) reduces to the ODE

$$\frac{\nu_{T0}}{\nu_C + \nu_{TC}}\ell_{10}^2\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(\frac{\mathrm{d}F}{\mathrm{d}\xi}\right)^2\right] - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi F(\xi)) = 0. \tag{5.175}$$

Equation (5.175) is another example of the double reduction theorem of Sjoberg [55]. The boundary conditions are from Equations (5.126) and (5.127)

$$F(\xi_b) = 0, \quad \frac{\mathrm{d}F(\xi_b)}{\mathrm{d}\xi} = 0, \quad \frac{\mathrm{d}F}{\mathrm{d}\xi}(0) = 0. \tag{5.176}$$

Integrating Equation (5.175) once and using the boundary condition at $\xi = 0$ gives

$$2\frac{\nu_{T0}}{\nu_C + \nu_{TC}}\ell_{10}^2\left(\frac{\mathrm{d}F}{\mathrm{d}\xi}\right)^2 - \xi F(\xi) = 0. \tag{5.177}$$

Making the transformation

$$F^* = AF, \quad \xi^* = B\xi, \tag{5.178}$$

transforms the ODE given by Equation (5.177) to

$$2\frac{\nu_{T0}}{\nu_C + \nu_{TC}}\ell_{10}^2\frac{A}{B^2}\left(\frac{\mathrm{d}F^*}{\mathrm{d}\xi^*}\right)^2 - \xi^* F^* = 0. \tag{5.179}$$

By choosing

$$2\frac{\nu_{T0}}{\nu_C + \nu_{TC}}\ell_{10}^2\frac{A}{B^2} = 1, \tag{5.180}$$

Equation (5.179) becomes

$$\left(\frac{\mathrm{d}F^*}{\mathrm{d}\xi^*}\right)^2 - \xi^* F^* = 0. \tag{5.181}$$

Unlike in the case for $\nu \neq 0$, Equation (5.181) can be solved analytically. Rewrite Equation (5.181) as

$$\frac{\mathrm{d}F^*}{\mathrm{d}\xi^*} = -\left(\xi^* F^*\right)^{1/2}, \tag{5.182}$$

where the negative root has again been taken for the upper half of the wake. Thus

$$\frac{\mathrm{d}F^*}{\sqrt{F^*}} = -\sqrt{\xi^*}\mathrm{d}\xi^*, \tag{5.183}$$

and integrating obtains

$$\sqrt{F^*(\xi^*)} = -\frac{1}{3}\xi^{*3/2} + c, \tag{5.184}$$

where c is a constant. It is seen that as $\xi^* \to \infty$, $\sqrt{F^*(\xi^*)} \to -\infty$ which is not physical because it is unbounded and negative. Hence ξ_b must be finite. From the boundary condition in Equation (5.176) it follows that

$$F^*(\xi_b^*) = 0 \quad \text{and hence} \quad c = \frac{1}{3}\xi_b^{*\,3/2}. \tag{5.185}$$

Thus

$$\sqrt{F^*(\xi^*)} = \frac{1}{3}\left(\xi_b^{*\,3/2} - \xi^{*\,3/2}\right), \quad 0 \le \xi \le \xi_b. \tag{5.186}$$

The solution in Equation (5.186) again shows that the wake is bounded in the y-direction. Squaring Equation (5.186) gives

$$F^*(\xi^*) = \frac{1}{9}\left(\xi_b^{*\,3/2} - \xi^{*\,3/2}\right)^2, \quad 0 \le \xi \le \xi_b. \tag{5.187}$$

The boundary constant ξ_b can be obtained from the conserved quantity given by Equation (5.161) which expressed in terms of F^* and ξ^* is

$$D = 2AB\int_0^{\xi_b^*} F^*(\xi^*)\,d\xi^*. \tag{5.188}$$

Substituting Equation (5.187) into Equation (5.188) and letting $u = \dfrac{\xi^*}{\xi_b^*}$ gives

$$D = \frac{2AB}{9}\xi_b^{*\,4}\int_0^1 \left(1 - u^{3/2}\right)^2 du \tag{5.189}$$

and hence

$$D = \frac{AB}{10}\xi_b^{*\,4}. \tag{5.190}$$

Thus

$$\xi_b^* = \left(\frac{10D}{AB}\right)^{1/4}. \tag{5.191}$$

5.4 Prandtl's extended mixing length model

In this section Prandtl's Extended mixing length model is applied to the wake. Two derivations of Prandtl's extended mixing length model were given. In one derivation the mixing lengths were shown to be proportional and the eddy viscosity was given by Equation (3.26). Here the two mixing lengths are taken to be distinct and it will be shown that for an invariant solution to exist the two mixing lengths must be proportional. The effective viscosity E is given by $E = \nu + \nu_T$ where, in dimensionless form and in terms of the mean velocity deficit,

$$E^* = \frac{\nu}{\nu_C + \nu_{TC}} + \frac{\nu_{T0}}{\nu_C + \nu_{TC}} \ell_{10}^{*\,2} \ell_1^{*2}(x^*) \sqrt{\overline{w}_y^{*2} + \ell_{20}^{*\,2} \ell_2^{*2}(x^*) \overline{w}_{yy}^{*\;2}}. \tag{5.192}$$

In order to simplify the notation let

$$\beta = \frac{\nu}{\nu_C + \nu_{TC}}, \quad \alpha = \frac{\nu_{T0}}{\nu_C + \nu_{TC}}, \quad M^2(x) = \ell_{10}^2 \ell_1^2(x), \quad N^2(x) = \ell_{20}^2 \ell_2^2(x), \tag{5.193}$$

where the overhead bars and stars $*$ have been suppressed to further simplify the notation. Equation (5.192) in terms of Equation (5.193) is now

$$E(x, w_y, w_{yy}) = \beta + \alpha M^2(x) \left(w_y^2 + N^2(x) w_{yy}^2 \right)^{1/2}. \tag{5.194}$$

The Lie point symmetry is from Equations (5.31) to (5.33)

$$X = \xi^1(x) \frac{\partial}{\partial x} + \xi^2(x, y) \frac{\partial}{\partial y} - w \frac{\partial \xi^2}{\partial y}(x, y) \frac{\partial}{\partial w},$$

with the remaining conditions in Equations (5.34) and (5.35).

5.4.1 The case $\nu \neq 0$

Consider first the case where the kinematic viscosity is not neglected. Substituting the effective viscosity given by Equation (5.194) into Equation (5.34) yields

$$\alpha M^2(x)N^2(x)w_y w_{yy}\left(w_y^2+N^2(x)w_{yy}^2\right)^{-1/2}\frac{\partial^3\xi^2}{\partial y^3}-\frac{\partial\xi^2}{\partial x}+ \\ \left(w_y^2+N^2(x)w_{yy}^2\right)^{-1/2}\left[\beta\left(w_y^2+N^2(x)w_{yy}^2\right)^{1/2}\right. \\ \left.+\,\alpha M^2(x)\left(2w_y^2+N^2(x)w_{yy}^2\right)\right]\frac{\partial^2\xi^2}{\partial y^2}=0. \quad (5.195)$$

Multiplying Equation (5.195) by $\left(w_y^2+N^2(x)w_{yy}^2\right)^{1/2}$ gives

$$\left(\frac{\partial\xi^2}{\partial x}-\beta\frac{\partial^2\xi^2}{\partial y^2}\right)\left[\left(w_y^2+N^2(x)w_{yy}^2\right)^{1/2}\right]=\alpha M^2(x)\left[\left(2w_y^2+N^2(x)w_{yy}^2\right)\frac{\partial^2\xi^2}{\partial y^2}\right. \\ \left.+N^2(x)w_y w_{yy}\frac{\partial^3\xi^2}{\partial y^3}\right]. \quad (5.196)$$

Squaring both sides of Equation (5.196) yields

$$\left(\frac{\partial\xi^2}{\partial x}-\beta\frac{\partial^2\xi^2}{\partial y^2}\right)^2\left[\left(w_y^2+N^2(x)w_{yy}^2\right)\right]=\alpha^2M^4(x)\left[N^4(x)w_y^2w_{yy}^2\left(\frac{\partial^3\xi^2}{\partial y^3}\right)\right. \\ +\left(4w_y^4+4N^2(x)w_y^2w_{yy}^2+N^4(x)w_{yy}^4\right)\left(\frac{\partial^2\xi^2}{\partial y^2}\right)^2 \\ \left.+2N^2(x)\frac{\partial\xi^2}{\partial y}\frac{\partial^3\xi^2}{\partial y^3}\left(2w_y^3w_{yy}+N^2(x)w_yw_{yy}^3\right)\right]. \quad (5.197)$$

Equation (5.197) is separated by powers of w_{yy}

$$w_{yy}^4: \qquad \alpha^2M^4(x)N^4(x)\left(\frac{\partial^2\xi^2}{\partial y^2}\right)=0. \quad (5.198)$$

Thus

$$\frac{\partial^2 \xi^2}{\partial y^2}(x,y) = 0 \tag{5.199}$$

and therefore

$$\xi^2(x,y) = yA(x) + B(x). \tag{5.200}$$

Using Equation (5.200), Equation (5.197) reduces to

$$\left(w_y^2 + N^2(x)\, w_{yy}^2\right)\left(\frac{\partial \xi^2}{\partial x}\right)^2 = 0. \tag{5.201}$$

Thus

$$\frac{\partial \xi^2}{\partial x}(x,y) = 0 \tag{5.202}$$

and therefore

$$y\frac{\mathrm{d}A}{\mathrm{d}x} + \frac{\mathrm{d}B}{\mathrm{d}x} = 0. \tag{5.203}$$

Hence separating Equation (5.203) by y,

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 0, \quad \frac{\mathrm{d}B}{\mathrm{d}x} = 0 \tag{5.204}$$

and therefore $A(x) = A_0$ and $B(x) = B_0$ where A_0 and B_0 are constants. Thus

$$\xi^2(y) = A_0 y + B_0. \tag{5.205}$$

The final condition is Equation (5.35) which by Equation (5.199) reduces to

$$\left(2\frac{\partial \xi^2}{\partial y} - \frac{\mathrm{d}\xi^1}{\mathrm{d}x}\right)E + \left(2w_y\frac{\partial E}{\partial w_y} + 3w_{yy}\frac{\partial E}{\partial w_{yy}}\right)\frac{\partial \xi^2}{\partial y} - \frac{\partial E}{\partial x}\xi^1(x) = 0. \tag{5.206}$$

