Introduction

Much research has been done on the transition from arithmetic to algebra and the quest for making such a transition smoother for the learner is ongoing. The question that remains is how we maintain an obstacle-free journey, especially with content such as long division that has proven to be challenging for learners.

Typically, many educators start a lesson with: “Today we are going to learn how to do long division. It is a difficult procedure, but you will need it later on, so please focus”. Then the algebraic process is discussed using various examples, and drilled over and over, not making mention of the arithmetic procedures that learners have become accustomed to in the earlier grades.

In this study I will explore how this familiarity with arithmetic procedures can assist the learners in the process of learning algebraic long division, which is a purely algebraic activity. From the outset it is assumed that learners are familiar with the arithmetic long division on naturals which they have engaged with since Grade 4 level. This familiar relationship will be investigated in the light of its conclusion in the *division algorithm*, and also in terms of the fluency of the learners in performing the division procedure. Having established learners’ cognitive base for division, the study will progress towards exploring multiple division procedures that can be used by learners. This is specifically done to enhance the progression towards- and the importance of - the *division algorithm*. I use the italics here, and will discuss this use in Chapter 2 by considering the arguments from Bates and Rousseau (1989) regarding the *real division algorithm*.

The next step will be to consider natural numbers as a decomposed unit in base ten using place value, in order to enhance the understanding of structural nature of natural numbers. This decomposition will only be explored in base ten for the purposes of the study, as it provides a sufficient precursor of the algebraic procedure of long division. Long division in base 10 has not been properly exposed as an algorithm in school mathematics across the world and therefore also not in the South African school curriculum, let alone in the training programs for pre-service teachers. It provides a powerful tool for teaching algebraic long division, but this potential will be explored later on in Chapters 4,5 and 6.

The important thing is for learners to really come to terms with the fact that, like all other standard algorithms, this is an algorithm that takes care of ONE DIGIT at a time. Once learners catch on to this idea, when they work with polynomials, it becomes natural to focus on ONE TERM at a time. The transition from arithmetic long division to algebraic long division is then almost seamless.

The work concludes by reflecting on this procedure, which was tested in this action research study, where suggestions are made for teaching and future research.

Chapter 1: Division – The jewel in the crown

Most learners have been engaging with long division and the *division algorithm* since Grade 4. However, by the time they need to use it in Grade 11, they have either forgotten ‘how to do it’, or may simply not have been taught the procedures that lead to the conceptual frame from which the *division algorithm* is defined. From what I have observed, many educators simply omit it from the Grade 9 syllabus, since the learners have great difficulty in grasping the concept of algebraic long division. The reasons for this difficulty that learners experience will be discussed in Chapter 2 when I will draw on the arguments of Tall (1992, 2000) and Thompson (1994) concerning cognitive roots.

The purpose of the pilot study (reported in Chapter 4) was to investigate how Grade 9 and Grade 11 learners respond to using arithmetic to understand and engage with division procedures. The focus was on Grades 9 and 11, since these are the two grades where long division on polynomials becomes important in the process of factoring polynomials of various degrees, in both additional mathematics and mathematics.

In this study my research will focus on the order in which the division process is presented to the learners at a Grade 8 level. I will focus specifically on the concepts that enhance division through multiple representations and lead up to the introduction of the *division algorithm*. It is important for learners to engage with the *division algorithm* whenever they perform division, be it on naturals or dividing polynomials in algebra. The *division algorithm* in this study becomes a guide by which learners can judge the correctness of their work at all times, and it also points to the ‘finality’ of the division procedure on naturals irrespective of the mathematical process that was followed to perform the division.

I will develop more understandable alternative ways of introducing this algorithm, based on learners’ arithmetic experience. This will offer teachers an alternative for introducing a seemingly difficult area of study (and procedure), using the principles of metacognition to build on existing knowledge in order to acquire new knowledge. This will ensure that the cognitive schema is built on a more solid foundation than mere procedural fluency. I will discuss these concepts in detail in Chapter 2.

Long division always appears to be a procedure that is very time consuming and often learners make senseless mistakes when engaging with long division on naturals as well as polynomials. Since the procedure is time consuming, they often do not bother to check their solution in search of the error they might have made, and in some cases they also may not even doubt the result. By using arithmetic division procedures, time spent on the algebraic procedure is almost halved and mistakes are very rarely made by learners. In this sense arithmetic strongly supports the algebraic process with which learners engage when dividing polynomials and makes it much easier as it consists of repetitive division-with-remainder procedures that are cognitively rooted in arithmetic division on (in most cases) two digit numbers. These division procedures will be enhanced by exploring multiple division procedures dealing with naturals and later with polynomials. The procedures are discussed in Chapter 5 where the choice of procedures will be justified by focusing on their interrelatedness.

Based on this rationale the algebraic division procedure temporarily abandons the variable (unknown) and operates on the coefficients (known) when performing the division. Abandoning the variable does in no way undermine the conceptual advancement toward polynomial structure as its cognitive roots are embedded in a process that emulates the algebraic procedure when performing division in base ten decomposition.

Cognitive architecture for algebraic division: Base ten decomposition – an emulation of the ‘real thing’.

Decomposing naturals into a power relationship in base ten has enormous benefits when teaching long division to learners at all levels. This type of long division has not been sufficiently explored before as an algorithm and it has valuable potential for teaching transition mathematics where the main aim is bridging the cognitive gap (Herscovics and Linchevski, 1994) that exists between arithmetic and algebra, especially as far as division is concerned. It forms the genesis of algebraic division by using the properties of arithmetic, with the algebra not residing in the problem but rather in the thinking. Learning long division in base ten decomposition raises the bar for division on naturals well above a highly procedural method, and ensures that new knowledge is built from a soundly established former knowledge base.

Firstly, base ten decomposition ensures that place value is fully preserved, and in fact enhanced, in the division procedure. The critics of the long division procedure argue for its exclusion from the curriculum on the grounds that it un-teaches place value and, among other objections 'encourage learners to give up their own thinking' (Kamii and Dominick, in Wu, 1999). Kamii and Dominick argue that un-teaching place value inhibits the development of number sense.

Klein and Milgram (2000) make some valuable suggestions on where to start when teaching long division to learners, and they argue the opposite of what Kamii and Dominick postulate. They suggest methods for teaching long division in such a way that the underlying mathematical concepts can be understood by students. They stress that before a teacher introduces long division, learners have to have a clear sense of place value. They suggest that numbers should be decomposed in decreasing order of magnitude and that division can play a role here in emphasizing the place value concepts. Their general prescription is based on the base 10 structure of our number system and reads as follows:

> An arbitrary whole number with the digit *k* in the nth decimal means that $k \cdot 10^n$ is less than or equal to the whole number minus the sum of multiples of higher powers of ten that are associated to the places to the left of *k*. But $(k+1) \cdot 10^n$ is bigger than this number minus the same sum of higher powers of ten associated to the places to the left of *k*. (p6).

In short then, using the argument that for an arbitrary whole number *a*, it is always true that $k \cdot 10^n \le a < (k+1) \cdot 10^n$ where *k* is the largest multiple of a power of 10 that is less than *a*. This highly algebraically structured argument was reported on in a study by

Du Plessis and Setati (2007) where learners were dealing with the exact concept of ‘place value division’, using Klein and Milgram’s suggestions.

Secondly, base ten decomposition also provides a strong rationale for performing algebraic long division synthetically or by using the method of detached coefficients while not devaluing the important conceptual issues with which learners have to engage when performing long division. Base ten decomposition successfully emulates the algebraic division procedure without the variable being present. Instead, the expression in base ten shows to learners that the ‘ten’ is only an indication of the place value of the digit in the number. The ten can thus be ignored once the basic skill of reducing its power every time the division shifts to the next digit is learnt. As this ten can be replaced later by any positive integer, its evolution toward the variable becomes seamless. This approach can be critiqued as also taking on a procedural method to teaching division, but this highly procedural approach, under the guidance of the teacher, should and must lead and challenge learners to investigate how and what they are learning in a deeply conceptual manner.

Lastly, base ten decomposition reflects (and changes) the cognitive architecture of division, and specifically long division, by creating a clear and yet challenging increase in the cognitive load involved in division on naturals. It maps the division on naturals more clearly and directly onto algebraic long division by offering teachers a new method that can be used in the transition through the cognitive gap (Herscovics and Linchevski, 1994) that exists between arithmetic and algebra. The process of increasing the cognitive load should begin at the intermediate phase, and should gradually increase as learners are introduced to the structure in arithmetic expressions by exploring basic mathematical principles in a more general manner. This would encourage teachers to make the pedagogical shift in basic arithmetic exploration towards generalising whenever the opportunity to do so emerges in the classroom. This will be discussed in detail in Chapter 3.

Division and its importance in the further learning of mathematics

Leaving the teaching of algebraic long division for the Grade 11 secondary school teacher or the university or college lecturer to complete nullifies the enormous potential for learning a skill embedded in a deeply conceptual landscape and reduces it to a mere procedure necessary for factoring cubic polynomials in order to sketch graphs. It remains a basic skill without which learning of more advanced subject matter cannot advance in a conceptual manner. Wu (1999) states boldly that “conceptual advances are invariably built on the bedrock of technique”. This indicates clearly that basic skills and techniques are sometimes an absolute necessity for learning and advancement to more sophisticated subject matter. This basic skill (specific to division) starts taking form when learners engage with algebraic division problems that divide polynomials with leading coefficients of 1, and linear divisors where the leading coefficient is also 1. Once this basic skill has been mastered and the concept has been applied successfully, the learner progresses on and acquires more advanced skills such as dividing a non-monic polynomial by a rational divisor.

When considering the *division algorithm*, the process of division has to be fully understood and the skill of dividing one quantity into another has to be mastered before we can introduce more advanced and generalised abstractions. These rules and basic skills are founded on arithmetic procedures and a firm knowledge and understanding of how and when the procedure takes a different course. All mathematics is bound to its context and to phenomena that may deviate in approach from the ‘usual’ path. Learners will not be able to recognise such fine pointers if they have not acquired the skill to consider division from a variety of different approaches, and not only as the inverse operation of multiplication. Here the basic skill of division can form fertile ground for understanding of fractions, factoring of numbers, the concept of a remainder, modular arithmetic, the *division algorithm*, the highest common factor, rationalising quadratic and higher order denominators … in short global issues in mathematics which rely on fluency in this basic skill to then develop a deeper conceptual framework within which to operate flexibly. When two fractions are divided into one another, we often teach through rote learning by instructing learners to “invert and multiply”. In arithmetic this procedure gives a quick and easy fix but what about algebra when the concept of a complex fraction is being studied? Here the objects that are divided sometimes require more than just inverting and multiplying. More often than not the process of factoring precedes the actual division.

When learners are engaging with the concept of division, it opens the door to introduce the concept of factoring not only larger numbers, but also polynomials. The recent trend of de-emphasising long division and factoring polynomials can be ascribed to the view that either such skills are no longer useful or necessary or that they are just too difficult for learners to engage with at Grade 9 level. The former argument reflects a deep lack of understanding of the prominent position that mathematics takes in the subject fields of science, economics, engineering and architecture. Often in mathematics many skills must be developed for many years before they can be used effectively or before applications become available, sometimes across subject fields. Here James Milgram (1999) takes the lead in the field when advocating that the long division procedure take its central place in mathematics when he points out that:

- *Learners cannot understand why rational numbers are either terminating or (ultimately) repeating decimals without understanding long division.*
- *Long division is essential in learning to manipulate and factor polynomials.*
- *Polynomial manipulation and factoring are skills critical in calculus and linear algebra, partial fractions and canonical forms*. (p7)

Upon inquiring from my Grade 11 additional mathematics learners what distinguishes an irrational number from a rational number, it was discovered to my amazement that they could not answer this very simple question that invokes the very core of number theory and number relations with which they have been dealing since Grade 4. After a short deliberation they responded with “*oh yes Mr X said so*”. The lack of conceptual framework was clearly evident. The base line concept that decimals represent rational numbers when such decimals terminate or ultimately repeat periodically was not part of their conceptual landscape. At the onset of the pilot study, it became very clear that these learners could not perform long division on all types of polynomials (see Chapter 4) and this could be directly attributed to their not understanding the fine pointers between rational and irrational numbers. Learners simply cannot build a strong conceptual ground for number theory without fully understanding the long division procedure. It is only in the understanding of the process of finding the remainder in long division that you see the periodicity or termination happen that defines a rational number (Milgram, 1999).

A major area where long division is essential in the development of a basic skill is the factoring and manipulation of polynomials. Without being able to factor a polynomial, any attempt at sketching a polynomial function will be superficial and a mere duplication of a procedural recipe that is supplied by the teacher. Finding turning points in polynomial functions involves forming a new polynomial through the derivative, and factoring such a polynomial through some division procedure. De-emphasising the importance of the long division procedure here will surely result in a complete section in mathematics that has been superficially learnt without addressing deeply conceptual issues. Here, for example, the quadratic formula is often (ab)used by both educators and learners, with educators often instructing the learners to use the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ if they struggle to solve (or factor) an equation of the form $ax^2 + bx + c = 0$. This might be an easy way out in solving the quadratic equation, but considering the general solution to the equations of the form $ax^3 + bx^2 + cx + d = 0$ the picture looks completely different with the formula

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

being the general formula for finding such roots. Incidentally this process can be simplified by creating a depressed polynomial which avoids the x^2 term by performing the following steps:

1. Apply the substitution $x = y - \frac{b}{3a}$
2. Substitute to get: $a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0$
3. Multiply out to get: $ay^3 + \left(c - \frac{b^2}{3a}\right)y + \left(d + \frac{2b^3}{27a^3} - \frac{bc}{3a}\right) = 0$
4. Solving the depressed cubic of the form $y^3 + Ay = B$:
    a. Find *s* and *t* such that $3st = A$ and $s^3 - t^3 = B$
    b. $y = s - t$ will be a solution to the depressed cubic.

The concept of a depressed polynomial was discovered by Scipione dal Ferro (1465 – 1526) as one possibility for finding the roots of a cubic polynomial. The mathematics is elegant. However, what is certain here is that engaging learners with this formula will have detrimental results for both the teacher and the learners. It will have to be 'delivered' as a procedure that learners will have to learn by rote in order that it be applied.

Irrespective of the approach from which one considers the factoring of polynomials, long division still remains central and a vital component of the factoring procedure.

<u>In conclusion:</u>

De-emphasising the long division procedure in the school mathematics curriculum is not a wise decision as it has multitudinous and diverse effects on mathematics for high school learners, as well as for tertiary learning across various fields of study. It remains a powerful conceptual tool that unpacks meaning and builds conceptual frames without which areas within mathematics cannot be explored concisely and on a deeply conceptual level. I stand in agreement with Wu (1999) when he says "… *why not consider the* *alternative approach of teaching these algorithms properly before advocating their banishment from classrooms…"* (p4)? If we advocate the de-emphasis of the long division procedure from our classrooms, it will surely result in superficial understanding of highly conceptual tools in the mathematics curriculum. In the United States fierce opposition was expressed to the decision to de-emphasise the long division in the school curriculum, and I strongly believe that we should listen to these petitions and take heed of what is being argued in order to avoid a repetition in the South African context.

Chapter 2 – A search for cognitive roots that connect new learning

Introduction

In this chapter I will explore key themes in mathematics education that inform this study. The eternal search in classroom instruction for cognitive roots that will connect new learning within well organised cognitive schemas while enhancing their applicability within an established cognitive unit is discussed in detail. The elements of metacognition aid learning for understanding and offer a comprehensive line of action for educators. These will assist educators to design learning plans that are consistent with the cognitive skills which learners have attained and their achieved level of productive disposition. I will discuss how learners file this new information within a cognitive schema and how a constructivist approach can aid learning with understanding.

As the current trend in the United States and recently in South Africa calls for the de-emphasis of the long division procedure in our curriculum, I will finally argue in agreement with Wu (1999) and Klein and Milgram (2000) that this basic skill is of utmost importance for meaningful and further learning in mathematics. This argument will be framed by the tensions that exist between conceptual knowledge and procedural knowledge, and these issues will be reviewed as they pertain to the long division procedures.

Establishing a cognitive root

By considering the cognitive roots with which learners have anchored and conceptualized subject content successfully, educators can ensure a solid foundation for any new learning that takes place (Tall, 1992 and 2000). According to Tall, sound cognitive roots form a comprehendible structure, called the cognitive unit, where learners make sense of the interrelatedness of new learning with already existing knowledge. If knowledge is firmly grounded in an established cognitive unit, this knowledge will be accessible in a highly flexible (Hiebert and Lefevre, 1986) manner for learners when they engage with new subject content, or when exploring an extension of already existing concepts.

To establish a comprehensive concept image (Thompson, 1994) requires the identification of a sound cognitive root that will later on develop into a cognitive unit as learners progress through the learning unit. This cognitive root embodies meaningful ideas and lays the groundwork for later theories to develop into a cognitive unit. Tall (1992) offers a more refined definition of a cognitive root. He argues that it is a concept that forms a meaningful cognitive unit of core knowledge at the beginning of a learning unit. It allows initial development through a strategy of cognitive expansion, contains the possibility of long term meaning in later developments and is robust enough to remain useful as more sophisticated understanding develops.

It therefore makes sense that arithmetic should be developed as a cognitive root that can come to the aid of the learner when complicated algebraic topics are being explored. Wheeler (1996) states that algebra is a language, and that we must remember that learners are already familiar with the arithmetic language. Abandoning these firmly grounded roots with which learners organised their cognitive schema results in a mere superficial mention of the conceptual links between algebra and arithmetic and does not aid learning in the intended manner. In this study the long division procedure on naturals will be used as the cognitive root for algebraic long division through a process of decomposing naturals in base ten.

Metacognition: How it assists learning for understanding

Learners in general see mathematics as a subject area with set rules and regulations, and they believe that mastering the subject depends on how well they can recall the formulas and recipes to apply based on coded cues (Boaler, 1999) that are given in the questions. The reasoning behind the applications does not form part of their cognitive schema, and often they blindly follow rules to obtain an answer. This dependence on cues has the result that it only guides their procedural disposition, and not their conceptual thinking. This tendency of learners to respond to cues and recall recipes is the direct consequence of how their teachers have instructed them. It has a very strong negative effect on the mathematical behaviour of learners. Metacognition is a process that increases the meaningfulness of the learning experience.

This process is facilitated by creating a forum in the classroom community that encourages debate, challenges one another's viewpoints, and in general creates an environment where learners can explore issues as a mathematical community. Learners start thinking mathematically and reflect on their approaches when solving problems. Mathematical thinking is a cognitive process and reflection and planning are metacognitive. The one without the other creates a void that is the cause of conflict within cognition as the sense of what is being done and why it is being done is absent.

According to Brown (1987), metacognitive experiences involve the use of metacognitive strategies or metacognitive regulation. These are sequential processes that one uses to control cognitive activities in order to ensure that a cognitive goal has been met. In the words of Livingstone (1997): "These processes help regulate learning".

Teachers of mathematics should bear in mind what Livingstone (1997) pointed to as the essential elements of metacognitive processes when they plan learning materials and classroom activities. According to Livingstone the process of metacognition has three basic elements:

a) Developing a plan of action:

   This requires thorough planning from the teacher, taking all the constraints that learners bring with them into the classroom into consideration. Very important is the starting point from which instruction will commence. Here the prior knowledge of learners becomes very important and planning has to include linking the prior learning and future learning with the new content that will be introduced. During this phase the teacher should continuously bear in mind the cognitive outcome that will be desired.

b) Maintaining and monitoring the plan:

   A constant quest for effective planning should guide implementation. The learners' cognition should be monitored continuously by asking guided questions which deliberately test their understanding and conceptual construction. Based on their responses, the direction and methods of the procedure have to be altered to ensure that gaps do not develop in their understanding of the concept under consideration. This might mean that the pace of the instruction has to vary depending on the cognitive demands of the task. Dialogue in the mathematics classroom is thus very important to gauge the needs of the learners. This can only take place effectively in a constructivist environment where reflection becomes primary to successful learning.

c) Evaluating the plan of action:

This involves reflection on how well the message was received by the learners and how the process can be applied to other problems. Were the questions that learners asked a result of a lack of understanding, suggesting that strategies need to be reconsidered and methods adjusted to overcome these difficulties? Or were they because the new learning interested them in other areas of learning that need subsequent exploration? How can this new learning be grouped under a global heading where the methods that were learnt are going to be useful? An example of this would be grouping, factoring and finding zeros of polynomials as a main heading where the *division algorithm* and long division skills are needed.

This process of metacognition involves thinking about the way we think, and specifically for the teacher, the way in which learners are thinking and learning. It should guide our instruction and, based on this, we should formulate our plan of action. When handling seemingly difficult content with children, we should always search for a cognitive root that can assist and build their conception and conceptual structuring in an integrated and interrelated manner.

Cognitive schema: The filing of new learning

Evolving from Piaget's theories, the cognitive schema is the data bank in which cognition is stored and organized as it relates to a specific content area or topic. The teacher should assist learners by offering constructive stimulus which is built on the specific cognitive roots with which learners have reached fluency, both in cognition and wider applicability. This will help to structure their schema in a sensible and flexible order. Because these schemas are constructed through experience, it is important that educators, through classroom reflection, create the opportunities for learners to resolve conflicts that may occur with cognition. This construction should provide a means to build on past experiences and provide the way for generalizing when newer learning is stored.

Classroom pedagogy should include known areas where conflict occurs, and should also caution learners as to why certain choices are not valid in a particular context. It should also serve to signal the way forward, indicating where the current learning will become applicable in future. Teaching long division as an isolated concept or method does not serve the purpose of organizing the cognitive schema in a sensible manner. In Grade 9 learners are taught how to divide a linear binomial into a quadratic or cubic polynomial. The process often ends at the point where learners have reached procedural fluency, thus neglecting the purpose and usefulness of the *division algorithm*. At this point it would be useful to introduce the *division algorithm*, since lessons usually conclude with the restructuring of the quotient, divisor, dividend and the remainder in a sentence that expresses the product and explains the division process in reverse. (See Figure 1)

$$\underbrace{\begin{array}{r} x-2 \\ (x+1)\overline{)x^2-x-6} \\ \underline{x^2+x} \\ -2x-6 \\ \underline{-2x-2} \\ -4 \end{array}}_{\text{The process of long division}}$$

Thus: $(x+1)(x-2)-4 \equiv x^2-x-6$

Figure 1

A discussion of the concept of a factor and non-factor will, at this point, also be a useful activity at Grade 9 level. In Grade 11, polynomial factorization comes to the fore again, but it is labelled as quadratic equations and its behaviour as a polynomial is completely ignored. Later on in the same year, learners learn how to factorize cubic polynomial expressions. Again the object that they are working with is ignored and the procedure is not generalised to encapsulate polynomial factoring in general, specifically its applicability to higher order polynomials.

Recently, in an additional mathematics class, when introducing the factoring of higher order polynomials, I asked the learners what a polynomial is. Much to my surprise, I found that they could factorize and solve any basic quadratic and cubic equation, but when asked what a polynomial is, no learner could provide a mathematically correct answer. It appeared that these learners were fluent in the procedures of factoring, but what they were factoring was absent from their schema.

In order for learners to operate flexibly within their schema it should be made explicit by the teacher where themes are interlinked and where they can be called upon in support of new learning. This can be achieved firstly by providing headings for global mathematical issues, designing learning units and instructing learners with these headings in mind. This will not only improve their cognition of the content but will also provide an organized cognitive schema through which later learning can find accessible roots to build and structure this schema. Secondly, it becomes apparent here that through the process of metacognition, cognitive roots will be identified by learners and recalled by them in support of the new learning which provides fertile ground for the teacher to manage the negotiation of meaning in this process.

In the case of long division on polynomials the ‘end goal’ of instruction will be informed by the building blocks necessary for crossing the cognitive divide that exists between arithmetic long division on naturals and algebraic long division on polynomials which can be accomplished by explicitly drawing on conceptual links that bind the knowledge systems together. These links need to be explored in classroom debate to create sense for learners when they attempt abstraction.

Constructivism: The road to teaching for understanding

The meaning of a mathematical concept is refined and defined by its use(fullness). From a constructivist standpoint, learners build meaning for a concept based on their interpretation of its use in a variety of contexts. This meaning is negotiated by the learners and the educator and the role of the teacher thus becomes crucial when instructing learners in mathematical processes. The challenge with mathematics teaching and learning in South African classrooms is that some educators still regard mathematical knowledge as a set of facts that were formulated and refined by the genius of various historical figures in the mathematics community, and that this knowledge can only be passed on effectively by refining procedures, through drilling and instructing learners to mindlessly follow these procedures. We find a huge amount of examination training taking place by teaching cue based responses founded in procedural recipes, instead of teaching for understanding. Wu (1999) is a great opponent of these types of practices:

> *"In Mathematics education, this debate takes on the form of 'basic skills or conceptual understanding'. This bogus dichotomy would seem to arise from a common misconception of mathematics held by a segment of the public and the education community: that the demand for precision and fluency in the execution of basic skills in school mathematics runs counter to the acquisition of conceptual understanding."* (Wu, H. 1999. Basic skills vs. Conceptual understanding. p1)

In the above quote, Wu emphasises the point that gaining fluency in mathematical procedures need not be at the expense of conceptual understanding. In fact the two are interdependent. Learners should be made explicitly aware of the uses of mathematical concepts, and the advantages they present to the learner in different problem solving contexts. Only when learners acquire the meaning of what is being studied will they have the liberty and cognition to confidently apply those skills in a much wider variety of situations.

The constructivist approach to teaching provides teachers with the opportunity to explore what their learners are thinking by using conversation and listening to that conversation critically. Heaton (2000) defines the teacher's role as that of an intellectual coach and a facilitator of learning. She describes the teacher as a conductor "orchestrating the oral and written discourse in ways that contribute to learners' understanding of mathematics"(p.4). Seen in this light, a mathematics classroom should be a conversation pit where ideas are challenged, explored, reflected upon and refined, bearing in mind the goals of learning which have to be reached. The goal of teaching should thus be to assist learners in constructing "powerful and reasonable understandings of why particular solutions and problem solving methods make sense" (Heaton, 2000, p5). Teaching should also focus on conceptual development, and not only on content knowledge or skill-drilling. One can argue that there is a symbiosis between considering skills and content, but the content should always be secondary to the primary learning that takes place when skills, which can be applied in a wider context than just the mathematics classroom, are developed. The two must take place in support of one another. Long division is a skill that can be applied in various contexts within mathematics and other subject fields. The application of this skill takes place in content specific environments such as polynomial division and sketching of polynomials.

Here the work of Kilpatrick, Swafford and Findell (2001) and the Rand Mathematics Study Panel (2002) on the five strands of mathematical proficiency are paramount when one teaches for understanding. These strands are interdependent in the development of proficiency in mathematics.

i) <u>Conceptual understanding</u> which deals with the comprehension of mathematical concepts, operations and relations. This entails the development of an integrated and functional grasp of mathematical ideas. Here division is understood as the inverse operation of multiplication or as a process of repeated subtraction. Learners develop an understanding that multiplication by a half is the same as division by two and they can simplify complex fractions by multiplication with a global LCD for the fraction.

ii) Procedural fluency which imparts to the learner the knowledge of procedures and how and when to use them appropriately. Here the skills of accuracy, flexibility and efficiency are developed when informed choices have to be made regarding the appropriate use of a basic skill. In this strand the knowledge of multiples and factors is essential to speed up the division procedure. Skills are applied based on a conscious choice that is rooted in the conceptual structures embedded in the problem.

iii) Strategic competence which reflects the ability to formulate mathematical problems, represent them in an appropriate model, and then solve them within a context. Here the application and relevance of the division skill become part of the learners' cognitive schema. They know that to factorise a higher order polynomial, a factor must be found and then removed from the polynomial by the process of long division. They also possess the ability to choose between different division procedures that can be applied in the context.

iv) Adaptive reasoning: where the ability to think logically about relationships among concepts and situations is developed to foster a culture of not only taking things at face value, but searching for lurking interrelatedness which runs broader than the mathematical content. Here the learner will realise that in order to decompose a fraction into its partial fractions, a division process has to take place to reverse the effect of multiplication on the denominator. They also choose the division skill in order to determine highest common factors to decompose a number into a product of primes or a polynomial into linear factors. The learner understands and recognises that in order to determine whether a rational function possesses an oblique asymptote, a division procedure can be followed.

v) Productive disposition: This is the end result towards which constructivist theory and activities help the learner. Here learners come to see the sense of the mathematics and experience it as worthwhile and useful. A productive disposition develops in sync with the other strands and is a major factor in determining learners' mathematical growth and educational success. Learners may not know a formula for factoring trinomials but understand the factoring procedure as a process that is embedded in division procedures.

These five strands form the foundation for effective and outstanding teaching which is essential in the creation of flexible knowledge. Developing each in isolation causes superficial learning to take place and often the broader picture stays framed inside an abyss of methods, recipes and cue-based responses.

Basic skills and conceptual understanding: A balancing act

Wu (1999) and Hiebert and Lefevre (1986) argue that basic skills and conceptual understanding work hand in hand to assist learners in cognition. Wu asserts that "conceptual advances are invariably built on the bedrock of technique" (p1). This indicates clearly that basic skills and techniques are an absolute necessity for learning and advancement to more sophisticated subject matter. For learners to insist on computing the derivative or the integral from first principles just does not make sense when they have the power rule that can compute these concisely. The concept of a derivative and an integral has been thoroughly explored in meaning and by lengthy calculation which enhanced the conceptual grounds for the rules to follow; now all that needs doing is the application of the rules. The basic skill has been mastered and the concept has been applied successfully. Now it is time to move on and learn about more advanced material.

When considering the *division algorithm*, the process of basic long division has to be understood fully and the skill of dividing one quantity into another has to be mastered before we can introduce more advanced and generalised abstractions. These rules and basic skills are founded in arithmetic procedures and a firm knowledge and understanding of how and when the procedure takes a different course. All mathematics is bound to its context and to phenomena that may deviate in approach from the 'usual' path. Learners will not be able to recognise such fine pointers if they have not acquired the skills to consider division from all different approaches, and not only as the inverse operation of multiplication. Here the basic skill of division can form fertile ground for understanding of fractions, factoring of numbers, the concept of a remainder, modular arithmetic, the *division algorithm*, the highest common factor – in short for global issues in mathematics which rely on fluency in this basic skill to then develop conceptual understanding.

When learners are engaging with the concept of division, it opens the door to introduce the concept of factoring not only larger numbers, but also polynomials. When two fractions are divided into one another, we often teach through rote learning by instructing learners to “invert and multiply”. In arithmetic this procedure gives a quick and easy fix but what about algebra, when the concept of a complex fraction is being studied? Here the objects that are divided sometimes require more than just inverting and multiplying. More often than not the process of factoring precedes the actual division. Yet other methods apply too, and often these make our calculations much simpler. Their roots can be found in arithmetic procedures. Consider the problem $\frac{1}{2} \div \frac{1}{3}$. Here we are measuring how many thirds there are in a half. It is easy to invert and multiply here and then $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \text{ x } \frac{3}{1} = \frac{3}{2}$. This procedure is conceptually rooted in the understanding of division as the inverse process of multiplication. But what if the problem was: $\frac{\frac{1}{2}+\frac{1}{3}}{\frac{3}{2}-\frac{1}{4}}$.

This complex fraction can then be approached in more than one way. Let us see how:

Using separate denominators and inverting, then multiplying	**Using a global LCD for the fraction**
$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{3}{2}-\frac{1}{4}}$ $=\frac{\frac{3+2}{6}}{\frac{6-1}{4}}$ (Separate LCD's) $=\frac{\frac{5}{6}}{\frac{5}{4}}$ $=\frac{5}{6}\times\frac{4}{5}$ (Invert and multiply) $=\frac{4}{6}$ (Simplify) $=\frac{2}{3}$	$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{3}{2}-\frac{1}{4}}$ $= \frac{\frac{1}{2}+\frac{1}{3}}{\frac{3}{2}-\frac{1}{4}} \text{ x } \frac{12}{12}$ (Global LCD) $=\frac{6+4}{18-3}$(Complex fraction dissolved) $=\frac{10}{15}$ (Simplify) $=\frac{2}{3}$

It is clear to see that the second option is a much better way as it gets rid of the complexity of the fraction as soon as possible and almost always averts calculation errors. This skill of deciding when the method is appropriate relies on the mastery of the first three strands of mathematical proficiency as mentioned by Kilpatrick et al (2001). The problems become even more complex when we consider exponential calculations. Changing $\frac{3^{-x}+3^{x}}{3^{-x}-3^{x}}$ into a friendlier equivalent is a skill that is needed to expedite the change in order to get to answer any questions that may follow.

This skill is taught in conjunction with the theory of a complex fraction. To some it might not make sense to apply it to the first problem, $\frac{1}{2}\div\frac{1}{3}$, since it is a simple one. But when the skill is mastered, one can then apply the concept to much more advanced subject matter in a more appropriate manner. Here it is evident that fluency in executing a basic skill is essential for progress. Wu (1999) could not have said it more clearly when he stated that "*when a skill is bypassed in favour of a conceptual approach, the resulting understanding is often too superficial*" (p2). He concludes by saying that a natural consequence of such an approach is that learners "*develop a sense of extreme insecurity upon the sight of any fraction other than the simplest possible*"(p2). We must, therefore, in the pursuit of sound teaching, find a balance between what we consider a basic skill and what we consider conceptual understanding, and define for ourselves a path where the two can complement one another. In this manner we will foster mathematically proficient learners who will not tremble at the sight of 'loaded' expressions where, actually, the mathematics underlying the procedures is rather basic.

This relationship between basic skills and conceptual proficiency is important to establish when teaching for understanding. Hiebert and Lefevre (1986) offer a more comprehensive discussion and analysis of the two types of knowledges. They view conceptual knowledge and procedural knowledge as embedded in the following characteristics:

Conceptual knowledge	Procedural knowledge
a. Knowledge that is rich in relationships both vertical and horizontal b. Connected web of knowledge c. A network in which the linking relationships are as prominent as the discrete pieces of information d. Knowledge is conceptual if the holder recognises its relationships to other pieces of information	Made up of two distinct parts: a. Formal language or symbol representation system of mathematics. This is often called the *form* of maths. Knowledge of the form of mathematics includes knowledge of the syntactic configurations of formal proofs. This implies knowledge of the surface features, and not knowledge of meaning. b. Algorithms, or rules, or procedures for completing mathematical tasks. They are step by step instructions that prescribe how to complete the task. A key feature is that this procedure is executed in a predetermined linear sequence.

According to Hiebert and Lefevre an important feature of the procedural system is that it is structured. Procedures are hierarchically arranged so that some are embedded in others as **subprocedures.** An entire sequence of step by step prescriptions or subprocedures can be characterised as a **superprocedure**. The advantage of creating superprocedures is that all subprocedures in a sequence can be accessed by retrieving a single superprocedure. The subprocedures are thus accessed as a sequential string once the superprocedure is identified.

The development of conceptual knowledge is achieved by constructing relationships between pieces of information. When previously independent networks of knowledge are related, there is a dramatic and significant cognitive reorganisation (Lawler, 1981 in Hiebert and Lefevre, 1986). Through the creation of relationships between existing knowledge and new information that is just entering the system, conceptual density of knowledge is significantly increased and stored within a flexible unit of cognitive recall.

Hiebert and Lefevre discuss extensively how the notions of conceptual and procedural knowledge are related to the issues of meaningful and rote learning. They argue that meaningful learning is generated as relationships between units of knowledge that are recognised and created by the learner. This emphasises that conceptual knowledge must be learned meaningfully. Procedures, on the other hand, may not be learnt with meaning, yet if procedures are learnt with meaning, such procedures are inevitably linked to conceptual knowledge.

They argue that rote learning produces knowledge that lacks relationships and is tied closely to the context in which it is learned. Conceptual knowledge cannot therefore be generated directly by rote learning. Procedures can be learnt by rote and are usually triggered by surface features similar to the original context in which they were learnt. They also infer that the sequential nature of procedures is not violated by rote learning.

Based on the above arguments from Hiebert and Lefevre (1986) it is therefore possible to consider procedures without concepts, but it is not so easy to imagine conceptual knowledge that is not linked with some procedures.

The importance of long division and the real division algorithm

For some time now, teachers and textbooks have referred to the procedure of long division as 'the division algorithm', when in fact it is not an algorithm in the true sense of the word. Bates and Rousseau (1989) argue that such descriptors are misleading, as there is more than one way to perform division. In their article aptly titled "Will the real division algorithm please stand up?", they argue that viewing long division or any division procedure as an algorithm, more so the division algorithm, "obscures a powerful tool for teaching division"(p1). They argue that any form of division is subordinate to the division algorithm, and once this is taught to learners, the actual *division algorithm* can open doors to understanding more advanced subject matter in elementary number theory as well as other areas of mathematics.

According to Bates and Rousseau, the *division algorithm* provides the vehicle for understanding division as a concept and as a process. Often secondary school learners are so fluent in simplifying expressions of the form $\frac{f(x)}{g(x)}$, that when posed with a division problem in the same form, the existence of a remainder is completely foreign to them. They immediately assume that $g(x)$ must contain some common factor – or even be a factor of $f(x)$. This is despite a fluency in arithmetic long division. This often needs to be overcome first when additional mathematics learners meet up with Euclid's Algorithm and various other content matters pertaining to the rational function theory. This is the case especially for learners who start additional mathematics at Grade 10 level.

The problem is not unique to additional mathematics learners. The current rational functions that learners simplify do not allow for denominators that are not factors of the numerator, or at least have one factor in common so that when simplified by 'division' there is no remainder, thus developing and indeed reinforcing a false expectation that algebraic fractions have a more streamlined solution that only requires the finding of a factor that is common to both the numerator and denominator. The matter compounds in the limit concept where learners engage with limits of the form $\left(\frac{0}{0}\right)$ in the very same manner with the teacher instructing learners to factorise and remove the common factor. This highly procedural approach does not serve any conceptual purpose as underlying issues of continuity and discontinuity do not enter the classroom discussion. Continuity of functions is one of the very complex and very important areas of mathematics as it forms the primary condition under which functions behave and also become differentiable. Is it possible that this void (or taught misconception) in our school curriculum could possibly inform our decision to de-emphasise the long division procedure in our classroom interaction, or are we blindly following the trend in the United States?

There is a movement in the United States that has urged the abandonment of the long division in the curriculum. As indicated by Klein and Milgram (2000), education leaders in the US have called into question the practice of requiring elementary school learners to master certain standard algorithms and procedures (p1). They indicate that specifically long division has been targeted for de-emphasis. This has been a view that has been widely supported in the press, offering calculators as an alternative to this desperately needed basic skill. Klein and Milgram (2000), quote a report that was published in February 1998 in an issue of the *Notices of the American Mathematical Society* which defends the place of such procedures and algorithms in the curriculum by noting the following:

> *"Standard Algorithms may be viewed analogously to spelling: to some degree they constitute a convention, and it is essential that learners operate with them from day one or even in their private thinking…"*
> "*We would like to emphasize that the standard algorithms of arithmetic are more than just 'ways to get the answer' – that is, they have a theoretical as well as practical significance. For one thing, all the algorithms for arithmetic are preparatory for algebra, since there are strong analogies between arithmetic of ordinary numbers and arithmetic of polynomials."* (p4)

If the purpose of a calculation or operation is to obtain an answer, then the calculator is available for that goal. What is necessary here is a shift in the purpose towards the understanding of the concept and the structure of operations. Different algorithms in arithmetic and algebra have clearly equivalent structures just as procedures that carry their conceptual bases may look different but have equivalent structures. In many cases some of the alternative algorithms in arithmetic are better precursors for their algebraic equivalent than the standard algorithm as many standard algorithms hide structure and are not preparatory for algebra.

Klein and Milgram are fierce opponents of this 'assault' on arithmetic, as they call it, and Wu is not far behind them in support of algorithms as providing learners with a conceptual understanding in preparation for algebra. He believes, and gives several examples to explain, that a deep understanding of mathematics ultimately lies within the mastering of basic skills. Wu is the voice of the group that values learners' fluency in basic mathematics skills. He passes harsh criticism on mathematics education reformers' inclination to de-emphasize basic skills and algorithms. Wu takes a more aggressive stance when he suggests that instead of abandoning a very important area of mathematics, we should revise the way in which teachers present these to the learners.

Kilpatrick et al, (2001) also adopts a composite and comprehensive view and stresses the interrelatedness between conceptual understanding, procedural fluency and other strands of mathematical proficiency, which can be used in support of the argument regarding the usefulness of algorithms and basic skill acquisition in the classroom practice. Wu reflects in his writing that standard algorithms embody conceptual understanding and are necessary to advance to more complex areas of mathematics. This argument holds particularly true for the transition into algebra, especially from arithmetic long division to algebraic long division. Successful completion of polynomial factorization and modular arithmetic are just two areas that require a level of fluency in division procedures.

This transition from arithmetic thinking to algebraic thinking will be greatly aided by helping learners to recognise similar core features in pieces of information that are superficially different. Why do learners have such a huge problem making this transition into algebraic division of polynomials when its roots are founded in the basic procedures of arithmetic long division? Does Wu hit the spot when he says "*…why not consider the alternative approach of teaching these algorithms properly before advocating their banishment from classrooms…"* (p4)? I am persuaded by Wu and will in the chapters to follow offer a possible didactic model to be followed to overcome this difficulty.

Focus on division problems

I will offer a hypothesis which informs this study and forms the rationale for proposing base ten decomposition as an alternative to the standard long division procedure taught in primary school.

The standard procedure by which division is taught at primary school is an algorithm that hides the semantic structure of the division procedure. The division procedure is often performed by using the standard long division procedure, and once learners are familiar with this procedure, or if they experience difficulties, they are *taught* to complete the division very concisely.

Long division

$$\begin{array}{r} 11 \\ 24\overline{)268} \\ \underline{24} \\ 28 \\ \underline{24} \\ 4 \end{array}$$

or 'concise' division

$$\begin{array}{r} 11r4 \\ 24\overline{)\cancel{26}\,{}^{2}8} \end{array}$$

What this study proposes is that the divisor and the dividend not be kept as 'whole' numbers, but rather decomposed into an additive relationship that is based on the place value of the numbers concerned. By the term 'whole' I mean any number that is written in sequential digit form, for example using the number 24 as is instead of 20 + 4 or $2\times10+4$ which expresses the number as an additive relationship. Just as the product of 32×24 can be written as:

$$\begin{aligned} &(30+2)(20+4) \\ &=30(20+4)+2(20+4) \\ &=600+120+40+8 \\ &=768 \end{aligned}$$

or

$$\begin{aligned} &(3\times10+2)(2\times10+4) \\ &=3\times10(2\times10+4)+2(2\times10+4) \\ &=6\times100+12\times10+4\times10+8 \\ &=600+120+40+8 \\ &=768 \end{aligned}$$

This enhances the distributive property that resembles post base ten decomposition of the equivalent algebraic multiplication of:

$(3x+2)(2x+4)=3x(2x+4)+2(2x+4)=6x^2+12x+4x+8=6x^2+16x+8$.

The division procedure will thus be completed as follows:

$$\begin{array}{r} 10+1 \\ (20+4)\overline{\big)200+60+8} \\ \underline{200+40} \\ +20+8 \\ \underline{+20+4} \\ +4 \end{array}$$

Here 28 > 24 so multiples of 20 + 4 can still be removed from the remainder. This can also be seen by looking at 8 > 4 since the 20 is the same for both remainder and divisor.

Here it becomes clear that the 20 has been eliminated by removing 20 + 4 from the previous remainder. So now 4 < 20 + 4 and the division has been completed.

or

$$\begin{array}{r} 10+1 \\ (2\times 10+4)\overline{\big)2\times 100+6\times 10+8} \\ \underline{2\times 100+4\times 10} \\ +2\times 10+8 \\ \underline{+2\times 10+4} \\ +4 \end{array}$$

Both the divisor and the dividend are written as a decomposition that enhances place value and works with power relations of ten. The number is thus rewritten as an additive relationship.
The objective becomes removing multiples of 2 x 10 by focusing on the 100 and the 10.

Here two multiples of 10 plus 8 are what is left to divide so we consider the fact that 2 x 10 +8 > 2 x 10 +4 since 8 > 4. The division thus continues.

Now it is evident that 4 < 2 x 10 + 4 since 4 = 4 and 0 < 2 x 10. Thus the division is complete.

or

$$\begin{array}{r} 1\cdot 10+1 \\ (2\cdot 10+4)\overline{\big)2\cdot 10^2+6\cdot 10+8} \\ \underline{2\cdot 10^2+4\cdot 10} \\ +2\cdot 10+8 \\ \underline{+2\cdot 10+4} \\ +4 \end{array}$$

The difference here is that the base ten is now written as a power, instead of as a product or a number. The structure here is syntactical and is experienced on the semantic level when the division is performed.

Here the structure in the arithmetic enhances the place value concept and emulates what is to become algebraic long division on polynomials. Considering these alternatives to the standard procedure requires that learners engage with both semantic and syntactic structures within the long division on naturals procedure.

The notation

Often when learners see the $\overline{)}$ symbol, it indicates (defines) the operation that needs to be performed as a division. There are in fact many different ways to complete a division which ignores this surface feature of division. Here are just a few:

a) Partial division:

$$
\begin{array}{l}
347 \div 12: \\
347 \\
\underline{120} \rightarrow 10\times 12 \\
227 \quad \downarrow \\
\underline{120} \rightarrow 10\times 12 \\
107 \quad \downarrow \\
\underline{\ 96} \rightarrow 8\times 12 \\
\ 11 \quad \downarrow \ \downarrow \ \downarrow \\
\therefore 347 = 28\times 12 + 11
\end{array}
$$

b) Removing multiples

$$
\begin{aligned}
347 &= 120+120+120-13 \\
&= 12\times 10+12\times 10+12\times 10-12\times 1-1 \\
&= 12(10+10+10-1)-1 \\
&= 12(29)-1 \\
&= 12(28)+12-1 \\
&= 12(28)+11
\end{aligned}
$$

c) Multiple divisions with remainder

$$
\begin{array}{r}
3 = 0\times 12+3 \\
34 = 2\times 12+10 \\
107 = 8\times 12+11 \\
\downarrow \quad \downarrow \quad \downarrow \\
\therefore 347 = 28\times 12+11
\end{array}
$$

d) Working backwards

$$
\begin{array}{lll}
120+ & \rightarrow 10\times 12 & \\
\underline{120}+ & \rightarrow 10\times 12 & \\
240+ & & \\
\underline{\ 60}+ & \rightarrow\ 5\times 12 & \\
300+ & & \\
\underline{\ 36}+ & \rightarrow 3\times 12 & \\
336+ & \llcorner\!\longrightarrow & 10+10+5+3=28 \\
\underline{\ 11}+ & \rightarrow 11<12 & \\
347 & \llcorner\!\longrightarrow & \text{remainder } 11
\end{array}
$$

$$\therefore 347 = 28\times 12+11$$

Exploring the multiple ways in which division can be performed on naturals enhances the semantic structure of the numbers and the division procedure itself. In fact the learners will interpret the structure of these problems at an advanced level even though they might not be aware of the hidden syntactical structure that they are using (Liebenberg et al, 1998).

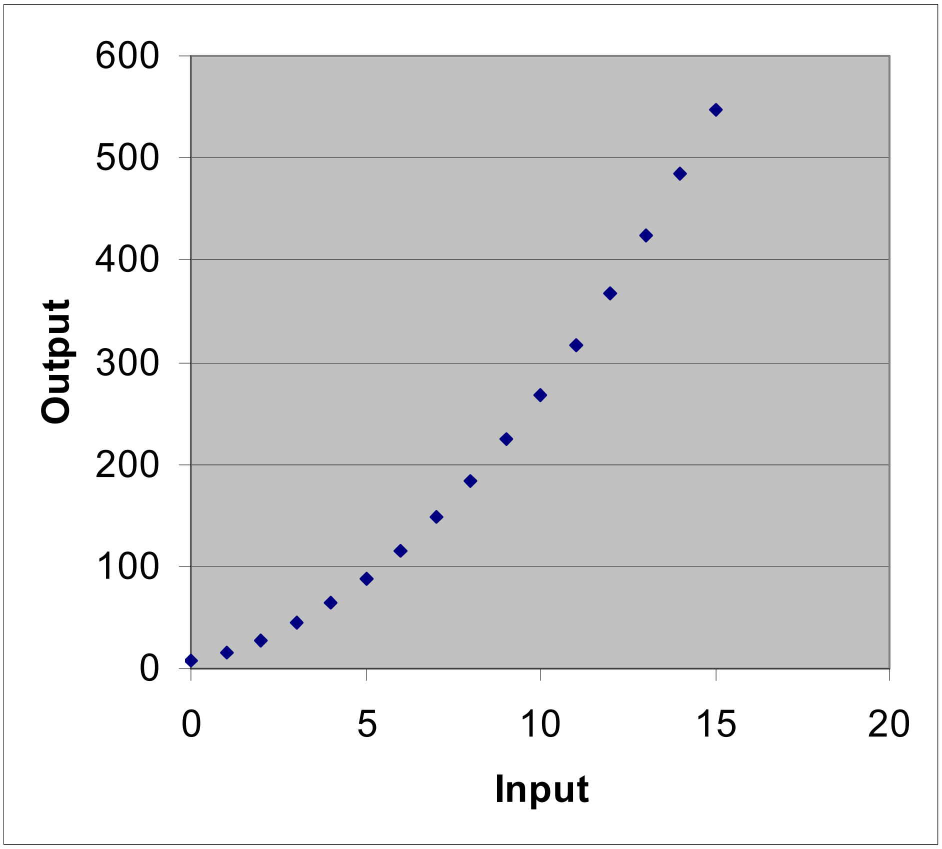
The functional relationship can be explored by investigating the algebraic structures within the division of naturals using the decomposition of numbers. Considering the division of 268 by 24, the result was:

$$
\begin{array}{rl}
 & 1\cdot 10+1 \\ \cline{2-2}
\left(2\cdot 10+4\right)\big) & 2\cdot 10^{2}+6\cdot 10+8 \\
 & \underline{2\cdot 10^{2}+4\cdot 10} \\
 & \qquad\quad +2\cdot 10+8 \\
 & \qquad\quad \underline{+2\cdot 10+4} \\
 & \qquad\qquad\qquad +4
\end{array}
$$

Writing this result as $2\cdot 10^{2}+6\cdot 10+8=\left(2\cdot 10+4\right)\left(1\cdot 10+1\right)+4$ establishes a basis from which to investigate the relationship that exists within the decomposition. When rewriting the division algorithm, using the variable x instead of the base 10, we get $2x^{2}+6x+8=\left(2x+4\right)\left(1x+1\right)+4$. For the purpose of discovering the general relationship that is embedded in this expression of equivalence we can explore the following:

If x = 1:	$2\left(1\right)^{2}+6\left(1\right)+8=\left(2\cdot 1+4\right)\left(1\cdot 1+1\right)+4$ $\therefore 16=6\times 2+4=12+4$
If x = 2:	$2\left(2\right)^{2}+6\left(2\right)+8=\left(2\cdot 2+4\right)\left(1\cdot 2+1\right)+4$ $\therefore 28=8\times 3+4=24+4$
If x = 3:	$2\left(3\right)^{2}+6\left(3\right)+8=\left(2\cdot 3+4\right)\left(1\cdot 3+1\right)+4$ $\therefore 44=10\times 4+4=40+4$
If x = 4:	$2\left(4\right)^{2}+6\left(4\right)+8=\left(2\cdot 4+4\right)\left(1\cdot 4+1\right)+4$ $\therefore 64=12\times 5+4=60+4$

When one completes a table of values, the following relationship can be plotted using the x values as input and the number as the output:



Input	Output
0	8
1	16
2	28
3	44
4	64
5	88
6	116
7	148
8	184
9	224
10	268
11	316
12	368
13	424
14	484
15	548

Here the *division algorithm* and the division procedure by decomposition can be used to explore the relationship that was established in the *division algorithm* after division was completed.

By following the above sequence in division the teacher can prepare the way for algebraic long division, as the decomposition clearly emulates the algebraic structure needed for algebraic long division on polynomials. It also creates the opportunity to graphically explore this relationship of equivalence as encapsulated in the *division algorithm* and extend it to a variety of different combinations in different bases. This generalised relationship naturally comes to resemble the structured polynomial in algebra.

Chapter 3 - Arithmetic and algebra: The undanced dance

"*We have allowed algebra's traditional role as gatekeeper to higher mathematics to make us believe that there is really something worthwhile in passing through the gate, rather than perhaps tearing down the whole fence." - James Kaput (1942-2005)*

In this chapter I will discuss the historical views on the dichotomous relationship that exists between algebra and arithmetic and then move towards the more current views that argue through various researchers' work for a symbiosis. I will consider the cognitive gap as defined by Herscovics and Linchevski (1994) and extend this to the work that various researchers have done to define this gap. The focus here will be on the two ideas of structural thinking and relational thinking within arithmetic. Lastly I will focus on the South African context.

Historical views of arithmetic and algebra – a dichotomy

Arithmetic is the science of numbers - the art of computation by figures. It is the study of numerical quantities and relationships that exist between them and the operations we perform on them. Arithmetic can, therefore, be seen as fundamental in the study of algebra, which is characterized as being generalised arithmetic (Wheeler, 1996 and Greenes, 2002). Arithmetic is a journey that leads to algebra while it contains no algebra. Algebra, on the other hand, contains arithmetic in the sense that all four of the basic operations are included in algebraic processes. When engaging with algebraic matter, arithmetic is preserved along with all the rules and patterns that develop in arithmetic exploration.

Wheeler (1996) argues that algebra is the completion of arithmetic. This fundamental use of arithmetic, when introducing new algebraic concepts, is often neither explored in school mathematics textbooks, nor is it used by educators to enhance learning. Without arithmetic, the evolution towards an algebraic way of thinking is impossible. Arithmetic forms the foundation on which all algebraic reasoning is founded. It offers a basis for abstraction and generalisation without which algebra cannot proceed meaningfully. The term algebra means "the reunion of broken parts" and one can classify this reunion at its origins of uniting arithmetic exploration with abstraction.

Wheeler (1996) describes the process of ‘algebrafying’ along the lines of two big ideas. The first is the concept of a variable *x* being an unknown or a general number that can be symbolized and operated upon as if it were a number, the basis of which lies primarily within arithmetic exploration which led to the necessity to generalise. The second idea describes algebra as the tool which introduces the learner to a new language that can be used to discover and invent new things. This is exactly where arithmetic is lacking in inventiveness, for it does not allow the learner to establish a firm and defined generalisation, mainly because it lacks the linguistic luxury of expressing itself in variable notation. These big ideas form the essential characteristics which are unique to algebraic thinking.

In earlier mathematics education, the relationship between arithmetic and algebra can as best be described as a dichotomous one. Herscovics and Linchevski (1994) argue that this dichotomous relationship does not serve algebra, let alone arithmetic. The view was taken that arithmetic instruction is completely distinct from algebra instruction and that the two fields have at best two separate purposes. Arithmetic was seen to deal with operations that involved particular numbers (Carraher, Schliemann and Brizuela, 2000), and is based on instructions that produce an answer (Subramaniam and Banerjee, 2004). Arithmetic can thus be said to be about product (obtaining an answer), whereas algebra is about process (Warren, 2004 and Malara and Navarra, 2003). The focus, until recently, in arithmetic instruction has been on procedural fluency and closure (Biggs, 1991 in Warren 2005) where getting to the correct answer was more important than the conceptual journey that produced the answer. Gazing through this historical lens, Carraher, Schliemann and Brizuela (2000) reveal that algebra deals with generalised numbers, variables and functions, but until recently this view was not at all considered when talking about arithmetic and the structure sense that emerges from it. They point out that this dichotomous view of arithmetic and algebra results in learners adopting the view that the start of algebra signals the point where reasoning surrenders to arbitrary rules and their work becomes manipulative. This sudden jump had the effect that learners could possibly do the work but had no underlying sense of what they were doing.

As a result of this dichotomous view of arithmetic and algebra, learners' conceptual base about numbers and number relationships is rather superficial and this could be a reason why they struggle with algebra in the later years. Carpenter and Levi (2000) hold the view that this artificial separation of arithmetic in algebra deprives learners of a powerful tool for later algebraic learning. The work of Kaput and Blanton (2001) supports this argument when they attribute learners' difficulties with algebra to the delay of algebra until high school, as well as the isolation of algebra from other fields within mathematics. Carpenter, Franke and Levi (2003) argue that keeping arithmetic and algebra separate results in a narrowly restricted knowledge of arithmetic and this has detrimental effects on later algebraic learning.

The current debate - a symbiosis:

Recent arguments echo the call for early algebraic instruction in which algebraic thinking and reasoning skills become part of the primary school mathematics curriculum focus. This debate has been ongoing for more than three decades, starting with the work done by R. Davis in 1971 and 1972 and taking an unexpected turn in the work carried out by Russian researcher F.G. Bodanskii which was reported on in 1991. Bodanskii collected data from learners who received instruction on algebraic representation of verbal problems from Grades 1- 4. The fourth Graders in the experimental group used algebraic notation to solve problems and performed better than their peers in the control group, who did not use algebraic notation, throughout their school years. They also performed better in algebra problem solving when compared to sixth and seventh Graders in traditional programmes of five years of arithmetic followed by algebraic instruction from Grade six. The work done by Bodanskii (1991) suggests that it is time to seriously consider significant changes in the primary mathematics curriculum. Bodanskii proved that algebraic thinking, embedded within the usual activities of arithmetic, is possible to teach at the beginning of the basic course of education.

