



Relaxed Inertial Subgradient Extragradient Algorithm for Solving Equilibrium Problems

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Abstract

We propose a relaxed inertial subgradient extragradient algorithm for solving equilibrium problems in a real Hilbert space. Under the assumption that the associated bivariate function is pseudomonotone and satisfies the Lipschitzness, we establish that the generated sequence of our proposed algorithm converges weakly to the equilibria set of the equilibrium problem. Furthermore, we obtain a linear convergence rate under the assumption that the bifunction is strongly pseudomonotone. We apply our proposed algorithm to variational inequality and fixed point problems. Finally, we compare our method with other schemes in the literature and the improvement brought by our proposed method is evident in the numerical experiments considered in this paper.

Keywords Equilibrium problems · Hilbert space · Self-adaptive stepsize · Relaxed inertial factors · Pseudomonotone · Bifunction · Linear convergence rate · Variational inequality problem · Fixed point problem

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1 Introduction

Let $\mathcal{H}$ denote a real Hilbert space and $\mathcal{C}$, a closed convex nonempty set in $\mathcal{H}$. Suppose $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathbb{R}$ is a bifunction such that $f(s, s) = 0$ for all $s \in \mathcal{C}$. The equilibrium problem (briefly, EP) for the bifunction f is defined as:

$$\text{find } \bar{r} \in \mathcal{C} \text{ such that } f(\bar{r}, z) \geq 0, \quad \forall z \in \mathcal{C}. \quad (1.1)$$

The solutions set of equilibrium problem (1.1) shall be designated as $EP(f, \mathcal{C}) := \{\bar{r} \in \mathcal{C} \mid f(\bar{r}, s) \geq 0 \quad \forall s \in \mathcal{C}\}$.

The term equilibrium problem first appeared in the work of Blum and Oettli (1994) while Muu and Oettli (1992) extended the term. Interestingly, different problems in mathematical sciences, engineering, sciences and even social sciences such as saddle point problems, variational inequality, optimization problems, fixed point problems, Nash equilibrium problems, complementarity problems, and others are all unified by EP(1.1) (example, see Blum and Oettli (1994); Hieu et al. (2020); Konnov (2007); Muu and Oettli (1992) and references quoted therein). Notice that if $f(s, z) = \langle \mathcal{B}s, z - s \rangle$, where $\mathcal{B} : \mathcal{C} \longrightarrow \mathcal{H}$, then EP(1.1) becomes the classical variational inequality of Fichera (1963), [13] and Stampacchia (1964). Thus, the problem is now defined as

$$\begin{cases} \text{find } s^* \in \mathcal{C} \text{ for which} \\ \langle \mathcal{B}s^*, z - s^* \rangle \geq 0 \quad \forall z \in \mathcal{C}, \end{cases} \quad (1.2)$$

and has been studied in Chu and Zhang (2024); Okeke and Bello (2024).

Due to the vast applications of the EP(1.1), and also as it unifies several problems in a simplified and convenient form, it has, therefore, become an interesting part of Optimization and, thus, has received great attentions from several researchers who have developed different effective iterative schemes for finding an equilibrium point (see, for instance Anh and Hai (2019); Flam and Antipin (1997); Hieu (2018); Santos and Scheimberg (2011)). But the method we will be focusing on in this paper is the subgradient extragradient method. Early in the study of EP(1.1), the Proximal Point Method (PPM), proposed by Moudafi (1999), became an effective method of estimating the solution of EP(1.1). The PPM generates the sequence $\{x_n\}$ defined by $x_{n+1} = \mathcal{T}_{f_\gamma}(x_n)$, $n \in \mathbb{N}$, $\gamma > 0$, $x_1 \in \mathcal{C}$, and the map $\mathcal{T}_{f_\gamma} : \mathcal{H} \longrightarrow \mathcal{C}$ is described by

$$\mathcal{T}_{f_\gamma}(x) = \left\{ u \in \mathcal{C} : f(u, w) + \frac{1}{\gamma} \langle w - u, u - x \rangle \geq 0, \quad \forall w \in \mathcal{C} \right\}. \quad (1.3)$$

This method became worrisome as the resolvent operator $\mathcal{T}_{f_\gamma}$ is not always easy to compute, and to overcome this computational ordeal, the extragradient method (EGM) was introduced by Flam and Antipin in Flam and Antipin (1997) as an effective method for solving EP(1.1). This method was further studied and extended by Quoc et al. (2008) to the bifunctions that are pseudomonotone and satisfy the Lipschitz-type condition. The EGM involves the solving of two optimization programs over each

iteration. Thus, this method generates, for solving EP(1.1), the sequences $\{y_n\}$ and $\{x_n\}$ as follows:

$$\begin{cases} y_n = \arg \min_{s \in \mathcal{C}} \left\{ \gamma f(x_n, s) + \frac{\|s - x_n\|^2}{2} \right\}, \\ x_{n+1} = \arg \min_{s \in \mathcal{C}} \left\{ \gamma f(y_n, s) + \frac{\|s - x_n\|^2}{2} \right\}, \end{cases} \text{ with } x_0 \in \mathcal{C} \text{ and } \gamma > 0. \quad (1.4)$$

This method posed another challenge since the computation of the two strong convex programs involved in the method is sometimes almost impossible when the convex set $\mathcal{C}$ has complex structure. Other EGM can be found in Dong et al. (2016); Hieu et al. (2018); Nguyen and Muu (2019); Vinh and Muu (2019) and other references cited therein.

The introduction of new iterative methods and the modification of existing ones help in refining the computational efficiency of existing algorithms. Hence, the extragradient method was modified and another effective method called the subgradient extragradient method (SGEGM) was introduced. In the SGEGM, a suitable half space is constructed and one of the strong convex programs is solved over this half space thereby reducing the cost of computations and improving the overall performance of the methods (see, Cholamjiak and Suparatulatorn (2023); Dadashi et al. (2019); Hieu (2017); Suparatulatorn et al. (2023)).

In recent years, the quest to obtain faster and efficient convergent algorithms for EP(1.1) has been of great interest. It was observed by researchers that the incorporation of the inertial extrapolation term $w_n = x_n + \theta_n(x_n - x_{n-1})$ and the relaxation technique $x_{n+1} = (1 - \alpha_n)w_n + \alpha_n \text{prox}_{\lambda_n g}(\cdot)$ (where $\text{prox}_{\lambda_n g}(\cdot)$ is the proximal point operator over the constructed half space) accelerates the rate of convergence for sequences generated by these algorithms. Algorithms of inertial-like can be traced to the work of Polyak (1964) who studied the second-order dynamic system called the Heavy Ball with Friction (HBF) and through an implicit time discretization of HBF, the inertial extrapolation term occurred naturally.

On this note, Moudafi (2003, Algorithm 1.5) introduced an inertial-proximal method for finding a solution to EP(1.1), thus:

Given the initial iterates x_{n-1} , $x_n \in \mathcal{C}$ and $\theta_n \in [0, 1)$, $\gamma_n > 0$, the next iterates x_{n+1} is generated by solving the following subproblem

$$\begin{cases} f(x_{n+1}, v) + \frac{1}{\gamma_n} \langle x_{n+1} - w_n, v - x_{n+1} \rangle \geq -\xi_n \quad \forall v \in \mathcal{C}, \quad \xi_n \geq 0, \quad \forall n \in \mathbb{N}, \\ w_n = x_n + \theta_n(x_n - x_{n-1}). \end{cases} \quad (1.5)$$

The author established under appropriate conditions on θ_n , γ_n that if $EF(f, \mathcal{C}) \neq \emptyset$, the generated sequence $\{x_n\}$ converges weakly to a point in $EF(f, \mathcal{C})$. Rehman et al. (2020) proposed an inertial subgradient extragradient method, and by incorporating the on-line rule, established a weak convergence of their method under mild conditions. Several other inertial and relaxed methods proposed by different authors who have assumed several conditions on the relaxation parameter α_n and/or the inertial factor

θ_n to ensure convergence of their methods can be found in Attouch and Cabot (2020); Hieu (2018, ?); Izuchukwu et al. (2021); Shehu et al. (2021) and the quoted references therein.

Motivated by the works of Cholakmiak and Suparatulatorn (2023); Rehman et al. (2020); Suparatulatorn et al. (2023) and Vinh and Muu (2019), we present a relaxed inertial subgradient extragradient method for solving EP(1.1). Our method will seek to improve the already existing methods in the literature in the following ways:

- The adaptive stepsize used in our method is easily updated using the previously generated sequences, thereby overcoming the challenge of estimating the Lipschitz constants which are sometimes costly to estimate. Therefore, our method improves the methods that use constant stepsizes or diminishing stepsizes considered in Pham et al. (2021). Also, our method improves the methods that rely on the prior information of the Lipschitz constants.
- Unlike the methods in Cholakmiak and Suparatulatorn (2023); Rehman et al. (2020); Suparatulatorn et al. (2023); Vinh and Muu (2019), our algorithm incorporates the relaxation technique to enhance these methods. This improvement is evident in the numerical experiments we considered in this work.
- Our method does not assume the summability condition $\sum_{i=1}^n \theta_i \|x_i - x_{i-1}\|^2$ directly nor involve the on-line rule in its convergence analysis.
- Under mild conditions, we obtain a weak convergence result and a linear convergence rate of our algorithm; we apply our method to variational inequality and fixed point problems. We carried out numerical experiment to demonstrate the effectiveness of our method.

The remaining of this paper is organized as follows: In Section 2, we collect some basic definitions and technical lemmas for further use. Section 3 presents the relaxed inertial subgradient extragradient algorithm with a self-adaptive stepsize for finding a solution to the equilibrium problems, and analyzes its convergence. In Section 4, we obtain a linear convergence rate result of our proposed method. In Section 5, we present the applications of our results to variational inequality and fixed point problems, while in Section 6, we report some numerical results which help us to determine the effectiveness of our proposed method by comparing it with other already existing methods in the literature.

2 Preliminaries

This section lays the groundwork for the convergence analysis of our proposed algorithm by presenting fundamental definitions and lemmas. Subsequently, we shall adapt $\rightarrow$ and $\rightharpoonup$ to respectively mean strong and weak convergence.

Suppose that for each $n \geq 1$, the sequence $\{x_n\}$ recursively satisfies $x_{n+1} = x_n + \theta_n(x_n - x_{n-1})$. Then, (see Attouch and Cabot (2020)):

$$x_{n+1} - x_n = \left(\prod_{j=1}^n \theta_j \right) (x_1 - x_0),$$

and

$$x_n = x_1 + \left(\sum_{l=1}^{n-1} \prod_{j=1}^l \theta_j \right) (x_1 - x_0).$$

Therefore, the sequence $\{x_n\}$ will converge if and only if $x_1 = x_0$ or if $\sum_{l=1}^{\infty} \prod_{j=1}^l \theta_j < \infty$. Thus, for the convergence analysis of our algorithm, we will subsequently assume that

$$\sum_{l=i}^{\infty} \left(\prod_{j=i}^l \theta_j \right) < \infty \text{ for each } i \geq 1, \quad (2.1)$$

and we define the real sequence $\{t_i\}$ by

$$t_i := \sum_{l=i-1}^{\infty} \left(\prod_{j=i}^l \theta_j \right) = 1 + \sum_{l=i}^{\infty} \left(\prod_{j=i}^l \theta_j \right). \quad (2.2)$$

Let $\{t_{i,n}\}$ be a sequence defined, for each $i, n \geq 1$, by

$$t_{i,n} := \begin{cases} \sum_{l=i-1}^{n-1} \left(\prod_{j=i}^l \theta_j \right) = 1 + \sum_{l=i}^{n-1} \left(\prod_{j=i}^l \theta_j \right), & i \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad (2.3)$$

Conventionally, for $l = i - 1$, we take $\prod_{j=i}^{l-1} \theta_j = 1$. One can obtain from (2.3) that

$$t_{i,n} = 1 + \theta_i t_{i+1,n} \text{ for each } i \geq 1, n \geq i + 1. \quad (2.4)$$

For every $i \geq 1$, the sequence $\{t_{i,n}\}$ converges increasingly to $\{t_i\}$. That is, $t_i = \lim_{n \rightarrow \infty} t_{i,n}$, hence, (2.1) guarantees that we can have from (2.4) a well defined sequence,

$$t_i = 1 + \theta_i t_{i+1} \text{ for each } i \geq 1. \quad (2.5)$$

Using ' $\sim$ ' to mean the asymptotic equivalence, the proposition below provides a criterion for ensuring that assumption (2.1) holds.

Proposition 2.1 (Attouch and Cabot 2020, Proposition 3.1, Izuchukwu et al. 2021, Proposition 2.3) *Let $\{\theta_n\}$ be a real sequence such that $\theta_n \in [0, 1)$ for each $n \geq 1$. Suppose, for some $c \in [0, 1)$, that*

$$\lim_{n \rightarrow \infty} \left(\frac{1}{1 - \theta_{n+1}} - \frac{1}{1 - \theta_n} \right) = c.$$

Then

(i) the assumption defined in (2.1) is true, and

$$t_{n+1} \sim \frac{1}{(1-c)(1-\theta_n)} \text{ as } n \rightarrow \infty,$$

(ii) the equivalence $1 - \theta_n \sim 1 - \theta_{n+1}$ holds true as $n \rightarrow \infty$. Hence, $t_{n+1} \sim t_{n+2}$ as $n \rightarrow \infty$.

Remark 2.2 (see also Izuchukwu et al. (2021))

One can see that the sequence $\theta_n = 1 - \frac{\bar{\theta}}{n}$, $\bar{\theta} > 1$, provides a good illustration of a sequence that fulfills assumption (2.1). In fact, we can get that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{1}{1 - \theta_{n+1}} - \frac{1}{1 - \theta_n} \right) &= \lim_{n \rightarrow \infty} \left(\frac{1}{\bar{\theta}}(n+1) - \frac{1}{\bar{\theta}}n \right) \\ &= \frac{1}{\bar{\theta}} \in [0, 1), \end{aligned}$$

satisfies the assumption of Proposition 2.1. Therefore, by Proposition 2.1(i), assumption (2.1) indeed holds.

Definition 2.3 Let $\mathcal{C}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$. For each $s, w, z \in \mathcal{C}$, a bifunction $f : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}$ is said to be:

i. monotone on $\mathcal{C}$ if

$$f(s, w) + f(w, s) \leq 0;$$

ii. strongly pseudomonotone on $\mathcal{C}$ if there exists a positive constant β such that

$$f(s, w) \geq 0 \implies f(w, s) + \beta \|s - w\|^2 \leq 0;$$

iii. pseudomonotone on $\mathcal{C}$ if

$$f(s, w) \geq 0 \implies f(w, s) \leq 0;$$

iv. satisfying Lipschitz-type conditions on $\mathcal{C}$ if there exist two constants $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$f(s, z) - f(s, w) - f(w, z) \leq \delta_1 \|s - w\|^2 + \delta_2 \|w - z\|^2.$$

Clearly, (i) $\implies$ (iii) and (ii) $\implies$ (iii) but the converse statements are not generally true.