Substituting Equation (5.194) for E into Equation (5.206) gives

$$\begin{aligned}\left(2\frac{\partial\xi^2}{\partial y}-\frac{\mathrm{d}\xi^1}{\mathrm{d}x}\right)&\left[\beta+\alpha M^2(x)\left(w_y^2+N^2(x)w_{yy}^2\right)^{1/2}\right]\\ &+\left[3\alpha M^2(x)N^2(x)w_{yy}^2\left(w_y^2+N^2(x)w_{yy}^2\right)^{-1/2}\right.\\ &\left.+2\alpha M^2(x)w_y^2\left(w_y^2+N^2(x)w_{yy}^2\right)^{-1/2}\right]\frac{\partial\xi^2}{\partial y}\\ &-\left[\alpha M^2(x)N(x)\frac{\mathrm{d}N}{\mathrm{d}x}w_{yy}^2\left(w_y^2+N^2(x)w_{yy}^2\right)^{-1/2}\right.\\ &\left.+2\alpha M(x)\frac{\mathrm{d}M}{\mathrm{d}x}\left(w_y^2+N^2(x)w_{yy}^2\right)^{1/2}\right]\xi^1(x)=0.\end{aligned}\qquad(5.207)$$

Multiply Equation (5.207) by $\left(w_y^2+N^2(x)w_{yy}^2\right)^{1/2}$ to obtain

$$\begin{aligned}\left(2\frac{\partial\xi^2}{\partial y}-\frac{\mathrm{d}\xi^1}{\mathrm{d}x}\right)&\left[\beta\left(w_y^2+N^2(x)w_{yy}^2\right)^{1/2}+\alpha M^2(x)\left(w_y^2+N^2(x)w_{yy}^2\right)\right]\\ &-\left[\alpha M^2(x)N(x)\frac{\mathrm{d}N}{\mathrm{d}x}w_{yy}^2+2\alpha M(x)\frac{\mathrm{d}M}{\mathrm{d}x}\left(w_y^2+N^2(x)w_{yy}^2\right)\right]\xi^1(x)\\ &+\left[3\alpha M^2(x)N^2(x)w_{yy}^2+2\alpha M^2(x)w_y^2\right]\frac{\partial\xi^2}{\partial y}=0.\end{aligned}\qquad(5.208)$$

Equation (5.208) is satisfied for all values of the independent variables x, w_y and w_{yy}. Set $w_{yy}=0$. Equation (5.208) reduces to

$$\begin{aligned}\left(2\frac{\partial\xi^2}{\partial y}-\frac{\mathrm{d}\xi^1}{\mathrm{d}x}\right)\left[\beta w_y+\alpha M^2(x)w_y^2\right]+2\alpha M^2(x)\frac{\partial\xi^2}{\partial y}w_y^2&\\ -2\alpha M(x)\frac{\mathrm{d}M}{\mathrm{d}x}\xi^1(x)w_y^2&=0.\end{aligned}\qquad(5.209)$$

Separate Equation (5.209) by powers of w_y

$$w_y:\qquad \beta\left(2\frac{\partial\xi^2}{\partial y}-\frac{\mathrm{d}\xi^1}{\mathrm{d}x}\right)=0,\qquad(5.210)$$

$$w_y^2:\qquad \alpha M^2(x)\left(2\frac{\partial\xi^2}{\partial y}-\frac{\mathrm{d}\xi^1}{\mathrm{d}x}\right)+2\alpha M^2(x)\frac{\partial\xi^2}{\partial y}-2\alpha M(x)\frac{\mathrm{d}M}{\mathrm{d}x}\xi^1(x)=0.\qquad(5.211)$$

Since $\nu \neq 0 \implies \beta \neq 0$, from Equation (5.210),

$$\frac{\mathrm{d}\xi^1}{\mathrm{d}x} = 2\frac{\partial \xi^2}{\partial y} = 2A_0 \tag{5.212}$$

by Equation (5.205) and therefore

$$\xi^1(x) = 2A_0 x + C_0 \tag{5.213}$$

where C_0 is a constant. Equation (5.211) becomes

$$\frac{1}{M}\frac{\mathrm{d}M}{\mathrm{d}x} = \frac{1}{\xi^1(x)}\frac{\partial \xi^2}{\partial y} \tag{5.214}$$

and therefore

$$\frac{1}{M}\frac{\mathrm{d}M}{\mathrm{d}x} = \frac{A_0}{2A_0 x + C_0}. \tag{5.215}$$

Hence

$$M(x) = M_0\,(2A_0 x + C_0)^{1/2} \tag{5.216}$$

where M_0 is a constant. Equation (5.208) reduces to

$$3M^2(x)N^2(x)\frac{\partial \xi^2}{\partial y} - \left(2M(x)\frac{\mathrm{d}M}{\mathrm{d}x}N^2(x) + M^2(x)N(x)\frac{\mathrm{d}N}{\mathrm{d}x}\right)\xi^1(x) = 0. \tag{5.217}$$

Dividing by $M^2(x)N^2(x)$ Equation (5.217) gives

$$3\frac{\partial \xi^2}{\partial y} - \left(\frac{2}{M(x)}\frac{\mathrm{d}M}{\mathrm{d}x} + \frac{1}{N(x)}\frac{\mathrm{d}N}{\mathrm{d}x}\right)\xi^1(x) = 0, \tag{5.218}$$

and therefore using Equation (5.214)

$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}x} = \frac{1}{\xi^1(x)}\frac{\partial \xi^2}{\partial y}. \tag{5.219}$$

Hence

$$N(x) = N_0\,(2A_0 x + C_0)^{1/2} \tag{5.220}$$

where N_0 is a constant. From Equation (5.193)

$$\ell_{10}\ell_1(x) = M_0\,(2A_0x + C_0)^{1/2}, \quad \ell_{20}\ell_2(x) = N_0\,(2A_0x + C_0)^{1/2}. \tag{5.221}$$

The constants M_0 and N_0 can be incorporated into ℓ_{10} and ℓ_{20}.

It is important to note that ℓ_1 and ℓ_2 share the same functional form of x and differ only by a constant of proportionality. This implies that for invariance the two mixing lengths must be proportional.

The Lie point symmetry associated with the elementary conserved vector is given by

$$X = (2A_0x + C_0)\frac{\partial}{\partial x} + (A_0y + B_0)\frac{\partial}{\partial y} - A_0w\frac{\partial}{\partial w}. \tag{5.222}$$

In the same manner as in Section 5.2, the constants B_0 and C_0 can be shown to be zero. Consider the general case in which $A_0 \neq 0$. The Lie point symmetry reduces to

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - w\frac{\partial}{\partial w}. \tag{5.223}$$

Now, $w = W(x,y)$ is an invariant solution of the PDE given by Equation (5.1) where $E(x, w_y, w_{yy})$ is given by Equation (5.194), provided

$$X(w - W(x,y))|_{w=W} = 0, \tag{5.224}$$

that is, provided

$$2x\frac{\partial W}{\partial x} + y\frac{\partial W}{\partial y} = -W. \tag{5.225}$$

The differential equations of the characteristic curves of Equation (5.225) are

$$\frac{dx}{2x} = \frac{dy}{y} = \frac{dW}{-W}. \tag{5.226}$$

Two first integrals are

$$\frac{y}{\sqrt{x}} = C_1, \quad \sqrt{x}W = C_2 \tag{5.227}$$

and the invariant solution is of the form

$$w = \frac{F(\xi)}{\sqrt{x}}, \quad \xi = \frac{y}{\sqrt{x}}. \tag{5.228}$$

The mixing lengths are given by Equation (5.221). Substituting the invariant solution into the PDE

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial y}\left[\left(\beta + \alpha M^2(x)\left(\left(\frac{\partial w}{\partial y}\right)^2 + N^2(x)\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right)^{1/2}\right)\frac{\partial w}{\partial y}\right] \tag{5.229}$$

gives

$$\frac{d}{d\xi}\left[\left(\beta + 2\alpha M_0^2\left(\left(\frac{dF}{d\xi}\right)^2 + N_0^2\left(\frac{d^2F}{d\xi^2}\right)^2\right)^{1/2}\right)\frac{dF}{d\xi}\right] + \frac{1}{2}\frac{d}{d\xi}(\xi F) = 0. \tag{5.230}$$

Choose

$$\sqrt{2}M_0 = \ell_{10}, \quad \sqrt{2}N_0 = \ell_{20} \tag{5.231}$$

so that by Equation (5.221)

$$\ell_1(x) = \sqrt{x}, \quad \ell_2(x) = \sqrt{x}. \tag{5.232}$$

Then

$$\frac{d}{d\xi}\left[\left(\beta + 2\alpha\ell_{10}^2\left(\left(\frac{dF}{d\xi}\right)^2 + \ell_{20}^2\left(\frac{d^2F}{d\xi^2}\right)^2\right)^{1/2}\right)\frac{dF}{d\xi}\right] + \frac{1}{2}\frac{d}{d\xi}(\xi F) = 0. \tag{5.233}$$

The boundary condition on the centre line is again

$$\frac{\partial w}{\partial y}(x,0)=0, \quad \frac{\mathrm{d}F}{\mathrm{d}\xi}(0)=0. \tag{5.234}$$

Integrating Equation (5.233) once gives

$$\left(\beta+2\alpha\ell_{10}^2\left(\left(\frac{\mathrm{d}F}{\mathrm{d}\xi}\right)^2+\ell_{20}^2\left(\frac{\mathrm{d}^2F}{\mathrm{d}\xi^2}\right)^2\right)^{1/2}\right)\frac{\mathrm{d}F}{\mathrm{d}\xi}+\frac{1}{2}(\xi F)=0. \tag{5.235}$$

5.4.2 The case $\nu = 0$

In this section Prandtl's Extended mixing length model is applied to the wake where the kinematic viscosity is neglected. Setting $\beta = 0$, the effective viscosity in Equation (5.194) is given by

$$\alpha M^2(x)\left(w_y^2+N^2(x)w_{yy}^2\right)^{1/2} \tag{5.236}$$

Substituting the effective viscosity given by Equation (5.236) into Equation (5.34) yields

$$\alpha M^2(x)N^2(x)w_y w_{yy}\left(w_y^2+N^2(x)w_{yy}^2\right)^{-1/2}\frac{\partial^3\xi^2}{\partial y^3}-\frac{\partial\xi^2}{\partial x}+ \\ \left(w_y^2+N^2(x)w_{yy}^2\right)^{-1/2}\alpha M^2(x)\left(2w_y^2+N^2(x)w_{yy}^2\right)\frac{\partial^2\xi^2}{\partial y^2}=0. \tag{5.237}$$

