Overpartition analogues of q-Binomial coefficients and their applications

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A Dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, South Africa, in fulfillment of the requirement for the award of the Degree of Master of Science (M.Sc.) in Pure Mathematics.

September 2018

Declaration

I declare that this Dissertation entitled "Overpartition analogues of q-Binomial coefficients and their Applications" is my own, unaided work carried out under the supervision of Prof A.O Munagi. It is being submitted for the award of the Degree of Masters of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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Date: ...28... day of ...May... 2019

Acknowledgement

I would like to express how thankful I am to my supervisor Prof. A.O Munagi for introducing me to partition theory. His guidance, support and motivation has helped me a lot during the period of this project. I would also like to thank the NRF for funding me.

Abstract

In partition theory it is well known that the Gaussian polynomials (or q-binomial coefficients) $\begin{bmatrix} m+n \\ n \end{bmatrix}$ generate integer partitions that fit inside an $m \times n$ rectangle, that is, partitions with largest part at most m and the number of parts does not exceed n. In this dissertation, we study the overpartition extension of these polynomials known as *over q-binomial coefficients.* These are generating functions that enumerate overpartitions fitting inside an $m \times n$ rectangle. First we discuss the ordinary q-binomial coefficients. Then we examine the over q-binomial coefficients with respect to their motivations, q-analogues of classical identities, and various combinatorial and analytic applications.

Contents

Chapter 1

Introduction

1.1 The q-binomial coefficients

Definition 1.1.1. A partition of a positive integer n is defined to be an integer sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

We call the summands λ_i *parts*.

Example 1.1.1. There are 7 partitions of 5:

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1).$$

The Gaussian polynomials (also called q-binomial coefficients or Gaussian coefficients) are defined, for non-negative m and n, as

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \frac{(1-q^{m+n})(1-q^{m+n-1})\cdots(1-q^{m+1})}{(1-q^n)(1-q^{n-1})\cdots(1-q)}. \tag{1.1}$$

These polynomials have many applications in mathematics. In partition theory, we interpret (1.1) as the generating function for partitions fitting inside an $m \times n$ rectangle, i.e., partitions that have parts of size $\leq m$ and number of parts $\leq n$.

In this dissertation, we study an extension of the q-binomial coefficient to the *over q-binomial coefficient* which was introduced recently by Dousse and Kim [16] and has to do with overpartitions.

1.2 Overpartitions

Corteel and Lovejoy [15] introduced a generalization of a partition called an *overpartition* which is defined as follows:

Definition 1.2.1. An overpartition of a positive integer N is an integer partition of N in which the last occurrence of a part may be overlined.

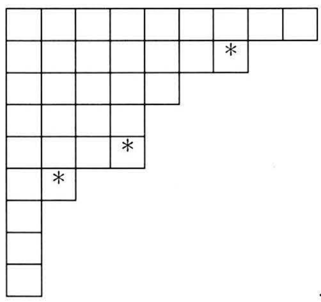
The number of overpartitions of N is denoted by $\overline{p}(N)$.

Example 1.2.1. We have $\overline{p}(4) = 14$; the overpartitions of 4 are

(4), $(\overline{4})$, $(3,1)$, $(\overline{3},1)$, $(3,\overline{1})$, $(\overline{3},\overline{1})$, $(2,2)$, $(2,\overline{2})$, $(2,1,1)$,
$(\overline{2},1,1)$, $(2,1,\overline{1})$, $(\overline{2},1,\overline{1})$, $(1,1,1,1)$, $(1,1,1,\overline{1})$.

Like ordinary partitions, we can represent overpartitions by means of Young diagrams and define conjugates.

The Young diagram (or Ferrers board) for an overpartition is the same as the one for the underlying ordinary partition with the exception that the last block of an overlined part is marked. For example, the Young diagram for $\lambda = (9,\overline{7},5,4,\overline{4},\overline{2},1,1,1)$ is



Definition 1.2.2. The Durfee square of an overpartition is the largest square that can fit in its Ferrers board.

The *conjugate* overpartition of λ is denoted by λ' and is obtained by reading the columns of the Ferrers board, i.e., $\lambda' = (9,\overline{6},5,\overline{5},3,2,\overline{2},1,1)$

We give two other methods of obtaining the conjugate of an overpartition below.

Second Method.

The conjugate of $\overline{p} = (p_1, p_2, \ldots, p_k)$ is given by $\overline{q} = (q_1, q_2, \ldots, q_s)$, where

$$q_j = \begin{cases} \overline{|\{r : p_r \geq j\}|} & \text{if } j = p_i \text{ is overlined} \\ \\ |\{r : p_r \geq j\}| & \text{otherwise.} \end{cases}$$

Example 1.2.2. Given $\overline{p} = (9,\overline{7},5,4,\overline{4},\overline{2},1,1,1)$, then q_2, q_4, and q_7 will be overlined. Hence $\overline{q} = (9,\overline{6},5,\overline{5},3,2,\overline{2},1,1)$.

Third Method.

Let $\overline{p} = (p_1^{x_1}, p_2^{x_2}, \ldots, p_t^{x_t})$ be an overpartition with $p_1 > p_2 > \cdots > p_t$ and $x_j > 0$, where each p_j appears x_j times. Then the conjugate of $\overline{p}$ is given by

$$\overline{q} = \Big((x_1 + x_2 + \cdots + x_t)^{p_t}, (x_1 + x_2 + \cdots + x_{t-1})^{p_{t-1}-p_t}, (x_1 + x_2 + \cdots + x_{t-2})^{p_{t-2}-p_{t-1}}, \ldots, (x_1 + x_2)^{p_2-p_3}, x_1^{p_1-p_2}\Big)$$

where the j^{th} part is overlined if $j = p_i$ is overlined.

Example 1.2.3. If $\overline{p} = (\overline{7}^2, 5^3, 3^1, \overline{2}^3, 1^2)$ then the 2^{nd} and 7^{th} parts will be overlined. Hence $\overline{q} = (11, \overline{9}, 6, 5^2, \overline{2}^2)$.

Definition 1.2.3. An overpartition λ is said to be self-conjugate if its conjugate is itself.

Example 1.2.4. The conjugate of $\lambda = (7, \overline{6}, 4, 4, 2, \overline{2}, 1)$ is $\lambda' = (7, \overline{6}, 4, 4, 2, \overline{2}, 1)$. Hence λ is self-conjugate.

We define the q-Pochhammer symbol (or q-shifted factorial) for $n \in \mathbb{N}$, $A \in \mathbb{C}$ and $|q| < 1$ by

$$(A; q)_n := (1 - A)(1 - Aq) \cdots (1 - Aq^{n-1}) = \prod_{j=0}^{n-1} (1 - Aq^j),$$

$$(A; q)_\infty := \lim_{n \to \infty} (A; q)_n = \prod_{j=0}^{\infty} (1 - Aq^j),$$

with $(A; q)_0 = 1$. We will use this symbol frequently when dealing with the generating functions of certain partition functions. For example, the generating function for ordinary partitions is given by

$$\prod_{j=1}^{\infty} \frac{1}{1 - q^j} = \frac{1}{(q; q)_\infty}.$$

Likewise, we can find the generating function for an overpartition function. Note that the overlined parts in an overpartition are generated by $(-q; q)_\infty$ since they form a partition into distinct parts. On the other hand, the non-overlined parts form an ordinary partition. Hence the generating function for overpartitions is given by

$$\sum_{N=0}^{\infty} \overline{p}(N) q^N = (-q; q)_\infty \times \frac{1}{(q; q)_\infty} = \frac{(-q; q)_\infty}{(q; q)_\infty}.$$

Theorem 1.2.1. *[15] For $N > 0$, $\overline{p}(N)$ is always even.*

Analytic proof. Note that $(1-q^k)^2 = 1 - 2q^k + q^{2k} \equiv 1 + q^{2k} \pmod 2$ and

$$\frac{1-q^{2k}}{1+q^{2k}} \equiv 1 \pmod 2.$$

So

$$\begin{aligned}
\sum_{N=0}^{\infty} \overline{p}(N) q^N &= \frac{(-q;q)_\infty}{(q;q)_\infty} \frac{(q;q)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} \\
&= \frac{(1-q^2)(1-q^4)(1-q^6)\cdots}{(1-q)^2(1-q^2)^2(1-q^3)^2\cdots} \\
&\equiv \frac{(1-q^2)(1-q^4)(1-q^6)\cdots}{(1+q^2)(1+q^4)(1+q^6)\cdots} \pmod 2 \\
&\equiv 1 \pmod 2
\end{aligned}$$

□

Combinatorial proof. Let $\lambda = (\lambda_1^{x_1}, \lambda_2^{x_2}, \lambda_3^{x_3}, \ldots, \lambda_k^{x_k})$ be an ordinary partition of N. Counting the number of overpartitions with underlying partition λ is equivalent to deciding, for each part λ_i, $1 \le i \le k$, whether its last occurrence will be overlined or not. There are 2^k possibilities. Hence $\overline{p}(N)$ is a multiple of 2. □

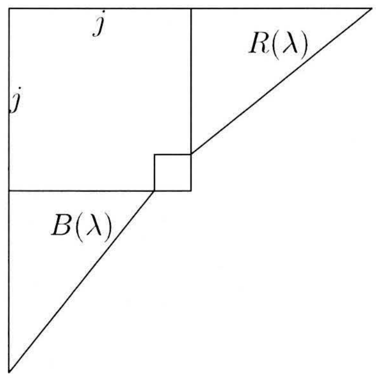
We state a new theorem for the generating function for the number of self-conjugate overpartitions below.

Theorem 1.2.2. *Let $\overline{sc}(N)$ be the number of self-conjugate overpartitions of N. Then*

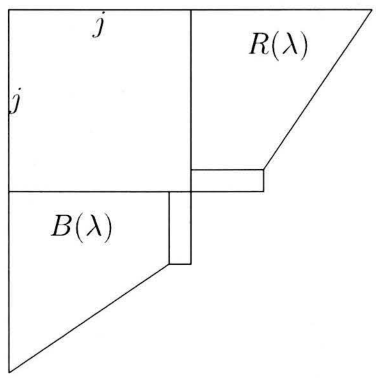
$$\sum_{N=0}^{\infty} \overline{sc}(N) q^N = 1 + \sum_{j=1}^{\infty} 2q^{j^2} \frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_j}.$$

Proof. Suppose we have a self-conjugate overpartition λ of N with a Durfee square of length $j > 0$. Then in the Ferrers graph of λ the jth part of λ, i.e., the last part of the Durfee square, may be overlined or not. We consider the cases when the j^{th} part is overlined and when it is not (see the two diagrams below).

1. If the j^{th} part is overlined (i.e., j^{th} part $= j$ and $(j+1)^{th}$ part $< j$).



2. If the j^{th} part is not overlined (i.e., j^{th} part $\geq j$ and $(j+1)^{th}$ part $\leq j$).



In the first case, the Durfee square is generated by q^{j^2}. Since λ is self-conjugate the overpartition $R(\lambda)$ represented by the boxes on the right of the Durfee square is the conjugate of the overpartition $B(\lambda)$ represented by the boxes below the Durfee square. Therefore each of these overpartitions is generated by $\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}$. Adding the rows of $B(\lambda)$ to the corresponding columns of $R(\lambda)$ gives an overpartition into even parts. So we deduce that these overpartitions are generated by

$$\frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j-1}}.$$

Similarly for the second case, the Durfee square is generated by q^{j^2} while the overpartitions represented on the right of and below the Durfee square are generated by

$$\frac{(-q^2;q^2)_j}{(q^2;q^2)_j}.$$

Hence, with 1 counting the (self-conjugate) empty overpartition, we have

$$\begin{aligned}
\sum_{N=0}^{\infty} \overline{sc}(N)q^N &= 1 + \sum_{j=1}^{\infty} q^{j^2} \frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j-1}} + \sum_{j=1}^{\infty} q^{j^2} \frac{(-q^2;q^2)_j}{(q^2;q^2)_j} \\
&= 1 + \sum_{j=1}^{\infty} q^{j^2} \frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_{j-1}} \left(1 + \frac{1+q^{2j}}{1-q^{2j}}\right) \\
&= 1 + \sum_{j=1}^{\infty} q^{j^2} \frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_j} \left(1 - q^{2j} + 1 + q^{2j}\right) \\
&= 1 + \sum_{j=0}^{\infty} 2q^{j^2} \frac{(-q^2;q^2)_{j-1}}{(q^2;q^2)_j}.
\end{aligned}$$

□

Given two overpartitions $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ in which there are no common overlined parts, we define their sum and union as

$$\lambda + \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \lambda_3 + \mu_3, \ldots);$$

$$\lambda \cup \mu := (\lambda_1, \lambda_2, \ldots, \mu_1, \mu_2, \ldots).$$

Example 1.2.5. If $\lambda = (5, 2, \overline{2}, 1, 1, 1)$ and $\mu = (\overline{7}, \overline{5}, 3, \overline{3}, \overline{1})$, then

$$\lambda + \mu := (\overline{12}, \overline{7}, \overline{5}, \overline{4}, \overline{2}, 1);$$

$$\lambda \cup \mu := (\overline{7}, 5, \overline{5}, 3, \overline{3}, 2, \overline{2}, 1, 1, 1, \overline{1}).$$

Then, in analogy to ordinary partitions, the following relation holds.