In the following definition, we will recall the concept of proximity operator originally introduced by Moreau (1962).

Definition 2.4 Let $g : \mathcal{C} \subset \mathcal{H} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function that satisfies properness, convexity and lower semicontinuity, and $\lambda > 0$. The Moreau envelope of g is the function

$$\lambda g(s) = \inf_{z \in \mathcal{C}} \left\{ g(z) + \frac{1}{2\lambda} \|z - s\|^2, \quad \forall s \in \mathcal{H} \right\}.$$

But for each $s \in \mathcal{H}$, the function

$$z \longmapsto g(z) + \frac{1}{2\lambda} \|z - s\|^2$$

is strongly convex and satisfies the properness and lower semicontinuity properties. Thus, it attains its infimum in $\mathbb{R}$. The unique minimizer of the function $z \longmapsto g(z) + \frac{1}{2\lambda} \|z - s\|^2$, denoted by $\text{prox}_g(s)$, is known as the proximal point operator of g at s . Hence, the operator $\text{prox}_{\lambda g} : \mathcal{H} \longrightarrow \mathbb{R}$ is defined by

$$\text{prox}_{\lambda g}(s) := \arg \min_{z \in \mathcal{C}} \left\{ \lambda g(z) + \frac{1}{2} \|s - z\|^2 \right\}.$$

If $g = \delta_{\mathcal{C}}$, where $\delta_{\mathcal{C}}$ is the indicator function of the set $\mathcal{C}$, then

$$\text{prox}_{\delta_{\mathcal{C}}}(s) = \mathcal{P}_{\mathcal{C}}(s) \quad \forall s \in \mathcal{H}.$$

Lemma 2.5 (Bauschke and Combettes 2011) For each $s \in \mathcal{H}$, $z \in \mathcal{C}$ and positive constant λ , the following inequality is true

$$\lambda(g(z) - g(\text{prox}_{\lambda g}(s))) \geq \langle s - \text{prox}_{\lambda g}(s), z - \text{prox}_{\lambda g}(s) \rangle.$$

Definition 2.6 i. Let $f : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$ be a convex function, then, for each $s, w \in \mathcal{H}$, the subdifferential of $f(s, \cdot)$ at w is defined by

$$\partial f(s, w) := \{u \in \mathcal{H} : f(s, z) \geq f(s, w) + \langle u, z - w \rangle, \quad \forall z \in \mathcal{H}\}.$$

And in particular,

$$\partial f(s, s) := \{u \in \mathcal{H} : f(s, z) \geq \langle u, z - s \rangle, \quad \forall z \in \mathcal{H}\}.$$

ii. The normal cone $\mathcal{N}_{\mathcal{C}}$ of the convex set $\mathcal{C}$ is defined at a point s by:

$$\mathcal{N}_{\mathcal{C}}(s) = \{v \in \mathcal{H} : \langle v, z - s \rangle \leq 0, \quad \forall z \in \mathcal{C}\}.$$

iii. Let $s \in \mathcal{H}$, there exists a unique closest point in $\mathcal{H}$ denoted by $\mathcal{P}_{\mathcal{C}}(s)$ such that

$$\mathcal{P}_{\mathcal{C}}(s) = \arg \min \{\|s - z\| : z \in \mathcal{C}\}.$$

$\mathcal{P}_{\mathcal{C}}$ is called the metric projection of $\mathcal{H}$ onto $\mathcal{C}$.

- Remark 2.7** i. Notice that for any convex function $f : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ with s fixed in $\mathcal{H}$ and $w \in \text{int}(\text{dom } f)$, the subdifferential $\partial f(s, w)$ (set of subgradients) is nonempty and compact in $\mathcal{H}$ (Mordukhovich and Nam 2014, Propositon 2.47), Rockafellar (1970).
- ii. For any point $s \in \mathcal{C}$, the normal cone $\mathcal{N}_{\mathcal{C}}(s)$ is nonempty (Mordukhovich and Nam 2014).

Lemma 2.8 (Attouch and Cabot 2020, Lemma B.1) Let $\{c_n\}$, $\{d_n\}$ and $\{\theta_n\}$ be real number sequences that fulfill

$$c_{n+1} \leq \theta_n c_n + d_n \text{ for each } n \geq 1.$$

Suppose $0 \leq \theta_n$ for each $n \geq 1$.

(i) For each $n \geq 1$, we have

$$\sum_{i=1}^n c_i \leq t_{1,n} c_1 + \sum_{i=1}^{n-1} t_{i+1,n} d_i,$$

where the sequence $\{t_{i,n}\}$ is defined by (2.3).

(ii) Under (2.1), assume that the sequence $\{t_i\}$ defined by (2.2) satisfies $\sum_{i=1}^{\infty} t_{i+1}[d_i]_+ < \infty$. Then the series $\sum_{i=1}^{\infty} [c_i]_+$ is convergent, and

$$\sum_{i=1}^{\infty} [c_i]_+ \leq t_1[c_1]_+ + \sum_{i=1}^{\infty} t_{i+1}[d_i]_+,$$

where $[s]_+ := \max\{s, 0\}$ for each $s \in \mathbb{R}$.

Lemma 2.9 (Rockafellar 1970) Let $g : \mathcal{C} \rightarrow \mathbb{R}$ be a convex, lower semicontinuous and subdifferentiable function on $\mathcal{C}$. Suppose $\text{int}(\mathcal{C}) \neq \emptyset$, then $\bar{r}$ is a solution of the convex minimization problem, $\min\{g(s) : s \in \mathcal{C}\}$ if and only if $\partial g(\bar{r}) + \mathcal{N}_{\mathcal{C}}(\bar{r}) \ni 0$.

Lemma 2.10 The following is true for each $s, z \in \mathcal{H}$ and $\beta \in \mathbb{R}$;

- (i) $\|\beta s + (1 - \beta)z\|^2 = \beta\|s\|^2 + (1 - \beta)\|z\|^2 - \beta(1 - \beta)\|s - z\|^2$;
(ii) $2\langle s, z \rangle = \|s\|^2 + \|z\|^2 - \|s - z\|^2 = \|s + z\|^2 - \|s\|^2 - \|z\|^2$.

Lemma 2.11 (Opial 1967)(Opial 1967) Let $\mathcal{C} \subset \mathcal{H}$ be nonempty, and $\{x_n\}$ a sequence in $\mathcal{H}$ such that:

- (i) $\lim_{n \rightarrow \infty} \|x_n - \bar{r}\|$ exists for each $\bar{r} \in \mathcal{C}$;
(ii) every sequentially weak cluster point of $\{x_n\}$ is found in $\mathcal{C}$.
Then the sequence $\{x_n\}$ weakly converges to a point in $\mathcal{C}$.

Definition 2.12 A sequence $\{x_n\}$ in $\mathcal{H}$ is said to converge R -linearly to a point $\bar{v}$ with rate $\rho \in [0, 1)$ if there is a constant $\kappa > 0$ such that

$$\|x_n - \bar{v}\| \leq \kappa \rho^n \text{ for all } n \in \mathbb{N}.$$

3 Main Results

We will assume that the bifunction f satisfies the following conditions:

Assumptions 3.1

- (A1) For each $s \in \mathcal{C}$, $f(s, s) = 0$ and f is pseudomonotone on $\mathcal{C}$;
- (A2) With two positive constants δ_1 and δ_2 , f satisfies the Lipschitz-type conditions on $\mathcal{H}$;
- (A3) $f(s, \cdot)$ is convex, lower semicontinuous and subdifferentiable on $\mathcal{C}$ for each $s \in \mathcal{C}$;
- (A4) For all fixed point $y \in \mathcal{C}$, $f(\cdot, y)$ is sequentially weakly upper semicontinuous on $\mathcal{C}$. That is, if $x_n \in \mathcal{C}$ is such that $x_n \rightarrow x^*$, as $n \rightarrow \infty$, then, $\limsup_{n \rightarrow \infty} f(x_n, y) \leq f(x^*, y)$ for each fixed $y \in \mathcal{C}$ and $x_n \in \mathcal{C}$;
- (A5) $EP(f, \mathcal{C}) \neq \emptyset$.

Now, we present our proposed method as follows:

Assumptions 3.2 Condition on the inertial and relaxation factors.

Suppose that $\theta_n \in [0, 1)$, $\alpha_n \in (0, 1]$ and $\liminf_{n \rightarrow \infty} \alpha_n > 0$. Assume for n sufficiently large enough, there exists $\epsilon \in (0, 1)$ such that

$$\begin{cases} (1 - \epsilon) \frac{1 - \alpha_{n-1}}{2\alpha_{n-1}} (1 - \theta_{n-1}) \\ \geq \theta_n t_{n+1} \left(1 + \theta_n + \left[\frac{1 - \alpha_n}{2\alpha_n} (1 - \theta_n) - \frac{1 - \alpha_{n-1}}{2\alpha_{n-1}} (1 - \theta_{n-1}) \right]_+ \right) & \text{if } \alpha_n \in (0, 1), \\ \left(\frac{1}{2} - \epsilon \right) (1 - \theta_{n-1}) \geq \theta_n t_{n+1} \left(1 + \theta_n + \frac{1}{2} [\theta_{n-1} - \theta_n]_+ \right) & \text{if } \alpha_n = 1. \end{cases} \quad (3.1)$$

Remark 3.4

- Since $y_n \in \mathcal{C}$ and f is convex in $\mathcal{C}$, we have by Remark 2.7 that $\partial f(w_n, \cdot)(y_n) \neq \emptyset$ and $\mathcal{N}_{\mathcal{C}}(y_n) \neq \emptyset$. Therefore, there exist $v_n \in \partial f(w_n, \cdot)(y_n)$ and $q_n \in \mathcal{N}_{\mathcal{C}}(y_n)$ such that step 2 of Algorithm 3.3 is well defined.
- If Algorithm 3.3 terminates after a finite number of iterations, that is, $y_n = w_n$, then y_n becomes a solution to the equilibrium problem. Indeed, by the definition of y_n and Lemma 2.9, it follows that

$$0 \in \partial \left(\lambda_n f(w_n, \cdot) + \frac{1}{2} \|w_n - \cdot\|^2 \right) (y_n) + \mathcal{N}_{\mathcal{C}}(y_n).$$

With $v_n \in \partial f(w_n, \cdot)(y_n)$ and $q_n \in \mathcal{N}_{\mathcal{C}}(y_n)$, then

$$\lambda_n v_n + y_n - w_n + q_n = 0.$$

Since by the assumption, $y_n = w_n$, it then follows that

$$\lambda_n v_n + q_n = 0.$$

Algorithm 3.3. *Relaxed Inertial Subgradient Extragradient Algorithm for Equilibrium Problems.*

Let $x_0, x_1 \in \mathcal{H}$, $\lambda_1 > 0$, $\mu \in]0, 1[$ and set $n = 1$.

STEP 1: Start with the present iterates x_{n-1} and x_n ,

compute

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \operatorname{argmin}_{y \in \mathcal{C}} \left\{ \lambda_n f(w_n, y) + \frac{1}{2} \|w_n - y\|^2 \right\}. \end{cases}$$

Stop the computation whenever $y_n = w_n$, and y_n becomes a solution, otherwise, move to step 2.

STEP 2:

Choose $v_n \in \partial f(w_n, \cdot)(y_n)$ in such a way there is $q_n \in N_{\mathcal{C}}(y_n)$ satisfying

$q_n = w_n - \lambda_n v_n - y_n$ and construct a half space

$T_n = \{z \in \mathcal{H} : \langle w_n - \lambda_n v_n - y_n, z - y_n \rangle \leq 0\}$.

Evaluate

$$z_n = \operatorname{argmin}_{y \in T_n} \left\{ \lambda_n f(y_n, y) + \frac{1}{2} \|w_n - y\|^2 \right\}.$$

STEP 3: Compute

$x_{n+1} = (1 - \alpha_n)w_n + \alpha_n z_n$, and

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\mu}{2} \frac{\|w_n - y_n\|^2 + \|z_n - y_n\|^2}{f(w_n, z_n) - f(w_n, y_n) - f(y_n, z_n)}, \lambda_n \right\}, & \text{if } f(w_n, z_n) - f(w_n, y_n) - f(y_n, z_n) > 0, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Set $n + 1 \leftarrow n$ and continue again from step 1.

This further implies that

$$\lambda_n \langle v_n, y - y_n \rangle + \langle q_n, y - y_n \rangle = 0, \quad \forall y \in \mathcal{C}. \quad (3.2)$$

But $q_n \in \mathcal{N}_{\mathcal{C}}(y_n)$ implies that $\langle q_n, y - y_n \rangle \leq 0$, $\forall y \in \mathcal{C}$. Thus, we obtain from (3.2) that

$$\lambda_n \langle v_n, y - y_n \rangle \geq 0, \quad \forall y \in \mathcal{C}.$$

Given that $v_n \in \partial f(w_n, \cdot)(y_n)$ and $\lambda_n > 0 \forall n \geq 1$, then we can use the definition of subdifferential to obtain

$$0 \leq \langle v_n, y - y_n \rangle \leq f(w_n, y) - f(w_n, y_n) = f(w_n, y), \quad \forall y \in \mathcal{C}.$$

Hence, $f(w_n, y) \geq 0$, $\forall y \in \mathcal{C}$. That is, $y_n = w_n \in EP(f, \mathcal{C})$. However, throughout this section, to ensure that Algorithm 3.3 generates an infinite sequence of iterations, we will assume that Algorithm 3.3 does not terminate in a finite number of iterations.

Remark 3.5 (1) Our Algorithm combines the inertial and relaxation parameters θ_n and α_n , respectively. These parameters (factors) play vital role in improving the convergence speed of iterative methods. Although, the algorithms in Pham et al. (2021); Yang and Liu (2020); Yao et al. (2024) require only one strongly convex minimization subproblem at each iteration, which is cost effective in situations

where solving strongly convex minimization subproblems is expensive, however, bearing in mind the importance of combining inertial and relaxation parameters in iterative algorithms which are lacking in Pham et al. (2021); Yang and Liu (2020); Yao et al. (2024), our algorithm has the potential of performing better, in terms of convergence speed, than those in Pham et al. (2021); Yang and Liu (2020); Yao et al. (2024), see Section 6 (Numerical experiments). Moreover, the influence of combining these parameters with regard to the numerical performance of iterative schemes was discussed in Iutzeler and Hendrickx (2019). Thus, our algorithm has the potential to perform numerically better than those in Pham et al. (2021); Yang and Liu (2020); Yao et al. (2024) as shown in Section 6 (Numerical experiments).

