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ORIGINAL PAPER

Dimensional Homogeneity in Classifying Second-Order Differential Invariant Systems for Four-Dimensional Lie Algebras

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Abstract

This research investigates a specific mathematical structure called a nonsingular differential invariant structure, which plays a crucial role in describing physical phenomena through differential equations. Focusing on a four-dimensional Lie algebra in (1+3)-dimensional space, this study classifies these structures. Here, a method is developed to identify second-order differential invariants and showed how they translate into systems of three second-order ordinary differential equations for the specific case of a four-dimensional Lie algebra, and these invariants are presented in the form of list in Table 1. To further refine the analysis, these systems are classified based on their "dimensional homogeneity structure," which reveals their behavior when scaling all variables. This sub-classification described in Table 2, goes beyond the initial broad categorization, potentially linking these structures to specific physical applications. **Novelty:** To best of our knowledge, the classification of nonsingular differential invariant structure in (1+3)-dimensional space of four-dimensional real Lie algebras and association of dimensional homogeneity structure in classification scheme are new contributions in algebraic classification problems in literature and never reported before this. Such type of classification scheme is fruitful in analysis of many physical systems. In addition, the algebraic properties of these peculiar invariant systems including integrability, are analysed in a detailed manner both mathematically and from a mechanics viewpoint.

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Introduction

The laws representing the natural and physical world are usually formulated and modelled in the form of differential equations. In this regard, nonsingular invariant structure is an influential feature of diverse problems of several physical systems. This feature is reflected in affiliated invariant differential equations of corresponding mathematical modeling. By utilizing these mathematical models, several physical phenomena of many applied sciences are analysed such as celestial motion, bridge design, interactions between neurons and mechanics of particles, etc. Thus, differential equations play a key role in many sciences, viz., chemistry, physics, biology, economics, mathematics and engineering (see [1, 3, 4, 7, 11, 12]). All of these disciplines are concerned with the properties of differential equations of various types. Consequently, any development in the theory of differential equations affects the diverse aspects of applied sciences. That is why different approaches and methods have been developed for the analysis of these underlying system of differential equations. One of the important methods which is usually utilized for analysis of differential equations is the Lie group method. This method was initially introduced as a tool to simplify and to reduce and solve ordinary as well as partial differential equations by Sophus Lie (1842–1899) in [8].

Nowadays, various aspects of differential equations such as linearization, invariant singularity, conservation laws, integrability and other related aspects are analysed by employing Lie group method (see [1–3, 5, 7, 9, 11–14]). In the Lie group method, the successive reduction of order via invariants by employing Lie symmetries for a system of ODEs is not well suited for utilization in a classical manner. The thirst of overcoming this hurdle led researchers towards alternative approaches in the frame of Lie symmetry method. In this era, one of the significant approaches is the Lie algebraic approach of classification for system of differential equations [15]. Consequently, the Lie algebraic approach is introduced for a system of two 2nd-order ODEs, not greater than four, which has been discussed in [1–3, 5]. In Lie algebraic approach, nonsingular invariant structure has a contributing role in both aspects namely structural formulation and applications. Thus classification of algebraic nonsingularity in differential invariant structure is the substantial part of research in the frame of Lie symmetry method which needs a deep insight of analysis. Invariant structure possessing nonsingularity is known as nonsingular invariant structure.

Like systems of two second-order ODEs, systems of three second-order ODEs have many applications in mechanics and other sciences as well [12–14]. But limited works have been performed on the analysis associated with the classification of systems of three second-order ODEs. Hence, there is a need to work in this direction. Dimensional homogeneity plays a key role in the mathematical modelling of physical systems and the corresponding equations. Thus, it is fruitful in diverse aspect of analysis of the investigated systems, including integrability.

The first part of this paper is devoted to formulation of a procedure to deduce a set of nonsingular kth-order ($k \geq 3$) differential invariants consisting bases of invariants associated with symmetry vectors in (1+3)-dimensional space of four-dimensional real Lie algebras. In addition a set of nonsingular differential invariants having order 2 for Lie algebras of dimension 4 is classified in (1+3)-dimensional space and presented in Table 1.

Moreover, in part two, the dimensional homogeneity structure is introduced in algebraic classification of nonsingular invariant systems by utilizing corresponding arbitrary functions in (1+3)-dimensional space. This is performed on classified invariant systems given in Table 1 and a new list is mentioned in Table 2. To the best of our knowledge, this is a new contribution in the classification problem in Lie symmetry approach. The underlying classification has several utilizations, both mathematical and physical as described in a detailed manner.

Notation

The succeeding notions are utilized for the underlying Lie algebra with corresponding realizations as well as associated invariant structure. In the follow-up, $\mathcal{A}$ is utilized as a place holder for the Lie algebra $\mathcal{A}_{i,j}^{a,b,n}$ which points toward the jth algebra of dimension i, where the superscripts a, b identify parameters on which the Lie algebra hinges on; the column N in Table 1 given at the end species the algebra realizations; the realization is referred to by a superscript n. As usual one has $\partial_t = \frac{\partial}{\partial t}$, $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$, $\partial_z = \frac{\partial}{\partial z}$ and X_i signifies the elements of a basis for the underlying Lie algebra of vector fields; here i is less than or equal to the dimension of the investigated Lie algebra.

Nonsingularity in Invariants Structure and Canonical Forms

In this section, by the utilization of realization of the underlying 4-dimensional Lie algebras in (1+3)-dimensional space, the construction of differential invariants are described. Then nonsingularity in underlying invariant structure is discussed and by employing these differential invariants, the nonsingular invariant system of differential equations; called canonical forms, are constructed. Here realizations are taken from [10] keeping the format of [2].

We emphasize on a system of three second-order ODEs in the general form

$$\begin{aligned} \ddot{x} &= f(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) \\ \ddot{y} &= g(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) \\ \ddot{z} &= h(t, x, y, z, \dot{x}, \dot{y}, \dot{z}). \end{aligned} \tag{1}$$

Our intention is to examine the nonsingularity in the invariant structure associated with the system (1) which possess four-dimensional Lie algebras. We utilize the procedure of [2, 3, 12] in constructing the invariant structure for the system (1). For completeness, we present some fundamental notions which are necessary for the follow-up.

Nonsingularity in Invariant Structure

If the rank of the coefficient matrix of vector fields on the solution manifold is equal to the rank of the coefficient matrix on the generic manifold, then nonsingularity occurs in the invariant structure of the associated underlying vector fields. Such type of invariants are known as nonsingular invariants. This nonsingularity has a substantial role in construction of the corresponding system of invariant DEs, which further affects their integrability [3, 4, 6]. Here we investigate such type of nonsingularity in invariant structure for the system (1) which admits a four-dimensional Lie algebra.

A differential function β is said to be invariant under the symmetry group generated by X if

$$X^{[q]}(\beta) = 0, \tag{2}$$

, where X is a symmetry vector and $X^{[q]}$ is its q-th prolongation.

Algorithm

For the description of the algorithmic procedure, here we are providing calculation details for some cases; related to the construction of nonsingular invariant representation admitting 4-dimensional Lie algebras in (1+3)-dimensional space from [2]. Here we adopt the same procedure as utilized in [2, 3, 12].

1. Let us consider the case of Lie algebra $A^4_{4,3}$ as mentioned in [2]; we have the realization

$$X_1 = \partial_y,\ X_2 = \partial_x + y\partial_y,\ X_3 = t\partial_x,\ X_4 = -t\partial_t \tag{3}$$

Using the invertible transformation

$$\bar{x} = \frac{x}{t},\ \bar{y} = \frac{1}{t},\ \bar{t} = y,\ \bar{z} = z \tag{4}$$

and removing the bars, we arrive at the following

$$X_1 = \partial_t,\ X_2 = y\partial_x + t\partial_t,\ X_3 = \partial_x,\ X_4 = x\partial_x + y\partial_y \tag{5}$$

The symmetry algebra (5) has the subalgebra $A^2_{3,2}$ with the operators X_1, X_2 and X_3. We obtain the associated set of invariants of $A^2_{3,2}$ presented in [12] and reverting to the original variables, we arrive at the following form

$$\begin{aligned}
&p = t,\ q = z,\ u = -\frac{1}{t^2\dot{y}}e^{(x-t\dot{x})},\ \frac{dq}{dp} = \dot{z},\ w = 2t + t^2\frac{\ddot{y}}{\dot{y}},\\
&\frac{du}{dp} = \left(\frac{\ddot{x}}{t\dot{y}} + \frac{2}{t^3\dot{y}} + \frac{\ddot{y}}{t^2\dot{y}^2}\right)e^{(x-t\dot{x})},\\
&\frac{d^2q}{dp^2} = \ddot{z}.
\end{aligned} \tag{6}$$

Applying the prolonged form of the operator $X_4 = -t\partial_t$ on the invariants (6), we find the following modified invariant representation

$$m_1 = p,\ m_2 = pu,\ m_3 = p\frac{dq}{dp},\ \ m_4 = \frac{w}{p},\ \ m_5 = p^2\frac{du}{dp},\ m_6 = p^2\frac{d^2q}{dp^2}, \tag{7}$$

and the related canonical form in terms of $p, q, u, v, w, \frac{dq}{dp}, \frac{du}{dp}, \frac{dv}{dp}, \frac{d^2q}{dp^2}$ invariants, is given by the system

$$\begin{aligned}
p^2\frac{du}{dp} &= f\left(p, pu, p\frac{dq}{dp}\right)\\
\frac{w}{p} &= g\left(p, pu, p\frac{dq}{dp}\right)\\
p^2\frac{d^2q}{dp^2} &= h\left(p, pu, p\frac{dq}{dp}\right).
\end{aligned} \tag{8}$$

In terms of (x, y, z) coordinates, the corresponding canonical form is provided by the system

$$\begin{aligned}
&\left(\frac{t\ddot{x}}{\dot{y}}+\frac{2}{t\dot{y}}+\frac{\ddot{y}}{\dot{y}^2}\right)e^{(x-t\dot{x})}=f\left(t,-\frac{1}{t^2z\dot{y}}e^{(x-t\dot{x})},\frac{\dot{z}}{z}\right)\\
&2+t\frac{\ddot{y}}{\dot{y}}=g\left(t,-\frac{1}{t^2z\dot{y}}e^{(x-t\dot{x})},\frac{\dot{z}}{z}\right)\\
&t^2\ddot{z}=h\left(t,-\frac{1}{t^2z\dot{y}}e^{(x-t\dot{x})},\frac{\dot{z}}{z}\right).
\end{aligned}\tag{9}$$

By invoking the procedure as described in the work of Ayub et al., [2], it is deduced that by employing (9), the rank of the coefficient matrix of vector fields on the solution manifold becomes the rank of generic manifold for $A_{4,3}^4$. Thus the nonsingularity occurs in invariants (7) and allied invariant system possessing Lie algebra $A_{4,3}^4$.

2. Consider the case of Lie algebra $A_{4,6}^{a,8}$ as given in [2]; the corresponding realization is

$$A_{4,6}^{a,8}\simeq "\partial_t,\ \partial_x,\ t\partial_t+ax\partial_x+\partial_y,\ \partial_z".$$

Proceeding in a similar manner as mentioned in the previous case 1, we arrive at the set of invariants

$$m_1=q,\ m_2=u,\ m_3=v,\ \ m_4=\frac{dq}{dp},\ \ m_5=\frac{du}{dp},\ m_6=\frac{dv}{dp},\tag{10}$$

where

$$\begin{aligned}
&p=z, q=e^y\dot{y},\ u=\dot{x}\dot{y}^{a-1},\ v=\frac{\dot{z}}{\dot{y}},\ \frac{dq}{dp}=e^y\left(\frac{\ddot{y}}{\dot{z}}+\frac{\dot{y}^2}{\dot{z}}\right),\\
&\frac{du}{dp}=\frac{\dot{y}^{(a-1)}\ddot{x}}{\dot{z}}+\frac{(a-1)\dot{x}\dot{y}^{(a-2)}\ddot{y}}{\dot{z}},\\
&\frac{dv}{dp}=\frac{\ddot{z}}{\dot{y}\dot{z}}-\frac{\ddot{y}}{\dot{y}^2}
\end{aligned}\tag{11}$$

The invariant representation in this case, in terms of $p, q, u, v, w, \frac{dq}{dp}, \frac{du}{dp}, \frac{dv}{dp}, \frac{d^2q}{dp^2}$ invariants, is given by

$$\begin{aligned}
\frac{du}{dp}&=f\,(q,u,v)\\
\frac{dq}{dp}&=g\,(q,u,v)\\
\frac{dv}{dp}&=h\,(q,u,v)\,.
\end{aligned}\tag{12}$$

The associated canonical form in terms of (x, y, z) coordinates is

$$\begin{aligned}
&\frac{\dot{y}^{(a-1)}\ddot{x}}{\dot{z}}+\frac{(a-1)\dot{x}\dot{y}^{(a-2)}\ddot{y}}{\dot{z}}=f\left(e^y\dot{y},\dot{x}\dot{y}^{a-1},\frac{\dot{z}}{\dot{y}}\right)\\
&e^y\left(\frac{\ddot{y}}{\dot{z}}+\frac{\dot{y}^2}{\dot{z}}\right)=g\left(e^y\dot{y},\dot{x}\dot{y}^{a-1},\frac{\dot{z}}{\dot{y}}\right)\\
&\frac{\ddot{z}}{\dot{y}\dot{z}}-\frac{\ddot{y}}{\dot{y}^2}=h\left(e^y\dot{y},\dot{x}\dot{y}^{a-1},\frac{\dot{z}}{\dot{y}}\right).
\end{aligned}\tag{13}$$

By adopting the same the procedure as described in the previous case 1, we also found that nonsingularity occurs for the invariants (10) and associated invariant system (13) admitting Lie algebra $A_{4,6}^{a,8}$.

By adopting the same procedure as utilized in preceding two cases of Lie algebras namely $A_{4,3}^{4}$ and $A_{4,6}^{a,8}$; invoking results of [2, 12], we classify nonsingular invariant system of three second-order ODEs admitting four-dimensional Lie algebras. A complete list of nonsingular 2nd-order differential invariants and their corresponding canonical forms for symmetry vectors of Lie algebras of dimension 4 and their associated nonsingular systems of three second-order ODEs are presented in Table 1. There are 87 such cases which are described in Table 1.

System of Three *kth*-Order ($k \geq 3$) ODEs

By employing a peculiar kind of invariant formulation that interrelates invariants via invariant differentiation, system of two kth-order ODEs admitting Lie algebras of dimension 4 have been constructed in [1]. We investigate the formulation of invariants for a system of three second order ODEs possessing Lie algebras of dimension 4, and utilize the notions $m_1,\ m_2,\ m_3,\ m_4,\ m_5$ and m_6. These contain $p, q, u, v, w, \frac{dq}{dp}, \frac{du}{dp}, \frac{dv}{dp}, \frac{d^2q}{dp^2}$ expressions similar to the system of three second order ODEs admitting Lie algebras of dimension 3 described in [12]. Here m_1 is a basic invariant, m_4, m_5 and m_6 are 2nd-order differential invariants in (1+3)-dimensional space.

We observe that if we differentiate the underlying three 2nd-order differential invariants; m_4, m_5 and m_6 with respect to the basic invariant m_1, we arrive at

$$\begin{aligned} \frac{dm_4}{dm_1} &= \frac{\frac{dm_4}{dp}}{\frac{dm_1}{dp}} \\ &= Dm_4, \\ \frac{dm_5}{dm_1} &= Dm_5, \\ \frac{dm_6}{dm_1} &= Dm_6 \end{aligned} \tag{14}$$

where $D = (D_p m_1)^{-1} D_p$ is the invariant differential operator. Thus the set of third-order differential invariants are

$$m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6,\ \frac{dm_4}{dm_1} = Dm_4,\ \frac{dm_5}{dm_1} = Dm_5,\ \frac{dm_6}{dm_1} = Dm_6. \tag{15}$$

In (15), $m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6,\ Dm_4,\ Dm_5,\ Dm_6$ consist of a set of invariants of the four-dimensional Lie algebra realization, where $Dm_4,\ Dm_5,\ Dm_6$ represent order three invariants. By utilizing these invariants and keeping the nonsingularity condition, nonsingular systems of three third-order ODEs can be constructed for the corresponding Lie algebras.

If we look at the symmetry algebra having realization $A_{4,3}^{4}$ in (1+3)-dimensional space, spanned by vector fields given by (3), the corresponding differential invariants are described in (6) and (7). We deduce the following:

$$\begin{aligned} Dm_4 &= f(m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6) \\ Dm_5 &= g(m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6) \\ Dm_6 &= h(m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6) \end{aligned} \tag{16}$$

In terms of $p, q, u, v, w, \frac{dq}{dp}, \frac{du}{dp}, \frac{dv}{dp}, \frac{d^2q}{dp^2}$ invariants, the system (16) assumes the following form

$$\begin{aligned} 2p\frac{du}{dp} + p^2\frac{d^2u}{dp^2} &= f(m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6) \\ \frac{1}{p}\frac{dw}{dp} - \frac{w}{p^2} &= g(m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6) \\ 2p\frac{d^2q}{dp^2} + p^2\frac{d^2q}{dp^2} &= h(m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6). \end{aligned} \tag{17}$$

In (x, y, z) coordinates, we find

$$\begin{aligned} \left(\frac{\ddot{x}}{\dot{y}} + \frac{t\dddot{x}}{\dot{y}} - \frac{t\ddot{x}\ddot{y}}{\dot{y}^2} - \frac{2}{t^2\dot{y}} - \frac{2\ddot{y}}{t\dot{y}^2} + \frac{\dddot{y}}{\dot{y}^2} - \frac{2\ddot{y}^2}{\dot{y}^3}\right)e^{(x-t\dot{x})} \\ - \left(\frac{t\ddot{x}}{t\dot{y}} + \frac{2\ddot{y}^2}{t\dot{y}^3\dot{z}^2} + \frac{2}{t\dot{y}} + \frac{t\ddot{y}}{\dot{y}^2}\right)t\ddot{x}e^{(x-t\dot{x})} &= f(m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6) \\ \left(\frac{\ddot{y}}{\dot{y}} - \frac{t\ddot{y}^2}{\dot{y}^2} + \frac{t\dddot{y}}{\dot{y}}\right) &= g(m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6) \\ 2t\ddot{z} + t^2\dddot{z} &= h(m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6) \end{aligned} \tag{18}$$

where (18) represents the canonical form for the system of three third-order ODEs admitting the Lie algebra of dimension 4 while (17) is the corresponding invariant representation. By employing the same the procedure as mentioned in the previous section (2), we also deduced that nonsingularity occurs for the invariants (15) and associated invariant system (18) possessing Lie algebra $A^4_{4,3}$. Proceeding in a similar manner, we can construct any nonsingular system of three kth-order ($k \geq 4$) ODEs which possesses a Lie algebra of dimension 4 as well.

Here $m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6$ form a basis of differential invariants for the system of three kth-order ($k \geq 4$) ODEs admitting Lie algebras of dimension 4. Moreover, these are formed by different combinations of $p, q, u, v, w, \frac{dq}{dp}, \frac{du}{dp}, \frac{dv}{dp}, \frac{d^2q}{dp^2}$ which forms a basis of differential invariants of kth-order for the associated Lie subalgebra of dimension 3. Hence, a basis of differential invariants for a Lie algebra of dimension 4 possessed by a canonical form for a system of three kth-order ODEs also admits a basis of invariants for the corresponding Lie subalgebra admitted by a canonical form for a system of three kth-order ODEs.

Dimensional Homogeneity Structure and Physical Systems

Generally, the aim of classification of differential equations can be categorized by two main types; one is related to its utilization in the mathematical sense and the second one is associated with its physical applications. In the former case, it is used to develop different mathematical formulas and theories in the applied sciences. In the latter case, it is employed in analysis of physical systems of several branches of applied sciences in the form of mathematical modelling. In the case of application of mathematical modelling to physical problems, dimensional homogeneity is required. Thus, if we want to utilize algebraic classification of differential equations in physical systems, then dimensional homogeneity structure plays a momentous role in the classified canonical forms.

In this work, we are introducing dimensional homogeneity structure in canonical forms for invariant systems of three second order ODEs possessing 4-dimensional Lie algebras as presented in Table 1. In these canonical forms, arbitrary functions are given in the form of f, g and h. We fix f, g and h as dimensional constants so that the corresponding dimensional homogeneity structure is satisfied in the underlying systems. In this way we obtain a list of canonical forms for invariant systems of three second order ODEs which may be used to depict physical systems presented in Table 2. To the best of our knowledge, this is performed for the first time in algebraic classification of differential equations. Moreover, this can be formulated for the classification problem of any system of differential equations provided that conditions of dimensional homogeneity are satisfied.

- In the case $A_{4,6}^{1,3}$ taken from Table 1, we have the nonsingular system of three second-order ODEs given by equations

$$\begin{aligned}\ddot{x} &= \dot{z} f\left(z, \dot{x}, \frac{\dot{y}}{\dot{z}}\right)\\ \ddot{y} &= \dot{y}^2 g\left(z, \dot{x}, \frac{\dot{y}}{\dot{z}}\right)\\ \ddot{z} &= \frac{\dot{z}\ddot{y}}{\dot{y}} + \frac{\dot{z}^3}{\dot{y}} h\left(z, \dot{x}, \frac{\dot{y}}{\dot{z}}\right)\end{aligned} \tag{19}$$

If we take in particular $f = k_1$ and $g = h = k_2$, then the system (19) takes the form

$$\begin{aligned}\ddot{x} &= k_1 \dot{z}\\ \ddot{y} &= k_2 \dot{y}^2\\ \ddot{z} &= k_2\left(\dot{z}\dot{y} - \frac{\dot{z}^3}{\dot{y}}\right)\end{aligned} \tag{20}$$

The system (20) represents the physical system of three interacting particles; in which the first particle is facing a resistance equal to linear velocity of the third particle, the 2nd particle is facing a drag resistance and the third particle is also facing a resistance which is a combination of the product of velocities of the 2nd and third particles, and also cubic power of velocity of the third particle. Here k_1 and k_2 are constants possessing dimensions of $\frac{1}{T}$, $\frac{1}{L}$, respectively.