Proposition 1.2.3. *If λ' and μ' are conjugate overpartitions of λ and μ, then $(\lambda + \mu)' = \lambda' \cup \mu'$ if and only if $(\lambda \cup \mu)' = \lambda' + \mu'$.*

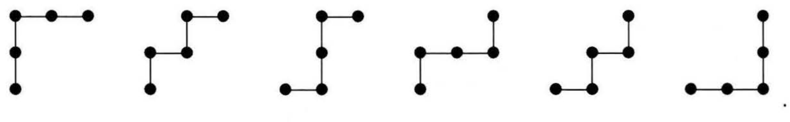
Chapter 2

Combinatorial Structure of q-Binomial Coefficients

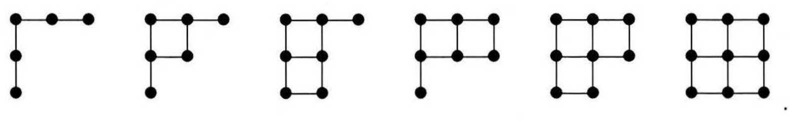
In this chapter we give motivations for the concept of q-binomial coefficient via three combinatorial structures. The q-binomial coefficients are q-analogues of the binomial numbers.

2.1 Lattice Paths motivation

In order to find the number of lattice paths on the plane from (0,0) to (m, n) using only unit steps (0,1) (upward) and (1,0) (to the right) we observe that any such path consists of m horizontal steps and n vertical steps. Thus elementary combinatorial reasoning implies that this number is equal to $\binom{m+n}{n}$ (see also [10]). For example there are 6 such paths from (0,0) to (2,2):



Suppose we apply a refinement by enclosing the top-left unit area in each path with squares:



Here we see that the last diagrams are Ferrers boards for a partitions into at most 2 parts where each part is at most 2. The generating function for these partitions is given by

$$\begin{bmatrix} 2+2 \\ 2 \end{bmatrix} = q^0 + q^1 + q^2 + q^2 + q^3 + q^4 = 1 + q + 2q^2 + q^3 + q^4.$$

Such generating functions are called q-Binomial coefficients or Gaussian polynomials. More generally, we make the following definition.

Definition 2.1.1. Let $p(N| \leq n \text{ parts, each} \leq m)$ denote the number of ways to partition N into at most n parts where each part is at most of size m. Then

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \sum_{N \geq 0} p(N| \leq n \text{ parts, each} \leq m) q^N.$$

Gaussian polynomials resemble binomial numbers with many similar properties. For example the following symmetry property follows from conjugation of partitions.

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \begin{bmatrix} m+n \\ m \end{bmatrix}.$$

Bijectively we can prove the following overpartition recurrences.

Proposition 2.1.1. *For positive integers m and n, we have*

(i) $$\begin{bmatrix} m+n \\ n \end{bmatrix} = \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix} + q^n \begin{bmatrix} m+n-1 \\ n \end{bmatrix};$$

(ii) $$\begin{bmatrix} m+n \\ n \end{bmatrix} = q^m \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix} + \begin{bmatrix} m+n-1 \\ n \end{bmatrix}.$$

Proof. Suppose we have the partitions of N with at most n parts, each of size at most m. We can separate these partitions into two groups:

1. Partitions with $< n$ parts, each $\leq m$.

2. Partitions with exactly n parts, each $\leq m$.

Let $p(N| \leq n \text{ parts, each} \leq m) = p(N, m, n)$, $N \geq 0$. Then the partitions in the first group are counted by $p(N, m, n-1)$.
If we subtract 1 from the parts of partitions in the second group, we get that the number of partitions in the second group is $p(N-n, m-1, n)$. Hence

$$p(N, m, n) = p(N, m, n-1) + p(N-n, m-1, n).$$

We multiply this expression by q^N and sum over N to get

$$\begin{aligned}\sum_{N\geq 0} p(N,m,n)q^N &= \sum_{N\geq 0} p(N,m,n-1)q^N + \sum_{N\geq n} p(N-n,m-1,n)q^N\\ &= \sum_{N\geq 0} p(N,m,n-1)q^N + q^n \sum_{N\geq 0} p(N,m-1,n)q^N.\end{aligned}$$

By definition 2.1.1., this can be written as

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix} + q^n \begin{bmatrix} m+n-1 \\ n \end{bmatrix}.$$

This proves part (i). As for part (ii), note that $p(N,m,n) = p(N,n,m)$ by conjugation. By the same reasoning as for part (i), we have

$$p(N,n,m) = p(N,n,m-1) + p(N-n,n-1,m).$$

Multiplying by q^N and summing over N will lead us to

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = q^m \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix} + \begin{bmatrix} m+n-1 \\ n \end{bmatrix}.$$

This complete our proof $\square$

By iterating these recurrences, we can deduce the following formal definition.

Definition 2.1.2. For m and n positive integers and $|q| < 1$, we have

$$\begin{bmatrix} m \\ n \end{bmatrix} = \begin{cases} \frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}} & 0 \leq n \leq m \\ 0 & \text{otherwise} \end{cases}.$$

Note. We can use L'Hopital's rule to show that

$$\begin{aligned}\lim_{q\to 1}(q;q)_n =& 1 \times 2 \times \cdots \times (n-1) \times n \lim_{q\to 1}(1-q)^n\\ =& n! \lim_{q\to 1}(1-q)^n.\end{aligned}$$

This gives

$$\begin{aligned}\lim_{q\to 1}\begin{bmatrix} m \\ n \end{bmatrix} =& \lim_{q\to 1}\frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}}\\ =& \frac{m!}{n!(m-n)!} \cdot \lim_{q\to 1}\frac{(1-q)^m}{(1-q)^n(1-q)^{m-n}}\\ =& \frac{m!}{n!(m-n)!} = \binom{m}{n}.\end{aligned}$$

So $\begin{bmatrix} m \\ n \end{bmatrix}$ is the q-analogue of $\binom{m}{n}$.

We will need the following important identity to prove the q-binomial theorem.

Theorem 2.1.2 (Cauchy's Identity). *(Gasper-Rahman [20])*
For $|q|, |z| < 1$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

In particular the case $a = 0$ *gives*

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}.$$

Theorem 2.1.3 (q-Binomial Theorem). *[10] For* $|q|, |z| < 1$, *we have*

$$\sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} z^k = \prod_{k=1}^{n} (1 + zq^k)$$

Proof. Note that

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})\frac{(1-aq^n)(1-aq^{n+1})\cdots}{(1-aq^n)(1-aq^{n+1})\cdots} = \frac{(a;q)_\infty}{(aq^n;q)_\infty};$$

$$\begin{aligned}
(q^{-n};q)_k &= (1-q^{-n})(1-q^{-n+1})\cdots(1-q^{-n+k-1}) \\
&= (-1)^k q^{-nk} q^{\binom{k}{2}} (1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1}) \\
&= \frac{(q;q)_n}{(q;q)_{n-k}} (-1)^k q^{\binom{k}{2}-nk}.
\end{aligned}$$

Hence by Theorem 2.1.1, with a set to be q^{-n} and z set to be aq^n, we have

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} = \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k}{(q;q)_k} (aq^n)^k.$$

$$\begin{aligned}
\implies (a;q)_n &= \sum_{k=0}^{\infty} \frac{1}{(q;q)_k} \frac{(q;q)_n}{(q;q)_{n-k}} (-1)^k q^{\binom{k}{2}-nk} (aq^n)^k \\
&= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k a^k q^{\binom{k}{2}}
\end{aligned}$$

Setting a to be $-zq$ we obtain

$$(-zq;q)_n = \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} z^k.$$

□

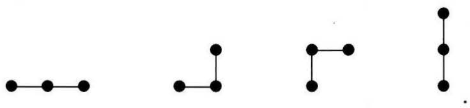
Combinatorial proof of Theorem 2.1.3 can be found in [7].

Note that if we take the limit as q tends to 1 in Theorem 2.1.3, we recover the standard binomial theorem:

$$\sum_{k=0}^{n} \binom{n}{k} z^k = (1+z)^n.$$

2.2 Non-commuting variables motivation

We are still working with lattice paths with steps (0,1) and (1,0), except this time we keep track of the area under path composed of a given number of unit steps along horizontal and vertical directions. The area is marked by q while the horizontal and vertical steps are marked by x and y respectively. For example consider the lattice paths when the total number of unit steps is 2, i.e.,



The first, second and fourth paths have no area under them while the third has a unit area under it.

The area is generated by $q^0 + q^0 + q^1 + q^0$. But the unit steps, which may incorporate areas under resulting paths, are generated by $(x+y)^2 = xx + xy + yx + yy$. The parameters q with x and y are combined by adding a unit area whenever yx is changed to xy, i.e., $xy \to qyx$. The following general rules apply:

$$\begin{aligned} yx &= qxy; \\ xq &= qx; \\ yq &= qy; \\ x^0 = y^0 &= q^0 = 1. \end{aligned} \tag{2.1}$$

Therefore the required generating function is obtained as follows:

$$\begin{aligned} (x+y)^2 =& xx + xy + yx + yy \\ =& x^2y^0 + xy + qxy + x^0y^2 \\ =& x^2y^0 + (1+q)xy + x^0y^2 \\ =& q^0x^2y^0 + (q^0+q^1)x^1y^1 + q^0x^0y^2. \end{aligned}$$

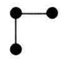


Figure 2.1: Path with area underneath

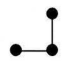


Figure 2.2: Path with no area underneath

Note. q plays the role of a *commutator* in the language of group theory. Since the monomial yx represents two unit steps that enclose a unit area, the transformation $yx \to qxy$ implies a rotation of figure 2.1 by 180 degrees into figure 2.2 such that the previous unit area is encoded by q.

For more general constructions we derive the following relations.

1. $yx^k = q^k x^k y$ since

$$\begin{aligned} yx \cdot x \cdots x =& qxyx \cdots x \\ =& qxqxyx \cdots x \\ &\vdots \\ =& qq \cdots qxx \cdots xy \\ =& q^k x^k y \end{aligned}$$

2. $y^k x = q^k xy^k$ since

$$\begin{aligned} yy \cdots yx =& yy \cdots yqxy \\ =& y \cdots yqyxy \\ &\vdots \\ =& qq \cdots qxyy \cdots y \\ =& q^k xy^k \end{aligned}$$

By a similar reasoning we obtain the generating function associated with 3 unit steps as follows:

$$\begin{aligned} (x+y)^3 &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy \\ &= x^3 + x^2y + xyx + xy^2 + yx^2 + yxy + y^2x + y^3 \\ &= x^3y^0 + x^2y + qx^2y + xy^2 + q^2x^2y + qxy^2 + q^2xy^2 + x^0y^3 \\ &= q^0x^3y^0 + (q^0 + q^1 + q^2)x^2y^1 + (q^0 + q^1 + q^2)x^1y^2 + q^0x^0y^3 \end{aligned}$$

The q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$ are now defined by [7]

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k, \tag{2.2}$$

subject to the rules (2.1).