Carraher, Brizuela and Schliemann (2000) hold the view that the instructional sequence of arithmetic first and then algebra, lends itself to discontinuities between arithmetic and algebra. They point out that there is currently a large gap between arithmetic and algebra, but then ask the question as to the legitimacy of the existence of such a gap. Herscovics and Linchevski (1994) refer to a *cognitive gap* that exists between arithmetic and algebra as a reason for learners' difficulties in mathematics. They specifically refer to the learners' "*inability to operate spontaneously with or on the unknown*"(p59).

Carraher, Schliemann and Brizuela (2000) points out that until recently, it was thought that difficulties with algebra could be minimized through a slow transition from arithmetic to algebra. Their suggestions include the proposal that algebra should be introduced from Grades five to eight as a generalisation of arithmetic, while focus on its own logical framework and consistency should be the goal of instruction in Grades nine to twelve.

This call for early algebra started in the *Curriculum and Evaluation Standards for School Mathematics* (1989), when the National Council of Teachers of Mathematics (NCTM) identified algebra as a major strand of the curriculum for learners in Grades K–12 in the United States of America. In the *Principles and Standards for School Mathematics*, published in 2000, the NCTM reiterated the need for the study of algebra at all levels of instruction.

The outcome of the research that Carraher, Brizuela and Schliemann conducted over the years regarding this 'gap' can be described as a revolution in mathematical instruction. Their work is ground-breaking in terms of translating theory into practice and proves that algebraic reasoning skills and algebraic representation are not such foreign concepts when it comes to primary school mathematics learning – in particular the emergence of algebraic reasoning skills embedded in arithmetic exploration in the early Grades. They present strong arguments and prove in their work that it is a mistake to ascribe the late emergence of algebraic thinking to developmental constraints. They strongly advocate the opinion that algebra and algebraic thinking enter the curriculum too late and that this could be the cause of learners' subsequent difficulties in algebra.

Their work includes among others the following foci:

- *Instantiating variables in addition and subtraction* – young learners reason about problems of an algebraic generalised nature
- *Solving algebra problems before algebra instruction* – third Grade learners use a logic principal in solving problems
- *Teaching operations as functions* – second and third Grade learners use functional notation in their problem-solving activities
- *Young learners operating on unknowns* – second and third Grade learners comfortably use letters to represent unknown values and also operate on representations involving letters and numbers without having to assign values. The introduction of the *variable number line* is made at this point.
- *The reification of additive differences in early algebra* – here the learners not only operate on unknowns, but understand the unknown to represent possible values that an entity can take on. They show an ability to reason algebraically as well as use algebraic symbols.

Substantial research conducted over the years points to the possibility of arithmetic and algebra coexisting from as early as Grade one in the mathematics school curriculum. Schliemann, Carraher, Brizuela and Pendexter (1998), Carraher, Schliemann and Brizuela (2000, 2001), Schliemann, Carraher, Brizuela, and Earnest(2003) , Carraher, Schliemann, Brizuela, and Earnest (2006), Carraher, Brizuela, and Earnest (2001), Carraher, Brizuela and Schliemann (2000), Carraher and Schliemann (2000), Brizuela, Carraher and Schliemann (2000), Kaput (1995, 1998), Davis (1985, 1989) , Vergnaud (1988), Carpenter and Levi (2000), Warren (2001, 2002, 2003, 2004, 2005), Mason (1996), Booth (1988), Brown and Coles (2001) and Bodanskii (1991) are among the researchers that strongly support the argument that arithmetic and algebra should not be treated as separate entities, but that they should be integrated throughout the mathematics curriculum from the very start of mathematical learning. After all, " *the focus of mathematics teaching should be on fostering fundamental skills in generalizing, expressing and systematically justifying mathematical generalisations"* (Kaput and Blanton, 2001).

Kaput (2000) advocates that by "algebrafying" arithmetic, we can fulfil the promise of algebra for all and eliminate the superficiality intrinsic in introducing algebra 'after' arithmetic. The idea of integrating algebraic thinking and reasoning throughout the mathematics curriculum has been advocated by many researchers.

Warren (2004) adds to this debate by pointing out that early algebra relies heavily on arithmetic. She advocates that the teaching of algebraic thinking with arithmetic thinking can enhance further learning in formal algebra. Warren bases her argument on the fact that learners experience difficulties moving from arithmetic world to algebraic world which seem to stem from a lack of appropriate foundation in arithmetic. This view is in line with Slavit (1999) who places the lack of true understanding of operations on the rough transition from arithmetic to algebra.

Teachers must find ways to support algebraic thinking and create a classroom culture that values learners' input on various levels (Blanton and Kaput, 2005). Blanton and Kaput further suggest that teachers "algebrafy" current curriculum materials by using existing arithmetic activities and contextualised problems, transforming them from problems with a single numerical answer to opportunities for discovering patterns and making conjectures or generalisations about mathematical facts and relationships and justifying these. This can be done by simply encouraging learners to discuss why they believe a mathematical statement or solution to a problem is correct.

Just how and when this debate will take on a practically implementable form is not known. The implication for curriculum reform worldwide, and especially in the South African context, is far reaching since we are hardly running the current reform process smoothly. One thing which is certain and is reflected in the work by Kaput and Blanton (2001) is that the most pressing factor for arithmetic algebraic reform is the ability of elementary teachers to 'algebrafy' arithmetic. This would imply that teachers need to develop in their learners the arithmetic underpinnings of algebra (Warren & Cooper, 2001). This journey will reach full circle when these underpinnings extend to the beginnings of algebraic reasoning (Carpenter & Franke, 2001).

Defining the cognitive gap: the foci of researchers

Various researchers have described the journey to bridge the divide between arithmetic and algebra in a variety of ways. A large number see the crossing of this great divide between algebra and arithmetic as a transition process (Filloy and Rojano, 1989; Rojano, 1996; Herscovics and Kieran, 1980; Kieran, 1985) which they have termed as '*transition mathematics'*. Some scholars advance the view that this transition should take place through an integrated curriculum where arithmetic and algebra co-exist from the early Grades, while others envisage that algebra should follow on to arithmetic in the later years - *"where arithmetic ends and algebra begins"* (Carraher, Schliemann, Brizuela and Earnest, 2006, p.2).

It would appear that there are two main foci in this debate: an integrated arithmetic curriculum which aims at extracting algebraic thought and the reasoning in the early Grades by the algebrafication of arithmetic and a second focus which emphasizes transitional processes.

The role of *operation sense* has been investigated by Slavit (1999) in the transition from arithmetic to algebraic thought. The role of *arithmetic structure* in this transition period is the focus of the work of (Elizabeth) Warren (2003, 2004). This focus is shared by Subramaniam and Banerjee (2004, 2005) as they argue for the development of a procedure and structure sense of arithmetic expressions to assist learners' growing understandings of arithmetic expressions and beginner algebra. Warren (2004) has particularly pointed to the difficulties that learners experience in moving from the arithmetic world to an algebraic world and she explicitly attributes this difficulty to a lack of appropriate foundation in arithmetic. She maintains that many learners have a narrow *understanding of the equal sign* and also struggle to express generalised thinking in their own words.

This claim is supported by the work done by Sfard (1991) who pointed out that the core of learners' difficulties with algebra and algebraic symbols lie at their inability to conceive the *operational-structural duality* of algebraic symbols. Sfard (1991) indicates a three phase process: i) interiorisation, ii) condensation of the sequence of operations into a whole, iii) *reification*, which is a qualitative change manifested by the ontological shift from operational thinking, where the focus is on mathematical processes, to structural thinking which shift this focus to the properties of, and relationships between, mathematical objects. Sfard and Linchevski (1994) and Kieran (1992) explore the difficulties which learners experience in adopting a structural conception of mathematics and maintain that these lead to the inability of learners to differentiate the process-object duality inherent in algebra and propose that algebra is first conceived as *process orientated* which only later becomes *reified* and conceived in terms of *objects and structures*. They connect the difficult progression from an operational conception (arithmetic) to a structural one (algebra) with the gap in the process of *reification.*

Gray and Tall (1994) focused on the role of mathematical symbolism within this process and introduced the idea of *procept* and *proceptual thinking* as part of this transition between the operational (arithmetic) and the structural (algebra). Here symbolism represents either a process to do or a concept to know. In an attempt to emphasise the duality of mathematical symbolism the term *procept* was introduced. Procepts thus begin as simple structures and then grow in interiority with the cognitive growth of the learner. The procept is intended for building a connection across the cognitive divide and operates on three levels. Starting with a process which eventually leads to the formation of a mathematical object and finally reaching a level where a symbol is defined in such a manner that it represents the process and the object. They point out that learners that continue to think in terms of processes will remain at an operational level of thinking.

Slavitt (1999) and Warren (2003) highlight the fact that learners often have a limited view of algebraic expressions and that this difficulty that they experience is rooted in grasping the *meaning of operations and literal symbols*, *algebraic expressions or equations*. The inability of learners to spontaneously *operate on or with the unknown* ‘constituted the cognitive obstacle that could be considered a gap between arithmetic and algebra’ (Herscovics and Linchevski, 1994, p. 41).

Learners show considerable hesitation about producing *written representations for unknown quantities* (Schliemann et al, 1998). They maintain that this hesitation stems from the challenge of finding the symbols to represent a quantity without constraining or making incorrect presumptions about the values that they may stand for.

Cognitive architecture: various approaches

"I think most everybody recognizes the importance of algebra. It is a question of how they introduce it and when…"

> *Johnny Lott, former president of the National Council of Teachers of Mathematics (NCTM).*

Blanton and Kaput (2005) suggest teachers can extend learner thinking by asking questions that explore their thinking (*Tell me what you are thinking?*), test their independent thinking (*Did you solve this in a different way?*), create a culture of justification and reflection on their hypotheses (*How do you know that it is true?*) and reflecting on the generalisability of their answers (*Does this always work?*). Blanton and Kaput (2005) proceeded to distinguish different categories of algebraic reasoning in a Grade 3 classroom as generalised arithmetic, functional relationship, properties of numbers and operations and the algebraic treatment of number.

The general consensus on the importance of algebraic language in mathematics has led some Italian researchers to study key processes which are present in the development of algebraic thinking. Bazzini, Gallo and Lemut (2006) studied algebraic thinking both as *register* for representing generalised ideas and as *tool* for solving problems.

Throughout the research that has been conducted over many years, two main themes emerge very strongly: that of structural thinking and, secondly, the focus on functional thinking.

Structural thinking:

Krishner (1989) advances the argument that the understanding of syntactic structure of algebraic expressions is primary to proficient performance in algebra. Yerushalmy (1992) maintains that transformation to algebra involves not only the mastery of algebraic rules, but relies heavily on analyzing structures within expressions. For Bell (1995 in Subramaniam and Banerjee, 2004) an appreciation of the structure of arithmetic expressions is useful for understanding algebraic expressions. The structure of arithmetic can be described as being composed of surface and systemic structure (Kieran, 1989). *Surface structure* includes recognizing that for the expression $a(b-c)$, c has to be subtracted from b and then finally multiplied by a, whereas *systemic structure* points to the ability to recognise equivalence between the forms $a(b-c)$, $(b-c)a$ and $ab-ac$ based on the properties of operations.

We can define structure sense as a collection of abilities, separate from manipulative abilities, which enables learners to use previously learned algebraic techniques more effectively (Hoch, 2003). Bell (in Subramaniam and Banerjee, 2004) argues that arithmetic expressions stand for number and thus focus on the value of an expression. This implies then that two expressions are equal if their values are equal. Bell continues by saying that this equality can also be judged by the relationship between the components or parts of the expressions. This would mean that learners must explore and identify relationships between the parts, and also all the parts that make up the whole. Kieran (1988) argues that structural knowledge implies the ability to identify all equivalent forms of an expression. Liebenberg, Linchevski, Olivier and Sasman (1998), however, maintain that “pupils will probably justify the equivalence of the expressions on the basis that they produce the same result and not on a structural justification” (p2). They maintain that viewing an expression structurally depends on whether the main focus in the structure is the numbers or the operations or both. According to Liebenberg et al, viewing an expression structurally requires the ability to identify both surface and hidden structure.

Hoch and Dreyfuss (2004, 2005, 2006) built on the definition of structure sense and explained its use to analyse learners’ problems when factoring expressions. They found that top achieving mathematics learners do not use a high level of structure sense when solving problems that require the use of algebraic techniques, but those learners that did use a structure sense made fewer mistakes in their work.

However Linchevski and Livneh (1999) have expressed their doubts whether focusing on structured arithmetic for algebra is a good pedagogic strategy as they found that many learners have a poor sense of the structure of arithmetic expressions and are not able to judge the equivalence of expressions without using computational tools.

It is therefore imperative that the teacher develops this focus with learners from the perspective of algebra as generalised arithmetic which symbolises and exploits the structure sense of arithmetic. Subramaniam and Banerjee (2004, 2005) make it very clear that algebra requires understanding of the threefold meaning of arithmetic expressions (process, product and relation) and that equality of expressions is the core notation around which structure of expressions can be grasped. They revealed preliminary evidence in favour of a structure-orientated approach which strengthened both procedural knowledge and structural understanding of arithmetic expressions.

Functional thinking:

Carraher, Schliemann and Brizuela (2000, 2001) believe that learners’ understanding of additive structures and functional notation provides the fruitful point of departure for algebraic arithmetic. This requires that learners develop an early awareness of negative numbers and quantities and their representation on number lines. They also advocate that multiple problems and representations for handling unknowns and variables, including algebraic notation, should become part of learners' repertoires as early as possible. Carraher, Schliemann, Brizuela and Earnest (2006) support the notion that algebraic meaning of arithmetic operations in the elementary school curriculum is integral. They argue that algebraic concepts and notation can be successfully integrated into arithmetic explorations, as arithmetic has an inherently algebraic character in that it concerns general cases and structures. They propose that giving functions a major role in the primary school mathematics curriculum will help facilitate the integration of algebra into the existing curriculum (p2).

Blanton and Kaput (2003) maintain that the focus of mathematics teaching should be on fostering fundamental skills in generalising, expressing and systematically justifying mathematical generalisations. Traditionally, primary school mathematics programmes do not explore relations and transformations as objects of study. Warren and Cooper (2005) argue that fundamental to relation and transformation is the concept of the function, an exploration into how the value of certain quantities relate to the value of other quantities and how values are changed or mapped to other quantities. Warren and Cooper suggest four levels of mathematical experience for learners:

> "*We are suggesting that these experiences go beyond simply finding and describing patterns of attribute change but also encompass ideas such as qualitative change (for example, I grew taller) quantitative change (for example, I grew 2 cm taller), relationships between these changes (for example, If everyone grew taller by the same amount and John's height changed from 143 cm to 145 cm, how much did the class grow by?) and using these relationships to solve problems (for example, If Alison's height is now 133 cm, what height was she before she grew?). Addition itself can also be represented as change: for example, If I have 3 and increase it by 2 how many do I have? This is the beginning of functional thinking.*"(p3)

These suggestions are in agreement with Kaput and Blanton (2001) and will support functional thinking later on in learners' mathematical journey. This enhances the conception of arithmetic as not being static and product orientated, but rather a dynamic journey where learners' arithmetic explorations can lead to various conjectures being made. Learners will also experience at an early age that any conjecture needs justification based on certain connections within the relationships that they explored. Warren and Cooper (2005) warn, however, that the exploration of functional thinking should be gradual and occur over a long period of time (p3).

The South African context

Despite the fact that research into early algebraic thinking has been reported on for more than three decades, very little is available from a South African research base regarding algebraic long division. The Mathematics Learning and Teaching Initiative (MALATI) has done extensive work in developing, piloting and spreading alternative approaches and tools for teaching and learning mathematics. The MALATI project was commissioned by the Education Initiative of the Open Society Foundation for South Africa in 1996 as a three-year co-operative project involving mathematics educators from the three Western Cape universities: The University of Cape Town (UCT), the University of Stellenbosch (US) and the University of the Western Cape (UWC). They have been actively developing and publishing materials in an effort to interpret Curriculum 2005.

Major reform policies that introduced South African teachers and learners to the constructivist approach to learning and teaching have been implemented. These reform policies addressed the imbalances that were embedded in the apartheid education policies on more than just the level of curriculum reform. International trends in mathematics education were carefully considered when designing the new curriculum, with particular consideration given to the theories on how learners learn mathematics. Extensive attention has been given to algebraic reasoning at primary school level in the curriculum design in South Africa.

Long division on naturals is explored at an early age, without paying the necessary attention to the *division algorithm* and the conditions that it sets on the divisor and the remainder. Thus this powerful tool, which could be used and operated on when discussing fractions and rational numbers, is often not explored at all in the primary school classroom.

As was internationally the case, learners found it exceptionally difficult to engage with algebraic long division when they reach Grade 9. The links to arithmetic long division are explored only on a superficial level, if at all. In its totality, the curriculum is lacking in its focus on fostering algebraic thinking and exploring arithmetic structure when learners engage with division on naturals. As far as the division on naturals and algebraic long division are concerned, the curriculum document explicitly states the following at the various levels on the required assessment standards:

Grade R: Uses the following techniques: doubling and halving to at least 10.
Grade 1: Uses the following techniques: doubling and halving.
Grade 2: Uses the following techniques: doubling and halving.
Grade 3: Can perform calculations, using appropriate symbols, to solve problems involving division of at least whole 2-digit by 1-digit numbers.
Grade 4: Estimates and calculates by selecting and using operations appropriate to solving problems that involve division of at least whole 3-digit by 1-digit numbers and equal sharing with remainders. Recognises, describes and uses the reciprocal relationship between multiplication and division (e.g. if 5 x 3 = 15 then 15 ÷ 3 = 5 and 15 ÷ 5 = 3); the equivalence of division and fractions (e.g. 1 ÷ 8 = 1/8).
Grade 5: Estimates and calculates by selecting and using operations appropriate to solving problems that involve division of at least whole 3-digit by 2-digit numbers. Recognises, describes and uses the reciprocal relationship between multiplication and division (e.g. if 5 x 3 = 15 then 15 ÷ 3 = 5 and 15 ÷ 5 = 3); the equivalence of division and fractions (e.g. 1 ÷ 8 = 1/8).

Grade 6: Estimates and calculates by selecting and using operations appropriate to solving problems that involve division of at least whole 4-digit by 3-digit numbers. Uses a range of techniques to perform written and mental calculations with whole numbers including long division. Recognises, describes and uses divisibility rules for 2, 5, 10, 100 and 1 000.

Grade 7: Estimates and calculates by selecting and using operations appropriate to solving problems that involve division of positive decimals with at least 3 decimal places by whole numbers.

Grade 8: No specific mention.

Grade 9: No specific mention.

What is even more interesting to note is the fact that the algebraic long division procedure has been completely de-emphasised in the FET curriculum as well. In South Africa this happened with very little resistance from the education sector, let alone the tertiary-level mathematics fraternity.

Considering the arguments of Hung-Shi Wu, David Klein and James Milgram discussed earlier in Chapters 1 and 2, it is only a matter of time before the implications of this exclusion of the division procedure on polynomials is felt by tertiary institutions. When Klein and Milgram made public their objections to the de-emphasis of long division, they were supported by 233 tertiary-level educators from across the United States who argued for the continued inclusion of the division procedure and the *division algorithm* in the curriculum. (See *An open letter to United States secretary of education, Richard Riley*, 2000 as published on http://mathematicallycorrect.com/riley.htm.).

Chapter 4 - The pilot study

In this chapter, the responses from learners and an analysis of the work they produced will be discussed, as this pilot data informed the way in which the current research was planned and framed.

Pilot data was collected in 2005 at a private school in Pretoria as part of an Honours research project. This school caters for middle to upper income families, and boasts a very cosmopolitan school society with learners from all over the world as well as the various African and international diplomats' children who attend the school. The grades focused on were Grade 11 learners and two groups of Grade 9 learners. The Grade 11 group all took the subject Additional Mathematics and their aggregates ranged from 96 % to 100 % with the top three learners in this group differing only by a decimal portion of a percentage and falling within the range of 99% to 100% for mathematics. One would therefore expect these Grade 11 learners to have grasped the concept of division fully, and that it would merely be a procedure for them. The reason for choosing the Grade 11 group was based on the fact that they experienced difficulties when factoring polynomials as part of their additional mathematics syllabus. The Grade 9 groups consisted of two mixed ability classes, who were chosen because this is the point at which algebraic long division is introduced in their curriculum. Both the Grade 11 and Grade 9 learners had already dealt with algebraic expressions prior to this.

The grade 11 group:

In order to establish a starting point, the Grade 11 group was tested on division, starting from basic arithmetic and including polynomial long division. Ten questions, ranging in difficulty, were presented to them and the answers were analyzed. This group of learners consists of the top 11 students in the grade.

The test items were as follows:

Item 1: $248 \div 7$

Item 2: $1039 \div 12$

Item 3: $2479 \div 11$

Item 4: $89304 \div 120$

Item 5: $(x^2 - 3x + 4) \div (x - 2)$

Item 6: $(x^3 - 4x^2 + 3x - 5) \div (x^2 - x - 6)$

Item 7: $(2x^3 - 3x^2 - 36x + 37) \div (x - 1)$

Item 8: $(6x^3 - 2x^2 + 4x - 1) \div (2x - 3)$

Item 9: $(5x^3 - 4x^2 + 3x + 6) \div (3x + 1)$

Item 10: $(-x^3 + 12x^2 - 45x + 54) \div (x - 3)$

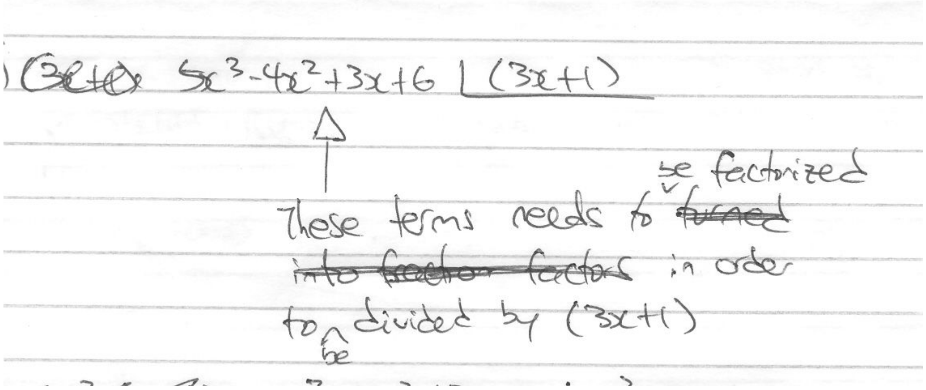
The following table indicates the success rate with which learners accomplished the task at hand.

Item no	**Correct**	**Incorrect**	**No attempt**
1	10	1	-
2	10	1	-
3	10	1	-
4	10	1	-
5	3	8	-
6	1	6	4
7	1	3	7
8	2	3	6
9	2	4	5
10	2	1	8

The Grade 11 group could perform basic algebraic long division, but could not handle cases where the divisor was not a factor of the leading coefficient. The results indicated to me that they could do arithmetic long division, but that there was a problem as far as extending this procedure to algebra, specifically polynomial algebra.

A few responses from this group are worth looking at.

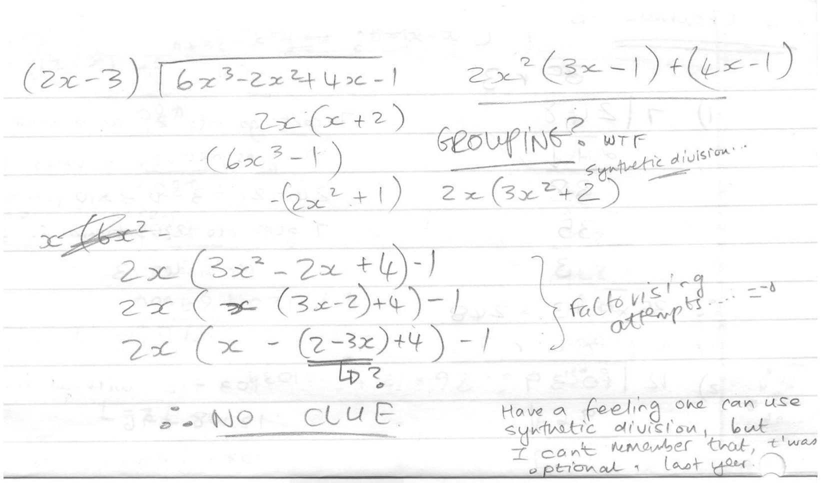
Learner 1:



~~(3x+1)~~ $5x^3-4x^2+3x+6 \;\underline{\lfloor (3x+1)}$

These terms needs to be factorized ~~turned into fraction factors~~ in order to be divided by (3x+1)

Learner 2:



$(2x-3)\overline{)6x^3-2x^2+4x-1}$

$\underline{2x^2(3x-1)+(4x-1)}$

$2x(x+2)$

$(6x^3-1)$

$-(2x^2+1)$

GROUPING? WTF synthetic division...

$2x(3x^2+2)$

~~$x(6x^2-$~~

$2x(3x^2-2x+4)-1$

$2x(\cancel{x}(3x-2)+4)-1$

$2x(x-\underline{(2-3x)}+4)-1$

↳?

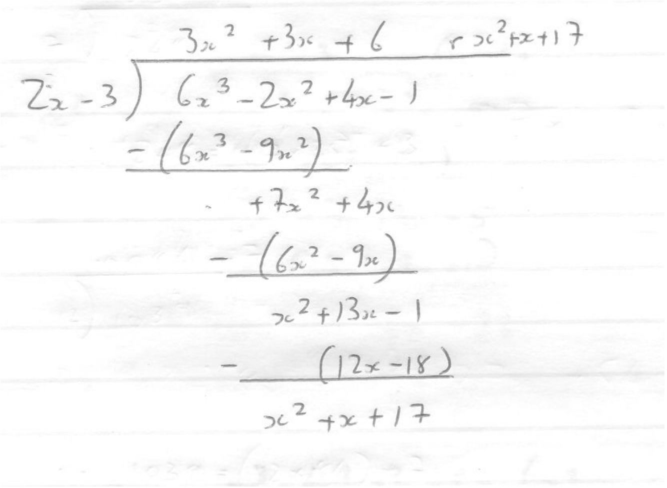
factorising attempts.... =-o

∴ NO CLUE

Have a feeling one can use synthetic division, but I can't remember that, t'was optional, last year.

Both these learners were looking to find common factors between the given polynomial (dividend) and the divisor by grouping terms. Learner 2 attempted all the algebraic problems by trying to group terms together and then complete the division. Her prior learning was based on simplifying fractions where it is always possible to identify the denominator as a factor of the numerator by simply factoring the numerator by which ever method is convenient. This knowledge was procedural to her, but did not extend to the concept of division with remainders.

Learner 3:



$3x^2 + 3x + 6 \quad r\ x^2 + x + 17$

$2x - 3 \,\big)\, 6x^3 - 2x^2 + 4x - 1$

$-(6x^3 - 9x^2)$

$+7x^2 + 4x$

$-(6x^2 - 9x)$

$x^2 + 13x - 1$

$-(12x - 18)$

$x^2 + x + 17$

This learner was convinced that he performed the division correctly, and seemed to ignore the duality of the division of algebraic expressions, involving both number and symbol as this short interview shows:

Teacher: How did you conclude that the remainder is $x^2 + x + 17$?

Learner 3: First of all, 2 could go into 6, 3 times. Then you subtract. Then 2 went into 6 another 3 times. Then you subtract again, and get $1x^2$ and 2 does not go into 1, so you go to the 13, and then 2 goes into 13, 6 times. So when you subtract, it leaves you with $x^2 + x + 17$.

Teacher: But do you think that this is correct?

Learner 3: I checked by multiplying $2x - 3$ with $3x^2 + 3x + 6$ and adding the remainder $x^2 + x + 17$. It gives you the divisor…so it must be right.

Teacher: But can your remainder have a higher degree than your divisor?

Learner: Yes, because it gave me the original polynomial when I checked.

Upon further questioning of the group, I learned that they had never before heard of the *division algorithm* and were therefore not familiar with the conditions set out in the extension of this algorithm to polynomials. Even though they were fluent at arithmetic division, they had not been formally introduced to these specific conditions. Bearing this in mind and knowing how important the conditions in the *division algorithm* are, I designed the new lesson plans in such a way that the details and conditions prevalent in the parts of this algorithm were foregrounded.

The Grade 9 group:

The Grade 9 classes were presented with a long division problem to which they had to respond and instruct me on how to proceed in completing the division. It was clear that both groups of Grade 9 learners were quite fluent when it came to procedural long division using only arithmetic methods. Both groups could tell me exactly what had to be done when we carried out long division, but as to the question "why are we doing what you are suggesting we do?", no clear answers were forthcoming. During classroom discussion my questions were centred around the issue of place value when performing arithmetical long division on naturals. What is interesting here is that the learners could easily engage with the concept of long division, but the act of moving from hundreds to tens and finally units, did not enter their minds. They also did not see that the *division algorithm* is an equation that relates the numbers and can be operated upon. So multiplying by 5 did not enter their thoughts at all. They were overly concerned with the value of 123 and how the different parts fit into the process, as the following interview shows:

Teacher: We are going to divide 123 by 5. Now cast your minds back to Grade 4. Where does the process start?
Student: You start with the second digit.
Teacher: Why the second, what is wrong with the first one?
Student: It's too small. You say 5 divides by twelve …I mean 12 divided by 5.
Teacher: And then what?
Student: You write it at the top and then subtract them once you have written the answer underneath the number.
Teacher: Why did you subtract?
Student: That's the way to find out the difference…which is two.
Teacher: Okay, what then?
Student: You bring down the 3 and then divide 23 by 5 in the same way.
Teacher: Why do you bring down the three?

Student: Because you need to…The number 2 is too small to divide by 5, and there is a 3 we haven't used yet. So it makes 23.

Teacher: Okay, then what?

Student: You subtract and the answer is three.

Teacher: So what now? There is a number left…

Students: (a)There is 3 left, (b)it is the remainder….(c)it is three fifths…(d)5 cannot divide into three…

Teacher: Okay but 5 could not divide into 1 as well, then you guys suggested that we move to the next digit remember.

Student: (a)Yes, but 3 is the remainder, so you can write $\frac{3}{5}$ … (b) yes because this is the remainder. $\frac{3}{5}$ is the remainder.

Teacher: Okay, we had a remainder of three just now and now we have another remainder of $\frac{3}{5}$. Let me help you direct your thoughts. If I were to ask you to conclude the procedure by writing a sentence of numbers at the end, what would you write?

Students: (a) $\frac{123}{5} = 24\text{rem}3$

(b) $\frac{123}{5} = 24 + \frac{3}{5}$

(c) $\frac{123}{5} = 24\frac{3}{5}$

Teacher: (Having written all the possibilities on the screen) Okay, if we keep the 123 by itself, what can we write?

Students: You can write $123 = 24 \text{ x } 5 + 3$.

Teacher: How did you get there?

Student: Because 24 times 5 plus 3 gives you the 123.

Learners in these two groups focused on the method in which the division procedure can be described as multiple 'division with remainder' within the same problem. The fact that 3 was the last digit in the number and was, therefore, the remainder because of its relation to the divisor was not mentioned. There was no focus whatsoever on the concept of place value, and no clear cut description could be given by these learners as to the role and the place of the remainder in this problem.

The plan of action that was followed in the pilot study included the various conceptual building blocks that increased the cognitive demand from basic arithmetic long division on naturals to decomposition in various bases of the dividend and the divisor. This plan of action was designed in order to construct the division procedure conceptually for the learners so that the gap between the primary school demands for division, and those of high school are narrowed significantly enough to make the transition seamless.

In Figure 2 below I indicate the steps which I perceive to be missing in what I will refer to as the ‘division ladder’. This forms the theoretical basis on which I attempted to build their cognition during the pilot study. This is an attempt to provide a graphical interpretation of which concepts should be covered to fill the gap between preparatory school division and secondary school needs:

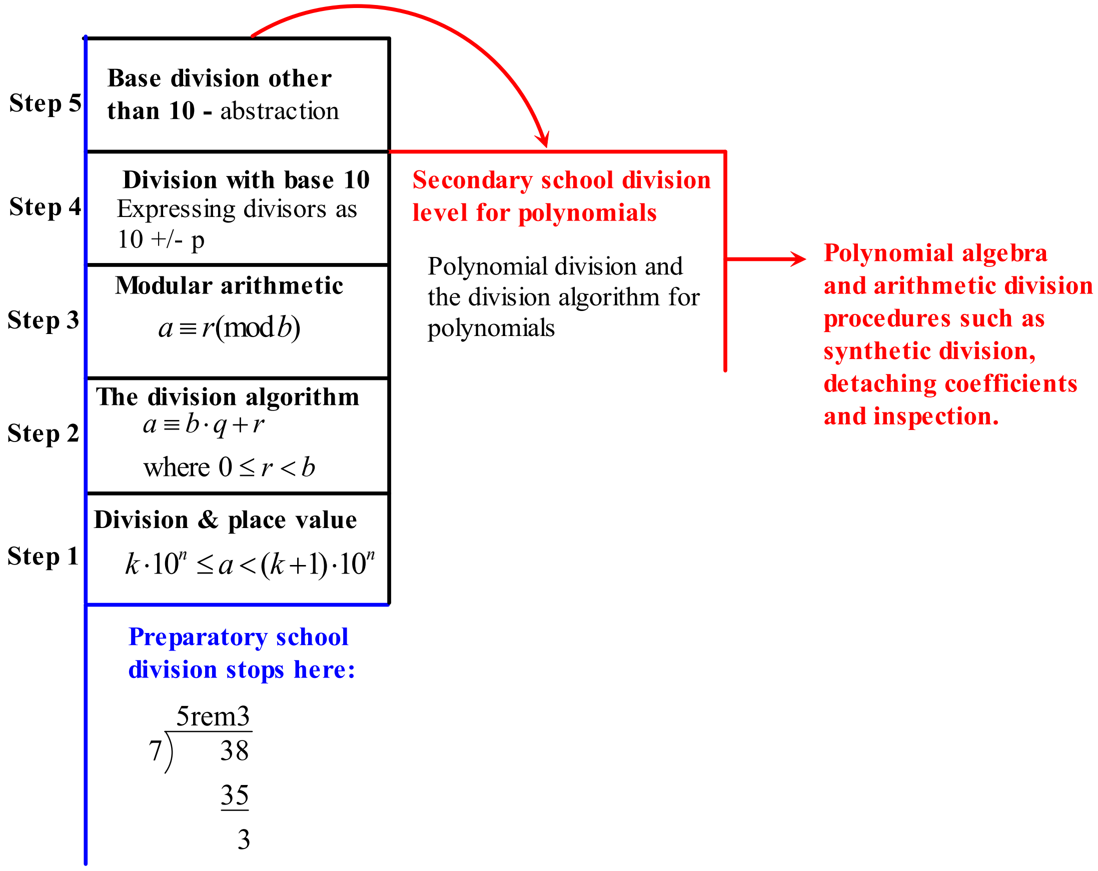


Figure 2

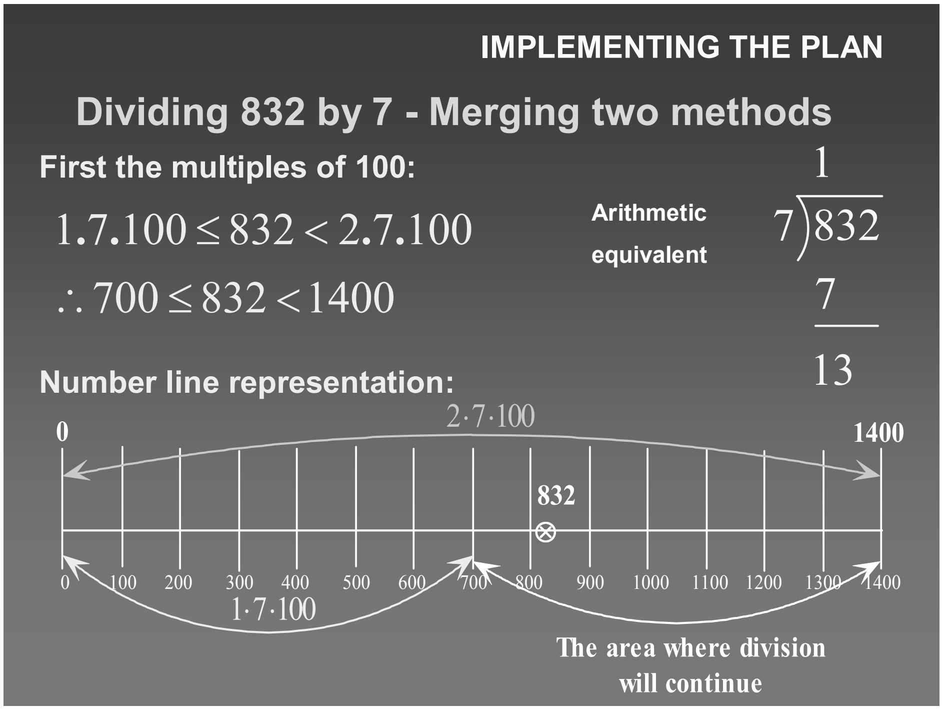
Step 1: Division that preserves the concept of place value

In response to the critics who advocate the exclusion of the long division procedure from the curriculum in the United States by arguing that this procedure un-teaches place value, Klein and Milgram (2000) and Wu (1999, 2000) suggest that the alternatives to teaching the long division procedure should be considered as in their view it is of vital importance that learners develop this skill. Klein and Milgram suggest that the division be retained by considering the following argument:

> For an arbitrary whole number *a*, it is always true that $k \cdot 10^n \leq a < (k+1) \cdot 10^n$ where *k* is the largest multiple of a power of 10 that is less than *a*. An arbitrary whole number with the digit *k* in the nth decimal means that $k \cdot 10^n$ is less than or equal to the whole number minus the sum of multiples of higher powers of ten that are associated to the places to the left of *k*. But $(k+1) \cdot 10^n$ is bigger than this number minus the same sum of higher powers of ten associated to the places to the left of *k*. (p6).

As this suggestion appeared at first glance to be overwhelming to a teacher of basic mathematics, structuring it with their second suggestion of using a number line as a further alternative provided more clarity on the procedure. The division was presented as follows:

Illustration 1



IMPLEMENTING THE PLAN

Dividing 832 by 7 - Merging two methods

First the multiples of 100:

$$1.7.100 \leq 832 < 2.7.100$$

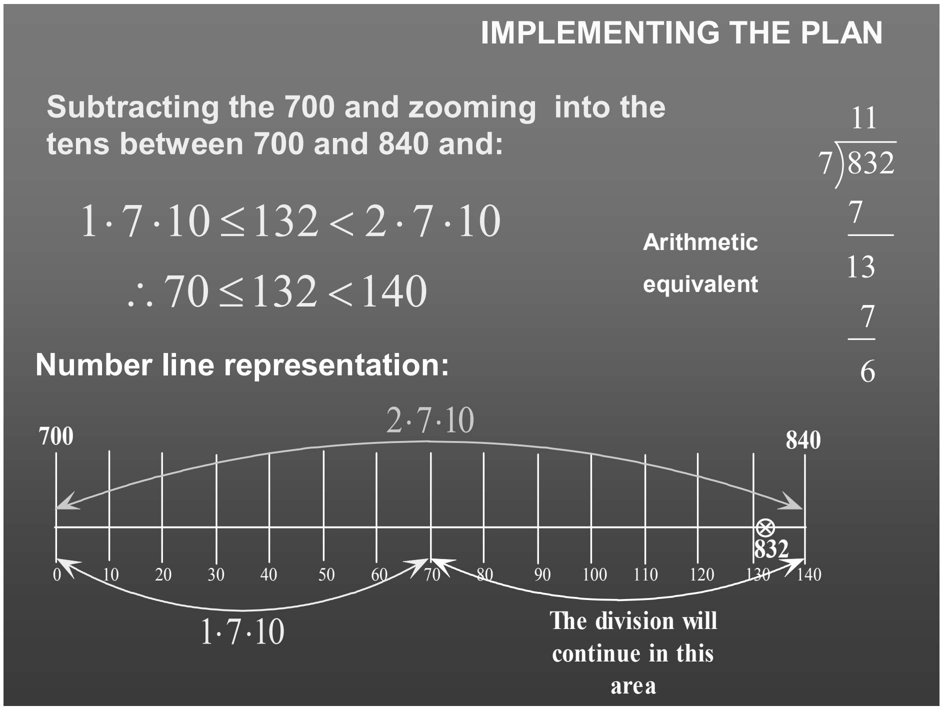
$$\therefore 700 \leq 832 < 1400$$

Arithmetic equivalent

$$\begin{array}{r} 1 \\ 7\overline{)832} \\ \underline{7} \\ 13 \end{array}$$

Number line representation:

Illustration 2



IMPLEMENTING THE PLAN

Subtracting the 700 and zooming into the tens between 700 and 840 and:

$$1\cdot 7\cdot 10 \le 132 < 2\cdot 7\cdot 10$$

$$\therefore 70 \le 132 < 140$$

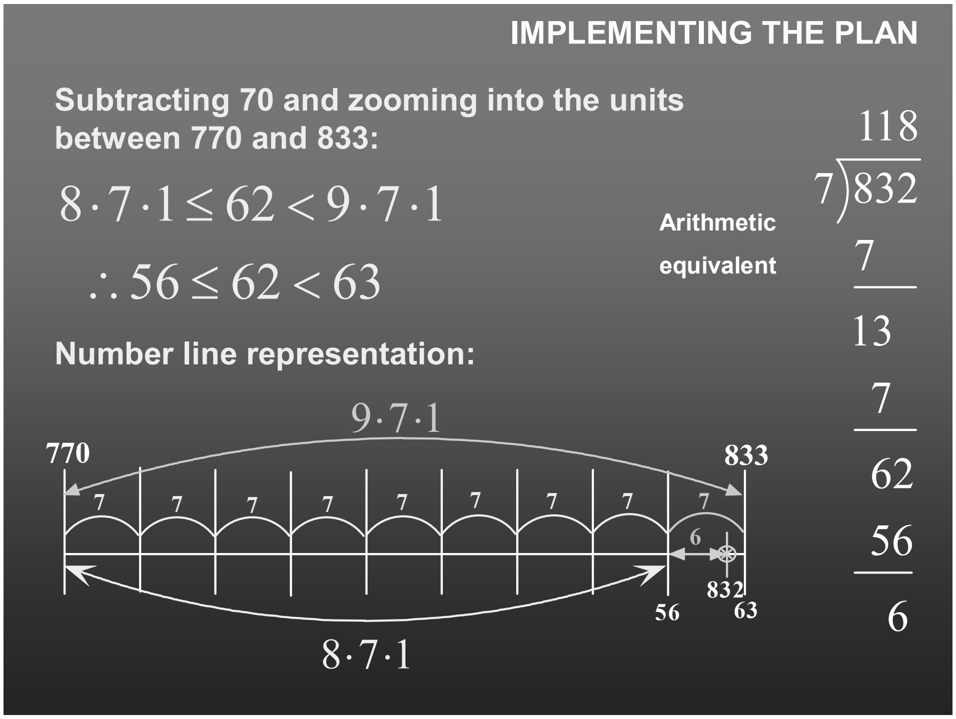
Arithmetic equivalent

$$\begin{array}{r} 11 \\ 7\overline{)832} \\ \underline{7} \\ 13 \\ \underline{7} \\ 6 \end{array}$$

Number line representation:

The division will continue in this area

Illustration 3



IMPLEMENTING THE PLAN

Subtracting 70 and zooming into the units between 770 and 833:

$$8\cdot 7\cdot 1 \le 62 < 9\cdot 7\cdot 1$$

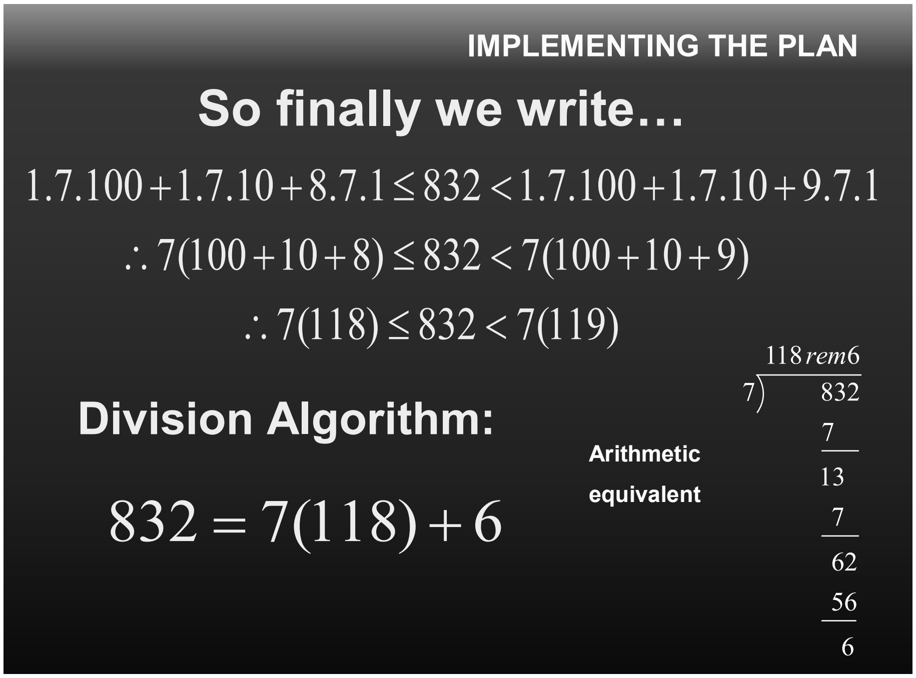
$$\therefore 56 \le 62 < 63$$

Arithmetic equivalent

$$\begin{array}{r} 118 \\ 7\overline{)832} \\ \underline{7} \\ 13 \\ \underline{7} \\ 62 \\ \underline{56} \\ 6 \end{array}$$

Number line representation:

Illustration 4



IMPLEMENTING THE PLAN

So finally we write…

$$1.7.100+1.7.10+8.7.1\leq 832<1.7.100+1.7.10+9.7.1$$

$$\therefore 7(100+10+8)\leq 832<7(100+10+9)$$

$$\therefore 7(118)\leq 832<7(119)$$

Division Algorithm:

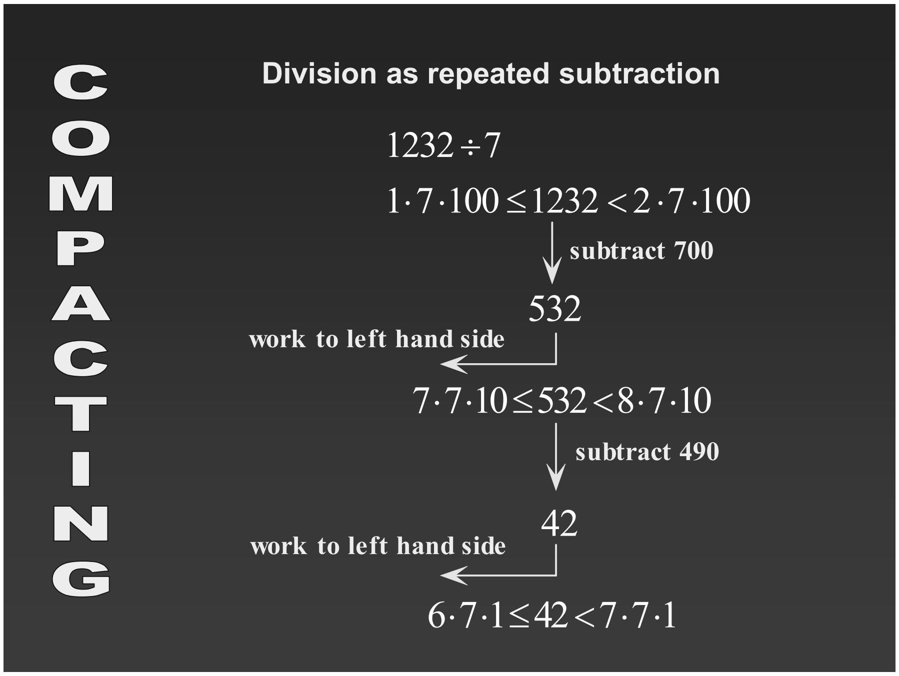
$$832=7(118)+6$$

Arithmetic equivalent

$$\begin{array}{r|r} & 118\,rem6 \\ \hline 7 & 832 \\ & \underline{7} \\ & 13 \\ & \underline{7} \\ & 62 \\ & \underline{56} \\ & 6 \end{array}$$

Compacting the argument through illustration by the teacher was necessary to avoid the ‘noise’ in which some learners lost the line of reasoning. This noise was mainly generated by arguing on a purely algebraic level first, and then after the first estimation was written down, representing our findings graphically.

Illustration 5



COMPACTING

Division as repeated subtraction

$$1232\div 7$$

$$1\cdot 7\cdot 100\leq 1232<2\cdot 7\cdot 100$$

subtract 700

$$532$$

work to left hand side

$$7\cdot 7\cdot 10\leq 532<8\cdot 7\cdot 10$$

subtract 490

$$42$$

work to left hand side

$$6\cdot 7\cdot 1\leq 42<7\cdot 7\cdot 1$$

The use of arithmetic here becomes two dimensional. At first we use division on naturals to draw on the principles of place value, and secondly we use arithmetic in conjunction with graphical representation to introduce the remainder as a number and as a concept. This shows clearly that the remainder has a position on the number line subject to what we chose as divisor.

Reflection informing the current study:

This procedure was chosen with the deliberate intention of ascertaining how learners would respond to the suggestions made by Klein and Milgram (2000) and to also test the impact of their suggestions on learner thinking. It took two one hour sessions to streamline this argument so that it made sense to the learners, after considering the cognitive challenges with which they were presented. Although learners eventually responded positively to this procedure, it was decided to abandon it for the current study based on the following reasons:

1. It was a time-consuming activity and needed careful planning in order to avoid an initial negative response from the learners. This negative response was caused by the fact that learners were faced with a highly academically formulated argument that needed careful unpacking from the teacher to convince the learners of its practical meaning.
2. The argument had created a dual problem – that of resetting each argument after the removal of the power of ten, and also the uncertainty that learners had when formulating their final statement (the division algorithm). Learners could not treat this final formulation as separate from the calculations that they made during the division procedure, mainly because they saw the resetting as tampering with the mathematical calculations.
3. Although the division was performed using a geometric as well as a generalised argument, using alternative division procedures will assist the teacher more effectively in creating a general approach that establishes and formulates the

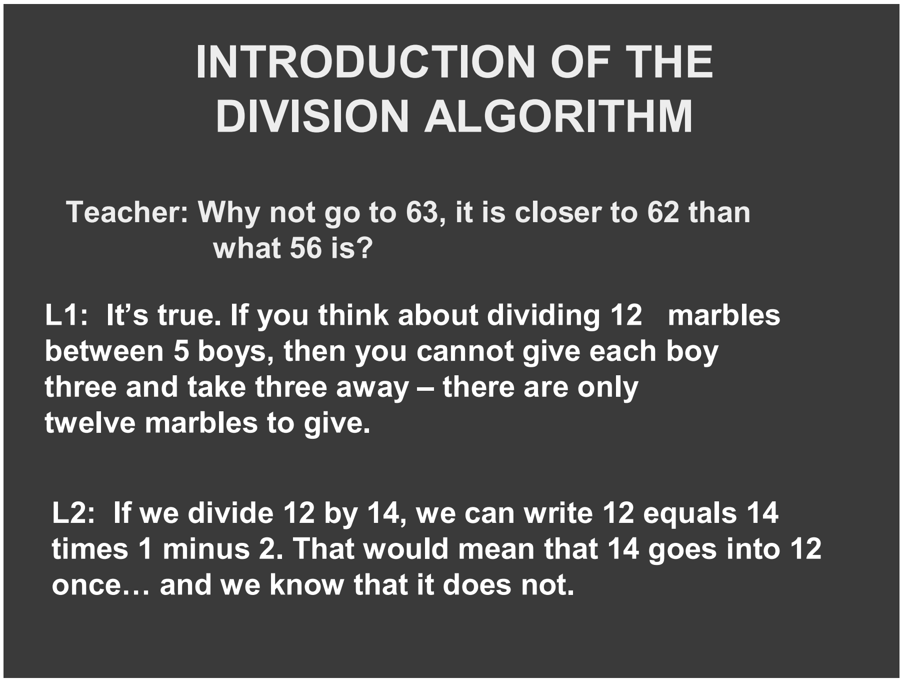
division algorithm, while preserving the concept of place value and addressing issues concerning the remainder more effectively.

I decided to approach the division procedure differently in this study (see Chapter 5) by using multiple representations that would emphasise the place value concept for learners more explicitly. This approach would also be more accessible to the Grade 8 learners than the work with which learners engaged during the pilot study.

Step 2: The introduction of the division algorithm

The next step in the division ladder was to introduce the formal *division algorithm* to the learners bearing in mind the arguments put forward by Bates and Rousseau (1989). Here the actual issues around 'over dividing' and the value of the remainder came into play in the classroom discussion as shown below when learners arrived at (the discussion refers to illustration 3):

Illustration 6



INTRODUCTION OF THE DIVISION ALGORITHM

Teacher: Why not go to 63, it is closer to 62 than what 56 is?

L1: It's true. If you think about dividing 12 marbles between 5 boys, then you cannot give each boy three and take three away – there are only twelve marbles to give.

L2: If we divide 12 by 14, we can write 12 equals 14 times 1 minus 2. That would mean that 14 goes into 12 once… and we know that it does not.

As all division procedures 'balance' in the division algorithm, it is important that this concept be made explicit to the learners. The conditions on the remainder take on a value that is less than the divisor which indicates to learners the finality of the division procedure. My assumptions were that the learners had not engaged previously with the *division algorithm* other than possibly summarising the division procedure. Indeed, I found that learners had not been afforded the opportunity of exploring the *division algorithm* as a tool upon which they can operate. Their usual conclusion after dividing 38 by 7 was that 7 divides into 38,5 times – remainder 3 (expressed as 5rem3). This limited the possibilities regarding division to an idea that it is final once this "5rem3" was concluded. Operating along this line, finding and making further conclusions about the numbers 7 and 38, was not a reality for these learners, as they had not been introduced formally to the division algorithm.

Dividing two numbers or quantities into one another will not make any sense to the learners, or any one else for that matter, if we do not consider the largest multiple of the dividend that can be removed from the divisor. In a problem-solving context the remainder must be smaller than the dividend and this characteristic must be explored explicitly by the learners.

The focus in the pilot study was to emphasise the conditions encapsulated in the division algorithm. After a lengthy analysis of the division procedure, the algorithm was introduced to learners formally through the following two statements:

a) In general: The division problem $a \div b$ is the search for whole number q and r for which $a \equiv b \cdot q + r$ where $0 \leq r < b$.

b) If we choose any whole number a (called the dividend) and any non zero whole number b (called the divisor), then it will always be possible to obtain a unique whole number q (called the quotient) and a unique whole number r (called the remainder) that is less than b, so that a, b, q and r are related as follows: $a \equiv b \cdot q + r$ where $0 \leq r < b$.

Here there are clearly two important rules that must be remembered:
i) $b \cdot q \leq a$ and ii) $0 \leq r < b$.

According to Bates and Rousseau (1989), emphasising these properties is very important as the *division algorithm* is closely related to many other advanced topics which develop from a clear understanding of the algorithm. This view is also supported by Rude (2004), Wu (1999, 2000, 2001) and Klein and Milgram (2000). Understanding this will, among other things, help learners to understand basic and advanced number theory, including modular arithmetic.

An example was given to the learners and was discussed in class, specifically highlighting the conditions in the algorithm as imposed on the quotient and the remainder. The aim of this exercise was to lead the learners to infer that although the divisor determines the quotient and the remainder, we can also work with the quotient and remainder to determine the divisor.

The example was $832 \equiv 7(118)+6$. Learners were made aware that they could have just composed the algorithm by using a different divisor, and the fact that they can obtain quite a few combinations by having chosen a different divisor was made explicit by the teacher by discussing a variety of divisors and different multiple combinations on the board while the learners made suggestions. Here are a few of the suggestions that the learners made and the teacher transcribed:

$$\begin{array}{ll}
832 \equiv 5(166)+2 & \text{where } 0 \le 2 < 5 \\
832 \equiv 12(69)+4 & \text{where } 0 \le 4 < 12 \\
832 \equiv 9(92)+4 & \text{where } 0 \le 4 < 9 \\
832 \equiv 15(55)+7 & \text{where } 0 \le 7 < 15 \\
832 \equiv 123(6)+94 & \text{where } 0 \le 94 < 123 \\
832 \equiv 13(64)+0 & \text{where } 0 \le 0 < 13 \\
832 \equiv 23(36)+4 & \text{where } 0 \le 4 < 23 \\
832 \equiv 11(75)+7 & \text{where } 0 \le 7 < 11 \\
832 \equiv 321(2)+190 & \text{where } 0 \le 190 < 321
\end{array}$$

After the third example was transcribed on the board, the learners immediately identified that the remainder is always less than the divisor and each divisor generates a unique quotient and remainder. The teacher explained that in mathematics the word *unique* means exactly one. So when a theorem (such as the division algorithm) states that "there is a unique r such that . . . ," it is telling us there *is* an r and *only one* such r.