Multiplying Equation (5.237) by $\left(w_y^2+N^2(x)w_{yy}^2\right)^{1/2}$ gives

$$\frac{\partial\xi^2}{\partial x}\left(w_y^2+N^2(x)w_{yy}^2\right)^{1/2}=\alpha M^2(x)\left[\left(2w_y^2+N^2(x)w_{yy}^2\right)\frac{\partial^2\xi^2}{\partial y^2}\right. \\ \left.+N^2(x)w_y w_{yy}\frac{\partial^3\xi^2}{\partial y^3}\right]. \tag{5.238}$$

Squaring both sides of Equation (5.238) yields

$$\begin{aligned}\left(\frac{\partial \xi^2}{\partial x}\right)^2 \left[\left(w_y^2 + N^2(x)\, w_{yy}^2\right)\right] = \alpha^2 M^4(x) \Bigg[& N^4(x)\, w_y^2 w_{yy}^2 \left(\frac{\partial^3 \xi^2}{\partial y^3}\right) \\ & + \left(4 w_y^4 + 4N^2(x)\, w_y^2 w_{yy}^2 + N^4(x)\, w_{yy}^4\right)\left(\frac{\partial^2 \xi^2}{\partial y^2}\right)^2 \\ & + 2N^2(x) \frac{\partial \xi^2}{\partial y}\frac{\partial^3 \xi^2}{\partial y^3}\left(2 w_y^3 w_{yy} + N^2(x)\, w_y w_{yy}^3\right)\Bigg].\end{aligned} \tag{5.239}$$

Equation (5.239) is separated by powers of w_{yy}

$$w_{yy}^4: \qquad \alpha^2 M^4(x) N^4(x)\left(\frac{\partial^2 \xi^2}{\partial y^2}\right) = 0. \tag{5.240}$$

Thus

$$\frac{\partial^2 \xi^2}{\partial y^2}(x, y) = 0 \tag{5.241}$$

and therefore

$$\xi^2(x, y) = yA(x) + B(x). \tag{5.242}$$

Using Equation (5.242), Equation (5.239) reduces to

$$\left(w_y^2 + N^2(x)\, w_{yy}^2\right)\left(\frac{\partial \xi^2}{\partial x}\right)^2 = 0. \tag{5.243}$$

Thus

$$\frac{\partial \xi^2}{\partial x}(x, y) = 0 \tag{5.244}$$

and therefore

$$y\frac{\mathrm{d}A}{\mathrm{d}x} + \frac{\mathrm{d}B}{\mathrm{d}x} = 0. \tag{5.245}$$

Hence separating Equation (5.245) by y,

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 0, \quad \frac{\mathrm{d}B}{\mathrm{d}x} = 0 \tag{5.246}$$

and therefore $A(x) = A_0$ and $B(x) = B_0$ where A_0 and B_0 are constants. Thus

$$\xi^2(y) = A_0 y + B_0. \tag{5.247}$$

The final condition is Equation (5.35) which by Equation (5.241) reduces to

$$\left(2\frac{\partial \xi^2}{\partial y} - \frac{\mathrm{d}\xi^1}{\mathrm{d}x}\right)E + \left(2w_y\frac{\partial E}{\partial w_y} + 3w_{yy}\frac{\partial E}{\partial w_{yy}}\right)\frac{\partial \xi^2}{\partial y} - \frac{\partial E}{\partial x}\xi^1(x) = 0. \tag{5.248}$$

Substituting Equation (5.236) for E into Equation (5.248) gives

$$\begin{aligned}
&\left(2\frac{\partial \xi^2}{\partial y} - \frac{\mathrm{d}\xi^1}{\mathrm{d}x}\right)\left[\alpha M^2(x)\left(w_y^2 + N^2(x)w_{yy}^2\right)^{1/2}\right] \\
&\qquad + \left[3\alpha M^2(x)N^2(x)w_{yy}^2\left(w_y^2 + N^2(x)w_{yy}^2\right)^{-1/2}\right. \\
&\qquad\qquad \left. + 2\alpha M^2(x)w_y^2\left(w_y^2 + N^2(x)w_{yy}^2\right)^{-1/2}\right]\frac{\partial \xi^2}{\partial y} \\
&\qquad - \left[\alpha M^2(x)N(x)\frac{\mathrm{d}N}{\mathrm{d}x}w_{yy}^2\left(w_y^2 + N^2(x)w_{yy}^2\right)^{-1/2}\right. \\
&\qquad\qquad \left. + 2\alpha M(x)\frac{\mathrm{d}M}{\mathrm{d}x}\left(w_y^2 + N^2(x)w_{yy}^2\right)^{1/2}\right]\xi^1(x) = 0.
\end{aligned} \tag{5.249}$$

Multiply Equation (5.249) by $\left(w_y^2 + N^2(x)w_{yy}^2\right)^{1/2}$ to obtain

$$\begin{aligned}
&\left(2\frac{\partial \xi^2}{\partial y} - \frac{\mathrm{d}\xi^1}{\mathrm{d}x}\right)\alpha M^2(x)\left(w_y^2 + N^2(x)w_{yy}^2\right) \\
&\qquad - \left[\alpha M^2(x)N(x)\frac{\mathrm{d}N}{\mathrm{d}x}w_{yy}^2 + 2\alpha M(x)\frac{\mathrm{d}M}{\mathrm{d}x}\left(w_y^2 + N^2(x)w_{yy}^2\right)\right]\xi^1(x) \\
&\qquad\qquad + \left[3\alpha M^2(x)N^2(x)w_{yy}^2 + 2\alpha M^2(x)w_y^2\right]\frac{\partial \xi^2}{\partial y} = 0.
\end{aligned} \tag{5.250}$$

Separating Equation (5.250) by w_{yy}

$$w_{yy}^0: \qquad 4M^2(x)\frac{\partial \xi^2}{\partial y} - M^2(x)\frac{\mathrm{d}\xi^1}{\mathrm{d}x} - 2M(x)\frac{\mathrm{d}M}{\mathrm{d}x}\xi^1(x) = 0. \tag{5.251}$$

$$w_{yy}^2: \quad 5\frac{\partial \xi^2}{\partial y}N^2(x)M^2(x) - M^2(x)N^2(x)\frac{\mathrm{d}\xi^1}{\mathrm{d}x} - \xi^1 M^2(x)N(x)\frac{\mathrm{d}N}{\mathrm{d}x}$$
$$-2M(x)\frac{\mathrm{d}M}{\mathrm{d}x}N^2(x)\xi^1(x) = 0. \tag{5.252}$$

Using Equation (5.247), Equation (5.251) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\xi^1(x)M^2(x)\right) = 4A_0M^2(x). \tag{5.253}$$

Hence

$$\xi^1(x) = \frac{1}{M^2(x)}4A_0\int^x M^2(\alpha)d\alpha + \frac{C_1}{M^2(x)}, \tag{5.254}$$

where C_1 is a constant. Using Equation (5.251), Equation (5.252) reduces to

$$A_0 = \xi^1(x)\frac{N'(x)}{N(x)}. \tag{5.255}$$

Using Equation (5.255), Equation (5.252) becomes

$$\frac{\mathrm{d}\xi^1}{\mathrm{d}x} + \xi^1(x)\left[2\frac{1}{M(x)}\frac{\mathrm{d}M}{\mathrm{d}x} - 4\frac{1}{N(x)}\frac{\mathrm{d}N}{\mathrm{d}x}\right] = 0. \tag{5.256}$$

Equation (5.256) is solvable by separation of variables to give

$$\xi^1(x) = C_2\frac{N^4(x)}{M^2(x)}, \tag{5.257}$$

where C_2 is a constant. Equating Equation (5.257) and Equation (5.254) results in

$$N^4(x) = \frac{1}{C_2}4A_0\int^x M^2(\alpha)d\alpha + \frac{C_1}{C_2}. \tag{5.258}$$

Letting $M(0) = 0$ in Equation (5.258) requires that $C_1 = 0$. Therefore Equation (5.254) reduces to

$$\xi^1(x) = \frac{1}{M^2(x)}4A_0\int^x M^2(\alpha)d\alpha, \tag{5.259}$$

and

$$N^4(x) = \frac{1}{C_2}4A_0\int^x M^2(\alpha)d\alpha. \tag{5.260}$$

Using $\xi^1(x)$ defined by Equation (5.259), the Lie point symmetry associated with the elementary conserved vector is given by

$$X = \left(\frac{1}{M^2(x)} 4A_0 \int^x M^2(\alpha)\,d\alpha\right)\frac{\partial}{\partial x} + (A_0 y + B_0)\frac{\partial}{\partial y} - A_0 w \frac{\partial}{\partial w}. \tag{5.261}$$

In the same manner as in Section 5.2, the constant B_0 can be shown to be zero. Consider the general case in which $A_0 \neq 0$. The Lie point symmetry reduces to

$$X = \left(\frac{1}{M^2(x)} 4 \int^x M^2(\alpha)\,d\alpha\right)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - w\frac{\partial}{\partial w}. \tag{5.262}$$

Keep $\xi^1(x)$ general so that Equation (5.262) is

$$X = \xi^1(x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - w\frac{\partial}{\partial w}. \tag{5.263}$$

Now, $w = W(x,y)$ is an invariant solution of the PDE given by Equation (5.1) where $E(x, w_y, w_{yy})$ is given by Equation (5.236), provided

$$X(w - W(x,y))|_{w=W} = 0, \tag{5.264}$$

that is, provided

$$\xi^1(x)\frac{\partial W}{\partial x} + y\frac{\partial W}{\partial y} = -W. \tag{5.265}$$

The differential equations of the characteristic curves of Equation (5.265) are

$$\frac{\mathrm{d}x}{\xi^1(x)} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}W}{-W}, \tag{5.266}$$

Two first integrals are

$$y = C_1 H(x), \quad H(x)W = C_2 \tag{5.267}$$

where

$$H(x) = \exp\left[\int^x \frac{dx}{\xi^1(x)}\right] \tag{5.268}$$

and the invariant solution is of the form

$$w = \frac{F(\xi)}{H(x)}, \quad \xi = \frac{y}{H(x)}. \tag{5.269}$$

Consider now the conserved quantity

$$\int_0^{y_b} w \mathrm{d}y = D/2. \tag{5.270}$$

The boundary, y_b, in terms of the variables defined in Equation (5.269) becomes

$$y_b = \xi_b H(x). \tag{5.271}$$

Using Prandtl's hypothesis, there are three cases to consider for the mixing lengths.