Now the known expression $(x+y)^{n+1} = (x+y)^n(x+y)$ implies that

$$\begin{aligned}
\sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix} x^{n+1-k} y^k &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k (x+y) \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \left(x^{n-k} y^k x + x^{n-k} y^{k+1} \right) \\
(\text{since } y^k x = q^k x y^k) \quad &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \left(x^{n-k} q^k x y^k + x^{n-k} y^{k+1} \right) \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^k x^{n+1-k} y^k + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^{k+1} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^k x^{n+1-k} y^k + \sum_{k=1}^{n+1} \begin{bmatrix} n \\ k-1 \end{bmatrix} x^{n+1-k} y^k
\end{aligned}$$

Taking the coefficients of $x^{n+1-k} y^k$ on both sides we get

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} q^k + \begin{bmatrix} n \\ k-1 \end{bmatrix}. \tag{2.3}$$

Similarly we can write

$$(x+y)^{n+1} = (x+y)(x+y)^n$$

and use a similar manipulation as above to obtain the dual relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{n+1-k}. \tag{2.4}$$

Subtract equation (2.3) from equation (2.4) to get

$$0 = \begin{bmatrix} n \\ k \end{bmatrix} (1-q^k) + \begin{bmatrix} n \\ k-1 \end{bmatrix} (q^{n+1-k} - 1)$$

which can be rewritten as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1-q^{n+1-k}}{1-q^k} \begin{bmatrix} n \\ k-1 \end{bmatrix}. \tag{2.5}$$

Iterating (2.5) we get

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix} &= \frac{(1-q^{n+1-k})(1-q^{n+2-k})\cdots(1-q^n)}{(1-q^k)(1-q^{k-1})\cdots(1-q)} \begin{bmatrix} n \\ 0 \end{bmatrix} \\
&= \frac{(1-q)(1-q^2)\cdots(1-q^{n-k})(1-q^{n+1-k})\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^{n-k})(1-q)\cdots(1-q^k)} \times 1 \\
&= \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}.
\end{aligned}$$

The q-binomial coefficients defined in this way turn out to be the same as those defined in the previous section.

2.3 Binary words motivation

In this section, we will connect binary words with partitions. A word can be easily defined as a sequence of letters. By binary word, we mean the only letters in the word are 1's and 0's. Let $B(N, m)$ denote the set of all binary words of length $N + m$ having exactly N letters equal to 1 and m letters equal to 0.

Example 2.3.1. 100101001 is a binary word of length $N + m = 9$ where $N = 4$ and $m = 5$.

Definition 2.3.1. Given a word $w = w_1w_2\cdots w_n$, the inversion number, inv w, of w is the number of pairs (i, j) such that $1 \leq i < j \leq n$ and $w_i > w_j$.

Example 2.3.2.

$$\begin{aligned}
w &= 100101001 \\
&= w_1w_2w_3w_4w_5w_6w_7w_8w_9
\end{aligned}$$

so

$w_1 > w_2, w_3, w_5, w_7, w_8$

$w_4 > w_5, w_7, w_8$

$w_6 > w_7, w_8$

Hence there are 10 pairs (i, j) with $i < j$ such that $w_i > w_j$, that is, inv $= 10$.

Note that if w has the form $w = 1^{x_1}0^{x_2}1^{x_3}0^{x_4}\cdots 1^{x_{2r-1}}0^{x_{2r}}1^{x_{2r+1}}$, where the exponent of each letter represents its frequency, then we have

$$\mathrm{inv}w = x_1x_2 + (x_1 + x_3)x_4 + \cdots + (x_1 + x_3 + \cdots + x_{2r-1})x_{2r}. \tag{2.6}$$

Proposition 2.3.1. *(Foata-Han [19]) For each pair of integers (N, m) we have*

$$\begin{bmatrix} N+m \\ m \end{bmatrix} = \sum_{w \in B(N,m)} q^{\mathrm{inv} w}.$$

Proof. Let $P_{N,m}$ be the set of partitions generated by $\begin{bmatrix} N+m \\ m \end{bmatrix}$ (i.e., partitions fitting inside $N \times m$ rectangle). Let $\nu(\lambda)$ be the number of parts of λ and let $|\lambda|$ denote the weight of λ. We want to construct a bijection $\varphi : P_{N,m} \to B(N, m)$ such that $\lambda \mapsto w$ if and only if $|\lambda| = \mathrm{inv} w$.

We can express λ in the form $\lambda = (\lambda_1^{x_1}, \lambda_2^{x_2}, \lambda_3^{x_3}, \ldots, \lambda_k^{x_k})$ with $N \geq \lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0$, $x_j > 0$ for all $1 \leq j \leq k$, and $x_1 + \cdots + x_k \leq m$.

If $\nu(\lambda) < m$, we let $x_k := m - \nu(\lambda)$ and $\lambda_k := 0$. Then λ consists of the parts $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ which are repeated $x_1, x_2, \ldots, x_{k-1}$ times, respectively. If $\nu(\lambda) = m$, then $\lambda_k \geq 1$ and λ consists of the parts $\lambda_1, \lambda_2, \ldots, \lambda_k$, which are repeated $x_1, x_2, \ldots, x_k$ times, respectively.

Therefore $\lambda \mapsto w$, where

$$w = 1^{\lambda_k} 0^{x_k} 1^{\lambda_{k-1} - \lambda_k} 0^{x_{k-1}} \cdots 0^{x_2} 1^{\lambda_1 - \lambda_2} 0^{x_1} 1^{N - \lambda_1}.$$

Note that the word w has $\lambda_k + (\lambda_{k-1} - \lambda_k) + \cdots + (\lambda_1 - \lambda_2) + (N - \lambda_1) = N$ letters equal to 1 and $x_k + x_{k-1} + \cdots + x_1 = m$ letters equal to 0. Moreover,

$$\begin{aligned}
\mathrm{inv} w =& \lambda_k x_k + (\lambda_k + \lambda_{k-1} - \lambda_k) x_{k-1} + \cdots + (\lambda_k + \lambda_{k-1} - \lambda_k + \cdots + \lambda_1 - \lambda_2) x_1 \\
=& \lambda_k x_k + \lambda_{k-1} x_{k-1} + \cdots + \lambda_1 x_1 \\
=& |\lambda|.
\end{aligned}$$

Clearly the mapping $\varphi : \lambda \mapsto w$ is one-to-one, and then bijective. $\square$

Example 2.3.3. As an illustration of Proposition 2.3.1 consider the partition $\lambda = (6, 5, 4, 4, 2, 2) \in P_{6,7}$. For the purpose of this bijection λ can also be written as $(0^1, 2^2, 4^2, 5^1, 6^1)$, with '7 parts'. Thus $\lambda \mapsto w$, where

$w = 1^0 0^1 1^{2-0} 0^2 1^{4-2} 0^2 1^{5-4} 0^1 1^{6-5} 0^1 1^{6-6} = 0110011001010 \in B(6, 7)$.

Note that using Equation (2.6) we have

$$\begin{aligned}
\mathrm{inv} w &= \mathrm{inv}(1^0 0^1 1^2 0^2 1^2 0^2 1^1 0^1 1^1 0^1) \\
&= 0 \cdot 1 + (0 + 2) \cdot 2 + (0 + 2 + 2) \cdot 2 + (0 + 2 + 2 + 1) \cdot 1 \\
&\quad + (0 + 2 + 2 + 1 + 1) \cdot 1 \\
&= 0 + 4 + 8 + 5 + 6 = 23.
\end{aligned}$$

But $|\lambda| = 6 + 5 + 4 + 4 + 2 + 2 = 23$.

2.4 The q-Multinomial coefficients

The binomial theorem can be extended to the multinomial theorem :

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1, k_2, \ldots, k_m \geq 0} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$$

where

$$k_1 + k_2 + \cdots + k_m = n$$

and

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}.$$

We call $\binom{n}{k_1, k_2, \ldots, k_m}$ the multinomial coefficients. Clearly these coefficients count the ways to permute $x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$. Here $x_j^{k_r}$ indicates that the letter x_j occurs k_r times $j \geq 1$, $r \geq 1$.
In this section we further extend the multinomial coefficients to q-multinomial coefficients. These are defined for each $m \geq 1$ and each sequence of nonnegative integers $(k_1, k_2, \ldots, k_m)$ by

$$\begin{bmatrix} k_1 + k_2 + \cdots + k_m \\ k_1, k_2, \ldots, k_m \end{bmatrix} := \frac{(q;q)_{k_1 + k_2 + \cdots + k_m}}{(q;q)_{k_1} (q;q)_{k_2} \cdots (q;q)_{k_m}}.$$

We have already studied the case of $m = 2$ which gave us the q-binomial coefficients

$$\begin{bmatrix} N+m \\ N, m \end{bmatrix} = \begin{bmatrix} N+m \\ m \end{bmatrix} = \begin{bmatrix} N+m \\ N \end{bmatrix} = \frac{(q;q)_{N+m}}{(q;q)_N (q;q)_m}.$$

Note. The limit $\lim_{q \to 1} (q;q)_n = n! \lim_{q \to 1} (1-q)^n$ implies that

$$\lim_{q \to 1} \begin{bmatrix} k_1 + k_2 + \cdots + k_m \\ k_1, k_2, \ldots, k_m \end{bmatrix} = \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1! k_2! \cdots k_m!} = \binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, \ldots, k_m}.$$

Thus q-multinomial coefficients are q-analogues of multinomial coefficients.

For $m \geq 1$ and each sequence $\mathbf{k} = (k_1, k_2, \ldots, k_m)$ of nonnegative integers, let $W(\mathbf{k}) = W(k_1, k_2, \ldots, k_m)$ denote the set of words of length $k = k_1 + k_2 + \cdots + k_m$ which are permutations of the word $1^{k_1} 2^{k_2} \cdots m^{k_m}$.
Clearly the cardinality of $W(k)$ is

$$|W(\mathbf{k})| = \binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, \ldots, k_m}.$$

Theorem 2.4.1. *We have*

$$\begin{bmatrix} k_1 + k_2 + \cdots + k_m \\ k_1, k_2, \ldots, k_m \end{bmatrix} = \sum_{w \in W(\boldsymbol{k})} q^{\mathrm{inv}\, w}. \tag{2.7}$$

Proof. This theorem can be proved by induction on m using the case of $m = 2$ (in Proposition 2.3.1) as a base case. We omit the details which may be found in Foata-Han [19, pp. 29-31]. $\square$

Chapter 3

Over q-Binomial Coefficients: Motivation and Basic Identities

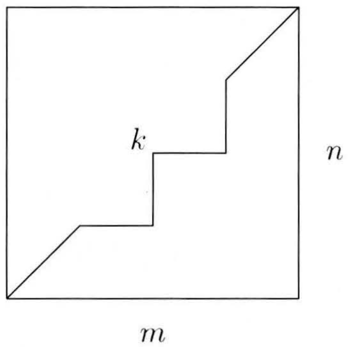
3.1 Lattice Paths

Consider the following problem: how many paths are there in the plane from the point $(0,0)$ to (m,n) using only unit steps vertically upward or horizontally to the right or diagonally to the right?

Let $P(m,n)$ count such paths. Then it can be shown by combinatorial reasoning [1, Exer. 1.24, p. 19] that

$$\begin{aligned} P(m,n) &= \sum_{k\geq 0} \binom{n+k}{m}\binom{m}{k} \\ &= \sum_{k\geq 0} \binom{m+k}{n}\binom{n}{k} \\ &= \sum_{k\geq 0} \binom{m+n-k}{k}\binom{m+n-2k}{m-k}. \end{aligned} \tag{3.1}$$

Suppose you have a path from $(0,0)$ to (m,n) where the number of diagonal steps is k.

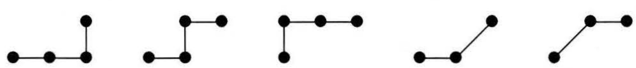


Then the number of total unit steps is $(m+n-k)$. Then there are $(n-k)$ vertical steps and $(m-k)$ horizontal steps, so we can choose $(n-k)$ vertical steps from $(m+n-k)$ steps then choose $(m-k)$ horizontal steps from m. Hence

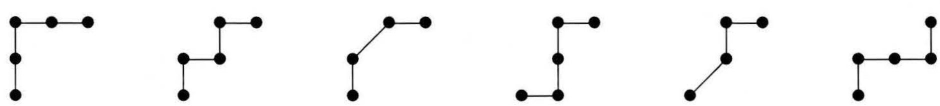
$$
\begin{aligned}
P(m,n) &= \sum_{k\geq 0} \binom{m+n-k}{n-k}\binom{m}{m-k} \\
&= \sum_{k\geq 0} \binom{m+n-k}{m}\binom{m}{m-k} \\
&= \sum_{k\geq 0} \binom{n+k}{m}\binom{m}{k} \quad \text{(change of variable } k \to m-k\text{)} \\
&= \sum_{k\geq 0} \binom{m+k}{n}\binom{n}{k} \qquad \Big(\text{Since } P(m,n) = P(n,m)\Big).
\end{aligned}
$$

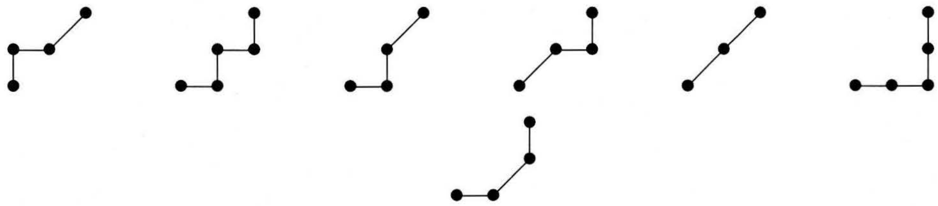
As for the third equality, we choose k diagonal steps from $(m+n-k)$ steps then choose $(m-k)$ horizontal from $m+n-2k$ remaining steps and sum over k.