The lesson was concluded by introducing a specific statement generated through the division algorithm. This introduction took place with the quotient given, and the divisor and remainder missing:

Rule 1: $b \cdot q \leq a$ **Rule 2**: $0 \leq r < b$

<table>
<tr><td colspan="3">Consider the statement: $13 \equiv (.....) \cdot 3 \ + (....)$</td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td>Quotient</td><td>Testing in $a \equiv b.q + r$</td><td>Conclusion</td></tr>
<tr><td>Trying 5:</td><td>$(5) \cdot 3 = 15$</td><td>Breaks rule 1</td></tr>
<tr><td>Trying 3:</td><td>$(3) \cdot 3 = 9$</td><td>Okay keep going</td></tr>
<tr><td></td><td>$\therefore 13 \equiv 3 \cdot 3 \ + 4$</td><td>Breaks rule 2</td></tr>
<tr><td>Trying 4:</td><td>$(4) \cdot 3 = 12$</td><td>Okay keep going</td></tr>
<tr><td></td><td>$\therefore 13 \equiv 4 \cdot 3 \ + 1$</td><td>Perfect</td></tr>
</table>

The learners engaged with the following problems at the end of the lesson, and completed the work at home. Here the questions focused on different numbers and their relationships with one another as defined in the division algorithm. The questions were as follows:

> ***Can you complete the following investigations regarding the properties of the division algorithm*** $a = b.q + r$ where $0 \leq r < b$, ***and use this to solve the missing values?***
>
> 1. $a \equiv 25(29) + 2$
> 2. $736 \equiv 25(29) + r$
> 3. $98 \equiv 5(q) + 3$
> 4. $a \equiv 5(2) + r$
> 5. $39 \equiv 4(q) + r$
> 6. $39 \equiv 9(q) + r$
> 7. $39 \equiv b(q) + 6$
> 8. $a \equiv 6(q) + 3$
> 9. $a \equiv 9(q) + 5$
> 10. $a \equiv 7(q) + 5$

The learners responded well to these questions and in most cases identified a value for the variable that meets the conditions. A few learners accurately observed that certain answers contain possible sets of values.

Reflection informing the current study:

Learners could correctly identify the key issues regarding the remainder after division had been completed. This, however, was only achieved by engaging the learners in one division procedure which was also presented in a geometric way that emphasised the correlation between the generalised argument and the actual implications for the number that was being divided. Approaching the arithmetic division through multiple algorithms will be beneficial to learners for the following reasons:

1. Using multiple division methods will clearly show to the learners that the final statement for all these procedures meets in the division algorithm; hence pointing out that the procedure is irrelevant and that learners can divide arithmetically in any way they feel comfortable.
2. It provides a tool with which they can check to see whether they have performed the division correctly.
3. It would provide an opportunity for them to see that whichever way they performed the division, comparing their remainders would become a way of checking whether, across the group, the division was performed without error.

The emphasis on the *division algorithm* will be shifted in this study for the following reasons:

1. The multiple division procedure focus will emphasise the conditions that are contained in the division algorithm.
2. The focus of the current study is more on the decomposition into base ten division as an algorithm, and therefore the *division algorithm* will only serve as a concluding statement for each procedure.

3. The learners responded intuitively to these conditions on the division procedure in the pilot study, before the formal algorithm was introduced.
4. The specific focus performs a two dimensional role as far as division is concerned which can be classified as summarising the procedure in a final statement and checking whether the division procedure is complete. Its use can be extended later on when more advanced mathematics comes into play in the field of fraction theory.

Step 3: The *division algorithm* and modular arithmetic

There is a definite link between the *division algorithm* and modular arithmetic. This was identified by the Grade 9 learners as they had already engaged with modular arithmetic in the learning area of computer studies. Learners revealed that the *division algorithm* helped them to make sense of what they had learnt in computer studies, and this relationship was explored during the classroom period. In this sense, modular arithmetic became a sensible application of the *division algorithm* for the learners and the conceptual framework within which the *division algorithm* and modular arithmetic enhanced the interconnectedness of the two products of division.

Modular arithmetic (commonly known as clock arithmetic) is currently a topic which is not included in the syllabus, yet it makes for an interesting extension of the *division algorithm* and elegantly rounds off the geometric approach to the division process of place value division: the idea that division by a number is in fact translated to removing modules / groups of this number from a bigger collective number which may contain various groupings of this specific number. Here the number that is divided by and the remainder is important, and not the quotient, as we are interested in the remainder more than anything else.

The link between these two products of division was made explicit after the classroom discussion and then the formal definitions were written on the board as follows:

> If a and b are both integers and $a,b \neq 0$, then there exist unique integers q and r such that: $a \equiv b \cdot q + r \quad where \quad 0 \leq r < b$.
>
> Linking this to modular division, we will write $a \equiv r\left(\text{mod}\, b\right)$.

Learners engaged with the long division procedure and were required to write a concluding statement in modular form as well as using the *division algorithm* in the following manner:

$$\begin{array}{r} 15 \\ 8\overline{)124} \\ \underline{8} \\ 44 \\ \underline{40} \\ 4 \end{array} \qquad \begin{array}{l} 124 \equiv \left(8 \cdot 15\right) + 4 \\ 124 \equiv 4\left(\text{mod}\, 8\right) \end{array}$$

Learners were given tasks to complete in class. They will not be discussed here since they are beyond the scope of this study.

Reflection informing the current study:

As modular arithmetic is a completely separate field of mathematics, it was decided to exclude it from the division ladder, despite the close conceptual base that it shares with the division algorithm. It was excluded from the current study based on the following reasons:

1. It is a branch of mathematics that is not necessary for the learners to engage with in order to understand the long division procedure.
2. Engaging with modular arithmetic shows the learners one application of the *division algorithm* and does not enhance the understanding of the division algorithm. It merely re-expresses the product of division more concisely.
3. It is only beneficial for the learners who do computer studies as a subject and could easily be left for the computer studies teacher to explore by using the *division algorithm* and building from there.

Step 4: Division using base ten decomposition

In the division ladder this is the step that I propose to bridge the cognitive gap between arithmetic long division and algebraic long division. However, in the pilot study the base ten decomposition was mainly focused on to introduce step 5 of the division ladder which introduces different base decompositions other than ten. Learners were introduced to this decomposition procedure using the following two examples:

Problem 1:

$$\begin{array}{r} 12 \\ 11\overline{)138} \\ \underline{11} \\ 28 \\ \underline{22} \\ 6 \end{array} \qquad \therefore 138 \equiv (11\cdot 12)+6$$

In base 10:

$$\begin{array}{r} 1\cdot 10+2 \\ (1\cdot 10+1)\overline{)1\cdot 10^2+3\cdot 10+8} \\ \underline{1\cdot 10^2+1\cdot 10} \\ 2\cdot 10+8 \\ \underline{2\cdot 10+2} \\ 6 \end{array} \qquad \therefore 138 \equiv (1\cdot 10+1)(12)+6$$

Problem 2:

$$\begin{array}{r} 15 \\ 8\overline{)123} \\ \underline{8} \\ 43 \\ \underline{40} \\ 3 \end{array} \qquad \therefore 123 \equiv (8\cdot 15)+3$$

In base 10:

$$\begin{array}{r} 1\cdot 10+4+1 \\ (1\cdot 10-2)\overline{)1\cdot 10^2+2\cdot 10+3} \\ \underline{1\cdot 10^2-2\cdot 10} \\ 4\cdot 10+3 \\ \underline{4\cdot 10-8} \\ 11 \\ \underline{8} \\ 3 \end{array} \qquad \therefore 124 \equiv (1\cdot 10-2)(15)+3$$

Learners could decompose the dividend easily as it is based on the base ten place value system within which our number system operates. Decomposing the divisor was not problematic when the divisor was greater than 10, but learners needed some explanation to write divisors that are smaller than ten, as a difference of ten and the number that was added to the divisor to get to one multiple of ten.

The most challenging aspect of the base ten decomposition division procedure was establishing the twofold discourse: to firstly develop the understanding of the procedure and secondly to focus on the correct sequence in the questions that are asked when the multiple division with remainder procedure is performed. I anticipated the difficulty that the learners would experience in performing the division, as it is as close as arithmetic division can come to the traditional algebraic division on polynomials – an area that has proven to be challenging for learners. The problems that learners in the pilot study experienced are the same as those difficulties that learners had in the current study, and these will be explored in Chapter 6.

Reflection informing the current study:

The first problem provided the opportunity to introduce the division in base ten decomposition without any major difficulty other than acquainting the learners with the division procedure. As mentioned above, establishing the division discourse was a two-fold procedure that proved challenging during the pilot study. Expressing the divisor as a multiple of ten will ensure that the intention of the question "*how many multiples of ten will go into one multiple of ten squared?"* becomes more explicit.

The second problem that arose was the difference in the remainders when performing arithmetic long division on naturals, and the case where the division was performed in base ten decomposition. In the pilot study, the equality of the division was reached by placing the emphasis on the long division procedure to continue after the last 'digit' was carried down from the dividend. The process of division continued by considering how the original divisor continues to divide the dividend. So the learners would simply ignore the base ten decomposition and continue to divide the partial remainder (which was 11) by the divisor 8. This was done to overcome the cognitive conflict created by the different remainder, and to show to learners that the division procedure would be completed when the two remainders are equal.

After careful consideration, I concluded that focusing the division within the long division procedure undermines the base ten decomposition procedure in the sense that learners could conclude that either the *division algorithm* does not hold true for all arithmetic division procedures or that the base ten division procedure does not yield the same result as dividing using the procedure that learners have been accustomed to since primary school.

Shifting the focus so that the division continues within the *division algorithm* will enhance the properties and conditions on the remainder and provide the learners with the opportunity to reason and construct meaning for themselves while considering why these remainders are indeed different. As an example of how division continues within the *division algorithm*, consider the following:

$$\begin{array}{r} 1\cdot 10+5 \\ (1\cdot 10-3)\overline{\big)\,1\cdot 10^2+2\cdot 10+9} \\ \underline{1\cdot 10^2-3\cdot 10\quad\quad} \\ +5\cdot 10+9 \\ \underline{+5\cdot 10-15} \\ +24 \end{array}$$

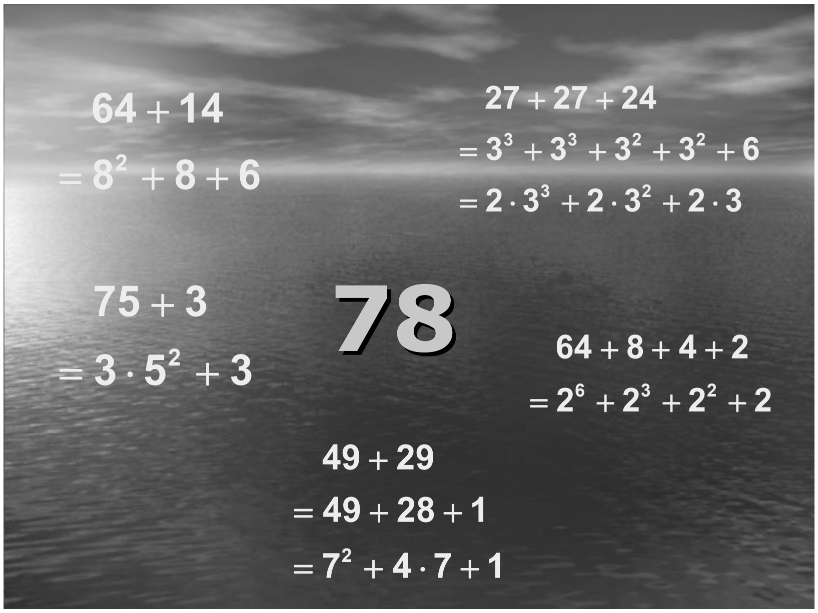
$$\begin{aligned} \therefore 1\cdot 10^2+2\cdot 10+9 &= (1\cdot 10-3)(1\cdot 10+5)+24 && \rightarrow 24>7 \\ &= (7)(15)+21+3 && \rightarrow \textit{remove multiples of } 7 \textit{ from remainder} \\ &= (7)(15)+(7)(3)+3 \\ &= (7)(15+3)+3 \\ &= (7)(18)+3 \end{aligned}$$

The case of 'over division' will also emerge more naturally in the case of obtaining a negative remainder. Learners will be given the opportunity to discover the tensions between the different procedures for themselves, and debate among themselves about the reasons for such differences. Shifting the division to the *division algorithm* will also provide a clear distinction between operating on the known (numbers) and later the unknown (variable) when they engage with algebraic long division. This distinction, if carefully drawn, will then provide the rationale for where and why algebraic long division will end.

Step 5: Division using a base other than base ten decomposition

The rationale for exploring decomposition beyond base ten was to raise the cognitive demand of arithmetic long division beyond the level which was necessary for introducing algebraic division in order to make the abstraction to algebra easier for the learners. This approach proved very interesting for the learners, but did take some exploration in order for them to feel comfortable with the fact that there was not one specific correct decomposition and that many different decompositions in one particular base could be obtained. Looking particularly at the number 78, one can decompose the number in a variety of bases and can then arrive at the following decompositions:

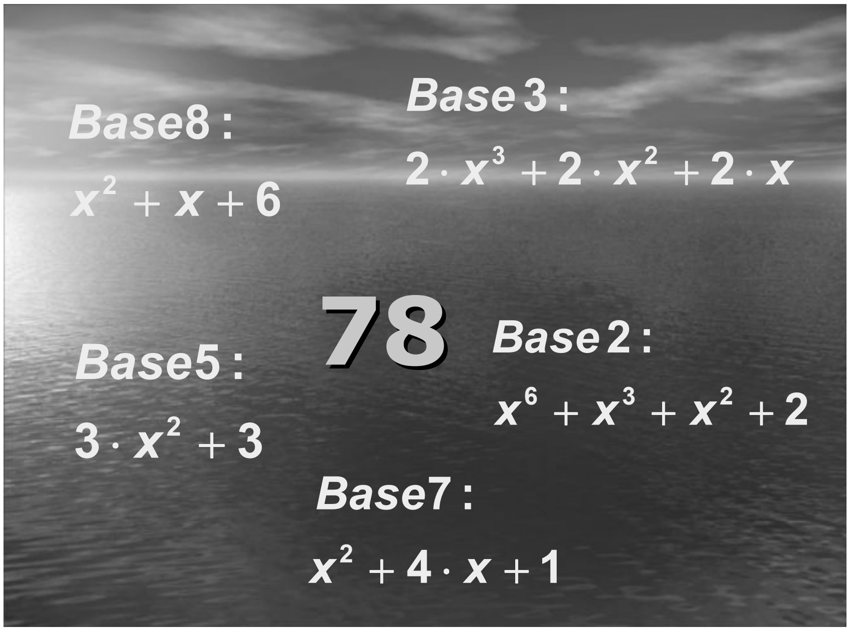
Illustration 7



In base 2 the decomposition is shown as $64+8+4+2=2^6+2^3+2^2+2$ but could easily have been decomposed as $3\cdot 2^4+2^3+4\cdot 2^2+3\cdot 2$ or $2\cdot 2^5+2^3+3\cdot 2$. This open-ended nature of the decomposition left the learners with a sense of uncertainty as they could not cross reference among one another to know for certain that their answer was right. To avoid such conflict hampering the procedure of decomposition, it was agreed that any decomposition containing a power relation of base b which is expressed as a multiple $a \cdot b^n$ will have to comply with the condition that $a < b$. This will still not guarantee a unique decomposition, but will limit the number of possibilities across the group and create the possibility that some decompositions will be the same.

The different base decompositions shown in the slide above will eventually be introduced as polynomials by replacing the base with the variable to obtain:

Illustration 8



Base8 :

$x^2 + x + 6$

Base3 :

$2 \cdot x^3 + 2 \cdot x^2 + 2 \cdot x$

78

Base5 :

$3 \cdot x^2 + 3$

Base2 :

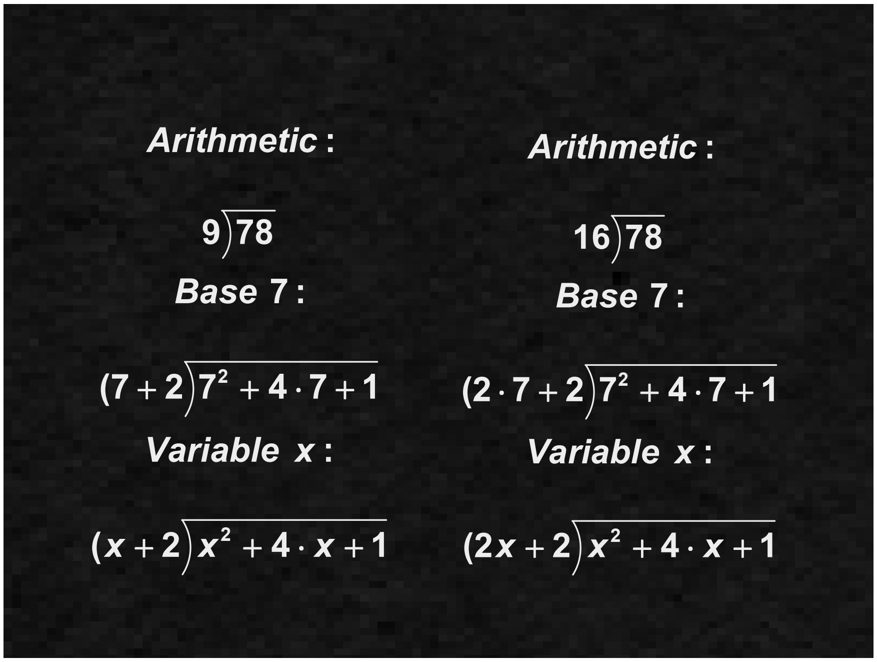
$x^6 + x^3 + x^2 + 2$

Base7 :

$x^2 + 4 \cdot x + 1$

The structure of each of the problems in different base representations will appear as follows:

Illustration 9



Arithmetic :

$9\overline{)78}$

Base 7 :

$(7+2\overline{)7^2+4\cdot7+1}$

Variable x :

$(x+2\overline{)x^2+4\cdot x+1}$

Arithmetic :

$16\overline{)78}$

Base 7 :

$(2\cdot7+2\overline{)7^2+4\cdot7+1}$

Variable x :

$(2x+2\overline{)x^2+4\cdot x+1}$

This provided a logical sequence with which the structure of arithmetic was enhanced by different base decompositions, and finally allowing the replacement of the base with the variable when it became possible to abstract.

The work that was introduced to the learners continuously drew on the building blocks that they had formed over the previous four days of exploration with arithmetic division on naturals, the division algorithm, modular arithmetic, base ten decomposition and further base decompositions.

What follows on the next six pages is an example of the work that was done during the pilot study in this step of the division ladder. (The learners adopted the title modular division when they referred to division based on base decompositions other than base ten.)

a) Let us consider $123 \div 8$ and use the divisor split of $8 = 5+3$. We will therefore have to write the number 123 as power multiples of 5. So let's begin:

$$\begin{aligned} 123 &\equiv 100+20+3 \\ &\equiv 25\cdot 4+4\cdot 5+3 \\ &\equiv 4\cdot 5^2+4\cdot 5+3 \end{aligned}$$

$$8\overline{)123} \quad \rightarrow 5+3\overline{)4\cdot 5^2+4\cdot 5+3\cdot 5^0}$$

$$\begin{array}{r} 4\cdot 5-8\cdot 5^0 +3 \\ 5+3\overline{)4\cdot 5^2+4\cdot 5+3\cdot 5^0} \\ \underline{4\cdot 5^2+12\cdot 5} \\ -8\cdot 5+3\cdot 5^0 \\ \underline{-8\cdot 5-24\cdot 5^0} \\ 27\cdot 5^0 \\ \underline{24} \\ 3 \end{array}$$

Thus: $123 \equiv (5+3)(4\cdot 5-8\cdot 5^0 +3)+3$
$\equiv 8\cdot 15+3$

$$\begin{aligned} 123 &\equiv 125-2 \\ &\equiv \underbrace{1\cdot 5^3-2}_{\substack{\text{This is the more difficult} \\ \text{choice as it involves a negative} \\ \text{remainder. The division} \\ \text{passed 123 in the} \\ \text{first estimation.}}} \end{aligned}$$

$$8\overline{)123} \quad \rightarrow 5+3\overline{)1\cdot 5^3+0\cdot 5^2+0\cdot 5-2\cdot 5^0}$$

$$\begin{array}{r} 1\cdot 5^2-3\cdot 5+9\cdot 5^0 -3 \\ 5+3\overline{)1\cdot 5^3+0\cdot 5^2+0\cdot 5-2\cdot 5^0} \\ \underline{1\cdot 5^3+3\cdot 5^2} \\ -3\cdot 5^2+0\cdot 5 \\ \underline{-3\cdot 5^2-9\cdot 5} \\ 9\cdot 5-2\cdot 5^0 \\ \underline{9\cdot 5+27\cdot 5^0} \\ -29\cdot 5^0 \\ \underline{-24} \\ -5 \end{array}$$

Thus: $123 = 1\cdot 5^3+0\cdot 5^2+0\cdot 5-2\cdot 5^0$
$= (5+3)(1\cdot 5^2-3\cdot 5+9\cdot 5^0 -3)-5$
$= 8\cdot 16-5$

b) Let us consider $123 \div 8$ and use the divisor split of $8 = 7+1$. We will therefore have to write the number 123 as power multiples of 7. So let's begin:

$$\begin{aligned}
123 &\equiv 70+53\\
&\equiv 7\cdot 10+7\cdot 7+4\\
&\equiv 7(7+3)+7^2+4\\
&\equiv 7^2+3\cdot 7+7^2+4\\
&\equiv 2\cdot 7^2+3\cdot 7+4
\end{aligned}$$

$$\begin{aligned}
123 &\equiv 84+39\\
&\equiv 7\cdot 12+7\cdot 5+4\\
&\equiv 7(7+5)+7\cdot 5+4\\
&\equiv 7^2+5\cdot 7+7\cdot 5+4\\
&\equiv 7^2+10\cdot 7+4\\
&\equiv 7^2+(7+3)7+4\\
&\equiv 7^2+7^2+3\cdot 7+4\\
&\equiv 2\cdot 7^2+3\cdot 7+4
\end{aligned}$$

Now for the division:

$$\begin{array}{r}
2\cdot 7+1\\
8\overline{)123} \;\rightarrow\; 7+1\overline{)2\cdot 7^2+3\cdot 7+4}\\
\underline{2\cdot 7^2+2\cdot 7}\quad\;\;\\
1\cdot 7+4\\
\underline{1\cdot 7+1}\\
3
\end{array}$$

Thus: $123 \equiv (7+1)(2\cdot 7+1)+3 \equiv 8(15)+3$

c) Let us consider $123 \div 8$ and use the divisor split of $8 = 6+2$. We will therefore have to write the number 123 as power multiples of 6.

$$\begin{aligned}
123 &\equiv 120+3\\
&\equiv 6\cdot 20+3\\
&\equiv 6(3.6+2)+3\\
&\equiv 3\cdot 6^2+2\cdot 6+3
\end{aligned}$$

$$\begin{array}{r}
3\cdot 6-4+1\\
8\overline{)123} \;\rightarrow\; 6+2\overline{)3\cdot 6^2+2\cdot 6+3}\\
\underline{3\cdot 6^2+6\cdot 6}\quad\;\;\\
-4\cdot 6+3\\
\underline{-4\cdot 6-8}\\
11\\
\underline{8}\\
3
\end{array}$$

So $123 \equiv (6+2)(3\cdot 6-4+1)+3 \equiv 8(15)+3$

Note the following about every division:

$123 \equiv (10-2)(1\cdot 10+4+1)+3 \equiv 8\cdot 15+3$

$123 \equiv (5+3)(4\cdot 5-8\cdot 5^0+3)+3 \equiv 8(15)+3$

$123 \equiv (7+1)(2\cdot 7+1)+3 \equiv 8(15)+3$

$123 \equiv (6+2)(3\cdot 6-4+1)+3 \equiv 8(15)+3$

In essence it is that same division that took place throughout as we divided 8 into 123.

<u>Generalising this idea to polynomial division</u>:

For base 10:

$$
\begin{array}{rr}
 & 1\cdot 10+4+1 \\ \cline{2-2}
10-2\big) & 1\cdot 10^2+2\cdot 10+3 \\
 & \underline{1\cdot 10^2-2\cdot 10} \\
 & 4\cdot 10+3 \\
 & \underline{4\cdot 10-8} \\
 & 11 \\
 & \underline{8} \\
 & 3
\end{array}
$$

<u>If we let *x* =10</u>:

$$
\begin{array}{rr}
 & 1\cdot x+4 \\ \cline{2-2}
x-2\big) & 1\cdot x^2+2\cdot x+3 \\
 & \underline{1\cdot x^2-2\cdot x} \\
 & 4\cdot x+3 \\
 & \underline{4\cdot x-8} \\
 & 11
\end{array}
$$

For Base 5:

$$
\begin{array}{rr}
 & 4\cdot 5-8\cdot 5^0+3 \\ \cline{2-2}
5+3\big) & 4\cdot 5^2+4\cdot 5+3\cdot 5^0 \\
 & \underline{4\cdot 5^2+12\cdot 5} \\
 & -8\cdot 5+3\cdot 5^0 \\
 & \underline{-8\cdot 5-24\cdot 5^0} \\
 & 27\cdot 5^0 \\
 & \underline{24} \\
 & 3
\end{array}
$$

<u>If we let *x* = 5</u>:

$$
\begin{array}{rr}
 & 4\cdot x-8 \\ \cline{2-2}
x+3\big) & 4\cdot x^2+4\cdot x+3 \\
 & \underline{4\cdot x^2+12\cdot x} \\
 & -8\cdot x+3 \\
 & \underline{-8\cdot x-24} \\
 & 27
\end{array}
$$

For base 7:

$$
\begin{array}{rr}
 & 2\cdot 7+1 \\ \cline{2-2}
7+1\big) & 2\cdot 7^2+3\cdot 7+4 \\
 & \underline{2\cdot 7^2+2\cdot 7} \\
 & 1\cdot 7+4 \\
 & \underline{1\cdot 7+1} \\
 & 3
\end{array}
$$

<u>If we let *x* = 7</u>:

$$
\begin{array}{rr}
 & 2x+1 \\ \cline{2-2}
x+1\big) & 2\cdot x^2+3\cdot x+4 \\
 & \underline{2\cdot x^2+2\cdot x} \\
 & 1\cdot x+4 \\
 & \underline{1\cdot x+1} \\
 & 3
\end{array}
$$

For base 6:

$$\begin{array}{rl}
 & 3\cdot 6-4+1 \\ \cline{2-2}
6+2\big) & 3\cdot 6^2+2\cdot 6+3 \\
 & \underline{3\cdot 6^2+6\cdot 6} \\
 & \qquad -4\cdot 6+3 \\
 & \qquad \underline{-4\cdot 6-8} \\
 & \qquad\qquad 11 \\
 & \qquad\qquad \underline{8} \\
 & \qquad\qquad 3
\end{array}$$

If we let *x* = 6:

$$\begin{array}{rl}
 & 3\cdot x-4 \\ \cline{2-2}
x+2\big) & 3\cdot x^2+2\cdot x+3 \\
 & \underline{3\cdot x^2+6\cdot x} \\
 & \qquad -4\cdot x+3 \\
 & \qquad \underline{-4\cdot x-8} \\
 & \qquad\qquad 11
\end{array}$$

More combinations of division showing both methods:

Considering $138 \div 11$. We have already considered 11 as 10 + 1, so we will continue with the modular versions.

Considering the split : 11 = 9 + 2

$$\begin{aligned}
138 &\equiv 9\cdot 10+48 \\
&\equiv 9(9+1)+(9\cdot 5)+3 \\
&\equiv 9^2+9+5\cdot 9+3 \\
&\equiv 9^2+6\cdot 9+3
\end{aligned}$$

So:

$$\begin{array}{rl}
 & 9+4 \\ \cline{2-2}
9+2\big) & 9^2+6\cdot 9+3 \\
 & \underline{9^2+2\cdot 9} \\
 & \qquad 4\cdot 9+3 \\
 & \qquad \underline{4\cdot 9+8} \\
 & \qquad\qquad -5
\end{array}$$

$\therefore 138 \equiv (9+2)(9+4)-5 \equiv 11(13)-5$

If we let *x* = 9:

$$\begin{array}{rl}
 & x+4 \\ \cline{2-2}
x+2\big) & x^2+6\cdot x+3 \\
 & \underline{x^2+2\cdot x} \\
 & \qquad 4\cdot x+3 \\
 & \qquad \underline{4\cdot x+8} \\
 & \qquad\qquad -5
\end{array}$$

So: $x^2+6x+3 \equiv (x+2)(x+4)-5$

Considering the split: 11 = 8 + 3

$$\begin{aligned}
138 &\equiv 8\cdot 10+58 \\
&\equiv 8(8+2)+(7\cdot 8)+2 \\
&\equiv 8^2+2\cdot 8+7\cdot 8+2 \\
&\equiv 8^2+9\cdot 8+2 \\
&\equiv 8^2+8(8+1)+2 \\
&\equiv 2\cdot 8^2+1\cdot 8+2
\end{aligned}$$

So:

$$\begin{array}{r r}
 & 2\cdot 8-5+1 \\
8+3 \big) & \overline{2\cdot 8^2+1\cdot 8+2} \\
 & \underline{2\cdot 8^2+6\cdot 8} \\
 & -5\cdot 8+2 \\
 & \underline{-5\cdot 8-15} \\
 & 17 \\
 & \underline{11} \\
 & 6
\end{array}$$

Thus:

$138 \equiv (8+3)(16-5+1)+6 \equiv 11\cdot 12+6$

<u>If we let *x* = 8</u>:

$$\begin{array}{r r}
 & 2\cdot x-5 \\
x+3 \big) & \overline{2\cdot x^2+1\cdot x+2} \\
 & \underline{2\cdot x^2+6\cdot x} \\
 & -5\cdot x+2 \\
 & \underline{-5\cdot x-15} \\
 & 17
\end{array}$$

Thus:

$2x^2+x+2 \equiv (x+3)(2x-5)+17$

<u>Considering the split: 11 = 7 + 4</u>:

$$\begin{aligned}
138 &\equiv 140-2 \\
&\equiv 7\cdot 20-2 \\
&\equiv 7(7\cdot 3-1)-2 \\
&\equiv 3.7^2-1\cdot 7-2
\end{aligned}$$

So:

$$\begin{array}{r r}
 & 3\cdot 7-13+4 \\
7+4 \big) & \overline{3\cdot 7^2-1\cdot 7-2} \\
 & \underline{3\cdot 7^2+12\cdot 7} \\
 & -13\cdot 7-2 \\
 & \underline{-13\cdot 7-52} \\
 & 50 \\
 & \underline{44} \\
 & 6
\end{array}$$

$138 \equiv (7+4)(21-13+4)+6 \equiv 11\cdot 12$

<u>If we let *x* =7</u>:

$$\begin{array}{r r}
 & 3\cdot x-13 \\
x+4 \big) & \overline{3\cdot x^2-1\cdot x-2} \\
 & \underline{3\cdot x^2+12\cdot x} \\
 & -13\cdot x-2 \\
 & \underline{-13\cdot x-52} \\
 & 50
\end{array}$$

$3x^2-x-2 \equiv (x+4)(3x-13)+50$

<u>Considering the split: 11 = 6 + 5</u>:

$$\begin{aligned}
138 &\equiv 120+18 \\
&\equiv 6(20)+3\cdot 6 \\
&\equiv 6(3\cdot 6+2)+3\cdot 6 \\
&\equiv 3\cdot 6^2+2\cdot 6+3\cdot 6 \\
&\equiv 3\cdot 6^2+5\cdot 6
\end{aligned}$$

So:

$$\begin{array}{r} 3\cdot 6-10+4 \\ 6+5\overline{)3\cdot 6^2+5\cdot 6+0} \\ \underline{3\cdot 6^2+15\cdot 6} \\ -10\cdot 6+0 \\ \underline{-10\cdot 6-50} \\ 50 \\ \underline{44} \\ 6 \end{array}$$

$138 \equiv (6+5)(18-10+4)+6 \equiv 11(12)+6$

If we let $x = 6$:

$$\begin{array}{r} 3\cdot x-10 \\ x+5\overline{)3\cdot x^2+5\cdot x+0} \\ \underline{3\cdot x^2+15\cdot x} \\ -10\cdot x+0 \\ \underline{-10\cdot x-50} \\ 50 \end{array}$$

$3x^2+5x \equiv (x+5)(3x-10)+50$

Does the algebra hold true for any x?

Using our last division: $3x^2+5x \equiv (x+5)(3x-10)+50$

If $x = 7$:

$LHS = 3(49)+35 = 182$ and

$RHS = (7+5)(21-10)+50 = 12(11)+50 = 182$

If $x = 13$:

$LHS = 3(13)^2+5(13) = 572$ *and*

$RHS = (13+5)(39-10)+50 = 18(29)+50 = 572$

Like any new content that learners need to engage with, it took some practice before they developed their personal procedures in order to complete the decompositions. At the end of the second day of working with base decompositions (base ten and other bases) and division, the learners felt comfortable enough to move to algebraic division and the next sample is an example of the work that was done with them over the following two days:

Learner 1:

a) i)

$$\begin{array}{r|l} & 2\ 4 \quad r1 \\ 8 & 1\ 9\ 3 \\ & 1\ 6 \downarrow \\ & \ \ 3\ 3 \\ & \ \ 3\ 2 \\ & \quad\ 1 \end{array}$$

$193 = 8(24) + 1$

ii)

$$\begin{array}{r|l} & 1 \times 10 \quad +11 \quad +3 \qquad r1 \\ 10 - 2 & 1 \times 10^2 + 9 \times 10^1 + 3 \times 10^0 \\ & 1 \times 10^2 - 2 \times 10^1 \downarrow \\ & \qquad 11 \times 10 + 3 \\ & \qquad 11 \times 10 - 22 \\ & \qquad\qquad - \ 25 \\ & \qquad\qquad\quad 24 \\ & \qquad\qquad\quad\ 1 \end{array}$$

$193 = (10-2)(24) + 1$

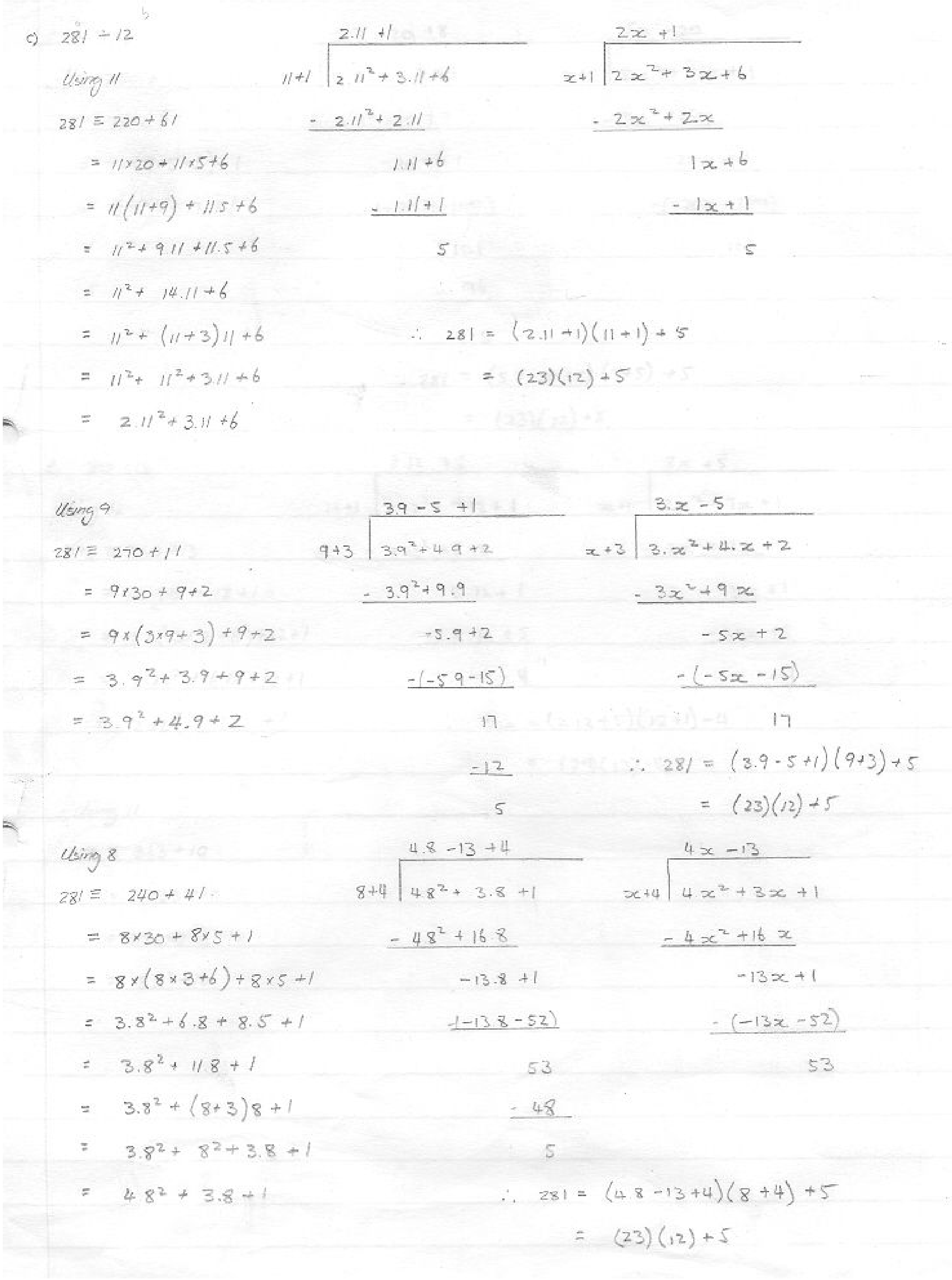
$x = 10$

iii)

$$\begin{array}{r|l} & x \quad +11 \\ x-2 & x^2 + 9x + 3 \\ & x^2 - 2x \downarrow \\ & \quad 11x + 3 \\ & \quad 11x - 22 \\ & \qquad\quad 25 \end{array}$$

$x2+9x+3 = (x-2)(x+11) + 25$

Learner 2:



c) $281 \div 12$

Using 11

$281 \equiv 220 + 61$

$= 11 \times 20 + 11 \times 5 + 6$

$= 11(11+9) + 11.5 + 6$

$= 11^2 + 9.11 + 11.5 + 6$

$= 11^2 + 14.11 + 6$

$= 11^2 + (11+3)11 + 6$

$= 11^2 + 11^2 + 3.11 + 6$

$= 2.11^2 + 3.11 + 6$

$11+1 \,\big|\, 2.11^2 + 3.11 + 6$ quotient $2.11 + 1$

$- \underline{2.11^2 + 2.11}$

$1.11 + 6$

$- \underline{1.11 + 1}$

5

$x+1 \,\big|\, 2x^2 + 3x + 6$ quotient $2x + 1$

$- \underline{2x^2 + 2x}$

$1x + 6$

$- \underline{1x + 1}$

5

$\therefore\ 281 = (2.11+1)(11+1) + 5$

$= (23)(12) + 5$

Using 9

$281 \equiv 270 + 11$

$= 9 \times 30 + 9 + 2$

$= 9 \times (3 \times 9 + 3) + 9 + 2$

$= 3.9^2 + 3.9 + 9 + 2$

$= 3.9^2 + 4.9 + 2$

$9+3 \,\big|\, 3.9^2 + 4.9 + 2$ quotient $3.9 - 5 + 1$

$- \underline{3.9^2 + 9.9}$

$-5.9 + 2$

$\underline{-(-5.9 - 15)}$

17

$\underline{-12}$

5

$x+3 \,\big|\, 3.x^2 + 4.x + 2$ quotient $3.x - 5$

$- \underline{3x^2 + 9x}$

$-5x + 2$

$\underline{-(-5x - 15)}$

17

$\therefore\ 281 = (3.9 - 5 + 1)(9+3) + 5$

$= (23)(12) + 5$

Using 8

$281 \equiv 240 + 41$

$= 8 \times 30 + 8 \times 5 + 1$

$= 8 \times (8 \times 3 + 6) + 8 \times 5 + 1$

$= 3.8^2 + 6.8 + 8.5 + 1$

$= 3.8^2 + 11.8 + 1$

$= 3.8^2 + (8+3)8 + 1$

$= 3.8^2 + 8^2 + 3.8 + 1$

$= 4.8^2 + 3.8 + 1$

$8+4 \,\big|\, 4.8^2 + 3.8 + 1$ quotient $4.8 - 13 + 4$

$- \underline{4.8^2 + 16.8}$

$-13.8 + 1$

$\underline{-(-13.8 - 52)}$

53

$\underline{-48}$

5

$x+4 \,\big|\, 4x^2 + 3x + 1$ quotient $4x - 13$

$- \underline{4x^2 + 16x}$

$-13x + 1$

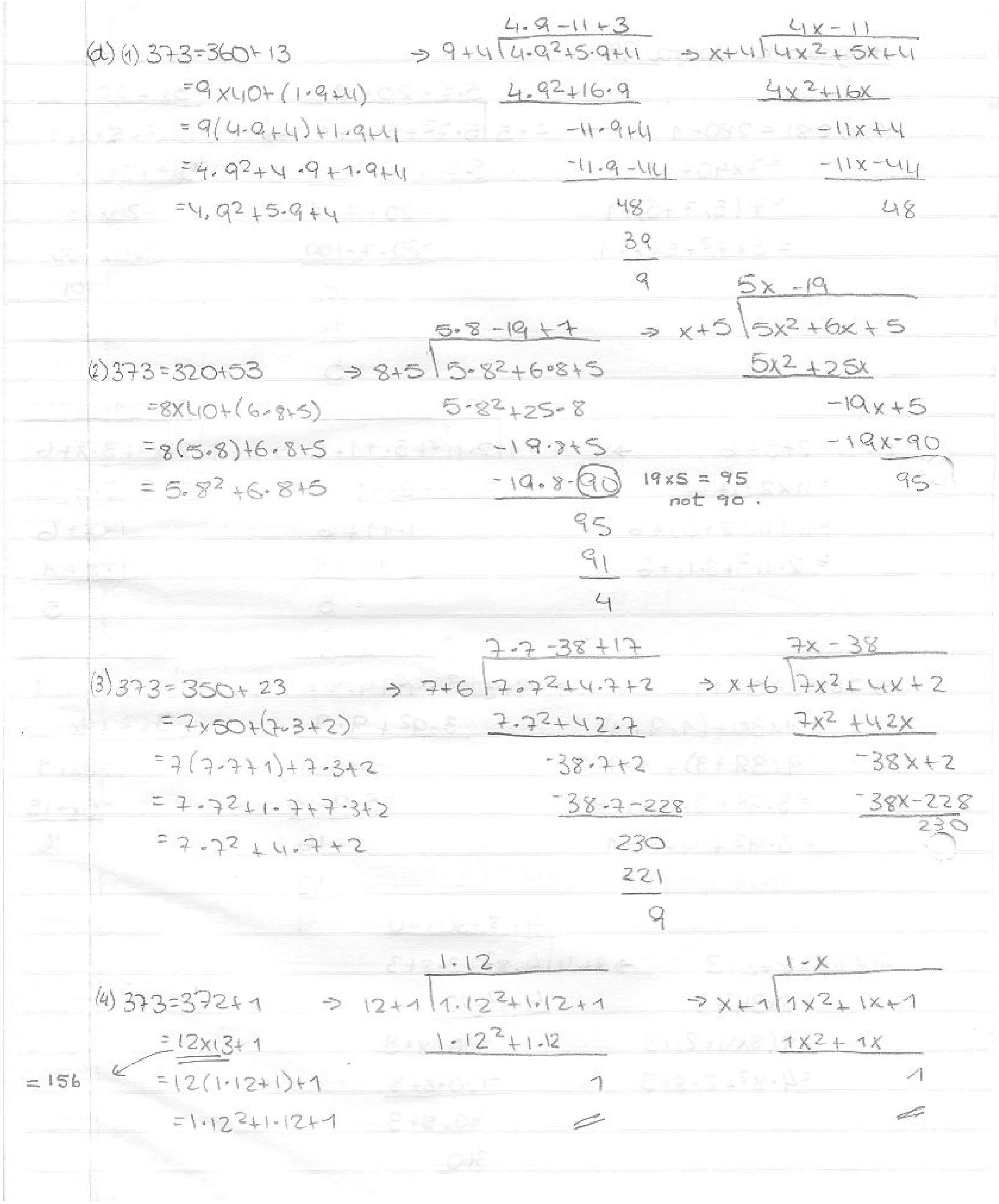
$\underline{-(-13x - 52)}$

53

$\therefore\ 281 = (4.8 - 13 + 4)(8+4) + 5$

$= (23)(12) + 5$

Learner 3:



(d) (1) $373 = 360 + 13$
$= 9 \times 40 + (1 \cdot 9 + 4)$
$= 9(4 \cdot 9 + 4) + 1 \cdot 9 + 4$
$= 4 \cdot 9^2 + 4 \cdot 9 + 1 \cdot 9 + 4$
$= 4 \cdot 9^2 + 5 \cdot 9 + 4$

$\rightarrow 9 + 4 \overline{)\, 4 \cdot 9^2 + 5 \cdot 9 + 4}$ quotient $4 \cdot 9 - 11 + 3$
$\underline{4 \cdot 9^2 + 16 \cdot 9}$
$-11 \cdot 9 + 4$
$\underline{-11 \cdot 9 - 44}$
48
$\underline{39}$
9

$\rightarrow x + 4 \overline{)\, 4x^2 + 5x + 4}$ quotient $4x - 11$
$\underline{4x^2 + 16x}$
$-11x + 4$
$\underline{-11x - 44}$
48

(2) $373 = 320 + 53$
$= 8 \times 40 + (6 \cdot 8 + 5)$
$= 8(5 \cdot 8) + 6 \cdot 8 + 5$
$= 5 \cdot 8^2 + 6 \cdot 8 + 5$

$\rightarrow 8 + 5 \overline{)\, 5 \cdot 8^2 + 6 \cdot 8 + 5}$ quotient $5 \cdot 8 - 19 + 7$
$5 \cdot 8^2 + 25 \cdot 8$
$-19 \cdot 8 + 5$
$\underline{-19 \cdot 8 - \boxed{90}}$ 19x5 = 95 not 90.
95
$\underline{91}$
4

$\rightarrow x + 5 \overline{)\, 5x^2 + 6x + 5}$ quotient $5x - 19$
$\underline{5x^2 + 25x}$
$-19x + 5$
$\underline{-19x - 90}$
95

(3) $373 = 350 + 23$
$= 7 \times 50 + (7 \cdot 3 + 2)$
$= 7(7 \cdot 7 + 1) + 7 \cdot 3 + 2$
$= 7 \cdot 7^2 + 1 \cdot 7 + 7 \cdot 3 + 2$
$= 7 \cdot 7^2 + 4 \cdot 7 + 2$

$\rightarrow 7 + 6 \overline{)\, 7 \cdot 7^2 + 4 \cdot 7 + 2}$ quotient $7 \cdot 7 - 38 + 17$
$\underline{7 \cdot 7^2 + 42 \cdot 7}$
$-38 \cdot 7 + 2$
$\underline{-38 \cdot 7 - 228}$
230
$\underline{221}$
9

$\rightarrow x + 6 \overline{)\, 7x^2 + 4x + 2}$ quotient $7x - 38$
$\underline{7x^2 + 42x}$
$-38x + 2$
$\underline{-38x - 228}$
230

(4) $373 = 372 + 1$
$= \underline{12 \times 13} + 1$ $\leftarrow = 156$
$= 12(1 \cdot 12 + 1) + 1$
$= 1 \cdot 12^2 + 1 \cdot 12 + 1$

$\rightarrow 12 + 1 \overline{)\, 1 \cdot 12^2 + 1 \cdot 12 + 1}$ quotient $1 \cdot 12$
$\underline{1 \cdot 12^2 + 1 \cdot 12}$
1
$=$

$\rightarrow x + 1 \overline{)\, 1x^2 + 1x + 1}$ quotient $1 \cdot x$
$\underline{1x^2 + 1x}$
1
$=$

<u>Reflection informing the current study</u>:

This final step in the division ladder was well received by the majority of learners (69%) in the pilot study despite the fact that they admitted to finding the decomposition the most challenging work that they had learnt over the six-day period. I was not convinced that with 31% of the learners not receiving this didactic model favourably, it would be a useful tool in the division ladder. The purpose of including the different base decompositions was to explore how the different polynomial structures evolve naturally from considering different base decompositions for the same number, thus illustrating to the learners that the same value can be reached using different base decompositions. After careful consideration of the necessity of base decomposition other than base ten for the division procedure, it was decided to exclude it from the division ladder for the following reasons:

1. As the main focus of the study is to explore the structure in arithmetic and the possibility of imitating the algebraic structure in polynomial division problems through base decomposition on naturals, it will be sufficient to explore only base ten decomposition.
2. The base ten decomposition is easy for the learners to grasp as it is fundamentally the base of the number system with which they are working and have worked throughout their schooling.
3. Once the base decomposition has evolved into an algebraic expression, the variable can be substituted with any number, without disturbing the balance in the division algorithm.
4. Lifting the cognitive demand past the level that is required only creates conflict that is not necessary in the construction of knowledge concerned with division.
5. Different base deconstructions will be more useful when introducing the concept of a polynomial and the various ways in which naturals in the decomposition process can be representative of different polynomials.

In conclusion:

As division in different base decompositions has never been explored as an algorithm, it was necessary to interrogate each step of the division ladder through three different lenses.

The first lens focused on how the learners received and dealt with the new information in terms of constructing new knowledge that is not in conflict with what they already know from their experience with natural numbers. Such conflict can be used constructively and is necessary when filing new information within the cognitive schema, but could also unnecessarily distract from the intended learning goals and should therefore be clear in its purpose and should be judged in its usefulness in developing the skill of algebraic long division.

The second lens provided a focus for the development and support of new ideas within the didactical model and how these ideas build vertically to enhance conceptual themes within the division procedure which has as its base arithmetic division and its end goal algebraic long division. The necessary structure of the division ladder will be analysed along the cognitive architectural design of each step, to judge the useful learning that can best be described through the learners' reaction to the content and their conception of key issues that are well reported on in the field of division.

The third lens focused on the structure within each step of the division ladder and explored whether the content, included in each step, is sufficient for extracting the key ideas which can be summarized as:

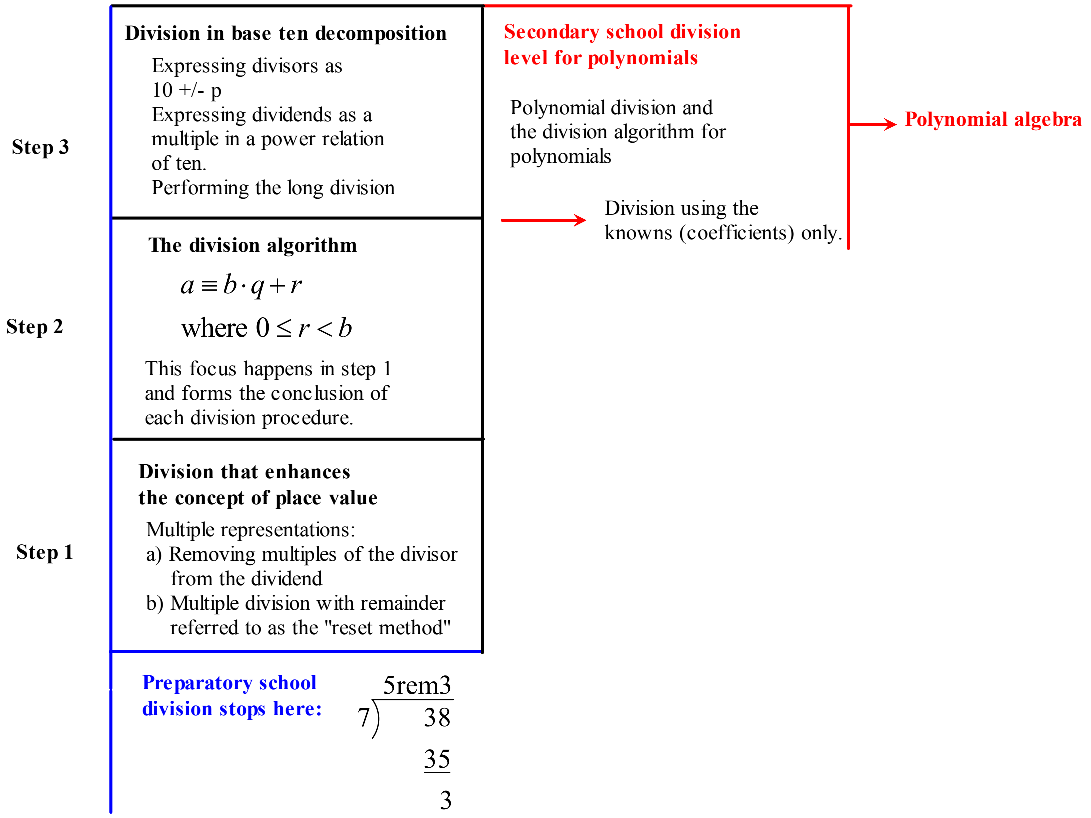
- division procedures and their natural progression for enhancing the *division algorithm* and its conditions on the remainder
- enhancing the structure sense of arithmetic to progress naturally towards abstraction in algebraic division on polynomials
- building sound links between arithmetic division and algebraic division

Based on the data collected during the pilot study, the division ladder for the didactic model in the current research will be discussed and motivated in Chapter 5.

Chapter 5: The didactical model and plan of action

There is a huge gap between where primary school stops in the instruction in division and where secondary school takes off. Finding a cognitive root that builds toward algebraic long division is important to lay the foundation on which exploration and new meaningful learning will be solidly rooted in the procedures and concepts that the learners are familiar with. This is followed by a building in difficulty and cognition from this base line. In refining the didactical model that was suggested in the pilot study, some of the steps had to be changed completely to accommodate the observations that were made during the pilot study.

In the following paragraphs I will explain each missing step of this ladder and motivate the theoretical basis on which I will build cognition. But let me start with a graphical interpretation of what will be done to fill the gap between primary school division and secondary school needs:



The learning programme will start with what the learners have done since primary school, namely ordinary long division on naturals. I will explore specifically whether they have reached procedural fluency in the basic skill of division on naturals and then specifically pose exploratory questions as they guide me through the steps when performing this division. This will establish how much of this process is purely procedural for them and to what extent they understand the concept in terms of its meaning. It is important to start by performing arithmetic division and build from there towards step 3 in the division ladder.

Step 1 – Focusing on multiple division procedures that enhance the concept of place value:

Multiple methods of division will develop more understandable alternative ways of exploring the concept of long division on naturals, based on the learners' division experience in arithmetic long division since primary school. This will offer teachers an alternative to introducing a seemingly difficult area of study by using the principles of metacognition that propose that one must build on existing knowledge in order to acquire new knowledge. Multiple division procedures that enhance place value differently within each method will ensure that the cognitive schema is built on a more solid foundation than just the procedure itself. In the following paragraphs I will discuss the division methods that were chosen, and will motivate the choice in terms of the rationale for building new knowledge.

Considering the division of 256 by 7:

A) By arithmetic long division:

$$\begin{array}{rl}
036 & \\
7\overline{)256} & \quad \therefore 256 = 7\left(36\right)+4 \\
\underline{0} & \\
25 & \\
\underline{21} & \\
46 & \\
\underline{42} & \\
4 &
\end{array}$$

B) By removing multiples of the divisor:

$$\begin{aligned}256 &= 140+70+42+4 && \text{(Breaking up the dividend in multiples of the divisor)}\\ &= 7\cdot 20+7\cdot 10+7\cdot 6+4 && \left(\text{Writing each partial sum as a factor of the divisor}\right)\\ &= 7\left(20+10+6\right)+4 && \left(\text{Removing the common divisor}\right)\\ &= 7\left(36\right)+4 && \left(\text{Division algorithm}\right)\end{aligned}$$

C) Multiple divisions with remainder (the reset method):

This method is based on moving from one digit to the next in the division procedure which means that the digit actually increases as a multiple of ten with each shift. Thus the move forwards from units to tens to hundreds involves an increase in pockets of ten each time. The journey backwards would involve a value that diminishes in pockets of ten to move back to the next column. Thus a remainder of two in the tens column translates to 20 in the units column.

The digit in the hundreds column represents the value 200 in terms of the units column, but 20 in terms of the tens column and can be expressed as ‘twenty tens’ which gives a value of 200.

Operating on the digit in the hundreds column:

$2 = 7\times 0+2$ which means that the 7 did not divide the digit 2, and left a remainder of 2 that can, therefore, be adjusted to form the number 20 in relation to the tens column. There is a 5 in the tens column of the dividend which represents 5 groupings of ten, and this needs to be added to the twenty groupings of ten that were transferred from the hundreds column to the tens column. This will reset the division to consider how many groupings of 7 can be removed from 25 groupings of ten.

This can be streamlined procedurally once the concept of movement between the digits in the dividend has been clearly understood by the learners. It would mean that each remainder gets multiplied by ten and added to the digit in the column to the right of it, to reset the division to the new number obtained in this manner.