- In the case $A_{4,2}^{2}$ taken from Table 1, we have the nonsingular system of three second-order ODEs given by equations

$$\begin{aligned}\ddot{x} &= \dot{x}^2 f\left(z, \frac{\dot{x}}{\dot{y}}, \frac{\dot{y}}{\dot{z}}\right)\\ \ddot{y} &= \frac{\dot{y}\ddot{x}}{\dot{x}} + \frac{\dot{y}^2\dot{z}}{\dot{x}} g\left(z, \frac{\dot{x}}{\dot{y}}, \frac{\dot{y}}{\dot{z}}\right)\\ \ddot{z} &= \frac{\ddot{y}}{\dot{y}\dot{z}} + \frac{\dot{z}^3}{\dot{y}} h\left(z, \frac{\dot{x}}{\dot{y}}, \frac{\dot{y}}{\dot{z}}\right)\end{aligned} \tag{21}$$

By taking $f = g = h = k_1$, then the system (21) takes the form

$$\begin{aligned}\ddot{x} &= \dot{x}^2 k_1\\ \ddot{y} &= \frac{\dot{y}\ddot{x}}{\dot{x}} + \frac{\dot{y}^2\dot{z}}{\dot{x}} k_1\\ \ddot{z} &= \frac{\ddot{y}}{\dot{y}\dot{z}} + \frac{\dot{z}^3}{\dot{y}} k_1\end{aligned} \tag{22}$$

where k_1 is a constant possessing dimensions of $\frac{1}{L}$. But the term $\frac{\ddot{y}}{\dot{y}\dot{z}}$ in the third equation of the system (22) has dimension $\frac{1}{L}$ different from the rest of the terms, so the system (22) does not possess dimensional homogeneity structure.

Remark

There are 25 such cases of canonical forms for nonsingular system of three second-order ODEs admitting Lie algebras of dimension 4 which possess dimensional homogeneity structure by setting arbitrary functions namely, f, g, h as dimensional constants. These are provided in the descriptive form of a list in Table 2. This description may be utilized in several physical problems of mechanics. The rest of the 62 cases given in Table 1, do not possess such type of dimensional homogeneity structure in the underlying restricted form of arbitrary functions namely, f, g, h. In this research work, we have described only those cases which have significance either classification point of view or dimensional homogeneity structure.

In view of the preceding discussions, the above observations for nonsingular system of three second-order ODEs admitting a Lie algebra of dimension 4 can be stated in the form of the following propositions.

Proposition 1 *A system of three second-order ODEs given in terms of nonsingular invariant equations is reducible to dimensional homogeneity structure by point transformation if and only if it admits one of the Lie algebras provided that the arbitrary functions namely, f, g, h are set as dimensional constants*

$A^4_{4,1}, A^3_{4,2}, A^4_{4,2}, A^5_{4,3}, A^1_{4,4}, A^3_{4,4}, A^{1,2}_{4,6}, A^{1,3}_{4,6}, A^{a,(2)}_{4,6}, A^{a,4}_{4,6}, A^{a,6}_{4,6}, A^2_{4,8}, A^7_{4,8}, A^2_{4,12}$, $A^4_{4,12}, A^{a,b,(2)}_{4,14}, A^{a,a,(2)}_{4,14}, A^{a,a,(3)}_{4,14}, A^{a,a,(4)}_{4,14}, A^{a,1,(2)}_{4,14}, A^{a,1,(3)}_{4,14}, A^{a,1,(4)}_{4,14}, A^{b,(1)}_{4,18}, A^{b,(2)}_{4,18}, A^5_{4,20}$.

The proofs of the above follow directly from Table 2.

Integrability and Canonical Forms

In the frame of Lie symmetry analysis, the classical approach of reduction of order does not work for the case of system of ODEs in same manner as it deployed for scalar ODEs [15]. On the other hand, the Lie algebraic classification approach as introduced by Lie himself [8] are equally valid for both scalar ODEs as well as system of ODEs, provided that corresponding equations are classified for their admittance of Lie algebras. In the case of a system of ODEs, algebraic classification for the underlying system of differential equations is introduced to tackle this hurdle as an alternative approach. Initially it was attempted by Wafo and Mahomd in [15], then its important aspects as realizations of Lie algebras were updated by [10]. Furthermore, by utilizing results of [10], the invariant representation is also modified and several applications of this approach has been performed by Ayub et al. (see [2–4]). These applications include system of two second-order ODEs admitting a Lie algebras of dimension not greater than four and the integrability aspect is also analysed in a detailed manner. Singularity in invariant structure for a four dimensional Lie algebras, is also tackled in [3] and [4] for both (1+2)- and (1+3)-dimensional spaces respectively. A lot of research in the literature has also been formulated in this direction due to the vast applications of systems of three second-order ODEs (see [12–14]).

There are two natural questions that arise in the process of adoption of the approach of algebraic classification of differential equations. The first one is how to enlarge the number

of equations in the underlying systems as well as dimension of Lie algebras and the second is how to extend its applications to physical systems?

In the current research, we are focussed on both these aspects; in the first case, we have classified nonsingular differential invariants of order two in (1+3)-dimensional space for a Lie algebras of dimension 4 and deduced the corresponding canonical forms for nonsingular system of three second-order ODEs. There are 81 such cases as presented in Table 1. Generally, the integrability of these classified canonical forms vary case to case due to arbitrariness in functions, namely f, g, h. However, we have introduced the dimensional homogeneity structure in these classified canonical forms by setting arbitrary functions f, g, h as dimensional constants. On the basis of this new structure, we can correlate them with physical systems of applied science and there are 25 such cases as described in Table 2.

On the basis of canonical forms presented in Table 2, we provide the following two approaches for their integration procedure.

(a). General integration approach
(b). Differential invariant approach

(a) General Integration Approach

In this approach, we prudently subdivide all the canonical forms of Table 2 into three types.

Type-I

In this type, those cases are considered which are completely integrable. We illustrate this by an example.

We take the case of $A_{4,6}^{1,3}$ with associated system of three second-order ODEs presented in (19), which after fixing the values of the arbitrary functions f, g and h, enables the simplified form (20).

After solving system (20) simultaneously, we arrive at the following solution of the system (20)

$$x = \frac{k_1}{k_2^2} e^{\frac{c_3}{2}} \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\sqrt{c_3 - 2\ln(c_1 - k_2 t)}\right) - \frac{k_1}{k_2^2}\sqrt{c_3 - 2\ln(c_1 - k_2 t)}\,(c_1 - k_2 t) + t c_5 + c_6$$

$$y = -\frac{1}{k_2}\ln(c_1 - k_2 t) + c_2$$

$$z = \frac{1}{k_2}\sqrt{c_3 - 2\ln(c_1 - k_2 t)} + c_4$$

where $c_i, i = 1, 2, ...6$ are arbitrary constants.

By careful observation, we deduce that the following cases of Lie algebras given in Table 2 are included in Type 1.

$$A_{4,1}^{4},\ A_{4,2}^{3},\ A_{4,2}^{4},\ A_{4,3}^{5},\ A_{4,6}^{1,3},\ A_{4,14}^{a,a,(3)},\ A_{4,14}^{a,a,(4)},\ A_{4,14}^{a,1,(3)},\ A_{4,14}^{a,1,(4)},\ A_{4,18}^{b,(1)},\ A_{4,18}^{b,(2)}.$$

Type-II

In this type, those cases of Table 2 are included in which one equation is integrable but the integrability of the canonical form for the underlying system of ODEs depends on the remaining system of two second-order ODEs. $A_{4,4}^{1}$ & $A_{4,20}^{5}$ are two such cases that exist in Table 2.

Type-III

In this type, the rest of the canonical forms of Table 2 are included in this type and their integrability vary from case to case.

(b) Differential Invariant Approach

In this approach, we invoke the basis of invariants denoting the canonical forms given in Table 1. For each nonsingular canonical form, we obtain the reduced system in terms of a first-order differential equation by utilizing the basis of invariants.

In this process, for each case of nonsingular canonical forms listed in Table 2, possess a set of differential invariants with underlying Lie algebra presented in Table 1. The invariant system for each case is expressed in terms of these classified differential invariants, when we take arbitrary functions f, g and h as constants (dimensional or non-dimensional), then we can formulate the second-order differential invariants as

$$\begin{aligned} m_4 &= k_1, \\ m_5 &= k_2, \\ m_6 &= k_3, \end{aligned} \tag{23}$$

where m_4, m_5, m_6 denote the second-order differential invariants and k_1, k_2, k_3 are constants (dimensional or non-dimensional) which may or may not be linearly independent of each other.

The classical reduction process is employed on the investigated invariant system (23). Ultimately we get a reduced system of first-order differential equations which can be further simplified and leads to the solution of the underlying nonsingular canonical form.

We explain this procedure by examples. Let us take system of three particles and x, y, z denote them, here we investigate the mechanics of these particles.

Problem-1

Let us investigate the system of three interacting particles denoted by x, y and z, whose motion is described by the following system of following three second-order ODEs

$$\begin{aligned} \ddot{x} &= -k_1\dot{x}^3 \\ \ddot{y} &= -k_1\dot{y}\dot{x}^2 - k_2\frac{\dot{z}\dot{y}^2}{\dot{x}} \\ \ddot{z} &= -k_2\left(\frac{\dot{z}^3}{\dot{y}} + \frac{\dot{y}\dot{z}^2}{\dot{x}}\right) - k_1\dot{z}\dot{x}^2 \end{aligned} \tag{24}$$

where k_1 and k_2 are arbitrary constants whose dimensions are $\frac{T}{L^2}$, $\frac{1}{L}$, respectively. The system (24) admits the symmetry Lie algebra $A_{4,4}^1$ spanned by generators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial t}, \quad X_4 = \frac{\partial}{\partial y}. \tag{25}$$

The system (24) is given in Table 2 and corresponding differential invariants are presented in Table 1 and its invariant form for (24) is written as

$$w = k_1$$

$$\frac{du}{dp} = k_2 \tag{26}$$
$$\frac{d^2q}{dp^2} = k_2.$$

Where $p,\ u,\ q,\ \frac{dq}{dp},\ w,\ \frac{du}{dp},\ \frac{d^2q}{dp^2}$ are described in Table 1.

From first equation of (3.4), we obtain $w = k_1$ (27)

By utilizing Table 1 and performing integration, we get

$$\dot{x} = \frac{1}{\sqrt{2}}(k_1 t + c_1)^{-\frac{1}{2}} \tag{28}$$

Further simplification leads to $x = \frac{\sqrt{2}}{k_1}(k_1 t + c_1)^{\frac{1}{2}} + c_2$ (29)

In similar manner, from (26), we deduced that

$$\frac{d^2q}{dp^2} = k_2 \tag{30}$$

Performing integration, we have $q = k_2\frac{p^2}{2} + c_3 p + c_4 pt$ (31)

Invoking Table 1, (31) becomes

$$y = k_2\frac{z^2}{2} + c_3 z + c_4 \tag{32}$$

Proceeding in similar manner, from (26), we have

$$\frac{du}{dp} = k_2$$

Integration gives that

$$u = k_2 p + c_5 \tag{33}$$

Using Table 1, we arrived at

$$\frac{\dot{x}}{\dot{y}} = k_2 z + c_5 \tag{34}$$

Invoking (32), we deduced that

$$\dot{y} = (k_2 z^2 + c_3)\dot{z} \tag{35}$$

By using (28) and (35) in (34), we get

$$\frac{1}{\sqrt{2}}(k_1 t + c_1)^{-\frac{1}{2}} = (k_2 z + c_5)(k_2 z + c_6)\dot{z} \tag{36}$$

Performing some manipulations on (36), we obtained

$$\frac{1}{3}k_2^2 z^3 + \frac{1}{2}(c_3 + c_5)k_2 z^2 + c_3 c_5 z = \frac{\sqrt{2}}{k_1}(k_1 t + c_1)^{\frac{1}{2}} + c_6, \tag{37}$$

Fig. 1 $k_2 = 0.01$

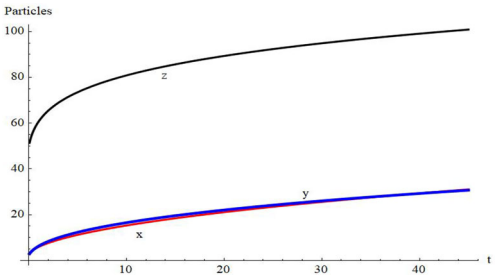


Fig. 2 $k_2 = 0.056$

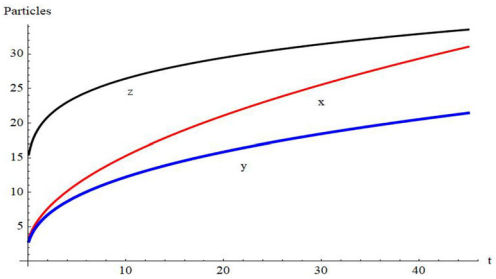


Fig. 3 $k_2 = 0.10$

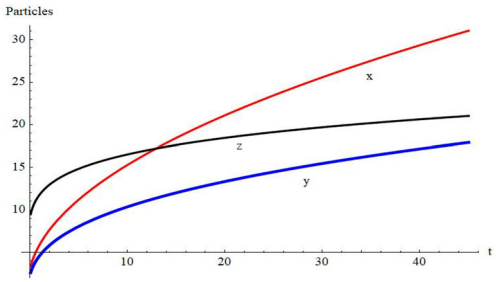
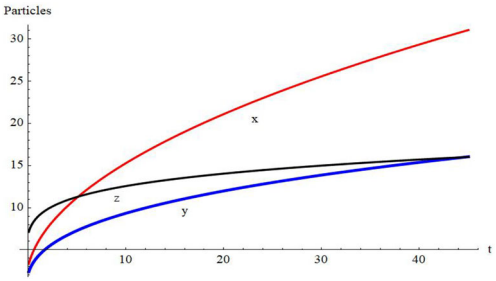


Fig. 4 $k_2 = 0.15$



where c_1, c_2, c_3, c_4, c_5 and c_6 are arbitrary constants with (29), (32) and (37) constituting the solution for system (24).

Graphical Analysis

Variation of k_2 on mechanics of particles for $c_1 = 0.2, c_2 = -c_1, c_3 = 0.01, c_4 = 0.02, c_5 = 1, c_6 = 0.1, k_1 = 0.1$

In the Problem-1, x, y and z denote the investigated system of three particles. For set of values namely $c_1 = 0.2, c_2 = -c_1, c_3 = 0.01, c_4 = 0.02, c_5 = 1, c_6 = 0.1, k_1 = 0.1$, we looked the variation of k_2 on mechanics of underlying particles.

Fig. 5 $k_2 = 0.7$

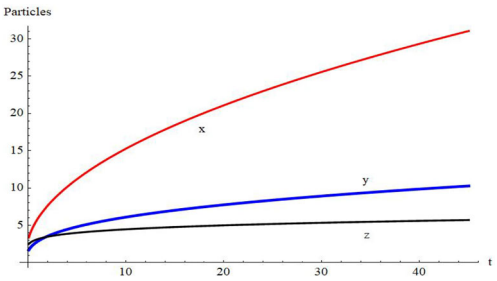


Fig. 6 $k_2 = 5.0$

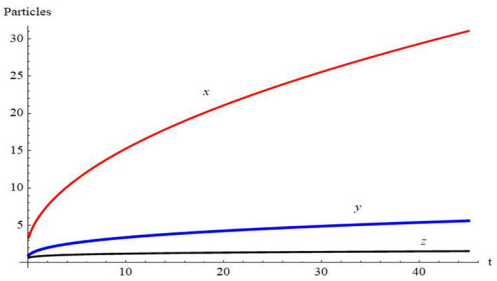
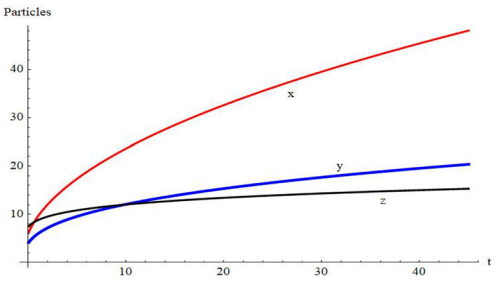


Fig. 7 $k_1 = 0.01$



In Fig. 1; we found that for $k_2 = 0.01$, the $x-$ and $y-$ particles are overlapping and while the $z-$ particle is uncoupled from others two. And also from Figs. 2, 3 and 4, the coupling process of underlying particles is started from $k_2 = 0.056$ up to $k_2 = 0.015$ and decoupling is started from $k_2 = 0.7$ to onwards (see Figs. 2, 3, 4, 5 and 6).

In similar manner for a set of values $c_1 = 0.2, c_2 = -c_1, c_3 = 0.01, c_4 = 0.02, c_5 = 1, c_6 = 0.1, k_2 = 0.2$, we inspected the variation of k_1 on motion of particles.

We deduced that the coupling process of underlying particles is started from $k_1 = 0.01$ up to $k_1 = 0.08$ and decoupling is started from $k_1 = 2.0$ to onwards (see Figs. 7, 8, 9, 10, 11 and 12).

Problem-2

Consider the system of three interacting particles denoted by x, y and z, whose motion is described by the following system of three second-order ODEs

$$\begin{aligned} \ddot{x} &= -k_1\dot{x}^2 \\ \ddot{y} &= -k_1\dot{x}\dot{y} + k_2\dot{x}^{(2-a)} \\ \ddot{z} &= -k_1\dot{x}\dot{z} + k_1\dot{x}^2. \end{aligned} \tag{38}$$

Fig. 8 $k_1 = 0.03$

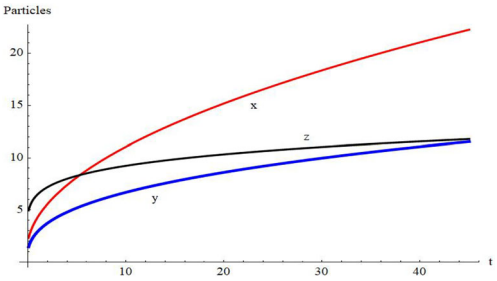


Fig. 9 $k_1 = 0.05$

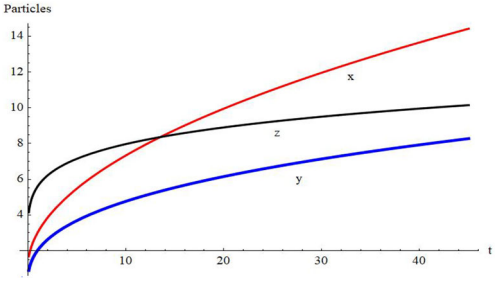


Fig. 10 $k_1 = 0.15$

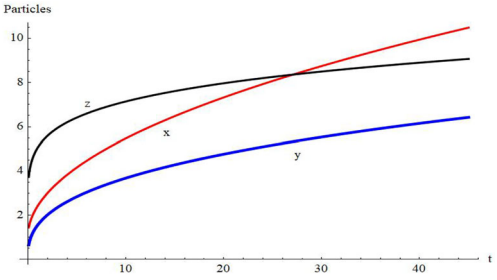
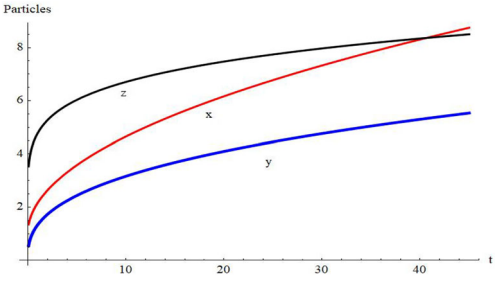
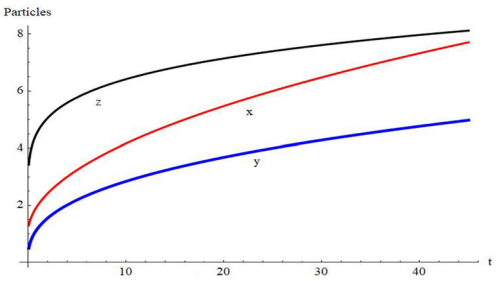


Fig. 11 $k_1 = 0.8$



where $-1 \leq a < 1$, $a \neq 0$ and k_1, k_2 are constants having dimensions $\frac{1}{L}$, $\frac{L^{a-1}}{T^a}$, respectively. The first particle is facing a drag resistance, the second a resistance equal to the product of velocities of the first and second particles as well as the "2-a" power of velocity of the first particle. The third particle is facing a resistance equal to the product of velocities of the first and third particles as well as a drag resistance of velocity of first particle.

Fig. 12 $k_1 = 2.0$



The system (38) admits the symmetry Lie algebra $A_{4,14}^{a,a}$ spanned by generators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial y}, \quad X_4 = t\frac{\partial}{\partial t} + ay\frac{\partial}{\partial y}. \tag{39}$$

By employing the following invertible transformations

$$\bar{t} = x, \ \bar{x} = t, \ \bar{y} = y, \ \bar{z} = z \tag{40}$$

we deduce the corresponding class of realization for this Lie algebra viz., $A_{4,14}^{a,a,4}$. Moreover, for the Lie algebra $A_{4,14}^{a,a,4}$, (38) the following form of the associated system occurs, viz.

$$\begin{aligned} \ddot{x} &= k_1\dot{x} \\ \ddot{y} &= k_2\dot{x}^a \\ \ddot{z} &= k_1, \end{aligned} \tag{41}$$

which is equivalent to the underlying system of $A_{4,14}^{a,a,4}$ presented in Table 2. The corresponding invariant form for (41) is

$$\begin{aligned} \frac{1}{u}\frac{du}{dp} &= k_1 \\ \frac{1}{u^a}w &= k_2 \\ \frac{d^2q}{dp^2} &= k_1. \end{aligned} \tag{42}$$

where p, u, q, $\frac{dq}{dp}$, w, $\frac{du}{dp}$, $\frac{d^2q}{dp^2}$ are described in Table 1. By utilizing Table 1 and adopting the procedure mentioned in Problem 1, we solve the invariant system (42). We find the solution of (41). Reverting back to the original variables via transformation (40), we obtain the following form

$$t = \frac{c_1}{k_1}e^{k_1x} \tag{43}$$

$$y = \frac{c_2c_1^2}{a^2k_1^2}e^{ak_1x} + c_5x + c_6 \tag{44}$$

$$z = \frac{k_1x^2}{2} + c_3x + c_4 \tag{45}$$

where c_1, c_2, c_4, c_5 and c_6 are arbitrary constants, with (43), (44) and (45) constituting the solution of the investigated system (38).