Case 1: $\ell_1 \propto y_b$

In this case, it must be that

$$\ell_1 = K\xi_b H(x), \tag{5.272}$$

where K is a constant of proportionality. Recall that $M^2(x) = \ell_{10}^2 \ell_1^2(x)$ and so Equation (5.272) in terms of $M(x)$ is

$$M(x) = K\ell_{10}\xi_b H(x). \tag{5.273}$$

Differentiating Equation (5.273) with respect to x gives

$$\frac{\mathrm{d}M(x)}{\mathrm{d}x} = \frac{K\ell_{10}\xi_b}{\xi^1} H(x) = \frac{M(x)}{\xi^1}. \tag{5.274}$$

That is

$$\xi^1 = \frac{M(x)}{M'(x)}. \tag{5.275}$$

But, from Equation (5.251) and substituting Equation (5.275) gives

$$\xi^1 = 2x + c, \tag{5.276}$$

where c can be shown to be zero. It is important to note that Equation (5.276) is the same result as for $v \neq 0$. Therefore, ℓ_1 and ℓ_2 will also have the same form as for the case when $v \neq 0$.

Case 2: $\ell_2 \propto y_b$

In this case, it must be that

$$\ell_2 = K\xi_b H(x), \tag{5.277}$$

where K is a constant of proportionality. Recall that $N^2(x) = \ell_{20}^2 \ell_2^2(x)$ and so Equation (5.277) in terms of $N(x)$ is

$$N(x) = K\ell_{20}\xi_b H(x). \tag{5.278}$$

Differentiating Equation (5.278) with respect to x gives

$$\frac{\mathrm{d}N(x)}{\mathrm{d}x} = \frac{K\ell_{20}\xi_b}{\xi^1} H(x) = \frac{N(x)}{\xi^1}. \tag{5.279}$$

However, using Equation (5.279), Equation (5.252) reduces to Equation (5.251) which is a repeat condition. This repeat condition leaves ξ^1 undetermined and thus the Lie point symmetry cannot produce an invariant solution and therefore it is not possible to continue.

Case 3: $\ell_1 \propto \ell_2$

In this case, it must be that

$$\ell_1 = K\ell_2, \tag{5.280}$$

where K is a constant of proportionality. This case is not Prandtl's hypothesis but rather verifies Prandtl's hypothesis for a special case. In terms of $M(x)$ and $N(x)$ Equation (5.281) becomes

$$M(x) = \kappa N(x), \tag{5.281}$$

where

$$\kappa = K\frac{\ell_{20}}{\ell_{10}}. \tag{5.282}$$

Substituting Equation (5.281) into Equation (5.251) results in the equation

$$\kappa N(x)\left(\frac{\partial \xi^1}{\partial x} - 4\right) + 2\xi^1\kappa\frac{\mathrm{d}N}{\mathrm{d}x} = 0. \tag{5.283}$$

That is

$$2\xi^1\frac{\mathrm{d}N}{\mathrm{d}x} = -N(x)\left(\frac{\partial \xi^1}{\partial x} - 4\right). \tag{5.284}$$

Substituting Equation (5.284) into Equation (5.255) gives

$$\xi^1 = 2x + c, \tag{5.285}$$

where c can be shown to be zero. It is important to note that Equation (5.285) is again the same result as for $\nu \neq 0$. It is also the same result as *Case* 1 where $\ell_1 \propto y_b$. Therefore, again, ℓ_1 and ℓ_2 will also have the same form as for the case when $\nu \neq 0$.

Proceeding using either *Case* 1 or *Case* 3, Equation (5.269) is now given by

$$\xi = \frac{y}{\sqrt{x}} \qquad \text{and} \qquad w = \frac{F(\xi)}{\sqrt{x}}. \tag{5.286}$$

Substituting Equations (5.221), (5.231), (5.232) and (5.286) into Equation (5.1) results in the nonlinear ODE

$$\ell_{10}^2 \frac{\nu_{T0}}{\nu_C + \nu_{TC}} F' \left(F'^2 + \ell_{20}^2 F''^2 \right)^{1/2} + \frac{1}{2}\xi F = 0. \tag{5.287}$$

5.5 Von Kármán Model

In this section the mixing length given by Equation (3.79) is substituted into Prandtl's mixing length model. The effective viscosity is given by $E = \nu + \nu_T$ where the turbulent viscosity in terms of the dimensionless variables is given by

$$\nu_T = \frac{\nu_{T0}}{\nu_C + \nu_{TC}} \left(\frac{\dfrac{\partial \overline{w}^*}{\partial y^*}}{\dfrac{\partial^2 \overline{w}^*}{\partial y^{*2}}} \right)^2 \left| \frac{\partial \overline{w}^*}{\partial y^*} \right|, \tag{5.288}$$

where ν_{T0} is defined as

$$\nu_{T0} = \left(\frac{c_2}{c_1} \right)^2 U\delta. \tag{5.289}$$

5.5.1 The case $\nu \neq 0$

Substituting the effective viscosity into Equation (5.34) yields

$$\frac{\partial A}{\partial y} \left[\frac{\nu}{\nu_C + \nu_{TC}} - 3\frac{\nu_{T0}}{\nu_C + \nu_{TC}} w_y^3 w_{yy}^{-2} \right] + \frac{\nu_{T0}}{\nu_C + \nu_{TC}} 2 w_y^4 w_{yy}^{-3} \frac{\partial^2 A}{\partial y^2} + \frac{\partial \xi^2}{\partial x} = 0. \tag{5.290}$$

Separating Equation (5.290) by powers of w_{yy} yields

$$w_{yy}^{-3}: \qquad \frac{\partial^2 A}{\partial y^2} w_y^4 = 0. \tag{5.291}$$

$$w_{yy}^{-2}: \qquad \frac{\partial A}{\partial y} w_y^3 = 0. \tag{5.292}$$

$$w_{yy}^{0}: \qquad \frac{\nu}{\nu_C + \nu_{TC}} \frac{\partial A}{\partial y} + \frac{\partial \xi^2}{\partial x} = 0. \tag{5.293}$$

Equation (5.292) identically satisfies Equation (5.291) and gives that

$$A = A(x). \tag{5.294}$$

Equation (5.294) together with Equation (5.32) results in

$$\xi^2 = \xi^2(y). \tag{5.295}$$

Using Equation (5.33) it can be seen that A is a constant, say, $-a$ and that

$$\xi^2 = ay + c, \tag{5.296}$$

where c is a constant.

Substituting the effective viscosity into Equation (5.35) yields

$$2A\frac{\nu}{\nu_C + \nu_{TC}} + \frac{\nu}{\nu_C + \nu_{TC}}\frac{\partial \xi^1}{\partial x} - 2Aw_y^3 \frac{\nu_{T0}}{\nu_C + \nu_{TC}} w_{yy}^{-2} - w_y^3 \frac{\nu_{T0}}{\nu_C + \nu_{TC}} w_{yy}^{-2} \frac{\partial \xi^1}{\partial x} = 0. \tag{5.297}$$

Separating Equation (5.297) by powers of w_{yy} gives

$$w_{yy}^{-2}: \qquad 2A + \frac{\partial \xi^1}{\partial x} = 0. \tag{5.298}$$

$$w_{yy}^{0}: \qquad 2A + \frac{\partial \xi^1}{\partial x} = 0. \tag{5.299}$$

Using either condition above gives

$$\xi^1 = 2ax + b, \tag{5.300}$$

where b is a constant.

Equation (5.33) is now also given by

$$\eta = -aw. \tag{5.301}$$

In the same manner as in Section 5.2, the constants b and c can be shown to be zero and the Lie point symmetry reduces to

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - w\frac{\partial}{\partial w}. \tag{5.302}$$

The similarity variables are defined by Equations (5.123) and (5.124), and the boundary conditions by Equations (5.126) and (5.127). Equation (4.29) is now given by

$$\xi F + \frac{\nu}{\nu_C + \nu_{TC}} F' - \frac{\nu_{T0}}{\nu_C + \nu_{TC}} F'^4 F''^{-2} = 0. \tag{5.303}$$

5.5.2 The case $\nu = 0$

Substituting the effective viscosity into Equation (5.34) yields

$$-4\frac{\partial A}{\partial y}\nu_{T0}w_y^3 w_{yy}^{-2} + 2\nu_{T0}w_y^4 w_{yy}^{-3}\frac{\partial^2 A}{\partial y^2} + \frac{\partial \xi^2}{\partial x} = 0. \tag{5.304}$$

Separating Equation (5.304) by powers of w_{yy} yields

$$w_{yy}^{-3}: \qquad \frac{\partial^2 A}{\partial y^2}w_y^4 = 0. \tag{5.305}$$

$$w_{yy}^{-2}: \qquad \frac{\partial A}{\partial y}w_y^3 = 0. \tag{5.306}$$

$$w_{yy}^{0}: \qquad \frac{\partial \xi^2}{\partial x} = 0. \tag{5.307}$$

Equation (5.306) identically satisfies Equation (5.305) and gives that

$$A = A(x). \tag{5.308}$$

Equation (5.307) gives

$$\xi^2 = \xi^2(y), \tag{5.309}$$

and together with Equation (5.33) it can be seen that A is a constant, say, $-a$ and that

$$\xi^2 = ay + c, \tag{5.310}$$

where c is a constant.

Substituting the effective viscosity into Equation (5.35) yields

$$2a - \frac{\partial \xi^1}{\partial x} = 0 \tag{5.311}$$

which gives

$$\xi^1 = 2ax + b, \tag{5.312}$$

where b is a constant.

Equation (5.33) is now also given by

$$\eta = -aw. \tag{5.313}$$

In the same manner as in Section 5.2, the constants b and c can be shown to be zero and the Lie point symmetry reduces to

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - w\frac{\partial}{\partial w}. \tag{5.314}$$

The similarity variables are defined by Equations (5.123) and (5.124), and the boundary conditions by Equations (5.126) and (5.127). Equation (4.29) is now given by

$$\xi F - \frac{\nu}{\nu_C + \nu_{TC}} F'^4 F''^{-2} = 0. \tag{5.315}$$

Chapter 6

Comparison of solutions for the classical wake with experimental results

In this chapter the solutions to the equations derived in Chapter 5 are presented and analysed. The velocity profiles obtained from each model are compared. This is achieved by assessing how well they perform against experimental observations. Any ranges that are visually approximated are indicated by the approximate inequality $\lessapprox$.