Example 3.1.1. $P(2,1) = \binom{1}{2}\binom{2}{0} + \binom{2}{2}\binom{2}{1} + \binom{3}{2}\binom{2}{2} = 0+2+3 = 5$ corresponds to the paths

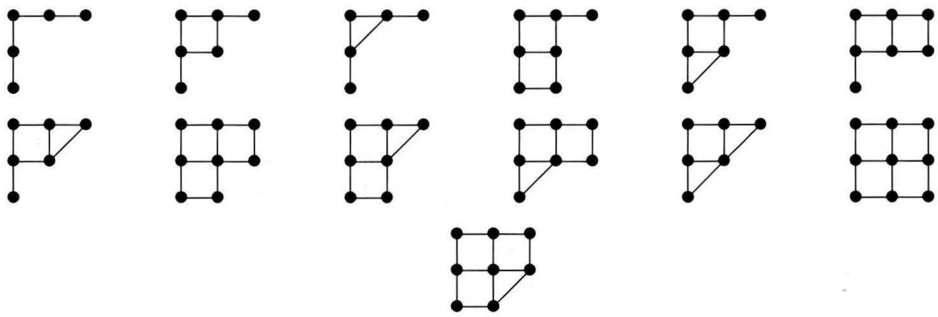


On the other hand $P(2,2) = \binom{2}{2}\binom{2}{0} + \binom{3}{2}\binom{2}{1} + \binom{4}{2}\binom{2}{2} = 1+6+6 = 13$ counts the following paths





We now refine these paths by enclosing the top-left corners with unit squares:



If we regard these new diagrams as Ferrers boards of overpartitions where a triangle represents an overlined box, we obtain the following overpartitions

$$\emptyset,\ (1),\ (\overline{1}),\ (1,1),\ (1,\overline{1}),\ (2),\ (\overline{2}),\ (2,1),\ (\overline{2},1),\ (2,\overline{1}),\ (\overline{2},\overline{1}),\ (2,2),\ (2,\overline{2}).$$

These are overpartitions with each part at most 2 and number of parts at most 2, i.e., fitting inside a 2×2 rectangle. We define the over q-binomial number (a generating function in the variable q) for these overpartitions as

$$\begin{aligned}\overline{\begin{bmatrix}2+2\\2\end{bmatrix}} &= q^0 + q^1 + q^{\overline{1}} + q^{1+1} + q^{1+\overline{1}} + q^2 + q^{\overline{2}} + q^{2+1} + q^{\overline{2}+1} + q^{2+\overline{1}} \\ &\quad + q^{\overline{2}+\overline{1}} + q^{2+2} + q^{2+\overline{2}} \\ &= q^0 + 2q^1 + 4q^2 + 4q^3 + 2q^4.\end{aligned}$$

Generally, we define the over q-binomial numbers as

$$\overline{\begin{bmatrix}m+n\\n\end{bmatrix}} := \sum_{n\geq 0} \overline{p}(N,m,n)q^n,$$

where $\overline{p}(N,m,n)$ counts overpartitions of N into at most n parts, where each part is at most m.

We have by conjugation, the symmetry property

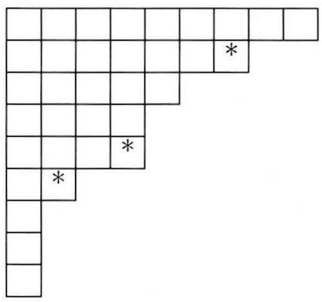
$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} = \overline{\begin{bmatrix} m+n \\ m \end{bmatrix}}. \tag{3.2}$$

We shall prove the following :

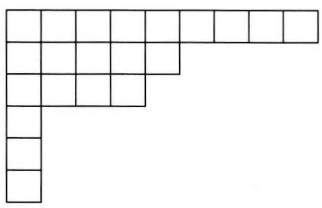
Theorem 3.1.1. *[16] For natural numbers N and m, we have*

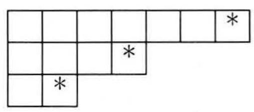
$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} = \sum_{k=0}^{min\{m,n\}} q^{\frac{k(k+1)}{2}} \frac{(q;q)_{m+n-k}}{(q;q)_k(q;q)_{m-k}(q;q)_{n-k}}.$$

Proof. Let $\overline{G}(N, m, n, k)$ be the generating function for the number of overpartitions of N fitting inside an $m \times n$ rectangle and having k overlined parts. Then the non-overlined parts in each overpartition will form a partition into at most $n-k$ parts, each at most m, while the overlined parts will form a partition into k distinct parts, each at most m. For example, the overpartition



can be decomposed into the two partitions depicted below.





The first partition can be viewed as a partition fitting inside an $m \times (n-k)$ rectangle. Hence it is generated by $\begin{bmatrix} m+n-k \\ n-k \end{bmatrix}$.
The partition into k distinct parts $\leq m$ can be "factored" into the partition $(k, k-1, \ldots, 1)$ and a partition into at most k parts, each $\leq m-k$ (i.e., fitting inside $(m-k) \times k$ rectangle). For example, the last Young diagram

above (with $k = 3, m \geq 9$) gives $(\overline{7}, \overline{4}, \overline{2}) \to (3, 2, 1) \cdot (4, 2, 1)$. Such partitions are generated by $q^{\frac{k(k+1)}{2}} \begin{bmatrix} m \\ k \end{bmatrix}$. Thus

$$\begin{aligned}
\overline{G}(N, m, n, k) &= q^{\frac{k(k+1)}{2}} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} m+n-k \\ n-k \end{bmatrix} \\
&= q^{\frac{k(k+1)}{2}} \cdot \frac{(q;q)_m}{(q;q)_k (q;q)_{m-k}} \cdot \frac{(q;q)_{m+n-k}}{(q;q)_{n-k}(q;q)_m} \\
&= q^{\frac{k(k+1)}{2}} \frac{(q;q)_{m+n-k}}{(q;q)_k (q;q)_{m-k} (q;q)_{n-k}}.
\end{aligned}$$

Since $\overline{G}(N, m, n, k) \geq 0$ if and only if $0 \leq k \leq \min\{m, n\}$, we have

$$\begin{aligned}
\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} &= \sum_{k=0}^{\min\{m,n\}} \overline{G}(N, m, n, k) \\
&= \sum_{k=0}^{\min\{m,n\}} q^{\frac{k(k+1)}{2}} \frac{(q;q)_{m+n-k}}{(q;q)_k (q;q)_{m-k} (q;q)_{n-k}}
\end{aligned}$$

□

Note that

$$\begin{aligned}
\lim_{q\to 1} \overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} &= \lim_{q \to 1} \sum_{k \geq 0} q^{\binom{k+1}{2}} \frac{(q;q)_{m+n-k}}{(q;q)_{n-k}(q;q)_m} \cdot \frac{(q;q)_m}{(q;q)_k (q;q)_{m-k}} \\
&= \lim_{q \to 1} \sum_{k \geq 0} q^{\binom{k+1}{2}} \begin{bmatrix} m+n-k \\ n-k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \\
&= \sum_{k \geq 0} \lim_{q \to 1} q^{\binom{k+1}{2}} \lim_{q \to 1} \begin{bmatrix} m+n-k \\ n-k \end{bmatrix} \lim_{q \to 1} \begin{bmatrix} m \\ k \end{bmatrix} \\
&= \sum_{k \geq 0} \binom{m+n-k}{n-k} \binom{m}{m-k} \\
&= P(m, n)
\end{aligned}$$

Interestingly, we see that $\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}}$ is the q-analogue of $P(m, n)$.

Example 3.1.2. The over-q binomial coefficient expansions, with $\begin{bmatrix} 7 \\ n \end{bmatrix}$ for $n = 0, 1, 2, 3, 4, 5, 6, 7$ are

<u>n = 0 :</u>

$$\begin{aligned}
\overline{\begin{bmatrix} 7 \\ 0 \end{bmatrix}} = \overline{\begin{bmatrix} 7+0 \\ 0 \end{bmatrix}} &= \sum_{k=0}^{0} q^{\frac{k(k+1)}{2}} \frac{(q;q)_{7-k}}{(q;q)_k (q;q)_{7-k} (q;q)_{0-k}} \\
&= 1
\end{aligned}$$

n = 1 :

$$\overline{\begin{bmatrix}7\\1\end{bmatrix}} = \overline{\begin{bmatrix}6+1\\1\end{bmatrix}} = \sum_{k=0}^{1} q^{\frac{k(k+1)}{2}} \frac{(q;q)_{7-k}}{(q;q)_k (q;q)_{6-k} (q;q)_{1-k}}$$
$$= 1 + 2q + 2q^2 + 2q^3 + 2q^4 + 2q^5 + 2q^6$$

n = 2 :

$$\overline{\begin{bmatrix}7\\2\end{bmatrix}} = \overline{\begin{bmatrix}5+2\\2\end{bmatrix}} = \sum_{k=0}^{2} q^{\frac{k(k+1)}{2}} \frac{(q;q)_{7-k}}{(q;q)_k (q;q)_{5-k} (q;q)_{2-k}}$$
$$= 1 + 2q + 4q^2 + 6q^3 + 8q^4 + 10q^5 + 10q^6 + 8q^7 + 6q^8 + 4q^9 + 2q^{10}$$

n = 3 :

$$\overline{\begin{bmatrix}7\\3\end{bmatrix}} = \overline{\begin{bmatrix}4+3\\3\end{bmatrix}} = \sum_{k=0}^{3} q^{\frac{k(k+1)}{2}} \frac{(q;q)_{7-k}}{(q;q)_k (q;q)_{4-k} (q;q)_{3-k}}$$
$$= 1 + 2q + 3q^2 + 7q^3 + 10q^4 + 14q^5 + 18q^6 + 19q^7 + 17q^8 + 14q^9$$
$$+ 8q^{10} + 4q^{11} + 2q^{12}$$

n = 4 :

$$\overline{\begin{bmatrix}7\\4\end{bmatrix}} = \overline{\begin{bmatrix}3+4\\4\end{bmatrix}} = \sum_{k=0}^{3} q^{\frac{k(k+1)}{2}} \frac{(q;q)_{7-k}}{(q;q)_k (q;q)_{3-k} (q;q)_{4-k}} = \overline{\begin{bmatrix}7\\3\end{bmatrix}}$$

n = 5 :

$$\overline{\begin{bmatrix}7\\5\end{bmatrix}} = \overline{\begin{bmatrix}2+5\\5\end{bmatrix}} = \sum_{k=0}^{2} q^{\frac{k(k+1)}{2}} \frac{(q;q)_{7-k}}{(q;q)_k (q;q)_{2-k} (q;q)_{5-k}} = \overline{\begin{bmatrix}7\\2\end{bmatrix}}$$

n = 6 :

$$\overline{\begin{bmatrix}7\\6\end{bmatrix}} = \overline{\begin{bmatrix}1+6\\6\end{bmatrix}} = \sum_{k=0}^{1} q^{\frac{k(k+1)}{2}} \frac{(q;q)_{7-k}}{(q;q)_k (q;q)_{1-k} (q;q)_{6-k}} = \overline{\begin{bmatrix}7\\1\end{bmatrix}}$$

n = 7 :

$$\overline{\begin{bmatrix}7\\7\end{bmatrix}} = \overline{\begin{bmatrix}0+7\\7\end{bmatrix}} = \sum_{k=0}^{0} q^{\frac{k(k+1)}{2}} \frac{(q;q)_{7-k}}{(q;q)_k (q;q)_{0-k} (q;q)_{7-k}} = \overline{\begin{bmatrix}7\\0\end{bmatrix}}$$

Remark. (i) The q-trinomial coefficient is defined by

$$\begin{bmatrix}a+b+c\\a,b,c\end{bmatrix} = \frac{(q;q)_{a+b+c}}{(q;q)_a (q;q)_b (q;q)_c}.$$

Hence the right-hand side of Theorem 3.1.1 can be re-written to give

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} = \sum_{k=0}^{\min\{m,n\}} q^{\frac{k(k+1)}{2}} \begin{bmatrix} m+n-k \\ m-k, k, n-k \end{bmatrix}. \tag{3.3}$$

(ii) The q-trinomial coefficients satisfy the recurrence, (see [8])

$$\begin{bmatrix} a+b+c \\ a, b, c \end{bmatrix} = \begin{bmatrix} a+b+c-1 \\ a-1, b, c \end{bmatrix} + q^{a} \begin{bmatrix} a+b+c-1 \\ a, b-1, c \end{bmatrix} + q^{a+b} \begin{bmatrix} a+b+c-1 \\ a, b, c-1 \end{bmatrix}. \tag{3.4}$$

Theorem 3.1.2. *[16] For positive integers m and n, we have*

(i) $\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} = \overline{\begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}} + q^{n} \overline{\begin{bmatrix} m+n-1 \\ n \end{bmatrix}} + q^{n} \overline{\begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix}}$;

(ii) $\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} = \overline{\begin{bmatrix} m+n-1 \\ n \end{bmatrix}} + q^{m} \overline{\begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}} + q^{m} \overline{\begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix}}$.

