THE SPECTRAL THEORY OF COMPLEX STURM-LIOUVILLE OPERATORS

by

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ABSTRACT

We begin this work by considering the existence of solutions of certain ordinary differential equations, the solutions only being required to satisfy the equations almost everywhere. The usual statement of existence theorems for such equations gives sufficiency conditions only. It is shown that under very general circumstances, including for a general system of linear differential equations, these conditions are also necessary for the local, and sometimes global, existence of solutions. These results are then extended to differential equations involving Banach-space-valued functions, making use of the Bochner integral. Such a situation arises in the study of stochastic differential equations.

Attention is thereafter focussed on the operators generated by the Sturm-Liouville differential expression

$$\ell(y) = \sum_{i=0}^{n} (-1)^i (p_i y^{(n-i)})^{(n-i)}$$

considered on a fixed but arbitrary real interval I, where the coefficients $p_i (0 \le i \le n)$ are complex-valued functions. The case when the coefficients are real-valued has been well investigated: the expression $\ell(\cdot)$ is then formally symmetric and has associated selfadjoint operators; there is a spectral theorem for such operators, their spectrum lies on the real axis, and there are many results available concerning the location of the various parts of the spectrum. It is the main purpose here to extend some of these

results to allow the coefficients to be complex-valued. However, when they are complex-valued, $\ell(\cdot)$ is in general no longer formally symmetric, there can be no associated selfadjoint operators, and no spectral theorem is available. Relatively little work has been done on the operator theory in this case. In chapter two the appropriate operator theory, analogous to M.A. Naimark's work [58, §17], is developed for the complex case. It is based on the idea of J-symmetric operators, where J is a conjugation operator. This idea was first introduced by I.M. Glazman [24] and is applicable here because $\ell(\cdot)$ is formally J-symmetric if J denotes complex conjugation.

In subsequent chapters we concentrate on the second order case, namely,

$$\ell(y) = -(py')' + qy$$

on the interval $[a,\infty)$, $a > -\infty$, where p and q are complex-valued. In this case, if the regularity field is non-empty, the defect number (which corresponds to the deficiency indices in the real case) may be either one or two, a dichotomy which parallels for the complex case, the Weyl limit-point, limit-circle alternatives of the real case. In chapter three, two general results are given which provide conditions under which, not all solutions of the equations $\ell(y) = \lambda y (\lambda \in \mathbb{C})$ are in $L^2[a,\infty)$, and hence certain operators associated with $\ell(\cdot)$ are J-selfadjoint. These results extend to the complex case a number of well-known limit-point criteria.

In chapter four a number of results concerning the location of the essential spectrum of operators associated with $\ell(\cdot)$ are obtained. Some of these are extensions or generalisations of results due to Birman, and Glazman, whilst others are new. These lead to criteria for the non-emptiness of the regularity field of the minimal operator, a condition which is needed in the foregoing theory of J-selfadjoint extensions. A complete determination of the regularity field is made when $\ell(y) = \lambda y$ has two linearly independent solutions in $L^2[a,\infty)$ for some $\lambda \in \mathbb{C}$.

The final chapter begins with a generalisation to the complex case of Kupcov's inequality, from which are obtained analogues of many well-known limit-circle criteria for $\ell(\cdot)$. We conclude with some results on the location of the part of the point spectrum (i.e. the eigenvalues) which is embedded in the essential spectrum of the operators under consideration. Some of these results are extensions of results known when p and q are real-valued, whilst others are new.

To the Glory of God, Father, Son and Holy Spirit.

STATEMENT OF ORIGINALITY

I certify that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and that, to the best of my knowledge and belief, it does not contain any material previously published or written by another person except where due reference is made in the text.

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TABLE OF CONTENTS

INTRODUCTION

We make a few comments here concerning notation and terminology used in the text. There are a number of occasions when it would have interrupted the flow of the text to define some of the basic terms being used and we therefore do so here. We also note the symbol # is often used in order to separate the end of a proof from the beginning of the following text.

If I denotes an interval on the real line with end-points a and b respectively, by $L^p(I)$ or $L^p(a,b)$ we mean the space of all Lebesgue-measurable functions f, for which the Lebesgue integral of $|f|^p$ over I exists and is finite. We use AC(I) to denote the set of functions which are absolutely continuous on I. When one of the above spaces is subscripted loc, the corresponding functions are only required to satisfy the relevant condition locally, i.e. on any compact sub-interval of I. Unless otherwise stated, all measure and integrals are in the sense of Lebesgue, and a.e. is used to mean 'almost everywhere with respect to Lebesgue measure'.

In the latter part of chapter I we work in Banach spaces and Banach algebras, using strong derivatives and Bochner integrals. More details of these terms may be found in [32], where use is also made of $L^p(\Omega)$, the set of functions on a probability space Ω which are integrable to the p th power.

In chapter II we work in Hilbert spaces, denoting the inner product of two elements x and y by (x,y). In the Hilbert space

$L^2(a,b)$ we use the usual inner product:

$$(x,y) = \int_a^b x\bar{y}$$

Here, and elsewhere, we omit specifying the dependent variable with respect to which we are integrating, in the interest of brevity. It will be obvious in such cases, what the appropriate variable is. We also refer to the adjoint of an operator and to symmetric and selfadjoint operators. In particular, we make use of the fact that when the adjoint A^* of an operator A exists, it is a closed operator. This and other related properties may be found in [13].

Throughout the present work we use K, K_1, K_2,... to denote constants; $\emptyset$ denotes the empty set; $Re(c)$, $Im(c)$ denote the real and imaginary parts of c respectively; supp f denotes the support of the function f i.e. the smallest connected set of values, outside of which f is identically zero; $[f]^+$, $[f]^-$ denote the positive and negative parts of f respectively, being defined by

$$[f]^+ = \max\{f,0\} \qquad [f]^- = \max\{-f,0\}$$

Thus $f = [f]^+ - [f]^-$. Furthermore, we say $x(t) \sim y(t)$ as $t \to a$ if $x(t)/y(t) \to 1$ as $t \to a$ and finally, a finite-dimensional extension of an operator is an extension whose domain is a finite dimensional enlargement of the original domain.

CHAPTER I

CONDITIONS FOR EXISTENCE OF SOLUTIONS TO CERTAIN DIFFERENTIAL EQUATIONS

This chapter is primarily concerned with finding necessary and sufficient conditions for certain differential equations to have solutions. For some time the following result has been known.

<u>Theorem 1.1</u>. *In order for the differential equation*

$$-(pu')' + qu = 0 \quad \text{a.e. on } I, \tag{1.1}$$

where I *is a real interval, to have a solution satisfying*

$$\begin{aligned} &u,\ pu' \in AC_{loc}(I) \\ &u(\xi) = \alpha, \quad pu'(\xi) = \beta \end{aligned} \tag{1.2}$$

for arbitrary $\xi \in I$ *and arbitrary complex numbers* α, β, *it is necessary and sufficient that*

$$\frac{1}{p},\ q \in L_{loc}(I) \tag{1.3}$$

The proof of the sufficiency is well-known (see e.g. [58,§16]). The proof of the necessity is due to W.N. Everitt and may be found in [61, p.61]. It transpires that a similar result is true for a general system of first order linear differential equations.

We firstly state and prove the usual existence theorem for such a system. The most well-known method of proof (see for example [58]) is that which uses the Picard approximations but there is a neater, lesser-known proof which uses a modification of a technique found in [12], together with the following lemma. Firstly we define a contraction mapping.

A mapping T of a subset X_0 of a Banach space X, into X is said to be a contraction, if there exists a real number θ, $0 < \theta < 1$, such that

$$|Tx_1 - Tx_2| \leq \theta |x_1 - x_2| \tag{1.4}$$

for all vectors x_1, x_2 in X_0.

<u>Lemma 1.2</u>. [5] *If* F *is a closed subset of a Banach space, any contraction mapping of* F *into itself has a unique fixed point.*

In what follows, column vectors are denoted by $\underline{Y}$, $\underline{Z}$ etc. and square matrices by Y, Z etc.. I is an interval on the real line $\mathbb{R}$. In particular, $A = \{a_{ij}\}$ is an $n \times n$ matrix whose entries are measurable, complex-valued functions of the independent variable $x \in I$; the matrix I_n is the $n \times n$ identity matrix. Y' denotes the derivative of Y with respect to the independent variable x. All integrals and measure are in the sense of Lebesgue.

We then have the following existence and uniqueness theorem:

<u>Theorem 1.3</u>. *If* I *is a real interval and* $\xi \in I$ *is arbitrary then for any constant vector* $\underline{Y}_0 \in \mathbb{C}^n$, *there is a (unique) solution of*

$$\underline{Y}' = A\underline{Y} + \underline{F} \quad \text{a.e. on } I \tag{1.5}$$

with

$$\underline{Y}(\xi) = \underline{Y}_0 \tag{1.6}$$

and

$$y_i \in AC_{loc}(I) \quad (1 \le i \le n) \tag{1.7}$$

where $\underset{\sim}{F} = \{f_i\}$, $(1 \le i \le n)$ *is a vector of complex-valued, measurable functions of the independent variable* x, *and* $A = (a_{ij})$ *is an* $n \times n$ *matrix of such functions, provided that*

$$a_{ij}(\cdot), f_i(\cdot) \in L_{loc}(I) \quad (1 \le i,j \le n)$$

<u>Proof</u>. We may assume without loss of generality, that I is a closed interval, $[x_0, x_1]$, and then denote by X the set of all n-dimensional column vectors of functions which are continuous on $[x_0, x_1]$. Following A. Bielecki [8], we define the norm of any $\underset{\sim}{Y} \in X$ by

$$\|\underset{\sim}{Y}\| = \sup_{\substack{x_0 \le x \le x_1 \\ 1 \le i \le n}} \{e^{-K(x-x_0)} |y_i(x)|\} \tag{1.8}$$

where K is a fixed, positive constant to be chosen later. It is easily seen that with this norm, X is a Banach space. Let $T: X \to X$ be defined by

$$(T\underset{\sim}{Y})_i(x) = (\underset{\sim}{Y}_0)_i + \int_{x_0}^{x} \{A(t)\underset{\sim}{Y}(t) + F(t)\}_i \, dt \quad (1 \le i \le n) \tag{1.9}$$

Then for $\underset{\sim}{Y}_1, \underset{\sim}{Y}_2 \in X$, and for any $1 \le i \le n$ and $x \in I$

$$e^{-K(x-x_0)} |(T\underset{\sim}{Y}_1)_i(x) - (T\underset{\sim}{Y}_2)_i(x)|$$

$$= e^{-K(x-x_0)} \left| \int_{x_0}^{x} \{A(t)(\underset{\sim}{Y}_1(t) - \underset{\sim}{Y}_2(t))\}_i \, dt \right|$$

$$\le \int_{x_0}^{x} a(t) e^{-K(x-t)} \|\underset{\sim}{Y}_1 - \underset{\sim}{Y}_2\| \, dt$$

$$= \|\underline{Y}_1 - \underline{Y}_2\| \int_{x_0}^{x} e^{-K(x-t)} . a(t)dt$$

where $a(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t)|$. Since $a(\cdot) \in L(I)$, it may be approximated by a continuous function $b(\cdot)$ such that

$$\int_{x_0}^{x} |a(t) - b(t)|dt < \varepsilon < \frac{1}{2}$$

and we then define $L = \sup_{x_0 \le x \le x_1} |b(x)|$. This gives

$$e^{-K(x-x_0)} |(T\underline{Y}_1)_i(x) - (T\underline{Y}_2)_i(x)|$$

$$\le \|\underline{Y}_1 - \underline{Y}_2\| . \{ \int_{x_0}^{x} e^{-K(x-t)} |a(t) - b(t)|dt + \int_{x_0}^{x} e^{-K(x-t)} |b(t)|dt \}$$

$$\le \|\underline{Y}_1 - \underline{Y}_2\| . \{\varepsilon + L \int_{x_0}^{x} e^{-K(x-t)}dt\}$$

$$\le \|\underline{Y}_1 - \underline{Y}_2\| (\varepsilon + \frac{L}{K})$$

So choosing $K > 2L$,

$$\|T\underline{Y}_1 - T\underline{Y}_2\| \le (\varepsilon + \frac{L}{K}) \|\underline{Y}_1 - \underline{Y}_2\|$$

where $\varepsilon + \frac{L}{K} < 1$, and hence, applying lemma 1.2, there is a unique $\underline{Y} \in X$ such that $T\underline{Y} = \underline{Y}$ on I. This vector function $\underline{Y}(x)$ clearly satisfies the requirements of the statement of the theorem. ∎

The converse of this existence theorem is as follows:

Theorem 1.4. *If for any $\xi \in I$ and any $\underline{Y}_0 \in \mathbb{C}^n$ there is a solution*

$\underset{\sim}{Y} = \{y_i\}$, $(1 \le i \le n)$ *of the differential equation* (1.5) *satisfying* (1.6) *and* (1.7), *where* $f_i(\cdot)$, $a_{ij}(\cdot)$, $(1 \le i,j \le n)$ *are complex-valued, measurable functions, then* $a_{ij}(\cdot)$, $f_i(\cdot) \in L_{loc}(I)$ $(1 \le i,j \le n)$.

Proof. Firstly, suppose $\underset{\sim}{F} \equiv \underset{\sim}{0}$ on I. Choose any $\xi \in I$ then there are solutions $\underset{\sim}{Y}_1, \underset{\sim}{Y}_2, \ldots, \underset{\sim}{Y}_n$ of

$$\underset{\sim}{Y}' = A\underset{\sim}{Y} \quad \text{a.e.} \quad \text{on} \quad I \tag{1.10}$$

with

$$(\underset{\sim}{Y}_j)_i(\xi) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1.11}$$

and

$$(\underset{\sim}{Y}_j)_i \in AC_{loc}(I) \qquad (1 \le i,j \le n) \tag{1.12}$$

Whence, defining the matrix Y by $Y = \{(\underset{\sim}{Y}_j)_i\}$ we have from (1.12),

$$\det Y(\xi) = \det I_n = 1 \tag{1.13}$$

and $Y' = AY$ a.e. on I from (1.10). Further, from (1.12) and (1.13), $Y(\cdot)$ is invertible on some neighbourhood of ξ in I and Y^{-1} is continuous on this neighbourhood. Hence we may write

$$A = Y'.Y^{-1}$$

But then since $(Y^{-1})_{ij}$ is continuous and $(Y')_{ij} \in L_{loc}(I)$ $(1 \le i,j \le n)$ so

$$a_{ij} = (Y'.Y^{-1})_{ij} \in L(J(\xi))$$

where $J(\xi) \subseteq I$ is some neighbourhood of ξ. Hence if J is any compact interval contained in I, and we consider $\{J(\xi);\ \xi \in J\}$, then by the Heine-Borel theorem, there is a finite set $\{J(\xi_k);\ 1 \le k \le N\}$ which covers J. So since $a_{ij} \in L(J(\xi_k))$, $(1 \le i,j \le n$ and $1 \le k \le N)$,

$a_{ij} \in L(J)$. So $a_{ij} \in L_{loc}(I)$, $(1 \le i,j \le n)$.

Suppose now $\underset{\sim}{F} \neq \underset{\sim}{0}$. Choose any $\xi \in I$ then there is a solution $\underset{\sim}{Z}$ of $\underset{\sim}{Z}' = A\underset{\sim}{Z} + \underset{\sim}{F}$ a.e. on I with $z_i \in AC_{loc}(I)$ and $z_i(\xi) = 0$ $(1 \le i \le n)$. Then given any $\underset{\sim}{Y}_0 \in \mathbb{C}^n$ there is a solution $\underset{\sim}{Y}$ satisfying the conditions (1.5), (1.6), (1.7) and so

$$(\underset{\sim}{Y}-\underset{\sim}{Z})' = A(\underset{\sim}{Y}-\underset{\sim}{Z}) \quad \text{a.e. on I}$$

with $(\underset{\sim}{Y}-\underset{\sim}{Z})(\xi) = \underset{\sim}{Y}_0$ and $(\underset{\sim}{Y}-\underset{\sim}{Z})_i \in AC_{loc}(I)$ $(1 \le i \le n)$. Thus applying the result proved above for the case $\underset{\sim}{F} \equiv \underset{\sim}{0}$, we have $a_{ij} \in L_{loc}(I)$ $(1 \le i,j \le n)$. Hence $(A\underset{\sim}{Z})_i \in L_{loc}(I)$ since $z_i \in AC_{loc}(I)$ $(1 \le i \le n)$. Also $(\underset{\sim}{Z}')_i \in L_{loc}(I)$ and so $f_i = (\underset{\sim}{Z}')_i - (A\underset{\sim}{Z})_i \in L_{loc}(I)$ $(1 \le i \le n)$. ∎

Thus theorems 1.3 and 1.4 prove that the L_{loc} properties of the coefficients in the differential equation (1.5) are both necessary and sufficient for the existence of locally absolutely continuous solutions. Two particular applications of theorems 1.3 and 1.4 are worthy of explicit mention:

<u>Corollary 1.5</u>. *There is a solution of*

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = F \quad \text{a.e. on I} \tag{1.14}$$

with

$$y^{(i)}(\xi) = C_{i+1} \quad (0 \le i \le n-1) \tag{1.15}$$

and

$$y^{(i)} \in AC_{loc}(I) \quad (0 \le i \le n-1) \tag{1.16}$$

where F, a_i: $I \to \mathbb{C}$ *are measurable* $(1 \le i \le n)$ *and* $a_0 \neq 0$ *except possibly on a set of measure zero, for any* $\xi \in I$ *and any* $\underset{\sim}{C} \in \mathbb{C}^n$, *if and only if*

$$\frac{f}{a_0}, \frac{a_i}{a_0} \in L_{loc}(I) \qquad (1 \le i \le n) \tag{1.17}$$

Proof. Equation (1.14) may be written in the form (1.5) by defining $y_i = y^{(i-1)}$ $(1 \le i \le n)$, $f_i = 0$ $(1 \le i \le n-1)$, $f_n = \frac{f}{a_0}$ and (see [11; p.82])

$$A = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ \frac{-a_n}{a_0} & \frac{-a_{n-1}}{a_0} & \frac{-a_{n-2}}{a_0} & . & . & . & \frac{-a_1}{a_0} \end{bmatrix}$$

Then, (1.15), (1.16) are equivalent to (1.6), (1.7) and the result follows directly by applying theorems 1.3 and 1.4. #

In the next corollary, $y^{[0]}, y^{[1]}, \dots, y^{[2n]}$ denote the quasi-derivatives of the function y related to the given differential expression, as defined by M.A. Naimark [58; sections 15 and 16] by the formulae:

$$y^{[i]} = \frac{d^i y}{dx^i} \quad (0 \le i \le n-1), \quad y^{[n]} = p_0 \frac{d^n y}{dx^n},$$

$$y^{[n+i]} = p_i \frac{d^{n-i} y}{dx^{n-i}} - \frac{d}{dx}(y^{[n+i-1]}) \quad (1 \le i \le n) \tag{1.18}$$

Corollary 1.6. *There is a solution of*

$$(-1)^n (p_0 y^{(n)})^{(n)} + (-1)^{n-1} (p_1 y^{(n-1)})^{(n-1)} + \dots p_n y = f \quad \text{a.e. on } I \tag{1.19}$$

with

$$y^{[i]}(\xi) = C_{i+1} \quad (0 \le i \le 2n-1) \tag{1.20}$$

and

$$y^{[i]} \in AC_{loc}(I) \quad (0 \le i \le 2n-1) \tag{1.21}$$

where $f, p_i : I \to \mathbb{C}$ *are measurable* $(0 \le i \le n)$ *and* $p_0 \ne 0$ *except possibly on a set of measure zero, for any* $\xi \in I$ *and any* $\underline{C} \in \mathbb{C}^{2n}$, *if and only if*

$$f, \frac{1}{p_0}, p_i \in L_{loc}(I) \quad (1 \le i \le n) \tag{1.22}$$

__Proof.__ Equation (1.19) may be written in the form (1.5) by defining $y_i = y^{[i-1]}$ $(1 \le i \le 2n)$, $f_i = 0$ $(1 \le i \le 2n-1)$, $f_{2n} = -f$ and (see [58, p.49-56])

$$A = \begin{bmatrix} 0 & 1 & & & & & & 0 \\ & & \ddots & & & \ddots & & \\ & & & \ddots & & & \ddots & \\ & & & 1 & 0 & 0 & & \\ & & & & 1/p_0 & & & \\ & & & p_1 & & -1 & & \\ & & p_2 & & & & -1 & \\ & \ddots & & & & & \ddots & \\ & p_{n-1} & & & & & & -1 \\ p_n & & & & & & & 0 \end{bmatrix}$$

Then (1.20), (1.21) are equivalent to (1.6), (1.7) and the result follows directly by applying theorem 1.3 and 1.4. #

Theorem 1.1 is the special case of this corollary, obtained by setting n = 1.

It is noteworthy that in theorem 1.4 and its applications, the question of uniqueness is nowhere needed, so the mere existence of solutions implies the L_{loc} properties obtained, and then these properties imply, by theorem 1.3, not only the existence but also the uniqueness of solutions. In other words, existence implies uniqueness.

Theorem 1.4 may be extended, for example to the m th-order system of equations

$$\underset{\sim}{Y}^{(m)} + A_1\underset{\sim}{Y}^{(m-1)} + A_2\underset{\sim}{Y}^{(m-2)} + \dots + A_m\underset{\sim}{Y} = 0 \qquad (1.23)$$

where $\underset{\sim}{Y}$ is an n-dimensional vector, and where we must now also specify initial values for the first m - 1 derivatives of $\underset{\sim}{Y}$. To see this we write (1.23) in the form

$$\underset{\sim}{\mathcal{Y}}' = \begin{bmatrix} 0 & I_m & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & I_m & I_m & \cdot & \cdot & \cdot & 0 \\ & & \cdot & & & & \\ & & \cdot & & & & \\ & & \cdot & & & & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & I_m \\ -A_m & -A_{m-1} & -A_{m-2} & \cdot & \cdot & \cdot & -A_1 \end{bmatrix} \cdot \underset{\sim}{\mathcal{Y}}$$

where $\underset{\sim}{\mathcal{Y}}_{(i-1)n+j} = (\underset{\sim}{Y}^{(i-1)})_j$ $(1 \le j \le n \text{ and } 1 \le i \le m)$, so that we can apply theorem 1.4 to obtain $(A_k)_{ij} \in L_{loc}(I)$ $(1 \le i,j \le n \text{ and } 1 \le k \le m)$. Similarly for the inhomogeneous case,

$$\underset{\sim}{Y}^{(m)} + A_1\underset{\sim}{Y}^{(m-1)} + \dots + A_m\underset{\sim}{Y} + \underset{\sim}{F} = \underset{\sim}{0}$$

we obtain $(A_k)_{ij} \in L_{loc}(I)$ $(1 \le i,j \le n$ and $1 \le k \le n)$ and $f_i \in L_{loc}(I)$ $(1 \le i \le n)$.

A further extension permits a weakening of the linearity requirement in theorem 1.4, though the existence proof of theorem 1.3 is no longer valid. We assume

$$\underline{Y}' = A.\underline{f}(\underline{Y}) \quad \text{a.e. on } I$$

where $\underline{f}(\underline{Y}) = (f_1(y_1),\ldots,f_n(y_n))^T$ and $f_i : \mathbb{C} \to \mathbb{C}$ $(1 \le i \le n)$. Using the notation of the proof of theorem 1.4 we obtain $Y' = A.(\underline{f}(\underline{Y}_1),\ldots,\underline{f}(\underline{Y}_n))$; so if f_i is continuous $(1 \le i \le n)$ and assuming also that

$$\det \begin{bmatrix} f_1(1) & f_1(0) & \cdot & \cdot & \cdot & f_1(0) \\ f_2(0) & f_2(1) & \cdot & \cdot & \cdot & f_2(0) \\ & \vdots & & & & \\ f_n(0) & f_n(0) & \cdot & \cdot & \cdot & f_n(1) \end{bmatrix} \neq 0 \qquad (1.24)$$

then we may invert the matrix $(\underline{f}(\underline{Y}_1),\ldots,\underline{f}(\underline{Y}_n))$ and proceed as in the original proof to obtain $a_{ij} \in L_{loc}(I)$ $(1 \le i,j \le n)$. This is valid if, for example, each $f_i(\cdot)$ is a polynomial and (1.24) is satisfied.

Remarks: (i) Initial values of 0 and 1 were chosen for simplicity but the results hold true if we choose any set of complex numbers $\{c_{ij};\ 1 \le i,j \le n\}$ provided that

$$\det \begin{bmatrix} f_1(c_{11}) & \cdot & \cdot & \cdot & f_1(c_{1n}) \\ \vdots & & & & \\ f_n(c_{n1}) & \cdot & \cdot & \cdot & f_n(c_{nn}) \end{bmatrix} \neq 0$$

(ii) This result extends further, to the case when $f_i: \mathbb{C}^n \to \mathbb{C}$ is dependent on $y_1,\dots,y_n$ provided f_i is continuous in each argument and the corresponding determinant is non-zero.

Theorem 1.4 together with the subsequent remarks, corollaries and extensions above were included in [21]. One may ask whether similar necessary and sufficient conditions may be found if instead of using the ideas of Lebesgue integrals and absolute continuity, one uses some other, non-absolutely convergent integral such as Perron's integral.

Differential and integral equations involving Perron's integral have been considered by M.N. Manougian [54] (see also H. Bauer [6], E. Kamke [35], E.J. McShane [56], S. Saks [65]), who proved

Theorem 1.7. *Suppose*

(i) $f(t,y)$ *is continuous in* y *for* t *a.e. on* I.

(ii) $f(t,y(t))$ *is Perron-integrable on* I *for* y *continuous on* I

(iii) $f(t,y(t)) \geq g(t)$ a.e. *on* I *where* g *is Perron-integrable on* I

(iv) $|f(t,y_1(t)) - f(t,y_2(t))| \leq v(t)|y_1(t) - y_2(t)|$ a.e. *on* I

where v *is Perron- (and hence Lebesgue-) integrable on* I.

Then for the problem

$$y'(t) = f(t,y(t)), \quad y(t_0) = \alpha$$

there exists a Picard sequence which yields a continuous, locally absolutely continuous solution $\psi(t)$, *only if the sequence* $\{\int_0^t (f_n-g)\}$ *is equi-absolutely continuous on* I.

However if we consider $y'(t) = a(t)y(t)$ then condition (iv) becomes

$$|a(t)| \le v(t) \quad \text{a.e. on } I$$

and hence (see [35, p.210]) $a(\cdot)$ is Lebesgue integrable on I, in which case theorem 1.3 holds and one is reduced to the Lebesgue integral once more.

The usual method of proof of theorem 1.3 using Picard approximations breaks down if the coefficients are assumed to be only Perron-integrable, since the product of a continuous (and hence bounded) function and a Perron-integrable function is not necessarily Perron-integrable, whereas the corresponding result for Lebesgue-integrable functions is trivially true and is vital to the existence of the sequence of Picard approximations. The same problem prevents the defining of the operator T in the proof of theorem 1.3 given above.

Since the product of a Perron-integrable function and a function of bounded variation is Perron-integrable, one might reasonably seek solutions which are of bounded variation and are equal to the Perron-integral of their derivative. However this reduces to the above situation in theorem 1.3 precisely, as is shown by the following lemma.

<u>Lemma 1.8</u>. *If* $y(x) = P\int_a^x f$, $x \in [a,b]$ *and* y *is of bounded variation, then* $y(x) = L\int_a^x f$.

($P\int$ and $L\int$ denote the Perron- and Lebesgue- integrals respectively.)

<u>Proof</u>. Suppose y is increasing on intervals $[\alpha_i, \beta_i]$ $i = 1,2,\ldots$. We note that there can be at most countably many such intervals,

since y is continuous (see [56, p. 337]) and of bounded variation. Likewise there are at most countably many intervals $[\gamma_j,\delta_j]$ on which y is decreasing. Then the total variation of y on [a,b] is given by

$$\sum_i (y(\beta_i) - y(\alpha_i)) + \sum_j (y(\gamma_j) - y(\delta_j))$$

$$= \sum_i P\int_{\alpha_i}^{\beta_i} f - \sum_j P\int_{\gamma_j}^{\delta_j} f$$

But the total variation is bounded, and the above expression is equal to

$$\sum_i P\int_{\alpha_i}^{\beta_i} |f| + \sum_j P\int_{\gamma_j}^{\delta_j} |f| = P\int_a^b |f|$$

since $f \geq 0$ a.e. on each $[\alpha_i,\beta_i]$ and $f \leq 0$ a.e. on each $[\gamma_j,\delta_j]$. Thus $|f|$ is Perron-integrable and hence also Lebesgue-integrable, since any positive Perron-integrable function is Lebesgue-integrable, and

$$P\int_a^x f = L\int_a^x f, \quad x \in [a,b]$$

∎

It therefore seems that the use of the *Lebesgue* integral and the idea of absolutely continuous solutions is the natural setting for the above theory concerning the existence of solutions of linear differential equations.

However, there is another direction in which one may generalize theorems 1.3 and 1.4, namely to consider Banach-space-valued functions instead of complex-valued functions, replacing the

Lebesgue integral by its counterpart, the Bochner integral. In fact we consider functions taking values in a Banach-algebra Y *with identity* e. We then have an existence theorem corresponding to theorem 1.3:

Theorem 1.9. *If* I *is a real interval and* $\xi \in I$ *is arbitrary then for any constant* $\underline{Y}_0 \in Y^n$, *there is a (unique) strongly differentiable solution* $\underline{Y}: I \to Y^n$ *of*

$$\underline{Y}' = A\underline{Y} + \underline{F} \quad \text{a.e. on } I \tag{1.25}$$

with

$$\underline{Y}(\xi) = \underline{Y}_0 \tag{1.26}$$

and

$$y_i(x) - y_i(a) = (B)\int_a^x y_i'(t)\,dt \quad (1 \le i \le n) \tag{1.27}$$

where $\underline{F} = \{f_i\}$, $(1 \le i \le n)$ *is a vector of* Y *- valued, measurable functions of the independent variable* x, *and* $A = \{a_{ij}\}$ *is an* $n \times n$ *matrix of such functions, provided that* a_{ij}, f_i $(1 \le i,j \le n)$ *are locally Bochner-integrable over* I.

Note: (i) We use $(B)\int \cdot dt$ to denote the Bocher integral with respect to t.

(ii) All derivatives are *strong* derivatives.

Proof. As in the proof of theorem 1.3 we assume $I = [x_0, x_1]$. We denote by X the set of all n-dimensional column vectors of Y-valued functions which are continuous on $[x_0, x_1]$. We define the norm of any $\underline{Y} \in X$ by

$$\|\underline{Y}\|_1 = \sup_{\substack{x_0 \le x \le x_1 \\ 1 \le i \le n}} \{e^{-K(x-x_0)} \|y_i(x)\|\} \tag{1.28}$$

where $\|\cdot\|_1$ denotes the norm in X and $\|\cdot\|$ denotes the norm of the Banach algebra V, and K is a fixed, positive constant to be chosen later. It is easily seen that this does define a *norm* on X. The completeness of X follows from theorem 3.7.7 on p.82 of [32], and the fact that the elements of X are continuous on $[x_0,x_1]$. Thus X is a Banach space. Let $T: X \to X$ be defined by

$$(T\underline{Y}(x))_i = (\underline{Y}_0)_i + (B)\int_{x_0}^{x} (A(t)\underline{Y}(t) + \underline{F}(t))_i\,dt \quad (1 \le i \le n) \qquad (1.29)$$

Each $a_{ij}(\cdot)$ is Bochner-integrable and each y_0 is continuous and hence bounded. Since the product of a Bochner-integrable function with a bounded function is Bochner-integrable and since V is a Banach-algebra, the expression in (1.29) is well-defined. Then for $\underline{Y}_1,\underline{Y}_2 \in X$ and any $1 \le i \le n$ and $x \in I$,

$$e^{-K(x-x_0)}\|(T\underline{Y}_1)_i(x) - (T\underline{Y}_2)_i(x)\|$$

$$= e^{-K(x-x_0)}\|(B)\int_{x_0}^{x} \{A(t)(\underline{Y}_1(t) - \underline{Y}_2(t))\}_i\,dt\|$$

$$\le e^{-K(x-x_0)}\int_{x_0}^{x} \|\{A(t)(\underline{Y}_1(t) - \underline{Y}_2(t))\}_i\|\,dt$$

(see [32, p.76-89])

$$\le \int_{x_0}^{x} e^{-K(x-t)}.a(t).\|\underline{Y}_1 - \underline{Y}_2\|_1\,dt.$$

Here, $a(t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\|a_{ij}(t)\|$, which is hence in $L_{loc}(I)$. We use here the fact that in a Banach algebra, $\|\beta.\gamma\| \le \|\beta\|.\|\gamma\|$. The remainder of the proof follows the proof of theorem 1.3 mutatis mutandis. #

In the above existence theorem, the existence of an identity in Y was not needed. However, it is needed in the converse theorem, since the proof requires the existence of inverses of certain elements of Y. We also need the following lemma.