Operating in the transferred number 25 in the tens column (known as the partial remainder):

$25 = 7 \times 3 + 4$ which tells us that only three times seven groupings of ten were removed when 25 groupings of ten were divided. This is equivalent to concluding that $250 = 210 + 40$ as the division was taking place in the tens column. In this column the remainder was 4, which translates to become 40 units in the furthest right column. There are already 6 units in this column and, therefore, the partial remainder has to be reset to 46 to include all the units in the column.

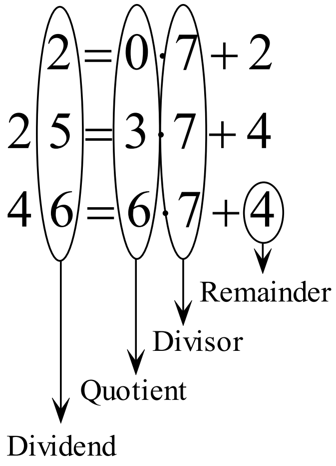
Operating on the transferred number 46 in the units column:

$46 = 7 \cdot 6 + 4$ clearly indicates that only 6 pockets of 7 units each could be formed in the units column, and this left 4 units that could not divide. This will form the final remainder in the division procedure.

Procedurally the reset method could be compacted as follows:

$$\begin{aligned} 2 &= 0 \cdot 7 + 2 \rightarrow \text{remainder } 2 \cdot 10 + 5 = 25 \\ 25 &= 3 \cdot 7 + 4 \rightarrow \text{remainder } 4 \cdot 10 + 6 = 46 \\ 46 &= 6 \cdot 7 + 4 \end{aligned}$$

For the division algorithm:



As Kieran (1990, 1992) and Sfard and Linchevski (1994) warn that learners find procedural interpretations more accessible than an approach which emphasises the structural character of the operation, this more procedural approach to the division will be adopted to enhance learners' cognition of the reset method.

Starting with division on naturals will remind the learners of the procedure in which they have most probably reached fluency at an early age. In fact my assumption was that many learners would most probably have a more compact division procedure that resembles the following:

$$\begin{array}{r} 36r4 \\ 7\big)\overline{\cancel{2}\,{}^{2}\cancel{5}\,{}^{4}6} \end{array}$$

This departure point will establish a sound cognitive root for the division procedure to be explored on the division ladder. The progression towards the removal of the multiples of the divisor from the dividend would thus provide those learners that had problems with this long division procedure with an alternative approach to division. This method might prove problematic to learners who experience difficulties with open-ended tasks which rely on their creative choices in order to progress towards a universally accepted solution. In fact the solutions will all, in the end, be identical in the division algorithm. For this method learners can possibly just remove those multiples of the divisor with which they are comfortable, or they can structure their thinking around multiples that work from hundreds to tens to units. This would suggest a question that asks "*can I remove 100 multiples of the divisor from the number?*" Then division moves to the next column in the place value structure until finally the units are the closest to the actual dividend. Here it is necessary for the teacher to guide the process of removing multiples so that the movement from hundreds to tens to units becomes explicit when considering the options.

The reset method would eventually evolve into a procedure with which the learners should be comfortable, within a very short span of time. The underlying conceptual structure of this method is at first glance very complicated to transfer, but once the sequential thinking has been paralleled with the actual steps that are performed for each column, the learners' understanding of this method will be unproblematic. This method was chosen above other methods as it compresses the multiple divisions with remainder procedures into separate short steps where the quotient, divisor, dividend and the remainder can easily be identified.

Step 2 – The division algorithm:

It is of utmost importance that the *division algorithm* be integrated in the first step of the division ladder when summarising each of the division procedures with which the learners will engage. The *division algorithm* provides the equalising factor to the different division procedures and learners will soon realise that it provides a check point for the correctness of their division procedures. When dividing arithmetically it will only be necessary to compare remainders against one another to determine whether the division procedures have been applied flawlessly.

The *division algorithm* focus of the division ladder is aimed at emphasising and exploring the conditions set on the remainder and the divisor within the algorithm. This will be done in preparation for the base ten decomposition division when the remainders will at times be different to those obtained during arithmetic long division. Learners will then be guided to explore why this is indeed the case, after having discussed the conflict that this different remainder might have created. The issues that will probably emerge will be based on the different value for the remainder as well as the possibility of obtaining a negative remainder.

The classroom discussion should then focus on the continuation of the division procedure based on the non-compliance of the remainder with the conditions set out in the division algorithm, whereby learners should then realise that not all multiples of the divisor have been removed from the dividend. This will then create the opportunity to continue removing multiples of the divisor from the remainder by operating on the *division algorithm* until it balances with the first finding obtained by arithmetic long division.

This will then result in a merger of the two division methods which were dealt with in step 1 and step 3.

Without placing such an emphasis on the *division algorithm* and the conditions on the divisor and the remainder, learners would not experience this application explicitly.

Step 3 – Base ten decomposition of the divisor and the dividend:

The first two steps of the division ladder can be thought of as preparatory in laying the foundation for abstraction. The third step is building the bridge that will cross the cognitive gap between arithmetic long division on naturals and algebraic long division on polynomials. This step of the division ladder will ensure that algebraic long division is not just presented to the learners as a ready-made object where the role of the variable and the steps are fixed within a rigid framework of conventions. Base ten decomposition is a conceptual tool which will evolve to algebraic division as a product of the sequence of division activities, and can be effectively used on two levels. The first level is imitating the algebraic procedure and, therefore, directing learner thinking to build conceptual discourses that will order their thinking away from number, to the structure in the numbers. This structural aspect of arithmetic and its usefulness in the division procedures of arithmetic has not been explored sufficiently in the past.

My expectation, both in the main and in the pilot study, is that long division that uses base ten decomposition will be the most challenging concept for the learners to understand. It is also the introductory step towards abstraction, which later prepares for the introduction of the variable into the division procedure - aiming towards a better conceptualization of what polynomial division is all about. A more fitting description would be to refer to this procedure as an imitation of algebraic long division. Lady (2000) warns that no matter how useful the language of mathematics may be, it does not represent the way in which learners develop understanding of concepts. For this reason, procedures with which learners may be familiar, such as arithmetic long division on naturals, may not be as familiar a process when described in an exact mathematical way, especially when abstraction is involved. Lady states that *"the mathematics of the twentieth century tends to place emphasis on the rules by which entities behave, rather than on the way they were constructed"*(p4). Fujii (2003) maintains that by "*utilizing the potentially algebraic nature of arithmetic, one forms a way of building a stronger bridge between early arithmetical experiences and the concept of a variable"* (p1). By breaking up the numbers (dividends) as well as the divisors into powers of ten, I will attempt to make the seemingly easy procedure of arithmetic long division more complicated for the learners, which will then hopefully transpire into a view that polynomial long division is actually easy. Here my idea is to separate what Skemp (1976) refers to as "instrumental understanding" and "relational understanding". Learners have a sound instrumental understanding about the process of arithmetic long division, but they do not necessarily understand why they are doing the same when performing algebraic long division. The understanding is thus shallow as Fujii (2003) and Wu (2000) argues. Aiming at relational understanding, I will challenge the learners with higher cognitive division exercises through base ten decomposition division, in order to create this smooth transition toward algebraic long division through successful cognition.

This process will be explored in a manner similar to the following example:
Considering the division $138 \div 11$:

$$\begin{array}{r} 12 \\ 11\overline{)138} \\ \underline{11} \\ 28 \\ \underline{22} \\ 6 \end{array}$$

By the *division algorithm* we conclude that $138 \equiv 11 \cdot 12 + 6$.

Decomposing the dividend 138 in base ten: $138 = 100 + 30 + 8 = 1 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0$
and the divisor in terms of 10 will be expressed as $11 = 10 + 1$.

So we can write:

$$\begin{array}{rl} & 1\cdot10+2 \\ \left(1\cdot10+1\right)\big) & \overline{1\cdot10^2+3\cdot10+8} \\ & \underline{1\cdot10^2+1\cdot10} \\ & \qquad\quad 2\cdot10+8 \\ & \qquad\quad \underline{2\cdot10+2} \\ & \qquad\qquad\quad 6 \end{array}$$

By the division algorithm: $138=(1\cdot10+1)(1\cdot10+2)+6=11\cdot12+6$.

In the final step the 10 will be replaced by the variable x and the division will be repeated alongside the division in base ten decomposition to finally reach algebraic long division. So the process will conclude with:

$$138=1\cdot10^2+3\cdot10^1+8\cdot10^0 \;\rightarrow\; x^2+3x+8$$
$$11=10+1 \;\rightarrow\; x+1$$

$$\begin{array}{rll} & \quad\;\; x+2 & \\ x+1\big) & \overline{x^2+3x+8} & \qquad \therefore x^2+3x+8\equiv\left(x+1\right)\left(x+2\right)+6 \\ & \underline{x^2+x} & \\ & \qquad 2x+8 & \\ & \qquad \underline{2x+2} & \\ & \qquad\quad\;\; 6 & \end{array}$$

These steps that are suggested in the division ladder are by no means claiming to be the solution to the division problems that learners experience when they engage with long division on polynomials, but it does provide a sound didactical model which could be followed to bridge the cognitive gap that exists between division on naturals and polynomial division. This gap could well create the cognitive chaos that some learners experience when attempting to unravel the conceptual building blocks of algebraic division. For these learners algebraic division is a replication of what the teacher said and did on the board, and often they cannot apply this highly procedural approach to division to more complicated problems.

Once the algebraic long division on naturals has been explored by the learners, the cognitive shift will be made horizontally to introduce the division by detached coefficients. Here the learners' attention will be drawn back to the base ten decomposition division procedure as well as the algebraic division procedure which was performed side by side by the learners. The instruction will then take the form of simply erasing both the tens and the *x* from each procedure, to look as follows:

$$329 = 3\cdot 10^2 + 2\cdot 10 + 9 \qquad\qquad 329 \rightarrow 3x^2 + 2x + 9$$

$$5 = 1\cdot 10 - 5 \qquad\qquad 5 \rightarrow x - 5$$

$$\begin{array}{rl}
 & 3 \;\; +17 \\
(1 \;\; -5) & \overline{\big)\,3 \;\; +2 \;\; +9} \\
 & \underline{3 \;\; -15} \\
 & 17 \;\; +9 \\
 & \underline{17 \;\; -85} \\
 & 94
\end{array}
\qquad\qquad
\begin{array}{rl}
 & 3 +17 \\
(1 -5) & \overline{\big)\,3 \; +2 \; +9} \\
 & \underline{3-15} \\
 & 17 +9 \\
 & \underline{17-85} \\
 & 94
\end{array}$$

This will clearly point to the learners that the ten and the *x* take on an irrelevant role during the long division procedure, and that the roles of these two place holders only become relevant when formulating the division algorithm. This method will enhance the conditions embodied in the algebraic *division algorithm* for polynomial division where the emphasis shifts to the degree of the remainder and the divisor.

Since the conditions stipulated in the algebraic *division algorithm* do not form part of the focus of this study, it is only mentioned here to complete the cycle of the division debate:

The *division algorithm* for naturals:

> If a and b are both integers and $a,b \neq 0$, then there exists unique integers q and r such that: $a \equiv b \cdot q + r \quad \textit{where} \quad 0 \leq r < b$.

The *division algorithm* for polynomials:

> For any two polynomials $f(x)$ and $g(x)$ where $g(x) \not\equiv 0$, there exists unique polynomials $q(x)$ and $r(x)$ such that $f(x) \equiv g(x) \cdot q(x) + r(x)$ where $\deg(r(x)) < \deg(g(x))$ or $r(x) = 0$.

The completion of the division cycle and the algebraic *division algorithm* will be discussed in Chapter 7 where a further model will be proposed for further research.

To conclude this chapter, the full division procedure will be given from step one to step three using the problem $329 \div 5$

Step 1 and 2:

Arithmetic division:

$$\begin{array}{r|lll} & 0 & 6 & 5 \\ \hline 5 & 3 & 2 & 9 \\ & \underline{0} & & \\ & 3 & 2 & \\ & \underline{3} & \underline{0} & \\ & & 2 & 9 \\ & & \underline{2} & \underline{5} \\ & & & 4 \end{array}$$

$$\therefore 329 = 5(65) + 4$$

Removing multiples of the divisor:

$$\begin{aligned} 329 = 300 + 20 + 9 &= 60 \cdot 5 + 4 \cdot 5 + 5 + 4 \\ &= 5(60 + 4 + 1) + 4 \\ &= 5(65) + 4 \end{aligned}$$

Reset Method:

$$3 = 0 \cdot 5 + 3 \ \rightarrow rem = 30 + 2 = 32$$

$$32 = 6 \cdot 5 + 2 \rightarrow rem = 20 + 9 = 29$$

$$29 = 5 \cdot 5 + 4$$

$$\therefore 329 = 5(65) + 4$$

Step 3:

Base ten decomposition:

$$329 = 3\cdot 10^2 + 2\cdot 10 + 9$$
$$5 = 1\cdot 10 - 5$$

$$\begin{array}{r@{}l}
 & \quad\quad\quad\quad\quad 3\cdot 10 + 17 \\ \cline{2-2}
(1\cdot 10 - 5)\big) & 3\cdot 10^2 + 2\cdot 10 + 9 \\
 & \underline{3\cdot 10^2 - 15\cdot 10} \\
 & \quad\quad\quad 17\cdot 10 + 9 \\
 & \quad\quad\quad \underline{17\cdot 10 - 85} \\
 & \quad\quad\quad\quad\quad\quad 94
\end{array}$$

$$\begin{aligned}
\therefore 3\cdot 10^2 + 2\cdot 10 + 9 &= (1\cdot 10 - 5)(3\cdot 10 + 17) + 94 \quad \rightarrow \text{ remainder differs} \\
&= (5)(47) + 50 + 40 + 4 \quad \rightarrow \text{ division continues} \\
&= (5)(47) + 10\cdot 5 + 8\cdot 5 + 4 \\
&= (5)(47) + 5(10 + 8) + 4 \\
&= (5)(47 + 18) + 4 \\
&= (5)(65) + 4
\end{aligned}$$

Algebraic long division:

$$329 \rightarrow 3x^2 + 2x + 9$$
$$5 \rightarrow x - 5$$

$$\begin{array}{r@{}l}
 & \quad\quad\quad 3x + 17 \\ \cline{2-2}
(x - 5)\big) & 3x^2 + 2x + 9 \\
 & \underline{3x^2 - 15x} \\
 & \quad\quad 17x + 9 \\
 & \quad\quad \underline{17x - 85} \\
 & \quad\quad\quad\quad\quad 94
\end{array}$$

$$\therefore 3x^2 + 2x + 9 = (x - 5)(3x + 17) + 94$$

Division by detached coefficients:

$$329 \rightarrow 3x^2 + 2x + 9$$
$$5 \rightarrow x - 5$$

$$\begin{array}{r@{}l}
 & \quad\quad 3 + 17 \\ \cline{2-2}
(1 - 5)\big) & 3 + 2 + 9 \\
 & \underline{3 - 15} \\
 & \quad\quad 17 + 9 \\
 & \quad\quad \underline{17 - 85} \\
 & \quad\quad\quad\quad 94
\end{array}$$

$$\therefore 3x^2 + 2x + 9 = (x - 5)(3x + 17) + 94$$

Chapter 6: Implementing the didactical model

Introduction

In this chapter I will give a detailed account of every lesson that was presented to the learners. Each lesson will be transcribed and, where necessary, screen shots of the work that was done on the overhead projector will also be included and transcribed. This is to ensure that every moment of this action research project is recorded in this chapter.

There were two adults – the researcher and the learners' mathematics teacher – in the class at all times. Collecting this data was a collaborative effort between the mathematics teacher (who helped film the lessons) and the researcher who prepared the materials and taught the lessons. The mathematics teacher taught one lesson which was filmed by the researcher.

Classroom discussions will be transcribed in detail to inform the reader of every debate that took place in the class.

At the end of each lesson a comprehensive reflection of the lesson will be included. This reflection is based on the class discussions and the work that the learners produced during the lesson, as well as the tasks that were given for homework. The learners' work will be reflected on and included in each lesson reflection to show the progress of the model that was implemented and the learners' cognition.

Learners' mistakes will be discussed and shown as evidence. The purpose is not to pathologise their attempts, but to show the common mistakes made by them. In most cases these were addition and multiplication errors which could have been avoided by the learners. When these mistakes occurred, they were included in the data as an incorrect attempt.

Lesson 1: Multiple representations

In order to establish a cognitive root (Tall, 1994) for the learning of algebraic long division, learners were required to divide a three digit number by a single digit number. The decision to start here was to establish whether learners can actually perform the procedure of arithmetic long division, and whether they are familiar with the various aspects as encapsulated in the formal division algorithm (Bates and Rousseau, 1989). What emerged from this classroom discussion provided the researcher with insights into the shared understanding in the group regarding arithmetic long division. The first lesson will be discussed in detail on the basis of the following issues of interest:

- The division algorithm
- Issues that emerged concerning the remainder
- Introducing the necessary vocabulary
- Multiple representation of the division procedure

Arithmetic long division:

The problem given to learners was to divide 437 by 3. It was interesting to notice that learners could immediately start this process by concisely dividing these numbers without expanding to the intermediate answers within this method (see Figure 1 below). This was not anticipated but it provided the researcher with fertile ground to discuss issues that were essential in preparing learners for algebraic long division. I will refer in this document to the short method as *concise division* and to the conventional method as *long division*.

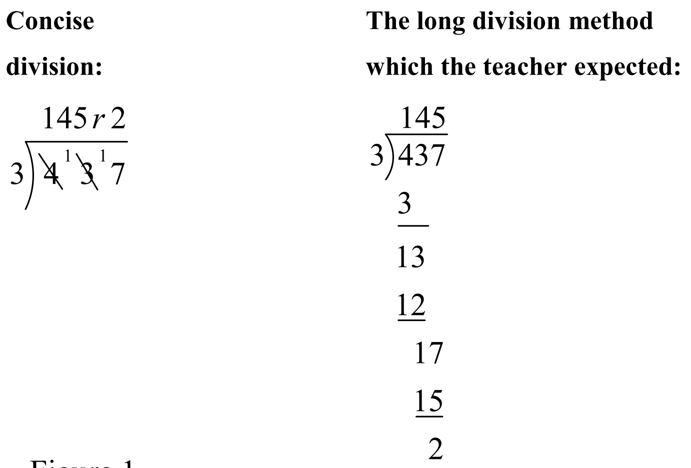
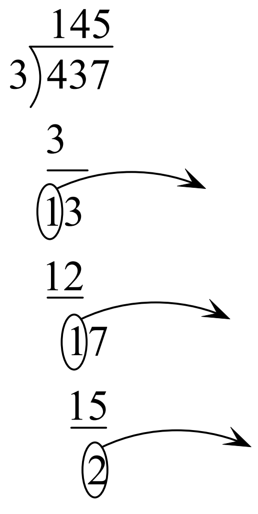


Figure 1

In the pilot study, conducted with Grade 9 and Grade 11 learners, various degrees of difficulty were incorporated in the numbers that were chosen for division on naturals during the first lesson. As mentioned earlier, the rationale behind this choice was to establish beyond any reasonable doubt, the level of fluency in the division procedure using arithmetic long division. As far as the Grade 11 students in the pilot study were concerned, algebraic long division on polynomials was also included as an introduction to establish their competence at long division on polynomials. The current study is focusing on the Grade 8 level of mathematics. Since these learners have had limited exposure to algebraic concepts, it was decided to adopt the same strategy as was used with the pilot group of Grade 9 learners.

It immediately became evident in the classroom debate that these learners experienced the very same difficulties regarding the formation and properties of the division algorithm and the role that the remainder plays within the algorithm as well as within the process of division. Since the learners suggested that the division process be performed in the concise manner (see Figure 1), this provided the researcher with a clear indication that they had reached a high level of fluency in the concise method. However, they were unable to summarise their activity by formulating this relationship in the division algorithm. After completing the discussion of the long division method as indicated in Figure 1, learners' attention was drawn to the partial remainders obtained after the division was performed on each digit. Without telling the learners what it was that they were supposed to see, the teacher requested information from them regarding the numbers that were obtained after subtraction (Figure 2).



Learners' attention were drawn to the relationship between these partial remainders and the divisor.

Figure 2

What follows is a short transcript of this classroom discussion:

Teacher: What is the relationship between the three, one, one and the two (*pointing at the divisor, and each remainder of the partial division problems*)

L1: They are all odd numbers.

Teacher: Is two odd? Look again … (*Continued to introduce the vocabulary needed to talk about the different parts of the division problem. This was necessary as the learners did not possess the necessary words to refer to these parts. These included remainder, quotient, divisor and dividend.*)

Teacher: Now try again: What can you tell me about the divisor (*Pointing at the three)* and these numbers over here (*pointing to the 1, 1 and 2*)

Teacher: Why am I so interested in those three numbers?

L2: (…a*fter a long silence…*) When you divide them by three you will always get a remainder.

Teacher: Did everybody hear that …. I'm going to reword that…now let me ask you: Can those numbers be divided by three?

Class: No.

Teacher: So what can you tell me?

L3: All those numbers are smaller than three!

Teacher: So …. What is so important about their being smaller than three?

L2: They are remainders … I think… Because they divided into…

Teacher: Good, stop there for a moment! For those of you that did not understand this, let's talk about it.

At this point in the classroom discussion it became necessary to describe what the meaning is of the term "*repeated division with remainder*". This was necessary to draw the attention to each remainder that emerged in the problem after division was performed on each digit in the number 437. By doing this the learners' attention would next be drawn to the concept of place value and to unpack the role (value) of the remainder when the next digits are carried down (Figure 3). Learners' attention was drawn to the fact that the remainder "resets" the number to a 13 and a 17 in this case.

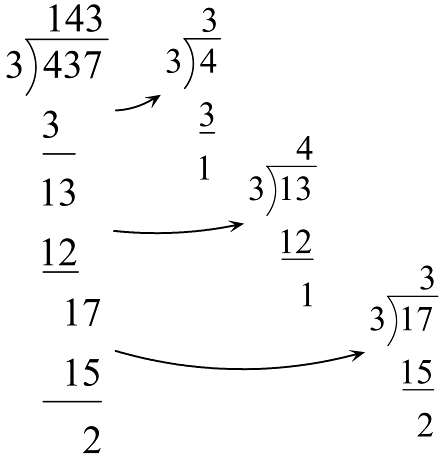


Figure 3

By making reference to the very first method (*concise division*) that the learners employed to do division, it was made very clear that the numbers that were carried over were indeed tens and not units as suggested by the value of the remainder. Emphasizing this was necessary to prepare the learners for the next two division methods that followed later in this lesson.

The next step in this lesson was to assess how familiar learners were with the division algorithm. By this I mean the "*real division algorithm*" as described by Bates and Rousseau (Bates and Rousseau, 1989). The reason for emphasizing the division algorithm was that the next two methods that would be discussed in the lesson would follow the structure as set out in the algorithm.

Teacher: Now when you completed a problem using this method of division in primary school, you wrote a final sentence… (*writing on the board* 437 = ….). Who wants to complete that sentence?

Teacher: I have gone through all this effort of dividing the numbers, what can I write about this number 437 in terms of the quotient, the remainder, the divisor and a dividend? How can I summarise what I have done? Anybody?

L1: $145 r 2 \times 3$ (*read: one hundred and forty five remainder 2 times 3*)

Teacher: (*after writing* $437 = 145 r 2 \times 3$ *on the board ...*)
Is this right? Look at it. Does anybody else have another idea?

Teacher: This is what L1 suggested: (*Putting brackets around 145 r 2)* : $437=(145r2)\times 3$ … Does anybody want to challenge this?

L2: Is it not $145\times 3+2$?

Teacher: Why? What is different about your suggestion? Is it not saying the same thing?

L2: Because it is a remainder … the 2.

L3: Because you can't multiply the remainder. It should be $437=(145\times 3)+2$.

Teacher: Why can't you multiply the remainder? What's wrong with doing that?

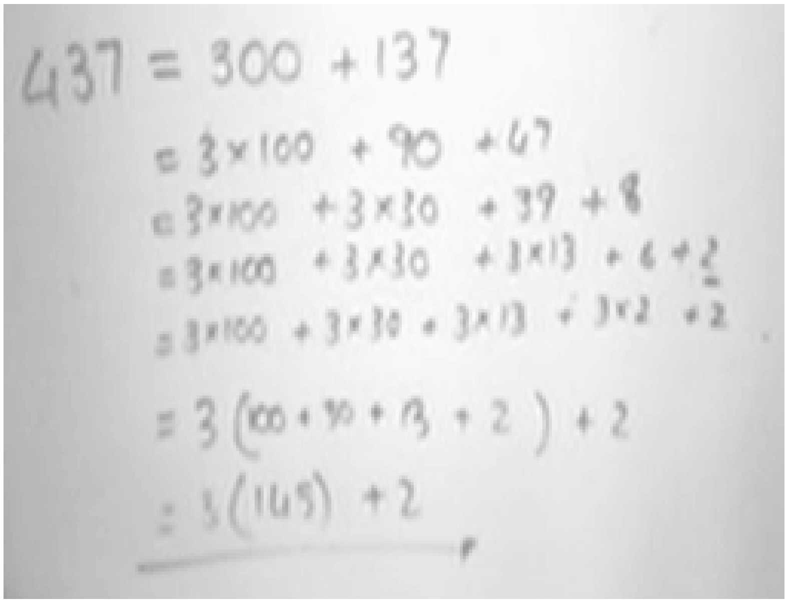
Teacher: (*after a silence)* It could not divide by the 3, it is what's left over after division was completed. Is that clear to everybody? L2?

L2: Yes I see now that you cannot remove the 3 from the 2, so it did not divide into the 2, and if we multiply the three by the two, we are inversing…reversing the division.

Teacher: Exactly! Does everybody understand that? I'll say it again. If we multiply the 2 with the 3 we are saying that the 3 divided into the 2 in the first place and it did not because the 2 is the remainder after the last multiple of 3 was removed from 437.

It was quite clear that these learners who were in Grade 8, and had dealt with arithmetic long division at primary school level, were not aware or simply forgot about the subtleties that surround the remainder. To make the learners more aware of the important issues with which they should be familiar, the teacher decided to follow up with multiple representations of division where the division algorithm and the structure in it stays preserved throughout each procedure. By doing this the learners' attention was immediately focused on place value and factoring of numbers.

<u>Removing multiples of the divisor from the dividend</u>:



$$437 = 300 + 137$$
$$= 3 \times 100 + 90 + 47$$
$$= 3 \times 100 + 3 \times 30 + 39 + 8$$
$$= 3 \times 100 + 3 \times 30 + 3 \times 13 + 6 + 2$$
$$= 3 \times 100 + 3 \times 30 + 3 \times 13 + 3 \times 2 + 2$$
$$= 3(100 + 30 + 13 + 2) + 2$$
$$= 3(145) + 2$$

The following transcript comes directly from the above activity:

Teacher: If we look at the number 437, what is the largest multiple of 3 that we can remove from this number? Now please do not use a calculator here, trust your intuition… Come guys try… (*after a while*) Is it 3, 30 or 300?

Class: 300

Teacher: Now what was that a multiple of?

L1: 100

Teacher: Does everybody understand what we are doing? Could I have used a multiple of ten and three?

L2: No because that gives you 30.

Teacher: Well is there something wrong with removing 30? It is after all smaller than 437.

L2: I suppose not.

L3: But it's too small. Then you might as well remove 90.

Teacher: OK so we return to my original question. What is the largest multiple of 3 that we can remove from this number?

Class: 300

Teacher: Good! Is there anybody that does not get that? Okay, let's move on (*writing on the board the equation* : $437 = 300 + 137$). Do we all agree with it?

L4: Where does the 137 come from now?

L2: It is what is left after you have taken out three hundred!

Teacher: Okay let's pick any multiple of three, say 90 … There might be bigger ones that will go into 137, but I just chose the first one that is smaller than 137 that came to mind. Everybody happy if I use 90? So let's write it as follows: $437 = 3\times100 + 90 + 47$.

L5: Ah … You can now write the 90 as 3 times 30.

Teacher: Let's do that: $437 = 3\times100 + 3\times30 + 39 + 8$. Could I have used 9 times 10 here guys? (*after no response*) What am I dividing by? (*pointing to the original question*)

Class: 3.

L5: No you can't, because you are not dividing by 9… and 9 times 10 is a multiple of nine.

Teacher: Okay let's zoom in on the 39 and the 8. What is now really the largest multiple of three that we can remove from 39 and from 8, for that matter?

L5: 3 times 12… that is 36…

Class: 3 times 13… It is exactly 39. (s*houting out*)

Teacher: And that 8?

L5: 6 and 2 …6 plus 2 gives 8…so you can write 6 as 2 times 3 then the plus 2.

Teacher: Okay let's write that down: $437 = 3\times100 + 3\times30 + 3\times13 + 6 + 2$. Can you see your remainder 2 pops up again? (*Circling the 2 on the board).* So, finally, we can say: $437 = 3\times100 + 3\times30 + 3\times13 + 3\times2 + 2$.

At this point the method becomes easier and more familiar for learners as they have already learned the skill of factorization and removing common factors on an algebraic level. As the method that the teacher above followed was introducing a new approach to division for the learners, quite a lot of scaffolding had to be done to introduce learners to this procedure. The learners were now required to find the quotient, as they could easily identify the divisor of 3, the dividend 437 as well as the remainder 2.

Teacher: Look at a final statement on the board: $437 = 3\times100+3\times30+3\times13+3\times2+2$. Where is my quotient? I know the quotient is 145, but where is it in all this mess? How can I write this line differently or what must I do to simplify it?

L1: You just calculate 3 times 100, 3 times 30, 3 times 13… and add the lot.

Teacher: What you are suggesting is that we multiply all these 3's out again … and that would mean that we get rid of ... or that we reverse this whole process. Then why did we take them out in the first place? Multiplying out will be taking the 3's away, and we don't want to do that because we are dividing by 3 and it makes our life easier to keep the 3's.

L1: Yes, but if we …(*thinking*)

L2: Don't you just like add up all the numbers that are *being* multiplied?

Teacher: Listen to what L2 is suggesting. He is saying that we just have to add 100, 30, 13 and 2. What about the 3 that all these numbers are multiplied with?

L2: You can ignore them.

Teacher: What do you mean we can ignore them?

L2: They are in each number, so you can ignore…remove them.

Teacher: Okay, I can see a few of you are not with us.
(*Writing on board*) : $437 = 3(......................)+2$.

L3: So you add up the 100, 30, 13 and the 2.

Teacher: Good! (*Writing on the board*): $437 = 3(100+30+13+2)+2 = 3(145)+2$.
And that gives us the required 145 which was the quotient if you look at the division algorithm that was used earlier on.

For the last method that was introduced to the learners, it was necessary for them to remind themselves of the partial division with remainder sums that they did when they performed long division at the beginning of the lesson. As this method draws heavily on the two methods that they started with, as well as the division algorithm both in structure and in method, it was necessary to just recap everything that had been done thus far.

<u>The ‘reset’ method</u>:

$$4 = 3\times 1+1 \quad \overrightarrow{remainder}\ 10+3=13$$

$$\therefore 13 = 3\times 4+1 \quad \overrightarrow{remainder}\ 10+7=17$$

$$\therefore 17 = 3\times 5+2$$

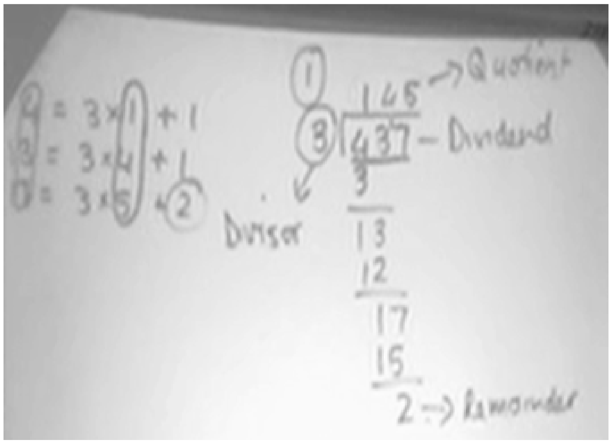


Figure 4

The introduction of the last method as well as the classroom discussion is indicated in the following transcript:

Teacher: If we want to write 4 in terms of 3, according to the division algorithm we can write it as $4 = 3\times 1+1$. Does everybody understand that? (*group confirms*)

L1: Why do you have to multiply by 1? Can’t you …? (*unclear audio*)

Teacher: I am cloning what the division algorithm suggests. We are not decomposing the number like we just did, nor are we doing long division on the number. We are using the division algorithm to look at the numbers as separate numbers and perform small divisions on each number. Is that clear to everybody? We are looking at each number and how we can decompose them using the divisor and the division algorithm only.

At this point in the process it was not clear to the learners where this new method was going, or how it was different from what we have done thus far. So the researcher decided to continue explaining this new approach, and only to take questions as the approach became clearer to the learners.

Teacher: Ok, let’s focus on the remainder of 1. If you cast your mind back to the very short method that we looked at right at the beginning (*referring to concise division*), you will remember that L1 at the time suggested that we carry ones to the next numbers. The one that was carried was indeed not 1 but …? Anybody?

L2: It was ten … Because the next division was done with number thirteen and not three.

Teacher: OK, but what if I tell you that it’s not 10 but actually 100 in the context of this problem…*writing* $400 = 3\times 100+100$, but we are moving one column at that time. So in that case the value of that 1 is not 100 but 10. That is what we did in the beginning when we transferred the 1 to the 3 and 3 became a 13… (*Pointing to the* 13 *in the long division problem and the* 13 *in the concise division problem*).
Right, so a remainder is one, and we change it to ten because we are moving it to the next column. So now we have to use 10 + 3, which is 13, for our next division.

After completing the division procedure with the class step by step, constantly drawing parallels with the methods with which they were familiar, learners were asked to identify where the quotient, remainder, divisor and the dividend lay in this calculation. At this point the layout on the board was as follows:

$$4 = 3\times 1 + 1$$
$$\therefore 13 = 3\times 4 + 1$$
$$\therefore 17 = 3\times 5 + 2$$

Teacher: Can anybody tell me where the quotient lies?

L3: It's your multiples with the three.

Teacher: (C*ircling the column which contains 145*) Where's the original number?

L4: It's next to the equal sign.

Teacher: (*Circling the column which contains 437*) It's the digits directly alongside the equal signs. Notice that it is ignoring the ones! Why are we ignoring the one? Somebody said it earlier…

L5: The ones were not part of the original sum – they're remainders.

It is important to note that at this point learners were encouraged to ask questions. One of the important questions that came up drew attention to the difficulty for learners to understand that the remainders change the value of the next number as the original digit is transferred next to the remainder. This was explained in detail using the long division, concise division and the most recent method. It was decided that this approach would be referred to as the *"reset method"* as the remainders reset the value of the digit that was transferred down. During this classroom discussion, it became evident that this was the first time that the learners had engaged with the division algorithm. They therefore had no knowledge of the conditions on the remainder in terms of the divisor within this algorithm. The teacher continued by giving the learners a new division problem, and requiring them to perform this division using all four methods to which they had been introduced.

The format of this activity was still a classroom based discussion, with the teacher transcribing on the board what the learners were suggesting. To highlight the similarities between the different methods, the *long division* was performed as multiple divisions with remainder problems, while the *reset method* was done alongside the *long division* to point out the similarities between the two methods. This seemed to provide more structure in the division process to the one learner who still had problems with arithmetic long division.

The problem that was discussed was 913 divided by 7. These numbers provided for some very interesting classroom discussion, as the remainder at one stage becomes zero, and this led to transfer issues in the problem layout and interpretation problems when the division algorithm was formulated from both the *reset method* and the method of *long division*. The learners correctly identified that a zero remainder implies that in the division algorithm a zero is added to the product of the quotient and the divisor, and for many of them the division process at this point was finalized. The reason for this was the interpretation of the fact that seven cannot divide into three. They did not immediately realise the effect that this would have on the quotient. The completed transcription of the problem, without the final division algorithm, appears as follows:

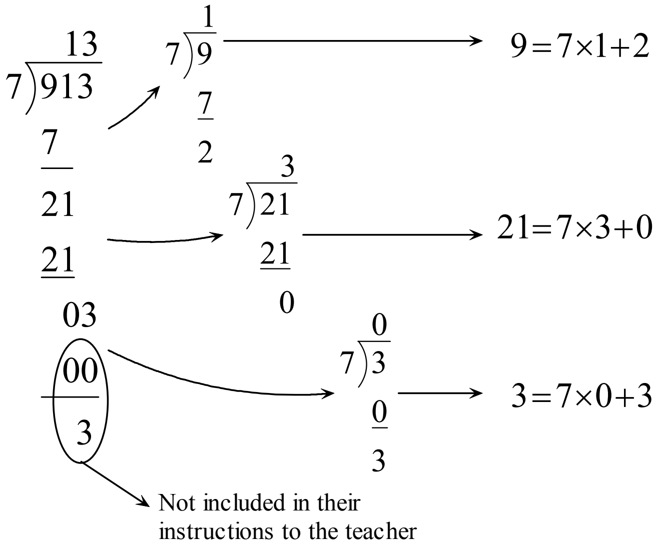
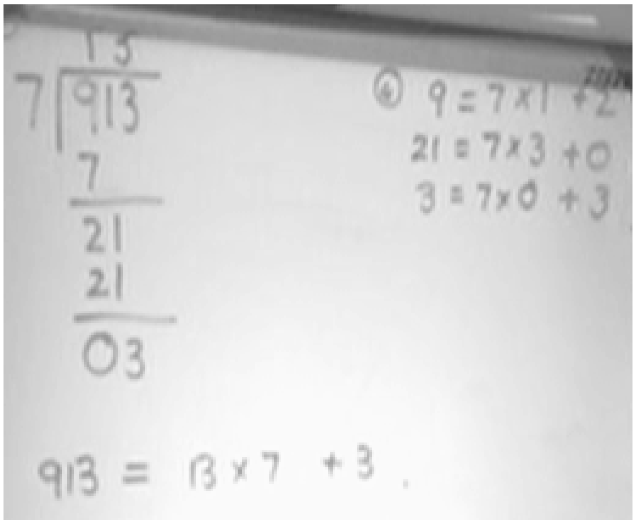


Figure 5

When asked to summarise the procedures of the division algorithm, one of the learners offered $931 = 7 \times 13 + 3$ which was based on their completion of the *long division* procedure, where they did not suggest that we record the fact that 7 does not divide into 3 with a zero in the quotient. This was immediately challenged by another learner who indicated that the 13 should be 130, validating her argument by referring to the *reset method* where the zero had been recorded.

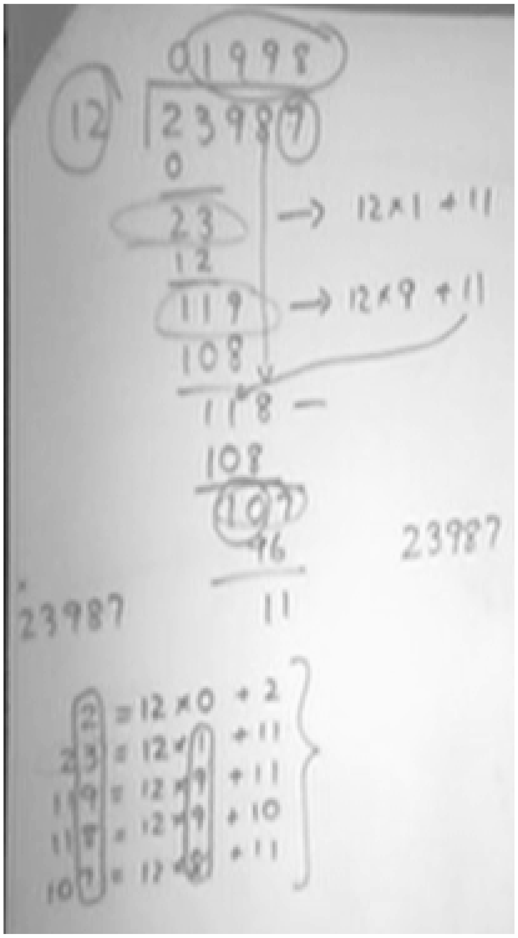


Making use of decomposing the number into multiples of seven, the learners suggested the following:

$$\begin{aligned} 913 &= 700 + 213 \\ &= 7 \times 100 + 210 + 3 \\ &= 7 \times 100 + 7 \times 30 + 3 \\ &= 7(100 + 30) + 3 \\ &= 7(130) + 3 \end{aligned}$$

One of the learners questioned why we were removing multiples of seven. The researcher explained that the purpose of the division was to remove as many sevens as possible from the number. However, as the learners continued with a problem in class, the same learner approached the teacher and posed the question again. Her conflict was based on the fact that on the board the division algorithm was recorded as $913 = 130 \times 7 + 3$ and this meant to her that multiples of 130 were removed from 913, as this was the order in which it had been written, and to her understanding the divisor should be written first. The class thus decided to adopt the convention of the divisor being written first and the quotient second.

As a final classroom activity, learners were required to divide 23987 by 12. As these learners are currently in Grade 8, the jump to ten thousands was chosen deliberately. Performing long division on this number came naturally to them. However, when asked to perform the *reset method*, the procedure became rather difficult for them. Conflict was again generated around the issues with the remainder. In previous problems, the remainder was a single digit number, and now they had to contend with two digit remainders. In practice this meant that the remainder of one translated into a ten when transferred to the next step in our division procedure. Having now obtained a remainder of 11, meant that this eleven changes to 110 plus the next digit for the necessary resetting in the division process.



Lesson 1 reflection:

Most of the work the learners produced was done entirely according to the instructions given to them by the teacher. The homework questions required learners to perform the following divisions:

1. $7\overline{)523}$
2. $2\overline{)1391}$
3. $13\overline{)2473}$
4. $29\overline{)32851}$ **(optional)**

Learners were given the following instructions:

a) Do the division using arithmetic *long division*.
b) Do the *reset method* alongside the *long division*.
c) Summarize the procedure in the *division algorithm*.
d) Do the same division by removing multiples of the divisor.

Keeping strictly to the sequence of the instructions, learners should be able to complete the work and have a clear understanding of what they are doing, since the arithmetic long division with which they are familiar will allow them to see whether they are correct or not. A second way to check their work is by comparing each answer with the division algorithm. Question 4 was included for the students as a challenge, and was optional.

Reasons for the choice of the four question types were twofold: to determine whether the learners grasped the concepts with which they dealt during the classroom period, and to test whether learners could deal with the different divisors (from single digits to two digit numbers, not larger than 19) as well as increased dividends. In the questions that were given to the learners the divisor was chosen not to be a factor of the first number in the dividend. As learners coped well with this in class, it was decided to keep the divisors large enough so that the division only started at the combined first two digits in the dividend.

Question 1: $7\overline{)523}$

$$a)\ \begin{array}{r} \underline{074} \\ 7\overline{)523} \\ \underline{0} \\ 52 \\ \underline{49} \\ 33 \\ \underline{28} \\ 5 \end{array}$$

$$\begin{aligned} b)\ 5 &= 7\cdot 0+5 \ \rightarrow 5\cdot 10+2 \\ 52 &= 7\cdot 7+3 \ \rightarrow 3\cdot 10+3 \\ 33 &= 7\cdot 4+5 \end{aligned}$$

$$c)\ 523 = 7\left(74\right)+5$$

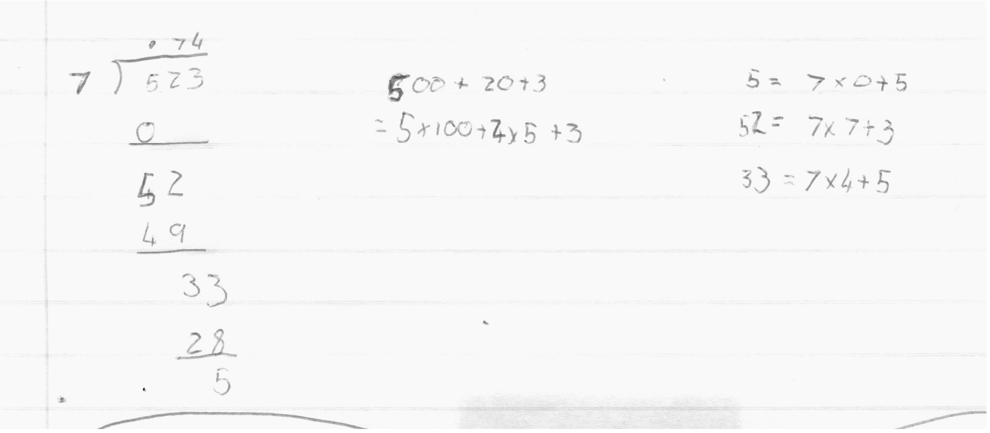
$$\begin{aligned} d)\ 523 &= 280+140+70+33 \\ &= 7\cdot 40+7\cdot 20+7\cdot 10+7\cdot 4+5 \\ &= 7\left(40+20+10+4\right)+5 \\ &= 7\left(74\right)+5 \end{aligned}$$

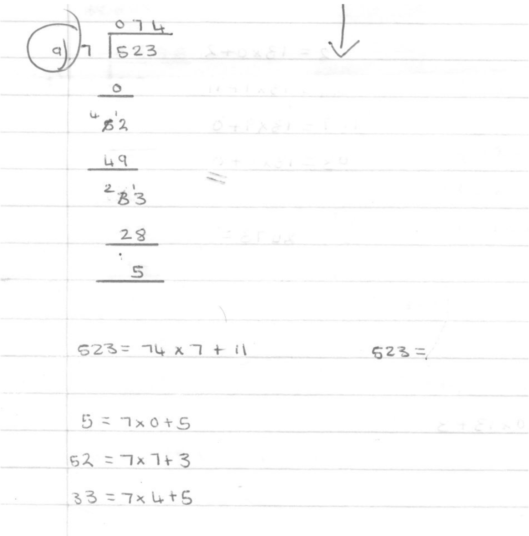
All the learners could answer this question correctly when they used arithmetic long division and the reset method. They could successfully write down their conclusions using the division algorithm. Problems started surfacing when learners were required to remove multiples of the divisor from the dividend and express the dividend as a sum of these multiples. When asked about this in class, there seemed to be an unfamiliarity with and discomfort among learners regarding questions that had an open approach with no, or at least a minimal, prescriptive procedure which is necessary to answer the question. When learners received instruction on these methods, it was made explicit that they could start wherever they wanted to when removing the multiples of the divisor, but the advice was given to remove the largest and easiest multiple of the divisor from the dividend first. This meant that one hundred – or ten multiples of the divisor should be the first choice. Having to make this choice themselves only found favour with sixteen learners. Seven learners solved the problem successfully while the remaining nine attempted this procedure without success. Six of the learners did not attempt this part of the question at all, and when asked what it was that made them hesitate, they replied that they did not know where to start. This was despite the teacher having indicated that 'ten' multiples of the divisor is usually a good starting point, provided that this is less than the value of the dividend. Learners clearly lacked confidence when it came to performing this procedure by themselves.

The results for this question were as follows:

	Correct	Partially Correct	Incorrect	No attempt
a	22	-	-	-
b	22			
c	21	1	-	-
d	7	-	9	6

Of the nine learners that did this completely wrong, eight focused on the place value of the digits in the number, in some sort of way, completely ignoring the dividend in the procedure. Below is an example of one such attempt where the learner used multiples of five with which to recompose the number.





Most of the learners who did not attempt the last method of division produced work that resembles the example alongside. What is interesting to notice about both these examples is that the learners could deal with the more cognitively challenging *reset method* which is filled with more sub- and super-procedures than merely removing multiples of the divisor. The learner alongside also made a minor error in transferring the information gathered by *long division* and by the *reset method*, to the *division algorithm* where she indicated the remainder as eleven and not five. When asked to write down the *division algorithm* again, she did so correctly.

Question 2: $2\overline{)1391}$

$$a)\ \begin{array}{r} 0695 \\ 2\overline{)1391} \\ \underline{0} \\ 13 \\ \underline{12} \\ 19 \\ \underline{18} \\ 11 \\ \underline{10} \\ 1 \end{array}$$

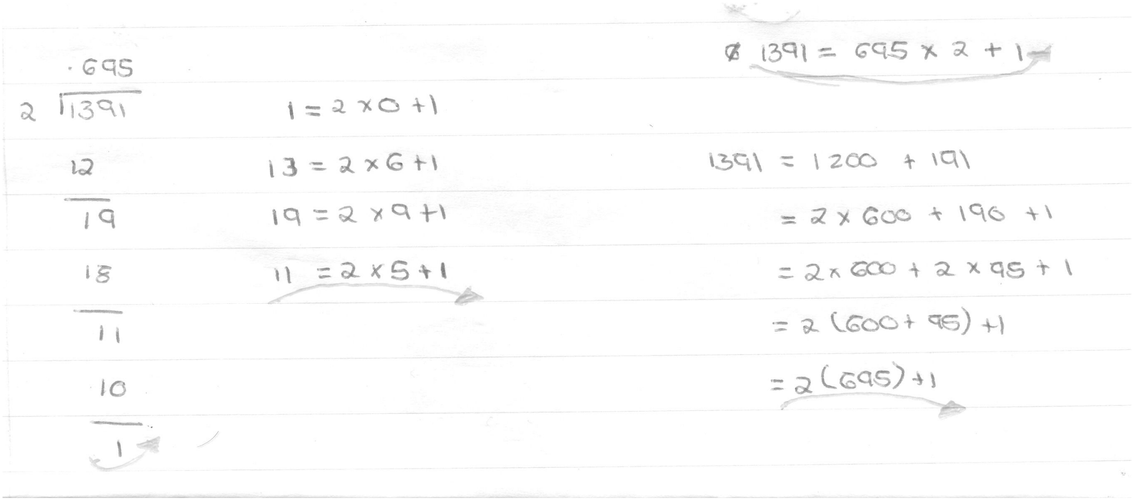
$$\begin{aligned} b)\ 1 &= 2\cdot 0+1 \rightarrow 1\cdot 10+3 \\ 13 &= 2\cdot 6+1 \rightarrow 1\cdot 10+9 \\ 19 &= 2\cdot 9+1 \rightarrow 1\cdot 0+1 \\ 11 &= 2\cdot 5+1 \end{aligned}$$

$$c)\ 1391 = 2\left(695\right)+1$$

$$\begin{aligned} d)\ 1391 &= 1000+300+90+1 \\ &= 2\cdot 500+2\cdot 150+2\cdot 45+1 \\ &= 2\left(500+150+45\right)+1 \\ &= 2\left(695\right)+1 \end{aligned}$$

Regarding this question, the researcher's expectation was that learners would perform better as the divisor (2) is a number that is small and even and learners had extensive experience in halving. The same issues emerged as those that arose in the question requiring the removal of multiples of the divisor. Two samples of work of learners who did everything correctly are shown below:

<u>Learner 1</u>:



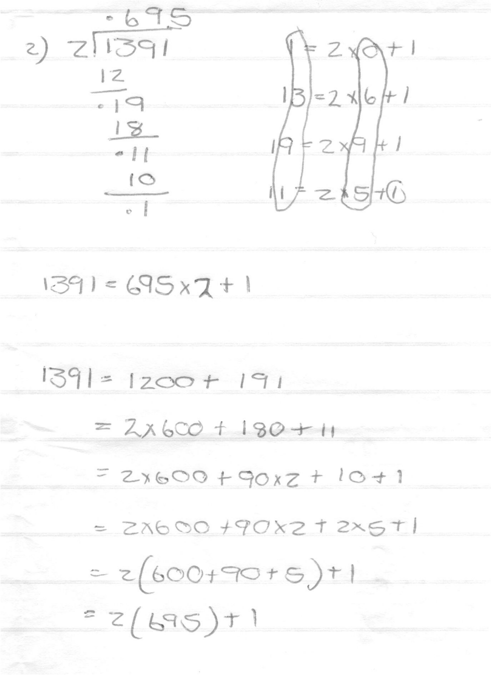
$$\begin{array}{r} \cdot 695 \\ 2\overline{)1391} \\ 12 \\ \overline{19} \\ 18 \\ \overline{11} \\ 10 \\ \overline{1} \end{array}$$

1 = 2 × 0 + 1
13 = 2 × 6 + 1
19 = 2 × 9 + 1
11 = 2 × 5 + 1

0 1391 = 695 × 2 + 1

1391 = 1200 + 191
= 2 × 600 + 196 + 1
= 2 × 600 + 2 × 95 + 1
= 2 (600 + 95) + 1
= 2 (695) + 1

Learner 2:



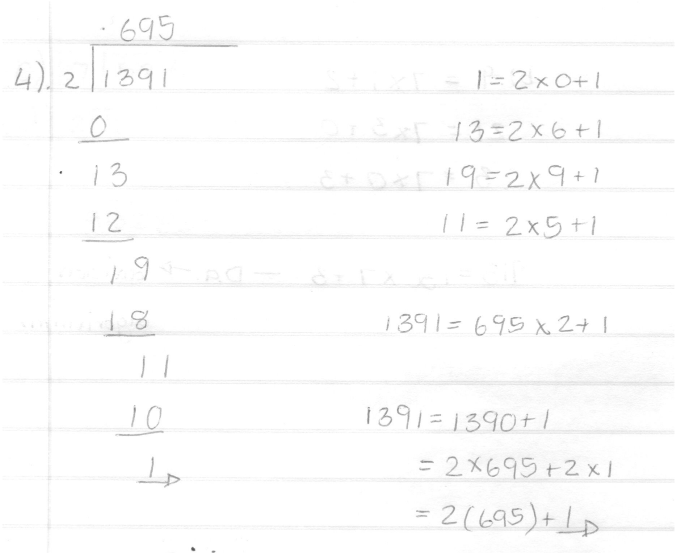
·695
2) 2|1391
12
·19
18
·11
10
·1

1 = 2×0+1
13 = 2×6+1
19 = 2×9+1
11 = 2×5+①

1391 = 695×2+1

1391 = 1200 + 191
= 2×600 + 180 + 11
= 2×600 + 90×2 + 10 + 1
= 2×600 + 90×2 + 2×5 + 1
= 2(600+90+5)+1
= 2(695)+1

Learner 3:



·695
4). 2|1391
0
·13
12
19
18
11
10
1

1 = 2×0+1
13 = 2×6+1
19 = 2×9+1
11 = 2×5+1

1391 = 695×2+1

1391 = 1390+1
= 2×695+2×1
= 2(695)+1

Having attempted the first question correctly, this learner had everything done perfectly except for the removal of the multiples of the divisor. Here he imitated the process of division, despite having already arrived at the same answer three times. Notice that the remainder was also multiplied by two when he wrote down the factors of 1390, so one clearly cannot judge his attempt as correct. In the next lesson it emerged that this learner continuously made the same 'error' before factoring the expression, as he believed that to be able to remove the common factor, each term had to have a two in it. He always checked his answer against the division algorithm, and constantly thought that he was doing the right thing by adding a multiple of the divisor in order to be able to remove the common factor.

The use of brackets was made explicit to him in order to correct this, as this interview illustrates:

Teacher: Look at what you wrote there (*pointing to the work*). Is there something wrong with what you are saying?

Learner: No … it's right … yeah if you check the division algorithm, it is the same (*proceeds to point out the divisor, the dividend, the quotient and the remainder in the long division and the reset method*).

Teacher: (*Covering the other work*) Look at the mathematics in the expression.

The learner again protests that it is right because it matches the division algorithm.

Okay, use your calculator and work out for me what the value is of two times six-nine-five plus two times one.

Learner: One-three-nine-two.

Teacher: And the dividend was one-three-nine-one … so something is definitely wrong here, do you agree?

Learner: No (*proceeds to multiply the final line again*) … yeah … I don't know.

Teacher: Okay, now tell me how you did this.

Learner: (*Not listening to the teacher*) Oh yeah, I have been wanting to ask you this sir. How can you remove the two from that line, when there is not a two in each number?

Teacher: Do we really need a two in each term before we can remove the two?

Learner: Yes, the two is a common factor so I put the two there so that I can have a common factor.

Teacher: But you cannot just put in a two. Surely you will then change the value of the number.

Learner: (*Proceeds to check both lines with the calculator*) I see, but how can I then remove the two … that is what I wanted to ask you today.

Teacher: You used brackets in the last step and the one is not in the bracket, which means it did not have a two with it, and is therefore separate from the 695 … Cast your mind back to the work we did yesterday. I wrote on the board something like $4\cdot 12+4\cdot 170+4\cdot 3+2$ and we removed the four to get $4(12+170+3)+2$. Can you see the two is by itself?

Learner: Yes, but it did not have a four, so you could not remove the four.

Teacher: Now what do you think the brackets are there for? Why did I use the brackets?

Learner: To show that you are multiplying (*thinking*) ... Oh, so you only use the brackets to show which had a four in them … and the two did not … I get it.

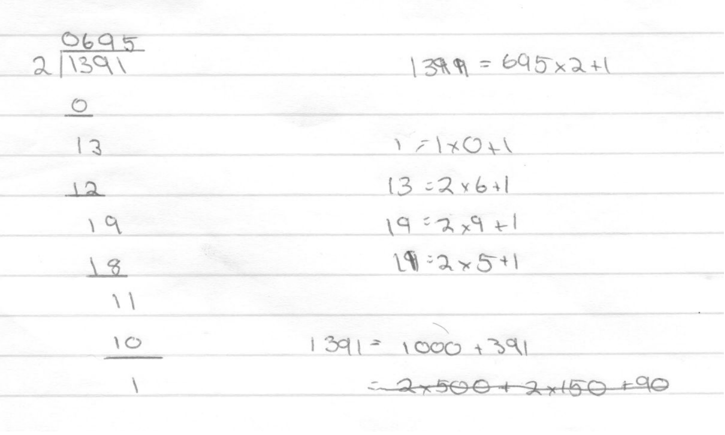
Teacher: What did you get?