Analytic proof of Theorem 3.1.2. We prove only part (ii), the proof of part (i) is similar. Using (3.3) and (3.4) we have

$$\begin{aligned}
\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} &= \sum_{k=0}^{\min\{m,n\}} q^{\frac{k(k+1)}{2}} \begin{bmatrix} m+n-k \\ m-k, k, n-k \end{bmatrix} \\
&= \sum_{k=0}^{\min\{m,n\}} q^{\frac{k(k+1)}{2}} \left(\begin{bmatrix} m+n-k-1 \\ m-k-1, k, n-k \end{bmatrix} \right. \\
&\quad \left. + q^{m-k} \begin{bmatrix} m+n-k-1 \\ m-k, k-1, n-k \end{bmatrix} + q^{m} \begin{bmatrix} m+n-k-1 \\ m-k, k, n-k-1 \end{bmatrix} \right) \\
&= \sum_{k=0}^{\min\{m-1,n\}} q^{\frac{k(k+1)}{2}} \begin{bmatrix} m+n-k-1 \\ m-k-1, k, n-k \end{bmatrix} \\
&\quad + q^{m} \sum_{k=0}^{\min\{m,n\}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} m+n-k-1 \\ m-k, k-1, n-k \end{bmatrix} \\
&\quad + q^{m} \sum_{k=0}^{\min\{m,n-1\}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} m+n-k-1 \\ m-k, k, n-k-1 \end{bmatrix} \\
&= \sum_{k=0}^{\min\{m-1,n\}} q^{\frac{k(k+1)}{2}} \begin{bmatrix} m+n-k-1 \\ m-k-1, k, n-k \end{bmatrix} + \\
&\quad q^{m} \sum_{k=0}^{\min\{m-1,n-1\}} q^{\frac{k(k+1)}{2}} \begin{bmatrix} m+n-k-2 \\ m-k-1, k, n-k-1 \end{bmatrix}
\end{aligned}$$

$$+ q^m \sum_{k=0}^{\min\{m,n-1\}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} m+n-k-1 \\ m-k, k, n-k-1 \end{bmatrix}$$

(By replacing k by $k+1$ in the second sum)

$$= \overline{\begin{bmatrix} m+n-1 \\ n \end{bmatrix}} + q^m \overline{\begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix}} + q^m \overline{\begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}}.$$

□

Combinatorial proof of Theorem 3.1.2. Let $O(N, m, n)$ count overpartitions of N into at most n parts $\leq m$. Then $O(N, m, n) - O(N, m, n-1)$ counts overpartitions λ into exactly n parts $\leq m$. We now subtract 1 from every part of λ in two cases as follows.

If 1 (or $\bar{1}$) is the smallest part, then we obtain an overpartition of $N - n$ fitting inside an $(m-1)\times(n-1)$ rectangle counted by $2O(N-n, m-1, n-1)$.

If the smallest part is not 1 (or $\bar{1}$), then we obtain an overpartition of $N - n$ having exactly n parts $\leq m-1$ and therefore counted by $O(N-n, m-1, n) - O(N-n, m-1, n-1)$.

Therefore

$$O(N, m, n) - O(N, m, n-1)$$

$$= 2O(N-n, m-1, n-1) + O(N-n, m-1, n) - O(N-n, m-1, n-1)$$
$$= O(N-n, m-1, n-1) + O(N-n, m-1, n).$$

We multiply by q^N and sum over N to obtain

$$\sum_{N\geq 0} \big(O(N, m, n) - O(N, m, n-1)\big) q^N = q^n \sum_{N\geq 0} O(N-n, m-1, n-1) q^{N-n}$$
$$+ q^n \sum_{N\geq 0} O(N-n, m-1, n) q^{N-n}.$$

Hence by the definition of the over q-binomial number we have

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} - \overline{\begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}} = q^n \overline{\begin{bmatrix} m+n-1 \\ n \end{bmatrix}} + q^n \overline{\begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix}}$$

which proves part (i).

Applying a similar argument to part (ii), we see that $O(N, m, n) - O(N, m-1, n)$ counts overpartitions π fitting inside an $m \times n$ rectangle with the largest part equal to m. If we remove the largest part m from π we get that:

If m is overlined, we obtain an overpartition of $N - m$ fitting inside an

$(m-1)\times(n-1)$ rectangle (since the last occurrence of a part may be overlined). This is counted by $O(N-m,m-1,n-1)$.
If m is not overlined, we obtain an overpartition of $N-m$ fitting inside an $m\times(n-1)$ rectangle counted by $O(N-m,m,n-1)$.
Hence, we have

$$O(N,m,n)-O(N,m-1,n)=O(N-m,m-1,n-1)+O(N-m,m,n-1).$$

As before, this translates into the recurrence

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}} - \overline{\begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}} = q^m \overline{\begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix}} + q^m \overline{\begin{bmatrix} m+n-1 \\ n \end{bmatrix}}$$

which is part (ii). □

Note that if we set $z=1$ on Theorem 2.1.3 (the q-binomial theorem), we obtain

$$(-q;q)_m=\sum_{k=0}^{m} q^{\frac{k(k+1)}{2}} \begin{bmatrix} m \\ k \end{bmatrix}. \tag{3.5}$$

Proposition 3.1.3. *[16] For a non-negative integer m, we have*

$$\lim_{N\to\infty} \overline{\begin{bmatrix} N \\ m \end{bmatrix}} = \frac{(-q;q)_m}{(q;q)_m}.$$

Proof. We have, by Theorem 3.1.1

$$\overline{\begin{bmatrix} N \\ m \end{bmatrix}} = \overline{\begin{bmatrix} (N-m)+m \\ m \end{bmatrix}} = \sum_{k=0}^{\min\{N-m,m\}} q^{\frac{k(k+1)}{2}} \frac{(q;q)_{N-k}}{(q;q)_k (q;q)_{N-m-k} (q;q)_{m-k}}.$$

Hence

$$\begin{aligned}
\lim_{N\to\infty} \overline{\begin{bmatrix} N \\ m \end{bmatrix}} &= \sum_{k=0}^{m} q^{\frac{k(k+1)}{2}} \frac{1}{(q;q)_k (q;q)_{m-k}} \\
&= \frac{1}{(q;q)_m} \sum_{k=0}^{m} q^{\frac{k(k+1)}{2}} \frac{(q;q)_m}{(q;q)_k (q;q)_{m-k}} \\
&= \frac{(-q;q)_m}{(q;q)_m}. \qquad \text{By equation (3.5)}
\end{aligned}$$

□

3.2 The over q-analogue of Chu-Vandermonde Identity

We consider the q-binomial analogue of the classical identity

$$\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}. \tag{3.6}$$

It is known that this identity has the q-analogue by q-binomial numbers [10]

$$\sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}^2 = \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

As an immediate application we state a new over q-analogue of (3.6) in terms the over q-binomial coefficients.

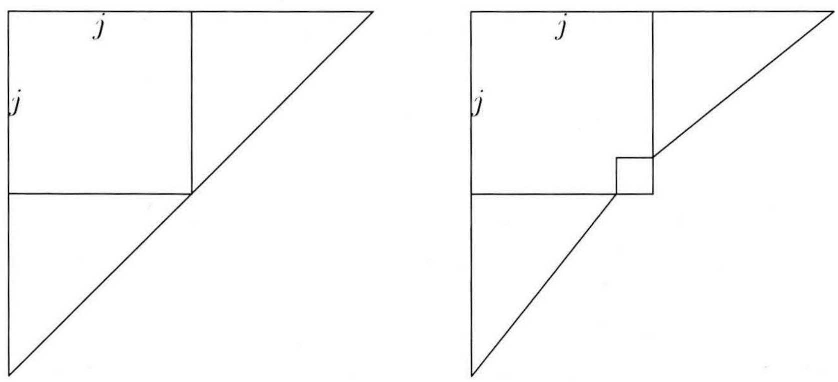
Proposition 3.2.1. *For any non-negative integer n, we have*

$$\overline{\begin{bmatrix} 2n \\ n \end{bmatrix}} = \sum_{j=0}^{n} q^{j^2} \left(\overline{\begin{bmatrix} n \\ j \end{bmatrix}}^2 + \overline{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}}^2 \right)$$

Proof. It is clear that overpartitions fitting inside an $n \times n$ square are generated by

$$\overline{\begin{bmatrix} 2n \\ n \end{bmatrix}}.$$

Now assume the overpartition has a Durfee square of length j.



The diagrams above represents overpartitions fitting inside $n \times n$ square with the Durfee square of length j. The first diagram (on the left) represents overpartitions where the last part of the Durfee square is overlined whereas

the second diagram represents overpartitions where the last part of the Durfee square is not overlined. In each of these diagrams, the Durfee square is generated by q^{j^2}.
If the last part of the Durfee square is overlined (as shown in the second diagram), then the diagram on its right is an overpartition fitting inside an $(n-j)\times(j-1)$ rectangle and the diagram below it is an overpartition fitting inside a $(j-1)\times(n-j)$ rectangle. Hence both diagrams are generated by

$$\overline{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}} \times \overline{\begin{bmatrix} n-1 \\ n-j \end{bmatrix}} = \overline{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}} \times \overline{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}} \qquad \text{By symmetry}$$
$$= \overline{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}}^2.$$

If the last part of the Durfee square is not overlined (as shown in the first diagram), then the diagram on the right is an overpartition fitting inside an $(n-j)\times j$ rectangle and the diagram below it is an overpartition fitting inside a $j\times(n-j)$ rectangle. Hence the diagrams are generated by

$$\overline{\begin{bmatrix} n \\ j \end{bmatrix}} \times \overline{\begin{bmatrix} n \\ n-j \end{bmatrix}} = \overline{\begin{bmatrix} n \\ j \end{bmatrix}} \times \overline{\begin{bmatrix} n \\ j \end{bmatrix}} = \overline{\begin{bmatrix} n \\ j \end{bmatrix}}^2.$$

Hence, the generating function for partitions into at most n parts $\leq n$ with Durfee square of side j is given by

$$q^{j^2}\left(\overline{\begin{bmatrix} n \\ j \end{bmatrix}}^2 + \overline{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}}^2\right).$$

Moreover, since every overpartition enumerated by $\overline{\left[\begin{smallmatrix} 2n \\ n \end{smallmatrix}\right]}$ has a Durfee square with size $0\leq j\leq n$, we obtain the stated identity. $\square$

Proposition 3.2.1 may be regarded as a finitized version of the following new identity.