Fig. 13 $k_1 = 0.01$

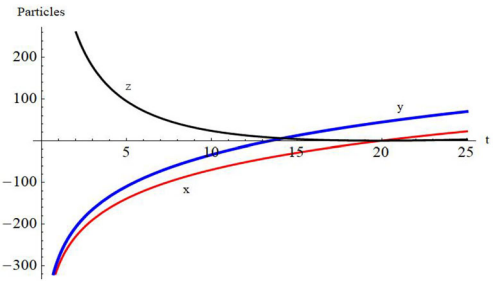


Fig. 14 $k_1 = 0.03$

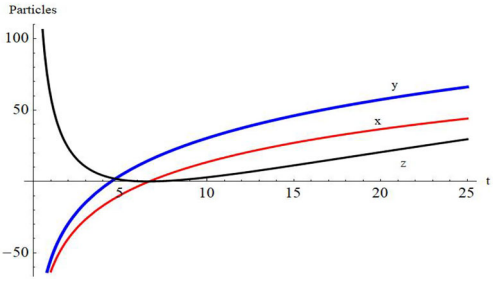


Fig. 15 $k_1 = 0.05$

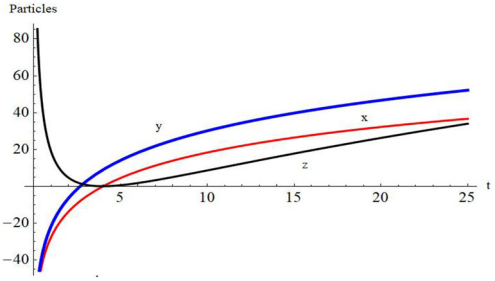
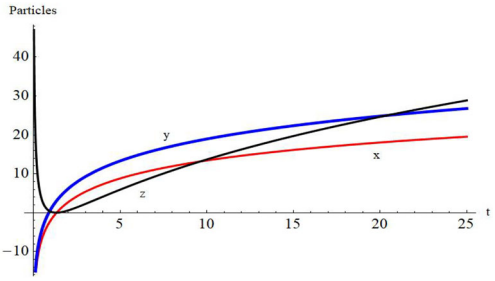


Fig. 16 $k_1 = 0.15$



Graphical Analysis

x, y and z representing the system of three particles under consideration. For a set of values $a = 0.3, c_1 = 0.2, c_2 = 1, c_3 = 0.01, c_4 = 0.02, c_5 = 1, c_6 = 0.1$, we examined the variation of k_1 on motion of underlying particles in this problem. We determined that the coupling process of underlying particles is started from $k_1 = 0.01$ up to $k_1 = 0.8$ and decoupling is started from $k_1 = 5.0$ to onwards, i.e, (see Figs. 13, 14, 15, 16, 17 and 18).

Fig. 17 $k_1 = 0.8$

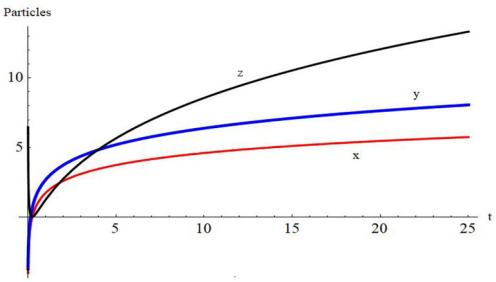
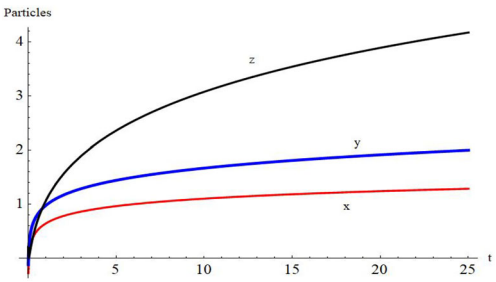


Fig. 18 $k_1 = 5.0$



Conclusions

Symmetry analysis is one of the important approaches which is widely used for the analysis of differential equations associated with underlying physical systems. The classical approach of symmetry analysis is not easily suited to successive reduction of order for a system of differential equations. That is why the classification approach in symmetry analysis is utilized to overcome reduction procedures [1–5, 9, 12, 13, 15].

In this study, nonsingularity in invariant structure for systems of three second order ODEs admitting four-dimensional Lie algebras has been investigated in detail. We have presented an approach comprising an algorithm which is utilized to obtain a complete set of kth-order ($k \geq 3$) differential invariants consisting of bases of invariants corresponding to vector fields in (1+3)-dimensional space of four-dimensional real Lie algebras. In addition, a set of nonsingular 2nd-order differential invariants associated to symmetry vectors of four-dimensional Lie algebras in (1+3)-dimensional space and their admitted corresponding canonical forms for nonsingular system of three second order ODEs are provided.

The formulation of dimensional homogeneity for the underlying system of equations provides a relation between mathematical modelling and physical application; this has a significant role in the analysis of many hidden aspects such as classification, linearization, singularity and analyticity. The nonsingular invariant systems presented in Table 1 are further classified on the basis of their dimensional homogeneity structure and presented in Table 2. These classified systems lead to description of several physical features namely mechanics of particles, small vibrations and biological systems, etc. The dimensional homogeneity structure is a new contribution in algebraic classifications of differential equations and fruitful in analysis of many physical systems. In this research work, we have classified nonsingular differential invariants in (1+3)-dimensional space on the basis of their dimensional homogeneity structure for 4 dimensional Lie algebras. We have determined that there are only 25 such cases of canonical forms for nonsingular system of three second-order ODEs admitting Lie algebras of dimension 4 which possess dimensional homogeneity structure by setting

Table 1 Second-order differential invariants and equations

Algebra	N	Invariants
$A_{4,1}$	1	$p=t,\ q=z,\ u=\dot{y}-\dot{x}\phi'(p),\ \frac{dq}{dp}=\dot{z},\ w=\ddot{x},\ \frac{du}{dp}=\ddot{y}-\ddot{x}\phi'(p)-\dot{x}\phi''(p),\ \frac{d^2q}{dp^2}=\ddot{z}$
		$m_1=p,\ m_2=q,\ m_3=\frac{dq}{dp},\ m_4=w-\frac{\psi''(p)u}{k'(p)-\psi'(p)\phi'(p)},\ m_5=\frac{du}{dp}-\frac{(k''(p)-\psi'(p)\phi''(p)-\psi''(p)\phi'(p))u}{k'(p)-\psi'(p)\phi'(p)}$
		$m_6=\frac{d^2q}{dp^2}$
		$\ddot{x}=\frac{-\phi'(t)\psi''(t)\dot{x}}{k'(t)-\psi'(t)\phi'(t)}+\frac{\psi''(t)\dot{y}}{k'(t)-\psi'(t)\phi'(t)}+f(p,q,\frac{dq}{dp}),$
		$\ddot{y}=\frac{(\phi''(t)k'(t)-\phi'(t)k''(t))\dot{x}}{k'(t)-\psi'(t)\phi'(t)}-\frac{(\phi''(t)\psi'(t)-k''(t))\dot{y}}{k'(t)-\psi'(t)\phi'(t)}+g(p,q,\frac{dq}{dp}),\ \ \ddot{z}=h(p,q,\frac{dq}{dp})$
	4	$p=z,\ q=\dot{x},\ u=\dot{y},\ v=\dot{z},\ \frac{dq}{dp}=\frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp}=\frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{z}}{\dot{z}}$
		$m_1=q,\ \ m_2=u,\ \ m_3=v,\ m_4=\frac{dq}{dp},\ m_5=\frac{du}{dp},\ m_6=\frac{dv}{dp}$
		$\ddot{x}=\dot{z}f(q,u,v),\ \ \ddot{y}=\dot{z}g(q,u,v),\ \ \ddot{z}=\dot{z}h(q,u,v)$
	5	$p=z,\ q=\dot{x},\ u=\dot{y},\ v=\dot{z},\ \frac{dq}{dp}=\frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp}=\frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{z}}{\dot{z}}$
		$m_1=p,\ \ m_2=\frac{q-\psi'(p)}{v},\ \ m_3=\frac{u-\psi'(p)}{v},\ m_4=\frac{1}{v^2}\frac{dv}{dp},\ m_5=\frac{1}{v}\frac{dq}{dp}-\frac{\psi''(p)}{v}-\frac{(q-\psi'(p))}{v^2}\frac{dv}{dp},$
		$m_6=\frac{1}{v}\frac{du}{dp}-\frac{\psi''(p)}{v}-\frac{(u-\psi'(p))}{v^2}\frac{dv}{dp}$
		$\dot{z}\ddot{x}-\dot{z}^2\psi''(t)-(\dot{x}-\psi'(t))=\dot{z}^3f(p,\frac{q-\psi'(p)}{v},\frac{u-\psi'(p)}{v}),$
		$\dot{z}\ddot{y}-\dot{z}^2\psi''(t)-(\dot{x}-\psi'(t))=\dot{z}^3g(p,\frac{q-\psi'(p)}{v},\frac{u-\psi'(p)}{v}),\ \ \ddot{z}=\dot{z}^3h(p,\frac{q-\psi'(p)}{v},\frac{u-\psi'(p)}{v})$
$A_{4,2}$	2	$p=z,\ q=y,\ \ u=\frac{\dot{x}}{\dot{y}},\ \frac{dq}{dp}=\frac{\dot{y}}{\dot{z}},\ w=\frac{\ddot{x}}{\dot{x}^2},\ \frac{du}{dp}=\frac{\ddot{x}}{\dot{y}\dot{z}}-\frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{d^2q}{dp^2}=\frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1=p,\ \ m_2=u,\ \ m_3=\frac{dq}{dp},m_4=w,\ h_5=\frac{du}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$\ddot{x}=\dot{x}^2f(p,u,\frac{dq}{dp}),\ \ \ddot{y}=\frac{\dot{y}\ddot{x}}{\dot{x}}+\frac{\dot{y}^2\dot{z}}{\dot{x}}g(p,u,\frac{dq}{dp}),\ \ \ddot{z}=\frac{\ddot{y}}{\dot{y}\dot{z}}+\frac{\dot{z}^3}{\dot{y}}h(p,u,\frac{dq}{dp})$
	3	$p=z,\ q=y,\ u=t\dot{y},\ v=t\dot{z},\ w=t\ddot{x},\ \frac{du}{dp}=\frac{\dot{y}}{\dot{z}}+\frac{t\ddot{y}}{\dot{z}},\ \frac{dv}{dp}=1+\frac{t\ddot{z}}{\dot{z}}$
		$m_1=p,\ \ m_2=u,\ \ m_3=v,\ m_4=w,\ m_5=\frac{du}{dp},\ m_6=\frac{dv}{dp}$
		$t\ddot{x}=f(p,u,v),\ \ \frac{\dot{y}}{\dot{z}}+\frac{t\ddot{y}}{\dot{z}}=g(p,u,v),\ \ 1+\frac{t\ddot{z}}{\dot{z}}=h(p,u,v)$

Table 1 continued

Algebra	N	Invariants
	4	$p=t,\ q=z,\ u=\dot{y},\ \frac{dq}{dp}=\dot{z},\ w=\frac{\ddot{y}}{\dot{y}^2}-\frac{\ddot{x}}{\dot{x}\dot{y}},\ \frac{du}{dp}=\ddot{y},\ \frac{d^2q}{dp^2}=\dddot{z}$
	*	$m_1=p,\ \ m_2=q,\ m_3=\frac{dq}{dp},\ \ m_4=-\ddot{\mu}\ln u+w,\ m_5=\frac{1}{u}\frac{du}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$\ddot{x}=\dot{x}f(p,u,\frac{dq}{dp}),\ \ \ddot{y}=\ddot{\mu}\ln\dot{x}+g(p,u,\frac{dq}{dp}),\ \ \dddot{z}=h(p,u,\frac{dq}{dp})$
	5	$p=z,\ q=\dot{x},\ u=\dot{y},\ v=\dot{z},\ \frac{dq}{dp}=\frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp}=\frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp}=\frac{\dddot{z}}{\dot{z}}$
	*	$m_1=qe^p,\ \ m_2=ue^p,\ m_3=ve^p,\ \ m_4=e^p\frac{dq}{dp},\ m_5=e^p\frac{du}{dp},\ m_6=e^p\frac{dv}{dp}$
		$\ddot{x}=e^{-z}\dot{z}f(qe^p,ue^p,ve^p),\ \ \ddot{y}=e^{-z}\dot{z}g(qe^p,ue^p,ve^p),\ \ \dddot{z}=e^{-z}\dot{z}h(qe^p,ue^p,ve^p)$
	6	$p=z,\ q=\dot{x},\ u=\dot{y},\ v=\dot{z},\ \frac{dq}{dp}=\frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp}=\frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp}=\frac{\dddot{z}}{\dot{z}}$
	*	$m_1=p,\ \ m_2=\frac{q}{v}+\ln v,\ m_3=\frac{u}{v}+\dot{\phi}(t)\ln v,\ \ m_4=\frac{1}{v}\frac{dv}{dp},$
		$m_5=\frac{1}{v}\frac{dq}{dp}+\frac{1}{v}\frac{dv}{dp}\ln v,\ m_6=\frac{1}{v}\frac{du}{dp}+(\frac{\dot{\phi}(t)}{v}\frac{dv}{dp}+\ddot{\phi}(t))\ln v$
		$\ddot{x}=-\dddot{z}\ln\dot{z}+\dot{z}^2f(p,\frac{q}{v}+\ln v,\frac{u}{v}+\dot{\phi}(t)\ln v),$
		$\ddot{y}=-\dot{\phi}(t)\dddot{z}-\dot{z}^2\ddot{\phi}(t)\ln\dot{z}+\dot{z}^2g(p,\frac{q}{v}+\ln v,\frac{u}{v}+\dot{\phi}(t)\ln v),\ \ \dddot{z}=\dot{z}^2h(p,\frac{q}{v}+\ln v,\frac{u}{v}+\dot{\phi}(t)\ln v)$
$A_{4,3}$	3	$p=z,\ u=\dot{y}e^y,\ u=\frac{\dot{x}}{\dot{y}},\ v=\frac{\dot{z}}{\dot{y}},\ \frac{dq}{dp}=\frac{\ddot{x}}{\dot{y}\dot{z}}-\frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{du}{dp}=(\ddot{y}+\dot{y}^2)\frac{e^y}{\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{z}}{\dot{z}^2}-\frac{\dot{x}\dddot{z}}{\dot{z}^3}$
		$m_1=p,\ m_2=\frac{u^c}{q},\ \ m_3=\frac{u}{v},\ \ m_4=q^{-1}\frac{dq}{dp},\ m_5=q^{-1/c}\frac{du}{dp},\ m_6=q^{-1/c}\frac{dv}{dp}$
		$\frac{\ddot{x}}{\dot{y}\dot{z}}-\frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}}=(\dot{y}e^y)^{1/c}f(p,\frac{u^c}{q},\frac{u}{v}),\ \ (\ddot{y}+\dot{y}^2)\frac{e^y}{\dot{z}}=(\dot{y}e^y)g(p,\frac{u^c}{q},\frac{u}{v}),\ \ \frac{\ddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{z}^3}=(\dot{y}e^y)^{1/c}h(p,\frac{u^c}{q},\frac{u}{v})$
	4	$p=t,\ q=z,\ u=-\frac{1}{t^2\dot{y}}e^{(x-t\dot{x})},\ \frac{dq}{dp}=\dot{z},\ w=2t+t^2\frac{\ddot{y}}{\dot{y}},$
		$\frac{du}{dp}=(\frac{\ddot{x}}{t\dot{y}}+\frac{2}{t^3\dot{y}}+\frac{\ddot{y}}{t^2\dot{y}^2})e^{(x-t\dot{x})},\ \frac{d^2q}{dp^2}=\dddot{z}$
		$m_1=p,\ m_2=pu,\ m_3=p\frac{dq}{dp},\ \ m_4=w/p,\ \ m_5=p^2\frac{du}{dp},\ m_6=p^2\frac{d^2q}{dp^2}$
		$(\frac{t\ddot{x}}{\dot{y}}+\frac{2}{t\dot{y}}+\frac{\ddot{y}}{\dot{y}^2})e^{(x-t\dot{x})}=f(q,pu,p\frac{dq}{dp}),\ \ 2+t\frac{\ddot{y}}{\dot{y}}=g(q,pu,p\frac{dq}{dp}),\ \ t^2\dddot{z}=h(q,pu,p\frac{dq}{dp})$
	5	$p=t,\ q=z,\ u=\dot{x},\ \frac{dq}{dp}=\dot{z},\ w=\frac{\ddot{x}}{\dot{x}^2}-\frac{\ddot{y}}{\dot{x}\dot{y}},\ \frac{du}{dp}=\ddot{x},\ \frac{d^2q}{dp^2}=\dddot{z}$

Table 1 continued

Algebra	N	Invariants
		$m_1 = p,\ m_2 = q,\ m_3 = \frac{dq}{dp},\ \ m_4 = uw,\ \ m_5 = \frac{1}{u}\frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = \dot{x}\dot{z}f(p, q, \frac{dq}{dp}),\ \ \ddot{y} = \frac{\dot{y}}{\dot{x}}\ddot{x} + \dot{y}g(p, q, \frac{dq}{dp}),\ \frac{-\ddot{z}}{\dot{z}^3} = h(p, q, \frac{dq}{dp})$
	6	$p = z,\ q = e^y\dot{y},\ u = \frac{\dot{x}}{\dot{y}},\ v = \frac{\dot{x}}{\dot{z}},\ \frac{dq}{dp} = e^y(\frac{\ddot{y}}{\dot{z}} + \frac{\dot{y}^2}{\dot{z}}),\ \frac{du}{dp} = \frac{\ddot{x}}{\dot{y}\dot{z}} - \frac{\dot{x}\ddot{y}}{\dot{z}\dot{y}^2},\ \frac{dv}{dp} = \frac{\ddot{x}}{\dot{z}^2} - \frac{\dot{x}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = q,\ m_3 = \frac{v}{u},\ \ m_4 = \frac{dq}{dp},\ \ m_5 = \frac{1}{u}\frac{du}{dp},\ m_6 = \frac{1}{u}\frac{dv}{dp}$
		$e^y(\frac{\ddot{y}}{\dot{z}} + \frac{\dot{y}^2}{\dot{z}}) = f(p, q, \frac{v}{u}),\ \ \frac{\ddot{x}}{\dot{x}\dot{z}} - \frac{\dot{x}\ddot{y}}{\dot{z}\dot{y}^2} = g(p, q, \frac{v}{u}),\ \frac{\dot{y}\ddot{x}}{\dot{x}\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3} = h(p, q, \frac{v}{u})$
$A_{4,4}$	1	$p = z,\ q = y,\ u = \frac{\dot{x}}{\dot{y}},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ w = -\frac{\ddot{x}}{\dot{x}^3},\ \frac{du}{dp} = \frac{\ddot{x}}{\dot{y}\dot{z}} - \frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = u,\ m_3 = \frac{dq}{dp},\ m_4 = w,\ \ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = -\dot{x}^3 f(p, q, \frac{dq}{dp}),\ \ \frac{\ddot{x}}{\dot{y}\dot{z}} - \frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}} = g(p, q, \frac{dq}{dp}),\ \ \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3} = h(p, q, \frac{dq}{dp})$
	2	$Case: \dot{\mu}(t) \neq 0,\ \ddot{\mu}(t) = 0,\ p = t,\ q = z,\ u = \frac{1}{2}\dot{y}^2 - \dot{x},\ \frac{dq}{dp} = \dot{z},\ w = \ddot{y},\ \frac{du}{dp} = \dot{y}\ddot{y} - \ddot{x},\ \frac{d^2q}{dp^2} = \dddot{z}$
		$m_1 = p,\ \ m_2 = q,\ \ m_3 = \frac{dq}{dp},\ m_4 = w,\ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\dot{y}\ddot{y} - \ddot{x} = f(p, q, \frac{dq}{dp}),\ \ \ddot{y} = g(p, q, \frac{dq}{dp}),\ \dddot{z} = h(p, q, \frac{dq}{dp})$
	3	$p = z,\ q = \dot{x},\ u = \dot{y},\ v = \dot{z},\ \frac{dq}{dp} = \frac{\ddot{x}}{\dot{x}},\ \frac{du}{dp} = \frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp} = \frac{\ddot{z}}{\dot{z}}$
	*	$m_1 = p - \frac{1}{u},\ \ m_2 = \frac{u}{q},\ m_3 = \frac{u}{v},\ \ m_4 = \frac{1}{u^2}\frac{du}{dp},\ m_5 = \frac{1}{u}\frac{dq}{dp} - \frac{q}{u^2}\frac{du}{dp},\ m_6 = \frac{1}{u}\frac{dv}{dp} - \frac{v}{u^2}\frac{du}{dp}$
		$\frac{\ddot{x}}{\dot{y}\dot{z}} - \frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}} = f(p - \frac{1}{u}, \frac{u}{q}, \frac{u}{v}),\ \ \ddot{y} = \dot{y}^2\dot{z}g(p - \frac{1}{u}, \frac{u}{q}, \frac{u}{v}),\ \ \frac{\ddot{z}}{\dot{y}\dot{z}} - \frac{\ddot{y}}{\dot{y}^2} = h(p - \frac{1}{u}, \frac{u}{q}, \frac{u}{v})$
$A_{4,5}$	2	$p = z,\ q = lnt - \frac{\dot{x}}{\dot{y}},\ u = \frac{1}{t\dot{y}}e^{\frac{\dot{x}}{\dot{y}}},\ v = t\dot{z},\ \frac{dq}{dp} = \frac{1}{t\dot{z}} + \frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}} - \frac{\ddot{x}}{\dot{y}\dot{z}},$