<u>Lemma 1.10</u>. *In the topology of the Banach-algebra Y, the points of a neighbourhood of e are invertible. If $y(t)$ belongs to this neighbourhood and is continuous for $t \in [t_0,t_1]$ then $y^{-1}(t)$ is continuous on $[t_0,t_1]$.*

<u>Proof</u>. Trivially, e itself is invertible. Consider any element $e + \alpha \in Y$ with $\|\alpha\| < 1$. Let $\beta = \sum_{n=0}^{\infty} (-1)^n \alpha^n$. Then $\beta \in Y$ since $\sum_{n=0}^{\infty} \|\alpha\|^n < \infty$ and Y is complete. But then

$$(e+\alpha)\beta = \beta(e+\alpha) = e$$

so $\beta = (e+\alpha)^{-1}$. The continuity of β when α is a continuous function of t follows from the definition of β. #

We may now prove the necessity of the integrability conditions on the coefficients of the differential equation, as in theorem 1.4.

<u>Theorem 1.11</u>. *If for any $\xi \in I$ there is a strongly differentiable solution $\underline{Y} = \{y_i\}$ $(1 \le i \le n)$ of the differential equation (1.25) which satisfies (1.27) and*

$$(Y)_i(\xi) = \delta_{ij}\, e \quad (1 \le i \le n) \tag{1.30}$$

for any given $1 \le j \le n$, where $\underline{F} = \{f_i\}$ $(1 \le i \le n)$ is a vector of Y-

valued, measurable functions of the independent variable x, then $a_{ij}(\cdot)$, $f_i(\cdot)$ $(1 \le i,j \le n)$ *are locally Bochner-integrable on* I.

Proof. As in the proof of theorem 1.4, we form the $n \times n$ fundamental matrix Y, then

$$\det Y(\xi) \neq 0$$

and since the elements of Y are continuous, detY and hence Y is invertible on a neighbourhood of ξ and the matrix Y^{-1} has continuous elements, applying lemma 1.10. So we may write

$$A = Y'.Y^{-1}$$

assuming $\underline{F} \equiv \underline{0}$. Following the proof of theorem 1.4 this yields the local Bochner-integrability of the elements of the matrix A, recalling that the product of a Bochner-integrable function and a bounded function is Bochner-integrable. The introduction of $\underline{F}$ also follows as before. ∎

Theorems 1.8 and 1.10 then combine to give necessary and sufficient conditions for the existence of solutions of differential equation (1.25) applied to V-valued functions where V is a Banach-algebra with identity. From this the obvious corollaries corresponding to cor. 1.5 and 1.6 follow. Likewise the remarks and extensions following cor. 1.5 and 1.6 also extend in an obvious way to Banach-algebra-valued functions.

Theorems 1.8 and 1.10 have certain applications in probability theory, where V is a space of functions on a probability space Ω. However, in that context, the space $L^p(\Omega)$, $1 \le p < \infty$ is often used, and since $L^p(\Omega)$ is not closed under multiplication and even if

$x, y, xy \in L^p(\Omega)$, in general $\|xy\|_p \not\leq \|x\|_p . \|y\|_p$, so it is not a Banach-algebra. The most important space used which *is* a Banach algebra is $L^\infty(\Omega)$ i.e. the set of functions on Ω which are measurable with respect to some probability measure μ on Ω and which are essentially bounded. f is essentially bounded if there exists some measurable set $N \subset \Omega$ such that $\mu(N) = 0$ and f is bounded on $\Omega - N$.

Since $L^p(\Omega)$ is not a Banach algebra and so the method of proof used in theorem 1.9 is no longer applicable, additional conditions of some sort are needed to obtain an existence theorem (see [71]). For example, if for $1 \leq i,j \leq n$,

(i) $a_{ij}(t) \in L^\infty(\Omega)$ for almost all $t \in I$

(ii) $a_{ij}(\cdot)$ is locally Bochner integrable on I with respect to $L^\infty(\Omega)$

(iii) $f_i(\cdot)$ is locally Bochner integrable on I with respect to $L^p(\Omega)$

then the method of proof of theorem 1.8 yields the existence (and uniqueness) of solutions of the differential equation, which take values in $(L^p(\Omega))^n$. This follows from the observation that

$$\|\alpha\beta\|_p \leq \|\alpha\|_\infty \|\beta\|_p$$

if $\alpha\beta, \beta \in L^p(\Omega)$ and $\alpha \in L^\infty(\Omega)$, and by letting

$$a(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \|a_{ij}(t)\|_\infty$$

which is well-defined for almost all $t \in I$. This existence and uniqueness theorem may also be obtained from theorem 9 of [71]. We note that the integrability of $a_{ij}(\cdot)$ $(1 \leq i,j \leq n)$ with respect to $L^\infty(\Omega)$ implies the integrability of $a_{ij}(\cdot)$ with respect to $L^p(\Omega)$.

It remains an open question whether the condition that the coefficients be locally Bochner integrable with respect to $L^\infty(\Omega)$ is also necessary when the assumption that they are pointwise essentially bounded elements of $L^p(\Omega)$ almost everywhere, is made. This results from the non-applicability of the method of proof used in theorem 1.11.

Concerning the applications of differential equations of the form considered above, we quote the following paragraph from T.T. Soong [69, p.217]:

'Intensive studies of differential equations with random coefficients have been undertaken only recently, and they have already exerted a profound influence on the analysis of a wide variety of problems in engineering and science. The earlier work in this area goes back to the investigation of the propagation of sound in an ocean of randomly varying refraction index [7]. Recently, a great number of physical problems which have been investigated falls into this class. In control and communication areas, for example, the behaviour of systems with randomly varying transfer characteristics or with random parametric excitations is described by differential equations with stochastic processes as coefficients. The same type of problems arises in chemical kinetics, economic systems, physiology, drug administration and many other areas. The study of systems and structures having imprecise parameter values or inherent imperfections also leads to differential equations of this type.'

In the remaining chapters of this present work however, we concentrate on the deterministic expression in (1.19) under the minimal conditions (1.22).

CHAPTER II

THE THEORY OF J-SYMMETRIC DIFFERENTIAL OPERATORS

If we consider the differential expression in (1.19), which we write as

$$\ell(y) = \sum_{i=0}^{n} (-1)^i (p_i y^{(n-i)})^{(n-i)} \tag{2.1}$$

on some real interval I, and assume each p_i to be real-valued, with $\frac{1}{p_0}, p_1, \ldots, p_n \in L_{loc}(I)$ then we may define the so-called 'minimal' closed operator, L_0, associated with $\ell(\cdot)$ and consider its extensions. It may be shown (see for example, [58,§18]) that all self-adjoint extensions of the minimal operator are characterized by suitable boundary conditions at the end-points of the interval. In particular, when n = 1, on the interval [a,∞) for any real a, the selfadjoint extensions of L_0 may be characterized by a boundary condition at a only, if and only if there is precisely one square-integrable solution of

$$\ell(y) = \lambda y \tag{2.2}$$

for some non-real number λ. This is known as the 'limit-point' case (see H. Weyl [76]). Most of the present chapter is devoted to the generalisation of this result to the situation when the coefficients are complex-valued, and to the background operator theory needed for this.

Initially we follow the methods of M.A. Naimark ([58,§17], see also [57]), making suitable modifications to allow for the fact that the coefficients are now complex-valued. These methods, however, do not readily give the characterization, in terms of boundary conditions, of the relevant extensions of L_0, and so for this we cite the results of M.A. Zhikhar, [77] and of I.W. Knowles, [46] in this direction.

Since the differential expression (2.1) is not in general formally symmetric, we can no longer expect to find self-adjoint operators associated with it. However, we may and do make use of the idea of a J-symmetric operator, introduced by I.M. Glazman in [24]. We firstly define an operator J in a Hilbert space H to be a *conjugation* if

(i) J is conjugate-linear

i.e. $J(\alpha x+\beta y) = \bar{\alpha}Jx + \bar{\beta}Jy$

for all scalars α, β and for all x, y in H

(ii) J is an involution i.e. $J^2 = I$

(iii) $(Jx,Jy) = (y,x)$ for all x, y in H

The particular such operator which we shall use is the operation of complex conjugation in the Hilbert space $L^2(a,b)$:

$$(Jf)(x) = \overline{f(x)} \tag{2.3}$$

for all f in $L^2(a,b)$.

A closed, densely defined operator A in a Hilbert space H is then defined to be *J-symmetric* if for some conjugation J,

$$JAJ \subset A^* \tag{2.4}$$

where A^* denotes the usual Hilbert space adjoint of A. Also, A is said to be *J-selfadjoint* if

$$JAJ = A^* \tag{2.5}$$

With these definitions, taking J to be the conjugation (2.3) and H to be $L^2(a,b)$, boundary conditions analogous to those of [58,§18] determine the J-selfadjoint extensions of L_0.

Following [58,p.49], we define the quasi-derivatives associated with (2.1) by (1.18). Initially we assume $\ell(\cdot)$ is defined on the interval (a,b) and work in $L^2(a,b)$. Following [58,§17] we define

$$\mathcal{D} = \{y \in L^2(a,b);\ y^{[k]} \in AC_{loc}(a,b),\ k=0,1,\dots,2n-1,$$
$$\ell(y) \in L^2(a,b)\} \tag{2.6}$$

and the operator L on $\mathcal{D}$ by

$$Ly = \ell(y),\ y \in \mathcal{D} \tag{2.7}$$

This operator is called the *maximal operator* in $L^2(a,b)$ corresponding to $\ell(\cdot)$. It is maximal in the sense that it cannot be extended as an operator in $L^2(a,b)$ generated by $\ell(\cdot)$. As is pointed out in [46], this is best seen by noting that

$$\mathcal{D} = \{y \in L^2(a,b);\ y \text{ is a solution of the equation}$$
$$\ell(y) = f \text{ for some } f \text{ in } L^2(a,b)\} \tag{2.8}$$

By a 'solution' is meant a function y which satisfies the differential equation $\ell(y) = f$ (regarded as a matrix differential equation, as in the proof of Corollary 1.6) on every compact sub-interval of (a,b). It will be seen later that L is a closed operator with a dense domain of definition.

We denote by $\mathcal{D}_0'$ the set of all functions y in $\mathcal{D}$ which vanish identically outside a finite interval $[\alpha,\beta] \subset (a,b)$; this interval may be different for different functions. In other words,

$$\mathcal{D}_0' = \{y \in \mathcal{D};\ \text{supp } y = [\alpha,\beta] \subset (a,b)\} \tag{2.9}$$

where supp y denotes the support of the function y. We denote the restriction of the operator L to $\mathcal{D}_0'$ by L_0' so then

$$L_0' \subset L \tag{2.10}$$

The following two properties of L_0' hold, where J is defined by (2.3) from now on, unless otherwise specified.

<u>Property 2.1</u>. *For arbitrary* $y \in \mathcal{D}_0'$, $\bar{z} \in \mathcal{D}$, *we have*

$$(L_0'y,z) = (y,JLJz) \tag{2.11}$$

<u>Proof</u>. Let $[\alpha,\beta] \subset (a,b)$ contain the support of the function $y \in \mathcal{D}_0'$ then for any $\bar{z} \in \mathcal{D}$,

$$\begin{aligned}\int_a^b \ell(y).\bar{z} &= \int_\alpha^\beta \ell(y).\bar{z} \\ &= [y,z]_\alpha^\beta + \int_\alpha^\beta y\ell(\bar{z}) \\ &= \int_a^b y.\ell(\bar{z})\end{aligned} \tag{2.12}$$

where

$$[y,z] = \sum_{k=1}^{n} \{y^{[k-1]}\bar{z}^{[2n-k]} - y^{[2n-k]}\bar{z}^{[k-1]}\} \tag{2.13}$$

by integrating by parts $2n$ times and using the fact that

$y^{[k]}, k = 0,1,\ldots,2n-1$ vanish outside (α,β). Since (2.11) and (2.12) are equivalent this proves that Property 2.1 holds. $\#$

If we knew $\mathcal{D}_0'$ was dense in $L^2(a,b)$ then equation (2.11) would imply

$$JLJ \subset L_0'^{*} \tag{2.14}$$

We now have

<u>Property 2.2</u>. *For arbitrary* $y, z \in \mathcal{D}_0'$,

$$(L_0'y, z) = (y, JL_0'Jz) \tag{2.15}$$

This follows from Property 2.1 since $\mathcal{D}_0' \subset \mathcal{D}$; thus L_0' is J-symmetric provided $\mathcal{D}_0'$ is dense in $L^2(a,b)$.

The differential expression $\ell(\cdot)$ is said to be *regular* if (a,b) is finite and $\frac{1}{p_0}, p_1,\ldots,p_n \in L(a,b)$; otherwise, $\ell(y)$ is said to be *singular*. For the moment we let $\ell(y)$ be regular in $[a,b]$, and denote

$$\mathcal{D}_0 = \{y \in \mathcal{D};\ y^{[k]}(a) = y^{[k]}(b) = 0,\ k = 0,1,\ldots,2n-1\} \tag{2.16}$$

Clearly,

$$\mathcal{D}_0' \subset \mathcal{D}_0 \tag{2.17}$$

and we may define the operator L_0 on $\mathcal{D}_0$ by

$$L_0y = Ly,\ y \in \mathcal{D}_0 \tag{2.18}$$

From the definitions of the operators L_0', L_0, L,

$$L_0' \subset L_0 \subset L \tag{2.19}$$

The following properties of L_0 may be derived in the same way as Properties 2.1, 2.2, the crucial factor being the boundary conditions in (2.16).

<u>Property 2.3</u>. *For arbitrary* $y \in \mathcal{D}_0$, $\bar{z} \in \mathcal{D}$, *we have*

$$(L_0 y, z) = (y, JLJz) \tag{2.20}$$

<u>Property 2.4</u>. *For arbitrary* y, $\bar{z} \in \mathcal{D}_0$ *we have*

$$(L_0 y, z) = (y, JL_0 Jz) \tag{2.21}$$

Thus, provided $\mathcal{D}_0$ is dense in $L^2[a,b]$, we have

$$JLJ \subset L_0^* \tag{2.22}$$

We now need the following two lemmas.

<u>Lemma 2.5</u>. *Let* $\ell(\cdot)$ *be regular in* $[a,b]$ *and let* f *be a function in* $L^2(a,b)$. *The equation*

$$\ell(y) = f \tag{2.23}$$

has a solution y *satisfying*

$$y^{[k]}(a) = y^{[k]}(b) = 0,\ k = 0,1,\ldots 2n-1 \tag{2.24}$$

if and only if f *is orthogonal to all solutions* y *of* $\ell(\bar{y}) = 0$.

<u>Proof</u>. Let y be a solution of (2.23) such that $y^{[k]}(a) = 0$, $k = 0,1,\ldots,2n-1$. By [58,§16.2, theorem 2] there is precisely one such solution. Denote by $z_1, z_2, \ldots, z_{2n}$ the functions which satisfy

$$\left.\begin{aligned} &\ell(\bar{z}_i) = 0 \\ &z_i^{[k-1]}(b) = \begin{cases} 0 & \text{for } i \neq k \\ 1 & \text{for } i = k \end{cases} \end{aligned}\right\} (1 \le i \le 2n) \tag{2.25}$$

Then

$$(f,z_i) = (\ell(y),z_i) = [y,z_i]_a^b + (y,\overline{\ell(\bar{z}_i)})$$

But then from (2.25), (2.13) and the boundary conditions on y,

$$(f,z_i) = [y,z_i](b)$$

$$= \begin{cases} -y^{[2n-i]}(b), & i = 1,\ldots,n \\ y^{[2n-i]}(b), & i = n+1,\ldots,2n \end{cases}$$

Thus $y^{[k]}(b) = 0$, $k = 0,1,\ldots,2n-1$ if and only if $(f,z_i) = 0$, $i = 1,\ldots,2n$ as required. #

Let $M = \{z;\ell(\bar{z}) = 0\}$ then $M \subset L^2(a,b)$ and is a finite-dimensional subspace in $L^2(a,b)$, of dimension $2n$. Denoting by R_0 the range of the operator L_0, Lemma 2.5 then states that $f \in R_0$ if and only if f is orthogonal to M. In other words,

$$L^2(a,b) = R_0 \oplus M \tag{2.26}$$

Lemma 2.6. *For arbitrary complex numbers* $\alpha_0, \alpha_1,\ldots,\alpha_{2n-1}$, $\beta_0, \beta_1,\ldots,\beta_{2n-1}$ *there is a function* $y \in \mathcal{D}$ *satisfying*

$$y^{[k]}(a) = \alpha_k,\ y^{[k]}(b) = \beta_k,\quad k = 0,1,\ldots,2n-1 \tag{2.27}$$

The proof of this lemma follows that of the same lemma for real-valued coefficients in [58,§17.3] noting that now, $(\ell(v),z) = (v,\overline{\ell(\bar{z})})$. We are now able to proceed:

Property 2.7. *$\mathcal{D}_0$ is dense in* $L^2(a,b)$

Proof. We need to show that every element h orthogonal to $\mathcal{D}_0$ is zero. Let h be such an element then $(h,y) = 0$, $y \in \mathcal{D}_0$. Let z be a solution of $\ell(\bar{z}) = \bar{h}$ then by Property 2.3, for any $y \in \mathcal{D}_0$,

$$(L_0y,z) = (y,JLJz) = (y,h) = \overline{(h,y)} = 0$$

so z is orthogonal to R_0, hence by formula (2.26), $z \in M$ i.e. $h = \overline{\ell(\bar{z})} = 0$.

#

From Properties 2.4, 2.7 it follows that L_0 is a J-symmetric operator and L_0^* exists.

<u>Property 2.8</u>. $L_0^* = JLJ$

<u>Proof</u>. Having [illegible] that (2.22) holds, it only remains to prove

$$L_0^* \subset JLJ \tag{2.28}$$

Let g be any element in the domain $\mathcal{D}_g^*$ of the operator L_0^* and put

$$h = L_0^*g \tag{2.29}$$

Let z be any particular solution of $\ell(\bar{z}) = \bar{h}$ then for every $y \in \mathcal{D}_0$,

$$(h,y) = (\overline{\ell(\bar{z})},y) = (JLJz,y) = (z,L_0y).$$

But,

$$(h,y) = (L_0^*g,y) = (g,L_0y)$$

by the definition of L_0^*. Thus

$$(z-g,L_0y) = 0, \quad y \in \mathcal{D}_0$$

and hence $z - g$ is orthogonal to R_0. Then by (2.26) we know $z - g \in M$. However, $JM \subset \mathcal{D}$ so $\bar{z} - \bar{g} \in \mathcal{D}$. Since $\bar{z} \in \mathcal{D}$ we must also have $\bar{g} \in \mathcal{D}$ and indeed $LJ(z-g) = 0$, therefore

$$L\bar{g} = L\bar{z} = \bar{h}$$

$$\text{i.e. } JLJg = h = L_0^*g$$

which proves (2.28) and hence Property 2.8.

Property 2.9. $L_0 = JL^*J$

Proof. From Property 2.8 we have

$$L_0 \subset L_0^{**} = (JLJ)^* \tag{2.30}$$

Firstly we show

$$(JLJ)^* \subset JL^*J \tag{2.31}$$

Let the domain of $(JLJ)^*$ be $\mathcal{D}_1$ and take any $f \in \mathcal{D}_1$. Then

$$(y,(JLJ)^*f) = (JLJy,f), \quad \bar{y} \in \mathcal{D}$$

$$\text{i.e.} \quad (y,(JLJ)^*f) = \overline{(L\bar{y},\bar{f})}$$

hence

$$(\bar{y},\overline{(JLJ)^*f}) = (L\bar{y},\bar{f}), \quad \bar{y} \in \mathcal{D}$$

Hence $\bar{f} \in \mathcal{D}^*$, the domain of the operator L^*, and $L^*\bar{f} = \overline{(JLJ)^*f}$ i.e. f is in the domain of JL^*J and $(JL^*J)f = (JLJ)^*f$. So (2.31) holds and hence

$$L_0 \subset JL^*J \tag{2.32}$$

So we need to prove

$$JL^*J \subset L_0 \tag{2.33}$$

Applying the operation * to both sides of the inclusion $L_0 \subset L$ we have that $L^* \subset L_0^*$ and hence by Property 2.8,

$$JL^*J \subset L \tag{2.34}$$

Now let z be in the domain of JL^*J i.e. $\bar{z} \in \mathcal{D}^*$. Then from (2.34), $z \in \mathcal{D}$ and $L^*\bar{z} = \overline{Lz}$ and

$$(L^*\bar{z},\bar{y}) = (\bar{z},L\bar{y}), \quad \bar{y} \in D$$

$$\text{i.e. } (\overline{Lz},\bar{y}) = (\bar{z},L\bar{y}), \quad \bar{y} \in D$$

But

$$(\overline{Lz},\bar{y}) = [\bar{z},\bar{y}]_a^b + (\bar{z},L\bar{y})$$

hence

$$[\bar{z},\bar{y}]_a^b = 0, \quad \bar{y} \in D \tag{2.35}$$

But then by Lemma 2.6, $\bar{y} \in D$ may be chosen with *any* boundary values so (2.35) is only possible if

$$z^{[k]}(a) = z^{[k]}(b) = 0, \ k = 0,1,\ldots,2n-1 \quad \text{i.e. } z \in D_0.$$

Hence (2.33) is proved, as required. #

Collecting the above results together we have

Theorem 2.10. *If the boundary points* a *and* b *are regular, then the operator* L_0 *is a closed, J-symmetric operator and*

$$L_0^* = JLJ.$$

Property 2.11. *The operator* L_0 *is the closure of* L_0': $L_0 = \overline{L_0'} = L_0'^{**}$

Proof. Since $L_0' \subset L_0$ and L_0 is closed, the closure $\overline{L_0'}$ of L_0' satisfies

$$\overline{L_0'} \subset L_0$$

so we need only prove $L_0 \subset \overline{L_0'}$. We let Δ be an interval $[\alpha,\beta] \subset (a,b)$ and consider $L^2(\alpha,\beta)$. We consider the expression $\ell(\cdot)$ on Δ, where it must be regular, and denote the corresponding operators by $L_{0,\Delta}$ and L_Δ. Then as above,

$$L_{0,\Delta}^* = JL_\Delta J \tag{2.36}$$

The space $L^2(\alpha,\beta)$ can be embedded in $L^2(a,b)$ by setting each

$y \in L^2(\alpha,\beta)$ equal to zero outside of Δ. In this process $\mathcal{D}_{0,\Delta}$ becomes part of $\mathcal{D}$. Every function so continued out of $\mathcal{D}_{0,\Delta}$ will belong to $\mathcal{D}_0'$. If then, $z \in \mathcal{D}_0'^*$,

$$(L_0'^* z, y) = (z, L_0' y), \quad y \in \mathcal{D}_{0,\Delta} \tag{2.37}$$

But $y = 0$ outside Δ so the above scalar products may be written as integrals over Δ i.e. are scalar products in $L^2(\alpha,\beta)$. Denoting such scalar products with the subscript Δ, we can rewrite (2.37) as

$$((L_0'^* z)_\Delta, y)_\Delta = (z_\Delta, L_{0,\Delta} y)_\Delta \tag{2.38}$$

where $(L_0'^* z)_\Delta$ and z_Δ denote the restrictions of $L_0'^* z$ and z to the interval Δ. Since (2.38) holds for all $y \in \mathcal{D}_{0,\Delta}$ so

$$z_\Delta \in \mathcal{D}^*_{0,\Delta}, \quad \bar{z}_\Delta \in \mathcal{D}_\Delta$$

and

$$(L_0'^* z)_\Delta = JL_\Delta Jz = (\overline{\ell(\bar{z})})_\Delta \tag{2.39}$$

Since these relations hold for an arbitrary interval Δ we conclude that

$$\bar{z} \in \mathcal{D}; \quad L_0'^* z = \overline{\ell(\bar{z})} = JLJz.$$

Thus

$$L_0'^* \subset JLJ \tag{2.40}$$

Hence

$$L_0'^{**} \supset (JLJ)^*$$

But $(JLJ)^* \supset JL^*J$ since if we take any $\bar{f} \in \mathcal{D}^*$ then

$$(y, JL^*Jf) = \overline{(\bar{y}, L^*\bar{f})} = \overline{(L\bar{y}, \bar{f})}$$

$$= (JLJy, f), \quad \bar{y} \in \mathcal{D}$$

and so f is in the domain of $(JLJ)^*$ and $(JLJ)^* f = JL^* Jf$. Thus

$$L_0^{'**} \supset JL^* J = L_0 \tag{2.41}$$

by Property 2.9, and since $\mathcal{D}_0^{'}$ is dense in $\mathcal{D}_0$ and hence in $L^2(a,b)$, $\bar{L}_0^{'} = L_0^{'**}$.

Thus $\bar{L}_0^{'} \supset L_0$ as required.

In the course of the above proofs we have proved the following lemma:

<u>Lemma 2.12</u>. *If* A *is an operator in a Hilbert space such that* A^* *exists then*

$$JA^* J = (JAJ)^* \tag{2.42}$$

So far we have concentrated on the regular case. We now consider the case when at least one of the points a and b is singular. It is no longer possible to define the operator L_0 directly by means of boundary conditions such as in (2.16). We therefore define L_0 in a different way by starting from the operator $L_0^{'}$. Firstly we prove

<u>Property 2.13</u>. *The domain* $\mathcal{D}_0^{'}$ *of* $L_0^{'}$ *is dense in* $L^2(a,b)$.

<u>Proof</u>. We need to show that any $h \in L^2(a,b)$ which is orthogonal to $\mathcal{D}_0^{'}$, is zero. Let h be such an element and take any fixed interval $\Delta = [\alpha,\beta] \subset (a,b)$. Every $y \in \mathcal{D}_{0,\Delta}$ can be regarded as an element of $\mathcal{D}_0^{'}$ so h is orthogonal to $\mathcal{D}_{0,\Delta}$. However, Property 2.7 implies $\mathcal{D}_{0,\Delta}$ is dense in $L^2(\alpha,\beta)$, so h vanishes almost everywhere in Δ. Since $\Delta \subset (a,b)$ was arbitrary, it follows that $h = 0$ a.e. in (a,b), as required.

Properties 2.2, 2.13 imply that $L_0^{'*}$ exists and $L_0^{'}$ is J-symmetric i.e. $L_0^{'} \subset JL_0^{'*}J$. Since J and $L_0^{'*}$ are closed operators it

follows that L_0' admits a closure. We denote the closure of L_0' by L_0

$$\text{i.e.} \quad L_0 = \overline{L_0'} \tag{2.43}$$

From this definition it follows that L_0 is a closed, J-symmetric operator, and in the regular case this definition of L_0 agrees with the definition given earlier by (2.16), (2.18). We now consider certain properties of L_0.

<u>Property 2.14</u>. $\qquad L_0^* = JLJ$

<u>Proof</u>. Since $\overline{L_0'} = L_0$, we have $L_0'^* = L_0^*$. It therefore suffices to show that

$$L_0'^* = JLJ \tag{2.44}$$

But since (2.14) holds, it only remains to prove

$$L_0'^* \subset JLJ \tag{2.45}$$

The proof is similar to that of Property 2.11. Let $z \in \mathcal{D}_0'^*$ and let $\Delta = [\alpha,\beta] \subset (a,b)$ be fixed, then as above,

$$((L_0'^* z)_\Delta, y)_\Delta = (z_\Delta, JL_{0,\Delta}Jy), \quad \bar{y} \in \mathcal{D}_{0,\Delta}$$

hence $z_\Delta \in \mathcal{D}_\Delta$ and $(L_0'^* z)_\Delta = JL_\Delta Jz_\Delta = \overline{\ell(\bar{z})}_\Delta$. Since $\Delta \subset (a,b)$ is arbitrary, it follows that

$$\bar{z} \in \mathcal{D}, \quad L_0'^* z = \overline{\ell(\bar{z})} = JLJz$$

and this implies (2.45) as required. $\qquad \#$

From Property 2.14 it follows that L is closed.

Thus, as in the regular case, we have

Theorem 2.15. *The operator* L_0 *is a closed, J-symmetric operator and* $L_0^* = JLJ$.

In order to find a more direct definition of L_0 we prove:

Property 2.16. *For any* $y,z \in \mathcal{D}$, *the limits* $[y,\bar{z}]_a = \lim_{x \to a}[y,\bar{z}]_x$ *and* $[y,\bar{z}]_b = \lim_{x \to b}[y,\bar{z}]_x$ *exist.*

Proof. We apply Lagrange's identity to the functions y, z in the interval $[\alpha,\beta] \subset (a,b)$, as in 2.12

$$\int_\alpha^\beta \ell(y).z = [y,\bar{z}]_\beta - [y,\bar{z}]_\alpha + \int_\alpha^\beta y.\ell(z) \tag{2.46}$$

Each integral has a limiting value as $\beta \to b$ so we infer that $\lim_{\beta \to b}[y,\bar{z}]_\beta$ also exists. Similarly $\lim_{\alpha \to a}[y,\bar{z}]_\alpha$ exists. ∎

Letting $\alpha \to a$ and $\beta \to b$ in (2.46) yields

$$\int_a^b \ell(y).z = [y,\bar{z}]_b - [y,\bar{z}]_a + \int_a^b y.\ell(z), \quad y,z \in \mathcal{D} \tag{2.47}$$

Property 2.17. $\mathcal{D}_0$ *consists precisely of those* $y \in \mathcal{D}$ *which satisfy*

$$[y,\bar{z}]_b - [y,\bar{z}]_a = 0 \quad \textit{for all} \quad z \in \mathcal{D} \tag{2.48}$$

Proof. Since L_0 is closed, we have from Property 2.14,

$$L_0 = L_0^{**} = (JLJ)^* \tag{2.49}$$

Therefore $\mathcal{D}_0$ consists of precisely those $y \in \mathcal{D}$ which satisfy

$$(y, JLJ\bar{z}) = (L_0 y, \bar{z}) \quad \text{for all } z \in \mathcal{D} \tag{2.50}$$

By formula (2.47) this is equivalent to (2.48). ∎

If we know that the end-point a is regular but b is singular then Property 2.17 may be sharpened to

Property 2.18. *If the end-point* a *is regular and* b *is singular then* $\mathcal{D}_0$ *consists precisely of those* $y \in \mathcal{D}$ *which satisfy*

(i) $y^{[k]}(a) = 0, \quad k = 0,1,\ldots,2n-1$

(ii) $[y,\bar{z}]_b = 0$ *for all* $z \in \mathcal{D}$

Proof. If these conditions hold then $y \in \mathcal{D}_0$ by Property 2.17, proving the sufficiency of (i) and (ii). To prove their necessity, let $y \in \mathcal{D}_0$ and $z \in \mathcal{D}$ be such that z is identically zero outside of some finite interval $[a,\beta] \subset [a,b)$. By Property 2.17, $[y,\bar{z}]_b - [y,\bar{z}]_a = 0$ and hence since $z(x) = 0$ for $x > \beta$, $[y,\bar{z}]_b = 0$ and so

$$[y,\bar{z}]_a = 0 \tag{2.51}$$

But the values of $z^{[k]}(a)$, $k = 0,1\ldots,2n-1$ can be prescribed arbitrarily (see Lemma 2.6); consequently (2.51) is possible only if (i) above is satisfied. It then follows from Property (2.17) that (ii) must also be satisfied.

In the case of a symmetric operator L_0, further results are obtained by making use of its deficiency indices. When L_0 is only known to be J-symmetric an analogue is needed. This analogue was provided by N.A. Zhikhar [77] (see also [46]), and makes use of the following definitions:

If A is a linear operator (in a Hilbert space) with domain $\mathcal{D}_A$ and λ is a complex number then the manifold $\Delta_\lambda(A)$ is defined to be the range of A - λI, I being the identity operator; thus

$$\Delta_\lambda(A) = (A-\lambda I)\mathcal{D}_A \tag{2.52}$$

A complex number λ is called a *point of regular type* of the closed operator A if for all $\phi \in \mathcal{D}_A$,

$$\|(A-\lambda I)\phi\| > k_\lambda \|\phi\| \tag{2.53}$$

where k_λ is independent of the choice of ϕ. The set of all such numbers λ is called the *regularity field* of A, and is denoted by $\pi(A)$. It follows from (2.52) and (2.53) that this set is open and that for $\lambda \in \pi(A)$, $\Delta_\lambda(A)$ is closed.

The following two theorems are due to Zhikhar [77], but the proofs may also be found in [46].