Corollary 3.2.2. *We have*

$$\frac{(-q;q)_\infty}{(q;q)_\infty} = \sum_{j=0}^{\infty} 2q^{j^2}\left(\frac{(-q;q)_{j-1}}{(q;q)_j}\right)^2 (1+q^{2j}).$$

Proof. Let $n \to \infty$ in Proposition 3.2.1. Then

$$\begin{aligned}
\lim_{n\to\infty} \overline{\begin{bmatrix} 2n \\ n \end{bmatrix}} &= \lim_{n\to\infty} \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \frac{(q;q)_{2n-k}}{(q;q)_k (q;q)_{n-k} (q;q)_{n-k}} \\
&= \sum_{k=0}^{\infty} \frac{q^{\frac{k(k+1)}{2}}}{(q;q)_k} \cdot \frac{1}{(q;q)_\infty} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty}.
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} \sum_{j=0}^{n} q^{j^2} \left(\overline{\begin{bmatrix} n \\ j \end{bmatrix}}^2 + \overline{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}}^2 \right) &= \sum_{j=0}^{\infty} q^{j^2} \left(\left(\frac{(-q;q)_j}{(q;q)_j} \right)^2 + \left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}} \right)^2 \right) \\
&= \sum_{j=0}^{\infty} q^{j^2} \left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}} \right)^2 \left(\left(\frac{1+q^j}{1-q^j} \right)^2 + 1 \right) \\
&= \sum_{j=0}^{\infty} q^{j^2} \left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}} \right)^2 \cdot \frac{2(1+q^{2j})}{(1-q^j)^2} \\
&= \sum_{j=0}^{\infty} 2q^{j^2} \left(\frac{(-q;q)_{j-1}}{(q;q)_j} \right)^2 (1+q^{2j}).
\end{aligned}$$

Hence the result. $\square$

Combinatorial proof of Corollary 3.2.2. Clearly

$$\frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{1}{(q;q)_k} q^{\frac{k(k+1)}{2}} = \frac{(-q;q)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty} \overline{p}(n) q^n$$

Now suppose we have an overpartition λ of N with Durfee square of side j. Separate λ into two classes as in the proof of Theorem 1.2.2. In both cases, the Durfee square is obviously generated by q^{j^2}. However, in the first, the top-right and bottom-left triangles are generated by $\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}$. Such overpartitions are generated by

$$q^{j^2} \frac{(-q;q)_{j-1}^2}{(q;q)_{j-1}^2}.$$

Similarly, in the second case, the top-right and bottom left triangles are generated by $\frac{(-q;q)_j}{(q;q)_j}$. Hence this class is generated by

$$q^{j^2} \frac{(-q;q)_j^2}{(q;q)_j^2}.$$

Hence we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{p}(n)q^n &= \sum_{j=0}^{\infty} q^{j^2} \frac{(-q;q)_{j-1}^2}{(q;q)_{j-1}^2} + \sum_{j=0}^{\infty} q^{j^2} \frac{(-q;q)_j^2}{(q;q)_j^2} \\ &= \sum_{j=0}^{\infty} q^{j^2} \left(\left(\frac{(-q;q)_j}{(q;q)_j} \right)^2 + \left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}} \right)^2 \right) \\ &= \sum_{j=0}^{\infty} 2q^{j^2} \left(\frac{(-q;q)_{j-1}}{(q;q)_j} \right)^2 (1+q^{2j}).\end{aligned}$$

$\square$

Chapter 4

Some Applications to Restricted Overpartitions

This chapter considers applications of over q-binomial coefficients to overpartitions which are described according to the number of parts, and according to the rank.

4.1 Overpartition with restricted number of parts

The next result will be proved by marking the number of parts by k.

Proposition 4.1.1. *[16] For a positive integer m, we have*

$$\frac{(-zq;q)_m}{(zq;q)_m} = 1 + \sum_{k\geq 1} z^k q^k \left(\overline{\begin{bmatrix} m+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} m+k-2 \\ k-1 \end{bmatrix}} \right).$$

Proof. Let $\overline{p}_m(N,k)$ count the number of overpartitions of N into k parts $\leq m$. Then we have

$$\sum_{N\geq 0}\sum_{k\geq 0} \overline{p}_m(N,k) z^k q^N = \frac{(-zq;q)_m}{(zq;q)_m}$$

since overlined parts form a partition into distinct parts $\leq m$ and the non-overlined parts form an ordinary partition with parts $\leq m$. But we showed in the combinatorial proof of Theorem 3.1.2 that

$$\begin{aligned} \overline{p}_m(N,k) &= O(N,m,k) - O(N,m,k-1) \\ &= O(N-k,m-1,k-1) + O(N-k,m-1,k). \end{aligned}$$

Multiply through by q^N and sum over N to get

$$\sum_{N\geq 0} \overline{p}_m(N,k)q^N = q^k \sum_{N\geq 0} \Big(O(N-k,m-1,k-1)+O(N-k,m-1,k)\Big)q^{N-k}$$
$$= q^k\left(\overline{\begin{bmatrix} m+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} m+k-2 \\ k-1 \end{bmatrix}}\right).$$

Hence

$$\begin{aligned}\frac{(-zq;q)_m}{(zq;q)_m} &= \sum_{k\geq 0}\sum_{N\geq 0} \overline{p}_m(N,k)q^n z^k \\ &= \sum_{k\geq 0} z^k q^k\left(\overline{\begin{bmatrix} m+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} m+k-2 \\ k-1 \end{bmatrix}}\right) \\ &= 1+\sum_{k\geq 1} z^k q^k\left(\overline{\begin{bmatrix} m+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} m+k-2 \\ k-1 \end{bmatrix}}\right).\end{aligned}$$

□

Proposition 4.1.1 is an overpartition analogue of the q-binomial series [10]:

$$\frac{1}{(zq;q)_m} = \sum_{j=0}^{\infty} z^j q^j \begin{bmatrix} m+j-1 \\ j \end{bmatrix}.$$

Corollary 4.1.2. *[16] Let $\overline{p}(N,k)$ count the overpartitions of N with k parts. Then,*

$$\sum_{N\geq 0}\sum_{k\geq 0} \overline{p}(N,k)z^k q^N = \frac{(-zq;q)_\infty}{(zq;q)_\infty} = 1+2\sum_{k\geq 1} z^k q^k \frac{(-q;q)_{k-1}}{(q;q)_k}.$$

Proof. We let $m\to\infty$ on Proposition 4.1.1. Then

$$\overline{p}_m(N,k)\to\overline{p}(N,k) \iff \frac{(-zq;q)_m}{(zq;q)_m} \to \frac{(-zq;q)_\infty}{(zq;q)_\infty}.$$

Lastly,

$$1+\sum_{k\geq 1} z^k q^k \left(\overline{\begin{bmatrix} m+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} m+k-2 \\ k-1 \end{bmatrix}} \right)$$

$$\begin{aligned}
&\to 1+\sum_{k\geq 1} z^k q^k \left(\frac{(-q;q)_k}{(q;q)_k} + \frac{(-q;q)_{k-1}}{(q;q)_{k-1}} \right) \\
&= 1+\sum_{k\geq 1} z^k q^k \left(\frac{(-q;q)_{k-1}}{(q;q)_{k-1}} \right) \left(\frac{1+q^k}{1-q^k} + 1 \right) \\
&= 1+\sum_{k\geq 1} z^k q^k \left(\frac{(-q;q)_{k-1}}{(q;q)_{k-1}} \right) \frac{2}{1-q^k} \\
&= 1+2\sum_{k\geq 1} z^k q^k \frac{(-q;q)_{k-1}}{(q;q)_k}.
\end{aligned}$$

□

Note that since $1+q^m \pmod 2 \equiv 1-q^m \pmod 2$, we have

$$\begin{aligned}
\sum_{k\geq 1} \frac{q^k(-q;q)_{k-1}}{(q;q)_k} &= \sum_{k\geq 1} q^k \cdot \frac{(1+q)(1+q^2)\cdots(1+q^{k-1})}{(1-q)(1-q^2)\cdots(1-q^{k-1})} \cdot \frac{1}{1-q^k} \\
&\equiv \sum_{k\geq 1} \frac{q^k}{1-q^k} \pmod 2 \\
&\equiv \sum_{N\geq 1} \tau(N) q^N \pmod 2, \qquad (4.1)
\end{aligned}$$

where $\tau(N)$ is the number of positive-integer divisors of N.

Corollary 4.1.3. *[16] For all non-negative integers N, we have*

$$\overline{p}(N) \equiv 2\tau(N) \pmod 4$$

Proof. Set $z=1$ in Corollary 4.1.2, then

$$\begin{aligned}
\sum_{N\geq 0} \overline{p}(N) q^N &= 1+\sum_{N\geq 1} \overline{p}(N) \\
&= 1+2\sum_{k\geq 1} \frac{q^k(-q;q)_{k-1}}{(q;q)_k}.
\end{aligned}$$

Hence

$$\sum_{N\geq 1} \overline{p}(N) q^N = 2\sum_{k\geq 1} \frac{q^k(-q;q)_{k-1}}{(q;q)_k}$$

giving, with (4.1),

$$\sum_{N\geq 1} \overline{p}(N)q^N \equiv 2\sum_{N\geq 1} \tau(N)q^N \pmod 4.$$

□

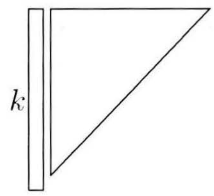
Theorem 4.1.4. *[17] For every positive integer m, we have*

$$\sum_{k=0}^{\infty} \overline{\begin{bmatrix} m+k-1 \\ k \end{bmatrix}} z^k q^k = \frac{(-zq^2;q)_{m-1}}{(zq;q)_m}. \tag{4.2}$$

Combinatorial proof . Note that

$$\frac{(-zq^2;q)_{m-1}}{(zq;q)_m} = \frac{(1+zq^2)(1+zq^3)\cdots(1+zq^m)}{(1-zq)(1-zq^2)\cdots(1-zq^m)}.$$

This is generating function for overpartitions where all parts are $\leq m$ and the smallest overlined part is > 1. Now for a fixed value of k, suppose we have such overpartition with exactly k parts.



The first column of this overpartition is generated by $z^k q^k$ and the triangle on the right is generated by $\overline{\begin{bmatrix} m-1+k \\ k \end{bmatrix}}$. Hence this overpartition is generated by $z^k q^k \overline{\begin{bmatrix} m-1+k \\ k \end{bmatrix}}$. Summing over k we get the results. □

In addition to the combinatorial proof, we give an analytic proof.

Analytic proof. We will use the fact that

$$(q;q)_{m+k} = (q;q)_m (q^{m+1};q)_k \tag{4.3}$$

Now we simplify the left-hand side of (4.2) (with $m-1$ set to be m) below. Note that in the second equality we set $\min(m,k) = m$ since $0 \leq j \leq m$ but

$j \leq k \to \infty$.

$$\begin{aligned}
\sum_{k=0}^{\infty} \overline{\begin{bmatrix} m+k \\ k \end{bmatrix}} z^k q^k &= \sum_{k=0}^{\infty} \sum_{j=0}^{\min(m,k)} q^{\binom{j+1}{2}} \frac{(q;q)_{m+k-j}}{(q;q)_j (q;q)_{m-j} (q;q)_{k-j}} (zq)^k \\
&= \sum_{j=0}^{m} \frac{q^{\binom{j+1}{2}}}{(q;q)_j (q;q)_{m-j}} \sum_{k=j}^{\infty} \frac{(q;q)_{m+k-j}}{(q;q)_{k-j}} (zq)^k \\
&= \sum_{j=0}^{m} \frac{q^{\binom{j+1}{2}}}{(q;q)_j (q;q)_{m-j}} \sum_{k=0}^{\infty} \frac{(q;q)_{m+k}}{(q;q)_k} (zq)^{k+j} \\
&= \sum_{j=0}^{m} \frac{q^{\binom{j+1}{2}}}{(q;q)_j (q;q)_{m-j}} (zq)^j \sum_{k=0}^{\infty} \frac{(q;q)_{m+k}}{(q;q)_k} (zq)^k \\
(\text{by } (4.3)) &= \sum_{j=0}^{m} \frac{(zq)^j q^{\binom{j+1}{2}}}{(q;q)_j (q;q)_{m-j}} (q;q)_m \sum_{k=0}^{\infty} \frac{(q^{m+1};q)_k}{(q;q)_k} (zq)^k \\
(\text{by Theorem 2.1.2}) &= \sum_{j=0}^{m} \frac{(q;q)_m}{(q;q)_j (q;q)_{m-j}} (zq)^j q^{\binom{j+1}{2}} \cdot \frac{(zq^{m+2};q)_\infty}{(zq;q)_\infty} \\
(\text{by Theorem 2.1.3}) &= \frac{(zq^{m+2};q)_\infty}{(zq;q)_\infty} (-zq \cdot q;q)_m \\
&= \frac{(-zq^2;q)_m}{(zq;q)_{m+1}}
\end{aligned}$$

This complete our proof ☐

4.2 An Analogue of Sylvester's Identity

We now consider Sylvester's series-product identity [22]:

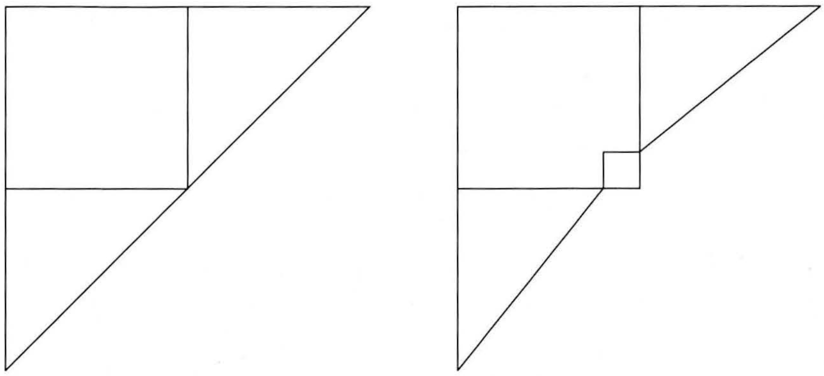
$$(-zq;q)_m = \sum_{j \geq 0} \left(\begin{bmatrix} m+1-j \\ j \end{bmatrix} + \begin{bmatrix} m-j \\ j \end{bmatrix} zq^{3j} \right) (-zq;q)_{j-1} z^j q^{\frac{3j(j-1)}{2}}.$$

We will show that the following expression is an over-q-binomial analogue.