Table 1 continued

Algebra	N	Invariants
		$\frac{du}{dp} = (-\frac{1}{t^2\dot{y}\dot{z}} - \frac{\dot{x}\ddot{y}}{t\dot{y}^3\dot{z}} + \frac{\ddot{x}}{t\dot{y}^2\dot{z}} - \frac{\ddot{y}}{t\dot{y}^2\dot{z}})e^{\frac{\dot{x}}{\dot{y}}},\ \frac{dv}{dp} = 1 + \frac{t\ddot{z}}{\dot{z}}$
		$m_1 = p,\ m_2 = \frac{1}{u}e^{-q},\ m_3 = v,\ m_4 = \frac{dq}{dp} + \frac{1}{u}\frac{du}{dp},\ m_5 = \frac{1}{u}\frac{du}{dp} + (\frac{dq}{dp} + \frac{1}{u}\frac{du}{dp})\ln u,\ m_6 = \frac{dv}{dp}$
		$-\frac{1}{t\dot{z}} - \frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}} + \frac{\ddot{x}}{\dot{y}\dot{z}} - \frac{\ddot{y}}{\dot{y}\dot{z}}\ln(\frac{e^{\frac{\dot{x}}{\dot{y}}}}{t\dot{y}}) = f(p, \frac{1}{u}e^{-q}, v),\ \ddot{y} = -\dot{y}\dot{z}g(p, \frac{1}{u}e^{-q}, v),\ 1 + \frac{t\ddot{z}}{\dot{z}} = h(p, \frac{1}{u}e^{-q}, v)$
	3	$p = t,\ q = z,\ u = \dot{y} - \dot{x}e^t,\ \frac{dq}{dp} = \dot{z},\ w = \ddot{x},\ \frac{du}{dp} = \ddot{y} - \ddot{x}e^t - \dot{x}e^t,\ \frac{d^2q}{dp^2} = \ddot{z}$
		$m_1 = q,\ m_2 = u,\ m_3 = \frac{dq}{dp},\ m_4 = we^p,\ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = e^{-t}f(q, u, \frac{dq}{dp}),\ \ddot{y} = \ddot{x}e^t + \dot{x}e^t + g(q, u, \frac{dq}{dp}),\ \ddot{z} = h(q, u, \frac{dq}{dp})$
	4	$p = z,\ q = y,\ u = \frac{\dot{y}}{\dot{x}}e^{\frac{1}{\dot{x}}},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ w = \frac{\ddot{x}}{\dot{x}^2\dot{y}},\ \frac{du}{dp} = \frac{\ddot{y}}{\dot{x}\dot{z}}e^{\frac{1}{\dot{x}}} - (1+\dot{x})\frac{\dot{y}\ddot{x}}{\dot{x}^3\dot{z}}e^{\frac{1}{\dot{x}}},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = u,\ m_3 = \frac{dq}{dp},\ m_4 = w,\ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = \dot{x}^2\dot{y}f(p, u, \frac{dq}{dp}),\ \ddot{y} = (1+\dot{x})\frac{\dot{y}\ddot{x}}{\dot{x}^3} + e^{\frac{-1}{\dot{x}}}\dot{x}\dot{z}g(p, u, \frac{dq}{dp}),\ \ddot{z} = \frac{\dot{z}\ddot{y}}{\dot{y}} + \frac{\dot{z}^3}{\dot{y}}h(p, u, \frac{dq}{dp})$
	5	$p = z,\ q = y,\ u = \frac{\dot{z}}{\dot{x}},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ w = -\frac{\ddot{x}}{\dot{x}^3}e^x,\ \frac{du}{dp} = \frac{\ddot{z}}{\dot{x}\dot{z}} - \frac{\ddot{x}}{\dot{x}^2},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = u,\ m_3 = \frac{dq}{dp},\ m_4 = w,\ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = \dot{x}^3e^{-x}f(p, u, \frac{dq}{dp}),\ \ddot{y} = \frac{\dot{y}\ddot{z}}{\dot{z}} + \dot{z}^2g(p, u, \frac{dq}{dp}),\ \ddot{z} = \frac{\dot{z}\ddot{x}}{\dot{x}} + \dot{x}\dot{z}h(p, u, \frac{dq}{dp})$
	6	$p = z,\ q = y - \frac{1}{\dot{x}},\ u = e^{\frac{1}{\dot{x}}}\frac{\dot{y}}{\dot{x}},\ v = e^{\frac{1}{\dot{x}}}\frac{\dot{z}}{\dot{x}},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}}(1 + \frac{\ddot{x}}{\dot{x}^2\dot{y}}),$
		$\frac{du}{dp} = e^{\frac{1}{\dot{x}}}(\frac{\ddot{y}}{\dot{x}\dot{z}} - \frac{(1+\dot{x})\ddot{x}\dot{y}}{\dot{x}^3\dot{z}}),\ \frac{dv}{dp} = e^{\frac{1}{\dot{x}}}(\frac{\ddot{z}}{\dot{x}\dot{z}} - \frac{(1+\dot{x})\ddot{x}}{\dot{x}^3})$
		$m_1 = q,\ m_2 = u,\ m_3 = v,\ m_4 = \frac{dq}{dp},\ m_5 = \frac{du}{dp},\ m_6 = \frac{dv}{dp}$

Table 1 continued

Algebra	N	Invariants
		$\frac{\dot{y}}{\dot{z}}(1+\frac{\ddot{x}}{\dot{x}^2\dot{y}})=f(q,u,v),\quad e^{\frac{1}{x}}(\frac{\ddot{y}}{\dot{x}\dot{z}}-\frac{(1+\dot{x})\ddot{x}\dot{y}}{\dot{x}^3\dot{z}})=g(q,u,v),\quad e^{\frac{1}{x}}(\frac{\ddot{z}}{\dot{x}\dot{z}}-\frac{(1+\dot{x})\ddot{x}}{\dot{x}^3})=h(q,u,v)$
$A^1_{4,6}$	2	$p=t,\ q=z,\ u=\dot{y},\ \frac{dq}{dp}=\dot{z},\ w=\ddot{x},\ \frac{du}{dp}=\ddot{y},\ \frac{d^2q}{dp^2}=\ddot{z}$
		$m_1=q,\ m_2=u,\ m_3=p\frac{dq}{dp},\ \ m_4=pw,\ m_5=p\frac{du}{dp},\ m_6=p^2\frac{d^2q}{dp^2}$
		$t\ddot{x}=f(q,u,p\frac{dq}{dp}),\ \ t\ddot{y}=g(q,u,p\frac{dq}{dp}),\ \ t^2\ddot{z}=h(q,u,p\frac{dq}{dp})$
	3	$p=z,\ q=y,\ u=\dot{x},\ \frac{dq}{dp}=\frac{\dot{y}}{\dot{z}},\ \frac{du}{dp}=\frac{\ddot{x}}{\dot{z}},\ w=\frac{\ddot{y}}{\dot{y}^2},\ \frac{d^2q}{dp^2}=\frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1=p,\ m_2=u,\ m_3=\frac{dq}{dp},\ \ m_4=\frac{du}{dp},\ \ m_5=w,\ m_6=\frac{d^2q}{dp^2}$
		$\ddot{x}=\dot{z}f(p,u,\frac{dq}{dp}),\ \ \ddot{y}=\dot{y}^2g(p,u,\frac{dq}{dp}),\ \ddot{z}=\frac{\dot{z}\ddot{y}}{\dot{y}}-\frac{\dot{z}^3}{\dot{y}}h(p,u,\frac{dq}{dp})$
	4	$Case:\psi'(x)\neq 0,\quad p=z,\ q=x,\ u=\frac{\dot{y}}{\dot{x}},\ \frac{dq}{dp}=\frac{\dot{x}}{\dot{z}},\ w=-\frac{\ddot{x}}{\dot{x}^3}e^{-y},\ \frac{du}{dp}=\frac{\ddot{y}}{\dot{x}^2}-\frac{\dot{y}\ddot{x}}{\dot{x}^3},\ \ \frac{d^2q}{dp^2}=\frac{\ddot{x}}{\dot{x}\dot{z}}-\frac{\dot{x}\ddot{z}}{\dot{x}^2\dot{z}}$
		$m_1=p,\ m_2=q,\ m_3=\frac{dq}{dp},\ \ m_4=we^{\frac{\psi(q)}{\psi'(q)}u},\ \ m_5=\frac{du}{dp}-\frac{\psi''(q)}{\psi'(q)u}\frac{dq}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$\ddot{x}=-\dot{x}^3e^{y-\frac{\psi(x)}{\psi'(x)}\frac{\dot{y}}{\dot{x}}}f(p,q,\frac{dq}{dp}),\ \ddot{y}=\frac{\dot{y}}{\dot{x}}\ddot{x}+\frac{\psi''(x)}{\psi'(x)}\dot{y}\dot{x}+\dot{x}\dot{z}g(p,q,\frac{dq}{dp}),\ \frac{\ddot{z}}{\dot{z}^2}-\frac{\dot{x}\ddot{x}}{\dot{z}^3}=h(p,q,\frac{dq}{dp})$
	5	$p=z,\ q=\dot{x},\ u=e^y\dot{y},\ v=\frac{\dot{z}}{\dot{y}},\ \frac{dq}{dp}=\frac{\ddot{x}}{\dot{z}},\ \frac{du}{dq}=e^y\frac{\dot{y}^2}{\dot{z}}+e^y\frac{\ddot{y}}{\dot{x}\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{z}}{\dot{y}\dot{z}}-\frac{\ddot{y}}{\dot{y}^2}$
		$m_1=q,\ m_2=u,\ m_3=v,\ \ m_4=\frac{dq}{dp},\ \ m_5=\frac{du}{dp},\ m_6=\frac{dv}{dp}$
		$\frac{\ddot{x}}{\dot{z}}=f(q,u,v),\ \ e^y\frac{\dot{y}^2}{\dot{z}}+e^y\frac{\ddot{y}}{\dot{z}}=g(q,u,v),\ \ \frac{\ddot{z}}{\dot{y}\dot{z}}-\frac{\ddot{y}}{\dot{y}^2}=h(q,u,v)$
	6	$p=z,\ q=x,\ u=\frac{\dot{y}}{\dot{x}},\ \frac{dq}{dp}=\frac{\dot{x}}{\dot{z}},\ w=-\frac{e^{-y}\ddot{x}}{\dot{x}^3},\ \frac{du}{dp}=\frac{\ddot{y}}{\dot{x}\dot{z}}-\frac{\dot{y}\ddot{x}}{\dot{x}^2\dot{z}},\ \ \frac{d^2q}{dp^2}=\frac{\ddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{z}^3}$
		$m_1=q,\ m_2=u,\ m_3=\frac{dq}{dp},\ \ m_4=w,\ \ m_5=\frac{du}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$-\frac{e^{-y}\ddot{x}}{\dot{x}^3}=f(p,u,\frac{dq}{dp}),\ \ \frac{\ddot{y}}{\dot{x}\dot{z}}-\frac{\dot{y}\ddot{x}}{\dot{x}^2\dot{z}}=g(p,u,\frac{dq}{dp}),\ \ \frac{\ddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{z}^3}=h(p,u,\frac{dq}{dp})$
$A^a_{4,6}$	2	$p=z,\ q=\frac{1}{\dot{x}},\ u=\frac{\dot{y}}{\dot{x}^{a-1}\dot{x}^a},\ v=\frac{\dot{z}}{\dot{y}},\ \frac{dq}{dp}=-\frac{\ddot{x}}{\dot{x}^2\dot{z}},$

Table 1 continued

Algebra	N	Invariants
		$\frac{du}{dp} = \frac{t^{1-a}\dot{x}^{-a}\ddot{y}}{\dot{z}} - a\frac{\dot{y}t^{1-a}\dot{x}^{1-a}\ddot{x}}{\dot{z}} - (a-1)\frac{\dot{y}t^{-a}\dot{x}^{-a}}{\dot{z}}$, $\frac{dv}{dp} = \frac{\dddot{z}}{\dot{y}\dot{z}} - \frac{\ddot{y}}{\dot{y}^2}$
$\lvert a\rvert \leq 1$,		$m_1 = p$, $m_2 = \frac{u}{q^a}$, $m_3 = v$, $m_4 = -\frac{\frac{dq}{dp}}{q^2}$, $m_5 = \frac{1}{q^a}\frac{du}{dp} - \frac{au}{q^{a+1}}\frac{dq}{dp}$, $m_6 = \frac{dv}{dp}$
$a \neq 0, 1$		$\ddot{x} = -\dot{z}f(p, \frac{u}{q^a}, v)$, $\ddot{y} = (a-1)\frac{\dot{y}}{t} + \dot{y}\ddot{x}g(p, \frac{u}{q^a}, v)$, $\ddot{z} = \frac{\dot{z}\ddot{y}}{\dot{y}} + \dot{y}\dot{z}h(p, \frac{u}{q^a}, v)$
	3	$p = z$, $q = y$, $u = \frac{t^{\frac{1}{a}-1}}{a\dot{x}}$, $v = \frac{\dot{y}}{a^{a-1}t^{a-1}\dot{x}^a}$, $\frac{dq}{dp} = t^{1/a}(\frac{(1-a)}{a^2t^2\dot{x}\dot{z}} - \frac{\ddot{x}}{a}t\dot{x}^2\dot{z})$,
		$\frac{du}{dp} = \frac{\ddot{y}}{\dot{x}^a a^{a-1}t^{a-1}\dot{z}} - \frac{a\dot{y}\ddot{x}}{\dot{x}^{a+1}a^{a-1}t^{a-1}\dot{z}} + \frac{(1-a)\dot{y}}{\dot{x}^a a^{a-1}t^a\dot{z}}$, $\frac{dv}{dp} = a(1 + t\frac{\dddot{z}}{\dot{z}})$
		$m_1 = p$, $m_2 = q$, $m_3 = v$, $m_4 = \frac{dq}{dp}$, $m_5 = q\frac{du}{dp} - au\frac{dq}{dp}$, $m_6 = \frac{dv}{dp}$
		$t^{1/a}(\frac{(1-a)}{a^2t^2\dot{x}\dot{z}} - \frac{\ddot{x}}{a}t\dot{x}^2\dot{z}) = f(p, q, v)$, $\frac{t^{1/a}\ddot{y}}{\dot{x}^{a+1}a^a t^a\dot{z}} = g(p, q, v)$, $a(1 + t\frac{\dddot{z}}{\dot{z}}) = h(p, q, v)$
	4	$p = z$, $q = y$, $u = \dot{x}\dot{y}^{a-1}$, $w = \frac{\ddot{x}}{\dot{x}\dot{y}}$, $\frac{dq}{dp} = \frac{\dot{y}}{\dot{z}}$, $\frac{du}{dp} = \frac{\ddot{x}\dot{y}^{a-1} + (a-1)\dot{x}\dot{y}^{a-2}\ddot{y}}{\dot{z}}$, $\frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p$, $m_2 = u$, $m_3 = \frac{dq}{dp}$, $m_4 = w$, $m_5 = \frac{du}{dp}$, $m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = \dot{x}\dot{y}f(p, u, \frac{dq}{dp})$, $\ddot{y} = \frac{\dot{y}}{(1-a)\dot{x}}\ddot{x} + \frac{\dot{z}}{(a-1)\dot{x}\dot{y}^{a-3}}g(p, u, \frac{dq}{dp})$, $\ddot{z} = \frac{\dot{z}\ddot{y}}{\dot{y}} + \frac{\dot{z}^3}{\dot{y}}h(p, u, \frac{dq}{dp})$
	5	$p = z$, $q = \dot{y}e^y$, $u = \dot{x}\dot{y}^{a-1}$, $v = \frac{\dot{z}}{\dot{y}}$, $\frac{dq}{dp} = e^y(\frac{\ddot{y}}{\dot{z}} + \frac{\dot{y}^2}{\dot{z}})$, $\frac{du}{dp} = \frac{\dot{y}^{(a-1)}\ddot{x}}{\dot{z}} + \frac{(a-1)\dot{x}\dot{y}^{(a-2)}\ddot{y}}{\dot{z}}$,
		$\frac{dv}{dp} = \frac{\dddot{z}}{\dot{y}\dot{z}} - \frac{\ddot{y}}{\dot{y}^2}$
		$m_1 = p$, $m_2 = \frac{u}{q^a} - \frac{a}{q}$, $m_3 = v$, $m_4 = \frac{1}{q^2}\frac{dq}{dp}$, $m_5 = \frac{1}{q^a}\frac{du}{dp} - (\frac{au}{q^{a+1}} - \frac{a^2}{q^2})\frac{dq}{dp}$, $m_6 = \frac{dv}{dp}$
		$\frac{\ddot{x}}{e^{ay}\dot{y}\dot{z}} + \frac{a^2\ddot{y}}{e^y\dot{y}^2\dot{z}} - \frac{a\dot{x}}{e^{ay}\dot{z}} - \frac{\dot{x}\ddot{y}}{e^{ay}\dot{y}^2\dot{z}} + \frac{a^2}{e^y\dot{y}^2\dot{z}} = f(p, \frac{u}{q^a} - \frac{a}{q}, v)$, $\frac{\ddot{y}}{\dot{y}^2\dot{z}} + \frac{1}{\dot{z}} = e^y g(p, \frac{u}{q^a} - \frac{a}{q}, v)$,
		$\frac{\dddot{z}}{\dot{y}\dot{z}} - \frac{\ddot{y}}{\dot{y}^2} = h(p, \frac{u}{q^a} - \frac{a}{q}, v)$
	6	$p = z$, $q = y$, $u = \frac{x\dot{z}}{\dot{x}}$, $\frac{dq}{dp} = \frac{\dot{y}}{\dot{z}}$, $w = -x^{\frac{2a-1}{a-1}}\frac{\ddot{x}}{\dot{x}^3}$, $\frac{du}{dp} = 1 + x(\frac{\dddot{z}}{\dot{x}\dot{z}} - \frac{\ddot{x}}{\dot{x}^2})$, $\frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$

Table 1 continued

Algebra	N	Invariants
	8	$m_1 = p,\ m_2 = u,\ m_3 = \frac{dq}{dp},\ \ m_4 = w,\ \ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$ $\ddot{x} = \dot{x}^3 x^{\frac{2a-1}{1-a}} f(p, u, \frac{dq}{dp}),\quad \ddot{z} = \frac{\dot{z}\ddot{x}}{\dot{x}} - \frac{\dot{x}\dot{z}}{x}(1 - h(p, u, \frac{dq}{dp})),\ \ \ddot{y} = \frac{\dot{y}\ddot{z}}{\dot{z}} + \dot{z}^2 g(p, u, \frac{dq}{dp})$ $p = z, q = e^y \dot{y},\ u = \dot{x}\dot{y}^{a-1},\ v = \frac{\dot{z}}{\dot{y}},\ \frac{dq}{dp} = e^y(\frac{\ddot{y}}{\dot{z}} + \frac{\dot{y}^2}{\dot{z}}),\ \frac{du}{dq} = \frac{\dot{y}^{(a-1)}\ddot{x}}{\dot{z}} + \frac{(a-1)\dot{x}\dot{y}^{(a-2)}\ddot{y}}{\dot{z}},$ $\frac{dv}{dp} = \frac{\ddot{z}}{\dot{y}\dot{z}} - \frac{\ddot{y}}{\dot{y}^2}$
		$m_1 = q,\ m_2 = u,\ m_3 = v,\ \ m_4 = \frac{dq}{dp},\ \ m_5 = \frac{du}{dp},\ m_6 = \frac{dv}{dp}$ $\frac{\dot{y}^{(a-1)}\ddot{x}}{\dot{z}} + \frac{(a-1)\dot{x}\dot{y}^{(a-2)}\ddot{y}}{\dot{z}} = f(q, u, v),\ e^y(\frac{\ddot{y}}{\dot{z}} + \frac{\dot{y}^2}{\dot{z}}) = g(q, u, v),\ \frac{\ddot{z}}{\dot{y}\dot{z}} - \frac{\ddot{y}}{\dot{y}^2} = h(q, u, v)$
$A^a_{4,7}$	1	$p = z,\ q = y,\ u = \frac{\dot{y}e^{-a\arctan\dot{x}}}{\sqrt{1+\dot{x}^2}},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ w = \frac{\ddot{x}e^{-a\arctan\dot{x}}}{(1+\dot{x}^2)^{\frac{3}{2}}},$ $\frac{du}{dp} = \frac{-(a+\dot{x})\dot{y}\ddot{x}e^{-a\arctan\dot{x}}}{\dot{z}(1+\dot{x}^2)^{\frac{3}{2}}} + \frac{\ddot{y}e^{-a\arctan\dot{x}}}{\dot{z}\sqrt{1+\dot{x}^2}},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
$a \geq 0$		$m_1 = p, m_2 = u,\ m_3 = \frac{dq}{dp},\ \ m_4 = w,\ \ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$ $\ddot{x} = (1 + \dot{x}^2)^{\frac{3}{2}} e^{a\arctan\dot{x}} f(p, u, \frac{dq}{dp}),$ $\ddot{y} = (a + \dot{x})\dot{y}\ddot{x}(1 + \dot{x}^2)^{-1} + \dot{z}\sqrt{1 + \dot{x}^2}e^{a\arctan\dot{x}} g(p, u, \frac{dq}{dp}),\ \ddot{z} = \frac{\dot{z}\ddot{y}}{\dot{y}} + \frac{\dot{z}^3}{\dot{y}} h(p, u, \frac{dq}{dp})$
	3	$p = z,\ q = y,\ u = (1 + x^2)\frac{\dot{z}}{\dot{x}},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ \ w = -\ddot{x}\dot{x}^{\frac{-3}{2}}\dot{y}^{\frac{-3}{2}} e^{a\arctan x},$ $\frac{du}{dp} = 2x + (1 + x^2)(\frac{\ddot{z}}{\dot{x}\dot{z}} - \frac{\ddot{x}}{\dot{x}^2}),\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = u,\ m_3 = \frac{dq}{dp},\ m_4 = w,\ \ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$ $\ddot{x} = -\dot{x}^{\frac{3}{2}}\dot{y}^{\frac{3}{2}} e^{-a\arctan x} f(p, u, \frac{dq}{dp}),\ \ddot{y} = \frac{\dot{y}\ddot{z}}{\dot{z}} + \dot{z}^2 g(p, u, \frac{dq}{dp}),\ \ \ddot{z} = \frac{\dot{z}}{\dot{x}}\ddot{x} - \frac{2x\dot{x}\dot{z}}{(1+x^2)} + \frac{\dot{x}\dot{z}}{(1+x^2)} h(p, u, \frac{dq}{dp})$