- (2) If $\alpha_n = 1$, our method reduces to an inertial subgradient extragradient algorithm for solving equilibrium problem. In this context, many weak convergence analysis like the one in Yao et al. (2024) require the inertia parameter to be restrictive and/or nondecreasing with an upper bound. Moreover, the condition on the inertial parameter in Yao et al. (2024) is not natural since the inertial parameter lies between -1 and 0 . One of the novelties of this paper is that we derived a general condition (Assumption 3.2) which is weaker than the above conditions. Another novelty of this paper is that the assumptions on the inertial and relaxation parameters of this paper, allow the case where θ_n converges to a point very close to 1 (see Remark 2.2), which is very crucial in the study of inertial methods. This is actually where the relaxation effect comes into play, as a crucial ingredient of our methods. Thus, the novelty of this paper is not only in improving the convergence speed of the methods in Pham et al. (2021); Yang and Liu (2020); Yao et al. (2024) but to provide weaker conditions on the inertial parameter in methods for solving equilibrium problems. Moreover, our study offers many possibilities for future research in this direction; like how to modify Assumption 3.2 so that θ_n is allowed to converge to 1 ?

Lemma 3.6 (Shehu et al. 2021) *The sequence $\{\lambda_n\}_{n \geq 1}$ generated by Algorithm 3.3 is a monotone non-increasing sequence with $\min \left\{ \lambda_1, \frac{\mu}{2 \max\{\delta_1, \delta_2\}} \right\}$ as its lower bound.*

The following propositions provides validation for Assumptions (2.1) and 3.2.

Proposition 3.7 *Let θ_n be an increasing sequence in $[0, 1)$ with $\lim_{n \rightarrow \infty} \theta_n = \theta$ such that*

$$1 - 3\theta > 0. \quad (3.3)$$

Then, Assumptions (2.1) and 3.2 hold true.

Proof From the fact that $\theta_n \in [0, 1)$ is an increasing sequence and $\theta_n \rightarrow \theta$ as $n \rightarrow +\infty$, we have that $\theta_n \leq \theta$ for all $n \in \mathbb{N}$. Hence, assumption (2.1) is satisfied and $t_n \leq \frac{1}{1 - \theta}$ for all $n \in \mathbb{N}$ (see Attouch and Cabot (2020); Izuchukwu et al. (2021)).

Notice that $(1 - \theta) > \frac{\theta(1 + \theta)}{1 - \theta}$ whenever $1 - 3\theta > 0$. Therefore, there is $\epsilon \in (0, 1)$ such that

$$\left(\frac{1}{2} - \epsilon \right) (1 - \theta) \geq \frac{\theta(1 + \theta)}{1 - \theta}. \quad (3.4)$$

Since $\theta_n \leq \theta$ and $t_n \leq \frac{1}{1-\theta}$ are true for all $n \geq 1$, it follows from (3.4) that

$$\left(\frac{1}{2} - \epsilon\right)(1 - \theta) \geq \frac{\theta(1 + \theta)}{1 - \theta} \geq \theta_n t_{n+1}(1 + \theta_n). \quad (3.5)$$

Using the fact that $\theta_{n-1} \leq \theta_n$ for all $n \geq 1$ and knowing that $[x]_+ := \max\{x, 0\}$, for any real number x , we obtain that

$$[\theta_{n-1} - \theta_n]_+ = \frac{1}{2}[\theta_{n-1} - \theta_n]_+ = 0. \quad (3.6)$$

Combining (3.6) with (3.5) establishes Assumption 3.2 when $\alpha_n = 1$.

Similarly, for $\alpha_n \in (0, 1)$ there is $\epsilon \in (0, 1)$ such that

$$(1 - \epsilon) \left(\frac{1 - \alpha_{n-1}}{2\alpha_n}\right)(1 - \theta) \geq \frac{\theta(1 + \theta)}{1 - \theta} \geq \theta_n t_{n+1}(1 + \theta_n). \quad (3.7)$$

From (3.6), we have that

$$\frac{1 - \alpha_n}{2\alpha_n} [(1 - \theta_n) - (1 - \theta_{n-1})]_+ = 0. \quad (3.8)$$

Thus, combining (3.7) with (3.8) gives Assumption 3.2 when $\alpha_n \in (0, 1)$. $\square$

Proposition 3.8 (see Izuchukwu et al. 2021, Proposition 3.7) Let $\theta_n \in [0, 1) \forall n \geq 1$, and suppose there is $c \in [0, 1)$ satisfying

$$\lim_{n \rightarrow \infty} \left(\frac{1}{1 - \theta_{n+1}} - \frac{1}{1 - \theta_n} \right) = c$$

and

$$\liminf_{n \rightarrow \infty} (1 - \theta_n)^2 > \limsup_{n \rightarrow \infty} \frac{\theta_n(1 + \theta_n)}{1 - c}.$$

Then assumption (2.1) and Assumption 3.2 hold true when $\alpha_n = 1$.

In a case where $\alpha_n \neq 1$, the next proposition presents a useful tool for verifying that Assumptions (2.1) and 3.2 are true.

Proposition 3.9 (see Attouch and Cabot 2020, Proposition 3.3, Izuchukwu et al. 2021, Proposition 3.8) Let $\alpha_n \in (0, 1)$ and $\theta_n \in [0, 1)$ for each $n \geq 1$. Suppose there exists $c \in [0, 1)$ and $c^* \in [c, 1)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{1}{1 - \theta_{n+1}} - \frac{1}{1 - \theta_n} \right) &= c, \\ \lim_{n \rightarrow \infty} \frac{\psi_{n+1} - \psi_n}{\psi_n(1 - \theta_n)} &= c^* \end{aligned}$$

and

$$\liminf_{n \rightarrow \infty} \psi_n (1 - \theta_n)^2 > \limsup_{n \rightarrow \infty} \frac{\theta_n (1 + \theta_n)}{1 - c^*},$$

where $\psi_n = \frac{1 - \alpha_n}{2\alpha_n}$. Then Assumptions (2.1) and 3.2 hold true.

Lemma 3.10 Suppose $\{x_n\}$ is a sequence in $\mathcal{H}$, and $\{\theta_n\}$ a real positive sequence. Given $\bar{r} \in \mathcal{H}$, define the sequence $\{\Phi_n\}$ by $\Phi_n := \frac{1}{2} \|x_n - \bar{r}\|^2$ and $w_n = x_n + \theta_n(x_n - x_{n-1})$. Then,

$$\begin{aligned} \Phi_{n+1} - \Phi_n - \theta_n(\Phi_n - \Phi_{n-1}) &= \frac{1}{2}(\theta_n + \theta_n^2) \|x_n - x_{n-1}\|^2 \\ &\quad + \frac{1}{2} \|x_{n+1} - \bar{r}\|^2 - \frac{1}{2} \|w_n - \bar{r}\|^2. \end{aligned} \quad (3.9)$$

Proof Using the definition of

$w_n = x_n + \theta_n(x_n - x_{n-1})$, we have,

$$\begin{aligned} \|w_n - \bar{r}\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - \bar{r}\|^2 \\ &= \|x_n - \bar{r} + \theta_n(x_n - x_{n-1})\|^2 \\ &= \|x_n - \bar{r}\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \langle x_n - \bar{r}, x_n - x_{n-1} \rangle \\ &= \|x_n - \bar{r}\|^2 + (\theta_n + \theta_n^2) \|x_n - x_{n-1}\|^2 + \theta_n (\|x_n - \bar{r}\|^2 - \|x_{n-1} - \bar{r}\|^2) \\ &= 2\Phi_n + (\theta_n + \theta_n^2) \|x_n - x_{n-1}\|^2 + 2\theta_n(\Phi_n - \Phi_{n-1}). \end{aligned} \quad (3.10)$$

Rearranging and adding $\Phi_{n+1} = \frac{1}{2} \|x_{n+1} - \bar{r}\|^2$ to both sides we have,

$$\begin{aligned} \Phi_{n+1} - \Phi_n - \theta_n(\Phi_n - \Phi_{n-1}) &= \frac{1}{2}(\theta_n + \theta_n^2) \|x_n - x_{n-1}\|^2 \\ &\quad + \frac{1}{2} \|x_{n+1} - \bar{r}\|^2 - \frac{1}{2} \|w_n - \bar{r}\|^2. \end{aligned} \quad (3.11)$$

□

Lemma 3.11 Suppose $\{x_n\}$ is a sequence generated by Algorithm 3.3 such that Assumption 3.1 is fulfilled. Then for $\bar{r} \in EP(f, C)$, it follows that

$$\begin{aligned} \|x_{n+1} - \bar{r}\|^2 &\leq \|w_n - \bar{r}\|^2 - \alpha_n \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) (\|w_n - y_n\|^2 + \|z_n - y_n\|^2) \\ &\quad - \frac{1 - \alpha_n}{\alpha_n} \|x_{n+1} - w_n\|^2 \quad \forall n \geq 1. \end{aligned}$$

Proof Using the definition of y_n and Lemma 2.9, we have that

$$0 \in \partial \left(\lambda_n f(w_n, \cdot) + \frac{1}{2} \|\cdot - w_n\|^2 \right) (y_n) + \mathcal{N}_C(y_n).$$

Thus, by Step 2, there is a vector $v_n \in \partial f(w_n, \cdot)(y_n)$ and $q_n \in \mathcal{N}_{\mathcal{C}}(y_n)$ such that

$$w_n - \lambda_n v_n - y_n - q_n = 0.$$

It follows from the expression above that

$$\begin{aligned} \langle w_n - y_n, y - y_n \rangle &= \langle \lambda_n v_n + q_n, y - y_n \rangle \\ &= \lambda_n \langle v_n, y - y_n \rangle + \langle q_n, y - y_n \rangle. \end{aligned} \quad (3.12)$$

But $q_n \in \mathcal{N}_{\mathcal{C}}(y_n)$ implies that $\langle q_n, y - y_n \rangle \leq 0 \quad \forall y \in \mathcal{C}$.

Thus we obtain from (3.12) that

$$\langle w_n - y_n, y - y_n \rangle \leq \lambda_n \langle v_n, y - y_n \rangle \quad \forall y \in \mathcal{C}. \quad (3.13)$$

Also, $v_n \in \partial f(w_n, y_n)$ implies that

$$f(w_n, y) - f(w_n, y_n) \geq \langle v_n, y - y_n \rangle \quad \forall y \in \mathcal{H}. \quad (3.14)$$

Using (3.13) and (3.14), we have

$$\lambda_n (f(w_n, y) - f(w_n, y_n)) \geq \langle w_n - y_n, y - y_n \rangle \quad \forall y \in \mathcal{C}. \quad (3.15)$$

Putting $z_n = y$ in (3.15), we see that

$$\lambda_n (f(w_n, z_n) - f(w_n, y_n)) \geq \langle w_n - y_n, z_n - y_n \rangle. \quad (3.16)$$

Again, from the definition of z_n and Lemma 2.5, we obtain

$$\lambda_n (f(y_n, y) - f(y_n, z_n)) \geq \langle w_n - z_n, y - z_n \rangle \quad \forall y \in T_n. \quad (3.17)$$

Let $y = \bar{r} \in EP(f, \mathcal{C})$ then,

$$\lambda_n (f(y_n, \bar{r}) - f(y_n, z_n)) \geq \langle w_n - z_n, \bar{r} - z_n \rangle \quad \forall \bar{r} \in EP(f, \mathcal{C}). \quad (3.18)$$

But $\bar{r} \in EP(f, \mathcal{C})$, implies that $f(\bar{r}, y_n) \geq 0$. This further implies by pseudomonotonicity of f that

$$f(y_n, \bar{r}) \leq 0.$$

Therefore, (3.18) becomes,

$$-\lambda_n f(y_n, z_n) \geq \langle w_n - z_n, \bar{r} - z_n \rangle. \quad (3.19)$$

Combining (3.16) and (3.19) implies that

$$\begin{aligned} 2\lambda_n (f(w_n, z_n) - f(w_n, y_n) - f(y_n, z_n)) &\geq 2\langle w_n - y_n, z_n - y_n \rangle \\ &\quad + 2\langle w_n - z_n, \bar{r} - z_n \rangle. \end{aligned} \quad (3.20)$$

Note that:

$$2\langle w_n - y_n, z_n - y_n \rangle = \|w_n - y_n\|^2 + \|z_n - y_n\|^2 - \|w_n - z_n\|^2. \quad (3.21)$$

Also,

$$2\langle w_n - z_n, \bar{r} - z_n \rangle = \|w_n - z_n\|^2 + \|z_n - \bar{r}\|^2 - \|w_n - \bar{r}\|^2. \quad (3.22)$$

Using (3.21) and (3.22) in (3.20), we obtain

$$\begin{aligned} 2\lambda_n(f(w_n, z_n) - f(w_n, y_n) - f(y_n, z_n)) &\geq \|w_n - y_n\|^2 \\ &+ \|z_n - y_n\|^2 + \|z_n - \bar{r}\|^2 - \|w_n - \bar{r}\|^2. \end{aligned} \quad (3.23)$$

By the definition of λ_n , we have

$$2(f(w_n, z_n) - f(w_n, y_n) - f(y_n, z_n)) \leq \frac{\mu(\|w_n - y_n\|^2 + \|z_n - y_n\|^2)}{\lambda_{n+1}}. \quad (3.24)$$

Using (3.24) in (3.23) gives us that

$$\begin{aligned} \mu \frac{\lambda_n}{\lambda_{n+1}} (\|w_n - y_n\|^2 + \|z_n - y_n\|^2) &\geq \|w_n - y_n\|^2 \\ &+ \|z_n - y_n\|^2 + \|z_n - \bar{r}\|^2 - \|w_n - \bar{r}\|^2. \end{aligned} \quad (3.25)$$

This further implies

$$\begin{aligned} \|z_n - \bar{r}\|^2 &\leq \|w_n - \bar{r}\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \|w_n - y_n\|^2 \\ &- \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \|z_n - y_n\|^2. \end{aligned} \quad (3.26)$$

Moreover, with $\alpha_n \in (0, 1]$, the definition of the sequence $\{x_{n+1}\}$ and Lemma 2.10 (i), we have

$$\begin{aligned} \|x_{n+1} - \bar{r}\|^2 &= (1 - \alpha_n) \|w_n - \bar{r}\|^2 + \alpha_n \|z_n \\ &- \bar{r}\|^2 - (1 - \alpha_n) \alpha_n \|z_n - w_n\|^2. \end{aligned} \quad (3.27)$$

Using (3.26) and the definition of $\{x_{n+1}\}$, that is, $z_n - w_n = \frac{1}{\alpha_n}(x_{n+1} - w_n)$ in (3.27), we establish the desired result. $\square$