6.1 Experimental results

Wygnanski et al. measured the mean velocity field in the self-preserving region of turbulent classical wakes generated in a wind tunnel [56]. The mainstream speed of the experiments varied between $7m/s$ and $20m/s$, far below subsonic levels where the compressibility of the air can be neglected. The Reynolds number of the experiment varied between 650 to 3200. The experimental data used to compare the results in this dissertation were produced by a long circular cylinder [56, pp 42]. A similarity profile is achieved sufficiently far downstream and the similarity variables are defined such that the normalised velocity profile, $F(\xi)$, is invariant for all wake generators. Although the experiment is conducted in a 3- dimensional setup and actual flows are turbulent in all three dimensions, it is assumed that the mean flow and fluctuations in the third dimension are negligible when considering the averaged quantities. The flow is therefore approximately 2- dimensional. If all three components of the flow are to be studied, then the 3- dimensional Navier-Stokes equations need to be considered. The mean velocity field was obtained experimentally and a best fit curve to the data was found to be

$$F(\xi) = \exp\left[-0.637\xi^2 - 0.056\xi^4\right]. \tag{6.1}$$

Note that this function also satisfies $F(\pm\infty) = 0$. In terms of the normalised velocity profile, the centre line velocity of the wake is

$$F(0) = 1. \tag{6.2}$$

The similarity variable ξ is defined such that

$$F(1) = \frac{1}{2}. \tag{6.3}$$

Various methods can be used to fit the models to the data. The method used in this dissertation ensures that three points are matched. These points, $(\xi, F(\xi))$, are $(0,1)$, $(1,1/2)$, and $(\infty,0)$.

6.2 Solutions to models

The equations derived in Chapter 5 are solved subject to the restrictions given in Equations (6.2) and (6.3).

6.2.1 Constant eddy viscosity

The similarity velocity profile corresponding to the constant eddy viscosity model is given by Equation (5.133)

$$F(\xi) = \frac{D}{\sqrt{2\pi\mathscr{E}}} \exp\left[\frac{-\xi^2}{2\mathscr{E}}\right]. \tag{6.4}$$

The normalised velocity profile that satisfies Equation (6.2) is

$$F_N(\xi) = \exp\left[\frac{-\xi^2}{2\mathscr{E}}\right]. \tag{6.5}$$

Equation (6.3) gives

$$F(1) = \exp\left[-\frac{1}{2\mathscr{E}}\right] = \frac{1}{2}, \tag{6.6}$$

resulting in $\frac{1}{2\mathscr{E}} = \ln(2)$. Finally the normalised velocity when using the constant eddy viscosity closure model is given by

$$F_N(\xi) = \exp\left[-\ln(2)\xi^2\right]. \tag{6.7}$$

As seen in Figure 6.1, an infinite wake is predicted. This is due to the nature of the solution which is an exponential function. The model's velocity profile is in close agreement with the experimental observations for $0 \leq \xi \lessapprox 1.5$. However, for $1.5 \lessapprox \xi < 2.9$ the model has far over estimated the velocity profile. Far from the centre line as $\xi \to \infty$ the profile tends toward the experimental observation as seen from the condition $F_N \to 0$ as $\xi \to \infty$.

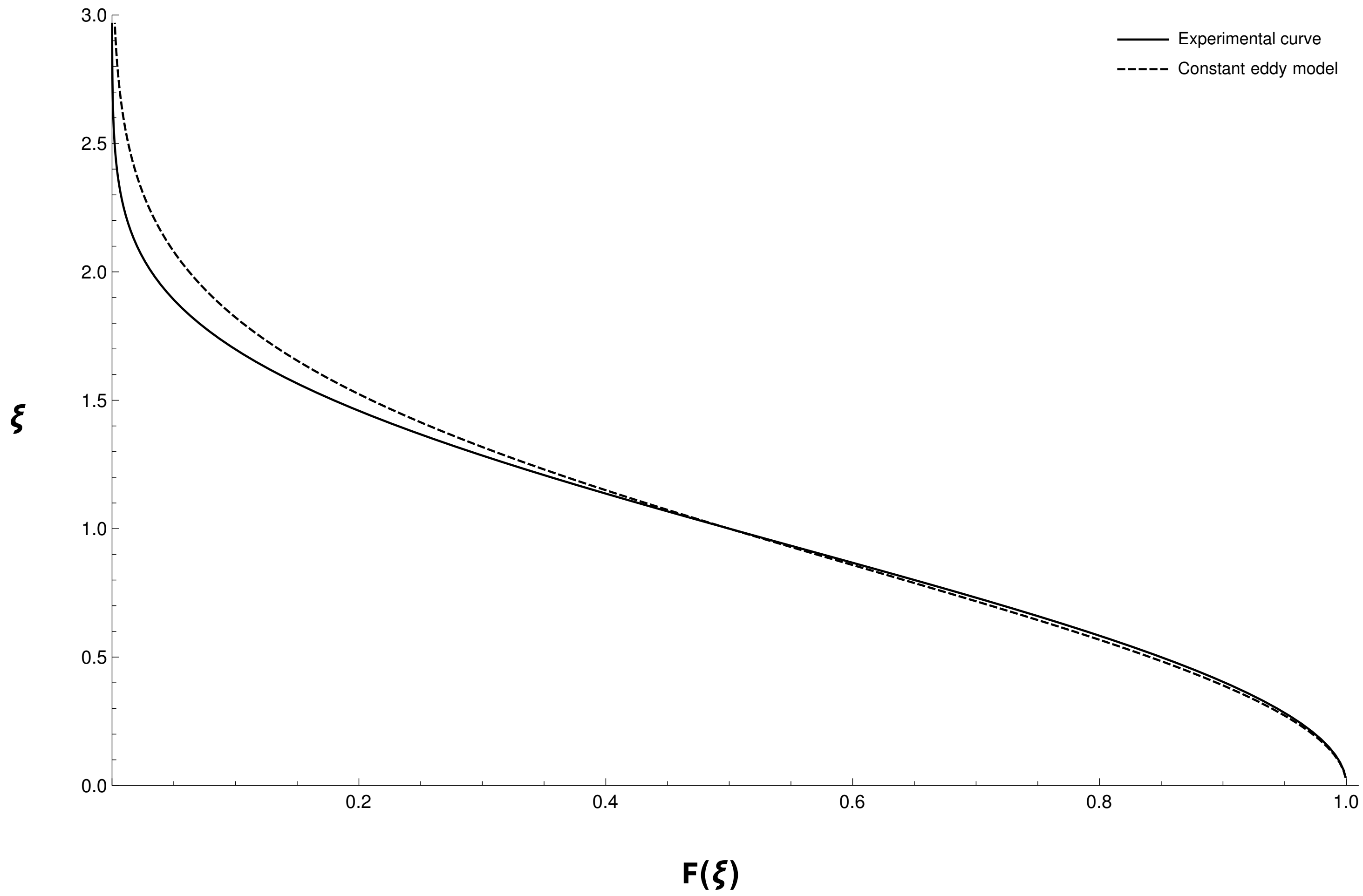


FIGURE 6.1: Constant eddy viscosity similarity velocity profile plotted on the same set of axes as the experimental curve profile.

6.2.2 Prandtl $\nu = 0$

The solution for Prandtl's mixing length model where the kinematic viscosity has been neglected is given by Equation (5.182) as

$$\frac{\mathrm{d}F}{\mathrm{d}\bar{\xi}} = -\left(\bar{\xi}F\right)^{1/2}. \tag{6.8}$$

Equation (6.8) can be solved by separation of variables to yield

$$F(\xi) = \frac{1}{36}\left(4\xi^3 - 12C_1\xi^{3/2} + 9C_1^2\right), \tag{6.9}$$

where C_1 is the constant of integration. In order to compare this solution to the experimental results, the similarity variable ξ has to be scaled to that used in the experiment. Define the scaled similarity variable $\bar{\xi}$ by

$$\bar{\xi} = \frac{\xi}{K}, \tag{6.10}$$

where K is some constant. Equation (6.9) is rewritten in terms of the scaled similarity variable

$$F(\xi) = \frac{1}{36}\left(4K^3\xi^3 - 12C_1K^{3/2}\xi^{3/2} + 9C_1^2\right). \tag{6.11}$$

where the bars have been omitted to keep the notation simple. Using the first condition given by Equation (6.2) requires that

$$F(0) = 1 = \frac{1}{4}C_1^2, \tag{6.12}$$

and thus $C_1 = \pm 2$.

Using the second condition given by Equation (6.3) results in an equation for K given by

$$K^3 \mp 6K^{3/2} + \frac{9}{2} = 0. \tag{6.13}$$

The solutions to Equation (6.13) are complex. The two alternative quadratics for $K^{3/2}$ have four real roots. Taking the $2/3$ power of these solutions for $K^{3/2}$ gives the two real roots

$$K = \left(3 - \frac{3}{\sqrt{2}}\right)^{2/3} \quad \text{and} \quad K = \left(3 + \frac{3}{\sqrt{2}}\right)^{2/3}. \tag{6.14}$$

As a finite wake is expected [26], it is required that

$$F(\xi_b) = \frac{1}{36}\left(4K^3\xi_b^3 - 24K^{3/2}\xi_b^{3/2} + 36\right) = 0, \tag{6.15}$$

where ξ_b is the boundary value.

Substituting $K = \left(3 - \frac{3}{\sqrt{2}}\right)^{2/3}$ from Equation (6.14) into Equation (6.15) and solving for ξ_b results in

$$\xi_b = \left(2\left(3 + 2\sqrt{2}\right)\right)^{(1/3)} \approx 2.2674. \tag{6.16}$$

The second value in Equation (6.14), $K = \left(3 + \frac{3}{\sqrt{2}}\right)^{2/3}$, is substituted into Equation (6.15) and solving for ξ_b results in

$$\xi_b = \left(2\left(3 - 2\sqrt{2}\right)\right)^{(1/3)} \approx 0.7001. \tag{6.17}$$

Recall that for $\xi = 1$, Equation (6.3) requires $F(1) = \frac{1}{2}$. Thus the boundary value found in Equation (6.17) predicts a finite wake that occurs before the point $\xi = 1$ and so the value of K that allows $\xi_b \approx 2.2674$ as calculated in Equation (6.16) is used. The final solution to Equation (6.11) subject to Equations (6.2) and (6.3) is the similarity velocity profile given by

$$F(\xi) = \frac{1}{36}\left(4\left(6-4\sqrt{2}\right)\xi^3 - 24\left(2-\sqrt{2}\right)\xi^{3/2} + 36\right). \tag{6.18}$$

As seen in Figure 6.2, a finite wake with a boundary value occurring at $\xi_b \approx 2.2674$ is plotted. The model under estimates the velocity profile for $0 \leq \xi \leq 1$ and over predicts the velocity profile for $1 \leq \xi \lessapprox 2$. It again under predicts the velocity profile for $2 \lessapprox \xi \leq \xi_b$.