Theorem 2.19. *Let A be a J-symmetric operator in a Hilbert space H. If $\Delta_\lambda(A) = H$ for some λ, then A is J-selfadjoint.*

Theorem 2.20. *Assume A is a J-symmetric operator with $\pi(A)$ not empty and $\lambda_0 \in \pi(A)$ given. Then there exists a J-selfadjoint extension A' of A such that $\lambda_0 \in \pi(A')$.*

If A is a closed, densely defined operator in a Hilbert space H, following [46] we define an extension $\tilde{A}$ of A to be *well-posed* if $\pi(\tilde{A})$ is not empty. From the definition of the regularity field it is clear that

$$\pi(\tilde{A}) \subset \pi(A) \tag{2.54}$$

for any extension $\tilde{A}$ of A. In [77] an extension $\tilde{A}$ of A is said to be *correct* if for some fixed λ_0 in $\pi(A)$, λ_0 is in $\pi(\tilde{A})$. This could more properly be termed 'well-posed relative to λ_0'. It seems likely, though no examples have yet been found, that a given extension $\tilde{A}$ of A could be well-posed relative to one value of λ_0, and not well-posed relative to other values. Clearly, $\tilde{A}$ is a well-posed extension of A if and only if it is well-posed with respect to some λ_0 in $\pi(A)$. Such extensions are 'well-posed' in the sense that there exist values of λ such that the problem $(\tilde{A}-\lambda I)x = f$ is at least normally solvable for x.

We now need the following result.

Theorem 2.21. ([77]; see also [46]). *Assume A is a J-symmetric operator with $\pi(A)$ not empty and for an arbitrary $\lambda_0 \in \pi(A)$ let A' be a J-selfadjoint extension of A for which $\lambda_0 \in \pi(A')$. Then the domain of JA^*J, $\mathcal{D}_{JA^*J}$ satisfies*

$$\mathcal{D}_{JA^*J} = \mathcal{D}_A \dotplus (A'-\lambda_0 I)^{-1} N_{\bar{\lambda}_0} \dotplus JN_{\bar{\lambda}_0} \qquad (2.55)$$

where $N_{\bar{\lambda}_0}$ is the subspace of solutions of

$$A^*u - \bar{\lambda}_0 u = 0 \qquad (2.56)$$

In (2.55), the symbol $\dotplus$ denotes a direct sum that need not be an orthogonal direct sum. The existence of at least one extension A' of A with λ_0 in $\pi(A')$ is guaranteed by Theorem 2.20. Decompositions of the form (2.55) seem to have been first introduced by M.I. Vishik [73] for a slightly different situation.

Let us now denote by $m(\lambda_0)$ the dimension of the subspace $N_{\bar{\lambda}_0}$. Clearly, $0 \le m(\lambda_0) \le \infty$. Since $(A'-\lambda_0 I)^{-1}$ is one-to-one (see [46]) it follows that the dimension of $\mathcal{D}(A^*)$ modulo $\mathcal{D}(A)$ is equal to $2m(\lambda_0)$, which must therefore be independent of the choice of λ_0 in

$\pi(A)$. Thus $m(\lambda_0)$ has a constant value over $\pi(A)$. We are now able to define the analogue for J-symmetric operators of the deficiency indices of a real symmetric operator.

Let A be a closed, densely defined, J-symmetric operator with $\pi(A)$ not empty. For any λ_0 in $\pi(A)$ we define the *defect number* of A, written def (A), to be the dimension of the subspace $N_{\bar{\lambda}_0}$, i.e. the number of linearly independent solutions of (2.56). This definition is given in [77]. In [25], def (A) is defined to be the dimension of the orthogonal complement of $(A-\lambda_0 I)\mathcal{D}_A$. That these two definitions are equivalent is shown by the following result, using $\oplus$ to denote the orthogonal direct sum.

<u>Lemma 2.22</u>. *If* A *is a closed, densely defined, J-symmetric operator in a Hilbert space* H *and if* $\lambda_0 \in \pi(A)$ *then*

$$H = (A-\lambda_0 I)\mathcal{D}_A \oplus N_{\bar{\lambda}_0} \tag{2.57}$$

<u>Proof</u>. If $u \in N_{\bar{\lambda}_0}$, i.e. u satisfies (2.56) then for any $v \in \mathcal{D}_A$,

$$0 = ((A^*-\bar{\lambda}_0 I)u,v) = (A^*u,v) - \bar{\lambda}_0(u,v)$$

$$= (u,Av) - (u,\lambda_0 v) = (u,(A-\lambda_0 I)v)$$

so u is orthogonal to $(A-\lambda_0 I)\mathcal{D}_A$.

On the other hand, supposing u to be orthogonal to $(A-\lambda_0 I)\mathcal{D}_A$ we obtain similarly that for any $v \in \mathcal{D}_A$,

$$(u,Av) = (\bar{\lambda}_0 u,v)$$

Thus $u \in \mathcal{D}_{A^*}$ and $A^*u = \bar{\lambda}_0 u$ as required. #

Let us compare the notion of the defect number of an operator with that of deficiency indices. Suppose an operator T is symmetric then T is said to be *J-real* if

(i) $Jf \in \mathcal{D}_T$ whenever $f \in \mathcal{D}_T$

(ii) $TJf = JTf$ for all $f \in \mathcal{D}_T$

Such an operator has equal deficiency indices and T is J-symmetric with defect number equal to those indices. (See [70, p.361] and [1,§78]).

We now return to the operator L_0 under consideration in the present chapter. From Theorems 2.10, 2.11 and the above definition of the defect number we have,

<u>Theorem 2.23</u>. *If $\pi(L_0)$ is not empty then* def (L_0) *is equal to the number of linearly independent solutions of*

$$Ly = \lambda_0 y \tag{2.58}$$

where λ_0 is any point in $\pi(L_0)$

This result is valid irrespective of whether the end-points a and b are regular or singular. Since 2.58 is a linear differential equation of 2n th order it can have at most 2n linearly independent solutions. Clearly, when (a,b) is a finite interval and a and b are regular points, there are precisely 2n such solutions. Hence we have

<u>Theorem 2.24</u>. *If $\pi(L_0)$ is not empty then*

(i) *In the regular case* def $(L_0) = 2n$

(ii) *In the singular case* $0 \le \text{def}(L_0) \le 2n$

It will be shown later, that in the regular case, the condition that $\pi(L_0)$ be not empty is automatically satisfied. However, in the singular case the assumption cannot in general be disregarded.

Before continuing, let us note that the operator $L_0 - \lambda_0 I$ is one-to-one for any λ_0 in $\pi(L_0)$, because of (2.53). We now consider the possible values of $\mathrm{def}(L_0)$, firstly in the case when a is regular and b is singular and hence we consider the interval [a,b), and secondly, in the case when both a and b are singular.

Theorem 2.25. ([77, theorem 6]). *If* (2.1) *is defined on* [a,b), $a \neq -\infty$, *and if* $\pi(L_0)$ *is not empty then the defect number of* L_0 *is defined and satisfies*

$$n \le \mathrm{def}(L_0) \le 2n \qquad (2.59)$$

Proof. Due to Theorem 2.24 we need only prove that $\mathrm{def}(L_0) \ge n$. We suppose to the contrary i.e. suppose

$$\mathrm{def}(L_0) < n \qquad (2.60)$$

and define the operator L'' by

$$L''y = \ell(y), \quad y \in \mathcal{D}''$$

where

$$\mathcal{D}'' = \{y \in \mathcal{D};\ y^{[k]}(a) = 0, k=1,\dots,2n-1,\ \ y \text{ has compact support}\}$$

Its closure is denoted by $\bar{L}''$. It can readily be seen that elements y of $\mathcal{D}_{\bar{L}''}$ also satisfy

$$y^{[k]}(a) = 0, \quad k = 1,2,\dots,2n-1 \qquad (2.61)$$

Furthermore we define the operator L_1' by

$$L_1'y = \ell(y), \quad y \in \mathcal{D}_1' \qquad (2.62)$$

where

$$\mathcal{D}_1' = \{y \in \mathcal{D};\ y^{[k]}(a) = 0,\ k=1,\dots,2n-2,\ y^{[2n-1]}(a) = h\, y^{[0]}(a),$$
$$y \text{ has compact support}\}$$

and h_1 is any non-zero complex number. Just as it was proved earlier that L_0', L_0 are J-symmetric so it can be proved that L'', $\overline{L''}$, L_1' and L_1 (the closure of L_1') are J-symmetric. From the above definitions, it is clear that

$$L_0' \subset L'' \subset L_1' \tag{2.63}$$

and hence

$$L_0 \subset \overline{L''} \subset L_1 \tag{2.64}$$

One may then analogously construct a J-symmetric extension L_2 of L_1 by replacing the conditions in (2.62) by

$$\left.\begin{aligned} &y^{[k]}(a) = 0,\quad k = 2,3,\dots,2n-3 \\ &y^{[2n-k]}(a) = h_k y^{[k-1]}(a),\quad k = 1,2 \end{aligned}\right\} \tag{2.65}$$

where h_2 is any non-zero complex number. Continuing this process we construct finally, the J-symmetric extension L_n of L_0. Each function y in the domain $\mathcal{D}_n$ of L_n will satisfy

$$y^{[2n-k]}(a) = h_k y^{[k-1]}(a),\quad k = 1,\dots,n \tag{2.66}$$

for some complex numbers $h_1,\dots,h_n$. Clearly the dimension of the manifold $\mathcal{D}_n \ominus \mathcal{D}_0$ is n. For any $\lambda_0 \in \pi(L_0)$, since $L_0 - \lambda_0 I$ is one-to-one, it then follows that

$$\dim(\Delta_{\lambda_0}(L_n) \ominus \Delta_{\lambda_0}(L_0)) = n \tag{2.67}$$

But (2.67) contradicts (2.60) and so the result follows.

The calculation of $\mathrm{def}(L_0)$ when both a and b are singular can be reduced to that for just one singular end-point. For, let c be any number in the interval (a,b) and let L_0^- and L_0^+ be the operators L_0 generated by the same differential expression $\ell(\cdot)$ in the intervals (a,c] and [c,b) respectively. Obviously the operators L_0^-, L_0^+ each have just one singular end-point, at a and b respectively. (The case of (a,b] can clearly be treated in the same way as that of [a,b) in Theorem 2.25.)

Theorem 2.26.

$$\mathrm{def}(L_0) = \mathrm{def}(L_0^-) + \mathrm{def}(L_0^+) - 2n$$

Proof. We define a restriction A' of L_0' by

$$\mathcal{D}(A') = \{y \in \mathcal{D}_0' : y^{[k]}(c) = 0,\ k = 0,1,\dots,2n-1\} \tag{2.68}$$

The closure of A' will be denoted by A. In the closure process, the conditions at c in (2.68) obviously remain valid for all $y \in \mathcal{D}_A$. Hence there are in $\mathcal{D}_0$ precisely 2n linearly independent functions such that no linear combination of them (exept the trivial one) lies in $\mathcal{D}_A$. This implies

$$\dim(\mathcal{D}_0 - \mathcal{D}_A) = 2n \tag{2.69}$$

Furthermore, since $L_0 - \lambda_0 I$ is one-to-one for any $\lambda_0 \in \pi(L_0)$ it then follows that

$$\dim((L_0-\lambda_0 I)\mathcal{D}_0 - (A-\lambda_0 I)\mathcal{D}_A) = 2n \tag{2.70}$$

On the other hand the operator A is the direct sum of the operators

L_0^- and L_0^+ and therefore from the definitions of the defect number, of L_0^-, L_0^+ and A,

$$\mathrm{def}(A) = \mathrm{def}(L_0^-) + \mathrm{def}(L_0^+) \tag{2.71}$$

Hence from (2.70), (2.71),

$$\mathrm{def}(L_0) = \mathrm{def}(A) - 2n = \mathrm{def}(L_0^+) + \mathrm{def}(L_0^-) - 2n.$$

If $\pi(L_0)$ is not empty then since $\pi(L_0^-)$ and $\pi(L_0^+)$ contain the set $\pi(L_0)$, they too are not empty.

Finally, we wish to determine the domains of the J-selfadjoint extensions of L_0. This work was begun in [77], but we cite here the more general and more explicit formulation given by I.W. Knowles [46], which is a consequence of his general theory on symmetric, conjugate-linear operators in a Hilbert space.

Theorem 2.27 ([46]). *Assume* $\pi(L_0)$ *is not empty and* $\mathrm{def}(L_0) = m$. *For arbitrary functions* $w_1, w_2, \ldots, w_m$ *in* $\mathcal{D}$ *which are linearly independent modulo* $\mathcal{D}_0$ *and which satisfy the relations*

$$[w_j, \bar{w}_k]_b - [w_j, \bar{w}_k]_a = 0, \quad j,k = 1,\ldots,m \tag{2.72}$$

the set of all functions y *in* $\mathcal{D}$ *which satisfy the conditions*

$$[y, \bar{w}_k]_b - [y, \bar{w}_k]_a = 0, \quad k = 1,\ldots,m \tag{2.73}$$

is the domain of definition of a J-selfadjoint extension of L_0. *Conversely, all J-selfadjoint extensions of* L_0 *are of this form.*

When $\ell(\cdot)$ is regular on the interval [a,b], the boundary conditions (2.72), (2.73) simplify to give

Theorem 2.28 ([46]). *Let $\ell(\cdot)$ be regular on $[a,b]$. Then the linear manifold in D determined by linearly independent boundary conditions of the form*

$$\sum_{k=1}^{2n} \alpha_{jk} y^{[k-1]}(a) + \sum_{k=1}^{2n} \beta_{jk} y^{[k-1]}(b) = 0, \quad j = 1,2,\dots,2n \tag{2.74}$$

with

$$\sum_{\nu=1}^{n} \alpha_{j\nu}\alpha_{k,2n-\nu+1} - \sum_{\nu=1}^{n} \alpha_{j,2n-\nu+1}\alpha_{k\nu} = \sum_{\nu=1}^{n} \beta_{j\nu}\beta_{k,2n-\nu+1} - \sum_{\nu=1}^{n} \beta_{j,2n-\nu+1}\beta_{k\nu}$$

$$j,k = 1,2,\dots,2n \tag{2.75}$$

is the domain of a J-selfadjoint extension of L_0. Conversely, every J-selfadjoint extension of L_0 is of this form.

In the case when one end-point, a, say, is regular and the other is singular we have

Lemma 2.29. *If $\ell(\cdot)$ is regular at* a *and singular at* b *and* $\mathrm{def}(L_0) = n$ *then for arbitrary* y *and* z *in* D

$$[y,\bar{z}]_b = 0 \tag{2.76}$$

The proof is virtually identical to the proof of the corresponding result in the real case (see [58,§18.3]). Under the conditions of Lemma 2.29, the conditions (2.72), (2,73) simplify to give

Theorem 2.30 ([46]). *Let $\ell(\cdot)$ be regular at* a *and singular at* b, *and assume* $\mathrm{def}(L_0) = n$. *Then the linear manifold in D determined by linearly independent boundary conditions at* a, *of the form*

$$\sum_{k=1}^{2n} \alpha_{jk} y^{[k-1]}(a) = 0, \quad j = 1,2,\ldots,n \tag{2.77}$$

with

$$\sum_{\nu=1}^{n} \alpha_{j\nu}\alpha_{k,2n-\nu+1} - \sum_{\nu=1}^{n} \alpha_{j,2n-\nu+1}\alpha_{k\nu} = 0 \quad j,k = 1,\ldots,n \tag{2.78}$$

is the domain of a J-selfadjoint extension of L_0. *Conversely, every J-selfadjoint extension of* L_0 *is of this form.*

So far in the present chapter we have been concerned with the 2n-th order differential expression, (2.1). Let us now specialise to the case when $n = 1$, in which case we write (2.1) as

$$L(y) = -(py')' + qy \tag{2.79}$$

with $\frac{1}{p}$, $q \in L_{loc}(I)$. Then in the regular case if $\pi(L_0)$ is not empty, $\mathrm{def}(L_0) = 2$ and in the singular case, $0 \le \mathrm{def}(L_0) \le 2$, by Theorem 2.24. If a is regular, b is singular, and $\pi(L_0)$ is not empty then $\mathrm{def}(L_0)$ is equal to either 1 or 2, by Theorem 2.25. However if a and b are both singular and $\pi(L_0) \neq \emptyset$ then Theorem 2.26 states that:

$$\mathrm{def}(L_0) = \begin{cases} 0 & \text{if } \mathrm{def}(L_0^-) = \mathrm{def}(L_0^+) = 1 \\ 1 & \text{if } \mathrm{def}(L_0^-) \neq \mathrm{def}(L_0^+) \\ 2 & \text{if } \mathrm{def}(L_0^-) = \mathrm{def}(L_0^+) = 2 \end{cases}$$

Furthermore, if $\mathrm{def}(L_0) = 0$ then $\Delta_\lambda(L_0) = L^2(a,b)$ and so by Theorem 2.19, L_0 is J-selfadjoint, and from Property 2.14 we have that $L_0 = L$.

When $n = 1$, condition (2.78) of Theorem 2.30 is trivially satisfied. This is very different from the characterization in the real case of the selfadjoint extensions of L_0 where the corresponding condition [58, equation (29), p.80] (there is a misprint in the text) essentially forces the coefficients α_{jk} to be real. With a slight change of notation, Theorem 2.30 becomes

<u>Theorem 2.31</u> ([46]). *If* L_0 *is the minimal operator corresponding to the formal operator* $\ell(\cdot)$ *defined by* (2.79) *on* $[a,\infty)$, $\pi(L_0)$ *is not empty and* $\mathrm{def}(L_0) = 1$, *then the J-selfadjoint extensions* L_γ *of* L_0 *are precisely given by*

$$\begin{aligned} \mathcal{D}(L_\gamma) &= \{y \in \mathcal{D};\ \gamma_1 y(a) + \gamma_2 py'(a) = 0\} \\ L_\gamma y &= \ell(y) \quad \textit{for} \quad y \in \mathcal{D}(L_\gamma) \end{aligned} \tag{2.80}$$

where $\gamma = (\gamma_1, \gamma_2)$ *is an arbitrary non-zero element of* $\mathbb{C}^2$.

We note that the characterization of J-selfadjoint extensions of L_0 given by [46] includes *all* such extensions including some which may not be well-posed. The authors of [77] sought only to characterize correct J-selfadjoint extensions.

The above results now tell us that when $n = 1$ and $I = [a,\infty)$, the J-selfadjoint extensions of L_0 are characterized by a boundary condition at the end-point a only, precisely when $\mathrm{def}(L_0) = 1$, which according to Theorem 2.23 occurs when there is only one solution of $Ly = \lambda_0 y$ in $L^2[a,\infty)$ for λ_0 in $\pi(L_0)$. Thus to summarize these results:

<u>Theorem 2.32</u>. *If* L_0 *is defined by* (2.79) *on* $[a,\infty]$, $a \neq -\infty$, *and if* $\pi(L_0)$ *is not empty, then the following are equivalent:*

(i) $\mathrm{def}(L_0) = 1$

(ii) *there is precisely one linearly independent solution of* $\ell(y) = \lambda_0 y$ *in* $L^2[a,\infty)$ *for some (and hence any)* λ_0 *in* $\pi(L_0)$.

(iii) *the operators* L_γ *defined by* (2.80) *are the J-selfadjoint extensions of* L_0.

This provides a very clear dichotomy between the case when $\mathrm{def}(L_0) = 1$ and the case when $\mathrm{def}(L_0) = 2$. This is a generalisation to complex coefficients of the well-known Weyl limit-point, limit-circle dichotomy in the real case. In [37] R.M. Kauffman defines an n-th order possibly non-selfadjoint ordinary differential expression to be in the limit point condition if the maximal operator in $L^2[a,\infty)$ is an n-dimensional extension of the minimal operator. For the expression (2.1) this is equivalent to requiring that $\mathrm{def}(L_0) = n$ when $I = [a,\infty)$ and hence for (2.79) this is the same as $\mathrm{def}(L_0) = 1$. This is the definition which we shall use for the 'limit point' case, although a geometrical argument led A.R. Sims [68] to define the limit-point and limit-circle cases in a slightly different way for the complex Sturm-Liouville expression (2.79). We shall say $\ell(\cdot)$ is in the 'limit-circle case' if $\mathrm{def}(L_0) = 2$.

It is unfortunately not possible to remove the restriction of the non-emptiness of $\pi(L_0)$ in the above work by Zhikhar and by Knowles, and we shall therefore look at this condition more closely in Chapter IV.

CHAPTER III

SOME J-SELFADJOINTNESS (OR LIMIT-POINT) CRITERIA

From now on we shall concentrate on the study of the operators associated with the differential expression $\ell(\cdot)$ given by (2.79). In the present chapter we shall derive certain criteria for the operators L_γ defined by (2.80) to be J-selfadjoint. Lemmas 3.1, 3.4 will also be needed in Chapter V. These results, together with Theorems 3.3, 3.5 are also contained in [47]. Following Theorem 3.5, a number of generalisations of limit-point criteria which are known for the real case, are given.

Lemma 3.1. *Assume that the real and imaginary parts of* p *are non-negative (with* $p(t) \neq 0$*), and monotone decreasing, and that there exists a positive, monotone decreasing, locally absolutely continuous function* w *on* $[a,\infty)$*, and a decomposition* $q = q_1 + q_2$ *of* q *satisfying:*

(i) $w^2|q_1| \leq k$,

(ii) $w(t)\left|\int_a^t q_2\right| \leq \eta|p(t)|, \quad t \in [a,\infty)$,

for some positive constants k *and* η. *Then* $wpu' \in L^2[a,\infty)$ *whenever* $u \in L^2[a,\infty)$ *and* $\ell(u) = 0$.

Remark. Condition (ii) above can be replaced by the condition

(ii)' $w(t)\left|\int_a^t \frac{q_2}{p}\right| \leq \eta, \quad t \in [a,\infty)$;

this involves only a straightforward modification of the proof to be given below. One can also replace the assumption that Re(p) and Im(p) be decreasing, with the requirement that each of Re(p), Im(p), and $w|p'|$ be bounded.

<u>Proof of Lemma 3.1</u>. The method of proof is an adaption of that used in [59, Théorème II].

Firstly, we integrate $(pu')'\bar{u}w^2$ by parts over $[a,t]$, rearrange, and use the fact that $\ell(u) = 0$ to obtain

$$\int_a^t |p|^2|u'|^2w^2$$

$$= O(1) + |p|^2u'\bar{u}w^2(t) - 2\int_a^t |p|^2u'\bar{u}ww'$$

$$- \int_a^t pu'\bar{p}'\bar{u}w^2 - \int_a^t \bar{p}q|u|^2w^2.$$

Taking real parts, and using the identity

$$\mathrm{Re}(ab) \le \tfrac{1}{2}(|a|^2 + |b|^2) \tag{3.1}$$

with $a = \bar{u}$ and $b = wu'$, we have

$$\int_a^t |p|^2|u'|^2w^2$$

$$\le O(1) + \tfrac{1}{2}|p|^2w(|u|^2 + w^2|u'|^2)(t)$$

$$- \mathrm{Re}\int_a^t pu'\bar{p}'\bar{u}w^2 - 2\,\mathrm{Re}\int_a^t |p|^2u'\bar{u}ww'$$

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$$- \mathrm{Re}\int_a^t \bar{p}q_1|u|^2w^2 - \mathrm{Re}\int_a^t \bar{p}q_2|u|^2w^2 \tag{3.2}$$

Consider now each of the integrals on the right hand side of (3.2) separately. We recall that Re(p) and Im(p) are decreasing, and let Re(p) = p_1 and Im(p) = p_2. Then

$$- \mathrm{Re}\int_a^t pu'\bar{p}'\bar{u}w^2$$

$$\le -\int_a^t p_1'|pu'\bar{u}|w^2 - \int_a^t p_2'|pu'\bar{u}|w^2$$

$$\le -\frac{1}{2}\int_a^t (p_1+p_2)'w(|u|^2 + |pu'|^2w^2)$$

$$\le O(1) + \frac{1}{2}\int_a^t (p_1+p_2)[w'|u|^2 + 2w\,\mathrm{Re}(\bar{u}u') + 3w^2w'|pu'|^2 + 2w^3\mathrm{Re}(\bar{p}qu\bar{u}')]$$

$$\le O(1) + \frac{1}{\varepsilon}\int_a^t |u|^2 + \varepsilon\int_a^t |p|^2w^2|u'|^2 + 2\int_a^t w^3|p|\mathrm{Re}(\bar{p}qu\bar{u}')$$

$$= O(1) + \varepsilon\int_a^t |p|^2w^2|u'|^2 + 2\int_a^t w^3|p|\mathrm{Re}(\bar{p}qu\bar{u}') \tag{3.3}$$

as $p_1 + p_2 \le 2|p|$ and $u \in L^2[a,\infty)$, and making use of the inequality

$$|a.b| \le \frac{1}{2}(\varepsilon|a|^2 + \frac{1}{\varepsilon}|b|^2) \tag{3.4}$$

which holds for any complex numbers a, b and for any $\varepsilon > 0$. We shall choose the value of $\varepsilon > 0$ later. We also have

$$-2\,\mathrm{Re}\int_a^t |p|^2 u'\bar{u}ww'$$

$$\leq -\int_a^t w'|p|^2(|u|^2+w^2|u'|^2)$$

$$= O(1) - w|p|^2(|u|^2+w^2|u'|^2)(t) +$$

$$+ \int_a^t w\{(|p|^2)'.|u|^2 + 2|p|^2\mathrm{Re}(\bar{u}u') + 2ww'|pu'|^2$$

$$+ 2w^2\mathrm{Re}(\bar{p}qu\bar{u}')\}$$

$$\leq O(1) - w|p|^2(|u|^2+w^2|u'|^2)(t)$$

$$+ \epsilon\int_a^t w^2|p|^2|u'|^2 + 2\int_a^t w^3\mathrm{Re}(\bar{p}qu\bar{u}') \qquad (3.5)$$

since $|p|$ is bounded, $w' < 0$, and $(|p|^2)' = 2(p_1p_1'+p_2p_2') < 0$. Clearly, from condition (i), and the fact that $u \in L^2[a,\infty)$,

$$-\mathrm{Re}\int_a^t \bar{p}q_1|u|^2w^2 = O(1). \qquad (3.6)$$

Also, if we set $Q(t) = \int_a^t q_2$,

$$-\mathrm{Re}\int_a^t \bar{p}q_2|u|^2w^2$$

$$= -\mathrm{Re}(Q\bar{p}w^2|u|^2)(t) + \mathrm{Re}\int_a^t Q(\bar{p}'|wu|^2 + 2\bar{p}|u|^2ww' + 2\bar{p}w^2\mathrm{Re}(\bar{u}u'))$$

$$\le O(1) + \eta w|p|^2|u|^2(t) + \frac{\varepsilon}{3}\int_a^t w^2|p|^2|u'|^2$$

$$- \eta K_1\int_a^t (p_1+p_2)'|u|^2 w - 2\eta\int_a^t |p|^2|u|^2 w',$$

assuming $|p| \le K_1$, and since $w|Q| \le \eta|p|$,

$$= O(1) + \eta w|p|^2|u|^2(t) + \frac{\varepsilon}{3}\int_a^t w^2|p|^2|u'|^2$$

$$- [\eta K_1(p_1+p_2)|u|^2 w]_a^t + \eta K_1\int_a^t (p_1+p_2)[w'|u|^2 + 2w\,\mathrm{Re}(\bar{u}u')]$$

$$- [2\eta w|p|^2|u|^2]_a^t + 2\eta\int_a^t w[(|p|^2)'.|u|^2 + 2\,\mathrm{Re}(u\bar{u}')|p|^2]$$

$$\le O(1) + \varepsilon\int_a^t w^2|p|^2|u'|^2 \qquad (3.7)$$

using (3.4) again. If we now combine (3.3), (3.5), (3.6) and (3.7) in (3.2) we obtain

$$(1-3\varepsilon)\int_a^t |p|^2 w^2|u'|^2$$

$$\le O(1) - \frac{1}{2}w|p|^2(|u|^2 + w^2|u'|^2)(t)$$

$$+ 2\,\mathrm{Re}\int_a^t w^3\bar{p}qu\bar{u}' + 2\,\mathrm{Re}\int_a^t w^3|p|\bar{p}qu\bar{u}' \qquad (3.8)$$

Also, using (3.8) and condition (i) it is not hard to see that

$$(1-4\varepsilon)\int_a^t |p|^2 w^2 |u'|^2$$

$$\le O(1) - \tfrac{1}{2} w |p|^2 (|u|^2 + w^2 |u'|^2)(t)$$

$$+ 2\,\mathrm{Re}\int_a^t \bar{p} q_2 w^3 u \bar{u}' + 2\,\mathrm{Re}\int_a^t |p| \bar{p} q_2 w^3 u \bar{u}' \qquad (3.9)$$

Now

$$2\,\mathrm{Re}\int_a^t \bar{p} q_2 w^3 u \bar{u}'$$

$$= 2\mathrm{Re}(Q\bar{p} w^3 u\bar{u})(t)$$

$$- 2\,\mathrm{Re}\int_a^t Q(3w^2 w' \bar{p}\bar{u}' u + u' \bar{p}\bar{u}' w^3 + \bar{q}|u|^2 w^3)$$

$$\le O(1) + 2\eta . w|p|^2 . (|u|^2 + w^2|u'|^2)(t)$$

$$+ 2\eta\int_a^t |p|^2 w^2 |u'|^2 - 2\,\mathrm{Re}\int_a^t Q\bar{q}_2 |u|^2 w^3$$

$$- 6\eta\int_a^t |p|^2 w |\bar{u}' u| w' \qquad (3.10)$$

where

$$- 2\,\mathrm{Re}\int_a^t Q\bar{q}_2 |u|^2 w^3$$

$$= -(|Q|^2 . |u|^2 w^3)(t)$$

$$+ \int_a^t |Q|^2 (3w^2 w' |u|^2 + 2w^3 \mathrm{Re}(\bar{u} u'))$$

$$\le O(1) + c\int_a^t |p|^2 w^2 |u'|^2 , \qquad (3.11)$$

and

$$- 6\eta\int_a^t |p|^2 w|\bar{u}'u|w'$$

$$\le -3\eta\int_a^t |p|^2 w'(|u|^2 + w^2|u'|^2)$$

$$\le 0(1) + 3\eta\int_a^t w\{(|p|^2)'|u|^2 + 2|p|^2 \operatorname{Re}(\bar{u}u')\}$$

$$+ 2\eta\int_a^t w^3\{\operatorname{Re}(\bar{p}qu\bar{u}')\}$$

$$\le 0(1) + \varepsilon\int_a^t |p|^2 w^2|u'|^2$$

$$+ 2\eta\int_a^t w^3 \operatorname{Re}(\bar{p}q_2u\bar{u}') \tag{3.12}$$

by (i), (3.4) and the fact that $(|p|^2)' \le 0$. We now make use of (3.11) and (3.12) in the inequality (3.10) and rearrange to obtain

$$(1-\eta)\operatorname{Re}\int_a^t \bar{p}q_2w^3u\bar{u}'$$

$$\le 0(1) + (\varepsilon+\eta).\int_a^t |p|^2 w^2|u'|^2$$

$$+ \eta\, w|p|^2(|u|^2 + w^2|u'|^2)(t). \tag{3.13}$$

The other integral on the right hand side of (3.9) can be estimated in a similar fashion:

$$2\,\mathrm{Re}\int_a^t w^3|p|\bar{p}q_2u\bar{u}'$$

$$= 2\,\mathrm{Re}(Q\bar{p}|p|w^3u\bar{u}')(t)$$

$$- 2\,\mathrm{Re}\int_a^t Q(3w^2w'|p|\bar{p}\bar{u}'u + |p|'w^3\bar{p}\bar{u}'u$$

$$+ w^3|p|\bar{q}|u|^2 + w^3|p|\bar{p}|u'|^2)$$

$$\leq O(1) + \eta K_1 w|p|^2(|u|^2+w^2|u'|^2)(t)$$

$$+ 2\eta K_1\int_a^t |p|^2w^2|u'|^2 - 2\,\mathrm{Re}\int_a^t Q\bar{q}_2w^3|p||u|^2$$

$$- 6\eta\int_a^t |p|^3w'.uw\bar{u}' - 2\eta K_1\int_a^t |p|'w^2|pu'u| \qquad (3.14)$$

where $|p| \leq K_1$ and $|p|' = \frac{1}{2}|p|^{-1}(|p|^2)' \leq 0$. Considering each of the last three integrals [illegible] separately we have firstly

$$- 2\,\mathrm{Re}\int_a^t (Q\bar{q}_2)w^3|p||u|^2$$

$$\leq \int_a^t |Q|^2(3w^2w'|p||u|^2 + w^3|p|'|u|^2$$

$$+ 2w^3|p|\,\mathrm{Re}(u\bar{u}'))$$

$$\leq O(1) + \int_a^t |p|^2w^2|u'|^2; \qquad (3.15)$$

also,

$$-6\eta\int_a^t |p|^3 w' u\bar{u}' \qquad (3.15)$$

$$\leq -3\eta\int_a^t |p|^3 w'(|u|^2 + w^2|u'|^2)$$

$$\leq O(1) + 3\eta\int_a^t w\{(|p|^3)'|u|^2 + 2|p|^3\mathrm{Re}(u\bar{u}')\}$$

$$+\eta\int_a^t w^3\{2|p|\mathrm{Re}(\bar{p}qu\bar{u}') + |p|''|p|^2|u'|^2\}$$

$$\leq O(1) + \varepsilon\int_a^t |p|^2 w^2 |u'|^2$$

$$+ 2\eta\int_a^t |p| w^3 \mathrm{Re}(\bar{p}q_2 u\bar{u}'); \qquad (3.16)$$

and finally,

$$2\eta K_1\int_a^t |p|' w^2 |pu'u|$$

$$\leq O(1) + \varepsilon\int_a^t |p|^2 w^2 |u'|^2 + 4\eta K_1\int_a^t w^3 |p| \mathrm{Re}(\bar{p}q_2 u\bar{u}') \qquad (3.17)$$

by the method used in proving (3.3). We now use (3.15), (3.16), and (3.17) in (3.14) to obtain

$$(1-\eta-2\eta K_1)\mathrm{Re}\int_a^t w^3 |p| \bar{p} q_2 u\bar{u}'$$

$$\leq O(1) + \tfrac{1}{2}\eta K_1 w|p|^2(|u|^2 + w^2|u'|^2)(t)$$

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$$+ (\eta K_1 + \tfrac{3}{2}\varepsilon)\int_a^t |p|^2 w^2 |u'|^2 \qquad (3.18)$$

Finally, returning to (3.9), we have from (3.13) and (3.18) that

$$(1 - 4\varepsilon - \frac{2\varepsilon+2\eta}{1-\eta} - \frac{3\varepsilon+2\eta K_1}{1-\eta-2\eta K_1})\int_a^t |p|^2 w^2 |u'|^2$$

$$\leq O(1) - (\frac{1}{2} - \frac{2\eta}{1-\eta} - \frac{\eta}{1-\eta-2\eta K_1}) w|p|^2(|u|^2 + w^2|u'|^2)(t).$$

If we now observe that one can choose η to be arbitrarily small, by replacing w in condition (ii) by a multiple of itself if necessary, then the result follows by choosing $\varepsilon > 0$ suitably small as well.