Theorem 4.2.1. *[16] For a positive integer m, we have*

$$\frac{(-zq;q)_m}{(zq;q)_m} = 1 + \sum_{j \geq 1} \left(\overline{\begin{bmatrix} m \\ j \end{bmatrix}} \frac{(-zq;q)_j}{(zq;q)_j} z^j q^{j^2} + \overline{\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}} \frac{(-zq;q)_{j-1}}{(zq;q)_{j-1}} z^j q^{j^2} \right).$$

Proof. Clearly $\dfrac{(-zq;q)_m}{(zq;q)_m}$ generates overpartitions into parts that are $\leq m$. Suppose you have such overpartition with a Durfee square of side j.



In both diagrams the Durfee square is generated by $z^j q^{j^2}$.

If the last part of the Durfee square is not overlined, then the overpartition on the top-right fits inside $(m-j)\times j$ rectangle, hence generated by $\begin{bmatrix} m \\ j \end{bmatrix}$. The bottom overpartition has parts $\leq j$, and therefore generated by $\frac{(-zq;q)_j}{(zq;q)_j}$. If the last part of the Durfee is overlined, then the overpartition on the top-right fit inside $(m-j)\times(j-1)$ rectangle, hence generated by $\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}$. The bottom overpartition has parts $\leq j-1$, and therefore generated by $\frac{(-zq;q)_{j-1}}{(zq;q)_{j-1}}$. Thus the right-hand-side of the stated result follows. □

Corollary 4.2.2. *[16] We have*

$$\frac{(-zq;q)_\infty}{(zq;q)_\infty} = 1 + \sum_{j\geq 1}\left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}\frac{(-zq;q)_{j-1}}{(zq;q)_{j-1}}z^j q^{j^2} + \frac{(-q;q)_j}{(q;q)_j}\frac{(-zq;q)_j}{(zq;q)_j}z^j q^{j^2}\right).$$

Proof. If we let $m\to\infty$ in Theorem 4.2.1, then the left-hand-side gives $\dfrac{(-zq;q)_m}{(zq;q)_m}\to\dfrac{(-zq;q)_\infty}{(zq;q)_\infty}$ while the right-hand-side becomes
$1+\sum_{j\geq 1}\left(\frac{(-q;q)_{j-1}}{(q;q)_{j-1}}\frac{(-zq;q)_{j-1}}{(zq;q)_{j-1}}z^j q^{j^2} + \frac{(-q;q)_j}{(q;q)_j}\frac{(-zq;q)_j}{(zq;q)_j}z^j q^{j^2}\right)$, where we have invoked Proposition 3.1.3. This completes our proof. □

4.3 Overpartitions with rank *r*

In 1944, Freeman Dyson [18] defined the rank of an ordinary partition as follows:

Definition 4.3.1. The rank r of a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$, $\lambda_1\geq\cdots\lambda_k$, is the difference between the largest part and the number of parts, i.e., the rank of λ is given by $r:=\lambda_1-k$, Note that the rank can be any integer. For example, given a partition $\lambda = (7, 7, 5, 3, 3, 3, 1)$, the rank of λ is r = 7 - 7 = 0. In [16], the rank of an overpartition is defined the same way as that of an ordinary partition. Let $\overline{R}_r(N)$ count overpartitions of N with rank r.

Theorem 4.3.1. *[16] For any integer r, we have*

$$\sum_{N=0}^{\infty} \overline{R}_r(N)q^N = 2q^{r+1} + \sum_{m=2}^{\infty} q^{2m+r-1}\left(\overline{\begin{bmatrix} 2m+r-2 \\ m-1 \end{bmatrix}} + \overline{\begin{bmatrix} 2m+r-3 \\ m-1 \end{bmatrix}} + \overline{\begin{bmatrix} 2m+r-3 \\ m-2 \end{bmatrix}} + \overline{\begin{bmatrix} 2m+r-4 \\ m-2 \end{bmatrix}}\right)$$

Proof. Suppose we have an overpartition λ with rank r. We decompose this overpartition into two cases :

1. λ has only 1 part;
2. λ has ≥ 2 parts.

For case 1, the only overpartitions are $r+1$ and $\overline{r+1}$, and they are generated by $2q^{r+1}$.

For case 2, the largest part is $r+m$ and the number of parts is m for some m. If we delete the first row and column in the Ferrers diagram of this overpartition, the deleted part is generated by $q^{r+m+m-1} = q^{2m+r-1}$. Now we want to know what happens to the remaining Ferrers diagram after deletion.

If $\overline{1}$ is a part and $r+m$ is overlined in λ, then the remaining overpartition is generated by $\overline{\begin{bmatrix} r+m-2+m-2 \\ m-2 \end{bmatrix}} = \overline{\begin{bmatrix} r+2m-4 \\ m-2 \end{bmatrix}}$.

If $\overline{1}$ is a part and $r+m$ is not overlined in λ, then the remaining overpartition is generated by $\overline{\begin{bmatrix} r+m-1+m-2 \\ m-2 \end{bmatrix}} = \overline{\begin{bmatrix} r+2m-3 \\ m-2 \end{bmatrix}}$.

If $\overline{1}$ is not a part and $r+m$ is overlined in λ, then the remaining overpartition is generated by $\overline{\begin{bmatrix} r+m-2+m-1 \\ m-1 \end{bmatrix}} = \overline{\begin{bmatrix} r+2m-3 \\ m-1 \end{bmatrix}}$.

If $\overline{1}$ is not a part and $r+m$ is not overlined in λ, then the remaining overpartition is generated by $\overline{\begin{bmatrix} r+m-1+m-1 \\ m-1 \end{bmatrix}} = \overline{\begin{bmatrix} r+2m-2 \\ m-1 \end{bmatrix}}$.

Hence case 2 is generated by

$$q^{2m+r-1}\left(\overline{\begin{bmatrix} 2m+r-2 \\ m-1 \end{bmatrix}} + \overline{\begin{bmatrix} 2m+r-3 \\ m-1 \end{bmatrix}} + \overline{\begin{bmatrix} 2m+r-3 \\ m-2 \end{bmatrix}} + \overline{\begin{bmatrix} 2m+r-4 \\ m-2 \end{bmatrix}}\right).$$

Hence an overpartition of rank r is generated by

$$2q^{r+1} + \sum_{m=2}^{\infty} q^{2m+r-1}\left(\overline{\begin{bmatrix} 2m+r-2 \\ m-1 \end{bmatrix}} + \overline{\begin{bmatrix} 2m+r-3 \\ m-1 \end{bmatrix}} + \overline{\begin{bmatrix} 2m+r-3 \\ m-2 \end{bmatrix}} + \overline{\begin{bmatrix} 2m+r-4 \\ m-2 \end{bmatrix}}\right). \quad \square$$

Chapter 5

Further Applications of Over-q binomial Coefficients

5.1 Introduction

Let $p(n,t)$ count the partitions of n where the difference between the largest and smallest parts is a fixed integer t. It has been shown by Andrews, Beck, and Robbins [9] that

$$\sum_{n\geq 1} p(n,t)q^n = \frac{q^t(1-q)}{(1-q^t)(1-q^{t-1})} + \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q;q)_t} + \frac{q^t}{(1-q^{t-1})(q;q)_t}. \tag{5.1}$$

It was later proved by Breur and Kronholm [12] that if we let $p_t(n)$ count partitions of n where the difference between the largest and smallest parts is at most t, then for $t \geq 1$

$$\sum_{n\geq 1} p_t(n)q^n = \frac{1}{1-q^t}\left(\frac{1}{(q;q)_t} - 1\right). \tag{5.2}$$

5.2 Overpartition analogue of the Breur-Kronholm results

Let $\overline{p}_t(n)$ count the overpartitions of n when the difference between the largest and smallest parts is at most t. Furthermore, we let $g_t(n)$ count the overpartitions of n counted by $\overline{p}_t(n)$ with the extra condition that: if the difference between the largest and smallest part is exactly t, then the largest part cannot be overlined.

Note. If the difference between the largest and smallest parts of an overpartition of n is zero, then all parts are equal and are divisors of n. Hence $\overline{p}_0(n) = 2\tau(n)$ and $g_0(n) = \tau(n)$.

We define

$$\overline{P}_t(q) := \sum_{n\geq 1} \overline{p}_t(n)q^n \qquad \text{and} \qquad G_t(q) := \sum_{n\geq 1} g_t(n)q^n.$$

Now we prove the overpartition q-analogue of (5.2).

Theorem 5.2.1. *[14] For $t \geq 1$, we have*

$$G_t(q) = \frac{1}{1-q^t}\left(\frac{(-q;q)_t}{(q;q)_t} - 1\right).$$

Combinatorial proof of Theorem 5.2.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ be an overpartition of n with exactly r parts, $\lambda_r = m \geq 1$, and $\lambda_1 \leq m+t$ so that $\lambda_1 - \lambda_r \leq m+t-m = t$ (i.e The difference between the largest and smallest parts is at most t). If we subtract m from each part of λ, we get the overpartition $\mu = (\lambda_1 - m, \lambda_2 - m, \ldots, \lambda_{r-1} - m)$ of $n - rm$ with at most $r-1$ parts and the largest part at most t. The overpartition μ is generated by $\overline{\begin{bmatrix} t+r-1 \\ t \end{bmatrix}}$. Hence the overpartition λ of $n - rm + rm = n$ is generated by $2q^{rm}\overline{\begin{bmatrix} t+r-1 \\ t \end{bmatrix}}$. Thus we obtain

$$\overline{P}_t(q) = 2\sum_{r\geq 1}\sum_{m\geq 1} q^{rm}\overline{\begin{bmatrix} t+r-1 \\ t \end{bmatrix}} = 2\sum_{r\geq 1}\frac{q^r}{1-q^r}\overline{\begin{bmatrix} t+r-1 \\ t \end{bmatrix}}.$$

Now note that overpartitions where the difference between the largest and smallest parts is at most t can be divided into three disjoint cases:

1. The largest part is at most t;

2. The largest part is greater than t, the difference between largest and smallest parts is exactly t, and the last occurrence of the smallest part is overlined;

3. The largest part is greater than t and the difference between largest and smallest parts is at most t such that if the difference between the largest and smallest parts is exactly t, the largest and smallest parts are not overlined.

Let us examine these cases separately

Case 1:
Overpartitions with the largest part at most t are generated by $\frac{(-q;q)_t}{(q;q)_t}$, including the empty overpartition of 0. Thus the effective generating function for this case is

$$\frac{(-q;q)_t}{(q;q)_t} - 1.$$

Case 2:
Suppose we have an overpartition $\overline{p} = (p_1, p_2, \ldots, p_r)$ with $p_1 \geq p_2 \geq \cdots \geq p_r$, $p_1 = m + t$, $p_r = m \geq 1$, and the last occurrence of the smallest part is overlined.

This overpartition comprises of the following:

- The smallest part m which can either be overlined or non-overlined generated by $2q^m$ and largest part $m+t$ which is generated by q^{m+t} since it can not be overlined.