Lemma 3.12 *Let $\{\Phi_n\}$ be as defined in Lemma 3.10, and $\{x_n\}$ a sequence generated by Algorithm 3.3. Suppose that Assumption 3.1 holds. Then for any $\bar{r} \in EP(f, \mathcal{C})$, it*

ensues that

$$\begin{aligned}\Phi_{n+1} - \Phi_n - \theta_n(\Phi_n - \Phi_{n-1}) &\leq \frac{1}{2} \left((\theta_n + \theta_n^2) - \psi_n(1 - \theta_n)^2 \right) \|x_n - x_{n-1}\|^2 \\ &\quad - \frac{\alpha_n}{2} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) \left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2 \right) \\ &\quad - \frac{1}{2} \psi_n(1 - \theta_n) \left(\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 \right),\end{aligned}$$

$$\text{where } \psi_n = \frac{1 - \alpha_n}{2\alpha_n}.$$

Proof Using the definition of $w_n = x_n + \theta_n(x_n - x_{n-1})$, we have

$$\begin{aligned}\|x_{n+1} - w_n\|^2 &= \|x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\|^2 \\ &= \|x_{n+1} - x_n - (x_n - x_{n-1}) + (1 - \theta_n)(x_n - x_{n-1})\|^2 \\ &= \|x_{n+1} - 2x_n + x_{n-1}\|^2 + (1 - \theta_n)^2 \|x_n - x_{n-1}\|^2 \\ &\quad + 2(1 - \theta_n) \langle x_{n+1} - 2x_n + x_{n-1}, x_n - x_{n-1} \rangle.\end{aligned}\quad (3.28)$$

But

$$\begin{aligned}2 \langle x_{n+1} - 2x_n + x_{n-1}, x_n - x_{n-1} \rangle &= \|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 \\ &\quad - \|x_{n+1} - 2x_n + x_{n-1}\|^2.\end{aligned}\quad (3.29)$$

Substituting (3.29) in (3.28) we, therefore, obtain

$$\begin{aligned}\|x_{n+1} - w_n\|^2 &= \theta_n \|x_{n+1} - 2x_n + x_{n-1}\|^2 + (1 - \theta_n)^2 \|x_n - x_{n-1}\|^2 \\ &\quad + (1 - \theta_n) \left(\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 \right) \\ &\geq (1 - \theta_n)^2 \|x_n - x_{n-1}\|^2 \\ &\quad + (1 - \theta_n) \left(\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 \right).\end{aligned}\quad (3.30)$$

Notice that $\frac{1 - \alpha_n}{2\alpha_n} \geq \frac{1 - \alpha_n}{4\alpha_n}$, $\forall n \geq 1$.

Thus, we have from (3.30) that

$$\begin{aligned}-\frac{1 - \alpha_n}{2\alpha_n} \|x_{n+1} - w_n\|^2 &\leq -\frac{1 - \alpha_n}{4\alpha_n} (1 - \theta_n)^2 \|x_n - x_{n-1}\|^2 \\ &\quad - \frac{1 - \alpha_n}{4\alpha_n} (1 - \theta_n) \left(\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 \right).\end{aligned}$$

That is,

$$\begin{aligned} -\psi_n \|x_{n+1} - w_n\|^2 &\leq -\frac{1}{2} \psi_n (1 - \theta_n)^2 \|x_n - x_{n-1}\|^2 \\ &\quad - \frac{1}{2} \psi_n (1 - \theta_n) \left(\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 \right). \end{aligned} \quad (3.31)$$

Hence, the desired result follows immediately from Lemmas 3.10, 3.11, and inequality (3.31). $\square$

Lemma 3.13 Suppose that Assumptions (2.1) and 3.1 hold and the sequence $\{t_i\}$ is given by the definition in (2.2). Let $\psi_n = \frac{1 - \alpha_n}{2\alpha_n}$ for $\{\alpha_n\}_{n \geq 1} \subset (0, 1]$. Given a sequence $\{x_n\}$ generated by Algorithm 3.3, then for each $n \geq 1$, we have

$$\begin{aligned} &\sum_{i=1}^{n-1} \left(\psi_{i-1} (1 - \theta_{i-1}) - \theta_i t_{i+1} \left(1 + \theta_i + \left[\psi_i (1 - \theta_i) \right. \right. \right. \\ &\quad \left. \left. \left. - \psi_{i-1} (1 - \theta_{i-1}) \right]_+ \right) \right) \|x_i - x_{i-1}\|^2 \\ &\leq 2\Phi_0 + 2t_1 |\Phi_1 - \Phi_0| + t_1 \psi_0 (1 - \theta_0) \|x_1 - x_0\|^2. \end{aligned}$$

Proof From Lemma 3.12, we have

$$\begin{aligned} \Phi_{n+1} - \Phi_n &\leq \theta_n (\Phi_n - \Phi_{n-1}) + \frac{1}{2} \left((\theta_n + \theta_n^2) - \psi_n (1 - \theta_n)^2 \right) \|x_n - x_{n-1}\|^2 \\ &\quad - \frac{1}{2} \psi_n (1 - \theta_n) \left(\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 \right). \end{aligned} \quad (3.32)$$

Let $c_n := \Phi_n - \Phi_{n-1}$,

$$d_n := \frac{1}{2} \left((\theta_n + \theta_n^2) - \psi_n (1 - \theta_n)^2 \right) \|x_n - x_{n-1}\|^2 - \frac{1}{2} \psi_n (1 - \theta_n) \left(\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 \right).$$

Thus, (3.32) is now in the form of $c_{n+1} \leq \theta_n c_n + d_n$, and applying Lemma 2.8(i), we have

$$\begin{aligned}
\Phi_n - \Phi_0 &= \sum_{i=1}^n c_i \\
&\leq t_1, n(\Phi_1 - \Phi_0) + \sum_{i=1}^{n-1} t_{i+1,n} \left(\frac{1}{2} \left((\theta_i + \theta_i^2) - \psi_i(1 - \theta_i)^2 \right) \|x_i - x_{i-1}\|^2 \right. \\
&\quad \left. - \frac{1}{2} \psi_i(1 - \theta_i) \left(\|x_{i+1} - x_i\|^2 - \|x_i - x_{i-1}\|^2 \right) \right) \\
&= t_1, n(\Phi_1 - \Phi_0) - \sum_{i=1}^{n-1} t_{i+1,n} \left(\frac{1}{2} \left(\psi_i(1 - \theta_i)^2 - (\theta_i + \theta_i^2) \right) \|x_i - x_{i-1}\|^2 \right) \\
&\quad - \sum_{i=1}^{n-1} t_{i+1,n} \left(\frac{1}{2} \psi_i(1 - \theta_i) \left(\|x_{i+1} - x_i\|^2 - \|x_i - x_{i-1}\|^2 \right) \right).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{1}{2} \left[\sum_{i=1}^{n-1} t_{i+1,n} \left(\psi_i(1 - \theta_i)^2 - (\theta_i + \theta_i^2) \right) \|x_i - x_{i-1}\|^2 \right. \\
&\quad \left. + \sum_{i=1}^{n-1} t_{i+1,n} \psi_i(1 - \theta_i) \left(\|x_{i+1} - x_i\|^2 - \|x_i - x_{i-1}\|^2 \right) \right] \\
&\leq t_1, n(\Phi_1 - \Phi_0) + \Phi_0 - \Phi_n.
\end{aligned} \tag{3.33}$$

Note that $\Phi_n \geq 0$, $t_{1,n} \leq t_1 \quad \forall n \in \mathbb{N}$. Thus

$$\begin{aligned}
&\sum_{i=1}^{n-1} t_{i+1,n} \left(\psi_i(1 - \theta_i)^2 - (\theta_i + \theta_i^2) \right) \|x_i - x_{i-1}\|^2 \\
&\quad + \sum_{i=1}^{n-1} t_{i+1,n} \psi_i(1 - \theta_i) \left(\|x_{i+1} - x_i\|^2 - \|x_i - x_{i-1}\|^2 \right) \\
&\leq 2t_1, n(\Phi_1 - \Phi_0) + 2\Phi_0 \\
&\leq 2t_1|\Phi_1 - \Phi_0| + 2\Phi_0.
\end{aligned} \tag{3.34}$$

Observe that

$$\begin{aligned}
&\sum_{i=1}^{n-1} t_{i+1,n} \psi_i(1 - \theta_i) \left(\|x_{i+1} - x_i\|^2 - \|x_i - x_{i-1}\|^2 \right) \\
&= \sum_{i=1}^{n-1} t_{i+1,n} \psi_i(1 - \theta_i) \|x_{i+1} - x_i\|^2 - \sum_{i=1}^{n-1} t_{i+1,n} \psi_i(1 - \theta_i) \|x_i - x_{i-1}\|^2 \\
&\quad + \sum_{i=1}^{n-1} t_{i,n} \psi_{i-1}(1 - \theta_{i-1}) \|x_i - x_{i-1}\|^2 - \sum_{i=1}^{n-1} t_{i,n} \psi_{i-1}(1 - \theta_{i-1}) \|x_i - x_{i-1}\|^2
\end{aligned}$$

$$= \sum_{i=1}^{n-1} \left(t_{i,n} \psi_{i-1}(1 - \theta_{i-1}) - t_{i+1,n} \psi_i(1 - \theta_i) \right) \|x_i - x_{i-1}\|^2 \\ + t_{n,n} \psi_{n-1}(1 - \theta_{n-1}) \|x_n - x_{n-1}\|^2 - t_{1,n} \psi_0(1 - \theta_0) \|x_1 - x_0\|^2.$$

Since $t_{n,n} \psi_{n-1}(1 - \theta_{n-1}) \|x_n - x_{n-1}\|^2 \geq 0$ and $t_{1,n} \leq t_1 \quad \forall n > 0$, it follows that

$$\sum_{i=1}^{n-1} t_{i+1,n} \psi_i(1 - \theta_i) \left(\|x_{i+1} - x_i\|^2 - \|x_i - x_{i-1}\|^2 \right) \\ \geq \sum_{i=1}^{n-1} \left(t_{i,n} \psi_{i-1}(1 - \theta_{i-1}) - t_{i+1,n} \psi_i(1 - \theta_i) \right) \|x_i \\ - x_{i-1}\|^2 - t_1 \psi_0(1 - \theta_0) \|x_1 - x_0\|^2. \quad (3.35)$$

The combination of (3.34) and (3.35) yields

$$\sum_{i=1}^{n-1} \left[t_{i+1,n} \left(\psi_i(1 - \theta_i)^2 - (\theta_i + \theta_i^2) \right) \right. \\ \left. + t_{i,n} \psi_{i-1}(1 - \theta_{i-1}) - t_{i+1,n} \psi_i(1 - \theta_i) \right] \|x_i - x_{i-1}\|^2 \\ \leq 2t_1 |\Phi_1 - \Phi_0| + 2\Phi_0 + t_1 \psi_0(1 - \theta_0) \|x_1 - x_0\|^2. \quad (3.36)$$

On the other hand, from the definition of $t_{i,n} = 1 + \theta_i t_{i+1,n}$ for each $i \geq 1$ and $n \geq i + 1$ in (2.4), we have

$$t_{i+1,n} \left(\psi_i(1 - \theta_i)^2 - (\theta_i + \theta_i^2) \right) + t_{i,n} \psi_{i-1}(1 - \theta_{i-1}) - t_{i+1,n} \psi_i(1 - \theta_i) \\ = t_{i+1,n} \left(\psi_i(1 - \theta_i)^2 - (\theta_i + \theta_i^2) \right) \\ + \psi_{i-1}(1 + \theta_i t_{i+1,n})(1 - \theta_{i-1}) - t_{i+1,n} \psi_i(1 - \theta_i) \\ = \psi_{i-1}(1 - \theta_{i-1}) - \theta_i t_{i+1,n} \left(1 + \theta_i \right. \\ \left. + \psi_i(1 - \theta_i) - \psi_{i-1}(1 - \theta_{i-1}) \right). \quad (3.37)$$

But $t_{i,n} \leq t_i \quad \forall i \geq 1, \quad n \geq i + 1$ and $[x]_+ \geq x \quad \forall x \in \mathcal{H}$. It follows that

$$t_{i+1,n} \left(\psi_i(1 - \theta_i)^2 - (\theta_i + \theta_i^2) \right) + t_{i,n} \psi_{i-1}(1 - \theta_{i-1}) - t_{i+1,n} \psi_i(1 - \theta_i) \\ \geq \psi_{i-1}(1 - \theta_{i-1}) - \theta_i t_{i+1} \left(1 + \theta_i + \left[\psi_i(1 - \theta_i) - \psi_{i-1}(1 - \theta_{i-1}) \right]_+ \right) \quad (3.38)$$

The combination of (3.38) and (3.36), therefore, establishes the desired result. $\square$

Lemma 3.14 Assume that Assumptions (2.1), 3.1 and 3.2 hold. Let $\{t_i\}$ be the sequence defined by (2.2) and $\{x_n\}$ the sequence generated by Algorithm 3.3. Then,

- a. (i.) $\lim_{n \rightarrow \infty} \sum_{i=1}^{n-1} \frac{1 - \alpha_i}{2\alpha_i} (1 - \theta_{i-1}) \|x_i - x_{i-1}\|^2 < \infty$, and as such $\lim_{n \rightarrow \infty} \sum_{i=1}^{n-1} \theta_i t_{i+1} \|x_i - x_{i-1}\|^2 < \infty$;
- (ii.) For each $\bar{r} \in EP(f, \mathcal{C})$, $\lim_{n \rightarrow \infty} \|x_n - \bar{r}\|$ exists, and hence $\{x_n\}$ is bounded;
- b. (i.) $\lim_{n \rightarrow \infty} \|w_n - y_n\| = 0$, $\lim_{n \rightarrow \infty} \|z_n - y_n\| = 0$;
- (ii.) $\lim_{n \rightarrow \infty} \|x_n - w_n\| = 0$;
- (iii.) $\lim_{n \rightarrow \infty} \|w_n - z_n\| = 0$.

Proof

- a. (i.) Assumption 3.2 holds for n sufficiently large enough, thus, we may assume without any loss of generality that it is true for $n \geq 1$.

