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Chromatic polynomials of some derived graphs

by

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science in mathematics at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

________________________________ (Signature of candidate)

24 day of march in 2025

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— Sunny Gudazi

Abstract

The chromatic polynomial of a graph, is well known and studied in the literature. Explicit expressions for the chromatic polynomials of some classes of graphs are well known. In addition, formulas for computing the chromatic polynomial of graphs derived from some graph operations are known. Interestingly, chromatic polynomials of graphs can be expressed in different forms, using different classes of graphs. In the literature, chromatic polynomials have been expressed and studied in null graph form, tree form and complete graph form. Although the chromatic polynomial of a graph can also be expressed in cycle graph form, this form has not been studied in great detail in the literature.

In this dissertation, we study the chromatic polynomials of some graphs derived from graph operations, namely, the wheel graph and the fan graph. These two graphs are derived from the vertex join of a cycle and a tree, respectively. The chromatic polynomial of a fan graph and a wheel graph found in the literature are factorizations of their null graph form. We studied the chromatic polynomial of a fan graph in cycle form and we found that the coefficients in cycle form result in an array where each row can be computed using the row above. Given this formula for the fan graph, we introduce the fan graph form and express the chromatic polynomial of the wheel graph in this form. We then extend this result to give a method to predict the coefficients of the chromatic polynomial of a wheel graph in cycle form.

— *Sunny Gudazi*

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Chapter 1

Basics in the theory of graphs

1.1 Introduction

This chapter begins with an overview of the dissertation. Following this, we define key concepts in graph theory that are pertinent to this study. We then select and define several well-known classes of graphs from the literature that are relevant to our work. Finally, we conclude the chapter by describing certain graph operations, with particular emphasis on how these operations can generate derived graphs.

1.2 Overview

After this Overview of this dissertation, Chapter 1 begins by introducing essential definitions and examples in Section 1.3 that are foundational to the topics discussed throughout the work. Section 1.4 covers several important classes of graphs that are frequently referenced in later chapters. In Section 1.5, various graph operations are defined, including the union and intersection of graphs, edge contraction, and graph complements, with examples provided to illustrate their application. Finally, Section 1.6 summarizes the key points covered in Chapter 1 and presents some initial conclusions.

Chapter 2 offers a comprehensive exploration of the chromatic polynomial and in particular the theory useful to this dissertation. Section 2.2 begins with a brief overview of the history of graph coloring problems. Section 2.3 introduces the chromatic polynomial, discussing its key properties and examining practical methods for computing the chromatic polynomail. In Sections 2.4 and 2.5, we present established formulas for calculating chromatic polynomials of certain derived graphs and provide explicit expressions for specific classes of graphs. The chapter concludes in Section 2.6 with a discussion of four distinct forms of presenting the chromatic polynomial of a graph, namely: the null graph form, complete graph form, tree graph form, and cycle graph form.

Chapter 3 explores some properties and the chromatic polynomial of fan graphs.

Section 3.2 provides a review of fan graphs, outlining some of their general characteristics. Section 3.3 introduces an explicit formula for calculating the chromatic polynomial of a fan graph and Section 3.4 presents a recursive approach to obtain this polynomial. The chapter's main findings are discussed in Section 3.5, focusing on the coefficients of the chromatic polynomial when the fan graph is represented in cycle graph form. Additionally, an array is provided to assist in predicting these coefficients in the cycle graph structure.

In Chapter 4, we explore the properties and chromatic polynomial of wheel graphs. Section 4.2 provides a review of wheel graphs, outlining some of their general characteristics. Section 4.3 gives the main result of this chapter, the introduction of a new form of representing the chromatic polynomial, the fan graph form. We then give the chromatic polynomial of a wheel graph in fan graph form. Finally, in Section 4.3, we outline a procedure on how to predict coefficients of the chromatic polynomial of a wheel graph in cycle graph form using an example to clarify.

1.3 Basic definitions

In this section, we define, discuss and clarify some chosen concepts in graph theory which will be used frequently and will be considered basic knowledge to understand this work. The reader is referred to [3] for further reading of these defined concepts, otherwise we state relevant references.

Definition 1.3.1. A *graph* is a mathematical structure consisting of a finite, non-empty set of elements V, and a set E, not necessarily non-empty, of 2-element subsets of V, usually denoted as $G = (V(G), E(G))$. The elements $u_i \in V$ are called *vertices* and the elements $\{u_i, u_j\} \in E$ are called *edges.* In this dissertation we call elements in V by u_i and u_j where i,j are indices indicating different vertices in V.

In a graph G, the number of vertices, denoted $|V(G)|$, is called the *order* and the number of edges, denoted $|E(G)|$ is called the *size* of the graph. For easy understanding and visualization of this structure, a graph G is represented graphically, where the vertices are given as tiny shaded circles and edges as lines joining two tiny shaded circles. An edge $\{u_i, u_j\}$ or u_iu_j, joins vertices u_i and u_j and hence the two vertices, u_i and u_j are said to be *adjacent.* Thus, the two adjacent vertices u_i and u_j are said to be *neighbours* of each other and u_i and u_j are said to be *endpoints* of an edge $\{u_i, u_j\}$. Furthermore, an edge $\{u_i, u_j\}$ is *incident* to vertices u_i and u_j. If there are two or more edges incident to the same vertices, u_i and u_j, then such edges are said to be *parallel* to the edge $\{u_i, u_j\}$. If $\{u_i, u_i\}$ is an edge, that is an edge incident with a single endpoint u_i, we call such an edge a *loop.* A graph with no parallel edges or loops is called a *simple* graph. In this work, when adjacency or incidency of vertices are not relevant, we will denote an edge by e to ease notation.

Definition 1.3.2. Let G be a graph and vertex $u_i \in V(G)$. The number of all the edges incident to u_i is called the *degree* of u_i denoted $\deg(u_i)$.

Alternatively, we can define the degree of the vertex u_i to be the number of vertices adjacent to u_i. The *maximum degree* of G is the largest $\deg(u_i)$ for $u_i \in V(G)$, and is denoted by $\Delta(G)$. Similarly, *the minimum degree* is denoted by $\delta(G)$.

Definition 1.3.3. Let G be a graph with $V(G) = \{u_0, u_1, u_2, \ldots, u_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. An alternating sequence of vertices and edges of the form $W = u_0, e_1, u_1, e_2, u_2, \ldots e_t, u_{t+1}$ such that $\{u_i, u_{i+1}\} = e_{i+1}$, is called a *walk*, in G.

A walk W which starts at vertex u_0 and ends at vertex u_t is also called a u_0-u_t walk. If no edges in W is repeated, then W is called a trail. If all the vertices u_i are distinct in W, then the walk is called a *path.* If u_0 and u_{t+1} are distinct in W, then the walk is called an *open* walk, otherwise the walk is *closed.* The *length* of a walk is the number of edges in W. A closed path is called a *cycle* in the graph. It is possible to get a cycle of length 1 in a graph and this corresponds to a loop defined earlier. In addition, G is said to be *connected* if, for every pair of vertices u_i, u_j, there exists a u_i-u_j path. Otherwise, if G is not connected the graph is said to be *disconnected.*

Definition 1.3.4. Let G and H be two graphs. H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, denoted $H \subseteq G$. If either $V(H) \neq V(G)$ or $E(H) \neq E(G)$, then we say H is a proper subgraph of G.

If $V(G) = V(H)$ then H is called a *spanning subgraph* of G. Let G be a graph and $Y \subseteq V(G)$. The subgraph of G, denoted by $G[Y]$ with vertex set $V(G[Y]) = Y$ and edge set $E(G[Y]) = \{u_iu_j \mid u_i, u_j \in Y \text{ and } u_iu_j \in E(G)\}$ is called the *subgraph* *induced by* Y. Furthermore, we refer to a connected subgraph H as a *component* of G if H is not a proper subgraph of any connected subgraph of G.

Definition 1.3.5. Let G and H be two graphs. Then G and H are isomorphic if there exists a function $f : V(G) \to V(H)$ which is both one to one and onto such that $\{u_i, u_j\} \in E(G)$ if and only if $\{f(u_i), f(u_j)\} \in E(H)$. If G is isomorphic to H, we write $G \cong H$.

Definition 1.3.6. Let G be a graph and $Y \subseteq V(G)$. The subset Y is called an *independent set* if no two vertices in Y are adjacent in G.

The subset Y is called a *maximal independent set* if there exists no other independent set $Y' \subseteq V(G)$ such that $Y \subset Y'$. The independence number of G, denoted as $\alpha(G)$ is $\max\{|Y|\ ,\ Y$ is an independent set in $G\}$.

Definition 1.3.7. Let $G = (V(G), E(G))$, $u_i \in V(G)$ and $e_j \in E(G)$. The process of removing a vertex u_i and the set of all edges, $\{e_j \in E(G) \mid e_j = \{u_i, u_j\}\}$, in the graph G is called *vertex-deletion.* The graph resulting from the process of vertex-deletion is denoted by $G \backslash u_i$. We say a vertex u_i is a *cut-vertex* if $G \backslash u_i$ is a disconnected graph.

Definition 1.3.8. Let $G = (V(G), E(G))$ and $e_j \in E(G)$. The process of removing an edge e_j and leaving the end points of e_j in the graph G, is called *edge-deletion.* The graph resulting from the process of edge-deletion is denoted by $G \backslash e_j$. We say an edge e_j is a *bridge* if $G \backslash e_j$ is a disconnected graph.

These concepts of deleting a vertex or an edge extend naturally to subsets of $V(G)$ or $E(G)$ respectively. Thus, if $Y \subset V(G)$ and $G\backslash Y$ is disconnected, then Y is called a *vertex-cut* of G. The order of a minimum vertex-cut of G is called the *vertex-connectivity* of G. To be specific, vertex-connectivity is simply called the connectivity of G, and is denoted by $\kappa(G)$. On the other hand, if $X \subset E(G)$ and $G\backslash X$ is disconnected, then we say X is an *edge-cut* of G. The *edge-connectivity* of G, denoted $\lambda(G)$, is the size of a minimum edge-cut of G. Furthermore, if $\lambda(G) \geq k$ for some positive integer k, then G is said to be k*-edge-connected*.

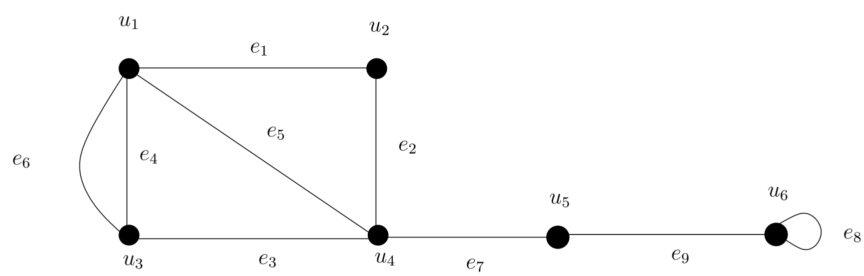

Figure 1.1: A graph G of order 6.

Example 1.3.9. In this example, we clarify some of the concepts defined in this section. The graph G is represented graphically by the diagram given in Figure 1.1. The vertex set $V(G) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and the edge set $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$. The order of the graph G is $|V| = 6$ and the size of the graph G is $|E| = 9$. The edge e_8 is a loop, while edges e_7 and e_9 are bridges. Edges e_6 and e_4 are parallel. The degrees of the vertices of G are $\deg(u_1) = 4$, $\deg(u_2) =$

$\deg(u_5) = 2$, $\deg(u_3) = \deg(u_6) = 3$ and $\deg(u_4) = 4$. The vertex sets $S_1 = \{u_1, u_5\}$, $S_2 = \{u_2, u_5\}$, $S_3 = \{u_3, u_5\}$, and $S_4 = \{u_2, u_3, u_5\}$, are independent sets. However, vertex sets S_2 and S_3 are not maximally independent, as they are proper subsets of S_4. On the other hand, both vertex sets S_1 and S_4 are maximally independent. The vertices u_4 and u_5 are cut-vertices in graph G, indicating that the vertex-connectivity of the graph G is $\kappa(G) = 1$. Furthermore, the edges e_7 and e_9 are bridges, which implies that the edge-connectivity of the graph G is $\lambda(G) = 1$. Additional examples of edge-cuts in graph G include $X_3 = \{e_1, e_3, e_5\}$ and $X_4 = \{e_2, e_4, e_5, e_6\}$. The sequence $u_1, e_4, u_3, e_4, u_1, e_6, u_3, e_3, u_4, e_7, u_5$ represents a walk, as consecutive vertices are adjacent to each other. The sequence $u_1, e_6, u_3, e_3, u_4, e_5, u_1, e_1, u_2$ is a trail which is not a path, given that all edges are distinct but some vertices are not distinct. The sequence $u_1, e_1, u_2, e_2, u_4, e_3, u_3$ is a path since all vertices and edges are distinct. The sequence u_1, e_6, u_3, e_4, u_1 is a cycle, as the first and last vertices are the same. $\square$

1.4 Classes of graphs

In this section, we define some well known classes of graphs. We then state some very basic properties of each class of graphs which may be used with or without reference in this work. Thereafter, we give an example of graphs for some classes discussed. The classes of graphs chosen will be useful in this work. For further details, we refer the reader to [3] unless otherwise stated.

Definition 1.4.1. Let $G = (V(G), E(G))$ such that the vertex set $V(G) = \{u_1, u_2, \ldots, u_q\}$ and edge set $E(G) = \{\}$. Then G is called a *null graph* of order q, denoted by N_q.