- The overlined parts whose size is between $m+1$ and $m+t-1$ generated by
$$(1+q^{m+1})(1+q^{m+2})\cdots(1+q^{m+t-1}).$$

- The non-overlined parts whose size is between m and $m+t$ generated by
$$\frac{1}{1-q^m}\cdot\frac{1}{1-q^{m+1}}\cdots\frac{1}{1-q^{m+t}}$$

Then $\overline{p}$ is generated by

$$\frac{2q^m}{1-q^m}\cdot\frac{1+q^{m+1}}{1-q^{m+1}}\cdot\;\cdots\;\cdot\frac{1+q^{m+t-1}}{1-q^{m+t-1}}\cdot\frac{q^{m+t}}{1-q^{m+t}}.$$

Hence this case is generated by

$$2\sum_{m\geq 1}\frac{q^m}{1-q^m}\cdot\frac{1+q^{m+1}}{1-q^{m+1}}\cdot\;\cdots\;\cdot\frac{1+q^{m+t-1}}{1-q^{m+t-1}}\cdot\frac{q^{m+t}}{1-q^{m+t}}.$$

On the other hand, $\overline{P}_t(q) - \overline{P}_{t-1}(q)$ generate overpartitions where the difference between largest and smallest parts is exactly t. Hence Case 2. is also generated by

$$\frac{\overline{P}_t(q) - \overline{P}_{t-1}(q)}{2}.$$

Thus

$$\frac{\overline{P}_t(q) - \overline{P}_{t-1}(q)}{2} = 2\sum_{m\geq 1} \frac{q^m}{1-q^m} \cdot \frac{1+q^{m+1}}{1-q^{m+1}} \cdot \dots \cdot \frac{1+q^{m+t-1}}{1-q^{m+t-1}} \cdot \frac{q^{m+t}}{1-q^{m+t}}.$$

<u>Case 3</u>:
Let $\overline{p} = (p_1, p_2, \dots, p_r)$ with $p_1 \geq p_2 \geq \dots \geq p_r$, $p_1 = m+t$, $m \geq 1$ be an overpartition of n that describes Case 3 . Then $\overline{q} = (p_1 - m, p_2 - m, \dots, p_r - m)$ is an overpartition of $n - rm$ with at most r parts and the largest part is exactly t (this is an overpartition fitting inside $t \times r$ box where the width is exactly t and the height is at most r). Recall that

$$\overline{\begin{bmatrix} t+r \\ t \end{bmatrix}} \quad \text{and} \quad \overline{\begin{bmatrix} t+r-1 \\ t-1 \end{bmatrix}}$$

respectively generates overpartitions fitting inside $t \times r$ and $t-1 \times r$ boxes. Hence $\overline{q}$ is generated by

$$\overline{\begin{bmatrix} t+r \\ t \end{bmatrix}} - \overline{\begin{bmatrix} t+r-1 \\ t-1 \end{bmatrix}}$$

and $\overline{p}$ is an overpartition of $n - rm + rm$ and so generated by

$$q^{rm}\left(\overline{\begin{bmatrix} t+r \\ t \end{bmatrix}} - \overline{\begin{bmatrix} t+r-1 \\ t-1 \end{bmatrix}}\right).$$

Thus the generating function for Case 3 (for $|q| < 1$) is

$$\begin{aligned}
\sum_{r\geq 1}\sum_{m\geq 1} q^{rm}\left(\overline{\begin{bmatrix} t+r \\ t \end{bmatrix}} - \overline{\begin{bmatrix} t+r-1 \\ t-1 \end{bmatrix}}\right) &= \sum_{r\geq 1} \frac{q^r}{1-q^r}\left(\overline{\begin{bmatrix} t+r \\ t \end{bmatrix}} - \overline{\begin{bmatrix} t+r-1 \\ t-1 \end{bmatrix}}\right) \\
&= \sum_{r\geq 1} \frac{q^r}{1-q^r} q^t \left(\overline{\begin{bmatrix} t+r-1 \\ t \end{bmatrix}} + \overline{\begin{bmatrix} t+r-2 \\ t-1 \end{bmatrix}}\right) \\
&= q^t \sum_{r\geq 1} \frac{q^r}{1-q^r} \overline{\begin{bmatrix} t+r-1 \\ t \end{bmatrix}} \\
&\quad + q^t \sum_{r\geq 1} \frac{q^r}{1-q^r} \overline{\begin{bmatrix} t+r-2 \\ t-1 \end{bmatrix}} \\
&= q^t \left(\frac{\overline{P}_t(q)}{2} + \frac{\overline{P}_{t-1}(q)}{2}\right) \\
&= q^t \frac{\overline{P}_t(q) + \overline{P}_{t-1}(q)}{2}.
\end{aligned}$$

Taking all the three cases above, therefore have

$$\overline{P}_t(q) = \left(\frac{(-q;q)_t}{(q;q)_t} - 1\right) + \frac{\overline{P}_t(q) - \overline{P}_{t-1}(q)}{2} + q^t \frac{\overline{P}_t(q) + \overline{P}_{t-1}(q)}{2}.$$

Rearranging this gives

$$\frac{\overline{P}_t(q) + \overline{P}_{t-1}(q)}{2} = \frac{1}{1-q^t}\left(\frac{(-q;q)_t}{(q;q)_t} - 1\right).$$

Lastly we note that the overpartition where the difference between the largest and smallest parts is exactly t, and the largest part is overlined is generated by

$$\sum_{m\geq 1} \frac{2q^m}{1-q^m} \cdot \frac{1+q^{m+1}}{1-q^{m+1}} \cdot \cdots \cdot \frac{1+q^{m+t-1}}{1-q^{m+t-1}} \cdot \frac{q^{m+t}}{1-q^{m+t}} = \frac{\overline{P}_t(q) - \overline{P}_{t-1}(q)}{2}.$$

Hence

$$G_t(q) = \frac{\overline{P}_t(q) - \overline{P}_{t-1}(q)}{2} + \overline{P}_{t-1}(q) = \frac{\overline{P}_t(q) + \overline{P}_{t-1}(q)}{2}.$$

This complete our proof. □

Theorem 5.2.2. *[14] For $t \geq 0$, we have*

$$\overline{P}_t(q) = 2(-1)^t \left(\sum_{n\geq 1} \tau(n) q^n + \sum_{n=1}^{t} (-1)^n G_n(q)\right).$$

Proof. Since $\overline{p}_0(n) = 2\tau(n)$, we have

$$\overline{P}_0(q) = 2\sum_{n\geq 1} \tau(n) q^n.$$

Now consider an overpartition where the smallest is $m \geq 1$ and the largest is at most $m+t$. Similarly to the previous proof this overpartition comprises of the following:

- The smallest part m which can either be overlined or non-overlined generated by $2q^m$.
- The overlined parts whose size is between $m+1$ and $m+t$ generated by
$$(1+q^{m+1})(1+q^{m+2})\cdots(1+q^{m+t}).$$

- The non-overlined parts whose size is between m and $m+t$ generated by

$$\frac{1}{1-q^m}\cdot\frac{1}{1-q^{m+1}}\cdots\frac{1}{1-q^{m+t}}$$

Hence this overpartition is generated by

$$\frac{2q^m}{1-q^m}\cdot\frac{1+q^{m+1}}{1-q^{m+1}}\cdots\frac{1+q^{m+t}}{1-q^{m+t}}.$$

Thus for $t \geq 0$, we have

$$\overline{P}_t(q) = \sum_{m\geq 1}\frac{2q^m}{1-q^m}\cdot\frac{1+q^{m+1}}{1-q^{m+1}}\cdots\frac{1+q^{m+t}}{1-q^{m+t}}.$$

We also have, for $n \geq 1$

$$\begin{aligned}
\overline{P}_n(q)+\overline{P}_{n-1}(q) &= \sum_{m\geq 1}\frac{2q^m}{1-q^m}\cdot\frac{1+q^{m+1}}{1-q^{m+1}}\cdots\frac{1+q^{m+n-1}}{1-q^{m+n-1}}\left(\frac{1+q^{m+n}}{1-q^{m+n}}+1\right)\\
&= \sum_{m\geq 1}\frac{2q^m}{1-q^m}\cdot\frac{1+q^{m+1}}{1-q^{m+1}}\cdots\frac{1+q^{m+n-1}}{1-q^{m+n-1}}\frac{2}{1-q^{m+n}}\\
&= 2G_n(q)
\end{aligned}$$

Hence we have

$$\overline{P}_n(q) = 2G_n(q) - \overline{P}_{n-1}(q),$$

iterating this equation we have

$$\begin{aligned}
\overline{P}_n(q) &= 2G_n(q) - \left(2G_{n-1} - \overline{P}_{n-2}(q)\right)\\
&= 2G_n(q) - 2G_{n-1}(q) + \overline{P}_{n-2}(q)\\
&= 2G_n(q) - 2G_{n-1}(q) + 2G_{n-2}(q) - \cdots\\
&\vdots\\
&= (-1)^0 2G_n(q) + (-1)^1 2G_{n-1}(q) + \cdots + (-1)^{n-1}\overline{P}_0(q)\\
&= \sum_{k=1}^{n}(-1)^{n-k}2G_k(q) + (-1)^{n-1}\overline{P}_0(q)\\
&= (-1)^n 2\left(\sum_{k=1}^{n}(-1)^k 2G_k(q) + \sum_{k\geq 1}\tau(k)q^k\right)
\end{aligned}$$

$\square$

The following statement is well known in Elementary Number Theory.

Lemma 5.2.3. *$\tau(n)$ is odd if and only if n is a perfect square.*

Corollary 5.2.4. *[14] For any $t \geq 0$ and $n \geq 1$, $\overline{p}_t(n)$ is an even integer. Furthermore, $\overline{p}_t(n)$ is divisible by 4 if and only if n is not a perfect square.*

Proof. From Theorem 5.2.2 , we have

$$\begin{aligned}\overline{P}_t(q) &= 2(-1)^t\left(\sum_{n\geq 1}\tau(n)q^n + \sum_{n=1}^{t}(-1)^n G_n(q)\right)\\ &= 2(-1)^t\sum_{n\geq 1}\tau(n)q^n + 2(-1)^t\sum_{n=1}^{t}(-1)^n G_n(q)\\ &= 2(-1)^t\sum_{n\geq 1}\tau(n)q^n + 2(-1)^t\sum_{n=1}^{t}(-1)^n 2\sum_{m\geq 1}q^m\frac{(-q^{m+1};q)_{t-1}}{(q^m;q)_{t+1}}\\ &= 2(-1)^t\sum_{n\geq 1}\tau(n)q^n + 4(-1)^t\sum_{n=1}^{t}\sum_{m\geq 1}(-1)^n q^m\frac{(-q^{m+1};q)_{t-1}}{(q^m;q)_{t+1}}.\end{aligned}$$

Hence we see that $\overline{p}_t(n)$ is an even integer and

$$\overline{p}_t(n) \equiv 2\tau(n) \pmod 4.$$

By Lemma 5.2.3, it follows that $\overline{p}_t(n)$ is divisible by 4 if and only if n is not a perfect square. □

Combinatorial proof of Corollary 5.2.4. By Theorem 1.2.1, the number $u_k(n)$ of overpartitions of $n > 0$ generated by an ordinary partition with k different parts is a power of 2, say $u_k(n) = 2^k$ for some positive integer k.
Now we consider the parity of $\overline{p}_t(n)$. Let the number of overpartitions counted by both $\overline{p}_t(n)$ and $u_k(n)$ be denoted by $\overline{p}_t(n,k)$. We decompose the set of overpartitions counted by $\overline{p}_t(n)$ into two sets:

Set 1 : where $t = 0$;

Set 2 : where $t \geq 1$.

Set 1 consists of overpartitions of n where all parts are equal in size, i.e., those counted by $\overline{p}_0(n,1)$. The number of the corresponding underlying ordinary partitions λ will be $\tau(n)$. Since the last part of λ can either be overlined or not, the cardinality of Set 1 is $\overline{p}_0(n,1) = 2\tau(n)$.
Set 2 consists of overpartitions of n containing at least two different part sizes, i.e., those counted by $\overline{p}_t(n,k)$, $t \geq 1, k \geq 1$. Since "at most t for $t \geq 1$"

implies "for all $t \in \{0, 1, 2, \ldots\} \supset \{0\}$" and $\overline{p}_t(n, 1) > 0$ if and only if $t = 0$, we have

$$\begin{aligned}
\overline{p}_t(n, k) &= \sum_{k \geq 1} \overline{p}_t(n, k) \\
&= \overline{p}_0(n, 1) + \sum_{k \geq 2} \overline{p}_t(n, k) \\
&= 2\tau(n) + \sum_{k \geq 2} 2^k \\
&\equiv 2\tau(n) \pmod 4.
\end{aligned}$$

Hence by Lemma 5.2.3, it follows that $\overline{p}_t(n)$ is divisible by 4 if and only if n is not a perfect square. $\square$

Chapter 6

Conclusion

The main theme of this dissertation is the over q-binomial coefficients, their derivations and applications. We first discussed the ordinary binomial coefficients by looking at the motivation behind them and how they are connected to ordinary partitions. We then looked at the over q-binomial coefficients and connected them to overpartitions.

In Chapter 3, we connected overpartitions to lattice paths where the diagonal step $(1, 1)$ is allowed. Various applications of over q-coefficients are given including the beautiful over q-analogue the Chu-Vandermonde identity. Combinatorial proofs were given for some results. Chapters 4 and 5 were devoted to the applications of over q-coefficients. We looked at the generating function for overpartitions with restricted number of parts and derived the generating function for overpartitions with bounded differences between largest and smallest parts. The over q-coefficients proved to be an interesting tool for the derivation of these generating functions.

It is clear that over q-coefficients are very special tools in the study of overpartitions. It will therefore be interesting to study further properties such as unimodality or the possibility of over q-multinomial coefficients, e.t.c. As seen, most of our results are proved combinatorially, it will therefore also be interesting to find analytic proofs.

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