In order to replace the condition of monoticity on $Re(p)$, $Im(p)$ by the boundedness of $Re(p)$, $Im(p)$ and $w|p'|$ it is sufficient to observe that

$$(|p|^2)' = (p_1^2+p_2^2)' = 2p_1p_1' + 2p_2p_2'$$

$$\leq 4|p|.|p'| \qquad (3.19)$$

and the corresponding inequalities for $|p|'$ and $(|p|^3)'$, in which case the above proof needs only very minor modification except for (3.3), which now becomes

$$- Re\int_a^t pu'\bar{p}'\bar{u}w^2$$

$$\leq \int_a^t (w|p'|)(|p|.|u'|.|u|.w)$$

$$\leq K\int_a^t |p|.|u'|.|u|.w$$

$$\leq \varepsilon\int_a^t |p|^2 w^2|u'|^2 + \frac{K^2}{4\varepsilon}\int_a^t |u|^2$$

$$= O(1) + \varepsilon\int_a^t |p|^2 w^2 |u'|^2 \qquad (3.20)$$

where $w|p'|$ is bounded by K.

The replacement of (ii) by (ii)' in the proof is immediate by integrating q_2/p instead of q_2 whenever Q occurs in the proof.

We now need the following result, which is the analogue in the complex case of H. Weyl's characterization of the limit-circle case for real Sturm-Liouville operators; his proof is valid for p and q complex.

Lemma 3.2 ([76], [11,p.225]). *If the equation $\ell(u) = \lambda u$ has two linearly independent solutions in $L^2[a,\infty)$ for some fixed value $\lambda = \lambda_0$, then all solutions of $\ell(u) = \lambda u$ are in $L^2[a,\infty)$ for all values of λ.*

As a first application of Lemma 3.1 we have

Theorem 3.3. *If, in addition to the conditions listed in Lemma 3.1 we assume $\pi(L_0)$ is not empty, and*

$$\int_a^\infty w(t)dt = \infty \qquad (3.21)$$

then the operators L_γ are J-selfadjoint.

Lemma 3.4. *Suppose that* $\mathrm{Re}(p) \geq 0$ *and* $\mathrm{Im}(p) \geq 0$, *with* $p(t) \neq 0$, *and*

(i) $\displaystyle\int_a^\infty |p|^{-1/2} = \infty.$

Suppose further that $\mathrm{Re}(q) = q_1 + q_2$ *and* $\mathrm{Im}(q) = q_3 + q_4$ *where* q_i, $i = 1, 2, 3, 4$ *are real, and that there is a positive locally absolutely continuous function* w *on* $[a,\infty)$ *such that*

(ii) $|p|(w')^2 \leq K_1$,

(iii) $-q_1 w^2 \leq K_2$,

(iv) $\displaystyle |p(t)|^{-1/2} w^\alpha(t) \left|\int_a^t q_2 w^{1-\alpha}\right| \leq K_3$

for some α, $0 \leq \alpha \leq 1$,

and if $\mathrm{Im}(p) \neq 0$ *that*

(v) $-q_3 w^2 \leq K_4$

(vi) $\displaystyle |p(t)|^{-1/2} w^\beta(t) \left|\int_a^t q_4 w^{1-\beta}\right| \leq K_5$

for some β, $0 \le \beta \le 1$.

Then $pw^2(u')^2 \in L[a,\infty)$ *whenever* $u \in L[a,\infty]$ *and* $\ell(u) = 0$.

Proof. The method of proof is essentially the same as that used in [42]. Let

$$\gamma(t) = \int_a^t |p|^{-1/2}$$

and

$$G(s,t) = 1 - \frac{\gamma(s)}{\gamma(t)},\quad a \le s < t$$

$$= 0 \qquad\qquad ,\quad s \ge t.$$

Then $0 \le G(s,t) \le 1$ for $s,t \in (a,\infty)$ and

$$\frac{\partial G}{\partial s} = -|p(s)|^{-1/2}(\gamma(t))^{-1},\quad a \le s < t$$

$$= 0 \qquad\qquad\qquad ,\quad s > t.$$

By condition (i) we can find a value $t_0 > a$ such that $\gamma(t) > 1$ for $t > t_0$. Consequently, from condition (ii) we have

$$K_1^{1/2}\gamma(s) \ge \int_a^s |w'| \ge w(s) - w(a)$$

and, for $t_0 \le s \le t$,

$$\frac{w(s)}{\gamma(t)} \le K_1^{1/2} + \frac{w(a)}{\gamma(t)} \le K_1^{1/2} + w(a) \tag{3.23}$$

We now let $u \in L^2[a,\infty)$ and $\ell(u) = 0$, integrate $(p(s)u'(s))'\bar{u}(s)w^2(s).G^2(s,t)$ by parts over $[a,t]$, $t \ge t_0$, and rearrange to obtain

$$\ldots \int_a q_1 \ldots |u|^2 \ldots + \int_a q_2 \ldots |u|^2 G^2 . \qquad (3.25)$$

As before, we estimate the integrals on the right hand side separately. Using (3.4) we have

$$2\int_a^t |pww'\bar{u}u'|G^2 \le \epsilon\int_a^t w^2|u'|^2|p|G^2 + \frac{1}{\epsilon}\int_a^t |p|(w')^2|u|^2G^2$$

$$= O(1) + \epsilon\int_a^t |p|w^2|u'|^2G^2 \qquad (3.26)$$

by virtue of condition (ii) and $u \in L^2[a,\infty)$. Here $\epsilon > 0$ will be chosen later. Also in a similar fashion, and using (3.23),

$$2\int_a^t \frac{t}{\gamma(t)} \cdot |p|^{1/2}w|u'\bar{u}|G$$

$$\leq 2(K_1^{1/2} + v(a))\int_a^t |p|^{1/2} w|u'\bar{u}|G$$

$$\leq O(1) + \epsilon\int_a^t |p|w^2|u'|^2G^2. \tag{3.27}$$

Using (iii), it is clear that

$$-\int_a^t q_1 w^2|u|^2G^2 = O(1) \tag{3.28}$$

Finally, if we set $Q(t) = \int_a^t q_2 w^{1-\alpha}$,

$$-\int_a^t q_2 w^2|u|^2G^2$$

$$= \int_a^t Q.\{(1+\alpha)w^\alpha w'|u|^2G^2 + 2w^{1+\alpha}G^2.Re(\bar{u}u')$$

$$- 2w^{1+\alpha}|u|^2G|p|^{-1/2}(\gamma(t))^{-1}\}$$

$$\leq O(1) + \epsilon\int_a^t |p|w^2|u'|^2G^2 \tag{3.29}$$

by condition (ii) and (iv), inequality (3.23), and the method of (3.26). Thus if we apply (3.26) - (3.29) to (3.25) we find that

$$\int_a^t Re(p)w^2|u'|^2G^2 \leq O(1) + 3\epsilon\int_a^t |p|w^2|u'|^2G^2$$

Similarly if $Im(p) \neq 0$, by using (v) and (vi) we may obtain for $t \geq t_0$,

$$\int_a^\infty |p| w^2 |u'|^2 < \infty,$$

as required.

#

Condition (i) is needed in the above proof because of the use of the function G, which seems to be necessary if p is complex valued. If p is real, then T.T. Read [64] has shown that condition (i) is not needed. We thus have

Theorem 3.5. *Suppose that* $\pi(L_0)$ *is not empty, and that conditions* (ii), (iii), *and* (iv) *of Lemma 3.4 are satisfied together with the condition*

$$\int_a^\infty |p|^{-1/2} w = \infty \tag{3.30}$$

If Im(p) $\neq 0$, *suppose further that conditions* (v) *and* (vi) *are also satisfied. Then the operators* L_γ *are J-selfadjoint.*

Proof. Firstly we note that by (3.23), for $t \geq t_0$,

$$\int_{t_0}^{t} |p|^{-1/2} w \le K \int_a^t |p(s)|^{-1/2} v(s)\,ds$$

$$\le K\Big(\int_a^t |p|^{-1/2}\Big)^2 \qquad (3.31)$$

Thus condition (3.30) ensures that condition (i) of Lemma 3.4 is satisfied. The proof is now identical with that of Theorem 3.3 with the exception that (3.22) is replaced by

$$|p|^{-1/2} w = u.p|p|^{-1/2} w' - v.p|p|^{-1/2} wu'.$$

Since Theorem 3.5 is the analogue in the complex case of the Read limit-point criterion (see [68]), from which may be obtained virtually all of the known limit-point conditions, it follows that these conditions all have complex analogues. Moreover, for $\mathrm{Im}(p) = 0$, the analogue is obtained by imposing the relevant conditions on p and $\mathrm{Re}(q)$ (always presuming that one can establish that $v(L_0)$ is not empty under the stated conditions.)

In order to obtain the first of these we need the following lemma:

Lemma 3.6 [2]. *Let μ be a positive continuous regular Borel measure on $[a,\infty)$ and let Q be a real-valued function defined on this interval. Then the following properties are equivalent.*

(i) *There is a constant C such that $-\int_I Q\,d\mu \le C$ for each interval I with $\mu(I) \le 1$.*

(ii) $Q = Q_1 + Q_2$ *where* $-Q_1 \le C_1$ *and* $\left|\int_a^x Q_2\,d\mu\right| \le C_2$ *for all* $x \ge a$.

This enables us to deduce the following result:

<u>Theorem 3.7</u>. *Suppose* $\pi(L_0)$ *is not empty,* $\mathrm{Re}\, p \geq 0$, $\mathrm{Im}\, p \geq 0$, $p(t) \neq 0$ *and that there is a positive, locally absolutely continuous function* w *such that*

(i) $|p|(w')^2 \leq K_1$

(ii) $\int_a^\infty |p|^{-1/2}.w = \infty$

(iii) *there exists a constant* C *such that* $-\int_I \mathrm{Re}\, q.w \leq C$ *for each interval* I *for which* $\int_I w^{-1} \leq 1$

If $\mathrm{Im}\, p \neq 0$, *suppose further that*

(iv) $-\int_I \mathrm{Im}\, q.w \leq C$ *for each interval* I *for which* $\int_I w^{-1} \leq 1$

Then the operators L_γ *are J-selfadjoint.*

<u>Proof</u>. In Theorem 3.5, set $\alpha = \beta = 0$, then by Lemma 3.6, with $d\mu = w(t)^{-1}dt$ and $Q = \mathrm{Re}\, q.w^2$, (iii) and (iv) of Theorem 3.5 are equivalent to (iii) above, and (v) and (vi) of Theorem 3.5 are equivalent to (iv) above. #

When $p = 1$ and q is real this result reduces to one originally due to I.W. Knowles [40] and deduced by Read from his more general result in [63]. This leads to the analogue of a limit-point criterion due to I. Brinck [10] when $p = 1$ and q is real (see [40]).

<u>Corollary 3.8</u>. *If* $\pi(L_0)$ *is not empty and*

(i) $\mathrm{Re}\, p \geq 0$, $\mathrm{Im}\, p \geq 0$, $p(t) \neq 0$.

(ii) $\int_a^\infty |p|^{-1/2} = \infty$

(iii) *there is a constant* $C > 0$ *such that* $-\int_I \mathrm{Re}(q) \leq C$ *and if*

Im $p \neq 0$, $-\int_I \mathrm{Im}(q) \le C$ *for all intervals* $I \subset [a,\infty]$ *of length at most* 1,

then the operators L_γ *are J-selfadjoint.*

Proof. Set $w = 1$ in Theorem 3.7. ∎

Another consequence of Theorem 3.5 is

Corollary 3.9. *Suppose* $\pi(L_0)$ *is not empty and that* Re $p \ge 0$, Im $p \ge 0$, $p(t) \neq 0$ *and there exists a positive differentiable function* M *such that*

(i) $\dfrac{|p|^{1/2}|M'|}{M^{3/2}} \le C$

(ii) $\int_a^\infty |pM|^{-1/2} = \infty$

(iii) Re $q \ge -k_1 M$

and if Im $p \neq 0$, Im $q \ge -k_2 M$

then the operators L_γ *are J-selfadjoint*

Proof. In Theorem 3.5, set $w = M^{-1/2}$, and $q_2 = q_4 = 0$. ∎

This result was first proved by N. Levinson [50] and D.B. Sears [67, theorem 2] for the case $p = 1$ and q real-valued. The generalisation to $p \neq 1$ and q real-valued is given in [11, p.229]. It was first extended to complex-valued q by W.D. Evans [19] assuming $p = 1$ and then by F.V. Atkinson and W.D. Evans [3] assuming p to be real and positive. Corollary 3.9 now extends this to complex-valued p.

The next result is a generalisation of a theorem of M.S.P. Eastham [15]. He used an entirely different method and assumed

$p = 1$. It was generalised to general real-valued p and q by T.T. Read [63] as a consequence of the real-valued version of Theorem 3.5 above. The proof for complex-valued p and q is obtained in the same way, replacing p by $|p|$ in Read's proof.

Corollary 3.10. *Suppose* $\pi(L_0)$ *is not empty,* $\mathrm{Re}\ p \geq 0$, $\mathrm{Im}\ p \geq 0$, $p(t) \neq 0$, *and that there is a sequence* $(I_n)_{n=1}^{\infty}$, $I_n = [a_n, b_n]$ *of pairwise disjoint intervals in* $[a,\infty)$ *and a sequence* $(V_n)_{n=1}^{\infty}$ *of positive numbers such that for each* n,

(i) $v_n P_n^2 \geq K > 0$ *where* $P_n = \int_{a_n}^{b_n} |p|^{-1/2}$

(ii) $\sum_{n=1}^{\infty} v_n^{-1} = \infty$

(iii) $\int_{a_n}^{b_n} (\mathrm{Re}\ q)^{-} \leq C v_n^2 P_n^3 \min_{I_n} |p|^{1/2}$ *where* $(\mathrm{Re}\ q)^{-}$ *is the negative part of* $\mathrm{Re}\ q$.

If $\mathrm{Im}\ p \neq 0$ *suppose further that*

(iv) $\int_{a_n}^{b_n} (\mathrm{Im}\ q)^{-} \leq C v_n^2 P_n^3 \min_{I_n} |p|^{1/2}$

Then the operators L_γ *are J-selfadjoint.*

In Theorems 3.3, 3.5, 3.7 and Corollaries 3.8, 3.9, 3.10, the conclusion that the operators L_γ are J-selfadjoint could be rephrased to say that the differential expression $\ell(\cdot)$ is in the limit-point case, using the results of Chapter II and R.M. Kauffman's definition (see [37]) of the limit-point case. If it is not known that $\pi(L_0)$ is not empty, then the results all remain valid if the conclusion is altered to the statement that there is a solution of $\ell(u) = 0$ which is not in $L^2[a,\infty)$ i.e. not all solutions of $\ell(u) = 0$

(and hence by Lemma 3.2, not all solutions of $\ell(u) = \lambda u$, for any complex number λ) are in $L^2[a,\infty)$. We now give a number of examples of functions p and q which satisfy the conditions of Theorem 3.5. We shall take $\ell(\cdot)$ to be defined on the interval $[1,\infty)$. Since it is in most cases not easy to determine whether or not $\pi(L_0)$ is empty, the conclusion which may be drawn for each example is that not all solutions of $\ell(u) = \lambda u$ lie in $L^2[a,\infty)$ and hence *if* $\pi(L_0)$ is not empty, $\ell(\cdot)$ is in the limit-point case and $\operatorname{def}(L_0) = 1$. Many of the examples given are based on examples taken from the folklore of the real-valued case.

<u>Examples</u>

(1) $p(t) \equiv 1$, $\operatorname{Re} q(t) = -(t^2+te^t\cos(e^t))$, $\operatorname{Im} q(t)$ may be any function of t

(2) $p(t) = t^\alpha + it^a$ where $\alpha, a \le 2$

$q(t) = -t^\beta - it^\gamma$ where $\beta,\gamma \le 2 - \max(a,\alpha)$

(3) $p(t) = t^\alpha + it^a$ where $\alpha, a \le 2$

$-\operatorname{Re} q(t) \le C_1 t^{2-\max(a,\alpha)}$, $-\operatorname{Im} q(t) \le C_2 t^{2-\max(a,\alpha)}$

(4) $p(t) \equiv 1$, $\operatorname{Re} q(t) = t^\alpha \sin(t^\beta)$, $\operatorname{Im} q(t)$ may be any function of t; and either

(a) $\alpha \le 2$ or (b) $\alpha > 2$ and $\beta \ge \alpha$.

(5) $p(t) = t$, $\operatorname{Re} q(t) = -(t+te^t\sin(e^t))$, $\operatorname{Im} q(t)$ may be any function of t.

(6) $p(t) = t$, $\operatorname{Re} q(t) = -(t+t^d\sin(t^d))$, $d \ne 0$, $\operatorname{Im} q(t)$ may be any function of t.

(7) $\operatorname{Re} p(t) = t$, $0 \le \operatorname{Im} p(t) \le t$, $\operatorname{Re} q(t) = -(t+te^t\sin(e^t))$,

$\text{Im } q(t) \geq -k$

(8) $\text{Re } p(t) = t$, $0 \leq \text{Im } p(t) \leq t$

$\text{Re } p(t) = -(t+t^d \sin(t^d))$, $d \neq 0$, $\text{Im } q(t) \geq -kt$

(9) $p(t) \equiv 1$, $\text{Re } q(t) = -t^{\alpha}\sin(t^{\beta})$, $\text{Im } q(t)$ may be any function of t; and either

(a) $\beta \leq 2$ or (b) $\beta > 2$ and $\alpha \leq \beta + 2$

(10) $p(t) \equiv 1$, $\text{Re } q(t) = -(t+t^2\sin^4 t)$, $\text{Im } q(t)$ may be any function of t

(11) $p(t) = t^{-a} + it^{-b}$, $a,b > 0$,

$\text{Re } q(t) = -t^{\gamma}\sin(t^{\beta})$, $\text{Im } q(t) = -t^{\delta}\sin(t^{\beta})$ where $0 < \beta \leq 2$

(12) $p(t) = t^{-a} + it^{-b}$, $a,b > 0$

$\text{Re } q(t) = -f(t)\sin(t^{\beta})$, $\text{Im } q(t) = -g(t)\sin(t^{\beta})$ where f, g are any non-negative functions of t.

<u>Proofs</u>. Examples (1) - (8) are proved by direct substitution into Theorem 3.5 as follows:

(1) Let $w(t) = t^{-1}$ then (3.30) is satisfied and in (ii), $|p|(w')^2 = t^{-4}$ which is bounded, and letting $q_1 = -t^2$, $-q_1 w^2 = 1$ and $q_2 = -te^t\cos(e^t)$ gives

$$\left|\int_1^t q_2 w\right| = \left|\int_1^t e^s\cos(e^s)ds\right| = |\sin(e^t) - \sin(e)|$$

$$\leq 2$$

so all the conditions of the theorem are satisfied.

(2) Let $w = t^{1/2\max(a,b)-1}$ so $|p|^{-1/2}.w \sim t^{-1}$ as $t \to \infty$ ensuring (3.30) is satisfied. Also, $|p|(w')^2 \sim t^{-4}$ and letting

$q_1 = \mathrm{Re}\, q$, $q_3 = \mathrm{Im}\, q$, $-q_3 w^2 = t^{\beta-2\max(a,\alpha)}$,
$-q_3 w^2 = t^{\gamma-2\max(a,\alpha)}$ all of which are bounded as required.

(3) The choice of w, q_1 and q_3, and the proof of this example are the same as example (2).

(4) (a) If $\alpha \le 2$, let $w = t^{-1}$ then (3.30) is satisfied, and $|p|(w')^2$ is bounded. Letting $q_1 = \mathrm{Re}\, q$, $-q_1 w^2 = t^{\alpha-2}\sin(t^{\beta})$, which is bounded, as required

(b) If $\alpha > 2$ and $\beta \ge \alpha$, let $w = t^{-1}$ and $q_2 = \mathrm{Re}\, q$ then in (iv),

$$|p|^{-1/2}\left|\int_1^t q_2 w\right| = \left|\int_1^t s^{\alpha-1}\sin(s^{\beta})ds\right|$$

$$= \left|\frac{t^{\alpha-\beta}}{\beta}\cos(t^{\beta})\Big]_1^t + \frac{1}{\beta}\int_1^t(\alpha-\beta)s^{\alpha-\beta-1}\cos(s^{\beta})ds\right|$$

which is bounded since $\beta \ge \alpha$.

(5) Let $w = t^{-1/2}$ then $|p|^{-1/2}w = t^{-1}$ so (3.30) is satisfied, and $|p|(w')^2 = \frac{1}{4}t^{-2}$ which is bounded. Letting $q_1 = -t$ then $-q_1 w^2 = 1$ and letting $q_2 = -te^t\sin(e^t)$, in (iv) we have:

$$|p|^{-1/2}w\left|\int_1^t q_2\right| = \frac{1}{t}\left|\int_1^t se^s\sin(e^s)ds\right|$$

$$= \frac{1}{t}\left|-s\cos(e^s)\Big]_1^t + \int_1^t\cos(e^s)ds\right|$$

$$\le 2 + \frac{1}{t}$$

which is bounded as required.

(6) As in (5), let $w = t^{-1/2}$ and $q_1 = -t$. Also let $q_2 = -t^d \sin(t^d)$, so that (iv) becomes

$$|p|^{-1/2} w \left|\int_1^t q_2\right| = \frac{1}{t}\left|\int_1^t s^d \sin(s^d)\,ds\right|$$

$$= \frac{1}{t}\left|\left[-\frac{s}{d}\cos(s^d)\right]_1^t + \frac{1}{d}\int_1^t \cos(s^d)\,ds\right|$$

$$\leq \frac{1}{d}\left(2 + \frac{1}{t}\right)$$

which is bounded since $d \neq 0$.

(7) The choice of w, q_1, q_2 is the same as in (5) and we let $q_3 = \operatorname{Im} q$ so that in (v), $-q_3 w^2 = -\frac{\operatorname{Im} q}{t} \leq k$

(8) The choice of w, q_1, q_2 is the same as in (6), and we let $q_3 = \operatorname{Im} q$ as in (7).

Examples (9) - (12) are proved by means of Corollary (3.10).

(9) Eastham gave this example in the case $\operatorname{Im} q = 0$ (see [15]). As Corollary (3.10) is a simple generalization of his result, so is this example.

(10) The same comment applies here as to example (9).

(11) By choosing $a_m = \{(2m-1)\pi\}^{1/\beta}$ and $b_m = (2m\pi)^{1/\beta}$, as in [15], $\operatorname{Re} q \geq 0$ and $\operatorname{Im} q \geq 0$ on each I_m and the result follows as in [15].

(12) The proof is the same as (11).

We note that in each of the above examples, the requirement that

$\frac{1}{p}$, $q \in L_{loc}[1,\infty)$ is also always needed.

For the sake of completeness we conclude the present chapter by quoting the following J-selfadjointness criterion given by I.M. Glazman in [25, p.80].

<u>Theorem 3.11</u>. *If* $p = 1$ *and* Re q *is bounded below then the minimal operator* L_0 *corresponding to expression* (2.79) *considered on the interval* $(-\infty,\infty)$, *is J-selfadjoint.*

CHAPTER IV

THE LOCATION OF THE ESSENTIAL SPECTRUM

Having given the necessary background operator theory associated with (2.79) on $[a,\infty)$, we now introduce the various parts of the spectrum of such operators. In the present chapter we prove a number of results concerning the location of the essential spectrum.

In Chapter II, the manifold $\Delta_\lambda(A)$ and the regularity field $\pi(A)$ of a general, linear, closed operator A in H were defined by (2.52), (2.53). We now define the *resolvent set* $\rho(A)$ of such an operator which is densely defined, to be the set of all λ in $\pi(A)$ for which $\Delta_\lambda(A) = H$. The *spectrum* of A, $\sigma(A)$, is the complement in the complex plane of the resolvent set of A. The set of eigenvalues of A is called the *point spectrum* of A and is denoted by $P\sigma(A)$. Further, we denote by $E\sigma(A)$ the *essential spectrum* of A, which is defined to be the set of values of λ for which there exists a bounded non-compact sequence $\{f_n\}$ in D_A (called a singular sequence) satisfying the condition

$$\lim_{n\to\infty}(A-\lambda I)f_n = 0 \tag{4.1}$$

Finally, we define the *residual spectrum*, $R\sigma(A)$ as the set of values of λ for which the closure of $\Delta_\lambda(A)$ is a strict subset of H, and which do not belong to $P\sigma(A)$. Thus $R\sigma(A)$ is the set of all complex numbers λ for which $\bar{\lambda} \in P\sigma(A^*)$ but $\lambda \notin P\sigma(A)$. ([25, p.7] erroneously has $\lambda \in P\sigma(A^*)$ instead of $\bar{\lambda} \in R\sigma(A^*)$.)

It is obvious that the sets $P\sigma(A)$, $E\sigma(A)$ and $R\sigma(A)$ belong to the spectrum $\sigma(A)$ of A. Conversely, each point of $\sigma(A)$ belongs to one of those sets. Thus the set $\sigma(A)$ may be written as the (not necessarily disjoint) union

$$\sigma(A) = P\sigma(A) \cup E\sigma(A) \cup R\sigma(A) \tag{4.2}$$

The above definitions are those given in [25] except for the set $E\sigma(A)$ which is called the continuous spectrum in [25], and has been given various other names (see [66, p.242]). Moreover, there are at least five other ways of defining the essential spectrum, which are not equivalent for general closed operators (see [26]). In particular, for J-selfadjoint operators arising from ordinary differential expressions (which cannot have eigenvalues of infinite multiplicity), $E\sigma(A)$ may conceivably be a proper subset of the set of non-isolated points of $\sigma(A)$.

In the present work we define the *continuous spectrum* $C\sigma(A)$ of A to be the set that remains when all eigenvalues and points of $R\sigma(A)$ are removed from $E\sigma(A)$. The spectrum of A may then be written as the disjoint union

$$\sigma(A) = P\sigma(A) \cup C\sigma(A) \cup R\sigma(A) \tag{4.3}$$

Clearly $\rho(A)$ is the set of values of λ for which $(A-\lambda I)^{-1}$ exists as a bounded operator, in agreement with the definition given in [14]. The definitions given above of the residual spectrum and the point spectrum are the same as those in [14], since $P\sigma(A)$ is the set of values of λ for which $(A-\lambda I)$ is not one-to-one.

In order to compare definitions of $E\sigma(A)$ we need the following result:

Lemma 4.1 [25, p.9]. *If* $\lambda \notin E\sigma(A)$ *then* $\Delta_\lambda(A)$ *is closed. Conversely if* $\Delta_\lambda(A)$ *is closed and* λ *is not an eigenvalue of infinite multiplicity then* $\lambda \notin E\sigma(A)$.

Hence if A has no eigenvalues of infinite multiplicity, $E\sigma(A)$ is the set of complex numbers λ for which $\Delta_\lambda(A)$ is not closed, which is the definition given in [14] for the essential spectrum of A. Furthermore, if A has no eigenvalues of infinite multiplicity, the continuous spectrum of A is the set of all $\lambda \in \sigma(A)$ for which $(A-\lambda I)$ is one-to-one and $\Delta_\lambda(A)$ is dense in H but not equal to H, in agreement with the definition given in [14].

For a J-selfadjoint operator A, it is clear that $\lambda \in P\sigma(A)$ if and only if $\bar{\lambda} \in P\sigma(A^*)$ and so we have

Lemma 4.2. *If* A *is J-selfadjoint then* $R\sigma(A)$ *is empty.*

However it is clear from the definitions that sets $\rho(A)$, $P\sigma(A)$ and $C\sigma(A)$ are disjoint and cover the complex plane if A is J-selfadjoint. The same is true of the sets $\pi(A)$, $P\sigma(A)$ and $C\sigma(A)$, thus

Lemma 4.3. *If* A *is J-selfadjoint then* $\pi(A) = \rho(A)$.

Indeed it is clear from the definitions that for any A we have

$$\pi(A) \cap \sigma(A) \subset R\sigma(A) \tag{4.4}$$

In the case of our differential operators we have the following important link between the L^2-solutions of the differential equation $\ell(u) = \lambda u$, and the spectra of extensions of L_0.

Theorem 4.4. *If there are no non-trivial solutions of the equation $\ell(u) = \lambda u$ in $L^2[a,\infty)$ then $\lambda \in C\sigma(\tilde{L})$ where $\tilde{L}$ is any closed extension of L_0.*

Proof. From the definition of $P\sigma(\tilde{L})$, it follows that $\lambda \notin P\sigma(\tilde{L})$. From Theorems 2.23, 2.25, $\lambda \notin \pi(L_0)$ and hence $\lambda \notin \pi(\tilde{L})$. Finally, since $L_0 \subset \tilde{L} \subset L$ so $L_0^* \supset \tilde{L}^* \supset L^*$ and from Theorem 2.15, Lemma 2.12 and (2.49), $JLJ \supset \tilde{L}^* \supset JL_0J$. Thus $\bar{\lambda} \notin P\sigma(\tilde{L}^*)$ and so $\lambda \notin R\sigma(\tilde{L})$. It then follows that $\lambda \in C\sigma(\tilde{L})$. #

In the case of the operator L_0, if we suppose $\ell(y) = \lambda y$ for some λ and some $y \in \mathcal{D}_0$, then from Property 2.18, $y(a) = py'(a) = 0$ and hence by the uniqueness of solutions of the linear differential equation, y must be identically zero on $[a,\infty)$. Thus L_0 can have no eigenvalues

i.e.
$$P\sigma(L_0) = \emptyset \tag{4.5}$$

Furthermore, it is clear from the definition of the residual spectrum of an operator A, that any point of $R\sigma(A)$ must be either an eigenvalue of A or a point in the essential spectrum, or the regularity field. Thus

$$R\sigma(A) \subset \pi(A) \cup P\sigma(A) \cup E\sigma(A) \tag{4.6}$$

So in view of (4.5) we have for the operator L_0, that the complex plane $\mathbb{C}$ is divided into two disjoint sets.

$$\mathbb{C} = \pi(L_0) \cup E\sigma(L_0) \tag{4.7}$$

In order to locate the essential spectrum of extensions of L_0 we need the following

Lemma 4.5 [25, p.10]. *If $\tilde{A}$ is a finite-dimensional linear extension of the closed operator A then $E\sigma(\tilde{A}) = E\sigma(A)$.*

By applying Theorem 2.21 to the operator L_0, (2.55) yields that if $\pi(L_0)$ is not empty then $L(=JL_0^*J)$ is a finite-dimensional extension of L_0, and hence Lemma 4.5 may be applied to any linear extension L_1 of L_0 which satisfies $L_0 \subset L_1 \subset L$. On the other hand if $\pi(L_0)$ is empty then from (4.7), $E\sigma(L_0)$ covers the whole complex plane, and hence if L_1 is an extension of L_0, $E\sigma(L_1)$ must also cover the complex plane. We have therefore proved

Theorem 4.6. *If L_1 is any linear extension of L_0 such that $L_0 \subset L_1 \subset L$ then $E\sigma(L_1) = E\sigma(L_0)$.*

From (2.54), we have that for any such extension L_1,

$$\pi(L_1) \subseteq \pi(L_0) \tag{4.8}$$

In Chapter II, the J-selfadjoint extensions of L_0 were characterised under the assumption that $\pi(L_0)$ be non-empty. However, even without such an assumption the following is known:

Theorem 4.7 [22]. *Every J-symmetric operator has a J-selfadjoint extension.*

This is a surprising result, as there is no direct analogue for general, symmetric operators.