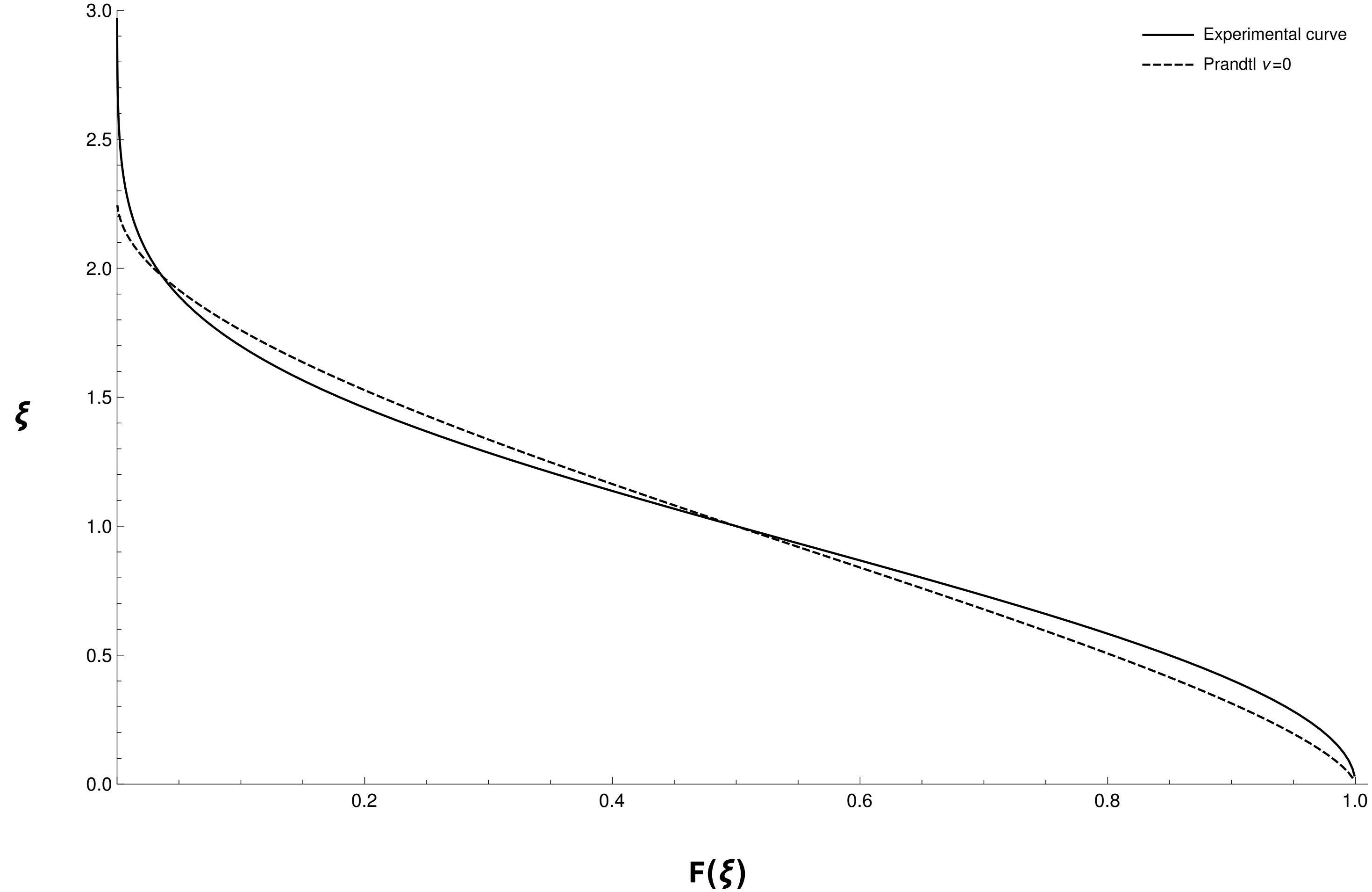


FIGURE 6.2: Prandtl's mixing length model when $\nu = 0$. The similarity velocity profile is plotted on the same set of axes as the experimental curve profile.

6.2.3 Prandtl $\nu \neq 0$

When the kinematic viscosity is not neglected, the solution given by Equation (5.147)

$$\frac{\mathrm{d}F}{\mathrm{d}\xi} = \frac{1-\sqrt{1+4\xi F}}{2}, \tag{6.19}$$

needs to be solved numerically subject to the conditions given by Equations (6.2) and (6.3). This is considered an initial value problem where the initial condition for $\xi = 0$ is given by Equation (6.2). A fourth-order Runge-Kutte method is chosen and implemented in Matlab. Again, a scaled similarity variable $\bar{\xi}$ is defined according to Equation (6.10) and Equation (6.19) transforms to

$$\frac{\mathrm{d}F}{\mathrm{d}\xi} = \frac{K-\sqrt{K^2+4K^3\xi F}}{2}, \tag{6.20}$$

where again the bars are omitted to keep the notation simple. The constant K is varied using a for loop until the condition given by Equation (6.3) is met. This ensures that an appropriate scaling of ξ matches those defined in the experimental results.

As seen in Figure 6.3, an infinite wake is predicted when the kinematic viscosity is not neglected. The model under predicts the velocity profile for $0 \leq \xi \leq 1$ and over predicts the velocity profile for $1 \leq \xi <\lessapprox 2.5$. Far from the centre line as $\xi \to \infty$ the profile tends toward the experimental observation as seen from the condition $F_N \to 0$ as $\xi \to \infty$.

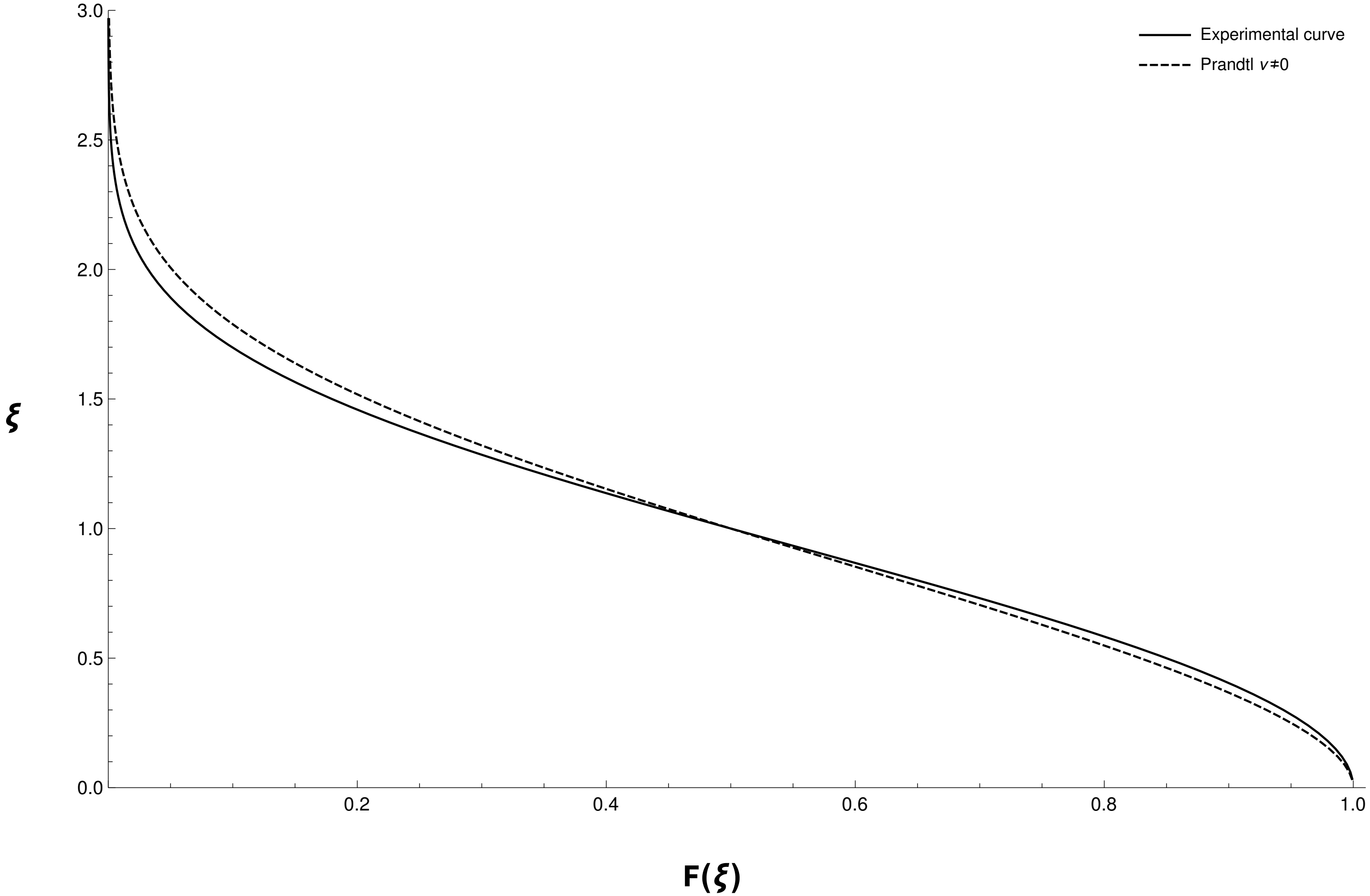


FIGURE 6.3: Prandtl's mixing length model when $\nu \neq 0$. The similarity velocity profile is plotted on the same set of axes as the experimental curve profile.

6.3 Conclusions

As seen in previous sections, all models tend to under predict the velocity profile from the centre line to $\xi = 1$ and over predict the velocity profile for $1 \leq \xi < \infty$. The goal of these comparisons is to determine where some models better fit the experimental curve profile. In order to make these comparisons, all velocity profiles are drawn on the same set of axes and displayed over four regions as seen in Figures 6.4 to 6.7.

In Figure 6.4 for $0 \leq \xi \lessapprox 1.4$, the experimental curve is best approximated by the constant eddy viscosity model. In Figure 6.5 for $1.3 \lessapprox \xi \lessapprox 1.65$ the constant eddy viscosity begins to over estimate the velocity profile of the experimental curve. Prandtl's mixing length model when the kinematic viscosity is not neglected is the best approximation for this region. In Figure 6.6 for $1.58 \lessapprox \xi \lessapprox 2.25$ the constant eddy viscosity model far over approximates the velocity profile of the experimental curve. Prandtl's mixing length model where the kinematic viscosity is neglected begins to best approximate the experimental curve. In Figure 6.7 for $1.95 \lessapprox \xi \lessapprox 3$, Prandtl's mixing length model where the kinematic viscosity is neglected results in a finite wake profile and the model when the kinematic viscosity is not neglected becomes the best approximation for the velocity profile as $\xi \to \infty$.