Now, we will examine two possible situations as follows:

CASE I:

Suppose for all $n \geq 1$ that $\alpha_n \in (0, 1)$, then, using inequality (3.1) and Lemma 3.13, noting that $\psi_n = \frac{1 - \alpha_n}{2\alpha_n}$, we have that:

$$(\epsilon - 1)\psi_{i-1}(1 - \theta_{i-1}) \leq -\theta_i t_{i+1} \left(1 + \theta_i + \left[\psi_i(1 - \theta_i) - \psi_{i-1}(1 - \theta_{i-1}) \right]_+ \right).$$

This implies

$$\begin{aligned} & \sum_{i=1}^{n-1} \epsilon \psi_{i-1}(1 - \theta_{i-1}) \|x_i - x_{i-1}\|^2 \\ & \leq \sum_{i=1}^{n-1} \left(\psi_{i-1}(1 - \theta_{i-1}) - \theta_i t_{i+1} \left(1 + \theta_i + \left[\psi_i(1 - \theta_i) - \psi_{i-1}(1 - \theta_{i-1}) \right]_+ \right) \right) \|x_i - x_{i-1}\|^2 \\ & \leq 2\Phi_0 + 2t_1|\Phi_1 - \Phi_0| + t_1\psi_0(1 - \theta_0)\|x_1 - x_0\|^2 \end{aligned}$$

This implies that

$$\begin{aligned} & \sum_{i=1}^{n-1} \psi_{i-1}(1 - \theta_{i-1}) \|x_i - x_{i-1}\|^2 \leq \frac{1}{\epsilon} \left(2\Phi_0 + 2t_1|\Phi_1 - \Phi_0| + t_1\psi_0(1 - \theta_0)\|x_1 - x_0\|^2 \right). \end{aligned} \quad (3.39)$$

Taking the limit of (3.39) as $n \rightarrow \infty$, we arrive at

$$\sum_{i=1}^{\infty} \psi_{i-1}(1 - \theta_{i-1}) \|x_i - x_{i-1}\|^2 < \infty. \quad (3.40)$$

Since $\liminf_{n \rightarrow \infty} \alpha_n > 0$ and $\lim_{n \rightarrow \infty} \theta_n = \theta$, (3.40) further implies that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

More so, using inequality (3.1) again, we have $\theta_i t_{i+1} \leq \psi_{i-1}(1 - \theta_{i-1}) - \theta_i t_{i+1} \left(\theta_i + \left[\psi_i(1 - \theta_i) + \psi_{i-1}(1 - \theta_{i-1}) \right]_+ \right)$.

And it ensues that

$$\begin{aligned} \sum_{i=1}^{n-1} \theta_i t_{i+1} \|x_i - x_{i-1}\|^2 &\leq \sum_{i=1}^{n-1} \left(\psi_{i-1}(1 - \theta_{i-1}) - \theta_i t_{i+1} \left(\theta_i + \left[\psi_i(1 - \theta_i) + \psi_{i-1}(1 - \theta_{i-1}) \right]_+ \right) \right) \|x_i - x_{i-1}\|^2 \\ &\leq \sum_{i=1}^{n-1} \psi_{i-1}(1 - \theta_{i-1}) \|x_i - x_{i-1}\|^2. \end{aligned} \quad (3.41)$$

Taking the limit of (3.41) and using (3.40), we conclude that:

$$\sum_{i=1}^{\infty} \theta_i t_{i+1} \|x_i - x_{i-1}\|^2 < \infty.$$

CASE II:

Assume that $\alpha_n = 1$, for every $n \geq 1$. Then, from the definitions of $\{x_{n+1}\}$ and ψ_n , we have that $x_{n+1} = z_n$ and $\psi_n = 0$ for each $n \geq 1$.

From Lemma 2.10 (ii), define $s := w_n - y_n$, $z := z_n - y_n$, then

$$\|w_n - y_n\|^2 + \|z_n - y_n\|^2 = \frac{1}{2} \left[\|w_n - y_n + x_{n+1} - y_n\|^2 + \|w_n - x_{n+1}\|^2 \right].$$

Using the above results in Lemma 3.12, it ensues that

$$\begin{aligned} &\Phi_{n+1} - \Phi_n - \theta_n(\Phi_n - \Phi_{n-1}) \\ &\leq \frac{1}{2} \left((\theta_n + \theta_n^2) - \psi_n(1 - \theta_n)^2 \right) \|x_n - x_{n-1}\|^2 \\ &\quad - \frac{\alpha_n}{2} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) (\|w_n - y_n\|^2 + \|z_n - y_n\|^2) \\ &\quad - \frac{1}{2} \psi_n(1 - \theta_n) (\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2) \\ &= \frac{1}{2} (\theta_n + \theta_n^2) \|x_n - x_{n-1}\|^2 \\ &\quad - \frac{1}{4} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) (\|w_n - y_n + x_{n+1} - y_n\|^2 + \|x_{n+1} - y_n\|^2) \\ &\leq \frac{1}{2} (\theta_n + \theta_n^2) \|x_n - x_{n-1}\|^2 - \frac{1}{4} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) \|x_{n+1} - y_n\|^2. \end{aligned} \quad (3.42)$$

Using (3.30) in (3.42), it follows that

$$\begin{aligned} & \Phi_{n+1} - \Phi_n - \theta_n(\Phi_n - \Phi_{n-1}) \\ & \leq \frac{1}{2}(\theta_n + \theta_n^2)\|x_n - x_{n-1}\|^2 - \frac{1}{4}\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\left((1 - \theta_n)^2\|x_n - x_{n-1}\|^2\right. \\ & \quad \left.+ (1 - \theta_n)(\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2)\right). \end{aligned}$$

Notice that $\lim_{n \rightarrow \infty} \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > 0$. Thus, there exists $N_0 \in \mathbb{N}$ such that $\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right) > 0 \quad \forall n \geq N_0$. So, since $\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right) > 0 \quad \forall n \geq N_0$, we have

$$\begin{aligned} \Phi_{n+1} - \Phi_n & \leq \theta_n(\Phi_n - \Phi_{n-1}) + \frac{1}{2}\left[(\theta_n + \theta_n^2) - \frac{1}{2}(1 - \theta_n)^2\right]\|x_n - x_{n-1}\|^2 \\ & \quad - \frac{1}{4}(1 - \theta_n)\left[\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2\right]. \end{aligned}$$

By letting $c_n := \Phi_n - \Phi_{n-1}$,

$$d_n := \frac{1}{2}\left[(\theta_n + \theta_n^2) - \frac{1}{2}(1 - \theta_n)^2\right]\|x_n - x_{n-1}\|^2 - \frac{1}{4}(1 - \theta_n)\left[\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2\right],$$

then, we can apply Lemma 2.8 (i) and proceed with a similar argument as in Lemma 3.13 to obtain

$$\begin{aligned} & \sum_{i=1}^{n-1} t_{i+1,n} \left[\frac{1}{2}(1 - \theta_i)^2 - (\theta_i + \theta_i^2) \right] \|x_i - x_{i-1}\|^2 \\ & \quad + \frac{1}{2} \sum_{i=1}^{n-1} t_{i+1,n} (1 - \theta_i) \left[\|x_{i+1} - x_i\|^2 - \|x_i - x_{i-1}\|^2 \right] \\ & \leq 2t_1|\Phi_1 - \Phi_0| + 2\Phi_0. \end{aligned} \tag{3.43}$$

In a similar way as it is in (3.35), we have

$$\begin{aligned} & \sum_{i=1}^{n-1} t_{i+1,n} (1 - \theta_i) \left[\|x_{i+1} - x_i\|^2 - \|x_i - x_{i-1}\|^2 \right] \\ & \geq \sum_{i=1}^{n-1} \left(t_{i,n} (1 - \theta_{i-1}) - t_{i+1,n} (1 - \theta_i) \right) \|x_i - x_{i-1}\|^2 - t_1 (1 - \theta_0) \|x_1 - x_0\|^2. \end{aligned} \tag{3.44}$$

Combining (3.43) and (3.44), yields that

$$\begin{aligned} & \sum_{i=1}^{n-1} t_{i+1,n} \left[\frac{1}{2} (1 - \theta_i)^2 - (\theta_i + \theta_i^2) \right] \|x_i - x_{i-1}\|^2 \\ & \quad + \frac{1}{2} \sum_{i=1}^{n-1} \left(t_{i,n} (1 - \theta_{i-1}) - t_{i+1,n} (1 - \theta_i) \right) \|x_i - x_{i-1}\|^2 \\ & \leq 2t_1 |\Phi_1 - \Phi_0| + 2\Phi_0 + \frac{1}{2} t_1 (1 - \theta_0) \|x_1 - x_0\|^2. \end{aligned} \quad (3.45)$$

Furthermore, following a similar argument as it is in (3.37) and (3.38) of Lemma 3.13, we arrive at

$$\begin{aligned} & t_{i+1,n} \left(\frac{1}{2} (1 - \theta_i)^2 - (\theta_i + \theta_i^2) \right) + \frac{1}{2} \left(t_{i,n} (1 - \theta_{i-1}) - t_{i+1,n} (1 - \theta_i) \right) \\ & \geq \frac{1}{2} (1 - \theta_{i-1}) - \theta_i t_{i+1} \left(1 + \theta_i + \frac{1}{2} [\theta_{i-1} - \theta_i]_+ \right). \end{aligned} \quad (3.46)$$

Combining (3.45) and (3.46), we have that

$$\begin{aligned} & \sum_{i=1}^{n-1} \left(\frac{1}{2} (1 - \theta_{i-1}) - \theta_i t_{i+1} \left(1 + \theta_i + \frac{1}{2} [\theta_{i-1} - \theta_i]_+ \right) \right) \|x_i - x_{i-1}\|^2 \\ & \leq 2t_1 |\Phi_1 - \Phi_0| + 2\Phi_0 + \frac{1}{2} t_1 (1 - \theta_0) \|x_1 - x_0\|^2. \end{aligned} \quad (3.47)$$

Using Assumption 3.2 in (3.47) and following the same argument as in **CASE I**, we have

$$\sum_{i=1}^{\infty} (1 - \theta_{i-1}) \|x_i - x_{i-1}\|^2 < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \theta_i t_{i+1} \|x_i - x_{i-1}\|^2 < \infty.$$

Thus, completing item a. (i).

(ii.) Using Lemmas 3.10 and 3.11, then

$$\Phi_{n+1} - \Phi_n - \theta_n (\Phi_n - \Phi_{n-1}) \leq \frac{1}{2} (\theta_n + \theta_n^2) \|x_n - x_{n-1}\|^2 - \frac{1 - \alpha_n}{2\alpha_n} \|x_{n+1} - w_n\|^2.$$

Note that $\theta_n^2 \leq \theta_n$, $\alpha_n > 0$, $\forall n \geq 1$.

Thus,

$$\Phi_{n+1} - \Phi_n \leq \theta_n (\Phi_n - \Phi_{n-1}) + \theta_n \|x_n - x_{n-1}\|^2. \quad (3.48)$$

Let $c_n := \Phi_n - \Phi_{n-1}$, $d_n := \theta_n \|x_n - x_{n-1}\|^2$, then, we utilize Lemma 2.8(ii) and item a. (i.) of Lemma 3.14 to obtain

$$\sum_{n=1}^{\infty} [\Phi_n - \Phi_{n-1}]_+ \leq t_1 [\Phi_1 - \Phi_0]_+ + \sum_{n=1}^{\infty} t_{n+1} [\theta_n \|x_n - x_{n-1}\|^2]_+ < \infty.$$

That is, $\sum_{n=1}^{\infty} [\Phi_n - \Phi_{n-1}]_+ < \infty$. In addition, this implies for each $\bar{r} \in \mathcal{C}$ that $\lim_{n \rightarrow \infty} \Phi_n = \frac{1}{2} \lim_{n \rightarrow \infty} \|x_n - \bar{r}\|^2$ exists. Thus, as claimed, $\{x_n\}$ is indeed bounded.

(b. i.) Using Lemma 3.12,

$$\begin{aligned} \text{define } \delta_n &:= \frac{\alpha_n}{2} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) \left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2 \right) - \frac{1}{2} \psi_n (1 - \theta_n) \left(\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 \right), \\ \text{then we have} \end{aligned}$$

$$\begin{aligned} \Phi_{n+1} - \Phi_n - \theta_n (\Phi_n - \Phi_{n-1}) &\leq \frac{1}{2} \left((\theta_n + \theta_n^2) - \psi_n (1 - \theta_n)^2 \right) \|x_n - x_{n-1}\|^2 - \delta_n \\ &\leq \frac{1}{2} (\theta_n + \theta_n^2) \|x_n - x_{n-1}\|^2 - \delta_n. \end{aligned}$$

Therefore,

$$\Phi_{n+1} - \Phi_n \leq \theta_n (\Phi_n - \Phi_{n-1}) + \theta_n \|x_n - x_{n-1}\|^2 - \delta_n. \quad (3.49)$$

Again, applying Lemma 2.8(i) to (3.49), we obtain that

$$\begin{aligned} \Phi_n - \Phi_0 &= \sum_{i=1}^n (\Phi_i - \Phi_{i-1}) \\ &\leq t_{1,n} (\Phi_1 - \Phi_0) + \sum_{i=1}^{n-1} t_{i+1,n} \left(\theta_i \|x_i - x_{i-1}\|^2 - \delta_i \right). \end{aligned}$$

This implies that

$$\sum_{i=1}^{n-1} t_{i+1,n} \delta_i \leq \Phi_0 + t_1 (\Phi_1 - \Phi_0) + \sum_{i=1}^{n-1} t_{i+1} \theta_i \|x_i - x_{i-1}\|^2. \quad (3.50)$$

Taking the limit as $n \rightarrow \infty$, and by Lemma 3.14 item a. (i.), thus

$$\sum_{i=1}^{\infty} t_{i+1,n} \delta_i \leq \Phi_0 + t_1 (\Phi_1 - \Phi_0) + \sum_{i=1}^{\infty} t_{i+1} \theta_i \|x_i - x_{i-1}\|^2 < \infty. \quad (3.51)$$

But $\{t_{i,n}\}$ tends increasingly to t_i , as $n \rightarrow \infty$, and by monotone convergence theorem $\sum_{i=1}^{\infty} t_{i+1} \delta_i < \infty$.

This implies that $\lim_{n \rightarrow \infty} t_{n+1} \delta_n = 0$.

But $t_n \geq 1 \ \forall n \geq 1$, and

$$\delta_n := \frac{\alpha_n}{2} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) \left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2 \right) - \frac{1}{2} \psi_n (1 - \theta_n) \left(\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 \right).$$

Thus,

$$\lim_{n \rightarrow \infty} \left(\frac{\alpha_n}{2} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) \left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2 \right) - \frac{1}{2} \psi_n (1 - \theta_n) \left(\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 \right) \right) = 0.$$

Using the facts that $\liminf_{n \rightarrow \infty} \alpha_n > 0$, $\lim_{n \rightarrow \infty} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) = 1 - \mu > 0$, and

$$\lim_{n \rightarrow \infty} \|x_n - x_{n-1}\| = 0,$$

we conclude that

$$\lim_{n \rightarrow \infty} \|w_n - y_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|z_n - y_n\| = 0.$$

- (ii.) With the definition of $w_n = x_n + \theta_n(x_n - x_{n-1})$ and Lemma 3.14 item a. (i.) we have:

$$\begin{aligned} \|x_n - w_n\|^2 &= \|\theta_n(x_n - x_{n-1})\|^2 \\ &= \theta_n^2 \|x_n - x_{n-1}\|^2 \\ &\leq \theta_n \|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.52}$$

This implies that

$$\sum_{n=1}^{\infty} t_{n+1} \|x_n - w_n\|^2 \leq \sum_{n=1}^{\infty} t_{n+1} \theta_n \|x_n - x_{n-1}\|^2 < \infty,$$

Hence, $\sum_{n=1}^{\infty} t_{n+1} \|x_n - w_n\|^2 < \infty$.