Proposition 1.4.2. *Let N_q be a null graph of order q. Then,*

(i) N_q is a graph of size 0.

(ii) N_q is a graph such that all vertices have degree 0.

(iii) N_q is a disconnected graph with q components.

Definition 1.4.3. Let $G = (V(G), E(G))$ such that the vertex set $V(G) = \{u_1, u_2, \ldots, u_q\}$ and edge set $E(G) = \{u_i u_{i+1} \mid i = 1, 2, \ldots, q-1\}$. Then G is called a *path* of order q, denoted by P_q.

Proposition 1.4.4. *Let P_q be a path graph of order q. Then,*

(i) P_q has size $q - 1$.

(ii) P_q has two vertices of degree one, called leaves, and $q - 2$ vertices of degree 2.

(iii) P_q is a connected graph.

Definition 1.4.5. Let $G = (V(G), E(G))$ such that the vertex set $V(G) = \{u_1, u_2, \ldots, u_q\}$ and edge set $E(G) = \{u_i u_{i+1} \mid i = 1, 2, \cdots, q-1\} \cup \{u_1 u_q\}$. Then G is called a *cycle graph* of order q, denoted by C_q.

Proposition 1.4.6. *Let C_q be a cycle graph of order q. Then,* *(i) All vertices in C_q are of degree* 2.

(ii) C_q is a connected graph.

(iii) $\displaystyle\sum_{u_i \in V(G)} deg(u_i) = 2q.$

(iv) $C_q \backslash e \cong P_q$ where e is any edge in $E(C_q)$.

Definition 1.4.7. Let G be a graph such that $V(G) = \{u_1, u_2, \ldots, u_q\}$ and $E(G) = \{u_o u_p \mid o, p \in \{1, 2, \ldots, q\}$ and $o \neq p\}$. Then G is called a *complete graph* of order q, denoted by K_q, and $K_1 \cong N_1$.

Proposition 1.4.8. *Let K_q be a complete graph of order q. Then,*

(i) all vertices of the graph K_q have degree $q-1$.

(ii) the size of the graph K_q is $\frac{q(q-1)}{2}$.

(iii) the sum of the degrees of the vertices of the graph K_q is $q(q-1)$.

Definition 1.4.9. A connected graph G with no subgraph isomorphic to a cycle is called a *tree*. Thus, a tree of order q is denoted by T_q. If a vertex u has degree one in a tree, then we say vertex u is a *leaf*.

Proposition 1.4.10. *Let T_q be a tree graph of order q. Then,*

(i) T_q has size $q-1$.

(ii) T_q has at least two leaves.

(iii) T_q *is a connected graph.*

(iv) Every edge of T_q *is a bridge.*

We define but skip properties of a wheel graph and a fan graph in this chapter. Since these graphs are of interest in this work, these properties will be discussed in detail in Chapter 3 and Chapter 4.

Definition 1.4.11. Let $G = (V(G), E(G))$ such that the vertex set $V(G) = \{u_1, u_2, \ldots, u_{q-1}, x\}$ and edge set $E(G) = \{u_l u_{l+1} \mid l = 1, 2, \ldots, q-2\} \cup \{u_1 u_{q-1}\} \cup \{u_l, x \mid l = 1, 2, \ldots, q-1\}$. Then G is called a *wheel graph* of order q, denoted by W_q.

Definition 1.4.12. Let $G = (V(G), E(G))$ such that the vertex set $V(G) = \{u_1, u_2, \ldots, u_{n-1}, x\}$ and edge set $E(G) = \{u_i u_{i+1} \mid i = 1, 2, \ldots, n-2\} \cup \{u_i, x \mid i = 1, 2, \ldots, n-1\}$. Then G is called a *fan graph* of order n, denoted by F_n.

Definition 1.4.13. Let $G = (V(G), E(G))$ such that the vertex set $V(G) = T \cup S$ such that $T \cap S = \emptyset$ and each element $e \in E(G)$ has an endpoint in T and an endpoint in S. Then G is called a *bipartite* graph. If each vertex of T is adjacent to all the vertices of S, then G is a *complete bipartite* graph denoted by $K_{|T|,|S|}$. T and S are independent sets meaning there are no edges between vertices within vertex set T and similarly, no edges between vertices within vertex S.

Example 1.4.14. In this example, we show graphical representation of some classes of graphs defined in this section. The diagrams in Figure 1.2 are examples of a null graph, a path, a cycle, a complete graph and a fan graph, all of order 4, a complete bipartite graph of order 8 and a wheel of order 5.

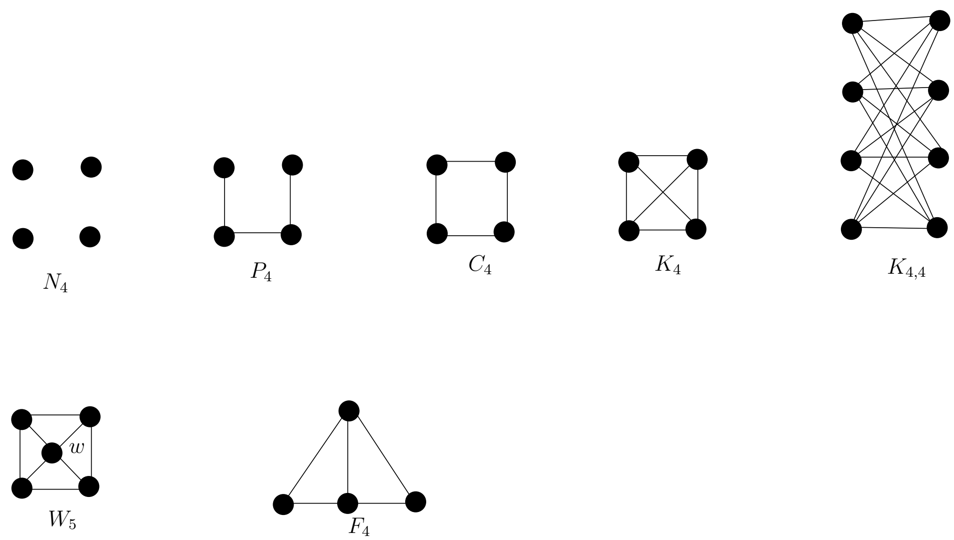

Figure 1.2: Classes of graphs.

1.5 Graph operations and derived graphs

In this section, we start by defining a derived graph. Thereafter, we discuss some graph operations found in the literature. We then give some examples illustrating some of the operations on graphs and the derived graphs obtained. Some of the graph operations discussed will be useful to our work. The reader can get further details in [3], unless otherwise stated.

Definition 1.5.1. A *derived* graph is a graph obtained after applying one or more graph operations to one or more graphs.

Definition 1.5.2. Let $G_2 = (V(G_2), E(G_2))$ and $G_1 = (V(G_1), E(G_1))$ be two graphs. Then the *union* of G_2 and G_1 is the graph G with $V(G) = V(G_2) \cup V(G_1)$ and $E(G) = E(G_2) \cup E(G_1)$, denoted by $G_2 \cup G_1$.

Definition 1.5.3. Let $G = (V(G), E(G))$. The graph denoted by $\overline{G}$ with vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G}) = \{u_i u_j \mid \text{if } u_i u_j \notin E(G)\}$ is called the *complement* of the graph G.

Definition 1.5.4. Let $G_2 = (V(G_2), E(G_2))$ and $G_1 = (V(G_1), E(G_1))$ be two graphs. Then the *join* of G_2 and G_1 is the graph $G = (V(G), E(G))$ with vertex set $V(G) = V(G_2) \cup V(G_1)$ and edge set $E(G) = E(G_2) \cup E(G_1) \cup \{e_{ij}\}$ where e_{ij} is the set of all edges between vertices from $V(G_2)$ to $V(G_1)$, denoted by $G_2 \vee G_1$. Thus the graph $G_2 \vee G_1$ is a derived graph by Definition 1.5.1

Definition 1.5.5. Let $G_2 = (V(G_2), E(G_2))$ and $G_1 = (V(G_1), E(G_1))$ be two graphs. Then the *cartesian product* of G_2 and G_1, denoted by $G_2 \Box G_1$, is the graph $G = (V(G), E(G))$ with vertex set $V(G) = V(G_2) \times V(G_1)$ such that two vertices (x, y) and (a, b) are adjacent if either $x = a$ with $yb \in E(G_1)$ or $y = b$ with $xa \in E(G_2)$.

Example 1.5.6. In this example, we use two graphs G_1 and G_2 to illustrate the graph operations union, join and cartesian product. The three diagrams in Figure 1.3 are labelled, (1), (2) and (3).

Diagram (1) shows the union of two graphs, $G = G_1 \cup G_2$, with $V(G) = \{x, y\} \cup \{a, b, c, d\}$ and $E(G) = \{xy\} \cup \{ab, bc, cd, da\}$.

Diagram (2) is an illustration of the join of graphs G_2 and G_1 to obtain $G = G_1 \vee G_2$, with $V(G_1 \vee G_2) = \{x, y, a, b, c, d\}$ and

$E(G_1 \vee G_2) = \{xy, ab, bc, cd, da, ax, bx, cx, dx, ay, by, cy, dy\}$.

Diagram (3) is an illustration of the cartesian product of G_2 and G_1, $G = G_1 \square G_2$, with $V(G_1 \square G_2) = \{(a, y), (b, y), (c, y), (d, y), (a, x), (b, x), (c, x), (d, x)\}$ and

$E(G_1 \square G_2) = \{\{(a, y), (b, y)\}, \{(b, y), (c, y)\}, \{(c, y), (d, y)\}, \{(d, y), (a, y)\},$

$\{(a, x), (b, x)\}, \{(b, x), (c, x)\}, \{(c, x), (d, x)\}, \{(d, x), (a, x)\}\},$

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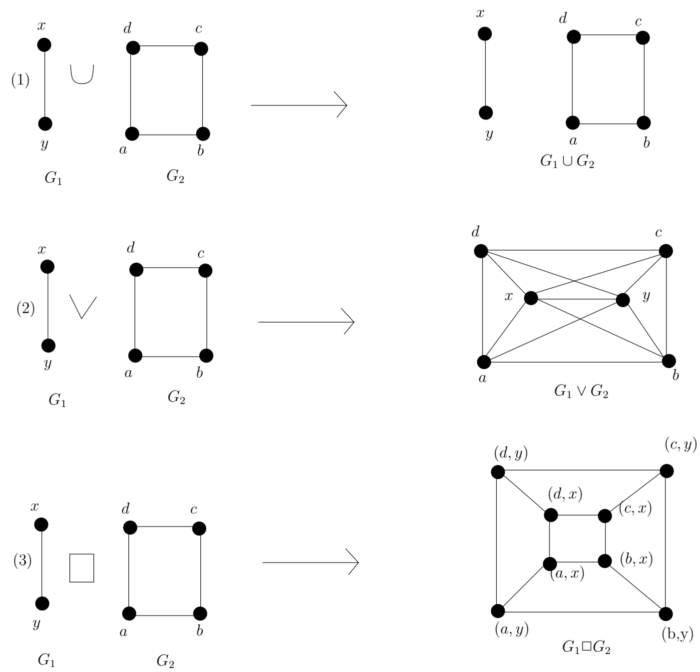

Figure 1.3: Some graph operations on G_1 and G_2.

Of particular interest to this dissertation is the join operation, from Definition 1.5.4. We define a special join and clarify with an example, since the derived graphs of interest in this work are obtained using this graph operation.

Definition 1.5.7. Let $G_2 = (V(G_2), E(G_2))$ be a graph of order n with $V(G_2) = \{w_1, w_2, \cdots, w_n\}$. A *vertex join* of G_2, denoted by $\widehat{G_2}$ is a graph of order $n+1$ obtained by taking the join of G_2 and K_1. The vertex join of G_2 has $V(\widehat{G_2}) = V(G_2) \cup \{x\}$ where $V(K_1) = \{x\}$ and $E(\widehat{G_2}) = E(G_2) \cup \{(w_1, x), (w_2, x), \ldots, (w_n, x)\}$.

Based on this concept of vertex join, we revisit Definition 1.4.11 of a wheel graph and Definition 1.4.12 of a fan graph giving alternative definitions. Thus, we see that a wheel graph and a fan graph are derived graphs.

Definition 1.5.8. The wheel graph of order n, W_n, is the derived graph obtained from a vertex join of the cycle graph C_{n-1}.

Definition 1.5.9. The fan graph of order n, F_n, is a vertex join of the path P_{n-1}.

Example 1.5.10. In this example, we illustrate how to obtain a vertex join of a graph G. We start with the graph G to obtain the $\widehat{G}$ as shown in Figure 1.4 where the vertex set $V(\widehat{G}) = \{u_1, u_2, u_3\} \cup \{u\}$ and the edge set $E(\widehat{G}) = E(G) \cup \{(u_1, u), (u_2, u), (u_3, u)\}$. $\square$

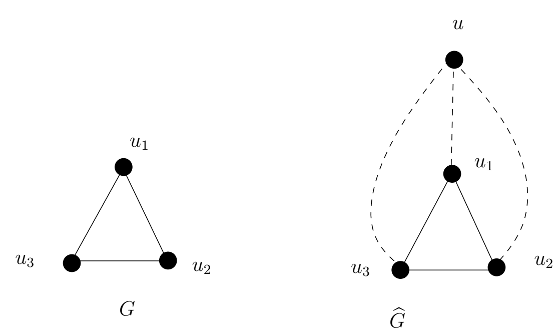

Figure 1.4: Graph G and vertex join $\widehat{G}$.