If, then, $\tilde{L}$ is a J-selfadjoint extension of L_0, the existence of such an extension being guaranteed by Theorems 2.15, 4.7, and we have $L_0 \subset \tilde{L} \subset L$. Furthermore, Lemma 4.2 yields that $R\sigma(\tilde{L})$ is empty and we have

<u>Theorem 4.8</u>. *Let $\tilde{L}$ be any J-selfadjoint extension of L_0 then $R\sigma(\tilde{L})$ is empty, $E\sigma(\tilde{L}) = E\sigma(L_0)$, and points of $\pi(L_0)$ are in either $\rho(\tilde{L})$ or $P\sigma(\tilde{L})$.*

Thus if we prove any result concerning the essential spectrum of J-selfadjoint extensions of L_0, this yields information about $E\sigma(L_0)$. Clearly, it is valuable to find results which locate the set $E\sigma(L_0)$, but in particular if we can prove that under certain conditions on p and q, $E\sigma(L_0)$ is restricted to some proper subset of the complex plane, then by (4.7), $\pi(L_0)$ will be non-empty and the results given in Chapter II will be applicable. Let us recall the dichotomy in that chapter, between the cases of $\mathrm{def}(L_0) = 1$ and $\mathrm{def}(L_0) = 2$ when $\pi(L_0)$ is not empty. If we consider having two linearly independent solutions of $\ell(y) = \lambda_0 y$ in $L^2[a,\infty)$ for some λ_0, without assuming $\pi(L_0) \neq \emptyset$, we have:

<u>Theorem 4.9</u> ([62]). *If for some $\lambda_0 \in \mathbb{C}$ there are two linearly independent solutions of $\ell(y) = \lambda_0 y$ in $L^2[a,\infty)$ (and hence all solutions are in $L^2[a,\infty)$) then $\lambda_0 \in \pi(L_0)$.*

<u>Proof</u>. Define $y_1(x)$, $y_2(x) \in L^2[a,\infty)$ to be the solutions of $\ell(y) = \lambda_0 y$ satisfying

and define the operator L_1 by

$$\mathcal{D}(L_1) = \{y \in \mathcal{D}(L): [y,\bar{y}_1](a) = 0, [y,\bar{y}_2](\infty) = 0\}$$

where

$$L_1 y = \ell(y), \quad y \in \mathcal{D}(L_1)$$

and

$$\begin{aligned} [u,v](t) &= u(t)p\bar{v}'(t) - pu'(t).\bar{v}(t), \quad t \in [a,\infty) \\ [u,v](\infty) &= \lim_{t\to\infty}[u,v](t). \end{aligned} \tag{4.10}$$

The existence of $[u\bar{v}](\infty)$ for $u,v \in \mathcal{D}(L)$ is guaranteed by the relation,

$$\int_a^t u\ell(v) = [u\bar{v}](t) - [u\bar{v}](a) + \int_a^t v\ell(u), \quad t \in [a,\infty]. \tag{4.11}$$

We note that for the linearly independent solutions y_1 and y_2,

$$[y_1,\bar{y}_2](t) = 1, \quad t \in [a,\infty). \tag{4.12}$$

We consider now whether λ_0 may be an eigenvalue of the operator L_1 by firstly supposing it is, with eigenfunction y. Then $\ell(y) = \lambda_0 y$ so

$$y = \alpha y_1 + \beta y_2$$

for some constants $\alpha,\beta \in \mathbb{C}$.

But since $y \in \mathcal{D}(L_1)$,

$$[y,\bar{y}_1](a) = 0, \quad [y,\bar{y}_2](\infty) = 0 \tag{4.13}$$

for any function u. So $y \equiv 0$ on $[a,\infty)$ contradicting its being an eigenfunction. Hence λ_0 is not an eigenvalue of L_1.

We now show L_1 is a closed extension of L_0. Clearly L_1 is an extension of L_0, since $\mathcal{D}(L_0) \subset \mathcal{D}(L_1)$. We use methods directly analogous to those of [58,§17]. Firstly, following [58], there exists $z_1 \in \mathcal{D}(L)$ with

$$z_1(a) = y_1(a) = 1,\ pz_1'(a) = py_1'(a) = 0,\ z_1 \equiv 0 \text{ near } \infty$$

and there exists a function $z_2 \in \mathcal{D}(L)$ defined on $[a,\infty)$ such that

$$\left.\begin{aligned} z_2(a) &= pz_2'(a) = 0 \\ z_2(a+1) &= y_2(a+1), pz_2'(a+1) = py_2'(a+1) \\ z_2(t) &= y_2(t),\quad t \in [a+1,\infty). \end{aligned}\right\} \tag{4.14}$$

This follows from Lemma 2.6. Now suppose a sequence $\{u_n\} \to y$ as $n \to \infty$ such that $\{L_1u_n\} \to z$ as $n \to \infty$. Then since $\{u_n\} \subset \mathcal{D}(L)$ and L is closed, $y \in \mathcal{D}(L)$ and $Ly = z$. Furthermore, using (4.11), (4.14) and the fact that $\{u_n\} \subset \mathcal{D}(L)$,

$$\lim_{n\to\infty}\left(\int_a^\infty z_1 \cdot \ell(u_n) - \int_a^\infty u_n \cdot \ell(z_1)\right) = \lim_{n\to\infty}\{[u_n,\bar{z}_1](\infty) - [u_n,\bar{z}_1](a)\} = 0.$$

But also

$$\lim_{n\to\infty}\left(\int_a^\infty z_1 \cdot \ell(u_n) - \int_a^\infty u_n \cdot \ell(z_1)\right) = \int_a^\infty z_1 \cdot \ell(y) - \int_a^\infty y \cdot \ell(z_1)$$

$$= [y,\bar{z}_1](\infty) - [y,\bar{z}_1](a)$$

$$= [y,\bar{y}_1](a)$$

To do this we define the Green's function,

$$G(x,\xi,\lambda_0) = \begin{cases} -y_1(x)y_2(\xi), & a \le x \le \xi, \\ -y_2(x)y_1(\xi), & \xi \le x < \infty. \end{cases} \tag{4.15}$$

By well-known methods if we define

$$\phi_f(x) = \int_a^\infty G(x,\xi,\lambda_0).f(\xi)d\xi, \quad f \in L^2[a,\infty) \tag{4.16}$$

which is well-defined for all $f \in L^2[a,\infty)$ since $y_1, y_2 \in L^2[a,\infty)$, then

$$\begin{gathered} (L-\lambda I)\phi_f = f \\ [\phi_f,\bar{y}_1](a) = 0, [\phi_f,\bar{y}_2](\infty) = 0 \end{gathered} \tag{4.17}$$

and ϕ_f is the unique solution of (4.17). Thus

$$\phi_f = (L_1-\lambda_0 I)^{-1}f$$

and since $y_1, y_2 \in L^2[a,\infty)$,

$$K = \int_a^\infty\int_a^\infty |G(x,\xi,\lambda_0)|^2 dx d\xi < \infty$$

so

$$\|(L_1-\lambda_0 I)^{-1}f\|^2 = \|\phi_f\|^2 \le K\|f\|^2, \quad f \in L^2[a,\infty).$$

Hence for any $g \in \mathcal{D}(L_1)$,

$$\|(L_1-\lambda_0 I)g\|^2 \geq \frac{1}{k}\|g\|^2$$

so $\lambda_0 \in \pi(L_1)$ as required. ∎

Combining Lemma 3.2 and Theorem 4.9 it follows that if there are two linearly independent solutions of $\ell(y) = \lambda_0 y$ for some $\lambda_0 \in \mathbb{C}$ then

$$\pi(L_0) = \mathbb{C} \quad \text{and} \quad E\sigma(L_0) = \emptyset \tag{4.18}$$

using (4.7). Hence the spectrum of any J-selfadjoint extension of L_0 consists only of eigenvalues.

By applying the above method of proof to the 2n-th order differential expression (2.1) it is now clear that the assumption that $\pi(L_0)$ be non-empty in the *regular* case in Chapter II is automatically satisfied, for example in Theorem 2.24(i), and that (4.18) is also always true in the regular case.

In the above situations it has been shown (see [77]) by methods similar to those used in the real case (see [58,§19]) that the resolvent of any *well-posed*, J-selfadjoint extension of L_0 is an integral operator with a kernel of Hilbert-Schmidt type, and hence the spectrum of well-posed, J-selfadjoint extensions is discrete i.e. consists of eigenvalues of finite multiplicity with no finite point of accumulation. Any J-selfadjoint extension which is not well-posed has empty essential spectrum, residual spectrum and regularity field and consequently its spectrum consists entirely of eigenvalues and covers the whole plane. Since we shall not need the above results concerning the nature of the resolvent and the consequent lack of points of accumulation of the spectrum in the present work we shall omit the proofs thereof. The above remarks apply only, of course, to the regular case and to the case when $\mathrm{def}(L_0) = 2$. Having completely determined the spectrum and regularity field of L_0 when all solutions of $\ell(y) = \mu_0 y$ are in $L^2[a,\infty)$ (or $L^2[a,b]$), we now consider results which confine the essential spectrum of L_0 to some proper subset of the complex plane, when the operators may not be in the 'limit-circle' case.

Firstly we mention a result due to V.B. Lidskii, who investigated the expression (2.79) with $p \equiv 1$ considered on the interval $(-\infty,\infty)$.

<u>Theorem 4.10</u> ([52],[53]). *Consider the expression*

$$\ell(y) = -y'' + qy \tag{4.19}$$

on the interval $(-\infty,\infty)$, *where* $\mathrm{Re}\, q$ *is bounded below and* $\mathrm{Im}\, q$ *is semi-bounded. Then for the complete continuity of the resolvent operator of any J-selfadjoint extension of* L_0, *and hence the discreteness of the spectrum, it is necessary and sufficient that for every sequence of disjoint intervals* $\{D_n\}$ *of equal length,*

$$\int_{D_n} (\mathrm{Re}\, q + |\mathrm{Im}\, q|) \to \infty$$

when the intervals recede to infinity.

Returning to the expression (2.79) on the interval $[a,\infty)$, the following result is useful in locating $E\sigma(L_0)$.

Theorem 4.11 ([25,p.75], [14,p.1438]). *For any closed linear operator A, the set of values of the quadratic functional* (Af,f) *for* $\|f\| = 1$, $f \in \mathcal{D}_A$, *is some convex set in the complex λ-plane. If the point λ does not belong to the closure* $K(A)$ *of this set then it is a regular-type point of the operator A and*

$$\|(A-\lambda I)f\| \geq \delta\|f\| \tag{4.20}$$

where δ is the distance of the point λ from the set $K(A)$.

In our case, since L_0 is the closure of L_0',

$$K(L_0') = K(L_0) \tag{4.21}$$

and so from (4.7) and Theorem 4.11,

$$E\sigma(L_0) \subset K(L_0') \tag{4.22}$$

and we need only consider the values of $(L_0'f,f)$, $f \in \mathcal{D}_0'$. We denote the complement of $K(A)$ in $\mathbb{C}$ by $R(A)$, which is called the *external regularity field* of A. We then also have

Theorem 4.12. [25,p.78]. *If A is a J-symmetric operator with* $\pi(A)$ *not empty and with a finite defect number, and* $\tilde{A}$ *is any J-self-adjoint extension of A, then:*

(i) $R(A)$ *contains no points of* $\sigma(\tilde{A})$ *other than its eigenvalues*

(ii) *Either each point of at least one of the connected components of* $R(A)$ *is an eigenvalue of* $\tilde{A}$, *or else* $\sigma(\tilde{A})$ *has no finite limit points inside* $R(A)$.

We may improve (4.22) by means of the so-called decomposition principle. Let $\beta > a$ be given and denote by L_1 the operator defined by

$$D_1 = \{y \in D_0 : y(\beta) = y'(\beta) = 0\}$$
$$L_1 y = L_0 y, \quad y \in D_1 \tag{4.23}$$

Now L_1 is obviously an orthogonal sum, of the minimal differential operators L_2 and L_β defined by $\ell(\cdot)$ in $L^2[a,\beta]$ and $L^2[\beta,\infty)$ respectively. It is clear that L_0 is a finite-dimensional extension of $L_2 \oplus L_\beta$, since $L_2 \oplus L_\beta$ has finite defect number (at most four), thus from Lemma 4.5,

$$E\sigma(L_0) = E\sigma(L_1) = E\sigma(L_2 \oplus L_\beta) \tag{4.24}$$

However (see [25,p.10]),

$$E\sigma(L_2 \oplus L_\beta) = E\sigma(L_2) \cup E\sigma(L_\beta) \tag{4.25}$$

and from (4.18), since L_2 is regular, $E\sigma(L_2)$ is empty so combining (4.24) and (4.25),

$$E\sigma(L_0) = E\sigma(L_\beta) \tag{4.26}$$

By applying Theorem 4.11 to L_β, using (4.22) and (4.26) and letting $\beta \to \infty$, we have

Theorem 4.13. *Let* $K_N(L_0')$ *be the closure of the set*

$$K_N'(L_0') = \{(L_0'y,y);\ y \in D_0',\ \|y\| = 1,\ \operatorname{supp} y \subset [N,\infty)\} \tag{4.27}$$

then

$$E\sigma(L_0) \subset \bigcap_{N=1}^{\infty} K_N(L_0') \tag{4.28}$$

However, if $y \in D_0'$, integrating by parts,

$$(L_0'y,y) = \int_a^\infty p|y'|^2 + \int_a^\infty q|y|^2 \tag{4.29}$$

Thus if p is real and non-negative and we let Q_N denote the smallest convex set containing the values of the function q(x) for $x \geq N$ and, if we let

$$Q_N^* = \{z + t;\ z \in Q_N,\ t \geq 0\} \tag{4.30}$$

then it is clear that

$$K_N(L_0') \subset Q_N^* \tag{4.31}$$

and

$$\bigcap_{N=1}^{\infty} K_N(L_0') \subset \bigcap_{N=1}^{\infty} Q_N^* \tag{4.32}$$

Defining Q_∞^* to be the set on the right in (4.32) and R_∞ to be its complement in $\mathbb{C}$, we now have from Theorems 4.12, 4.13 that

$$E\sigma(L_0) \subset Q_\infty^* \tag{4.33}$$

and also

<u>Theorem 4.14</u>. *If $\pi(L_0)$ is not empty and if p is real and non-negative then the spectrum of any J-selfadjoint extension $\tilde{L}$ of L_0 satisfies*

(i) *The set R_∞ contains no points of $\sigma(\tilde{L})$ other than its eigenvalues.*

(ii) *Either each point of at least one of the connected components of R_∞ is an eigenvalue of $\tilde{L}$, or else $\sigma(\tilde{L})$ has no finite limit points* *inside* R_-.

Theorem 4.14 is given in [25,p.122] when the interval considered is $(-\infty,\infty)$ rather than $[a,\infty)$, as is the following result. By applying the above theory we obtain

<u>Theorem 4.15.</u> *If* $p \geq 0$ *and for some* $z \in \mathbb{C}$ *and* $\alpha \in \mathbb{R}$, *the values of the potential* q *lie inside some angle,*

$$\alpha < \arg(q(t)-z) < \alpha + \pi - \delta \tag{4.34}$$

which does not contain the negative semi-axis, and

$$\lim_{t\to\infty} |q(t)| = \infty \tag{4.35}$$

then

$$E\sigma(L_0) = \emptyset$$

and the spectrum of any J-selfadjoint extension $\hat{L}$ *of* L_0 *consists of eigenvalues only. Further, either* $P\sigma(\tilde{L})$ *covers the whole complex plane or else, if* $\tilde{L}$ *is well-posed, the spectrum of* $\tilde{L}$ *has no finite limit points.*

In particular, if Re $q(t) \to \infty$ or Im $q(t) \to \infty$ as $t \to \infty$, the conditions of Theorem 4.15 are satisfied, and the spectrum of well-posed, J-selfadjoint extensions $\tilde{L}$ is discrete, a result first proved by V.B. Lidskii ([52]). A further consequence of the above analysis is

<u>Theorem 4.16.</u> *Let* p *be real and non-negative and suppose* q *takes values in the half-plane:*

$$\mathrm{Re}\{(q(t)-z)e^{i\alpha}\} \geq 0 \tag{4.36}$$

for some $z \in \mathbb{C}$ *and* $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, *then*

$$E\sigma(L_0) \subset \{\lambda \in \mathbb{C};\ \mathrm{Re}((\lambda - z)e^{i\alpha}) \ge 0\} \tag{4.37}$$

Setting $\alpha = \pm\frac{\pi}{2}$ in Theorem 4.16, and using (4.35), we have

Corollary 4.17. *Let* p *be real and non-negative then*

$$E\sigma(L_0) \subset \{\lambda \in \mathbb{C};\ \liminf_{t\to\infty} \mathrm{Im}\, q(t) \le \mathrm{Im}(\lambda) \le \limsup_{t\to\infty} \mathrm{Im}\, q(t)\}$$

In particular if $\mathrm{Im}\, q(t) \to k$ *as* $t \to \infty$, *then*

$$E\sigma(L_0) \subset \{\lambda \in \mathbb{C};\ \mathrm{Im}(\lambda) = k\}.$$

The results above serve to reinforce the conclusion of [25, p.29] that the essential spectrum is determined solely by the behaviour of the coefficients at infinity. On the other hand, it is not hard to show by the reasoning of [25,p.28-29] that the behaviour of the set of eigenvalues of extensions of L_0 is affected by the behaviour of the coefficients at finite values of $t \in [a,\infty)$. This explains, for example, why one cannot easily locate such eigenvalues via 'integral conditions' on Im q. We shall look at the question of locating eigenvalues, in more detail in Chapter V.

From now on we shall for simplicity assume $p \equiv 1$ *on* $[a,\infty)$. We firstly have the following two simple results.

Theorem 4.18. *If* L_0 *is the usual, minimal operator and* T_0 *is the minimal operator corresponding to the expression*

$$-y'' + \mathrm{Re}\, q.y \tag{4.38}$$

on $[a,\infty)$, *and if*

$$\mathrm{Im}\, q(t) \to 0 \text{ as } t \to \infty \tag{4.39}$$

then $E\sigma(T_0) = E\sigma(L_0)$ *and the J-selfadjoint extensions of* T_0 *and* L_0 *have identical domains.*

Proof. We note that $\mathrm{Im}\, q.y \in L^2[a,\infty)$ whenever $y \in L^2[a,\infty)$, since $\mathrm{Im}\, q$ is bounded, and also

$$(i\,\mathrm{Im}\, q.y,y) = (y,J(i\,\mathrm{Im}\, q)Jy), \quad y \in L^2[a,\infty)$$

hence the domain of T_0 and L_0 are identical, as are those of their J-selfadjoint extensions. Hence T_0 and L_0 are either both in the limit-point case, or both in the limit-circle case, since $\pi(L_0)$, $\pi(T_0)$ are non-empty, by Corollary 4.17.

Now by means of the decomposition principle (see(4.26)), $E\sigma(L_0) = E\sigma(L_\beta)$ and $E\sigma(T_0) = E\sigma(T_\beta)$. Choose any fixed $\lambda \in \pi(L_0) = \pi(L_\beta)$ (for any $\beta \geq a$) and choose β sufficiently large so that

$$|\mathrm{Im}\, q(t)| < \tfrac{1}{2}k_\lambda, \quad t \geq \beta \tag{4.40}$$

where k_λ is as defined in (2.53). Now for any $y \in \mathcal{D}(T_\beta)$

$$\begin{aligned} \|(T_\beta-\lambda I)y\| &\geq \|(L_\beta-\lambda I)y\| - \|(T_\beta-L_\beta)y\| \\ &\geq k_\lambda\|y\| - \|\mathrm{Im}\, q.y\| \\ &\geq \tfrac{1}{2}k_\lambda\|y\| \end{aligned}$$

by (4.40). Hence $\lambda \in \pi(T_\beta)$ $(=\pi(T_0))$. Hence $\pi(L_0) \subset \pi(T_0)$. Similarly, the reverse inclusion may be proved, so $\pi(L_0) = \pi(T_0)$. The conclusion of the theorem now follows from (4.7). ∎

Theorem 4.19. *If* [illegible]

$$q(t) \to 0 \quad \text{as} \quad t \to \infty \tag{4.41}$$

then the operators L_γ *as defined in Chapter II are J-selfadjoint and* $E\sigma(L_0) = [0,\infty)$. *Outside of this semi-axis either* $P\sigma(L_\gamma)$ *covers the rest of the plane, or* $\sigma(L_\gamma)$ *consists of eigenvalues which accumulate only at points on the semi-axis* $[0,\infty)$.

Proof. Apply Theorem 4.18, using the fact that if $\mathrm{Re}\, q(t) \to 0$ as $t \to \infty$, then T_0 is in the limit-point case and $E\sigma(T_0) = [0,\infty)$. (See [25,p.85,106]). The latter part of the conclusion follows from Theorem 4.14. ∎

Theorems 4.18 and 4.19 were proved in a slightly different form for the case when the interval $(-\infty,\infty)$ is considered, using other methods, by Lidskii [53]. We now give a generalisation of Theorem 4.19, which in the real case is due to M. Sh Birman [9] (see also [25,p.112]), and which determines the exact location of $E\sigma(L_0)$. In order to prove it we need the concept of relatively compact (or relatively completely continuous) operators. We use the definition given by T. Kato [36,p.194].

Let K and A be linear operators in a Hilbert space H with $\mathcal{D}(A) \subset \mathcal{D}(K)$. Assume that for any sequence $\{u_n\} \subset \mathcal{D}(A)$ with both $\{u_n\}$ and $\{Au_n\}$ bounded, $\{Ku_n\}$ contains a convergent subsequence. (i.e. K is compact in $\mathcal{D}(A)$ with respect to the graph norm of A). Then K is said to be *relatively compact with respect to A* or simply *A-compact*.

In [25,p. [illegible]], Glazman defines relative compactness slightly differently, but it may readily be seen that if K is A-compact according to Glazman's definition, then so is K A-compact by Kato's definition. We note also (see [36,p.194]) that if K is A-compact then K is A-bounded i.e. there exist non-negative numbers a and b such that

$$\|Ku\| \leq a\|u\| + b\|Au\|, \quad u \in \mathcal{D}(A). \tag{4.42}$$

We need this fact in the following lemma, which is used in the proof of the main result of this section.

<u>Lemma 4.20.</u> *If K is A-compact where K and A are closeable, linear operators with $\mathcal{D}(A) \subset \mathcal{D}(K)$, then $\bar{K}$ is $\bar{A}$-compact.*

<u>Proof.</u> Take any $x \in \mathcal{D}(\bar{A})$ then there is a sequence $\{x_n\} \subset \mathcal{D}(A)$ such that $x_n \to x$ and $Ax_n \to \bar{A}x$ as $n \to \infty$. Since (4.42) is satisfied,

$$\|K(x_n - x_m)\| \leq a\|A(x_n - x_m)\| + b\|x_n - x_m\|$$

for any n and m. Therefore $\{Kx_n\}$ is a Cauchy sequence and from the definition of $\bar{K}$, $x \in \mathcal{D}(\bar{K})$. Letting $n \to \infty$ in the inequality

$$\|Kx_n\| \leq a\|Ax_n\| + b\|x_n\|$$

yields

$$\|\bar{K}x\| \leq a\|\bar{A}x\| + b\|x\|.$$

Thus $\mathcal{D}(\bar{A}) \subset \mathcal{D}(\bar{K})$ and $\bar{K}$ is $\bar{A}$-bounded.

Now taking a bounded sequence $\{u_n\} \subset \mathcal{D}(\bar{A})$ such that $\{\bar{A}u_n\}$ is bounded, for each n there is a sequence $\{v_{nm}\}$ in $\mathcal{D}(A)$ such that

$$\|v_{nm} - u_n\| \leq \frac{1}{m}, \; \|Av_{nm} - \bar{A}u_n\| \leq \frac{1}{m} \tag{4.43}$$

so consider the bounded sequences $\{v_{nn}\}$, $\{Kv_{nn}\}$. Since K is A-compact there is a convergent subsequence, $\{Kv_{n_k n_k}\}$ of $\{Kv_{nn}\}$. Let

$$y = \lim_{k\to\infty} Kv_{n_k n_k}.$$

Then since $\tilde{K}$ is $\tilde{A}$-bounded, and from (4.43), we have

$$\begin{aligned}\|\tilde{K}u_{n_k} - y\| &\le \|\tilde{K}u_{n_k} - Kv_{n_k n_k}\| + \|Kv_{n_k n_k} - y\| \\ &\le a\|\tilde{A}u_{n_k} - Av_{n_k n_k}\| + b\|u_{n_k} - v_{n_k n_k}\| \\ &\quad + \|Kv_{n_k n_k} - y\| \\ &\le \frac{a}{n_k} + \frac{b}{n_k} + \|Kv_{n_k n_k} - y\| \qquad (4.44)\end{aligned}$$

Letting $k \to \infty$ in (4.44), the right hand side tends to zero, so $\tilde{K}u_{n_k} \to y$ and hence $\{\tilde{K}u_n\}$ contains a convergent subsequence, as required to prove $\tilde{K}$ is $\tilde{A}$-compact. ∎

(I am grateful to M. Faierman for indicating the details of the above proof, the basis of which may be found in [36]).

We are now able to prove the following theorem, first obtained by [illegible]rman [9] for the case when q is real-valued.

<u>Theorem 4.21</u>. *If for some (and consequently any) $\omega > 0$,*

$$\lim_{x\to\infty} \int_x^{x+\omega} |q(t)|\,dt = 0 \qquad (4.45)$$

then $E\sigma(L_0)$ coincides with the semi-axis $\lambda \ge 0$.

Proof. Define the operators K_1, K_2 and A by

$$K_1 y = (\mathrm{Re}\, q).y, \quad K_2 y = (\mathrm{Im}\, q).y, \tag{4.46}$$

$$Ay = -y'' + y$$

where

$$\mathcal{D}(K_1) = \mathcal{D}(K_2) = \mathcal{D}(A) = \mathcal{D}(L_0').$$

Following Birman [9] we have that K_1 and K_2 are both A-compact. It is then clear from our definition of relative compactness, that the operator K defined by

$$Ky = K_1 y + iK_2 y = q.y \tag{4.47}$$

on $\mathcal{D}(K) = \mathcal{D}(L_0')$ is A-compact. Hence by Lemma 4.20 since A and K are closable (see [57]), $\bar{K}$ is $\bar{A}$-compact. Then by Kato [36, p.244] we have $E\sigma(\bar{A}) = E\sigma(\bar{K}+\bar{A})$. Now $E\sigma(\bar{A}) = [1,\infty)$ and $E\sigma(\bar{K}+\bar{A}) = E\sigma(L_0+I)$, so $E\sigma(L_0) = [0,\infty)$. ∎

From this we have

Corollary 4.22. *If* $\int_a^\infty |q|^p < \infty$ *for some* $p \geq 1$ *then* $E\sigma(L_0) = [0,\infty)$.

Proof.

$$\int_X^{X+\omega} |q| \leq \left(\int_X^{X+\omega} |q|^p\right)^{1/p} . \left(\int_X^{X+\omega} 1\right)^{1-1/p}$$

$$= \left(\int_X^{X+\omega} |q|^p\right)^{1/p} . \omega^{1-1/p}.$$

Since $\int_a^\infty |q|^p < \infty$, the right hand side tends to zero as $X \to \infty$ so condition (4.45) of Theorem 4.21 is satisfied.

We observe that the above theorem is also a generalization of the application of a result of Balslev and Gamelin [[illegible], Theorem 4] to the particular operator L under consideration here. The condition on q (which is complex-valued) which they have is

$$\lim_{X\to\infty}\int_X^{X+\omega}|q|^2 = 0$$

which implies condition (4.45), as is seen from the proof of Corollary 4.22.

Theorems 4.15, 4.18, 4.19, 4.21 and Corollary 4.22 all stated precisely where in the complex plane, the set $E\alpha(L_\theta)$ lies. In general it is not possible to be so precise. However, we now prove a number of results which force the set $E\alpha(L_\theta)$ to lie in a quadrant, a half-plane, or a half-line in the complex plane. The first of these is

<u>Theorem 4.23</u>. *If* Re(q) *satisfies one of the following:*

(i) $$\int_{M_\delta}(\text{Re } q)^- < \infty \quad \textit{for any } \delta > 0 \tag{4.48}$$

where $M_\delta = \{t \in [a,\infty): (\text{Re } q(t))^- \geq \delta\}$

(ii) $$\int_a^\infty\{(\text{Re } q)^-\}^r < \infty \quad \textit{for some } r \geq 1 \tag{4.49}$$

(iii) $$\lim_{X\to\infty}\int_X^{X+\omega}(\text{Re } q)^- = 0 \quad \textit{for some } \omega \neq 0 \tag{4.50}$$

(and consequently for any $\omega \neq 0$*)*

(iv) $\int_J \operatorname{Re} q \geq -C$ *for some real number C, and for all intervals*

J of length ≤ 1 (4.51)

and if

$$\int_0^\infty |\operatorname{Im}(q)| < \infty \tag{4.52}$$

then $\operatorname{Ess}(L_0) \subset \Lambda,$

where $\Lambda = [0,\infty)$ *if* (i), (ii) *or* (iii) *is satisfied*

and $\Lambda = [-2C^2,\infty)$ *if* (iv) *is satisfied.*

The case when q is real and satisfies (4.48) or (4.49) was proved by Glazman [25, p.85-87]. The result for real-valued q satisfying (4.50) was proved by Birman ([9]) and quoted in [25, p.87]. The condition (4.51) was used by Brinck ([10]) in proving a different result.

We give the proof in detail under the assumption that Re(q) satisfies (4.48) and then give the necessary modifications for each of the other cases.

Proof.

(i) By Theorem 4.13 it suffices to consider values of the quadratic form $(L'y,y)$ for N-finite functions in $\mathcal{D}_0'$ i.e. functions in $\mathcal{D}_0'$ whose support is contained in (N,∞). Let m be any real number then setting p = 1 in (4.29),

$$(\operatorname{Re} + m \operatorname{Im})(L_0'y,y) = \int_N^\infty |y'|^2 + \int_N^\infty (\operatorname{Re} + m \operatorname{Im})(q).|y|^2$$

$$\geq \int_N^\infty |y'|^2 - \int_N^\infty (\operatorname{Re} q)^-.|y|^2 + m\int_N^\infty \operatorname{Im}(q).|y|^2$$

$$\geq \int_N^\infty |y'|^2 - \int_N^\infty (\operatorname{Re} q)^-.|y|^2 - |m|\int_N^\infty |\operatorname{Im}(q)|.|y|^2 \tag{4.53}$$

But then by means of a method used in [20],

$$y(t)^2 - y(a)^2 = 2\int_a^t yy', \quad t \in [a,\infty) \tag{4.54}$$

and $y(a) = 0$ so

$$|y(t)|^2 \leq 2\int_a^\infty |y|.|y'| \leq 2\left(\int_N^\infty |y|^2\right)^{1/2}.\left(\int_N^\infty |y'|^2\right)^{1/2} \tag{4.55}$$

So in (4.53)

$$(\operatorname{Re} + m \operatorname{Im})(L_0'y,y)$$

$$\geq \int_N^\infty |y'|^2 - 2|m|\left(\int_N^\infty |\operatorname{Im}(q)|\right)\left(\int_N^\infty |y|^2\right)^{1/2}\left(\int_N^\infty |y'|^2\right)^{1/2}$$

$$- \int_N^\infty (\operatorname{Re} q)^-.|y|^2. \tag{4.56}$$

Let $0 < \varepsilon < \frac{1}{4}$ be arbitrary and define $M(N,\varepsilon)$ and $N(N,\varepsilon)$ by

$$M = \{x \geq N: (\operatorname{Re} q)^- < \varepsilon\}$$

$$N = \{x \geq N: (\operatorname{Re} q)^- \geq \varepsilon\}$$

then choose $N(\varepsilon)$ sufficiently large so that

$$\int_N (\mathrm{Re}\ q)^- < \varepsilon \quad \text{and} \quad \int_N^\infty |\mathrm{Im}\ q| \le \frac{\varepsilon}{|m|} .$$

We then have

$$- \int_N (\mathrm{Re}\ q)^- . |y|^2 \ge -\varepsilon \int_N |y|^2 \ge -\varepsilon \int_N^\infty |y|^2$$

and, applying (4.55),

$$- \int_N (\mathrm{Re}\ q)^- . |y|^2 \ge -2 | \int_N (\mathrm{Re}\ q)^- | . (\int_N^\infty |y|^2)^{1/2} . (\int_N^\infty |y'|^2)^{1/2}$$

$$\ge -2\varepsilon (\int_N^\infty |y|^2)^{1/2} (\int_N^\infty |y'|^2)^{1/2} .$$

Thus in (4.56)

$$(\mathrm{Re} + m\ \mathrm{Im}) (L_0' y, y) \ge \int_N^\infty |y'|^2 - 4\varepsilon (\int_N^\infty |y'|^2)^{1/2} (\int_N^\infty |y|^2)^{1/2}$$

$$- \varepsilon \int_N^\infty |y|^2$$

$$= [(\int_N^\infty |y'|^2)^{1/2} - 2\varepsilon (\int_N^\infty |y|^2)^{1/2}]^2 - (\varepsilon + 4\varepsilon^2) \int_N^\infty |y|^2$$

$$\ge -2\varepsilon \int_N^\infty |y|^2$$

$$= -2\varepsilon \quad \text{if} \quad \|y\| = 1.$$

So by Theorem 4.13

$$E\sigma(L_0) \subset \{ \lambda : \mathrm{Re}\ \lambda + m\ \mathrm{Im}\ \lambda \ge -2\varepsilon \}$$

for $\varepsilon > 0$ arbitrarily small. Hence

$$E\sigma(L_0) \subset \{\lambda : \mathrm{Re}\,\lambda + m\,\mathrm{Im}\,\lambda \geq 0\}. \tag{4.57}$$

Geometrically this says $E\sigma(L_0)$ is contained in the half-plane $\{x + iy : x + my \geq 0\}$ to the right of the line $x + my = 0$ through the origin.