Learner: When you take the four out, and you multiply it back, you have to get the same as what was written there (*pointing to the previous line*), and the bracket tells you what to multiply; only those inside need to be multiplied … to get to that line.

The teacher gave this learner another combination of factored numbers to work with, and the process was completed successfully.

The use of brackets in the factorization process is a super procedure with which learners have to be familiar and they must understand how this is used when common factors are removed from terms in an expression. When formalizing the factoring process in this method, it needs to be addressed in class that not all the terms need to have this common multiple, and that the terms that do contain the common factor need to be grouped together.

Learner 4:



0695
2 | 1391
0
13
12
19
18
11
10
1

$1391 = 695 \times 2 + 1$

$1 = 1 \times 0 + 1$

$13 = 2 \times 6 + 1$

$19 = 2 \times 9 + 1$

$11 = 2 \times 5 + 1$

$1391 = 1000 + 391$

~~$= 2 \times 500 + 2 \times 150 + 90$~~

This is yet another example of the insecurity that learners face when given an activity that requires of them to make a creative choice. The learner started with the correct approach, but abandoned her attempt as she was not sure if she was right.

The results for this question are as follows:

	Correct	Partially Correct	Incorrect	No attempt
a	22	-	-	-
b	22	-	-	-
c	22	-	-	-
d	5	2	7	8

Question 3: $13\overline{)2473}$

$$
\begin{array}{rl}
 & 0190 \\
a)\ 13 & \overline{)2473} \\
 & \underline{0} \\
 & 24 \\
 & \underline{13} \\
 & 117 \\
 & \underline{117} \\
 & 03 \\
 & \underline{00} \\
 & 3
\end{array}
$$

$$
\begin{aligned}
b)\ \ 2 &= 13\cdot 0+2 \to 2\cdot 10+4 \\
24 &= 13\cdot 1+11 \to 11\cdot 10+7 \\
117 &= 13\cdot 9+0 \to 0\cdot 10+3 \\
3 &= 13\cdot 0+3
\end{aligned}
$$

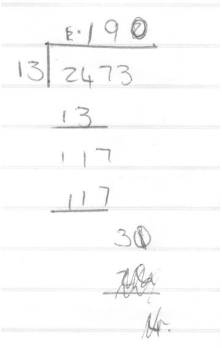
$$c)\ \ 2473 = 13(190)+3$$

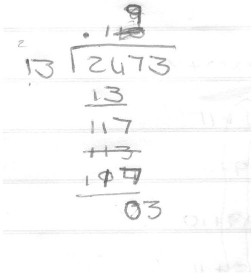
$$
\begin{aligned}
d)\ \ 2473 &= 1300+1300-127 \\
&= 13\cdot 100+13\cdot 100-13\cdot 10+3 \\
&= 13(100+100-10)-3 \\
&= 13(190)+3
\end{aligned}
$$

This question was different as it covered two subtleties in the remainder; that of a zero remainder and its transfer to the quotient in the long division procedure. This problem also included a two digit remainder and engaged learners in issues of how this would transfer in the *reset method*. When learners were introduced to division problems, a great deal of time was spent on discussing these two important issues. Learners should be able to predict whether an answer makes sense, by examining the product of the quotient and the divisor. One important way to check the correctness of an answer derived from using one of the methods is to compare the answer with the outcome achieved by using the division algorithm.

Both the following learners only completed the long division procedure, and left the answer there without completing the rest of the procedures as required by the teacher's instructions.

The one learner did realise that the zero had to be placed in the quotient, but erased the steps from the problem, after realising that he incorrectly added a zero after the three to form the number 30. This could have been done by the learner as a continuation of the division into the decimal system, something that they learnt in earlier grades.

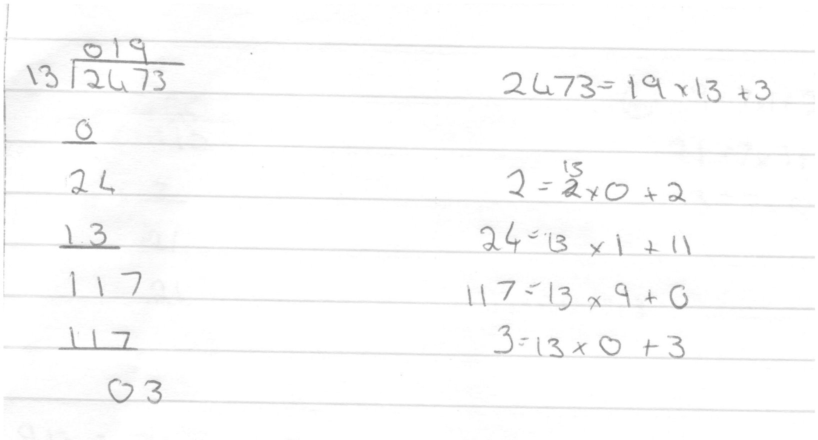




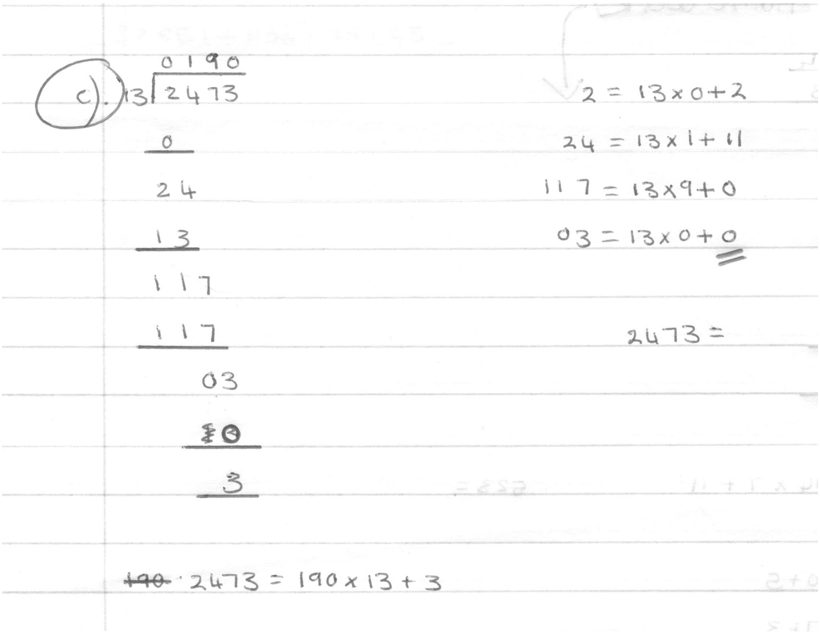
A second learner stopped the division procedure immediately after realising that thirteen does not divide three, and the result was that the quotient remained nineteen.

This mistake was also made by six other learners who omitted to transfer the zero to the quotient. One of these learners (see below) continued with the *reset method* and still did not realise that the long division procedure does not get to the same answer as this more challenging method which was performed flawlessly. This learner did not remove multiples of the divisor. If she had done this, she most probably would have realised her mistake immediately. The order in which she carried out the instructions was not followed clearly, as she formulated the *division algorithm* directly after having done the long division.

The instructions were given to do the *reset division* directly after the long division, in order to compare whether the two division procedures are indeed executed correctly.



The following learner mistakenly wrote that $3 = 13 \cdot 0 + 0$ instead of $3 = 13 \cdot 0 + 3$.



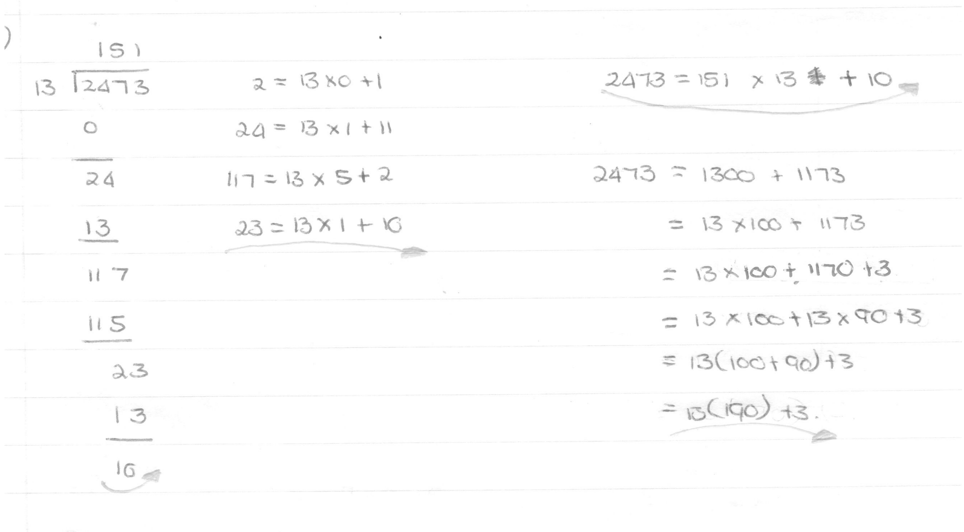
c). $13 \overline{)2473}$ → 0190

0
24
13
117
117
03
0
3

2 = 13x0+2
24 = 13x1+11
117 = 13x9+0
03 = 13x0+0
2473 =

~~190~~ 2473 = 190 x 13 + 3

An interesting phenomenon that learners reveal in their work is the failure to check whether the results to each of the methods agree with one another. The last two examples of learners' work, and indeed the next one below, could all have been corrected if the learners checked for this balance. The learner below mistakenly wrote that $13 \times 5 = 115$ without recognising the obvious discrepancy of that result after she has removed multiples of the divisor from the dividend. With this mistake, the concept of a zero remainder did not emerge in the long division procedure.



$13 \overline{)2473}$ → 151

0
24
13
117
115
23
13
10

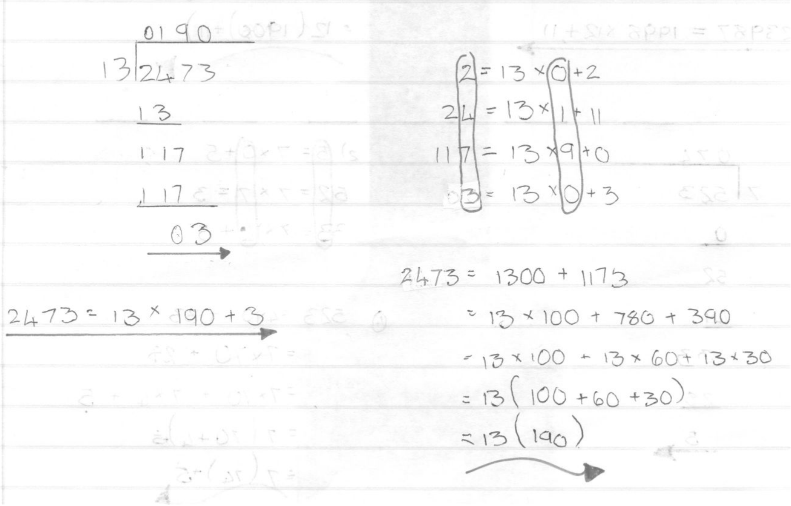
2 = 13x0 +1
24 = 13x1 + 11
117 = 13 x 5 + 2
23 = 13 x 1 + 10

2473 = 151 x 13 + 10

2473 = 1300 + 1173
= 13 x100 + 1173
= 13 x 100 + 1170 + 3
= 13 x 100 + 13 x 90 + 3
= 13(100+90) + 3
= 13(190) + 3.

Only three of the learners completed this division without mistakes. Two of the three learners' work is shown below:

Learner 1:



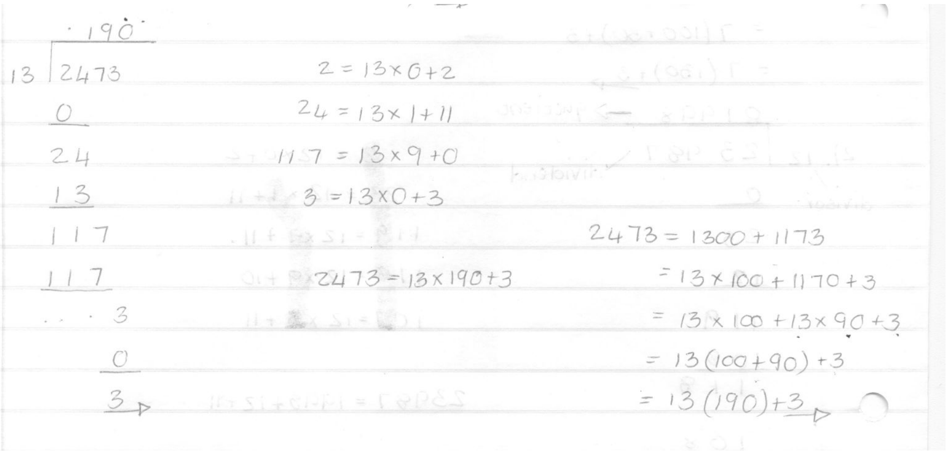
0190
13 | 2473
13
117
117
03

2473 = 13 × 190 + 3

2 = 13 × 0 + 2
24 = 13 × 1 + 11
117 = 13 × 9 + 0
03 = 13 × 0 + 3

2473 = 1300 + 1173
= 13 × 100 + 780 + 390
= 13 × 100 + 13 × 60 + 13 × 30
= 13 (100 + 60 + 30)
= 13 (190)

Learner 2:



190
13 | 2473
0
24
13
117
117
3
0
3

2 = 13×0+2
24 = 13× 1+11
117 = 13×9+0
3 = 13×0+3

2473 = 13×190+3

2473 = 1300 + 1173
= 13 × 100 + 1170 + 3
= 13 × 100 + 13× 90 + 3
= 13 (100+90) + 3
= 13 (190)+3

The results for this question were poorer than the previous two questions.

	Correct	Partially Correct	Incorrect	No attempt
a	12	5	2	3
b	17	2	-	3
c	14			8
d	3	-	-	19

Question 4: $29\overline{)32851}$ **(optional)**

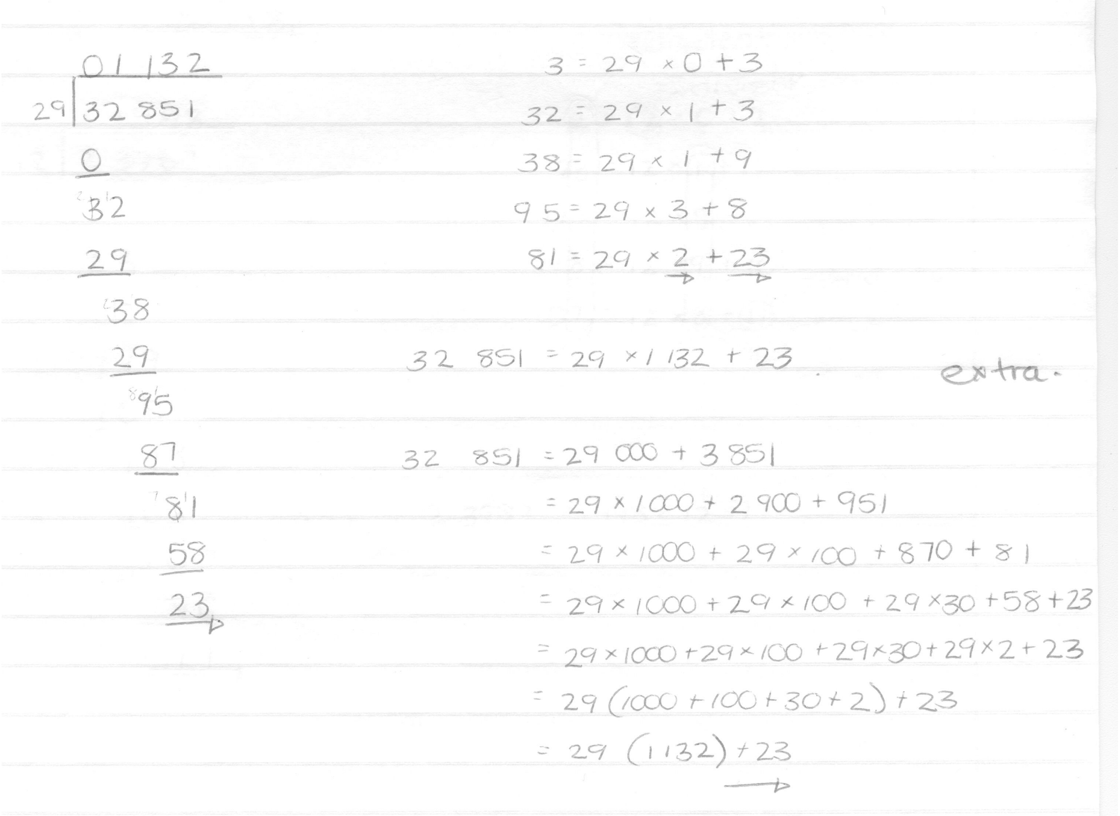
$$a)\quad \begin{array}{r} 1132 \\ 29\overline{)32851} \\ \underline{29} \\ 38 \\ \underline{29} \\ 95 \\ \underline{87} \\ 81 \\ \underline{58} \\ 23 \end{array}$$

$$b)\quad \begin{aligned} 3 &= 29\cdot 0+3 \rightarrow 3\cdot 10+2 \\ 32 &= 29\cdot 1+3 \rightarrow 3\cdot 10+8 \\ 38 &= 29\cdot 1+9 \rightarrow 9\cdot 10+5 \\ 95 &= 29\cdot 3+8 \rightarrow 8\cdot 10+1 \\ 81 &= 29\cdot 2+23 \end{aligned}$$

$$c)\quad 32851 = 29\left(1132\right)+23$$

$$d)\quad \begin{aligned} &32851 \\ &= 29000+3851 \\ &= 29\cdot 1000+2900+870+58+23 \\ &= 29\cdot 1000+29\cdot 100+29\cdot 30+29\cdot 2+3 \\ &= 29\left(1000+100+30+2\right)+3 \\ &= 29\left(1132\right)+3 \end{aligned}$$

This question was given as a challenge to those learners who commented in class that *'this is easy'*. A mere six learners attempted the question with only one completing every method that had been learnt successfully. His work is shown below:



$$\begin{array}{r} 01132 \\ 29\,|\,32\,851 \\ \underline{0} \\ 32 \\ \underline{29} \\ 38 \\ \underline{29} \\ 95 \\ \underline{87} \\ 81 \\ \underline{58} \\ 23 \end{array}$$

$3 = 29 \times 0 + 3$

$32 = 29 \times 1 + 3$

$38 = 29 \times 1 + 9$

$95 = 29 \times 3 + 8$

$81 = 29 \times 2 + 23$

$32\,851 = 29 \times 1\,132 + 23$ extra.

$32\,851 = 29\,000 + 3\,851$

$= 29 \times 1000 + 2\,900 + 951$

$= 29 \times 1000 + 29 \times 100 + 870 + 81$

$= 29 \times 1000 + 29 \times 100 + 29 \times 30 + 58 + 23$

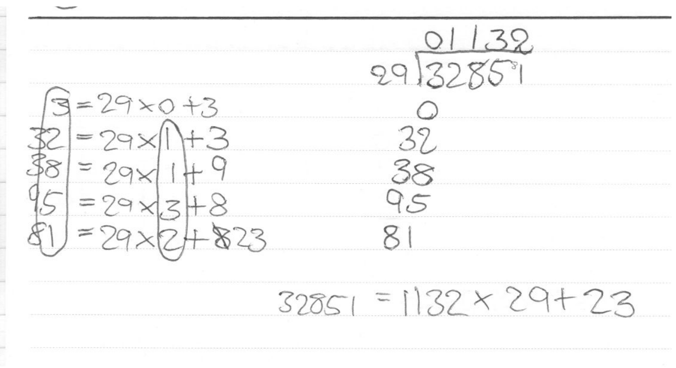
$= 29 \times 1000 + 29 \times 100 + 29 \times 30 + 29 \times 2 + 23$

$= 29\,(1000 + 100 + 30 + 2) + 23$

$= 29\,(1132) + 23$

The difference in this question lies in the divisor which now was above twenty, and the dividend was now a five digit number, both of which make the division procedures much longer, but not necessarily more complicated.

One of the learners performed each division problem correctly, but failed to remove multiples of the divisor from the dividend. His work was done very concisely, as the example alongside shows clearly.



01132
29)32851

3 = 29×0 + 3   0
32 = 29×1 + 3   32
38 = 29×1 + 9   38
95 = 29×3 + 8   95
81 = 29×2 + ~~8~~23   81

32851 = 1132 × 29 + 23

The overall results for this question is as follows:

	Correct	Partially Correct	Incorrect	No attempt
a	6			16
b	6	-	-	16
c	6	-	-	16
d	2	-	-	20

In summary:

There are definite issues that emerged when analysing the work of the learners. The key problem was that learners did not feel comfortable enough to engage with the method which required them to remove multiples of the divisor from the dividend. The main reason for this inability to complete a basic factoring procedure is learners' lack of confidence in performing a procedure that is totally dependent on the choice that they make. Another possibility could be that they could not accept the fact that two different decomposition procedures could finally lead to the same result, despite having debated this at length in class. The data shows clearly that the learners could complete the *long division* and the *reset method* quite competently and that in most cases there were no problems using these two methods together in order to check their answers before summarizing the procedures in the division algorithm. It is also very clear that the *reset method* was performed more successfully in more complex division problems.

The fact that the learners went into automatic mode when doing *long division*, provided an obstacle free journey for them to perform the division. This resulted in important concepts surrounding place value and the remainder not always coming to the fore. For learners to have a deep conceptual understanding of the underlying tensions, it is necessary to create situations where cognitive conflict is generated in order for flexible learning and understanding to take place.

After analyzing the work of the learners and reviewing the video data, it could also be argued that the cognitive bombardment of alternative approaches could be overwhelming for some learners as the cognitive load becomes too great for the weaker learners to bear. This lesson would be better received by the learners if it took place over two sessions.

Lesson 2: The arithmetic – algebra link

The purpose of this lesson was to establish the link between arithmetic long division and algebraic long division by emulating the algebraic process. This process of imitation was designed around the decomposition of the numbers used in the division process, into numbers constructed as power relationships in base ten. This decomposition not only foregrounds the concept of coefficients, but it focuses the learners' attention on the fact that the digits on which and with which they perform the division occupy a specific place within the number, hence enhancing the concept of place value in division. In the pilot study, this proved to be the most difficult part of the journey of learning for the students. There are many super- and sub-procedures (Hiebert and Lefevre, 1986) with which the learners have to be familiar and in which they must develop fluency, in order to complete this part of the division process successfully. A thorough knowledge of exponential laws, the distributive property, place value, basic factorization, symbolic representation and the use of parentheses in mathematical symbolization, the *division algorithm* and division with the remainder are necessary. Due to the importance of this lesson, it will be discussed in full in this chapter. There are many issues that surfaced during the lesson, but the main challenges that learners experienced were with the remainder, syntax and basic exponential laws.

A) The decomposition of numbers:

At the commencement of this lesson, learners were required to write 523 as a sum of the power relations in base ten. All of the learners did this successfully, but the difficulties that arose were related to syntax. Learners were not sure whether a dot " ." or the conventional symbol "$\times$" should be used to represent multiplication. There were a few learners that had problems writing 100 as 10^2 and this was solely based on their lack of retention of basic exponential laws.

The teachers commenced the lesson by decomposing the number 523 as follows:

$$\begin{aligned} 523 &= 500+20+3 \\ &= 5\times100+2\times10+3 \\ &= (5\times100)+(2\times10)+3 \rightarrow \text{explaining}\, 100 = 10\times10 = 10^2 \rightarrow a^1\cdot a^1 = a^2 \\ &= 5\cdot10^2+2\cdot10+3 \rightarrow \text{decomposing the number in base 10} \end{aligned}$$

At this point the attention of the learners was drawn to the fact that the number 523 is still clearly visible in this representation. The only difference here is that place value has become important when decomposing the number in base ten.

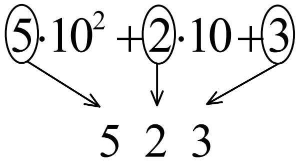


Figure 6

A second, larger number was then decomposed in the same way:

$$\begin{aligned} &28321 \\ &= 2\times10000+8\times1000+3\times100+2\times10+1 \rightarrow see* \\ &= 2\times10^4+8\times10^3+3\times10^2+2\times10+1 \rightarrow see** \\ &= 2\cdot10^4+8\cdot10^3+3\cdot10^2+2\cdot10+1 \rightarrow see*** \\ &= 2\left(10^4\right)+8\left(10^3\right)+3\left(10^2\right)+2\left(10\right)+1 \rightarrow see**** \end{aligned}$$

* Here a learner did not identify 20000 as 2×10000, but suggested instead that it was 2×1000. This was explained by that teacher as a relationship between the number of digits after the 2, and the number of zeros that need to be included when this number is expressed as a multiple in base ten.

** Teacher: How can we write the 10000 as a power of ten?

L1: 10^4, because there are four zeroes after the one, and that would mean that the power of 10 has to be 4.

Teacher: There are four zeroes, and that means that ten has been multiplied by itself every time. . . Ten times ten times ten times ten … So the power of the ten has to be four.

L2: Can we not say that it is 10000 squared?

Teacher: If you say that it is 10000 squared, then this will mean that there have to be eight zeros…(*writing* $10000^2 = \left(10^4\right)^2 = 10^8$) . The number of zeros determines the power of the ten.

*** Here the teacher asked the learners what the alternative convention is for indicating a product between two numbers. Learners immediately responded that one can use a dot or alternatively a bracket.

**** Some learners were still not satisfied by representing multiplication with a single dot. The teacher explained that it is a convention in mathematics to present the operation of multiplication by one of the following symbols: $"\times" or\ "\cdot"\ or\ \left(...\right)$. Learners were made aware that mathematics is a language in which things and operations should be expressed concisely, and that the use of the dot to represent multiplication affords mathematicians the advantage of conciseness.

B) Multiple representations:

After learners had gained confidence in decomposing numbers and writing them as a sum of the powers of ten, the teacher revisited lesson one so as to incorporate this new method. The problem was solved in the following manner:

ii) Removing multiples of 5: $\rightarrow see*$

$$\begin{aligned} 87 &= 10 + 70 + 7 \\ &= 2\times 5 + 14\times 5 + 5 + 2 \\ &= 5\left(2 + 14 + 1\right) + 2 \\ &= 5\left(17\right) + 2 \end{aligned} \qquad or \qquad \begin{aligned} 87 &= 25 + 25 + 25 + 10 + 2 \\ &= 5\times 5 + 5\times 5 + 5\times 5 + 5\times 2 + 2 \\ &= 5\left(5 + 5 + 5 + 2\right) + 2 \\ &= 5(17) + 2 \end{aligned}$$

i) Long division:

$$\begin{array}{r} 17 \\ 5\overline{)87} \\ \underline{5} \\ 37 \\ \underline{35} \\ 2 \end{array}$$

$\therefore 87 = 5\left(17\right) + 2$

Figure 7

* The teacher decided to show the learners that one can choose more than one set of partial sums to decompose the number. The learners' response was rather unexpected, as is shown in the following transcript of the classroom dialogue:

Teacher: If we decompose our number differently, must the final answer be the same?

L1: I would say no, because you're working with different numbers.

Teacher: OK, does anybody have another opinion?

L2: You're going to add something else after the bracket … equal to 87.

Teacher: Say you guys are arguing that it's not going to be the same. Let's take one final argument.

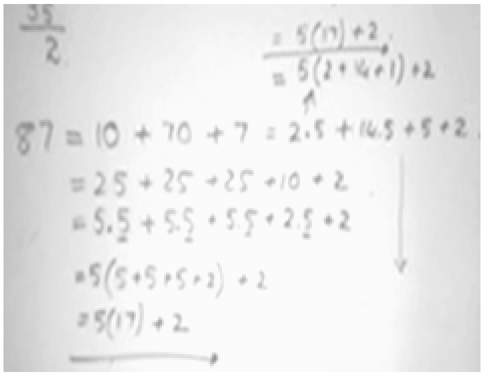
L3: I think it should be the same, because if you have like a different number in the bracket (*referring to the quotient*), then it will equal something different.

Teacher: So you are saying that it will be exactly the same, no matter what the numbers are that I chose. Do I understand you correctly?

L3: Yes.

Teacher: Right, we have got one for and three against. Let's do it and then we'll talk to the answer.

One could clearly observe that the learners were struggling with this concept. They could not see that expressing the partial sum differently will lead to the same composition at the end of the process. They clearly viewed this activity in terms of a ‘product’ and ignored the process completely. (Sfard, 1991).



Teacher: How is it possible to come to the same conclusion having taken the different route?

L1: Isn’t it because all your factors are still the same?

Teacher: They are the same at the end of the day when we summarise in the division algorithm. (*Pointing to the division algorithm in each of the two problems*). But if we look at the partial sums, the numbers that I have added are different. Why are they different?

At this point the teacher proceeded to explain to the learners that the roads they take to school every day are different, yet they all arrived at the same place. After waiting for while, it became evident that the learners did not understand why the answer was the same even though the sum parts were different. The teacher continued as follows:

Teacher: We are ultimately removing multiples of five from this specific number, which in our case is 87. There can only be one amount of multiples of five that can go into this number. It cannot be more than one amount of multiples of five that goes into 87. If I had broken up my partial sum as a sum of 50 and 37, then I would have ten multiples of five, but the 37 still contains seven multiples of five. If I had chosen 70 and 17 as my partial sums, then the 70 contains fourteen multiples of five with the 17 still holding three multiples of five. So no matter how I go about this, there can only be 17 multiples of five in the number 87 in the end. So we can say that the partial sums are different, because we chose the numbers differently. But always remember in the end they all add up to 87.

iii) Decomposition into a sum of powers of ten:

The dividend:

$$87 = 80 + 7$$
$$= 8 \cdot 10 + 7$$

The divisor:

$$5 = 10 - 5 \rightarrow see*$$

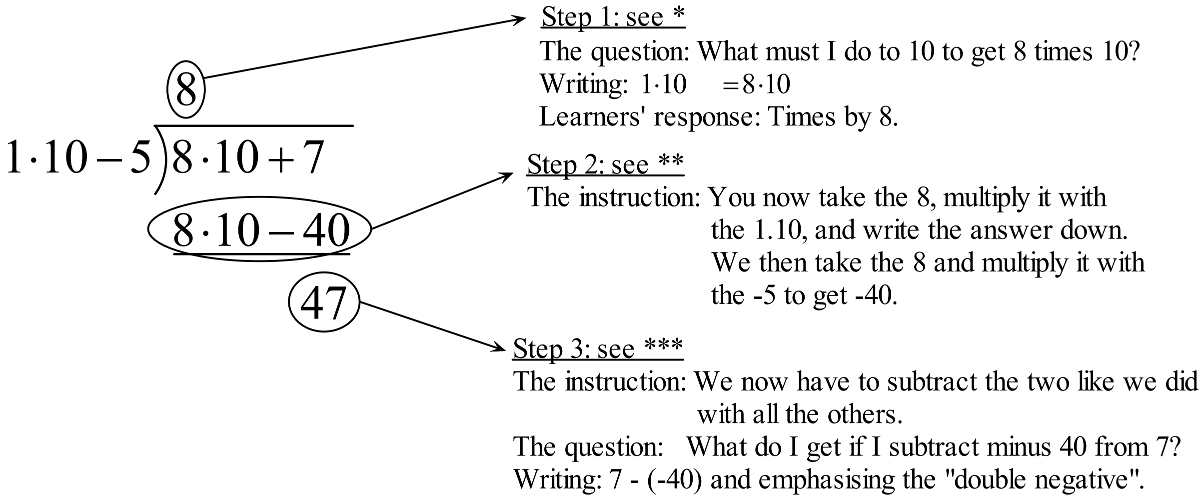
* At this point learners were asked the question: “What must I do to 10 to get back to 5?”

The immediate answer from the group was “divide by two” and only seconds later the alternative of subtracting five was suggested. The teacher pointed out that opting for division by two is not wrong, only problematic since it will involve fractions and mathematicians always try to avoid forming fractions unnecessarily.

<u>The division using base 10 decomposition</u>:

$$\begin{array}{r} 8 \\ 1\cdot10-5\overline{\smash{)}8\cdot10+7} \\ \underline{8\cdot10-40} \\ 47 \end{array}$$

<u>The classroom instruction</u>:



<u>Step 1: see *</u>
The question: What must I do to 10 to get 8 times 10?
Writing: 1·10 =8·10
Learners' response: Times by 8.

<u>Step 2: see **</u>
The instruction: You now take the 8, multiply it with the 1.10, and write the answer down. We then take the 8 and multiply it with the -5 to get -40.

<u>Step 3: see ***</u>
The instruction: We now have to subtract the two like we did with all the others.
The question: What do I get if I subtract minus 40 from 7?
Writing: 7 - (-40) and emphasising the "double negative".

Figure 8

* First the teacher said: “I have one multiple of ten and I need to have eight multiples of ten. What must I do now?” There was no response for some time and the question was rephrased to: “What must I do to ten to get eight times ten?” To this the learners immediately responded with the answer “multiply by

8”. This is important to note, as the same issue emerges later on during classroom discussion. It seemed to the teacher that focusing on the concept of coefficient at this stage would increase the cognitive load for learners as they might not cope with all this information at once. The teacher decided to address this concept at a later stage when another example would be discussed.

** Again the concept of the coefficient was deliberately ignored so as not to increase the cognitive load what beyond the learners could cope with simultaneously. Performing the product $1\cdot 10\times 8$, the teacher multiplied the eight and the one and then wrote down the answer as $8\cdot 10$.

*** When learners subtracted the two numbers from one another, a small minority of them neglected to change the sign of the 40 to a positive because of the double negative. This was soon corrected by the other learners. The problem that emerged at step three was that the remainder was now 47 instead of 2 as we concluded after performing the long division. The teacher pointed this difference out to the learners, by indicating that the remainders are not the same and yet there were no digits to bring down in order to continue the division process. The division process was then summarized using the division algorithm.

<u>Dealing with the different remainder</u>:

$$\begin{aligned} 8\cdot 10+7 &= (10-5)8+47 \rightarrow \mathit{see}* \\ &= 5(8)+47 \rightarrow \mathit{see}** \\ &\left.\begin{aligned}&=5(8)+45+2\\ &=5(8)+5(9)+2\end{aligned}\right\}\rightarrow \mathit{see}*** \\ &= 5(8+9)+2 \\ &= 5(17)+2 \end{aligned}$$

* After writing on the board $8\cdot 10+7=(10-5)8+47$, the teacher wrote alongside this the result of the previous division, which was $5(17)+2$, to point out the difference between the two answers. Learners were made aware of the fact that

this method progresses to a point, but could not continue beyond that point as there were no more digits to bring down. To continue the process of division, learners were instructed to use the *division algorithm* and the process of removing multiples of the divisor simultaneously, to complete the division. The tensions with the remainder became very prevalent at this point, as it was emphasized in lesson one that the remainder should always be the same irrespective of the method that was used, and here this was clearly not the case.

** As this step is not necessary for algebraic long division, learners were instructed to revert back to the numbers in order for the resemblance in the *division algorithm* to be recognized more easily. It would also make it easier for them to remove multiples of the divisor and perform the necessary factorization if they reverted back to the original divisor.

*** It was pointed out to the learners that our division could continue from this point onwards, as we knew the full value of the divisor and the dividend. The teacher was fully aware that if this was an algebraic problem, the division would have been completed at this point. The transition between arithmetic and algebra appears to be problematic at this juncture, and this will be discussed later.

After the problem was completed, the teacher indicated to the learners that the division could continue because the numbers were known. At the onset of this research project, learners were made aware that they would have to complete division on expressions that contained variables. The teacher again drew their attention to the fact that the ‘numbers’ (polynomial expressions) are not always known, and therefore the procedure with such division problems will be finalized at the point when no digits could be brought down. Sharing this information with the learners at this point served to prepare them for the generalization that lay ahead, and to open the cognitive doorway to the abstract world.

After this discussion was completed, learners were encouraged to ask questions. What follows is a transcript of the classroom discussion:

L1: I don’t understand the one dot ten minus five (*referring to the divisor*).

Teacher: We are writing our numbers as a relationship in base ten because we are reconstructing our numbers as power relations of ten. We want to divide these decomposed numbers into one another. If I leave my divisor as five, it will be more difficult to establish its relationship with the dividend which is $8\cdot 10+7$. Writing my divisor as $1\cdot 10-5$ makes it easy to see this relationship as a relationship with the ten.

L2: Why do you have to multiply ten by one if you know that it is ten already?

Teacher: Because our focus is on the number that is in front of the ten. If you look at the factorisation that we performed earlier. (*Pointing to* $2\times 5+14\times 5+5+2=5(2+14+1)+2$ *in Figure 7*), the 1 inside the brackets is actually a placeholder for the number five which stands in front of the bracket. You guys often forget to write this 1 as a placeholder for five in your bracket when you factorise, since the five is in fact one times five. In our problem we put the one in front of the ten to remind us that we are working with *one multiple of ten*.

<u>A second problem was considered where 124 had to be divided by 13:</u>

- Decomposition into a sum of powers of ten:

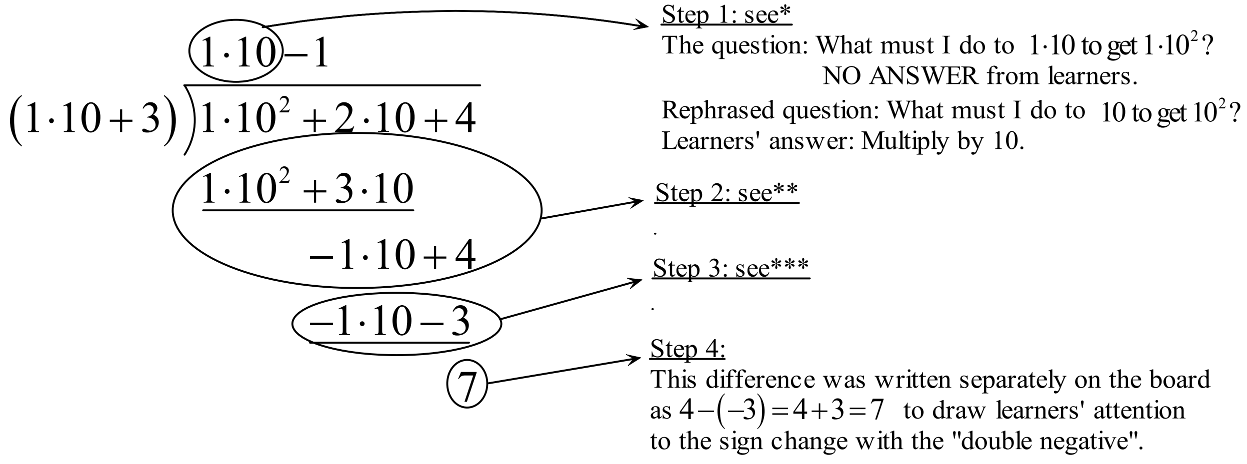
The dividend:

$$124=100+20+4$$
$$=10^2+2\cdot 10+4$$

The divisor:

13=10+3

- The division using base 10 decomposition:



Step 1: see*
The question: What must I do to $1\cdot 10$ to get $1\cdot 10^2$?
NO ANSWER from learners.
Rephrased question: What must I do to 10 to get 10^2?
Learners' answer: Multiply by 10.

Step 2: see**

Step 3: see***

Step 4:
This difference was written separately on the board as $4-(-3)=4+3=7$ to draw learners' attention to the sign change with the "double negative".

Figure 9

* The learners still seemed to have problems when determining the quotient. The problem seemed to emerge when the words *"What must I do to one multiple of ten to get to one multiple of ten square?"* are used. Rephrasing this to *"What must I do to ten to get ten square?"* made immediate sense to them. As these coefficients of ten are important to understand when it comes to the coefficients of the variable in algebra, it is necessary that this concept be understood by all learners. In an attempt to help learners cross this cognitive gap easily, it was explained to them in two ways. The first approach involves using multiplication and requires the learners to ask themselves what they must multiply the first term in the divisor with to obtain the first term in the dividend. As this question phrase has been used all along by the teacher, it was decided to show them the alternative by looking at this step as a division operation. Learners were shown the following: $\frac{1\cdot 10^2}{1\cdot 10} = \frac{\cancel{1}.\cancel{10}.10}{\cancel{1}.\cancel{10}} = 10$.

** The product $1\cdot 10\left(1\cdot 10+3\right)$ was not discussed in depth at this point. The learners were made aware that the purpose of this procedure is to get the numbers, which are the first terms in the divisor and the dividend, to be the same so that when they are subtracted the result is zero. The teacher discussed this product in a very leading way, not paying attention to the mathematical details in this product. Again this was done to reduce the cognitive load for the learners. By working only with the coefficients of ten, the teacher pointed out to the learners that the tens were almost irrelevant as they only indicated place value at this stage. The teacher used the following language when subtracting: *"what are two multiples of ten minus three multiples of ten?"* and the learners responded by saying *"minus one"*. The teacher rephrased this by saying that it is 'minus one *multiple of ten'*. To convince the learners that the ten was redundant here, this step was repeated using an analogy with trees by saying: *"what are two trees minus three trees…* minus one *tree*". This seemed to make sense to the learners as they were convinced that they were doing exactly the same as what they had done previously when they divided.

*** The product $-1(1\cdot 10+3)$ was discussed now, paying particular attention to the mathematical detail. Learners were reminded that it is a product that contains one term only, and that they therefore only had to multiply the -1 with either the 1 or the 10, so that the product became -1 times 10 $\left(write\ (-1\cdot 10)\right)$. The focus was on the distributive property and the possible mistakes that learners could make here. The teacher used the following language when multiplying: "*minus one times one is* (*pointing only to the units and ignoring the ten*) *… minus one … multiple of ten*" pointing out to the learners that the ten was again ignored in the operation.

Summarising the work in the division algorithm:

$$\begin{aligned} 1\cdot 10^2+2\cdot 10+4 &= (1\cdot 10+3)(1\cdot 10-1\ )+7 \\ &= 13(9)+7 \end{aligned}$$

After the division was completed, it was emphasised to the learners that the remainder seven is indeed smaller than the number ten. The division was therefore completed and one did not need to use the *division algorithm* to further remove multiples of ten from this remainder. To check whether the answer was indeed correct, the teachers suggested that learners use normal arithmetic long division.

- ➢ Checking this result using long division:

$$\begin{array}{rr}
 & 009\\
13\big) & \overline{124}\\
 & \underline{0}\\
 & 12\\
 & \underline{0}\\
 & 124\\
 & \underline{117}\\
 & 7
\end{array}$$

$$\therefore 124 = 13(9)+7$$

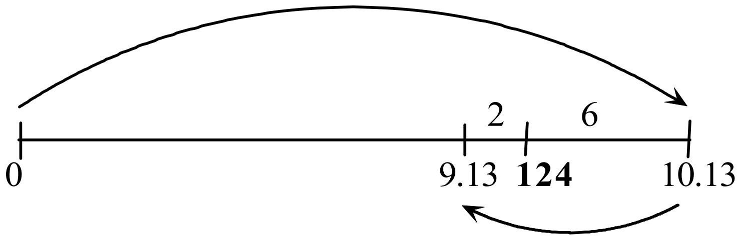
- Removing multiples of 13:

$$\begin{aligned} 124 &= 130-6 && \rightarrow \text{ Going bigger than the number*} \\ &= 10\cdot 13-6 && \rightarrow \text{The negative remainder**} \\ &= 9\cdot 13+13-6 && \rightarrow \text{Changing remainder to become positive ***} \\ &= 9\cdot 13+7 \end{aligned}$$

* The process of removing multiples of thirteen was approached in a different way. Instead of looking for the largest number of multiples of thirteen that could be removed from 124, the teacher suggested that the learners work with the easiest multiple of thirteen, in this case 130, as it is six units away from the number.

** While being discouraged from using their calculators, learners were introduced to the concept of choosing multiples of thirteen that lie closest to the number. These multiples were chosen as ten multiples of thirteen, and then six units needed to be removed to get back to the number 124. This problem was specifically chosen to explain to learners why a remainder can be negative, and what this implies about the division concept.

*** The concept the teacher focused on was explaining that if the remainder was negative, then too many multiples of thirteen were removed from the number. The teacher explained division as ending before the number was “crossed”, by using the following diagram to explain what was meant by this:



Thus for the division process to be complete, the multiples that are removed from the number must be smaller than the number but as close to it as possible in order to leave a positive remainder. Learners were instructed to remove one multiple of thirteen from the quotient and add the remainder to this multiple that was removed in order to get a positive remainder. This was done to correct the ‘over division’ which took the product of the divisor and the quotient to a value larger than the dividend.

In order to ensure that the learners understood this procedure, the example below was then discussed on the board.

$$\begin{aligned}128 &= 130-2\\ &= 10\cdot 13-2\\ &= 9\cdot 13+13-2\\ &= 9\cdot 13+11\end{aligned}$$

After completing the work, learners were encouraged to ask questions about the work.

L1: Why is that number (*referring to the remainder*) always smaller than the thirteen (*referring to the divisor*)?

Teacher: Did everybody hear the question? L1 is asking why the remainder is always smaller than the divisor.

L2: Because if it's not it's going to make the 128 and bigger.

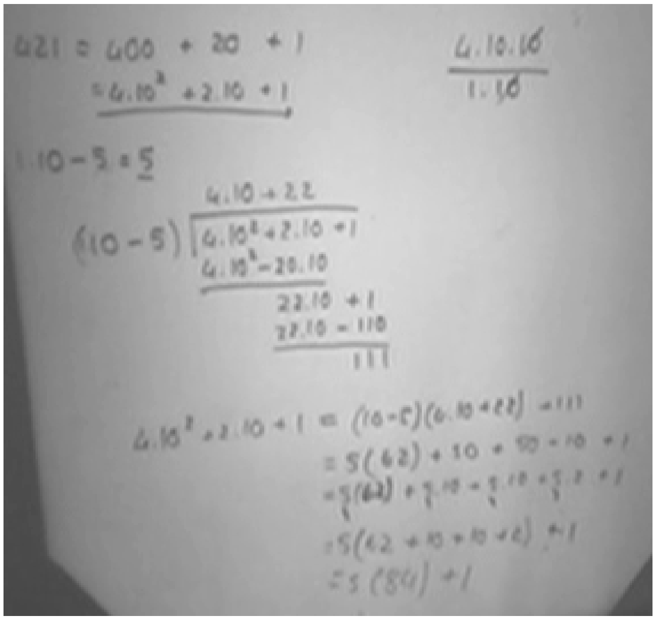
Teacher: Does it have anything to do with the 128? If we reached the remainder of 32, what would this mean?

L3: That you can still remove multiples of thirteen from the 32.

This discussion highlighted the importance of the relationship between the remainder and the divisor yet again. It became clearer for the learners that this relationship determines whether the division has been completed or whether division can and must continue. Another interesting comment was made by one of the learners to her classroom teacher (who sat in on the lessons) when she pointed out that *"all answers meet in the division algorithm"*.

A final example was given to the learners who instructed the teacher on what should be done to complete the division. This example required the number 421 to be divided by 5. What follows is a transcription of what was written on the board by the teacher.

<u>421 divided by 5</u>:



421 = 400 + 20 + 1
= 4.10² + 2.10 + 1

4.10.16
1.10

1.10 − 5 = 5

4.10 + 22
(10 − 5) 4.10² + 2.10 + 1
4.10² − 20.10
22.10 + 1
22.10 − 110
111

Long division:

$$\begin{array}{r} 084 \\ 5\overline{)421} \\ \underline{0} \\ 42 \\ \underline{40} \\ 21 \\ \underline{20} \\ 1 \end{array}$$

$\therefore 421 = 5(84) + 1$

Reconstructing in base 10:

The divisor: $5 = 1 \cdot 10 - 5$

The dividend: $421 = 4 \cdot 10^2 + 2 \cdot 10 + 1$

$$\begin{array}{r} 4 \cdot 10 + 22 \\ (1 \cdot 10 - 5)\overline{)4 \cdot 10^2 + 2 \cdot 10 + 1} \\ \underline{4 \cdot 10^2 - 20 \cdot 10} \\ 22 \cdot 10 + 1 \\ \underline{22 \cdot 10 - 110} \\ 111 \end{array}$$

$$\begin{aligned} \therefore 4 \cdot 10^2 + 2 \cdot 10 + 1 &= (1 \cdot 10 - 5)(4 \cdot 10 + 22) + 111 \\ &= (5)(62) + 50 + 50 + 10 + 1 \\ &= (5)(62) + 5 \cdot 10 + 5 \cdot 10 + 2 \cdot 5 + 1 \\ &= (5)(62 + 10 + 10 + 2) + 1 \\ &= 5(84) + 1 \end{aligned}$$

Lesson 2 reflection:

Learners were given five new problems to complete and the instructions were given to the learners as follows:

1. Remove the multiples of the divisor from the dividend first.
2. Perform arithmetic long division and formulate the findings in the division algorithm.
3. Decompose the divisor and the dividend as a power relation in base ten, and repeat the long division procedure until the result is the same as indicated in the division algorithm.

The problems that the learners were given were chosen to consist of smaller numbers as dividends, as this had proven to be the most challenging part of the division procedure for the learners to complete.

The problems:

1. $14\overline{)238}$
2. $7\overline{)191}$
3. $13\overline{)471}$
4. $11\overline{)527}$
5. $23\overline{)438}$

The work proved to be very difficult for the learners to do independently at home. The method of removing multiples of the dividend was repeated in two problems in class where the issue of decomposition into multiples of the divisor was discussed thoroughly and thereafter learners engaged with two decompositions. This was done to illustrate to them that all decompositions lead to the same conclusion as summarised in the division algorithm. It became evident, by the work they produced, that this method made more sense to them now.

Hereafter the work progressed toward decomposing the numbers in base 10. The learners could decompose the dividend quite effectively, as it was a mere decomposition of a number that already contained hundreds, tens and units. When the divisor was a single digit number, however, they experienced problems as there was not a multiple of ten in the number, and they had to subtract to decompose the divisor. This took some time for the learners to conceptualise and remembering to always refer to one ten as '*one multiple of ten*' by writing $1\cdot 10$ often eluded the learners. These characteristics were emphasised in class as they would lead to the concept of the coefficient later on in the programme.

Problem 1: $14\overline{)238}$

$$\begin{aligned} a)\quad 238 &= 140+70+28 \\ &= 14\cdot 10+14\cdot 5+14\cdot 2 \\ &= 14(10+5+2) \\ &= 14(17)+0 \end{aligned}$$

$$b)\quad \begin{array}{r} 017 \\ 14\overline{)238} \\ \underline{0} \\ 23 \\ \underline{14} \\ 98 \\ \underline{98} \\ 0 \end{array} \qquad \therefore 238 = 14(17)+0$$

$$c)\quad 14 = 1\cdot 10+4$$

$$238 = 2\cdot 10^2+3\cdot 10+8$$

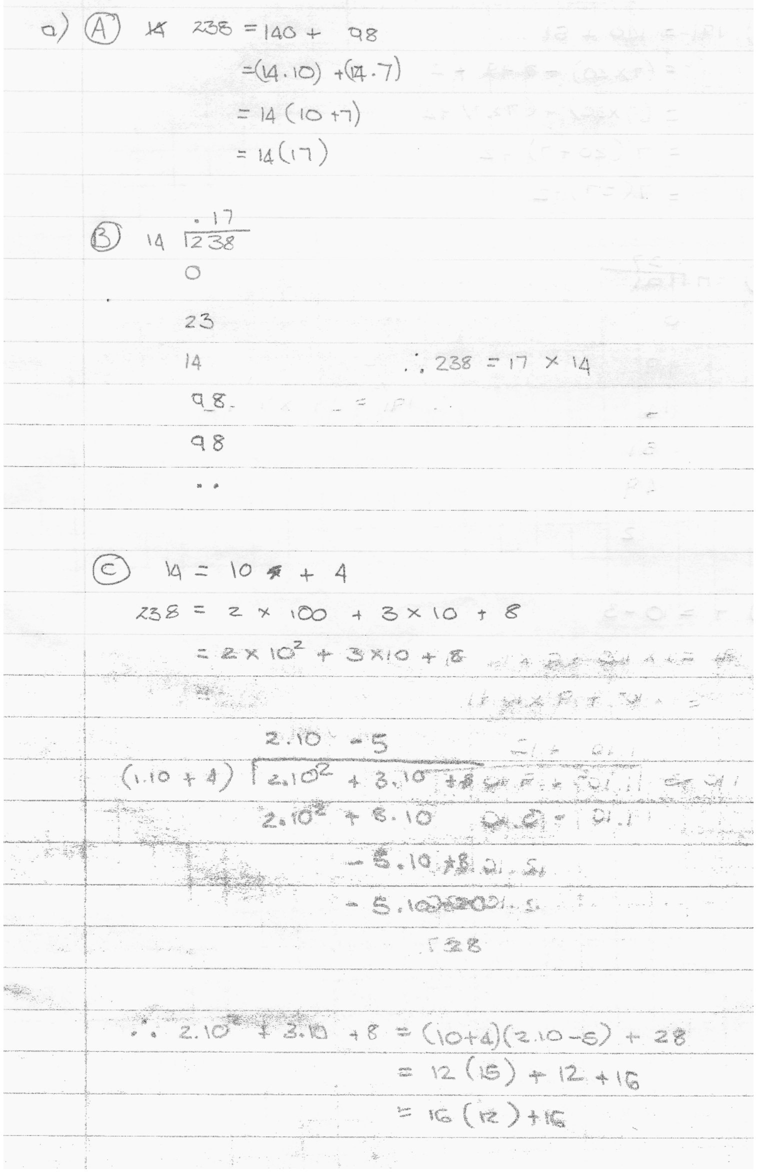
$$\begin{array}{r l} & 2\cdot 10-5 \\ (1\cdot 10+4) & \overline{\big)2\cdot 10^2+3\cdot 10+8} \\ & \underline{2\cdot 10^2+8\cdot 10} \\ & -5\cdot 10+8 \\ & \underline{-5\cdot 10-20} \\ & +28 \end{array}$$

$$\begin{aligned} \therefore 2\cdot 10^2+3\cdot 10+4 &= (1\cdot 10+4)(2\cdot 10-5)+28 \\ &= (14)(15)+14\cdot 2+0 \\ &= 14(15+2)+0 \\ &= 14(17)+0 \end{aligned}$$

In this problem, the divisor was a factor of the dividend which resulted in a zero remainder. Some learners handled this perfectly well in the first and second parts of the instructions, but the moment they decomposed the numbers into power relations in base ten and completed the division, the picture looked different as the remainder was now 28. Five of the learners who did the division in base ten, stopped here and did not complete the last instruction. When this was queried the next day these learners claimed that they forgot how to proceed with the division. Only three learners out of the twenty-two, attempted to complete the division by removing multiples of the divisor from the remainder.

Learner 1:

This learner was one of the stronger learners in the class who always worked very fast and made careless mistakes in the process. She performed parts (a) and (b) without error, but her response to part (c) was faulty. She mistakenly wrote that 10 + 4 is 12, and proceeded to remove multiples of 12 from the remainder at which point she abandoned the problem, most probably since it did not match what she already knew from part (a) and (b).



a) Ⓐ ~~14~~ $238 = 140 + 98$

$= (14.10) + (14.7)$

$= 14(10+7)$

$= 14(17)$

Ⓑ $14 \overline{)238}$ 17

0

23

14

98

98

..