In addition to the vertex join of a graph, we will use the following two graph operations, deletion and contraction, in our computation in this work.

Definition 1.5.11. Let $G = (V(G), E(G))$ with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$

and let an edge $e = \{u_i, u_j\} \in E(G)$. Then the *contraction* of an edge e is the process of deleting $e \in E(G)$ and identifying the two vertices u_i and u_j that is letting $u_i = u_j$, denoted by G/e.

Definition 1.5.12. A graph G' obtained from G after a succession of edge or vertex deletions and edge contractions is called a *minor* of G.

Example 1.5.13. In this example, we illustrate deletion of an edge e and contraction of an edge e in the graph G. The graphs, G, G/e, and $G\backslash e$ are shown in Figure 1.5. Since $G \cong C_4$, we obtain the graph $G\backslash e \cong P_4$ by choosing to delete edge $e = v_2v_4$. Similarly, we obtain the graph $G/e \cong C_3$ by contracting the edge $e \in E(G)$.

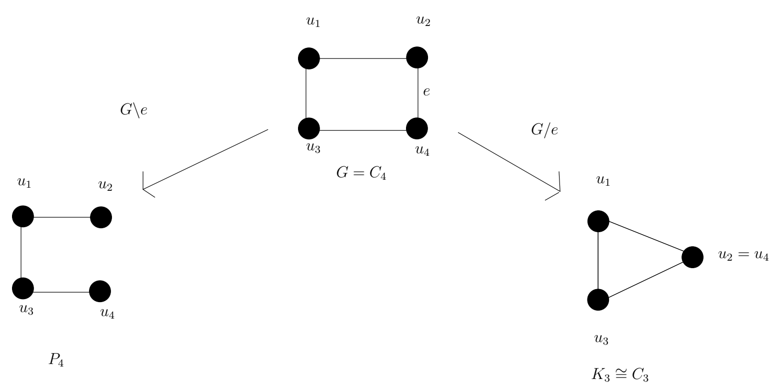

Figure 1.5: The process of deletion and contraction.

Definition 1.5.14. Let $G = (V(G), E(G))$ such that $e = u_iu_j \in E(G)$. The graph with vertex set $V(G) \cup \{x\}$ and edge set $E(G) \cup \{u_ix, u_jx\} - \{u_iu_j\}$ is called a *subdivision* of G. In particular, we say that the edge u_iu_j has been subdivided and the graph obtained from G by a sequence of subdivisions is called a *subdivision* of G.

Example 1.5.15. In this example, we illustrate the graph operation of subdivision. The graph G, and two graphs, subdivisions G_1 and G_2 obtained from graph G are shown in Figure 1.6. The graph G_1 is obtained by subdividing edge u_1u_4, while graph G_2 is obtained subsequently subdividing edge zu_4. $\square$

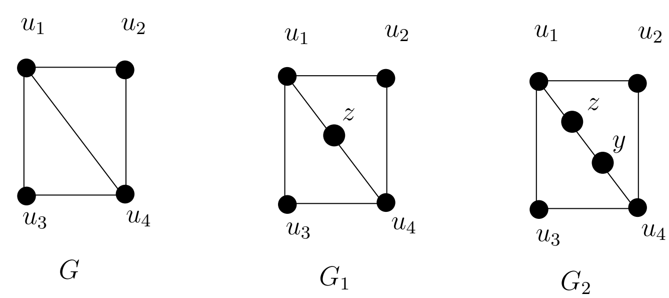

Figure 1.6: Subdivisions of graph G.

Definition 1.5.16. Let G_1 and G_2 be two distinct graphs both having a complete graph K_r as a subgraph. The *derived graph* G obtained by identifying the two $K_r \subset G_1$ and $K_r \subset G_2$ is called the K_r-gluing of G_1 and G_2 is denoted by $G = G_1 \underset{K_r}{\cup} G_2$.

Note that some K_r-gluings have special names. A K_1-gluing is called a vertex-gluing and a K_2-gluing is called an edge-gluing.

1.6 Conclusion

In this chapter, we provided definitions, notations, and certain classes of graphs that are relevant to this dissertation. Subsequently, we defined several graph operations, including the union, join, and cartesian product of graphs, the deleting and contracting of an edge, the complement of a graph, as well as subdivisions.

Chapter 2

Chromatic polynomial of a graph

2.1 Introduction

This chapter gives a detailed literature review on the chromatic polynomial of a graph. We start with a brief background on the genesis of the colouring problem in graph theory. Thereafter, we introduce the chromatic polynomial of a graph which is associated with the colouring problem. We then discuss some known methods for computing the chromatic polynomial and some known properties. In addition, we present some known explicit expressions of the chromatic polynomial for some of the classes of graphs discussed in Chapter 1 and some derived graphs. We conclude this chapter by presenting different ways of representing the chromatic polynomial of a graph.

2.2 Background of the colouring problem

In this section we give a brief background on topics related to graph colouring problems, which are a class of problems in graph theory concerned with assigning colours to the vertices or edges of a graph under certain constraints.

In 1852 F. Guthrie observed that colouring a map of England, with the condition that neighbouring regions must have different colours, could be achieved using only four colours. This observation led to the question: Can any map be coloured using only four colours so that regions sharing a boundary have different colours?, see [16].

An answer to this question came in the form of a proof proposed by A. Kempe in 1879. In this proof, Kempe claimed that any map could be coloured with four or fewer colours to ensure that regions sharing a boundary have different colours, see [11]. From 1879 to 1889 it was believed that the Four Colour Problem was solved. In 1890 a serious mistake was found in Kempe's proof by P. Heawood. Despite this, Heawood successfully applied Kempe's techniques to show that any map could be coloured with five colors or fewer. He also extended the problem beyond spheres to other surfaces, we refer to [16].

A paper titled "A Determinant Formula for Calculating the Total Ways to Color a Map" by G. Birkhoff in 1912, was a result of trying to resolve the Four Color Problem. This paper is a very rich source of the colouring problems. In particular, it is in this paper that a function which gives the number of proper colourings of a graph G using the colours from a set of t colours, was introduced, see [14]. This function was limited to planar graphs and was later called the chromatic polynomial of a graph, denoted by $P(G;t)$ by Whitney. We refer to [5]. In response to Birkhoff's work, H. Whitney generalized this function, the chromatic polynomial, for any graph, establishing foundational results for this function, we refer to [15, 11].

A paper titled "An Introduction to Chromatic Polynomials," [13], written by R. Read in 1968, generated widespread interest in chromatic polynomials. In this work, Read posed the question: What are the necessary and sufficient conditions for two non-isomorphic graphs to have the same chromatic polynomial? From 1968, the research on chromatic polynomials picked up in many different directions. Since then, researchers have explored many different aspects of the chromatic polynomials. Some of these areas include the coefficients of the chromatic polynomial, chromatically equivalent graphs, roots of chromatic polynomials, chromatic factorization and many more, we refer to [6, 8, 10, 13].

Irrespective of the many results in the chromatic polynomial of a graph, following Read's paper, the Four Color Theorem was still outstanding. Using computer-assisted methods, K. Appel and W. Haken proved the Four Color Theorem in 1976. Their approach involved analyzing nearly 2,000 configurations, each needing verification for reducibility, a process that required substantial computational support. We refer to [2]. Later, in 1996, N. Robertson, D. Sanders, P. Seymour, and R. Thomas revisited the Appel-Haken proof. They demonstrated that a minimum counterexample to the Four Color Theorem could not include any of 633 specific configurations, each of which could be shown to be reducible. Verifying this manually remained too complex and time-consuming. We refer to [14].

2.3 Chromatic polynomial of a graph

In this section, we start by introducing the concept of colouring a graph. Thereafter, we define the chromatic number and the chromatic polynomial of a graph. Then we give some known methods for computing the chromatic polynomial of a graph. In addition, we give some basic properties of the chromatic polynomial of a graph. Finally, we give explicit expressions of the chromatic polynomials of some classes of graphs. For further details about this section, we refer the reader to [7, 8, 13], unless otherwise stated.

Definition 2.3.1. Let $t \geq 1$ be a natural number and let $G = (V(G), E(G))$ be a graph. A proper colouring of graph G is an assignment of a colour to each vertex $u_i \in V(G)$ such that if $u_i u_j \in E(G)$, then vertex u_i and vertex u_j are assigned different colours. If a total of t colours are used in a proper colouring of a graph, we call that a t-colouring of the graph and say that the graph is t-colourable.

Definition 2.3.2. The chromatic number of graph G, denoted by $\chi(G)$, is the minimum number of colours required for a proper colouring of G.

Definition 2.3.3. The chromatic polynomial of graph G is the function $P(G;t)$ that counts the number of ways of properly colouring the vertices of graph G using t colours.

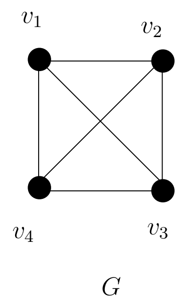

Figure 2.1: Graph G.

Example 2.3.4. In this example, we discuss how to count the number of ways of properly colouring a graph. Let G be the graph shown in Figure 2.1. All pairs of vertices of G are adjacent. Therefore, we assign different colours to each vertex of G by Definition 2.3.1 of proper colouring. If we have a set of t colours, and we start assigning vertex v_1 with any of the t colours, then vertex v_2 can be assigned any of the colours except for the colour assigned to vertex v_1, resulting in $t-1$ colours for vertex v_2. Continuing this process for vertex v_3, we again choose one of the remaining colours from the set of $t-2$ colours available after vertices v_1 and v_2 have been coloured. Finally, to colour vertex v_4, we select a colour from the remaining $t-3$ colours. Thus the number of ways of colouring G with t colours is $t(t-1)(t-2)(t-3)$ which, by

Definition 2.3.3, gives the chromatic polynomial of graph G.

This way of computing the chromatic polynomial is only possible for very small graphs with small connectivity. Hence, it is necessary to find efficient methods for finding chromatic polynomials of graphs. $\square$

2.3.1 Methods for computing the chromatic polynomial

We discuss some methods for computing the chromatic polynomial of graph G. The process of deletion and contraction, or addition and contraction, of edges allows us to break down the chromatic polynomial of the original graph G into the chromatic polynomials of null graphs and complete graphs, respectively.

The notation introduced by A. Zykov, see [13], which involves using a diagram to calculate $P(G;t)$, is a useful tool that will help us calculate the chromatic polynomial of G.

Unless otherwise stated, we refer the reader to [8, 13], for further details. Recall the graph operations of deleting and contracting an edge, discussed in Chapter 1, Section 1.5. We have that G/e, the contraction of an edge $e = u_i u_j$ is the process of deleting e and identifying the two vertices u_i and u_j and $G \backslash e$, the deletion of an edge $e = u_i u_j$ is the process of deleting e.

Theorem 2.3.5 (Fundamental Theorem)**.** *Let* $G = (V(G), E(G))$ *with* $u_i, u_j \in V(G)$

and an edge $e = u_iu_j \notin E(G)$, Then

$$P(G;t) = P(G+e;t) + P(G+e/e;t)$$

where $G+e$ is obtained by adding an edge e to G and $G+e/e$ is the graph obtained by contracting the edge e in $G+e$.

Proof. Let u_i and u_j be vertices in $V(G)$ such that $u_iu_j \notin E(G)$. In proper coloring of graph G, there are two cases:

(i) Where u_i and u_j are assigned the same colours and

(ii) u_i and u_j are assigned different colours.

Let G be the graph obtained by adding the edge u_iu_j, denoted by $G+e$. then colouring of $G+e$ is the same as colouring G with u_i and u_j have different colours. On the other hand in the graph $G+e$ we merge the vertices u_i and u_j and denote this by $(G+e)/e$ is the same as colouring G with u_i and u_j having the same colour. Hence, $P(G;t) = P(G+e;t) + P(G+e/e;t)$. □

We now demonstrate how to apply the Fundamental Theorem 2.3.5 in the computation of the chromatic polynomial. The idea of this method is to apply the theorem recursively on each graph until we get complete graphs.

Example 2.3.6. In this example, the graphs given in the diagram of Figure 2.2 are representing the chromatic polynomials of the graphs. We illustrate addition and contraction of an edge method. Thus, $P(G;t) = P(G+e;t) + P(G/e;t) = P(K_4;t) + P(K_3;t)$ where in this example $G + e/e$ is simplified. $\square$

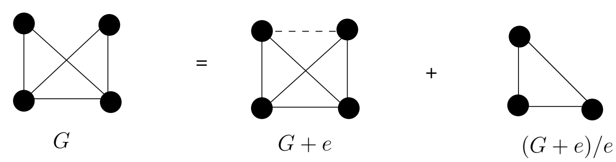

Figure 2.2: Addition and contraction of an edge.

The process of addition and contraction is repeated until all the chromatic polynomials are those of complete graphs, which are easy to compute.

Proposition 2.3.7. *Let K_n be the complete graph of order n. Then the chromatic polynomial, $P(K_n;t) = t(t-1)(t-2)\ldots(t-n+1)$.*

$P(K_n;t)$ is denoted by $(t)_r$ called the factorial form of the chromatic polynomial of a complete graph of order r.

The Fundamental Theorem 2.3.5, can be rearranged in a more usable and simple way for computing the chromatic polynomial of graph G stated in the following corollary.