Giving us that

$$\lim_{n \rightarrow \infty} t_{n+1} \|x_n - w_n\|^2 = 0.$$

And since $t_n \geq 1 \ \forall n \geq 1$, we have that $\lim_{n \rightarrow \infty} \|x_n - w_n\| = 0$, establishing (ii.).

- (iii.) Recall that $x_{n+1} = (1 - \alpha_n)w_n + \alpha_n z_n$.

This implies

$$\begin{aligned} \alpha_n \|z_n - w_n\| &= \|x_{n+1} - w_n\| \\ &\leq \|x_n - w_n\| + \|x_{n+1} - x_n\|. \end{aligned} \tag{3.53}$$

That is,

$$0 \leq \alpha_n \|z_n - w_n\| \leq \|x_n - w_n\| + \|x_{n+1} - x_n\|.$$

Taking limit as $n \rightarrow \infty$ and, using items a. (i.), b. (ii.) of Lemma 3.14 and the fact that $\liminf_{n \rightarrow \infty} \alpha_n > 0$, we conclude that $\lim_{n \rightarrow \infty} \|z_n - w_n\| = 0$. $\square$

Theorem 3.15 Assume that Assumptions (2.1), 3.1 and 3.2 hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.3 converges weakly to an element of $EP(f, \mathcal{C})$.

Proof Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to $\bar{v} \in \mathcal{C}$.

From items b. (i.) and (ii.) of Lemma 3.14, it follows that $\|x_n - y_n\| \rightarrow 0$, $n \rightarrow \infty$.

This further implies that $\{y_{n_k}\} \rightharpoonup \bar{v} \in \mathcal{C}$.

Moreso, we have that $\{w_{n_k}\}$ and $\{z_{n_k}\}$ converge weakly to $\bar{v} \in \mathcal{C}$.

We now show that $\bar{v} \in EP(f, \mathcal{C})$.

Notice from (3.24) that

$$\lambda_n(f(w_n, z_n) - f(w_n, y_n)) - \frac{\mu\lambda_n(\|w_n - y_n\|^2 + \|z_n - y_n\|^2)}{2\lambda_{n+1}} \leq \lambda_n f(y_n, z_n). \quad (3.54)$$

So that inequalities (3.17) and (3.54) give us that

$$\begin{aligned} \lambda_n(f(w_n, z_n) - f(w_n, y_n)) - \frac{\mu\lambda_n(\|w_n - y_n\|^2 + \|z_n - y_n\|^2)}{2\lambda_{n+1}} \\ \leq \lambda_n f(y_n, z_n) \leq \lambda_n f(y_n, y) - \langle w_n - z_n, y - z_n \rangle \quad \forall y \in \mathcal{C}. \end{aligned} \quad (3.55)$$

Furthermore, from (3.16), we have

$$\lambda_n(f(w_n, y_n) - f(w_n, z_n)) \leq -\langle w_n - y_n, z_n - y_n \rangle. \quad (3.56)$$

Adding (3.55) and (3.56) together gives

$$\begin{aligned} \lambda_n f(y_n, y) \geq \langle w_n - y_n, z_n - y_n \rangle - \frac{\mu\lambda_n(\|w_n - y_n\|^2 + \|z_n - y_n\|^2)}{2\lambda_{n+1}} \\ + \langle w_n - z_n, y - z_n \rangle \quad \forall y \in \mathcal{C}. \end{aligned} \quad (3.57)$$

By Lemma 3.6, it follows that $\lim_{n \rightarrow \infty} \lambda_n$ exists. That is,

$$\lim_{n \rightarrow \infty} \lambda_n = \lambda > 0.$$

Thus, from (3.57) we obtain

$$\begin{aligned} \lambda_{n_k} f(y_{n_k}, y) \geq \langle w_{n_k} - y_{n_k}, z_{n_k} - y_{n_k} \rangle - \frac{\mu\lambda_{n_k}(\|w_{n_k} - y_{n_k}\|^2 + \|z_{n_k} - y_{n_k}\|^2)}{2\lambda_{n_k+1}} \\ + \langle w_{n_k} - z_{n_k}, y - z_{n_k} \rangle \quad \forall y \in \mathcal{C}. \end{aligned} \quad (3.58)$$

Taking the limit of (3.58) as $k \rightarrow \infty$, and using conditions (A1), (A4), and the facts that $\{w_{n_k}\}$, $\{y_{n_k}\}$, $\{z_{n_k}\}$ weakly converge to $\bar{v}$, we obtain that $f(\bar{v}, y) \geq 0 \forall y \in \mathcal{C}$. Thus, showing that $\bar{v} \in EP(f, \mathcal{C})$, and by Lemma 2.11, we conclude that $\{x_n\}$ converges weakly to $\bar{v} \in EP(f, \mathcal{C})$. $\square$

Remark 3.16 (i) Notice that the crucial function of Assumption 3.2 is to ensure the validity of the summability condition

$$\sum_{i=1}^{\infty} \theta_i t_{i+1} \|x_i - x_{i-1}\|^2 < \infty \quad (3.59)$$

obtained in Lemma 3.14 a.(i). Hence, the rest of Lemma 3.14 holds true undoubtedly if we assume (3.59) directly without relying on Assumption 3.2.

- (ii) Note that in a case where $\alpha_n = 1$, for each $n \geq 1$, (our method will become an inertial subgradient extragradient method) and $0 \leq \theta_n \leq \theta \forall n \geq 1$, where $0 \leq \theta < 1$, then, for all $n \geq 1$, $t_n \leq \frac{1}{1-\theta}$. In this settings, the summability condition

$$\sum_{i=1}^{\infty} \theta_i \|x_i - x_{i-1}\|^2 < \infty \quad (3.60)$$

suffices to ensure that (3.59) holds. Therefore, for the convergence of our method when $\alpha_n = 1$, we can replace Assumption 3.2 with the direct assumptions that $\theta_n \in [0, \theta]$ for all $n \geq 1$ and that condition (3.60) is satisfied. Several authors (see, for instance Alvarez and Attouch 2001; Choleamjiak and Suparatulatorn 2023; Rehman et al. 2020; Suparatulatorn et al. 2023; Vinh and Muu 2019 and other quoted references therein) have utilized condition (3.60) when $\theta_n \in [0, \theta]$ for all $n \geq 1$, where $\theta \in [0, 1)$, to guarantee the convergence of inertial schemes.

4 Convergence Rate

In this section, we will show that the iterative sequence $\{x_n\}$ generated by Algorithm 3.3 converges to the unique solution of EP (1.1) with an R -linear rate under the assumption that f is β -strongly pseudomonotone.

Theorem 4.1 *Let $f : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}$ be a β -strongly pseudomonotone bifunction satisfying (A2) and (A3) of Assumption 3.1. Suppose that $EP(f, \mathcal{C}) \neq \emptyset$. Let $\gamma, \rho, \mu, \sigma \in (0, 1)$, $\alpha_n \in (0, 1]$ and $\theta_n \in [0, 1)$ be a non decreasing sequence such that*

$$0 \leq \theta_n \leq \min \left\{ \frac{\rho}{2}, \frac{1}{1 + \frac{2}{\gamma} + \frac{1}{\rho}} \right\}.$$

Assume moreover that

$$0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1.$$

Then, the sequence $\{x_n\}$ produced by Algorithm 3.3 converges to the unique solution $\bar{r}$ of EP(1.1) with an R -linear rate.

Proof In a similar way as it is in Lemma 3.11, if we use the assumption that f is β -strongly pseudomonotone in the place of the pseudomonotonicity condition of f , we have

$$\begin{aligned} \|x_{n+1} - \bar{r}\|^2 &\leq \|w_n - \bar{r}\|^2 - \alpha_n \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) (\|w_n - y_n\|^2 + \|z_n - y_n\|^2) \\ &\quad - \frac{1 - \alpha_n}{\alpha_n} \|x_{n+1} - w_n\|^2 - 2\alpha_n \lambda_n \beta \|y_n - \bar{r}\|^2. \end{aligned}$$

We know that $\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) > 0 \quad \forall n \geq N_0$ and $\alpha_n \in (0, 1]$.

Therefore,

$$\begin{aligned} \|x_{n+1} - \bar{r}\|^2 &\leq \|w_n - \bar{r}\|^2 - \alpha_n \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \|w_n - y_n\|^2 \\ &\quad - \frac{1 - \alpha_n}{\alpha_n} \|x_{n+1} - w_n\|^2 - 2\alpha_n \lambda_n \beta \|y_n - \bar{r}\|^2 \quad \forall n \geq N_0. \end{aligned} \tag{4.1}$$

Since $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\liminf_{n \rightarrow \infty} \alpha_n > 0$, therefore, there exists $\sigma \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = \sigma$. Let

$$\gamma := \min \left\{ \frac{\sigma(1 - \mu)}{2}, \sigma \lambda \beta, \frac{1 - \sigma}{\sigma} \right\},$$

where $\lambda := \lim_{n \rightarrow \infty} \lambda_n$, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_n \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) &= \sigma(1 - \mu) \geq 2\gamma, \\ \lim_{n \rightarrow \infty} \alpha_n \lambda_n \beta &= \sigma \lambda \beta \geq \gamma \quad \text{and} \\ \lim_{n \rightarrow \infty} \frac{1 - \alpha_n}{\alpha_n} &= \frac{1 - \sigma}{\sigma} \geq \gamma. \end{aligned}$$

Therefore, there exists $N_\gamma \in \mathbb{N}$ with $N_\gamma \geq N_0$ such that

$$\begin{aligned} \alpha_n \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) &\geq 2\gamma \quad \forall n \geq N_\gamma, \\ \alpha_n \lambda_n \beta &\geq \gamma \quad \forall n \geq N_\gamma \quad \text{and} \\ \frac{1 - \alpha_n}{\alpha_n} &\geq \gamma \quad \forall n \geq N_\gamma. \end{aligned}$$

Thus, using (4.1), we have that

$$\begin{aligned}
 \|x_{n+1} - \bar{r}\|^2 &\leq \|w_n - \bar{r}\|^2 - \alpha_n \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \|w_n - y_n\|^2 \\
 &\quad - \frac{1 - \alpha_n}{\alpha_n} \|x_{n+1} - w_n\|^2 - 2\alpha_n \lambda_n \beta \|y_n - \bar{r}\|^2 \\
 &\leq \|w_n - \bar{r}\|^2 - 2\gamma \|w_n - y_n\|^2 - 2\gamma \|y_n - \bar{r}\|^2 - \gamma \|x_{n+1} - w_n\|^2 \\
 &\leq \|w_n - \bar{r}\|^2 - \gamma \|w_n - \bar{r}\|^2 - \gamma \|x_{n+1} - w_n\|^2 \\
 &= \rho \|w_n - \bar{r}\|^2 - \gamma \|x_{n+1} - w_n\|^2 \quad \forall n \geq N_\gamma,
 \end{aligned} \tag{4.2}$$

where $\rho := (1 - \gamma) \in (0, 1)$.

Furthermore, we have

$$\begin{aligned}
 \|w_n - \bar{r}\|^2 &= \|(1 + \theta_n)(x_n - \bar{r}) - \theta_n(x_{n-1} - \bar{r})\|^2 \\
 &= (1 + \theta_n)\|x_n - \bar{r}\|^2 - \theta_n\|x_{n-1} - \bar{r}\|^2 + \theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2.
 \end{aligned} \tag{4.3}$$

And also,

$$\begin{aligned}
 \|x_{n+1} - w_n\|^2 &= \|x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\|^2 \\
 &= \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 - 2\theta_n \langle x_{n+1} - x_n, x_n - x_{n-1} \rangle \\
 &\geq \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 - 2\theta_n \|x_{n+1} - x_n\| \|x_n - x_{n-1}\| \\
 &\geq \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 - \theta_n \|x_{n+1} - x_n\|^2 - \theta_n \|x_n - x_{n-1}\|^2 \\
 &= (1 - \theta_n)\|x_{n+1} - x_n\|^2 - \theta_n(1 - \theta_n)\|x_n - x_{n-1}\|^2.
 \end{aligned} \tag{4.4}$$

Using (4.3) and (4.4) in (4.2), we have

$$\begin{aligned}
 \|x_{n+1} - \bar{r}\|^2 &\leq \rho \left[(1 + \theta_n)\|x_n - \bar{r}\|^2 - \theta_n\|x_{n-1} - \bar{r}\|^2 + \theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2 \right] \\
 &\quad - \gamma \left[(1 - \theta_n)\|x_{n+1} - x_n\|^2 - \theta_n(1 - \theta_n)\|x_n - x_{n-1}\|^2 \right] \quad \forall n \geq N_\gamma.
 \end{aligned}$$

Or equivalently

$$\begin{aligned}
 &\|x_{n+1} - \bar{r}\|^2 - \rho\theta_n\|x_n - \bar{r}\|^2 + \gamma(1 - \theta_n)\|x_{n+1} - x_n\|^2 \\
 &\leq \rho \left[\|x_n - \bar{r}\|^2 - \theta_n\|x_{n-1} - \bar{r}\|^2 + \gamma(1 - \theta_n)\|x_n - x_{n-1}\|^2 \right] \\
 &\quad - \left[\rho\gamma(1 - \theta_n) - \rho\theta_n(1 + \theta_n) - \gamma\theta_n(1 - \theta_n) \right] \|x_n - x_{n-1}\|^2 \quad \forall n \geq N_\gamma.
 \end{aligned}$$

Since θ_n is non decreasing and $\rho^2 \leq \rho$, hence, we obtain

$$\begin{aligned} & \|x_{n+1} - \bar{r}\|^2 - \rho\theta_{n+1}\|x_n - \bar{r}\|^2 + \gamma(1 - \theta_{n+1})\|x_{n+1} - x_n\|^2 \\ & \leq \rho \left[\|x_n - \bar{r}\|^2 - \rho\theta_n\|x_{n-1} - \bar{r}\|^2 + \gamma(1 - \theta_n)\|x_n - x_{n-1}\|^2 \right] \\ & \quad - \left[\rho\gamma(1 - \theta_n) - \rho\theta_n(1 + \theta_n) - \gamma\theta_n(1 - \theta_n) \right] \|x_n - x_{n-1}\|^2 \quad \forall n \geq N_\gamma. \end{aligned} \quad (4.5)$$

Define

$$\varphi_n := \|x_n - \bar{r}\|^2 - \rho\theta_n\|x_{n-1} - \bar{r}\|^2 + \gamma(1 - \theta_n)\|x_n - x_{n-1}\|^2,$$

then (4.5) can be written as

$$\varphi_{n+1} \leq \rho\varphi_n - \left[\rho\gamma(1 - \theta_n) - \rho\theta_n(1 + \theta_n) - \gamma\theta_n(1 - \theta_n) \right] \|x_n - x_{n-1}\|^2 \quad \forall n \geq N_\gamma.$$