Now since in (4.57), m is an arbitrary real number, we may let m run through all real numbers to obtain $E\sigma(L_0) \subset [0,\infty)$ as required to prove the theorem under the assumption that (i) holds.

(ii) If Re q satisfies (4.49) then for $x \in M_\delta$,

$$\frac{1}{\delta}\cdot(\mathrm{Re}\, q)^- \geq 1$$

so

$$\frac{1}{\delta}(\mathrm{Re}\, q)^- \leq \frac{1}{\delta^r}\{(\mathrm{Re}\, q)^-\}^r$$

hence

$$\int_{M_\delta}(\mathrm{Re}\, q)^- \leq \frac{1}{\delta^{r-1}}\int_{M_\delta}\{(\mathrm{Re}\, q)^-\}^r < \infty$$

and Re q satisfies (4.48). The result then follows from the proof of (i).

(iii) If Re q satisfies (4.50) we again consider expression (4.53) and use the proof given in [15, p.87] in order to estimate the term involving Re q. Replacing q^* by $(\mathrm{Re}\, q)^-$ and choosing

$$2\eta(1 + C_\eta) < \varepsilon < \frac{1}{2}$$

in that proof yields

$$(\mathrm{Re} + m\,\mathrm{Im})(L_0' y,y) \ge \int_N^\infty |y'|^2 + m\int_N^\infty \mathrm{Im}(q).|y|^2 - \varepsilon\int_N^\infty (|y'|^2 + |y|^2)$$

$$\ge \frac{1}{2}\int_N^\infty |y'|^2 + m\int_N^\infty \mathrm{Im}(q).|y|^2 - \varepsilon\int_N^\infty |y|^2. \qquad (4.58)$$

Then the proof may proceed as above to estimate the term involving $\mathrm{Im}(q)$.

(iv) Suppose now that Re q satisfies (4.51). Although the interval $(-\infty,\infty)$ is used in [10], the same methods apply to the operator ℓ on $[a,\infty)$, giving

$$\int_N^\infty |y'|^2 + \int_N^\infty \mathrm{Re}(q).|y|^2 \ge -2C^2\int_N^\infty |y|^2 \qquad (4.59)$$

for any N-finite function y in $\mathcal{D}(L_0')$. By replacing $\mathrm{Im}(q)$ by $\frac{1}{\eta}.\mathrm{Re}(q)$ in (4.51) and (4.59) we have

$$\eta\int_N^\infty |y'|^2 + \int_N^\infty \mathrm{Re}(q).|y|^2 \ge -\frac{2}{\eta}C^2\int_N^\infty |y|^2 \qquad (4.60)$$

for any $0 < \eta < 1$. Hence, following the proof of part (i), we have

$$(\mathrm{Re} + m\,\mathrm{Im})(L_0' y,y) \ge (1-\eta)\int_N^\infty |y'|^2 - 2\iota\left(\int_N^\infty |y'|^2\right)^{1/2}\left(\int_N^\infty |y|^2\right)^{1/2}$$

$$- \frac{2}{\eta}C^2\int_N^\infty |y|^2$$

$$\ge -\left(\frac{\iota^2}{(1-\eta)} + \frac{2}{\eta}C^2\right)\int_N^\infty |y|^2$$

for any $\varepsilon > 0$ and $0 < \eta < 1$. So

$$E\sigma(L_0) \subset \{\lambda : \operatorname{Re}\lambda + m \operatorname{Im}\lambda \geq -\frac{\varepsilon^2}{(1-\eta)} - \frac{2}{\eta}C^2\}.$$

Since ε is arbitrarily small, we therefore have

$$E\sigma(L_0) \subset \{\lambda : \operatorname{Re}\lambda + m \operatorname{Im}\lambda \geq -\frac{2}{\eta}C^2\}$$

for any $0 < \eta < 1$, and hence

$$E\sigma(L_0) \subset \{\lambda : \operatorname{Re}\lambda + m \operatorname{Im}\lambda \geq -2C^2\}$$

from which the result follows. ∎

We also have the following three results

<u>Theorem 4.24</u>. *Suppose that*

$$\int_a^\infty (\operatorname{Im} q)^+ < \infty \tag{4.61}$$

If Re(q) *satisfies one of* (4.48) - (4.50) *then*

$$E\sigma(L_0) \subset \{\lambda : \operatorname{Re}\lambda \geq 0, \operatorname{Im}\lambda \leq 0\} \tag{4.62}$$

If Re(q) *satisfies* (4.51) *then*

$$E\sigma(L_0) \subset \{\lambda : \operatorname{Re}\lambda \geq -2C^2, \operatorname{Im}\lambda \leq 0\} \tag{4.63}$$

<u>Theorem 4.25</u>. *Suppose that*

$$\int_a^\infty (\operatorname{Im} q)^- < \infty \tag{4.64}$$

If Re(q) *satisfies one of* (4.48) - (4.50) *then*

$$\mathrm{Ess}(L_0) \subset \{\lambda\colon \mathrm{Re}\,\lambda \geq -2C^2,\ \mathrm{Im}\,\lambda \leq 0\} \tag{4.66}$$

<u>Theorem 4.26.</u> *If* Re(q) *satisfies one of* (4.48) - (4.50) *then*

$$\mathrm{Ess}(L_0) \subset \{\lambda\colon \mathrm{Re}\,\lambda \geq 0\} \tag{4.67}$$

and if Re(q) *satisfies* (4.51) *then*

$$\mathrm{Ess}(L_0) \subset \{\lambda\colon \mathrm{Re}\,\lambda \geq -2C^2\} \tag{4.68}$$

<u>Proof of Theorems 4.24 - 4.26.</u> To prove these theorems it is sufficient to observe that the proof given for Theorem 4.23 is still valid, where m is any *non-positive* real number when (4.61) is satisfied, where m is any *non-negative* real number when (4.64) is satisfied, and where m = 0 when no condition is placed on Im(q). #

As an example, we note that Re(q) satisfies (4.48) if for some $\alpha \geq 1$, $\int_a^\infty |\mathrm{Re}\ q|^\alpha < \infty$.

In each of the results of this section so far, separate conditions have been placed on the real and imaginary parts of q. The next result involves a 'mixed condition' connecting Re(q) and Im(q).

<u>Theorem 4.27.</u> *If for some particular real number* m,

$$\int_a^\infty |(\mathrm{Re}\ q + m\ \mathrm{Im}\ q)^-| < \infty \tag{4.69}$$

then

$$\mathrm{Em}(L_0) \subset \{\lambda : \mathrm{Re}\,\lambda + m\,\mathrm{Im}\,\lambda \geq 0\}$$

Proof. The method of proof of Theorem 4.23 is valid for the particular value of m for which (4.69) holds. ∎

The geometrical meaning of condition (4.69) is that q may take any values in the complex plane to the right of the line $x + my = 0$ but may have only 'integrable excursions' to the left of that line.

Remarks:

(i) In Theorems 4.23 - 4.[illegible], if in the conditions (4.48) - (4.50) and (4.69), q is replaced by $q - a$ for some complex number a then the corresponding results follow with λ replaced by $\lambda - a$.

(ii) The method of proof [illegible] in proving Theorems 4.23 - 4.27 remain valid if the assumption $p(t) = 1$ on $[a,\infty)$ is weakened to $p(t) \geq 1$ for t sufficiently large.

(iii) Lemma 4.20 - Theorem 4.27 were included in [62].

In considering relaxing conditions on p, let us note also that in considering the valuesof (4.29) to obtain Theorem 4.16 the conclusion remains valid if p is allowed to be complex-valued but takes values only in the half-plane $\mathrm{Re}(pe^{i\alpha}) \geq 0$. Thus we have

Theorem 4.28. *Let* p *and* q *satisfy*

$$\mathrm{Re}\{(q(t)-z)e^{i\alpha}\} \geq 0,\ \mathrm{Re}\{p(t)e^{i\alpha}\} \geq 0 \qquad (4.70)$$

for some $z \in \mathbb{C}$, $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ *then*

$$E\sigma(L_0) \subset \{\lambda \in \mathbb{C};\ \mathrm{Re}((\lambda - r)e^{i\theta}) \geq 0\}$$

We illustrate some of the above results by the following examples.

<u>Examples</u> (when $p = 1$)

(1) $q(t) = a\dfrac{\sqrt{(|\sec t|)}}{t^{\alpha}} + ib\dfrac{\sqrt{(|\sec t|)}}{t^{\beta}}$

where a, b are real numbers and $0 < \alpha,\beta \leq 1$.

(2) $q(t) = a\dfrac{(\sec t)^{1/3}}{t^{\alpha}} + ib\dfrac{(\sec t)^{1/3}}{t^{\beta}}$

where a, b are real numbers and $0 < \alpha,\beta \leq 1$.

(3) $q(t) = af(t) + ibf(t)$

where a, b are real numbers and

$f(t) = \exp((1-n^2(t-n)^2)^{-1}),\quad n - \frac{1}{n} \leq t \leq n + \frac{1}{n}$

$\quad\quad = 0 \quad$ elsewhere

(4) $q(t) = at^{\alpha}f(t) + ibt^{\beta}f(t)$

where $0 \leq \alpha,\beta < 1$ and a, b, $f(\cdot)$ are defined as in (3)

In examples (1) - (4), $E\sigma(L_0) = [0,\infty)$.

(5) $q(t) = t - \dfrac{\sqrt{(|\sec t|)}}{t} + i\dfrac{\sqrt{(|\sec t|)}}{t^{\gamma}}$

where $\gamma > 1$

(6) $q(t) = t^{\alpha}\sin(t^{\beta}) + i\dfrac{\sqrt{(|\sec t|)}}{t^{\gamma}}$

where $\gamma > 1$ *and* $\alpha < \beta - 1,\ \beta > 0$

In examples (5), (6), $E\sigma(L_0) \subset [0,\infty)$

(7) If $q(t) = e^t \sin(e^t) + \frac{i\sqrt{(|\sec t|)}}{t^\gamma}$

where $\gamma > 1$ then $\mathrm{Ee}[L_q] \subset [-\delta, \infty]$

(8) If $\operatorname{Re} q(t) = \frac{(\sec t)^{1/3}}{t^\gamma} + f(t) - m \operatorname{Im} q(t)$

where m is any real number, $\gamma > 1$ and $f(\cdot)$ is a positive function, then

$$\mathrm{Ee}(L_q) \subset \{\lambda : \operatorname{Re} \lambda + m \operatorname{Im} \lambda \geq 0\}$$

<u>Proofs</u>. We show that in each of examples (1) - (8), the given potential function q satisfies the requirement of Theorem 4.21. We use the property of $\sec t$, that

$$\sec t \sim \frac{1}{t-(2n+1)\frac{\pi}{2}} \quad \text{as} \quad t \to (2n+1)\frac{\pi}{2} \tag{4.71}$$

and hence $|\sec t|^\alpha$ is locally integrable if $0 < \alpha < 1$. In each of the examples given $q(t)$ does not tend to zero as the case $q(t) \to 0$ is dealt with by the simple Theorem 4.19.

It is clear that for any fixed $\omega > 0$, by the periodicity of $\sec t$, there is a positive constant k_ω such that for any $x \in [a, \infty)$,

$$\int_x^{x+\omega} \sqrt{(|\sec t|)}\, dt \leq k_\omega \tag{4.72}$$

We now show that (4.45) is satisfied by each of examples (1) - (4).

(1) From (4.72),

$$\int_x^{x+\omega} |q(t)|\, dt \leq (|a|+|b|)k_\omega . \max(x^{-\alpha}, x^{-\beta})$$

the right-hand-side of which tends to zero as $n \to \infty$, as req-uired. Furthermore, since $\alpha, \beta < 1$, $\operatorname{Im} q \notin L(a,\infty)$, so q does not satisfy condition (4.52) of Theorem 4.22.

(2) Inequality (4.72) also holds with $\sqrt{(|\sec t|)}$ replaced by $|\sec t|^{\alpha}$, $0 < \alpha < 1$ or by $(\sec t)^{\frac{1}{2n+1}}$ where n is a positive integer, and in particular when $n = 1$. The conclusion now follows as in (1). We note that these various powers of $\sec t$ are also interchangeable in the other examples given.

(3) By substituting $u = n(x-n)$, we have

$$\int_{n-\frac{1}{n}}^{n+\frac{1}{n}} f(t)dt = \frac{1}{n}\int_{-1}^{1} \exp((1-u^2)^{-1})du \to 0 \qquad (4.73)$$

$$\text{as } n \to \infty$$

hence $q(t)$ satisfies (4.45) of Theorem 4.21.

(4) Multiplying by t^{α}, $0 \le \alpha < 1$ in (4.73) yields

$$\int_{n-\frac{1}{n}}^{n+\frac{1}{n}} t^{\alpha}f(t) \sim n^{\alpha-1}\int_{-1}^{1} \exp((1-u^2)^{-1})du \to 0 \text{ as } n \to \infty$$

so again $q(t)$ satisfies (4.45). We note that $t^{\alpha}f(t) = n^{\alpha}$ when $t = n$ so the values of $\operatorname{Re} q(t)$ and $\operatorname{Im} q(t)$ are unbounded in this example, as $t \to \infty$. In examples (5) - (7) it is shown that q satisfies one of the conditions of Theorem 4.23. In each case, q does not satisfy (4.45); however

$$\int_{n}^{n+1} |\operatorname{Im} q| \le k_1 \cdot \frac{1}{n^{\gamma}}, \quad n \ge a$$

Therefore

$$\int_a^\infty |\mathrm{Im}\, q| \le K_1 \sum_{n \ge N} n^{-\gamma} < \infty$$

i.e. $\mathrm{Im}\, q \in L[a,\infty)$.

(5) By the foregoing analysis, it is clear that since $t > 0$, $\mathrm{Re}\, q$ satisfies (4.50) so part (iii) of Theorem 4.23 is applicable.

(6) Consider the interval $I = [c,d]$:

$$\int_I \mathrm{Re}\, q = \int_c^d t^\alpha \sin(t^\beta)\,dt$$

$$= -\frac{1}{\beta}\cos(t^\beta).t^{\alpha-\beta+1}\Big]_c^d + \frac{\alpha-\beta+1}{\beta}\int_c^d t^{\alpha-\beta}\cos(t^\beta)\,dt$$

If $\alpha - \beta + 1 < 0$, this tends to zero as $c,d \to \infty$. So by applying part (iv) of Theorem 4.23 and using the decomposition principle, the result follows.

(7) Since

$$\int_c^d e^t \sin(e^t)\,dt = -\cos e^t\Big]_c^d \ge -2$$

so part (iv) of Theorem 4.23 is applicable with $C = 2$. Similarly, if in (6), $\alpha = \beta - 1$, part (iv) would be applicable with $C = 2/\beta$.

(8) In this case

$$(\mathrm{Re}\, q + m\, \mathrm{Im}\, q)^- = \{\frac{(\sec t)^{1/3}}{t^\gamma} + f(t)\}^-$$

$$= \{\frac{(\sec t)^{1/3}}{t^\gamma}\}^-$$

and since this is integrable over (a,∞), the conclusion follows from Theorem 4.27. An example of this type of potential q, is

$$q(t) = \frac{(\sec t)^{1/3}}{t^2} - ax\sin x + ix\sin x \qquad (4.74)$$

where a is any real number. #

In examples (5), (6) and (7) above, by adding to $q(t)$, the function $-it^{\alpha}$, $\alpha \geq -1$ examples to which Theorem 4.24 but not Theorem 4.23 may be applied, are provided. Similarly by adding it^{α}, $\alpha \geq -1$, examples are provided to which Theorem 4.25 but not Theorem 4.23 may be applied.

From the above theorems, due to equation (4.7), we are now able to compile the following five conditions under which $E\sigma(L_0) \neq \mathbb{C}$ and hence $\pi(L_0)$ is not empty. As has been mentioned earlier, this property is the one prerequisite for the applicability of the theory of J-selfadjoint extensions of L_0, which was given in Chapter II.

<u>Theorem 4.29</u>. *The regularity field $\pi(L_0)$ of the operator L_0 is non-empty if there is a complex constant z such that one of the following is satisfied for all large values of t:*

(a) $\mathrm{Re}((q(t)-z)e^{i\alpha}) \geq 0$ *and* $\mathrm{Re}(p(t)e^{i\alpha}) \geq 0$ *for some* $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

(b) $p(t) \equiv 1$ *and* $q(\cdot)$ *satisfies one of the conditions:*

(i) $\int_t^{\infty} (\mathrm{Re}(q-z))^- < \infty$

and since this is integrable over $[a,\infty)$, the conclusion follows from Theorem 4.27. An example of this type of potential q, is

$$q(t) = \frac{(\sec t)^{1/3}}{t^2} - m\sin t + im\sin t \qquad (4.74)$$

where m is any real number. #

In examples (5), (6) and (7) above, by adding to $q(t)$, the function $-it^{\alpha}$, $\alpha \geq -1$ examples to which Theorem 4.24 but not Theorem 4.23 may be applied, are provided. Similarly by adding it^{α}, $\alpha \geq -1$, examples are provided to which Theorem 4.25 but not Theorem 4.23 may be applied.

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(a) $\mathrm{Re}((q(t)-z)e^{i\alpha}) \geq 0$ *and* $\mathrm{Re}(p(t)e^{i\alpha}) \geq 0$ *for some* $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

(b) $p(t) \geq 1$ *and* $q(\cdot)$ *satisfies one of the conditions:*

(i) $\int_t^{\infty} (\mathrm{Re}(q-z))^{-} < \infty$

(ii) $\lim\limits_{X\to\infty} \int_X^{X+\omega} (\mathrm{Re}(q-s))^- = 0$ *for some* $\omega \neq 0$

(iii) $\int_t^{\infty} ((\mathrm{Re}+m\,\mathrm{Im})(q-s))^- < \infty$ *for some* $m \in \mathbb{R}$

(iv) $\int_J \mathrm{Re}(q) \geq -C$ *for some* $C \in \mathbb{R}$ *and for all intervals* J *of length* ≤ 1.

We note that by the introduction of s, the results concerning integrals over the set M_0 reduce to condition (i) and that this in turn is a special case of (iii), obtained by setting $m = 0$. Furthermore when $p(t) \geq 0$, (iii) includes the case when (a) holds, except when $\alpha = \pm\frac{\pi}{2}$. Also, (iv) includes both (i) and (ii), though in practice it is usually easier to check condition (i) or (ii), given some particular function q. Putting $\alpha = \pm\frac{\pi}{2}$ in (a) gives the case when Im p, Im q are semi-bounded.

In the proofs of Theorems 4.24 – 4.27, it is possible to relax the requirements that p be real and that $p \geq 1$. However we still require that (Re $p + m$ Im p) ≥ 1 for the relevant values of m. This is equivalent to the following assumptions: In Theorem 4.24, Re $p \geq 1$ and Im $p \leq 0$; in Theorem 4.25, Re $p \geq 1$ and Im $p \geq 0$; in Theorem 4.26, Re $p \geq 1$; in Theorem 4.27, Re $p + m$ Im $p \geq 1$. Similarly we may allow m to be complex in Theorem 4.29(b) provided we assume Re $p \geq 1$ in parts (i), (ii), (iv) and Re $p + m$ Im $p \geq 1$ in part (iii). In each case, the conclusion of the theorem remains unaltered.

Looking at the examples given at the end of Chapter III, it can readily be seen that none of them is covered by the conditions given in Theorem 4.29 without specifying additional conditions in those examples.

Furthermore, if $\sigma(L_0) = \emptyset$ it is not known whether it can happen that there is precisely one $L^2[a,\infty)$ solution of $\ell(y) = \lambda y$ for every $\lambda \in \mathbb{C}$, or that for some λ there is one whilst for other values of λ there is none. However J.B. McLeod [55] has produced the following example to show that it can occur that there is no $L^2[a,\infty)$ solution for any $\lambda \in \mathbb{C}$, making $C\sigma(L_0) = \mathbb{C}$ and $C\sigma(\tilde{L}) = \mathbb{C}$ for any J-selfadjoint extension $\tilde{L}$ of L_0. Let $p = 1$ and $q(x) = -2i\, e^{2(1+i)x}$ so that

$$\ell(y) = -y'' - 2ie^{2(1+i)x}.y, \quad x \in [0,\infty) \tag{4.75}$$

Substituting $\xi = e^{(1+i)x}$, the equation $\ell(y) = \lambda y$ becomes

$$\frac{d^2y}{d\xi^2} + \frac{1}{\xi}\frac{dy}{d\xi} + \left(1 + \frac{\lambda}{2i\xi^2}\right)y = 0$$

which has linearly independent solutions $J_\alpha(\xi)$ and $J_{-\alpha}(\xi)$ where $\alpha = \left(-\frac{\lambda}{2i}\right)^{1/2}$ (We assume α non-integral; the adjustment to the case where α is an integer is straightforward.) As $x \to \infty$, $|\xi| \to \infty$ and $\arg \xi = x$ so from [75, p.202],

$$J_\alpha(\xi) \sim c_1\left(\frac{2}{\pi\xi}\right)^{1/2} \exp\{i(\xi - \tfrac{1}{2}\alpha\pi - \tfrac{1}{4}\pi)\} + c_2\left(\frac{2}{\pi\xi}\right)^{1/2} \exp\{-i(\xi - \tfrac{1}{2}\alpha\pi - \tfrac{1}{4}\pi)\}$$

where, for $(2p-1)\pi < x < (2p+1)\pi$,

$$c_1 = c_2 = \tfrac{1}{2}\exp\{2p(\alpha + \tfrac{1}{2})\pi i\}$$

and, for $\qquad 2p\pi < x < (2p+2)\pi$

$$c_1 = \frac{1}{2}\exp\{2(p+1)(\alpha+\tfrac{1}{2})\pi i\}, \quad c_2 = \frac{1}{2}\exp\{2p(\alpha+\tfrac{1}{2})\pi i\}$$

p being a positive integer. Considering the interval,
$(2p-\frac{1}{2})\pi \le x \le (2p-\frac{1}{6})\pi$,

$$|e^{i\xi}| = \exp(-e^x \sin x) \ge \exp(\tfrac{1}{2}e^x)$$

and so

$$|J_\alpha(\xi)| \ge \left(\frac{2}{\pi}\right)^{1/2} |e^{-1/2\alpha\pi i}| . |c_1| . \exp(\tfrac{1}{2}e^x - \tfrac{1}{2}x)$$

$$\to \infty \quad \text{as} \quad x \to \infty$$

Hence $J_\alpha(\xi) \notin L^2(0,\infty)$. Similarly, $J_{-\alpha}(\xi) \notin L^2(0,\infty)$. Also, if any linear combination of $J_\alpha(\xi)$ and $J_{-\alpha}(\xi)$ were in $L^2(0,\infty)$, we would have to choose the linear combination which eliminates the term in $e^{i\xi}$. But similarly, the term in $e^{-i\xi}$ must be eliminated; thus since no linear combination can eliminate both $e^{i\xi}$ and $e^{-i\xi}$, there is no solution of (4.75) in $L^2(0,\infty)$.

It also remains an open question as to what domains the J-selfadjoint extensions of L_0 have when $\pi(L_0)$ is empty, such as in the above example.

It is also not clear what the set $\pi(L_0)$ looks like when $p = 1$ and

$$q(t) = -t^\alpha(1+i\sin t), \quad 0 < \alpha \le 2 \tag{4.76}$$

Setting $w = t^{-\alpha/2}$ in Theorem 3.5 shows that for any $\lambda \in \mathbb{C}$ there provide certain conditions on q, under which a certain set such as the positive real axis, is known to belong to $E\sigma(L_0)$ and hence to the essential spectrum of any J-selfadjoint extension of L_0. The first of these is the generalisation to complex-valued q, of [25, theorem 22, p.108]. The statement and proof are in fact identical to the real case.

<u>Theorem 4.30</u>. *If $p = 1$ and for some disjoint sequence of intervals Δ_k, whose length $|\Delta_k|$ increases without bound,*

$$\lim_{k\to\infty} \int_{\Delta_k} |q|^2 = 0 \tag{4.77}$$

then the positive semi-axis, $\lambda \geq 0$ belongs to $E\sigma(L_0)$.

<u>Proof</u>. Consider the function $\phi(x;\lambda,\eta)$ defined for $\lambda \geq 0$ and $\varepsilon > 0$ by

$$\phi(x;\lambda,\eta) = \sqrt{\varepsilon}.\phi_0(\varepsilon x)\exp(ix\sqrt{\lambda}) \tag{4.78}$$

where $\eta \to 0$ is such that (see [25, p.107])

$$\|-\phi'' - \lambda\phi\| < \eta\|\phi\| \tag{4.79}$$

and ϕ_0 is an arbitrarily assigned, twice continuously differentiable function with compact support contained in $|x| < 1$, whose norm and maximum modulus are equal to one. Clearly, for a fixed $\lambda \geq 0$, the value of η tends to zero as $\varepsilon \to 0$.

Define functions ϕ_k, $k = 1,2,\ldots$ by setting

$$\varepsilon = \varepsilon_k = \frac{1}{2|\Delta_k|}$$

in (4.78) and letting

$$\phi_k(x) = \phi_k(x;\lambda,\eta) = \phi(x-\alpha_k;\lambda,\eta) \tag{4.80}$$

where α_k is the mid-point of the interval Δ_k. Then we have $\phi_k \in D_0$ and

$$\|(L_0-\lambda I)\phi_k\| \le \eta_k\|\phi_k\| + \|q\phi_k\|$$

$$\le \eta_k + \left(\frac{1}{2|\Delta_k|}\int_{\Delta_k} |q|^2\right)^{1/2}$$

from (4.79). Thus letting $k \to \infty$,

$$\lim_{k\to\infty}\|(L_0-\lambda I)\phi_k\| = 0$$

and the sequence $\{\phi_k\}$ is a bounded, non-compact sequence (clearly $\|\phi_k\| = 1$ and the supports of the sequence ϕ_k are disjoint), so from the definition of $E\sigma(L_0)$ we have that $\lambda \in E\sigma(L_0)$ [for any $\lambda \ge 0$). #

The next result uses slightly different test functions $\{\phi_k\}$. Let ϕ_0 be as in the above proof and

$$\phi_k(x;\lambda,\varepsilon) = \sqrt{\varepsilon}.\phi_0(\varepsilon(x-\alpha_k)).e_k(x;\lambda) \tag{4.81}$$

where $e_k(x;\lambda)$ is the solution of $\ell(y) = \lambda y$ satisfying

$$e_k(\alpha_k;\lambda) = 1, \quad e_k'(\alpha_k;\lambda) = i\sqrt{\lambda} \tag{4.82}$$

for $\lambda \ge 0$. We need the following simple lemma.

Lemma 4.31. *If for some* $b \in (a,\infty)$, *the sequence of functions* $q_n(x)$ *satisfies the condition*

$$\lim_{m\to\infty}\int_a^b |q_m| = 0 \tag{4.83}$$

then the solutions $\phi_m(x;\lambda)$ *of*

$$-y'' + q_m y = \lambda y \tag{4.84}$$

satisfying

$$\phi_m(a;\lambda) = 1, \quad \phi_m'(a;\lambda) = i\sqrt{\lambda} \tag{4.85}$$

have the following limits, the convergence being uniform in $[a,b]$:

$$\lim_{m\to\infty}\phi_m(x;\lambda) = e^{i\sqrt{\lambda}x}, \quad \lim_{m\to\infty}\phi_m'(x;\lambda) = i\sqrt{\lambda}e^{i\sqrt{\lambda}x} \tag{4.86}$$

The proof when the (q_m) are complex-valued is the same as that given in [25, p.109] when the (q_m) are real-valued. Thus we are now able to prove the following theorem, first proved for the case of real-valued q by I.M. Glazman [23] (see also [25, p.110]).

Theorem 4.32. *Suppose* p = 1 *and that for some disjoint sequence of intervals* Δ_k *whose length* $|\Delta_k|$ *increases without bound, we have*

$$\lim_{k\to\infty}\int_{\Delta_k} |q| = 0$$

then the positive semi-axis, $\lambda \geq 0$ *belongs to* $E\sigma(L_0)$.

Proof. In (4.81) set

$$\varepsilon = \varepsilon_k = \frac{1}{2|\Delta_k|}$$

and let a_k be the mid-point of the interval Δ_k; then $\phi_k \in$ ... for each k and the sequence $\{\phi_k\}$ have disjoint compact supports, so that it is a bounded, non-compact sequence. Clearly $\phi_k \in \mathcal{D}_0$ for each k, and by direct computation,

$$(L_0-\lambda I)\phi_k = -\sqrt{\epsilon}\,(\epsilon^2\phi_0''.\phi_k + 2\epsilon\phi_0'\phi_k') \tag{4.87}$$

From this and from (4.86) of Lemma 4.31, for any $\lambda \geq 0$ we have

$$\lim_{k\to\infty}\|(L_0-\lambda I)\phi_k\| = 0$$

and hence by the definition of essential spectrum, $\lambda \in E\sigma(L_0)$. $\blacksquare$

To conclude the present chapter, we have the following generalisation to complex-valued q, of a result of D. Hinton [33].

<u>Theorem 4.33</u>. *If* $p = 1$, $\mathrm{Re}\, q(t) \to -\infty$ *as* $t \to \infty$, $\mathrm{Re}\, q'(t)|\mathrm{Re}\, q(t)|^{-1/2} \to 0$ *as* $t \to \infty$ *and* $\mathrm{Im}\, q \in L^2[a,\infty)$ *then the whole real axis belongs to* $E\sigma(L_0)$.

<u>Proof</u>. As in [33] it suffices to show that for any $\lambda \in \mathbb{R}$, there exists for each $\eta > 0$ and $N \geq a$ a non-trivial $y \in \mathcal{D}_0'$ such that the support of y is contained in $[N,\infty)$ and $\|(L_0-\lambda I)y\| \leq \eta\|y\|$, since we may then construct a sequence $\{y_i\}$ of functions with disjoint supports, which will hence be a non-compact sequence e.g. if $\mathrm{supp}\, y_1 \subset (1,N_1)$, we may choose y_2 such that $\mathrm{supp}\, y_2 \subset (N_1,N_2)$ and so on. Given $\eta > 0$, we may assume without loss of generality that N is sufficiently large so that

$$\int_N^\infty |\mathrm{Im}\, q|^2 \leq \eta \tag{4.88}$$

Choose $0 < \varepsilon < \frac{\eta^2}{25}$, $K > \frac{8\cdot 400}{\eta^2}$ (assume $\eta < 1$) then there is a number B such that if $t, s \geq B$ and $|t-s| \leq K(-\mathrm{Re}\, q + \lambda)^{-1/2}$ then

$\left|\frac{\mathrm{Re}\ q(t)-\lambda}{\mathrm{Re}\ q(s)-\lambda} - 1\right| < \varepsilon$ and $\mathrm{Re}\ q(t) - \lambda \leq -\varepsilon^2$. (See [33]). Now let $s_0 \geq \max(N,B)$ be such that $|\mathrm{Re}\ q'(t)|(-\mathrm{Re}\ q + \lambda)^{-3/2} < \varepsilon$ for $t \geq s_0$ and define $s_1, s_2, \ldots$ by

$$s_{i+1} = s_i + K(-\mathrm{Re}\ q(s_i) + \lambda)^{-1/2} \quad i = 0,1,\ldots$$

and denote $(-\mathrm{Re}\ q(s_i) + \lambda)^{1/2}$ by α_i. As in [33], it is then possible to define a function $y \in D_0'$ whose compact support is contained in (N,∞) and having the properties that

$$|y(t)| \leq \frac{\sqrt{2}}{\alpha_0}, \quad t \in [0,\infty) \tag{4.89}$$

$$\|y\|^2 \geq \frac{K}{8\alpha_0} \tag{4.90}$$

$$\|-y'' + \mathrm{Re}\ q.y - \lambda y\| < \eta\|y\| \tag{4.91}$$

But then from (4.89) and (4.90),

$$\frac{|y(t)|}{\|y\|} \leq \frac{4}{\alpha_0^{1/2}K^{1/2}} < \frac{\eta}{20} < \eta^{1/2} \tag{4.92}$$

and so from (4.88), (4.91) and (4.92)

$$\|(L_0-\lambda I)y\| \leq \|-y'' + \mathrm{Re}\ q.y - \lambda y\| + \|\mathrm{Im}\ q.y\|$$

$$< \eta\|y\| + \left(\int_N^\infty |\mathrm{Im}\ q|^2\right)^{1/2} . \sup_{(N,\infty)} |y(t)|$$

$$\leq 2\eta\|y\|$$

which proves the theorem. ∎

CHAPTER V

THE LOCATION OF THE POINT AND CONTINUOUS SPECTRUM

As was mentioned earlier, we concentrate on the location of eigenvalues of extensions of L_0 in the present chapter. It is possible for eigenvalues (i.e. points of $P\sigma(A)$) of an operator A to be embedded in the essential spectrum $E\sigma(A)$, a phenomenon first noticed by J. von Neumann and E. Wigner [74]. In this chapter our main aim is to find bounds for the location of such eigenvalues for closed extensions of the operator L_0 associated with (2.79), the interval under consideration being $[a,\infty)$. This work is an extension to the case of complex-valued coefficients, of results due to I.W. Knowles et al (see [43], [44]) and most of it is included in [47].