Corollary 2.3.8. *Let $G = (V(G), E(G))$ and let $e = u_iu_j \in E(G)$, Then*

$$P(G;t) = P(G\backslash e;t) - P(G/e;t).$$

where $G\backslash e$ is the graph obtained by deleting the edge e and G/e is the graph obtained by contracting the edge e.

The following corollary follows from Definition 2.3.3 of the chromatic polynomial and the Fundamental Theorem 2.3.5.

Corollary 2.3.9. *Let G be a graph with the chromatic polynomial $P(G;t)$, then*

(i) $P(K_1;t) = t$.

(ii) $P(G;t) = (t-1)P(G/e;t)$ *if e is a bridge of G.*

(iii) $P(G;t) = 0$ *if G has a loop.*

(iv) $P(G;t) = P(G_1;t)P(G_2;t)\dots P(G_k;t)$, *if G is a disjoint union of connected graphs $G_1, G_2, \dots, G_k$.*

(v) If $G_1 \cong G_2$ up to parallel classes, then $P(G_1;t) = P(G_2;t)$

The following corollary follows from Corollary 2.3.9 Part(i) and Part(iv).

Corollary 2.3.10. *Let N_n be the null graph of order n. Then the chromatic polynomial, $P(N_n;t) = t^n$.*

The idea of this method of computing the chromatic polynomial stated in Corollary 2.3.8 and Corollary 2.3.9, is to apply the procedures recursively on each minor until we end up with null graphs. Then we substitute the chromatic polynomial of a null graph given in Corollary 2.3.10.

Example 2.3.11. In this example, we illustrates the computation of the chromatic polynomial of graph G using Corollary 2.3.8, the method of deleting and contracting edges. The graphs given in the diagram of Figure 2.3 are representing the chromatic polynomials of the graphs. In each step, the labelled edge with a dotted line is the edge that is being deleted and contracted from that graph. We repeat this algorithm until we obtain the chromatic polynomial of graph G in terms of null graphs. Then we substitute the chromatic polynomials of the null graphs given in Corollary 2.3.10. Hence, $P(G;t) = t^4 - 5t^3 + 8t^2 - 4t$. □

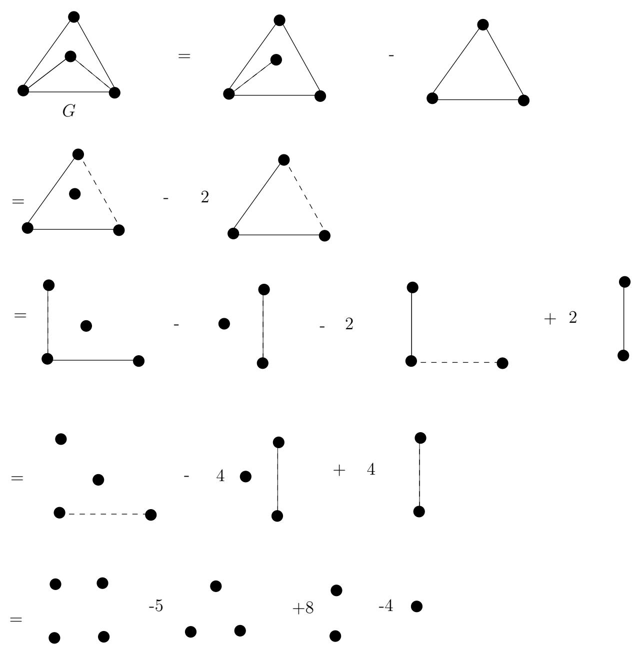

Figure 2.3: Deletion and contraction of an edge method.

To conclude this section, we summarize some well known properties of the chromatic polynomial of a graph which are basic, in one proposition. Since the literature is vast, we only give properties that will be used in this work.

To ease notation where applicable, the chromatic polynomial of a graph $P(G;t) = b_n t^n + b_{n-1}t^{n-1} + \ldots + b_1 t$ will be written as a summation, $P(G;t) = \sum_{i=1}^{n} b_i t^i$. We state the following basic properties without proof.

Proposition 2.3.12. *Let $G = (V(G), E(G))$ with $|V(G)| = n$ and $|E(G)| = m$ and let the chromatic polynomial, $P(G;t) = \sum_{i=1}^{n} b_i t^i$ where $b_i \in \mathbb{Z}$, for $i = 1, 2, 3, \ldots, n$. Then,*

(i) the degree of $P(G;t)$ is n.

(ii) there is no constant term in $P(G;t)$.

(iii) the coefficient of t^n, $b_n = 1$.

(iv) the absolute value of the coefficient t_{n-1} is $|b_{n-1}| = m$.

(v) the coefficients of $P(G;t)$ alternate in signs.

2.4 Chromatic polynomial of some derived graphs

In this section, we discuss some chromatic polynomials of derived graphs whose formula can be expressed in terms of the chromatic polynomials of the original graph or graphs.

Recall Definition 1.5.2 which states that the union of two graphs G_2 and G_1, denoted by $G_2 \cup G_1$, is obtained by combining the two graphs G_2 and G_1 as components of the new graph $G_2 \cup G_1$, provided there are no edges connecting the vertex sets of G_2 and G_1.

Theorem 2.4.1. *Let G be the union of two graphs G_1 and G_2. Then the chromatic polynomial of G,*

$$P(G;t) = P(G_1;t)P(G_2;t).$$

Example 2.4.2. In this example, we give the chromatic polynomial of the union of two graphs. The diagram in Figure 2.4 is the union of graphs G_2 and G_1 denoted by $G = G_1 \cup G_2$. By Theorem 2.4.1, the chromatic polynomial of G, $P(G;t) = P(G_2;t)P(G_1;t)$.

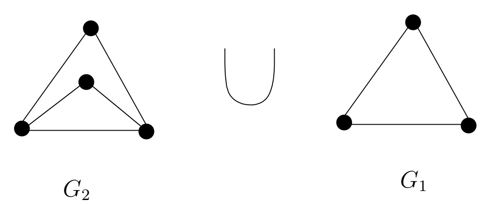

Figure 2.4: Graph $G_2 \cup G_1$.

$\square$

Recall from Definition 1.5.4 that the join of two graphs G_2 and G_1, denoted by $G_2 \vee G_1$ is obtained by connecting every vertex in graph G_2 with an edge to all vertices in graph G_1.

It is known that $P(G_2 \vee G_1; t)$ can be given in terms of the chromatic polynomials $P(G_2; t)$ and $P(G_1; t)$ using the Umbral product. To perform the umbral product, we start by computing the chromatic polynomials $P(G_2; t)$ and $P(G_1; t)$ using the Fundamental Theorem 2.3.5 until we get chromatic polynomials of complete graphs. Re-writing these chromatic polynomials in factorial form, the umbral product of $P(G_2; t)$ and $P(G_1; t)$, denoted $P(G_2; t) \otimes P(G_1; t)$ is given by multiplying the polynomials given in factorial form where $(t)_r \otimes (t)_s = (t)_{r+s}$.

Theorem 2.4.3. *Let G_2 and G_1 be any two graphs with $P(G_2; t)$ and $P(G_1; t)$ expressed in factorial form. Then the chromatic polynomial of the join $G_2 \vee G_1$,*

$$P(G_2 \vee G_1; t) = P(G_2; t) \otimes P(G_1; t).$$

Example 2.4.4. In this example we show how to find the chromatic polynomial of the join of graphs G_2 and G_1. The diagram in Figure 2.5 gives the graphs G_2 and G_1 and show how to get the join of G_2 and G_1. The dotted lines represent the edges connecting G_2 and G_1. The Chromatic polynomial of G_1 in factorial form is $(t)_3 + (t)_2$ and the Chromatic polynomial of $G_2 = K_2$ is $(t)_2$.

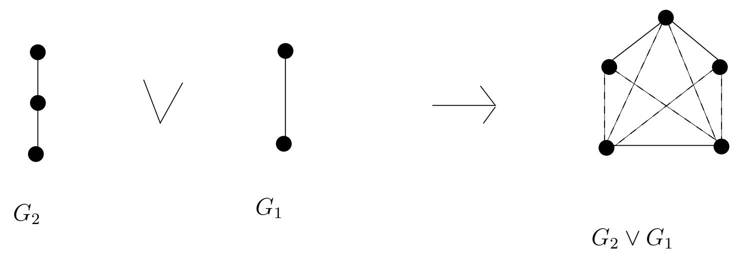

Figure 2.5: Graph $G_2 \vee G_1$.

Hence, by applying Theorem 2.4.3,

$$\begin{aligned} P(G_2 \vee G_1; t) &= P(G_2; t) \otimes P(G_1; t) \\ &= ((t)_3 + (t)_2)(t)_2 = (t)_5 + (t)_4 \\ &= t(t-1)(t-2)(t-3)(t-4) + t(t-1)(t-2)(t-3) \\ &= t(t-1)(t-2)(t-3)^2 \\ &= t^5 - 9t^4 + 29t^3 - 39t^2 + 18t. \end{aligned}$$

In this work, we are interested in derived graphs obtained from the special join, given in Definition 1.5.7 called the vertex join of a graph G denoted by $\widehat{G}$. In short, $\widehat{G}$ is the join of G and K_1. The chromatic polynomial of the derived graph $\widehat{G}$ is given in terms of the chromatic polynomial in any form, which makes the chromatic polynomial more attractive and simple. □

Theorem 2.4.5. *Let $\widehat{G}$ be the vertex join of graph G. Then*

$$P(\widehat{G}; t) = tP(G; t-1).$$

Example 2.4.6. In this example, we demonstrate how to find the chromatic polynomial of the vertex join of G. The graphs in the diagram of Figure 2.6 are graph G and the derived graph $\widehat{G}$.

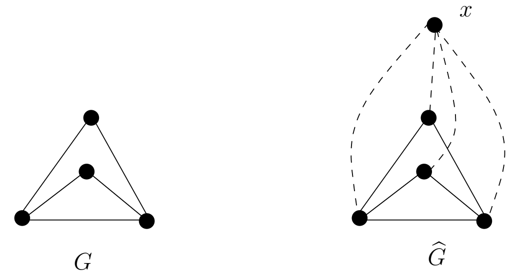

Figure 2.6: Graph G and graph $\widehat{G}$.

By applying Theorem 2.4.5 and recalling the chromatic polynomial of the graph G, $P(G;t) = t^4 - 5t^3 + 8t^2 - 4t$ from Example 2.3.11, we get;

$$\begin{aligned} P(\widehat{G};t) &= tP(G;t-1) \\ &= t[(t-1)^4 - 5(t-1)^3 + 8(t-1)^2 - 4(t-1)] \\ &= t[t^4 - 9t^3 + 29t^2 - 39t + 18] \\ &= t^5 - 9t^4 + 29t^3 - 39t^2 + 18t. \end{aligned}$$

We recall from Definition 1.5.16, that the operation of a K_r-gluing of two graphs G_2 and G_1, $G_1 \underset{K_r}{\cup} G_2$, is obtained by identifying K_r in both G_2 and G_1. Then the chromatic polynomial of the derived graph $G_1 \underset{K_r}{\cup} G_2$ can be given in terms of the chromatic polynomials $P(G_2;t)$ and $P(G_1;t)$.

Theorem 2.4.7. *Let G_1 and G_2 be two graphs, both having K_r as a subgraph. Let G be the derived graph obtained by the K_r-gluing of G_1 and G_2. The chromatic polynomial of G,*

$$P(G;t) = \frac{P(G_1;t)P(G_2;t)}{P(K_r;t)}.$$

Example 2.4.8. In this example, we illustrate how to find the chromatic polynomial of a derived graph from a K_r-gluing. The graphs in the diagram of Figure 2.7 are G_2 and G_1, both having K_2 as a subgraph and the graph G which is obtained by the K_2-gluing of G_2 and G_1. □

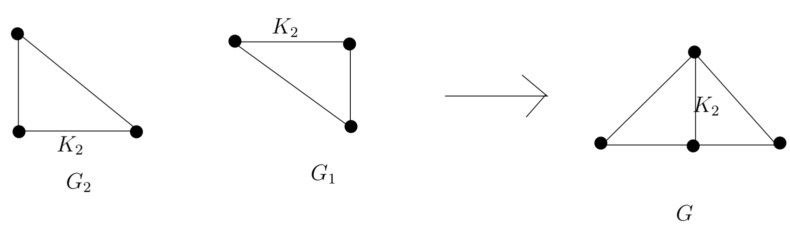

Figure 2.7: Graph $G_1 \underset{K_2}{\cup} G_2$.

By applying Theorem 2.4.7 and Theorem 2.3.7,

$$\begin{aligned} P(G;t) &= \frac{P(K_3;t)P(K_3;t)}{P(K_2;t)} \\ &= \frac{t(t-1)(t-2)t(t-1)(t-2)}{t(t-1)} \\ &= t(t-1)(t-2)^2 \\ &= t^4 - 5t^3 + 8t^2 - 4t. \end{aligned}$$

□

2.5 Chromatic polynomial of some classes of graphs

In this section, we discuss explicit expressions of the chromatic polynomials of some of the classes of graphs defined in Section 1.4. We refer the reader to [1, 8], unless otherwise stated.

Theorem 2.5.1. *Let T_q be any tree of order q. The chromatic polynomial,*

$$P(T_q;t) = t(t-1)^{q-1}$$

where $q \in \mathbb{N}$.

Proof. We prove the formula by induction on order of the tree T_q.