Table 1 continued

Algebra	N	Invariants
$A_{4,8}$	1	$p = z,\ q = y,\ u = \frac{\dot{x}}{(t-x)^2\dot{y}^2},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ w = \frac{2}{(t-x)\dot{y}} + \frac{\ddot{y}}{\dot{y}^2},$
		$\frac{du}{dp} = \frac{\ddot{x}}{(t-x)^2\dot{y}^2\dot{z}} - \frac{2(1-\dot{x})\dot{x}}{(t-x)^3\dot{y}^2\dot{z}} - \frac{2\dot{x}\ddot{y}}{(t-x)^2\dot{y}^3\dot{z}},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = u,\ m_3 = \frac{dq}{dp},\ m_4 = w,\ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = \frac{2\dot{x}\ddot{y}}{\dot{y}} + \frac{2(1-\dot{x})\dot{x}}{(t-x)} + (t-x)^2\dot{y}^3 f(p, u, \frac{dq}{dp}),\quad \ddot{y} = -\frac{2\dot{y}}{(t-x)} + \dot{y}^2 g(p, u, \frac{dq}{dp}),\ \ \ddot{z} = \frac{\dot{z}\ddot{y}}{\dot{y}} + \frac{\dot{z}^3}{\dot{y}} h(p, u, \frac{dq}{dp})$
	2	$p = z,\ q = y,\ u = \frac{t\dot{x}-x}{\dot{y}},\ w = \frac{\ddot{x}}{\dot{y}^3},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ \frac{du}{dp} = \frac{t\ddot{x}}{\dot{y}\dot{z}} - \frac{(t\dot{x}-x)\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = u,\ m_3 = \frac{dq}{dp},\ m_4 = w,\ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = \dot{y}^3 f(p, u, \frac{dq}{dp}),\quad \ddot{y} = \frac{t\dot{y}\ddot{x}}{(t\dot{x}-x)} - \frac{\dot{z}\dot{y}^2}{(t\dot{x}-x)} g(p, u, \frac{dq}{dp}),\ \ \ddot{z} = \frac{\dot{z}\ddot{y}}{\dot{y}} + \frac{\dot{z}^3}{\dot{y}} h(p, u, \frac{dq}{dp})$
	5	$p = z,\ q = y,\ u = \frac{x\dot{z}}{\sqrt{1+\dot{x}^2}},\ w = \frac{(1+\dot{x}^2)}{x^3\dot{y}^3} + \frac{\ddot{x}}{x^2\dot{y}^3},\ \ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},$
		$\frac{du}{dp} = \frac{\dot{x}(1+\dot{x}^2-x\ddot{x})}{(1+\dot{x}^2)^{\frac{3}{2}}} + \frac{x\ddot{z}}{\dot{z}\sqrt{1+\dot{x}^2}},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = u,\ m_3 = \frac{dq}{dp},\ m_4 = w,\ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = -\frac{(1+\dot{x}^2)}{x} + x^2\dot{y}^3 f(p, u, \frac{dq}{dp}),\ \ \ddot{y} = \frac{\dot{y}\ddot{z}}{\dot{z}} + \dot{z}^2 g(p, u, \frac{dq}{dp}),\ \ \ddot{z} = \frac{\dot{x}\dot{z}(1+\dot{x}^2-x\ddot{x})}{x(1+\dot{x}^2)} + \frac{\dot{z}\sqrt{1+\dot{x}^2}}{x} h(p, u, \frac{dq}{dp})$
	6	$p = z,\ q = y,\ u = -\frac{1}{x\dot{y}},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ w = \frac{\ddot{x}}{2x^2\dot{y}^3} - \frac{\dot{x}^2}{4x^3\dot{y}^3},\ \ \frac{du}{dp} = \frac{\ddot{y}}{x\dot{y}^2\dot{z}} + \frac{\dot{x}}{x^2\dot{y}\dot{z}},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = q,\ m_3 = \frac{dq}{dp},\ \ m_4 = (\frac{1}{u}\frac{du}{dp} + \frac{1}{q}\frac{dq}{dp} ln(\frac{1}{\lvert u\rvert}))$
		$m_5 = \frac{w}{u} - \frac{3}{4q^2}\ln(\frac{1}{u})^2 - \frac{\frac{du}{dp}\ln u}{2qu\frac{dq}{dp}},\ m_6 = \frac{d^2q}{dp^2}$

Table 1 continued

Algebra	N	Invariants
		$\frac{\dot{x}^2}{4x^2\dot{y}^2} - \frac{\ddot{x}}{2x\dot{y}^2} + \frac{\dot{y}}{y\dot{z}} \ln\lvert x\dot{y}\rvert = f(p,q,\frac{dq}{dp})$,
		$\frac{\dot{x}^2 4x^2\dot{y}^2 - \frac{\ddot{x}}{2x\dot{y}^2} - \frac{3}{4y^2}}{\ln\lvert x\dot{y}\rvert^2} + \frac{x\dot{z}}{2y} \ln(\frac{1}{\lvert x\dot{y}\rvert})(\frac{\ddot{y}}{x\dot{y}^2\dot{z}} + \frac{\dot{x}}{x^2\dot{y}\dot{z}}) = g(p,q,\frac{dq}{dp})$, $\frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3} = h(p,q,\frac{dq}{dp})$
	7	$p = z,\ q = y,\ u = -\frac{1}{x\dot{y}},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ w = \frac{\ddot{x}}{2x^2\dot{y}^3} - \frac{\dot{x}^2}{4x^3\dot{y}^3},\ \frac{du}{dp} = \frac{\ddot{y}}{x\dot{y}^2\dot{z}} + \frac{\dot{x}}{x^2\dot{y}\dot{z}},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = q,\ m_3 = \frac{dq}{dp},\ m_4 = \frac{1}{u}w,\ m_5 = \frac{1}{u}\frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\frac{\dot{x}^2}{4x^2\dot{y}^2} - \frac{\ddot{x}}{2x\dot{y}^2} = f(p,q,\frac{dq}{dp}),\ -\frac{\ddot{y}}{\dot{y}\dot{z}} - \frac{\dot{x}}{x\dot{z}} = g(p,q,\frac{dq}{dp}),\ \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3} = h(p,q,\frac{dq}{dp})$
$A_{4,9}$	1	$p = z,\ q = y,\ u = \frac{1+\dot{x}^2+(t\dot{x}-x)^2}{(1+t^2+x^2)^2\dot{y}^2},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ w = \frac{\ddot{x}}{(1+t^2+x^2)^{\frac{3}{2}}\dot{y}^3},$
		$\frac{du}{dp} = -\frac{2(1+\dot{x}^2+(t\dot{x}-x)^2)\ddot{y}}{\dot{z}\dot{y}^3(1+t^2+x^2)^2} - \frac{4(t+x\dot{x})(1+\dot{x}^2+(t\dot{x}-x)^2)}{\dot{z}\dot{y}^2(1+t^2+x^2)^3} - \frac{2(xt-(1+t^2)\dot{x})\ddot{x}}{\dot{z}\dot{y}^2(1+t^2+x^2)^2},\quad \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = u,\ m_3 = \frac{dq}{dp},\ m_4 = w,\ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\frac{\ddot{x}}{(1+t^2+x^2)^{\frac{3}{2}}\dot{y}^3} = f(p,u,\frac{dq}{dp}),$
		$\frac{2(1+\dot{x}^2+(t\dot{x}-x)^2)\ddot{y}}{\dot{z}\dot{y}^3(1+t^2+x^2)^2} + \frac{4(t+x\dot{x})(1+\dot{x}^2+(t\dot{x}-x)^2)}{\dot{z}\dot{y}^2(1+t^2+x^2)^3} + \frac{2(xt-(1+t^2)\dot{x})\ddot{x}}{\dot{z}\dot{y}^2(1+t^2+x^2)^2} = -g(p,u,\frac{dq}{dp}),\ \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3} = h(p,u,\frac{dq}{dp})$
	2	$p = z,\ q = \arctan(\dot{x}\sec x) - y,\ u = \frac{\dot{y}\sec x + \tan x}{\sqrt{1+\dot{x}^2\sec^2 x}},\ v = \frac{\dot{z}\sec x}{\sqrt{1+\dot{x}^2\sec^2 x}},\ \frac{dq}{dp} = \frac{\sec x(\ddot{x}+\dot{x}^2\tan x)}{\dot{z}(1+\dot{x}^2\sec x^2)} - \frac{\dot{y}}{\dot{z}}$
		$\frac{du}{dp} = \frac{(\sqrt{1+\dot{x}^2\sec^2 x})}{z\dot{\sec} x}(\frac{\ddot{y}}{\cos^2 x(1+\dot{x}^2\sec^2 x)} + \frac{(1+\dot{x}^2+\dot{y}\sin x)\dot{x}}{\cos^3 x(1+\dot{x}^2\sec^2 x)^2} - \frac{\dot{x}\ddot{x}(\dot{y}\sec x+\tan x)}{\cos^3 x(1+\dot{x}^2\sec^2 x)^2}),$
		$\frac{dv}{dp} = \frac{(\sqrt{1+\dot{x}^2\sec^2 x})}{z\dot{\sec} x}(\frac{\ddot{z}}{\cos^2 x(1+\dot{x}^2\sec^2 x)} + \frac{\dot{x}\dot{z}\tan x}{\cos^2 x(1+\dot{x}^2\sec^2 x)^2} - \frac{\dot{x}\ddot{x}\dot{z}\sec^2 x}{\cos^2 x(1+\dot{x}^2\sec^2 x)^2})$

Table 1 continued

Algebra	N	Invariants
		$m_1 = p,\ \ m_2 = u,\ m_3 = v,\ \ m_4 = \frac{dq}{dp},\ m_5 = \frac{du}{dp},\ m_6 = \frac{dv}{dp}$
		$\frac{\sec x(\ddot{x}+\dot{x}^2\tan x)}{\dot{z}(1+\dot{x}^2\sec x^2)} - \frac{\dot{y}}{\dot{z}} = f(p,u,v)$
		$\frac{(\sqrt{1+\dot{x}^2\sec^2 x})}{z\dot{\sec} x}(\frac{\ddot{y}}{\cos^2 x(1+\dot{x}^2\sec^2 x)} + \frac{(1+\dot{x}^2+\dot{y}\sin x)\dot{x}}{\cos^3 x(1+\dot{x}^2\sec^2 x)^2} - \frac{\dot{x}\ddot{x}(\dot{y}\sec x+\tan x)}{\cos^3 x(1+\dot{x}^2\sec^2 x)^2}) = g(p,u,v),$
		$\frac{(\sqrt{1+\dot{x}^2\sec^2 x})}{z\dot{\sec} x}(\frac{\ddot{z}}{\cos^2 x(1+\dot{x}^2\sec^2 x)} + \frac{\dot{x}\dot{z}\tan x}{\cos^2 x(1+\dot{x}^2\sec^2 x)^2} - \frac{\dot{x}\ddot{x}\dot{z}\sec^2 x}{\cos^2 x(1+\dot{x}^2\sec^2 x)^2}) = h(p,u,v)$
$A_{4,10}$	2	$p = z.\ q = t - \frac{\dot{x}}{\dot{y}},\ u = \dot{y},\ v = \frac{\dot{y}}{\dot{z}},\ \frac{dq}{dp} = \frac{1}{\dot{z}} + \frac{\ddot{y}}{\dot{y}^2\dot{z}} - \frac{\ddot{x}}{\dot{y}\dot{z}},\ \frac{du}{dp} = \frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = qu,\ \ m_3 = \frac{v}{u},\ m_4 = \frac{du}{dp},\ m_5 = u\frac{dq}{dp} + \frac{du}{dp}\ln u,\ \ m_6 = u\frac{dv}{dp} - v\frac{du}{dp}$
		$\ddot{x} = \frac{\ddot{y}}{\dot{y}} + \ln\dot{y} + \dot{y} + \dot{z}f(p,qu,\frac{v}{u}),\ \ \ddot{y} = \dot{z}g(p,qu,\frac{v}{u}),\ \ddot{z} = -\frac{\dot{z}^3}{\dot{y}^2}h(p,qu,\frac{v}{u})$
	3	$p = z,\ q = \dot{x},\ u = \dot{y},\ v = \dot{z},\ \frac{dq}{dp} = \frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp} = \frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp} = \frac{\ddot{z}}{\dot{z}}$
		$m_1 = p,\ m_2 = q^2 - 2u,\ \ m_3 = v,\ m_4 = \frac{dq}{dp},\ \ m_5 = \frac{du}{dp} - q\frac{dq}{dp},\ m_6 = \frac{dv}{dp}$
		$\ddot{x} = \dot{z}f(p,q^2-2u,v),\ \ \ddot{y} = \dot{x}\ddot{x} + \dot{z}g(p,q^2-2u,v),\ \ddot{z} = \dot{z}h(p,q^2-2u,v)$
	4	$p = z,\ q = y,\ u = \frac{\dot{x}}{\dot{y}},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ w = \ddot{x}\frac{(2+x^2\dot{y})}{2\dot{x}^3} - \frac{x^2}{2\dot{x}^2}\ddot{y} - \frac{2x\dot{y}}{\dot{x}} - y,\ \frac{du}{dp} = \frac{\ddot{x}}{\dot{y}\dot{z}} - \frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = u,\ m_3 = \frac{dq}{dp},\ \ m_4 = w + q,\ \ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = \frac{x^2\dot{x}\ddot{y}+4x\dot{x}^2\dot{y}}{(2+x^2\dot{y})} + \frac{2\dot{x}^3}{(2+x^2\dot{y})}f(p,u,\frac{dq}{dp}),\ \ \ddot{y} = \frac{\dot{y}}{\dot{x}}\ddot{x} + \frac{\dot{z}\dot{y}^2}{\dot{x}}g(p,u,\frac{dq}{dp}),\ \ \ddot{z} = \frac{\dot{z}\ddot{y}}{\dot{y}} + \frac{\dot{z}^3}{\dot{y}}h(p,u,\frac{dq}{dp})$
$A^a_{4,11}$	2	$p = z,\ q = xe^{-at},\ u = -\dot{z},\ \frac{dq}{dp} = -\frac{(a\dot{x}-\ddot{x})e^{-at}}{\dot{z}},\ w = -\ddot{y}e^{-t},\ \frac{du}{dp} = \frac{-\ddot{z}}{\dot{z}},$
		$\frac{d^2q}{dp^2} = \frac{e^{-at}}{\dot{z}^3}(\dot{z}(a^2x - 2a\dot{x}) + \ddot{x}) + \ddot{z}(ax - \dot{x})$

Table 1 continued

Algebra	N	Invariants
$a \neq 0$		$m_1 = p,\ m_2 = u,\ m_3 = u\frac{dq}{dp} - aq,\ \ m_4 = w,\ \ m_5 = \frac{du}{dp},\ m_6 = u\frac{d^2q}{dp^2} - a^2q + auq\frac{du}{dp}$
		$(-a^2x + 2a\dot{x} - \ddot{x})\dot{z} + (\dot{x} - ax) + a^2x\dot{z}^2 + \dot{z}^2ax\ddot{z} = f(p, u, u\frac{dq}{dp}),\ \ \ddot{y} = e^t g(p, u, u\frac{dq}{dp}),$
		$\ddot{z} = -\dot{z}h(p, u, u\frac{dq}{dp})$
	3	$p = z,\ q = \dot{x},\ u = \dot{y},\ v = \dot{z},\ \frac{dq}{dp} = \frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp} = \frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp} = \frac{\ddot{z}}{\dot{z}}$
		$m_1 = p,\ m_2 = ue^{(a-1)\frac{q}{u}},\ m_3 = vu^{\frac{a}{1-a}},\ \ m_4 = \frac{1}{u}\frac{du}{dp},\ \ m_5 = \frac{1}{u}\frac{dq}{dp} - \frac{\ln u}{u(1-a)}\frac{du}{dp},\ m_6 = u^{\frac{a}{1-a}}\frac{dv}{dp}$
		$\frac{\ddot{x}}{\dot{x}\dot{z}} - \frac{\ddot{y}\ln\dot{x}}{(1-a)\dot{x}\dot{z}} = f(p, ue^{(a-1)\frac{q}{u}}, vu^{\frac{a}{1-a}}),\ \frac{-\ddot{y}}{\dot{x}\dot{z}} = g(p, ue^{(a-1)\frac{q}{u}}, vu^{\frac{a}{1-a}}),$
		$\frac{\dot{x}^{\frac{a}{1-a}}\ddot{z}}{\dot{z}} = h(p, ue^{(a-1)\frac{q}{u}}, vu^{\frac{a}{1-a}})$
	4	$p = z,\ q = \frac{1}{\dot{y}}e^t,\ u = \frac{\dot{x}}{\dot{y}^a},\ v = \dot{z},\ \frac{dq}{dp} = e^t(\frac{1}{\dot{y}\dot{z}} - \frac{\ddot{y}}{\dot{y}^2\dot{z}}),\ \frac{du}{dp} = \frac{\dot{y}\ddot{x} - a\dot{x}\ddot{y}}{\dot{y}^a\dot{z}},\ \frac{dv}{dp} = \frac{\ddot{z}}{\dot{z}}$
		$m_1 = p,\ m_2 = \frac{u+(a-1)q^{a-1}}{q^a},\ m_3 = v,\ \ m_4 = \frac{v\frac{dq}{dp} - q}{vq^2},$
		$m_5 = \frac{1}{q^a}\frac{du}{dp} + (a-1)^2\frac{1}{qv} + \frac{1}{q}(\frac{au}{q^a} + \frac{a(a-1)}{q})(\frac{dq}{dp} - \frac{q}{v}),\ m_6 = \frac{dv}{dp}$
		$\frac{\ddot{x}}{\dot{z}e^{at}} - \frac{a\dot{x}\ddot{y}}{\dot{z}e^{at}} - \frac{(a-1)^2}{\dot{y}\dot{z}^2}(\frac{a\dot{x}}{e^{at}} + \frac{a(a-1)\dot{y}}{e^t}) = f(p, \frac{u+(a-1)q^{a-1}}{q^a}, v),\ \ \ddot{y} = -e^t\dot{z}g(p, \frac{u+(a-1)q^{a-1}}{q^a}, v),$
		$\ddot{z} = \dot{z}h(p, \frac{u+(a-1)q^{a-1}}{q^a}, v)$
$A_{4,12}$	2	$p = z,\ q = y,\ u = e^{\frac{\dot{x}}{\dot{y}}}\dot{y},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ w = \frac{\ddot{y}}{\dot{y}^2},\ \frac{du}{dp} = e^{\frac{\dot{x}}{\dot{y}}}(\frac{\ddot{y}}{\dot{z}} + \frac{(\dot{y}-\dot{x})\ddot{y}}{\dot{y}\dot{z}}),\ \ \frac{d^2q}{dp^2} = \frac{\ddot{z}}{\dot{z}\dot{y}} - \frac{\ddot{y}}{\dot{y}^2}$
		$m_1 = p,\ m_2 = q,\ m_3 = u,\ \ m_4 = \frac{dq}{dp},\ \ m_5 = \frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$e^{\frac{\dot{x}}{\dot{y}}}(\frac{\ddot{y}}{\dot{z}} + \frac{(\dot{y}-\dot{x})\ddot{x}}{\dot{y}\dot{z}}) = f(p, q, u),\ \ \ddot{y} = \dot{y}^2g(p, q, u),\ \ \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3} = h(p, q, u)$

Table 1 continued

Algebra	N	Invariants
	3	$p=z,\ q=\frac{e^t}{\dot{x}},\ u=\dot{y},\ v=\frac{\dot{y}}{\dot{z}},\ \frac{dq}{dp}=e^t(-\frac{\ddot{x}}{\dot{z}\dot{x}^2}+\frac{1}{\dot{x}\dot{z}}),\ \frac{du}{dp}=\frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3},\quad c\in\{0,1\}$
		$m_1=p,\ m_2=cu+\frac{1}{q},\ m_3=\frac{u}{v},\ m_4=q^{-2}\frac{dq}{dp},\ m_5=\frac{du}{dp},\ m_6=u\frac{dv}{dp}-u^{-2}q\frac{du}{dp}$
		$\ddot{x}=\dot{x}-\dot{z}e^t f(p,cu+\frac{1}{q},\frac{u}{v}),\ \ \ddot{y}=\dot{z}g(p,cu+\frac{1}{q},\frac{u}{v}),\ \ddot{z}=-\dot{z}^3h(p,cu+\frac{1}{q},\frac{u}{v})$
	4	$p=z,\ q=lnx-\frac{(x-t\dot{x})}{x^2\dot{y}},\ u=\frac{x\dot{y}}{\dot{x}},\ v=\frac{\dot{y}}{\dot{z}},\ \frac{dq}{dp}=\frac{\dot{x}}{x\dot{z}}+\frac{t\ddot{x}}{x^2\dot{y}\dot{z}}+\frac{(x-t\dot{x})}{\dot{z}}(\frac{2\dot{x}}{x^3\dot{y}}+\frac{\ddot{y}}{x^2\dot{y}^2}),$
		$\frac{du}{dp}=\frac{x\ddot{y}}{\dot{x}\dot{z}}+\frac{\dot{y}}{\dot{z}}-\frac{x\dot{y}\ddot{z}}{\dot{x}^2\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1=p,\ \ m_2=u,\ \ m_3=v,\ m_4=\frac{du}{dp}.\ m_5=u^3\frac{dq}{dp}+q(u\frac{du}{dp}+v),\ m_6=\frac{dv}{dp}$
		$\ddot{x}=\frac{\dot{x}}{\dot{y}}\ddot{y}+\frac{\dot{x}^2}{x}-\frac{\dot{x}^5}{x^4\dot{y}^3}f(p,u,v),\ \frac{x\ddot{y}}{\dot{x}\dot{z}}+\frac{\dot{y}}{\dot{z}}-\frac{x\dot{y}\ddot{z}}{\dot{x}^2\dot{z}}=g(p,u,v),\ \frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}=h(p,u,v)$
$A_{4,13}$	2	$p=z,\ q=\dot{x},\ u=\dot{y},\ v=\dot{z},\ \frac{dq}{dp}=\frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp}=\frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{z}}{\dot{z}}$
		$m_1=p,\ m_2=q^2-2u,\ m_3=ve^q,\ \ m_4=\frac{dq}{dp},\ \ m_5=\frac{du}{dp}-q\frac{dq}{dp},\ m_6=e^q\frac{dv}{dp}$
		$\ddot{x}=\dot{z}f(p,q^2-2u,ve^q),\ \ \ddot{y}=\dot{x}\ddot{x}+\dot{z}g(p,q^2-2u,ve^q),\ \ \ddot{z}=e^{-\dot{x}}\dot{z}h(p,q^2-2u,ve^q)$
	4	$p=z,\ q=ye^x,\ u=\frac{\dot{z}}{\dot{x}},\ \frac{dq}{dp}=(\frac{\dot{y}+y\dot{x}}{\dot{z}})e^x,\ w=(\frac{x^2\ddot{y}}{2\dot{x}^2}-\frac{(x^2\dot{y}+2)}{2\dot{x}^3}\ddot{x}+\frac{2x\dot{y}}{\dot{x}}+y)e^x,$
		$\frac{du}{dp}=\frac{\ddot{z}}{\dot{x}\dot{z}}-\frac{\ddot{x}}{\dot{x}^2},\ \frac{d^2q}{dp^2}=(\frac{\ddot{y}+2\dot{x}\dot{y}+y\ddot{x}+y\dot{x}^2-\dot{y}\ddot{z}-y\dot{x}\ddot{z}}{\dot{z}^2})e^x$
		$m_1=p,\ m_2=u,\ m_3=u\frac{dq}{dp}-q,\ \ m_4=w-q,\ \ m_5=\frac{du}{dp},\ m_6=u^2\frac{d^2q}{dp^2}-(1-\frac{du}{dp})q$
		$\ddot{x}=\frac{x^2\dot{x}\ddot{y}}{x^2\dot{y}+2}+\frac{4x\dot{x}^2\dot{y}}{x^2\dot{y}+2}-\frac{2\dot{x}^3e^{-x}}{x^2\dot{y}+2}f(p,u,u\frac{dq}{dp}-q),\ \ \frac{\ddot{y}}{\dot{x}^2}+\frac{2\dot{y}}{\dot{x}}-\frac{\dot{y}\ddot{z}}{\dot{x}^2\dot{z}}=e^{-x}g(p,u,u\frac{dq}{dp}-q),$
		$\frac{\ddot{z}}{\dot{x}\dot{z}}-\frac{\ddot{x}}{\dot{x}^2}=h(p,u,u\frac{dq}{dp}-q)$
$A^{a,b}_{4,14}$	2	$p=z,\ q=\dot{x},\ u=\dot{y},\ v=\dot{z},\ \frac{dq}{dp}=\frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp}=\frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{z}}{\dot{z}}$
$ab\neq 0$		$m_1=p,\ m_2=qu^{\frac{a-1}{1-b}},\ m_3=qv^{(a-1)},\ \ m_4=\frac{1}{q}\frac{dq}{dp},\ \ m_5=q^{\frac{1-b}{a-1}}\frac{du}{dp},\ m_6=q^{\frac{1}{a-1}}\frac{dv}{dp}$