We firstly obtain an analogue of the Kupcov inequality (see [49]) for solutions of $\ell(u) = 0$ when p and q may be complex-valued.

Let θ, ϕ, and γ denote arbitrary locally absolutely continuous functions on $[a,\infty)$, with θ and γ complex-valued, $\theta(t) \neq 0$ for $a \leq t < \infty$, and $\phi > 0$. Define

$$E(t) = |\theta(t)p(t)u'(t) + \gamma(t)u(t)|^2 + \phi^2(t)|u(t)|^2 \tag{5.1}$$

where u is a non-trivial solution of $\ell(u) = 0$. We note that for $E(t)$ to vanish, it would be necessary for both $u(t)$ and $pu'(t)$ to vanish which could only happen if u were identically zero, which it is not.

Then

$$
\begin{aligned}
E'(t) &= 2\mathrm{Re}\{(\theta' p u' + \gamma' u)\overline{(\theta p u' + \gamma u)} \\
&\quad + (\theta q u + \gamma u')\overline{(\theta p u' + \gamma u)} \\
&\quad + \phi\phi' u\bar{u} - \phi^2 u'\bar{u}\} \\
&= 2\mathrm{Re}\{(\frac{\theta'}{\theta} + \frac{\gamma}{p\theta})|\theta p u' + \gamma u|^2 + (\frac{\phi'}{\phi} - \frac{\gamma}{p\theta})|\phi u|^2 \\
&\quad + (\gamma' + \theta q + \frac{\phi^2}{p\theta} - \frac{\bar{\gamma}\theta'}{\theta} - \frac{\bar{\gamma}^2}{p\theta})(\theta p u' + \gamma u)\bar{u}\} \\
&= (\theta p u' + \gamma u, \phi u) Z \overline{(\theta p u' + \gamma u, \phi u)}^T . \qquad (5.2)
\end{aligned}
$$

Here

$$
Z(t) = \begin{vmatrix} A(t) & B(t) \\ \bar{B}(t) & C(t) \end{vmatrix}
$$

where

$$A = 2\ \mathrm{Re}(\frac{\theta'}{\theta} + \frac{\gamma}{p\theta})$$

$$B = \frac{\gamma'}{\phi} + \frac{\theta q}{\phi} + \frac{\phi}{p\theta} - \frac{\gamma\theta'}{\phi\theta} - \frac{\gamma^2}{p\theta\phi}$$

$$C = 2\ \mathrm{Re}(\frac{\phi'}{\phi} - \frac{\gamma}{p\theta}).$$

Hence, noting that $E(t) \neq 0$,

$$\frac{E'(t)}{E(t)} = (\xi, \eta) Z \overline{(\xi, \eta)}^T \qquad (5.5)$$

where $\xi = (\theta p u' + \gamma u)E^{-1/2}$, $\eta = \phi u E^{-1/2}$, and $\xi^2 + \eta^2 = 1$. Since Z is a Hermitian matrix the value of $\frac{E'(t)}{E(t)}$ lies between the values of the two eigenvalues of the matrix $Z(t)$:

$$\lambda_0(t) \le \frac{E'(t)}{E(t)} \le \lambda_1(t) \tag{5.4}$$

where

$$\lambda_1(t),\ \lambda_0(t) = \frac{A+C}{2} \pm \left(\left(\frac{A-C}{2}\right)^2 + |B|^2\right)^{1/2}$$

$$= \mathrm{Re}\left(\frac{\theta'(t)}{\theta(t)} + \frac{\phi'(t)}{\phi(t)}\right) \pm F(t) \tag{5.5}$$

with

$$F^2 = \left[\mathrm{Re}\left(\frac{\theta'}{\theta} - \frac{\phi'}{\phi} + \frac{2w}{p^2}\right)\right]^2 + \left(\frac{\phi}{\theta}\right)^2 \cdot \left| q + \frac{\phi'^2}{p|\phi|^2} - \frac{1}{p}\left(\frac{\phi}{\theta}\right)^2 + \left(\frac{\theta}{\phi}\right)' \right|^2 \tag{5.6}$$

On integrating (5.4) we obtain the desired inequality:

$$K|\theta(t)|\phi(t)\exp\left[-\int_a^t F\right] \le E(t) \le K|\theta(t)|\phi(t)\exp\left[\int_a^t F\right] \tag{5.7}$$

where $K = E(a)(|\theta(a)|\phi(a))^{-1}$. Finally, integrating (5.7) over $[a,\infty)$ we obtain

$$K\int_a^\infty |\theta(t)|\phi(t)\exp\left[-\int_a^t F\right]dt \le \int_a^\infty E(t)dt \le K\int_a^\infty |\theta(t)|\phi(t)\exp\left[\int_a^t F\right]dt \tag{5.8}$$

We shall later use the left-hand-side of (5.8) to find bounds on the eigenvalues of operators associated with $\ell(\cdot)$. However, let us firstly consider the right-hand-side. If we set $\phi = 1$ in (5.1) then we see that $|u(t)|^2 \le E(t)$ and hence

$$\int_a^\infty |u(t)|^2 dt \le \int_a^\infty E(t)dt \tag{5.9}$$

for all solutions u of $\ell(u) = 0$.

Comparing (5.8) and (5.9) we see that if θ and γ may be chosen so that the expression on the right is finite then we can conclude that all solutions of $\ell(u) = 0$ are in $L^2[a,\infty)$. This gives the case when $\mathrm{def}(L_0) = 2$, which we called the limit-circle case. We have thus proved:

<u>Theorem 5.1</u> (c.f. [48], [43], [44]). *If there exist locally absolutely continuous, complex-valued functions $\theta(t)$, $\gamma(t)$ defined on $[a,\infty)$ with $\theta(t) \neq 0$ for any t, such that*

$$\int_a^\infty |\theta(t)| \exp\left[\int_a^t F(s)\,ds\right] dt < \infty \tag{5.10}$$

where

$$F^2 = \left[\mathrm{Re}\left(\frac{\theta'}{\theta} + \frac{\gamma}{p\theta}\right)\right]^2 + |\theta|^2 \cdot \left|q + \frac{1}{\bar{p}|\theta|^2} - \frac{1}{p}\left(\frac{\gamma}{\theta}\right)^2 + \left(\frac{\gamma}{\theta}\right)'\right|^2 \tag{5.11}$$

then $\ell(\cdot)$ is in the limit-circle case.

In the real case this is the same as a result in [43] if we set $\gamma = -(p\phi'\phi^{-1}).\theta$, and is the same as a result in [48] if we set $\theta = r^{-1/2}$ and $\gamma = p\theta.\phi$. Since this result includes many of the known limit-circle criteria in the real case, by suitable choices of γ and θ, it follows that these have complex analogues. We now give some of the main analogues.

By setting $\theta = \phi^2$ and $\gamma = -p\phi'\phi$ we have from Theorem 5.1,

<u>Theorem 5.2</u> (c.f. [41], [33]). *If there exists a non-vanishing complex-valued function $\phi(t)$ defined on $[a,\infty)$ with ϕ and $p\phi'$ locally absolutely continuous such that*

$$\int_a^\infty |\psi|^2 < \infty$$

and either (i) $$\int_a^\infty |\psi|^2 \cdot \left| q + \frac{1}{\bar{p}|\psi|^4} - \frac{(p\psi')'}{\psi} \right| < \infty$$

or

(ii) $$\left| q + \frac{1}{\bar{p}|\psi|^4} - \frac{(p\psi')'}{\psi} \right| \leq K < \infty$$

then $\ell(\cdot)$ *is in the limit-circle case.*

Here, γ and θ were chosen so that the first term in (5.11) vanishes identically. If now, $p = 1$ and we set $\theta = \psi^2$, where ψ is real and $\gamma = - Q'Q^{-1} \cdot \psi^2$ where Q may be complex-valued, we obtain

Theorem 5.3 (c.f. [41]). *If $p = 1$ and there exists a real-valued, non-vanishing function $\psi(t)$ and a complex-valued function $Q(t)$, defined on $[a,\infty)$ with ψ, Q, Q' locally absolutely continuous, such that*

$$\int_a^\infty \psi^2(t) \exp\left[\int_a^t \left(2\left|\frac{\psi'}{\psi} - \mathrm{Re}\left(\frac{Q'}{Q}\right)\right| + \psi^2 \cdot \left| q + \psi^{-4} - \frac{Q''}{Q}\right|\right)\right] dt < \infty \qquad (5.12)$$

then $\ell(\cdot)$ *is in the limit-circle case.*

(In [41] the factor of 2 in (5.12) was erroneously omitted.) As in the real case, if in Theorem 5.3, we set $Q = P$ and $\psi = P(1+\phi)^{-1/4}$, we obtain

Theorem 5.4 (c.f. [16]). *Suppose $p = 1$ and let $P(t)$, $\phi(t)$ be real-valued and $h(t)$ a complex-valued function, defined on $[a,\infty)$ such that*

(i) $P(t) > 0$, $P \in L^2[a,\infty)$

(ii) P, P', ϕ are locally absolutely continuous

(iii) $\phi(t) \to 0$ as $t \to \infty$

(iv) $hP, \frac{\phi'}{P} \in L^2[a,\infty)$

If q *is defined by*

$$q = \frac{h}{P} + \frac{P''}{P} - \frac{1+\phi}{P^4} \tag{5.13}$$

then $\ell(\cdot)$ *is in the limit-circle case.*

The following gives a more explicit condition on q:

<u>Theorem 5.5</u> (c.f. [51]). *If* $p = 1$ *and* Re q *is negative and locally absolutely continuous, such that*

$$\frac{\mathrm{Re}\, q'(t)}{\mathrm{Re}\, q(t)} \geq \frac{2+\varepsilon}{t}, \quad t \geq a \tag{5.14}$$

and if

$$\int_a^\infty \frac{|\mathrm{Im}\, q|}{|\mathrm{Re}\, q|^{1/2}} < \infty \tag{5.15}$$

then $\ell(\cdot)$ *is in the limit-circle case.*

<u>Proof</u>. In Theorem 5.1 set $p = 1$, $P = (-\mathrm{Re}\, q)^{-1/2}$, $\gamma = \frac{1}{2}(1+\frac{\varepsilon}{2})t^{-1} > 0$. Condition (5.14) implies Re q' is negative and for some $k > 0$,

$$|\mathrm{Re}\, q| \geq kt^{2+\varepsilon} \tag{5.16}$$

By (5.11),

$$F^2(t) = \left(-\frac{1}{2}\frac{\mathrm{Re}\, q'}{\mathrm{Re}\, q} + \frac{(1+\varepsilon/2)}{t}\right)^2 + \frac{1}{|\mathrm{Re}\, q|}\cdot\left|i\, \mathrm{Im}\, q - \frac{(1+\varepsilon/2)^2}{4t^2} - \frac{(1+\varepsilon/2)}{2t^2}\right|^2$$

So, using (5.14),

$$F(t) \le \frac{\operatorname{Re} q'}{2\operatorname{Re} q} - \frac{1+\varepsilon/2}{t} - \frac{|\operatorname{Im} q|}{|\operatorname{Re} q|^{1/2}} + O(t^{-1})$$

Integrating over $[a,t]$, using (5.19),

$$\int_a^t F(s)ds \le \log(|\operatorname{Re} q|^{1/2} t^{-(1+\varepsilon/2)}) + O(1)$$

Thus in (5.10),

$$\int_a^\infty |\theta| \exp[\int_a^t F] \le O(1).\int_a^\infty t^{-(1+\varepsilon/2)} < \infty$$

as required. ∎

Once again we make use of the substitution $\theta = (-\operatorname{Re} q)^{-1/2}$ when $\operatorname{Re} q < 0$ to obtain

<u>Theorem 5.6</u> (c.f. [14], [60]). *Let* $p = 1$ *and* $\operatorname{Re} q$ *be negative. Suppose* $\operatorname{Re} q$ *and* $\operatorname{Re} q'$ *are locally absolutely continuous and that*

$$\int_a^\infty |\operatorname{Re} q|^{-1/2} < \infty, \quad \int_a^\infty |\operatorname{Im} q|.|\operatorname{Re} q|^{-1/2} < \infty$$

and either

(i) $$\int_a^\infty \left| \frac{\operatorname{Re} q''}{(-\operatorname{Re} q)^{3/2}} + \frac{5}{4} \cdot \frac{(\operatorname{Re} q')^2}{(-\operatorname{Re} q)^{5/2}} \right| < \infty$$

or

(ii) $$\left| \frac{\operatorname{Re} q''}{\operatorname{Re} q} - \frac{5}{4} \left(\frac{\operatorname{Re} q'}{\operatorname{Re} q}\right)^2 \right| \le K < \infty$$

then $\ell(\cdot)$ *is in the limit-circle case.*

<u>Proof</u>. In Theorem 5.1 set $p = 1$; $\theta = (-\operatorname{Re} q)^{-1/2}$, $\gamma = \frac{1}{4} \operatorname{Re} q'.(\operatorname{Re} q)^{-1}.\theta$ then

the first term in (5.11) vanishes and we have

$$F = |\mathrm{Re}\,q|^{-1/2}.\left|i\,\mathrm{Im}\,q - \left(\frac{\mathrm{Re}\,q'}{4\mathrm{Re}\,q}\right)^2 + \left(\frac{\mathrm{Re}\,q''}{4\mathrm{Re}\,q}\right)\right|$$

$$\leq |\mathrm{Im}\,q|.|\mathrm{Re}\,q|^{-1/2} + \frac{1}{4}|\mathrm{Re}\,q|^{-1/2}.\left|\frac{\mathrm{Re}\,q''}{\mathrm{Re}\,q} - \frac{5}{4}\left(\frac{\mathrm{Re}\,q'}{\mathrm{Re}\,q}\right)^2\right|$$

The conditions imposed above then imply $F \in L^1[a,\infty)$ and $G \in L^1[a,\infty)$, so (5.10) is satisfied. Alternatively we could have set $\psi = (-\mathrm{Re}\,q)^{-1/4}$ in Theorem 5.2. #

It is known (see [39]) that condition (i) of Theorem 5.6 is satisfied if it is further assumed that $\mathrm{Re}\,q' \leq 0$, $\mathrm{Re}\,q'' \leq 0$, $\lim_{t\to\infty} \mathrm{Re}\,q(t) = -\infty$ and $\mathrm{Re}\,q' = O(|\mathrm{Re}\,q|^{\alpha})$ for some $0 < \alpha < \frac{3}{2}$, thus we have

<u>Theorem 5.7</u> (c.f.[72, p.125]). *Let* $p = 1$ *and suppose* $\mathrm{Re}\,q$, $\mathrm{Re}\,q'$ *are locally absolutely continuous with* $\mathrm{Re}\,q' \leq 0$, $\mathrm{Re}\,q'' \leq 0$, $\mathrm{Re}\,q(t) \to -\infty$ *as* $t \to \infty$ *and* $\mathrm{Re}\,q' = O(|\mathrm{Re}\,q|^{\alpha})$ *for some* $0 < \alpha < \frac{3}{2}$. *Suppose further that*

$$\int_a^{\infty}|\mathrm{Re}\,q|^{-1/2} < \infty, \quad \int_a^{\infty}|\mathrm{Im}\,q|.|\mathrm{Re}\,q|^{-1/2} < \infty$$

then $\ell(\cdot)$ *is in the limit-circle case.*

We now seek to generalize two limit-circle conditions proved in the real case by S.G. Halvorsen [27]. In introducing $\mathrm{Im}(q)$, either a pointwise- or an integral- condition may be imposed as follows.

<u>Theorem 5.8</u> (c.f.[27]). *If* $p = 1$ *and there exist positive constants*

k, n, δ *such that either*

(i) $\quad |\mathrm{Im}\, q(t)| + |\mathrm{Re}\, q(t) + k^2 t^{2(n+1)}| \le (n-\delta)kt^n$

or

(ii) $\quad |\mathrm{Im}\, q(t)| + |\mathrm{Re}\, q(t) + k^2 e^{nt}| \le ke^{nt/2}\left(\frac{n}{2} - \frac{1+\delta}{t}\right)$

then $\ell[\cdot]$ *is in the limit-circle case.*

Proof. If (i) holds, set $\theta = \frac{1}{k}t^{-(n+1)}$, $\gamma = \frac{n+1}{2t}\cdot\theta$, then the first term in (N.11) vanishes and

$$F = \frac{1}{k}t^{-(n+1)}\cdot\left|q + k^2t^{2(n+1)} - \frac{(n+1)^2}{4t^2} - \frac{(n+1)}{2t^2}\right|$$

$$\le (n-\delta)t^{-1} + O(t^{-2})$$

Hence

$$\int_a^t F \le (n-\delta)\log t + O(1)$$

and

$$\int_a^\infty |\theta| \exp[\int_a^t F] \le O(1).\int_a^\infty t^{-(n+1)}.t^{n-\delta} = O(1).\int_a^\infty t^{-1-\delta} < \infty.$$

Similarly if (ii) holds, set $\theta = \frac{1}{k}e^{-nt/2}$ and $\gamma = \frac{n}{4}\cdot\theta$, giving

$$F = \frac{1}{k}e^{-nt/2}\cdot\left|q + k^2e^{nt} - \frac{n^2}{16} - \frac{n}{4}\right|$$

so that

$$\int_a^t F \le \frac{nt}{2} - (1+\delta)\log t + O(1)$$

and

$$\int_a^\infty |\theta| \exp[\int_a^t F] \le O(1)\int_a^\infty \frac{1}{k}e^{-nt/2}.e^{nt/2}.t^{-(1+\delta)} < \infty$$

Thus by applying Theorem 5.1 the required result follows. #

In considering the term involving Im q in the above proof it is clear that we can impose an integral condition to obtain:

Theorem 5.9 (c.f. [27]). *If p = 1 and there exist positive constants k, n, δ such that either*

(i) $\left|\mathrm{Re}\, q(t) + k^2 t^{2(n+1)}\right| \le (n-\delta) k t^n$ and $\int_a^\infty \frac{|\mathrm{Im}\, q|}{t^{n+1}} < \infty$

or

(ii) $\left|\mathrm{Re}\, q(t) + k^2 e^{nt}\right| \le k n e^{nt/2} \cdot \left[\frac{3}{2} - \frac{(1+\delta)}{k}\right]$ and $\int_a^\infty |\mathrm{Im}\, q| e^{-nt/2} < \infty$

then $\ell(\cdot)$ is in the limit-circle case.

We may use another result of Halvorsen [28] to enable us to classify $\ell(\cdot)$ when Im q is bounded:

Theorem 5.10 [28, theorem 1]. *If we define* $\ell_1(y) = -y'' + qy$, $\ell_2(y) = -y'' + \mathrm{Re}\, q.y$ *and if* Im q *is bounded then* $\ell_1(\cdot)$ *is in the limit-circle case if and only if* $\ell_2(\cdot)$ *is in the limit-circle case.*

Let us now return to inequality (5.8) and make use of the left-hand-side. Recalling the definition of $E(\cdot)$ in (5.1), it is clear that if

(A) $\int_a^\infty |\theta(t)|\phi(t)\exp\left[-\int_a^t F\right]dt = \infty$

(B) $p\theta u' \in L^2[a,\infty)$ whenever $\ell(u) = 0$ and $u \in L^2[a,\infty)$

(C) γ and ϕ are bounded

then $\ell(u) = 0$ has no solutions in $L^2[a,\infty)$. But Lemmas 3.1 and 3.4 give two sets of conditions under which (B) occurs.

Firstly, applying Lemma 5.2 with $w = |\theta|$ we have

Theorem 5.11. *Suppose that* $\mathrm{Re}(p)$ *and* $\mathrm{Im}(p)$ *are decreasing and non-negative (with* $p(t) \neq 0$*), and that there exist complex-valued functions* θ *and* γ*, and a positive function* ϕ*, all absolutely continuous on* $[a,\infty)$ *with* $\theta(t) \neq 0$ *for* $t \in [a,\infty)$*, and a decomposition* $q = q_1 + q_2$ *of* q *such that on* $[a,\infty)$

(i) $|\theta|$ *is monotone decreasing,*

(ii) ϕ, γ *are bounded,*

(iii) $|\theta^2 q_1| \le k_1$,

(iv) $\left|\frac{\theta(t)}{p(t)}\int_a^t q_2\right| \le k_2$ *or* $\left|\theta(t)\int_a^t \frac{q_2}{p}\right| \le k_2$

(v) $\int_a^\infty |\theta(t)|\phi(t)\exp\left[-\int_a^t F\right]dt = \infty$,

where k_1 *and* k_2 *are constants. Then* $\ell(u) = 0$ *has no non-trivial solutions in* $L^2[a,\infty)$.

From condition (v) and the fact that ϕ is bounded it is clear that

$$\int_a^\infty |\theta| = \infty \tag{5.17}$$

and so from Theorem 3.5, provided $\pi(L_\beta) \neq \emptyset$ the operators L_γ are automatically J-selfadjoint under the conditions of Theorem 5.11.

Secondly, applying Lemma 5.4 with $w = |p|^{1/2}.|\theta|$, we obtain

Theorem 5.12. *Suppose* $\mathrm{Re}(p) > 0$*, and that there exist complex-valued functions* γ *and* θ *and a positive function* ϕ*, all locally* *absolutely continuous on* $[a,\infty)$, *with* $\theta(t) \neq 0$ *on* $[a,\infty)$, *and a decomposition* $\mathrm{Re}\ q = q_1 + q_2$ *of* $\mathrm{Re}\ q$ *such that on* $[a,\infty)$

(i) $|p|((|p|^{1/2}|\theta|)')^2 \le K_1$,

(ii) γ *and* ϕ *are bounded*,

(iii) $-q_1|p\theta^2| \le K_2$,

(iv) $|p(t)|^{\frac{1}{2}(\alpha-1)}|\theta(t)|^{\alpha}\left|\int_a^t q_2|p\theta^2|^{\frac{1}{2}(1-\alpha)}\right| \le K_3$

for some $0 \le \alpha \le 1$,

(v) $\int_a^\infty |\theta(t)|\phi(t)\exp[-\int_a^t \gamma]dt = \infty$.

If $\mathrm{Im}(p) \not\equiv 0$, *suppose further that* $\mathrm{Im}(p) \ge 0$ *and* $\mathrm{Im}(q) = q_3 + q_4$ *where*

(vi) $-q_3|p\theta^2| \le K_4$,

(vii) $|p(t)|^{\frac{1}{2}(\beta-1)}|\theta(t)|^{\beta}\left|\int_a^t q_4|p\theta^2|^{\frac{1}{2}(1-\beta)}\right| \le K_5$

for some $0 \le \beta \le 1$.

Then $\ell(u) = 0$ *has no solutions in* $L^2[a,\infty)$.

The derivation of Theorem 5.12 is routine except for the verification of condition (i) of Lemma 3.4. For this, we observe that the operators L_γ are J-selfadjoint under the conditions of Theorem 5.12.

Recalling from Theorem 4.4 that when there are no non-trivial solutions of $\ell(u) = \lambda u$ in $L^2[a,\infty)$, $\lambda \in C\sigma(\tilde{L})$ for any closed extension $\tilde{L}$ of L_0, we have proved

<u>Corollary 5.13.</u> *If the conditions of either Theorem 5.11 or Theorem 5.12 are satisfied and $\tilde{L}$ is any closed extension of L_0, then 0 cannot be an eigenvalue of $\tilde{L}$ and indeed $0 \in C\sigma(\tilde{L})$.*

Theorems 5.11 and 5.12 are exceedingly general results but quite difficult to use due to the wide choice available for the functions θ, ϕ, γ. Our task now is to derive some of the more accessible corollaries, and to use these to locate the point spectra of closed extensions of L_0, including the J-selfadjoint operators L_γ when $\pi(L_0)$ is not empty. We shall henceforward use $\tilde{L}$ to denote any arbitrary, closed linear extension of L_0, which will then include the operators L_γ. We begin with some consequences of Corollary 5.13, using Theorem 5.11.

<u>Theorem 5.14.</u> *If* $\mathrm{Re}(p)$, $\mathrm{Im}(p)$, *and* $\mathrm{Re}(q)$ *are locally absolutely continuous and monotone decreasing on* $[a,\infty)$, $\mathrm{Re}(p)$ *and* $\mathrm{Im}(p)$ *are non-negative (with* $p(t) \neq 0$*), and*

(a) $$\int_a^\infty \frac{|\mathrm{Im}(q)|}{|p|\,|\mathrm{Re}(q)|^{1/2}} < \infty,$$

(b) $$\int_a^\infty \frac{|\mathrm{Re}(q)|^{1/2}\,|\mathrm{Im}(p)|}{|p|^{3/2}} < \infty,$$ *and*

(c) $\displaystyle\int_a^\infty \frac{|p|}{|\mathrm{Re}(q)|} = \infty,$

then $(q_\infty, \infty) \subset C\sigma(\tilde{L})$, where $q_\infty = \lim_{t\to\infty} \mathrm{Re}(q(t))$.

<u>Proof</u>. We assume for the moment that $\mathrm{Re}\, q < 0$ on $[a,\infty)$ and set $\gamma = 0$, $\phi = |p|^{1/2}$, $q_1 = \mathrm{Re}(q)$, $q_2 = i\,\mathrm{Im}(q)$ and $\theta = (-\mathrm{Re}(q))^{-1/2}$ in Theorem 5.11. Conditions (i) - (iii) of the theorem are clearly satisfied. Also

$$\left|\theta(t)\int_a^t (q_2/p)\right| \le \int_a^t |\mathrm{Im}(q)|\,|p|^{-1}|\mathrm{Re}\; q|^{-1/2}$$

$$\le K \quad \text{by (a)},$$

and thus (iv) is also satisfied. Finally, to verify (v), we note firstly that

$$\frac{\theta}{\phi}\left|q + \frac{\phi^2}{\bar{p}\theta^2}\right|$$

$$= |p|^{-1/2}|q_1|^{-1/2}|q_1 + q_2 + p|p|^{-1}(-q_1)|$$

$$\le |p|^{-1/2}|q_1|^{-1/2}|q_2| + |p|^{-3/2}|q_1|^{1/2}|\mathrm{Im}(p)|$$

$$+ |p|^{-1/2}|q_1|^{1/2}|1 - \mathrm{Re}(p)\cdot|p|^{-1}|$$

$$\le K|p|^{-1}|q_1|^{-1/2}|q_2| + 2|p|^{-3/2}|q_1|^{1/2}|\mathrm{Im}(p)| \tag{5.18}$$

Here we have used the fact that $|p|$ is decreasing, and the inequality

$$\left\{1 - \frac{a}{\sqrt{a^2+b^2}}\right\} \geq \frac{b}{\sqrt{a^2+b^2}}$$

which is valid for all $a, b \geq 0\,(a^2+b^2 \neq 0)$

By (a) and (b) the term on the left of (5.18) is in $L[a,\infty)$. Thus, as θ and ϕ are both decreasing,

$$\int_a^t F \leq O(1) - \int_a^t (\theta'\theta^{-1} + \phi'\phi^{-1})$$

$$= O(1) - \log[\theta\phi](t)$$

and hence

$$\int_a^\infty \theta(t)\phi(t)\exp[-\int_a^t F]$$

$$\geq K\int_a^\infty \theta^2(t)\phi^2(t)\,dt$$

$$= K\int_a^\infty |p|.[\mathrm{Re}(q)]^{-1}$$

$$= \infty \quad \text{by (c).}$$

We now have that the equation $\ell(u) = 0$ has no non-trivial solutions in $L^2[a,\infty)$. Also it is not hard to see that one can repeat the above argument with $q - \lambda$ replacing q, provided $\lambda > q_\infty$, and the end-point a is chosen suitably large. This means that the equation $\ell(u) = \lambda u$ has no solution in $L^2[a,\infty)$ for these values of λ. Finally, we show that the assumption $\mathrm{Re}(q) < 0$ is not

$$\lim_{t\to\infty} \operatorname{Im} q(t) = 0 \qquad (5.19)$$

which is a not unreasonable assumption under conditions (a) and (b) above, then from Corollary 4.17 and Theorem 4.16, the complement of $[q_\infty,\infty)$ in the complex plane contains no points of $\sigma(L_\gamma)$ other than eigenvalues. Similar remarks apply to subsequent results.

In the real case Theorem 5.14 reduces to

<u>Corollary 5.15</u>. *If $p > 0$ and q are real-valued, monotone decreasing and locally absolutely continuous on $[a,\infty)$ and*

$$\int_a^\infty \frac{p}{|q|} = \infty,$$

then

$$(q_\infty,\infty) \subset \mathrm{Ce}(\bar{L}) \qquad (5.20)$$

where $q_\infty = \lim_{t\to\infty} q(t)$.

This is a generalization (at least for q locally absolutely continuous) of a result of Hartman and Wintner [31]. We note in passing that there is in [31] a further condition guaranteeing (5.20): namely, if $p = 1$, q is negative and monotone and there exists a sequence $\{a_n\}$, $a_n \to \infty$ as $n \to \infty$, such that

$$\liminf_{n\to\infty} q(a_n)/a_n > -\infty.$$

That this is a consequence of the above result of Hartman and Wintner may be seen from the following argument. We can assume that q is monotone non-increasing, and that for some $K > -\infty$, $q(a_n) \geq Ka_n$ for all n. Then for $a_n \leq t \leq a_{n+1}$, $0 > q(t) \geq q(a_{n+1}) \geq Ka_{n+1}$, and hence $|q(t)|^{-1} \geq |K|^{-1}a_{n+1}^{-1}$. Finally, by Abel's theorem, [28, p.290],

$$\int_{a_1}^{\infty} |q(t)|^{-1}dt \geq |K|^{-1} \sum_{n=1}^{\infty} (a_{n+1}-a_n)a_{n+1}^{-1} = \infty.$$

<u>Example</u>. If $p(t) = bt^{-\beta} + ict^{-s}$ and $q(t) = -ht^{\alpha} + idt^{-r}$ where h, a, b and α are positive constants and c, β and r are non-negative constants, then $q_{\infty} = -\infty$ and so $(-\infty,\infty) \subset C\sigma(\overline{L})$ provided that

$$\alpha + \beta \leq 1 \tag{5.21}$$

and

$$s > \frac{1}{2}(\alpha+5\beta) + 1 \quad \text{and} \quad r > \beta - \frac{1}{2}\alpha + 1 \tag{5.22}$$

<u>Proof</u>. Applying Theorem 5.14; from (5.21) and (5.22),

(a) $\dfrac{|\operatorname{Im} q|}{|p|.|\operatorname{Re} q|^{1/2}} \sim t^{\beta-r-\frac{1}{2}\alpha} \in L\,[a,\infty)$

(b) $\dfrac{|\mathrm{Re}\ q|^{1/2}|\mathrm{Im}\ q|}{|p|^{3/2}} = [illegible]$, $t \in [a,\infty)$

(c) $\dfrac{|p|}{|\mathrm{Re}\ q|} = t^{-\beta-\alpha}$, $t \in [a,\infty)$.

We now give two variations of Theorem 5.14

Theorem 5.16 (i). *Suppose that condition (c) of Theorem 5.14 is replaced by the requirements that* $\mathrm{Re}(q) < 0$, $|p|\mathrm{Re}(q)$ *is monotone decreasing, and*

$$\int_a^\infty [illegible] = \infty.$$

Then $(q_-,0] \subset C\sigma(\hat{L})$, *where* $q_- = \lim_{t\to\infty} \mathrm{Re}(q(t))$.

(ii) *If we replace condition (c) by the requirements that* $|p|\mathrm{Re}(q)$ *be monotone increasing,* $\mathrm{Re}(q) > 0$, *and*

$$\int_a^\infty |p| = \infty$$

then $[0,\infty) \subset C\sigma(\bar{L})$.