If a tree has order 2, then $T_2 \cong K_2$. Thus the chromatic polynomial $P(T_2;t) = P(K_2;t) = t(t-1)$, by Theorem 2.3.7. Hence, rewriting

$$P(T_2;t) = t(t-1)^1 = t(t-1)^{2-1}.$$

We assume that the formula is true for every tree of order less than q.

We now consider any tree of order q. By Corollary 2.3.8, we get

$$P(T_q;t) \;=\; P(T_q\backslash e) - P(T_q/e;t). \tag{2.1}$$

We consider the chromatic polynomials of the two parts of Equation 2.1 separately.

Let $e \in E(T_q)$ and consider the minor $T_q\backslash e$. Since every edge of T_q is a bridge by Proposition 1.4.10 (v) of a tree, then $T_q\backslash e$ has two components by Definition 1.3.8 of a bridge. It is clear that the two components of $T_q\backslash e$ are also trees, say T_o and T_p where both $o,\ p < q$. But $o+p=q$ since deletion of an edge does not affect the number of vertices. By induction assumption, $P(T_o;t) = t(t-1)^{o-1}$ and $P(T_p;t) = t(t-1)^{p-1}$. Thus,

$$\begin{aligned}
P(T_q\backslash e;t) &= P(T_o;t)P(T_p;t) \text{ by Corollary 2.3.9 (iv)} \\
&= [t(t-1)^{o-1}][t(t-1)^{p-1} \\
&= t^2(t-1)^{o-1+p-1} \\
&= t^2(t-1)^{o+p-2} \\
&= t^2(t-1)^{q-2} \text{ since } o+p=q. \qquad (2.2)
\end{aligned}$$

Let $e \in E(T_q)$ and consider the minor T_q/e. Since every edge of T_q is a bridge by Proposition 1.4.10(v) of a tree, then $T_q/e \cong T_{q-1}$. Hence, by induction assumption,

$$\begin{aligned}
P(T_q/e;t) &= P(T_{q-1};t) \\
&= t(t-1)^{q-2}. \qquad (2.3)
\end{aligned}$$

Now substituting Equation 2.2 and Equation 2.3 in Equation 2.1, we get

$$\begin{aligned} P(T_q;t) &= P(T_q \backslash e) - P(T_q/e;t) \\ &= t^2(t-1)^{q-2} - t(t-1)^{q-2} \\ &= t(t-1)^{q-2}[t-1] \\ &= t(t-1)^{q-1}. \end{aligned}$$

Thus the formula is true for a tree of order q. $\square$

We now state the explicit expression of the chromatic polynomial for cycle graphs without proof, since the proof is similar to the proof of Theorem 2.5.1.

Theorem 2.5.2. *Let C_q be any cycle graph of order $q \in \mathbb{N}$. The chromatic polynomial,*

$$P(C_q;t) = (t-1)^q + (-1)^q(t-1).$$

Theorem 2.5.3. *Let W_q be a wheel graph of order q. The chromatic polynomial,*

$$P(W_q;t) = t[(t-2)^{q-1} + (-1)^{q-1}(t-2)].$$

Proof. By Definition 1.5.8, W_q is a vertex join of graphs C_{q-1} and K_1. Hence, by

Theorem 2.4.5, we have

$$\begin{aligned} P(W_q;t) &= tP(C_{q-1};t-1) \\ &= t[((t-1)-1)^{q-1} + (-1)^{q-1}((t-1)-1)] \\ &= t[(t-2)^{q-1} + (-1)^{q-1}(t-2)]. \end{aligned}$$

□

2.6 Different forms of the chromatic polynomial

In this section we discuss different ways of presenting the chromatic polynomials of graphs. We start by presenting the chromatic polynomial as a summation of the chromatic polynomials of null graphs, then complete graphs, tree graphs and finally cycle graphs. We use the graph G from Example 2.3.11 to compute $P(G;t)$ and verify that we get the same polynomial after expansion. We refer the reader to [4, 9, 12, 13], unless otherwise stated.

2.6.1 Null graph form or power form

We discussed the chromatic polynomial of a graph in null graph form in Section 2.3. Note that the null graph form is also known as the power form of the chromatic polynomial.

Definition 2.6.1. Let G be a graph of order q and $P(G;t)$ the chromatic polynomial of G. If the chromatic polynomial of G is expressed as $P(G;t) = \sum_{i=1}^{q} s_i P(N_i;t) = \sum_{i=1}^{q} s_i t^i$ where $s_i \in \mathbb{Z}$, then we say $P(G;t)$ is in *null graph form* or *power form.*

An example of the chromatic polynomial of a graph G in null graph form or power form, see Example 2.3.11 is given by

$$\begin{aligned} P(G;t) &= P(N_4;t) - 5P(N_3;t) + 8P(N_2;t) - 4P(N_1;t) \\ &= t^4 - 5t^3 + 8t^2 - 4t. \end{aligned} \tag{2.4}$$

2.6.2 Complete graph form or factorial form

We discussed the chromatic polynomial of a graph in complete graph form in Section 2.3. Note that the complete graph form is also known as the factorial form of the chromatic polynomial.

Definition 2.6.2. Let G be a graph of order q and $P(G;t)$ the chromatic polynomial of G. If the chromatic polynomial of G is expressed as $P(G;t) = \sum_{i=1}^{q} r_i P(K_i;t) = \sum_{i=1}^{q} r_i t_i$ where t_i is the factorial and $r_i \in \mathbb{Z}$, then we say $P(G;t)$ is in *complete graph form* or *factorial form.*

An example of $P(G;t)$ in complete graph form or factorial form, see Example 2.3.6, is given by

$$
\begin{aligned}
P(G;t) &= P(K_4;t) + P(K_3;t) \\
&= (t)_4 + (t)_3 \\
&= t(t-1)(t-2)(t-3) + t(t-1)(t-2) \qquad (2.5)
\end{aligned}
$$

Now expanding Equation 2.5 we get

$$
\begin{aligned}
P(G;t) &= t(t-1)(t-2)[(t-3)+1] \\
&= t(t-1)(t-2)^2 \\
&= t^4 - 5t^3 + 8t^2 - 4t.
\end{aligned}
$$

This is equal to the Formula given in Equation 2.4, thus verifying that the factorial form of G is correct.

2.6.3 Tree graph form

We now define the chromatic polynomial of graph G in tree form.

Definition 2.6.3. Let G be a graph of order q and $P(G;t)$ the chromatic polynomial of G. If the chromatic polynomial of G is expressed as $P(G;t) = \sum_{i=1}^{q} x_i P(T_i;t) = \sum_{i=1}^{q} x_i[t(t-1)^{i-1}]$ where $x_i \in \mathbb{Z}$, then we say $P(G;t)$ is in *tree form.*

Example 2.6.4. Let G be the graph given in Example 2.3.11. We apply recursively the deletion and contraction method stated in Corollary 2.3.8 to G. We demonstrate how to achieve the tree form, see the diagram in Figure 2.8.

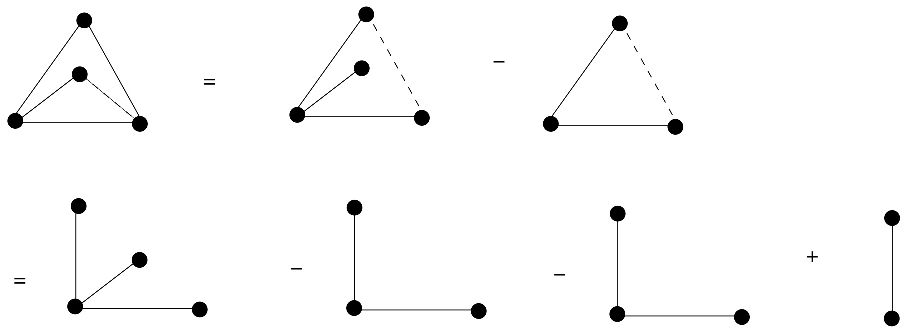

Figure 2.8: $P(G;t)$ in tree graph form.

Hence, the chromatic polynomial of G in tree graph form is given by,

$$
\begin{aligned}
P(G;t) &= P(T_4;t) - 2P(T_3;t) + P(T_2;t) \\
&= t(t-1)^3 - 2t(t-1)^2 + t(t-1). \qquad (2.6)
\end{aligned}
$$

Expanding Equation 2.6 we get

$$
\begin{aligned}
P(G;t) &= (t^4 - 3t^3 + 3t^2 - t) - (2t^3 - 4t^2 + 2t) + (t^2 - t) \\
&= t^4 - 5t^3 + 8t^2 - 4t,
\end{aligned}
$$

verifying that the chromatic polynomial of graph G in tree graph form is equal to Equation 2.4 for the chromatic polynomial of graph G in null graph form. $\square$

2.6.4 Cycle graph form

We now define the chromatic polynomial of graph G in cycle graph form.

Definition 2.6.5. Let G be a graph of order q and $P(G;t)$ the chromatic polynomial of G. If the chromatic polynomial of G is expressed as $P(G;t) = \sum_{i=1}^{q} y_i P(C_i;t) = \sum_{i=1}^{q} y_i[(t-1)^i + (-1)^i(t-1)]$ where $y_i \in \mathbb{Z}$, then we say $P(G;t)$ is in *cycle graph form.*

Example 2.6.6. Let G be the graph given in Example 2.3.11. We apply recursively the deletion and contraction method stated in Corollary 2.3.8 and the Fundamental Theorem 2.3.5 to the graph G. We demonstrate how to achieve the cycle graph form, see the diagram in Figure 2.9

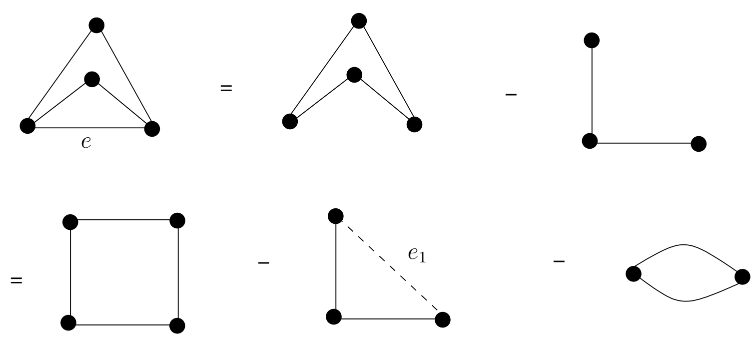

Figure 2.9: $P(G;t)$ in cycle graph form.

Hence, the chromatic polynomial of G in cycle graph form is given by,

$$\begin{aligned} P(G;t) &= P(C_4;t) - P(C_3;t) + P(C_2;t) \\ &= \left[t-1)^4 + (-1)^4(t-1)\right] - \left[t-1)^3 + (-1)^3(t-1)\right] \\ &+ \left[(t-1)^2 + (-1)^2(t-1)\right]. \end{aligned} \tag{2.7}$$

Expanding Equation 2.7 we get

$$\begin{aligned} P(G;t) &= \left[t-1)^4 + (-1)^4(t-1)\right] - \left[t-1)^3 + (-1)^3(t-1)\right] \\ &+ \left[t-1)^2 + (-1)^2(t-1)\right]. \\ &= t^4 - 4t^3 + 6t^2 - 4t + 1 - t^3 + 3t^2 - 3t + 1 + 2t - 2 - t^2 + t \\ &= t^4 - 5t^3 + 8t^2 - 4t, \end{aligned}$$

verifying that the chromatic polynomial of graph G in cycle graph form is equal to Equation 2.4 for $P(G;t)$ in null graph form. □

2.7 Conclusion

This chapter has established a foundation for the work ahead by providing a brief history of the graph coloring problem. The chromatic polynomial of a graph, along with its properties and explicit expressions, was introduced. Recognizing that the chromatic polynomial's definition alone does not aid computation, we explored several methods for calculating the chromatic polynomial that are relevant to this study.

Finally, we presented definitions and examples of four distinct forms in which the chromatic polynomial can be expressed: the null graph form, complete graph form, tree graph form, and cycle graph form.

Chapter 3

Chromatic polynomials of fan graphs

3.1 Introduction

In this chapter, we begin by revisiting the definition of a fan graph which was introduced in Chapter 1. We then present an alternative approach for constructing fan graphs using wheel graphs, offering a different perspective on their structure. Following this, we derive an explicit formula for the chromatic polynomial of a fan graph and introduce a recursive formula to further generalize this polynomial. This leads us to a primary result of the chapter: the chromatic polynomial of a fan graph expressed in cycle form. We also introduce a triangular array that predicts and calculates the coefficients of the chromatic polynomial in this form. To conclude, we provide a step-by-step example using this triangular array, verifying the formulas and demonstrating their practical applications.

3.2 General properties of fan graphs

In this section, we recall the definition of a fan graph given in Chapter 1 and give some properties of this class of graphs. Thereafter, we recall the definition of the wheel graph and show that a fan graph is a minor of a wheel graph, to allow us to borrow some terminology.

Recall Definition 1.4.12 that a *fan* graph, F_q, is a graph with $V(G) = \{u_1, u_2, \ldots, u_{q-1}, x\}$ and $E(G) = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{q-2}, u_{q-1}\}\} \cup \{\{u_1, x\}, \{u_2, x\}, \ldots, \{u_{q-1}, x\}\}$. Alternatively, by Definition 1.5.9, F_q is a vertex join of a path graph P_{q-1} and a complete graph K_1,. Thus $F_q \cong P_{q-1} \vee K_1$ showing that a fan graph is a derived graph.