Now, we show that

$$\rho\gamma(1 - \theta_n) - \rho\theta_n(1 + \theta_n) - \gamma\theta_n(1 - \theta_n) \geq 0 \quad \forall n \geq N_\gamma.$$

With $\theta_n \in [0, 1)$, we have $1 + \theta_n \leq 2$ and $\theta_n(1 - \theta_n) \leq \theta_n$. Thus,

$$\begin{aligned} \rho\gamma(1 - \theta_n) - \rho\theta_n(1 + \theta_n) - \gamma\theta_n(1 - \theta_n) & \geq \rho\gamma(1 - \theta_n) - 2\rho\theta_n - \gamma\theta_n \\ & = \rho\gamma - \theta_n(\rho\gamma + 2\rho + \gamma) \\ & \geq 0, \end{aligned}$$

$$\text{since } \theta_n \leq \frac{1}{1 + \frac{2}{\gamma} + \frac{1}{\rho}}.$$

Therefore,

$$\varphi_{n+1} \leq \rho\varphi_n \quad \forall n \geq N_\gamma.$$

Next, we show that $\varphi_n \geq 0$ for all n .

$$\text{Notice that } \theta_n \leq \frac{1}{1 + \frac{2}{\gamma} + \frac{1}{\rho}} \leq \frac{\gamma\rho}{\gamma\rho + 2\rho} = \frac{\gamma}{2 + \gamma} \leq \frac{\gamma}{2 - \gamma}.$$

Using this fact, we obtain

$$\begin{aligned}
 \varphi_n &= \|x_n - \bar{r}\|^2 - \rho\theta_n\|x_{n-1} - \bar{r}\|^2 + \gamma(1 - \theta_n)\|x_n - x_{n-1}\|^2 \\
 &\geq \|x_n - \bar{r}\|^2 - 2\rho\theta_n\|x_n - \bar{r}\|^2 - 2\rho\theta_n\|x_n - x_{n-1}\|^2 + \gamma(1 - \theta_n)\|x_n - x_{n-1}\|^2 \\
 &= (1 - 2\rho\theta_n)\|x_n - \bar{r}\|^2 + [\gamma - \theta_n(\gamma + 2\rho)]\|x_n - x_{n-1}\|^2 \\
 &= (1 - 2\rho\theta_n)\|x_n - \bar{r}\|^2 + [\gamma - \theta_n(2 - \gamma)]\|x_n - x_{n-1}\|^2 \\
 &\geq (1 - 2\rho\theta_n)\|x_n - \bar{r}\|^2 \\
 &\geq 0,
 \end{aligned}$$

since $\theta_n \leq \frac{\rho}{2}$.

Thus,

$$\begin{aligned}
 \varphi_{n+1} &\leq \rho\varphi_n \leq \dots \leq \rho^{n-N_\gamma+1}\varphi_{N_\gamma}. \\
 \|x_n - \bar{r}\|^2 &\leq \frac{\varphi_{N_\gamma}}{\rho^{N_\gamma-1}}\rho^n.
 \end{aligned}$$

Hence, $\{x_n\}$ converges R -linearly to $\bar{r}$. □

5 Application to Variational Inequality and Fixed Point Problems

5.1 Variational Inequality Problem

In relation to the variational inequality problem defined in (1.2) where the cost bifunction is $f(s, z) = \langle \mathcal{B}s, z - s \rangle$, $\forall s, z \in \mathcal{C}$ and the operator $\mathcal{B} : \mathcal{C} \rightarrow \mathcal{H}$ has the Lipschitz constant $L = 2\delta_1 = 2\delta_2$. Using the fact that $\arg \min\{h(z) : z \in \mathcal{C}\} = \arg \min\{h(z) + b : z \in \mathcal{C}\}$, the sequences $\{y_n\}$ and $\{z_n\}$ of Algorithm 3.3 can be respectively translated into

$$\begin{aligned}
 y_n &= \arg \min_{y \in \mathcal{C}} \left\{ \frac{1}{2}\|y - w_n\|^2 + \lambda_n f(w_n, y) \right\} \\
 &= \arg \min_{y \in \mathcal{C}} \left\{ \frac{1}{2}\|y - w_n\|^2 + \lambda_n \langle \mathcal{B}w_n, y - w_n \rangle + \frac{1}{2}\lambda_n^2 \|\mathcal{B}w_n\|^2 \right\} \\
 &= \arg \min_{y \in \mathcal{C}} \left\{ \frac{1}{2}\|(w_n - \lambda_n \mathcal{B}w_n) - y\|^2 \right\} \\
 &= \mathcal{P}_{\mathcal{C}}(w_n - \lambda_n \mathcal{B}w_n).
 \end{aligned}$$

And similarly,

$$z_n = \mathcal{P}_{T_n}(w_n - \lambda_n \mathcal{B}y_n).$$

Assumptions 5.1 *Assumption on the operator $\mathcal{B}$.*

- B1. For some $L > 0$, $\mathcal{B}$ is L -Lipschitz continuous, that is $\|\mathcal{B}s - \mathcal{B}z\| \leq L\|s - z\|$, $\forall s, z \in \mathcal{C}$.
- B2. $\mathcal{B}$ is pseudomonotone on $\mathcal{C}$, meaning, $\langle \mathcal{B}s, z - s \rangle \geq 0 \implies \langle \mathcal{B}z, s - z \rangle \leq 0$, $\forall s, z \in \mathcal{C}$; and the solution set $\mathcal{VI}(\mathcal{B}, \mathcal{C})$ is nonempty.
- B3. $\limsup_{n \rightarrow \infty} \langle \mathcal{B}s_n, z - s_n \rangle \leq \langle \mathcal{B}\bar{r}, z - \bar{r} \rangle$ for each $z \in \mathcal{C}$ and $\{s_n\} \subset \mathcal{C}$ satisfying $\{s_n\} \rightharpoonup \bar{r}$.

Corollary 5.2 Let $\mathcal{B} : \mathcal{C} \longrightarrow \mathcal{H}$ satisfy Assumption 5.1. Assume that Assumptions (2.1), 3.2 hold. Let $\{t_i\}$ be the sequence defined by (2.2). Iteratively generate the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ as follows:

Let $x_0, x_1 \in \mathcal{H}$, $\lambda_1 > 0$, $\mu \in]0, 1[$.

Given the current iterates x_{n-1} and x_n , construct a half space

$$T_n = \{z \in \mathcal{H} : \langle w_n - \lambda_n v_n - y_n, z - y_n \rangle \leq 0\}.$$

$$\text{Compute} \quad \begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \mathcal{P}_{\mathcal{C}}(w_n - \lambda_n \mathcal{B}w_n), \\ z_n = \mathcal{P}_{T_n}(w_n - \lambda_n \mathcal{B}y_n), \\ x_{n+1} = (1 - \alpha_n)w_n + \alpha_n z_n. \end{cases}$$

Update the stepsize λ_{n+1} as follows:

$$\lambda_{n+1} = \begin{cases} \min \left\{ \lambda_n, \frac{\mu}{2} \frac{\|w_n - y_n\|^2 + \|z_n - y_n\|^2}{\langle \mathcal{B}w_n - \mathcal{B}y_n, z_n - y_n \rangle} \right\} & \text{if } \langle \mathcal{B}w_n - \mathcal{B}y_n, z_n - y_n \rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$

Then, the sequences $\{x_n\}$, $\{w_n\}$ and $\{y_n\}$ weakly converge to $\bar{v} \in \mathcal{VI}(\mathcal{B}, \mathcal{C})$.

5.2 Fixed Point Problem

Let $\mathcal{H} = \mathcal{H}^*$ be a Hilbert space and $\mathcal{T} : \mathcal{C} \longrightarrow \mathcal{C}$ be a given mapping. The fixed point problem is described as:

$$\text{obtain } s^* \in \mathcal{C} \text{ for which } s^* = \mathcal{T}s^*. \quad (5.1)$$

Set $f(s, z) := \langle s - \mathcal{T}s, z - s \rangle$, therefore s^* solves EP(1.1) if and only if s^* is a solution of (5.1).

It is obviously clear that (5.1) implies EP(1.1). Also, suppose that EP(1.1) is fulfilled, then select $z := \mathcal{T}s^*$ to have

$$\begin{aligned} 0 \leq f(s^*, z) &= \langle s^* - \mathcal{T}s^*, \mathcal{T}s^* - s^* \rangle \\ &= -\|s^* - \mathcal{T}s^*\|^2 \\ &\leq 0, \end{aligned}$$

hence, $s^* = Ts^*$. Therefore, EP(1.1) implies (5.1). In this case, f is a monotone bifunction if and only if T is non-expansive, that is

$$\langle T(s - z), s - z \rangle \leq \|s - z\|^2, \quad \forall s, z \in \mathcal{C}.$$

Definition 5.3 The mapping $T : \mathcal{C} \subset \mathcal{H} \longrightarrow \mathcal{C} \subset \mathcal{H}$ is said to be α -strict pseudo-contractive if there is a constant $\alpha \in (0, 1)$ for which

$$\|Ts - Tz\|^2 \leq \|s - z\|^2 + \alpha \|\mathbf{I} - T)(s - z)\|^2, \quad \forall s, z \in \mathcal{C}, \quad (5.2)$$

or, equivalently,

$$\langle T(s - z), s - z \rangle \leq \|s - z\|^2 - \frac{1 - \alpha}{2} \|\mathbf{I} - T)(s - z)\|^2, \quad \forall s, z \in \mathcal{C}. \quad (5.3)$$

Theorem 5.4 If T is α -strict pseudo-contractive, then f is a pseudomonotonic bifunction.

Proof Let $f(s, z) = \langle s - Ts, z - s \rangle \quad \forall s, z \in \mathcal{C}$.

Using (5.3) we have

$$\begin{aligned} f(z, s) &= \langle z - Tz, s - z \rangle \\ &= \langle z - s + s - Ts + Ts - Tz, s - z \rangle \\ &= -\|s - z\|^2 + \langle s - Ts, s - z \rangle + \langle Ts - Tz, s - z \rangle \\ &\leq -\|s - z\|^2 + \langle Ts - Tz, s - z \rangle \quad \forall s, z \in \mathcal{C} \\ &= -\|s - z\|^2 + \|s - z\|^2 - \frac{1 - \alpha}{2} \|\mathbf{I} - T)(s - z)\|^2 \\ &= -\frac{1 - \alpha}{2} \|\mathbf{I} - T)(s - z)\|^2 \\ &\leq 0. \end{aligned}$$

Thus, f is pseudomonotone. □

Moreover, using Cauchy-Schwarz and triangle inequalities, and since

$\mathcal{T}$ is $\frac{2-\alpha}{1-\alpha}$ -Lipschitz continuous, we have that,

$$\begin{aligned}
 f(s, w) &= \langle s - \mathcal{T}s, w - s \rangle \\
 &= \langle s - \mathcal{T}s, z - s \rangle + \langle z - \mathcal{T}z, w - z \rangle + \langle s - \mathcal{T}s + \mathcal{T}z - z, w - z \rangle \\
 &= f(s, z) + f(z, w) + \langle s - \mathcal{T}s + \mathcal{T}z - z, w - z \rangle \\
 &\leq f(s, z) + f(z, w) + (\|s - z\| + \|\mathcal{T}z - \mathcal{T}s\|)\|w - z\| \\
 &\leq f(s, z) + f(z, w) + \|s - z\|\|w - z\| + \frac{2-\alpha}{1-\alpha}\|s - z\|\|w - z\| \\
 &= f(s, z) + f(z, w) + \frac{3-2\alpha}{1-\alpha}\|s - z\|\|w - z\| \\
 &\leq f(s, z) + f(z, w) + \frac{3-2\alpha}{2(1-\alpha)}(\|s - z\|^2 + \|w - z\|^2).
 \end{aligned}$$

Hence, f satisfies Lipschitz-type conditions on $\mathcal{C}$ with constants $\delta_1 = \delta_2 = \frac{3-2\alpha}{2(1-\alpha)}$, where $\alpha \in (0, 1)$.

From (5.1) where the cost bifunction $f(s, z) = \langle s - \mathcal{T}s, z - s \rangle$, the strongly convex programs one needs to solve in algorithm 3.3 are as follows:

$$\begin{aligned}
 y_n &= \arg \min_{y \in \mathcal{C}} \left\{ \frac{1}{2} \|y - w_n\|^2 + \lambda_n f(w_n, y) \right\} \\
 &= \arg \min_{y \in \mathcal{C}} \left\{ \frac{1}{2} \|y - w_n\|^2 + \lambda_n \langle w_n - \mathcal{T}w_n, y - w_n \rangle \right\} \\
 &= \arg \min_{y \in \mathcal{C}} \left\{ \frac{1}{2} \|y - (w_n - \lambda_n(w_n - \mathcal{T}w_n))\|^2 \right\} \\
 &= \mathcal{P}_{\mathcal{C}}[w_n - \lambda_n(w_n - \mathcal{T}w_n)] \\
 &= \mathcal{P}_{\mathcal{C}}[(1 - \lambda_n)w_n + \lambda_n \mathcal{T}w_n].
 \end{aligned}$$

And,

$$z_n = \arg \min_{y \in \Omega_n} \left\{ \frac{1}{2} \|y - w_n\|^2 + \lambda_n f(y_n, y) \right\}$$

which using similar argument as above gives

$$z_n = \mathcal{P}_{\Omega_n}[w_n - \lambda_n(y_n - \mathcal{T}y_n)].$$

Corollary 5.5 *Let $\mathcal{C}$ be a closed convex nonempty subset of $\mathcal{H}$ and $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{C}$ be weakly continuous α -strict pseudo-contractive mapping with $\text{Fix}(\mathcal{T}) \neq \emptyset$. Suppose that Assumptions (2.1) and 3.2 hold. Let $\{t_i\}$ be the sequence defined by (2.2). Iteratively generate the sequences $\{x_n\}$, $\{y_n\}$, $\{w_n\}$, $\{z_n\}$ as follows:*

Let $x_0, x_1 \in \mathcal{H}$, $\lambda_1 > 0$, $\mu \in]0, 1[$.