Proof. The proof is almost identical with that of Theorem 5.14; the only change is the observation that

$$\int_a^t |\theta'\theta^{-1} - \phi'\phi^{-1}| = \int_a^t |(\phi/\theta)'(\phi/\theta)^{-1}|$$

$$= \pm\log(\phi/\theta)(t) + O(1)$$

depending on whether $\phi/\lambda = |q|^{1/2}(-\mathrm{Re}(q))^{1/2}$ is increasing or decreasing.

Example. For p and q as in the previous example, it follows from Theorem 5.16 that if (5.22) is satisfied, then $[0,\infty) \subset Co(\tilde{L})$ if

$$\alpha \le \beta < 1 \tag{5.23}$$

and $(-\infty,0] \subset Co(\tilde{L})$ if

$$\beta \le \alpha \le 1 \tag{5.24}$$

Thus if $\alpha = \beta \le 1$, $r > \frac{1}{2}\alpha - 1$ and $s > 2\alpha + 1$, then $(-\infty,\infty) \subset Co(\tilde{L})$.

The proof is obvious. One can now easily see from (5.21), (5.23), (5.24) that Theorems 5.14 and 5.16 are independent.

The second variation of Theorem 5.14 involves the function q^* where

$$q^*(t) = \inf_{s \ge t} \mathrm{Re}(q(s)). \tag{5.25}$$

Theorem 5.17. *Suppose that p and q are locally absolutely continuous on $[a,\infty)$, with p positive and monotone decreasing and*

(a) $\displaystyle\int_a^\infty \frac{|\mathrm{Im}(q)|}{p|q^*|^{1/2}} < \infty$,

(b) $\displaystyle\int_a^\infty \frac{|\mathrm{Re}(q) - q^*|}{p^{1/2}|q^*|^{1/2}} < \infty$, *and*

(c) $\displaystyle\int_a^\infty \frac{p}{|q^*|} = \infty$.

Then $(q_\infty,\infty) \subset Co(\bar{L})$, *where* $q_\infty = \lim_{t\to\infty} \mathrm{Re}(q(t))$.

The proof is very similar to that of Theorem 5.14, since q^* is monotone decreasing and locally absolutely continuous on $[a,\infty)$.

The final application of Theorem 5.11 gives is a rather different result and complements the limit-circle result of Theorem 5.5 above.

Theorem 5.18. *If* $p = 1$ *and* $\mathrm{Re}\, q$ *is negative and locally absolutely continuous, such that*

$$0 \le \frac{\mathrm{Re}\, q'(t)}{\mathrm{Re}\, q(t)} \le \frac{2}{t}, \quad t \ge a \tag{5.26}$$

and if

$$\int_a^\infty \frac{|\mathrm{Im}\, q|}{|\mathrm{Re}\, q|^{1/2}} < \infty \tag{5.27}$$

Then $[0,\infty) \subset Co(\bar{L})$.

Proof. As in the proof of Theorem 5.5 we set $p = \xi = 1$, $\theta = (-\mathrm{Re}(q))^{-1/2}$, $\gamma(t) = \frac{1}{2}\theta(t).t^{-1}$ and $q_1 = \mathrm{Re}(q)$, $q_2 = i\ \mathrm{Im}(q)$. Since θ is monotone decreasing, it is clear that conditions (i) - (iii) of Theorem 5.11 are satisfied. In (iv), from (5.27),

$$|\mathrm{Re}\, q|^{-1/2}\int_a^t |\mathrm{Im}(q)| \le \int_a^t \frac{|\mathrm{Im}\, q|}{|\mathrm{Re}\, q|^{1/2}} < \infty$$

Finally, similar to the proof of Theorem 5.5 we have

$$\int_a^t F = \log(|\mathrm{Re}\, q|^{-1/2}.t) + O(1)$$

using (5.26) and (5.27). Consequently

$$\int_a^\infty \theta(t)\exp\left(-\int_a^t \beta\right)dt \ge C\int_a^\infty t^{-1}\,dt = \infty$$

and therefore $0 \in Co(\tilde{L})$. We can now conclude the proof by observing that (5.26) and (5.27) are also valid if q is replaced by $q - \lambda$ for any $\lambda \ge 0$. ∎

We now consider some consequences of Corollary 5.15, using Theorem 5.12. The next four results are straightforward analogues of the corresponding results for the real case.

<u>Theorem 5.19</u> (c.f.[44]). *Suppose* $p = 1$, $\mathrm{Re}(q) = r + v$ *where* $v(t)$ *is locally absolutely continuous and there exist constants* K, L, M *such that*

$$\limsup_{t\to\infty} t|r(t)| = K \tag{5.28}$$

$$\limsup_{t\to\infty} t|v'(t)| = L \tag{5.29}$$

$$\limsup_{t\to\infty} t|\mathrm{Im}\, q(t)| = M \tag{5.30}$$

Suppose also that $v(t) \to 0$ *as* $t \to \infty$. *Then* $(\Lambda_0, \infty) \subset Co(\tilde{L})$ *where*

$$\Lambda_0 = \frac{1}{2}\{(K^2+M^2) + \{(K^2-M^2)^2 + L^2\}^{1/2}\} \tag{5.31}$$

<u>Proof</u>. Setting $\alpha = 1$, $\gamma = 0$, $\theta = (\lambda - v(t))^{1/2}$, $q_1 = \mathrm{Re}(q) - \lambda$ and $q_2 = 0$ in Theorem 5.12, the proof proceeds exactly as in the real case (see [44, theorem 3.1]). ∎

The remarks following Theorem 5.19 are applicable in the above case, since (5.30) implies (5.29). Thus the operators L_γ are J-selfadjoint. Since (4.48) is satisfied, we may apply Theorem 4.19 to obtain $\sigma_e(\tilde{L}) = \sigma_e(L_\gamma) = [0,\infty)$. Hence under the conditions of Theorem 5.19, the essential spectrum of $\tilde{L}$ is the half-line $[0,\infty)$ and there are no eigenvalues embedded in it beyond Λ_0, where Λ_0 is given by (5.31).

The conditions (5.28) and (5.30) may be replaced by

$$\limsup_{t\to\infty} t|r(t) - r_0| = K$$

$$\limsup_{t\to\infty} t|\operatorname{Im} q(t) - k| = M$$

provided the conclusion is altered to read that $\lambda + ik \in C\sigma(\tilde{L})$ whenever $\lambda > r_0 + \Lambda_0$, where Λ_0 is given by (5.31).

<u>Theorem 5.20</u> (c.f.[44]). *Suppose* $n = 1$ *and* $\operatorname{Re}(q) = r + v$ *where* $v(t)$ *is locally absolutely continuous,* $v(t) \to 0$ *as* $t \to \infty$ *and*

$$\limsup_{t\to\infty} (\log t)^{-1}\int_0^t |r| = K, \tag{5.32}$$

$$\limsup_{t\to\infty} (\log t)^{-1}\int_a^t |dv| = L, \tag{5.33}$$

$$\limsup_{t\to\infty} (\log t)^{-1}\int_a^t |\operatorname{Im}(q)| = M \tag{5.34}$$

Then $(\Lambda_0,\infty) \subset C\sigma(\tilde{L})$ *where*

$$\Lambda_0 = \tfrac{1}{2}\{L + (K+M)^2 + (K+M)(2L + (K+M)^2)^{1/2}\} \tag{5.35}$$

<u>Proof</u>. Assume without loss of generality that $a > 1$. Setting $\theta(t) = (\log t)^{-1}$, $\omega(t) = \theta(t)(\lambda - v(t))^{1/2}$, $\gamma = 0$, $q_1 = v - \lambda$, $q_2 = r$ in Theorem 5.12, the proof proceeds exactly as in the real case (see [44, theorem 3.2]). #

<u>Corollary 5.21</u>. *If* $p = 1$ *and* $q \in L[a,\infty)$, *then* $(0,\infty) \subset C\sigma(\bar{L})$.

<u>Proof</u>. In Theorem 5.20, set $v = 0$ and $r = \mathrm{Re}(q)$, and note that since $\mathrm{Re}(q)$, $\mathrm{Im}(q) \in L[a,\infty)$ so (5.32) and (5.34) hold for arbitrarily small K and M. #

Under the condition of this corollary, the point $\lambda = 0$ could be an eigenvalue. However, setting $\theta(t) = t$, $p = \phi = 1$, $\gamma = 0$, $q_1 = 0$, $q_2 = \mathrm{Re}(q)$ and $\alpha = 0$, in Theorem 5.12 we see that this cannot happen if q satisfies the stronger condition,

$$\int_a^\infty t|q(t)|dt < \infty \tag{5.36}$$

In this case, by Corollary 4.22, $E\sigma(\bar{L}) = C\sigma(\bar{L}) = [0,\infty)$.

We also have the following extension to the complex case of a theorem due to M.S.P. Eastham [17]:

<u>Theorem 5.22</u>. *Suppose* $p = 1$ *and* $\mathrm{Re}(q) = r + v$ *where* $v(t)$ *is locally absolutely continuous,* $v(t) \to 0$ *as* $t \to \infty$ *and*

$$\limsup_{t\to\infty} t|r(t)| = K,$$

$$\limsup_{t\to\infty} tv'(t) = L,$$

$$\limsup_{t\to\infty} t|\mathrm{Im}(q)| = M$$

Then $(\Lambda_0, \infty) \subset Co(\tilde{L})$, *where*

$$\Lambda_0 = \tfrac{1}{4}\{L + (K+M)^2 + (K+M)(2L + (K+M)^2)\}^{1/2}$$

Proof. We observe that in the derivation of condition (A) preceding Theorem 5.11, we may replace $C = 2\,\mathrm{Re}(\phi'\phi^{-1} - \gamma(p\theta)^{-1})$ by any function which minorizes it; this can be seen easily from the quadratic form (5.3). In particular, if in Theorem 5.12 we set $p = \theta = 1$, $4\gamma = \frac{-L_1}{t(\lambda-\varepsilon)}$, $\phi = (\lambda - v)^{1/2}$, $q_1 = \mathrm{Re}(q) - \lambda$, and $q_2 = 0$, we have

$$2\,\mathrm{Re}\left(\frac{\phi'}{\phi} - \frac{\gamma}{p\theta}\right) \geq \frac{-L_1}{2t(\lambda-\varepsilon)},$$

where L_1 is arbitrarily close to L and $\lambda > \lambda - \varepsilon > \Lambda_0$. Thus condition (v) becomes

$$\text{(v)}' \qquad \int_a^\infty \exp\int_a^t \mathrm{Re}\left[\frac{\theta'}{\theta} - \frac{L_1}{2t(\lambda-\varepsilon)}\right] . \exp\left[-\int_a^t F_1\right] dt = \infty, \tag{5.37}$$

where

$$F_1 = \left|\frac{\bar{\theta}}{\phi}\right| . \left|q + \frac{\phi^2}{\bar{p}|\theta|^2} - \left(\frac{\gamma}{\theta}\right)^2 + \left(\frac{\gamma}{\theta}\right)'\right|$$

The proof now follows that of Theorem 5.20. ∎

The results of Eastham in [18] also have complex analogues:

Theorem 5.23 (c.f. [18]). *Let* $p = 1$. *Suppose that* $\mathrm{Re}(q) = r + s$ *where* s *is continuously differentiable in* $[a, \infty)$, *and*

(a) $s(t) \to -\infty$ as $t \to \infty$,

(b) $s(t) = O(t^{J})$ as $t \to \infty$,

(c) $\limsup_{t\to\infty} (\log t)^{-1} \int_a^t |r||s|^{-1/2} = K$,

(d) $\limsup_{t\to\infty} (\log t)^{-1}\int_a^t S|s|^{-1} = L$

where $S(t) = \max(s'(t),0)$.

(e) $\limsup_{t\to\infty} (\log t)^{-1}\int_a^t |\operatorname{Im}(q)||s|^{-1/2} = M$.

Then $(-\infty,\infty) \subset Co(\tilde{L})$ *provided that*

$$J + K + L + M < 1. \tag{5.38}$$

Proof. Assume without loss of generality that $a > 1$ and choose a so that $\lambda - s(t) > 0$ on $[a,\infty)$, and then in Theorem 5.12 set $\theta(t) = (t^{J/2}\log t)^{-1}$, $\phi(t) = (\lambda - s(t))^{1/2}\theta(t)$, $\gamma \equiv 0$, $q_1 = s$, $q_2 = r$, and $\alpha = 1$. As usual, we need only check condition (v). As in (5.37), observe that when $\gamma = 0$ and we minorize the term $-s'/2(\lambda-s)$ in ϕ'/ϕ by $-S/2(\lambda-s)$, condition (v) of Theorem 5.12 becomes

$$(v)' \quad \int_a^\infty \exp\int_a^t \left(2\frac{\theta'}{\theta} + \frac{S}{2(\lambda-s)}\right).\exp\left[-\int_a^t \left(\left|\frac{S}{2(\lambda-s)}\right| + |r|.|\lambda-s|^{-1/2} + |\operatorname{Im} q|.|\lambda-s|^{-1/2}\right)\right] = \infty$$

Using conditions (c) - (e) above, this becomes

$$\int_a^\infty t^{-(J+K_1+L_1+M_1)}(\log t)^{-2}dt = \infty$$

where K_1, L_1, and M_1 may be chosen arbitrarily close to K, L, and M respectively. Clearly, this holds if (and only if) (5.38) holds.

This completes the proof.

A similar extension of Theorem 4 of [19] is possible by replacing (c) - (e) above by

(c)' $\quad \limsup_{t\to\infty} t|r(t)||s(t)|^{-1/2} = K$

(d)' $\quad \limsup_{t\to\infty} tS(t)|s(t)|^{-1} = L$

(e)' $\quad \limsup_{t\to\infty} t|\mathrm{Im}\, q(t)||s(t)|^{-1} = M.$

In this case $(-\infty,\infty) \subset C\sigma(\tilde{L})$, provided that $J + K + M + \frac{L}{2} < 1$. Note that in this case, as $J < 1$, it follows from (e)' that $\mathrm{Im}(q(t)) \to 0$ as $t \to \infty$, and hence $E\sigma(\tilde{L}) = C\sigma(\tilde{L}) = (-\infty,\infty)$, by the usual reasoning.

In the next few applications of Theorem 5.12, we make use of the function γ.

Theorem 5.24. *Suppose that* q *is bounded and complex-valued, such that* $\sqrt{q}$ *is well-defined and locally absolutely continuous on* $[a,\infty)$. *Suppose that* $\mathrm{Re}(p) > 0$ *and* $\mathrm{Im}(p) \geq 0$ *(and hence* $\sqrt{p}$ *is well-defined), and that* $\sqrt{p}$ *is also locally absolutely continuous. If*

(a) $\quad \displaystyle\int_a^\infty \left|\mathrm{Re}\left(\frac{\sqrt{q}}{\sqrt{p}}\right)\right| < \infty,$

(b) $\quad \displaystyle\int_a^\infty \frac{|q'|}{|q|^{1/2}} < \infty,$

(c) $\quad \displaystyle\int_a^\infty \left|\frac{p'}{p}\right| < \infty,$

(d) $\displaystyle\int_a^\infty \frac{\big||p| - p\big|}{|p|^{3/2}} < \infty,$ *and*

(e) $\displaystyle\int_a^\infty |p|^{-1/2} = \infty,$

then $(0,\infty) \subset C\sigma(\tilde{L})$.

Proof. Replacing q by $q - \lambda$ in (5.6) and setting $\theta = p^{-1/2}$, $\gamma = q^{1/2}$, $\phi = \lambda^{1/2}$, $q_1 = \mathrm{Re}(q)$, $q_3 = \mathrm{Im}(q)$ and $q_2 = q_4 = 0$ in Theorem 5.12, the only condition that needs checking is (v):

$$F \le \left|\mathrm{Re}\left(-\tfrac{1}{2}p'p^{-1} + 2q^{1/2}p^{-1/2}\right)\right| + (\lambda|p|)^{-1/2}\left|-\lambda + \lambda|p|(\bar{p})^{-1} + (\sqrt{pq})'\right|$$

$$\le |(p'/2p)| + 2\left|\mathrm{Re}(q^{1/2}p^{-1/2})\right| + \lambda^{1/2}\big||p| - \bar{p}\big|.|p|^{-3/2}$$

$$+ \tfrac{1}{2}\lambda^{-1/2}\left|\sqrt{q}(p'/p) + q'q^{-1/2}\right|$$

and hence from conditions (a) - (d) above, $F \in L[a,\infty)$; since $|\theta|.\phi \notin L[a,\infty)$ by (e), the result follows.

We note that $\sqrt{q}$ is well defined if, for example, the range of q is a subset of a simply connected region not containing the origin. Conditions (d) and (e) are satisfied for example if $p(t) \to c$ as $t \to \infty$ where c is a positive real constant, and $\mathrm{Im}(p) \in L[a,\infty)$ (see example below). The following Corollaries are almost immediate consequences of the Theorem:

Corollary 5.25. *Let* $p = 1$. *If* q *is positive and locally absolutely continuous on* $[a,\infty)$, *and* $\sqrt{q} \in L[a,\infty)$ *and of bounded variation on* $[a,\infty)$, *then* $(0,\infty) \subset C\sigma(\tilde{L})$.

Proof. It is enough to show that q is bounded. But, for all $t \geq a$,

$$\sqrt{q(t)} = \sqrt{q(a)} + \int_a^t (\sqrt{q})' = O(1).$$

The result now follows easily from Theorem 5.24. ∎

Corollary 5.26. *Let* $p = 1$. *If* q *is negative and locally absolutely continuous on* $[a,\infty)$, *and*

$$\int_a^\infty \frac{|q'|}{|q|^{1/2}} < \infty,$$

then $(0,\infty) \subset C\sigma(\tilde{L})$.

Proof. Since $\mathrm{Re}(\sqrt{q}) = 0$, once again it suffices to show that q is bounded. But

$$\sqrt{-q(t)} = \sqrt{-q(a)} + \int_a^t \sqrt{-q}\,'$$

$$= \sqrt{-q(a)} + \frac{1}{2}\int_a^t |q'|\,|q|^{-1/2}$$

$$= O(1),$$

which shows that q is bounded below, as required. ∎

Example. If $p(t) = c + \frac{b}{t} + i(\frac{d}{t})^2$ on $[a,\infty)$ where $a > 0$, $c > 0$, b and d are constants, and if q satisfies either of the conditions of Corollaries 5.25 and 5.26, then $(0,\infty) \subset C\sigma(\tilde{L})$.

Proof. Applying Theorem 5.24, since $p(t) \to c > 0$ as $t \to \infty$, (d) and (e) are satisfied. As in the above Corollaries, (a) and (b) are satisfied, so we need only consider (c):

$$\left|\frac{p'}{p}\right| = \frac{1}{|p|}\left|\frac{b}{t^2} + 2i\frac{d^2}{t^3}\right| = O(t^{-2})$$

hence (c) is indeed satisfied.

$\blacksquare$

Finally we give two applications of Theorem 5.12 when p and q are real-valued.

<u>Theorem 5.27</u>. *If $p > 0$, $q < 0$, pq is monotone and $(pq)^{-1}$ is locally absolutely continuous such that*

$$\int_a^\infty \frac{1}{|pq|} = \infty, \quad p\frac{(q')^2}{|q|^3} < c$$

for some constant c, then $0 \in Co(\tilde{L})$.

<u>Proof</u>. Setting $\gamma = 0$, $\phi = 1$, $\theta = |pq|^{-1/2}$, $q_1 = q$ and $q_2 = 0$ in Theorem 5.12, the only condition we need to check is (v). In the expression (5.6) for F^2, the second term vanishes and we have

$$F = \left|\frac{\theta'}{\theta}\right|$$

Since θ is monotone, we then have

$$\int_a^\infty |\theta(t)|.\phi(t)\exp[-\int_a^t F]dt = \begin{cases} \int_a^\infty \theta^2 \\ \\ \int_a^\infty 1 \end{cases}$$

according as s is monotone decreasing or increasing. In either case the right-hand side is infinite.

∎

<u>Theorem 5.28.</u> *If $p > 0$ and locally absolutely continuous, $q < 0$, pq is monotone and locally absolutely continuous, such that p' and pq are bounded and*

$$\int_a^\infty |pq| = \infty$$

then $0 \in Ce(\bar{L})$.

<u>Proof.</u> Setting $\gamma = \Pi$, $\theta = 1$, $\phi = |pq|^{1/2}$, $q_1 = q$ and $q_2 = 0$ in Theorem 5.12, the proof is similar to the proof of Theorem 5.27 except we now require

$$\int_a^\infty \phi^2 = \infty$$

which is indeed the case.

∎

To conclude the present work we prove two generalisations of the following result of P. Hartman [30].

<u>Theorem 5.29.</u> *Let q be real and let $Q(t)$ be a positive, non-decreasing function satisfying*

$$-q(t) \le Q(t) \quad on \quad [a,\infty) \tag{5.39}$$

and

$$\int_a^\infty \frac{1}{Q} = \infty \tag{5.40}$$

then if all solutions of $-y'' + qy = 0$ are bounded, there is no *solution in* $L^2[a,\infty)$.

The conclusion of Theorem 5.29 is equivalent to saying $0 \in C\sigma(\tilde{L})$ and so in particular, 0 is not an eigenvalue. We seek firstly, to generalise this result to a more general function p, then secondly, to include an integral as well as a pointwise condition on q. We note that a summary of most of the known criteria for bounded solutions appears in [45]. The theorem depends on two lemmas which must therefore also be generalised. The first of these was originally proved when p = 1 by D.B. Sears [67]:

Lemma 5.30. *Let* $Q(t)$ *be a positive, non-decreasing function satisfying* (5.39) *and let* $p(t)$ *be positive and satisfy*

$$p(t) = O(t^2) \quad \text{on} \quad [a,\infty) \tag{5.41}$$

Suppose $y \in L^2[a,\infty)$ *is a solution of* $-(py')' + qy = 0$ *then*

$$\int_a^\infty \frac{p(t)|y'(t)|^2}{Q(2t)}dt < \infty \tag{5.42}$$

Proof. Integrating $p(t)|y'(t)|^2(1-\frac{t}{T})^2$ by parts, for $T > 2a$, over the interval $[a,T]$.

$$\int_a^T (1-\tfrac{t}{T})^2 p(t)|y'(t)|^2 dt = y(t)p\bar{y}'(t)(1-\tfrac{t}{T})^2\Big]_a^T$$

$$- \int_a^T y(t)\{(1-\tfrac{t}{T})^2 (p\bar{y}'(t))' - \tfrac{2}{T}(1-\tfrac{t}{T})p\bar{y}'(t)\}dt$$

$$= O(1) - \int_a^T \left(1-\frac{t}{T}\right)^2 q(t)|y(t)|^2 dt + \frac{2}{T}\int_a^T \left(1-\frac{t}{T}\right) p\bar{y}'(t)y(t)dt$$

$$\le O(1) + \int_a^T \left(1-\frac{t}{T}\right)^2 Q(t)|y(t)|^2 dt + \int_a^T p\left\{\frac{2|y|^2}{T^2} + \frac{1}{2}\left(1-\frac{t}{T}\right)^2 |y'|^2\right\} \qquad (5.43)$$

using (5.39), and (3.4) with $\epsilon = 2$. Thus from (5.41) and (5.43) since $y \in L^2[a,\infty)$ we have for some constant K,

$$\frac{1}{2}\int_a^T \left(1-\frac{t}{T}\right) p(t)|y'(t)|^2 dt \le \frac{K}{8} + \int_a^T \left(1-\frac{t}{T}\right)^2 Q(t)|y(t)|^2 dt \qquad (5.44)$$

However,

$$\int_a^T \left(1-\frac{t}{T}\right)^2 p(t)|y'(t)|^2 dt \ge \int_a^{\frac{1}{2}T} \left(1-\frac{t}{T}\right)^2 p(t)|y'(t)|^2 dt$$

$$\ge \frac{1}{4}\int_a^{\frac{1}{2}T} p(t)|y'(t)|^2 dt$$

Thus replacing T by 2T in (5.44),

$$\int_a^T p(t)|y'(t)|^2 dt \le K + 8\int_a^{2T} Q(t)|y(t)|^2 dt$$

Writing

$$\xi(T) = K + 8\int_a^{2T} Q(t)|y(t)|^2 dt, \quad \eta(T) = \int_a^T p(t)|y'(t)|^2 dt$$

we have $\eta(T) < \xi(T)$ and

$$\int_a^T p(t)|y'(t)|^2 (Q(2t))^{-1} dt = \int_a^T (Q(2t))^{-1} d\eta(t)$$

$$= \frac{\eta(T)}{Q(2T)} + \int_a^T \eta(t)\, d\left(-\frac{1}{Q(2t)}\right)$$

Since $Q(\cdot)$ is non-decreasing,

$$\int_a^T p(t)|y'(t)|^2 (Q(2t))^{-1} dt < \frac{\xi(T)}{Q(2T)} + \int_a^T \xi(t)\, d\left(\frac{-1}{Q(2t)}\right)$$

$$< \frac{K}{Q(2a)} + \frac{8}{Q(2a)} \int_a^{2a} Q(t)|y(t)|^2 dt + \int_a^T \frac{d\xi(t)}{Q(2t)}$$

$$< O(1) + 16 \int_a^T |y(2t)|^2 dt$$

$$= O(1) \quad \text{as} \quad T \to \infty.$$

as required to prove the lemma. $\#$

The second of the lemmas we need was originally proved when $p = 1$ by P. Hartman [29]. It seems that we need a stronger condition on p here than (5.41).

<u>Lemma 5.31</u>. *Suppose* p, q *satisfy*

$$0 < p(t) \le 1, \quad -q(t) \le C^2 \tag{5.45}$$

on $[a,\infty)$ *and* $y = y(t)$ *is a real solution of* $-(py')' + qy = 0$, *and* $\max(a,\frac{2}{C}) \le T < \infty$. *If* $y'(T) \ne 0$ *let* $T^* = T^*(T)$ *be defined as the largest (smallest) possible value satisfying both* $|T^*-T| \le \frac{2}{C}$ *and* $yy' \ge 0 (\le 0)$ *for* $T \le t \le T^* (T^* \le t \le T)$. *Then*

$$(py'(T))^2 \le 4C^2y^2(T^*) \tag{5.46}$$

Proof. Consider the case $yy'(t) \ge 0$ for $T \le t \le T^*$. Since y and y' cannot vanish simultaneously, yy' changes sign at a zero of y. Hence y and therefore y' do not change sign on the interval (T,T^*). It can be supposed that $y \ge 0$ and $y' \ge 0$ (otherwise replace y by $-y$). Thus

$$0 < y(t) \le y(T^*) \quad \text{for} \quad T < t \le T^*$$

Since $-(py')' + qy = 0$ and $-q \le C^2$ so

$$(py')'(t) \ge -C^2y(T^*)$$

which, when integrated over (T,t) gives

$$py'(t) \ge -C^2y(T^*)(t-T) + py'(T)$$

We suppose (5.46) does not hold, then

$$py'(t) > -C^2y(T^*)(t-T) + 2Cy(T^*) \tag{5.47}$$

The right hand side is non-negative for $T \le t \le T + \frac{2}{C}$ so by definition, $T^* = T + \frac{2}{C}$. Since $p(t) \le 1$ and $y'(t)$ is non-negative $py'(t) \le y'(t)$ and we may integrate (5.47) over (T,T^*) using this fact, to obtain

$$y(T^*) > -2y(T^*) + 4y(T^*)$$

since $y(T) \ge 0$. But this leads to the contradiction

$$y(T^*) > 2y(T^*) > 0$$

and proves the lemma for this case. The proof for the case $yy' \le 0$ is similar. ∎

We are now able to prove

<u>Theorem 5.32</u>. *Let $Q(t)$ be a positive, non-decreasing function satisfying (5.39) and (5.40) and suppose $0 < p(t) \le 1$ on $[a,\infty)$ then if all solutions of $-(py')' + qy = 0$ are bounded, there is no solution in $L^2[a,\infty)$ and hence $0 \in Co(\tilde{L})$.*

<u>Proof</u>. We show firstly that if y is a bounded solution of $\ell(y) = 0$ then $py'(t)Q^{-1/2}(2t)$ is also bounded. Let $T > \max(a, 2Q^{-1/2}(a))$ then the conditions of Lemma 5.31 hold on the interval $[T - \frac{2}{C}, T + \frac{2}{C}]$ where $C = Q^{1/2}(T + 2Q^{-1/2}(T))$ provided

$$Q(T + \tfrac{2}{C}) \le C^2 = Q(T + 2Q^{-1/2}(T))$$

But this follows from the monotonicity of $Q(t)$. Thus

$$|py'(T)| \le 2Q^{1/2}(T + 2Q^{-1/2}(T)).|y(T^*)|$$

and so if t is sufficiently large, since y is bounded and Q is monotone, we have that $py'(t)Q^{-1/2}(2t)$ is bounded.

Suppose now that $\ell(y) = 0$ has a solution y_1 in $L^2[a,\infty)$ and let y_2 be a linearly independent solution such that the Wronskian of y_1 and y_2 is 1, i.e.

$$py_1'.y_2 - py_2'.y_1 = 1 \tag{5.48}$$

Dividing (5.48) by $Q^{1/2}(2t)$, since $y_1 \in L^2[a,\infty)$, $p^{1/2}y_2$ is bounded,

$py_2'(t).Q^{-1/2}(2t)$ is bounded, and by Lemma 5.39, $p^{1/2}(t)y'(t)Q^{-1/2}(2t) \in L^2[a,\infty)$, so the left hand side is in $L^2[a,\infty)$ and hence $Q^{-1/2}(2t) \in L^2[a,\infty)$, contradicting (5.40). Hence there can be no solution of $\ell(y) = 0$ in $L^2[a,\infty)$. #

We now seek a similar result when q satisfies a different condition, and prove firstly the lemma analogous to Lemma 5.31.

<u>Lemma 5.33.</u> *Suppose* $0 < p(t) \le 1$ *on* $[a,\infty)$ *and* $q = q_1 + q_2$ *such that*

$$-q_1 \le C^2, \quad \int_a^t |q_2| \le C \tag{5.49}$$

Let $y = y(t)$ *be a solution of* $-(py')' + qy = 0$ *and* $\max(a,\frac{2}{C}) \le T < \infty$. *If* $y'(T) \neq 0$ *let* T^* *be defined as in Lemma 5.31, then*

$$(py'(T))^2 \le 9C^2y^2(T^*) \tag{5.50}$$

<u>Proof.</u> Proceeding as in the proof of Lemma 5.31, we may suppose $y \ge 0$ and $y' \ge 0$ on (T,T^*), then

$$(py')'(t) = q(t)y(t) = (q_1(t)+q_2(t)).y(t)$$

$$\ge -C^2y(T^*) + q_2(t)y(t)$$

Integrating over (T,t),

$$py'(t) \ge -C^2y(T^*)(t-T) + py'(T) - y(T^*)\int_T^{T^*}|q_2(t)|dt$$

$$\ge -C^2y(T^*)(t-T) + py'(T) - Cy(T^*)$$

Suppose (5.50) does not hold then since $p(t) \le 1$,

$$y'(t) > -C^2y(T^*)(t-T) + 3Cy(T^*) - Cy(T^*)$$

$$= -C^2y(T^*)(t-T) + 2Cy(T^*).$$

$$= C.y(T^*)\{2-C(t-T)\}$$

Since the right-hand side is positive, $T^* = T + \frac{2}{C}$ by definition. Integrating as in (5.47) again yields

$$y(T^*) > 2y(T^*) > 0$$

which is a contradiction. Again the case $yy' \le 0$ may be dealt with similarly.

We now need a result which corresponds to Lemma 5.30 when q satisfies a condition of the form of (5.49). Such a result has not been found to be readily accessible so instead we use Lemma 3.1 with $w(t) = Q^{-1/2}(t)$ which, instead of (5.49) requires $q = q_1 + q_2$ to satisfy

$$|q_1(t)| \le kQ(t), \quad \left|\int_a^t q_2\right| \le \eta|p(t)|.Q^{1/2}(t), \quad t \in [a,\infty) \qquad (5.51)$$

As in the proof of Theorem 5.32, we now have

<u>Theorem 5.34</u>. *If* $0 < p(t) \le 1$ *is non-increasing and there exists a locally absolutely continuous, non-decreasing function* $Q(t)$ *on* $[a,\infty)$ *such that* $q = q_1 + q_2$ *satisfies*

$$|q_1| \le kQ(t), \quad \int_a^t |q_2| \le \eta|p(t)|Q^{1/2}(t) \qquad (5.52)$$

for some positive constants k, c, then if all solutions of $-(py')' + qy = 0$ *are bounded and* $Q(t)$ *satisfies* (5.40), *there is no solution in* $L^2(a,\infty)$ *and hence* $0 \in C\sigma(\tilde{L})$.

We notice that whereas (5.39) was a one-sided bound on q, q_1 must satisfy the two-sided bound given in (5.52) in order to use Lemma 3.1. Also, whereas Lemma 3.1 requires a bound on the integral of q_2, the use of Lemma 5.33 requi[illegible] bound on the integral of $|q_2|$.

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