We now show that a fan graph can be constructed from a wheel graph. We start by recalling from Chapter 1, Definition 1.4.11, that a wheel graph of order q, W_q, is a graph with $V(G) = \{u_1, u_2, u_3, \ldots, u_{q-1}, x\}$ and $E(G) = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{q-2}, u_{q-1}\}, \{u_{q-1}, u_1\}\} \cup \{\{u_1, x\}, \{u_2, x\}, \ldots, \{u_{q-1}, x\}\}$. Vertex x is known as the central vertex and an edge of the form $\{u_i, x\}$ is known as a spoke. Furthermore, the vertex set $V' = \{u_1, u_2, u_3, \ldots, u_{q-1}\}$ and edge set

$E' = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{q-2}, u_{q-1}\}, \{u_{q-1}, u_1\}\}$ is known as the rim of W_q. An edge $e \in E'$ is called a *rim edge*.

It is clear that a fan graph F_q can be constructed from a wheel graph W_q by deleting a single rim edge $e = \{u_{q-1}, u_1\}$. Hence, based on this construction we call an edge of the form $\{u_i, x\}$ a spoke and vertex x is the central vertex in F_q.

We now state some properties of F_q which will be used in this work.

Proposition 3.2.1. *Let F_q be a fan graph of order q. Then,*

(i) the degree sequence is $2,\ 2,\ \underbrace{3,\ 3,\ \ldots\ 3,}_{q-3\ times}\ q-1.$

(ii) The size of F_q is $2q-3$.

(iii) F_q is a connected graph.

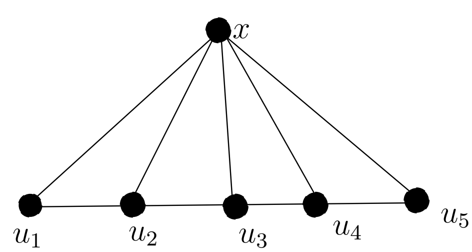

Figure 3.1: A fan graph F_6

Example 3.2.2. In this example, we clarify the concepts of a spoke and central vertex of a fan graph. The diagram in Figure 3.1 is a fan graph F_6 with vertex set $V(F_6) = \{u_1, u_2, u_3, u_4, u_5, x\}$ and edge set $E(F_6) = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_4\}, \{u_4, u_5\}\} \cup \{\{u_1, x\}, \{u_2, x\}, \{u_3, x\}, \{u_4, x\}, \{u_5, x\}\}$.

An edge $e \in \{\{u_1, x\}, \{u_2, x\}, \{u_4, x\}, \{u_5, x\}\}$ in $E(F_6)$ is called a spoke.

3.3 Chromatic polynomial of a fan graph, the explicit expression

In this section, we state one of the basic results of this chapter. We give an explicit expression of the chromatic polynomial of a fan graph.

We recall Definition 1.5.16 in Chapter 1 to define a special K_r-gluing.

Definition 3.3.1. Let G_1 and G_2 be two distinct graphs both having a complete graph K_r as a subgraph. If $r = 1$, then a K_1-gluing is also called a vertex-gluing, or a *bowtie* of G_1 and G_2 denoted by $G_1 \bowtie G_2$.

The minors of F_q constructed in the following lemma will be useful in some of the proofs in this section.

Lemma 3.3.2. *Let F_q be a fan graph with $V(F_q) = \{u_1, u_2, u_3, \ldots, u_{q-1}, x\}$ and $E(F_q) = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{q-2}, u_{q-1}\}\} \cup \{\{u_1, x\}, \{u_2, x\}, \{u_3, x\}, \ldots, \{u_{q-1}, x\}\}$. Let $e = \{u_{q-2}, u_{q-1}\}$. Then,* *(i)* $F_q \backslash e \cong F_{q-1} \bowtie K_2$.

(ii) $F_q/e \cong F_{q-1}$, *up to parallel class.*

Proof. (i) Let $F_q \backslash e$ be a graph obtained by deleting the edge $e = \{u_{q-2}, u_{q-1}\}$. Then $V(F_q \backslash e) = \{u_1, u_2, \ldots, u_{q-1}, x\}$ and

$E(F_q \backslash e) = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{q-3}, u_{q-2}\}\} \cup \{\{u_1, x\}, \{u_2, x\}, \ldots, \{u_{q-1}, x\}\}$.

We know that, $\{u_{q-1}, x\} \cong K_2$. By the definition of a fan graph and K_1-gluing, the subgraph given by $\{V(F_q \backslash e), E(F_q \backslash e)\}$ is isomorphic to a K_1-gluing of K_2 and F_{q-1} on the vertex x. Hence the result,

$$F_n \backslash e \cong F_{n-1} \bowtie K_2.$$

(ii) Let F_q/e be a fan graph with edge $e = \{u_{q-2}, u_{q-1}\}$ contracted. Then vertices u_{q-2} and u_{q-1} are identified. Thus $V(F_n/e) = \{u_1, u_2, \ldots, u_{q-2}, x\}$ and

$$\begin{aligned} E(F_n/e) &= \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{n-3}, u_{n-2}\}\} \\ &\cup \{\{u_1, x\}, \{u_2, x\}, \ldots, \{u_{q-2}, x\}, \{u_{q-2}, x\}\}. \end{aligned}$$

Hence, by Definition 1.4.12, $F_q/e \cong F_{q-1}$, up to parallel class.

□

Now that we know the minors obtained by the deletion of an edge and contracting of an edge in a fan graph, we are in a position to state and prove the formula for the explicit expression of the chromatic polynomial of a fan graph.

Proposition 3.3.3. *Let F_q be a fan graph of order q, then the chromatic polynomial of F_q,*

$$P(F_q;t) = t(t-1)(t-2)^{q-2}.$$

Proof. By Definition 1.5.9, the fan graph of order q, F_q, is the derived graph obtained from a vertex join of path P_{q-1} and the complete graph K_1. Thus, $F_q \cong P_{q-1} \vee K_1$ which is equal to $F_q \cong \widehat{P_{q-1}}$. By Proposition 2.4.5, the chromatic polynomial,

$$P(\widehat{P_{q-1}};t) \quad = \quad tP(P_{q-1};t-1). \tag{3.1}$$

By Proposition 2.5.1 the chromatic polynomial,

$$P(P_{q-1};t) = t(t-1)^{q-2}. \tag{3.2}$$

Substituting Equation 3.2 into Equation 3.1, we get $P(F_q;t) = t(t-1)(t-2)^{q-2}$. $\square$

Example 3.3.4. In this example, we compute the chromatic polynomial of F_5 in tree form to verify Proposition 3.3.3.

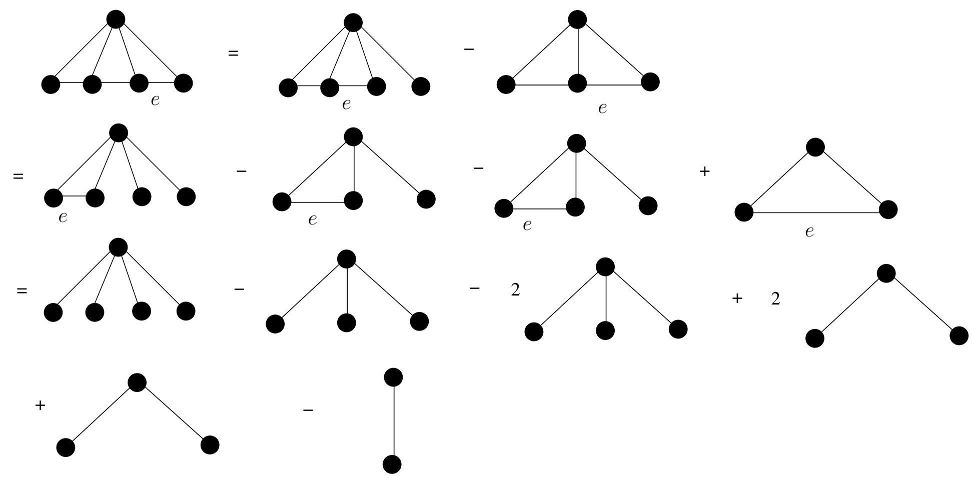

Figure 3.2: Chromatic polynomial of F_5 in tree form.

Hence,

$$
\begin{aligned}
P(F_5;t) &= P(T_5;t) - 3P(T_4;t) + 3P(T_3;t) - P(T_2;t) \\
&= t(t-1)^4 - 3t(t-1)^3 + 3t(t-1)^2 - t(t-1) \text{ by Theorem 2.5.1(3.3)}
\end{aligned}
$$

Factorizing Equation 3.3, we get

$$
\begin{aligned}
P(F_5;t) &= t(t-1)[t^3 - 3t^2 + 3t - 1 - 3t^2 + 6t - 3 + 3t - 3 - 1] \\
&= t(t-1)[t^3 - 6t^2 + 12t - 8] \\
&= t(t-1)(t-2)^3.
\end{aligned}
$$

□

3.4 A recursive formula for the chromatic polynomial of a fan graph

In this section, we state and prove a recursive formula for the chromatic polynomial of a fan graph. This is an alternative way of computing the chromatic polynomial of a fan graph.

Proposition 3.4.1. *The chromatic polynomial of a fan graph F_q, can be expressed recursively as*

$$P(F_q;t) = (t-2)P(F_{q-1};t).$$

Proof. Let F_q be a fan graph with $V(F_q) = \{u_1, u_2, u_3, \ldots, u_{q-1}, x\}$ and $E(F_q) = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{q-2}, u_{q-1}\}\} \cup \{\{u_1, x\}, \{u_2, x\}, \ldots, \{u_{q-1}, x\}\}$. Let $e = \{u_{q-2}, u_{q-1}\}$. Then $F_q \backslash e \cong F_{q-1} \bowtie K_2$ by Lemma 3.3.2 and has a bridge $\{u_{q-1}, x\}$. By Corollary 2.3.9, the chromatic polynomial of $F_q \backslash e$ is,

$$P(F_q \backslash e;t) = (t-1)P(F_{q-1};t).$$

By Lemma 3.3.2, $F_q/e \cong F_{q-1}$, up to parallel class. Then the chromatic polynomial of F_q/e, $P(F_q/e;t) = P(F_{q-1};t)$. Hence we get,

$$\begin{aligned}
P(F_q;t) &= P(F_q \backslash e;t) - P(F_q/e;t) \text{ by Theorem 2.3.8} \\
&= (t-1)P(F_{q-1};t) - P(F_{q-1};t) \\
&= [(t-1)-1]P(F_{q-1};t) \\
&= (t-2)P(F_{q-1};t).
\end{aligned}$$

□

Example 3.4.2. In this example, we compute the chromatic polynomial of fan graph F_5 recursively to verify Proposition 3.4.1.

$$\begin{aligned}
P(F_5;t) &= (t-2)P(F_4;t) \\
&= (t-2)[(t-2)P(F_3;t)] \\
&= (t-2)(t-2)[t(t-1)(t-2)] \\
&= t(t-1)(t-2)^3.
\end{aligned}$$

□

3.5 Chromatic polynomial of a fan graph in cycle graph form

In this section, we present one of the results of this chapter, the chromatic polynomial of a fan graph in cycle graph form. We demonstrate the computation of the chromatic polynomial of fan graphs F_4 and F_5. Thereafter, we present a triangular array whose entries are coefficients of the chromatic polynomial in cycle graph form. The entries in the rows of this triangular array can be used to predict the coefficients of the chromatic polynomial of the next fan graph in cycle graph form. We conclude the section by verifying the use of the triangular array in predicting the coefficients of the chromatic polynomial of F_5 and F_{10}.

Example 3.5.1. In this example, we illustrates the process of computing the chromatic polynomial of F_q in cycle graph form. We use a combination of the Fundamental Theorem 2.3.5, the addition and contraction of an edge and Corollary 2.3.8, the deletion and contraction of an edge. At each step in the diagram, we indicate an edge e is to be deleted, contracted or added by a dotted line. We repeat this process until we obtain the chromatic polynomial of a fan graph in terms of cycle graphs. Without loss of generality, we use F_4 and F_5.

We start with the computation of $P(F_4; t)$ as shown in the diagram of Figure 3.3.

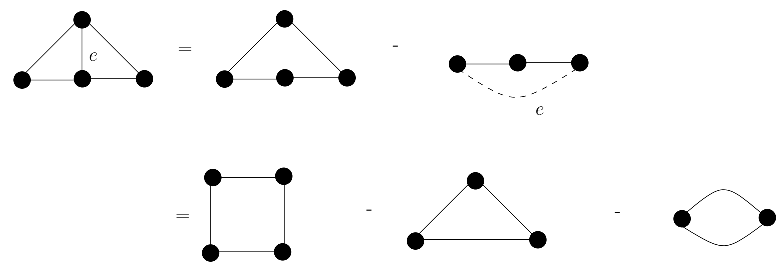

Figure 3.3: Chromatic polynomial of F_4 in cycle form.

Thus we get in cycle graph form,

$$P(F_4;t) \;=\; P(C_4;t) - P(C_3;t) - P(C_2;t). \tag{3.4}$$

Similarly, we compute $P(F_5;t)$ as shown in the diagram of Figure 3.4.