$\therefore 238 = 17 \times 14$

Ⓒ $14 = 10 + 4$

$238 = 2 \times 100 + 3 \times 10 + 8$

$= 2 \times 10^2 + 3 \times 10 + 8$

$2.10 - 5$

$(1.10 + 4) \overline{)2.10^2 + 3.10 + 8}$

$2.10^2 + 8.10$

$-5.10 + 8$

$-5.10 - 20$

28

$\therefore 2.10^2 + 3.10 + 8 = (10+4)(2.10-5) + 28$

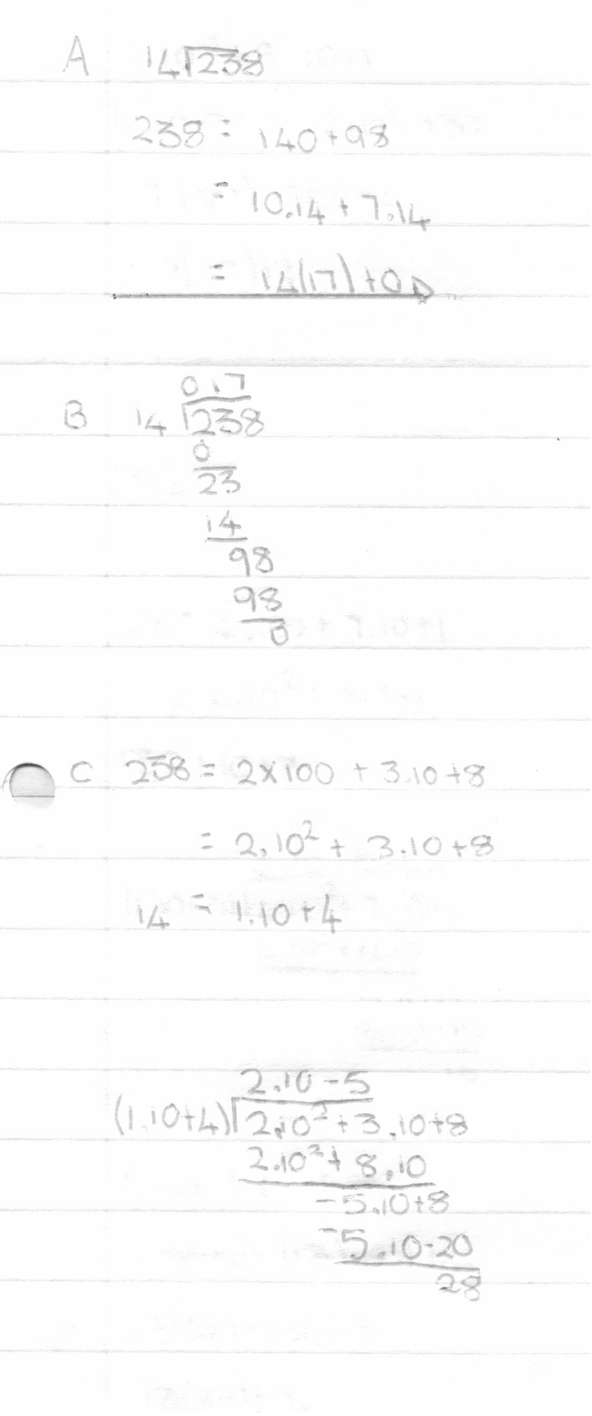
$= 12(15) + 12 + 16$

$= 16(12) + 16$

Learner 2:

Five of the learners could complete each part of the problem as set out in the instructions given by the teacher. In general there was an improvement in part (a) of the problem among all the learners. As the learners were reminded over the previous two lessons to always check whether the answers were the same as what was written in the division algorithm, learners failed to complete the final instruction in full. This happened despite the conflict that arose in class when this difference in remainders appeared for the first time.

This final part of the division procedure is really not a necessary concept to build toward the algebraic division procedure, as this procedure reaches its end the moment the last partial remainder has been attained. It is, therefore, only a useful tool to again point to the fact that all division leads to and ends in the division algorithm.



A $14\overline{)238}$

$238 = 140 + 98$

$= 10.14 + 7.14$

$= 14(17) + 0$

B $\begin{array}{r} 017 \\ 14\overline{)238} \\ \underline{0} \\ 23 \\ \underline{14} \\ 98 \\ \underline{98} \\ 0 \end{array}$

C $238 = 2 \times 100 + 3.10 + 8$

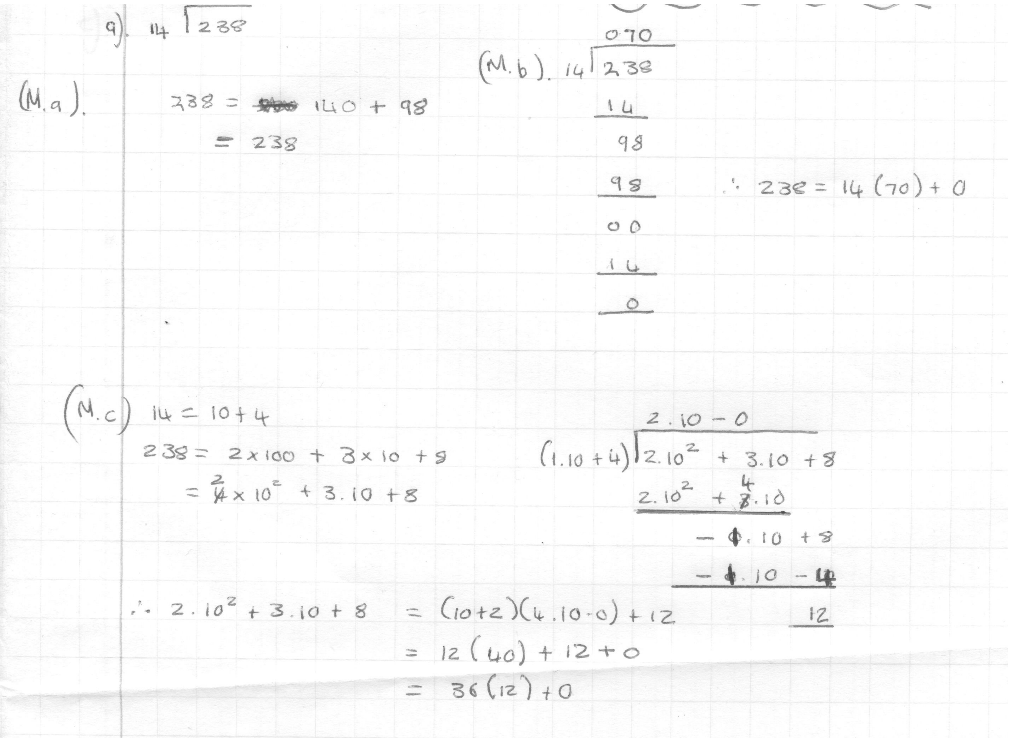
$= 2.10^2 + 3.10 + 8$

$14 = 1.10 + 4$

$\begin{array}{r} 2.10 - 5 \\ (1.10+4)\overline{)2.10^2 + 3.10 + 8} \\ \underline{2.10^2 + 8.10} \\ -5.10 + 8 \\ \underline{-5.10 - 20} \\ 28 \end{array}$

<u>Learner 3</u>:

This learner clearly had a problem with establishing basic multiples and factors of numbers within the context of the long division problem. The learner failed to perform the removal of multiples of the divisor from the dividend in part (a), and merely decomposed the dividend into a sum. He then failed to transfer the 1 to the quotient in the long division. The long division was carried out efficiently, with the only problem being the non-transferral of the 1, yet the one multiple of 14 was removed from the dividend. As a result of this omission, the *division algorithm* does not reflect what really was done in the long division. The phenomenon of not checking whether an answer makes sense or not was again apparent in his work.



9). $14\overline{)238}$

(M.a). $238 = 140 + 98$

$= 238$

(M.b). $14\overline{)238}$

070

14

98

98

00

14

0

$\therefore 238 = 14(70) + 0$

(M.c) $14 = 10 + 4$

$238 = 2 \times 100 + 3 \times 10 + 8$

$= \overset{2}{\cancel{1}} \times 10^2 + 3.10 + 8$

$(1.10 + 4)\overline{)2.10^2 + 3.10 + 8}$

$2.10 - 0$

$2.10^2 + \overset{4}{\cancel{8}}.10$

$-1.10 + 8$

$-1.10 - 4$

12

$\therefore 2.10^2 + 3.10 + 8 = (10+2)(4.10-0) + 12$

$= 12(40) + 12 + 0$

$= 36(12) + 0$

This learner had problems with performing basic products within the procedure. He failed to complete the product $2\cdot10(1\cdot10+4)=2\cdot10^2+8\cdot10$ correctly, and also did not realise that in order to obtain $-1\cdot10+\ldots$, he had to multiply $1\cdot10$ with -1, as he wrote a 0 down in the quotient instead. He did not, however, experience a problem with subtracting in the procedure.

Problem 2: $7\overline{)191}$

$$
\begin{aligned}
a)\quad 191 &= 210-19 \\
&= 7\cdot 30-7\cdot 2-5 \\
&= 7(30-2)-5 \\
&= 7(28)-5 \\
&= 7(27)+7-5 \\
&= 7(27)+2
\end{aligned}
$$

$$
b)\quad \begin{array}{r}
027 \\
7\overline{)191} \\
\underline{0} \\
19 \\
\underline{14} \\
51 \\
\underline{49} \\
2
\end{array}
\qquad \therefore 191 = 7(27)+2
$$

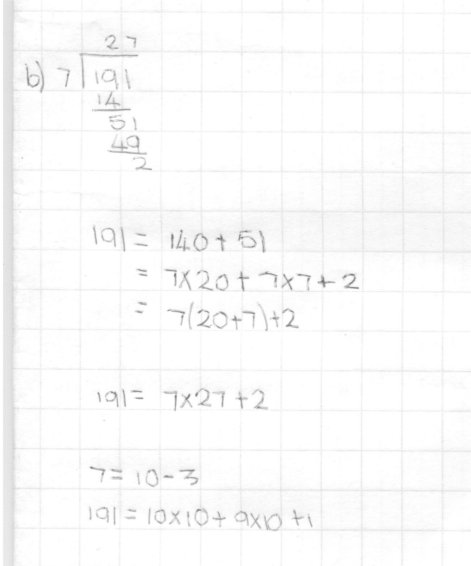
$$
c)\quad 7 = 1\cdot 10-3
$$

$$
191 = 1\cdot 10^2+9\cdot 10+1
$$

$$
\begin{array}{r}
1\cdot 10+12 \\
(1\cdot 10-3)\overline{)1\cdot 10^2+9\cdot 10+1} \\
\underline{1\cdot 10^2-3\cdot 10} \\
+12\cdot 10+1 \\
\underline{+12\cdot 10-36} \\
+37
\end{array}
$$

$$
\begin{aligned}
\therefore 1\cdot 10^2+9\cdot 10+1 &= (1\cdot 10-3)(1\cdot 10+12)+37 \\
&= (7)(22)+37 \\
&= 7(22)+7\cdot 5+2 \\
&= 7(22+5)+2 \\
&= 7(27)+2
\end{aligned}
$$

This question did not have any particular problems, except for the remainder again being different and thus indicating that the division procedure is not completed. The divisor is also expressed as a subtraction from one multiple of ten, instead of as an addition.



$$
b)\quad \begin{array}{r}
27 \\
7\overline{)191} \\
\underline{14} \\
51 \\
\underline{49} \\
2
\end{array}
$$

191 = 140 + 51

= 7X20 + 7X7 + 2

= 7(20+7)+2

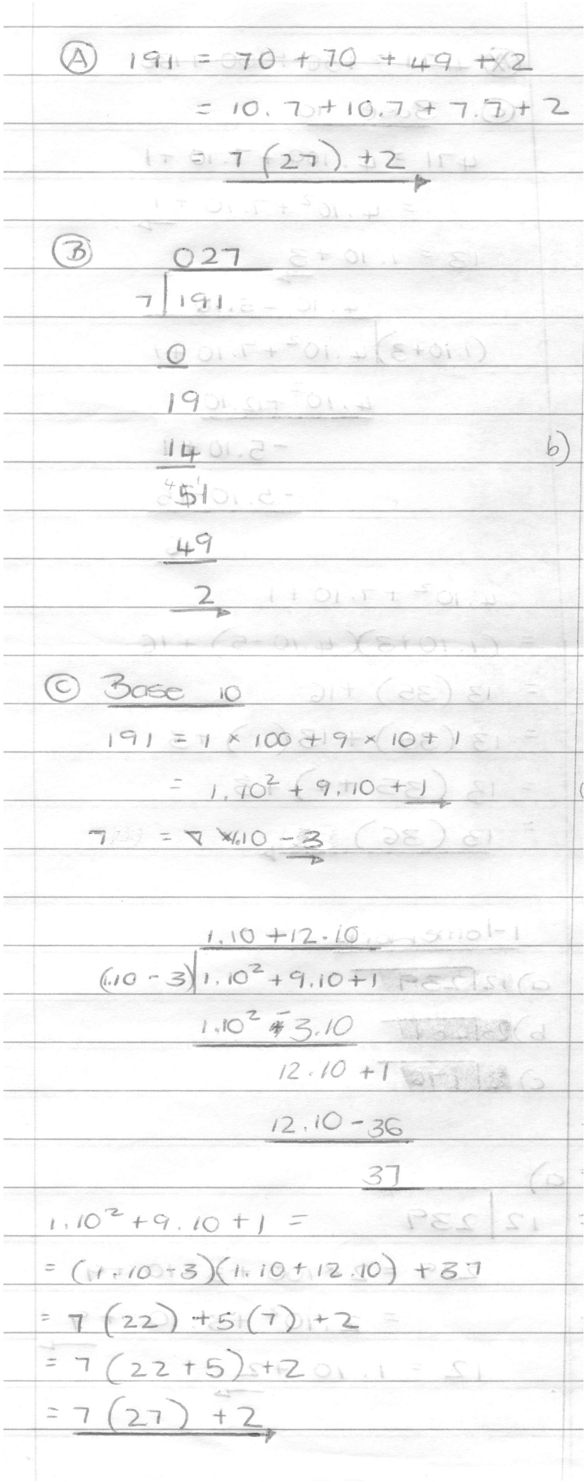
191 = 7X27 + 2

7 = 10 - 3

191 = 10X10 + 9X10 +1

The vast majority of learners performed only the first two parts of each question, as evidenced by the sample of work alongside. They also went ahead and decomposed each dividend and divisor into a base ten relationship. Besides this, no long division was performed using the base ten long division procedure. Only three learners attempted to do the long division in base ten and achieved success in doing so.

Here is an example of two of these learners' work, the one showing the complete calculation and the second featuring only the base ten division:



Ⓐ $191 = 70 + 70 + 49 + 2$

$= 10 \cdot 7 + 10 \cdot 7 + 7 \cdot 7 + 2$

$= 7(27) + 2$

Ⓑ 027

$7 \overline{)191}$

0

19

14

51

49

2

b)

Ⓒ Base 10

$191 = 1 \times 100 + 9 \times 10 + 1$

$= 1 \cdot 10^2 + 9 \cdot 10 + 1$

$7 = 1 \times 10 - 3$

$1 \cdot 10 + 12 \cdot 10$

$(1 \cdot 10 - 3)\overline{)1 \cdot 10^2 + 9 \cdot 10 + 1}$

$1 \cdot 10^2 - 3 \cdot 10$

$12 \cdot 10 + 1$

$12 \cdot 10 - 36$

37

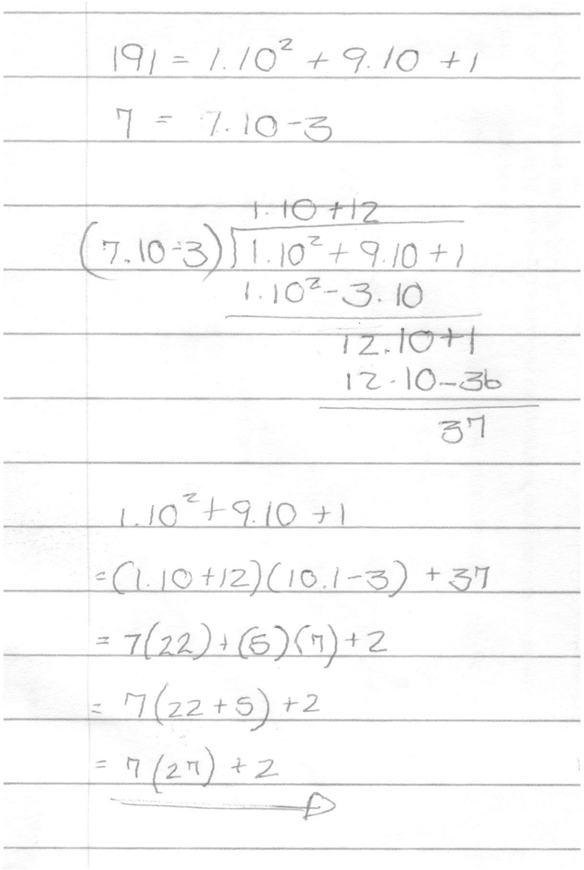
$1 \cdot 10^2 + 9 \cdot 10 + 1 =$

$= (1 \cdot 10 - 3)(1 \cdot 10 + 12 \cdot 10) + 37$

$= 7(22) + 5(7) + 2$

$= 7(22 + 5) + 2$

$= 7(27) + 2$



$191 = 1 \cdot 10^2 + 9 \cdot 10 + 1$

$7 = 7 \cdot 10 - 3$

$1 \cdot 10 + 12$

$(7 \cdot 10 - 3)\overline{)1 \cdot 10^2 + 9 \cdot 10 + 1}$

$1 \cdot 10^2 - 3 \cdot 10$

$12 \cdot 10 + 1$

$12 \cdot 10 - 36$

37

$1 \cdot 10^2 + 9 \cdot 10 + 1$

$= (1 \cdot 10 + 12)(10 \cdot 1 - 3) + 37$

$= 7(22) + (5)(7) + 2$

$= 7(22 + 5) + 2$

$= 7(27) + 2$

At this point it became very clear that the base ten division procedure was something with which the learners were not confident, as they only decomposed the dividend and the divisor in base ten, and then were unable to continue further.

It was decided to discuss each of these homework problems with the learners in the next lesson, focusing mainly on base ten decomposition and long division in base ten. For the remaining part of this chapter, the answers to the homework problems will be provided along with a break down of learners' success in the various stages of the homework exercises.

Problem 3: $13\overline{)471}$

$$\begin{aligned} a)\quad 471 &= 390+81 \\ &= 13\cdot 30+78+3 \\ &= 13\cdot 30+13\cdot 6+3 \\ &= 13(30+6)+3 \\ &= 13(36)+3 \end{aligned}$$

$$b)\quad \begin{array}{r} 036 \\ 13\overline{)471} \\ \underline{0} \\ 47 \\ \underline{39} \\ 81 \\ \underline{78} \\ 3 \end{array} \qquad \therefore 471 = 13(36)+3$$

$$c)\quad 13 = 1\cdot 10+3$$

$$471 = 4\cdot 10^2+7\cdot 10+1$$

$$\begin{array}{rl} & 4\cdot 10-5 \\ (1\cdot 10+3)\overline{)} & 4\cdot 10^2+7\cdot 10+1 \\ & \underline{4\cdot 10^2+12\cdot 10} \\ & -5\cdot 10+1 \\ & \underline{-5\cdot 10-15} \\ & +16 \end{array}$$

$$\begin{aligned} \therefore 4\cdot 10^2+7\cdot 10+1 &= (1\cdot 10+3)(4\cdot 10-5)+16 \\ &= (13)(35)+13\cdot 1+3 \\ &= 13(35+1)+3 \\ &= 13(36)+3 \end{aligned}$$

Problem 4: $11\overline{)527}$

$$\begin{aligned} a)\ 527 &= 440+87 \\ &= 11\cdot 40 + 11\cdot 8 - 1 \\ &= 11(40+8)-1 \\ &= 11(48)-1 \\ &= 11(47)+11-1 \\ &= 11(47)+10 \end{aligned}$$

$$b)\ \begin{array}{r} 047 \\ 11\overline{)527} \\ \underline{0} \\ 52 \\ \underline{44} \\ 87 \\ \underline{77} \\ 10 \end{array} \qquad \therefore 527 = 11(47)+10$$

$$c)\ 11 = 1\cdot 10 + 1$$

$$527 = 5\cdot 10^2 + 2\cdot 10 + 7$$

$$\begin{array}{rl} & 5\cdot 10 - 3 \\ (1\cdot 10+1) & \overline{)5\cdot 10^2 + 2\cdot 10 + 7} \\ & \underline{5\cdot 10^2 + 5\cdot 10} \\ & -3\cdot 10 + 7 \\ & \underline{-3\cdot 10 - 3} \\ & +10 \end{array}$$

$$\begin{aligned} \therefore 5\cdot 10^2 + 2\cdot 10 + 7 &= (1\cdot 10+1)(5\cdot 10-3)+10 \\ &= 11(47)+10 \end{aligned}$$

Problem 5: $23\overline{)438}$

$$\begin{aligned} a)\ 438 &= 230+230-22 \\ &= 23\cdot 10 + 23\cdot 10 - 22 \\ &= 23(10+9)+23-22 \\ &= 23(19)+1 \end{aligned}$$

$$b)\ \begin{array}{r} 019 \\ 23\overline{)438} \\ \underline{0} \\ 43 \\ \underline{23} \\ 208 \\ \underline{207} \\ 1 \end{array} \qquad \therefore 438 = 23(19)+1$$

$$c)\ 23 = 2\cdot 10 + 3$$

$$438 = 4\cdot 10^2 + 3\cdot 10 + 8$$

$$\begin{array}{rl} & 2\cdot 10 - \tfrac{3}{2} \\ (2\cdot 10+3) & \overline{)4\cdot 10^2 + 3\cdot 10 + 8} \\ & \underline{4\cdot 10^2 + 6\cdot 10} \\ & -3\cdot 10 + 8 \\ & \underline{-3\cdot 10 - \tfrac{9}{2}} \\ & +\tfrac{25}{2} \end{array}$$

$$\begin{aligned} \therefore 4\cdot 10^2 + 3\cdot 10 + 8 &= (2\cdot 10+3)\left(2\cdot 10-\tfrac{3}{2}\right)+\tfrac{25}{2} \\ &= (23)\left(\tfrac{40-3}{2}\right)+\tfrac{25}{2} \\ &= 23\left(\tfrac{37}{2}\right)+\tfrac{25}{2} \\ &= \frac{23(37)+25}{2} \\ &= \frac{23(37)+23+2}{2} \\ &= \frac{23(38)+2}{2} \\ &= 23(19)+1 \end{aligned}$$

The results of learners' success with the problems:

		Correct	Partially correct	Incorrect	No attempt
1	a	18	3	1	-
	b	22	-	-	-
	c	3	16	-	3
2	a	19	2	1	-
	b	22	-	-	-
	c	3	16	-	3
3	a	18	3	-	1
	b	22	-	-	-
	c	3	16	-	3
4	a	18	3	-	1
	b	22	-	-	-
	c	3	16	-	3
5	a	18	3	-	1
	b	14	1	-	7
	c	-	-	-	22

***Partially correct in this table indicates the learners who only decomposed the divisor and the dividend in base ten and did not perform the long division in base ten.*

The percentages of learners that could successfully complete each of the sections over all three instructions:

	Correct	Partially correct	Incorrect	No attempt
Removing multiples of the divisor from the dividend	82%	13%	2%	3%
Arithmetic long division and the division algorithm	93%	1%	-	6%
Base ten decomposition and base ten long division	11%	58%	-	31%

Based on these statistics, the focus of the next lesson shifted to only base ten decomposition and the long division procedure in base ten. The remainder and the *division algorithm* for the long division procedure on naturals were de-emphasised, as this equality in the remainder was not necessary for algebraic division procedures.

In summary:

The learners experienced tremendous hurdles when removing multiples of the divisor from the dividend. This division procedure was included in the division ladder as it is needed when division is done in base ten. Here the division procedure will continue beyond the base ten long division as the remainder at times does not resemble the same remainder as what the learners obtained when performing alternative division procedures. This would demand that learners would have to proceed with the division in a manner which focuses on the divisor and the remainder in the *division algorithm* by removing multiples of the divisor from this remainder. This procedure was decided upon as the different remainders were bound to cause serious conflict with the learners. The teacher continuously emphasised the fact that all answers to the different methods meet in the *division algorithm* and that if an answer does not match, learners should revise what they did when they performed the division procedure.

Lesson 3: Reinforcing base ten decomposition

After reflecting on the work that the learners produced for homework, I decided to aim this lesson at assisting them to develop a complete understanding of the process of base ten division. Without a thorough understanding, any attempt at introducing the variable in the division problem would be problematic. The approach of this lesson was to revise the problems that they had completed for homework and explain issues that were still unclear to them. The matters that became important in this lesson were:

- determining the leading term in the quotient and then multiplying this term with the leading term and the 'constant' in the divisor
- the subtraction of a negative term
- the factoring step in the *division algorithm* where multiples of the divisor are removed from the remainder, as the learners had lacked confidence in continuing the division procedure beyond the remainder.

There were two teachers in the class all the time, the one teacher being the researcher and the second teacher the regular mathematics teacher for these learners. Initially, the researcher explained the first homework problem, and then a learner was asked to explain the procedure to her classmates using the second homework question. Finally, the mathematics teacher explained the third homework problem to the learners. They were required to solve each division problem using the three methods that they had been taught thus far, namely: (a) long division, (b) removing multiples of the divisor and (c) base ten long division.

The first problem: 238 divided 14

The complete solution of this problem, which was written on the board, will be given below and only the issues that were pressing at this point will be discussed.

Removing multiples of the divisor :

$$\begin{aligned} 238 &= 140 + 98 \\ &= 10\cdot 14 + 7\cdot 14 \\ &= 14\left(10+7\right)+0 \\ &= 14\left(17\right)+0 \end{aligned}$$

The learners appeared confident with this method and rarely made mistakes. The only mistake that occurred was incorrect addition.

Long Division :

$$\begin{array}{r} 017 \\ 14\overline{)238} \\ \underline{0} \\ 23 \\ \underline{14} \\ 98 \\ \underline{98} \\ 0 \end{array}$$

$$\therefore 238 = 14\left(17\right)+0$$

As this is the method that they used in primary school when they dealt with arithmetic long division, all the learners completed this division successfully. A few learners omitted the zero remainder in the division algorithm. The only variation was a layout difference where learners did not indicate the zero in front of the seventeen in the quotient.

Base 10 reconstruction :

Divisor: $14 = 1\cdot 10 + 4$

Dividend: $238 = 2\cdot 10^2 + 3\cdot 10 + 8$

$$\begin{array}{r} 2\cdot 10 - 5 \\ \left(1\cdot 10+4\right)\overline{)2\cdot 10^2 + 3\cdot 10 + 8} \\ \underline{2\cdot 10^2 + 8\cdot 10} \\ -5\cdot 10 + 8 \\ \underline{-5\cdot 10 - 20} \\ +28 \end{array}$$

$$\begin{aligned} \therefore 2\cdot 10^2 + 3\cdot 10 + 8 &= \left(1\cdot 10+4\right)\left(2\cdot 10-5\right)+28 \\ &= \left(14\right)\left(15\right)+14\cdot 2 \\ &= 14\left(15+2\right)+0 \\ &= 14\left(17\right)+0 \end{aligned}$$

This has proven to be the most difficult part of the division process. It is designed to be more complex than normal arithmetic long division, in order to prepare learners for algebraic long division. One can infer it to be an imitation of algebra, at least resembling the traditional long division procedure.

In the classroom discussion learners had problems with finding the first term in the quotient from the first terms in both the divisor and the dividend. This was explained again using both multiplication and division.

The second problem that emerged was when learners had to subtract a negative number and neglected to change it to a positive quantity. The teacher continued by giving them a procedure for doing this by instructing them to:

“*always change the second sign and add* ”.

After completing the problem on the board, the teacher proceeded to motivate to the learners why this method was important to understand. This was done in response to a learner who protested that it was very difficult. The difficulty seems to arise from the fact that, after the long division had been completed, the division process proceeds in the division algorithm. The teacher explained that, when the variable is brought into these calculations, the division process will terminate once the remainder has been determined in the normal long division. He made the learners aware of the fact that the numbers are known when one performs arithmetic long division, but that once the variable is in place, the numbers will not be known. It is, therefore, possible to get a negative remainder when doing algebraic long division, whereas this is not possible when performing long division on natural numbers that are known.

When the learner was asked to explain the second problem to her classmates, she adopted a more procedural approach by giving her classmates a step by step instruction sequence with which to perform the base ten division.

The second problem: 191 divided by 7

$$\text{Dividend}: 7 = 1\cdot 10 - 3$$

$$\text{Divisor}: 191 = 1\cdot 10^2 + 9\cdot 10 + 1$$

$$\begin{array}{r l}
 & \quad 1\cdot 10+12 \\ \cline{2-2}
(1\cdot 10-3) & \big)\, 1\cdot 10^2+9\cdot 10+1 \\
 & \underline{1\cdot 10^2-3\cdot 10} \\
 & \qquad +12\cdot 10+1 \\
 & \qquad \underline{+12\cdot 10-36} \\
 & \qquad\qquad\quad +37
\end{array}$$

$$\begin{aligned}
\therefore 1\cdot 10^2+9\cdot 10+1 &= (1\cdot 10-3)(1\cdot 10+12)+37 \\
&= 7(22)+35+2 \\
&\left.\begin{aligned} &= 7(22)+5\cdot 7+2 \\ &= 7(22+5)+2 \end{aligned}\right\}** \\
&= 7(27)+2
\end{aligned}$$

The learners responded very positively to this with the only challenge that emerged being the order in which the factors were written in the *division algorithm* (see ** above). One of the learners asked what happened to the second fourteen, to which she responded by swapping the five and the seven around. She then wrote:

$$\begin{aligned}
&7(22)+7\cdot 5+2 \\
&= 7(22+5)+2
\end{aligned}$$

This learner obviously forgot the basic laws of multiplication where $a\cdot b = b\cdot a$ and once both the sevens were written first, it made sense to her. This step by step instructional sequence that the learner gave to her classmates seemed to help those learners who still had a problem with performing this division problem.

Lastly, the mathematics teacher requested that she explain the last homework problem to the learners. She was familiar with the learners' abilities, and focused on the weaker learners in the class. She challenged them to respond to the questions that she formulated while completing the problem.

The third problem: 471 divided by 13

$$\begin{array}{r} 4\cdot 10 - 5 \\ (1\cdot 10+3)\overline{\big)\,4\cdot 10^2+7\cdot 10+1} \\ \ominus\ \underline{4\cdot 10^2+12\cdot 10} \\ -5\cdot 10+1 \\ \ominus\ \underline{-5\cdot 10-15} \\ +16 \end{array}$$

$$\begin{aligned} \therefore 4\cdot 10^2+7\cdot 10+1 &= (1\cdot 10+3)(4\cdot 10-5)+16 \\ &= 13(35)+13(1)+3 \\ &= 13(36)+3 \end{aligned}$$

In order to overcome the problem with the double negative, the mathematics teacher introduced a negative in front of each of the necessary steps. In her classroom instruction she emphasized this negative when the subtraction was done. This was introduced very successfully since this mistake did not emerge this time around. This procedure was carried out with fluency and the learners chorused the steps to the teacher, with her acting as scribe for their instructions.

This mathematics teacher's approach was slightly more procedural than the research teacher's approach. Although key conceptual issues were discussed and pointed out during her classroom instruction, most of the procedures, in the base ten long division, were cued in by the instruction. At this stage of the process it is useful to emphasize the procedure, including the sub- and super-procedures, to the learners. Many of their problems originated when they did not know what to do next. Presenting the long division to them in a procedural manner highlighted the repetitive nature of the division process and also pointed out clearly where pitfalls may arise.

This was the fifth example that was discussed with the learners regarding specifically base ten division and, having summarized it in a procedural manner, this brought closure to this part of the division process.

Lesson 3 reflection:

These problems were chosen carefully to include each remainder type: remainders that are smaller and larger then the divisor and also the possibility of a negative remainder. At the onset of the lesson, the learners requested that the procedure with the continual division of remainder be explained again by discussing the previous day's problems. This lesson was the final focus on the remainder in terms of the division of naturals continuing beyond the division algorithm. The learners generally performed better in the base ten division after this lesson. The first problem was solved by learners during the class period, while the teachers assisted individual learners where necessary.

Problem 1: $12\overline{)239}$

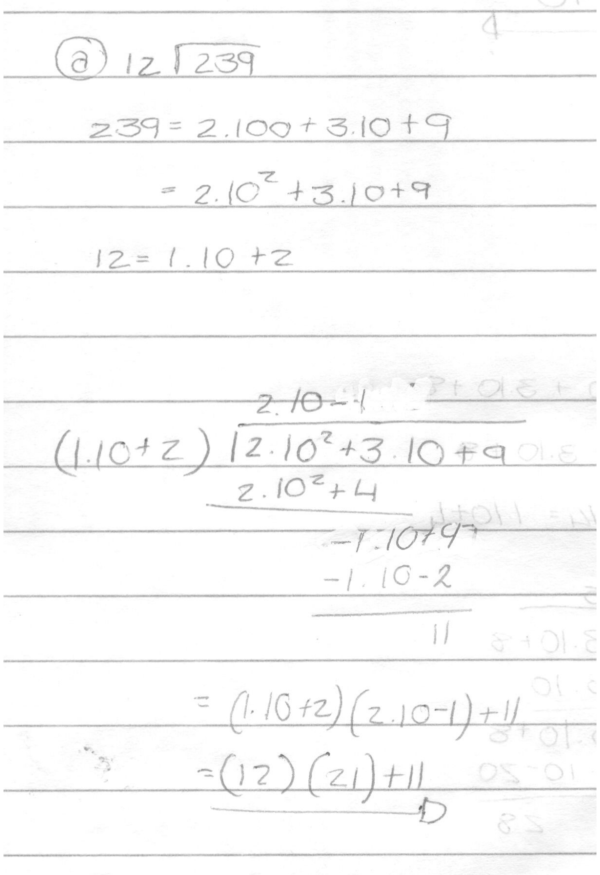
$$\begin{aligned} 12 &= 1\cdot 10 + 2 \\ 239 &= 2\cdot 10^2 + 3\cdot 10 + 9 \end{aligned}$$

$$\begin{array}{r} 2\cdot 10 - 1 \\ (1\cdot 10 + 2)\overline{)2\cdot 10^2 + 3\cdot 10 + 9} \\ \underline{2\cdot 10^2 + 4\cdot 10} \quad\quad \\ -1\cdot 10 + 9 \\ \underline{-1\cdot 10 - 2} \\ +11 \end{array}$$

$$\begin{aligned} \therefore 2\cdot 10^2 + 3\cdot 10 + 9 &= (1\cdot 10 + 2)(2\cdot 10 - 1) + 11 \\ &= (12)(19) + 11 \end{aligned}$$

The learners did not experience any major difficulties with this question. The problem areas will be discussed over the following pages. There were a few learners who made careless mistakes, yet the application of the base ten division procedure was faultless in these instances. In total, 87% of the learners performed this procedure successfully.

All but one of the learners proceeded with the decomposition of both the divisor and the dividend without making any mistakes. Alongside is an example of a learner's work where careless mistakes were made when formulating the procedure in the division algorithm. Three other learners made the same error when they calculated the value of the quotient, which is indicated correctly to be $2 \cdot 10 - 1$. They, however, solved this as $20 + 1$, leading them to arrive at a value of 21. It was later observed that these three learners worked together on the problem in class and did not notice their error while doing the problem.



ⓐ $12 \overline{)239}$

$239 = 2.100 + 3.10 + 9$

$= 2.10^2 + 3.10 + 9$

$12 = 1.10 + 2$

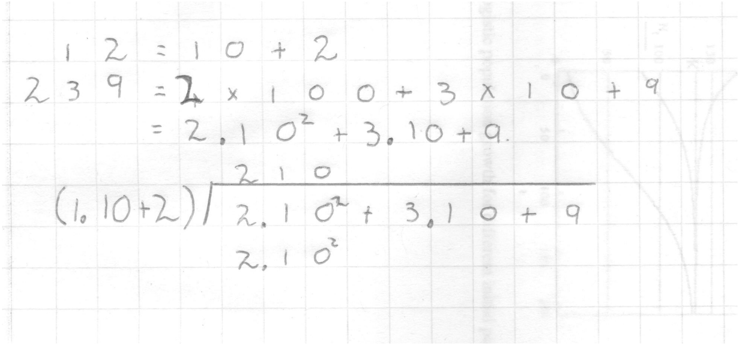
$$\begin{array}{r} 2.10 - 1 \\ (1.10+2)\overline{)2.10^2 + 3.10 + 9} \\ 2.10^2 + 4 \\ \hline -1.10 + 9 \\ -1.10 - 2 \\ \hline 11 \end{array}$$

$= (1.10+2)(2.10-1) + 11$

$= (12)(21) + 11$

The base ten division procedure was performed by the majority of learners without any major problems. This improvement was clear when learners were asked to exchange their work with a partner to assess one another's work. Of the twenty-two students in the class there were only three errors that were picked up by the learners when checking one another's work. In order to establish whether an answer was indeed correct, the class was asked the value of the remainder to which they responded eleven. In the event of a learner not obtaining eleven as a remainder, it was a clear indication to them that they needed to revisit their work and search for a mistake in the calculations.

One of the learners could perform the decomposition of the divisor and the dividend in base ten without any mistake, but the learner experienced great difficulty distinguishing the first term in the quotient from the first term in the divisor and the dividend. The work was explained to this learner in two ways: the first method entailed asking what needed to be done to the first term in the divisor to get to the first term in the dividend, using multiplication, and the second method was explained by performing a division between the first term in the dividend and the first term in the divisor. The learners (below) could not progress beyond the first operation in the division procedure.



$12 = 10 + 2$

$239 = 2 \times 100 + 3 \times 10 + 9$

$= 2.10^2 + 3.10 + 9.$

$$\begin{array}{r l} & 2\,1\,0 \\ (1.10+2) \,\big) & \overline{2.10^2 + 3.10 + 9} \\ & 2.10^2 \end{array}$$

What follows is a transcript of that explanation:

Teacher: How did you find the $2 \cdot 10$ at the top? (*pointing at the quotient*)

Learner: I divided $2 \cdot 10^2$ by $1 \cdot 10$.

Teacher: Show me what you did.

Learner: *Writing the following*:

$$\frac{2 \cdot 10^2}{1 \cdot 10} = \frac{2 \cdot \cancel{10} \cdot 10}{1 \cdot \cancel{10}} = \frac{2 \cdot 10}{1}$$

and that is $2 \cdot 10$ because if you divide by one, it stays what it was.

Teacher: Now after that, what did you do? Where does this $2 \cdot 10^2$ come from?

Learner: It always has to be the same … that's what you told us.

Teacher: Why does it have to be the same? Is it always going to be the same?

Learner: I don't know … you multiply? (*pointing to the quotient and the divisor*) But I don't know how to get … that one. (*pointing to the term left blank*)

Teacher: Okay, let's look at it in terms of multiplication. What must I multiply $1 \cdot 10$ with to get $2 \cdot 10^2$?

No response. The teacher reworded the question and wrote down the following:

Let's break up the product: $(2 \cdot 10 \cdot 10) = (.......)(1 \cdot 10)$. What must I multiply the one with to get two?

Learner: Times by two.

Teacher: … and what do I do to ten to get ten squared?

Learner: Times by ten … I know that, I just don't know what to do to get that term.

Teacher: Hang on … If you decided to multiply $(1 \cdot 10)$ by $(2 \cdot 10)$ to get to the $2 \cdot 10^2$, where does the $(1 \cdot 10)$ belong? Is it part of the divisor?

Learner: It comes from there … the divisor.

Teacher: Is $(1 \cdot 10)$, in other words, the divisor ten?

Learner: No it's twelve … it's 10 + 2.

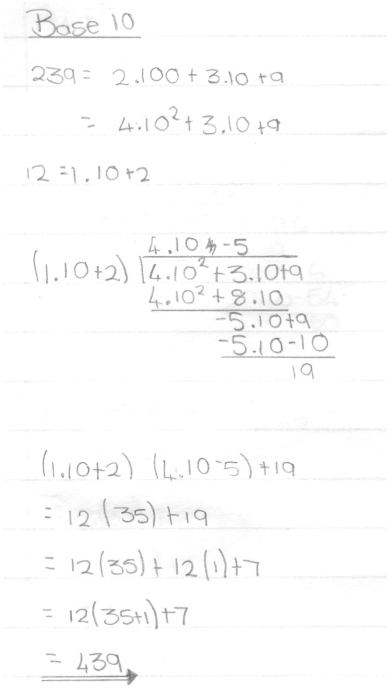
Teacher: So if you multiply back to write down the next step, must you also multiply the two?

Learner: You ignore the two.

Teacher: But then you would be dividing by ten and the divisor is twelve.

Learner: Oh, yes … (*realising what must be done*)

Teacher: What we are saying is that your divisor is twelve which you broke up into 10 + 2, so let's see … what is $(1 \cdot 10 + 2)(2 \cdot 10)$? *The learner continued to the correct product of* $2 \cdot 10^2 + 4 \cdot 10$.



Base 10

$239 = 2 . 100 + 3 . 10 + 9$

$= 4 . 10^2 + 3 . 10 + 9$

$12 = 1 . 10 + 2$

$$\begin{array}{r|l} & 4 . 10 \cancel{4} - 5 \\ \hline (1 . 10 + 2) & 4 . 10^2 + 3 . 10 + 9 \\ & \underline{4 . 10^2 + 8 . 10} \\ & -5 . 10 + 9 \\ & \underline{-5 . 10 - 10} \\ & 19 \end{array}$$

$(1 . 10 + 2)\ (4 . 10 - 5) + 19$

$= 12\ (35) + 19$

$= 12 (35) + 12 (1) + 7$

$= 12 (35 + 1) + 7$

$= 439$

The learner, whose work is reproduced alongside, incorrectly decomposed the dividend 239 into $4 \cdot 10^2 + 9 \cdot 10 + 3$ which is the decomposition for 439. She continued to do the long division in base ten without mistakes and even continued to remove multiples of the divisor (12) from the remainder which was 19. This mistake was queried by the mathematics teacher. It appears that the error was the result of negligence.

Problem 2: $23\overline{)481}$

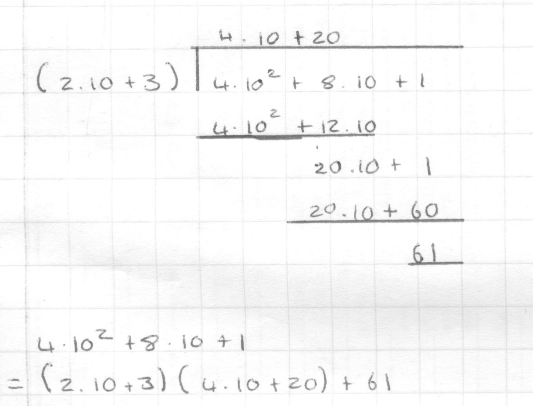
$$23 = 2\cdot10+3$$

$$481 = 4\cdot10^2+8\cdot10+1$$

$$\begin{array}{r} \overline{2\cdot10+1} \\ (2\cdot10+3)\big)4\cdot10^2+8\cdot10+1 \\ \underline{4\cdot10^2+6\cdot10} \\ +2\cdot10+1 \\ \underline{+2\cdot10+3} \\ -2 \end{array}$$

$$\begin{aligned} \therefore 4\cdot10^2+8\cdot10+1 &= (2\cdot10+3)(2\cdot10+1)-2 \\ &= (23)(21)-2 \\ &= 23(20)+23-2 \\ &= 23(20)+21 \end{aligned}$$

In this problem, the divisor was chosen as a number larger than twenty which would divide into the first two digits of the dividend, thus avoiding the formation of fractions. The second issue with which the learners would have to engage was the negative remainder. This had been explained to them as 'having gone past the number' where multiples of the divisor were removed from the dividend. Another way of looking at this in base ten long division is by considering it as 'over division', indicating that the division has removed more multiples of the divisor than was necessary. This time round 87% of the learners were successful with the base ten division process.

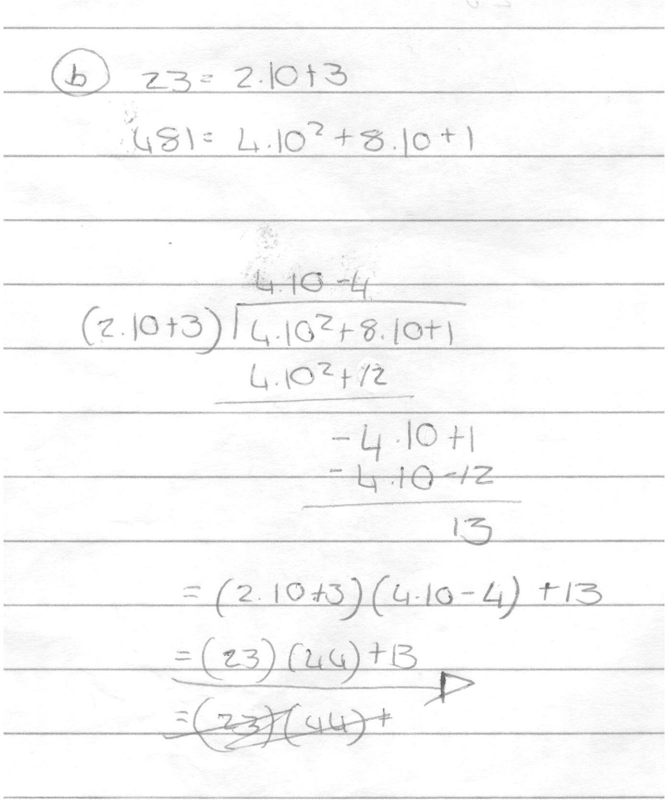


$$\begin{array}{r} \overline{4\cdot10+20} \\ (2\cdot10+3)\big|\,4\cdot10^2+8\cdot10+1 \\ \underline{4\cdot10^2+12\cdot10} \\ 20\cdot10+1 \\ \underline{20\cdot10+60} \\ \underline{61} \end{array}$$

$$4\cdot10^2+8\cdot10+1$$
$$=(2\cdot10+3)(4\cdot10+20)+61$$

The main problem occurred with the divisor being $2\cdot10+3$ since some learners still used $1\cdot10$ to find the first term in the quotient. Their first term in the quotient read $4\cdot10$ and they multiplied this back as $4\cdot10^2+12\cdot10$, thus ignoring the 2 in the divisor. Only two of the learners did this in their work. This mistake is indicated alongside. This learner also neglected to change the sign when subtracting and added instead, a mistake that is profoundly detrimental in the long division procedure. Division can be seen as repeated subtraction of the multiples of the divisor from the dividend. What might have created this conflict was the instruction from the teacher to *'change the sign and add'* when learners are supposed to subtract.

The second learner that made the same mistake with the quotient performed the division in base procedure without difficulty (see alongside), despite having omitted to indicate the $12 \cdot 10$ in the second line. This mistake results from carelessness, but could potentially create problems later on. Learners need to see the interconnectedness of the terms in the divisor and the dividend as well as the interconnection between the divisor, dividend and the quotient. The two learners who made this mistake paid no attention to these links when they performed the division.



ⓑ $23 = 2\cdot10+3$

$481 = 4\cdot10^2+8\cdot10+1$

$4\cdot10-4$

$(2\cdot10+3)\overline{)4\cdot10^2+8\cdot10+1}$

$\underline{4\cdot10^2+12}$

$-4\cdot10+1$

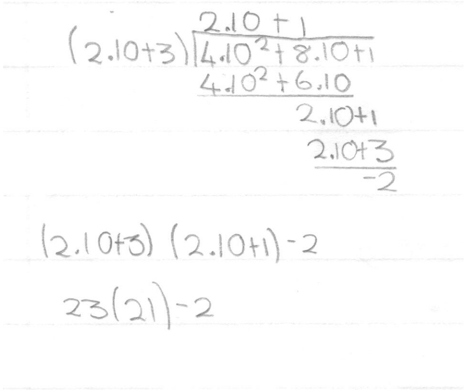
$\underline{-4\cdot10-12}$

13

$=(2\cdot10+3)(4\cdot10-4)+13$

$=(23)(44)+13$

~~$=(23)(44)+$~~



$2\cdot10+1$

$(2\cdot10+3)\overline{)4\cdot10^2+8\cdot10+1}$

$\underline{4\cdot10^2+6\cdot10}$

$2\cdot10+1$

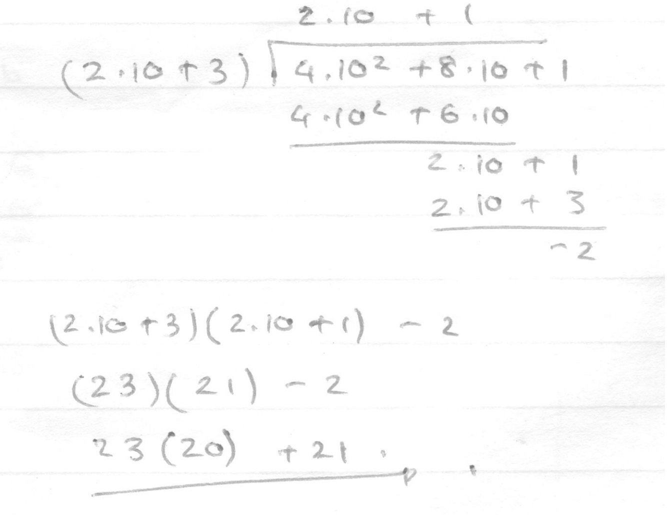
$\underline{2\cdot10+3}$

-2

$(2\cdot10+3)(2\cdot10+1)-2$

$23(21)-2$

The problem alongside is an example of the procedure that most learners followed when they performed the long division in base ten. Five of the learners ended their division at the negative remainder and fourteen proceeded to 'give back' one multiple of the divisor in order to make the remainder positive.

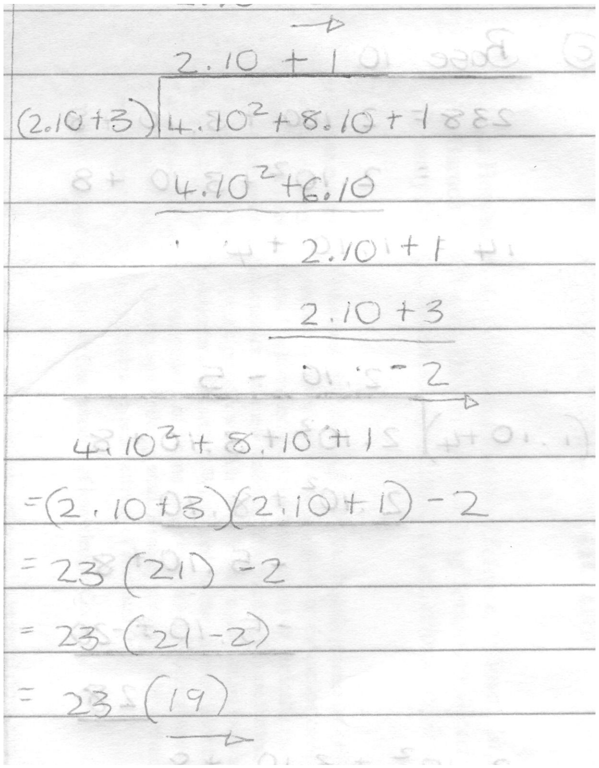


$$2.10 + 1$$
$$(2.10+3)\overline{)\,4.10^2 + 8.10 + 1}$$
$$\underline{4.10^2 + 6.10}$$
$$2.10 + 1$$
$$\underline{2.10 + 3}$$
$$-2$$

$$(2.10+3)(2.10+1) - 2$$
$$(23)(21) - 2$$
$$\underline{23(20) + 21}$$

The calculation alongside represents the work of one of the fourteen learners who proceeded with the division by removing multiples of the divisor from the remainder.

Some of the learners might not have attended to this problem of making the remainder positive, since the teacher revealed to the learners, at the end of the third lesson, that the remainders in each of the problems were 11, -2 and 24 respectively. This disclosure was made during the lesson as some of the learners had completed the work in class and wanted to know what the remainders were in order to check their answers.



$$2.10 + 1$$
$$(2.10+3)\overline{)\,4.10^2+8.10+1}$$
$$\underline{4.10^2+6.10}$$
$$2.10+1$$
$$\underline{2.10+3}$$
$$-2$$
$$4.10^3+8.10+1$$
$$=(2.10+3)(2.10+1)-2$$
$$=23(21)-2$$
$$=23(21-2)$$
$$=23(19)$$

The next learner completed the division with no mistakes but failed in her attempt to make the remainder positive. She included the -2, which is the remainder, in the same bracket as the 21, which is the quotient. This violation of the basic laws of multiplication and factorization could lead to serious misconceptions when the *division algorithm* is extended to algebraic representation. This learner was reminded that each of the numbers in the *division algorithm* represent a specific number that was formed during the long division procedure. The remainder represents the amount that was ‘left over’ after the maximum number of multiples of the divisor was removed from the dividend and can, therefore, not be drawn into the value of the quotient. The teacher proceeded to explain the mistake in a mathematical manner by making reference to the distributive law. It is important that learners are reminded of this sub-procedure when dealing with the division algorithm. Learners’ inability to perform a basic multiplication problem such as

$$\begin{aligned}
&9\cdot 10(1\cdot 10+5)-2 \\
&=9\cdot 10\cdot 1\cdot 10+9\cdot 10\cdot 5-2 && \rightarrow a(b+c)+d=ab+ac+d \\
&=9\cdot 10^2+45\cdot 10-2 && \rightarrow a\cdot b\cdot a\cdot d=a^2bd
\end{aligned}$$

could, in all probability, be the reason for their inability to perform the division procedure faultlessly. As we have seen previously, the shared understanding regarding a common factor can skew a learner’s application of the basic factoring methods.

Problem 3: $8\overline{)176}$

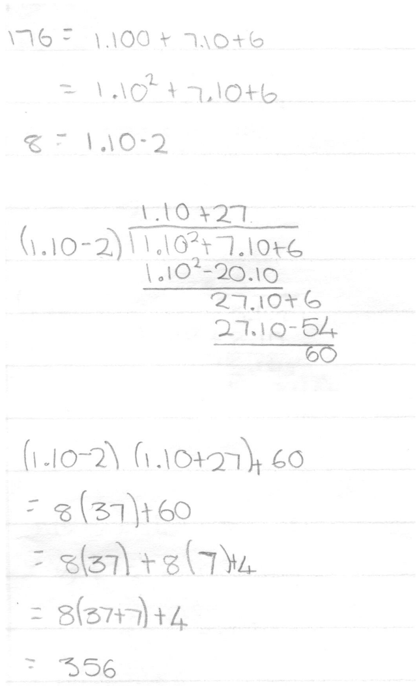
$$\begin{aligned}
8&=1\cdot 10-2 \\
176&=1\cdot 10^2+7\cdot 10+6
\end{aligned}$$

$$\begin{array}{rr}
 & 1\cdot 10+9 \\
(1\cdot 10-2) & \overline{)1\cdot 10^2+7\cdot 10+6} \\
 & \underline{1\cdot 10^2-2\cdot 10}\qquad\quad \\
 & +9\cdot 10+6 \\
 & \underline{+9\cdot 10-18} \\
 & +24
\end{array}$$

$$\begin{aligned}
\therefore 1\cdot 10^2+7\cdot 10+6&=(1\cdot 10-2)(1\cdot 10+9)+24 \\
&=(8)(19)+24 \\
&=8(19)+8(3) \\
&=8(19+3)+0 \\
&=8(22)+0
\end{aligned}$$

Here the division in base ten leaves a remainder of 24, despite the fact that 176 is a multiple of 8, or that 8 is a factor of 176 and the remainder thus has to be zero. Learners should be able to recognise this only when they formulate the *division algorithm* for the particular division problem.

82% of the learners completed this problem correctly. Again those mistakes that learners made can be attributed to carelessness. For example, two learners concluded respectively that 10 – 2 = 12 and that $24 = 4 \times 8$.



$176 = 1.100 + 7.10 + 6$

$= 1.10^2 + 7.10 + 6$

$8 = 1.10 - 2$

$1.10 + 27$

$(1.10-2)\overline{)1.10^2 + 7.10 + 6}$

$1.10^2 - 20.10$

$27.10 + 6$

$27.10 - 54$

60

$(1.10-2)(1.10+27) + 60$

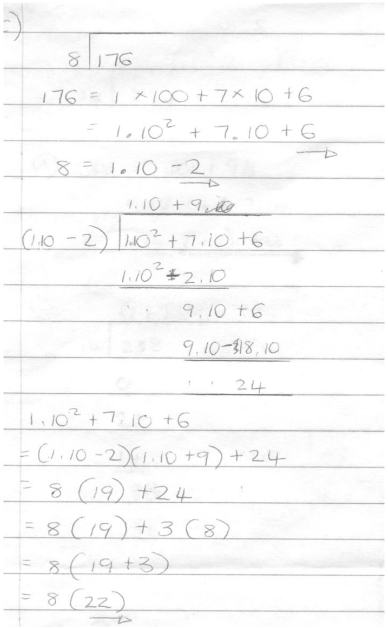
$= 8(37) + 60$

$= 8(37) + 8(7) + 4$

$= 8(37+7) + 4$

$= 356$

A third learner (alongside) stated that $-2(1 \cdot 10) = -20 \cdot 10$, despite having performed the base ten divisions in the previous two problems proficiently. This is the same learner who changed the number 238 to 438 earlier in problem 1.



c)

$8 \,|\, 176$

$176 = 1 \times 100 + 7 \times 10 + 6$

$= 1.10^2 + 7.10 + 6$

$8 = 1.10 - 2$

$1.10 + 9.\cancel{10}$

$(1.10 - 2) \,|\, 1.10^2 + 7.10 + 6$

$1.10^2 \cancel{+} 2.10$

$9.10 + 6$

$9.10 - \cancel{3}18.10$

24

$1.10^2 + 7.10 + 6$

$= (1.10 - 2)(1.10 + 9) + 24$

$= 8(19) + 24$

$= 8(19) + 3(8)$

$= 8(19 + 3)$

$= 8(22)$

Alongside is an example of the work that the majority of learners, (13 of the 22), produced. The remaining 9 learners completed the division but did not continue to remove multiples of the divisor from the dividend.

As this is the final step in the arithmetic division ladder, it is important that the learners establish a clear understanding of the base ten decomposition procedures and how division is performed in base ten.

In summary:

A better approach would definitely have been to allow learners time in class to engage with the new concepts before giving any homework assignments. Those learners who did not understand fully the concepts that they had learned in lesson two found the time allocated to them in class for working through the new material very beneficial. It afforded them the opportunity to engage with the teachers and with one another and ask questions regarding the difficulties that they experienced.