Table 1 continued

Algebra	N	Invariants
$-1 \le a < b < 1$		$\ddot{x} = \dot{x}\dot{z} f(p, qu^{\frac{a-1}{1-b}}, qv^{(a-1)}),\ \ \ddot{y} = \dot{z}\dot{x}^{\frac{b-1}{a-1}} g(p, qu^{\frac{a-1}{1-b}}, qv^{(a-1)}),\ \ \ddot{z} = \dot{z}\dot{x}^{\frac{1}{1-a}} h(p, qu^{\frac{a-1}{1-b}}, qv^{(a-1)})$
	5	$p = z,\ q = \frac{e^t}{\dot{x}},\ u = \frac{\dot{y}}{\dot{x}^a},\ v = \dot{z},\ \frac{dq}{dp} = e^t(\frac{1}{\dot{x}\dot{z}} - \frac{\ddot{x}}{\dot{x}^2\dot{z}}),\ \frac{du}{dp} = \frac{\ddot{y}}{\dot{x}^a\dot{z}} - \frac{a\dot{y}\ddot{x}}{\dot{x}^{a+1}\dot{z}},\ \frac{dv}{dp} = \frac{\ddot{z}}{\dot{z}}$
		$m_1 = p, m_2 = \frac{u}{q^a} + \frac{(a-b)}{(b-1)q}, m_3 = v, m_4 = \frac{1}{q^{2-2b}}\frac{dq}{dp} + v\frac{bq^{2b-1}}{1-2b}, m_5 = \frac{1}{q^a}(\frac{du}{dp} - \frac{au}{q}\frac{dq}{dp} + \frac{bu}{v}), m_6 = \frac{dv}{dp}$
		$\ddot{x} = \frac{(1-b)\dot{x}}{(1-2b)} + \dot{z}\dot{x}^{2b}e^{(2b-1)t} f(p, \frac{u}{q^a} + \frac{(a-b)}{(b-1)q}, v),\ \ \ddot{y} = (a-b)\dot{y} + \dot{z}e^{at} g(p, \frac{u}{q^a} + \frac{(a-b)}{(b-1)q}, v),$
		$\ddot{z} = \dot{z}h(p, \frac{u}{q^a} + \frac{(a-b)}{(b-1)q}, v)$
$A^{a,a}_{4,14}$	2	$p = z,\ q = \dot{x},\ u = \dot{y},\ v = \dot{z},\ \frac{dq}{dp} = \frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp} = \frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp} = \frac{\ddot{z}}{\dot{z}}$
$a \neq 0$		$m_1 = p,\ m_2 = \frac{q}{u},\ \ m_3 = qv^{(a-1)},\ \ m_4 = \frac{1}{q}\frac{dq}{dp},\ m_5 = \frac{1}{q}\frac{du}{dp},\ m_6 = \frac{1}{q^{\frac{1}{(a-1)}}}\frac{dv}{dp}$
$-1 \le a < 1$		$\ddot{x} = \dot{x}\dot{z} f(p, \frac{q}{u}, qv^{(a-1)}),\ \ddot{y} = \dot{z}\dot{x} g(p, \frac{q}{u}, qv^{(a-1)}),\ \ \ddot{z} = \dot{x}^{\frac{1}{1-a}}\dot{z}h(p, \frac{q}{u}, qv^{(a-1)})$
	3	$p = z,\ q = \frac{1}{(1-a)t^{1-\frac{1}{1-a}}\dot{x}},\ u = \frac{\dot{y}}{(1-a)^{a-1}t^{a-1}\dot{x}^a},\ v = (1-a)t\dot{z},$
		$\frac{dq}{dp} = t^{1/1-a}(\frac{1}{(1-a)^2}t^2\dot{x}\dot{z}) - \frac{(\dot{x}+t\ddot{x})}{(1-a)t^2\dot{x}^2\dot{z}},\ \frac{du}{dp} = \frac{t\dot{x}\dot{y} - at\dot{y}\ddot{x} - (a-1)\dot{x}\dot{y}}{((a-1)^{a-1}t^a\dot{x}^{a+1}\dot{z}},\ \ \frac{dv}{dp} = (1-a)(1+\frac{t\ddot{z}}{\dot{z}})$
		$m_1 = p,\ m_2 = \frac{u}{q^a},\ \ m_3 = v,\ \ m_4 = \frac{1}{q^2}\frac{dq}{dp} + \frac{(a-2)}{vq},\ m_5 = \frac{1}{q^a}\frac{du}{dp} - \frac{au}{q^{a+1}}\frac{du}{dp} - \frac{a(a-2)u}{vq^a},\ m_6 = \frac{dv}{dp}$
		$\ddot{x} = -\frac{\dot{z}t^{\frac{1}{1-a}-1}}{(1-a)} f(p, \frac{u}{q^a}, v),\ \ t(a-1)\ddot{y} - a(a-1)\dot{y} = \dot{z}t^{\frac{a}{a-1}} g(p, \frac{u}{q^a}, v),\ (1-a)(1+\frac{t\ddot{z}}{\dot{z}}) = h(p, \frac{u}{q^a}, v)$
	4	$p = t,\ q = z,\ u = \dot{x},\ \frac{dq}{dp} = \dot{z},\ w = \ddot{y},\ \frac{du}{dp} = \ddot{x},\ \frac{d^2q}{dp^2} = \ddot{z}$
		$m_1 = p,\ m_2 = q,\ m_3 = \frac{dq}{dp},\ \ m_4 = \frac{d^2q}{dp^2},\ m_5 = \frac{1}{u^a}w,\ m_6 = \frac{1}{u}\frac{du}{dp}$
		$\ddot{x} = \dot{x} f(p, q, \frac{dq}{dp}),\ \ddot{y} = \dot{x}^a g(p, q, \frac{dq}{dp}),\ \ddot{z} = h(p, q, \frac{dq}{dp})$
$A^{a,1}_{4,14}$	2	$p = z,\ q = \dot{x},\ u = \dot{y},\ v = \dot{z},\ \frac{dq}{dp} = \frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp} = \frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp} = \frac{\ddot{z}}{\dot{z}}$
$a \neq 0$		$m_1 = p,\ m_2 = u,\ m_3 = vq^{\frac{1}{(a-1)}},\ \ m_4 = \frac{1}{q}\frac{dq}{dp},\ \ m_5 = \frac{du}{dp},\ m_6 = q^{\frac{1}{a-1}}\frac{dv}{dp}$

Table 1 continued

Algebra	N	Invariants
$-1 \le a < 1$		$\ddot{x} = \dot{x}\dot{z} f(p,u,vq^{\frac{1}{(a-1)}}),\ \ddot{y} = \dot{z} g(p,u,vq^{\frac{1}{(a-1)}}),\ \ddot{z} = \dot{x}^{\frac{1}{1-a}} h(p,u,vq^{\frac{1}{(a-1)}})$
	3	$p = t,\ q = z,\ u = \frac{\dot{y}}{\dot{x}^a},\ \frac{dq}{dp} = \dot{z},\ w = \frac{\ddot{y}}{\dot{y}} - \frac{\ddot{x}}{\dot{x}},\ \frac{du}{dp} = \frac{\dddot{y}}{\dot{x}^a} - \frac{a\dot{y}\ddot{x}}{\dot{x}^{a+1}},\ \frac{d^2q}{dp^2} = \ddot{z}$
		$m_1 = p,\ m_2 = q,\ m_3 = \frac{dq}{dp},\ m_4 = \frac{1}{u^{\frac{a+1}{a}}(a-1)}(uw - \frac{du}{dp}),\ m_5 = \frac{1}{1-a}\left(\frac{1}{u}\frac{du}{dp} - aw\right),\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = \dot{y}^{\frac{1}{a}} f(p,q,\frac{dq}{dp}),\ \ddot{y} = \dot{y} g(p,q,\frac{dq}{dp}),\ \ddot{z} = h(p,q,\frac{dq}{dp})$
	4	$p = z,\ q = \frac{1}{(a-1)t^{1-\frac{1}{a-1}}\dot{x}},\ u = \frac{\dot{y}}{(a-1)^{a-1}t^{a-1}\dot{x}^a},\ v = (a-1)t\dot{z},$
		$\frac{dq}{dp} = \dot{x}t^{1/1-a}(\frac{1}{(1-a)^2}t^2\dot{x}^2\dot{z}) - \frac{(\dot{x}+t\ddot{x})}{(1-a)t^2\dot{x}^3\dot{z}},\ \frac{du}{dp} = \frac{(t\dot{x}\dddot{y} - at\dot{y}\ddot{x} - (a-1)\dot{x}\ddot{y})}{(a-1)^{a-1}\dot{x}^{a+1}t^a\dot{z}},\ \frac{d^2q}{dp^2} = (a-1)(1+\frac{t\ddot{z}}{\dot{z}})$
		$m_1 = p,\ m_2 = q,\ m_3 = v,\ m_4 = \frac{dq}{dp},\ m_5 = qv\frac{du}{dp} - (av\frac{dq}{dp} - q)u,\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = -\frac{(2-a)\dot{x}}{t(a-1)} + \dot{x}^2 t(a-1)\dot{z} f(p,q,v),\ \ddot{y} = t^{\frac{a}{a-1}-a}(a-1)^{a-1}\dot{x}^{a+1} g(p,q,v),\ (1+\frac{t\ddot{z}}{\dot{z}}) = \frac{h(p,q,v)}{(a-1)}$
$A_{4,15}$	1	$p = z,\ q = \dot{x},\ u = \dot{y},\ v = \dot{z},\ \frac{du}{d;} = \frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp} = \frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp} = \frac{\ddot{z}}{\dot{z}}$
		$m_1 = p,\ m_2 = q,\ m_3 = u,\ m_4 = \frac{dq}{dp},\ m_5 = \frac{du}{dp},\ m_6 = e^v\frac{dv}{dp}$
		$\ddot{x} = \dot{z} f(p,q,u),\ \ddot{y} = \dot{z} g(p,q,u),\ \ddot{z} = e^{-z}\dot{z} h(p,q,u)$
	3	$p = t,\ q = z,\ u = \dot{y} - \dot{x} f'(t),\ \frac{dq}{dp} = \dot{z},\ w = \ddot{x},\ \frac{du}{dp} = \ddot{y} - \ddot{x} f'(t) - \dot{x} f''(t),\ \frac{d^2q}{dp^2} = \ddot{z}$
		$m_1 = p,\ m_2 = q,\ m_3 = \frac{dq}{dp},\ m_4 = \frac{w}{u},\ m_5 = \frac{1}{u}\frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = (\dot{y} - \dot{x} f'(t)) f(p,q,\frac{1}{u}\frac{du}{dp}),\ \ddot{y} = \ddot{x} f'(t) + \dot{x} f''(t) + (\dot{y} - \dot{x} f'(t)) g(p,q,\frac{1}{u}\frac{du}{dp}),\ \ddot{z} = h(p,q,\frac{1}{u}\frac{du}{dp})$
$A^{a,b}_{4,16}$	2	$p = z,\ q = \dot{x},\ u = \dot{y},\ v = \dot{z},\ \frac{dq}{dp} = \frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp} = \frac{\ddot{y}}{\dot{z}},\ \frac{dv}{dp} = \frac{\ddot{y}}{\dot{z}}$
$a > 0$		$m_1 = p, m_2 = \frac{u}{\sqrt{1+q^2}} e^{(b-a)\arctan q},\ m_3 = \frac{v}{\sqrt{1+q^2}} e^{(b)\arctan q},\ \ m_4 = \frac{(1+q^2)\frac{du}{dp}}{u\frac{dq}{dp}} - q$

Table 1 continued

Algebra	N	Invariants
		$m_5 = \frac{1}{1+q^2}\frac{dv}{du},\ m_6 = \frac{(1+q^2)\frac{dv}{dp}}{v\frac{dq}{dp}} - q$
		$\ddot{x} = \dot{z}(1+\dot{x}^2) f(p, \frac{u}{\sqrt{1+q^2}} e^{(b-a)\arctan q}, \frac{v}{\sqrt{1+q^2}} e^{(b)\arctan q}),$
		$\ddot{y} = \frac{\dot{x}\dot{y}\ddot{x}}{(1+\dot{x}^2)} + \frac{\dot{y}\ddot{x}}{(1+\dot{x}^2)} g(p, \frac{u}{\sqrt{1+q^2}} e^{(b-a)\arctan q}, \frac{v}{\sqrt{1+q^2}} e^{(b)\arctan q}),$
		$\ddot{z} = \frac{\dot{x}\dot{z}\ddot{x}}{(1+\dot{x}^2)} + \frac{\dot{z}\ddot{x}}{(1+\dot{x}^2)} h(p, \frac{u}{\sqrt{1+q^2}} e^{(b-a)\arctan q}, \frac{v}{\sqrt{1+q^2}} e^{(b)\arctan q})$
	3	$p = \tan t,\ q = z,\ u = \dot{x}\cos^2 t + \dot{y}(\cos^4 t) g'(t),\ \frac{dq}{dp} = \dot{z}\cos^2 t,\ w = -(\ddot{y} - 2\dot{y}\tan t)\cos^4 t,$
$c \in \mathcal{R}$		$\frac{du}{dp} = \ddot{x}\cos^4 t - 2\dot{x}\cos^3 t \sin t + (\ddot{y} - 4\dot{y}\tan t)(\cos^6 t) g'(t) + \dot{y}(\cos^6 t) g''(t),$
		$\frac{d^2q}{dp^2} = \ddot{z}\cos^4 t - 2\dot{z}\cos^3 t \sin t$
		$m_1 = q,\ m_2 = u(1+p^2)e^{-a\arctan p},\ m_3 = (1+p^2)\frac{dq}{dp},\ m_4 = w(1+p^2)^{\frac{3}{2}} e^{-b\arctan p}$
		$m_5 = (1+u^2)\frac{dv}{du} e^{-a\arctan u} + \frac{2uv}{1+u^2} e^{-a\arctan u} + w((a-b+u)(1+u^2)^{\frac{3}{2}} - \frac{u}{(1+u^2)^{\frac{5}{2}}})ce^{-b\arctan u},$
		$m_6 = (1+u^2)^2\frac{d^2q}{dp^2} + 2p(1+p^2)\frac{dq}{dp}$
		$\ddot{x} = -\dot{y}(\cos^2 t) g''(t) - c(\ddot{y} - 2\dot{y}\tan t)(\sin t) e^{(a-b)t} + e^{at} f(q, u(1+p^2)e^{-a\arctan p}, (1+p^2)\frac{dq}{dp})$
		$\ddot{y} = 2\dot{y}\tan t - (\sec t) e^{bt} g(q, u(1+p^2)e^{-a\arctan p}, (1+p^2)\frac{dq}{dp}),$
		$\ddot{z} = h(q, u(1+p^2)e^{-a\arctan p}, (1+p^2)\frac{dq}{dp}),\quad where\ g(t) = ce^{(a-b)t}\sec t.$
$A_{4,17}$	1	$p = z,\ q = y,\ u = \frac{1}{\dot{y}},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ w = -\ddot{x},\ \frac{du}{dp} = -\frac{\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p, m_2 = q,\ m_3 = \frac{dq}{dp},\ m_4 = w - ln(u^2),\ m_5 = \frac{1}{u}\frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = ln(\dot{y}^2) - f(p, q, \frac{dq}{dp}),\ \ddot{y} = -\dot{y}\dot{z} g(p, q, \frac{dq}{dp}),\ \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3} = h(p, q, \frac{dq}{dp})$

Table 1 continued

Algebra	N	Invariants
	2	$p=z,\ q=y,\ u=\frac{1}{\dot{y}},\ \frac{dq}{dp}=\frac{\dot{y}}{\dot{z}},\ w=-\ddot{x},\ \frac{du}{dp}=-\frac{\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{d^2q}{dp^2}=\frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1=p,\ m_2=ue^{-q},\ m_3=\frac{dq}{dp},\ m_4=w-2q,\ m_5=e^{-q}\frac{du}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$\ddot{x}=-2y-f(p,ue^{-q},\frac{dq}{dp}),\ \ddot{y}=-\dot{y}^2\dot{z}e^y g(p,ue^{-q},\frac{dq}{dp}),\ \frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}=h(p,ue^{-q},\frac{dq}{dp})$
	3	$p=z,\ q=t-\frac{\dot{y}}{2\dot{x}},\ u=2\dot{x},\ v=\frac{2\dot{x}}{\dot{z}},\ \frac{dq}{dp}=\frac{1}{\dot{z}}+\frac{\dot{y}\ddot{x}}{2\dot{x}^2\dot{z}}-\frac{\ddot{y}}{2\dot{x}\dot{z}},\ \frac{du}{dp}=\frac{2\ddot{x}}{\dot{z}},\ \frac{dv}{dp}=2(\frac{\ddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{z}^3})$
		$m_1=p,\ m_2=uqe^{-u/2},\ m_3=\frac{v}{u}e^{-u/2},\ m_4=\frac{du}{dp},\ m_5=ue^{-v/2}\frac{dq}{dp}+\frac{1}{u}\frac{du}{dp},\ m_6=\frac{1}{u^2}\frac{dv}{dp}-\frac{1}{2u^2}\frac{du}{dp}$
		$\ddot{x}=\frac{\dot{z}}{2}f(p,uqe^{-u/2},\frac{v}{u}e^{-u/2}),\ \ddot{y}=\frac{\dot{y}+\ddot{x}e^{\dot{x}}}{\dot{x}}+2\dot{x}-\dot{z}e^{\dot{x}}g(p,uqe^{-u/2},\frac{v}{u}e^{-u/2}),$
		$\ddot{z}=\frac{\dot{z}\ddot{x}}{\dot{x}}-\frac{\dot{z}^2\ddot{x}}{2\dot{x}}-2\dot{x}\dot{z}^3h(p,uqe^{-u/2},\frac{v}{u}e^{-u/2})$
	4	$p=t,\ q=z,\ u=\frac{1}{2}\dot{x}^2+\dot{y},\ \frac{dq}{dp}=\dot{z},\ w=\ddot{x},\ \frac{du}{dp}=\dot{x}\ddot{x}+\ddot{y},\ \frac{d^2q}{dp^2}=\ddot{z}$
		$m_1=q,\ m_2=ue^{=2p},\ m_3=\frac{dq}{dp},\ m_4=we^{-p},\ m_5=e^{-2p}\frac{du}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$\ddot{x}=e^t f(p,ue^{-2p},\frac{dq}{dp}),\ \ddot{y}=-\dot{x}\ddot{x}+e^{2t}g(p,ue^{-2p},\frac{dq}{dp}),\ \ddot{z}=h(p,ue^{-2p},\frac{dq}{dp})$
$A^b_{4,18}$	1	$p=t,\ q=z,\ u=\dot{x},\ \frac{dq}{dp}=\dot{z},\ w=\frac{\dot{y}\ddot{x}}{\dot{x}^3}-\frac{\ddot{y}}{\dot{x}^2},\ \frac{du}{dp}=\ddot{x},\ \frac{d^2q}{dp^2}=\ddot{z}$
$\lvert b\rvert\le 1$		$m_1=p,\ m_2=q,\ m_3=\frac{dq}{dp},\ m_4=wu^{(1-b)},\ m_5=\frac{1}{u}\frac{du}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$\ddot{x}=\dot{x}\dot{z}f(p,q,\frac{dq}{dp}),\ \ddot{y}=\frac{\dot{y}}{\dot{x}}\ddot{x}-\dot{x}^{(1+b)}g(p,q,\frac{dq}{dp}),\ \ddot{z}=h(p,q,\frac{dq}{dp})$
	2	$Case: b\neq 0,\ \ p=z,\ q=y,\ u=\frac{1}{\dot{y}},\ \frac{dq}{dp}=\frac{\dot{y}}{\dot{z}},\ w=-\ddot{x},\ \frac{du}{dp}=-\frac{\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{d^2q}{dp^2}=\frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}$
$-1<b<1$		$m_1=p,\ m_2=q,\ m_3=\frac{dq}{dp},\ m_4=wu^{((b-1)/b)},\ m_5=\frac{1}{u}\frac{du}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$\ddot{x}=-\dot{y}^{(b-1)/b}f(p,q,\frac{dq}{dp}),\ \ddot{y}=-\dot{y}\dot{z}g(p,q,\frac{dq}{dp}),\ \frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}=h(p,q,\frac{dq}{dp})$