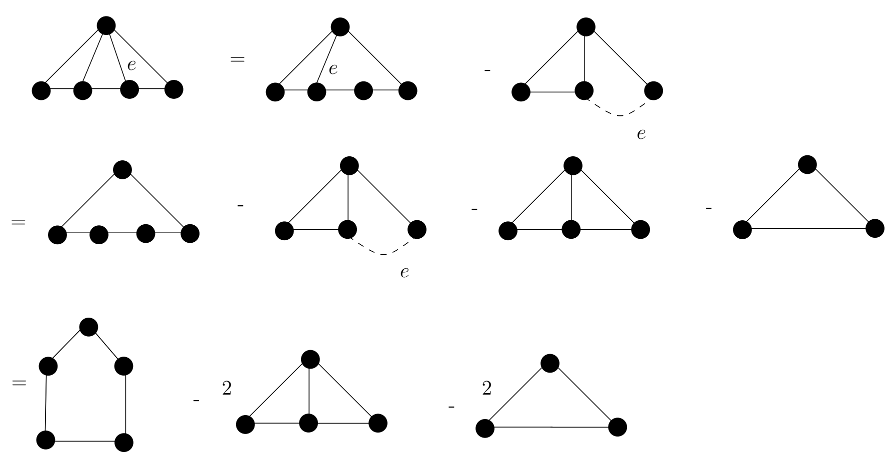

Figure 3.4: Chromatic polynomial of F_5 in cycle graph form.

Thus we get in cycle graph form,

$$
\begin{aligned}
P(F_5;t) &= P(C_5;t) - 2P(F_4;t) - 2P(C_3;t) \\
&= P(C_5;t) - 2[P(C_4;t) - P(C_3;t) - P(C_2;t)] - 2P(C_3;t) \text{ by Equation 3.4} \\
&= P(C_5;t) - 2P(C_4;t) + 2P(C_3;t) + 2P(C_2;t) - 2P(C_3;t) \\
&= P(C_5;t) - 2P(C_4;t) + 0P(C_3;t) + 2P(C_2;t).
\end{aligned}
$$

□

In this research, we collected data by computing the chromatic polynomials for fan graphs in cycle graph form, up to F_{10}. From this data, we observed that the coefficients in cycle graph form of a fan graph form a pattern, which can be arranged in a triangular array presented in the following Table 3.1.

	$P(C_{10};t)$	$P(C_9;t)$	$P(C_8;t)$	$P(C_7;t)$	$P(C_6;t)$	$P(C_5;t)$	$P(C_4;t)$	$P(C_3;t)$	$P(C_2;t)$
$P(F_2;t)$									1
$P(F_3;t)$								1	0
$P(F_4;t)$							1	-1	-1
$P(F_5;t)$						1	-2	0	2
$P(F_6;t)$					1	-3	2	2	-3
$P(F_7;t)$				1	-4	5	0	-5	4
$P(F_8;t)$			1	-5	9	-5	-5	9	-5
$P(F_9;t)$		1	-6	14	-14	0	14	-14	6
$P(F_{10};t)$	1	-7	20	-28	14	14	-28	20	-7

Table 3.1: Coefficients of the chromatic polynomials in cycle graph form.

We can easily verify the first 5 rows of the triangular array. Note that in the procedure of computing the chromatic polynomial in cycle graph form we stop reducing when we get to C_2, since C_1 is a loop and $P(C_1;t) = 0$. We start with Row 1, which is representing $P(F_2;t)$ in cycle graph form.

- By the definition of a fan graph, we know that $F_2 = K_2$ thus $F_2 \cong C_2$ up to parallel class implying that $P(F_2;t) = P(C_2;t)$ by Corollary 2.3.9 (v). Thus $P(F_2;t)$ in cycle graph form has one term $P(C_2;t)$, with a coefficient of 1, corresponding to Row 1 in the triangular array.

- By the definition of a fan graph, we know that $F_3 = C_3$ implying that $P(F_3;t) = P(C_3;t) + 0P(C_2;t)$. Thus $P(F_3;t)$ in cycle graph form has two terms, $P(C_3;t)$ and $P(C_2;t)$ with coefficients 1 and 0 respectively, corresponding to row 2 in the triangular array.

- From Example 3.5.1, the chromatic polynomial of F_4 in cycle graph form has three terms, $P(C_4;t)$, $P(C_3;t)$ and $P(C_2;t)$ with coefficients 1, -1, -1 respectively, corresponding to row 3 in the triangular array.

- From Example 3.5.1, the chromatic polynomial of F_5 in cycle graph form has four terms, $P(C_5;t)$, $P(C_4;t)$, $P(C_3;t)$ and $P(C_2;t)$ with coefficients 1, -2, 0, 2, respectively, corresponding to row 4 in the triangular array.

Since we collected data for the coefficients up to row 10 of Table 3.1, we make the following claims without proofs.

Claim 3.5.1. *Let the triangular array be given by Table 3.1 and let $P(F_q;t)$ be the chromatic polynomial in cycle graph form. Then*

(i) Row q in the triangular array, represent the coefficients of $P(F_{q+1};t)$.

(ii) The first entry in each Row is 1.

(iii) Row $q+1$ has one entry more than Row q.

(iv) Coefficients of $P(F_q;t)$ do not alternate is sign.

The pattern in the triangular array enables us to systematically predict the entries of Row q using the entries in the previous row, to be specific, Row $q-1$. Hence, we can predict coefficients of the chromatic polynomials for larger fan graphs.

Claim 3.5.2. *Let $\{a_1, a_2, \ldots, a_k\}$ represent the entries of the k-th row in Table 3.1, where a_i denotes the i-th entry in the row. Let the entries of the $k+1$-th row be given by $\{b_1, b_2, \ldots, b_{k+1}\}$. Then*

(i) $b_1 = 1$.

(ii) $b_{i+1} = -a_i + a_{i+1}$ for $i = 2, 3, \ldots, k-1$.

(iii) The last coefficient $b_{k+1} = -a_k \pm a_1$, where the sign depends on the sign of a_k, if a_k is positive, subtract a_1, if a_k is negative, add a_1.

Example 3.5.2. In this example, we illustrate the claims by determining the coefficients $\{b_1, b_2, b_3, b_4\}$ of $P(F_5;t)$ corresponding to Row 5, by using the the coefficients

$\{a_1, a_2,\ a_3\}$ of $P(F_4;t)$ corresponding to Row 4 in the triangular array, Table 3.1. Thus the coefficients of $P(F_4;t)$ are $a_1 = 1,\ a_2 = -1,\ a_3 = -1$. By Claim 3.5.1 and Claim 3.5.2, we get

(i) $b_1 = 1$.

(ii) Using $b_{i+1} = -a_i + a_{i+1}$. we get $b_2 = -a_1 + a_2 = -1 + -1 = -2$.

(iii) Using $b_{i+1} = -a_i + a_{i+1}$. We get $b_3 = -a_2 + a_3 = -(-1) + (-1) = 0$.

(iv) Using $b_{k+1} = -a_k \pm a_1$. We get $b_4 = -a_3 + a_1 = -(-1) + 1 = 2$.

$\square$

Example 3.5.3. In this example, we compute the chromatic polynomial of F_{10}, which corresponds to Row 9 of the triangular array, Table 3.1. Thus the predicted chromatic polynomial of F_{10} in cycle graph form is given by:

$$\begin{aligned} P(F_{10};t) &= P(C_{10};t) - 7P(C_9;t) + 20P(C_8;t) - 28P(C_7;t) + 14P(C_6;t) \\ &+ 14P(C_5;t) - 28P(C_4;t) + 20P(C_3;t) - 7P(C_2;t) \end{aligned} \tag{3.5}$$

Expanding Equation 3.5 to power form, we get

$$\begin{aligned}
P(F_{10};t) &= [(t-1)^{10}+(t-1)]-7[(t-1)^9-(t-1)]+20[(t-1)^8+(t-1)] \\
&- 28[(t-1)^7-(t-1)]+14[(t-1)^6+(t-1)]+14[(t-1)^5-(t-1)] \\
&- 28[(t-1)^4+(t-1)]+20[(t-1)^3-(t-1)]-7[(t-1)^2+(t-1)] \\
&= t^{10}-17t^9+128t^8-560t^7+1568t^6-2912t^5+3584t^4 \\
&- 2816t^3+1280t^2-256t. \qquad (3.6)
\end{aligned}$$

To verify, we now use the explicit expression stated in Proposition 3.3.3, to compute $P(F_{10};t)$ in power form.

$$\begin{aligned}
P(F_{10};t) &= t(t-1)(t-2)^{10-2} \\
&= t(t-1)(t-2)^8 \\
&= t^{10}-17t^9+128t^8-560t^7+1568t^6-2912t^5+3584t^4 \\
&- 2816t^3+1280t^2-256t,
\end{aligned}$$

verifying our result stated in Equation 3.6. □

3.6 Conclusion

In this chapter, we explored key properties of a fan graph and presented foundational results, including an explicit expression and a recursive formula for $P(F_q;t)$. The primary result was a method to predict the coefficients of the chromatic polynomial of a fan graph when represented in cycle graph form, utilizing an array. Although we did not provide a formal proof of this computational procedure, we verified the accuracy through examples. Establishing a complete proof remains an open problem for further research.

Chapter 4

Chromatic polynomial of a wheel graph

4.1 Introduction

In this chapter, we express the chromatic polynomial of a wheel graph in fan form. We then use the chromatic polynomial of a fan graph in cycle form to compute the chromatic polynomial of a wheel graph in cycle graph form. We generate a table consisting of the coefficients of the chromatic polynomial of a fan graph in cycle form, found in Section 3.5, to compute the chromatic polynomial of a wheel graph in cycle form. We use our array to calculate the coefficients of W_{10} in cycle form and verify that this does give the chromatic polynomial of W_{10}.

4.2 General properties of wheel graphs

In this section, we recall the definitions of a wheel graph given in Chapter 1 and give some properties of this class of graphs.

Recall from Chapter 1, Definition 1.4.11, that a wheel graph of order q, W_q, is a graph with $V(G) = \{u_1, u_2, u_3, \ldots, u_{q-1}, x\}$ and

$E(G) = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{q-2}, u_{q-1}\}, \{u_{q-1}, u_1\}\} \cup \{\{u_1, x\}, \{u_2, x\}, \ldots, \{u_{q-1}, x\}\}$.

The vertex x is known as the central vertex, an edge of the form $\{u_i, x\}$ is known as a spoke and an edge of the form $\{u_i, u_{i+1}\}$ is called a rim edge.

Wheel graphs is one of the well known and widely studied classes of graphs. We now state some properties of W_q.

Proposition 4.2.1. *Let W_q be a wheel graph of order $q \geq 4$ Then,*

(i) the degree sequence is $\underbrace{3,\ 3,\ \cdots\ 3,}_{q-1 \text{ times}}\ q - 1$.

(ii) The size of W_q is $2(q - 1)$.

(iii) W_q is a connected graph.

The minors of W_q constructed in the following lemma will be useful in some of the proofs in this section.

Lemma 4.2.2. *Let W_q be a wheel graph with $V(W_q) = \{u_1, u_2, u_3, \ldots, u_{q-1}, x\}$ and*

$E(W_q) = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{q-2}, u_{n-1}\}, \{u_{q-1}, u_1\}\} \cup \{\{u_1, x\}, \{u_2, x\}, \ldots, \{u_{q-1}, x\}\}$.

Let $e = \{u_{q-1}, u_1\}$ be a rim edge of W_q. Then,

(i) $W_q \backslash e \cong F_q$.

(ii) $W_q/e \cong W_{q-1}$*, up to parallel class.*

Proof. (i) Let $G_1 = W_q \backslash e$ be a minor obtained from W_q by deleting the rim edge $e = \{u_{q-1}, u_1\}$. Then, G_1 has vertex set $V(G_1) = \{u_1, u_2, \ldots, u_{q-1}, x\}$ and edge set

$$E(G_1) = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{q-2}, u_{q-1}\}\} \cup \{\{u_1, x\}, \{u_2, x\}, \ldots, \{u_{q-1}, x\}\}.$$

Hence, by Definition 1.4.12, $G_1 = W_q \backslash e \cong F_q$.

(ii) Let $G_2 = W_q/e$ be a minor obtained from W_q by contracting the rim edge $e = \{u_{q-1}, u_1\}$. Thus, vertex u_{q-1} is identified with vertex u_1.

Then, $V(G_2) = \{u_1, u_2, \ldots, u_{q-2}, x\}$ and $E(G_2) = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{q-2}, u_1\}\} \cup \{\{u_1, x\}, \{u_2, x\}, \ldots, \{u_{q-2}, x\}, \{u_1, x\}\}$. Thus, $\{u_1, x\}$ is a parallel class. Hence if we simplify G_2 we get $W_q/e \cong W_{q-1}$, up to parallel class.

□

Corollary 4.2.3. *The chromatic polynomial,* $P(W_q \backslash e; t) = P(F_q; t)$.

Corollary 4.2.4. *The chromatic polynomial,* $P(W_q/e; t) = P(W_{q-1}; t)$.

4.3 Chromatic polynomial of a wheel graph in fan graph form

In this section, we state and prove one of the results of this chapter, the chromatic polynomial of a wheel graph in fan graph form. We verify the result with an example.

In Chapter 2, Section 2.6, we discussed the chromatic polynomial as a summation of the chromatic polynomials of null graphs, complete graphs, tree graphs and cycle graphs, which we found in the literature. A fan graph has a very simple explicit formula of the chromatic polynomial, see Proposition 3.3.3. Hence, we introduce another form of representing the chromatic polynomial, the fan graph form.