The constant eddy viscosity model is a good approximator for the velocity profile of a classical wake near the centre line. Since the constant eddy viscosity is not dependent on the kinematic viscosity, the kinematic viscosity does not play a large role for higher velocity scales which corresponds to $0.3 \lessapprox F(\xi) \leq 1$ in the normalised scale. Since the kinematic viscosity is responsible for the transfer of momentum, this becomes more important towards the edge of the wake in the lower velocity scales and hence Prandtl's mixing length model when the kinematic viscosity is not neglected is a better predictor in this region which corresponds to $0 \leq F(\xi) \lessapprox 0.3$ in the normalised scale. It is unclear why exactly Prandtl's mixing length model when the kinematic viscosity is neglected is a good predictor in the velocity scales for $0.005 \lessapprox F(\xi) \lessapprox 0.155$. The shear stresses need to be included in the analysis for further insight into why this was the case.

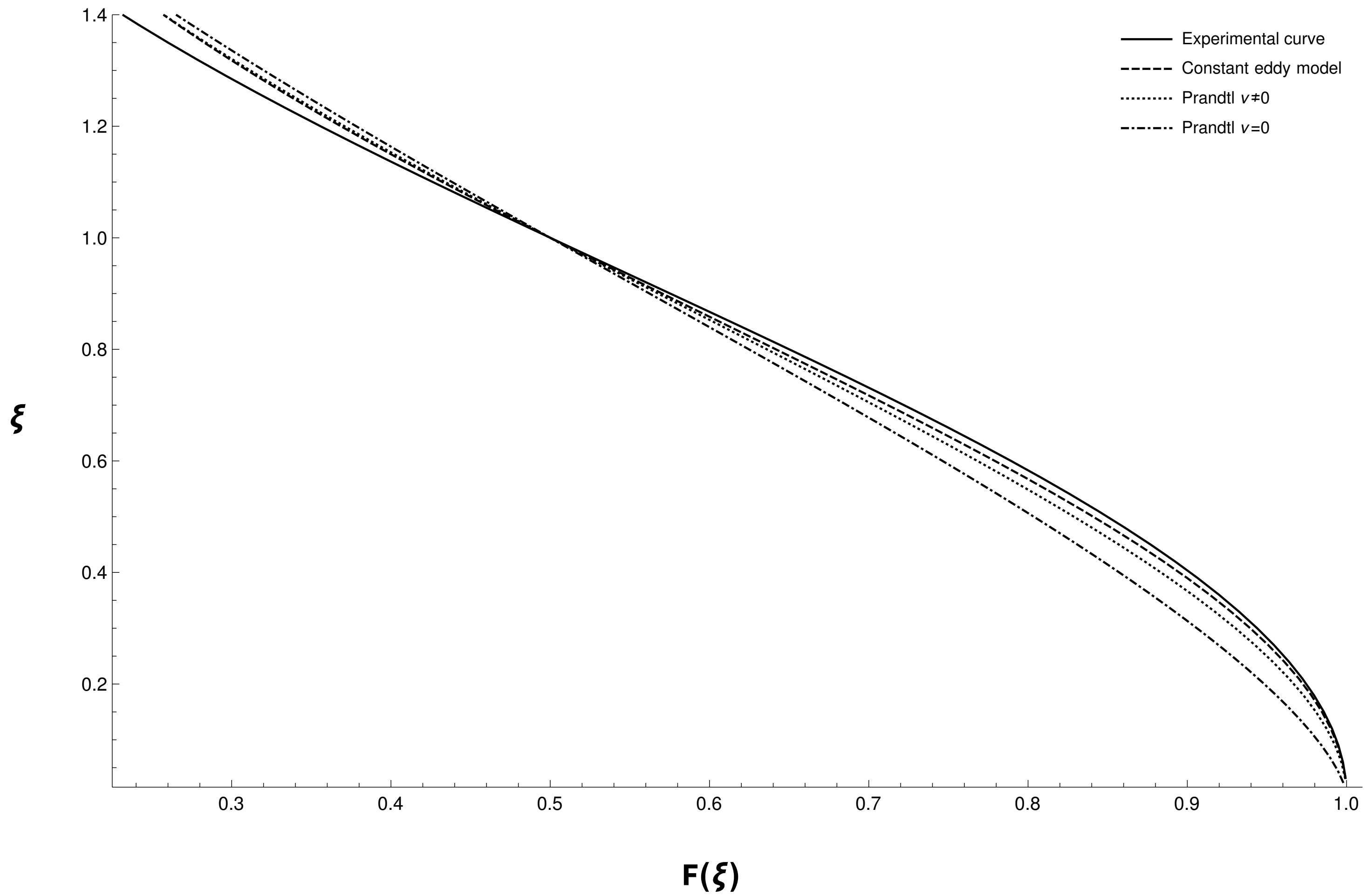


FIGURE 6.4: All models under consideration plotted on the same set of axes as the experimental curve for $0 \leq \xi \lessapprox 1.4$

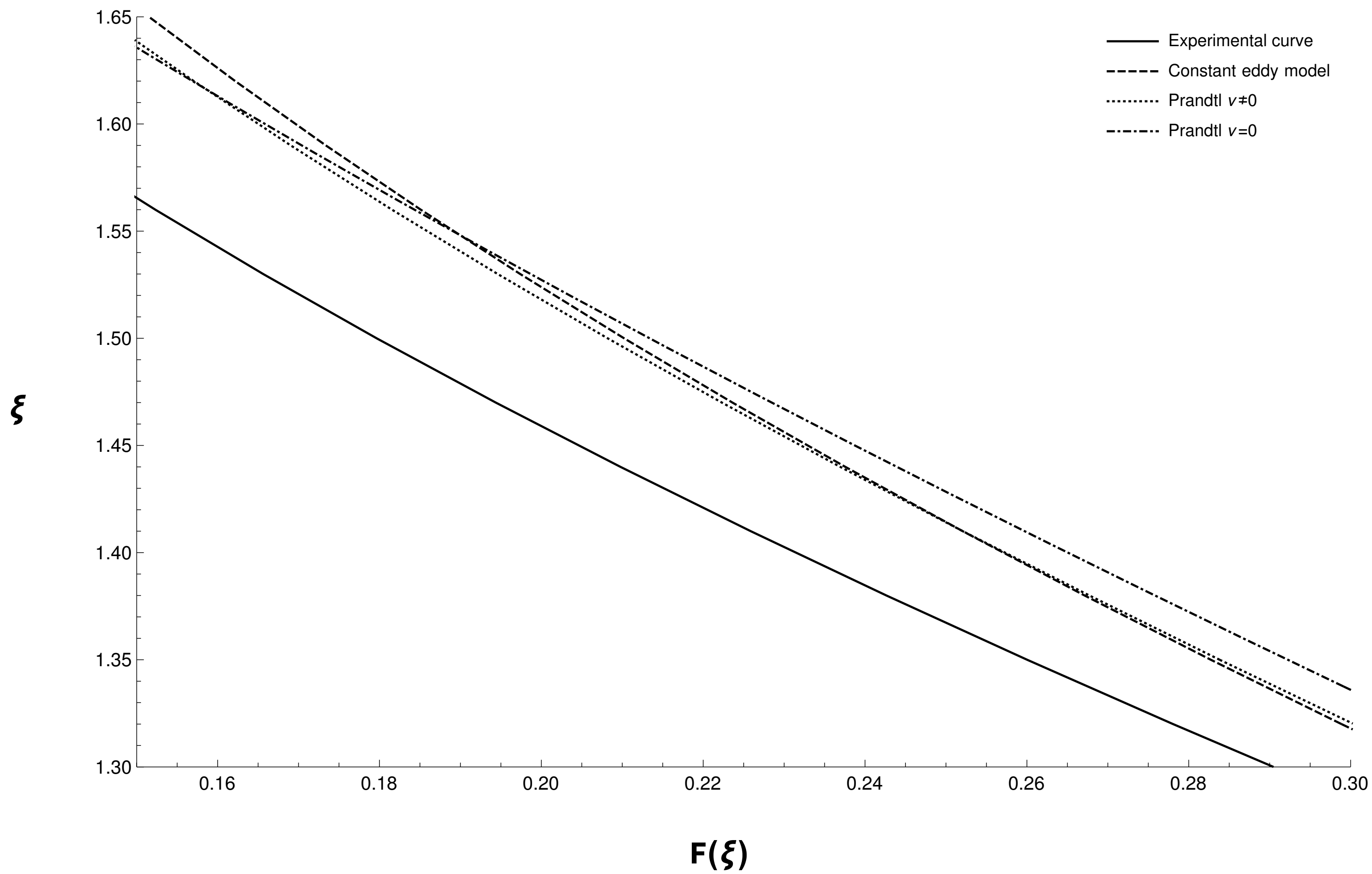


FIGURE 6.5: All models under consideration plotted on the same set of axes as the experimental curve for $1.3 \lessapprox \xi \lessapprox 1.65$