The multiplication of the partial quotient with the whole divisor should be made explicit to the learners through an in-depth analysis of the reasoning behind this product. The teacher should have avoided drawing the learners' attention to the fact that in determining the first term in the quotient, one 'ignores' the constant in the divisor, without pointing out to the learners that this constant cannot be ignored when multiplying back. By this means, the teacher made the learners aware of the instances where only the multiple of ten in the divisor is used, and when it is necessary to use both terms.

In lesson two, conflict emerged regarding the new remainders that differ from the remainders which were obtained by removing multiples of the divisor from the dividend as opposed to those remainders which were derived from doing arithmetic long division and the *reset method*. The teacher was fully aware that those remainders would be different and continued the division process beyond the long division by focusing on the remainder in the division algorithm. This conflict was overcome in the third lesson, but should possibly have been clarified explicitly in the second lesson. The teacher did indicate, in the second lesson, that the division continues because we are dividing naturals and not unknown quantities where there are variables included.

The different remainders at first seemed not to make sense to a large group of the learners, and the issues with remainder in general possibly needs revision regarding its appropriateness in the sequence of the division ladder.

Lesson 4: Detaching coefficients

In this lesson the transition to algebra and algebraic representation was introduced to the learners. The learners were familiarized with the concept of a variable *x* by decomposing the numbers in base ten and then substituting the ten with an *x*. Using this method, they formed polynomials in *x* that had to be divided into one another. Two options were given to the learners, that of doing algebraic long division with *x* present and the second option of working only with the coefficients of the *x*'s in the polynomial. The latter is referred to as the method of *detached coefficient* division. In this lesson similarities were highlighted between the two methods in order for the learners to decide which one of the methods they wanted to use when they performed any algebraic division on polynomials.

The lessons started by looking at dividing 298 by 13. This was approached using only base ten division before algebraic notation was introduced.

The problem:

The following two problems were put on the board next to one another. Below is a transcription of the discussion that took place.

$$13\overline{)298} \rightarrow$$

Divisor: $1\cdot 10+3$

Dividend: $2\cdot 10^2+9\cdot 10+8$

The division:

$$\begin{array}{r} 2\cdot 10+3 \\ (1\cdot 10+3)\overline{)2\cdot 10^2+9\cdot 10+8} \\ \underline{2\cdot 10^2+6\cdot 10} \\ +3\cdot 10+8 \\ \underline{+3\cdot 10+9} \\ -1 \end{array}$$

See *

See **

Without the tens:

$$\begin{array}{r} 2\ \ +\ 3 \\ (1+3)\overline{)2\ \ +9\ \ +8} \\ \underline{2\ \ +6} \\ +3\ \ +8 \\ \underline{+3\ \ +9} \\ -1 \end{array}$$

See ***

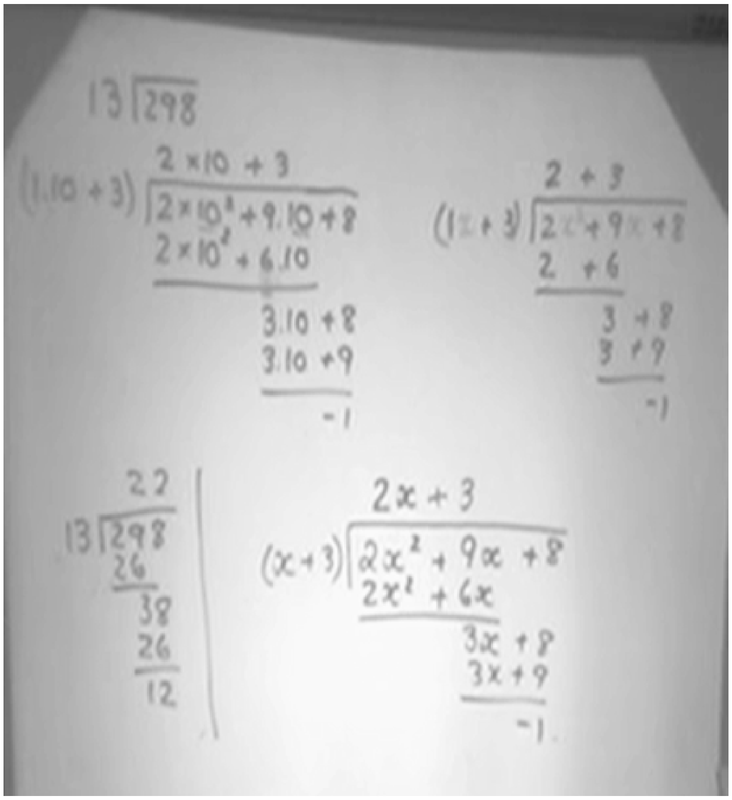


Figure 9

* The two forms of this question were written on the board next to one another. The traditional division was done in base ten first. Then the teacher introduced the very same problem but this time focused the learners' attention on the fact that the tens had been removed. The learners were shown that all the coefficients of ten were still present. The discussion was as follows:

Teacher: I am now going to ignore the tens in this problem. If I covered the tens, and only wrote down the rest of the numbers, then I can write … (writing *down the problem as indicated on the right hand side in Figure 9*). Can you see that the two are very similar? The right hand one just does not have the powers of ten indicated. This is because the tens are unnecessary as the only purpose that they serve is to indicate to us the actual value of the digits that we are dividing … It's called place value. So the tens reminded you all the time that you were working with hundreds, tens and units. All along these tens could've been anything. They could've been twos, nines or even fives … anything …

L1: Even an *x*?

Teacher: Yes, we could have substituted anything in the place of the ten … even an *x*. Now that is what you'll see today. Remember I told you guys three days ago that this is what's coming … The *x* …

** This division procedure was done faultlessly by the learners, with the teacher merely acting as scribe in the situation. The mistakes that learners had made over the past two days seemed to have been overcome as they performed this procedure without error.

*** The teacher continued to do the same division problem, asking the same questions while performing the procedure and merely transcribing the learners' responses. What follows is the transcription of this discussion:

Teacher: Let's see how the base ten division process links up with what I have written down here … (*Pointing to the problem as written on the right hand side of Figure 9*). Can you see that I have taken out the tens?

L1: One times two…

Teacher: (*To the group*) What must I do to one to get to two?

Class: Times by two.

Teacher: Okay then … multiply.

Class: Two plus six …

Teacher: Now subtract for me.

Class: Three …

Teacher: Okay, let's bring down the positive eight … now …

Class: Times by three …

Teacher: Positive or negative?

Class: Positive …

Teacher: Multiply … three and positive nine … okay now?

Class: Minus one

L2: Wow!

L3: Could that now become the x ? *(Referring to the empty spaces between the numbers)*

Teacher: That could be anything; it's the power relationship that is important here.

L1: This is *so* easy …

Teacher: (*To* L1) Do you find it easy? Who finds it easier without the tens? (*The whole class by indication of hands*). What made it difficult? Let's talk about it …

L1: The tens are confusing us … (S*ome learners indicate their agreement.*)

Teacher: Why do you think the tens confused you?

L4: (*Immediately*) Too many numbers …

L1: It doesn't really *(unclear*) me… I get it right, but that is *much* easier.

Teacher: I call it *noise*. There is a lot of *noise* is in this one … *(Referring to the base 10 problem*).

It is important to note at this point that the division procedure was terminated the moment the remainder was obtained. This was done deliberately as this remainder in algebraic long division does not divide beyond this point. Judging by the learners' reaction to this type of division, the teacher believed they were ready to start with algebraic long division.

Introducing the *x*:

Starting off with the arithmetic long division problem, the teacher introduced the variable alongside the arithmetic division problem.

Arithmetic long division:	**Algebraic equivalent:**
$\begin{array}{r} 22 \\ 13\overline{)298} \\ \underline{26} \\ 38 \\ \underline{26} \\ 12 \end{array}$	$(x+3)\overline{)2x^2+9x+8}$

Figure 10

One of the learners responded negatively to the introduction of the x in the algebraic version and this prompted the teacher to return to the problem on the right hand side in Figure 9. Using a different colour pen, the teacher introduced the x in the very same place as in Figure 10, as shown alongside. The teacher pointed out to the learners that what was written in the two representations only differed by the presence of the x in the latter representation. It was important to ensure that the learners were comfortable with the x.

$$\begin{array}{rl} & \;\;2 \quad + \;\; 3 \\ (1\boldsymbol{x}+3) & \overline{\big)\,2\boldsymbol{x}^2+9\boldsymbol{x}+8} \\ & \;\;\underline{2 \quad +6} \\ & \qquad +3 \quad +8 \\ & \qquad \underline{+3 \quad +9} \\ & \qquad\qquad -1 \end{array}$$

Teacher: I can see some eyes widening! But what is written over here *(Figure 9 RHS*) is exactly what is written over here *(Figure 10 RHS*). The first one is just without x.

At this point the teacher introduced the x with a different colour pen as indicated above.

It just appears *ugly* with the x's … Just as you guys felt that it was ugly with the tens in the base ten division. There is too much … But part of that *too much* is irrelevant … I can ignore it. We did not have the x's there when we performed this division (*pointing to Figure 9 RHS*). There were no x's there when we divided, yet when I put in the x's now, some people looked quite shocked … Let's continue.

L1: Don't you go x times $2x$ equals the $2x^2$

Teacher: Why?

L1: Because if you times x it becomes the number… (*unclear*)

Teacher: Did you guys hear what Ll suggested? He's saying if you times x by $2x$ you get to $2x^2$. Just as we said that if we times one ten by two tens, we get two-ten-squared (*pointing to Figure 9 LHS*). Remember, the ten is now just an x.

Writing:
$$\begin{array}{rl} & \;2x \\ (x+3) & \overline{\big)\,2x^2+9x+8} \end{array}$$

Now let's multiply…we get … $2x^2+6x$ (*learners first omitted the x in* 6*x)*
Writing:

$$\begin{array}{rl} & \;2x \\ (x+3) & \overline{\big)\,2x^2+9x+8} \\ & \;\underline{2x^2+6x} \end{array}$$

L1: Because there are two x's … (*Referring to the* x^2)

Teacher: Let's subtract …

Class: Three x …

Teacher: Exactly the same answer that we got when we did the base ten subtraction, but ten is just an x. Bring down the eight … What must be done to x to get $3x$?

Writing:

$$\begin{array}{rl} & \quad 2x \\ (x+3) & \overline{)\,2x^2+9x+8} \\ & \quad \underline{2x^2+6x} \\ & \qquad\quad +3x+8 \end{array}$$

Class: Three … multiply by three.

Teacher: We get …

Class: Three x plus nine.

Writing:

$$\begin{array}{rl} & \quad 2x+3 \\ (x+3) & \overline{)\,2x^2+9x+8} \\ & \quad \underline{2x^2+6x} \\ & \qquad\quad +3x+8 \\ & \qquad\quad \underline{+3x+9} \end{array}$$

Teacher: Subtract … you get minus one (*class*). Can you guys see that we get exactly the same remainder for all three of the calculations?

Writing:

$$\begin{array}{rl} & \quad 2x+3 \\ (x+3) & \overline{)\,2x^2+9x+8} \\ & \quad \underline{2x^2+6x} \\ & \qquad\quad +3x+8 \\ & \qquad\quad \underline{+3x+9} \\ & \qquad\qquad\quad -1 \end{array}$$

Naming the methods:

In the light of the learners' positive response to the division method that was performed by detaching the coefficients, the teacher decided to explore the concept of coefficient with the learners for future reference. These learners had already engaged with algebra since they were in Grade 8, and it was not necessary to discuss this concept in depth as they already knew what was meant by the term coefficient. For the purposes of the lesson, the teacher focused learners' attention on the fact that "*whatever is in front of the* ***unknown*** *is referred to as the coefficient of that* **unknown**". Emphasis was placed on the word 'unknown' and the teacher followed up by referring to the method in Figure 9 (RHS) by saying that this method operates on what is ***known*** **(which is the coefficient)**, therefore ignoring the unknown x. This method is well known in mathematics, and is referred to as division by the *detached coefficients*.

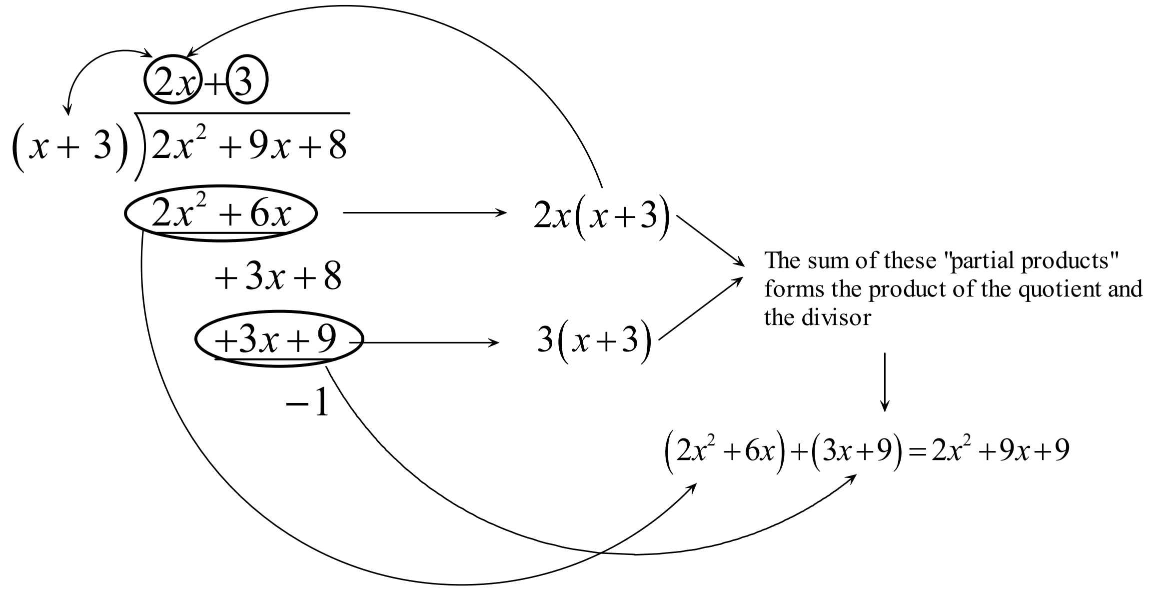
At this point one of the learners queried the fact that the *division algorithm* was not referred to in any one of the calculations. This was a deliberate omission by the teacher to avoid the conflict that could arise since the remainders were different. This question was answered by writing down the *division algorithm* for each calculation and then discussing whatever conflict the learners may have experienced.

For base ten division: $2\cdot 10^2+9\cdot 10+8=(1\cdot 10+3)(2\cdot 10+3)-1$

For algebraic long division: $2x^2+9x+8=(x+3)(2x+3)-1$

The teacher instructed the learners to decide which one of these methods was the easier for them, and then to work only with the method of their choice. Without concentrating on the subtle details in the definition of the concept of a polynomial, the teacher pointed to the dividend in the algebraic long division and made the learners aware there was a power relationship contained in the variable x and it was, therefore, called a polynomial.

A final question was asked by one of the learners, regarding the partial divisors in the algebraic long division problem. This was the third time that the same question arose, and the teacher decided to discuss the issue with the class as follows by referring to the example on the board:



After this explanation the learners were given a problem to work through in class. The instructions that were given to the learners were:

1. Write the numbers as power relations in base 10 first.
2. Replace the tens with an *x.*
3. Do the detached coefficient division first.
4. Repeat the process by algebraic long division including the *x*.

The problem: 624 divided by 15

In order to start the learners off and for the sake of uniformity in the work, the teacher commenced by writing the following on the board:

$15\overline{)624}$:

$15 = 10 + 5 \rightarrow (x + 5)$

$624 = 6 \cdot 10^2 + 2 \cdot 10 + 4 \rightarrow (6x^2 + 2x + 4)$

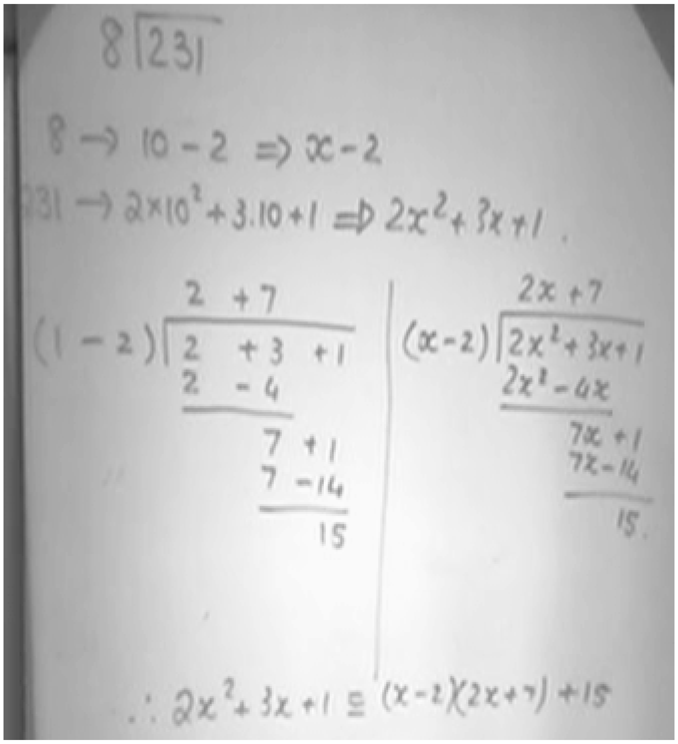
Algebraic long division: **Detached coefficients:**

$(x+5)\overline{)6x^2 + 2x + 4}$ $(1 \ +5)\overline{)6 \ +2 \ +4}$

The learners were instructed to complete the process after which the teacher transcribed the problem on the board while engaging the learners all the time. After observing that the learners understood the process, the teacher asked them to comment on the method which was preferable to them. The majority of learners chose to do this algebraic division by using the method of detached coefficients. The reason for that choice could possibly be explained in the light of the arithmetic nature of this method as it employs basic multiplication operations to complete a complex process. This process appeared to be more complex for the majority of learners when the *x* was included. In an earlier discussion of base ten division, the learners pointed out that the process contained too many numbers and keeping the *x* just replaces the ten with a variable. The cognitive load thus remained the same with the learners having to focus on two issues at the same time: number and variable. The learners were given a second problem to do in class while the teachers assisted individual learners where necessary.

<u>The second problem: 231 divided by 8</u>

The instructions given to the learners were the same as with the first problem. However, this time around there was no introduction given by the teacher. The learners were given ten minutes to complete this problem, after which the teacher asked the learners what value they arrived at for the remainder. With the exception of one student who got a remainder of five, the class calculated the remainder to be fifteen. This was indicated by a show of hands. The teacher then commenced to discuss the problem with the class.



There were no major problems that emerged during this classroom discussion except for a few learners omitting the signs in front of the numbers when they divided using detached coefficients.

$$231 \div 8$$

$$8 \rightarrow 10-2 \rightarrow x-2$$

$$231 \rightarrow 2\cdot 10^2 + 3\cdot 10 + 1 \rightarrow 2x^2 + 3x + 1$$

Detached coefficients

$$\begin{array}{rrrr} & 2 & +7 & \\ \hline (1\ \ -2)\big) & 2 & +3 & +1 \\ & \underline{2} & \underline{-4} & \\ & & +7 & +1 \\ & & \underline{+7} & \underline{-14} \\ & & & +15 \end{array}$$

Algebraic long division

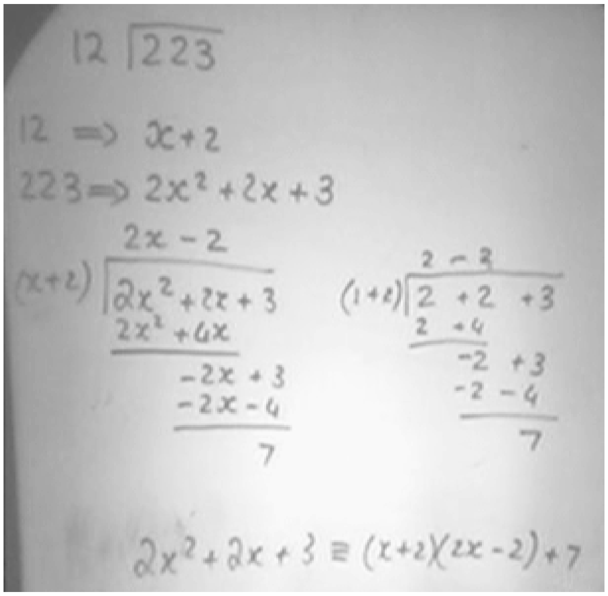
$$\begin{array}{rrrr} & 2x & +7 & \\ \hline (x-2)\big) & 2x^2 & +3x & +1 \\ & \underline{2x^2} & \underline{-4x} & \\ & & +7x & +1 \\ & & \underline{+7x} & \underline{-14} \\ & & & +15 \end{array}$$

$$\therefore 2x^2 + 3x + 1 = (x-2)(2x+7) + 15$$

One of the learners enquired why the division did not continue as the divisor (eight) was smaller than the remainder (fifteen) in this case. The teacher explained that the divisor is eight and is thus known, and the eight was decomposed in base ten, to obtain ten minus 2, after which the ten was replaced with an *x*. This meant that the *x*'s in this problem represented a value of ten but that need not always be the case. The researcher explained that learners would not always start with a number that needs to be decomposed, but that their work from that point onwards would start with a polynomial that would be given to them. Then the division process would stop the moment the remainder was reached, as there the value of *x* would not be known. The teacher concluded by saying: "just as the numbers were decomposed as a sum of power relations of ten, the polynomial is an expression of a sum of power relations of *x* with the value of *x* not always known, thus representing any number.

Problem three: 223 divided by 12

The learners were instructed as per earlier instructions, but this time around they were not allowed to ask anyone, including the teachers, for help. They were, therefore, required to solve the problem completely by themselves and, after working for ten minutes, the class announced the remainder to be seven. Only two learners disagreed. This was checked on the board by the teacher with input from the learners:



$$12\overline{)223}$$

$$12 \Rightarrow x+2$$

$$223 \Rightarrow 2x^2+2x+3$$

$$\begin{array}{r} 2x-2 \\ (x+2)\overline{)2x^2+2x+3} \\ \underline{2x^2+4x} \\ -2x+3 \\ \underline{-2x-4} \\ 7 \end{array} \qquad \begin{array}{r} 2\ -2 \\ (1+2)\overline{)2\ +2\ +3} \\ \underline{2\ +4} \\ -2\ +3 \\ \underline{-2\ -4} \\ 7 \end{array}$$

$$2x^2+2x+3 \equiv (x+2)(2x-2)+7$$

$$12 \to x+2$$

$$223 \to 2x^2+2x+3$$

Algebraic long division:

$$\begin{array}{r} 2x-2 \\ (x+2)\overline{)2x^2+2x+3} \\ \underline{2x^2+4x} \\ -2x+3 \\ \underline{-2x-4} \\ +7 \end{array}$$

Detached coefficients:

$$\begin{array}{r} 2\ -2 \\ (1\ +2)\overline{)\ 2\ +2\ +3} \\ \underline{2\ +4} \\ -2\ +3 \\ \underline{-2\ -4} \\ +7 \end{array}$$

$$\therefore 2x^2+2x+3=(x+2)(2x-2)+7$$

An interesting observation here was that the learners omitted the step where they decompose the divisor and the dividend in base ten, and commenced immediately to a representation that contained *x*. The learners responded with surprise as a classmate, who was not usually good at mathematics, provided an answer for the teacher that was flawless.

It would appear at this point that the learners understood this procedure both with and without the variable. The learners were given two problems to prepare in the same way for the next day.

Lesson 4 reflection:

The learners were given time in class to complete the two division problems by themselves. This was done after the teacher had explained the two procedures in depth, namely: division by detached coefficients and algebraic long division. Learners were allowed to ask one another questions and engage the help of the teachers when solving the first two problems in class. These questions were handled on an individual basis by the teachers and no classroom discussions were initiated by either of the teachers regarding the work that learners were doing. At the end of each question the teacher asked the class what the remainder was that they had obtained through division as this was a way for learners to check whether their answers were accurate. Learners were instructed to correct their work in a different colour while the teacher transcribed the solutions on the board.

Class work problems: $8\overline{)231}$ and $12\overline{)223}$

These two problems have already been examined in the description of the lesson.

$$231 \div 8$$

$$8 \rightarrow 10-2 \rightarrow x-2$$

$$231 \rightarrow 2\cdot 10^2+3\cdot 10+1 \rightarrow 2x^2+3x+1$$

Detached coefficients

$$\begin{array}{rrrr}
 & 2 & +7 & \\ \cline{2-4}
(1\ \ -2)\big) & 2 & +3 & +1 \\
 & \underline{2} & \underline{-4} & \\
 & & +7 & +1 \\
 & & \underline{+7} & \underline{-14} \\
 & & & +15
\end{array}$$

Algebraic long division

$$\begin{array}{rrrr}
 & 2x & +7 & \\ \cline{2-4}
(x-2)\big) & 2x^2 & +3x & +1 \\
 & \underline{2x^2} & \underline{-4x} & \\
 & & +7x & +1 \\
 & & \underline{+7x} & \underline{-14} \\
 & & & +15
\end{array}$$

$$\therefore 2x^2+3x+1=(x-2)(2x+7)+15$$

$$223 \div 12$$

$$12 \rightarrow x+2$$

$$223 \rightarrow 2x^2+2x+3$$

Detached coefficients:

$$\begin{array}{rrrr}
 & 2 & -2 & \\ \cline{2-4}
(1\ \ +2)\big) & 2 & +2 & +3 \\
 & \underline{2} & \underline{+4} & \\
 & & -2 & +3 \\
 & & \underline{-2} & \underline{-4} \\
 & & & +7
\end{array}$$

Algebraic long division:

$$\begin{array}{rrrr}
 & 2x & -2 & \\ \cline{2-4}
(x+2)\big) & 2x^2 & +2x & +3 \\
 & \underline{2x^2} & \underline{+4x} & \\
 & & -2x & +3 \\
 & & \underline{-2x} & \underline{-4} \\
 & & & +7
\end{array}$$

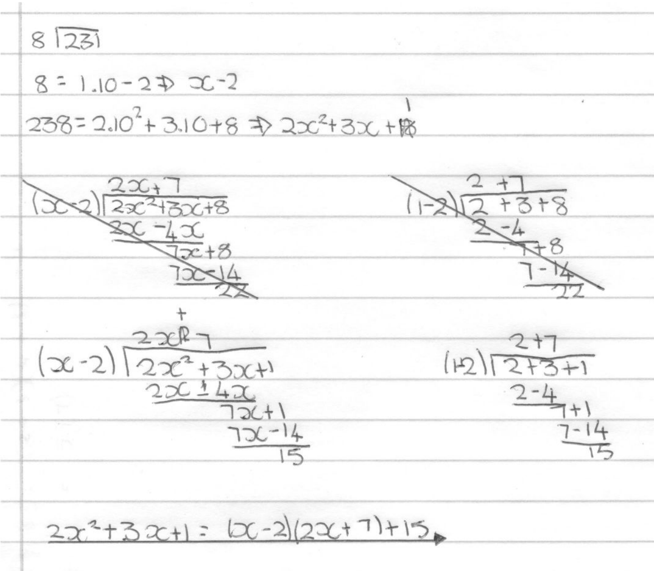
$$\therefore 2x^2+2x+3=(x+2)(2x-2)+7$$

About 65% of the learners got this right the first time around, with mistakes ranging from negative signs that were placed inappropriately to subtractions performed incorrectly. These negative signs arose due to learners indicating that they were changing the sign to add and then still subtracting with the result that their remainder was incorrect. Some mistakes resulted from negligence as indicated by the work of learner one below.

<u>Problem 1</u>:

Learner 1:

This learner mistakenly decomposed the number 238 instead of the number 231. Despite this mistake the division was performed without error.



8 ⟌231

8 = 1.10 − 2 ⇒ x − 2

238 = 2.10² + 3.10 + 8 ⇒ 2x² + 3x + ~~8~~ 1

~~(x − 2) ⟌ 2x² + 3x + 8 ; 2x + 7 ; 2x − 4x ; 7x + 8 ; 7x − 14 ; 22~~

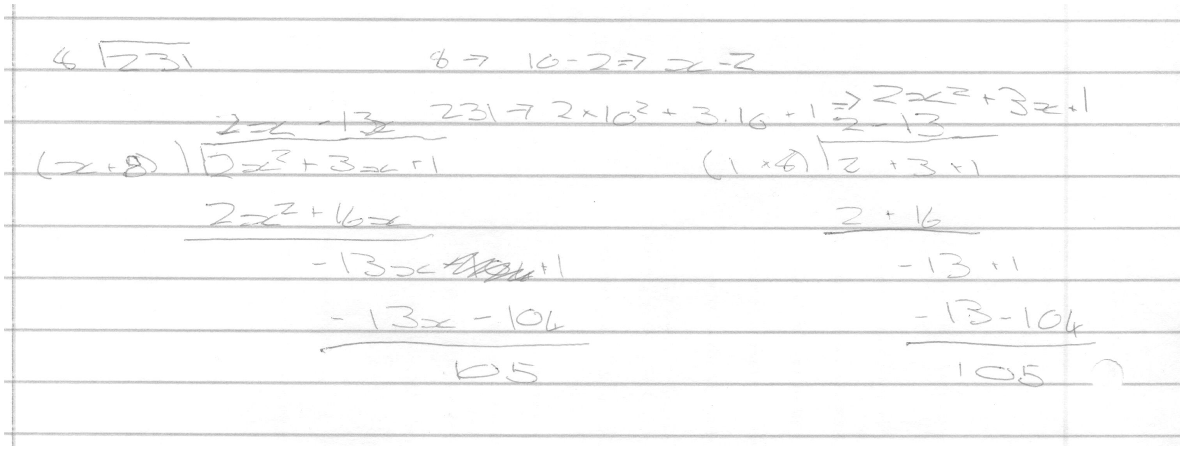
~~(1 − 2) ⟌ 2 + 3 + 8 ; 2 + 7 ; 2 − 4 ; 7 + 8 ; 7 − 14 ; 22~~

(x − 2) ⟌ 2x² + 3x + 1 ; 2x + 7 ; 2x ± 4x ; 7x + 1 ; 7x − 14 ; 15

(1 − 2) ⟌ 2 + 3 + 1 ; 2 + 7 ; 2 − 4 ; 7 + 1 ; 7 − 14 ; 15

2x² + 3x + 1 = (x − 2)(2x + 7) + 15

Learner 2:



8 ⟌231

8 ⇒ 10 − 2 ⇒ x − 2

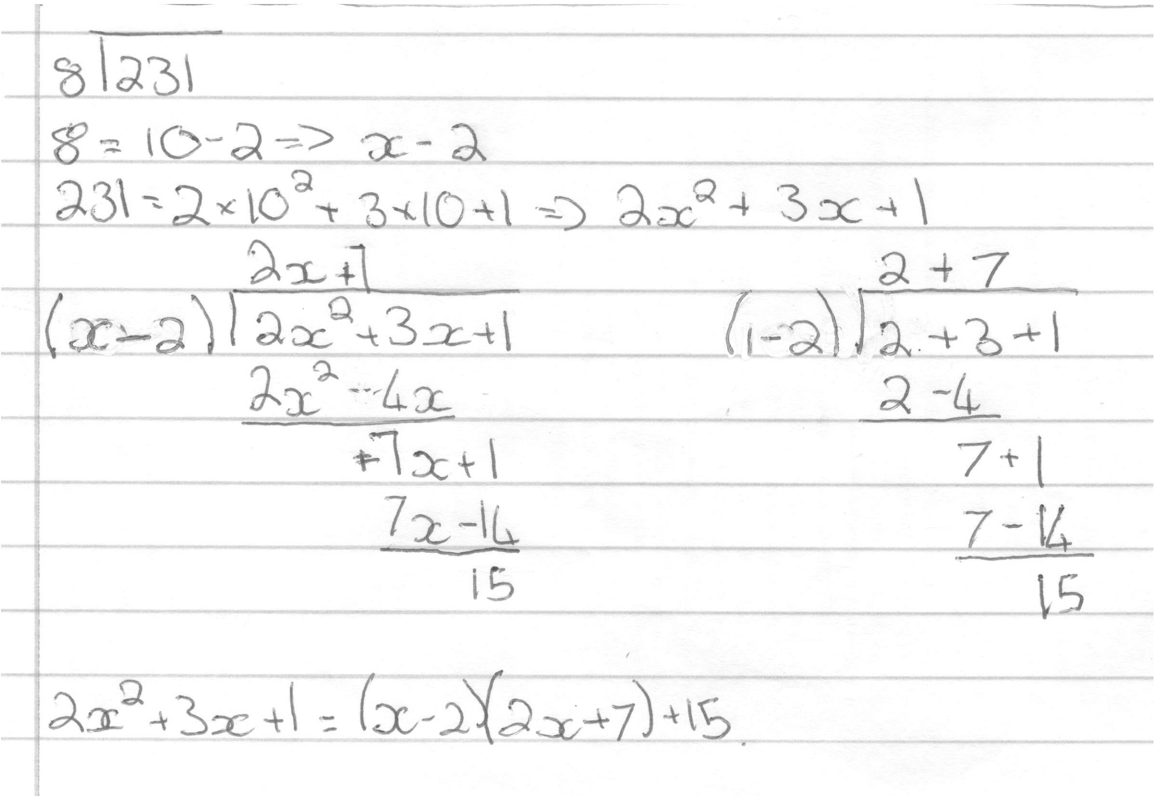
231 ⇒ 2×10² + 3.10 + 1 ⇒ 2x² + 3x + 1

(x + 8) ⟌ 2x² + 3x + 1 ; 2x − 13x ; 2x² + 16x ; −13x + 1 ; −13x − 104 ; 105

(1 + 8) ⟌ 2 + 3 + 1 ; 2 − 13 ; 2 + 16 ; −13 + 1 ; −13 − 104 ; 105

This learner decomposed the divisor 8 correctly to get $10-2=x-2$ but wrote down the divisor in the division as $x+8$. The division with this error was completed correctly to get a remainder of 105. If these types of error are ignored and the focus is shifted to the division procedures, then 77% of the learners successfully completed the first problem.

Learner 3:



$8 \overline{)231}$

$8 = 10-2 \Rightarrow x-2$

$231 = 2\times10^2 + 3\times10+1 \Rightarrow 2x^2+3x+1$

$2x+7$

$(x-2)\overline{)2x^2+3x+1}$

$2x^2-4x$

$+7x+1$

$7x-14$

15

$2+7$

$(1-2)\overline{)2+3+1}$

$2-4$

$7+1$

$7-14$

15

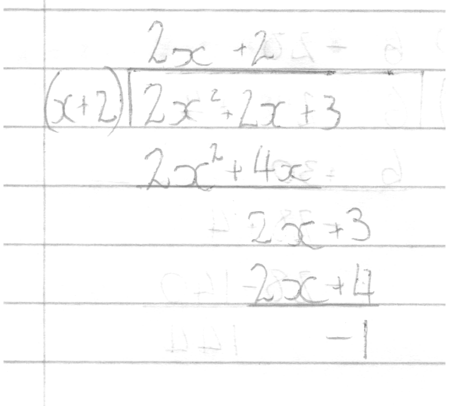
$2x^2+3x+1 = (x-2)(2x+7)+15$

The work above is an example of what the majority of learners, who were successful in solving the first algebraic division problem, did.

<u>Problem 2</u>:

This question was answered with a success rate of 67%. The learners continued to make careless mistakes when they performed the division procedure. A mistake which occurred with a higher frequency here was that learners neglected to change the sign or failed to subtract in the long division procedure. An interesting phenomenon is the variety of different remainders that learners got when they made this mistake. The fact that the learners had to subtract was emphasized by both teachers during the lessons, with the mathematics teacher explicitly indicating a negative sign where the subtraction had to take place.

Here are a few examples of these mistakes that learners made when solving this particular problem.



$2x+2$

$(x+2)\,|\,2x^2+2x+3$

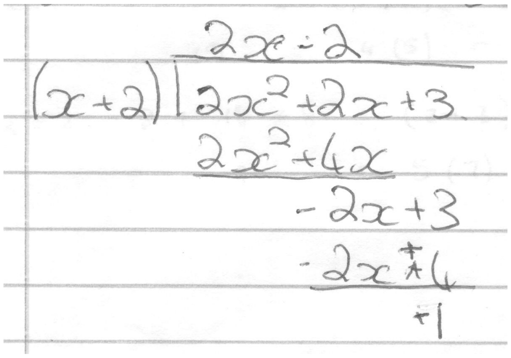
$2x^2+4x$

$2x+3$

$2x+4$

-1

This learner did not subtract correctly in the very first step when he calculated that $2x-4x=2x$. Notice that the same mistake did not occur when subtracting $3-4=-1$.



$2x-2$

$(x+2)\,|\,2x^2+2x+3$

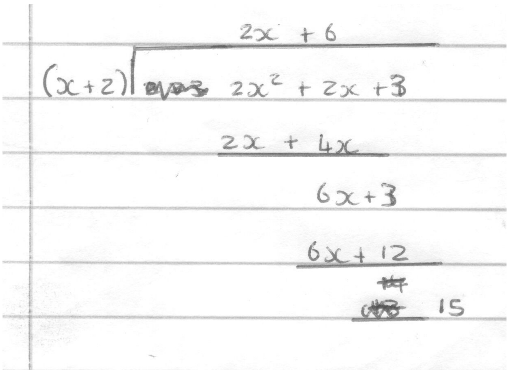
$2x^2+4x$

$-2x+3$

$-2x+4$

$+1$

This learner multiplied incorrectly and stated that $(-2)(2)=4$ in the second last step and then also subtracted incorrectly to say that $3-4=1$.



$2x+6$

$(x+2)\,|\,2x^2+2x+3$

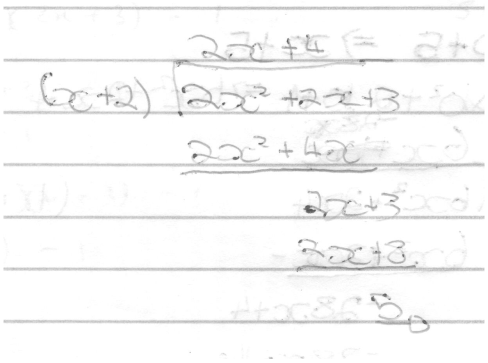
$2x+4x$

$6x+3$

$6x+12$

15

The learner forgot to subtract and added to obtain $2x+4x=6x$ instead of $2x-4x=-2x$ and later on $3+12=15$ instead of $3-12=-9$.



$2x+4$

$(x+2)\,|\,2x^2+2x+3$

$2x^2+4x$

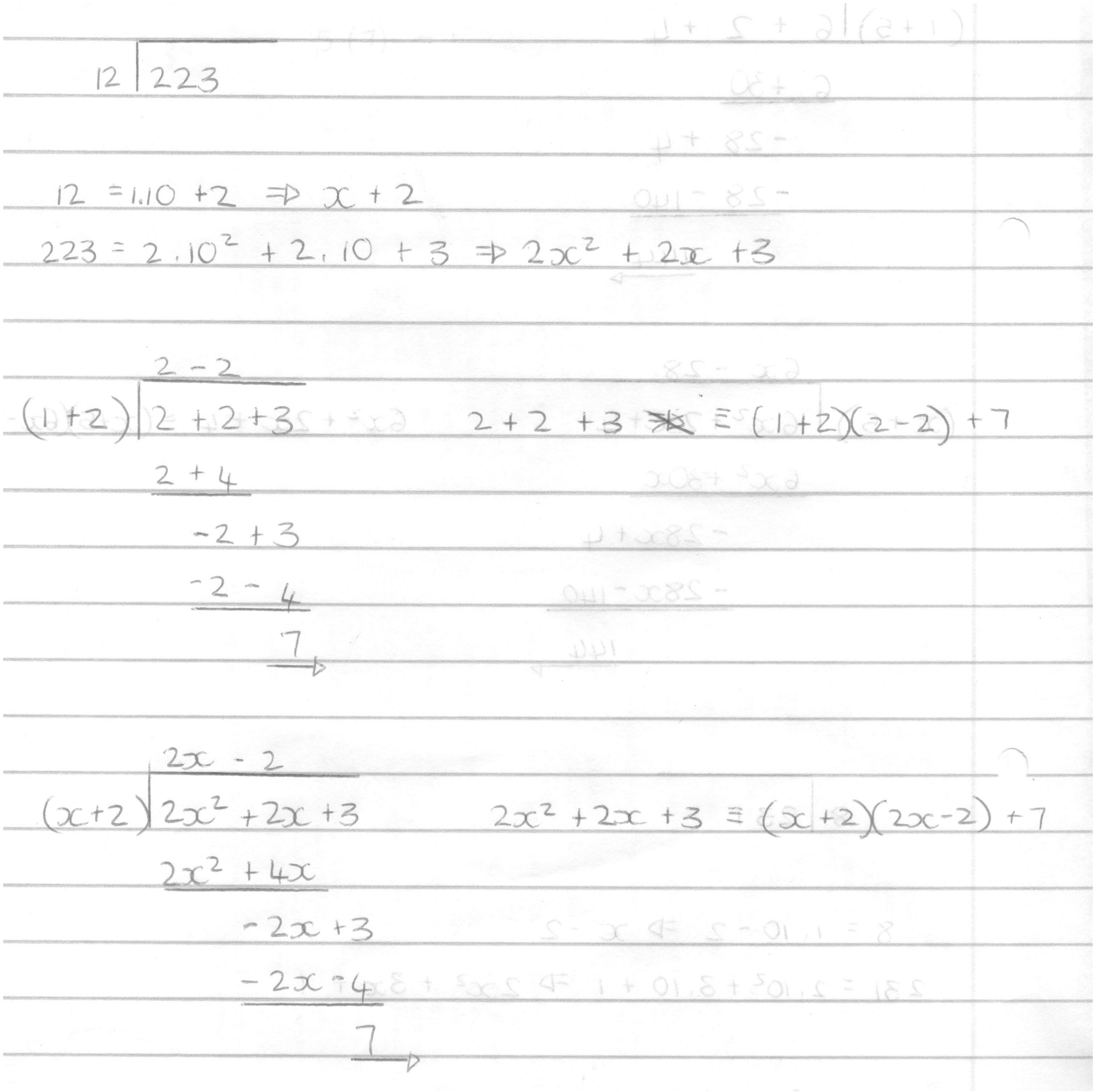
$2x+3$

$3x+3$

5

The learner also subtracted incorrectly to conclude that $2x-4x=2x$ and also later on in the problem $3-8=5$.

The majority of the class members did the work proficiently and without any mistakes or problems that could be attributed to carelessness. Here is an example of one such answer:



$12\overline{)223}$

$12 = 1.10 + 2 \Rightarrow x + 2$

$223 = 2.10^2 + 2.10 + 3 \Rightarrow 2x^2 + 2x + 3$

$$\begin{array}{r|l} & 2 - 2 \\ \hline (1+2) & 2 + 2 + 3 \\ & \underline{2 + 4} \\ & \quad -2 + 3 \\ & \quad \underline{-2 - 4} \\ & \qquad 7 \end{array}$$

$2 + 2 + 3 \cancel{x} \equiv (1+2)(2-2) + 7$

$$\begin{array}{r|l} & 2x - 2 \\ \hline (x+2) & 2x^2 + 2x + 3 \\ & \underline{2x^2 + 4x} \\ & \quad -2x + 3 \\ & \quad \underline{-2x - 4} \\ & \qquad 7 \end{array}$$

$2x^2 + 2x + 3 \equiv (x+2)(2x-2) + 7$

Learners were required to use the dividend and the divisor, decompose these in base ten and finally replace the tens with an *x* to complete the process. Thereafter, they had to perform the division using detached coefficients as well as keeping the *x*'s in the division. A large number of the learners commented that the method of detached coefficients was easier as there was less writing and fewer numbers present within the problem. By a show of hands, the majority of learners indicated their preference for this method of division.

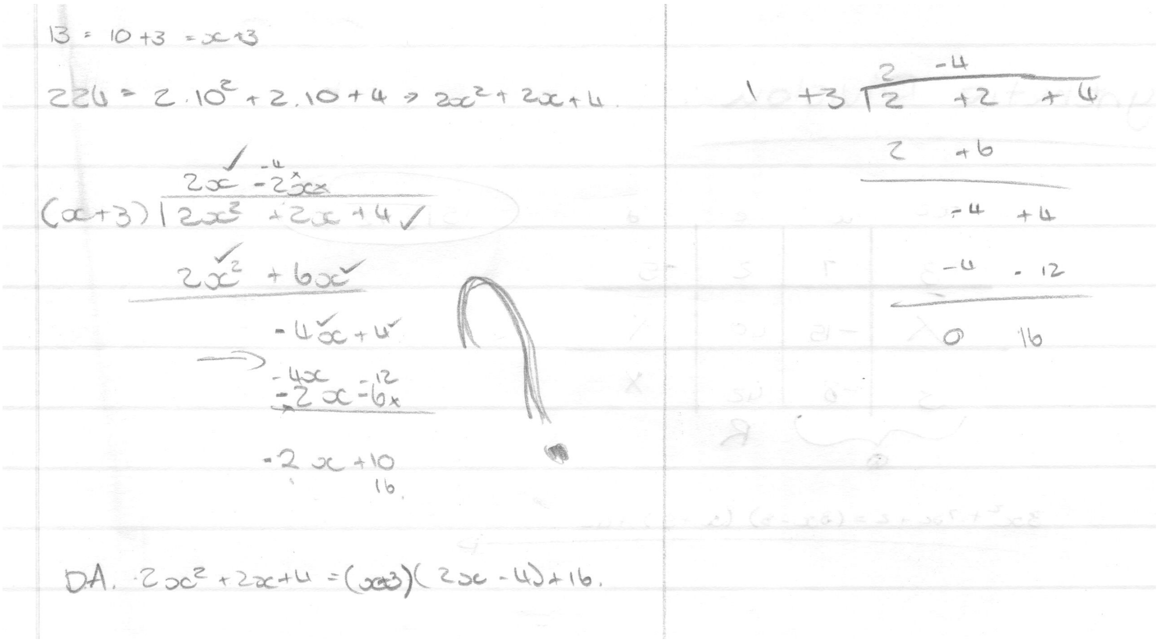
Problem 3: $13\overline{)224}$

$$13 = 10 + 3 \rightarrow x + 3$$
$$224 = 2\cdot 10^2 + 2\cdot 10 + 4 \rightarrow 2x^2 + 2x + 4$$

$$\begin{array}{rl} & 2x-4 \\ (x+3) & \overline{)2x^2+2x+4} \\ & \underline{2x^2+6x} \\ & \qquad -4x+4 \\ & \qquad \underline{-4x-12} \\ & \qquad\qquad +16 \end{array} \qquad\qquad \begin{array}{rl} & 2\ -4 \\ (1\ +3) & \overline{)2\ +2\ +4} \\ & \underline{2\ +6} \\ & \quad -4+4 \\ & \quad \underline{-4-12} \\ & \qquad +16 \end{array}$$

$$\therefore 2x^2 + 2x + 4 = (x+3)(2x-4) + 16$$

The learners were required to decompose both the divisor and dividend in base ten and then replace the ten with an *x*. The division procedure had to be completed using both methods as indicated above. The success rate for this question was 77%. The mistakes that occurred in five learners' work were mainly related to incorrect subtraction and making sign mistakes, with one of these learners multiplying incorrectly. The learner below had problems performing the traditional long division with the *x*'s present and checked her work using the detached coefficients method.



13 = 10 +3 = x+3

224 = 2.10² + 2.10 + 4 → 2x² + 2x + 4.

2x - 2x (-4)

(x+3) | 2x² + 2x + 4 ✓

2x² + 6x ✓

-4x + 4

-4x -12

-2x -6x

-2x +10

16.

DA. 2x² + 2x + 4 = (x+3)(2x - 4) + 16.

2 -4

1 +3 | 2 +2 +4

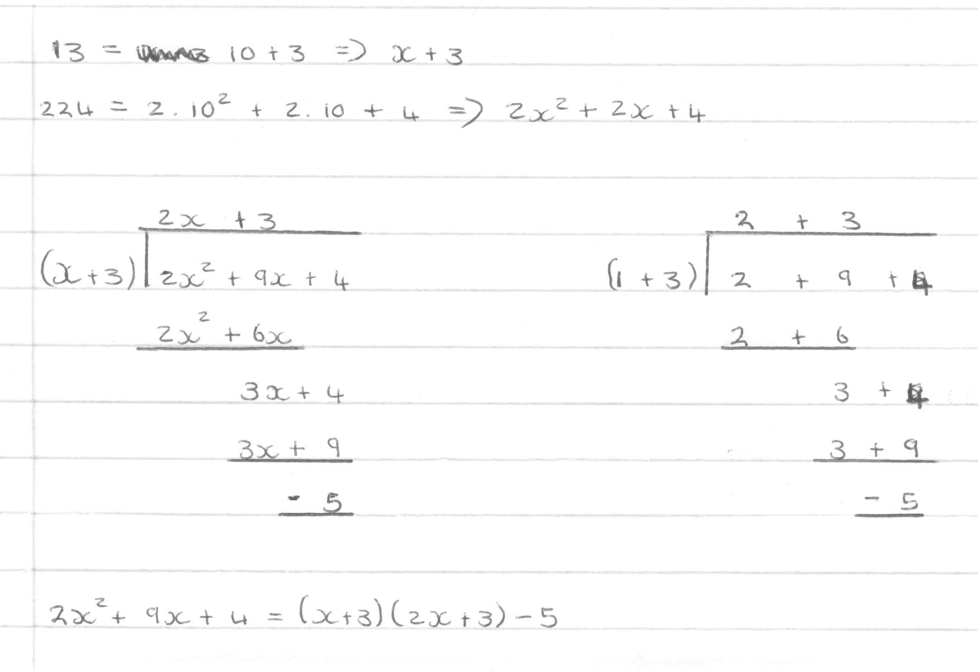
2 +6

-4 +4

-4 -12

0 16

Another learner performed the division perfectly in both methods, but again transferred the numbers incorrectly as she wrote 294 instead of 224.



$13 = 10 + 3 \Rightarrow x + 3$

$224 = 2.10^2 + 2.10 + 4 \Rightarrow 2x^2 + 2x + 4$

$$(x+3)\overline{)\,2x^2 + 9x + 4\,}\quad \text{quotient } 2x + 3$$

$2x^2 + 6x$

$3x + 4$

$3x + 9$

-5

$$(1+3)\overline{)\,2 + 9 + 4\,}\quad \text{quotient } 2 + 3$$

$2 + 6$

$3 + 4$

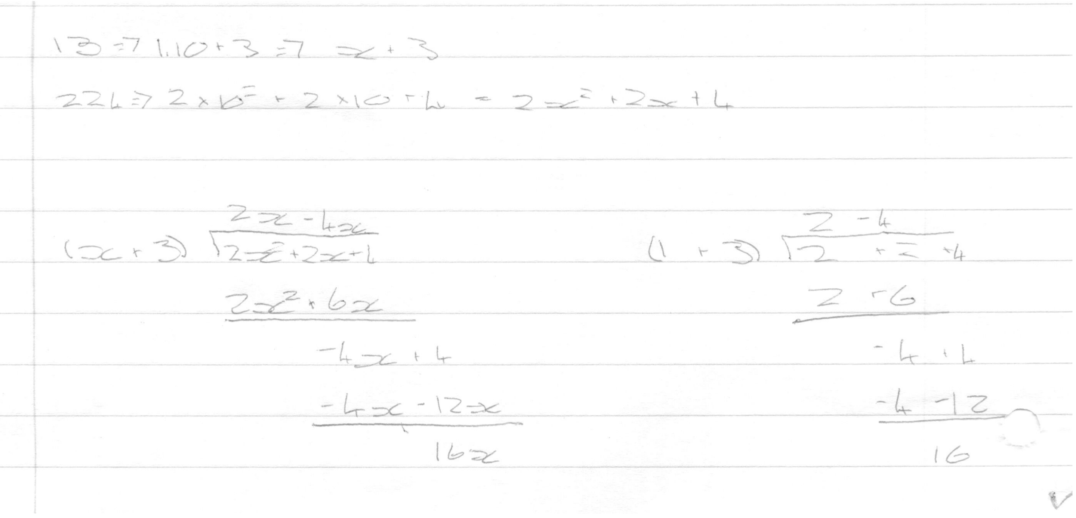
$3 + 9$

-5

$2x^2 + 9x + 4 = (x+3)(2x+3) - 5$

Learners also showed some confusion when they divided with the *x* present in the division procedure. All three learners below made the same mistake, yet managed to do the division perfectly when using detached coefficients:

Learner 1:



$13 \Rightarrow 1.10 + 3 \Rightarrow x + 3$

$224 \Rightarrow 2 \times 10^2 + 2 \times 10 + 4 = 2x^2 + 2x + 4$

$$(x+3)\overline{)\,2x^2 + 2x + 4\,}\quad 2x - 4x$$

$2x^2 + 6x$

$-4x + 4$

$-4x - 12x$

$16x$

$$(1+3)\overline{)\,2 + 2 + 4\,}\quad 2 - 4$$

$2 + 6$

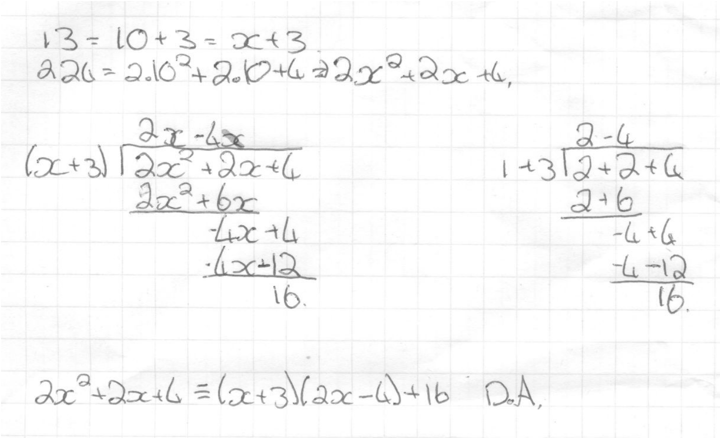
$-4 + 4$

$-4 - 12$

16

The division that learners 1 and 2 performed with detached coefficients is proof of their understanding of the division procedure when dividing with only the coefficients, yet when the *x* was present in the division procedure, the learners made some fundamental mistakes. The comment from the learners was that '*there are too many numbers, just like the base ten division*'.

Learner 2:



$13 = 10+3 = x+3$

$226 = 2.10^2+2.10+6 \Rightarrow 2x^2+2x+6,$

$2x - 4x$

$(x+3)\ \overline{)\,2x^2+2x+6}$

$2x^2+6x$

$-4x+4$

$-4x-12$

$16.$

$2-4$

$1+3\ \overline{)\,2+2+6}$

$2+6$

$-4+6$

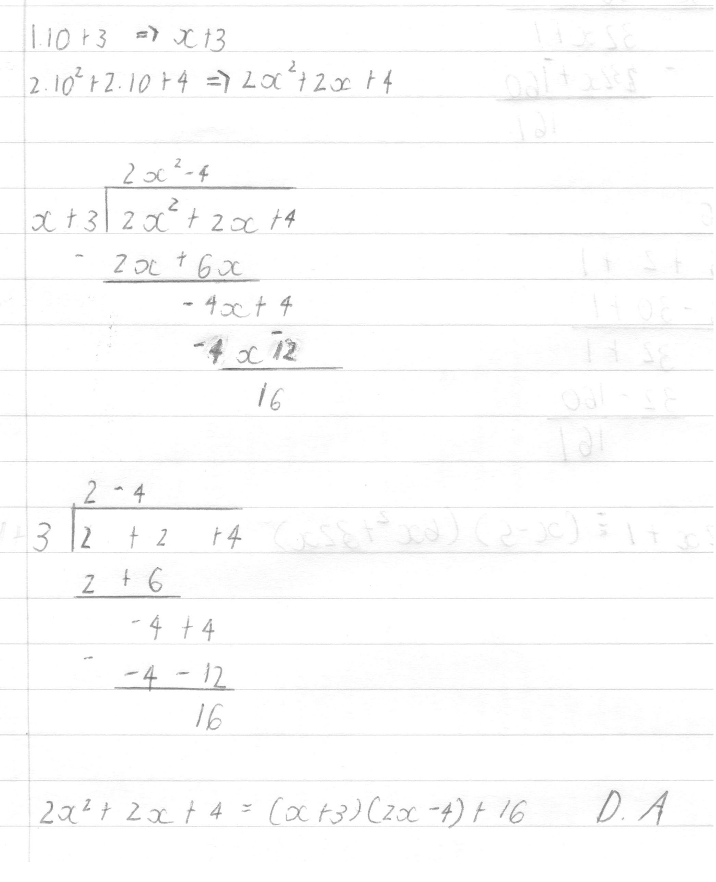
$-4-12$

$16.$

$2x^2+2x+6 \equiv (x+3)(2x-4)+16$ D.A,

Learner 3:

This learner experienced problems placing the *x*'s in the quotient and the partial difference within the first two steps of the division procedure. She placed the x^2 in the quotient, to get a quotient of $2x^2-4$ instead of $2x-4$, and then realised her mistake once she had completed the division with detached coefficients. In questioning how she formulated the division algorithm, she pointed out the she had used the detached coefficients to write it down.