Start with the current iterates x_{n-1} and x_n , construct a half space

$$\Omega_n = \{z \in \mathcal{H} : \langle w_n - \lambda_n v_n - y_n, z - y_n \rangle \leq 0\}.$$

$$\text{Evaluate} \quad \begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \mathcal{P}_{\mathcal{C}} \left[(1 - \lambda_n)w_n + \lambda_n \mathcal{T} w_n \right], \\ z_n = \mathcal{P}_{\Omega_n} \left[w_n - \lambda_n(y_n - \mathcal{T} y_n) \right], \\ x_{n+1} = (1 - \alpha_n)w_n + \alpha_n \mathcal{P}_{\Omega_n}(u_n), \\ u_n = w_n - \lambda_n(y_n - \mathcal{T} y_n). \end{cases}$$

Update the stepsize λ_{n+1} as follows:

$$\lambda_{n+1} = \begin{cases} \min \left\{ \lambda_n, \frac{\mu}{2} \frac{\|w_n - y_n\|^2 + \|z_n - y_n\|^2}{\langle (\mathbf{I} - \mathcal{T})(w_n - y_n), z_n - y_n \rangle} \right\} & \text{if } \langle (\mathbf{I} - \mathcal{T})(w_n - y_n), z_n - y_n \rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$

Then the generated sequences $\{x_n\}$, $\{w_n\}$, $\{y_n\}$ converge weakly to $\bar{v} \in \text{Fix}(\mathcal{T})$.

6 Numerical experiments

In this section, we conduct some numerical experiments to illustrate the effectiveness, efficiency and implementability of our proposed algorithm. We compare its (Algorithm 3.3) performance to that of four state-of-the-art algorithms: Algorithm 1 in Rehman et al. (2020), Algorithm 1 in Vinh and Muu (2019), Algorithm 2 in Cholanjiak and Suparatulatorn (2023), Algorithm 1 in Suparatulatorn et al. (2023), Algorithm 2 in Yao et al. (2024), Algorithm 3.1 in Pham et al. (2021) and Algorithm 3.1 in Yang and Liu (2020). Throughout this section, we shall for brevity refer to these algorithms as RKSPW (Alg. 1), VM (Alg. 1), CS (Alg. 2), SCJ (Alg. 1), YASY (Alg. 2), HTV (Alg. 4.1) and YL (Alg. 3.1), respectively.

Example 6.1 We consider the Nash-Cournot oligopolistic equilibrium model in Facchinei and Pang (2007) where the bifunction f in $\mathbb{R}^N$ is of the form:

$$f(x, y) = \langle Px + Qy + q, y - x \rangle,$$

where $q \in \mathbb{R}^N$ and P, Q are two matrices of order N such that Q is symmetric positive semi-definite and $Q - P$ is symmetric negative semi-definite. The feasible set is defined as $\mathcal{C} = \{x \in \mathbb{R}^N : -5 \leq x_i \leq 5\}$. The bifunction satisfies Assumption 3.1 with $\delta_1 = \delta_2 = \frac{1}{2} \|P - Q\|$ (see Quoc et al. 2008). The vectors x_0, x_1, q are generated randomly and uniformly in $[-N, N]$ and the two matrices P, Q are generated randomly such that their properties are satisfied.

Example 6.2 Let $\mathcal{H} = (\ell_2(\mathbb{R}), \|\cdot\|_2)$ be the linear spaces whose elements are all 2-summable sequences $\{x_i\}_{i=1}^\infty$ of scalars in $\mathbb{R}$, that is

$$\ell_2(\mathbb{R}) = \mathcal{H} = \left\{ x = (x_1, x_2, \dots, x_i, \dots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$$

with inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \rightarrow \mathbb{R}$ and norm $\|\cdot\| : \ell_2 \rightarrow \mathbb{R}$ defined by $\langle x, y \rangle := \sum_{j=1}^{\infty} x_j y_j$ and $\|x\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}$, for $x = \{x_i\}_{i=1}^\infty$, $y = \{y_i\}_{i=1}^\infty \in \ell_2(\mathbb{R})$.

Let $C = \left\{ x \in \mathcal{H} : \|x\| \leq 1 \right\}$. Define the bifunction $f : C \times C \rightarrow \mathbb{R}$ by $f(x, y) = (3 - \|x\|)\langle x, y - x \rangle, \forall x, y \in C$. It is easy to show that f is a pseudomonotone bifunction which is not monotone and f satisfies the Lipschitz-type condition with constants $\delta_1 = \delta_2 = \frac{5}{2}$. Also, f satisfies Assumptions 3.1 ((A3)-(A5)). We consider the following cases for the numerical experiments of this example

Case 1: Take $x_1 = \left(\frac{5}{7}, \frac{1}{7}, \frac{1}{35}, \dots \right)$ and $x_0 = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \dots \right)$.

Case 2: Take $x_1 = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \dots \right)$ and $x_0 = \left(\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots \right)$.

Case 3: Take $x_1 = \left(\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots \right)$ and $x_0 = \left(\frac{2}{3}, \frac{1}{5}, \frac{1}{10}, \dots \right)$.

Throughout the computation, we utilize the following:

- Algorithm 3.3: $\theta_n = \frac{19}{20} - \frac{1}{(n+1)^{\frac{1}{2}}}$, $\alpha_n = \frac{1}{20} + \frac{1}{(n+1)^2}$ (which satisfy assumption (2.1) and Assumption 3.1 by Proposition 3.9), $\lambda_1 = 0.9 \min\{0.5\delta_1, 0.5\delta_2\}$, and $\mu = 0.1$.
- RKSPW (Alg. 1) in Rehman et al. (2020): $\lambda_1 = 0.1, \delta = 0.9, \sigma = 0.9 \min\{1, 0.5a_1, 0.5a_2\}$, $\mu = 0.9\sigma$ and $\epsilon_n = \frac{100}{(n+1)^2}$.
- VM (Alg. 1) in Vinh and Muu (2019): $\lambda = 0.9 \min\{0.5a_1, 0.5a_2\}$, $\theta = 0.9$ and $\epsilon_n = \frac{100}{(n+1)^2}$.
- CS (Alg. 2) in Chalamjiak and Suparatulatorn (2023): $\lambda_1 = 0.9 \min\{0.5a_1, 0.5a_2\}$, $\alpha_n = \frac{1}{2n+1}$, $\mu = 0.1$, $\beta_n = \frac{n}{100n+1}$, $\nabla\phi(x) = \rho x$ (where $\rho \in [0, 1)$), and $\delta_n := \begin{cases} \frac{0.5}{n^2 \|x_n - x_{n-1}\|}, & \text{if } x_n \neq x_{n-1} \\ 0.5, & \text{otherwise.} \end{cases}$
- SCJ (Alg. 1) in Suparatulatorn et al. (2023): $\mu_n = \mu = 0.9 \min\{0.5a_1, 0.5a_2\}$, $\eta_n = \frac{1}{n+1}$, $\nabla\phi(x) = \rho x$ (where $\rho \in [0, 1)$), and $\delta_n := \begin{cases} \frac{1}{n \|x_n - x_{n-1}\|}, & \text{if } x_n \neq x_{n-1} \\ 0.5, & \text{otherwise.} \end{cases}$
- YASY (Alg. 1) in Yao et al. (2024): $\mu = 10^{-5}$, $\lambda_0 = 0.001$, $\delta = \frac{2}{3}$, $\theta = 0.001$, $\alpha_n = -0.1$ and $t_n = \frac{1}{(p+1)^4}$.
- HTV (Alg. 4.1) in Pham et al. (2021): $\varphi = \frac{\sqrt{5}+1}{2}$ and $\lambda_n = \frac{1}{p+1}$.
- YL (Alg. 3.1) in Yang and Liu (2020): $\mu = 10^{-5}$, $\delta = 0.5$, $\alpha = 0.6$, $\theta = 0.2$, and $\lambda_1 = 0.001$.

Table 1 Numerical results for Example 6.1 with $\varepsilon = 10^{-4}$

Algorithms	N=40		N=80		N=120	
	Iter	CPU	Iter	CPU	Iter	CPU
Algorithm 3.3	59	0.0391	48	0.0094	25	0.0051
YASY (Alg. 2)	90	0.0475	65	0.0186	30	0.0151
RKSPW (Alg. 1)	120	0.0690	110	0.0227	49	0.0231
HTV (Alg. 4.1)	152	0.0721	146	0.0522	69	0.0313
YL (Alg. 3.1)	165	0.8131	220	1.0161	165	0.9436
VM (Alg. 1)	220	1.0214	142	0.0497	84	0.6131
CS (Alg. 2)	325	2.0121	220	1.0222	110	0.7313
SCJ (Alg. 3.1)	380	2.4325	240	1.7161	125	0.8436

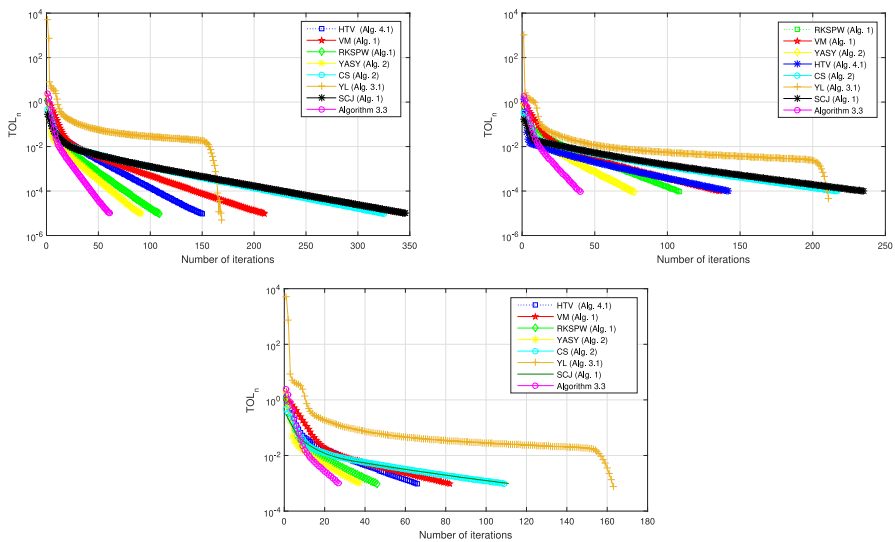


Fig. 1 The behavior of TOL_n with $\varepsilon = 10^{-4}$: Top Left: $N = 40$; Top Right: $N = 80$; Bottom: $N = 120$

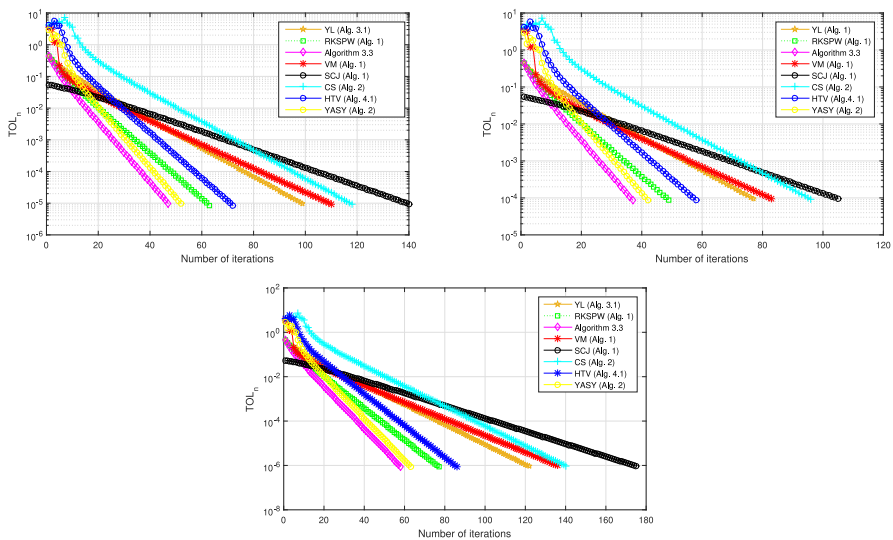
We then use the stopping criterion; $TOL_n := \|y_n - w_n\| < \varepsilon$, where ε is the pre-determined error.

All the computations are performed using Matlab R2023b which is running on a personal computer with an Intel(R) Core(TM) i5-10210U CPU at 2.11GHz and 8.00 Gb-RAM.

In the tables below, “Iter” means the number of iterations. Also, in the tables and figures, Alg. 3.3, RKSPW (Alg. 1), VM (Alg. 1), CS (Alg. 2), SCJ (Alg. 1), YASY (Alg. 2), HTV (Alg. 4.1) and YL (Alg. 3.1) represent Algorithm 3.3, Algorithm 1 in Rehman et al. (2020), Algorithm 1 in Vinh and Muu (2019) Algorithm 2 in Cholamjiak and Suparatulatorn (2023), Algorithm 1 in Suparatulatorn et al. (2023), Algorithm 2 in Yao et al. (2024), Algorithm 4.1 in Pham et al. (2021) and Algorithm 3.1 in Yang and Liu (2020), respectively.

Table 2 Numerical results for Example 6.2 with $\varepsilon = 10^{-4}$

Algorithms	Case 1		Case 2		Case 3	
	Iter	CPU	Iter	CPU	Iter	CPU
Algorithm 3.3	47	0.0221	37	0.0194	58	0.0351
YASY (Alg. 2)	55	0.0325	45	0.0286	64	0.0451
RKSPW (Alg. 1)	65	0.0690	50	0.0327	78	0.0831
HTV (Alg. 4.1)	75	0.0721	58	0.0522	87	0.0913
YL (Alg. 3.1)	99	0.8131	77	0.6161	125	1.0436
VM (Alg. 1)	110	1.0214	85	0.7497	137	1.2131
CS (Alg. 2)	118	1.121	96	0.8222	141	1.3713
SCJ (Alg. 3.1)	141	2.005	110	1.0061	176	1.4436

**Fig. 2** The behavior of TOL_n with $\varepsilon = 10^{-4}$: Top Left: Case 1; Top Right: Case 2; Case 3

Remark 6.3 It is clear from the tables and figures above that our proposed Method 3.3 outperforms (Cholamjiak and Suparatulatorn 2023, Algorithm 2), (Rehman et al. 2020, Algorithm 1), (Suparatulatorn et al. 2023, Algorithm 1), (Vinh and Muu 2019, Algorithm 1), (Yao et al. 2024, Algorithm 2), (Pham et al. 2021, Algorithm 4.1) and (Yang and Liu 2020, Algorithm 3.1) in terms of efficiency and performance, thanks to the added inertial extrapolation step and relaxation technique. These lead to substantial savings in iterations and CPU time for our test problems.

7 Conclusion

In this paper, we have introduced a subgradient extragradient iterative method for solving equilibrium problems in real Hilbert spaces. We combined the inertial technique with the relaxation parameter to obtain a weak convergence result of our proposed method when the cost bivariate function is pseudomonotone and satisfies Lipschitz condition. We further obtained a linear convergence rate under the assumption that the bifunction is strongly pseudomonotone. The Lipschitz constants need not to be known for our method to be effective. Thus, this paper distinguishes itself from the majority of research works in the field by applying techniques that depart from the conventional approaches typically used. In further study, we tend to investigate on a case where the inertial coefficient is allowed to converge to 1.

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Declarations

Competing interests The authors declare that they have no competing interests.

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