Definition 4.3.1. Let G be a graph of order q and $P(G;t)$ the chromatic polynomial of G. If the chromatic polynomial of G is expressed as $P(G;t) = \sum_{i=1}^{q} h_i P(F_i;t) = \sum_{i=1}^{q} h_i \left[t(t-1)(t-2)^{q-2}\right]$ where $h_i \in \mathbb{Z}$, then we say $P(G;t)$ is in *fan graph form.*

The chromatic polynomial of a wheel graph is well known in the literature, we recall the following Theorem 2.5.3 from Chapter 2. Expansion of this formula gives the chromatic polynomial of a wheel graph in null graph form.

Theorem 4.3.2. *Let W_q be a wheel graph of order q. Then, the chromatic polynomial, $P(W_q;t) = t[(t-2)^{q-1} + (-1)^{q-1}(t-2)]$.*

We are now in a position to give one of the results of this chapter, the chromatic polynomial of a wheel graph in fan graph form, Recall Definition 1.5.9, that F_q is a vertex join of a path graph P_{q-1}.

Proposition 4.3.3. *Let W_q be a wheel graph of order $q \geq 3$. Then,*

$$\begin{aligned} P(W_q;t) &= P(F_q;t) - P(F_{q-1};t) + P(F_{q-2};t)\ldots + (-1)^{q-3}P(F_3;t) \\ &= \sum_{i=3}^{q}(-1)^{q-i}P(F_i;t). \end{aligned}$$

Proof. Let W_q be a wheel graph of order q and let e be a rim edge, then by repeated application of Lemma 4.2.2, once we get a minor which is isomorphic to a wheel graph, up to parallel class, we get,

$$\begin{aligned} P(W_q;t) &= P(W_q \backslash e;t) - P(W_q/e;t) \\ &= P(F_q;t) - P(W_{q-1};t) \\ &= P(F_q;t) - [P(F_{q-1};t) - P(W_{q-2};t)] \\ &= P(F_q;t) - P(F_{q-1};t) + [P(F_{q-2};t) - P(W_{q-3};t)] \\ &\vdots \\ &= P(F_q;t) - P(F_{q-1};t) + P(F_{q-2};t)\ldots + (-1)^{q-3}P(F_3;t) \end{aligned}$$

□

Example 4.3.4. In this example, we illustrate how to compute the chromatic polynomial of W_6 in fan graph form. In the diagram of Figure 4.1, at each step, we label the rim edge to be deleted and contracted in the next step.

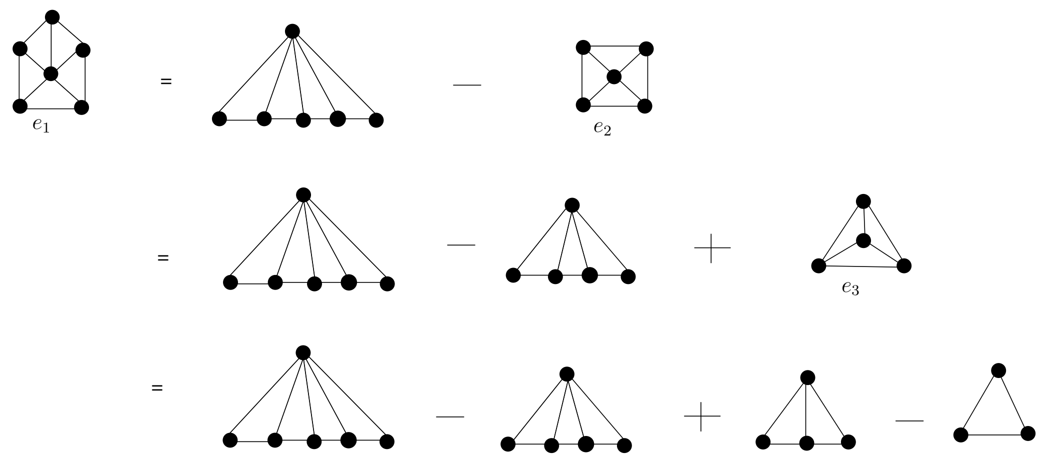

Figure 4.1: $P(W_6;t)$ in fan graph form

Thus we get in fan graph form, the chromatic polynomial,

$$\begin{aligned} P(W_6;t) &= P(F_6;t) - P(F_5;t) + P(F_4;t) - P(F_3;t) \\ &= t(t-1)(t-2)^4 - t(t-1)(t-2)^3 \\ &+ t(t-1)(t-2)^2 - t(t-1)(t-2). \end{aligned} \tag{4.1}$$

Expanding the Formula in Equation 4.1, we get

$$\begin{aligned} P(W_6;t) &= t(t-1)(t-2)^4 - t(t-1)(t-2)^3 + t(t-1)(t-2)^2 - t(t-1)(t-2) \\ &= t^6 - 10t^5 + 40t^4 - 80t^3 + 79t^2 - 30t. \end{aligned}$$

We now verify this chromatic polynomial using the null form. By Theorem 4.3.2,

$$\begin{aligned} P(W_6;t) &= t[(t-2)^5 + (-1)^5(t-2)] \\ &= t^6 - 10t^5 + 40t^4 - 80t^3 + 79t^2 - 30t. \end{aligned}$$

□

4.4 Chromatic polynomial of a wheel graph in cycle form

In this section, we present another result of this chapter. We apply Proposition 4.3.3 together with the triangular array of coefficients of the chromatic polynomials of a fan graph in cycle form as outlined in Section 3.5, to create another new array. This new array will predict coefficients of the chromatic polynomial of a wheel graph in cycle form. We conclude the section with an example verifying our results.

Without loss of generality, we illustrate the creation of and use the new array by computing the chromatic polynomial of W_9 in cycle form.

We know by Proposition 4.3.3, the chromatic polynomial of W_9 in fan form is $P(W_9;t) = P(F_9;t) - P(F_8;t) + \ldots + P(F_3;t)$. Then, we use the triangular array from Section 3.5 for the coefficients of the chromatic polynomial of $P(F_i;t),\ i \in \{3, \ldots, 9\}$. The following Table 4.1 shows all the coefficients to be added in the computation of the chromatic polynomial of W_9.

	$P(C_9;t)$	$P(C_8;t)$	$P(C_7;t)$	$P(C_6;t)$	$P(C_5;t)$	$P(C_4;t)$	$P(C_3;t)$	$P(C_2;t)$
$+P(F_9;t)$	1	-6	14	-14	0	14	-14	6
$-P(F_8;t)$		-1	5	-9	5	5	-9	5
$+P(F_7;t)$			1	-4	5	0	-5	4
$-P(F_6;t)$				-1	3	-2	-2	3
$+P(F_5;t)$					1	-2	0	2
$-P(F_4;t)$						-1	1	1
$+P(F_3;t)$							1	0
$P(W_9;t)$	1	-7	20	-28	14	14	-28	21

Table 4.1: Coefficients of $P(W_9;t)$ in cycle graph form.

From Table 4.1, we get that

$$\begin{aligned}
P(W_9;t) &= P(C_9;t) - 7P(C_8;t) + 20P(C_7;t) - 28P(C_6;t) + 14P(C_5;t) \\
&+ 14P(C_4;t) - 28P(C_3;t) + 21P(C_2;t) \\
&= [(t-1)^9 - (t-1)] - 7[(t-1)^8 + (t-1)] + 20[(t-1)^7 - (t-1)] \\
&- 28[(t-1)^6 + (t-1)] + 14[(t-1)^5 - (t-1)] + 14[(t-1)^4 + (t-1)] \\
&- 28[(t-1)^3 - (t-1)] + 21[(t-1)^2 + (t-1)],
\end{aligned}$$

which, when expanded to null graph form, gives $P(W_9;t) = t^9 - 16t^8 + 112t^7 - 448t^6 + 1120t^5 - 1792t^4 + 1792t^3 - 1023t^2 + 254t$.

We verify this by computing the chromatic polynomial of W_9 using the well known formula given in Theorem 2.5.3 and then expand to null graph form. Thus,

$$\begin{aligned} P(W_9;t) &= t[(t-2)^8 + (-1)^8(t-2)] \\ &= t^9 - 16t^8 + 112t^7 - 448t^6 + 1120t^5 - 1792t^4 + 1792t^3 - 1023t^2 + 254t. \end{aligned}$$

We now give an array for computing the coefficients of $P(W_q;t)$ in cycle graph form using a combination of the triangular array in Table 4.1 and coefficients of the wheel W_{q-1}. Recall the formula that $P(W_q;t) = P(F_q;t) - P(W_{q-1};t)$. To start the new array, we recall that $W_2 \cong C_2$ up to parallel class and $W_3 \cong C_3$, implying $P(F_2;t) = P(C_2;t)$ and $P(F_3;t) = P(C_3;t) + 0P(C_2;t)$, respectively.

	$P(C_9;t)$	$P(C_8;t)$	$P(C_7;t)$	$P(C_6;t)$	$P(C_5;t)$	$P(C_4;t)$	$P(C_3;t)$	$P(C_2;t)$
$P(W_2;t)$								1
$P(W_3;t)$							1	0
$P(W_4;t)$						1	-2	-1
$P(W_5;t)$					1	-3	2	3
$P(W_6;t)$				1	-4	5	0	-6
$P(W_7;t)$			1	-5	9	-5	-5	10
$P(W_8;t)$		1	-6	14	-14	0	14	-15
$P(W_9;t)$	1	-7	20	-28	14	14	-28	21

Table 4.2: The coefficients of $P(W_2;t)$ up to $P(W_9;t)$ in cycle form.

We now predict the next row of the array. We predict $P(W_{10};t)$ using the coefficients of the chromatic polynomial of F_{10} in cycle graph form and the coefficients of the chromatic polynomial of W_9 in cycle graph form.

	$P(C_{10})$	$P(C_9)$	$P(C_8;t)$	$P(C_7;t)$	$P(C_6;t)$	$P(C_5;t)$	$P(C_4;t)$	$P(C_3;t)$	$P(C_2;t)$
$P(F_{10};t)$	1	−7	20	−28	14	14	−28	20	−7
$-P(W_9;t)$		−1	7	−20	28	−14	−14	28	−21
$P(W_{10};t)$	1	−8	27	−48	42	0	−42	48	−28

Table 4.3: The coefficients of $P(W_{10};t)$ in cycle graph form.

Using the predicted coefficients in cycle graph form,

$$\begin{aligned} P(W_{10};t) &= P(C_{10};t) - 8P(C_9;t) + 27P(C_8;t) - 48P(C_7;t) \\ &+ 42P(C_6;t) + 0P(C_5;t) - 42P(C_4;t) + 48P(C_3;t) - 28P(C_2;t) \\ &= [(t-1)^{10} + (t-1)] - 8[(t-1)^9 - (t-1)] + 27[(t-1)^8 + (t-1)] \\ &- 48[(t-1)^7 - (t-1)] + 42[(t-1)^6 + (t-1)] - 42[(t-1)^4 + (t-1)] \\ &+ 48[(t-1)^3 - (t-1)] - 28[(t-1)^2 + (t-1)]. \end{aligned}$$

Expanding, we get

$$\begin{aligned} P(W_{10};t) &= t^{10} - 18t^9 + 144t^8 - 672t^7 + 2016t^6 - 4032t^5 + 5376t^4 \\ &- 4608t^3 + 2303t^2 - 510t. \end{aligned}$$

We verify this by computing the chromatic polynomial of W_{10} using the well known formula given in Theorem 2.5.3 and then expand it to null graph form. Thus

$$\begin{aligned}
P(W_{10};t) &= t[(t-2)^9 + (-1)^9(t-2)] \\
&= t^{10} - 18t^9 + 144t^8 - 672t^7 + 2016t^6 \\
&- 4032t^5 + 5376t^4 - 4608t^3 + 2303t^2 - 510t.
\end{aligned}$$

These arrays or tables representing the chromatic polynomials of fan graphs F_q and wheel graphs W_q in cycle form reveals some patterns that can be used to predict chromatic polynomials of larger graphs in these classes. However, we have not found a way to predict the coefficients in the next row of the chromatic polynomial of a wheel graph when presented in cycle graph form, independent of the fan graph form.

4.5 Conclusion

In this chapter, we gave some well known properties of a wheel graph. Then we gave the chromatic polynomial of a wheel graph in fan graph form. Finally, we showed how to construct an array which can be used to predict coefficients of $P(W_q;t)$ in cycle graph form.

Chapter 5

Conclusion

This dissertation examines the chromatic polynomials of specific graphs derived from graph operations. The derived graphs which have been studied are the wheel graph and the fan graph. These graphs are constructed through the vertex join of a cycle and a tree, respectively. Existing literature presents the chromatic polynomial of these graphs in their null graph forms. However, our study explored the chromatic polynomial of the fan graph in cycle form, revealing that the coefficients in this form produce an array in which each row is calculable from the previous one. Building on this, we developed the fan graph form and applied this form to express the chromatic polynomial of the wheel graph accordingly. This formulation allowed us to generalize our approach, providing a method for predicting the coefficients of the chromatic polynomial of a wheel graph in cycle form.

In conclusion, we reflect on the potential avenues for extending this research. An interesting direction would be to generalize the fan graph form to other well-studied graphs and compare the resulting chromatic polynomial coefficients. This approach raises intriguing questions: for instance, can certain coefficients of the chromatic polynomial of a general graph (of order greater than three) expressed in fan graph form be zero? Additionally, the triangular array identified in this study invites further exploration—can this array be interpreted in a meaningful way within number theory, or does it follow a particular formula? These questions point to rich possibilities for further investigation.

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