$1.10+3 \Rightarrow x+3$

$2.10^2+2.10+4 \Rightarrow 2x^2+2x+4$

$2x^2-4$

$x+3\ \overline{)\,2x^2+2x+4}$

$-\ 2x+6x$

$-4x+4$

$-4x-12$

16

$2-4$

$3\ \overline{)\,2+2+4}$

$2+6$

$-4+4$

$-\ -4-12$

16

$2x^2+2x+4 = (x+3)(2x-4)+16$ D. A

Problem 4: $5\overline{)621}$

$$5 = 10 - 5 \rightarrow x - 5$$
$$621 = 6 \cdot 10^2 + 2 \cdot 10 + 1 \rightarrow 6x^2 + 2x + 1$$

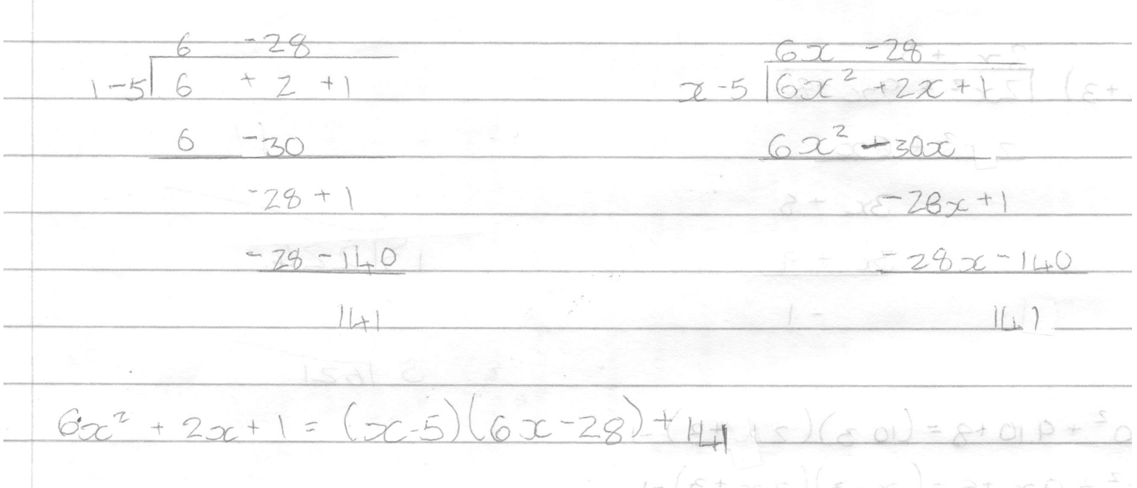
$$\begin{array}{r} 6x+32 \\ (x-5)\overline{)6x^2+2x+1} \\ \underline{6x^2-30x} \\ +32x+1 \\ \underline{+32x-160} \\ +161 \end{array} \qquad \begin{array}{r} 6\ \ +32 \\ (1\ \ -5)\overline{)6\ \ +2\ \ +1} \\ \underline{6\ -30} \\ +32\ +1 \\ \underline{+32-160} \\ +161 \end{array}$$

$$\therefore 6x^2 + 2x + 1 = (x-5)(6x+32) + 161$$

This problem was not any different from the other problems that the learners solved during class time, except for the remainder that was large and they had not had to deal with that previously. This did not create doubt and conflict with the learners.

The sign mistakes were the most frequent errors in this problem. Some of the learners who made these mistakes had not previously experienced difficulty. Below are two samples of learners' work which reflect these errors:

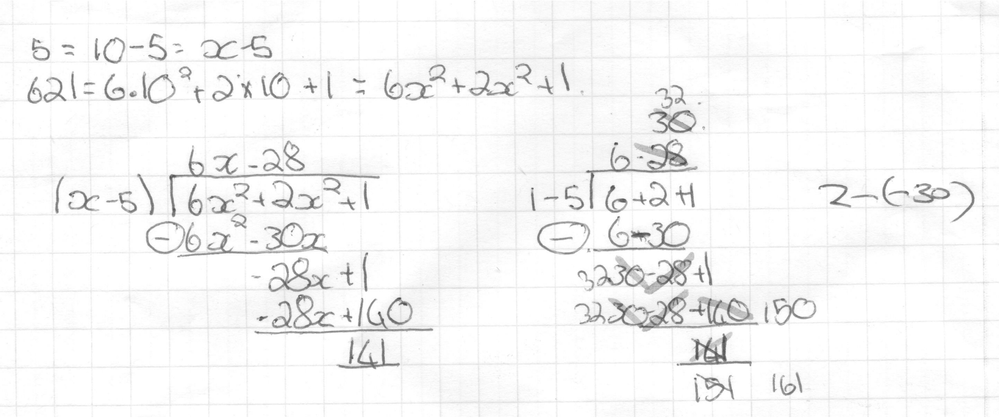
Learner 1:



$$\begin{array}{r} 6\ \ -28 \\ 1-5\overline{)6\ \ +2\ \ +1} \\ \underline{6\ -30} \\ -28+1 \\ \underline{-28-140} \\ 141 \end{array} \qquad \begin{array}{r} 6x\ \ -28 \\ x-5\overline{)6x^2+2x+1} \\ \underline{6x^2-30x} \\ -28x+1 \\ \underline{-28x-140} \\ 141 \end{array}$$

$$6x^2 + 2x + 1 = (x-5)(6x-28) + 141$$

Learner 2:



$5 = 10-5 = x-5$

$621 = 6.10^2 + 2\times 10 + 1 = 6x^2+2x^2+1.$

$6x-28$

$(x-5)\overline{)6x^2+2x^2+1}$

$\ominus 6x^2-30x$

$-28x+1$

$-28x+160$

141

32

$\cancel{30}$

$6-\cancel{28}$

$1-5\overline{)6+2+1}$

$\ominus\ 6+30$

$3230-\cancel{28}+1$

$3230-28+\cancel{160}\ 150$

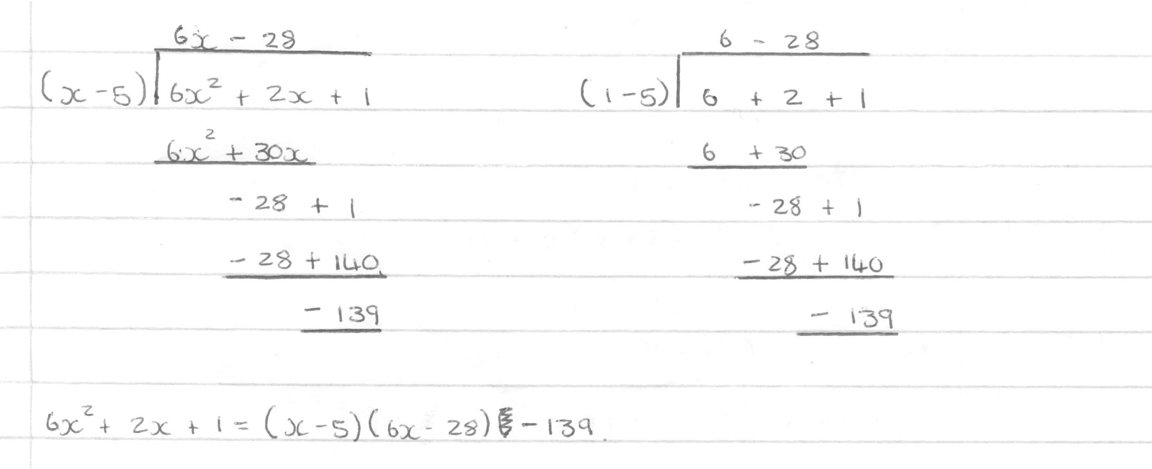
$\cancel{141}$

$151\ 161$

$2-(-30)$

Both these learners multiplied correctly, that is $-5\times 6x=-30x$, went on to make a subtraction error, that is $2x-(-30x)=-28x$. Learner 1 then repeated the mistake by saying that $-28\times -5=-140$ whilst learner 2 subtracted to conclude that $1-140=141$. These sign errors were problematic throughout this series of lessons and were the cause of many wrong remainders. This problem did not emerge during arithmetic long division and only became problematic when the algebraic structure was imitated in the arithmetic division problems in base ten.

Learner 3:



$6x - 28$

$(x-5)\overline{)6x^2 + 2x + 1}$

$6x^2 + 30x$

$-28 + 1$

$-28 + 140$

-139

$6 - 28$

$(1-5)\overline{)6 + 2 + 1}$

$6 + 30$

$-28 + 1$

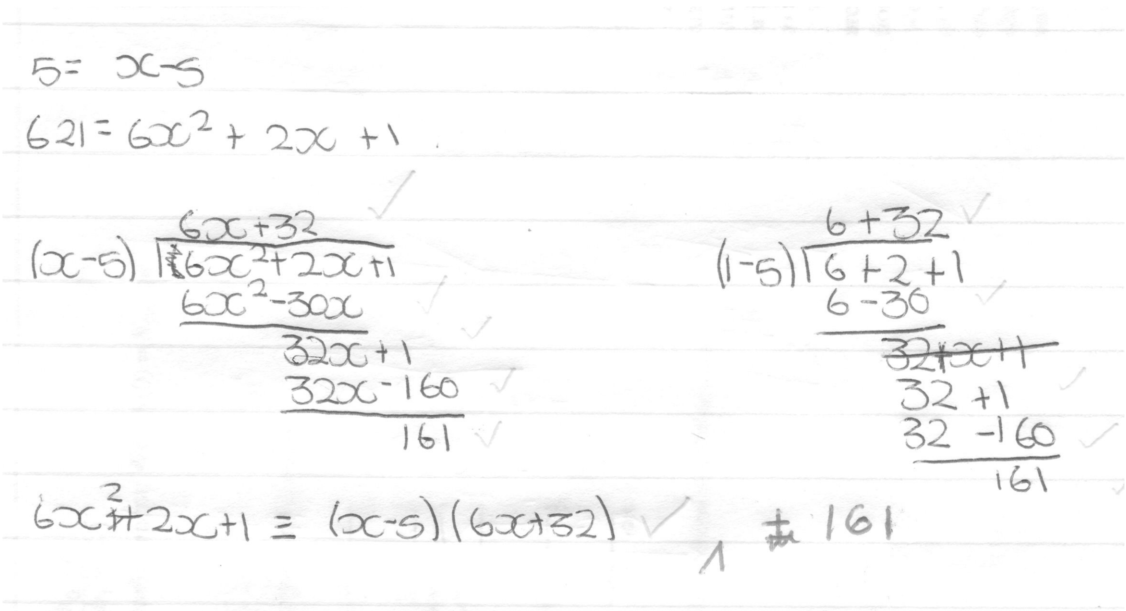
$-28 + 140$

-139

$6x^2 + 2x + 1 = (x-5)(6x-28) - 139.$

Learner 3 incorrectly concluded that $-5\times 6x=30x$ but the rest of the division was free of errors.

The following learner performed the division without difficulty but omitted the remainder from the division algorithm. 77% of the learners completed the division correctly. Below is an example of the procedure that was generally followed:



$5 = x - 5$

$621 = 6x^2 + 2x + 1$

$$\begin{array}{r} 6x+32 \\ (x-5)\overline{)6x^2+2x+1} \\ \underline{6x^2-30x} \\ 32x+1 \\ \underline{32x-160} \\ 161 \end{array} \qquad \begin{array}{r} 6+32 \\ (1-5)\overline{)6+2+1} \\ \underline{6-30} \\ \cancel{32+x+1} \\ 32+1 \\ \underline{32-160} \\ 161 \end{array}$$

$6x^2 + 2x + 1 = (x-5)(6x+32) \checkmark \quad 161$

In summary:

The main problem that was identified in learners' work, was the careless mistakes that they made when multiplying and subtracting within the division procedure. These mistakes did not occur repeatedly in particular learners' work, with learners often performing the division procedure perfectly in the first problem, only to stumble in performing basic procedures in the following problem. During classroom discussions, nearly all the learners showed a clear understanding of the work that was discussed, with only minor questions that were posed regarding the structure in the division problems. However, their homework assignments showed evidence of careless mistakes.

A distinct preference for the detached coefficients division procedure was shown by learners. Reasons that the learners offered for this preference was that this procedure appeared to require less writing, and is based on basic multiplication, division, subtraction and addition problems. At this stage of the division ladder, learners showed a remarkable proficiency in using division language, with learners constantly referring to the terms ‘divisor’, ‘dividend’, ‘quotient’, ‘remainder’ and the ‘division algorithm’ in their classroom discourse. Not only could they use these terms, but they also learned to identify the positions of the x when doing long division in detached coefficient form.

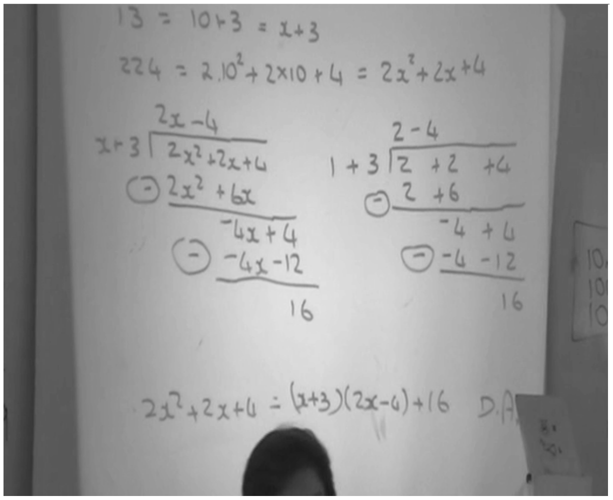
It is of utmost importance that an approach be devised to assist the learners who made sign errors in their work. During the lessons, the mathematics teacher suggested that learners indicate the subtraction by placing a “-” sign alongside the step where this is supposed to take place, in order to remind themselves that the signs are going to change in the process. This strategy was not followed by any of the learners who made this mistake and it could be a wise strategy to implement when dealing with long division procedures.

Lesson 5 and reflection

The purpose of this lesson was to synthesize the work that had been done with the learners over the previous two weeks. The learners had homework in which they needed to express numbers in terms of x in order to form polynomials that should be divided into one another. These divisions had to be done in two ways. Firstly, learners were required to perform algebraic long division where the x is present and, secondly, they had to perform the procedure of detached coefficients in order to complete the division. The work that the learners did at home was handed in to the teacher and, thereafter, the mathematics teacher discussed the problems with the learners while recording their responses on the board. The two homework problems were 224 divided by 13 and 621 divided by 5.

When the learners engaged with division by detached coefficients, they emphasized the sign of the numbers when doing the division. The problems that the learners had previously when subtracting a negative number seemed to have been overcome when the teacher again introduced the negative at the step where the subtraction should take place. Throughout the dictation of the solution to the teacher, they paid meticulous attention to the detail in the problem where they previously made careless mistakes with the sign.

Problem one: 224 divided by 13



$$224 \to 2\cdot10^2+2\cdot10+4 \to 2x^2+2x+4$$

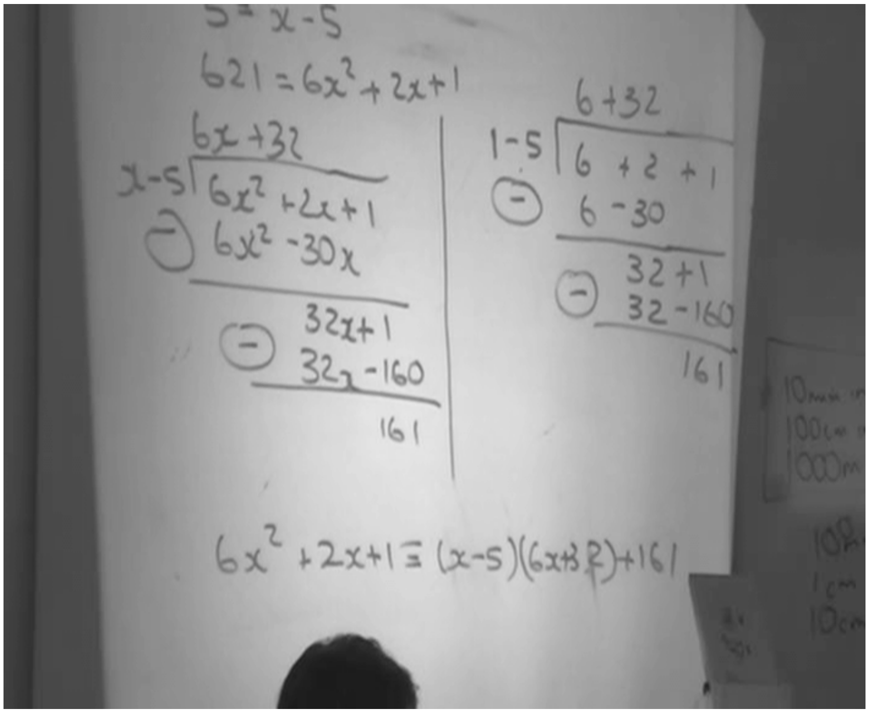
$$13 \to 10+3 \to x+3$$

$$\begin{array}{r}
2x-4\\
(x+3)\overline{)2x^2+2x+4}\\
\underline{2x^2+6x}\\
-4x+4\\
\underline{-4x-12}\\
+16
\end{array}
\qquad
\begin{array}{r}
2\ \ -4\\
(1\ \ +3)\overline{)\,2\ \ +2\ \ +4}\\
\underline{2\ \ +6}\\
-4\ \ +4\\
\underline{-4\ \ -12}\\
+16
\end{array}$$

$$\therefore 2x^2+2x+4=(x+3)(2x-4)+16$$

Problem two: 621 divided by 5

The learners were given ten minutes to complete the problem and they were not allowed to ask for help from their classmates or the teachers. The mathematics teacher asked the learners what the difference was between this problem and the one that they had done previously. The class immediately replied that this time round the divisor was smaller than 10 and the teacher asked how this affected their calculations. They responded by saying that they now had to subtract when they formed the divisor.



$$5 \rightarrow x-5$$

$$621 \rightarrow 6x^2+2x+1$$

$$\begin{array}{rl}
 & 6+32 \\
(1\ -5)\big) & \overline{6\ +2\ +1} \\
 & \underline{6\ -30} \\
 & \quad +32\ +1 \\
 & \quad \underline{+32\ -160} \\
 & \qquad\quad 161
\end{array}
\qquad\qquad
\begin{array}{rl}
 & 6x+32 \\
(x-5)\big) & \overline{6x^2+2x\ +1} \\
 & \underline{6x^2-30} \\
 & \qquad +32x+1 \\
 & \qquad \underline{+32x-160} \\
 & \qquad\qquad\quad 161
\end{array}$$

$$\therefore 6x^2+2x+1=(x-5)(6x+32)+161$$

After this question was completed, one of the learners asked what would happen if the divisor was 27. This was an important observation from the learner, as all the calculations that were performed by the learners had divisors between zero and twenty. The teacher asked the class what the divisor would be in terms of x if its numerical value was 27. A learner immediately replied that it would be $x^2 + 7$, and when asked to motivate his suggestion he immediately corrected himself by saying $2x + 7$. The following is a transcript of that classroom debate:

L1: What happens when it is like … 27 … then is it x^2 minus something?

Teacher: A very important question. Who wants to answer that?

Class: It's $x^2 + 7 \ldots 2x \ldots x^2 \ldots . x^2 - \ldots$ (*all together)*

Teacher: Hang on. (*to* L2) Why do you say x^2 ?

L2: Because it is two x's … no, it must be $2x + 7$.

Teacher: And what does the $2x$ represent here?

L3: The two tens.

Teacher: Good, I want to ask you what you think will make this problem more difficult than the ones you have done so far? There are two of your classmates that are busy doing some of these problems for me. Can one of the two tell us?

L4: (*Who was one of the two*) It is not just x, but it is $2x$ …

Teacher: So what does it form? (*leading the learner to the fraction concept*)

L4: It depends … (*She hesitates*)

Teacher: If the dividend started with a seven – then I have to ask myself what I must do to two to get to exactly seven, which would mean that I need to form a fraction.

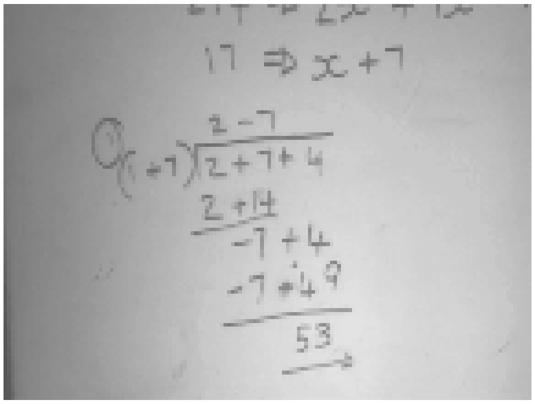
L1: No, but that is difficult.

Teacher: That is why we deliberately chose our divisors in the way we did. We will do more advanced problems with you next year, once you have become fluent at performing this type of division problem.

One of the learners requested that the teacher give her the problem to do with the class on the board. This learner was one of the stronger mathematics learners in the class. As she was busy explaining the problem to the learners, the teacher interrupted at crucial stages to ask the learners questions on what was being done. These questions were asked to gauge whether all the learners fully understood the division procedure. The result of this exercise was sufficient for the teachers to conclude that the learners were comfortable with the procedure.

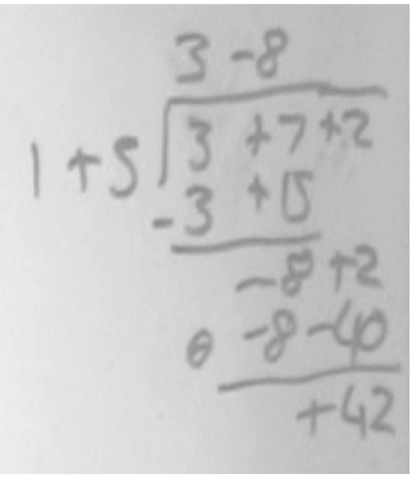
<u>Problem 3: 274 divide by 17</u>

A screenshot of the problem as it was performed by the learner is given below.



One of the important aspects of this division is to multiply in such a way that the first numbers (i.e. the 2 in 2 + 7 + 4 and the 2 in 2 + 14 as well as the -7 in $-7+4$ and the -7 in $-7-49$) are always the same in order to continue with the division to obtain a remainder. The learners were asked what they noticed about these numbers and they correctly identified that they were the same. The teacher followed this up with the question "*Why do you think they are the same?*" to which a learner replied that one must multiply to get them the same. Another learner then commented that these numbers were the numbers that formed the coefficients in the quotient.

A second learner requested to lead the class in solving a problem and the mathematics teacher gave him the division problem $372 \div 15$. This learner was one of the weaker students in the class who continuously made sign errors in his work. He started off by completing the division using detached coefficients and also emphasized the subtraction by indicating it in the problem.



The learners were encouraged to ask questions of their peer at any time while he completed the division. One such question was "*why do you write* 1 + 5 *for the divisor?*" The learner immediately responded by saying that it was one multiple of x that they were dividing. The teacher asked him to write down the *division algorithm* from the detached coefficients, without performing the division again with the x present. He immediately wrote the following without the x's:

$$3+7+2 \equiv (1+5)(3-8)+42$$

This was followed up by the teacher who pointed out that the learners should choose whichever one of the two methods they preferred to use for division, but that the *division algorithm* should reveal the x's once the division was completed. The teacher reminded the learners that the 3 was actually 300, the 7 was 70 and the 2 was simply a 2, and then asked the learner to insert the x and the x^2 in their appropriate positions. The learner continued to write the following:

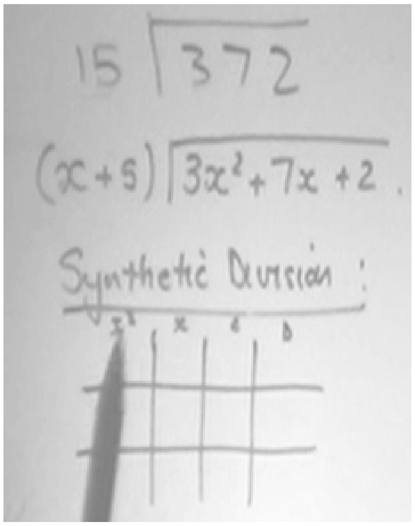
$$3x^2+7x+2 \equiv (1x+5)(3x-8)+42$$

What became very important here was the use of the congruent sign within the division algorithm. This use was introduced by the teacher in lesson 3, but was not emphasised at all. The learner continuously used the words "*is nothing other than* ..." when writing down the congruency sign in the algorithm.

As a final method for division, learners were introduced to the concept of synthetic division. The teacher indicated clearly that this method would be focused on in Grade 9 when the learners would return to long division with their mathematics teacher. She then gave a step by step instructional sequence to set up the division, and followed that with certain key questions that replicated the processes with which the learners had become accustomed over the previous two weeks. Learners responded quite well to these questions and identified the key concepts successfully during this discussion.

To set up the division grid:

The teacher pointed out that the highest power of the x was two, so the learners needed to draw 2 + 1 vertical lines, and follow that up with only two horizontal lines. He then indicated that the highest power plus one would always determine the number of columns and that there would always be three rows in the grid. The teacher continued to indicate what each column represented, namely the positions occupied by the terms in the polynomial and the position of the dividend.



Filling in the blanks:

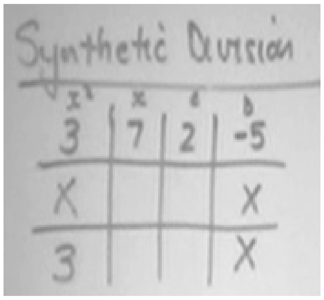
The teacher indicated that the grid should contain the coefficients of the variables only, as the learners were operating only on the 'known' when they performed synthetic division. The learners continued to prompt the teacher as to which coefficients would be inserted into the appropriate blocks. A learner enquired as to why the divisor had changed position to the right hand side, and the teacher indicated that it could be placed either on the left or the right, since the position did not matter. The teacher continued to explain to the learners how to determine the divisor by writing down and solving the equation $x+5=0 \rightarrow x=-5$. Here the negative sign with the divisor was emphasized.

Teacher: Now we want to use -5 instead of 5 as our divisor. What effect do you think that negative will have in the procedure?

L1: It changes the sign.

Teacher: Great … can you guys see that it changes the sign so that you don't have to constantly change that sign in your head when you subtract.

Finally, the learners were shown that certain blocks would not be used and should therefore always be crossed out. The teacher referred to the procedure up to this point as 'monkey

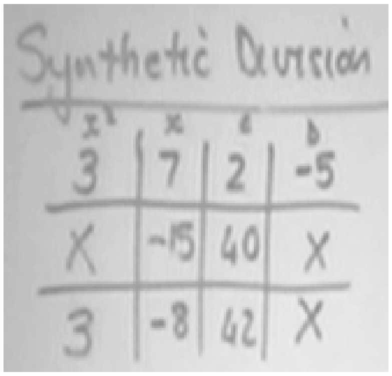

Synthetic Division

x²	x	c	b
3	7	2	-5
X			X
3			X

see, monkey do' as it merely required copying the information into the grid.

Performing the division:

As this study did not focus on this procedure, the method will not be discussed in detail, but the final screenshot will be included, and the questions from the teacher with learner responses will be indicated. The reason for doing this is to indicate how the knowledge that the learners gained over the two week research period helped them to master this new method.



Synthetic Division

x^2	x	[illegible]	b
3	7	2	-5
X	-15	40	X
3	-8	42	X

The teacher pointed to the 42 in the grid and the learners responded immediately by pointing out that it was the remainder. Pointing to the 3 and -8, the learners responded that they were the coefficients of the quotient. Being able to recognize these positions within the grid was a positive indication that the learners could separate procedural information from the conceptual information in the grid quite clearly.

The learners reacted with amazement to the conciseness of the synthetic division procedure. At the end of the lesson learners were asked which one of the two methods they preferred to use to perform the algebraic long division, the choice being between the methods of the attached coefficients or doing algebraic long division with the *x*. With the exception of two learners, everybody indicated a preference for performing long division by the method of detached coefficients, and they expressed a keen interest in completing the division by synthetic procedures.

Additional work

Two of the academically stronger learners, who had mastered the work, asked the teachers whether they could be given more difficult problems on which to practise. They were asked to perform the divisions $23\overline{)463}$, $23\overline{)8463}$ and $41\overline{)6023}$. The first problem was similar to the problems that learners solved in lessons 3 and 4 and will, therefore, not be discussed in this chapter. The second and third problems were clear extensions into the thousands and with a divisor that did not resemble only one multiple of ten. The construction of a polynomial from the base ten decomposition would mean that these two learners would have to divide a cubic polynomial using a rational divisor. The first cubic polynomial had a leading coefficient of 8 while the second had a leading coefficient of 6. This was clearly an extension that reached far beyond the work that the two learners had done in class over the previous two weeks.

The learners were required to perform the division using both the methods with which they had become acquainted and they were allowed to consult with one another in completing the task.

Problem 1:

$23 \rightarrow 2x+3$

$8463 \rightarrow 8x^3+4x^2+6x+3$

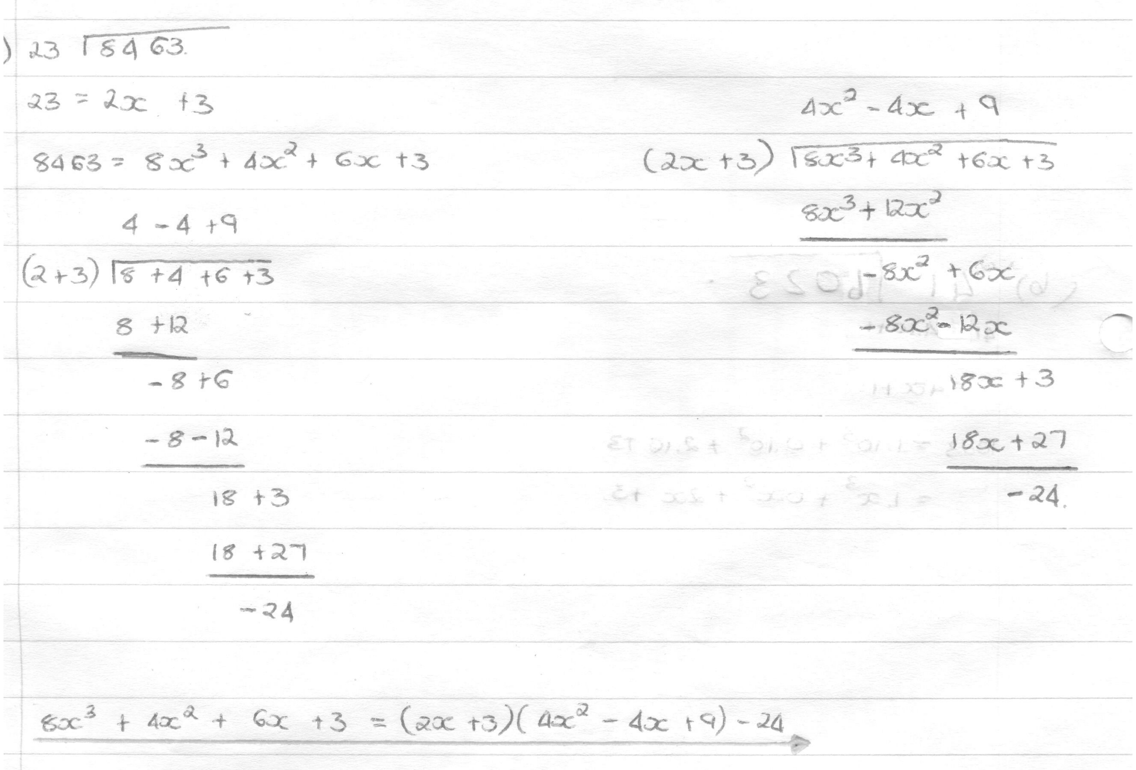
$$\begin{array}{r} 4x^2-4x+9 \\ (2x+3)\overline{)8x^3+4x^2+6x+3} \\ \underline{8x^3+12x^2} \\ -8x^2+6x \\ \underline{-8x^2-12x} \\ +18x+3 \\ \underline{+18x+27} \\ -24 \end{array} \qquad \begin{array}{r} 4\ -4\ +9 \\ (2\ +3)\overline{)8\ +4\ +6\ +3} \\ \underline{8\ +12} \\ -8\ +6 \\ \underline{-8\ -12} \\ +18\ +3 \\ \underline{+18\ +27} \\ -24 \end{array}$$

$\therefore 8x^3+4x^2+6x+3=(2x+3)(4x^2-4x+9)-24$

This problem challenged the two learners to apply the skills that they had learned to cubic polynomials. The decomposition of the numbers would form a cubic relationship once the $8 \cdot 10^3$ becomes $8x^3$ after replacing the ten with an x. This would also leave a quadratic quotient, a concept with which the learners had not previously dealt.

<u>Learners' work</u>

The two learners worked together and produced the following answer:



) $23 \overline{)8463}$.

$23 = 2x + 3$

$8463 = 8x^3 + 4x^2 + 6x + 3$

$4 \ -4 \ +9$

$(2+3) \overline{)8 \ +4 \ +6 \ +3}$

$\underline{8 \ +12}$

$-8 \ +6$

$\underline{-8 - 12}$

$18 \ +3$

$\underline{18 \ +27}$

-24

$4x^2 - 4x + 9$

$(2x+3) \overline{)8x^3 + 4x^2 + 6x + 3}$

$\underline{8x^3 + 12x^2}$

$-8x^2 + 6x$

$\underline{-8x^2 - 12x}$

$18x + 3$

$\underline{18x + 27}$

-24.

$8x^3 + 4x^2 + 6x + 3 = (2x+3)(4x^2 - 4x + 9) - 24$

They managed to complete the division correctly and without any help from the teachers who were busy engaging with the class on the work as described in lesson 5. These two learners were at the top of their class and both of them had achieved marks of above 90% for mathematics. In their answer they also preferred to complete the division procedure with detached coefficients first, before attempting to do the division with the x.

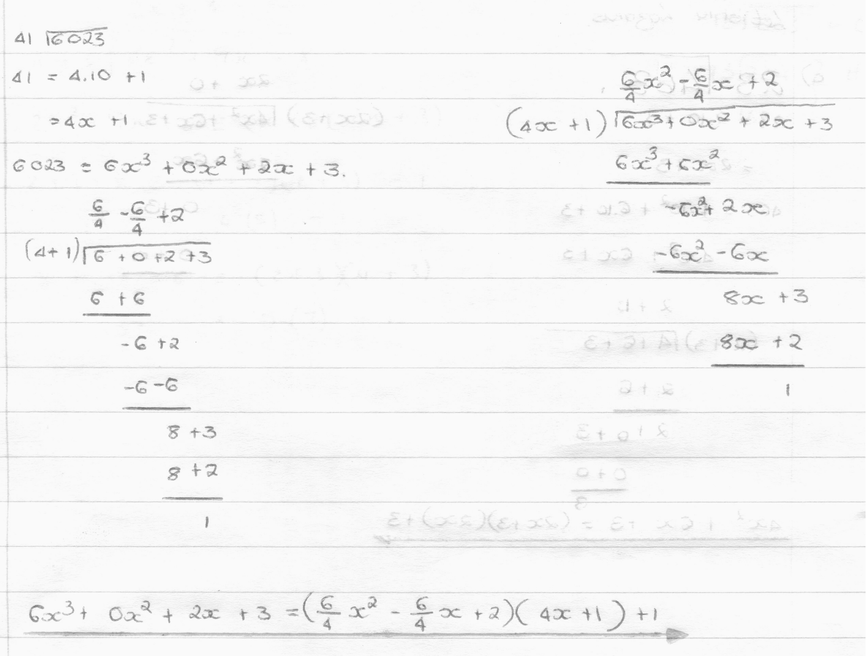
The teacher checked their answer and gave them the third problem to complete. This polynomial also did not have a squared term as there was a zero in the hundreds column. The learners had not dealt with a zero digit at all in the class. What made this problem extremely difficult was the fact that 4 does not divide 6, and that the learners would have to revert to the introduction of a fraction in their problem.

Problem 2: (Only the detached coefficients procedure is shown)

$$41 \rightarrow 4x+1$$
$$6023 \rightarrow 6x^3+0x^2+2x+3$$

$$
\begin{array}{r l}
 & \quad \tfrac{6}{4} - \tfrac{6}{16} + \tfrac{38}{64} \\
(4+1) & \overline{)6 \; +0 \; +2 \; +3} \\
 & \underline{6+\tfrac{6}{4}} \\
 & \quad -\tfrac{6}{4}+2 \\
 & \quad \underline{-\tfrac{6}{4}-\tfrac{6}{16}} \\
 & \qquad +\tfrac{38}{16}+3 \\
 & \qquad \underline{+\tfrac{38}{16}+\tfrac{38}{64}} \\
 & \qquad\quad +\tfrac{154}{64}
\end{array}
$$

Learners' work:



$41 \overline{)6023}$

$41 = 4.10 + 1$

$= 4x + 1$

$6023 = 6x^3 + 0x^2 + 2x + 3.$

$$
\begin{array}{r l}
 & \quad \tfrac{6}{4} - \tfrac{6}{4} + 2 \\
(4+1) & \overline{)6 \; +0 \; +2 \; +3} \\
 & \underline{6 \; +6} \\
 & \quad -6 \; +2 \\
 & \quad \underline{-6 \; -6} \\
 & \qquad 8 \; +3 \\
 & \qquad \underline{8 \; +2} \\
 & \qquad\quad 1
\end{array}
$$

$$
\begin{array}{r l}
 & \quad \tfrac{6}{4}x^2 - \tfrac{6}{4}x + 2 \\
(4x+1) & \overline{)6x^3 + 0x^2 + 2x + 3} \\
 & \underline{6x^3 + 6x^2} \\
 & \quad -6x^2 + 2x \\
 & \quad \underline{-6x^2 - 6x} \\
 & \qquad 8x + 3 \\
 & \qquad \underline{8x + 2} \\
 & \qquad\quad 1
\end{array}
$$

$6x^3 + 0x^2 + 2x + 3 = \left(\frac{6}{4}x^2 - \frac{6}{4}x + 2\right)(4x + 1) + 1$

The problem was not completed with much success, but this was anticipated as the learners had not dealt with fractions in their work, and this problem was very complicated since the fractions that had to be used did not simplify to integers in the division. The learners could identify correctly that 4 must be multiplied with $\frac{6}{4}$ to give a product of 6, but then incorrectly stated, twice in the problem, that $\frac{6}{4}\times 1 = 6$, which made the division procedure for the rest of the problem easier as the fractions dissolved. The learners did, however, identify the use of a zero placeholder for the zero digit correctly. This was something that the researcher had not anticipated.

When this work is continued in Grade 9, the teacher will have to structure the problems containing fractions very carefully, so that the necessary scaffolding is provided where the divisibility of the numbers is concerned; otherwise the learners may give up on the division with fractions. Here the latest technology should be used in conjunction with either the detached coefficients or synthetic division, especially when fractions come into the division ladder. All the latest calculators have the fraction function and teachers should make a concerted effort to encourage and guide learners in the use of this technology. By using a calculator when fractions are divided, learners will avoid having to work with tedious paper and pencil calculations in order to multiply or divide during the sub-procedural steps within algebraic long division.

Chapter 7 – Reflections and future research

In this chapter I will discuss the findings in summaries that will focus on various issues that emerged during the study. I will start by considering the success with which learners engaged with base ten decomposition division as an algorithm and how this process made sense for the learners as it progressed towards algebraic long division. I will then continue to discuss the importance of the division algorithm within long division procedures as well as across themes within mathematics. Here I will point to the cognitive conflict that emerged when learners obtained different remainders for the seemingly similar division problem and how the base ten decomposition and the division algorithm complemented one another in resolving this conflict. I will conclude my reflection by arguing the case for long division within the South African context.

Next I will focus on the shortcomings of this study, and will recommend areas that need more research in future. I will end this chapter with concluding remarks that will focus on the symbiotic nature of algebra and arithmetic.

The research focus

This study focuses on the cognitive gap that exists between arithmetic long division on naturals and algebraic long division on polynomials. There is currently no connection between the standard procedure used in primary school division problems and the standard long division procedure in high school mathematics. This study developed and implemented a didactical model in an endeavour to cross this cognitive divide between long division in primary school and long division in high school. The didactical model focused on the implementation of base ten decomposition on naturals as a division procedure which has its conceptual structure embedded in the concept of place value. Base ten decomposition was used in this study as it serves as a precursor for the algebraic equivalent by extracting the algebraic structure of the long division procedure on naturals. This model was implemented and reported on in this study.

Base ten division as an algorithm

Investigating base ten decomposition division as an algorithm contributes in preparing learners for the transition across the cognitive gap that exists between arithmetic long division on naturals and algebraic long division on polynomials. In this study the learners reflected on this experience in a positive manner pointing out that base ten division was *'full of numbers'* that made it appear more difficult. Their response to algebraic division on polynomials was positive and they repeatedly pointed out that *'this is much easier'* (referring to algebraic division on polynomials) as it does not contain *'so many numbers'*. The reason for this was that the power relationships in base ten was substituted with the variable x and learners now had to focus mainly on the coefficients of the x's to perform the division. Previously the focus was on two aspects of the decomposed number: i) the power relation of ten which emphasised place value in the division procedure and ii) the coefficients of these powers which were used to perform the actual division. Raising the cognitive load for arithmetic long division by considering long division in base ten decomposition appeared to have had the desired effect on their conceptual basis for division on naturals. The effect of increasing the cognitive load was twofold. Firstly, it challenged learners to deal with more than just the digits within the number (dividend) by shifting their focus to the specific place value of each digit, and secondly, it emulated the algebraic process with which they later divided polynomials into one another. For both the pilot group of Grade 9 learners and the current group of Grade 8 learners it made the transition to algebraic division a seamless exercise which had the effect that algebraic division on polynomials appeared easier for them than what is usually the case for learners. Traditionally this introduction to algebraic long division was not rooted in the cognitive grounds of arithmetic long division, but merely glanced at across the cognitive divide in terms of similarity of procedure. This superficial reference to arithmetic long division could be creating some of the problems experienced by learners when engaging with algebraic long division for the first time as the two processes are currently vastly different in their structure (variable) and their focus (coefficients). The cognitive architecture that base ten decomposition provides as a conceptual building block structures the two procedures closer by a process of emulation.

There are issues with the subtraction of one binomial expression from another within the long division procedure, and the sign change that must accompany this. Traditionally this happens more procedurally compared to the transition through base ten decomposition division. In this study the importance of multiple division representations emphasised the character of division as repeated subtraction, and learners were required to use this in a number of representations. When number was replaced with symbol this repeated use of subtraction consequently lead to a natural understanding of the subtraction procedure since the process emerged repeatedly during the base ten decomposition accompanied by the multiple representations of the division procedure.

The division algorithm and the remainder

One of the most challenging situations arose when learners were required to complete the traditional long division on naturals alongside the base ten decomposition division procedure. Throughout the different stages of the division lessons, it was emphasised that when learners summarise their findings, the division algorithm provides a balancing *tool* to which all different representations converge. Different remainders were arrived at when learners performed the division, and this created conflicting outcomes for the division procedure. It provided me with an unexpected opportunity to highlight to the learners yet again the special conditions set on the remainder and the divisor within the division algorithm. The outcome of these discussions was measured in the transition to symbol, where the learners soon realised that the division does not continue beyond the remainder as the variable which was introduced can now take on any value and is unknown to them. It was clear to me that, for this group of learners, the concept of the remainder across this transition from number to symbol was preserved and enriched by the conflict that arose when different remainders were obtained. This conflict also served to emphasise the transition from number to variable – thus from *known* to *unknown*, accompanied by the manner in which this transition affects the division procedure in the two domains.

This break in conceptual continuity and flow will not be possible to negotiate constructively without base ten decomposition division on naturals, as the cognitive leap is too vast without the proper didactical sequencing of division. It is more probable that the issues of the different remainders that are obtained by arithmetic and algebraic long division will not emerge naturally, let alone be discussed on a deeply conceptual level, without forcing such a discussion on a superficial level. Consider the problem $123 \div 12$ and the algebraic equivalent $\left(x^2+2x+3\right) \div \left(x+2\right)$. Both these problems lead to a remainder of three after the division has been completed:

$$\begin{array}{r} 11 \\ 11\overline{)123} \\ \underline{11} \\ 13 \\ \underline{11} \\ 2 \end{array} \qquad \text{and} \qquad \begin{array}{r} x+1 \\ \left(x+1\right)\overline{)x^2+2x+3} \\ \underline{x^2+x} \\ x+3 \\ \underline{x+1} \\ 2 \end{array}$$

Changing the divisor to 17, gives two different remainders and quotients:

$$\begin{array}{r} 7 \\ 17\overline{)123} \\ \underline{119} \\ 4 \end{array} \quad \text{and} \quad \begin{array}{r} x-5 \\ \left(x+7\right)\overline{)x^2+2x+3} \\ \underline{x^2+7x} \\ -5x+3 \\ \underline{-5x-35} \\ 38 \end{array}$$

Learners' experience with base ten decomposition will enable them to realise that for the algebraic equivalent, there are still multiples of 17 that can be removed from the remainder, assuming that $x = 10$ in this case. Such a conception of the long division procedure can only emerge once the learners have engaged with base ten decomposition as an algorithm for division.

Throughout the didactical model, the division algorithm takes central position within the division procedures that were presented to the learners as alternatives to arithmetic long division on naturals. The division algorithm was also used as a tool that can be operated upon within the clearly defined parameters as encapsulated in its definition. This continuously reminded the learners that these parameters must be taken seriously. These precisely defined parameters formed the bedrock on which their division attempts rested. They experienced the remainder as more than just "what is left over" and could judge the division process purely on the value of such a remainder. For this group of learners it became essential to validate the existence of the remainder in the light of the definition of the division algorithm, where the defining property of the remainder as a number which satisfies the condition $0 \le r < b$ was continuously used by them to validate their answers.

The division algorithm provides a powerful tool for the teaching of rational and irrational numbers, fractions, ratio and the decimal system. Conceptually supportive structures between arithmetic and algebra can be employed to assist learners to reason in the abstract and to validate their choices in this domain, and one such a universal tool is the division algorithm. It can be employed across various topics within algebra and arithmetic, but without the emphasis that is needed when learners engage with division on naturals for the first time, it is rendered powerless. This lack of appreciation of the division algorithm is a historical reality in the South African mathematics classroom. Could this possibly be the reason why concepts such as fractions, ratio, decimals, different base representations and long division find teachers encountering severe difficulty when teaching these concepts, with learners struggling to make sense of these attempts?

Consider the following problem: Convert 1234 to a base 7 number. First we divide 1234 by 7 and record our findings in the division algorithm.

$$\begin{gathered} 1234 = 7(176) + 2 \\ 176 = 7(25) + 1 \\ 25 = 7(3) + 4 \\ 3 = 7(0) + 3 \end{gathered}$$

Now by reading the remainders in reverse we get the answer. So 1234 = $(3412)_7$. Here the division algorithm is used to compute conceptually challenging content in a more procedural manner for those learners who experience difficulties with base work. This example is merely one of many where the division algorithm can be applied with great success.

The South African curriculum

Although the long division procedure on polynomials has been de-emphasised in the South African curriculum, the consequences of such a decision can only be measured once the learners have progressed onto higher learning at tertiary institutions where mathematics is a prerequisite for a specific course of study. This will only take place from 2008 onwards, as the current Grade 12 group will be the first cohort of learners who would have experienced the new curriculum in its totality. The problem that I foresee will be with those areas of study where polynomial division will be key to certain domains within a certain course, and it will be interesting to measure such difficulties that South African students may experience with the findings of researchers such as Klein and Milgram (2000) and Wu (1999, 2000, 2001) in the United States where this de-emphasis was severely criticised by a number of mathematics educators. I have pointed to these arguments in Chapter 2 of this study which indicates clearly that without fluency in the long division procedure, conceptual growth to more advanced areas of mathematics becomes problematic.

Within the study of algebraic fractions, learners will engage with the division of polynomials on a highly superficial level which will without a doubt reinforce general misconceptions regarding division. Here the issue that comes to mind is the simplification of algebraic fractional expressions where the equivalent division procedure does not leave a remainder and fully divides into the numerator to purposefully and superficially simplify such a fraction. This issue emerged in a 4th year class of pre-service mathematics teachers at the InterSen level. When asked to divide $x^2-4x-12$ by $x-1$, three of the fourteen students argued that the quadratic polynomial has to be factored in order to 'cancel' the denominator. They wrote the expression as $\frac{x^2-4x-12}{x-1}$ and were convinced that there was an error with the denominator. Their conception of this algebraic fraction and its relation to the division procedure was one of simplification, where the division with remainder procedure did not feature at all. To compound the underlying conceptual misunderstanding, these students declared that they have never dealt with the long division procedure or with the division algorithm throughout their schooling, and that this was all new to them. These three students have only studied mathematics up to Grade 9 level under the previous curriculum, and even though they could perform division on naturals using ordinary long division, there seemed to be a clear (cognitive) gap in their understanding of algebraic long division.

De-emphasising algebraic long division in the South African curriculum will certainly affect other areas of study in the new curriculum, and surely in the level at which certain thrusts, such as functions, are studied. In this context, methods of solving certain classes of problems will be carefully categorized and then taught separately so as to avoid division issues emerging. Consider the problem where the multiplication of $(x-3)(x+1)$ is related to learners as a rigid process called "foiling" (first, outer, inner, last) instead of a direct consequence of the distributive rule. Learners can immediately write x^2-2x-3 as an answer, but apply the principle of reversibility to the very same problem, and learners will deal with this as a factoring exercise instead of addressing division procedures as the inverse of multiplication. Adjusting the problem slightly to $(x-3)(x+1)+5$ will yield an answer of x^2-2x+2 but without the long division procedure on polynomials, is not that easily reversible. Opportunities for engaging with transformations will only happen at a very superficial level of "moving 5 units up" when relating x^2-2x-3 to x^2-2x+2 on the Cartesian plane. The problem might not seem so vast when engaging with quadratic expressions, but compounds enormously when higher order polynomials are studied.

Thus, when the focus of classroom discussion deals with underlying conceptual structure (as is specified in the curriculum documents) learners will, at least initially, be likely to be totally confused. In many South African classrooms students were expected to embrace algebraic long division without the concept development stages that occurred with the other three operations. This happened mainly at Grade 9 level, after learners had considerable experience with abstraction and symbol.

This study takes the position that the exclusion of the algebraic long division procedure will limit conceptual growth in the study of functions and the concept of factoring polynomials of degree higher than two. The de-emphasis of the division procedure completely removes the possibility of the division algorithm being employed in various contexts to enhance conceptually rich areas of study within mathematics and across subject fields. In agreement with Wu (1999), I urge that we should revise the way in which we teach division, both arithmetically and algebraically, before accepting its banishment from our classrooms.

Suggestions for the curriculum:

At Grade 6 level learners would engage with division of at least whole 4-digit by 3-digit numbers. It is my opinion that teachers should be discouraged from teaching the *concise division* method as this completely steers away from arithmetic long division and also the natural evolution towards the division algorithm. It performs the division procedure flawlessly and very compactly, but disregards place value in its totality. Place value division can therefore not be enhanced in preparation of the algebraic equivalent in division procedures through base ten decomposition. At Grade 6 level, I suggest that learners operate with base ten decomposition to solve problems by using the following:

$$\begin{array}{rccc}
 & 0 & 1 & 9 \\ \cline{2-4}
7\big) & 1 & 3 & 8 \\
 & \underline{0} & & \\
 & 1 & 3 & \\
 & & \underline{7} & \\
 & & 6 & 8 \\
 & & \underline{6} & \underline{3} \\
 & & & 5
\end{array}
\qquad \therefore 138 = 7(19) + 5$$

The second step would be to divide by the base ten decomposition:

$$7 = 10 - 3$$

$$138 = 100 + 30 + 8$$

$$\begin{array}{rl}
 & \qquad\quad 10+6 \\
(10-3) & \overline{)100+30+8} \\
 & \underline{100-30} \\
 & \qquad\quad 60+\ 8 \\
 & \qquad\quad \underline{60-18} \\
 & \qquad\qquad +26
\end{array}
\qquad\qquad \therefore 100+30+8=(10-3)(10+6)+26$$

Here the possibilities of discussing the different remainder emerge naturally as the two formulations in the division algorithm are not the same. This will open the possibility of operating on the division algorithm by considering the conditions that were not satisfied regarding the remainder and the divisor, thus resulting in the division continuing in the division algorithm:

$$\begin{aligned}
\therefore 100+30+8 &= (10-3)(10+6)+26 \\
&= 7(16)+21+5 \\
&= 7(16)+7(3)+5 \\
&= 7(16+3)+5 \\
&= 7(19)+5
\end{aligned}$$

At Grade 7 level, this process can be taken to decomposing the number as a power relation of ten which then could possibly introduce the concept of the coefficient (of ten) and its appropriate use in the division procedure. The natural flow to division by detached coefficients would be made possible here, as the redundancy of the base ten would become increasingly obvious to the learners when they compare the two different procedures.

Recommendations

Research has increasingly shown that utilizing the potentially algebraic nature of arithmetic is one way of building a stronger bridge between early arithmetical experiences and the concept of a variable (Kirshner, 1989; Kaput, 1992; Kieran, 1989; Subramaniam and Banerjee, 2004; Carraher et al, 2006; Sfard, 1991). This study was undertaken at the Grade 8 level where learners have already engaged with variable and algebraic representation. Despite positioning the division procedures alongside arithmetic division on naturals as the cognitive root, it would be useful to introduce the base ten decomposition division at an earlier stage of their schooling. This could be researched at Grade 6 level to draw more conclusively on the experience of learners with this algorithm.

At Grade 6 level, learners have had ample exposure to the concept of place value, division on naturals, base work and the division algorithm. Further research could focus on the base ten division in Grade 6 and follow this through to Grade 7 level where the division could return to detaching the coefficients from the base ten to divide synthetically and then return to the division algorithm reflecting the power relations in base ten yet again. The question remains then whether the increased cognitive load will be within the reach of the learners' conceptual understanding of the division concept that emulates the algebraic division procedure. This didactical sequence should work towards a natural progression that purposefully creates the need for abstraction, rather than structuring arithmetic in such a way that it prepares the learner for algebraic long division on polynomials. This approach to the division concept will support the under-explored symbiotic relationship that exists between algebra and arithmetic more explicitly.

Limitations of the study

This study and the pilot project were both conducted with learners that attend private schools and are mainly English first language speakers, with the exception of approximately 25% who spoke Mandarin or Afrikaans as a home language. These learners were mainly from an advantaged background. Although the implementation of the didactical model was successful with this group of learners, I do not presume to claim that the results will be generalisable. Further research will have to be conducted to include learners from the public school system and those who have been historically disadvantaged.

This study was conducted with Grade 8 learners who had experience in dealing with unknowns and variables in various contexts and the explicit use of the *x* in the division procedure was not as challenging for them to conceptualise as had been anticipated. Future research should focus at Grade 6 and Grade 7 level in an attempt to measure how the extraction of algebraic structure by base ten decomposition transfers to the explicit use of variable in the long division procedure.

In this study variable was introduced into the division procedure rather than evolving from the redundancy of the base ten power relations within the division by detached coefficients procedure. The decision to introduce the variable *x* was based on the learners' experience with variable and abstraction at Grade 8 level and learners were constantly reminded that the process is leading to divisions that contain a variable. The possible use of a variable was identified by two learners towards the end of the study, but was not the 'shared' experience of the group. Only after the variable was introduced to replace the powers of ten did the group experience the advantages and sense in introducing the *x*.

Concluding remarks

Bednarz et al (1996, in Van Amerom 2002) classify four different thrusts in algebraic curriculum design. *Learning algebra through generalising* emphasizes the product characteristic of arithmetic exploration, based on the generalization of the properties and relationships among numbers. Learners can explore the structure sense of arithmetic in order to generalize using algebraic thinking. *Algebra as a problem solving tool* takes the form of constructing and solving equations from contextual problems that are posed to learners. This forms a major theme in transition mathematics. *Learning algebra through* *modeling* is based on the need to create and develop flexible knowledge with learners where they make sense of the reality around them. By doing so, their sense of variable is enhanced through among others multiple representations of the same phenomena. In the *functional approach to algebra* learners analyze data and uncover the underlying mathematical structure to describe any trend in the data through an algebraic equation. It is important to consider these in classroom instructional design, and in assessment practices that are aimed at developing algebraic reasoning skills in the early grades. Even more important is the support and training that teachers receive from policymakers in order to guide them in the pedagogical shifts that are necessary in the development of learners' algebraic reasoning skills.

Polynomial structure which features both variable and coefficients of the variable, evolves naturally by considering base decompositions of natural numbers. Yet when learning about functions, polynomials of degrees one, two and three usually form the genesis of analyzing certain power relations, without referring to the conceptual ground that gave existence to these polynomials. Instead the idea of a function is used to explain a linear, quadratic or cubic relationship that finds its expression in a formalized polynomial structure.

Learning algebra through generalization as a problem solving tool that finds its application in modelling and ends by expressing this journey as a final elegantly defined mathematical function is an exciting journey. This journey has to emerge from a sound cognitive root which lies in the field of arithmetic exploration and nowhere else. Through such a journey mathematics becomes a powerful continuum with which learners' chaotic experiences become flexible and valuable. It forms knowledge which is structured sensibly and meaningfully. It is my opinion that this journey towards algebraic long division finds its meaning in base ten decomposition as an algorithm for division that successfully crosses the cognitive divide.

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