Table 1 continued

Algebra	N	Invariants
	3	$p=z,\ q=y,\ u=\frac{1}{\dot{y}},\ \frac{dq}{dp}=\frac{\dot{y}}{\dot{z}},\ w=-\ddot{x},\ \frac{du}{dp}=-\frac{\ddot{y}}{\dot{y}^2\dot{z}}$
		$m_1=p,\ m_2=ue^{-bq},\ m_3=\frac{dq}{dp},\ m_4=we^{(b-1)q},\ m_5=e^{-bq}\frac{du}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$\ddot{x}=-e^{(1-b)y}f(p,ue^{-bq},\frac{dq}{dp}),\ \ddot{y}=-e^{by}\dot{z}\dot{y}^2g(p,ue^{-bq},\frac{dq}{dp}),\ \frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}h(p,ue^{-bq},\frac{dq}{dp})$
	4	$p=z,\ q=y,\ u=\frac{\dot{x}}{\dot{y}},\ \frac{dq}{dp}=\frac{\dot{y}}{\dot{z}},\ w=\frac{\ddot{x}}{\dot{x}^3},\ \frac{du}{dp}=\frac{\ddot{x}}{\dot{y}\dot{z}}-\frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{d^2q}{dp^2}=\frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1=p,\ m_2=ue^{-q},\ m_3=\frac{dq}{dp},\ m_4=we^{(1-b)q},\ m_5=e^{-q}\frac{du}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$\frac{\ddot{x}}{\dot{x}^3}e^{(1-b)y}=f(p,ue^{-q},\frac{dq}{dp}),\ (\frac{\ddot{x}}{\dot{y}\dot{z}}-\frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}})e^{-y}=g(p,ue^{-q},\frac{dq}{dp}),\ \frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}=h(p,ue^{-q},\frac{dq}{dp})$
$A^b_{4,18}$	5	$Case: b\neq 1,\ \ p=t,\ q=z,\ u=\frac{1}{2}\dot{x}^2-\dot{y},\ \frac{dq}{dp}=\dot{z},\ w=-\ddot{x},\ \frac{du}{dp}=\dot{x}\ddot{x}-\ddot{y},\ \frac{d^2q}{dp^2}=\ddot{z}$
$\lvert b\rvert\leq 1$		$m_1=p,\ m_2=up^{\frac{2b}{b-1}},\ m_3=p\frac{dq}{dp},\ m_4=wp^{\frac{1-2b}{1-b}},\ m_5=p^{\frac{3b-1}{b-1}}\frac{du}{dp},\ m_6=\frac{1}{p^2}\frac{d^2q}{dp^2}$
		$\ddot{x}=-t^{\frac{1-2b}{b-1}}f(p,uq^{\frac{2b}{b-1}},p\frac{dq}{dp}),\ \ddot{y}=\dot{x}\ddot{x}-\dot{z}t^{\frac{1-3b}{b-1}}g(p,uq^{\frac{2b}{b-1}},p\frac{dq}{dp}),\ \ddot{z}=-t^2h(p,uq^{\frac{2b}{b-1}},p\frac{dq}{dp})$
		$Case: b=1,\ \ p=t,\ q=z,\ u=\frac{1}{2}\dot{x}^2-\dot{y},\ \frac{dq}{dp}=\dot{z},\ w=\ddot{x},\ \frac{du}{dp}=\dot{x}\ddot{x}-\ddot{y},\ \frac{d^2q}{dp^2}=\ddot{z}$
		$m_1=p,\ m_2=q,\ m_3=\frac{dq}{dp},\ m_4=wu^{\frac{-1}{2}},\ m_5=\frac{1}{u}\frac{du}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$\ddot{x}=(\frac{1}{2}\dot{x}^2-\dot{y})^{\frac{1}{2}}f(p,q,\frac{dq}{dp}),\ \ddot{y}=\dot{x}\ddot{x}-(\frac{1}{2}\dot{x}^2-\dot{y})g(p,q,\frac{dq}{dp}),\ \ddot{z}=h(p,q,\frac{dq}{dp})$
	6	$Case: b\neq 1,\ \ p=z,\ q=t-\frac{\dot{x}}{\ddot{x}},\ u=\dot{x},\ v=\frac{\dot{x}}{\dot{z}},\ \frac{dq}{dp}=\frac{1}{\dot{z}}+\frac{\dot{y}\ddot{x}}{\dot{x}^2\dot{z}}-\frac{\ddot{y}}{\dot{x}\dot{z}},\ \frac{du}{dp}=\frac{\ddot{x}}{\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{x}^3}$
		$m_1=p,\ m_2=qu^{\frac{b}{b-1}},\ m_3=qv^{-b},\ m_4=u^{\frac{b}{b-1}}\frac{dq}{dp},\ m_5=\frac{1}{u}\frac{du}{dp},\ m_6=u^{\frac{1}{b-1}}\frac{dv}{dp}$
		$\ddot{x}=\dot{x}\dot{z}f(p,qu^{\frac{b}{b-1}},qv^{-b}),\ \ \frac{1}{\dot{z}}+\frac{\dot{y}\ddot{x}}{\dot{x}^2\dot{z}}-\frac{\ddot{y}}{\dot{x}\dot{z}}=\dot{x}^{\frac{b}{b-1}}g(p,qu^{\frac{b}{b-1}},qv^{-b}),$
		$\frac{\ddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{x}^3}=\dot{x}^{\frac{1}{1-b}}h(p,qu^{\frac{b}{b-1}},qv^{-b})$

Table 1 continued

Algebra	N	Invariants
		$Case: b=1,\ \ p=z,\ q=t-\frac{\dot{y}}{\dot{x}},\ u=\dot{x},\ v=\frac{\dot{x}}{\dot{z}},\ \frac{dq}{dp}=\frac{1}{\dot{z}}+\frac{\dot{y}\ddot{x}}{\dot{z}}-\frac{\ddot{y}}{\dot{x}\dot{z}},\ \frac{du}{dp}=\frac{\ddot{x}}{\dot{z}},\ \frac{dv}{dp}=\frac{\dddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{z}^3}$
		$m_1=p,\ m_2=u,\ m_3=\frac{q}{v},\ \ m_4=\frac{1}{q}\frac{dq}{dp},\ \ m_5=\frac{du}{dp},\ m_6=\frac{1}{q}\frac{dv}{dp}$
		$\ddot{x}=\dot{z}f(p,u,\frac{q}{v}),\ \frac{1}{\dot{z}}+\frac{\dot{y}\ddot{x}}{\dot{z}}-\frac{\ddot{y}}{\dot{x}\dot{z}}=(t\dot{x}-\dot{y})\dot{x}g(p,u,\frac{q}{v}),\ \frac{\dddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{z}^3}=h(p,u,\frac{q}{v})$
	7	$Case: b=-1::p=t,\ q=z,\ u=\dot{x},\ \frac{dq}{dp}=\dot{z},\ w=\frac{\dot{y}\ddot{x}}{\dot{x}^3}-\frac{\ddot{y}}{\dot{x}^2},\ \frac{du}{dp}=\ddot{x},\ \frac{d^2q}{dp^2}=\dddot{z}$
		$m_1=p,\ m_2=q,\ m_3=\frac{dq}{dp},\ \ m_4=\frac{1}{u}\frac{du}{dp},\ \ m_5=wu^2-\frac{1}{u}(ln(u))\frac{du}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$\ddot{x}=\dot{x}f(p,q,\frac{dq}{dp}),\quad \ddot{y}=\frac{\dot{y}\ddot{x}}{\dot{x}}-\frac{\ddot{x}}{\dot{x}}\ln(\dot{x})+g(p,q,\frac{dq}{dp}),\quad \dddot{z}=h(p,q,\frac{dq}{dp})$
	8	$Case: b=0,\ \ p=z,\ q=y-\frac{1}{\dot{x}},\ u=\frac{\dot{x}}{\dot{y}},\ v=\frac{\dot{x}}{\dot{z}},\ \frac{dq}{dp}=\frac{\dot{y}}{\dot{z}}+\frac{\ddot{x}}{\dot{x}^2\dot{z}},\ \frac{du}{dp}=\frac{\ddot{x}}{\dot{y}\dot{z}}-\frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{dv}{dp}=\frac{\dot{y}\ddot{x}}{\dot{x}\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}$
$0\neq c\in\mathcal{R}$		$m_1=p,\ m_2=ue^{-cq},\ m_3=\frac{u}{v},\ \ m_4=\frac{dq}{dp},\ \ m_5=\frac{1}{u}\frac{du}{dp},\ m_6=\frac{1}{u}\frac{dv}{dp}$
		$\frac{\dot{y}}{\dot{z}}+\frac{\ddot{x}}{\dot{x}^2\dot{z}}=f(p,ue^{-cq},\frac{u}{v}),\quad \frac{\ddot{x}}{\dot{x}\dot{z}}-\frac{\ddot{y}}{\dot{y}\dot{z}}=g(p,ue^{-cq},\frac{u}{v}),\quad \frac{\dot{y}\ddot{x}}{\dot{x}\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}=h(p,ue^{-cq},\frac{u}{v})$
	9	$p=z,\ q=y-\frac{1}{\dot{x}},\ u=\frac{\dot{x}}{\dot{y}},\ v=\frac{\dot{x}}{\dot{z}},\ \frac{dq}{dp}=\frac{\ddot{z}}{\dot{x}^2\dot{z}}+\frac{\dot{y}}{\dot{z}},\ \frac{du}{dp}=\frac{\ddot{x}}{\dot{y}\dot{z}}-\frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{z}^3}$
		$m_1=ue^{-p},\ m_2=bp-q,\ m_3=\frac{v}{p},\ \ m_4=e^{-bp}\frac{dq}{dp},\ \ m_5=e^{(b-1)p}\frac{du}{dp},\ m_6=e^{-p}\frac{dv}{dp}$
		$e^{-bz}\left(\frac{\ddot{x}}{\dot{x}\dot{z}}+\frac{\dot{y}}{\dot{z}}\right)=f(ue^{-p},bp-q,\frac{v}{p}),\quad e^{(b-1)z}\left(\frac{\ddot{x}}{\dot{y}\dot{z}}-\frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}}\right)=g(ue^{-p},bp-q,\frac{v}{p}),$
		$e^{-z}\left(\frac{\ddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{z}^3}\right)=h(ue^{-p},bp-q,\frac{v}{p})$
	10	$Case: b=1,p=z,q=y,u=\frac{\dot{x}^2}{2\dot{y}^2}-\frac{1}{\dot{y}},\frac{dq}{dp}=\frac{\dot{y}}{\dot{z}},w=\frac{\ddot{x}}{\dot{y}^2}-\frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}},\frac{du}{dp}=\frac{\dot{x}\ddot{x}}{\dot{y}^2\dot{z}}+\frac{(\dot{y}-\dot{x}^2)\ddot{y}}{\dot{y}^3\dot{z}},\ \frac{d^2q}{dp^2}=\frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1=q,\ m_2=\frac{dq}{dp},\ m_3=ue^{-p},\ \ m_4=we^{-p},\ \ m_5=e^{-2p}\frac{du}{dp},\ m_6=\frac{d^2q}{dp^2}$
		$e^{-z}\left(\frac{\ddot{x}}{\dot{y}^2}-\frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}}\right)=f(q,\frac{dq}{dp},ue^{-p}),\quad e^{-2z}\left(\frac{\dot{x}\ddot{x}}{\dot{y}^2\dot{z}}+\frac{(\dot{y}-\dot{x}^2)\ddot{y}}{\dot{y}^3\dot{z}}\right)=g(q,\frac{dq}{dp},ue^{-p}),$
		$\frac{\ddot{y}}{\dot{z}^2}-\frac{\dot{y}\ddot{z}}{\dot{z}^3}=h(q,\frac{dq}{dp},ue^{-p})$

Table 1 continued

Algebra	N	Invariants
	11	$Case: b=-1,\ p=z,\ q=y-\frac{1}{\dot{x}},\ u=\frac{\dot{x}}{\dot{y}},\ v=\frac{\dot{x}}{\dot{z}},\ \frac{dq}{dp}=\frac{\dddot{z}}{\dot{x}^2\dot{z}}+\frac{\dot{y}}{\dot{z}},\ \frac{du}{dp}=\frac{\ddot{x}}{\dot{y}\dot{z}}-\frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{z}^3}$ $m_1=p,\ m_2=\frac{v^2}{u},\ m_3=\frac{q}{u^{1/2}}-\frac{1}{2vu^{1/2}},\ m_4=\frac{1}{u}\frac{du}{dp},\ m_5=u^{1/2}\frac{dv}{dp},\ m_6=u^{1/2}\left(\frac{dq}{dp}-\frac{v^2}{2}\frac{dv}{dp}\right)$ $\frac{\dddot{x}}{\dot{x}\dot{z}}-\frac{\ddot{y}}{\dot{y}\dot{z}}=f(p,\frac{v^2}{u},\frac{q}{u^{1/2}}-\frac{1}{2vu^{1/2}}),\quad \frac{\dot{x}^{1/2}}{\dot{y}^{1/2}}\left(\frac{\ddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{y}^3}\right)=g(p,\frac{v^2}{u},\frac{q}{u^{1/2}}-\frac{1}{2vu^{1/2}}),$ $\frac{\dot{x}^{1/2}}{\dot{y}^{1/2}}\left(\frac{(2\dot{z}^3-\dot{x}^4)\ddot{x}}{2\dot{x}^2\dot{z}}+\frac{\dot{y}}{\dot{z}}+\frac{\dot{x}^3\ddot{z}}{2\dot{z}^5}\right)=h(p,\frac{v^2}{u},\frac{q}{u^{1/2}}-\frac{1}{2vu^{1/2}})$
	12	$Case: b=0,\ p=z,\ q=y-\frac{1}{\dot{x}},\ u=\frac{\dot{x}}{\dot{y}},\ v=\frac{\dot{x}}{\dot{z}},\ \frac{dq}{dp}=\frac{\dddot{z}}{\dot{x}^2\dot{z}}+\frac{\dot{y}}{\dot{z}},\ \frac{du}{dp}=\frac{\ddot{x}}{\dot{y}\dot{z}}-\frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{z}^3}$ $m_1=p,\ m_2=e^{-\frac{q}{p}}v,\ m_3=e^{-\frac{v}{u}}v,\ \ m_4=\frac{dq}{dp}-\frac{q}{p},\ \ m_5=\frac{1}{v}\frac{dv}{dp},\ m_6=\frac{v}{u^2}\frac{du}{dp}-\frac{1}{u}\frac{dv}{dp}$ $\frac{\ddot{x}}{\dot{x}^2\dot{z}}+\frac{\dot{y}}{\dot{z}}-\frac{y}{z}+\frac{1}{xz}=f(p,e^{-\frac{q}{p}}v,e^{-\frac{v}{u}}v),\quad \frac{\dddot{x}}{\dot{y}\dot{z}}-\frac{\dot{x}\ddot{z}}{\dot{y}\dot{z}^2}=g(p,e^{-\frac{q}{p}}v,e^{-\frac{v}{u}}v),$ $\frac{\dot{y}(1-\dot{z})\ddot{x}}{\dot{x}\dot{z}^3}-\frac{\ddot{y}}{\dot{z}^3}+\frac{\dot{y}\ddot{z}}{\dot{z}^3}=h(p,e^{-\frac{q}{p}}v,e^{-\frac{v}{u}}v)$
$A_{4,19}$	1	$p=z,\ q=t-\frac{\dot{y}}{\dot{x}},\ u=\dot{x},\ v=\frac{\dot{x}}{\dot{z}},\ \frac{dq}{dp}=\frac{1}{\dot{z}}+\frac{\ddot{x}\dot{y}}{\dot{x}^2\dot{z}}-\frac{\ddot{y}}{\dot{x}\dot{z}},\ \frac{du}{dp}=\frac{\ddot{x}}{\dot{z}},\ \frac{dv}{dp}=\frac{\ddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{z}^3},$ $m_1=p,\ \ m_2=\frac{uqe^{-a\arctan u}}{\sqrt{1+u^2}},\ m_3=\frac{v\sqrt{1+u^2}e^{-a\arctan u}}{u},\ m_4=\frac{1}{(1+u^2)}\frac{du}{dp},\ \ m_5=\frac{(1+u^2)}{qu}\frac{dq}{dp}+\frac{1}{u^2}\frac{du}{dp}$ $m_6=e^{-\arctan u}\frac{dv}{dp}$ $\ddot{x}=\dot{z}(1+\dot{x}^2)f(p,\frac{uqe^{-a\arctan u}}{\sqrt{1+u^2}},\frac{v\sqrt{1+u^2}e^{-a\arctan u}}{u}),$ $\frac{(1+\dot{x}^2)}{(t-\frac{\dot{y}}{\dot{x}})\dot{x}}(\frac{1}{z}+\frac{\ddot{x}\dot{y}}{\dot{x}^2\dot{z}}-\frac{\ddot{y}}{\dot{x}\dot{z}})+\frac{\dddot{x}}{\dot{x}^2\dot{z}}=g(p,\frac{uqe^{-a\arctan u}}{\sqrt{1+u^2}},\frac{v\sqrt{1+u^2}e^{-a\arctan u}}{u}),$ $e^{-\arctan u}(\frac{\dddot{x}}{\dot{z}^2}-\frac{\dot{x}\ddot{z}}{\dot{z}^3})=h(p,\frac{uqe^{-a\arctan u}}{\sqrt{1+u^2}},\frac{v\sqrt{1+u^2}e^{-a\arctan u}}{u})$

Table 1 continued

Algebra	N	Invariants
	2	$p = t,\ q = z,\ u = \frac{\dot{x}^2}{2} + \dot{y},\ \frac{dq}{dp} = \dot{z},\ w = \ddot{x},\ \frac{du}{dp} = \dot{x}\ddot{x} + \ddot{y},\ \frac{d^2q}{dp^2} = \ddot{z}$
		$m_1 = q,\ m_2 = u(1+q^2)e^{-2a\arctan p},\ m_3 = (1+p^2)\frac{dq}{dp},\ \ m_4 = we^{-a\arctan p}(1+p^2)^{\frac{3}{2}}$
		$m_5 = e^{-2a\arctan p}(1+p^2)((1+p^2)\frac{du}{dp} + 2up),\ m_6 = (1+p^2)((1+p^2)\frac{d^2q}{dp^2} + 2p\frac{dq}{dp})$
		$\ddot{x} = e^{a\arctan p}(1+t^2)^{\frac{-3}{2}} f(q, e^{-2a\arctan p}u(1+p^2), (1+p^2)\frac{dq}{dp}),$
		$e^{-2a\arctan t}(1+t^2)((1+t^2)(\dot{x}\ddot{x} + \ddot{y}) + 2t(\frac{\dot{x}^2}{2} + \dot{y})) = g(q, e^{-2a\arctan p}u(1+p^2), (1+p^2)\frac{dq}{dp}),$
		$(1+t^2)^2\ddot{z} + 2t\dot{z}(1+t^2) = h(q, e^{-2a\arctan p}u(1+p^2), (1+p^2)\frac{dq}{dp})$
$A_{4,20}$	2	$p = z,\ q = y,\ u = \dot{x},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ \frac{du}{dp} = \frac{\ddot{x}}{\dot{z}},\ w = \frac{\ddot{y}}{\dot{y}^2},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ \ m_2 = q,\ \ m_3 = \frac{dq}{dp},\ \ m_4 = \frac{1}{(1+u^2)}\frac{du}{dp},\ \ m_5 = w - \frac{u(1+u^2)^{-1}}{\frac{dq}{dp}}\frac{du}{dp},\ m_6 = \frac{d^2q}{dp^2}$
		$\ddot{x} = \dot{z}(1+\dot{x}^2)f(p, q, \frac{dq}{dp}),\ \ \ddot{y} = \frac{\dot{x}\dot{y}}{(1+\dot{x}^2)}\ddot{x} + \dot{y}^2 g(p, q, \frac{dq}{dp}),\ \ \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3} = h(p, q, \frac{dq}{dp})$
	3	$p = t,\ q = z,\ u = \frac{\dot{y}}{y},\ \frac{dq}{dp} = \dot{z},\ w = \frac{\ddot{x}}{y},\ \frac{du}{dp} = \frac{\ddot{y}}{y} - \frac{\dot{y}^2}{y^2},\ \frac{d^2q}{dp^2} = \ddot{z}$
		$m_1 = q,\ m_2 = u(1+q^2),\ m_3 = (1+q^2)\frac{dq}{dp},\ \ m_4 = w(1+p^2)^{\frac{3}{2}}$
		$m_5 = (1+p^2)((1+p^2)\frac{du}{dp} + 2up),\ m_6 = (1+p^2)((1+p^2)\frac{d^2q}{dp^2} + 2p\frac{dq}{dp})$
		$\ddot{x} = y(1+t^2)^{\frac{-3}{2}} f(q, u(1+q^2), (1+q^2)\frac{dq}{dp}),\ \ddot{y} = \frac{\dot{y}^2}{y} - \frac{2t\dot{y}}{(1+t^2)} + \frac{y}{(1+t^2)^2}g(q, u(1+q^2), (1+q^2)\frac{dq}{dp}),$
		$\ddot{z} = \frac{2t\dot{z}}{(1+t^2)} + \frac{1}{(1+t^2)^2}h(q, u(1+q^2), (1+q^2)\frac{dq}{dp})$
	4	$p = z,\ q = y,\ u = \dot{x},\ \frac{dq}{dp} = \frac{\dot{y}}{\dot{z}},\ w = \frac{\ddot{y}}{\dot{y}^2},\ \frac{du}{dp} = \frac{\ddot{x}}{\dot{z}},\ \frac{d^2q}{dp^2} = \frac{\ddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3}$
		$m_1 = p,\ m_2 = q + \arctan u,\ m_3 = \frac{dq}{dp},\ \ m_4 = \frac{1}{(1+u^2)}\frac{du}{dp},\ m_5 = w - \frac{u}{\frac{dq}{dp}(1+u^2)}\frac{du}{dp},\ \ m_6 = \ \frac{d^2q}{dp^2}$
		$\ddot{x} = \dot{z}(1+\dot{x}^2)f(p, q + \arctan u, \frac{dq}{dp}),\ \ \ddot{y} = \frac{\dot{x}\dot{y}}{(1+\dot{x}^2)}\ddot{x} + \dot{y}^2 g(p, q + \arctan u, \frac{dq}{dp})$