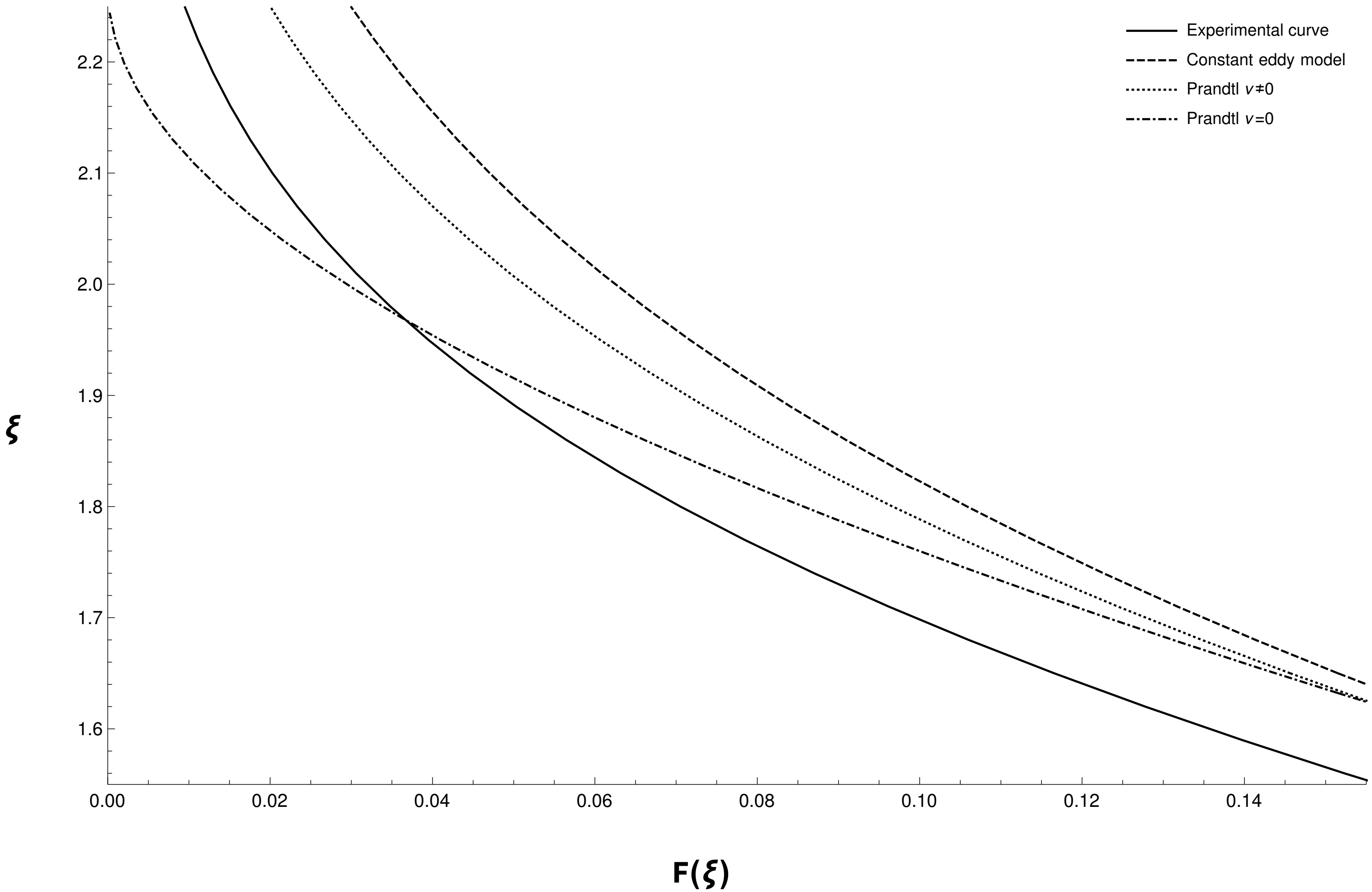


FIGURE 6.6: All models under consideration plotted on the same set of axes as the experimental curve for $1.58 \lessapprox \xi \lessapprox 2.25$

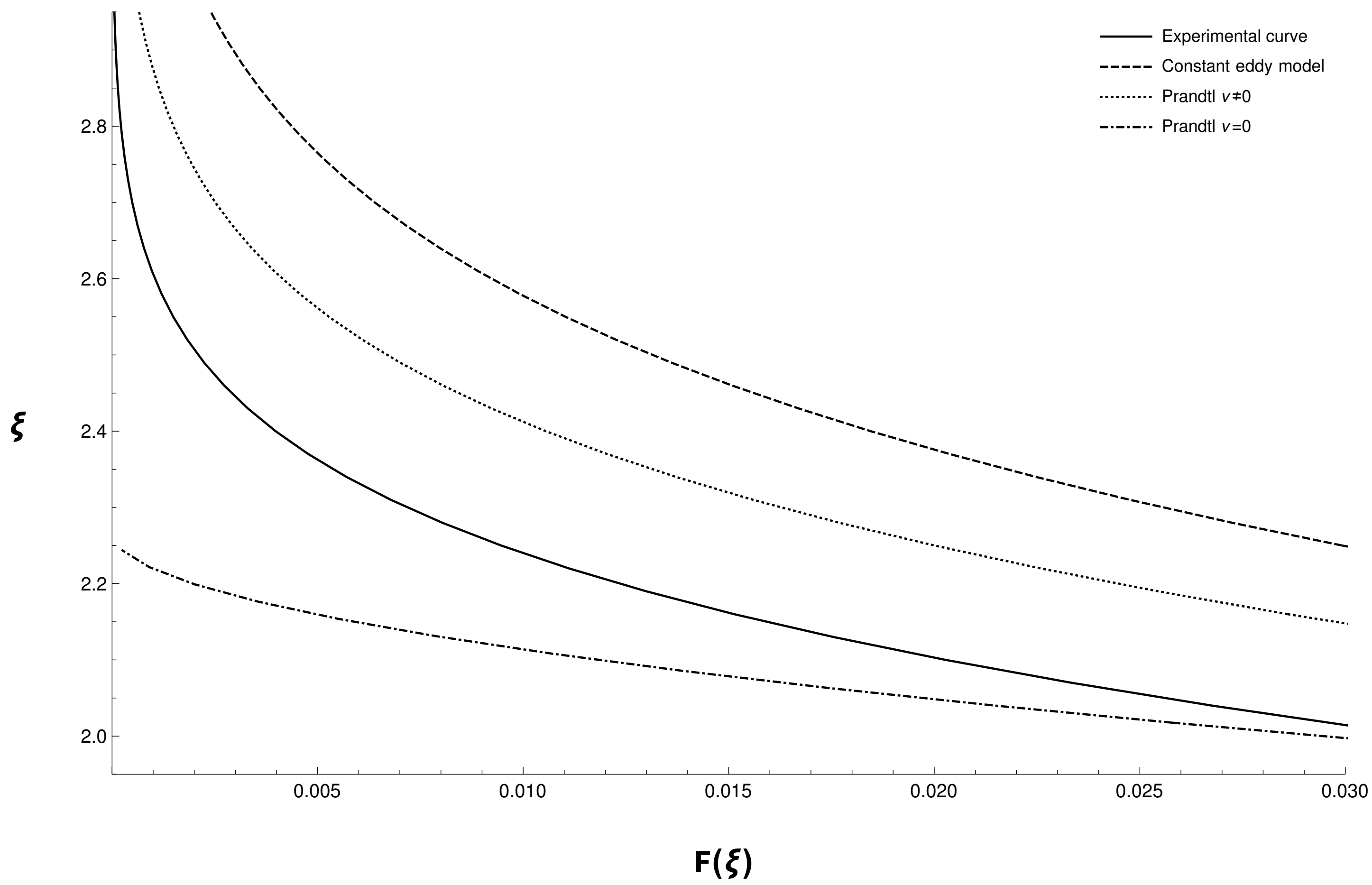


FIGURE 6.7: All models under consideration plotted on the same set of axes as the experimental curve for $1.95 \lessapprox \xi \lessapprox 3$

6.4 Prandtl's extended mixing length model and von Kármán's mixing length model

Both Prandtl's extended mixing length model and von Kármán's mixing length model resulted in 2nd order nonlinear ODEs. The 2nd order ODEs were reduced to a system of two 1st order ODEs and a final value Runge-Kutte method developed in [57] was used to solve the system. The resulting numerical solutions were not in agreement with the expected velocity profiles for the two-dimensional turbulent classical wake. It is expected that more complex or implicit numerical methods are needed to solve the equations. This is left as future work.

Chapter 7

CONCLUSIONS

A brief history of the study of turbulence was given in Chapter 1. It was established that turbulence is abundant in both flows in nature and industry and that developing theories to understand turbulence has practical implications.

In Chapter 2 various shear flows were explored and existing studies on the classical far wake, for both laminar and turbulent cases, were provided. The notable differences between shear flows and free shear flows were highlighted. A justification to study the classical wake was given in the context of energy production.

A method of studying turbulence using the Reynolds decomposition of the flow into a mean component and a fluctuation was considered in Chapter 3. The various algebraic closure models used to model the Reynolds stresses in this dissertation were then presented. It was stated that Boussinesq both closed and averaged the Navier-Stokes equations before Reynolds. The averaged terms containing products of velocity fluctuations were later called the Reynolds stresses. Prandtl's mixing length model and Prandtl's extended mixing length model were then derived using physical interpretations of the fluid flow in the context of parallel fluid flows. Several derivations of Prandtl's extended mixing length model were explored. In some derivations there are two distinct mixing lengths while other derivations have the same or proportional mixing lengths. Two different derivations of the extended model were given. Von Kármán's mixing length model was derived by physical interpretations. A translation of Betz's derivation for the von Kármán mixing length based on a vorticity model was given. Most importantly, von Kármán's mixing length was derived from boundary layer theory which motivated the application of the model to the classical wake. This derivation appears to be a novel contribution.

In Chapter 4 the governing equations for the turbulent classical far wake were derived from the boundary layer equations using Reynolds decomposition. The approximation was made that the second order terms in smallness could be neglected. The derived equations were nonlinear due to the presence of the eddy viscosity term. The equations as well as the boundary conditions and conserved quantity, which for the classical wake is the drag, were described using the mean velocity deficit $\overline{w}$.

Lie symmetry methods were used to derive the invariant solution associated with the conserved vector for any arbitrary effective viscosity E of the form $E = E(x, w_y, w_{yy})$ in Chapter 5. Models for which E is independent of w_{yy} were analysed fully. In Prandtl's extended mixing length model, the two mixing lengths were chosen to be distinct. When the kinematic viscosity was not neglected, it was found that the two mixing lengths were proportional. When the kinematic viscosity was neglected, Prandtl's hypothesis was used and it was shown that for an invariant solution to exist the mixing lengths had to be proportional. A third case where the two mixing lengths were made proportional to each other verified Prandtl's hypothesis for a special case. The coefficient ξ^1 depended on the turbulent model considered whereas ξ^2 and η were the same for all models analysed. This illustrated that turbulence seems to affect mostly ξ^1. The resultant PDEs were closed by the various algebraic closure models resulting in ODEs.

Analytic and numerical solutions to the ODEs were provided in Chapter 6. A best fit curve that runs through experimental data was considered. The problems were formulated in terms of the similarity variables used in the experimental setup so that the resultant mean velocity deficit profiles could be plotted against the experimental curve. The following observations were made.

A constant eddy viscosity closure model best approximated the velocity profile from the centre line to $\xi \approx 1.4$. By neglecting the kinematic viscosity, the fluid is inviscid and fluid elements are unable to affect neighbouring fluid elements that would otherwise transfer the disturbances to infinity. Therefore, Prandtl's mixing length model where the kinematic viscosity was neglected resulted in a finite boundary. The velocity profile of the wake was similar to that of the experimental curve which motivated Prandtl's decision to neglect the kinematic viscosity in his calculations. Including the kinematic viscosity slightly improved the accuracy of the velocity profile near the centre line of the wake and it better approximated the profile far from the centre line. It should be noted that it is not sufficient to only compare the velocity profile with experimental data. To judge a model's effectiveness, the shear stresses should also be considered and this is left for future work. The experimental curve used in these comparisons was derived from experimental data performed in a wind tunnel. The kinematic and dynamic viscosities of air and water both differ considerably and may affect each model's prediction of the velocity profile differently for varying Reynolds numbers. Future work is required to compare the models for which the effective viscosity depends on w_{yy}. More experimental data is needed to perform further studies pertaining to the accuracy of each closure model.

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