Table 1 continued

Algebra	N	Invariants
		$\frac{\dddot{y}}{\dot{z}^2} - \frac{\dot{y}\ddot{z}}{\dot{z}^3} = h(p, q + \arctan u, \frac{dq}{dp})$
	5	$p = z,\ q = \frac{\dot{x}}{\dot{y}},\ u = \frac{1}{\dot{y}},\ v = t\dot{z},\ \frac{dq}{dp} = \frac{\ddot{x}}{\dot{y}\dot{z}} - \frac{\dot{x}\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{du}{dp} = \frac{-\ddot{y}}{\dot{y}^2\dot{z}},\ \frac{dv}{dp} = 1 + \frac{t\ddot{z}}{\dot{z}},\quad c \in \mathcal{R}$
		$m_1 = p,\ m_2 = \frac{u}{\sqrt{1+q^2}} e^{c \arctan q},\ m_3 = v,\ \ m_4 = \frac{\frac{dq}{dp}}{1+q^2}$
		$m_5 = ((1+q^2)\frac{du}{dp} - uq\frac{dq}{dp})\frac{1}{\sqrt{1+u^2}} e^{c \arctan u},\ \ m_6 = \frac{dv}{dp}$
		$\frac{(\dot{y}\ddot{x} - \dot{x}\ddot{y})}{\dot{z}} = (\dot{x}^2 + \dot{y}^2) f(p, \frac{u}{\sqrt{1+q^2}} e^{c \arctan q}, v)$
		$-\frac{\ddot{y}}{\dot{y}^2\dot{z}} - \dot{x}\frac{(\dot{y}\ddot{y} + \dot{x}\ddot{x})}{(\dot{x}^2+\dot{y}^2)\dot{y}^2\dot{z}} = e^{-c \arctan(\frac{\dot{x}}{\dot{y}})} g(p, \frac{u}{\sqrt{1+q^2}} e^{c \arctan q}, v),\ \ 1 + \frac{t\ddot{z}}{\dot{z}} = h(p, \frac{u}{\sqrt{1+q^2}} e^{c \arctan q}, v)$
	6	$p = z,\ q = \dot{x},\ u = e^y\dot{y},\ v = \frac{\dot{z}}{\dot{y}},\ \frac{dq}{dp} = \frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp} = \frac{e^y\dot{y}^2}{\dot{z}} + \frac{e^y\ddot{y}}{\dot{z}},\ \frac{dv}{dp} = \frac{\ddot{z}}{\dot{y}\dot{z}} - \frac{\ddot{y}}{\dot{y}^2}$
		$m_1 = p + \arctan q,\ m_2 = u(1+q^2)^{\frac{1}{2}},\ m_3 = v,\ \ m_4 = \frac{1}{(1+q^2)}\frac{dq}{dp}$
		$m_5 = \frac{1}{(1+q^2)^{\frac{1}{2}}}\frac{du}{dp} - \frac{uq}{(1+q^2)^{\frac{3}{2}}}\frac{dq}{dp},\ \ m_6 = \frac{dv}{dp}$
		$\frac{1}{(1+\dot{x}^2)}\frac{\ddot{x}}{\dot{z}} = f(p + \arctan u, u(1+q^2)^{\frac{1}{2}}, v),\ \ \frac{\ddot{z}}{\dot{y}\dot{z}} - \frac{\ddot{y}}{\dot{y}^2} = g(p + \arctan u, u(1+q^2)^{\frac{1}{2}}, v)$
		$\frac{1}{(1+\dot{x}^2)^{\frac{1}{2}}}\left(\frac{e^y\dot{y}^2}{\dot{z}} + \frac{e^y\ddot{y}}{\dot{z}}\right) - \frac{\dot{x}\ddot{x}}{\dot{x}(1+\dot{x}^2)^{\frac{3}{2}}} = h(p + \arctan u, u(1+q^2)^{\frac{1}{2}}, v)$
	7	$p = z,\ q = \dot{x},\ u = e^y\dot{y},\ v = \frac{\dot{z}}{\dot{y}},\ \frac{dq}{dp} = \frac{\ddot{x}}{\dot{z}},\ \frac{du}{dp} = \frac{e^y\dot{y}^2}{\dot{z}} + \frac{e^y\ddot{y}}{\dot{z}},\ \frac{dv}{dp} = \frac{\ddot{z}}{\dot{y}\dot{z}} - \frac{\ddot{y}}{\dot{y}^2}$
		$m_1 = p,\ m_2 = \arctan q + \frac{1}{v},\ m_3 = e^{p(\arctan q)}\frac{uv}{(1+q^2)^{\frac{1}{2}}},\ \ m_4 = \frac{1}{(1+q^2)}\frac{dq}{dp}$
		$m_5 = \frac{1}{v}\frac{dv}{dp},\ \ m_6 = \frac{dv}{dp}$
		$\frac{1}{(1+\dot{x}^2)}\frac{\ddot{x}}{\dot{z}} = f(p, \arctan q + \frac{1}{v}, \frac{e^{p(\arctan q)}uv}{(1+q^2)^{\frac{1}{2}}}),\ \ \frac{\ddot{z}}{\dot{z}^3} - \frac{\ddot{y}}{\dot{z}^2} = g(p, \arctan q + \frac{1}{v}, \frac{e^{p(\arctan q)}uv}{(1+q^2)^{\frac{1}{2}}}), v)$
		$\frac{\ddot{z}}{\dot{z}^2} - \frac{\dot{x}\ddot{x}}{\dot{z}(1+\dot{x}^2)} = h(p, \arctan q + \frac{1}{v}, \frac{e^{p(\arctan q)}uv}{(1+q^2)^{\frac{1}{2}}})$

Table 2 Special cases

S. no.	Algebra	Values of f, g, h	System of equations	Dimension
1	$A_{4,1}^4$	$f = g = h = k_1$	$\ddot{x} = k_1\dot{z},\ \ \ddot{y} = k_1\dot{z},\ \ \ddot{z} = k_1\dot{z}$	$k_1 = \frac{1}{T}$
2	$A_{4,2}^3$	$f = k_1,$	$\ddot{x} = \frac{1}{t}k_1,\ \ \ddot{y} = -\frac{\dot{y}}{t} + \frac{\dot{z}}{t}k_2$	$k_1 = \frac{L}{T},$ where as k_2 and k_3 are
		$g = k_2,\ h = k_3$	$\ddot{z} = \frac{\dot{z}}{t}(k_3 - 1)$	arbitrary dimensionless constants
3	$A_{4,2}^4$	$f = k_1,\ g = h = k_2$	$\ddot{x} = \dot{x}k_1,\ \ \ddot{y} = k_2,\ \ \ddot{z} = k_2$	$k_1 = \frac{1}{T},\ \ k_2 = \frac{L}{T^2}$
		$\ddot{\mu}(t) = 0$		
4	$A_{4,3}^5$	$f = k_1,\ g = k_2,$	$\ddot{x} = k_1\dot{x}\dot{z},\ \ddot{y} = (k_2 + k_1\dot{z})\dot{y}$	$k_1 = \frac{1}{L},\ \ k_2 = \frac{1}{T},\ \ k_3 = \frac{T}{L^2}$
		$h = k_3$	$\ddot{z} = -k_3\dot{z}^3$	
5	$A_{4,4}^1$	$f = k_1,\ g = h = k_2$	$\ddot{x} = -k_1\dot{x}^3,\ \ \ddot{y} = -k_1\dot{y}\dot{z}^2 - k_2\frac{\dot{z}\dot{y}^2}{\dot{x}}$	$k_1 = \frac{T}{L^2},\ \ k_2 = \frac{1}{T}$
			$\ddot{z} = -k_2(\frac{\dot{z}^3}{\dot{y}} + \frac{\dot{y}\dot{z}^2}{\dot{x}}) - k_1\dot{z}\dot{x}^2$	
6	$A_{4,4}^3$	$f = h = k_1,$	$\ddot{x} = (k_2\dot{x} + k_1)\dot{z}\dot{y},\ \ddot{y} = k_2\dot{y}^2\dot{z}$	$k_1 = \frac{1}{L},\ \ k_2 = \frac{T}{L^2}$
		$g = k_2$	$\ddot{z} = (k_2\dot{z} + k_1)\dot{z}\dot{y}$	
7	$A_{4,6}^{1,2}$	$f = g = k_1,\ h = k_2$	$t\ddot{x} = k_1,\ \ t\ddot{y} = k_1,\ \ t^2\ddot{z} = k_2$	$k_1 = \frac{L}{T},\ \ k_2 = L$
8	$A_{4,6}^{1,3}$	$f = k_1,\ g = h = k_2$	$\ddot{x} = k_1\dot{z},\ \ddot{y} = k_2\dot{y}^2$	$k_1 = \frac{1}{T},\ \ k_2 = \frac{1}{L}$
			$\ddot{z} = k_2(\dot{z}\dot{y} - \frac{\dot{z}^3}{\dot{y}})$	
9	$A_{4,6}^{a,2}$	$f = k_1,\ g = k_2,$	$\ddot{x} = -k_1\dot{z},\ \ddot{y} = (a-1)\frac{\dot{y}}{t} - k_1k_2\dot{y}\dot{z}$	$k_1 = \frac{1}{T},\ \ k_2 = \frac{T}{L},\ \ k_3 = \frac{1}{L}$
		$h = k_3$	$\ddot{z} = (a-1)\frac{\dot{z}}{t} - k_1k_2\dot{z}^2 + k_3\dot{y}\dot{z}$	
10	$A_{4,6}^{a,4}$	$f = k_1 = h,\ g = k_2$	$\ddot{x} = k_1\dot{x}\dot{y},\ \ddot{y} = k_1\frac{\dot{y}^2}{(1-a)} + k_2\frac{\dot{z}\dot{y}^{3-a}}{(a-1)\dot{x}}$	$k_1 = \frac{1}{L},\ \ k_2 = \frac{T^{1-a}}{L^{2-a}}$
			$\ddot{z} = k_1\frac{\dot{z}\dot{y}}{(1-a)} + k_2\frac{\dot{z}^2}{(a-1)\dot{x}\dot{y}^{a-2}} + k_1\frac{\dot{z}^3}{\dot{y}}$	

Table 2 continued

S. no.	Algebra	Values of f, g, h	System of equations	Dimension
11	$A_{4,6}^{a,6}$	$f = k_1,\ g = k_2,$	$\ddot{x} = k_1 \dot{x}^3 x^{\frac{2a-1}{1-a}},\ \ \ddot{y} = \frac{\dot{y}\ddot{z}}{\dot{z}} + k_2 \dot{z}^2$	$k_1 = \frac{T}{L^{\frac{1}{1-a}}},\ \ k_2 = \frac{1}{L},$ where as k_3 is
		$h = k_3$	$\ddot{z} = \frac{\dot{z}\ddot{x}}{\dot{x}} + \frac{\dot{x}\dot{z}}{x}(k_3 - 1)$	an arbitrary constant
12	$A_{4,8}^{2}$	$f = k_1,\ g = k_2,$	$\ddot{x} = k_1 \dot{y}^3,\ \ \ddot{y} = \frac{t\dot{y}\ddot{x}}{(t\dot{x}-x)} - k_2 \frac{\dot{z}\dot{y}^2}{(t\dot{x}-x)}$	$k_1 = \frac{T}{L^2},\ \ k_2 = \frac{T}{L},\ \ k_3 = \frac{1}{L}$
		$h = k_3$	$\ddot{z} = \frac{\dot{z}\ddot{y}}{\dot{y}} + k_3 \frac{\dot{z}^3}{\dot{y}}$	
13	$A_{4,8}^{7}$	$f = k_1,\ g = h = k_2$	$\ddot{x} = \frac{\dot{x}^2}{2x} - 2k_1 x \dot{y}^2,\ \ \ddot{y} = -\frac{\dot{x}\dot{y}}{x} - k_2 \dot{y}\dot{z}$	$k_1 = \frac{1}{L^2},\ \ k_2 = \frac{1}{L}$
			$\ddot{z} = \frac{\dot{z}\ddot{y}}{\dot{y}} - k_2 \frac{\dot{z}^3}{\dot{y}}$	
14	$A_{4,12}^{2}$	$f = k_1,\ g = h = k_2$	$\ddot{x} = \frac{-\dot{y}\ddot{y}}{(\dot{y}-\dot{x})} + \frac{k_1 \dot{y}\dot{z} e^{-\frac{\dot{x}}{\dot{y}}}}{(\dot{y}-\dot{x})},\ \ \ddot{y} = k_2 \dot{y}^2$	$k_1 = \frac{1}{T},\ \ k_2 = \frac{1}{L}$
			$\ddot{z} = \frac{\dot{z}\ddot{y}}{\dot{y}} - k_2 \frac{\dot{z}^3}{\dot{y}}$	
15	$A_{4,12}^{4}$	$f = k_1,\ g = k_2,$	$\ddot{x} = \frac{\dot{x}\ddot{y}}{\dot{y}} + \frac{\dot{x}^2}{x} - k_1 \frac{\dot{x}^5}{x^4 \dot{y}^3}$	$k_1 = L^3,\ \ k_2 = \frac{1}{T},\ \ k_3 = \frac{1}{L}$
		$h = k_3$	$\ddot{y} = -\frac{\dot{x}\dot{y}}{x} + \frac{\dot{y}\ddot{z}}{\dot{z}} + k_2 \frac{\dot{y}\dot{z}}{x},\ \ddot{z} = \frac{\dot{z}\ddot{y}}{\dot{y}} - k_3 \frac{\dot{z}^3}{\dot{y}}$	
16	$A_{4,14}^{(a,b),2}$	$-1 \le a < b < 1,\ f = k_1$	$\ddot{x} = \dot{x}\dot{z}k_1,\ \ \ddot{y} = \dot{x}^{\frac{(b-1)}{(a-1)}} \dot{z}k_2$	$k_1 = \frac{1}{L},\ \ k_2 = \frac{T^{\frac{(b-a)}{(a-1)}}}{L^{\frac{(b-1)}{(a-1)}}},\ \ k_3 = \frac{T^{\frac{a}{(1-a)}}}{L^{\frac{1}{(1-a)}}}$
		$ab \ne 0. g = k_2, h = k_3$	$\ddot{z} = \dot{x}^{\frac{1}{1-a}} \dot{z}k_3$	
17	$A_{4,14}^{(a,a),2}$	$-1 \le a < 1, a \ne 0$	$\ddot{x} = \dot{x}\dot{z}k_1,\ \ddot{y} = \dot{x}\dot{z}k_1$	$k_1 = \frac{1}{L},\ \ k_2 = \frac{T^{\frac{a}{(1-a)}}}{L^{\frac{1}{(1-a)}}}$
		$f = g = k_1,\ h = k_2$	$\ddot{z} = \dot{x}^{\frac{1}{1-a}} \dot{z}k_2$	
18	$A_{4,14}^{(a,a),3}$	$-1 \le a < 1, a \ne 0$	$\ddot{x} = -\frac{\dot{z}t^{\frac{a}{1-a}}}{(1-a)} k_1,\ \ddot{y} = \frac{a\dot{y}}{t} + \frac{\dot{z}t^{\frac{1}{a-1}}}{(a-1)} k_2$	$k_1 = T^{\frac{1}{a-1}},\ \ k_2 = T^{\frac{-a}{a-1}},$ where as

Table 2 continued

S. no.	Algebra	Values of f, g, h	System of equations	Dimension
		$f = k_1, g = k_2, h = k_3$	$\ddot{z} = -\frac{\dot{z}}{t} + \frac{\dot{z}}{(1-a)t}k_3$	k_3 is an arbitrary constant
19	$A_{4,14}^{(a,a),4}$	$-1 \leq a < 1, a \neq 0$	$\ddot{x} = \dot{x}k_1, \quad \ddot{y} = \dot{x}^a k_2, \quad \ddot{z} = k_3$	$k_1 = \frac{1}{T}, \quad k_2 = \frac{L^{1-a}}{T^{2-a}}, \quad k_3 = \frac{L}{T^2}$
		$f = k_1, g = k_2, h = k_3$		
20	$A_{4,14}^{(a,1),2}$	$-1 < a < 1, a \neq 0$	$\ddot{x} = \dot{x}\dot{z}k_1, \quad \ddot{y} = \dot{z}k_2, \quad \ddot{z} = \dot{x}^{\frac{1}{1-a}}k_3$	$k_1 = \frac{1}{L}, \quad k_2 = \frac{1}{T}, \quad k_3 = \frac{L^{1-\frac{1}{1-a}}}{T^{2-\frac{1}{1-a}}}$
		$f = k_1, g = k_2, h = k_3$		
21	$A_{4,14}^{(a,1),3}$	$-1 < a < 1, a \neq 0$	$\ddot{x} = \dot{y}^{\frac{1}{a}}k_1, \quad \ddot{y} = \dot{y}k_2, \quad \ddot{z} = k_3$	$k_1 = \frac{L^{1-\frac{1}{a}}}{T^{2-\frac{1}{a}}}, \quad k_2 = \frac{1}{T}, \quad k_3 = \frac{T}{L^2}$
		$f = k_1, g = k_2, h = k_3$		
22	$A_{4,14}^{(a,1),4}$	$-1 < a < 1, a \neq 0$	$\ddot{x} = \frac{(a-2)\dot{x}}{(a-1)t} + \frac{t\dot{z}\dot{x}^2k_1}{(a-1)^{-1}}$	$k_1 = \frac{1}{L^2}, \quad k_2 = \frac{T^{\frac{2a}{1-a}-1}}{L^a}$, where as
		$f = k_1, g = k_2, h = k_3$	$\ddot{y} = \frac{t^{\frac{a}{a-1}-a}\dot{x}^{a+1}k_2}{(a-1)^{1-a}}$	k_3 is an arbitrary constant
			$\ddot{z} = -\frac{\dot{z}}{t} + \frac{\dot{z}}{(a-1)t}k_3$	
23	$A_{4,18}^{b,1}$	$f = k_1, g = k_2, h = k_3$	$\ddot{x} = \dot{x}\dot{z}k_1, \quad \ddot{y} = \dot{y}\dot{z}k_1 - \dot{x}^{(1+b)}k_2$	$k_1 = \frac{1}{L}, \quad k_2 = \frac{L^{-b}}{T^{1-b}}, \quad k_3 = \frac{L}{T^2}$
	$\lvert b\rvert \leq 1$		$\ddot{z} = k_3$	
24	$A_{4,18}^{b,2}$	$-1 < b < 1, b \neq 0$	$\ddot{x} = -\dot{y}^{\frac{(b-1)}{b}}k_1, \quad \ddot{y} = -\dot{y}\dot{z}k_2$	$k_1 = \frac{L^{1-\frac{(b-1)}{b}}}{T^{2-\frac{(b-1)}{b}}}, \quad k_2 = \frac{1}{L}, \quad k_3 = \frac{L}{T^2}$
		$f = k_1, g = k_2, h = k_3$	$\ddot{z} = -\dot{z}^2k_2 - k_3$	
25	$A_{4,20}^{5}$	$f = k_1, g = k_2, h = k_3$	$\frac{\dot{y}\ddot{x}}{\dot{z}} - \frac{\dot{x}\ddot{y}}{\dot{z}} = (\dot{x}^2 + \dot{y}^2)k_1$	$k_1 = \frac{1}{L}, \quad k_2 = \frac{T}{L^2}$, where as k_3
			$\frac{\ddot{y}}{\dot{y}^2\dot{z}} + \frac{\dot{x}(\dot{y}\ddot{y}+\dot{x}\ddot{x})}{(\dot{x}^2+\dot{y}^2)\dot{y}^2\dot{z}} = -e^{-c\arctan(\frac{\dot{x}}{\dot{y}})}k_2$	is an arbitrary constant
			$1 + \frac{t\ddot{z}}{\dot{z}} = k_3$	

arbitrary functions namely, f, g, h as dimensional constants. In addition, the algebraic properties of these peculiar canonical forms including integrability, are analysed in a detailed manner both mathematically and in physical terms. To best of our knowledge, these are new contributions.

The algebraic properties of these classified canonical forms including integrability, are investigated in detail: two physical examples from mechanics are provided to illustrate the results related to classification of nonsingular systems of three second-order ODEs possessing Lie algebra of dimension 4. Moreover, the graphical discussion is also is included for mechanics of these particles.

Remarks on Table 1 and Table 2

- f, g, h, k, ψ and ϕ are arbitrary functions with specified conditions as stated in the corresponding realization.
- p, q, u, v, w, $\frac{dq}{dp}$, $\frac{du}{dp}$, $\frac{dv}{dp}$ and $\frac{d^2q}{dp^2}$ "form a set of differential invariants for the associated three-dimensional subalgebras [12] in each case of the Lie algebra of dimension 4 discussed."
- m_1, m_2, m_3, m_4, m_5 and m_6 form a set of second-order nonsingular differential invariants for the corresponding system of three second-order ODEs which possess Lie algebras of dimension 4 in each case.
- k_1, k_2, k_3 are constants (dimensional or non-dimensional) which may or may not be linearly independent of each other.

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Declarations

Conflict of interest The authors (Muhammad Ayub, Zahida Sultan, F.M Mahomed and Saima Ijaz) declare that they have no Conflict of interest and no identified competing financial interests or personal associations that could have appeared to influence the work described in this paper.

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